CALCULUS-BASED MECHANICS (WORKBOOK)



Andrew Totah-McCarty

Chabot College

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1: KINEMATICS

1.1: Position and Velocity
1.2: Kinematics Graphs
1.3: Kinematics with Calculus and Vectors
1.4: Projectiles

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1.1: Position and Velocity

Standard practiced: K1 (motion maps and motion graphs). This activity builds position and velocity on motion maps. The next activity adds changing velocity.

A motion map shows only positions: one dot for each photo of an object, labeled with its clock reading. In this workbook, the photos are always 1 second apart. You will mark up the motion map with arrows to show changes in position and velocity.

Whiteboard first: in class, your group drafts its key work on the whiteboard. When the group agrees, everyone copies the final version into their notes.

Colors: use 3 or 4 colors to highlight. Using the same color for the same quantity in a diagram and in an equation shows where that quantity appears in each.

Before class: complete each Explore section on your own. Every Explore answer can be found in the Model above it.

The kinematics model. In every kinematics activity, "Model the scenario" means building these parts so that each one agrees with the others:

- **Physical story:** what happens, in words, with a positive direction chosen.
- **Motion map:** positions at equal time intervals, marked up with change in position, velocity, and change in velocity arrows.
- **Graphs:** position, velocity, and acceleration against clock reading.
- **Equations:** update equations (initial + change = final), and later, functions grouped by powers of t.
- **Math solution:** circle each unknown, solve, and check the answer against the other parts.

This activity builds the physical story, the motion map, and the update equations. Later activities add the other parts.

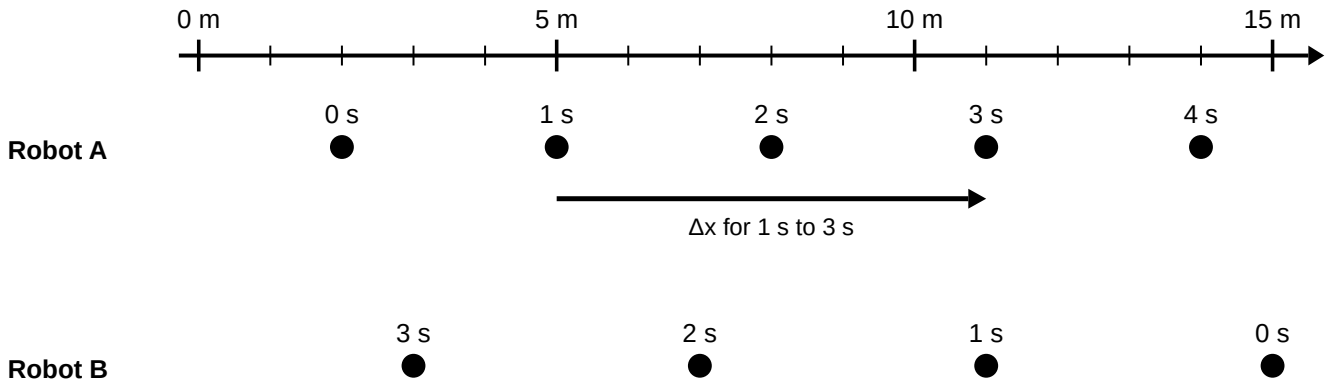
1. Where and When

Model

Physical story

Two delivery robots roll along a straight sidewalk, each at a steady pace. A camera above the sidewalk takes a photo every second. Positions are measured from a crack in the sidewalk, and to the right is positive.

Motion map



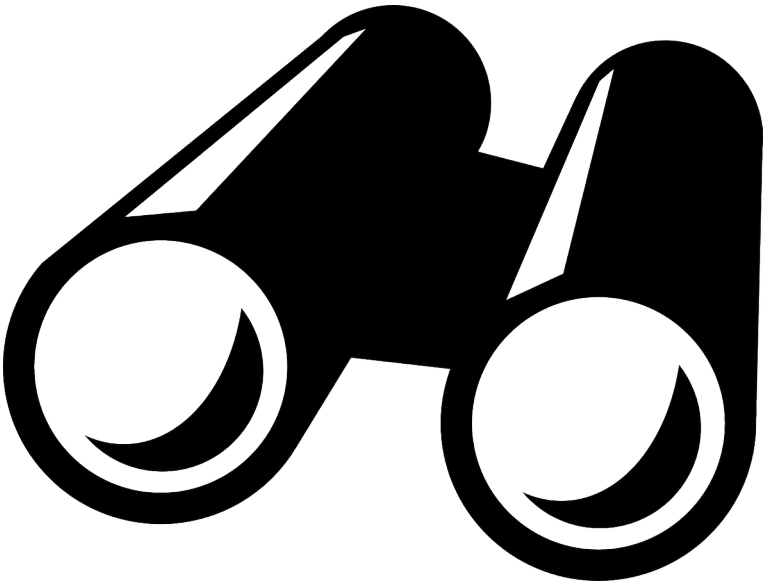
Position at a clock reading

$\vec{x}\,(1\,\mathrm{s}) = 5\,\mathrm{m}$ means Robot A's position at the clock reading 1 s is 5 m.

Initial, final, and change

An interval runs from one clock reading to a later one. Robot A's position at the start of the interval is its **initial position**, $\vec{x}_i$. Its position at the end is its **final position**, $\vec{x}_f$. Its **change in position**, or displacement, $\Delta\vec{x}$, is an arrow from the initial dot to the final dot. An arrow to the right is positive, and an arrow to the left is negative.

Interval	Initial position	Final position	Change in position
Robot A, 1 s to 3 s	$\vec{x}\,(1\,\mathrm{s}) = 5\,\mathrm{m}$	$\vec{x}\,(3\,\mathrm{s}) = 11\,\mathrm{m}$	$+6\,\mathrm{m}$
Robot A, 0 s to 2 s			
Robot A, 2 s to 4 s			
Robot B, 1 s to 3 s			



Explore

1. Find each position: Robot A at 0 s, Robot A at 4 s, and Robot B at 2 s. Write each one like $\vec{x}\,(1\,\mathrm{s}) = 5\,\mathrm{m}$.

2. For the interval 2 s to 4 s, which clock reading gives the initial position and which gives the final position? How do you know?

3. Complete the table in the Model. For each row, mark the Δx arrow on the motion map, from the initial dot to the final dot, and label its length.



Critical thinking & Concept Invention

Launch: Manager, read the physical story aloud. Then have each person point to one dot and say the position and clock reading, like "x of 1 second is 5 meters."

Compare: Check your table and Δx arrows with your group.

1. Highlight Robot A's initial position, change in position, and final position for 1 s to 3 s in three colors, in the table and on the motion map (initial dot, arrow, final dot). Then answer on your whiteboard: how do you find a final position if you know the initial position and the change? Write your answer in words, then as an equation using $\vec{x}_i$, $\Delta\vec{x}$, and $\vec{x}_f$. Check it on every row of the table, and highlight its terms in the same three colors. Finalize it on your whiteboard as a group, then copy it into your notes below.

2. Robot B's Δx arrow from 1 s to 3 s is 8 m long, but its change in position is −8 m. Does your equation still work for Robot B? What does the sign of Δx tell you? Finalize it on your whiteboard as a group, then copy it into your notes below.

3. On your whiteboard, rearrange your equation to find the change in position from the initial and final positions. Use it to find Robot A's change in position from 0 s to 4 s, then check your answer with an arrow on the motion map. Finalize it on your whiteboard as a group, then copy it into your notes below.

4. A group member writes $\vec{x}\,(1\,\mathrm{s}) + \Delta\vec{x} = \vec{x}\,(4\,\mathrm{s})$. Is this the same equation as yours? Which term is the initial position, and which is the final? Write the same kind of equation for Robot B from 0 s to 3 s, with numbers.

   _____ + _____ = _____

5. A group member says that Robot B's change in position from 0 s to 3 s must be positive, because its final position, 3 m, is positive. Write the Claim, Evidence, Reasoning template on your whiteboard and answer the question: do you agree? Finalize it on your whiteboard as a group, then copy it into your notes below.

   Claim

   Evidence

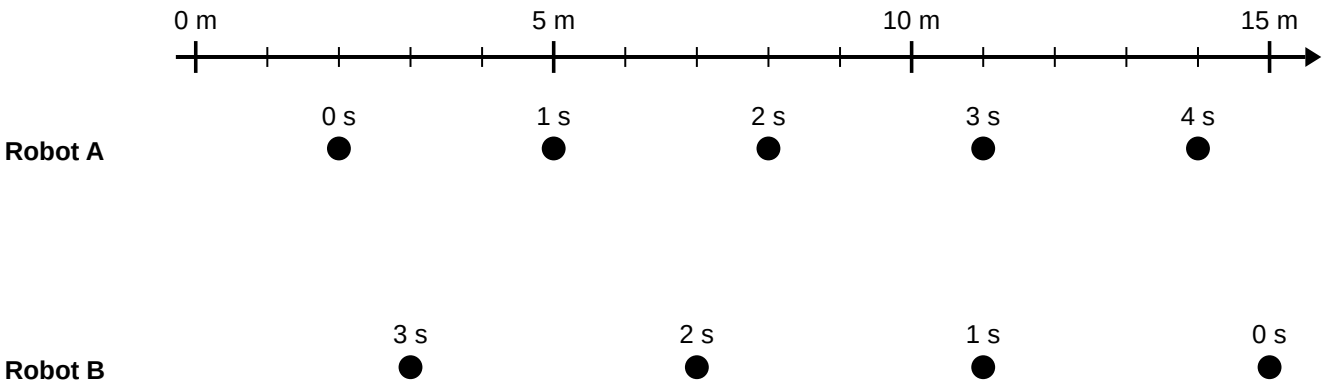
   Reasoning

Check in: Reflector, check that everyone can explain how to find a final position from an initial position and a change. Set your cup.

2. How Fast and Which Way

Model

These are the same two robots from Part 1. Each robot rolls at a steady pace.

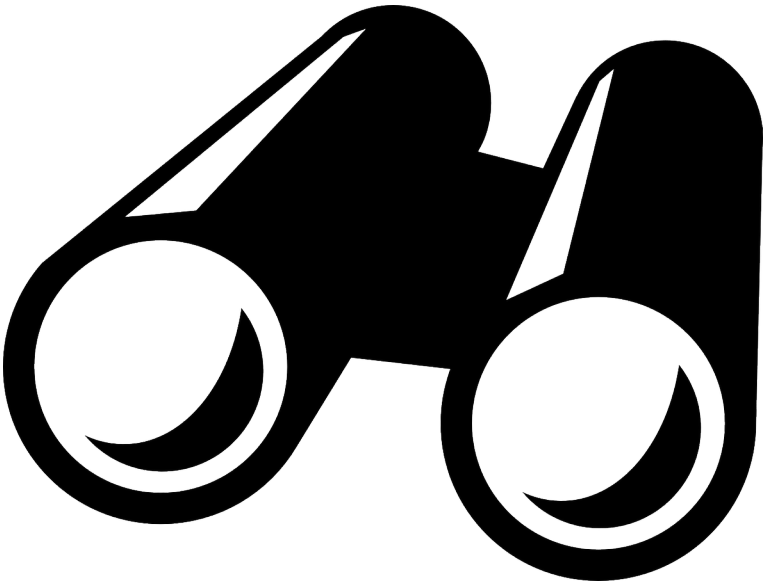


Time interval

The time interval, Δt, is the final clock reading minus the initial clock reading. From 1 s to 3 s, $\Delta t = 2\,\text{s}$.

Robot A

Interval	Δx	Δt	Δx/Δt
0 s to 1 s	$+3\,\text{m}$	$1\,\text{s}$	$+3\,\text{m/s}$
1 s to 2 s			
0 s to 2 s			
1 s to 4 s			
0 s to 4 s			



Explore

1. Complete Robot A's table in the Model. Use the motion map or your equation from Part 1 to find each Δx.
2. Which interval has the largest Δx? Which has the largest $\Delta x/\Delta t$?



Critical thinking & Concept Invention

Launch: Manager, have each person say one $\Delta x/\Delta t$ value from the table and its interval.

1. What do you notice about $\Delta x/\Delta t$ for Robot A? On your whiteboard, use the spacing of the dots on the motion map to explain why. Finalize it on your whiteboard as a group, then copy it into your notes below.

2. The ratio Δx/Δt tells how far and which way an object moves per second. It is called the **velocity**: $\vec{v} = \frac{\Delta \vec{x}}{\Delta t}$. On your whiteboard, make a Δx, Δt, and Δx/Δt table for Robot B, using three intervals of your choice. What is Robot B's velocity, with its sign? Finalize it on your whiteboard as a group, then copy it into your notes below.

3. Mark velocity arrows on the motion map: just below each dot, draw that robot's velocity arrow, using 1 m on the ruler for each 1 m/s. On your whiteboard, compare a velocity arrow with Robot A's Δx arrow for a 1 s interval, and with its Δx arrow for a 2 s interval. Finalize it on your whiteboard as a group, then copy it into your notes below.

4. A group member says Robot B's velocity (−4 m/s) is smaller than Robot A's velocity (+3 m/s), because −4 is less than 3. Write the Claim, Evidence, Reasoning template on your whiteboard and answer the question: do you agree? Finalize it on your whiteboard as a group, then copy it into your notes below.

   Claim

   Evidence

   Reasoning

5. How can you find Δx if you know the velocity and the time interval? On your whiteboard, answer that, then put it into your position update equation from Part 1. Use your new equation to find Robot A's position at 10 s and the clock reading when Robot B reaches 0 m. Circle each unknown. Finalize it on your whiteboard as a group, then copy it into your notes below.

Check in: Reflector, check that everyone can explain why $\Delta x/\Delta t$ is the same for every interval of Robot A. Set your cup.

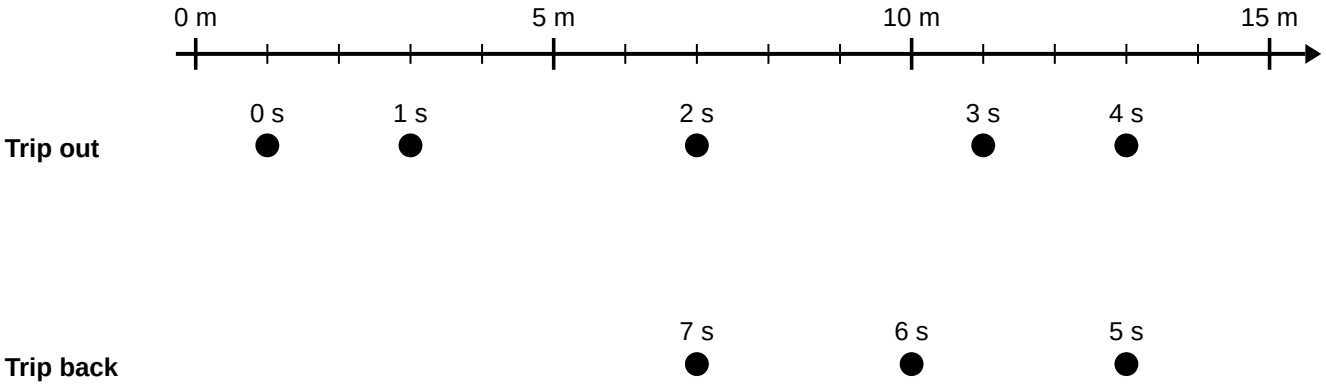
3. When the Pace Changes

Model

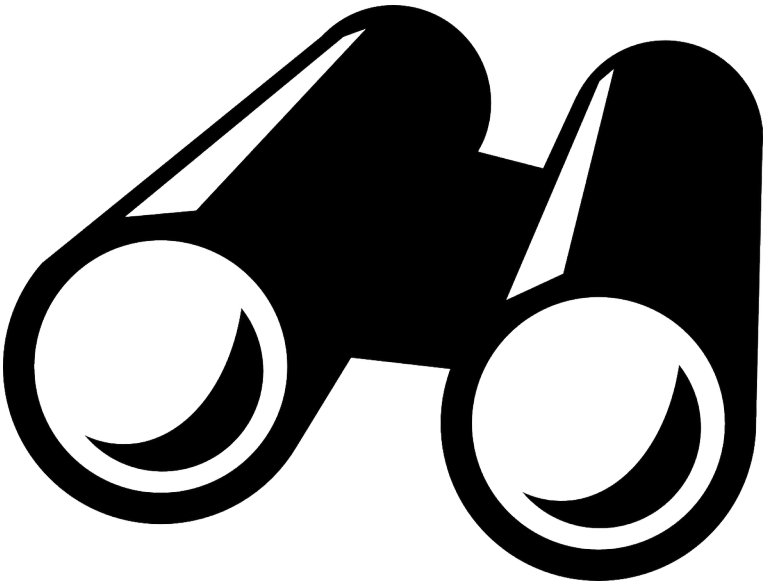
Physical story

A third delivery robot makes a delivery. It speeds up, slows down to stop at a doorstep for a moment, then starts back. The camera still takes a photo every second, and to the right is positive.

Motion map



Interval	Δx	Δt	$\Delta x/\Delta t$
0 s to 1 s	$+2\,\text{m}$	$1\,\text{s}$	$+2\,\text{m/s}$
1 s to 2 s			
0 s to 2 s			
0 s to 4 s			
4 s to 5 s			
4 s to 6 s			



Explore

1. Mark a Δx arrow on the motion map for each second, and label its length. If the robot did not move, write 0 instead of drawing an arrow.

2. Complete the table in the Model.



Critical thinking & Concept Invention

Launch: Manager, read the physical story aloud. Then have each person point to a part of the motion map where the robot was fast, slow, or at rest.

Compare: Check your table with your group.

1. Is Δx/Δt the same for every interval of this robot? On your whiteboard, explain what is different about this motion map compared with Robot A's. Finalize it on your whiteboard as a group, then copy it into your notes below.

2. When the velocity changes, Δx/Δt for an interval is the **average velocity** for that interval: $\vec{v}_{\text{avg}} = \frac{\Delta \vec{x}}{\Delta t}$. From 0 s to 4 s, the average velocity is +3 m/s. Does +3 m/s describe the robot's motion from 0 s to 1 s? On your whiteboard, decide which intervals describe the robot's motion best, moment by moment, and explain. Finalize it on your whiteboard as a group, then copy it into your notes below.

3. From 0 s to 6 s, the robot rolled 12 m out and 3 m back, 15 m in all. A group member says its average velocity from 0 s to 6 s was (15 m)/(6 s) = 2.5 m/s. Write the Claim, Evidence, Reasoning template on your whiteboard and answer the question: do you agree? Finalize it on your whiteboard as a group, then copy it into your notes below.

   Claim

   Evidence

   Reasoning

4. Use the average velocity from 0 s to 4 s and $\vec{x}_i + \vec{v}\Delta t = \vec{x}_f$ to predict the robot's position at 1 s. Compare it with the motion map. On your whiteboard, explain when this equation gives the right position for any time interval. Finalize it on your whiteboard as a group, then copy it into your notes below.

Check in: Reflector, check that everyone can explain the difference between velocity and average velocity. Set your cup.

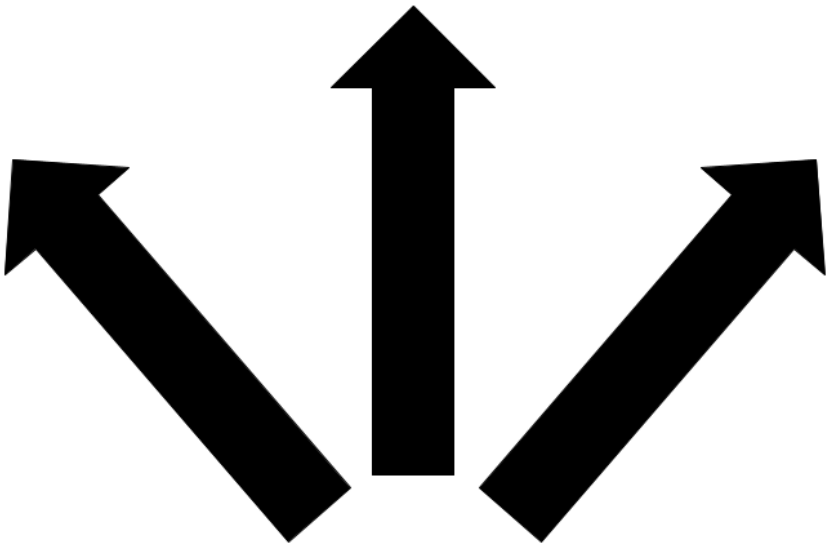
Summary

- A motion map shows positions, one dot per clock reading. $\vec{x}\,(1\,\mathrm{s})$ is the position at the clock reading 1 s. For an interval, the earlier reading gives the initial position and the later one gives the final position.
- The change in position, Δx, is an arrow from the initial dot to the final dot. Its sign shows its direction. The position update equation and its rearranged form are

$$\vec{x}_i + \Delta\vec{x} = \vec{x}_f$$

$$\Delta\vec{x} = \vec{x}_f - \vec{x}_i$$

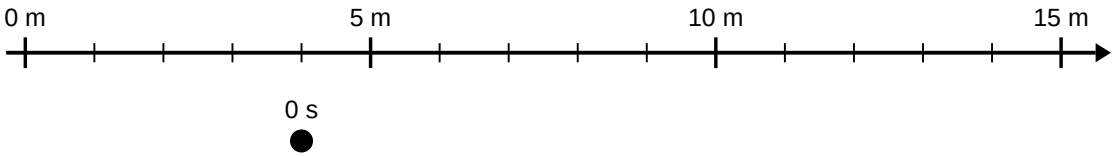
- Velocity is the change in position per time, $\vec{v} = \frac{\Delta\vec{x}}{\Delta t}$. When the dots are equally spaced, the velocity is constant, and Δx/Δt is the same for every interval. The length of a velocity arrow is the speed, and in one dimension the sign shows the direction.
- With a constant velocity, $\vec{x}_i + \vec{v}\Delta t = \vec{x}_f$ works for any time interval.
- When the velocity changes, Δx/Δt is the average velocity for that interval. Shorter intervals describe the motion better. Average velocity uses the change in position, not the total distance.
- Update equations follow one pattern: final = initial + change. You will use this pattern for velocity in the next activity, and for momentum later in the course.



Apply

Build each piece on your whiteboard, finalize it as a group, then copy it into your notes before starting the next piece.

1. Model the scenario. A skateboarder starts at 4 m and rolls at +3 m/s for 2 s, stops for 1 s, then rolls back at –2 m/s for 3 s. Draw the motion map, with a dot for each second. Then mark a Δx arrow for each second. Write a position update equation to find the skateboarder's position at 6 s, and circle the unknown.



2. The skateboarder's average velocity from 0 s to 6 s is 0 m/s. A friend says that means the skateboarder never moved. Write the Claim, Evidence, Reasoning template on your whiteboard and answer the question: do you agree? Finalize it on your whiteboard as a group, then copy it into your notes below.

   Claim

   Evidence

   Reasoning

Check in: Reflector, before the consensus share-out, check that your group agrees on both answers.

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1.2: Kinematics Graphs

Standard practiced: K1 (motion maps and motion graphs). This activity connects motion maps to graphs. The K1 quiz covers this activity and Motion Maps: Position and Velocity.

A graph of position or velocity against clock reading tells the same story as a motion map. In this activity, you will find the slopes and areas of graphs and connect them to the arrows on motion maps.

Whiteboard first: in class, your group drafts its key work on the whiteboard. When the group agrees, everyone copies the final version into their notes.

Colors: use 3 or 4 colors to highlight. Using the same color for the same quantity in a diagram and in an equation shows where that quantity appears in each.

Before class: complete each Explore section on your own. Every Explore answer can be found in the Model above it.

The kinematics model: physical story, motion map, graphs, equations, and math solution. This activity adds the graphs.

1. Position Graphs

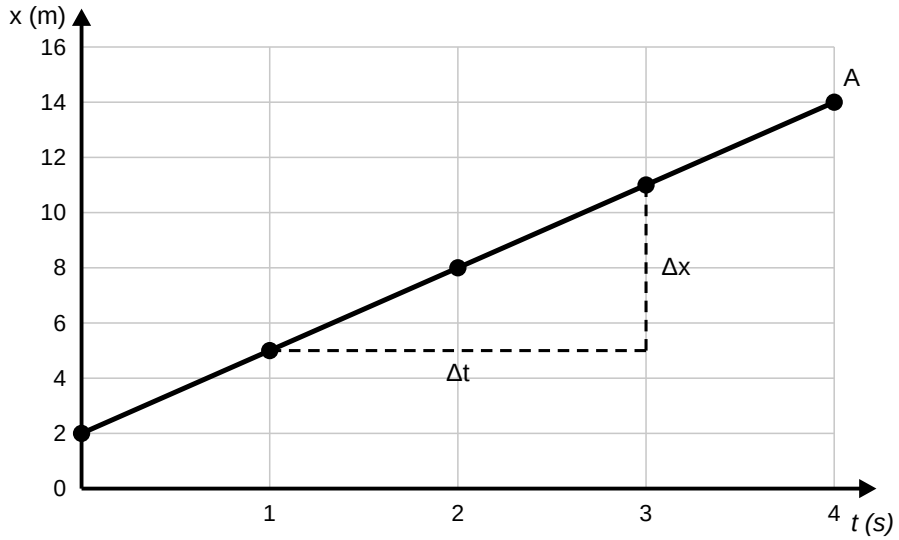
Model

These are the two delivery robots from Motion Maps: Position and Velocity. Robot A's positions are plotted on the position graph below.

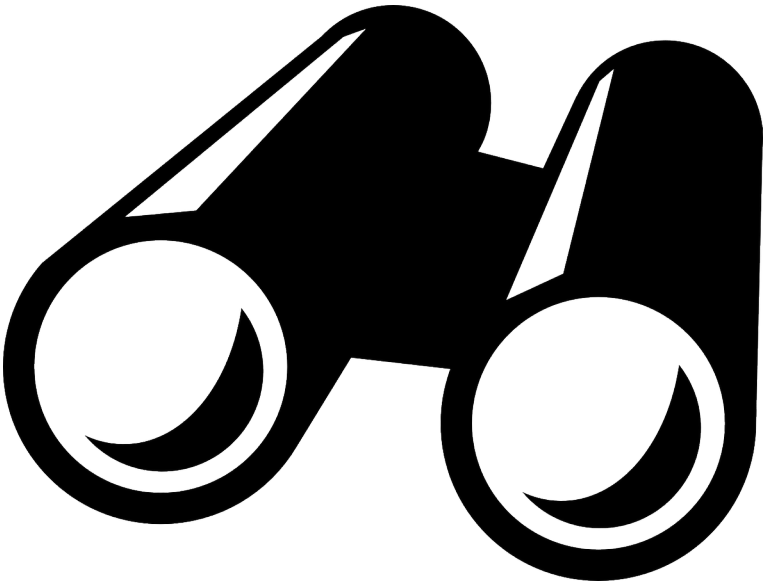
Motion map

Robot A at 2, 5, 8, 11, 14 m (0 to 4 s). Robot B at 15, 11, 7, 3 m (0 to 3 s).

Position graph



The **slope** of a line is its rise over its run. On a position graph, the rise is a change in position, Δx, and the run is a time interval, Δt.



Explore

1. Use the graph to find Robot A's position at 2 s. Check it with the motion map.

2. Find the rise (Δx) and the run (Δt) of the slope triangle. Highlight the Δx side of the triangle in one color, then draw Robot A's Δx arrow for 1 s to 3 s on the motion map and highlight it in the same color.

3. Plot Robot B's positions on the position graph and connect them.



Critical thinking & Concept Invention

Launch: Manager, have each person point to one dot on the graph and the matching dot on the motion map.

1. On your whiteboard, find the slope of Robot A's line and the slope of Robot B's line, with units. What quantity does the slope of a position graph tell you? Then, connecting to calculus: is a slope a derivative or an integral, and of which quantity? Finalize it on your whiteboard as a group, then copy it into your notes below.

2. A third robot is parked at 9 m the whole time. Draw its line on the graph. On your whiteboard, write a rule: how do the steepness and the direction of a line on a position graph show an object's velocity? Finalize it on your whiteboard as a group, then copy it into your notes below.

3. At 3 s, Robot A's point is higher on the graph than Robot B's. A group member says that means Robot A is moving faster. Write the Claim, Evidence, Reasoning template on your whiteboard and answer the question: do you agree? Finalize it on your whiteboard as a group, then copy it into your notes below.

   Claim

   Evidence

   Reasoning

Check in: Reflector, check that everyone can find a velocity from a position graph. Set your cup.

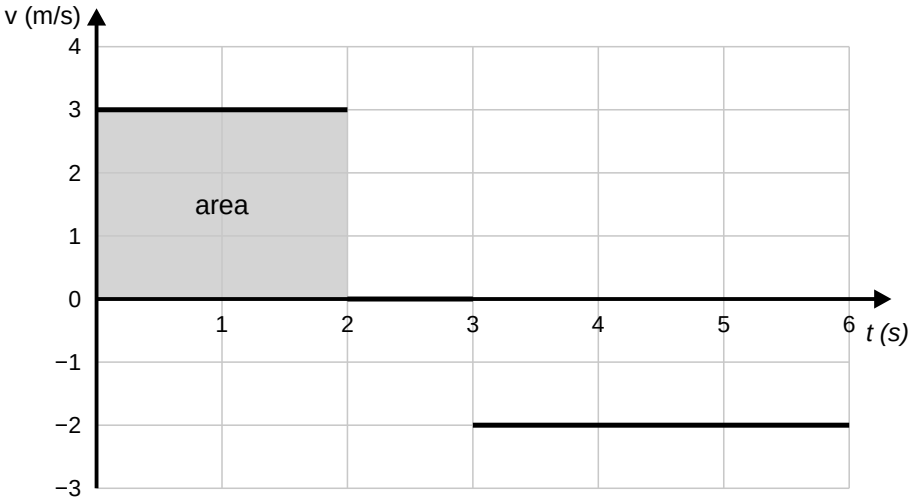
2. Velocity Graphs and Area

Model

Physical story

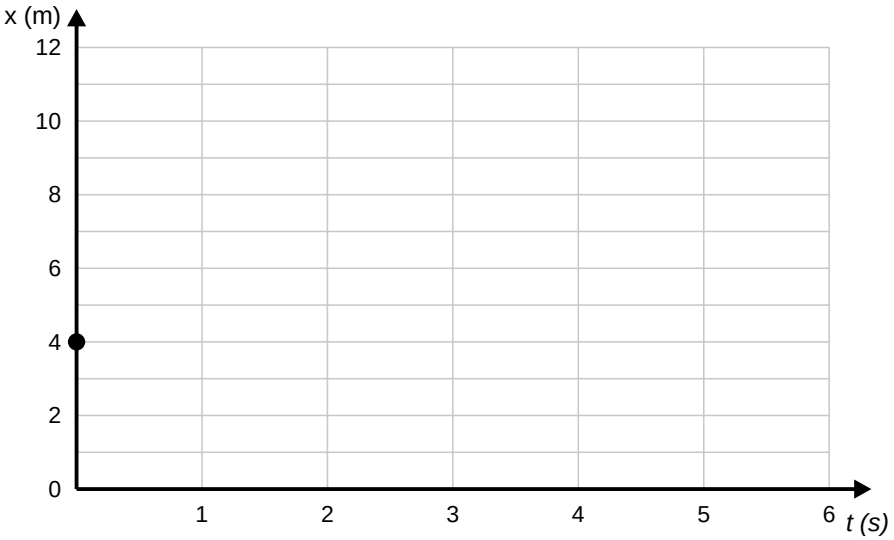
This is the skateboarder from Motion Maps: Position and Velocity. The skateboarder starts at 4 m, rolls at +3 m/s for 2 s, stops for 1 s, then rolls back at –2 m/s for 3 s.

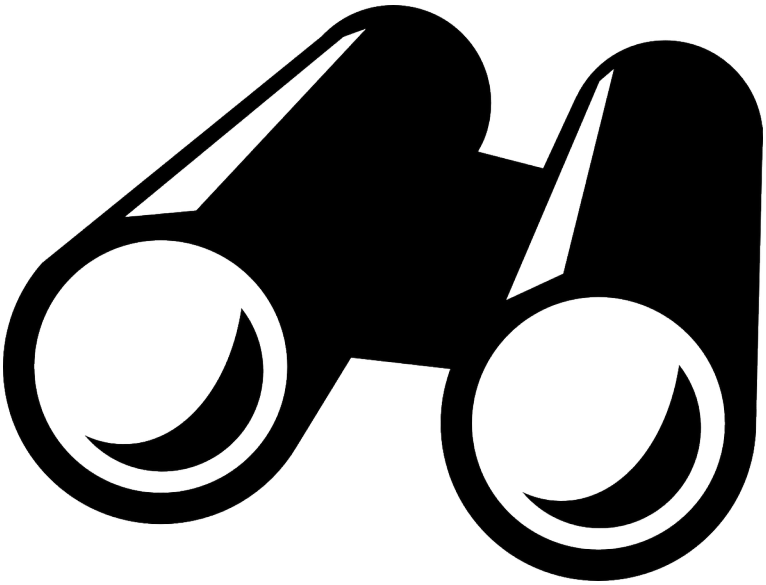
Velocity graph



The **area** under a velocity graph is the area between the graph and the t axis. Area below the t axis counts as negative.

Position graph





Explore

1. Use the velocity graph to find the skateboarder's velocity at 1 s and at 4 s.

2. Find the shaded area, from 0 s to 2 s. Multiply the height by the width, and keep the units.

3. Shade the rest of the area under the velocity graph, and find the area from 2 s to 3 s and from 3 s to 6 s.



Critical thinking & Concept Invention

Launch: Manager, read the physical story aloud. Then have each person say one area and its units.

Compare: Check your areas with your group.

