

CONCEPTUAL PHYSICS



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Conceptual Physics

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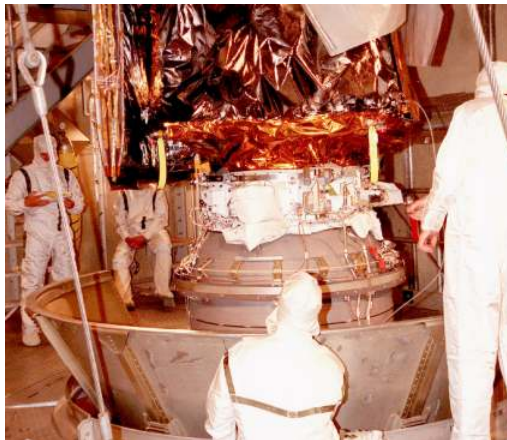
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CHAPTER OVERVIEW

2.1: Introduction and Review



The Mars Climate Orbiter is prepared for its mission. The laws of physics are the same everywhere, even on Mars, so the probe could be designed based on the laws of physics as discovered on earth. There is unfortunately another reason why this spacecraft is relevant to the topics of this chapter: it was destroyed attempting to enter Mars' atmosphere because engineers at Lockheed Martin forgot to convert data on engine thrusts from pounds into the metric unit of force (newtons) before giving the information to NASA. Conversions are important!

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2.1.1: Introduction and Review

If you drop your shoe and a coin side by side, they hit the ground at the same time. Why doesn't the shoe get there first, since gravity is pulling harder on it? How does the lens of your eye work, and why do your eye's muscles need to squash its lens into different shapes in order to focus on objects nearby or far away? These are the kinds of questions that physics tries to answer about the behavior of light and matter, the two things that the universe is made of.

The Scientific Method

Until very recently in history, no progress was made in answering questions like these. Worse than that, the *wrong* answers written by thinkers like the ancient Greek physicist Aristotle were accepted without question for thousands of years. Why is it that scientific knowledge has progressed more since the Renaissance than it had in all the preceding millennia since the beginning of recorded history? Undoubtedly the industrial revolution is part of the answer. Building its centerpiece, the steam engine, required improved techniques for precise construction and measurement. (Early on, it was considered a major advance when English machine shops learned to build pistons and cylinders that fit together with a gap narrower than the thickness of a penny.) But even before the industrial revolution, the pace of discovery had picked up, mainly because of the introduction of the modern scientific method. Although it evolved over time, most scientists today would agree on something like the following list of the basic principles of the scientific method:

(1) *Science is a cycle of theory and experiment.* Scientific theories are created to explain the results of experiments that were created under certain conditions. A successful theory will also make new predictions about new experiments under new conditions. Eventually, though, it always seems to happen that a new experiment comes along, showing that under certain conditions the theory is not a good approximation or is not valid at all. The ball is then back in the theorists' court. If an experiment disagrees with the current theory, the theory has to be changed, not the experiment.

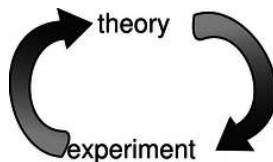


Figure 2.1.1.1: Science is a cycle of theory and experiment



Figure 2.1.1.2: A satirical drawing of an alchemist's laboratory. H. Cock, after a drawing by Peter Brueghel the Elder (16th century).

(2) *Theories should both predict and explain.* The requirement of predictive power means that a theory is only meaningful if it predicts something that can be checked against experimental measurements that the theorist did not already have at hand. That is, a theory should be testable. Explanatory value means that many phenomena should be accounted for with few basic principles. If you answer every “why” question with “because that's the way it is,” then your theory has no explanatory value. Collecting lots of data without being able to find any basic underlying principles is not science.

(3) *Experiments should be reproducible.* An experiment should be treated with suspicion if it only works for one person, or only in one part of the world. Anyone with the necessary skills and equipment should be able to get the same results from the same experiment. This implies that science transcends national and ethnic boundaries; you can be sure that nobody is doing actual science who claims that their work is “Aryan, not Jewish,” “Marxist, not bourgeois,” or “Christian, not atheistic.” An experiment cannot be reproduced if it is secret, so science is necessarily a public enterprise.

As an example of the cycle of theory and experiment, a vital step toward modern chemistry was the experimental observation that the chemical elements could not be transformed into each other, e.g., lead could not be turned into gold. This led to the theory that chemical reactions consisted of rearrangements of the elements in different combinations, without any change in the identities of the elements themselves. The theory worked for hundreds of years, and was confirmed experimentally over a wide range of pressures and temperatures and with many combinations of elements. Only in the twentieth century did we learn that one element could be transformed into one another under the conditions of extremely high pressure and temperature existing in a nuclear bomb or inside a star. That observation didn't completely invalidate the original theory of the immutability of the elements, but it showed that it was only an approximation, valid at ordinary temperatures and pressures.

self-check:

A psychic conducts seances in which the spirits of the dead speak to the participants. He says he has special psychic powers not possessed by other people, which allow him to “channel” the communications with the spirits. What part of the scientific method is being violated here?

The scientific method as described here is an idealization, and should not be understood as a set procedure for doing science. Scientists have as many weaknesses and character flaws as any other group, and it is very common for scientists to try to discredit other people's experiments when the results run contrary to their own favored point of view. Successful science also has more to do with luck, intuition, and creativity than most people realize, and the restrictions of the scientific method do not stifle individuality and self-expression any more than the fugue and sonata forms stifled Bach and Haydn. There is a recent tendency among social scientists to go even further and to deny that the scientific method even exists, claiming that science is no more than an arbitrary social system that determines what ideas to accept based on an in-group's criteria. I think that's going too far. If science is an arbitrary social ritual, it would seem difficult to explain its effectiveness in building such useful items as airplanes, CD players, and sewers. If alchemy and astrology were no less scientific in their methods than chemistry and astronomy, what was it that kept them from producing anything useful?

Discussion Questions

withintro{ Consider whether or not the scientific method is being applied in the following examples. If the scientific method is not being applied, are the people whose actions are being described performing a useful human activity, albeit an unscientific one? }

◇ Acupuncture is a traditional medical technique of Asian origin in which small needles are inserted in the patient's body to relieve pain. Many doctors trained in the west consider acupuncture unworthy of experimental study because if it had therapeutic effects, such effects could not be explained by their theories of the nervous system. Who is being more scientific, the western or eastern practitioners?

◇ Goethe, a German poet, is less well known for his theory of color. He published a book on the subject, in which he argued that scientific apparatus for measuring and quantifying color, such as prisms, lenses and colored filters, could not give us full insight into the ultimate meaning of color, for instance the cold feeling evoked by blue and green or the heroic sentiments inspired by red. Was his work scientific?

◇ A child asks why things fall down, and an adult answers “because of gravity.” The ancient Greek philosopher Aristotle explained that rocks fell because it was their nature to seek out their natural place, in contact with the earth. Are these explanations scientific?

◇ Buddhism is partly a psychological explanation of human suffering, and psychology is of course a science. The Buddha could be said to have engaged in a cycle of theory and experiment, since he worked by trial and error, and even late in his life he asked his followers to challenge his ideas. Buddhism could also be considered reproducible, since the Buddha told his followers they could find enlightenment for themselves if they followed a certain course of study and discipline. Is Buddhism a scientific pursuit?

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2.1.2: Scaling and Order-of-Magnitude Estimates

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2.1.3: Footnotes

For as knowledges are now delivered, there is a kind of contract of error between the deliverer and the receiver; for he that delivereth knowledge desireth to deliver it in such a form as may be best believed, and not as may be best examined; and he that receiveth knowledge desireth rather present satisfaction than expectant inquiry. -- *Francis Bacon*

Many students approach a science course with the idea that they can succeed by memorizing the formulas, so that when a problem is assigned on the homework or an exam, they will be able to plug numbers in to the formula and get a numerical result on their calculator. Wrong! That's not what learning science is about! There is a big difference between memorizing formulas and understanding concepts. To start with, different formulas may apply in different situations. One equation might represent a definition, which is always true. Another might be a very specific equation for the speed of an object sliding down an inclined plane, which would not be true if the object was a rock drifting down to the bottom of the ocean. If you don't work to understand physics on a conceptual level, you won't know which formulas can be used when.

Most students taking college science courses for the first time also have very little experience with interpreting the meaning of an equation. Consider the equation $w=A/h$ relating the width of a rectangle to its height and area. A student who has not developed skill at interpretation might view this as yet another equation to memorize and plug in to when needed. A slightly more savvy student might realize that it is simply the familiar formula $A=wh$ in a different form. When asked whether a rectangle would have a greater or smaller width than another with the same area but a smaller height, the unsophisticated student might be at a loss, not having any numbers to plug in on a calculator. The more experienced student would know how to reason about an equation involving division --- if h is smaller, and A stays the same, then w must be bigger. Often, students fail to recognize a sequence of equations as a derivation leading to a final result, so they think all the intermediate steps are equally important formulas that they should memorize.

When learning any subject at all, it is important to become as actively involved as possible, rather than trying to read through all the information quickly without thinking about it. It is a good idea to read and think about the questions posed at the end of each section of these notes as you encounter them, so that you know you have understood what you were reading.

Many students' difficulties in physics boil down mainly to difficulties with math. Suppose you feel confident that you have enough mathematical preparation to succeed in this course, but you are having trouble with a few specific things. In some areas, the brief review given in this chapter may be sufficient, but in other areas it probably will not. Once you identify the areas of math in which you are having problems, get help in those areas. Don't limp along through the whole course with a vague feeling of dread about something like scientific notation. The problem will not go away if you ignore it. The same applies to essential mathematical skills that you are learning in this course for the first time, such as vector addition.

Sometimes students tell me they keep trying to understand a certain topic in the book, and it just doesn't make sense. The worst thing you can possibly do in that situation is to keep on staring at the same page. Every textbook explains certain things badly --- even mine! --- so the best thing to do in this situation is to look at a different book. Instead of college textbooks aimed at the same mathematical level as the course you're taking, you may in some cases find that high school books or books at a lower math level give clearer explanations. The three books listed on the left are, in my opinion, the best introductory physics books available, although they would not be appropriate as the primary textbook for a college-level course for science majors.

Finally, when reviewing for an exam, don't simply read back over the text and your lecture notes. Instead, try to use an active method of reviewing, for instance by discussing some of the discussion questions with another student, or doing homework problems you hadn't done the first time.

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2.1.4: Problems

1. Correct use of a calculator: (a) Calculate $\frac{74658}{53222+97554}$ on a calculator. [Self-check: The most common mistake results in 97555.40.] (answer check available at lightandmatter.com)

(b) Which would be more like the price of a TV, and which would be more like the price of a house, $\$3.5 \times 10^5$ or $\$3.5^5$?

2. Compute the following things. If they don't make sense because of units, say so.

(a) 3 cm + 5 cm

(b) 1.11 m + 22 cm

(c) 120 miles + 2.0 hours

(d) 120 miles / 2.0 hours

3. Your backyard has brick walls on both ends. You measure a distance of 23.4 m from the inside of one wall to the inside of the other. Each wall is 29.4 cm thick. How far is it from the outside of one wall to the outside of the other? Pay attention to significant figures.

4. The speed of light is 3.0×10^8 m/s. Convert this to furlongs per fortnight. A furlong is 220 yards, and a fortnight is 14 days. An inch is 2.54 cm. (answer check available at lightandmatter.com)

5. Express each of the following quantities in micrograms:

(a) 10 mg, (b) 10^4 g, (c) 10 kg, (d) 100×10^3 g, (e) 1000 g. (answer check available at lightandmatter.com)

6. Convert 134 mg to units of kg, writing your answer in scientific notation.

7. In the last century, the average age of the onset of puberty for girls has decreased by several years. Urban folklore has it that this is because of hormones fed to beef cattle, but it is more likely to be because modern girls have more body fat on the average and possibly because of estrogen-mimicking chemicals in the environment from the breakdown of pesticides. A hamburger from a hormone-implemented steer has about 0.2 ng of estrogen (about double the amount of natural beef). A serving of peas contains about 300 ng of estrogen. An adult woman produces about 0.5 mg of estrogen per day (note the different unit!). (a) How many hamburgers would a girl have to eat in one day to consume as much estrogen as an adult woman's daily production? (b) How many servings of peas? (answer check available at lightandmatter.com)

8. The usual definition of the mean (average) of two numbers a and b is $(a + b)/2$. This is called the arithmetic mean. The geometric mean, however, is defined as $(ab)^{1/2}$ (i.e., the square root of ab). For the sake of definiteness, let's say both numbers have units of mass. (a) Compute the arithmetic mean of two numbers that have units of grams. Then convert the numbers to units of kilograms and recompute their mean. Is the answer consistent? (b) Do the same for the geometric mean. (c) If a and b both have units of grams, what should we call the units of ab ? Does your answer make sense when you take the square root? (d) Suppose someone proposes to you a third kind of mean, called the superduper mean, defined as $(ab)^{1/3}$. Is this reasonable?

9. In an article on the SARS epidemic, the May 7, 2003 New York Times discusses conflicting estimates of the disease's incubation period (the average time that elapses from infection to the first symptoms). "The study estimated it to be 6.4 days. But other statistical calculations ... showed that the incubation period could be as long as 14.22 days." What's wrong here?

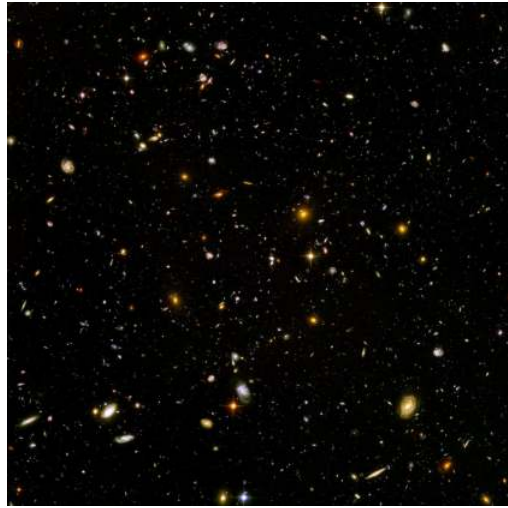
10. The distance to the horizon is given by the expression $\sqrt{2rh}$, where r is the radius of the Earth, and h is the observer's height above the Earth's surface. (This can be proved using the Pythagorean theorem.) Show that the units of this expression make sense. (See example 2 on p. 26 for an example of how to do this.) Don't try to prove the result, just check its units.