1. The skateboarder's Δx arrows from Motion Maps: Position and Velocity were 6 m right from 0 s to 2 s, 0 from 2 s to 3 s, and 6 m left from 3 s to 6 s. On your whiteboard, explain what the area under a velocity graph tells you. Use the units and the sign of the areas. Then, connecting to calculus: is an area under a graph a derivative or an integral, and of which quantity? Finalize it on your whiteboard as a group, then copy it into your notes below.

2. On your whiteboard, write the position update equation using the area under the velocity graph. Use it to find the skateboarder's position at 2 s, 3 s, and 6 s, then draw the position graph in the Model. Highlight each area on the velocity graph and the matching rise on the position graph in the same color. Finalize it on your whiteboard as a group, then copy it into your notes below.

3. A group member says the skateboarder's position at 5 s is negative, because the velocity graph is below the t axis at 5 s. Write the Claim, Evidence, Reasoning template on your whiteboard and answer the question: do you agree? Finalize it on your whiteboard as a group, then copy it into your notes below.

   Claim

   Evidence

   Reasoning

Check in: Reflector, check that everyone can find a change in position from a velocity graph. Set your cup.

3. Changing Velocity and Acceleration

Model

Physical story

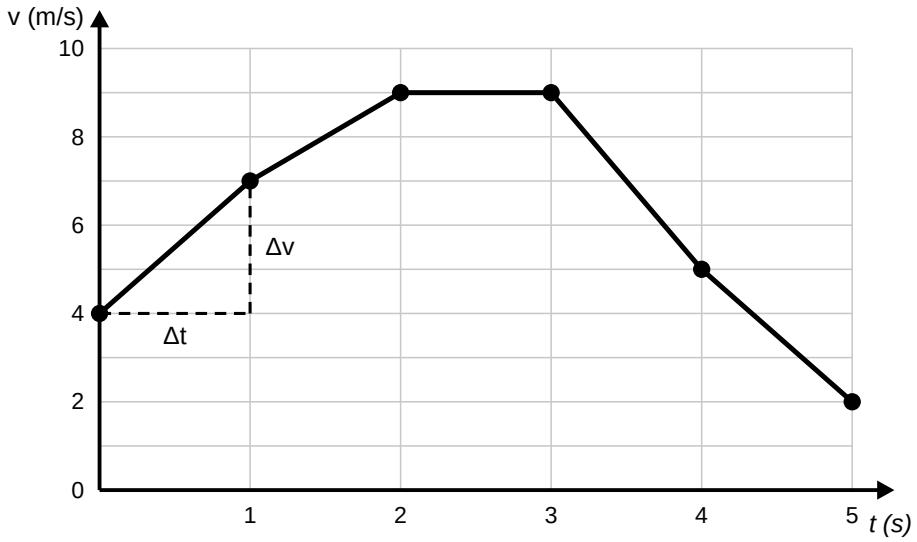
A car drives along a straight road. Its dashcam records the speedometer every second. The car speeds up, holds steady, then slows down for a crosswalk. East is positive. Between readings, assume the velocity changes steadily.

Motion map with velocity arrows

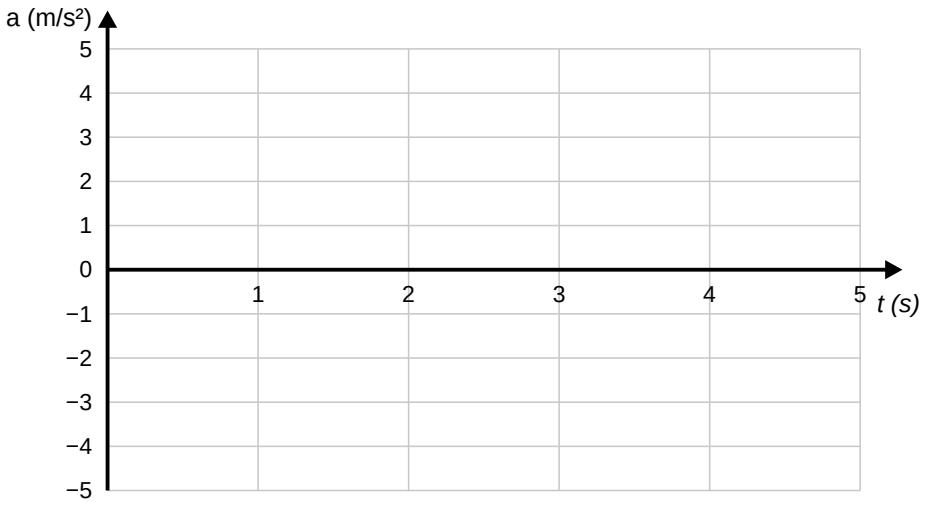
Car at 0, 5.5, 13.5, 22.5, 29.5, 33 m (0 to 5 s); velocity arrows 4, 7, 9, 9, 5, 2 m/s east; 2 Δv arrows drawn.

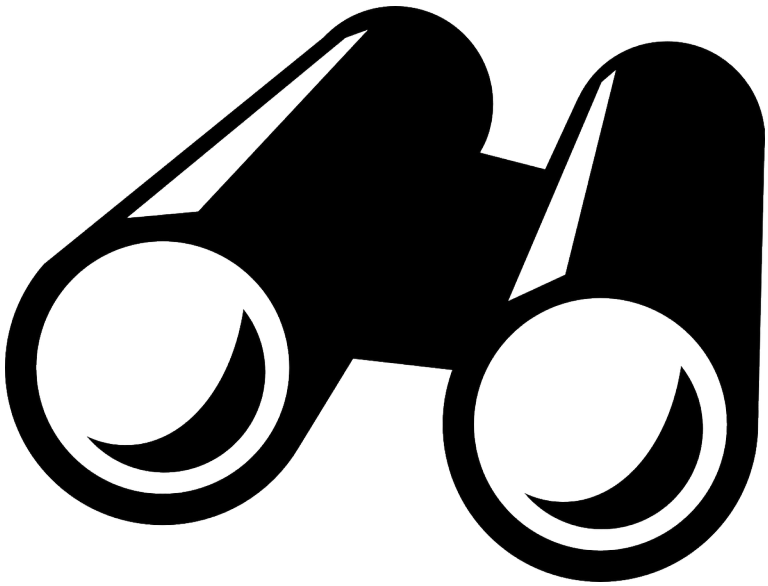
Each velocity arrow shows the speedometer reading at that instant, with its direction. Each **change in velocity** arrow, Δv, shows how the velocity changed during one second.

Velocity graph



Acceleration graph (to build)





Explore

1. Complete the Δv arrows on the motion map. Label each one, and write 0 if the velocity did not change.

2. Use the velocity graph to find Δv for each second. Highlight the Δv side of the slope triangle and the 0 s to 1 s Δv arrow on the motion map in the same color.



Critical thinking & Concept Invention

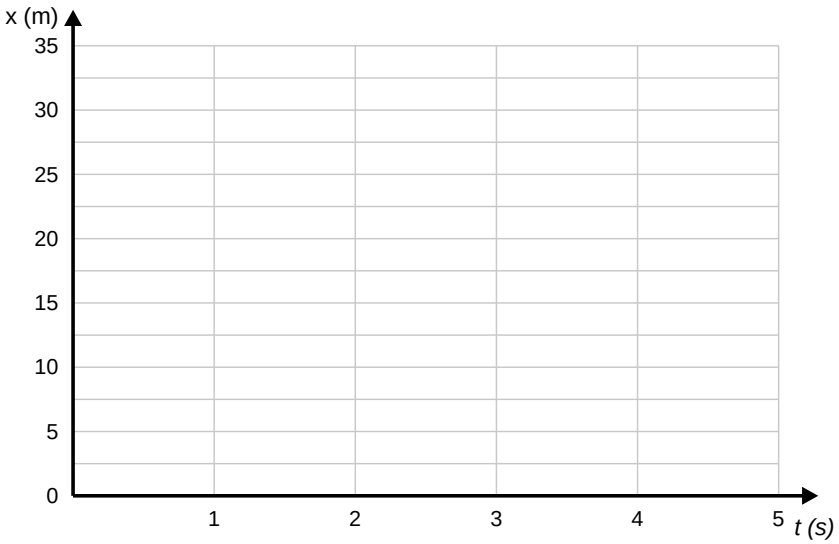
Launch: Manager, read the physical story aloud. Then have each person point to one part of the velocity graph where the car speeds up, holds steady, or slows down.

1. The position update equation is $\vec{x}_i + \Delta\vec{x} = \vec{x}_f$. On your whiteboard, write a velocity update equation with the same pattern, using $\vec{v}_i$, $\Delta\vec{v}$, and $\vec{v}_f$. Check it on the velocity graph from 0 s to 1 s, then write it for 3 s to 4 s with numbers and signs. Finalize it on your whiteboard as a group, then copy it into your notes below.

_____ + _____ = _____

2. The slope of a velocity graph, the change in velocity per time, is called the **acceleration**: $\vec{a} = \frac{\Delta\vec{v}}{\Delta t}$. Its units are (m/s)/s, written m/s^2. On your whiteboard, find the acceleration for each second. Then write a rule: how do the signs of the velocity and the acceleration tell you whether the car is speeding up or slowing down? Then, connecting to calculus: is the slope of a velocity graph a derivative or an integral, and of which quantity? Finalize it on your whiteboard as a group, then copy it into your notes below.

3. From the motion map, the car's positions at 0 s to 5 s are 0, 5.5, 13.5, 22.5, 29.5, and 33 m. Plot them on the position graph below and connect them with a smooth curve. On your whiteboard, explain how the steepness of the curve shows the car speeding up, holding steady, and slowing down. Use the meaning of the slope of a position graph from Part 1. Then, connecting to calculus: is that slope a derivative or an integral, and of which quantity? Finalize it on your whiteboard as a group, then copy it into your notes below.



4. On your whiteboard, build the acceleration graph: draw a flat segment for each second at that second's acceleration. Shade the area between each segment and the t axis. Finalize it on your whiteboard as a group, then copy it into your notes below. Then draw the graph in the Model.

5. What does the area under the acceleration graph from 0 s to 2 s tell you? Find it, with units, and check your answer with the velocity graph. Then, connecting to calculus: is this area a derivative or an integral, and of which quantity?

Check in: Reflector, check that everyone can find an acceleration from a velocity graph. Set your cup.

4. Ball Toss

Model

Physical story

You toss a ball straight up at 10 m/s from the edge of a cliff. It rises, stops for an instant, then falls past you toward the beach below. A camera records the ball every second. Up is positive, and air resistance is small enough to ignore.

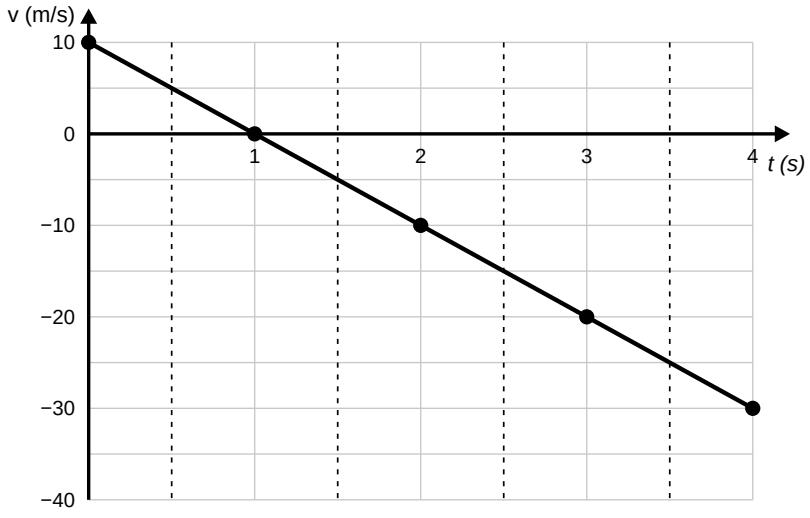
A note on g: in this course, we round the acceleration of a falling object near Earth's surface to 10 m/s^2. The measured value is

closer to 9.8 m/s^2, and you may see 9.8 m/s^2 in other books, classes, and websites.

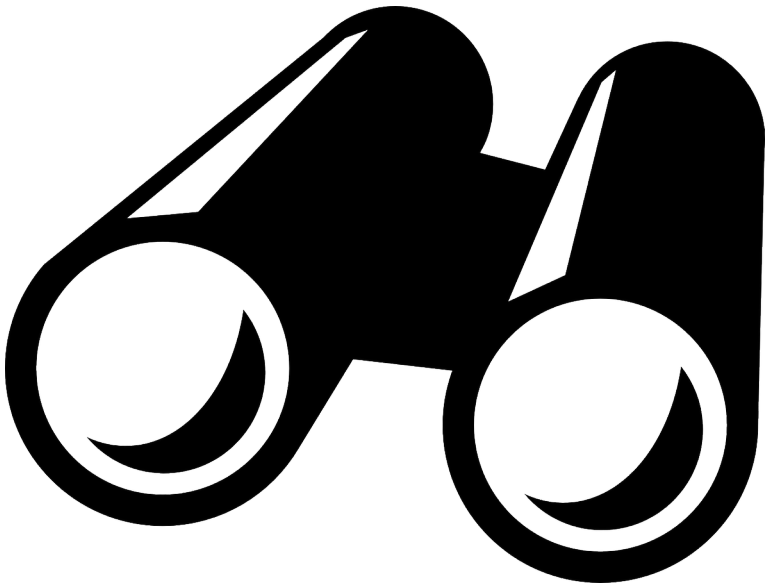
Motion map with velocity arrows

Ball positions 0 s to 4 s: 0, 5, 0, –15, –40 m. Velocities: 10 m/s up, 0, then 10, 20, 30 m/s down.

Velocity graph



The dashed lines mark the middle of each second.



Explore

1. Use the motion map to find the ball's displacement, Δy, for each second. Then find the average velocity for each second, $\vec{v}_{\text{avg}} = \frac{\Delta \vec{y}}{\Delta t}$.

	0 s to 1 s	1 s to 2 s	2 s to 3 s	3 s to 4 s
Δy				
Average velocity				

2. Use the velocity graph to find the ball's velocity at 1 s and at 2 s.



Critical thinking & Concept Invention

Launch: Manager, read the physical story aloud. Then have each person predict the ball's acceleration at its highest point.

Compare: Check your Explore table with your group.

1. Plot each second's average velocity on the velocity graph, on the dashed line in the middle of that second. What do you notice? On your whiteboard, explain why the middle of the second is the right place to plot an average velocity when the velocity changes steadily. Finalize it on your whiteboard as a group, then copy it into your notes below.

2. At 1 s, the ball's velocity is 0. A group member says the ball's acceleration must also be 0 at the top. Write the Claim, Evidence, Reasoning template on your whiteboard and answer the question: is the acceleration 0 at the top? Finalize it on your whiteboard as a group, then copy it into your notes below.

   Claim

Evidence

Reasoning

3. Shade the area under the velocity graph from 2 s to 3 s. Split it into a rectangle, with height $\vec{v}_i$, and a triangle, with height $\Delta\vec{v}$. On your whiteboard, write the area of each piece using $\vec{v}_i$, $\Delta\vec{v}$, and Δt, then add them to write an equation for Δy. Check it against your Explore table. Then, connecting to calculus: is the area you found a derivative or an integral, and of which quantity? Finalize it on your whiteboard as a group, then copy it into your notes below.

4. On your whiteboard, use your equation for Δy over the whole interval from 0 s to 4 s, then use the position update equation to find the ball's position at 4 s. Circle each unknown, and check your answer with the motion map. Finalize it on your whiteboard as a group, then copy it into your notes below.

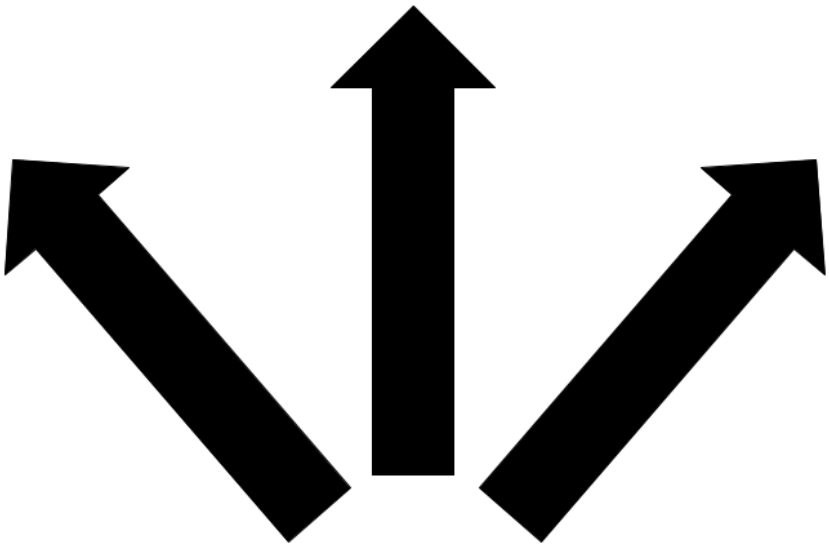
Check in: Reflector, check that everyone can explain why the acceleration is not 0 at the top. Set your cup.

Summary

- The slope of a position graph is the velocity. Steeper means faster, and a flat line means at rest.
- The area under a velocity graph is the change in position, Δx. Area below the t axis is negative. $\vec{x}_i + \text{area} = \vec{x}_f$.
- The slope of a velocity graph is the acceleration, $\vec{a} = \frac{\Delta\vec{v}}{\Delta t}$, in m/s². When the velocity and the acceleration have the same sign, the object speeds up. When they have opposite signs, it slows down.
- The area under an acceleration graph is the change in velocity, Δv. $\vec{v}_i + \Delta\vec{v} = \vec{v}_f$.
- When the velocity changes steadily, the average velocity for an interval equals the velocity at the middle of the interval, and $\Delta\vec{x} = \vec{v}_i\Delta t + \frac{1}{2}\Delta\vec{v}\Delta t$ (a rectangle plus a triangle).
- In calculus language, a slope is a derivative and an area is an integral: velocity is the derivative of position, acceleration is

the derivative of velocity, Δx is the integral of velocity, and Δv is the integral of acceleration.

- Near Earth's surface, with air resistance ignored, a tossed or dropped object has an acceleration of about 10 m/s^2 downward, on the way up, at the top, and on the way down. We use 10 m/s^2 in this course; the measured value is closer to 9.8 m/s^2, which you may see elsewhere.

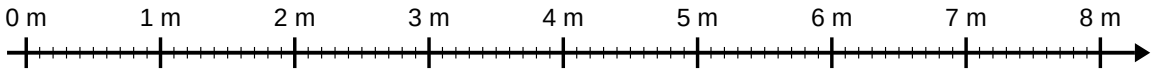


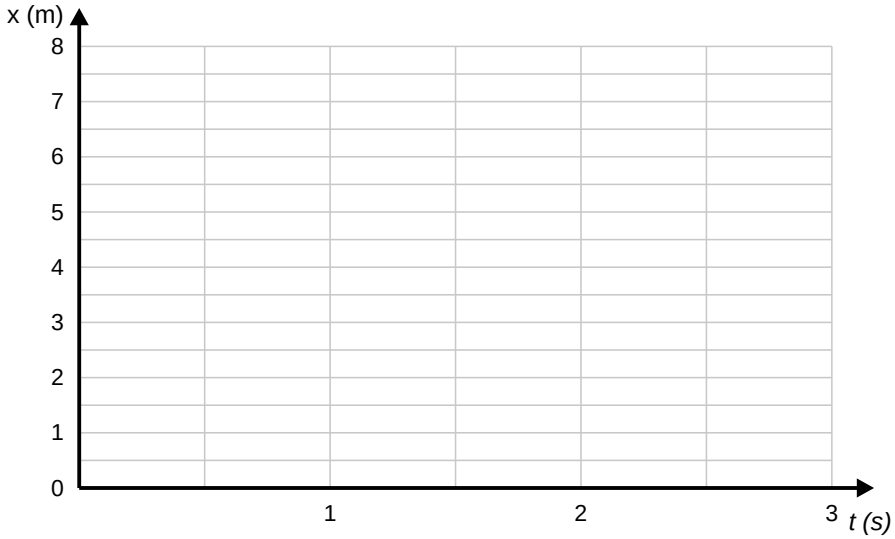
Apply

Build each piece on your whiteboard, finalize it as a group, then copy it into your notes before starting the next piece. Write each equation with variables before you put in numbers.

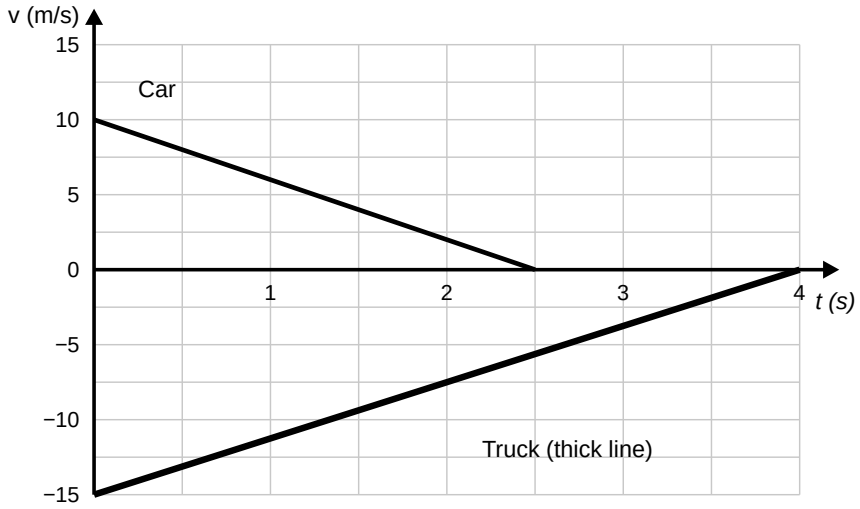
1. Model the scenario. A cart has an initial velocity of +3 m/s and an acceleration of −0.6 m/s^2. Sketch its velocity graph from 0 s to 3 s. Then find its velocity and its displacement at 3 s. Circle each unknown.

   The cart starts at x = 0. Use the area under your velocity graph for each second to draw its motion map, with a dot for each second. Mark a velocity arrow at each dot and a change in velocity arrow for each second. Then sketch its position graph, and check that your motion map and position graph agree.





2. Model the scenario. The graph shows the velocities of a car and a truck. Find the acceleration and the displacement of each from 0 s until it stops.



3. A group member says the truck speeds up from 0 s to 4 s, because its acceleration is positive. Write the Claim, Evidence, Reasoning template on your whiteboard and answer the question: do you agree? Finalize it on your whiteboard as a group, then copy it into your notes below.

   Claim

   Evidence

   Reasoning

Check in: Reflector, before the consensus share-out, check that your group agrees on all three answers.

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1.3: Kinematics with Calculus and Vectors

Standard practiced: K2 (vector kinematics equations), K3 (kinematics vectors, including trig components), and X1 (solving the equations). The week 2 quiz covers this activity and Projectiles.

In Kinematics Graphs, you saw that velocity is the derivative of position and that a change in position is an integral of velocity. In this activity, you will use calculus on polynomial functions to move between position, velocity, and acceleration. Then you will do the same thing in two dimensions, one row at a time, with column vectors. Keep unrounded values in your calculator during a problem, and round final answers to 2 decimal places.

Whiteboard first: in class, your group drafts its key work on the whiteboard. When the group agrees, everyone copies the final version into their notes.

Colors: use 3 or 4 colors to highlight. Using the same color for the same quantity in a diagram and in an equation shows where that quantity appears in each.

Before class: complete each Explore section on your own. Every Explore answer can be found in the Model above it.

The kinematics model: physical story, motion map, graphs, equations, and math solution. This activity adds functions grouped by powers of t to the equations, and column vectors for two dimensions.

1. From Position to Velocity

Model

Physical story

This is the ball toss from Kinematics Graphs. You toss a ball straight up at 10 m/s from the edge of a cliff. Up is positive, and the ball starts at y = 0.

Ball positions 0 s to 4 s: 0, 5, 0, –15, –40 m. Velocities: 10 m/s up, 0, then 10, 20, 30 m/s down.

Position function

In one dimension, we write the component of each vector, like y or v_y. Its sign shows its direction. The ball's position, in meters, with t in seconds, is written grouped by powers of t:

$$y(t) = -5t^2 + 10t + 0$$

Calculus tools

Derivatives from Kinematics Graphs:

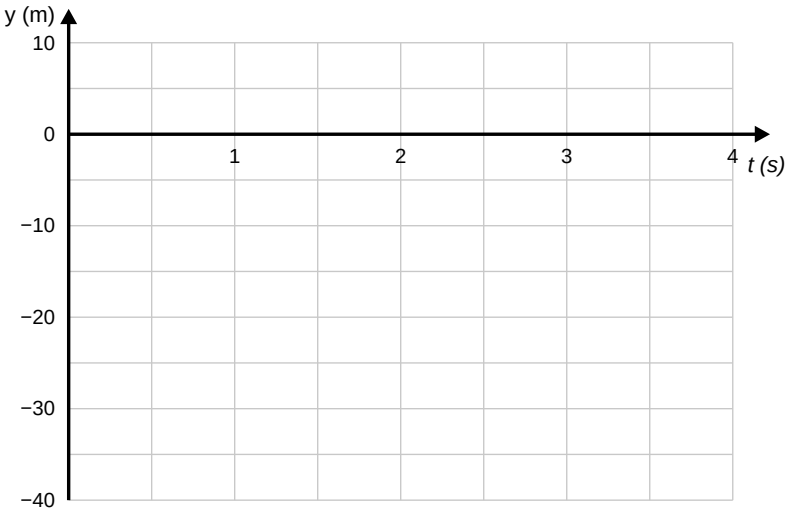
$$v_y(t) = \frac{dy}{dt}$$

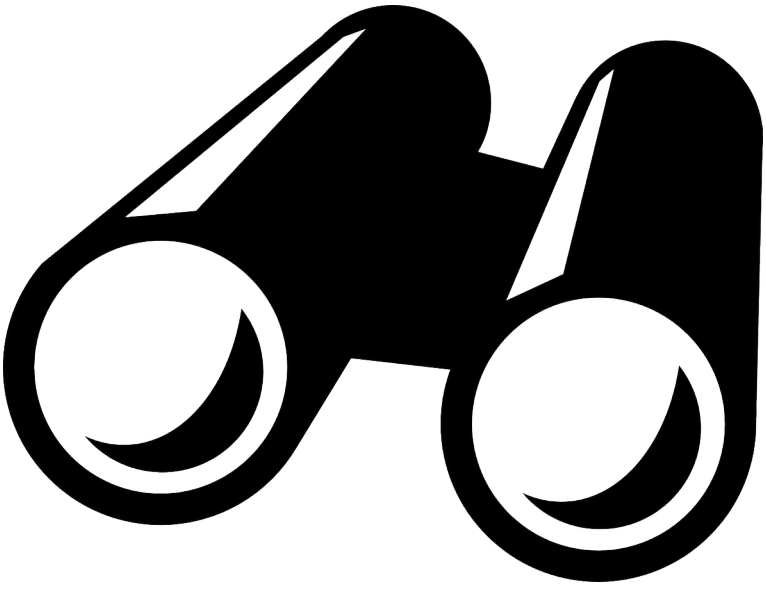
$$a_y(t) = \frac{dv_y}{dt}$$

Power rule for derivatives: $\frac{d(ct^n)}{dt} = nct^{n-1}$. The derivative of a constant is 0.

t (s)	y(t) (m)	v_y(t) (m/s)
0	0	
1	5	
2		
3		
4		

Position graph (to build)





Explore

1. Use the position function to complete the y(t) column of the table. Check each value with the motion map.
2. What is the coefficient of each power of t in y(t)?

3. Use the power rule to take the derivative of each term of y(t), one term at a time.



Critical thinking & Concept Invention

Launch: Manager, have each person say one term of y(t) and its derivative.

1. On your whiteboard, write $v_y(t)$ grouped by powers of t. Use it to complete the v_y column of the table, and check it with the velocity arrows on the motion map. Then take one more derivative to find a_y. Finalize it on your whiteboard as a group, then copy it into your notes below.

2. Each coefficient of y(t) matches a physical quantity of the ball at t = 0. Highlight the constant term, the t coefficient, and the t^2 coefficient in three colors. Highlight the matching quantity on the motion map, or in your answer to question 1, in the same color. On your whiteboard, write a rule for the meaning of each coefficient. Finalize it on your whiteboard as a group, then copy it into your notes below.

3. Plot y(t) from the table on the position graph in the Model, and connect the points with a smooth curve. Draw the tangent line at t = 2 s, a straight line that touches the curve at only that point, and find its slope. Then draw a straight line from the 1 s point to the 3 s point and find its slope. On your whiteboard, explain what each slope represents. Then, connecting to calculus: is the slope of the tangent line a derivative or an integral, and of which quantity? Finalize it on your whiteboard as a group, then copy it into your notes below.

4. A group member says the ball's acceleration is −5 m/s², because −5 is the coefficient of t². Write the Claim, Evidence, Reasoning template on your whiteboard and answer the question: do you agree? Finalize it on your whiteboard as a group, then copy it into your notes below.

   Claim

   Evidence

   Reasoning

Check in: Reflector, check that everyone can explain what each coefficient of y(t) means. Set your cup.

2. From Acceleration to Position

Model

Physical story

A cart is pushed up a ramp and let go. Up the ramp is +x. At the clock reading t = 0, the cart is at x = 2 m and moving up the ramp at 3 m/s. The ramp gives the cart a constant acceleration of 0.6 m/s² down the ramp.

Given

$$a_x = -0.6\ \mathrm{m/s^2}$$

$$v_x(0) = 3\ \mathrm{m/s}$$

$$x(0) = 2\ \mathrm{m}$$

Process

1. Start from what is given.
2. Integrate one term at a time.
3. Find the constant of integration from the value at t = 0.
4. Write the result grouped by powers of t.

Calculus tools

Power rule for integrals: $\int ct^n \ dt = \frac{c}{n+1}t^{n+1} + C$. A constant c integrates to ct + C.

Worked step: velocity

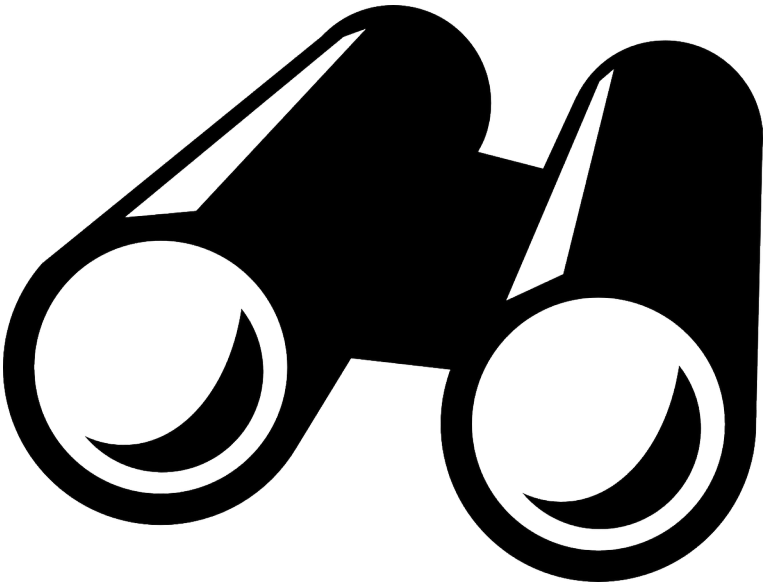
$v_x(t)$	$=$	$\int a_x \ dt$
	$=$	$\int(-0.6) \ dt$
	$=$	$-0.6t + C$

At t = 0, $v_x(0) = C$, so C = 3 m/s. The velocity is

$$v_x(t) = -0.6t + 3$$

Position, grouped by powers of t





Explore

1. Highlight each given quantity in the physical story and in the Given line in the same color. Which given quantity was used to find C in the worked step?

2. Follow the process to integrate $v_x(t)$ one term at a time. Use x(0) to find the constant, then fill in the boxes for x(t) in the Model.

3. Use your functions to find the cart's velocity and position at t = 2 s.



Critical thinking & Concept Invention

Launch: Manager, have each person read one step of the process aloud and point to where it happens in the Model.

Compare: Check your x(t) with your group.

1. Check your rule from Part 1 on x(t): is the constant the initial position, the t coefficient the initial velocity, and the t^2 coefficient half the acceleration? On your whiteboard, explain where the ½ comes from when you integrate. Finalize it on your whiteboard as a group, then copy it into your notes below.

2. On your whiteboard, find when the cart stops for an instant and how far up the ramp it is then. Circle each unknown. Finalize it on your whiteboard as a group, then copy it into your notes below.

3. A group member finds the cart's position at 5 s with x = 2 m + (3 m/s)(5 s) = 17 m. Write the Claim, Evidence, Reasoning template on your whiteboard and answer the question: is 17 m correct? Finalize it on your whiteboard as a group, then copy it into your notes below.

   Claim

Evidence

Reasoning

4. A second cart has a motor that pushes harder and harder: $a_x(t) = 1.2t$ (in m/s², with t in s). It starts from rest at x = 0. On your whiteboard, use the same process to find $v_x(t)$ and $x(t)$, then find its velocity and position at 3 s. Could you use a rectangle plus a triangle on the velocity graph instead? Explain. Finalize it on your whiteboard as a group, then copy it into your notes below.

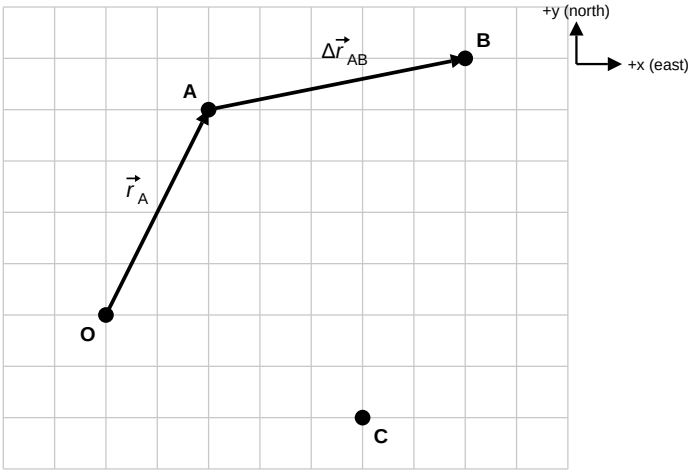
Check in: Reflector, check that everyone can say the four steps of the process. Set your cup.