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CHAPTER OVERVIEW

2.2: Conservation of Mass



It took just a moment for that head to fall, but a hundred years might not produce another like it. -- *Joseph-Louis Lagrange, referring to the execution of Lavoisier on May 8, 1794*

Topic hierarchy

- [2.2.1: Mass](#)
- [2.2.2: Equivalence of Gravitational and Inertial Mass](#)
- [2.2.3: 1.3 Galilean Relativity](#)
- [2.2.4: A Preview of Some Modern Physics](#)
- [2.2.5: Footnotes](#)
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2.2.1: Mass

Change is impossible, claimed the ancient Greek philosopher Parmenides. His work was nonscientific, since he didn't state his ideas in a form that would allow them to be tested experimentally, but modern science nevertheless has a strong Parmenidean flavor. His main argument that change is an illusion was that something can't be turned into nothing, and likewise if you have nothing, you can't turn it into something. To make this into a scientific theory, we have to decide on a way to measure what "something" is, and we can then check by measurements whether the total amount of "something" in the universe really stays constant. How much "something" is there in a rock? Does a sunbeam count as "something?" Does heat count? Motion? Thoughts and feelings.

If you look at the table of contents of this book, you'll see that the first four chapters have the word "conservation" in them. In physics, a conservation law is a statement that the total amount of a certain physical quantity always stays the same. This chapter is about conservation of mass. The metric system is designed around a unit of distance, the meter, a unit of mass, the kilogram, and a time unit, the second. Numerical measurement of distance and time probably date back almost as far into prehistory as counting money, but mass is a more modern concept. Until scientists figured out that mass was conserved, it wasn't obvious that there could be a single, consistent way of measuring an amount of matter, hence jiggers of whiskey and cords of wood. You may wonder why conservation of mass wasn't discovered until relatively modern times, but it wasn't obvious, for example, that gases had mass, and that the apparent loss of mass when wood was burned was exactly matched by the mass of the escaping gases.

Once scientists were on the track of the conservation of mass concept, they began looking for a way to define mass in terms of a definite measuring procedure. If they tried such a procedure, and the result was that it led to nonconservation of mass, then they would throw it out and try a different procedure. For instance, we might be tempted to define mass using kitchen measuring cups, i.e., as a measure of volume. Mass would then be perfectly conserved for a process like mixing marbles with peanut butter, but there would be processes like freezing water that led to a net increase in mass, and others like soaking up water with a sponge that caused a decrease. If, with the benefit of hindsight, it seems like the measuring cup definition was just plain silly, then here's a more subtle example of a wrong definition of mass. Suppose we define it using a bathroom scale, or a more precise device such as a postal scale that works on the same principle of using gravity to compress or twist a spring. The trouble is that gravity is not equally strong all over the surface of the earth, so for instance there would be nonconservation of mass when you brought an object up to the top of a mountain, where gravity is a little weaker.

Although some of the obvious possibilities have problems, there do turn out to be at least two approaches to defining mass that lead to its being a conserved quantity, so we consider these definitions to be "right" in the pragmatic sense that what's correct is what's useful.

One definition that works is to use balances, but compensate for the local strength of gravity. This is the method that is used by scientists who actually specialize in ultraprecise measurements. A standard kilogram, in the form of a platinum-iridium cylinder, is kept in a special shrine in Paris. Copies are made that balance against the standard kilogram in Parisian gravity, and they are then transported to laboratories in other parts of the world, where they are compared with other masses in the local gravity. The quantity defined in this way is called *gravitational mass*.

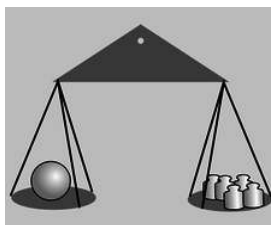


Figure b: A measurement of gravitational mass: the sphere has a gravitational mass of five kilograms.

A second and completely different approach is to measure how hard it is to change an object's state of motion. This tells us its *inertial mass*. For example, I'd be more willing to stand in the way of an oncoming poodle than in the path of a freight train, because my body will have a harder time convincing the freight train to stop. This is a dictionary-style conceptual definition, but in physics we need to back up a conceptual definition with an operational definition, which is one that spells out the operations required in order to measure the quantity being defined. We can operationalize our definition of inertial mass by throwing a standard kilogram at an object at a speed of 1 m/s (one meter per second) and measuring the recoiling object's velocity. Suppose we want to measure the mass of a particular block of cement. We put the block in a toy wagon on the sidewalk, and throw a standard

kilogram at it. Suppose the standard kilogram hits the wagon, and then drops straight down to the sidewalk, having lost all its velocity, and the wagon and the block inside recoil at a velocity of 0.23 m/s . We then repeat the experiment with the block replaced by various numbers of standard kilograms, and find that we can reproduce the recoil velocity of 0.23 m/s with four standard kilograms in the wagon. We have determined the mass of the block to be four kilograms.¹ Although this definition of inertial mass has an appealing conceptual simplicity, it is obviously not very practical, at least in this crude form. Nevertheless, this method of collision is very much like the methods used for measuring the masses of subatomic particles, which, after all, can't be put on little postal scales!

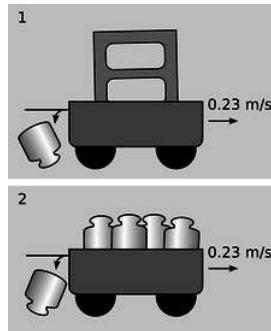


Figure c: A measurement of inertial mass: the wagon recoils with the same velocity in experiments 1 and 2, establishing that the inertial mass of the cement block is four kilograms.

Astronauts spending long periods of time in space need to monitor their loss of bone and muscle mass, and here as well, it's impossible to measure gravitational mass. Since they don't want to have standard kilograms thrown at them, they use a slightly different technique (figures d and e). They strap themselves to a chair which is attached to a large spring, and measure the time it takes for one cycle of vibration.

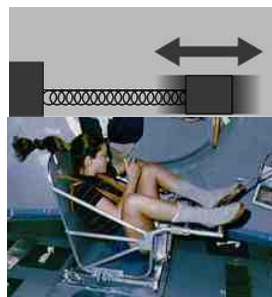


Figure d (left): The time for one cycle of vibration is related to the object's inertial mass.

Figure e (right): Astronaut Tamara Jernigan measures her inertial mass aboard the Space Shuttle.

1.1.1 Problem-solving techniques

How do we use a conservation law, such as conservation of mass, to solve problems? There are two basic techniques.

As an analogy, consider conservation of money, which makes it illegal for you to create dollar bills using your own inkjet printer. (Most people don't intentionally destroy their dollar bills, either!) Suppose the police notice that a particular store doesn't seem to have any customers, but the owner wears lots of gold jewelry and drives a BMW. They suspect that the store is a front for some kind of crime, perhaps counterfeiting. With intensive surveillance, there are two basic approaches they could use in their investigation. One method would be to have undercover agents try to find out how much money goes in the door, and how much money comes back out at the end of the day, perhaps by arranging through some trick to get access to the owner's briefcase in the morning and evening. If the amount of money that comes out every day is greater than the amount that went in, and if they're convinced there is no safe on the premises holding a large reservoir of money, then the owner must be counterfeiting. This inflow-equals-outflow technique is useful if we are sure that there is a region of space within which there is no supply of mass that is being built up or depleted.

Example 1: A stream of water

If you watch water flowing out of the end of a hose, you'll see that the stream of water is fatter near the mouth of the hose, and skinnier lower down. This is because the water speeds up as it falls. If the cross-sectional area of the stream was equal all along its length, then the rate of flow (kilograms per second) through a lower cross-section would be greater than the rate of flow through a cross-section higher up. Since the flow is steady, the amount of water between the two cross-sections stays constant. Conservation of mass therefore requires that the cross-sectional area of the stream shrink in inverse proportion to the increasing speed of the falling water.



f / Example 1.

self-check:

Suppose the you point the hose straight up, so that the water is rising rather than falling. What happens as the velocity gets smaller? What happens when the velocity becomes zero?

(answer in the back of the PDF version of the book)

How can we apply a conservation law, such as conservation of mass, in a situation where mass might be stored up somewhere? To use a crime analogy again, a prison could contain a certain number of prisoners, who are not allowed to flow in or out at will. In physics, this is known as a *closed system*. A guard might notice that a certain prisoner's cell is empty, but that doesn't mean he's escaped. He could be sick in the infirmary, or hard at work in the shop earning cigarette money. What prisons actually do is to count all their prisoners every day, and make sure today's total is the same as yesterday's. One way of stating a conservation law is that for a closed system, the total amount of stuff (mass, in this chapter) stays constant.

Example 2: Lavoisier and chemical reactions in a closed system

The French chemist Antoine-Laurent Lavoisier is considered the inventor of the concept of conservation of mass. Before Lavoisier, chemists had never systematically weighed their chemicals to quantify the amount of each substance that was undergoing reactions. They also didn't completely understand that gases were just another state of matter, and hadn't tried performing reactions in sealed chambers to determine whether gases were being consumed from or released into the air. For this they had at least one practical excuse, which is that if you perform a gas-releasing reaction in a sealed chamber with no room for expansion, you get an explosion! Lavoisier invented a balance that was capable of measuring milligram masses, and figured out how to do reactions in an upside-down bowl in a basin of water, so that the gases could expand by pushing out some of the water. In a crucial experiment, Lavoisier heated a red mercury compound, which we would now describe as mercury oxide (HgO), in such a sealed chamber. A gas was produced (Lavoisier later named it "oxygen"), driving out some of the water, and the red compound was transformed into silvery liquid mercury metal. The crucial point was that the total mass of the entire apparatus was exactly the same before and after the reaction. Based on many observations of this type, Lavoisier proposed a general law of nature, that mass is always conserved. (In earlier experiments, in which closed systems were not used, chemists had become convinced that there was a mysterious substance, phlogiston, involved in combustion and oxidation reactions, and that phlogiston's mass could be positive, negative, or zero depending on the situation!)



a / Portrait of Monsieur Lavoisier and His Wife, by Jacques-Louis David, 1788. Lavoisier invented the concept of conservation of mass. The husband is depicted with his scientific apparatus, while in the background on the left is the portfolio belonging to Madame Lavoisier, who is thought to have been a student of David's.

1.1.2 Delta notation

A convenient notation used throughout physics is Δ , the uppercase Greek letter delta, which indicates "change in" or "after minus before." For example, if b represents how much money you have in the bank, then a deposit of \$100 could be represented as $\Delta b = \$100$. That is, the change in your balance was \$100, or the balance after the transaction minus the balance before the transaction equals \$100. A withdrawal would be indicated by $\Delta b < 0$. We represent "before" and "after" using the subscripts i (initial) and f (final), e.g., $\Delta b = b_f - b_i$. Often the delta notation allows more precision than English words. For instance, "time" can be used to mean a point in time ("now's the time"), t , or it could mean a period of time ("the whole time, he had spit on his chin"), Δt .

This notation is particularly convenient for discussing conserved quantities. The law of conservation of mass can be stated simply as $\Delta m = 0$, where m is the total mass of any closed system.

self-check:

If x represents the location of an object moving in one dimension, then how would positive and negative signs of Δx be interpreted?

(answer in the back of the PDF version of the book)

Discussion Questions

If an object had a straight-line $x - t$ graph with $\Delta x = 0$ and $\Delta t \neq 0$, what would be true about its velocity? What would this look like on a graph? What about $\Delta t = 0$ and $\Delta x \neq 0$?

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2.2.2: Equivalence of Gravitational and Inertial Mass

We find experimentally that both gravitational and inertial mass are conserved to a high degree of precision for a great number of processes, including chemical reactions, melting, boiling, soaking up water with a sponge, and rotting of meat and vegetables. Now it's logically possible that both gravitational and inertial mass are conserved, but that there is no particular relationship between them, in which case we would say that they are separately conserved. On the other hand, the two conservation laws may be redundant, like having one law against murder and another law against killing people!

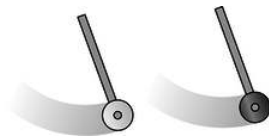


Figure a: The two pendulum bobs are constructed with equal gravitational masses. If their inertial masses are also equal, then each pendulum should take exactly the same amount of time per swing.

Here's an experiment that gets at the issue: stand up now and drop a coin and one of your shoes side by side. I used a 400-gram shoe and a 2-gram penny, and they hit the floor at the same time as far as I could tell by eye. This is an interesting result, but a physicist and an ordinary person will find it interesting for different reasons.

The layperson is surprised, since it would seem logical that heavier objects would always fall faster than light ones. However, it's fairly easy to prove that if air friction is negligible, any two objects made of the same substance must have identical motion when they fall. For instance, a 2-kg copper mass must exhibit the same falling motion as a 1-kg copper mass, because nothing would be changed by physically joining together two 1-kg copper masses to make a single 2-kg copper mass. Suppose, for example, that they are joined with a dab of glue; the glue isn't under any strain, because the two masses are doing the same thing side by side. Since the glue isn't really doing anything, it makes no difference whether the masses fall separately or side by side.²

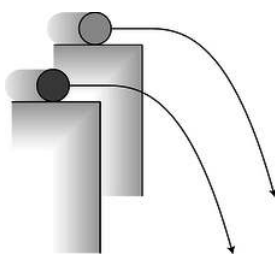


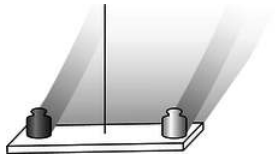
Figure b: If the cylinders have slightly unequal ratios of inertial to gravitational mass, their trajectories will be a little different.

What a physicist finds remarkable about the shoe-and-penny experiment is that it came out the way it did even though the shoe and the penny are made of *different* substances. There is absolutely no theoretical reason why this should be true. We could say that it happens because the greater gravitational mass of the shoe is exactly counteracted by its greater inertial mass, which makes it harder for gravity to get it moving, but that just leaves us wondering why inertial mass and gravitational mass are always in proportion to each other. It's possible that they are only approximately equivalent. Most of the mass of ordinary matter comes from neutrons and protons, and we could imagine, for instance, that neutrons and protons do not have exactly the same ratio of gravitational to inertial mass. This would show up as a different ratio of gravitational to inertial mass for substances containing different proportions of neutrons and protons.

Galileo did the first numerical experiments on this issue in the seventeenth century by rolling balls down inclined planes, although he didn't think about his results in these terms. A fairly easy way to improve on Galileo's accuracy is to use pendulums with bobs made of different materials. Suppose, for example, that we construct an aluminum bob and a brass bob, and use a double-pan balance to verify to good precision that their gravitational masses are equal. If we then measure the time required for each pendulum to perform a hundred cycles, we can check whether the results are the same. If their inertial masses are unequal, then the one with a smaller inertial mass will go through each cycle faster, since gravity has an easier time accelerating and decelerating it. With this type of experiment, one can easily verify that gravitational and inertial mass are proportional to each other to an accuracy of 10^{-3} or 10^{-4} .