3. Column Vectors

Model

Physical story

A robotic lawn mower moves around a yard. Point O is the origin. East is +x and north is +y.



Each square is 1 m.

Column vectors

A column vector lists the x component on top and the y component below. Each row is a one-dimensional component, and its sign shows its direction. The arrow from O to A is A's position:

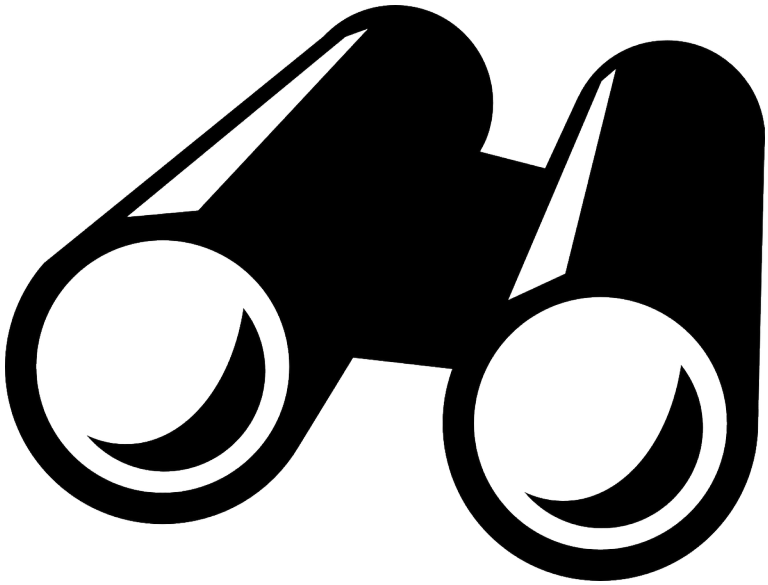
$$\vec{r}_A = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \text{ m}$$

Multiplying a column vector by a number multiplies each row:

$$3\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \end{bmatrix}$$

The position update equation works one row at a time:

$$\begin{matrix} \vec{r}_A & \Delta\vec{r}_{AB} & \vec{r}_B \\ \begin{bmatrix} 2 \\ 4 \end{bmatrix} + & \begin{bmatrix} 5 \\ 1 \end{bmatrix} & = \begin{bmatrix} 7 \\ 5 \end{bmatrix} \end{matrix}$$



Explore

1. Write the positions of B and C as column vectors, and draw their position arrows from O on the grid.

$\vec{r}_B$

$\vec{r}_C$

2. Draw the displacement arrow from B to C on the grid. Count squares to write it as a column vector.

$\Delta\vec{r}_{\mathrm{BC}}$

3. Highlight the top row of each column vector in the Model's update equation in one color, and the bottom row in another. Write the top row as a one-dimensional update equation.

4. The mower drives the displacement from A to B three times in a row. Write $3\Delta\vec{r}_{\mathrm{AB}}$ as a column vector.

$3\Delta\vec{r}_{\mathrm{AB}}$



Critical thinking & Concept Invention

Launch: Manager, have each person trace one arrow on the grid with a finger and say its column vector.

1. On your whiteboard, write the position update equation from B to C with column vectors, and check each row. Then rearrange it to find $\Delta\vec{r}_{\mathrm{AC}}$ from the positions of A and C, one row at a time. Check your answer by adding $\Delta\vec{r}_{\mathrm{AB}}$ and $\Delta\vec{r}_{\mathrm{BC}}$ tip to tail on the grid. Finalize it on your whiteboard as a group, then copy it into your notes below.

2. A group member says that if the origin moved to point A, the displacement from B to C would change too. Write the Claim, Evidence, Reasoning template on your whiteboard and answer the question: do you agree? Finalize it on your whiteboard as a group, then copy it into your notes below.

   Claim

   Evidence

   Reasoning

3. On your whiteboard, sketch $\Delta\vec{r}_{\mathrm{BC}}$ with its x and y components as the sides of a right triangle. Find its magnitude (length) and its direction, as an angle measured from south. Finalize it on your whiteboard as a group, then copy it into your notes below.

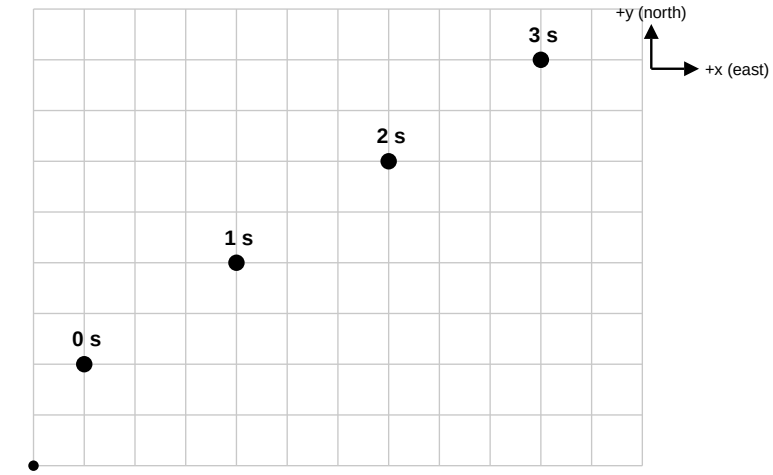
Check in: Reflector, check that everyone can find a magnitude and angle from a column vector. Set your cup.

4. Velocity in Two Dimensions

Model

Physical story

A delivery drone flies at a constant velocity over a field. A camera above the field takes a photo every second. East is +x and north is +y.



Each square is 1 m. Dots are 1 s apart.

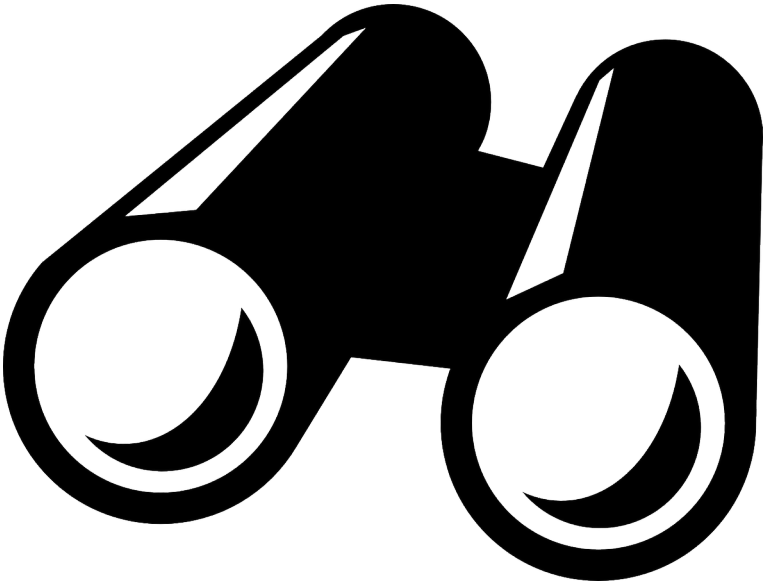
Position function, grouped by powers of t

$$\begin{bmatrix} \vec{r}(t) \\ x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} t + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} \vec{v}(t) \\ v_x(t) \\ v_y(t) \end{bmatrix} =$$

You will build $\vec{v}(t)$ in the space under $\vec{r}(t)$ in Critical thinking question 1.

Positions are in meters, with t in seconds. To take a derivative or an integral of a column vector, do it one row at a time.



Explore

1. Use the position function to find the drone's position at t = 2 s. Check it on the grid.

$$\vec{r}(2) = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

2. Draw the displacement arrow from the 0 s dot to the 1 s dot, and write it as a column vector.

$$\Delta\vec{r} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$



Critical thinking & Concept Invention

Launch: Manager, have each person point to one dot and read its position from the function.

1. On your whiteboard, take the derivative of $\vec{r}(t)$ one row at a time to find $\vec{v}(t)$, and write it under $\vec{r}(t)$ in the Model. Compare it with your displacement for 1 s. Then use your rule from Part 1 to explain what each column vector in $\vec{r}(t)$ means. Finalize it on your whiteboard as a group, then copy it into your notes below.

2. Draw the drone's velocity arrow at each dot. On your whiteboard, find the drone's speed (the magnitude of its velocity) and its direction, as an angle from east. Finalize it on your whiteboard as a group, then copy it into your notes below.

3. A group member says the drone's speed is 3 m/s + 2 m/s = 5 m/s. Write the Claim, Evidence, Reasoning template on your whiteboard and answer the question: do you agree? Finalize it on your whiteboard as a group, then copy it into your notes below.

   Claim

   Evidence

   Reasoning

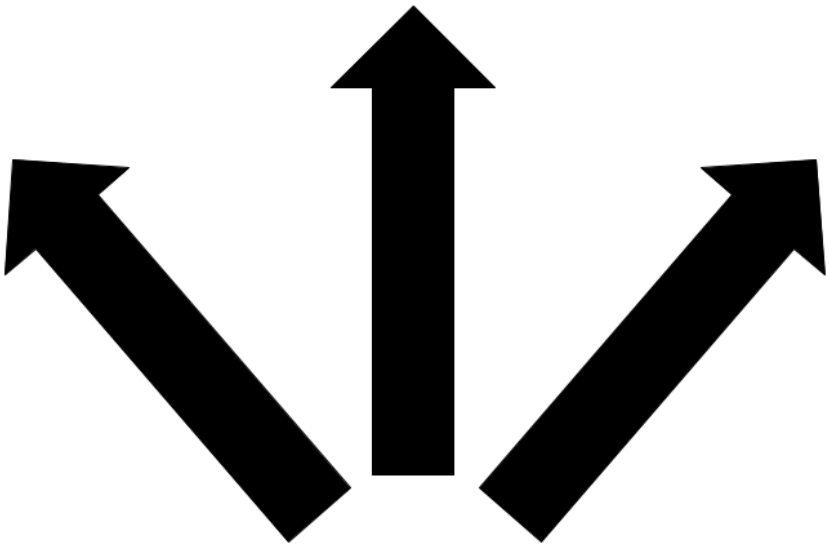
4. A second drone leaves O at t = 0, flying at a constant 5.0 m/s at 28° east of north. On your whiteboard, sketch its velocity with its components, write its velocity as a column vector, and write its position function grouped by powers of t. Finalize it on your whiteboard as a group, then copy it into your notes below.

$$\vec{r}(t) \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} =$$

Check in: Reflector, check that everyone can go from a magnitude and angle to a column vector and back. Set your cup.

Summary

- For a polynomial, take derivatives and integrals one term at a time with the power rule. A derivative goes from position to velocity to acceleration. An integral goes back, with a constant you find from the value at t = 0.
- The process: start from what is given, integrate (or differentiate), find each constant from t = 0, and group by powers of t. It works for any polynomial, so there are no separate equations to memorize.
- When the acceleration is constant, the grouped position function has the form $x(t) = \left(\frac{1}{2}a_x\right)t^2 + (v_{x,i})\,t + x_i$.
- A column vector lists the x component on top and the y component below. Update equations, sums, differences, multiplying by a number, derivatives, and integrals all work one row at a time.
- A vector's components form the sides of a right triangle. The magnitude is the hypotenuse, and right triangle trigonometry connects the components to the angle.
- Positions depend on the origin. Displacements, velocities, and accelerations do not.



Apply

Build each piece on your whiteboard, finalize it as a group, then copy it into your notes before starting the next piece.

1. An elevator's position is $y(t) = 2t^3 - 6t + 1$ (in m, with t in s). Find $v_y(t)$ and $a_y(t)$. When is the elevator at rest? What is its acceleration at t = 2 s?

2. An object's velocity is given below, grouped by powers of t (in m/s, with t in s). It starts at $\vec{r}(0) = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$ m. Find $\vec{a}(t)$ and $\vec{r}(t)$

, each grouped by powers of t. Then find its speed and direction (angle from +x) at t = 1 s.

$$\begin{bmatrix} \vec{v}(t) \\ v_x(t) \\ v_y(t) \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix} t^2 + \begin{bmatrix} 3 \\ 0 \end{bmatrix} t + \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} \vec{a}(t) \\ a_x(t) \\ a_y(t) \end{bmatrix} =$$

$$\begin{bmatrix} \vec{r}(t) \\ x(t) \\ y(t) \end{bmatrix} =$$

$$\begin{bmatrix} \vec{v}(1) \\ \square \\ \square \end{bmatrix} =$$

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1.4: Projectiles

Standard practiced: K2 (vector kinematics equations), K3 (kinematics vectors, including trig components), and X1 (solving the equations). The week 2 quiz covers this activity and Kinematics with Calculus and Column Vectors.

A projectile is an object moving through the air with only gravity changing its motion. In this activity, you will see that a projectile's motion is two one-dimensional motions at once, one in each row of a column vector. You will use the same calculus process from the last activity. There are no separate projectile equations to memorize. Keep unrounded values in your calculator during a problem, and round final answers to 2 decimal places.

Whiteboard first: in class, your group drafts its key work on the whiteboard. When the group agrees, everyone copies the final version into their notes.

Colors: use 3 or 4 colors to highlight. Using the same color for the same quantity in a diagram and in an equation shows where that quantity appears in each.

Before class: complete each Explore section on your own. Every Explore answer can be found in the Model above it.

Column vectors on both sides: $\vec{r}(t)$ and $\vec{v}(t)$ are column vectors too. When you write an equation for a special moment, like the top of the path or the landing, write the left side as a column vector and fill in what you know in each row.

The kinematics model: physical story, motion map, graphs, equations, and math solution. This activity uses all of the parts in two dimensions.

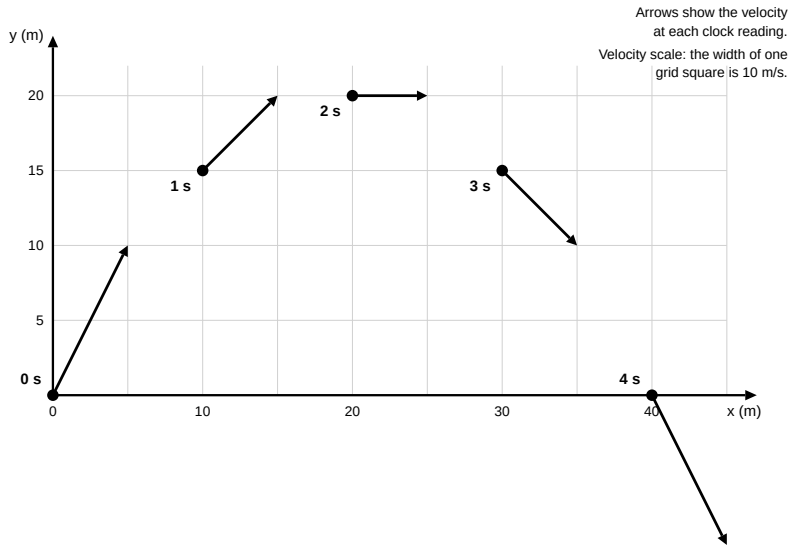
1. Two Motions at Once

Model

Physical story

A soccer ball is kicked from the ground across a flat field. A camera at the side of the field records the ball every second. +x is forward along the field and +y is up. Air resistance is small enough to ignore.

Motion map with velocity arrows



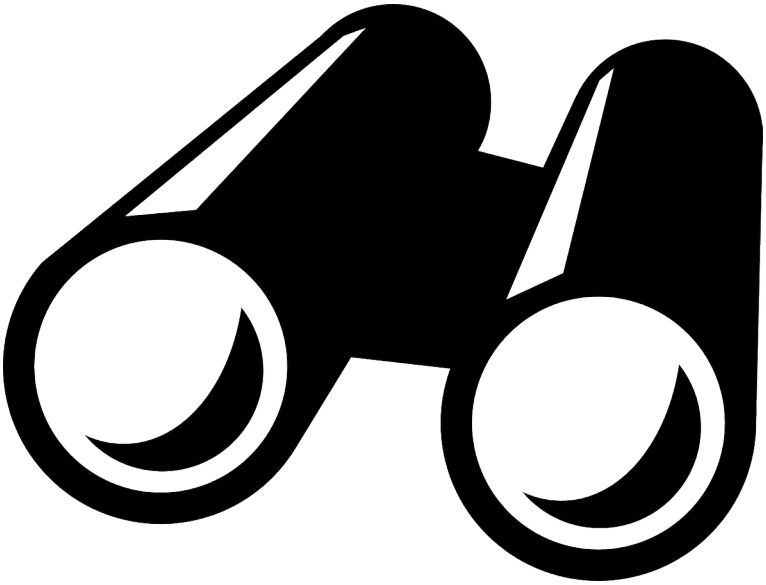
Velocity at each clock reading (in m/s)

$$\begin{matrix} \vec{v}\,(0\,\mathrm{s}) \\ \begin{bmatrix} 10 \\ 20 \end{bmatrix} \end{matrix} \quad \vec{v}\,(1\,\mathrm{s}) \quad \vec{v}\,(2\,\mathrm{s}) \quad \vec{v}\,(3\,\mathrm{s}) \quad \vec{v}\,(4\,\mathrm{s})$$

Velocity update equation

$$\vec{v}_i + \Delta\vec{v} = \vec{v}_f$$

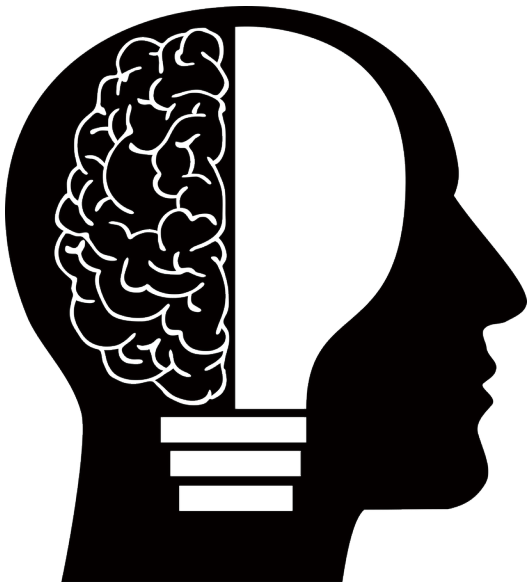
Like every column vector equation, it works one row at a time.



Explore

1. Fill in the velocity column vectors in the Model. Count grid squares along each arrow: the width of one grid square is 10 m/s.
2. Write the velocity update equation for 0 s to 1 s with column vectors, and find Δv.

$$\begin{bmatrix} \\ \\ \end{bmatrix}_{\vec{v}(0)} + \begin{bmatrix} \\ \\ \end{bmatrix}_{\Delta\vec{v}} = \begin{bmatrix} \\ \\ \end{bmatrix}_{\vec{v}(1)}$$



Critical thinking & Concept Invention

Launch: Manager, have each person trace the ball's path with a finger and say its velocity at one dot as a column vector.

1. On your whiteboard, draw the arrow equation for 0 s to 1 s: v(0 s), then Δv tip to tail, then v(1 s) from the first tail to the last tip. Then find Δv for every second. What pattern do you see, and what is the ball's acceleration as a column vector? Finalize it on your whiteboard as a group, then copy it into your notes below.

2. Highlight the x row of every velocity column vector in the Model in one color and the y row in another. On your whiteboard, describe the motion in each row, using a motion from an earlier activity. Finalize it on your whiteboard as a group, then copy it into your notes below.

3. A group member says the ball's velocity at the top of its path (2 s) is zero. Write the Claim, Evidence, Reasoning template on your whiteboard and answer the question: do you agree? Finalize it on your whiteboard as a group, then copy it into your notes below.

   Claim

Evidence

Reasoning

Check in: Reflector, check that everyone can explain why only the y row of the velocity changes. Set your cup.

2. The Process in Two Dimensions

Model

This is the same soccer ball. Positions are in meters and velocities in m/s, with t in seconds.

Given

$$\vec{a} = \begin{bmatrix} 0 \\ -10 \end{bmatrix} \text{ m/s}^2$$

$$\vec{v}(0) = \begin{bmatrix} 10 \\ 20 \end{bmatrix} \text{ m/s}$$

$$\vec{r}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ m}$$

A note on g: in this course, we round the acceleration of a falling object near Earth's surface to 10 m/s². The measured value is closer to 9.8 m/s², and you may see 9.8 m/s² in other books, classes, and websites.

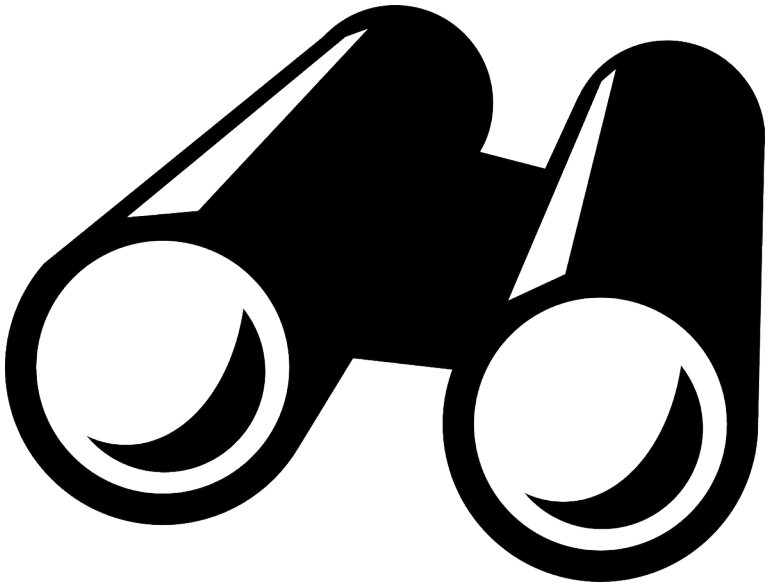
Process (one row at a time)

1. Start from what is given.
2. Integrate one term at a time.
3. Find each constant of integration from the value at t = 0.
4. Write the result grouped by powers of t.

Velocity and position, grouped by powers of t

$$\begin{matrix} \vec{v}(t) \\ \begin{bmatrix} v_x(t) \\ v_y(t) \end{bmatrix} \end{matrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix} t + \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

$$\begin{bmatrix} \vec{r}(t) \\ x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \square \\ \square \end{bmatrix} t^2 + \begin{bmatrix} \square \\ \square \end{bmatrix} t + \begin{bmatrix} \square \\ \square \end{bmatrix}$$



Explore

1. Integrate the acceleration one row at a time, use the initial velocity to find the constants, and fill in the velocity in the Model.
2. Integrate the velocity one row at a time, use the initial position to find the constants, and fill in the position in the Model.
3. Use your functions to find the ball's position and velocity at t = 2 s. Check them with the motion map.

$$\vec{r}(2) \begin{bmatrix} \square \\ \square \end{bmatrix}$$

$$\vec{v}(2) \begin{bmatrix} \square \\ \square \end{bmatrix}$$



Critical thinking & Concept Invention

Launch: Manager, have each person point to one column vector in the position function and say what it means.

Compare: Check your velocity and position functions with your group.

1. At the top of the path, which row of $\vec{v}(t)$ do you know? On your whiteboard, write $\vec{v}(t_{\text{top}})$ with its column vector filled in: the known value in one row and the unknown in the other. Circle each unknown. Solve the row with only one unknown first. Then write $\vec{r}(t_{\text{top}})$ the same way to find the top of the path. Finalize it on your whiteboard as a group, then copy it into your notes below.

$$\vec{v}(t_{\text{top}}) \begin{bmatrix} \\ \end{bmatrix} =$$

$$\vec{r}(t_{\text{top}}) \begin{bmatrix} \\ \end{bmatrix} =$$

2. At the landing, which row of $\vec{r}(t)$ do you know? On your whiteboard, write $\vec{r}(t_{\text{land}})$ with its column vector filled in, circle each unknown, and find when and where the ball lands. The row you solve first gives two times. Which one is the landing, and what does the other one mean? Finalize it on your whiteboard as a group, then copy it into your notes below.

$$\vec{r}(t_{\text{land}}) \begin{bmatrix} \\ \end{bmatrix} =$$

Check in: Reflector, check that everyone can say which row they know at the top and at the landing. Set your cup.

3. Launch Angle

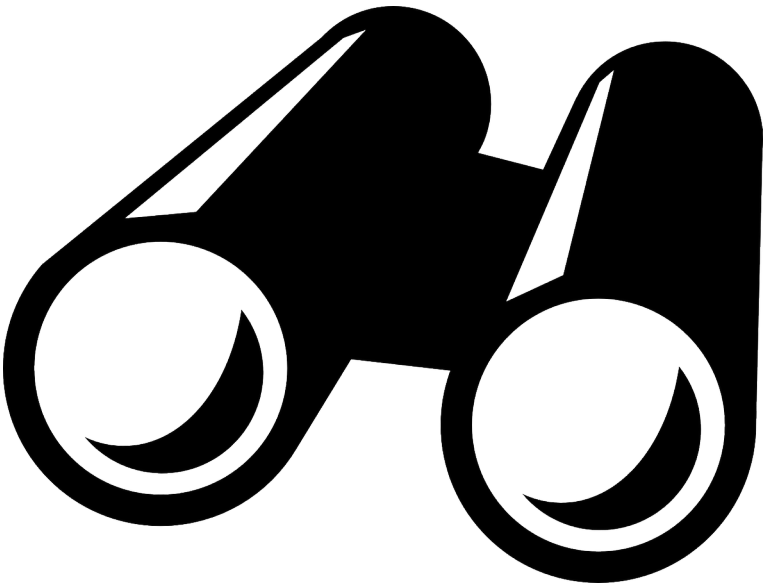
Model

Physical story

Two players each kick a soccer ball from the ground at 30 m/s. Kick A leaves the ground at 25° above the horizontal. Kick B leaves at 70° above the horizontal. +x is forward and +y is up, and the acceleration is $\begin{bmatrix} 0 \\ -10 \end{bmatrix}$ m/s².

Initial velocity sketches

Kick A	**Kick B**



Explore

1. In each sketch box in the Model, draw the initial velocity with its x and y components as the sides of a right triangle, and label the angle.

2. Find the components of each initial velocity, and write each as a column vector.

$$\vec{v}_{\mathrm{A}}(0) = \begin{bmatrix} \\ \end{bmatrix} \ \mathrm{m/s}$$

$$\vec{v}_{\mathrm{B}}(0) = \begin{bmatrix} \\ \end{bmatrix} \ \mathrm{m/s}$$



Critical thinking & Concept Invention

Launch: Manager, have each person say which kick they predict will land farther away, and why.

Compare: Check your initial velocity column vectors with your group.

1. On your whiteboard, build kick A's position function grouped by powers of t. Then write $\vec{r}(t_{\text{land}})$ with its column vector filled in, and find how long kick A is in the air and how far away it lands. Finalize it on your whiteboard as a group, then copy it into your notes below.

$$\begin{bmatrix} \vec{r}(t) \\ x(t) \\ y(t) \end{bmatrix} =$$

$$\begin{bmatrix} \vec{r}(t_{\text{land}}) \\ \square \\ \square \end{bmatrix} =$$

2. On your whiteboard, do the same for kick B. Finalize it on your whiteboard as a group, then copy it into your notes below.

$$\begin{bmatrix} \vec{r}(t) \\ x(t) \\ y(t) \end{bmatrix} =$$

$$\begin{bmatrix} \vec{r}(t_{\text{land}}) \\ \square \\ \square \end{bmatrix} =$$

3. A group member says kick B lands farther away, because it stays in the air longer. Write the Claim, Evidence, Reasoning template on your whiteboard and answer the question: do you agree? Finalize it on your whiteboard as a group, then copy it into your notes below.

   Claim

   _____

   _____

   Evidence

Reasoning

4. On your whiteboard, find kick B's maximum height. Write $\vec{v}(t_{\text{top}})$ and $\vec{r}(t_{\text{top}})$ with their column vectors filled in, and circle each unknown. Finalize it on your whiteboard as a group, then copy it into your notes below.

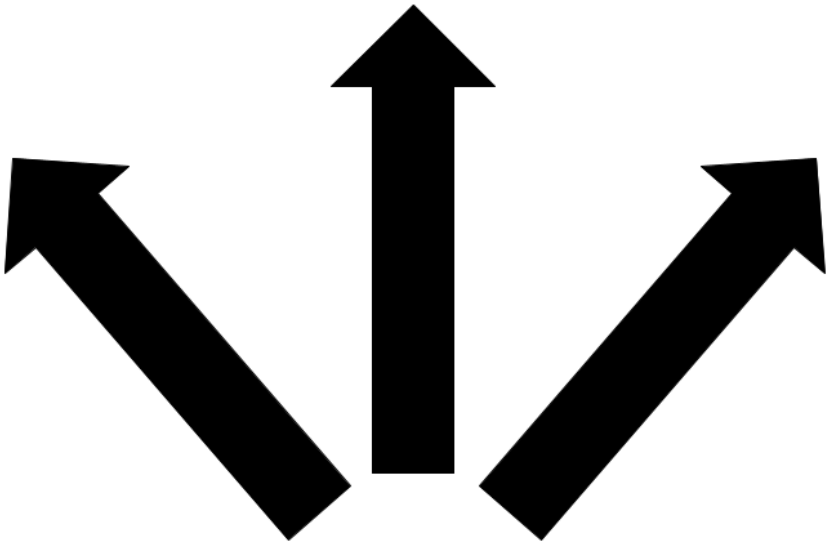
$$\vec{v}(t_{\text{top}}) = \begin{bmatrix} \\ \end{bmatrix} =$$

$$\vec{r}(t_{\text{top}}) = \begin{bmatrix} \\ \end{bmatrix} =$$

Check in: Reflector, check that everyone writes the left side as a column vector before solving. Set your cup.

Summary

- A projectile's acceleration is $\begin{bmatrix} 0 \\ -10 \end{bmatrix}$ m/s², with +y up. We use 10 m/s² in this course; the measured value is closer to 9.8 m/s², which you may see elsewhere. Its x row is a constant velocity motion, and its y row is a ball toss. Δv is 10 m/s straight down every second.
- At the top of the path, only the y row of the velocity is 0.
- Use the same process as any other motion: start from what is given, integrate one row and one term at a time, find each constant from t = 0, and group by powers of t. There are no separate projectile equations.
- $\vec{r}(t)$ and $\vec{v}(t)$ are column vectors. For a special moment, write the left side as a column vector, fill in what you know in each row, circle the unknowns, and solve the row with only one unknown first.
- Find the components of a launch velocity from its speed and angle with a right triangle sketch.



Apply

Build each piece on your whiteboard, finalize it as a group, then copy it into your notes before starting the next piece. Write each left side as a column vector.

1. Model the scenario. A golf ball is hit from the ground at 54° above the horizontal. It reaches a maximum height of 6.0 m. A tree 3.5 m tall stands 14 m away. Does the ball clear the tree?

$$\begin{bmatrix} \vec{r}(t_{\text{tree}}) \\ \\ \\ \end{bmatrix} =$$

2. Model the scenario. A ball is thrown from the edge of a cliff 20 m above the beach, at 15 m/s and 35° above the horizontal. Put the origin at the edge of the cliff, with +x forward and +y up. Where does the ball land, and what is its velocity (speed and direction) just before it lands? Math tool: the solutions of

$$at^2 + bt + c = 0$$

are

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\vec{r}(t) \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} =$$

$$\vec{r}(t_{\text{land}}) \begin{bmatrix} \\ \end{bmatrix} =$$

$$\vec{v}(t_{\text{land}}) \begin{bmatrix} \\ \end{bmatrix} =$$

3. Model the scenario. A hockey stick deflects a puck. Before the hit, the puck slides at 9.0 m/s at 20° south of east. After the hit, it slides at 6.0 m/s at 55° north of east. East is +x and north is +y. Sketch each velocity with its components and write each as a column vector. Then write the velocity update equation with column vectors, find Δv one row at a time, and draw the arrow equation tip to tail. Finally, find the magnitude and direction of Δv.

$$\vec{v}_i \begin{bmatrix} \\ \end{bmatrix}$$

$$\vec{v}_f \begin{bmatrix} \\ \end{bmatrix}$$

$$\vec{v}_i \begin{bmatrix} \\ \end{bmatrix} + \Delta\vec{v} \begin{bmatrix} \\ \end{bmatrix} = \vec{v}_f \begin{bmatrix} \\ \end{bmatrix}$$

4. From the same height, one ball is dropped and another is thrown straight forward (horizontally). A group member says the thrown ball lands later, because it travels farther. Write the Claim, Evidence, Reasoning template on your whiteboard and

answer the question: do you agree? Finalize it on your whiteboard as a group, then copy it into your notes below.

Claim

Evidence

Reasoning

Check in: Reflector, before the consensus share-out, check that your group agrees on every answer.

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2: ENERGY AND MOMENTUM IN CLOSED SYSTEMS

2.1: Momentum in One Dimension
2.2: Momentum in Two Dimensions
2.3: Energy Conservation
2.4: Applying the Energy Conservation Model
2.5: Combined Conservation Laws

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2.1: Momentum in One Dimension

2.1 Momentum Conservation in One Dimension

2.1.1 Motion Quantities: Velocity and Momentum

Model

The charts below represent the motion of two cars. The two cars are initially separate and then Car A collides and sticks to Car B.

v-chart

Objects	$\vec{v}_i$	$\Delta\vec{v}$	$\vec{v}_f$
Car A	→	←	→
Car B	0	→	→
Total System	→	←	→

p-chart

Objects	$\vec{p}_i$	$\Delta\vec{p}$	$\vec{p}_f$
Car A	→	←	→
Car B	0	→	→
Total System	→	0	→

Explore

1. Which car is initially moving? How can you tell?
2. What is the direction of the change in velocity vector $\Delta \mathrm{v}$ or change in momentum vector $\Delta \mathrm{p}$ for each object? Are the arrows the same length?
3. Is the total change in system to the right, left, or zero?
4. Is the final arrow the same or different length for each car?
5. How many different length arrows are shown?

Critical Thinking & Concept Invention

Use the Evidence in your notes from the Explore section to make claims from the two charts. Use the Claim-Evidence-Reasoning structure.

Claim: The motion of Car A is initially ...

Evidence:

- The velocity vector for Car A...
- The momentum vector for Car A...

Reasoning:

Claim: The final velocity of each cart should be the same.

Evidence:

Reasoning:

Claim: The final momentum of each cart might not be the same, even if the velocity is the same.

Evidence:

Reasoning:

Claim: Velocity _____ conserved in collisions.

Evidence:

Reasoning:

Claim: Momentum _____ conserved in collisions.

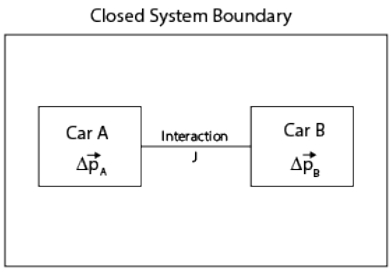
Evidence:

Reasoning:

2.1.2 Open and Closed Momentum Systems

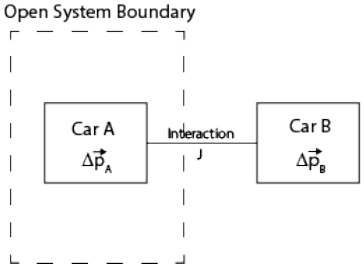
Model

The charts below represent the motion of two cars. The two cars are initially separate and then Car A collides and sticks to Car B. Car A has a known, nonzero initial velocity and momentum and Car B is initially at rest. The Open and Closed systems both accurately model the same phenomenon, just in different ways.



Closed System p-chart

Objects	$\vec{p}_i$	$\Delta\vec{p}$	$\vec{p}_f$
Car A	→		
Car B	0		
Total System		0	



Open System p-chart

Objects	$\vec{p}_i$	$\Delta\vec{p}$	$\vec{p}_f$
Car A	→	?	

Explore

1. Examine the closed system model (top) to complete the System Interaction Story:
   - There are _____ objects inside the system boundary.
   - This system includes _____.
   - There is/are _____ $\Delta \mathbf{p}$ terms inside the boundary.
   - There is/are _____ internal interactions (interaction lines fully enclosed in the system) and _____ external interactions (interaction lines that cross the system boundary).
   - The system boundary is (solid / dashed).
   - The total change in momentum of the system is (zero / nonzero).
2. Examine the open system model (bottom) to complete the System Interaction Story:
   - There are _____ objects inside the system boundary.
   - This system includes _____.
   - There is/are _____ $\Delta \mathbf{p}$ terms inside the boundary.
   - There is/are _____ internal interactions (interaction lines fully enclosed in the system) and _____ external interactions (interaction lines that cross the system boundary).
   - The system boundary is (solid / dashed).
   - The total change in momentum of the system is (zero / nonzero).

Critical Thinking & Concept Invention

1. If there are zero external interactions, is the system an open or closed system?
   If there are zero external interactions, the system is _____.
2. If there is an external interaction, is the system an open or closed system?
   If there is an external interaction, the system is _____.
3. How do we indicate whether the system is open or closed when we draw the system boundary?
4. Which part(s) of a p-chart are impacted by the decision to identify the system as open or closed?

Summary

The Momentum Principle is the first of the three key dynamics principles that we will use in our study of Mechanics. This principle is represented by the equation:

$$\Delta \mathbf{p}_{system} = \sum \mathbf{J}$$

“The total change in momentum of a system” = “The net impulse acting on the system”

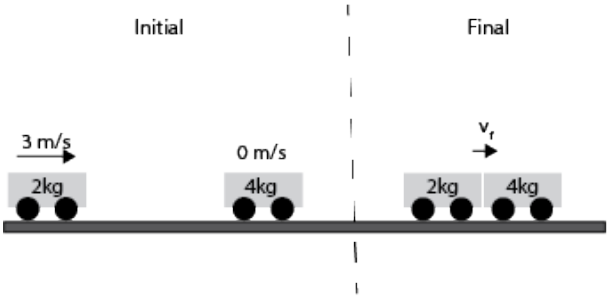
The choice of a **system** is arbitrary in that any choice can be made and the physics still works. The physics equations will be different (but still accurate) for different choices of system, so it is necessary to identify your chosen system in every problem. For now, we will choose only **closed systems** so that $\sum \mathbf{J} = 0$, which means the momentum of the system is **conserved**.

2.1.3 Building a Momentum Chart (p-chart)

Model

The models below represent the motion of two cars. The two cars are initially separate and then Car A collides and sticks to Car B.

Physical Scenario Diagram (PSD)



p-chart

Objects	$\vec{p}_i$	$\Delta\vec{p}$	$\vec{p}_f$
Car A	6 kg m/s →		
Car B	0		
Total System	→	0	

Explore

1. Examine the Physical Scenario Diagram (left) to complete the Physical Story:
   - There are two Cars, A and B.
   - Car _____ has more mass than Car _____.
   - In the beginning, Car A has a velocity of _____ to the _____ and Car B has a velocity of _____.
   - The two cars collide and (stick together / bounce apart).
   - After the collision, Car A has a velocity of _____ to the _____ and Car B has a velocity of _____ to the _____.
2. Examine the p-chart (right) to tell a Momentum Story:
   - In the beginning, Car A has a momentum of _____ to the _____ and Car B has a momentum of _____.
   - The total momentum of the system is _____ to the _____.

Critical Thinking & Concept Invention

1. Evaluate the system momentum row to finish your momentum story:
   - The two cars collide and _____, and the total change in momentum of the system is _____.
   - This means the system is (open / closed).
   - The final momentum of the system should be _____ to the _____.
   - Car _____ should have more momentum than Car _____ because...
   - Proportionally, Car A should have _____% of the total momentum and Car A should have _____% of the total momentum.
2. Summarize: How can you determine the change in momentum (Δp) for each car? Describe a process here, then follow that process to complete those rows of the p-chart.

Apply

Create a Physical Scenario Diagram (PSD), Object Interaction Diagram (OID), and Momentum Chart (p-chart) for the following scenarios. Choose a closed system for each scenario. Use your diagrams as evidence to answer the question.

1. Victoria initially stands on a boat in still water. Neither Victoria nor the boat are moving. Then, Victoria begins to walk towards the right at 1 m/s. What will happen to the boat?
2. A spaceship is filled with gas and moves through space at 10 m/s towards the right.
    - **Timeline 1:** The spaceship turns on its rockets, which send gas out towards the left at 100 m/s. What will happen to the rocket?
    - **Timeline 2:** After sending the gas out, the spaceship turns off its engines. What will happen to the spaceship after the rocket engines have been off for a while?

2.1.4 Equations from a Momentum Chart (p-chart)

Model

The equations below represent the same collision of two cars in the previous model.

$\mathbf{p}_{i,A} + \mathbf{p}_{i,B} = \mathbf{p}_{i,system}$

$\mathbf{p}_{i,A} + \Delta\mathbf{p}_{A} = \mathbf{p}_{f,A}$

$\mathbf{p}_{f,A} + \mathbf{p}_{f,B} = \mathbf{p}_{f,system}$

$\mathbf{p}_{i,B} + \Delta\mathbf{p}_{B} = \mathbf{p}_{f,B}$

$\Delta\mathbf{p}_{A} + \Delta\mathbf{p}_{B} = \Delta\mathbf{p}_{system}$

$\mathbf{p}_{i,system} + \Delta\mathbf{p}_{system} = \mathbf{p}_{f,system}$

Explore

1. How many equations represent the interaction of these two carts?
2. What are the different labels before the commas in the subscripts? Identify equations that have the same letter before the comma in the subscript for each term in the equation. Circle each of these equations.
3. What are the different labels after the commas in the subscripts? Identify equations that have the same letter/word after the comma in the subscript for each term in the equation. Box each of these equations.
4. There should be one equation left. Does it more closely resemble the boxed or circled equations? Circle/box it based on your choice.

Critical Thinking & Concept Invention

Each column and each row represents an equation.

1. Which sets of equations (circled or boxed) represent the columns and which represent the rows?
    - The columns are represented by the equations that were _____.
    - The rows are represented by the equations that were _____.
2. Label each equation by the row or column it represents (e.g., Row 1, Row 2, Column 1, Column 2, etc.).
3. Could there ever be more than 3 rows (and 3 row equations)? If so, explain how.
4. Could there ever be more than 3 columns (and 3 column equations)? If so, explain how.

Summary

All of our physics dynamics equations will be for the system that we select in our models. The physics equations of motion that we will solve using the Momentum Principle are represented by the bottom row of the momentum chart (p-chart):

$$\mathbf{p}_{i,system} + \Delta\mathbf{p}_{system} = \mathbf{p}_{f,system}$$

In each specific scenario, we will create a specific system momentum equation that aligns with the scenario and the chosen system. This generally takes the form of:

$$\mathbf{p}_{i,A} + \mathbf{p}_{i,B} + ... + \Delta\mathbf{p}_{system} = \mathbf{p}_{f,A} + \mathbf{p}_{f,B} + ...$$

with the subscripts A, B, etc. labeling the different interacting objects that are within the system.

Apply

1. Create a system momentum equation for the scenario of Carts A and B colliding. Then put a column vector structure under each vector in that equation and fill in values to solve the equation.
2. Look back at your models for Victoria and the Boat. Generate momentum equations from that p-chart. Leave unknown quantities as variables (e.g., $m_{Victoria}$). Then answer the questions:
   - How would the final velocity of the boat compare to the final velocity of Victoria if Victoria and the boat had the same mass?
   - How would the final velocity of the boat compare to the final velocity of Victoria if Victoria had more mass than the boat?
   - How would the final velocity of the boat compare to the final velocity of Victoria if Victoria had less mass than the boat?
3. Look back at your models for the spaceship. Generate momentum equations from that p-chart. Leave unknown quantities as variables (e.g., m_{gas}). Then answer the question: Does the mass of the gas affect the change in velocity for the rocket? If so, how? Explain your thinking.

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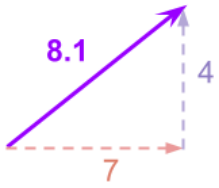
2.2: Momentum in Two Dimensions

2.2 Momentum Conservation: 2 Dimensions

2.2.1 Vector Dot Product and Magnitude

Model

Pythagorean Theorem



$A^2 + B^2 = C^2$
$7^2 + 4^2 = C^2$
$49 + 16 = C^2$
$8.1 = C$

Vector Dot Product

$$\begin{pmatrix} 7 \\ 4 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} 7 \\ 4 \\ 0 \end{pmatrix} = \begin{matrix} 49 \\ 16 \\ + \quad 0 \\ \hline 65 \end{matrix}$$

To get the magnitude, take the square root

$$R = \sqrt{65}$$

$$R = 8.06 \sim 8.1$$

Explore

Examine the Pythagorean Theorem:

1. What part of the vector is represented by A, B, and C in the Pythagorean theorem?
2. What math operations are performed in the process of calculating C?

Examine the Vector Dot Product:

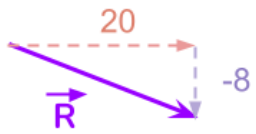
1. How do the components of the vector interact for multiplication and division?
2. What math operations are performed in the process of calculating R? Describe all steps.

Critical Thinking & Concept Invention

1. How are the two processes similar?
2. What might be a benefit of the vector dot product?

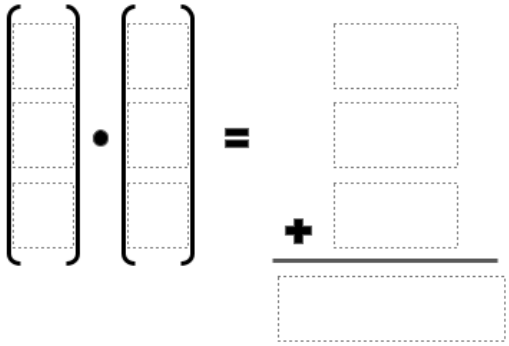
Apply

Find the magnitude of the vector



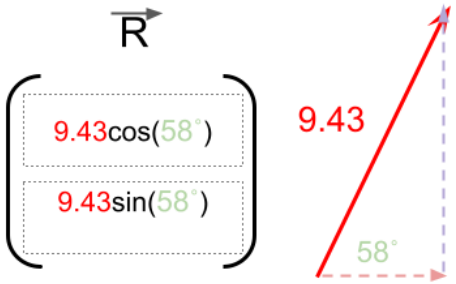
R =

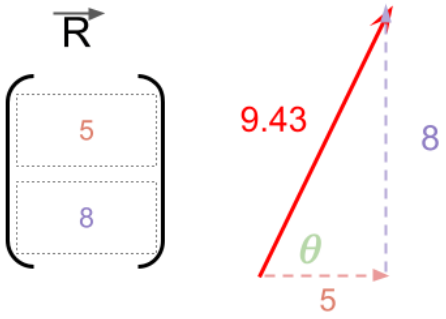
Vector Dot Product



2.2.2 Vector Magnitude and Angle

Model





$R_x = R\cos(58°) = 5$

$R_y = R\sin(58°) = 8$

$\theta = \tan^{-1}(R_y / R_x)$

$\theta = \tan^{-1}(8 / 5) = 58°$

Explore

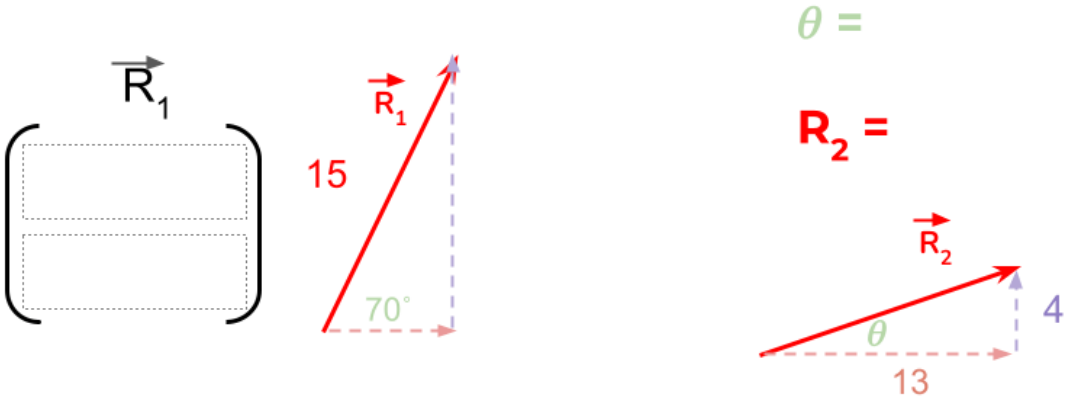
1. What is the expression for R_x?
2. What is the expression for R_y?
3. What is the expression for the angle θ?

Critical Thinking & Concept Invention

1. The angle θ represents the angle between which two vectors?
2. Is R_y next to the angle θ or opposite that angle?
3. Is R_x next to the angle θ or opposite that angle?
4. Explain why sine and cosine are used to find the components.
5. Explain why tangent is used to find the angle.

Apply

Identify the column vector or unknown angle using the right triangle geometry relationships.

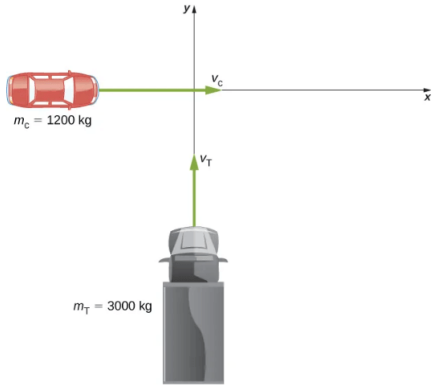


2.2.3 90° Car Collision

Model

A small car of mass 1200 kg traveling east at 60 km/hr collides at an intersection with a truck of mass 3000 kg that is traveling due north at 40 km/hr. The two vehicles are locked together. What is the velocity of the combined wreckage? **Example Credit:** OpenStax University Physics Vol 1 (https://openstax.org/details/books/university-physics-volume-1)

Physical Scenario



p-chart	$\vec{p}_i$	$\Delta\vec{p}$	$\vec{p}_f$
System			

Physical Story

System Momentum Equation

$$\vec{p}_{i,\,car} + \vec{p}_{i,\,truck} + \Delta\vec{p}_{system} = \vec{p}_{f,\,car} + \vec{p}_{f,\,truck}$$

$$\begin{bmatrix} \\ \\ \end{bmatrix} + \begin{bmatrix} \\ \\ \end{bmatrix} + \begin{bmatrix} \\ \\ \end{bmatrix} = \begin{bmatrix} \\ \\ \end{bmatrix} + \begin{bmatrix} \\ \\ \end{bmatrix}$$

Explore

1. Read the prompt and examine the Physical Scenario Diagram to write a complete the Physical Story underneath the model:
   - There are two objects, a car with mass _____ and a truck with mass _____.
   - In the beginning, the car travels in the (+ / –) (x / y) direction and the truck travels in the (+ / –) (x / y) direction.
   - The two objects collide and (stick together / bounce apart).
   - The two objects (do / do not) have the same final velocity because _____.

Critical Thinking & Concept Invention

1. Which component (x or y) of the initial momentum for the car should be zero? Fill that into the equation.
2. Which component (x or y) of the initial momentum for the truck should be zero? Fill that into the equation.
3. Make claims and provide evidence and reasoning to each of the following questions:
   - How does the final velocity of the car and the truck compare?
   - How should the final momentum of the car and the truck compare?
4. Solve the problem:
   - Fill in the momentum chart with the assumption that there are no net external interactions with the two vehicles. Tell

your group members what part of the equation you can fill in based on this assumption.
- Fill in the empty spaces in the momentum equations. It can be helpful to start with components that are zero.

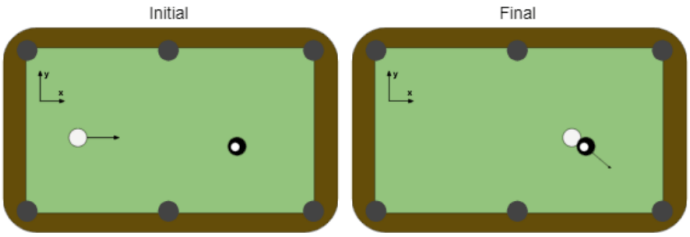
5. Why was it necessary to use column vectors to solve this problem?

2.2.4 Pool Shot

Model

You are playing pool with friends and are trying to sink the eight ball (black, 0.16 kg). You shoot the cue ball (white, 0.17 kg) and it initially has a velocity of 1.5 m/s towards the right. The eight ball is initially at rest and needs to travel at an angle that is 45° below horizontal in order to roll towards the pocket.

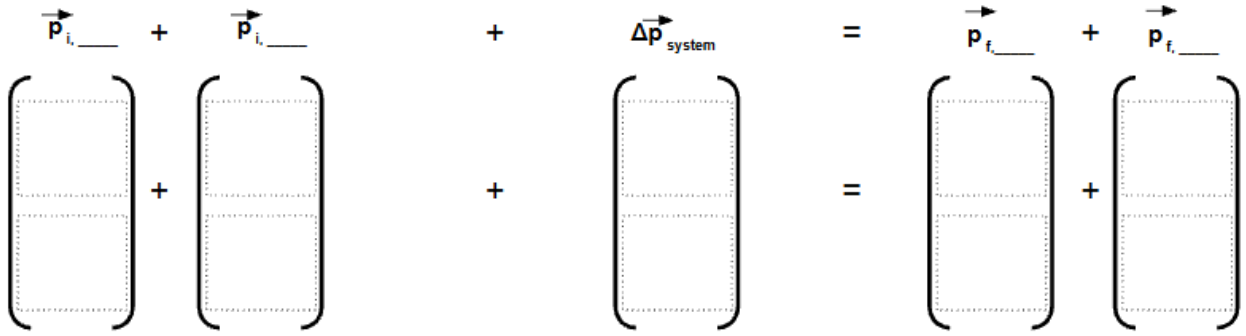
Physical Scenario



p-chart

	$\vec{p}_i$	$\Delta\vec{p}$	$\vec{p}_f$
System			

Momentum Story



Explore

1. Read the prompt and examine the Physical Scenario Diagram to complete the Physical Story:
   - There are two objects, with mass _____ and with mass _____.
   - In the beginning, the initial speed of _____ is _____ in the _____ direction and the initial speed of _____ is _____.
   - The two objects collide and (stick together / bounce apart).
   - After the collision, _____ moves in the direction _____° below horizontal.
   - The two objects (do / do not) have the same final velocity because _____.

Critical Thinking & Concept Invention

1. Construct a Momentum Chart that is consistent with the problem. Underneath the charge, create a Momentum Story to describe each object before, during, and after the collision.

Apply

2. Complete the System Momentum Equation that is consistent with the cue ball and eight ball as a closed system. Fill in as much as you can, leaving unknown quantities as variables.
3. Circle the unknown quantities in your momentum equations. How many unknowns are there? Can you solve the equation? Explain why or why not.

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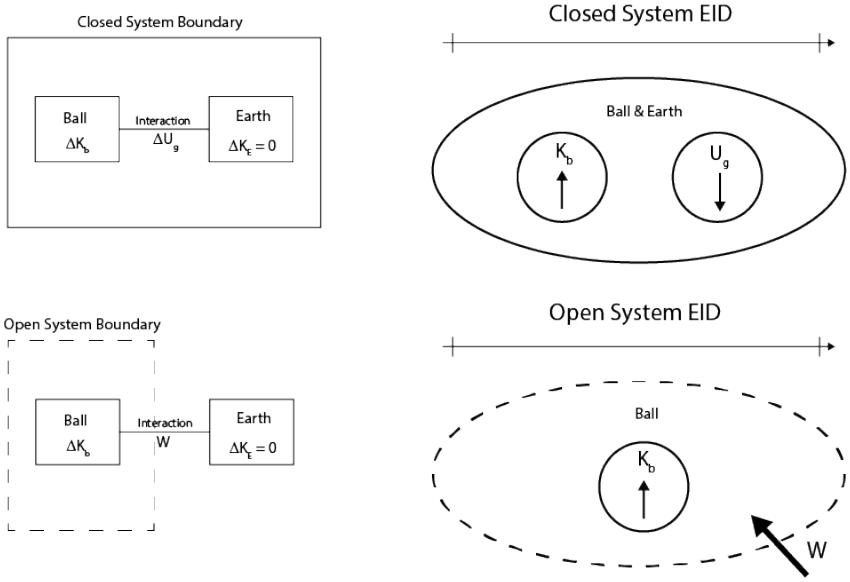
2.3: Energy Conservation

2.3 Energy Conservation Model

2.3.1 Open and Closed Energy Systems

Model

The model below shows two choices of system for the same phenomenon of a ball falling towards the ground. The ball speeds up as it falls.



Explore

1. **Tell a System Story for the closed system (top):**
    - This system includes objects: _____ and _____.
    - There is/are _____ internal interactions and _____ external interactions.
    - This interaction results _____ (copy the symbol under the interaction line) of the system.
    - This is a(n) (open / closed) system and has a (solid / dashed) boundary.
2. **Tell an Energy Story for the closed system (top):**
    - This system includes different types of energy that could change: _____, _____, and _____, but _____ is equal to zero.
    - In this system, (energies only increase / energies only decrease / some energies increase and some decrease).
    - There is an increase in _____ and a decrease in _____.
3. **Tell a System Story for the open system (bottom):**

- This system includes object: the _____.
- There is/are _____ internal interactions and _____ external interactions.
- This interaction results _____ (copy the symbol under the interaction line) of the system.
- This is a(n) (open / closed) system and has a (solid / dashed) boundary.

4. **Tell an Energy Story for the open system (bottom):**

   - This system includes different types of energy that could change: _____.
   - In this system, (energies only increase / energies only decrease / some energies increase and some decrease).

Critical Thinking & Concept Invention

1. For each system, is it possible that the total energy of the system does not change? Write a CER response where you take the evidence from parts of your previous stories or descriptions of the model.

   Claim: In the closed system, it (is possible / is not possible) that the total energy of the system does not change.

   Evidence:

   Reasoning:

   Claim: In the open system, it (is possible / is not possible) that the total energy of the system does not change.

   Evidence:

   Reasoning:

2. Do you think Gravitational Potential Energy (U_g) is associated with a single object? If not, what is this energy associated with?

   Claim: Gravitational Potential Energy is associated with...

   Evidence:

   Reasoning:

3. **Summarize some steps for making these diagrams**

   - You only include ΔU_g when ...
   - The number of energy circles in the EID matches ...
   - Energy is conserved (total energy does not change) when ...

Summary

The Energy Principle is the third of the three key dynamics principles that we will use in our study of Mechanics. The more general version of this principle is called the First Law of Thermodynamics and is given by the equation:

$$\Delta E_{system} = Q + W \tag{2.3.1}$$

"The total change in energy of a system" = "The net Heat exchange (Q) plus the net Work (W) between the system and the

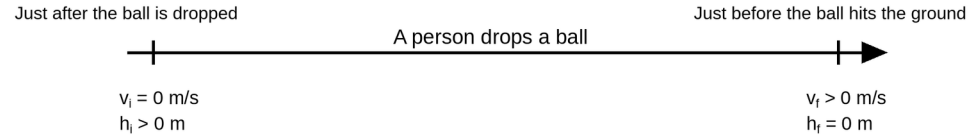
surroundings”

This course (Mechanics) will focus on scenarios where the heat exchange between the system and the surroundings is zero ($Q = 0$). Heat will be explored in more detail when you learn about thermodynamics.

The choice of a system is arbitrary in that any choice can be made and the physics still works. The physics equations will be different (but still accurate) for different choices of system, so it is necessary to identify your chosen system in every problem. For now, we will choose only closed systems so that $W = 0$ as well, which means the energy of the system is conserved.

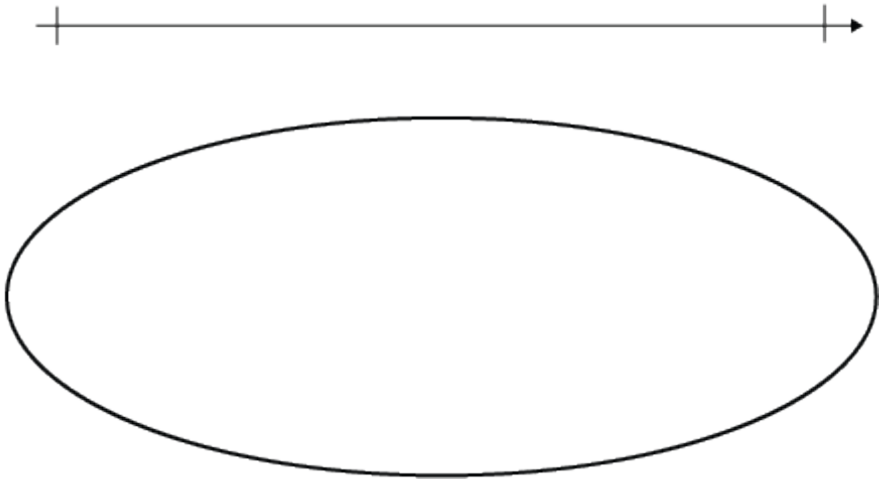
2.3.2 Dropping a Ball

Model





Object Interaction Diagram



Energy Story

Explore

1. Tell a Physical Story for this scenario as a few sentences. Use the following prompts:
    - **This timeline begins with:**
        - Copy the description above the initial mark on the timeline.
        - State initial energy indicators (e.g., speed, height) for each object (in this case the ball) as values, variables, or descriptions.
    - **Over this timeline:**
        - Identify which energy indicators increase (if any).
        - Identify which energy indicators decrease (if any).
        - Identify any interactions between objects (if any).
    - **At the end of the timeline:**
        - Copy the description above the final mark on the timeline.
        - State final energy indicators (e.g., speed, height) for each object (in this case the ball) as values, variables, or descriptions.

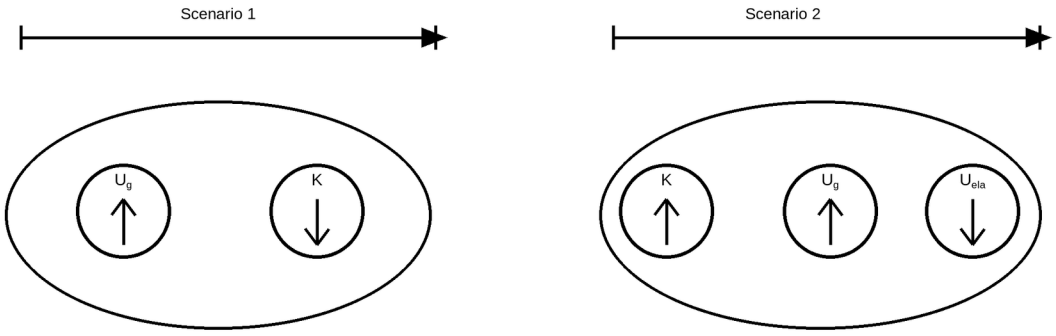
Critical Thinking & Concept Invention

1. Why do you think the initial velocity is zero if the timeline starts just after the ball is dropped?
2. Why do you think the final height is zero if the timeline ends just before the ball hits the ground?
3. How many energy systems should change in the EID?
4. What type of energy should increase? What type of energy should decrease?
    - _____ should increase.
    - _____ should decrease.
5. Construct an Object Interaction Diagram and Energy Interaction Diagram for this scenario. Model it as a closed system.
6. Tell an energy story.

2.3.3 Energy Equations

Model

The EIDs and energy equations below represent two different phenomena. The details of the phenomena are unknown, but the energy changes are shown in the models.



$\Delta E_{system} = Q + W$

(+) (−)

$\Delta U_g + \Delta K = 0$

$\Delta E_{system} = Q + W$

(+) (+) (−)

$\Delta K + \Delta U_g + \Delta U_{ela} = 0$

Explore

1. In scenario 1, how many energy systems (circles / bubbles) in the EID change? How many terms (parts of an equation separated by + or – signs) are included on the ΔE_{system} side of the equation (left-hand side)?
   - In the EID, _____ energy systems change.
   - In the equation, ΔE_{system} is made up of _____ terms.
2. In scenario 2, how many energy systems (circles / bubbles) in the EID change? How many terms (parts of an equation separated by + or – signs) are included on the ΔE_{system} side of the equation (left-hand side)?
   - In the EID, _____ energy systems change.
   - In the equation, ΔE_{system} is made up of _____ terms.
3. In both scenarios, what do Work (W) and Heat (Q) equal?
   - $W =$ _____
   - $Q =$ _____

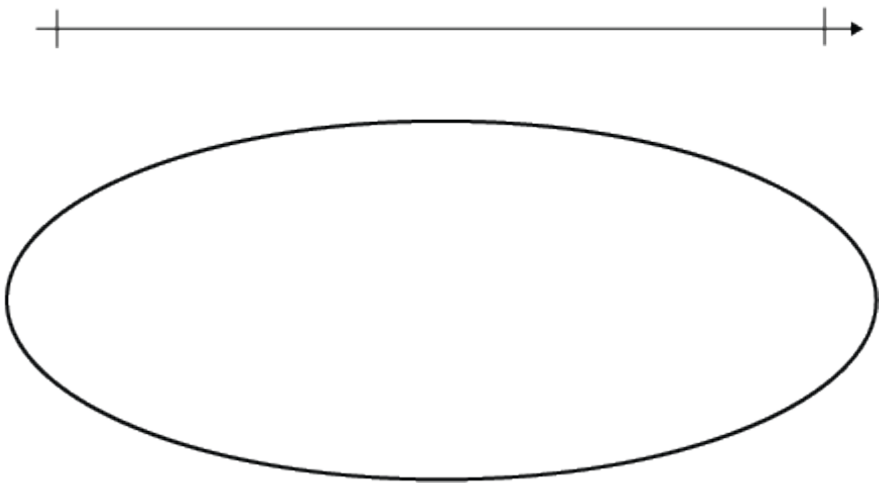
Critical Thinking & Concept Invention

1. Compare the arrows on the Energy Interaction Diagrams to the signs (+ or –) on the Energy Equations.
   - When an energy system increases, the change in that type of energy is: + / –
   - When an energy system decreases, the change in that type of energy is: + / –
2. Describe how to generate the equation for ΔE_{system} (including the + and – signs) from an Energy Interaction Diagram.

Apply

A spring launches a car down a ramp. Create an Object Interaction Diagram, timeline, Energy Interaction Diagram, and energy equation to model the change in energy of the car if it starts at rest with the spring compressed and ends moving at a lower height than it started. Be sure to include + / – signs above the terms in your energy equation.

Object Interaction Diagram



$$\Delta E_{system} = Q + W \tag{2.3.2}$$

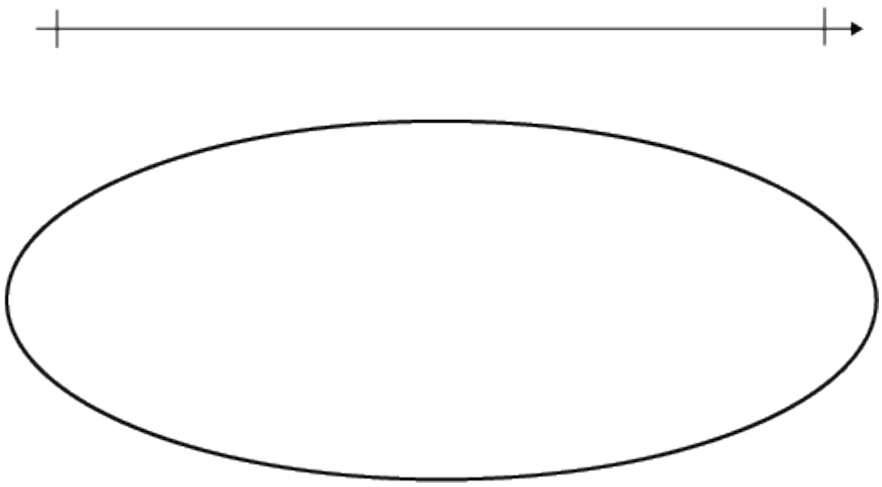
2.3.4 Three Thrown Rocks

Model

Three rocks are thrown off a cliff at the same speed but in different directions. They have a speed of v_0 just after they are thrown. The rocks are thrown from a height H above the ground below the cliff. They land on flat ground below the cliff. How does the speed of each rock just before they hit the ground compare with one another? Which rock (if any) is traveling the fastest?