In 1889, the Hungarian physicist Roland Eötvös used a slightly different approach to verify the equivalence of gravitational and inertial mass for various substances to an accuracy of about 10^{-8} , and the best such experiment, figure d, improved on even this phenomenal accuracy, bringing it to the 10^{-12} level.³ In all the experiments described so far, the two objects move along similar trajectories: straight lines in the penny-and-shoe and inclined plane experiments, and circular arcs in the pendulum version. The

Eötvös-style experiment looks for differences in the objects' trajectories. The concept can be understood by imagining the following simplified version. Suppose, as in figure b, we roll a brass cylinder off of a tabletop and measure where it hits the floor, and then do the same with an aluminum cylinder, making sure that both of them go over the edge with precisely the same velocity. An object with zero gravitational mass would fly off straight and hit the wall, while an object with zero inertial mass would make a sudden 90-degree turn and drop straight to the floor. If the aluminum and brass cylinders have ordinary, but slightly unequal, ratios of gravitational to inertial mass, then they will follow trajectories that are just slightly different. In other words, if inertial and gravitational mass are not exactly proportional to each other for all substances, then objects made of different substances will have different trajectories in the presence of gravity.



c / A simplified drawing of an Eötvös-style experiment. If the two masses, made out of two different substances, have slightly different ratios of inertial to gravitational mass, then the apparatus will twist slightly as the earth spins.

A simplified drawing of a practical, high-precision experiment is shown in figure c. Two objects made of different substances are balanced on the ends of a bar, which is suspended at the center from a thin fiber. The whole apparatus moves through space on a complicated, looping trajectory arising from the rotation of the earth superimposed on the earth's orbital motion around the sun. Both the earth's gravity and the sun's gravity act on the two objects. If their inertial masses are not exactly in proportion to their gravitational masses, then they will follow slightly different trajectories through space, which will result in a very slight twisting of the fiber between the daytime, when the sun's gravity is pulling upward, and the night, when the sun's gravity is downward. Figure d shows a more realistic picture of the apparatus.

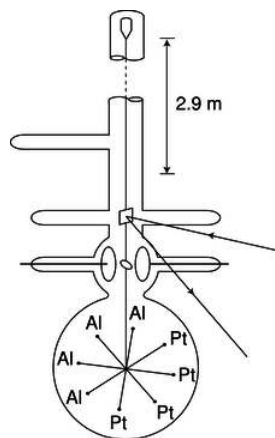


Figure d: A more realistic drawing of Braginskii and Panov's experiment. The whole thing was encased in a tall vacuum tube, which was placed in a sealed basement whose temperature was controlled to within 0.02°C . The total mass of the platinum and aluminum test masses, plus the tungsten wire and the balance arms, was only 4.4 g. To detect tiny motions, a laser beam was bounced off of a mirror attached to the wire. There was so little friction that the balance would have taken on the order of several years to calm down completely after being put in place; to stop these vibrations, static electrical forces were applied through the two circular plates to provide very gentle twists on the ellipsoidal mass between them. After Braginskii and Panov.

This type of experiment, in which one expects a null result, is a tough way to make a career as a scientist. If your measurement comes out as expected, but with better accuracy than other people had previously achieved, your result is publishable, but won't be considered earth-shattering. On the other hand, if you build the most sensitive experiment ever, and the result comes out contrary to expectations, you're in a scary situation. You could be right, and earn a place in history, but if the result turns out to be due to a defect in your experiment, then you've made a fool of yourself.

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2.2.3: 1.3 Galilean Relativity

I defined inertial mass conceptually as a measure of how hard it is to *change* an object's state of motion, the implication being that if you don't interfere, the object's motion won't change. Most people, however, believe that objects in motion have a natural tendency to slow down. Suppose I push my refrigerator to the west for a while at 0.1 m/s, and then stop pushing. The average person would say fridge just naturally stopped moving, but let's imagine how someone in China would describe the fridge experiment carried out in my house here in California. Due to the rotation of the earth, California is moving to the east at about 400 m/s. A point in China at the same latitude has the same speed, but since China is on the other side of the planet, China's east is my west. (If you're finding the three-dimensional visualization difficult, just think of China and California as two freight trains that go past each other, each traveling at 400 m/s.) If I insist on thinking of my dirt as being stationary, then China and its dirt are moving at 800 m/s to my west. From China's point of view, however, it's California that is moving 800 m/s in the opposite direction (my east). When I'm pushing the fridge to the west at 0.1 m/s, the observer in China describes its speed as 799.9 m/s. Once I stop pushing, the fridge speeds back up to 800 m/s. From my point of view, the fridge "naturally" slowed down when I stopped pushing, but according to the observer in China, it "naturally" sped up!

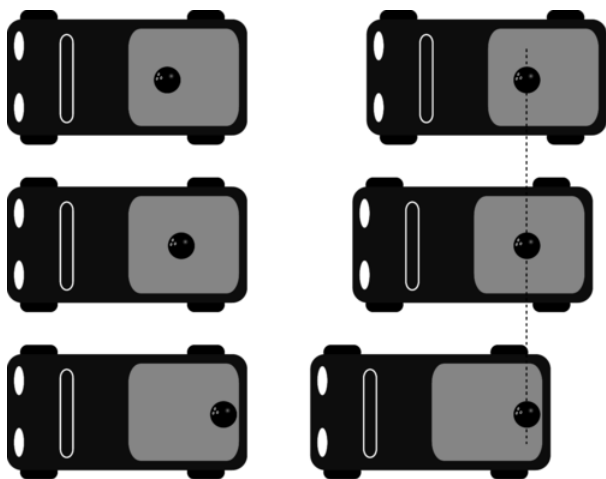
What's really happening here is that there's a tendency, due to friction, for the fridge to stop moving *relative to the floor*. In general, only relative motion has physical significance in physics, not absolute motion. It's not even possible to define absolute motion, since there is no special reference point in the universe that everyone can agree is at rest. Of course if we want to measure motion, we do have to pick some arbitrary reference point which we will say is standing still, and we can then define x , y , and z coordinates extending out from that point, which we can define as having $x = 0$, $y = 0$, $z = 0$. Setting up such a system is known as choosing a *frame of reference*. The local dirt is a natural frame of reference for describing a game of basketball, but if the game was taking place on the deck of a moving ocean liner, we would probably pick a frame of reference in which the deck was at rest, and the land was moving.



a / Galileo Galilei (1564-1642).

Galileo was the first scientist to reason along these lines, and we now use the term Galilean relativity to refer to a somewhat modernized version of his principle. Roughly speaking, the principle of Galilean relativity states that the same laws of physics apply in any frame of reference that is moving in a straight line at constant speed. We need to refine this statement, however, since it is not necessarily obvious which frames of reference are going in a straight line at constant speed. A person in a pickup truck pulling away from a stoplight could admit that the car's velocity is changing, or she could insist that the truck is at rest, and the meter on the dashboard is going up because the asphalt picked that moment to start moving faster and faster backward! Frames of reference are not all created equal, however, and the accelerating truck's frame of reference is not as good as the asphalt's. We can tell, because a bowling ball in the back of the truck, as in figure c, appears to behave strangely in the driver's frame of reference: in her rear-view mirror, she sees the ball, initially at rest, start moving faster and faster toward the back of the truck. This goofy behavior is evidence that there is something wrong with her frame of reference. A person on the sidewalk, however, sees the ball as standing still. In the sidewalk's frame of reference, the truck pulls away from the ball, and this makes sense, because the truck is burning gas and using up energy to change its state of motion.

We therefore define an *inertial frame of reference* as one in which we never see objects change their state of motion without any apparent reason. The sidewalk is a pretty good inertial frame, and a car moving relative to the sidewalk at constant speed in a straight line defines a pretty good inertial frame, but a car that is accelerating or turning is not a inertial frame.



c / Left: In a frame of reference that speeds up with the truck, the bowling ball appears to change its state of motion for no reason. Right: In an inertial frame of reference, which the surface of the earth approximately is, the bowling ball stands still, which makes sense because there is nothing that would cause it to change its state of motion.

The principle of Galilean relativity states that inertial frames exist, and that the same laws of physics apply in all inertial frames of reference, regardless of one frame's straight-line, constant-speed motion relative to another.⁴

Another way of putting it is that all inertial frames are created equal. We can say whether one inertial frame is in motion or at rest relative to another, but there is no privileged “rest frame.” There is no experiment that comes out any different in laboratories in different inertial frames, so there is no experiment that could tell us which inertial frame is really, truly at rest.



b / The earth spins. People in Shanghai say they're at rest and people in Los Angeles are moving. Angelenos say the same about the Shanghainese.

Example 3: The speed of sound

▷ The speed of sound in air is only 340 m/s, so unless you live at a near-polar latitude, you're moving at greater than the speed of sound right now due to the Earth's rotation. In that case, why don't we experience exciting phenomena like sonic booms all the time? ▷ It might seem as though you're unprepared to deal with this question right now, since the only law of physics you know is conservation of mass, and conservation of mass doesn't tell you anything obviously useful about the speed of sound or sonic booms. Galilean relativity, however, is a blanket statement about all the laws of physics, so in a situation like this, it may let you predict the results of the laws of physics without actually knowing what all the laws are! If the laws of physics predict a certain value for the speed of sound, then they had better predict the speed of the sound relative to the air, not their speed relative to some special “rest frame.” Since the air is moving along with the rotation of the earth, we don't detect any special phenomena. To get a sonic boom, the source of the sound would have to be moving relative to the air.

Example 4: The Foucault pendulum



d / Foucault demonstrates his pendulum to an audience at a lecture in 1851.

Note that in the example of the bowling ball in the truck, I didn't claim the sidewalk was *exactly* a Galilean frame of reference. This is because the sidewalk is moving in a circle due to the rotation of the Earth, and is therefore changing the direction of its motion continuously on a 24-hour cycle. However, the curve of the motion is so gentle that under ordinary conditions we don't notice that the local dirt's frame of reference isn't quite inertial. The first demonstration of the noninertial nature of the earth-fixed frame of reference was by Foucault using a very massive pendulum (figure d) whose oscillations would persist for many hours without becoming imperceptible. Although Foucault did his demonstration in Paris, it's easier to imagine what would happen at the north pole: the pendulum would keep swinging in the same plane, but the earth would spin underneath it once every 24 hours. To someone standing in the snow, it would appear that the pendulum's plane of motion was twisting. The effect at latitudes less than 90 degrees turns out to be slower, but otherwise similar. The Foucault pendulum was the first definitive experimental proof that the earth really did spin on its axis, although scientists had been convinced of its rotation for a century based on more indirect evidence about the structure of the solar system.



e / Galileo's trial.

Although popular belief has Galileo being prosecuted by the Catholic Church for saying the earth rotated on its axis and also orbited the sun, Foucault's pendulum was still centuries in the future, so Galileo had no hard proof; Galileo's insights into relative versus absolute motion simply made it more plausible that the world could be spinning without producing dramatic effects, but didn't disprove the contrary hypothesis that the sun, moon, and stars went around the earth every 24 hours. Furthermore, the Church was much more liberal and enlightened than most people believe. It didn't (and still doesn't) require a literal interpretation of the Bible, and one of the Church officials involved in the Galileo affair wrote that "the Bible tells us how to go to heaven, not how the heavens go." In other words, religion and science should be separate. The actual reason Galileo got in trouble is shrouded in mystery, since Italy in the age of the Medicis was a secretive place where unscrupulous people might settle a score with poison or a false accusation of heresy. What is certain is that Galileo's satirical style of scientific writing made many enemies among the powerful Jesuit scholars who were his intellectual opponents --- he compared one to a snake that doesn't know its own back is broken. It's also possible that the Church was far less upset by his astronomical work than by his support for atomism (discussed further in the next section). Some theologians perceived atomism as contradicting transubstantiation, the Church's doctrine that the holy bread and wine were literally transformed into the flesh and blood of Christ by the priest's blessing.

self-check:

What is incorrect about the following supposed counterexamples to the principle of inertia?

- (1) When astronauts blast off in a rocket, their huge velocity does cause a physical effect on their bodies --- they get pressed back into their seats, the flesh on their faces gets distorted, and they have a hard time lifting their arms.
- (2) When you're driving in a convertible with the top down, the wind in your face is an observable physical effect of your absolute motion.

(answer in the back of the PDF version of the book)

◇ Solved problem: a bug on a wheel — problem 12

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2.2.4: A Preview of Some Modern Physics

“Mommy, why do you and Daddy have to go to work?” “To make money, sweetie-pie.” “Why do we need money?” “To buy food.” “Why does food cost money?” When small children ask a chain of “why” questions like this, it usually isn't too long before their parents end up saying something like, “Because that's just the way it is,” or, more honestly, “I don't know the answer.”

The same happens in physics. We may gradually learn to explain things more and more deeply, but there's always the possibility that a certain observed fact, such as conservation of mass, will never be understood on any deeper level. Science, after all, uses limited methods to achieve limited goals, so the ultimate reason for all existence will always be the province of religion. There is, however, an appealing explanation for conservation of mass, which is atomism, the theory that matter is made of tiny, unchanging particles. The atomic hypothesis dates back to ancient Greece, but the first solid evidence to support it didn't come until around the eighteenth century, and individual atoms were never detected until about 1900. The atomic theory implies not only conservation of mass, but a couple of other things as well.

First, it implies that the total mass of one particular element is conserved. For instance, lead and gold are both elements, and if we assume that lead atoms can't be turned into gold atoms, then the total mass of lead and the total mass of gold are separately conserved. It's as though there was not just a law against pickpocketing, but also a law against surreptitiously moving money from one of the victim's pockets to the other. It turns out, however, that although chemical reactions never change one type of atom into another, transmutation can happen in nuclear reactions, such as the ones that created most of the elements in your body out of the primordial hydrogen and helium that condensed out of the aftermath of the Big Bang.

Second, atomism implies that mass is *quantized*, meaning that only certain values of mass are possible and the ones in between can't exist. We can have three atoms of gold or four atoms of gold, but not three and a half. Although quantization of mass is a natural consequence of any theory in which matter is made up of tiny particles, it was discovered in the twentieth century that other quantities, such as energy, are quantized as well, which had previously not been suspected.

self-check:

Is money quantized?

(answer in the back of the PDF version of the book)

If atomism is starting to make conservation of mass seem inevitable to you, then it may disturb you to know that Einstein discovered it isn't really conserved. If you put a 50-gram iron nail in some water, seal the whole thing up, and let it sit on a fantastically precise balance while the nail rusts, you'll find that the system loses about 6×10^{-12} kg of mass by the time the nail has turned completely to rust. This has to do with Einstein's famous equation $E = mc^2$. Rusting releases heat energy, which then escapes out into the room. Einstein's equation states that this amount of heat, E , is equivalent to a certain amount of mass, m . The c in the c^2 is the speed of light, which is a large number, and a large amount of energy is therefore equivalent to a very small amount of mass, so you don't notice nonconservation of mass under ordinary conditions. What is really conserved is not the mass, m , but the mass-plus-energy, $E + mc^2$. The point of this discussion is not to get you to do numerical exercises with $E = mc^2$ (at this point you don't even know what units are used to measure energy), but simply to point out to you the empirical nature of the laws of physics. If a previously accepted theory is contradicted by an experiment, then the theory needs to be changed. This is also a good example of something called the *correspondence principle*, which is a historical observation about how scientific theories change: when a new scientific theory replaces an old one, the old theory is always contained within the new one as an approximation that works within a certain restricted range of situations. Conservation of mass is an extremely good approximation for all chemical reactions, since chemical reactions never release or consume enough energy to change the total mass by a large percentage. Conservation of mass would not have been accepted for 110 years as a fundamental principle of physics if it hadn't been verified over and over again by a huge number of accurate experiments.