Object Interaction Diagram



$$\Delta E_{system} = Q + W \tag{2.3.3}$$

Explore

1. Make a Prediction: Without constructing any models or using any equations, make a prediction. Which rock do you think will have the greatest speed just before it hits the ground?
2. Examine the written prompt: What is the initial height and speed of the rocks? (Your response could be a number or a labeled variable.)
   - The initial speed is _____.
   - The initial height is _____.
3. Examine the Physical Scenario Diagram. What is the final height of the rocks? (Your response could be a number or a labeled variable.)
   - The final height is _____.
4. What is the unknown variable that you are going to use your physics models to solve for?
   - The unknown is _____.
5. Add the indicators you identified to this timeline. For the unknown, identify a variable for that indicator and make it equal to "?" on the timeline to identify it as the unknown.

Critical Thinking & Concept Invention

1. What written descriptions should be given to the beginning and end of the timeline? Add it to the timeline above.
2. What indicators change from the beginning to the end of this timeline?
3. Based on those indicators, how many energy systems will change? Which types of energy will change?
4. **Your instructor will assign you one of the three rocks to model.** Make an Energy Interaction Diagram and equations for your assigned rock. Solve the equation for the unknown that you identified in Exploration Question 3.

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2.4: Applying the Energy Conservation Model

2.4 Applying the Energy Conservation Model

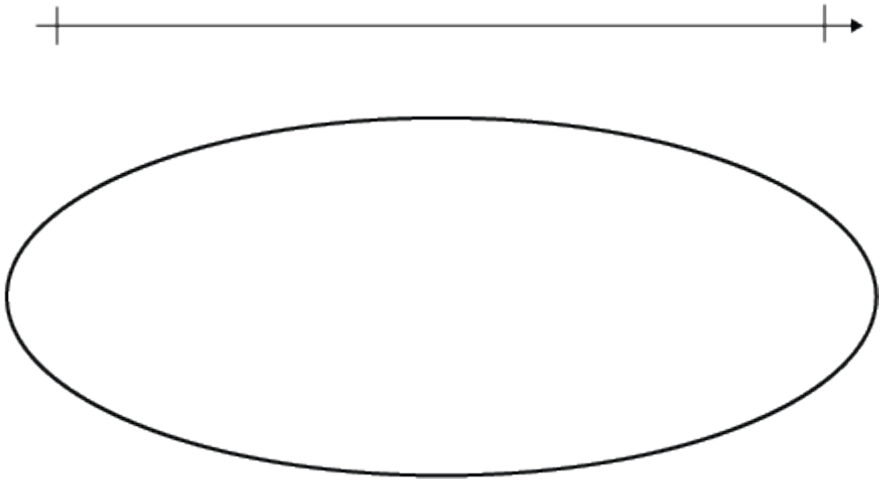
2.4.1 Throwing a Ball to Maximum Height

Model

With your new knowledge of energy, you wonder if you can determine how high you can throw a ball. Let's say you can throw a ball upward with a speed of 15 m/s. How high will it go before it starts to fall back down? Assume the ball is 1.5 meters above the ground just after it leaves your hand.



Object Interaction Diagram



$$\Delta E_{system} = Q + W \tag{2.4.1}$$

Explore

1. What indicator values are given in the prompt?
2. What qualitative description represents the moment on the timeline for those indicators? Place a dot labeled **i** for initial and **f** for final to represent the physical location of the ball at the beginning and end of this timeline.
3. What is the unknown that you are trying to determine?
4. Tell a Physical Story for this scenario as a few sentences:
   - This timeline starts when the ball...
   - The initial speed is _____ and the initial height is _____.
   - Over this timeline, the height of the ball _____ and the speed of the ball _____.
   - This timeline ends when the ball reaches _____ and the speed of the ball is _____.

Critical Thinking & Concept Invention

1. What do you think the speed of the ball is just as it reaches its maximum height? Check that this makes sense to everyone in the group.
2. At the maximum height, the ball switches from going up to going down. We often call this a turnaround point. What is the speed of an object at a turnaround point?

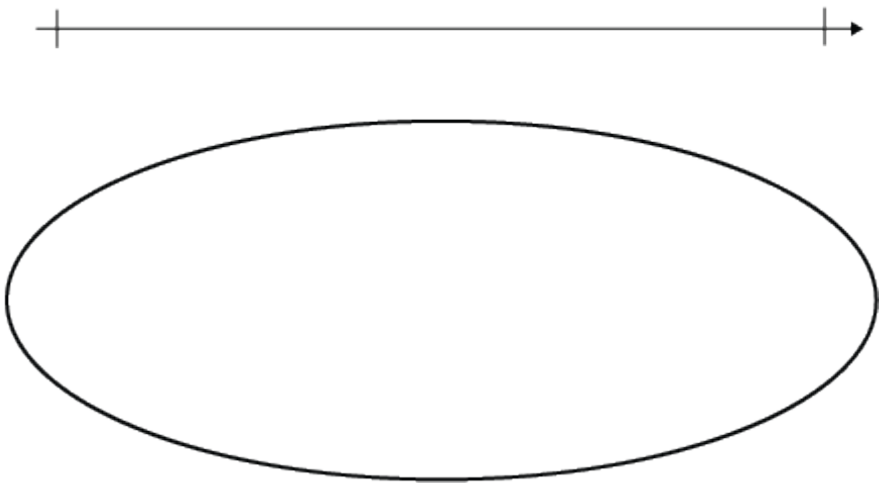
   At a turnaround point, the speed of an object is _____.
3. Construct a Timeline using the information from the previous questions. Then, construct an Energy Interaction Diagram and a corresponding Energy Equation. Solve the equation for the maximum height.

Apply

Use the same process as before, but solve for the speed of the ball when it is 10 meters above the ground on the way up and 10 meters above the ground on the way down. Mark these points on the diagram below (not fully to scale) to show the different final states for each timeline.



Object Interaction Diagram



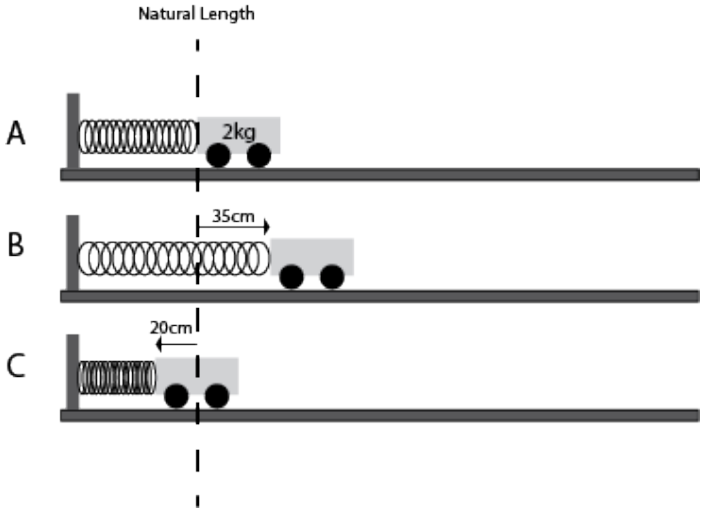
$$\Delta E_{system} = Q + W \tag{2.4.2}$$

2.4.2 Horizontal Spring

Model

A cart is attached to a spring ($k = 10\ \mathrm{J/m^2}$) and is able to move side to side on a horizontal tabletop. The cart is initially not moving and the spring is at its natural length. Then, the cart is pulled 35 cm to the right, causing the spring to stretch. Then the cart is released from rest. How fast will the cart be moving when it is 20 cm to the left of where it started (i.e., the spring is compressed 20 cm)?

Physical Scenario Diagram (PSD)



Explore

1. How many different positions is the spring and cart shown in the Physical Scenario Diagram?
2. What is the value for the indicator for Elastic Potential Energy in this scenario at:

- Time A: _____
- Time B: _____
- Time C: _____

3. What do these indicator values measure?

   These values measure _____.
4. What other indicator do you know the value for at some (possibly not all) times? What is the unknown that you will need to solve for?

   We know the value of _____ at times _____.
   We need to determine the value of _____.

Critical Thinking & Concept Invention

1. Why do you think there are three positions of the spring and cart shown in the Physical Scenario Diagram?
2. What does the vertical dashed line represent? Why is it important to show this across the different images of the spring position?
3. Explain what the indicator for Elastic Potential Energy is and how to model it on a Physical Scenario Diagram.
4. Create two sets of models, one for the timeline from A to B and one from the timeline B to C. Create equations for each model and solve for any unknown variables:

B to C



$$\Delta E_{system} = Q + W \tag{2.4.3}$$

A to B



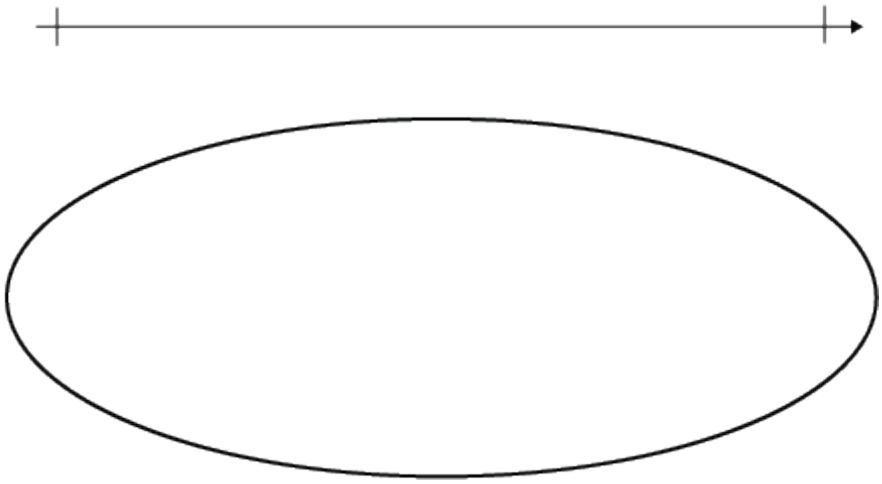
$$\Delta E_{system} = Q + W \tag{2.4.4}$$

Apply

A spring ($k = 5\ \mathrm{J/m^2}$) with a natural length of 20 cm hangs vertically above the ground. A 200 gram mass is connected to the end of the spring and then released. Identify how far down it will fall before the spring pulls it back up (i.e., find the lower turnaround point). To do this, draw a Physical Scenario Diagram, make a timeline with relevant indicators identified, construct an Energy Interaction Diagram to show the energies that change in this process, and construct an Energy Equation that is consistent with the EID and timeline.

Physical Scenario Diagram

Object Interaction Diagram

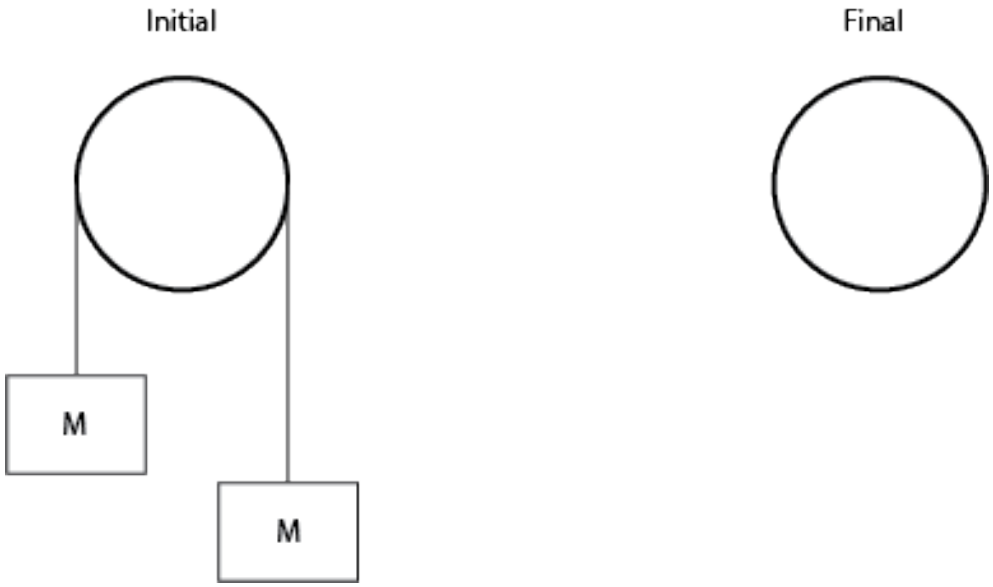


$$\Delta E_{system} = Q + W \tag{2.4.5}$$

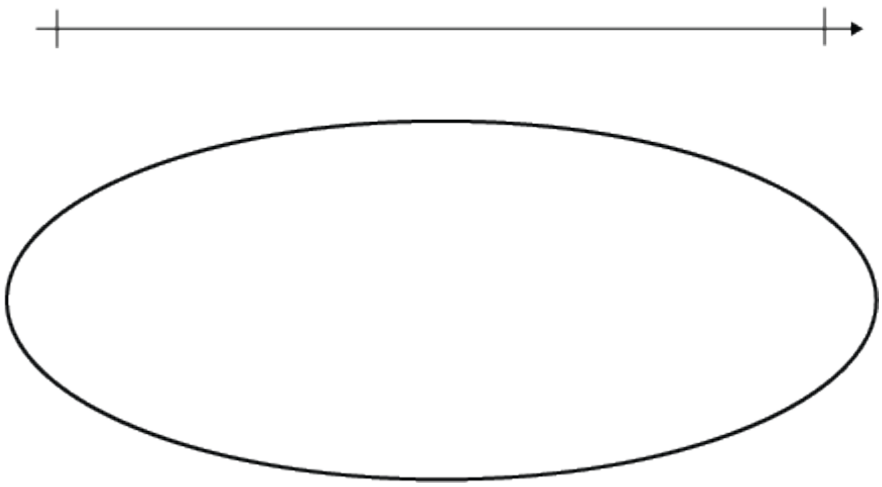
2.4.3 Equal Masses on a Pulley

Model

Two blocks with the same mass (M) are connected by a string that crosses over a pulley that is free to rotate. Assume the pulley has no mass. The two blocks are initially held in the position shown below and are then released.



Object Interaction Diagram



$$\Delta E_{system} = Q + W \tag{2.4.6}$$

Explore

1. Identify how the following quantities and indicators compare for each block before they are released. Respond "Left Block", "Right Block", or "The same for each block":
   - Which block has more mass?
   - Which block has a greater speed?
   - Which block is at a greater height?
2. Identify how the following energies compare for each block before they are released. Respond "Left Block", "Right Block", or "The same for each block":
   - Which block has more gravitational potential energy?
   - Which block has more kinetic energy?
3. What do you think will happen to the blocks when they are released?

Critical Thinking & Concept Invention

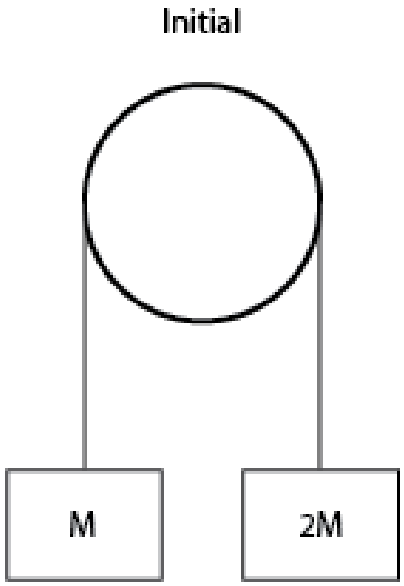
1. Record a tally of everyone's responses to what will happen to the blocks when released. Discuss any disagreements, then draw the blocks on the diagram for the final moment on the timeline as they will appear a few seconds later, based on what you said would happen. If the blocks are moving, draw arrows to indicate their direction. If not, write $v = 0$ next to the block.
2. Fill in a timeline for these blocks that aligns with your drawing. For indicators, you can provide qualitative descriptions (e.g., "faster/slower" or "higher/lower"). Then construct an Energy Interaction Diagram and Energy Equation with +/– symbols. You do not need to solve the energy equation.

3. Does your EID and equation make sense? Is the final state that you modeled possible or does it violate energy conservation? Explain.

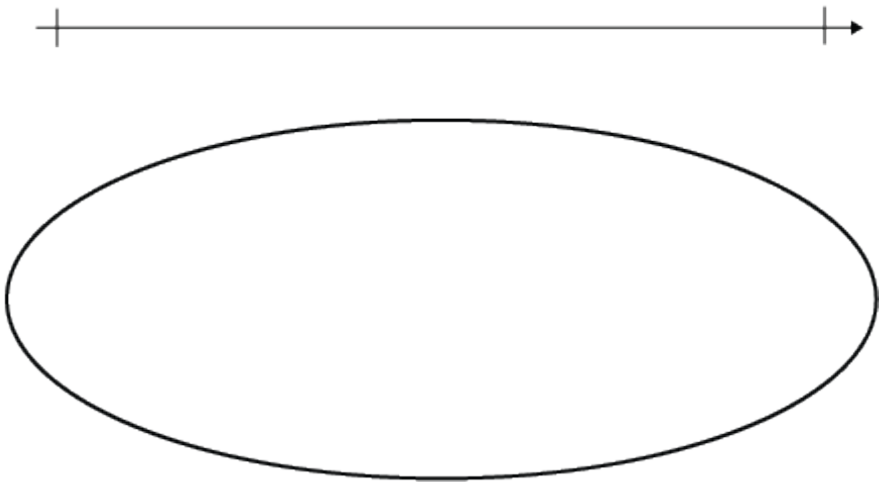
2.4.4 Unequal Masses on a Pulley

Model

Two blocks with different masses (M and $2M$) are connected by a string that crosses over a pulley that is free to rotate. Assume the pulley has no mass. The two blocks are initially held in the position shown below and are then released.



Object Interaction Diagram



$$\Delta E_{system} = Q + W \tag{2.4.7}$$

Explore

1. Identify how the following quantities and indicators compare for each block before they are released. Respond "Left Block", "Right Block", or "The same for each block":
   - Which block has more mass?
   - Which block has a greater speed?
   - Which block is at a greater height?

2. Identify how the following energies compare for each block before they are released. Respond "Left Block", "Right Block", or "The same for each block":
    - Which block has more gravitational potential energy?
    - Which block has kinetic energy?

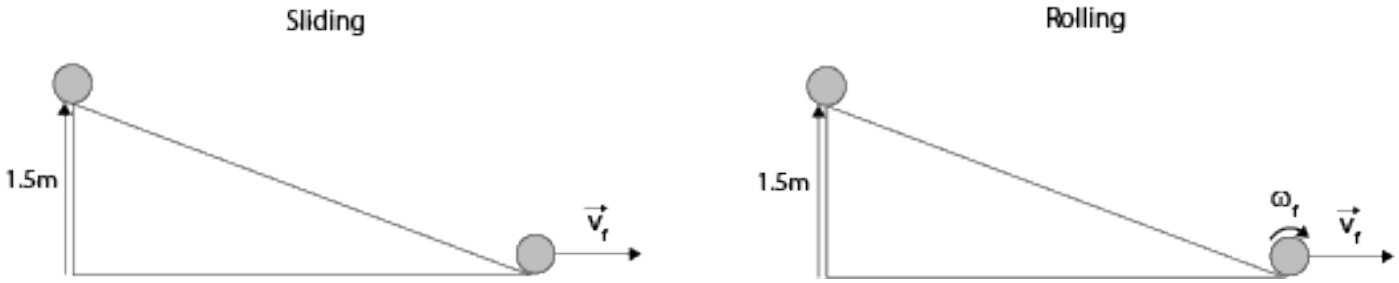
Critical Thinking & Concept Invention

1. What do you think will happen to the blocks when they are released? Explain your thinking and draw the blocks on the diagram for the final moment on the timeline as they will appear a few seconds later. If the blocks are moving, draw arrows to indicate their direction. If not, write $v = 0$ next to the block.
2. Your instructor will assign your group a possible final state of the system. Fill in a timeline for these blocks that aligns with that assignment. For indicators, you can provide qualitative descriptions (e.g., "faster/slower" or "higher/lower"). Then construct an Energy Interaction Diagram and Energy Equation with +/– symbols. You do not need to solve the energy equation.
3. Does your EID and equation make sense? Is the final state that you modeled possible or does it violate energy conservation? Explain.

2.4.5 Rotational Energy

Model

A ball travels down a ramp, starting from rest at the top of the ramp. Consider two different ways this could happen: The ball could slide down the ramp without rolling/spinning OR the ball could roll down the ramp.



Explore

1. Tell a physical story about each scenario. Be sure to list how the object(s) change (motion, height, etc.).

Ball Slides Down Ramp	Ball Rolls Down Ramp

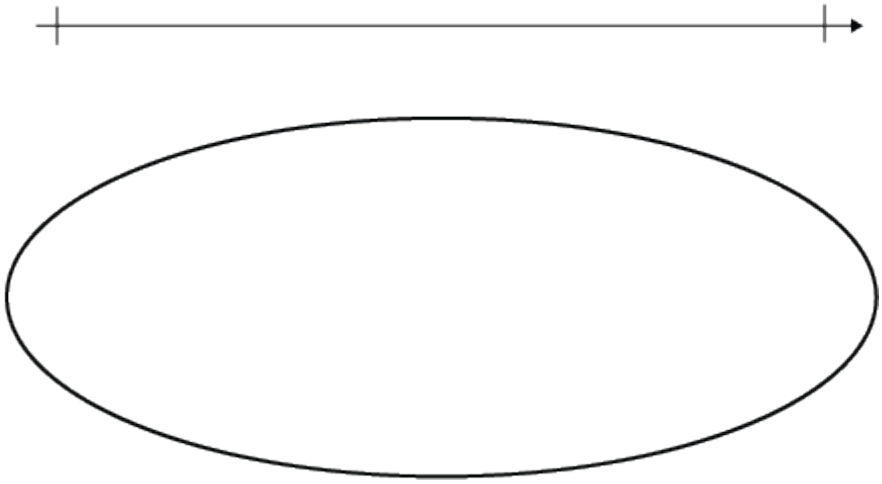
2. Identify the energies that increase/decrease when the ball slides down the ramp:
    - **Increase:**
    - **Decrease:**
3. Identify the energies that increase/decrease when the ball rolls down the ramp:
    - **Increase:**
    - **Decrease:**

Critical Thinking & Concept Invention

1. How do you think the final speed (v_f) of the ball that slides will compare to the ball that rolls? Explain your thinking.
2. Model both types of motion as a closed energy system. Construct an Object Interaction Diagram, an Energy Interaction Diagram, and an Energy Equation for each. You do not need to solve the equation.

Sliding

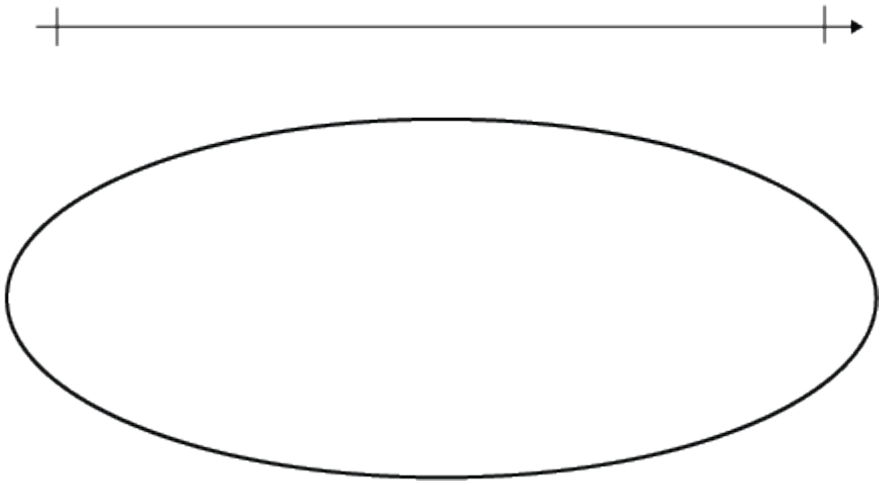
Object Interaction Diagram



$$\Delta E_{system} = Q + W \tag{2.4.8}$$

Rolling

Object Interaction Diagram



$$\Delta E_{system} = Q + W \tag{2.4.9}$$

3. Use your EID Model and Energy Equation to revisit Question 1. How will the final speeds of the ball compare in each scenario? Construct a CER response:

Claim: The final speed of the ball that slides will be _____ than the final speed of the ball that rolls.

Evidence:

Reasoning:

Summary

Translational Kinetic Energy

$$K_{trans} = \tag{2.4.10}$$

$$\Delta K_{trans} = \tag{2.4.11}$$

Rotational Kinetic Energy

$$K_{rot} = \tag{2.4.12}$$

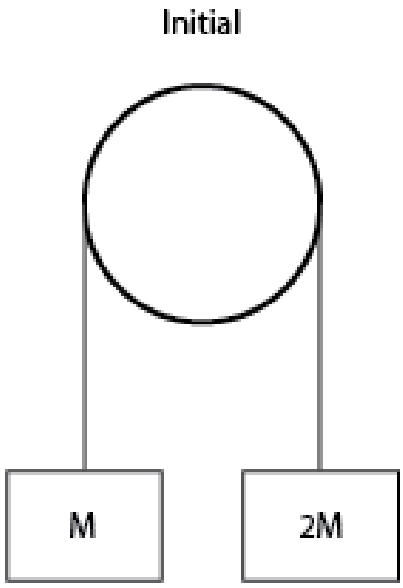
$$\Delta K_{rot} = \tag{2.4.13}$$

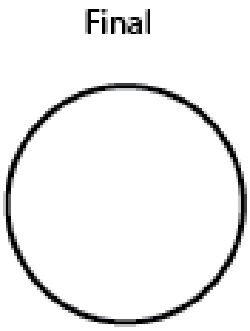
We can assume a moving/spinning object does not have either type of kinetic energy if _____ is zero (or really small compared to other objects).

Apply

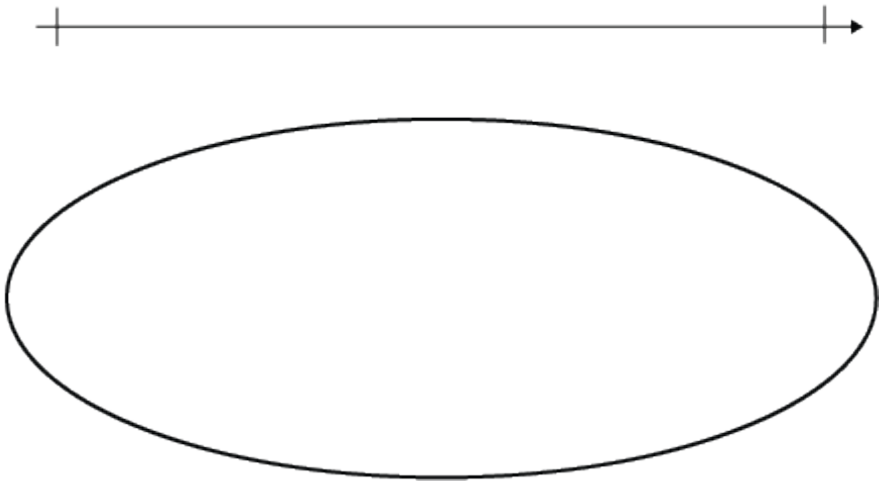
Let's revisit the unequal masses on a pulley from earlier in this activity. This time, we will assume the pulley is a disk with mass $3M$, which means it can have rotational kinetic energy. Make a new EID for this scenario, then describe how the motion will be different than when the pulley had no mass.

Pulley with Unequal Masses





Object Interaction Diagram



$$\Delta E_{system} = Q + W \tag{2.4.14}$$

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2.5: Combined Conservation Laws

1 Combined Conservation Laws

1.1 Pool Shot Revisited

Model

Let's return to the Pool Shot scenario from a few weeks ago. The original prompt and diagram are below:

You are playing pool with friends and are trying to sink the eight ball (black, 0.16 kg). You shoot the cue ball (white, 0.17 kg) and it initially has a velocity of 1.5 m/s toward the right. The eight ball is initially at rest and needs to travel at an angle that is 45° below horizontal in order to roll toward the pocket. Assume that **only kinetic energy changes** during this interaction.

Image needed here with Alt Text: Cue ball striking eight ball that travels 45 degrees below horizontal toward a pocket

Please copy your momentum equations from the original problem. Circle unknown quantities.

Explore

1. How many different unknown quantities did you have in your equations? Do not double count the same variable that might show up in different equations.
2. How many momentum equations did you have for this problem? Each row of a vector equation counts as a separate equation unless all entries are zeros for that row.
3. Did you have more unknowns, more equations, or an equal amount of each?

Critical Thinking & Concept Invention

1. What energy indicators change for the objects in this problem?
2. Construct an Energy Interaction Diagram and from it an Energy Equation. Circle unknown quantities in the energy equation.
3. With your new energy equation, how many different unknown quantities did you have in your equations? Do not double count the same variable that might show up in different equations.
4. Did you have more unknowns, more equations, or an equal amount of each?
5. Solve your equations to identify the value of all unknown quantities. Then compute the magnitude and direction of the final momentum for both the cue ball and the eight ball. Draw them to scale and graphically check to see if momentum was conserved.

Summary

In all brief collisions, _____ is a conserved quantity. Sometimes (such as in this example), _____ is also conserved. When that is the case, we call the collision an _____ collision.

Collisions that result in the two objects moving as one (i.e., they have the same velocity) are called _____ collisions. These collisions are easier to solve because there are fewer unknowns.

Collisions can also fit into neither of the two previous categories. They are called _____ collisions and are difficult to solve. Usually we need some additional information about the collision.

This example highlights the need to sometimes use more than one Key Dynamics Principle in order to generate enough independent equations to solve for all unknown quantities. In order to compute all unknown quantities, you need to have the same amount (or more) of independent equations as unknowns.

Apply

This problem comes from *University Physics* by Young and Freedman. Below is a top-down view of two air hockey pucks that experience an elastic collision. Identify the final momentum vectors for both pucks. Report these as both column vectors and vector magnitudes with angles α and β as identified in the diagram. Be careful about positive and negative signs.

Image needed here with Alt Text: Top view of two air hockey pucks undergoing an elastic collision

---

1.2 Unequal Masses on a Pulley: Energy and Angular Momentum

Model

Two masses hang from a massive pulley. As the masses move, the string rotates the pulley without slipping. Consider a timeline that ends when the blocks have moved two meters (up or down, as shown in the diagram). Two choices of system are shown and should be considered throughout the analysis of this scenario.

Image needed here with Alt Text: Two masses connected by a string over a massive rotating pulley with system boundaries

Explore

1. Mark up the OID to add a $\Delta\mathbf{p}$ on each object that could change its momentum, a $\Delta\mathbf{L}$ on each object that could change its angular momentum, and a ΔK on each object that could change its kinetic energy (translational or rotational).
2. Examine the open system model (B) to complete the System Interaction Story:
   - There are _____ objects inside the system boundary.
   - This system includes _____.
   - There is/are _____ $\Delta\mathbf{p}$ terms inside the boundary.

- There is/are _____ internal interactions (interaction lines fully enclosed in the system) and _____ external interactions (interaction lines that cross the system boundary).
- The system boundary is (solid / dashed).
- The total change in momentum of the system should/could be (zero / nonzero).
- The total change in angular momentum of the system should/could be (zero / nonzero).
- The total change in energy of the system should/could be (zero / nonzero).

3. Examine the closed system model (A) to complete the System Interaction Story:
   - There are _____ objects inside the system boundary.
   - This system includes _____.
   - There is/are _____ $\Delta \mathbf{p}$ terms inside the boundary.
   - There is/are _____ internal interactions (interaction lines fully enclosed in the system) and _____ external interactions (interaction lines that cross the system boundary).
   - The system boundary is (solid / dashed).
   - The total change in momentum of the system should/could be (zero / nonzero).
   - The total change in angular momentum of the system should/could be (zero / nonzero).
   - The total change in energy of the system should/could be (zero / nonzero).

Critical Thinking & Concept Invention

1. The final speed of the masses can be determined from an Energy Approach. Which system (A or B) will be easiest to model the energy of and solve for the final speed of the blocks? Explain.
2. What energy indicators change for the objects in the chosen system? List all energy indicators for all objects that change.
3. Construct a timeline, Energy Interaction Diagram (EID), and energy equation to determine the speed of the blocks.

$$\Delta E_{system} = Q + W \tag{2.5.1}$$

4. How many objects in system B will experience a change in angular momentum? List those objects.
5. Now that you know the final speed of the blocks, construct an angular momentum chart (L-chart) and determine the change in angular momentum of System B ($\Delta \mathbf{L}_{System\ B}$).

Angular Momentum Vector Equations

Summary

Sometimes it can be difficult to observe or know the changes in momentum or angular momentum for large objects, such as the Earth. This makes it difficult to model closed momentum and angular momentum systems that involve interactions between the Earth and other objects.

In general, the kinetic energy of the Earth does not change, so we can model closed energy systems. This can allow us to determine speeds of objects at different points on a timeline, which we can use to solve open momentum and angular momentum systems.

The three Key Dynamics Principles provide independent ways to approach a physical scenario. In complex scenarios, you might need to use two or even all three principles to solve for unknown quantities.

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3: OPEN MOMENTUM AND ENERGY SYSTEMS

3.1: Impulse and Work Graphs

3.2: Impulse - A Systems Approach

3.3: Work Dot Products

3.4: First Law of Thermodynamics - A Systems Approach

3.5: Constraint Forces - Normal and Tension

3.6: Friction

3.7: Integral Impulse and Work

3.8: Integral Work

3.9: Center of Mass

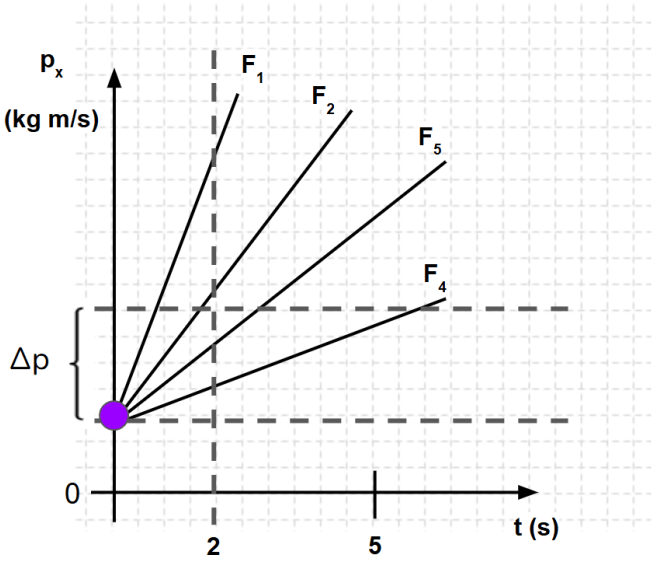
3.10: Comparing Center of Mass Equation (CM) and First Law of Thermodynamics (FLT)

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3.1: Impulse and Work Graphs

3.2.1 Force

Model



$$F_x = \Delta p_x / \Delta t$$

$$\Delta p_x = F\Delta t$$

$$p_{xf} = p_{xi} + \Delta p_x$$

Linear: $y = mx + b$

$$p_x(t) = Ft + p_x(0s)$$

Explore

1. What is the x-axis label? What are the units?
2. What is the y-axis label? What are the units?
3. At t = 2s, which force resulted in the greatest change in momentum?
4. Which force took longer to change the momentum of the object by an amount Δp?
5. What is the equation for Force (F)? Copy it here.

Critical Thinking & Concept Invention

1. The slope of a line or section of a graph is often written as $\frac{RISE}{RUN}$ and can be written as $\frac{\Delta y}{\Delta x}$ or $\frac{dy}{dx}$. Rewrite these equations with ”y” and ”x” replaced by the variables listed on those axes.
2. The area under a line or curve of a graph is often written can be written as $y\Delta x$ for a constant function or $\int ydx$ for a general function. Rewrite these equations with ”y” and ”x” replaced by the variables listed on those axes.
3. Does the equation for force represent a slope or area on this graph? Explain your thinking.
4. Which force is greatest? Least?
5. How does a force, and the amount (magnitude of that force) affect an object’s momentum?

Summary

When objects interact, the strength of that interaction is called **Force**. Force is defined as the rate the momentum of an object changes based on that interaction alone. This definition is represented by the equation:

$$\mathbf{F} \equiv \frac{\Delta \mathbf{p}}{\Delta t} \tag{3.1.1}$$

Or more precisely...

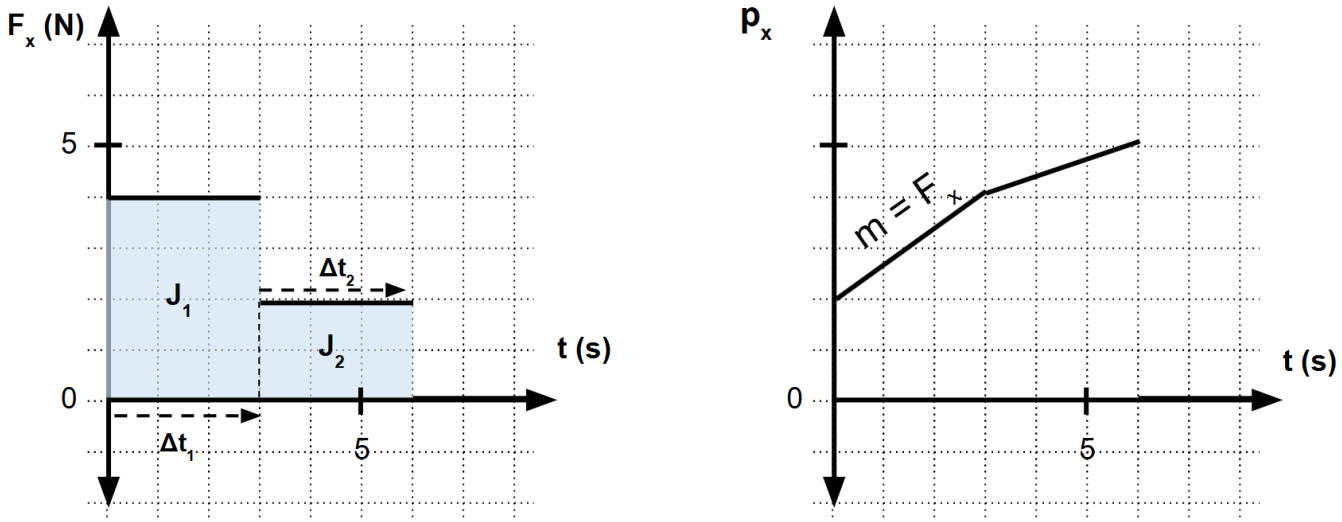
$$\mathbf{F} = \frac{d\mathbf{p}}{dt} \tag{3.1.2}$$

"Force is the time rate of change of momentum"

This relationship is known as **Newton's Second Law**. Force is a vector and points in the same direction as the change in momentum. This might not be the same direction as the actual momentum of the object while the force is applied.

3.2.2 Graphical Description of Impulse

Model



Explore

1. What is the x-axis label? What are the units?
2. What is the y-axis label? What are the units?
3. How many shaded areas are shown? What shapes are those areas?
4. What is the height (symbols/variables) and width (symbols/variables) for each area?
5. In this problem, what are the values for those heights and widths (numbers and units)?
6. What is the equation for Impulse (J)? Copy it here

Critical Thinking & Concept Invention

1. Does the equation for Impulse represent a slope or area on this graph? Explain your thinking.
2. Compute both areas. Identify J_1, J_2, and ΣJ.
3. If you know the force acting on an object over time. Explain the process to compute the change in momentum (Δp)

Summary

Impulse is a product of **Force (**$\vv{F}$**)** and the **time-interval (**Δt**)** over which that force is applied. Specifically, Impulse is the **Area** on a **Force vs time graph**. When forces are constant, this area is a rectangle, and the impulse is given by the equation:

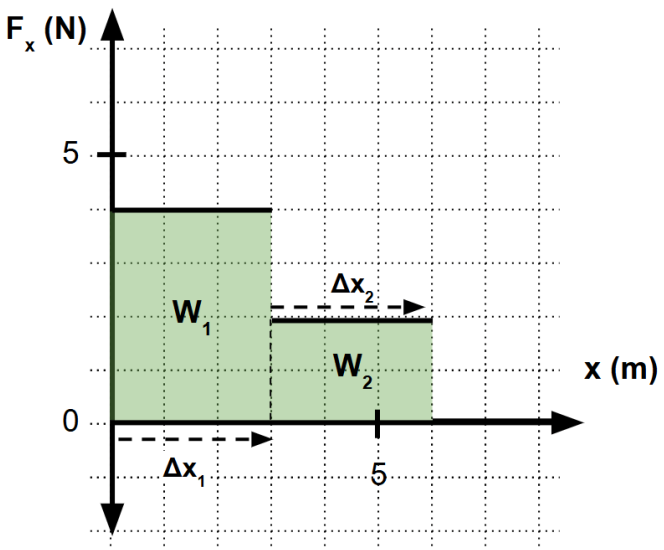
$$\mathbf{J} = \mathbf{F}_{ext}\Delta t \tag{3.1.3}$$

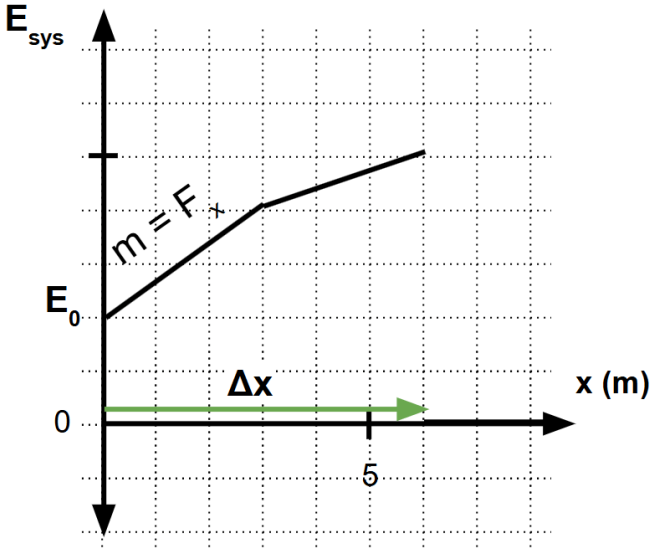
When forces change with time, we need to use an integral to calculate this area. The general form of Impulse is therefore represented by the equation:

$$\mathbf{J} \equiv \int \mathbf{F}_{ext}\, dt \tag{3.1.4}$$

3.2.3 Graphical Description of Positive Work

Model





Explore

1. Underneath each graph, write a description of the graph:
   - This is a (y-label) vs (x-label) graph...
   - There is a (positive / negative) relationship between (y-variable) and (x-variable).
   - The (y-variable) (is constant / changes linearly / changes nonlinearly) with (x-variable).
   - The Slope / Area on the (y-label) vs (x-label) graph represents ...

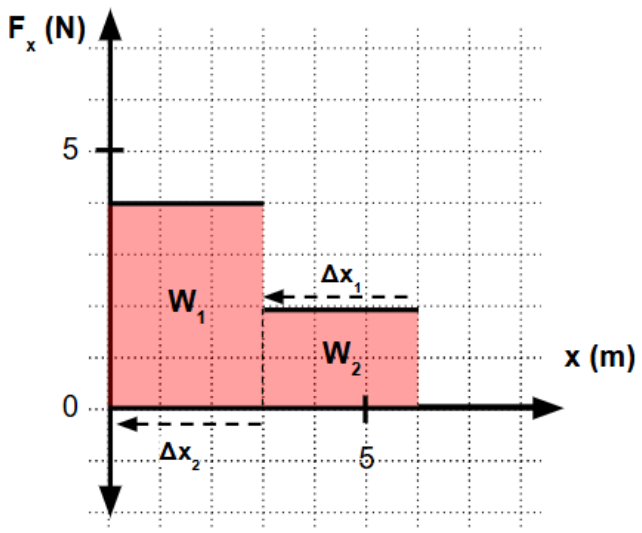
Critical Thinking & Concept Invention

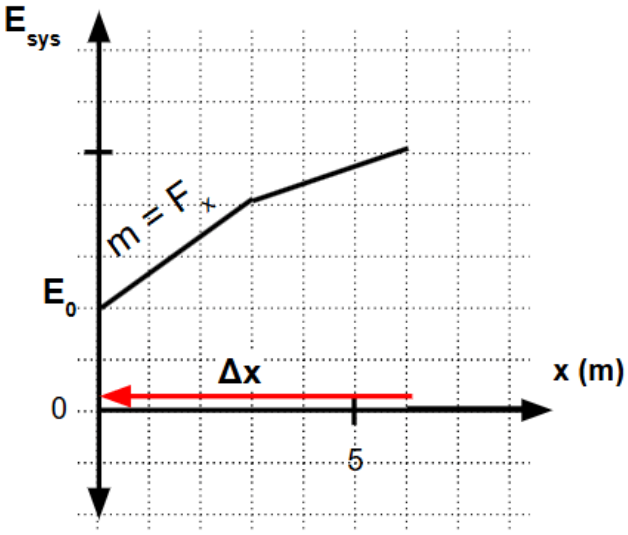
1. Summarize the graph that shows Work:
   a. Identify an equation that would be used to determine Work based on the graph that represents this quantity. Use variables for the x and y axes, not numbers.
   b. Write a statement about how Work relates to the variable shown on the graph and in your equation.
   c. Compute the work by substituting numbers into your equation. Identify W_1, W_2, and add them together to get the total work, W. Note: We do not typically write $\sum W$ to represent total work, but you can think of W for the system as $\sum W$.
   d. If the forces were not constant, how would you determine work? Write a statement and propose an equation to represent the process.
2. Summarize the graph that shows Force (on the graph, not the axis):

a. Did the y-variable increase or decrease over the shown displacement? Would that be a positive or negative change (Δ)?
b. Identify an equation that would be used to determine Force based on the graph that represents this quantity. Use variables for the x and y axes, not numbers.
c. Write a statement about how Force relates to the variables shown on the graph and in your equation.
d. If the energy did not change linearly with position, how would you determine Force? Write a statement and propose an equation to represent the process.

3.2.4 Graphical Description of Negative Work

Model





Explore

1. Circle the cause of the negativity of Work. It might help to consider:
   - What are the ways an integral can be negative?
   - What is different about this graph compared to the previous one?
2. Examine the force as it is shown on each graph. Is it positive or negative? Explain how you know by citing graphical features (x or y values, slopes, areas, minima/maxima, etc).

Critical Thinking & Concept Invention

1. **Summarize:** How Work can be negative using the concept of area/integrals
2. **Draw four Force vs Position graphs** that show the two ways to make a negative work/area and two ways to make a positive work/area. For each one, make the corresponding Energy vs Position graph.
3. Explain each graph and clearly state how it results in positive/negative work and increases/decreases in energy.

Summary

Work is a product of Force ($\mathbf{F}$) and the displacement over which that force is applied ($\Delta \mathrm{s}$). Here we use s as a generic displacement vector that can have x, y, and z components. Specifically, Work is the area on a Force vs. Position graph for each component of that force. When forces are constant, this area is a rectangle, and the Work is given by:

$$W = \mathbf{F}_{ext} \cdot \Delta \mathbf{s} \tag{3.1.5}$$

When forces change with position, we need to use an integral to calculate this area. The defining equation for Work is:

$$W \equiv \int \mathbf{F}_{ext} \cdot d\mathbf{s} \tag{3.1.6}$$

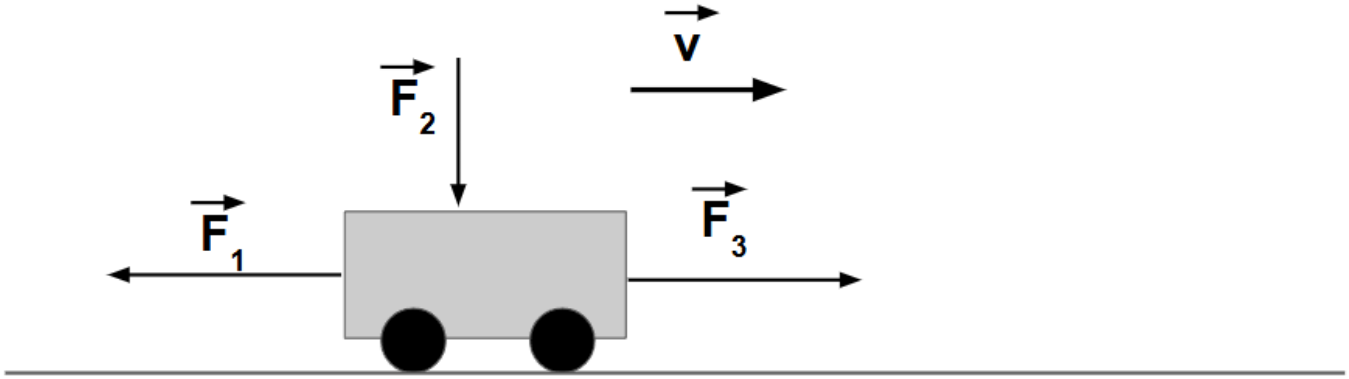
Force can be thought of as the rate energy changes with position. This is given by the equations:

$$F_x = \frac{\Delta E_{system}}{\Delta x} \qquad F_y = \frac{\Delta E_{system}}{\Delta y} \qquad F_z = \frac{\Delta E_{system}}{\Delta z} \tag{3.1.7}$$

$$F_x = \frac{dE_{system}}{dx} \qquad F_y = \frac{dE_{system}}{dy} \qquad F_z = \frac{dE_{system}}{dz} \tag{3.1.8}$$

3.2.5 Work, Force, and Displacement

Model



Explore

1. Identify the direction ($+/-\hat{x}$, $+/-\hat{y}$, $+/-\hat{z}$) of:
   - $\mathbf{v}$
   - $\Delta \mathbf{s}$
   - $\mathbf{F}_1$
   - $\mathbf{F}_2$
   - $\mathbf{F}_3$

Critical Thinking & Concept Invention

1. Identify whether each force will:
   a. Speed up, slow down, or not change the speed of the car.
   b. Add, remove, or not change the energy of the system (car).

   Provide reasoning for each:

Force	Effect on Speed	Effect on Energy	Reasoning
$\mathbf{F}_1$			
$\mathbf{F}_2$			

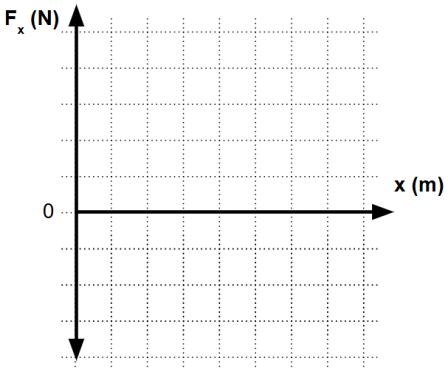
$\mathbf{F}_3$			

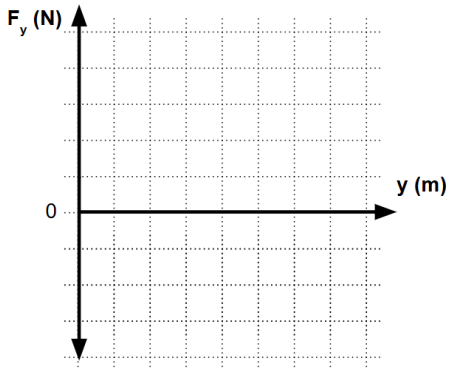
2. Using the relationship

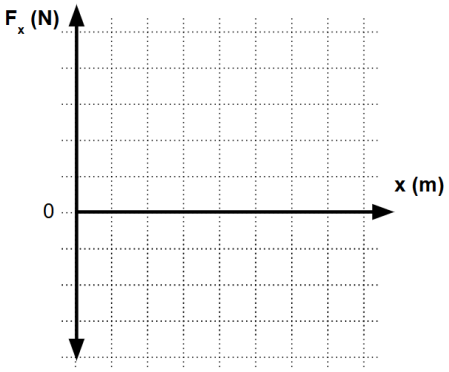
$$\Delta E_{system} = W, \tag{3.1.9}$$

under what circumstances is Work (W) positive (+), negative (–), or zero (0)?

3. Draw a separate Force vs Position and Energy vs Position graph for each of the three forces. The axes can be qualitative (ie, without numbers) but should be consistent with the assumption that the forces are equal in magnitude and the displacement is the same for each force. **Note:** One of the graphs below is marked with **y-components** not **x-components** so be sure to use that one for the correct force.





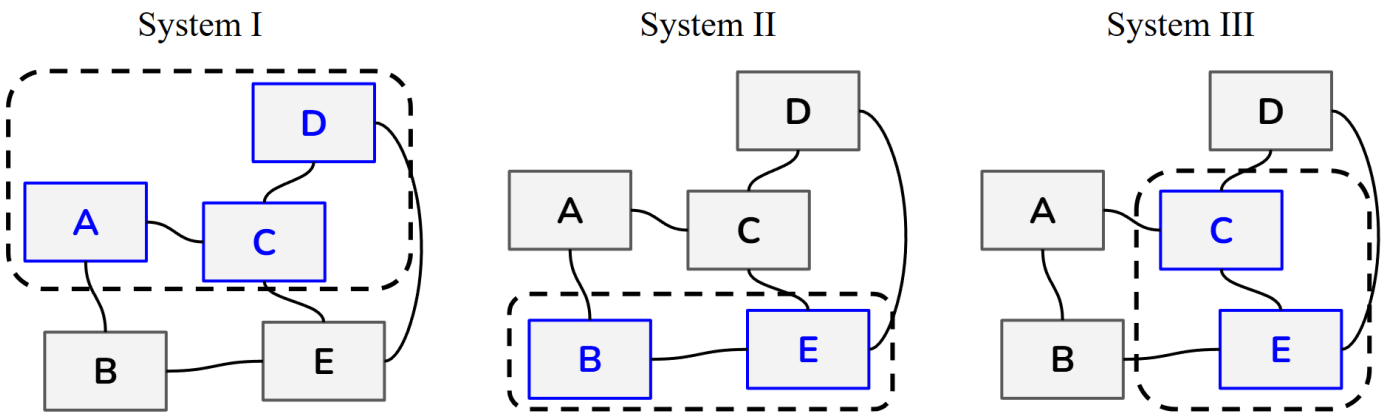


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3.2: Impulse - A Systems Approach

3.2.1 External Interactions on an Object Interaction Diagram

Model



$$\Delta \mathbf{p}_I = \Sigma \mathbf{F}_{ext} \qquad \Delta \mathbf{p}_{II} = \Sigma \mathbf{F}_{ext} \qquad \Delta \mathbf{p}_{II} = \Sigma \mathbf{F}_{ext} \qquad (3.2.1)$$

Explore

For each system:

1. Circle the external interactions as the lines cross the system boundary
2. Write a system story:
    1. There are ___ objects in the system (list them).
    2. There are ___ external interactions acting on the system

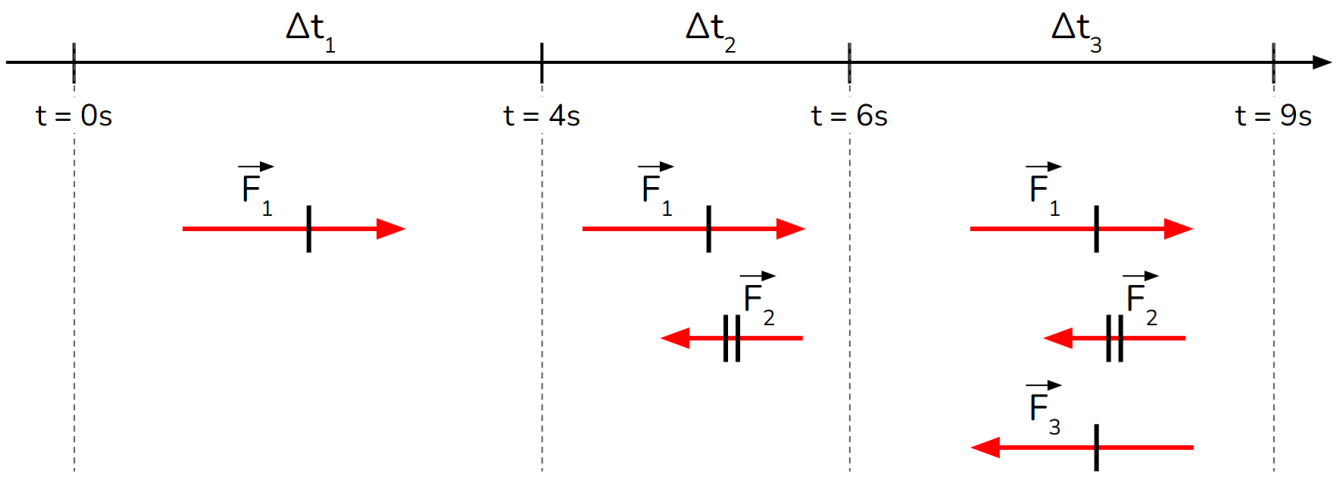
Critical Thinking & Concept Invention

1. Use the general structure $\Delta \mathbf{p}_{system} = \Sigma \mathbf{F}_{ext}$ to create a specific system momentum equation for each system. You can use the letters for each object as subscripts on the momentum side and Identify forces as "$\mathbf{F}_{A\ on\ B}$" with substitutions for the appropriate object labels.
2. Put an empty column vector under each vector in your equation. You will not fill these in, but this will help you practice creating a template for yourself
3. **Summarize:** How does the choice of a system affect how you solve a physics problem?

3.2.2 Net Impulse

Model

The time line below shows different forces that act on an object for intervals of time. We will assume that individual forces do not change and are present for the entire time-interval when they are shown.



Explore

1. Over what time intervals is each force applied to an object?
   - $\mathbf{F}_1$: _____ to _____.
   - $\mathbf{F}_2$: _____ to _____.
   - $\mathbf{F}_3$: _____ to _____.
2. Fill in the column vectors with the force label to create an equation for the total impulse. Be sure to add +/- signs in front of each one based on axes with $\hat{x}$ to the right.

$$\sum \mathbf{J} = [\quad]\Delta t_1 + [\quad]\Delta t_2 + [\quad]\Delta t_2 + [\quad]\Delta t_3 + [\quad]\Delta t_3 + [\quad]\Delta t_3 \tag{3.2.2}$$

3. Group the terms in your equation above by factoring the vector portion (Force). Put in the + or - in front of each force label in the vectors and fill in the time intervals with Δt_1, Δt_2, and/or Δt_3.

$$\sum \mathbf{J} = [\pm F_1](____) + [\pm F_2](____) + [\pm F_3](____) \tag{3.2.3}$$

4. Group the terms in your equation above by factoring the time portion. Put in the + or - in front of each force label in the vectors and list all forces for each time interval.

$$\sum \mathbf{J} = [\quad]\Delta t_1 + [\quad]\Delta t_2 + [\quad]\Delta t_3 \tag{3.2.4}$$

Critical Thinking & Concept Invention

1. Review your equation in Explore question 4. Do any forces cancel (equal but opposite) during any time intervals? If so, cross them out and then rewrite the equation below.

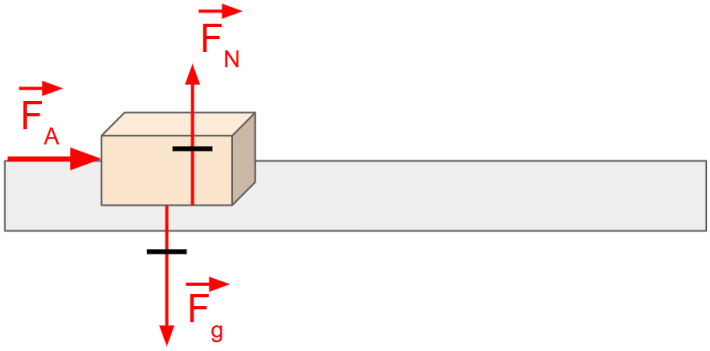
2. What might be the benefit of grouping according to force, as in Explore question 3?
3. What might be the benefit of grouping according to time, as in Explore question 4?
4. Summarize: How do you figure out the net impulse ($\sum \mathbf{J}$) on an object/system when there are multiple forces acting on the system for different time intervals?

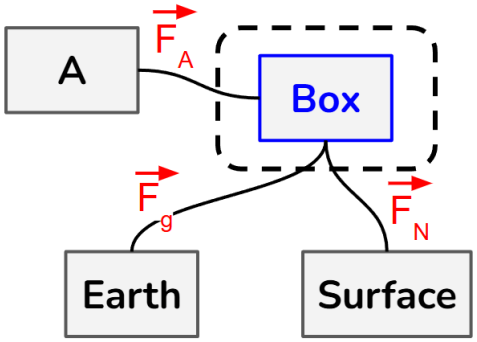
Apply

1. Determine the total impulse on the object in the model if $F_1 = F_3 = 8$ N and $F_2 = 6$ N. Do this by rewriting the equations from Explore 3 AND Explore 4 with all variables filled in. Solve both (the answer should be the same).
2. A fan cart turns on its fan for 6 seconds and experiences a force of 2.5 N towards the right from the interaction between the fan and the air. After traveling for 3 seconds, a gust of wind applies an additional force of 1.8 N towards the left for 1.5 seconds. Write out an equation for the total impulse on the fan cart over the 6 seconds.

3.2.3 Net External Interactions

Model





Explore

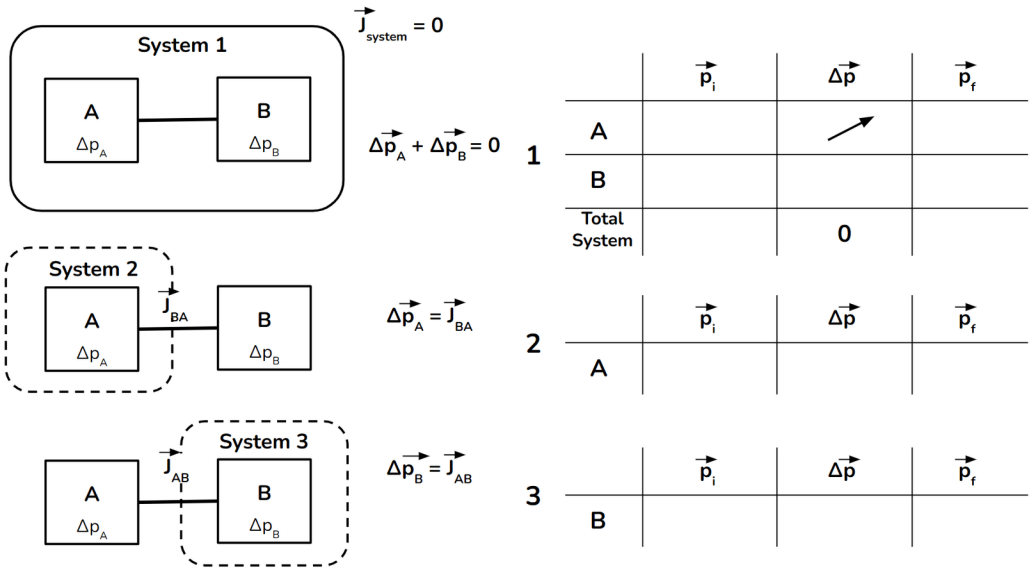
1. Circle the external interactions as they cross the system boundary
2. Write a system story:
    1. There is/are ___ object(s) in the system and ___ external interaction(s) on the system
3. Use the general structure $\Delta \mathbf{p}_{system} = \Sigma \mathbf{F}_{ext}$ to create a specific system momentum equation for this system
4. Do any forces cancel each other? Explain how you know and why this makes sense.

Critical Thinking & Concept Invention

1. Put an X through the circle on the system boundary for any interactions that cancel
2. Create a new system story:
    1. There is/are ___ object(s) in the system and ___ **net external interaction(s)** on the system
3. Cross out any terms from your system momentum equation to show only the net external interaction
4. **Reflect:** Why might it be useful to identify forces that cancel before solving a system momentum equation?

3.2.4 Impulse between objects

Model



Explore

Tell a System Story for each system to the right of the p-charts. Be sure to include:

- The number and names of the objects in the system.
- The number of internal interactions.
- The number of external interactions.
- The label of the external interaction(s), if any.

Critical Thinking & Concept Invention

1. Complete the middle column of the momentum chart for System 1. Explain how you completed it here.
2. Use the momentum chart to complete the $\Delta\mathbf{p}$ cell for Systems 2 and 3. How do those two vectors compare?
3. Summarize: When two objects interact, how does the impulse each object experiences compare? Explain why this has to be the case.

Summary

When two objects interact, they exchange equal amounts of impulse with opposite directions. This is given by the equation:

$$\mathbf{J}_{A \text{ on } B} = -\mathbf{J}_{B \text{ on } A} \tag{3.2.5}$$

Which is equivalent to...

$$\mathbf{F}_{A \text{ on } B}\Delta t = -\mathbf{F}_{B \text{ on } A}\Delta t \tag{3.2.6}$$

Since interactions between objects occur for the same length of time for each object, this leads to:

$$\mathbf{F}_{A \text{ on } B} = -\mathbf{F}_{B \text{ on } A} \tag{3.2.7}$$

This is called the **Action-Reaction Law** and is also known as **Newton's Third Law**.

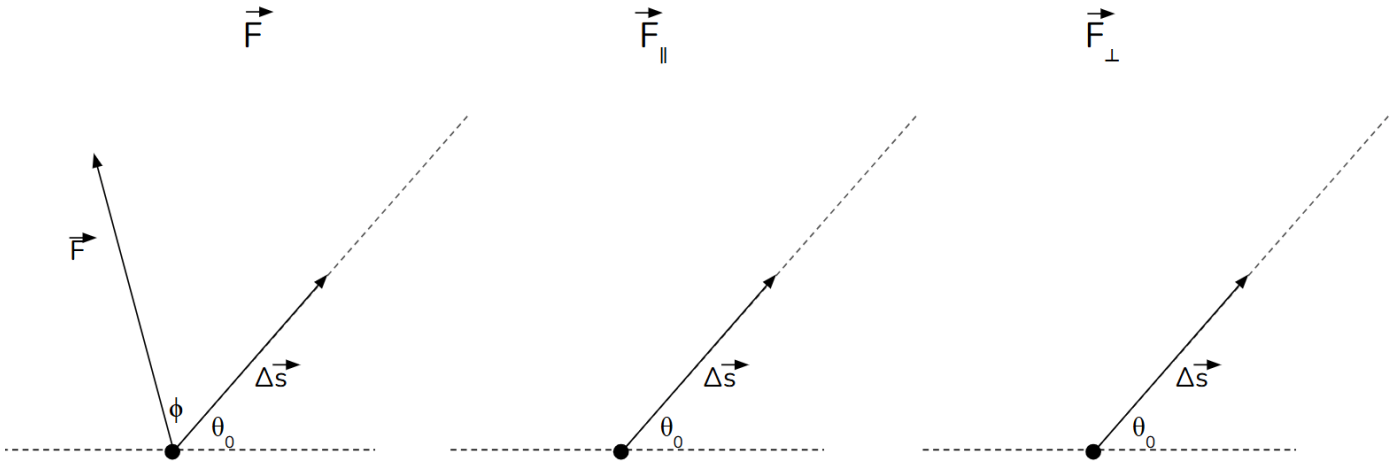
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3.3: Work Dot Products

3.3 Work Dot Products

3.3.1 Parallel and Perpendicular Products

Model



Explore

1. Identify the components of $\mathbf{F}$ that are **parallel** and **perpendicular** to $\Delta \mathrm{s}$ then draw them at the ends of the $\Delta \mathrm{s}$ vector
2. Use the angle ϕ and the force vector magnitude **F** to write an expression for the vector you drew using sine and cosine. Put this expression next to the arrow.