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2.2.5: Footnotes

1. You might think intuitively that the recoil velocity should be exactly one fourth of a meter per second, and you'd be right except that the wagon has some mass as well. Our present approach, however, only requires that we give a way to test for equality of masses. To predict the recoil velocity from scratch, we'd need to use conservation of momentum, which is discussed in a later chapter.
2. The argument only fails for objects light enough to be affected appreciably by air friction: a bunch of feathers falls differently if you wad them up because the pattern of air flow is altered by putting them together.
3. V.B. Braginskii and V.I. Panov, Soviet Physics JETP 34, 463 (1972).
4. The principle of Galilean relativity is extended on page 190.

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2.2.6: Problems

1. Thermometers normally use either mercury or alcohol as their working fluid. If the level of the fluid rises or falls, does this violate conservation of mass?

2. The ratios of the masses of different types of atoms were determined a century before anyone knew any actual atomic masses in units of kg. One finds, for example, that when ordinary table salt, NaCl, is melted, the chlorine atoms bubble off as a gas, leaving liquid sodium metal. Suppose the chlorine escapes, so that its mass cannot be directly determined by weighing. Experiments show that when 1.00000 kg of NaCl is treated in this way, the mass of the remaining sodium metal is 0.39337 kg. Based on this information, determine the ratio of the mass of a chlorine atom to that of a sodium atom (answer check available at lightandmatter.com)

3. An atom of the most common naturally occurring uranium isotope breaks up spontaneously into a thorium atom plus a helium atom. The masses are as follows:

uranium	$3.95292849 \times 10^{25} \text{ kg}$
thorium	$3.88638748 \times 10^{25} \text{ kg}$
helium	$6.646481 \times 10^{27} \text{ kg}$

Each of these experimentally determined masses is uncertain in its last decimal place. Is mass conserved in this process to within the accuracy of the experimental data? How would you interpret this?

4. If two spherical water droplets of radius b combine to make a single droplet, what is its radius? (Assume that water has constant density.)

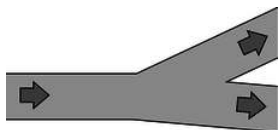
5. Make up an experiment that would test whether mass is conserved in an animal's metabolic processes.

6. The figure shows a hydraulic jack. What is the relationship between the distance traveled by the plunger and the distance traveled by the object being lifted, in terms of the cross-sectional areas?

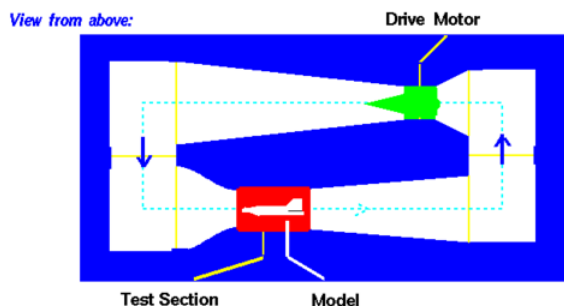


7. In an example in this chapter, I argued that a stream of water must change its cross-sectional area as it rises or falls. Suppose that the stream of water is confined to a constant-diameter pipe. Which assumption breaks down in this situation?

8. A river with a certain width and depth splits into two parts, each of which has the same width and depth as the original river. What can you say about the speed of the current after the split?



9. The diagram shows a cross-section of a wind tunnel of the kind used, for example, to test designs of airplanes. Under normal conditions of use, the density of the air remains nearly constant throughout the whole wind tunnel. How can the speed of the air be controlled and calculated? (Diagram by NASA, Glenn Research Center.)

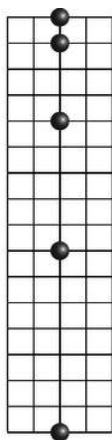


10. A water wave is in a tank that extends horizontally from $x = 0$ to $x = a$, and from $z = 0$ to $z = b$. We assume for simplicity that at a certain moment in time the height y of the water's surface only depends on x , not z , so that we can effectively ignore the z coordinate. Under these assumptions, the total volume of the water in the tank is

$$V = b \int_0^a y(x) dx \quad (2.2.6.1)$$

Since the density of the water is essentially constant, conservation of mass requires that V is always the same. When the water is calm, we have $y = h$, where $h = V/ab$. If two different wave patterns move into each other, we might imagine that they would add in the sense that $y_{total} - h = (y_1 - h) + (y_2 - h)$. Show that this type of addition is consistent with conservation of mass.

11. The figure shows the position of a falling ball at equal time intervals, depicted in a certain frame of reference. On a similar grid, show how the ball's motion would appear in a frame of reference that was moving horizontally at a speed of one box per unit time relative to the first frame.



12. (solution in the pdf version of the book) The figure shows the motion of a point on the rim of a rolling wheel. (The shape is called a cycloid.) Suppose bug A is riding on the rim of the wheel on a bicycle that is rolling, while bug B is on the spinning wheel of a bike that is sitting upside down on the floor. Bug A is moving along a cycloid, while bug B is moving in a circle. Both wheels are doing the same number of revolutions per minute. Which bug has a harder time holding on, or do they find it equally difficult?



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CHAPTER OVERVIEW

2.3: Conservation of Energy

Do you pronounce it Joule's to rhyme with schools,

Joule's to rhyme with Bowls,

or Joule's to rhyme with Scowls?

Whatever you call it, by Joule's,

or Joule's,

or Joule's, it's good! -- Advertising slogan of the Joule brewery. The name, and the corresponding unit of energy, are now usually pronounced so as to rhyme with "school."

Topic hierarchy

[2.3.1: Energy](#)

[2.3.2: Numerical Techniques](#)

[2.3.3: Gravitational Phenomena](#)

[2.3.4: Atomic Phenomena](#)

[2.3.5: Oscillations](#)

[2.3.6: Footnotes](#)

[2.3.7: Problems](#)

Thumbnail: Roller coaster "Blue Fire" at Europa Park. (CC SA 3.0; [Coaster J](#)).

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2.3.1: Energy

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2.3.2: Numerical Techniques

Engineering majors are a majority of the students in the kind of physics course for which this book is designed, so most likely you fall into that category. Although you surely recognize that physics is an important part of your training, if you've had any exposure to how engineers really work, you're probably skeptical about the flavor of problem-solving taught in most science courses. You realize that not very many practical engineering calculations fall into the narrow range of problems for which an exact solution can be calculated with a piece of paper and a sharp pencil. Real-life problems are usually complicated, and typically they need to be solved by number-crunching on a computer, although we can often gain insight by working simple approximations that have algebraic solutions. Not only is numerical problem-solving more useful in real life, it's also educational; as a beginning physics student, I really only felt like I understood projectile motion after I had worked it both ways, using algebra and then a computer program. (This was back in the days when 64 kilobytes of memory was considered a lot.)

In this section, we'll start by seeing how to apply numerical techniques to some simple problems for which we know the answer in "closed form," i.e., a single algebraic expression without any calculus or infinite sums. After that, we'll solve a problem that would have made you world-famous if you could have done it in the seventeenth century using paper and a quill pen! Before you continue, you should read Appendix 1 on page 908 that introduces you to the Python programming language.

First let's solve the trivial problem of finding how much time it takes an object moving at speed v to travel a straight-line distance dist . This closed-form answer is, of course, dist/v , but the point is to introduce the techniques we can use to solve other problems of this type. The basic idea is to divide the distance up into n equal parts, and add up the times required to traverse all the parts. The following Python function does the job. Note that you shouldn't type in the line numbers on the left, and you don't need to type in the comments, either. I've omitted the prompts `>>>` and `...` in order to save space.

```
import math
def time1(dist,v,n):
    x=0 # Initialize the position.
    dx = dist/n # Divide dist into n equal parts.
    t=0 # Initialize the time.
    for i in range(n):
        x = x+dx # Change x.
        dt=dx/v # time=distance/speed
        t=t+dt # Keep track of elapsed time.
    return t
```

How long does it take to move 1 meter at a constant speed of 1 m/s? If we do this,

```
>>> print(time1(1.0,1.0,10)) # dist, v, n
0.99999999999999989
```

Python produces the expected answer by dividing the distance into ten equal 0.1-meter segments, and adding up the ten 0.1-second times required to traverse each one. Since the object moves at constant speed, it doesn't even matter whether we set n to 10, 1, or a million:

```
>>> print(time1(1.0,1.0,1)) # dist, v, n
1.0
```

Now let's do an example where the answer isn't obvious to people who don't know calculus: how long does it take an object to fall through a height h , starting from rest? We know from example 8 on page 83 that the exact answer, found using calculus, is $\sqrt{2h/g}$. Let's see if we can reproduce that answer numerically. The main difference between this program and the previous one is that now the velocity isn't constant, so we need to update it as we go along. Conservation of energy gives $mgh = (1/2)mv^2 + mgy$ for the velocity v at height y , so $v = -\sqrt{2g(h-y)}$. (We choose the negative root because the object is moving down, and our coordinate system has the positive y axis pointing up.)

```
import math
def time2(h,n):
    g=9.8 # gravitational field
    y=h # Initialize the height.
    v=0 # Initialize the velocity.
    dy = -h/n # Divide h into n equal parts.
    t=0 # Initialize the time.
    for i in range(n):
        y = y+dy # Change y. (Note dy<0.)
        v = -math.sqrt(2*g*(h-y)) # from cons. of energy
        dt=dy/v # dy and v are <0, so dt is >0
        t=t+dt # Keep track of elapsed time.
    return t
```

For $h=1.0$ m, the closed-form result is $\sqrt{2 \cdot 1.0 \text{ m}/9.8 \text{ m/s}^2} = 0.45$ s. With the drop split up into only 10 equal height intervals, the numerical technique provides a pretty lousy approximation:

```
>>> print(time2(1.0,10)) # h, n
0.35864270709233342
```

But by increasing n to ten thousand, we get an answer that's as close as we need, given the limited accuracy of the raw data:

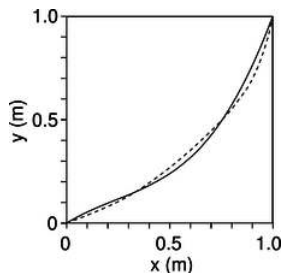
```
>>> print(time2(1.0,10000)) # h, n
0.44846664060793945
```

A subtle point is that we changed y in line 9, and *then* on line 10 we calculated v , which depends on y . Since y is only changing by a ten-thousandth of a meter with each step, you might think this wouldn't make much of a difference, and you'd be almost right, except for one small problem: if we swapped lines 9 and 10, then the very first time through the loop, we'd have $v=0$, which would produce a division-by-zero error when we calculated dt ! Actually what would make the most sense would be to calculate the velocity at height y and the velocity at height $y+dy$ (recalling that dy is negative), average them together, and use that value of y to calculate the best estimate of the velocity between those two points. Since the acceleration is constant in the present example, this modification results in a program that gives an exact result even for $n=1$:

```
import math
def time3(h,n):
    g=9.8
    y=h
    v=0
    dy = -h/n
    t=0
    for i in range(n):
        y_old = y
        y = y+dy
        v_old = math.sqrt(2*g*(h-y_old))
        v = math.sqrt(2*g*(h-y))
        v_avg = -(v_old+v)/2.
        dt=dy/v_avg
```

```
t=t+dt
return t
```

```
>>> print(time3(1.0,1)) # h, n
0.45175395145262565
```



a / Approximations to the brachistochrone curve using a third-order polynomial (solid line), and a seventh-order polynomial (dashed). The latter only improves the time by four milliseconds.

Now we're ready to attack a problem that challenged the best minds of Europe back in the days when there were no computers. In 1696, the mathematician Johann Bernoulli posed the following famous question. Starting from rest, an object slides frictionlessly over a curve joining the point (a,b) to the point $(0,0)$. Of all the possible shapes that such a curve could have, which one gets the object to its destination in the least possible time, and how much time does it take? The optimal curve is called the *brachistochrone*, from the Greek “short time.” The solution to the brachistochrone problem evaded Bernoulli himself, as well as Leibniz, who had been one of the inventors of calculus. The English physicist Isaac Newton, however, stayed up late one night after a day's work running the royal mint, and, according to legend, produced an algebraic solution at four in the morning. He then published it anonymously, but Bernoulli is said to have remarked that when he read it, he knew instantly from the style that it was Newton --- he could “tell the lion from the mark of his claw.”

Rather than attempting an exact algebraic solution, as Newton did, we'll produce a numerical result for the shape of the curve and the minimum time, in the special case of $a=1.0$ m and $b=1.0$ m. Intuitively, we want to start with a fairly steep drop, since any speed we can build up at the start will help us throughout the rest of the motion. On the other hand, it's possible to go too far with this idea: if we drop straight down for the whole vertical distance, and then do a right-angle turn to cover the horizontal distance, the resulting time of 0.68 s is quite a bit longer than the optimal result, the reason being that the path is unnecessarily long. There are infinitely many possible curves for which we could calculate the time, but let's look at third-order polynomials,

$$y = c_1x + c_2x^2 + c_3x^3,$$

where we require $c_3 = (b - c_1a - c_2a^2)/a^3$ in order to make the curve pass through the point (a,b) . The Python program, below, is not much different from what we've done before. The function only asks for c_1 and c_2 , and calculates c_3 internally at line 4. Since the motion is two-dimensional, we have to calculate the distance between one point and the next using the Pythagorean theorem, at line 16.

```
import math
def timeb(a,b,c1,c2,n):
    g=9.8
    c3 = (b-c1*a-c2*a**2)/(a**3)
    x=a
    y=b
    dx = -a/n
    t=0
    for i in range(n):
        y_old = y
```

```
x = x+dx
y = c1*x+c2*x**2+c3*x**3
dy = y-y_old
v_old = math.sqrt(2*g*(b-y_old))
v = math.sqrt(2*g*(b-y))
v_avg = (v_old+v)/2.
ds = math.sqrt(dx**2+dy**2) # Pythagorean thm.
dt=ds/v_avg
t=t+dt
return t
```

As a first guess, we could try a straight diagonal line, $y = x$, which corresponds to setting $c_1 = 1$, and all the other coefficients to zero. The result is a fairly long time:

```
>>> a=1.
>>> b=1.
>>> n=10000
>>> c1=1.
>>> c2=0.
>>> print(timeb(a,b,c1,c2,n))
0.63887656499994161
```

What we really need is a curve that's very steep on the right, and flatter on the left, so it would actually make more sense to try $y = x^3$:

```
>>> c1=0.
>>> c2=0.
>>> print(timeb(a,b,c1,c2,n))
0.59458339947087069
```

This is a significant improvement, and turns out to be only a hundredth of a second off of the shortest possible time! It's possible, although not very educational or entertaining, to find better approximations to the brachistochrone curve by fiddling around with the coefficients of the polynomial by hand. The real point of this discussion was to give an example of a nontrivial problem that can be attacked successfully with numerical techniques. I found the first approximation shown in figure a,

$$y = (0.62)x + (-0.93)x^2 + (1.31)x^3$$

by using the program listed in appendix 2 on page 911 to search automatically for the optimal curve. The seventh-order approximation shown in the figure came from a straightforward extension of the same program.

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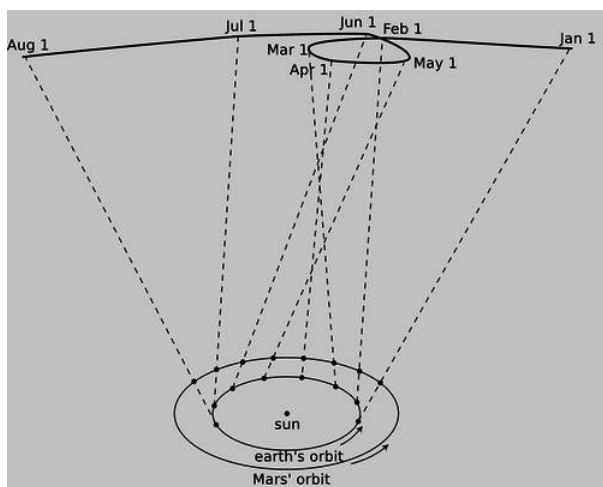
2.3.3: Gravitational Phenomena

Cruise your radio dial today and try to find any popular song that would have been imaginable without Louis Armstrong. By introducing solo improvisation into jazz, Armstrong took apart the jigsaw puzzle of popular music and fit the pieces back together in a different way. In the same way, Newton reassembled our view of the universe. Consider the titles of some recent physics books written for the general reader: **The God Particle**, **Dreams of a Final Theory**. When the subatomic particle called the neutrino was recently proven for the first time to have mass, specialists in cosmology began discussing seriously what effect this would have on calculations of the evolution of the universe from the Big Bang to its present state. Without the English physicist Isaac Newton, such attempts at universal understanding would not merely have seemed ambitious, they simply would not have occurred to anyone.

This section is about Newton's theory of gravity, which he used to explain the motion of the planets as they orbited the sun. Newton tosses off a general treatment of motion in the first 20 pages of his **Mathematical Principles of Natural Philosophy**, and then spends the next 130 discussing the motion of the planets. Clearly he saw this as the crucial scientific focus of his work. Why? Because in it he showed that the same laws of nature applied to the heavens as to the earth, and that the gravitational interaction that made an apple fall was the same as the one that kept the earth's motion from carrying it away from the sun.

2.3.1 Kepler's laws

Newton wouldn't have been able to figure out *why* the planets move the way they do if it hadn't been for the astronomer Tycho Brahe (1546-1601) and his protege Johannes Kepler (1571-1630), who together came up with the first simple and accurate description of *how* the planets actually do move. The difficulty of their task is suggested by the figure below, which shows how the relatively simple orbital motions of the earth and Mars combine so that as seen from earth Mars appears to be staggering in loops like a drunken sailor.

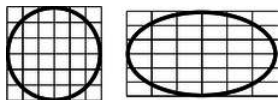


d / As the earth and Mars revolve around the sun at different rates, the combined effect of their motions makes Mars appear to trace a strange, looped path across the background of the distant stars.

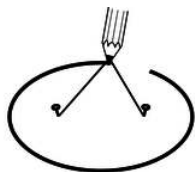
Brahe, the last of the great naked-eye astronomers, collected extensive data on the motions of the planets over a period of many years, taking the giant step from the previous observations' accuracy of about 10 minutes of arc (10/60 of a degree) to an unprecedented 1 minute. The quality of his work is all the more remarkable considering that his observatory consisted of four giant brass protractors mounted upright in his castle in Denmark. Four different observers would simultaneously measure the position of a planet in order to check for mistakes and reduce random errors.

With Brahe's death, it fell to his former assistant Kepler to try to make some sense out of the volumes of data. After 900 pages of calculations and many false starts and dead-end ideas, Kepler finally synthesized the data into the following three laws:

Kepler's elliptical orbit law: The planets orbit the sun in elliptical orbits with the sun at one focus.

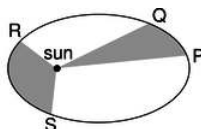


a / An ellipse is circle that has been distorted by shrinking and stretching along perpendicular axes.



b / An ellipse can be constructed by tying a string to two pins and drawing like this with a pencil stretching the string taut. Each pin constitutes one focus of the ellipse.

Kepler's equal-area law: The line connecting a planet to the sun sweeps out equal areas in equal amounts of time.



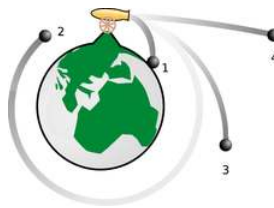
c / If the time interval taken by the planet to move from P to Q is equal to the time interval from R to S, then according to Kepler's equal-area law, the two shaded areas are equal. The planet is moving faster during time interval RS than it was during PQ, because gravitational energy has been transformed into kinetic energy.

Kepler's law of periods: The time required for a planet to orbit the sun, called its period, T , is proportional to the long axis of the ellipse raised to the $3/2$ power. The constant of proportionality is the same for all the planets.

Although the planets' orbits are ellipses rather than circles, most are very close to being circular. The earth's orbit, for instance, is only flattened by 1.7% relative to a circle. In the special case of a planet in a circular orbit, the two foci (plural of "focus") coincide at the center of the circle, and Kepler's elliptical orbit law thus says that the circle is centered on the sun. The equal-area law implies that a planet in a circular orbit moves around the sun with constant speed. For a circular orbit, the law of periods then amounts to a statement that the time for one orbit is proportional to $r^{3/2}$, where r is the radius. If all the planets were moving in their orbits at the same speed, then the time for one orbit would simply depend on the circumference of the circle, so it would only be proportional to r to the first power. The more drastic dependence on $r^{3/2}$ means that the outer planets must be moving more slowly than the inner planets. Our main focus in this section will be to use the law of periods to deduce the general equation for gravitational energy. The equal-area law turns out to be a statement on conservation of angular momentum, which is discussed in chapter 4. We'll demonstrate the elliptical orbit law numerically in chapter 3, and analytically in chapter 4.2.3.2 Circular orbits

Kepler's laws say that planets move along elliptical paths (with circles as a special case), which would seem to contradict the proof on page 90 that objects moving under the influence of gravity have parabolic trajectories. Kepler was right. The parabolic path was really only an approximation, based on the assumption that the gravitational field is constant, and that vertical lines are all parallel. In figure e, trajectory 1 is an ellipse, but it gets chopped off when the cannonball hits the earth, and the small piece of it that is above ground is nearly indistinguishable from a parabola. Our goal is to connect the previous calculation of parabolic trajectories, $y = (g/2v^2)x^2$, with Kepler's data for planets orbiting the sun in nearly circular orbits. Let's start by thinking in terms of an orbit that circles the earth, like orbit 2 in figure e. It's more natural now to choose a coordinate system with its origin at the center of the earth, so the parabolic approximation becomes $y = r - (g/2v^2)x^2$, where r is the distance from the center of the earth. For small values of x , i.e., when the cannonball hasn't traveled very far from the muzzle of the gun, the parabola is still a good approximation to the actual circular orbit, defined by the Pythagorean theorem, $r^2 = x^2 + y^2$, or $y = r\sqrt{1 - x^2/r^2}$. For small values of x , we can use the approximation $\sqrt{1 + \epsilon} \approx 1 + \epsilon/2$ to find $y \approx r - (1/2r)x^2$. Setting this equal to the equation of the parabola, we have $g/2v^2 = (1/2r)$, or

$$v = \sqrt{gr}[\text{condition for a circular orbit}].$$



e / A cannon fires cannonballs at different velocities, from the top of an imaginary mountain that rises above the earth's atmosphere. This is almost the same as a figure Newton included in his **Mathematical Principles**.

Example 14: Low-earth orbit

To get a feel for what this all means, let's calculate the velocity required for a satellite in a circular low-earth orbit. Real low-earth-orbit satellites are only a few hundred km up, so for purposes of rough estimation we can take r to be the radius of the earth, and g is not much less than its value on the earth's surface, 10 m/s^2 . Taking numerical data from Appendix 5, we have

$$\begin{aligned} v &= \sqrt{gr} \\ &= \sqrt{(10 \text{ m/s}^2)(6.4 \times 10^3 \text{ km})} \\ &= \sqrt{(10 \text{ m/s}^2)(6.4 \times 10^6 \text{ m})} \\ &= \sqrt{6.4 \times 10^7 \text{ m}^2/\text{s}^2} \\ &= 8000 \text{ m/s} \end{aligned}$$

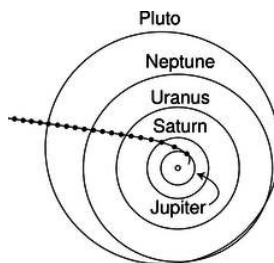
(about twenty times the speed of sound). In one second, the satellite moves 8000 m horizontally. During this time, it drops the same distance any other object would: about 5 m. But a drop of 5 m over a horizontal distance of 8000 m is just enough to keep it at the same altitude above the earth's curved surface.

2.3.3 The sun's gravitational field

We can now use the circular orbit condition $v = \sqrt{gr}$, combined with Kepler's law of periods, $T \propto r^{3/2}$ for circular orbits, to determine how the sun's gravitational field falls off with distance.⁸ From there, it will be just a hop, skip, and a jump to get to a universal description of gravitational interactions. The velocity of a planet in a circular orbit is proportional to r/T , so

$$\begin{aligned} r/T &\propto \sqrt{gr} \\ r/r^{3/2} &\propto \sqrt{gr} \\ g &\propto 1/r^2 \end{aligned}$$

If gravity behaves systematically, then we can expect the same to be true for the gravitational field created by any object, not just the sun. There is a subtle point here, which is that so far, r has just meant the radius of a circular orbit, but what we have come up with smells more like an equation that tells us the strength of the gravitational field made by some object (the sun) if we know how far we are from the object. In other words, we could reinterpret r as the distance from the sun.



i / The Pioneer 10 space probe's trajectory from 1974 to 1992, with circles marking its position at one-year intervals. After its 1974 slingshot maneuver around Jupiter, the probe's motion was determined almost exclusively by the sun's gravity.

2.3.4 Gravitational energy in general

We now want to find an equation for the gravitational energy of any two masses that attract each other from a distance r . We assume that r is large enough compared to the distance between the objects so that we don't really have to worry about whether r is measured from center to center or in some other way. This would be a good approximation for describing the solar system, for

example, since the sun and planets are small compared to the distances between them --- that's why you see Venus (the "evening star") with your bare eyes as a dot, not a disk.

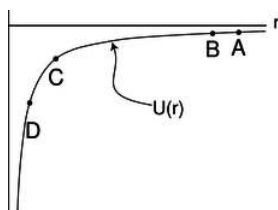
The equation we seek is going to give the gravitational energy, U , as a function of m_1 , m_2 , and r . We already know from experience with gravity near the earth's surface that U is proportional to the mass of the object that interacts with the earth gravitationally, so it makes sense to assume the relationship is symmetric: U is presumably proportional to the product $m_1 m_2$. We can no longer assume $\Delta U \propto \Delta r$, as in the earth's-surface equation $\Delta U = mg\Delta y$, since we are trying to construct an equation that would be valid for all values of r , and g depends on r . We can, however, consider an infinitesimally small change in distance dr , for which we'll have $dU = m_2 g_1 dr$, where g_1 is the gravitational field created by m_1 . (We could just as well have written this as $dU = m_1 g_2 dr$, since we're not assuming either mass is "special" or "active.") Integrating this equation, we have

$$\begin{aligned}\int dU &= \int m_2 g_1 dr \\ U &= m_2 \int g_1 dr \\ U &\propto m_1 m_2 \int \frac{1}{r^2} dr \\ U &\propto -\frac{m_1 m_2}{r},\end{aligned}$$

where we're free to take the constant of integration to be equal to zero, since gravitational energy is never a well-defined quantity in absolute terms. Writing G for the constant of proportionality, we have the following fundamental description of gravitational interactions:

$$U = -\frac{Gm_1 m_2}{r} \text{ [gravitational energy of two masses separated by a distance } r\text{]}$$

We'll refer to this as Newton's law of gravity, although in reality he stated it in an entirely different form, which turns out to be mathematically equivalent to this one.



f / The gravitational energy $U = -Gm_1 m_2 / r$ graphed as a function of r .

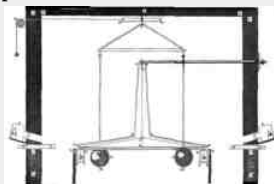
Let's interpret his result. First, don't get hung up on the fact that it's negative, since it's only differences in gravitational energy that have physical significance. The graph in figure f could be shifted up or down without having any physical effect. The slope of this graph relates to the strength of the gravitational field. For instance, suppose figure f is a graph of the gravitational energy of an asteroid interacting with the sun. If the asteroid drops straight toward the sun, from A to B, the decrease in gravitational energy is very small, so it won't speed up very much during that motion. Points C and D, however, are in a region where the graph's slope is much greater. As the asteroid moves from C to D, it loses a lot of gravitational energy, and therefore speeds up considerably. This is due to the stronger gravitational field.