Critical Thinking & Concept Invention

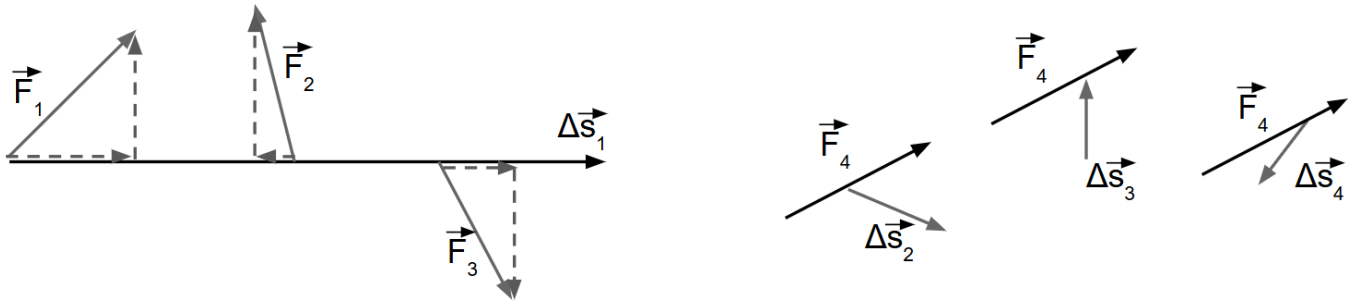
The dot (and cross) products are a product of the parallel/perpendicular component of one vector and the other vector.

1. Which product (parallel or perpendicular) represents an **area** (looks like a length and width)? Shade in this area as a rectangle.
2. Which product (parallel or perpendicular) represents a number but not an area? This number is **signed (+/-)**. Create a rule to help determine the sign.

3.3.2 $F_{\parallel}$ and $\Delta s_{\parallel}$

Model

On the left there is a displacement vector ($\Delta \mathbf{s}_1$) for an object and three force vectors ($\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$) shown next to the displacement vector. On the right is a different force vector ($\mathbf{F}_4$) shown three times, with displacement vectors ($\Delta \mathbf{s}_2$, $\Delta \mathbf{s}_3$, and $\Delta \mathbf{s}_4$) next to each copy of the force vector.



Explore

1. When comparing one vector to another, we can decompose one of the vectors into parallel and perpendicular components. Label the dashed vectors of $\mathbf{F}_1$ as $\mathbf{F}_{1,\parallel}$ and $\mathbf{F}_{1,\perp}$, then do the same for $\mathbf{F}_2$ and $\mathbf{F}_3$.
2. On the right part of the model, draw in dashed lines and add labels to show the perpendicular and parallel components of the displacements ($\Delta \mathbf{s}_{\parallel}$ and $\Delta \mathbf{s}_{\perp}$) with respect to $\mathbf{F}_4$.

Critical Thinking & Concept Invention

1. Do any of the first three forces do negative work? How can you tell?
2. If the displacement ($\Delta \mathbf{s}_1$) is the same for all three forces, rank the forces in terms of the amount of work that each does over that displacement.
3. Rank the work done by $\mathbf{F}_4$ over the three displacements.
4. Explain how to compare and rank the work done by different forces over different displacements.

Summary

Work is a parallel product or dot product between the Force vector ($\mathbf{F}$) and the displacement vector ($\mathbf{s}$). The dot product produces a scalar that can be positive, negative, or zero depending on the direction of each vector. Dot products are at a maximum when the two vectors are parallel or anti-parallel to one another and are zero when the two vectors are perpendicular to one another. You can compute this value from the parallel components of either vector.

$$W = \mathbf{F}_{avg} \cdot \Delta \mathbf{s} \tag{3.3.1}$$

$$W = F_{\parallel} \Delta s \tag{3.3.2}$$

$$W = F \Delta s_{\parallel} \tag{3.3.3}$$

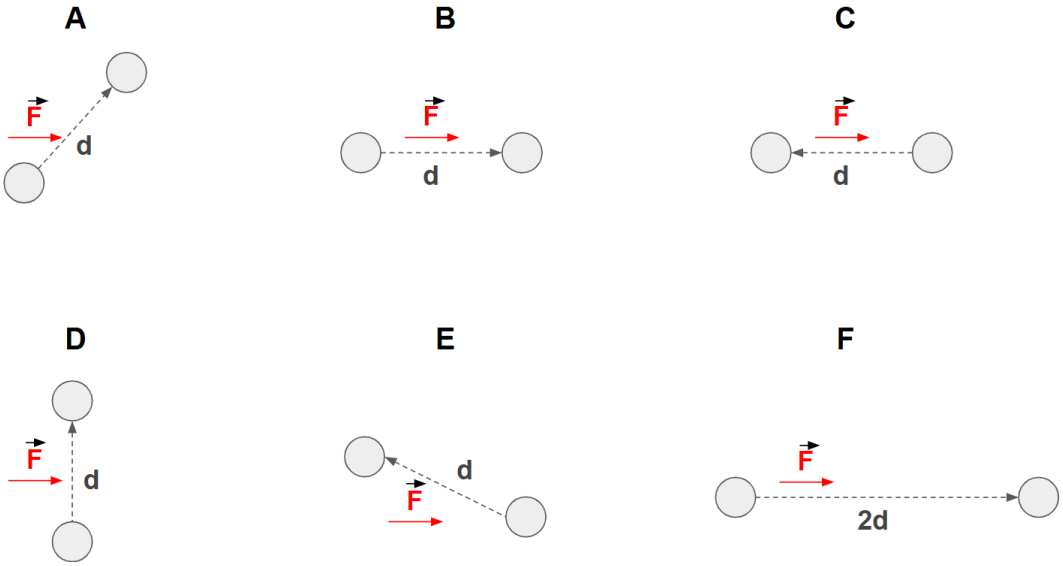
$$W = F\Delta s\cos(\theta) \tag{3.3.4}$$

Depending on the problem, one version of this equation might be easier to use than another. When a force is constant over the displacement, $F_{avg} = F$. When forces are functions of position, we need to integrate.

Apply

In each scenario below, the same force is applied to an object that experiences a different displacement. Identify each Work as positive, negative, or zero. Then rank the Work from most negative, through zero, to most positive.

On each example, identify either the $F_{\parallel}$ or $\Delta s_{\parallel}$. Or write 0 if either of those quantities are zero.



3.3.3 Column Vector Dot Products

Model

$$W = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} \cdot \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} = \begin{matrix} F_x \Delta x \\ F_y \Delta y \\ + \quad F_z \Delta z \\ \hline F_x \Delta x + F_y \Delta y + F_z \Delta z \end{matrix}$$

Explore

1. For dot products of column vectors, do you multiple like components or unlike components?
2. What do you have to do after you take the products of the components?

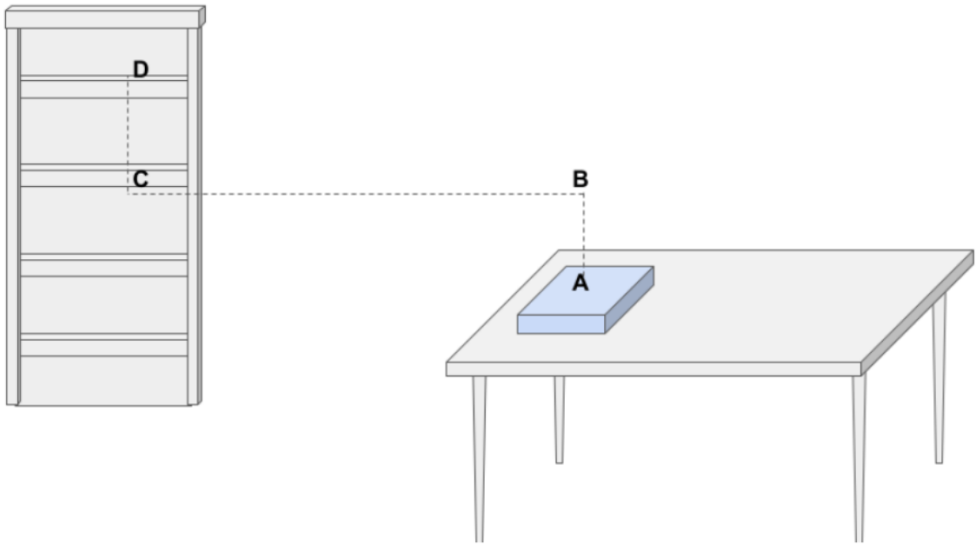
Critical Thinking & Concept Invention

1. Is the result of a dot product a vector or a scalar? How can you tell?
2. Is it possible that the result of a dot product is negative? What would cause this?
3. The individual components of these column vectors depend on your coordinate system. Do you think the result of the dot product is different for different coordinate systems or is it the same (invariant) no matter how you choose your system? Explain your thinking

3.3.4 Applying Work Dot Products

Model

A person picks a book up off a table, carries it to the bookshelf, then lifts the book to the top shelf. Assume the book is at rest at each letter on the path (transition points).



Explore

For each segment of this motion:

- Draw forces on the book.
- Draw a vector to show the displacement of the book for that interval.
- Identify the whether work is positive, negative, or zero (+/-/0) from each force acting on the book.

Critical Thinking & Concept Invention

1. Use a coordinate system to identify Force and displacement vectors for each segment of the path and determine work using the vector dot product and column vectors. For the force from the person, use **average force** as a variable so you do not need to integrate.

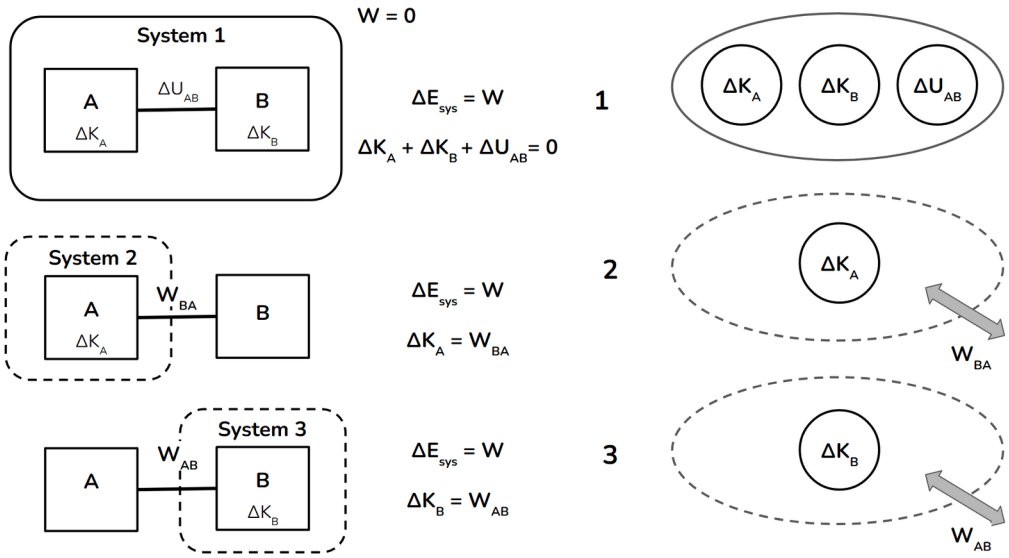
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3.4: First Law of Thermodynamics - A Systems Approach

1 The First Law of Thermodynamics

3.4.1 Open Systems and First Law of Thermodynamics

Model



Explore

Question	System A	System B	System C
What objects are in the system?			
Is the interaction shown between objects an internal or external interaction?			
What is the label on the interaction?			
How many ΔE terms (ΔK, ΔU, etc.) are inside the system boundary?			
According to the energy equation, what does ΔE_{system} equal?			

1. Draw a line connecting the ΔE terms in the system to the ΔE terms in the energy equation.

Critical Thinking & Concept Invention

1. Which system is open? Which is closed? How can you tell?
2. How does your choice of system affect the ΔE_{system} side of the equation (left side)?
3. How does your choice of system affect the other side of the equation (right side)?
4. Are both equations valid? How can that be possible if they are different?

Summary

Interactions are modeled differently when they are internal or external to our system.

- **Internal interactions** can result in changes in potential energies for the system.
- **External interactions** can do Work on the system.

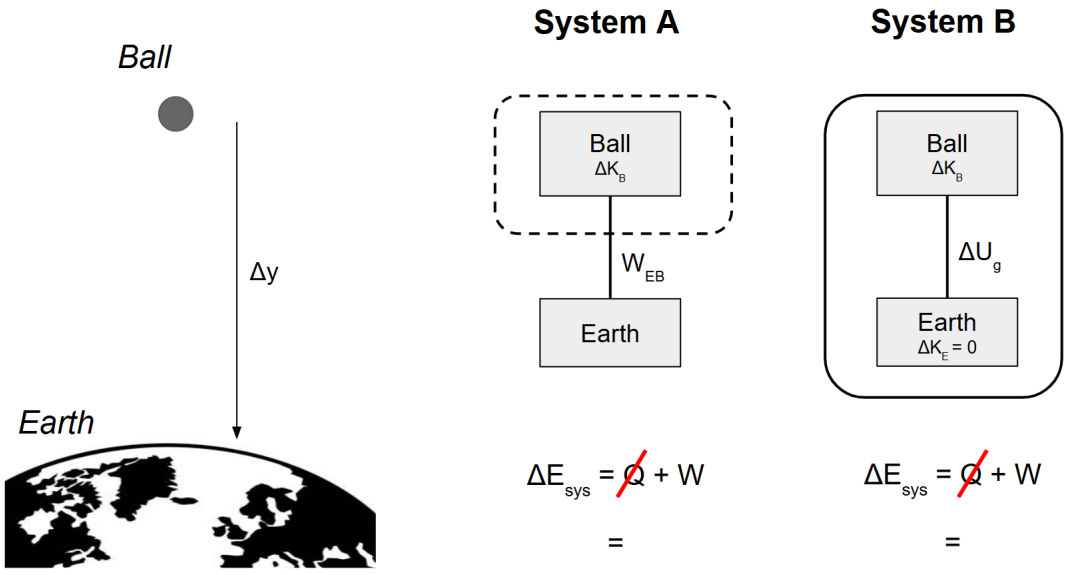
Different choices of systems will result in different equations that are all equally valid. We can compare two system choices to discover potential energy equations from work or force equations from potential energies.

In order to interpret the validity of a First Law of Thermodynamics equation, it is essential that a system is chosen and explicitly stated.

1.1 Ball & Earth

Model

A ball at some height above the ground interacts with the Earth. This scenario is modeled with two different choices of systems. Both interaction lines are initially drawn as dashed. If you determine that a particular system is closed, you should then make the boundary solid.



Explore

1. Examine the Object Interaction Diagrams: Tell a system story for each system. Be sure to include the following in each story:
   - The number and names of the items in the system.
   - The number and types of energy that could change in the system. Be clear on changes in energy for objects and changes in energy associated with interactions.
   - Is the system an open or a closed system? How do you know?
   - The number and names of external objects that interact with the system.
   - The number of external interactions that do work on the system.

System A	System B

2. **Examine the Energy Equations:** Draw a line from each change in energy label in the system to its corresponding term in the energy equation.
3. **Examine the Physical Scenario Diagram:** What will Δs equal if the ball falls straight down?

Critical Thinking & Concept Invention

1. Which change in energy terms $(\Delta K,\ \Delta U,\ \Delta E,\ W,\ \text{etc.})$ are labeled in an object and which are labeled for interactions?

Objects	Interactions

2. How does the label on the interaction change when it is an external interaction rather than an internal interaction?

Internal	External

3. Compare the equations that result from different choices of system:
    - Subtract the equation (System A equation – System B equation).
    - Interpret the resulting equation: What does W equal? Why might that make sense?
    - Replace W with $F\Delta s$ and ΔU for its potential energy equation. Solve for F.

Summary

We can identify equations for the magnitude of forces that result in potential energies by comparing a system where that force is an internal interaction and where that force is an external interaction. Each system generates an accurate, but different equation for the dynamics of the system.

In cases where the two equations result in the relationship:

$$W = -\Delta U \tag{3.4.1}$$

the interaction is called a conservative interaction. All conservative interactions are represented as potential energies when they are internal interactions.

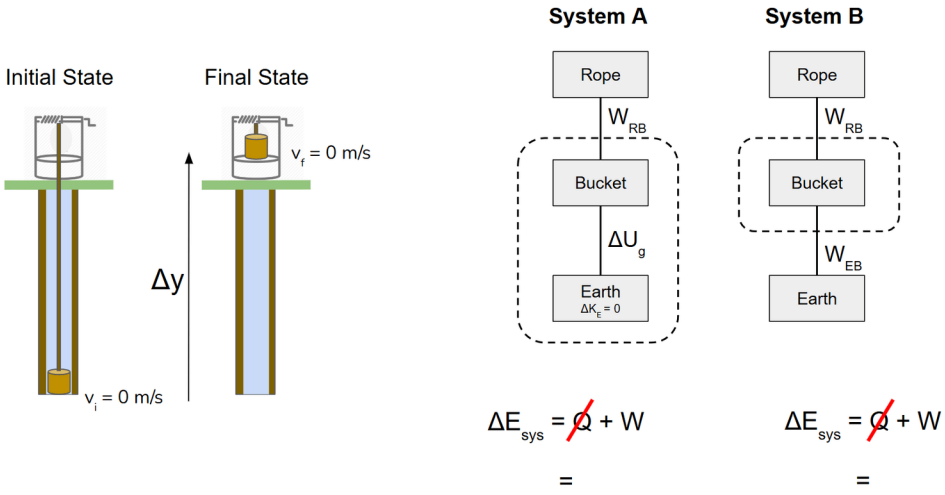
From this process, we identified that the magnitude of the force of the Earth acting on an object is given by the equation:

$$F_{\text{Earth on Object}} = \tag{3.4.2}$$

1.2 Pulling Up a Bucket

Model

The model below shows a person who pulls a bucket up a well using a rope that is attached to the bucket. We consider the timeline where the bucket begins and ends at rest.



Explore

1. Describe the motion at the beginning and end of the timeline. Does this motion change?
2. What object(s) change position? What is Δs?
3. Examine the Object Interaction Diagrams: Tell a system story for each system. Be sure to include the following in each story:
   - The number and names of the items in the system.
   - The number and types of energy that could change in the system. Be clear on changes in energy for objects and changes in energy associated with interactions.
   - Is the system an open or a closed system? How do you know?
   - The number and names of external objects that interact with the system.
   - The number of external interactions that do work on the system.

System A	System B

Critical Thinking & Concept Invention

1. What is the value of ΔK_{bucket} for this timeline? How do you know?

$$\Delta K_{\text{bucket}} = \tag{3.4.3}$$

because

2. Identify and mark up the following on the Object Interaction Diagrams:

- If any objects do not have a change in Kinetic Energy, write $\Delta K = 0$ inside the box for that object.
- Next to each external interaction that does work, put a (+) or (–) above the W to identify the sign of that work.

3. Apply the First Law of Thermodynamics to explore the scenario from the perspective of each system:
   - Create an energy equation for each system with appropriate (+) or (–) signs over each term.
   - Substitute in $\mathbf{F} \cdot \Delta \mathbf{s}$ with appropriate labels for each W in your equations.
   - Identify the direction of each force and the parallel displacement of each force to evaluate the dot product.
   - Solve for $F_{\text{Person on Bucket}}$ in terms of $F_{\text{Earth on Bucket}}$.

System A
$\Delta E_{sys} = W$ (3.4.4)

4. Why does the $F_{\text{Person on Bucket}}$ make sense?
5. What can we say about the forces acting on an object when

$$\Delta K = 0 \tag{3.4.6}$$

   ?
6. Below is a diagram that shows the force of the Earth on the bucket. Draw an arrow to represent the force of the person on the bucket. Explain how you drew your arrow (length and direction).

Summary

When there are multiple external interactions, each one can do Work on the system. The W in the First Law of Thermodynamics equation $\Delta E_{system} = Q + W$ is the net work on the system. This becomes a sum of each individual Work:

$$\Delta E_{system} = Q + W \tag{3.4.7}$$

$$\Delta E_1 + \Delta E_2 + \cdots + \Delta E_n = W_1 + W_2 + W_3 + \cdots + W_n \tag{3.4.8}$$

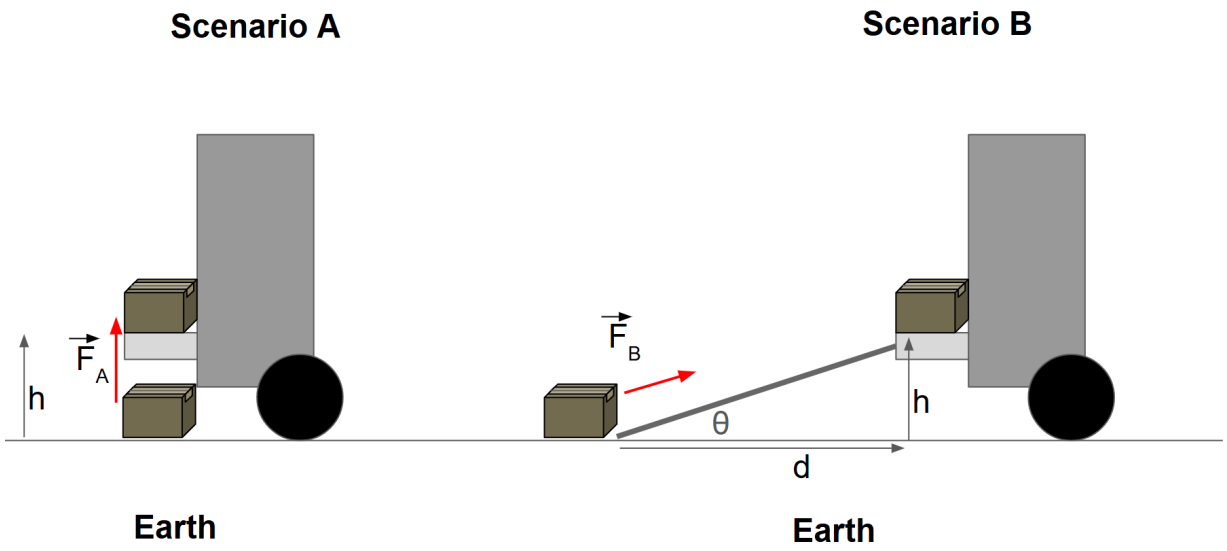
When $\Delta K = 0$ for an object and all of those forces are applied over the same displacement $\Delta \mathbf{s}$, the forces on that object are _____ and the motion of the object _____ change.

1.3 Pushing a Box up a Ramp

Model

In this model, a person (Person A or Person B) lifts a box or pushes a box up a ramp to get the box to the bed of the truck. Assume you can model this scenario with only forces of the person (A or B) and gravity doing work on the box. That means there might be

other forces, but they do not affect the energy dynamics so you can ignore them. The vectors shown for forces A and B represent average Force applied to the box over its displacement.



Explore

1. In each case, what is the initial and final velocity of the box?
2. Draw a vector to show $\Delta \mathbf{s}$ of the box for each scenario. Remember that the displacement vector is drawn from where the box started to where the box ended. Identify the magnitude of $\Delta \mathbf{s}$ in terms of variables in the scenario (h, d, θ).
3. Draw a vector to show the force of gravity on the box in each scenario.

For Scenario A

4. Compare the direction of each force to the different position-based vectors:

	$\Delta \mathbf{s}$	$\mathbf{h}$
$\mathbf{F}_A$	+ / – / $\perp$ / $\parallel$ / neither	+ / – / $\perp$ / $\parallel$ / neither
$\mathbf{F}_g$	+ / – / $\perp$ / $\parallel$ / neither	+ / – / $\perp$ / $\parallel$ / neither

For Scenario B

5. Compare the direction of each force to the different position-based vectors:

	$\Delta \mathbf{s}$	$\mathbf{h}$	$\mathbf{d}$
$\mathbf{F}_B$	+ / – / $\perp$ / $\parallel$ / neither	+ / – / $\perp$ / $\parallel$ / neither	+ / – / $\perp$ / $\parallel$ / neither
$\mathbf{F}_g$	+ / – / $\perp$ / $\parallel$ / neither	+ / – / $\perp$ / $\parallel$ / neither	+ / – / $\perp$ / $\parallel$ / neither

6. How does the magnitude of

$$\Delta \mathbf{s}_B \tag{3.4.9}$$

compare to

$$\Delta \mathbf{s}_A \tag{3.4.10}$$

?

Critical Thinking & Concept Invention

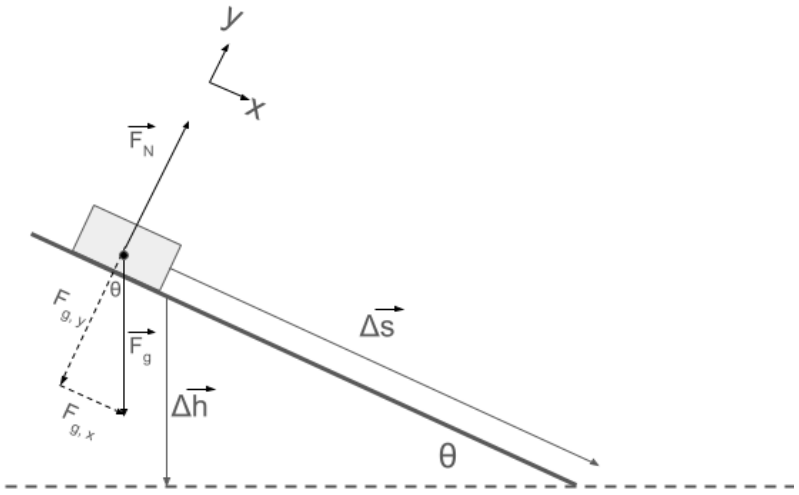
1. Should W_g be + or –? Explain.
2. Should W_A be + or –? Explain.
3. Should W_B be + or –? Explain.
4. Model each scenario using the First Law of Thermodynamics:
   - Write an equation for the Work done to the box in each case. Put a (+) or (–) over the terms based on your responses above.
   - Determine the Force from the person in each case. Your answer might be in terms of variables m, g, h, and/or θ.

System A	System B
$\Delta E_{sys} = W$	$\Delta E_{sys} = W$

5. Which method of putting the box on the truck requires less force from the person? Explain why this is the case.

1.4 Ramps

Model



Explore

1. Examine the coordinate axes. What direction is:
   - $+\hat{x}$?
   - $+\hat{y}$?
2. How many forces act on the block (do not count components of a force)? List them.
3. What direction ($+/-\hat{x}$ and/or $+/-\hat{y}$; OR vertically up/down or horizontally left/right) do each of the following point?

$\Delta\mathbf{s}$	$\Delta\mathbf{h}$	$\mathbf{F}_g$	$F_{g,x}$	$F_{g,y}$	$\mathbf{F}_N$

Critical Thinking & Concept Invention

1. How does $\Delta\mathbf{h}$ relate to $\Delta\mathbf{s}$? Explain.
2. Which is parallel ($F_\parallel$) to $\Delta\mathbf{s}$? Circle one.
   - $\mathbf{F}_g$
   - $F_{g,x}$
   - $F_{g,y}$
3. Write $F_\parallel \Delta s$ as an equation.
4. Which is parallel ($\Delta s_\parallel$) to $\mathbf{F}_g$? Circle one.
   - $\Delta\mathbf{s}$

- $\Delta \mathbf{h}$

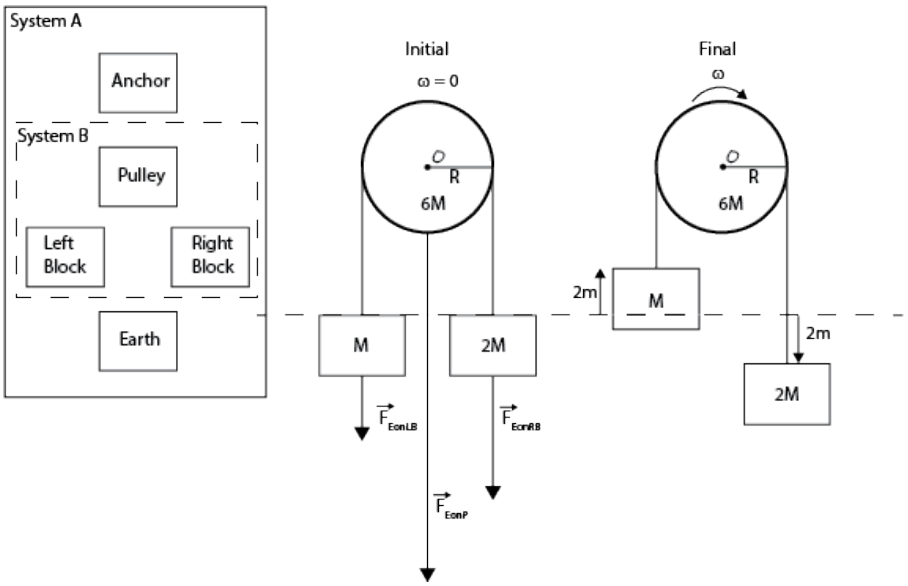
5. Write $F\Delta s_{\parallel}$ as an equation.
6. What is $\mathbf{F}_N \cdot \Delta \mathbf{s}$? Does the Normal force do work on the object?
7. Construct an Energy Interaction Diagram for the object that models only the object as the system. From it, construct an energy equation and solve for the speed of the block at the bottom of the ramp. Assume no friction and no rotation.

$$\Delta E_{sys} = W + Q \tag{3.4.11}$$

1.5 Atwood Machine

Model

Below is a model you have previously explored through the lens of a System A for energy. For this activity, focus only on System B. In this first activity, we will assume the pulley is massless instead of having a mass of $6M$.



Explore

1. Draw the interactions between the objects in the Object Interaction Diagram.
2. Draw a $\Delta \mathbf{s}$ vector on any object that you expect to move that shows the direction the object moves. Pay attention to how this direction relates to the forces on the objects (parallel, antiparallel, perpendicular, etc.).
3. Tell a system story for System B. Be sure to include the following in each story:
   - The number and names of the items in the system.
   - The number and types of energy that could change in the system. Be clear on changes in energy for objects and changes in energy associated with interactions.
   - Is the system an open or a closed system? How do you know?
   - The number and names of external objects that interact with the system.
   - The number of external interactions that do work on the system, including whether the work is positive or negative.

Critical Thinking & Concept Invention

1. How should the Work on different objects in the system compare? Consider sign and magnitude.
2. Apply the First Law of Thermodynamics to explore the scenario from the perspective of System B:
   - Create an energy equation with appropriate (+) or (−) signs over each term.
   - Substitute in $\mathbf{F} \cdot \Delta \mathbf{s}$ with appropriate labels for each W in your equations.
   - Identify the direction of each force and the parallel displacement of each force to evaluate the dot product.
   - Solve the equation to determine the final speed of each mass after the more massive object has experienced the displacement of 2 meters below its initial position. Your solution will be in terms of the variables given in the problem, not a final number.
3. Use the final speed you determined from modeling the dynamics of System B to determine the force of the pulley (string) on each individual mass. Do this by modeling two new systems:
   - System L, which is just the Left Block.
   - System R, which is just the Right Block.

   Separately set up and solve a First Law of Thermodynamics equation for each system.

Apply

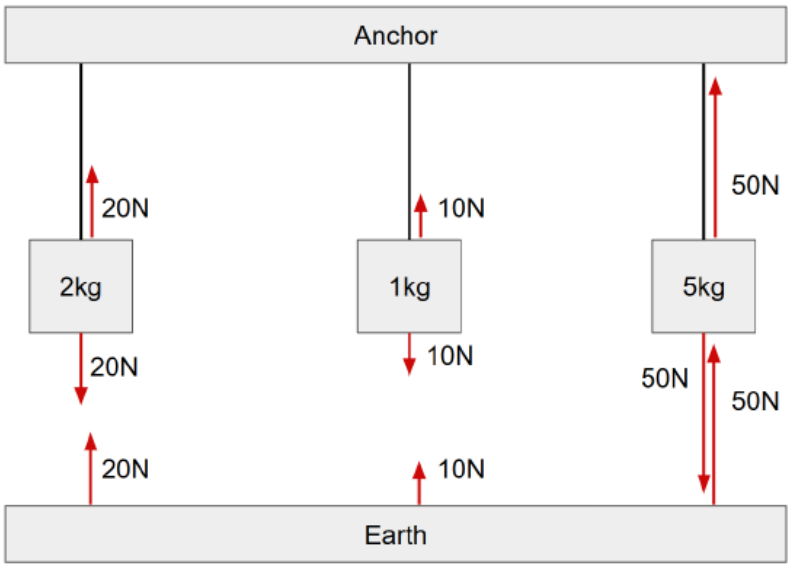
1. Model the energy changes for this Atwood machine setup (two masses on a string over a pulley) as the previous model, but this time assume the pulley has a mass of $6M$ (as shown in the diagram) and a radius R. The pulley rotates as the string moves so that the speed of the rim of the pulley matches the speed of the string. Identify the final speed of the string/masses for the same displacement as the previous model.
2. Using Systems R and L, identify the force of the pulley (string) on each mass for the case where the pulley has a mass of $6M$.