Example 15: Determining G

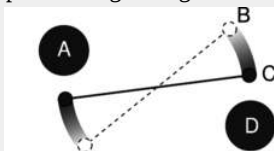
The constant G is not easy to determine, and Newton went to his grave without knowing an accurate value for it. If we knew the mass of the earth, then we could easily determine G from experiments with terrestrial gravity, but the only way to determine the mass of the earth accurately in units of kilograms is by finding G and reasoning the other way around! (If you estimate the average density of the earth, you can make at least a rough estimate of G .) Figures g and h show how G was first measured by Henry Cavendish in the nineteenth century. The rotating arm is released from rest, and the kinetic energy of the two moving balls is measured when they pass position C. Conservation of energy gives

$$-2\frac{GMm}{r_{BA}} - 2\frac{GMm}{r_{BD}} = -2\frac{GMm}{r_{CA}} - 2\frac{GMm}{r_{CD}} + 2K,$$

where M is the mass of one of the large balls, m is the mass of one of the small ones, and the factors of two, which will cancel, occur because every energy is mirrored on the opposite side of the apparatus. (As discussed on page 102, it turns out that we get the right result by measuring all the distances from the center of one sphere to the center of the other.) This can easily be solved for G . The best modern value of G , from later versions of the same experiment, is $6.67 \times 10^{-11} \text{ J} \cdot \text{m}/\text{kg}^2$.



g / Cavendish's original drawing of the apparatus for his experiment, discussed in example 15. The room was sealed to exclude air currents, and the motion was observed through telescopes sticking through holes in the walls.



h / A simplified drawing of the Cavendish experiment, viewed from above. The rod with the two small masses on the ends hangs from a thin fiber, and is free to rotate.

Example 16: Escape velocity

- ▷ The Pioneer 10 space probe was launched in 1972, and continued sending back signals for 30 years. In the year 2001, not long before contact with the probe was lost, it was about $1.2 \times 10^{13} \text{ m}$ from the sun, and was moving almost directly away from the sun at a velocity of $1.21 \times 10^4 \text{ m/s}$. The mass of the sun is $1.99 \times 10^{30} \text{ kg}$. Will Pioneer 10 escape permanently, or will it fall back into the solar system?
- ▷ We want to know whether there will be a point where the probe will turn around. If so, then it will have zero kinetic energy at the turnaround point:

$$\begin{aligned} K_i + U_i &= U_f \\ \frac{1}{2}mv^2 - \frac{GMm}{r_i} &= -\frac{GMm}{r_f} \\ \frac{1}{2}v^2 - \frac{GM}{r_i} &= -\frac{GM}{r_f}, \end{aligned}$$

where M is the mass of the sun, m is the (irrelevant) mass of the probe, and r_f is the distance from the sun of the hypothetical turnaround point. Plugging in numbers on the left, we get a positive result. There can therefore be no solution, since the right side is negative. There won't be any turnaround point, and Pioneer 10 is never coming back.

The minimum velocity required for this to happen is called *escape velocity*. For speeds above escape velocity, the orbits are open-ended hyperbolas, rather than repeating elliptical orbits. In figure i, Pioneer's hyperbolic trajectory becomes almost indistinguishable from a line at large distances from the sun. The motion slows perceptibly in the first few years after 1974, but later the speed becomes nearly constant, as shown by the nearly constant spacing of the dots.

The gravitational field

We got the energy equation $U = -Gm_1m_2/r$ by integrating $g \propto 1/r^2$ and then inserting a constant of proportionality to make the proportionality into an equation. The opposite of an integral is a derivative, so we can now go backwards and insert a constant of proportionality in $g \propto 1/r^2$ that will be consistent with the energy equation:

$$\begin{aligned}
 dU &= m_2 g_1 dr \\
 g_1 &= \frac{1}{m_2} \frac{dU}{dr} \\
 &= \frac{1}{m_2} \frac{d}{dr} \left(-\frac{Gm_1 m_2}{r} \right) \\
 &= -Gm_1 \frac{d}{dr} \left(\frac{1}{r} \right) \\
 &= \frac{Gm_1}{r^2}
 \end{aligned}$$

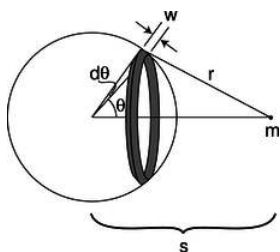
This kind of inverse-square law occurs all the time in nature. For instance, if you go twice as far away from a lightbulb, you receive 1/4 as much light from it, because as the light spreads out, it is like an expanding sphere, and a sphere with twice the radius has four times the surface area. It's like spreading the same amount of peanut butter on four pieces of bread instead of one --- we have to spread it thinner.

Discussion Questions

◇ A bowling ball interacts gravitationally with the earth. Would it make sense for the gravitational energy to be inversely proportional to the distance between their surfaces rather than their centers?

2.3.5 The shell theorem

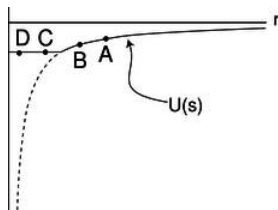
Newton's great insight was that gravity near the earth's surface was the same kind of interaction as the one that kept the planets from flying away from the sun. He told his niece that the idea came to him when he saw an apple fall from a tree, which made him wonder whether the earth might be affecting the apple and the moon in the same way. Up until now, we've generally been dealing with gravitational interactions between objects that are small compared to the distances between them, but that assumption doesn't apply to the apple. A kilogram of dirt a few feet under his garden in England would interact much more strongly with the apple than a kilogram of molten rock deep under Australia, thousands of miles away. Also, we know that the earth has some parts that are more dense, and some parts that are less dense. The solid crust, on which we live, is considerably less dense than the molten rock on which it floats. By all rights, the computation of the total gravitational energy of the apple should be a horrendous mess. Surprisingly, it turns out to be fairly simple in the end. First, we note that although the earth doesn't have the same density throughout, it does have spherical symmetry: if we imagine dividing it up into thin concentric shells, the density of each shell is uniform.



j / A spherical shell of mass M interacts with a pointlike mass m .

Second, it turns out that a uniform spherical shell interacts with external masses as if all its mass were concentrated at its center.

\mythmhdr{The shell theorem} The gravitational energy of a uniform spherical shell of mass M interacting with a pointlike mass m outside it equals $-GMm/s$, where s is the center-to-center distance. If mass m is inside the shell, then the energy is constant, i.e., the shell's interior gravitational field is zero.



k / The gravitational energy of a mass m at a distance s from the center of a hollow spherical shell of mass.

\mythmhdr{Proof} Let b be the radius of the shell, h its thickness, and ρ its density. Its volume is then $V = (\text{area})(\text{thickness}) = 4\pi b^2 h$, and its mass is $M = \rho V = 4\pi \rho b^2 h$. The strategy is to divide the shell up into rings as shown in figure j, with each ring extending from θ to $\theta + d\theta$. Since the ring is infinitesimally skinny, its entire mass lies at the same distance, r , from mass m . The width of such a ring is found by the definition of radian measure to be $w = b d\theta$, and its mass is $dM = (\rho)(\text{circumference})(\text{thickness})(\text{width}) = (\rho)(2\pi b \sin \theta)(h)(b d\theta) = 2\pi \rho b^2 h \sin \theta d\theta$. The gravitational energy of the ring interacting with mass m is therefore

$$dU = -\frac{GmdM}{r}$$

$$= -2\pi G \rho b^2 h m \frac{\sin \theta d\theta}{r}.$$

Integrating both sides, we find the total gravitational energy of the shell:

$$U = -2\pi G \rho b^2 h m \int_0^\pi \frac{\sin \theta d\theta}{r}$$

The integral has a mixture of the variables r and θ , which are related by the law of cosines,

$$r^2 = b^2 + s^2 - 2bs \cos \theta,$$

and to evaluate the integral, we need to get everything in terms of either r and dr or θ and $d\theta$. The relationship between the differentials is found by differentiating the law of cosines,

$$2rdr = 2bs \sin \theta d\theta,$$

and since $\sin \theta d\theta$ occurs in the integral, the easiest path is to substitute for it, and get everything in terms of r and dr :

$$U = -\frac{2\pi G \rho b h m}{s} \int_{s-b}^{s+b} dr$$

$$= -\frac{4\pi G \rho b^2 h m}{s}$$

$$= -\frac{GMm}{s}$$

This was all under the assumption that mass m was on the outside of the shell. To complete the proof, we consider the case where it's inside. In this case, the only change is that the limits of integration are different:

$$U = -\frac{2\pi G \rho b h m}{s} \int_{b-s}^{b+s} dr$$

$$= -4\pi G \rho b h m$$

$$= -\frac{GMm}{b}$$

The two results are equal at the surface of the sphere, $s = b$, so the constant-energy part joins continuously onto the $1/s$ part, and the effect is to chop off the steepest part of the graph that we would have had if the whole mass M had been concentrated at its center. Dropping a mass m from A to B in figure k releases the same amount of energy as if mass M had been concentrated at its center, but there is no release of gravitational energy at all when moving between two interior points like C and D. In other words, the internal gravitational field is zero. Moving from C to D brings mass m farther away from the nearby side of the shell, but closer to the far side, and the cancellation between these two effects turns out to be perfect. Although the gravitational field has to be zero at the center due to symmetry, it's much more surprising that it cancels out perfectly in the whole interior region; this is a special mathematical characteristic of a $1/r$ interaction like gravity.

Example 17: Newton's apple

Over a period of 27.3 days, the moon travels the circumference of its orbit, so using data from Appendix 5, we can calculate its speed, and solve the circular orbit condition to determine the strength of the earth's gravitational field at the moon's distance from the earth, $g = v^2/r = 2.72 \times 10^{-3} \text{ m/s}^2$, which is 3600 times smaller than the gravitational field at the earth's surface. The center-to-center distance from the moon to the earth is 60 times greater than the radius of the earth. The earth is, to a very good approximation, a sphere made up of concentric shells, each with uniform density, so the shell theorem tells us that its external gravitational field is the same as if all its mass was concentrated at its center. We already know that a gravitational energy that varies as $-1/r$ is equivalent to a gravitational field proportional to $1/r^2$, so it makes sense that a distance that is greater by a factor of 60 corresponds to a gravitational field that is $60 \times 60 = 3600$ times weaker. Note that the calculation didn't require knowledge of the earth's mass or the gravitational constant, which Newton didn't know.

In 1665, shortly after Newton graduated from Cambridge, the Great Plague forced the college to close for two years, and Newton returned to the family farm and worked intensely on scientific problems. During this productive period, he carried out this calculation, but it came out wrong, causing him to doubt his new theory of gravity. The problem was that during the plague years, he was unable to use the university's library, so he had to use a figure for the radius of the moon's orbit that he had memorized, and he forgot that the memorized value was in units of nautical miles rather than statute miles. Once he realized his mistake, he found that the calculation came out just right, and became confident that his theory was right after all. ⁹

Example 18: Weighing the earth

▷ Once Cavendish had found $G = 6.67 \times 10^{-11} \text{ J} \cdot \text{m/kg}^2$ (p. 101, example 15), it became possible to determine the mass of the earth. By the shell theorem, the gravitational energy of a mass m at a distance r from the center of the earth is $U = -GMm/r$, where M is the mass of the earth. The gravitational field is related to this by $mgdr = dU$, or $g = (1/m)dU/dr = GM/r^2$. Solving for M , we have

$$\begin{aligned} M &= gr^2/G \\ &= \frac{(9.8 \text{ m/s}^2)(6.4 \times 10^6 \text{ m})^2}{6.67 \times 10^{-11} \text{ J} \cdot \text{m/kg}^2} \\ &= 6.0 \times 10^{24} \frac{\text{m}^2 \cdot \text{kg}^2}{\text{J} \cdot \text{s}^2} \\ &= 6.0 \times 10^{24} \text{ kg} \end{aligned}$$

Example 19: Gravity inside the earth

▷ The earth is somewhat more dense at greater depths, but as an approximation let's assume it has a constant density throughout. How does its internal gravitational field vary with the distance r from the center?

▷ Let's write b for the radius of the earth. The shell theorem tell us that at a given location r , we only need to consider the mass $M_{<r}$ that is deeper than r . Under the assumption of constant density, this mass is related to the total mass of the earth by

$$\frac{M_{<r}}{M} = \frac{r^3}{b^3},$$

and by the same reasoning as in example 18,

$$g = \frac{GM_{<r}}{r^2},$$

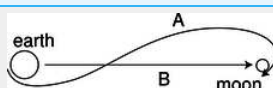
so

$$g = \frac{GMr}{b^3}.$$

In other words, the gravitational field interpolates linearly between zero at $r = 0$ and its ordinary surface value at $r = b$.

The following example applies the numerical techniques of section 2.2.

Example 20: From the earth to the moon



1 / The actual trajectory of the Apollo 11 spacecraft, A, and the straight-line trajectory, B, assumed in the example.

The Apollo 11 mission landed the first humans on the moon in 1969. In this example, we'll estimate the time it took to get to the moon, and compare our estimate with the actual time, which was 73.0708 hours from the engine burn that took the ship out of earth orbit to the engine burn that inserted it into lunar orbit. During this time, the ship was coasting with the engines off, except for a small course-correction burn, which we neglect. More importantly, we do the calculation for a straight-line trajectory rather than the real S-shaped one, so the result can only be expected to agree roughly with what really happened. The following data come from the original press kit, which NASA has scanned and posted on the Web:

initial altitude	3.363 X 10 ⁵ m
initial velocity	1.083 X 10 ⁴ m/s

The endpoint of the the straight-line trajectory is a free-fall impact on the lunar surface, which is also unrealistic (luckily for the astronauts).

The ship's energy is

$$E = -\frac{GM_em}{r} - \frac{GM_m m}{r_m - r} + \frac{1}{2}mv^2,$$

but since everything is proportional to the mass of the ship, m, we can divide it out

$$\frac{E}{m} = -\frac{GM_e}{r} - \frac{GM_m}{r_m - r} + \frac{1}{2}v^2,$$

and the energy variables in the program with names like e, k, and u are actually energies per unit mass. The program is a straightforward modification of the function time3 on page 93.

```
import math
def tmoon(vi,ri,rf,n):
    bigg=6.67e-11      # gravitational constant
    me=5.97e24         # mass of earth
    mm=7.35e22         # mass of moon
    rm=3.84e8          # earth-moon distance
    r=ri
    v=vi
    dr = (rf-ri)/n
    e=-bigg*me/ri-bigb*mm/(rm-ri)+.5*vi**2
    t=0
    for i in range(n):
        u_old = -bigg*me/r-bigb*mm/(rm-r)
        k_old = e - u_old
        v_old = math.sqrt(2.*k_old)
        r = r+dr
        u = -bigg*me/r-bigb*mm/(rm-r)
        k = e - u
        v = math.sqrt(2.*k)
        v_avg = .5*(v_old+v)
        dt=dr/v_avg
        t=t+dt
    return t
```

```
>>> re=6.378e6 # radius of earth
>>> rm=1.74e6  # radius of moon
>>> ri=re+3.363e5 # re+initial altitude
>>> rf=3.8e8-rm # earth-moon distance minus rm
```

```
>>> vi=1.083e4 # initial velocity
>>> print(tmoon(vi,ri,rf,1000)/3600.) # convert seconds to hours
59.654047441976552
```

This is pretty decent agreement, considering the wildly inaccurate trajectory assumed. It's interesting to see how much the duration of the trip changes if we increase the initial velocity by only ten percent:

```
>>> vi=1.2e4
>>> print(tmoon(vi,ri,rf,1000)/3600.)
18.177752636111677
```

The most important reason for using the lower speed was that if something had gone wrong, the ship would have been able to whip around the moon and take a “free return” trajectory back to the earth, without having to do any further burns. At a higher speed, the ship would have had so much kinetic energy that in the absence of any further engine burns, it would have escaped from the earth-moon system. The Apollo 13 mission had to take a free return trajectory after an explosion crippled the spacecraft.

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2.3.4: Atomic Phenomena

Variety is the spice of life, not of science. So far this chapter has focused on heat energy, kinetic energy, and gravitational energy, but it might seem that in addition to these there is a bewildering array of other forms of energy. Gasoline, chocolate bars, batteries, melting water --- in each case there seems to be a whole new type of energy. The physicist's psyche rebels against the prospect of a long laundry list of types of energy, each of which would require its own equations, concepts, notation, and terminology. The point at which we've arrived in the study of energy is analogous to the period in the 1960's when a half a dozen new subatomic particles were being discovered every year in particle accelerators. It was an embarrassment. Physicists began to speak of the "particle zoo," and it seemed that the subatomic world was distressingly complex. The particle zoo was simplified by the realization that most of the new particles being whipped up were simply clusters of a previously unsuspected set of fundamental particles (which were whimsically dubbed quarks, a made-up word from a line of poetry by James Joyce, "Three quarks for Master Mark.") The energy zoo can also be simplified, and it's the purpose of this section to demonstrate the hidden similarities between forms of energy as seemingly different as heat and motion.

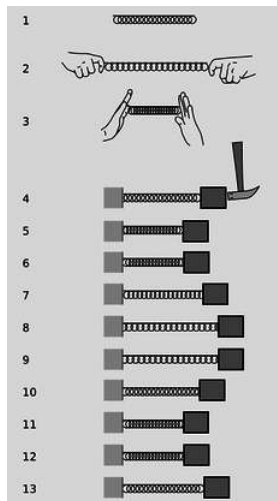
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2.3.5: Oscillations

Let's revisit the example of the stretched spring from the previous section. We know that its energy is a form of electrical energy of interacting atoms, which is nice conceptually but doesn't help us to solve problems, since we don't know how the energy, U , depends on the length of the spring. All we know is that there's an equilibrium (figure a/1), which is a local minimum of the function U . An extremely important problem which arises in this connection is how to calculate oscillatory motion around an equilibrium, as in a/4-13. Even if we did special experiments to find out how the spring's energy worked, it might seem like we'd have to go through just as much work to deal with any other kind of oscillation, such as a sapling swinging back and forth in the breeze.



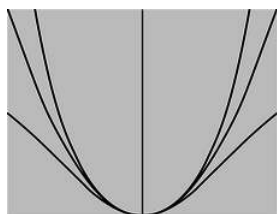
a / The spring has a minimum-energy length, 1, and energy is required in order to compress or stretch it, 2 and 3. A mass attached to the spring will oscillate around the equilibrium, 4-13.

Surprisingly, it's possible to analyze this type of oscillation in a very general and elegant manner, as long as the analysis is limited to *small* oscillations. We'll talk about the mass on the spring for concreteness, but there will be nothing in the discussion at all that is restricted to that particular physical system. First, let's choose a coordinate system in which $x = 0$ corresponds to the position of the mass where the spring is in equilibrium, and since interaction energies like U are only well defined up to an additive constant, we'll simply define it to be zero at equilibrium:

$$U(0) = 0$$

Since $x = 0$ is an equilibrium, $U(x)$ must have a local minimum there, and a differentiable function (which we assume U is) has a zero derivative at a local minimum:

$$\frac{dU}{dx}(0) = 0$$



b / Three functions with the same curvature at $x=0$.

There are still infinitely many functions that could satisfy these criteria, including the three shown in figure b, which are $x^2/2$, $x^2/2(1+x^2)$, and $(e^{3x} + e^{-3x} - 2)/18$. Note, however, how all three functions are virtually identical right near the minimum. That's because they all have the same curvature. More specifically, each function has its second derivative equal to 1 at $x = 0$, and the second derivative is a measure of curvature. We write k for the second derivative of the energy at an equilibrium point,

$$\frac{d^2U}{dx^2}(0) = k.$$

Physically, k is a measure of stiffness. For example, the heavy-duty springs in a car's shock absorbers would have a high value of k . It is often referred to as the spring constant, but we're only using a spring as an example here. As shown in figure b, any two functions that have $U(0) = 0$, $dU/dx = 0$, and $d^2U/dx^2 = k$, with the same value of k , are virtually indistinguishable for small values of x , so if we want to analyze small oscillations, it doesn't even matter which function we assume. For simplicity, we'll just use $U(x) = (1/2)kx^2$ from now on.

Now we're ready to analyze the mass-on-a-spring system, while keeping in mind that it's really only a representative example of a whole class of similar oscillating systems. We expect that the motion is going to repeat itself over and over again, and since we're not going to include frictional heating in our model, that repetition should go on forever without dying out. The most interesting thing to know about the motion would be the period, T , which is the amount of time required for one complete cycle of the motion. We might expect that the period would depend on the spring constant, k , the mass, m , and the amplitude, A , defined in figure c.¹¹



c / The amplitude would usually be defined as the distance from equilibrium to one extreme of the motion, i.e., half the total travel.

In examples like the brachistochrone and the Apollo 11 mission, it was generally necessary to use numerical techniques to determine the amount of time required for a certain motion. Once again, let's dust off the time3 function from page 93 and modify it for our purposes. For flexibility, we'll define the function $U(x)$ as a separate Python function. We really want to calculate the time required for the mass to come back to its starting point, but that would be awkward to set up, since our function works by dividing up the distance to be traveled into tiny segments. By symmetry, the time required to go from one end to the other equals the time required to come back to the start, so we'll just calculate the time for half a cycle and then double it when we return the result at the end of the function. The test at lines 16-19 is necessary because otherwise at the very end of the motion we can end up trying to take the square root of a negative number due to rounding errors.

```
import math
def u(k,x):
    return .5*k*x**2

def osc(m,k,a,n):
    x=a
    v=0
    dx = -2.*a/n
    t=0
    e = u(k,x)+.5*m*v**2
    for i in range(n):
        x_old = x
        v_old = v
        x = x+dx
        kinetic = e-u(k,x)
        if kinetic<0. :
            v=0.
            print "warning, K=",kinetic,"<0"
        else :
            v = -math.sqrt(2.*kinetic/m)
        v_avg = (v+v_old)/2.
        dt=dx/v_avg
```

```
t=t+dt
return 2.*t
```

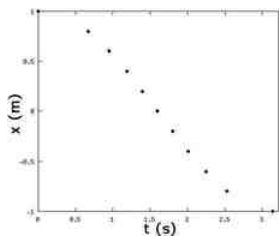
```
>>> print(osc(1.,1.,1.,100000))
warning, K= -1.43707268307e-12 <0
6.2831854132667919
```

The first thing to notice is that with this particular set of inputs ($m=1$ kg, $k = 1$ J/m², and $A = 1$ m), the program has done an excellent job of computing $2\pi = 6.2831853...$. This is Mother Nature giving us a strong hint that the problem has an algebraic solution, not just a numerical one. The next interesting thing happens when we change the amplitude from 1 m to 2 m:

```
>>> print(osc(1.,1.,2.,100000))
warning, K= -5.7482907323e-12 <0
6.2831854132667919
```

Even though the mass had to travel double the distance in each direction, the period is the same to within the numerical accuracy of the calculation!

With these hints, it seems like we should start looking for an algebraic solution. For guidance, here's a graph of x as a function of t , as calculated by the osc function with $n=10$.

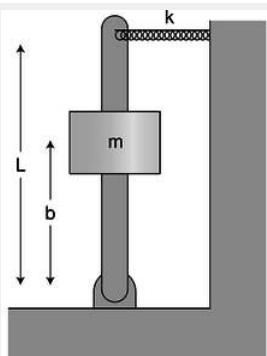


This looks like a cosine function, so let's see if a $x = A \cos(\omega t + \delta)$ is a solution to the conservation of energy equation --- it's not uncommon to try to “reverse-engineer” the cryptic results of a numerical calculation like this. The symbol $\omega = 2\pi/T$ (Greek omega), called angular frequency, is a standard symbol for the number of radians per second of oscillation. Except for the factor of 2π , it is identical to the ordinary frequency $f = 1/T$, which has units of s^{-1} or Hz (Hertz). The phase angle δ is to allow for the possibility that $t = 0$ doesn't coincide with the beginning of the motion. The energy is

$$\begin{aligned}
 E &= K + U \\
 &= \frac{1}{2}mv^2 + \frac{1}{2}kx^2 \\
 &= \frac{1}{2}m \left(\frac{dx}{dt} \right)^2 + \frac{1}{2}kx^2 \\
 &= \frac{1}{2}m [-A\omega \sin(\omega t + \delta)]^2 + \frac{1}{2}k [A \cos(\omega t + \delta)]^2 \\
 &= \frac{1}{2}A^2 [m\omega^2 \sin^2(\omega t + \delta) + k \cos^2(\omega t + \delta)]
 \end{aligned}$$

According to conservation of energy, this has to be a constant. Using the identity $\sin^2 + \cos^2 = 1$, we can see that it will be a constant if we have $m\omega^2 = k$, or $\omega = \sqrt{k/m}$, i.e., $T = 2\pi\sqrt{m/k}$. Note that the period is independent of amplitude.

Example 23: A spring and a lever



d / Example 23. The rod pivots on the hinge at the bottom.

▷ What is the period of small oscillations of the system shown in the figure? Neglect the mass of the lever and the spring. Assume that the spring is so stiff that gravity is not an important effect. The spring is relaxed when the lever is vertical.

▷ This is a little tricky, because the spring constant k , although it is relevant, is *not* the k we should be putting into the equation $T = 2\pi\sqrt{m/k}$. The k that goes in there has to be the second derivative of U with respect to the position, x , of the mass that's moving. The energy U stored in the spring depends on how far the *tip* of the lever is from the center. This distance equals $(L/b)x$, so the energy in the spring is

$$U = \frac{1}{2}k\left(\frac{L}{b}x\right)^2 \\ = \frac{kL^2}{2b^2}x^2,$$

and the k we have to put in $T = 2\pi\sqrt{m/k}$ is

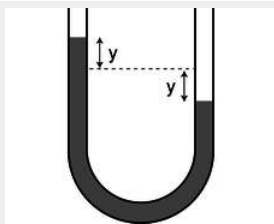
$$\frac{d^2U}{dx^2} = \frac{kL^2}{b^2}.$$

The result is

$$T = 2\pi\sqrt{\frac{mb^2}{kL^2}} \\ = \frac{2\pi b}{L}\sqrt{\frac{m}{k}}$$

The leverage of the lever makes it as if the spring was stronger, and decreases the period of the oscillations by a factor of b/L .

Example 24: Water in a U-shaped tube



e / Water in a U-shaped tube.

▷ What is the period of oscillation of the water in figure e?

▷ In example 13 on p. 89, we found $U(y) = \rho g A y^2$, so the “spring constant,” which really isn't a spring constant here at all, is

$$k = \frac{d^2 U}{dy^2} = 2\rho g A.$$

This is an interesting example, because k can be calculated without any approximations, but the kinetic energy requires an approximation, because we don't know the details of the pattern of flow of the water. It could be very complicated. There will be a tendency for the water near the walls to flow more slowly due to friction, and there may also be swirling, turbulent motion. However, if we make the approximation that all the water moves with the same velocity as the surface, dy/dt , then the mass-on-a-spring analysis applies. Letting L be the total length of the filled part of the tube, the mass is $\rho L A$, and we have

$$\begin{aligned} T &= 2\pi \sqrt{m/k} \\ &= 2\pi \sqrt{\frac{\rho L A}{2\rho g A}} \\ &= 2\pi \sqrt{\frac{L}{2g}}. \end{aligned}$$

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2.3.6: Footnotes

1. An entertaining account of this form of quackery is given in **Voodoo Science: The Road from Foolishness to Fraud**, Robert Park, Oxford University Press, 2000. Until reading this book, I hadn't realized the degree to which pseudoscience had penetrated otherwise respectable scientific organizations like NASA.
2. Although the definition refers to the Celsius scale of temperature, it's not necessary to give an operational definition of the temperature concept in general (which turns out to be quite a tricky thing to do completely rigorously); we only need to establish two specific temperatures that can be reproduced on thermometers that have been calibrated in a standard way. Heat and temperature are discussed in more detail in section 2.4, and in chapter 5. Conceptually, heat is a measure of energy, whereas temperature relates to how concentrated that energy is.
3. It's not at all obvious that the solution would work out in the earth's frame of reference, although Galilean relativity states that it doesn't matter which frame we use. Chapter 3 discusses the relationship between conservation of energy and Galilean relativity.
4. From Joule's point of view, the point of the experiment was different. At that time, most physicists believed that heat was a quantity that was conserved separately from the rest of the things to which we now refer as energy, i.e., mechanical energy. Separate units of measurement had been constructed for heat and mechanical energy, but Joule was trying to show that one could convert back and forth between them, and that it was actually their sum that was conserved, if they were both expressed in consistent units. His main result was the conversion factor that would allow the two sets of units to be reconciled. By showing that the conversion factor came out the same in different types of experiments, he was supporting his assertion that heat was not separately conserved. From Joule's perspective or from ours, the result is to connect the mysterious, invisible phenomenon of heat with forms of energy that are visible properties of objects, i.e., mechanical energy.
5. If you've had a previous course in physics, you may have seen this presented not as an empirical result but as a theoretical one, derived from Newton's laws, and in that case you might feel you're being cheated here. However, I'm going to reverse that reasoning and derive Newton's laws from the conservation laws in chapter 3. From the modern perspective, conservation laws are more fundamental, because they apply in cases where Newton's laws don't.
6. Système International
7. There is a mathematical loophole in this argument that would allow the object to hover for a while with zero velocity and zero acceleration. This point is discussed on page 910.
8. There is a hidden assumption here, which is that the sun doesn't move. Actually the sun wobbles a little because of the planets' gravitational interactions with it, but the wobble is small due to the sun's large mass, so it's a pretty good approximation to assume the sun is stationary. Chapter 3 provides the tools to analyze this sort of thing completely correctly --- see p. 142.
9. Some historians are suspicious that the story of the apple and the mistake in conversions may have been fabricated by Newton later in life. The conversion incident may have been a way of explaining his long delay in publishing his work, which led to a conflict with Leibniz over priority in the invention of calculus.
10. Subsection 6.1.5 presents some evidence for the Big Bang theory.
11. Many kinds of oscillations are possible, so there is no standard definition of the amplitude. For a pendulum, the natural definition would be in terms of an angle. For a radio transmitter, we'd use some kind of electrical units.

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11. The following table gives the amount of energy required in order to heat, melt, or boil a gram of water.

heat 1 g of ice by 1°C	2.05 J
melt 1 g of ice	333 J
heat 1 g of liquid by 1°C	4.19 J
boil 1 g of water	2500 J
heat 1 g of steam by 1°C	2.01 J

(a) How much energy is required in order to convert 1.00 g of ice at -20°C into steam at 137°C ? (answer check available at lightandmatter.com)

(b) What is the minimum amount of hot water that could melt 1.00 g of ice? (answer check available at lightandmatter.com)

12. Anya climbs to the top of a tree, while Ivan climbs half-way to the top. They both drop pennies to the ground. Compare the kinetic energies and velocities of the pennies on impact, using ratios.

13. Anya and Ivan lean over a balcony side by side. Anya throws a penny downward with an initial speed of 5 m/s. Ivan throws a penny upward with the same speed. Both pennies end up on the ground below. Compare their kinetic energies and velocities on impact.

14. (a) A circular hoop of mass m and radius r spins like a wheel while its center remains at rest. Let ω (Greek letter omega) be the number of radians it covers per unit time, i.e., $\omega = 2\pi/T$, where the period, T , is the time for one revolution. Show that its kinetic energy equals $(1/2)m\omega^2r^2$.

(b) Show that the answer to part a has the right units. (Note that radians aren't really units, since the definition of a radian is a unitless ratio of two lengths.)

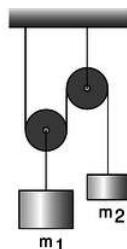
(c) If such a hoop rolls with its center moving at velocity v , its kinetic energy equals $(1/2)mv^2$, plus the amount of kinetic energy found in part a. Show that a hoop rolls down an inclined plane with half the acceleration that a frictionless sliding block would have.

15. On page 83, I used the chain rule to prove that the acceleration of a free-falling object is given by $a = -g$. In this problem, you'll use a different technique to prove the same thing. Assume that the acceleration is a constant, a , and then integrate to find v and y , including appropriate constants of integration. Plug your expressions for v and y into the equation for the total energy, and show that $a = -g$ is the only value that results in constant energy.



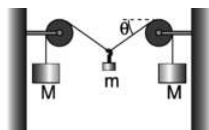
b / Problem 16.

16. The figure shows two unequal masses, m_1 and m_2 , connected by a string running over a pulley. Find the acceleration.
 \hwhint{hwhint:atwood-energy-sn}(answer check available at lightandmatter.com)



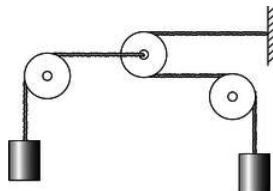
c / Problem 17.

17. What ratio of masses will balance the pulley system shown in the figure?
 \hwhint{hwhint:pulley}



d / Problem 18.

18. (a) For the apparatus shown in the figure, find the equilibrium angle θ in terms of the two masses. (answer check available at lightandmatter.com)
 (b) Interpret your result in the case of $M \gg m$ (M much greater than m). Does it make sense physically?
 (c) For what combinations of masses would your result give nonsense? Interpret this physically. \hwhint{hwhint:tightropish}



e / Problem 19.

19. In the system shown in the figure, the pulleys on the left and right are fixed, but the pulley in the center can move to the left or right. The two hanging masses are identical, and the pulleys and ropes are all massless. Find the upward acceleration of the mass on the left, in terms of g only. \hwhint{hwhint:funkyatwood} (answer check available at lightandmatter.com)
 20. Two atoms will interact via electrical forces between their protons and electrons. One fairly good approximation to the electrical energy is the Lennard-Jones formula,

$$U(r) = k \left[\left(\frac{a}{r} \right)^{12} - 2 \left(\frac{a}{r} \right)^6 \right],$$

where r is the center-to-center distance between the atoms and k is a positive constant. Show that (a) there is an equilibrium point at $r = a$,

- (b) the equilibrium is stable, and
 (c) the energy required to bring the atoms from their equilibrium separation to infinity is k . \hwhint{hwhint:lennardjones}

21. The International Space Station orbits at an altitude of about 360 to 400 km. What is the gravitational field of the earth at this altitude? (answer check available at lightandmatter.com)

22. (a) A geosynchronous orbit is one in which the satellite orbits above the equator, and has an orbital period of 24 hours, so that it is always above the same point on the spinning earth. Calculate the altitude of such a satellite. (answer check available at lightandmatter.com)

(b) What is the gravitational field experienced by the satellite? Give your answer as a percentage in relation to the gravitational field at the earth's surface. \hwhint{hwhint:geosynch} (answer check available at lightandmatter.com)

23. Astronomers calculating orbits of planets often work in a nonmetric system of units, in which the unit of time is the year, the unit of mass is the sun's mass, and the unit of distance is the astronomical unit (A.U.), defined as half the long axis of the earth's orbit. In these units, find an exact expression for the gravitational constant, G . (answer check available at lightandmatter.com)

24. The star Lalande 21185 was found in 1996 to have two planets in roughly circular orbits, with periods of 6 and 30 years. What is the ratio of the two planets' orbital radii? (answer check available at lightandmatter.com)

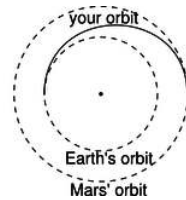
25. A projectile is moving directly away from a planet of mass M at exactly escape velocity. (a) Find r , the distance from the projectile to the center of the planet, as a function of time, t , and also find $v(t)$. (answer check available at lightandmatter.com)

(b) Check the units of your answer.

(c) Does v show the correct behavior as t approaches infinity? \hwhint{hwhint:escape2}

26. The purpose of this problem is to estimate the height of the tides. The main reason for the tides is the moon's gravity, and we'll neglect the effect of the sun. Also, real tides are heavily influenced by landforms that channel the flow of water, but we'll think of the earth as if it was completely covered with oceans. Under these assumptions, the ocean surface should be a surface of constant U/m . That is, a thimbleful of water, m , should not be able to gain or lose any gravitational energy by moving from one point on the

ocean surface to another. If only the spherical earth's gravity was present, then we'd have $U/m = -GM_e/r$, and a surface of constant U/m would be a surface of constant r , i.e., the ocean's surface would be spherical. Taking into account the moon's gravity, the main effect is to shift the center of the sphere, but the sphere also becomes slightly distorted into an approximately ellipsoidal shape. (The shift of the center is not physically related to the tides, since the solid part of the earth tends to be centered within the oceans; really, this effect has to do with the motion of the whole earth through space, and the way that it wobbles due to the moon's gravity.) Determine the amount by which the long axis of the ellipsoid exceeds the short axis. \hwhint{hwhint:tides}



f / Problem 27.

27. You are considering going on a space voyage to Mars, in which your route would be half an ellipse, tangent to the Earth's orbit at one end and tangent to Mars' orbit at the other. Your spacecraft's engines will only be used at the beginning and end, not during the voyage. How long would the outward leg of your trip last? (The orbits of Earth and Mars are nearly circular, and Mars's is bigger by a factor of 1.52.) (answer check available at lightandmatter.com)

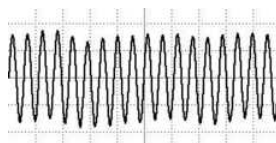
28. When you buy a helium-filled balloon, the seller has to inflate it from a large metal cylinder of the compressed gas. The helium inside the cylinder has energy, as can be demonstrated for example by releasing a little of it into the air: you hear a hissing sound, and that sound energy must have come from somewhere. The total amount of energy in the cylinder is very large, and if the valve is inadvertently damaged or broken off, the cylinder can behave like bomb or a rocket.

Suppose the company that puts the gas in the cylinders prepares cylinder A with half the normal amount of pure helium, and cylinder B with the normal amount. Cylinder B has twice as much energy, and yet the temperatures of both cylinders are the same. Explain, at the atomic level, what form of energy is involved, and why cylinder B has twice as much.

29. Explain in terms of conservation of energy why sweating cools your body, even though the sweat is at the same temperature as your body. Describe the forms of energy involved in this energy transformation. Why don't you get the same cooling effect if you wipe the sweat off with a towel? Hint: The sweat is evaporating.

30. [This problem is now problem 3-73.]

31. All stars, including our sun, show variations in their light output to some degree. Some stars vary their brightness by a factor of two or even more, but our sun has remained relatively steady during the hundred years or so that accurate data have been collected. Nevertheless, it is possible that climate variations such as ice ages are related to long-term irregularities in the sun's light output. If the sun was to increase its light output even slightly, it could melt enough Antarctic ice to flood all the world's coastal cities. The total sunlight that falls on Antarctica amounts to about 1×10^{16} watts. In the absence of natural or human-caused climate change, this heat input to the poles is balanced by the loss of heat via winds, ocean currents, and emission of infrared light, so that there is no net melting or freezing of ice at the poles from year to year. Suppose that the sun changes its light output by some small percentage, but there is no change in the rate of heat loss by the polar caps. Estimate the percentage by which the sun's light output would have to increase in order to melt enough ice to raise the level of the oceans by 10 meters over a period of 10 years. (This would be enough to flood New York, London, and many other cities.) Melting 1 kg of ice requires 3×10^3 J.



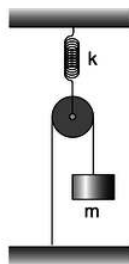
g / Problem 32.

32. The figure shows the oscillation of a microphone in response to the author whistling the musical note "A." The horizontal axis, representing time, has a scale of 1.0 ms per square. Find the period T , the frequency f , and the angular frequency ω . (answer check available at lightandmatter.com)

33. (a) A mass m is hung from a spring whose spring constant is k . Write down an expression for the total interaction energy of the system, U , and find its equilibrium position. \hwhint{hwhint:hangfromspring}

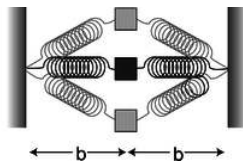
(b) Explain how you could use your result from part a to determine an unknown spring constant.

34. A certain mass, when hung from a certain spring, causes the spring to stretch by an amount h compared to its equilibrium length. If the mass is displaced vertically from this equilibrium, it will oscillate up and down with a period T_{osc} . Give a numerical comparison between T_{osc} and T_{fall} , the time required for the mass to fall from rest through a height h , when it isn't attached to the spring. (You will need the result of problem 33).(answer check available at lightandmatter.com)



h / Problem 35.

35. Find the period of vertical oscillations of the mass m . The spring, pulley, and ropes have negligible mass.\hwhint{hwhint:pulleyandspring}(answer check available at lightandmatter.com)



i / Problem 36.

36. The equilibrium length of each spring in the figure is b , so when the mass m is at the center, neither spring exerts any force on it. When the mass is displaced to the side, the springs stretch; their spring constants are both k .

(a) Find the energy, U , stored in the springs, as a function of y , the distance of the mass up or down from the center.(answer check available at lightandmatter.com)

(b) Show that the period of small up-down oscillations is infinite.



j / Problem 37.

37. Two springs with spring constants k_1 and k_2 are put together end-to-end. Let x_1 be the amount by which the first spring is stretched relative to its equilibrium length, and similarly for x_2 . If the combined double spring is stretched by an amount b relative to its equilibrium length, then $b = x_1 + x_2$. Find the spring constant, K , of the combined spring in terms of k_1 and k_2 . \hwhint{hwhint:springsseries}\hwans{hwans:springsseries} (answer check available at lightandmatter.com)

38. A mass m on a spring oscillates around an equilibrium at $x = 0$. Any function $U(x)$ with an equilibrium at $x = 0$ can be approximated as $U(x) = (1/2)kx^2$, and if the energy is symmetric with respect to positive and negative values of x , then the next level of improvement in such an approximation would be $U(x) = (1/2)kx^2 + bx^4$. The general idea here is that any smooth function can be approximated locally by a polynomial, and if you want a better approximation, you can use a polynomial with more terms in it. When you ask your calculator to calculate a function like $\sin$ or e^x , it's using a polynomial approximation with 10 or 12 terms. Physically, a spring with a positive value of b gets stiffer when stretched strongly than an "ideal" spring with $b = 0$. A spring with a negative b is like a person who cracks under stress --- when you stretch it too much, it becomes more elastic than an ideal spring would. We should not expect any spring to give totally ideal behavior no matter no matter how much it is stretched. For example, there has to be some point at which it breaks.

Do a numerical simulation of the oscillation of a mass on a spring whose energy has a nonvanishing b . Is the period still independent of amplitude? Is the amplitude-independent equation for the period still approximately valid for small enough amplitudes? Does the addition of a positive x^4 term tend to increase the period, or decrease it? Include a printout of your program and its output with your homework paper.

39. An idealized pendulum consists of a pointlike mass m on the end of a massless, rigid rod of length L . Its amplitude, θ , is the angle the rod makes with the vertical when the pendulum is at the end of its swing. Write a numerical simulation to determine the period of the pendulum for any combination of m , L , and θ . Examine the effect of changing each variable while manipulating the others.
40. A ball falls from a height h . Without air resistance, the time it takes to reach the floor is $\sqrt{2h/g}$. A numerical version of this calculation was given in program time2 on page 92. Now suppose that air resistance is not negligible. For a smooth sphere of radius r , moving at speed v through air of density ρ , the amount of energy dQ dissipated as heat as the ball falls through a height dy is given (ignoring signs) by $dQ = (\pi/4)\rho v^2 r^2 dy$. Modify the program to incorporate this effect, and find the resulting change in the fall time in the case of a 21 g ball of radius 1.0 cm, falling from a height of 1.0 m. The density of air at sea level is about 1.2 kg/m^3 . Turn in a printout of both your program and its output. Answer: 0.34 ms.
41. The factorial of an integer n , written $n!$, is defined as the product of all the positive integers less than or equal to n . For example, $3! = 1 \times 2 \times 3 = 6$. Write a Python program to compute the factorial of a number. Test it with a small number whose factorial you can check by hand. Then use it to compute $30!$. (Python computes integer results with unlimited precision, so you won't get any problems with rounding or overflows.) Turn in a printout of your program and its output, including the test.
42. Estimate the kinetic energy of a buzzing fly's wing.

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CHAPTER OVERVIEW

2.4: Conservation of Momentum



Forces transfer momentum to the girl.

"I think, therefore I am." I hope that posterity will judge me kindly, not only as to the things which I have explained, but also to those which I have intentionally omitted so as to leave to others the pleasure of discovery. -- *René Descartes*

[2.4.1: Momentum In One Dimension](#)

[2.4.2: Force In One Dimension](#)

[2.4.3: Resonance](#)

[2.4.4: Motion In Three Dimensions](#)

[2.4.5: Footnotes](#)

[2.4.E: Problems](#)

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2.4.1: Momentum In One Dimension

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2.4.2: Force In One Dimension

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