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3.5: Constraint Forces - Normal and Tension

3.5.1 Tension and Weight

Model



Explore

1. The Force of Gravity is directly based on the mass of each object. Identify a mathematical relationship between mass and gravity.

$$F_g = \tag{3.5.1}$$

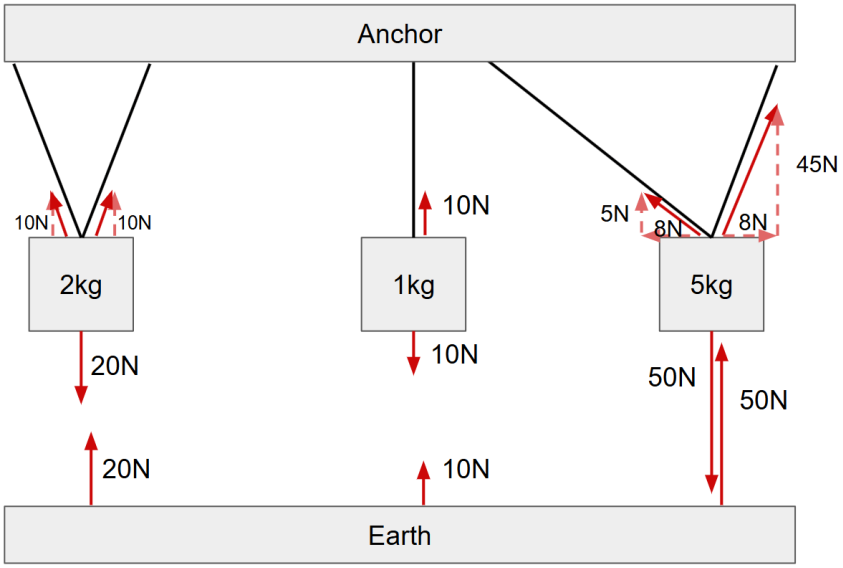
2. Add lines / hash marks on the vector to indicate which vectors have the same magnitude.

Critical Thinking & Concept Invention

1. The force on each mass from the rope is called **Tension**. How does the force of tension relate to the force of gravity acting on each object?
2. If you added the two force vectors that act on each object together, what would you get?
3. Imagine that the model had no forces drawn and no force magnitude given. Describe a process to determine these values and draw these forces

3.5.2 Tension Forces at Angles

Model



Explore

1. Draw lines / hash marks on the vector to indicate which vectors have the same magnitude. Be sure to do this for vector components (dashed arrows) as well

Critical Thinking & Concept Invention

1. How do the horizontal components of the two tension forces on an object relate to each other?
2. How do the vertical components of the two tension forces on an object relate to each other?
3. Imagine that the model had no forces drawn and no force magnitude given, but did include the angles of the cables. Describe a process to determine these values and draw these forces.

Summary

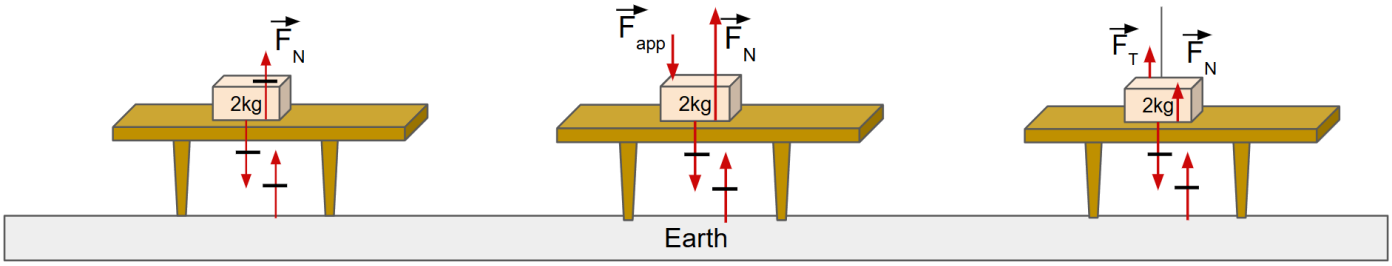
Ropes/Strings and similar objects are often modeled as a **Force of Tension** (rather than an object), which can be written $\mathbf{F}_T$ if we assume the rope/string has **no mass.** In this case, both objects experience the same magnitude of force based on Newtons ___ Law. The directions might not be opposite as ropes interact with things like pulleys.

For both cases, tension (or string-object interactions) are reactionary forces and do not have their own equation. They will react to produce the force necessary based on the motion of the objects and other forces that interact with the object. If they cannot produce the necessary force, they break.

3.5.3 Normal Force

Model

The same book is shown on a table in all three images and remains motionless. In one case, an applied force (for example, a person) pushes the book downward. In another case, a rope (tension) pulls the book upward.



Explore

1. How does the force of gravity compare across all three scenarios?
2. How does the Normal Force $\mathbf{F}_N$ compare across all three scenarios?
3. The normal force is from the table. What angle does the vector make with the surface of the table?

Critical Thinking & Concept Invention

1. If the book remains motionless, what should be true about the forces on the book in each scenario?
2. Does the normal force appear to be a reactionary force? Explain your thinking.
3. Describe how the magnitude of the normal force changes when an additional downward force is applied to the book and when an upward tension force is applied.

Summary

The force between a surface and an object is often called the Normal Force. This force is always perpendicular (normal) to the surface. The Normal Force is a reactionary force, which means the value depends on the rest of the interactions and the changes in motion that are possible.

Different textbooks label this force in different ways:

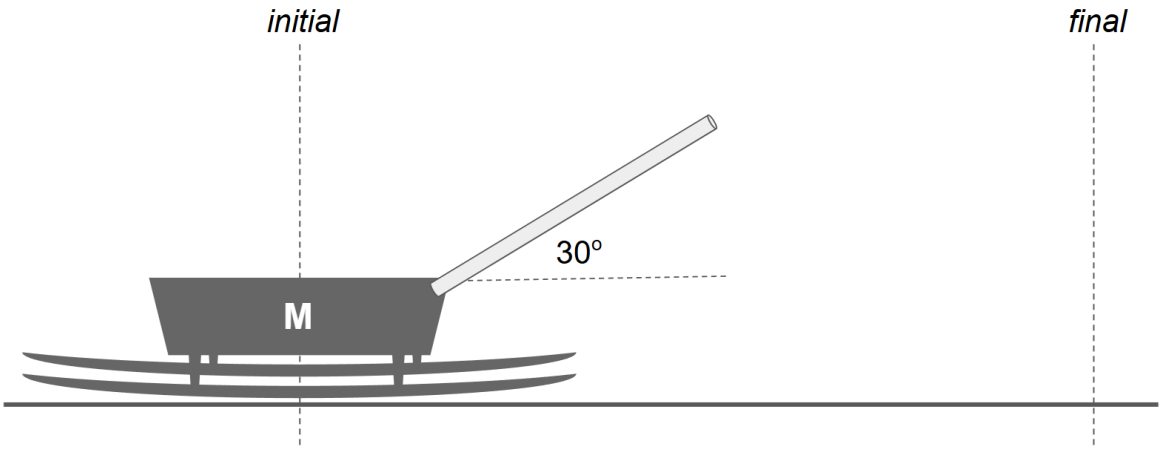
$$\mathbf{F}_N \quad \text{or} \quad \mathbf{N} \tag{3.5.2}$$

These are the most common representations.

3.5.4 Applying Forces to an Open System Problem

Model

A sled of mass M is pulled by a rope at a 30° angle along frictionless icy ground. It is pulled from an initial position to a final position, as shown in the diagram.



Explore

1. To the right of the image, create an Object Interaction Diagram for the sled and choose only the sled as the system. Draw lines to show all of the interactions between the sled and other objects. Circle external interactions as they cross the system boundary.
2. Tell a system interaction story for the sled. Be sure to include:
   - The number and names of the objects in the system.
   - The number and types of energy that could change in the system.
   - Whether the system is open or closed and how you know.
   - The number and names of external objects that interact with the system.
   - The number of external interactions that could do impulse and/or work on the system.

Critical Thinking & Concept Invention

1. For each interaction, draw a force vector on the model diagram. Highlight (or circle) any force or force component that is parallel to the displacement.
2. Consider whether this problem is more suited to analyzing time intervals or displacements. Based on that decision, identify which equation you expect to use:

$$\Delta \mathbf{p}_{system} = \sum \mathbf{J} \tag{3.5.3}$$

OR

$$\Delta E_{system} = W \tag{3.5.4}$$

3. Explain your problem-solving approach:

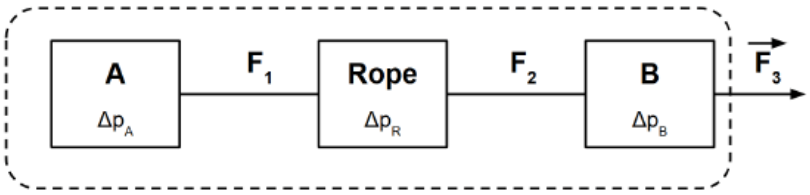
We are using (Energy / Momentum) to solve this problem because...

4. Consider which interactions do non-zero impulse or work on the sled. Label the interaction lines with a variable (for example, J_1) or indicate that the interaction does not do impulse/work (for example, $W = 0$).
5. Solve for the speed of the sled if:
   - The mass is $5\ \text{kg}$.
   - The tension force is $35\ \text{N}$.
   - The displacement from initial to final position is $6\ \text{m}$ to the right.

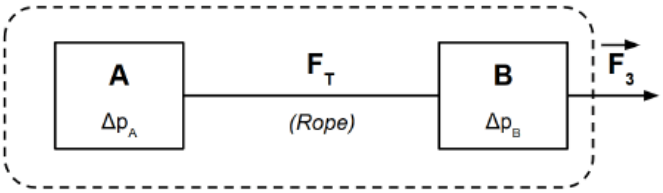
3.5.5 Tension and Mass

Model

Model 1: Rope has mass



Model 2: Rope has no mass



Explore

1. For each model, identify the number of:

	Model 1	Model 2
Objects		
External Interactions		
Internal Interactions		
$\Delta \mathbf{p}$ Terms		

2. What direction is the force for the external interaction on both systems?

Critical Thinking & Concept Invention

1. What should the direction of $\Delta \mathbf{p}$ be for all of the objects? Explain your reasoning.
2. How should the magnitudes of $\mathbf{F}_1$ and $\mathbf{F}_2$ compare? Explain your thinking.
3. How should the magnitude of the tension force $\mathbf{T}$ on objects A and B compare? Explain your thinking.
4. How should the magnitude of $\mathbf{T}$ and $\mathbf{F}_3$ compare? Explain your thinking.
5. If $\mathbf{F}_3$ is the same and the masses of objects A and B are the same across both models, how should the change in momentum of Object B compare across each model? Construct an impulse-momentum equation for each model and use that equation to justify your response.

Summary

When the rope or string has significant mass compared to the other objects, we need to include changes in momentum and kinetic energy of the rope or string in our models.

In this case, the force of the rope or string on the objects it connects is not necessarily equal and opposite.

For both Normal Force and Tension, the magnitudes of the forces react to the other forces acting on the object or system. The sum of all of the forces determines the change in motion of the object or system by adding impulse, work, or twirl to the object or system.

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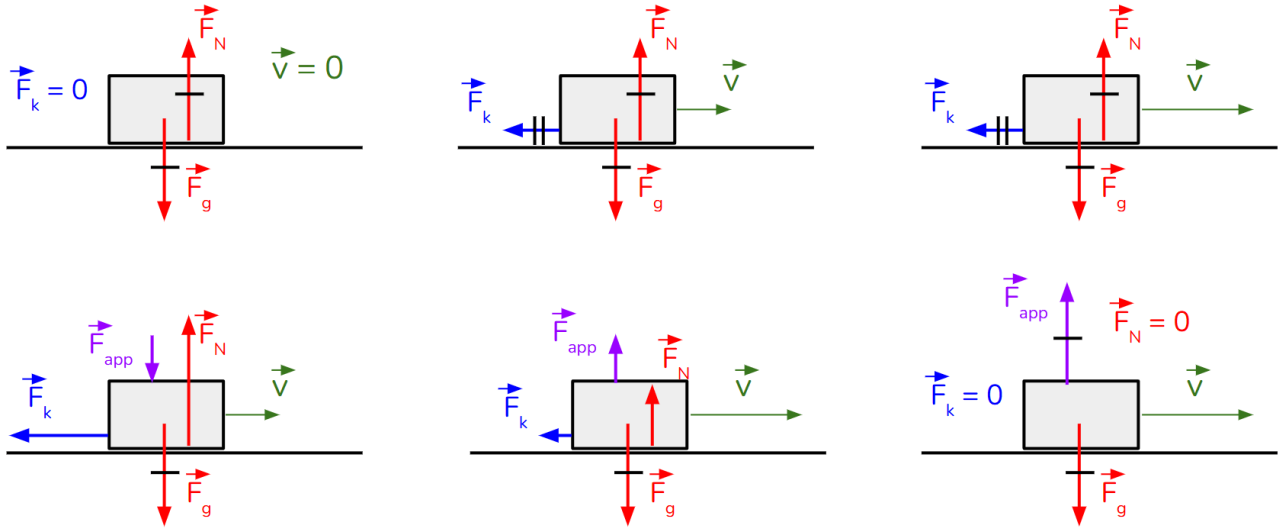
3.6: Friction

3.6 Frictional Forces

3.6.1 Kinetic Friction

Model

Below are six different examples of an identical mass on a surface that can result in friction. The type of friction we are exploring is Kinetic Friction and is represented by the variable $\mathbf{F}_k$.



Explore

1. Circle the two examples that have the same magnitude of kinetic friction. What other vectors are the same in those examples?
2. Find an example that has a larger force of friction than the examples you circled. What other vector is also larger?
3. Find an example that has a smaller force of friction (but not zero) than the examples you circled. What other vector is also smaller?
4. Two examples have $\mathbf{F}_k = 0$. What other quantities are zero for those examples?

Critical Thinking & Concept Invention

1. Propose a proportional relationship for the magnitude of the force of kinetic friction:

$$F_k \propto \tag{3.6.1}$$

2. Does friction depend on velocity? Give clear reasoning.

3. When friction is nonzero, what direction does it point?
4. In situations with kinetic friction, would $\Delta \mathbf{s}$ be in the same direction as the force of kinetic friction, opposite the direction of the force of kinetic friction, or equal to zero? Explain your thinking.
5. Based on your response above, would the Work done by kinetic friction be positive, negative, or zero? Use the definition of work $W \equiv \int \mathbf{F} \cdot d\mathbf{s}$ to justify your thinking.
6. Imagine your truck is stuck in a wet pothole and the wheel keeps spinning instead of gripping the ground. What could you do to increase the friction and get out?

Summary

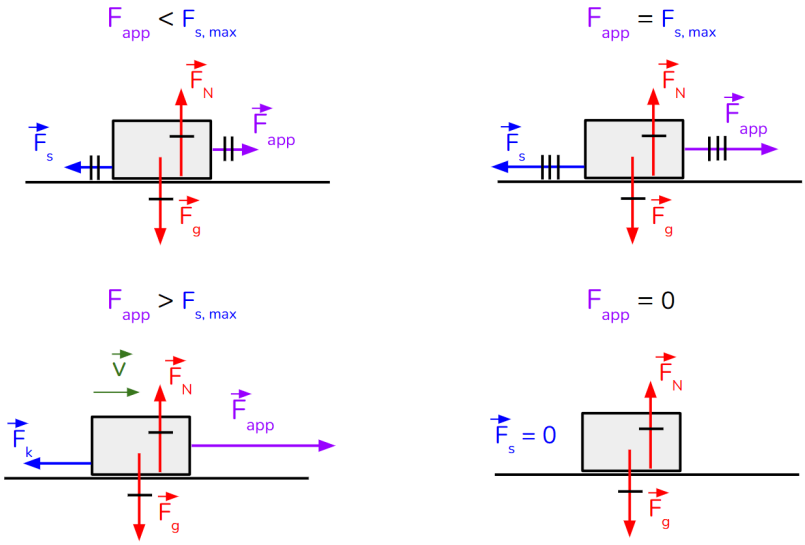
Kinetic Friction only exists when an object is moving against another surface (sliding/slipping) and always points opposite the direction that the surface is moving relative to the other surface. This can sometimes be different than the direction of the velocity of the object as a whole $\mathbf{v}_{CM}$. Kinetic Friction (F_k) is proportional to the normal force (F_N).

$$F_k \propto F_N \tag{3.6.2}$$

3.2 Static Friction

Model

Below are four different examples of friction, some of which have static friction $\mathbf{F}_s$.



Explore

1. If there was no static friction $\mathbf{F}_s$, what direction would these boxes slide?
2. What direction does the force of static friction point when present?
3. What vector is equal in magnitude to the force of static friction in each case?
4. Circle the example that has kinetic friction instead of static friction. What is different about this example?

Critical Thinking & Concept Invention

1. What would the net force be on each model if all of the static frictional forces were zero? Consider the vector sum of the arrows. Do any cancel? If so, what is left?
2. What is the net force on all of the models with static friction?
3. Is Static Friction a reactionary force like the Normal Force? Explain your thinking and cite examples from this model.
4. When do you use static friction and when do you use kinetic friction? Will you ever have both on the same force diagram?
5. In situations with static friction, would $\Delta \mathrm{s}$ be in the same direction as the force of static friction, opposite the direction of the force of static friction, or equal to zero? Explain your thinking.
6. Based on your response above, would the Work done by static friction be positive, negative, or zero? Use the definition of work to justify your thinking.
7. What do you think determines the maximum value of static friction?

Summary

Static Friction only exists when an object is not sliding against a surface. It points in the opposite direction as the object would move if the static friction force was not present. Static Friction is a reactionary force up to a maximum value, so its equation is an inequality. The maximum force of static friction is proportional to the normal force between the surfaces, similar to kinetic friction.

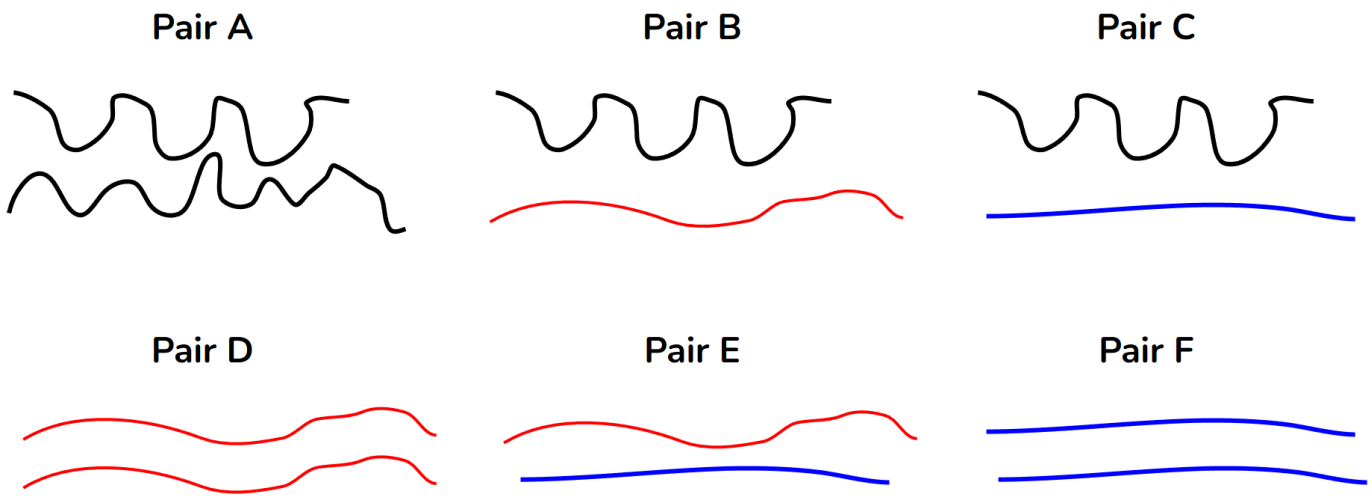
$$F_s \leq F_{s,\max} \tag{3.6.3}$$

$$F_{s,\max} \propto F_N \tag{3.6.4}$$

Static friction often does not do work because the displacement is typically zero. However, static friction can do work if the other surface also moves. Static just means no relative motion between the surfaces (no sliding/slipping).

3.6.3 Coefficients of Friction

Model



Explore

1. Label each surface in each pair as:
    - Rough
    - Smooth
    - Medium
2. Which pair do you think will result in the greatest horizontal force if they slide against each other?
3. Which pair do you think will result in the least horizontal force if they slide against each other?

Critical Thinking & Concept Invention

1. As a group, identify reasoning for the pairs you labeled as greatest and least horizontal force.
2. Make a claim about how the surface of an object affects the force of friction:

Claim: The force of friction is based on ...

Evidence:

Reasoning:

Summary

The molecular properties of both surfaces affect how much frictional force they can apply to each other. Each pair of surfaces has a coefficient of friction

$$(\mu) \tag{3.6.5}$$

that determines how much friction is present (or could be present) for a given normal force between the surfaces. These coefficients are different for kinetic and static friction, with the coefficient of static friction being higher.

$$F_k = \mu_k F_N \tag{3.6.6}$$

$$F_s \leq \mu_s F_N \tag{3.6.7}$$

$$F_{s,\max} = \mu_s F_N \tag{3.6.8}$$

Because static friction is an inequality, we often calculate its value based on what the force needs to be in order to prevent motion. If this value is less than or equal to the maximum value, the objects will not start slipping.

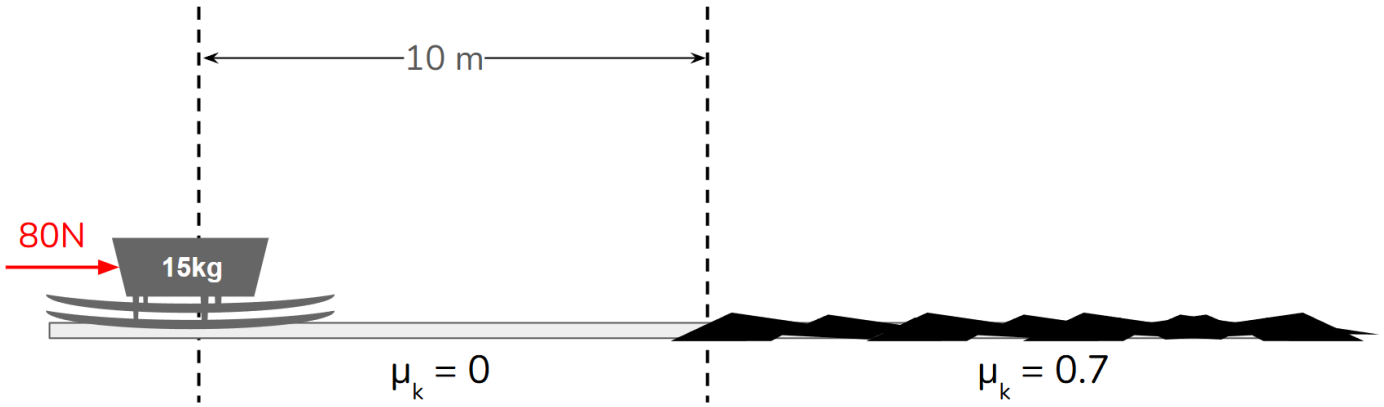
Apply

1. Your 1000 kg car is traveling 20 m/s on wet concrete ($\mu_s = 0.3,\ \mu_k = 0.25$) and you slam on the brakes. Determine the forces acting on the car to determine how long it will take to come to a stop:
    - With anti-lock brakes (static friction)
    - Skidding (kinetic friction)
2. Your 1000 kg car is traveling 20 m/s on dry concrete ($\mu_s = 2,\ \mu_k = 0.8$) and you slam on the brakes. Determine the forces acting on the car to determine how long it will take to come to a stop:
    - With anti-lock brakes (static friction)
    - Skidding (kinetic friction)

3.6.4 Work and Impulse with Friction

Model

A 15 kg sled that is initially at rest is pushed with a force of 80 N toward the right. The sled is initially on a frictionless surface $(\mu_k = \mu_s = 0)$for the first 10 meters that it is pushed. Then it encounters a surface with coefficients of friction $\mu_k = 0.7$ and $\mu_s = 1.2$. How long will it take for the sled to come to a stop on that surface?



Explore

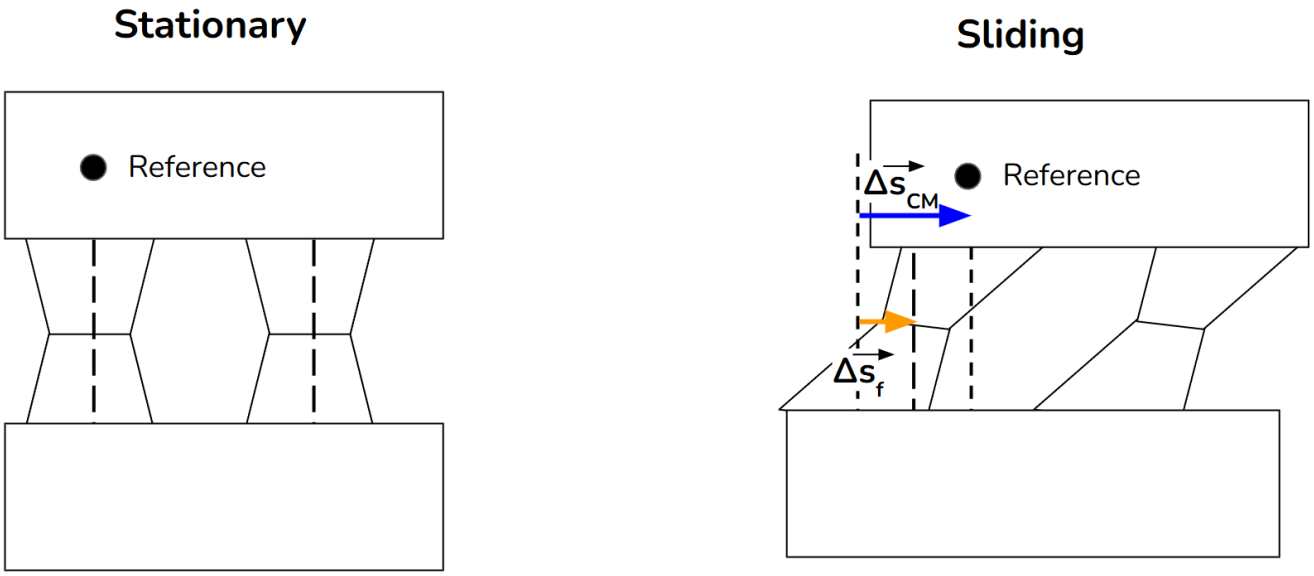
1. Draw in missing forces. If any are equal in magnitude, indicate that with hash marks.
2. Draw a displacement vector Δs for the frictionless part of the problem.
3. To the right of the model, construct an Object Interaction Diagram with just the sled as the system. Circle external interactions (if present) as they cross the system boundary.

Critical Thinking & Concept Invention

1. Examine the Physical Story:
   - In one color, highlight the portion of the story that is framed based on position/displacement.
   - In a different color, highlight the portion of the story that is framed based on time.
2. Model this storyline and solve for the time it takes for the sled to stop:
   - Complete the timeline for each phase of the story.
   - Summarize that portion of the story at the top of the timeline.
   - Write information about speed under the beginning and end tick marks.
   - Write Δs or Δt under the middle of the timeline. If you know the value, add it there.
   - Select the appropriate equation to model that phase of the story and write that equation in the box.
   - Write a short statement that explains your problem-solving approach.
   - Substitute the specifics of this story into the general equation and solve for necessary measurements.

3.6.5 Friction at a Microscopic Level

Model



Explore

1. What happens to the "teeth" of each object's surface as they slide against each other?
2. What direction is:
   - $\Delta \mathrm{s}_{CM}$
   - $\Delta \mathrm{s}_f$
3. How does the magnitude of $\Delta \mathrm{s}_{CM}$ compare to $\Delta \mathrm{s}_f$?
4. Where do you think the force of friction occurs between these objects? Draw an arrow at that location on the picture in the direction of the force of friction on the top block.

Critical Thinking & Concept Invention

1. Do you think the example above represents static or kinetic friction? Explain your reasoning.
2. Which displacement $(\Delta \mathrm{s}_{CM} \text{ or } \Delta \mathrm{s}_f)$ do you think should be used in the calculation of the Work done by friction? Explain your thinking.
3. If you used the other displacement to calculate work, would you overestimate or underestimate the amount of work done by friction? Explain your thinking.
4. Why might it be challenging (often impossible) to accurately calculate the work done by friction in physics problems?

Summary

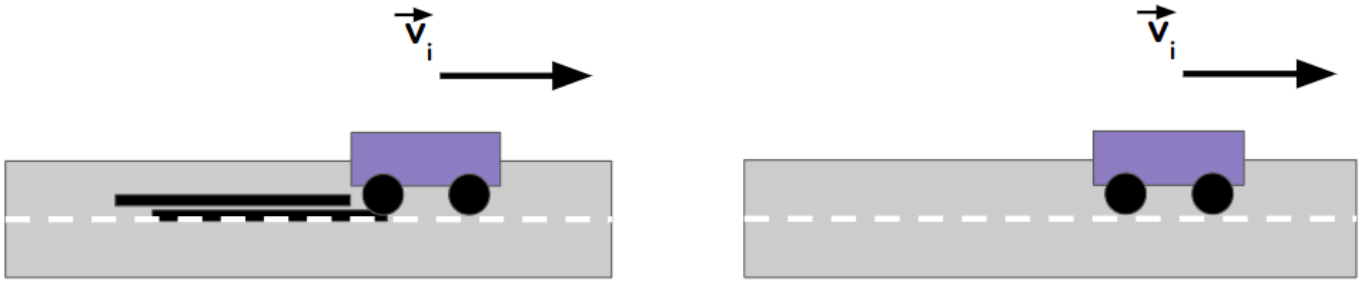
When calculating Work, the displacement in that calculation is the displacement of the force acting on the object, not the displacement of the object as a whole. Since friction is a microscopic effect, we often do not know the displacement of the force of friction. Unless we are given that information, we cannot calculate the work done by kinetic friction. We either need to make

the frictional force an internal interaction or avoid using energy methods to solve problems with kinetic friction.

Additional Notes

Sometimes, a moving object experiences static friction instead of kinetic friction. This is because the surfaces are not sliding against each other. Always consider where the force is applied and ask yourself if those surfaces are sliding against each other or not. When you walk, you experience static friction if your foot does not slip.

Wheels that roll without slipping experience static friction (or no friction), not kinetic friction. As a result, the frictional force between the ground and the wheel does zero work on the wheel. Static Friction can do work if the ground is also moving, such as on a treadmill. Both Kinetic and Static Frictional Forces can add impulse or twirl to an object because those quantities depend on time, not motion.



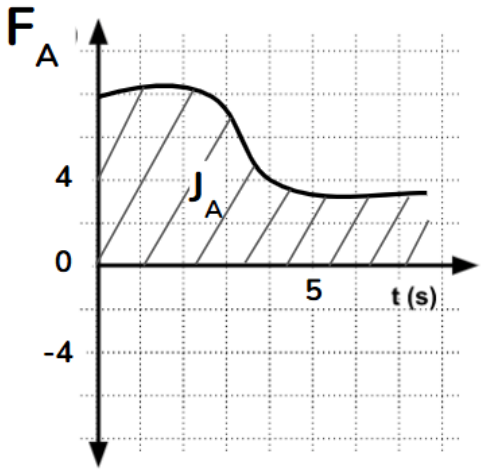
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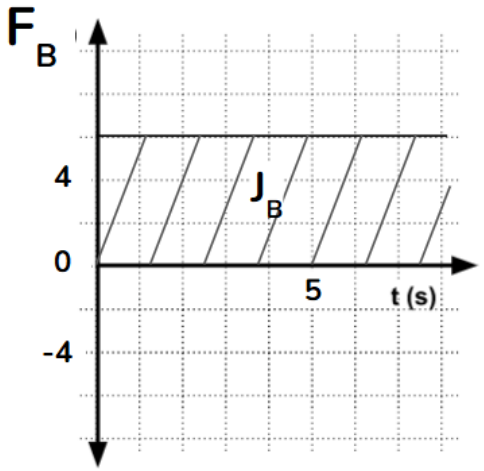
3.7: Integral Impulse and Work

3.7 Integral Impulse

3.7.1 Impulse from a Changing Force

Model





$$\mathbf{J} = \int \mathbf{F}_{ext}\, dt \qquad \mathbf{J} = \int \mathbf{F}_{ext}\, dt \tag{3.7.1}$$

Explore

1. Which Impulse or Impulses can be solved using an integral calculation $\int F\ dt$? Explain your reasoning.
2. Which Impulse or Impulses can be solved without integrating using simple geometry? Explain your reasoning.
3. Which force can be pulled out of the integral? Rewrite the integral expression underneath the current one with the **F** in front of the integral sign.

Critical Thinking & Concept Invention

1. Why can the $\mathbf{F}_{ext}$ come out of the integral in some cases? What needs to be true for that mathematical manipulation to be valid?
2. What is $\int_{t_1}^{t_2} dt$?
3. Why are you not able to use the equation $\mathbf{J} = F\Delta t$ for time-varying forces?

Summary

Impulse is calculated as the area on a Force vs. time graph. When Force is not constant, that area is not a simple product. We

can calculate the area under any function by integrating that function.

$$\mathbf{J}_{1\to 2} \equiv \int_{t_1}^{t_2} \mathbf{F}\, dt \tag{3.7.2}$$

Impulse is defined as the integral of Force with respect to time.

In all cases, the total impulse divided by the time interval gives the average force applied during that time interval:

$$\mathbf{J} = \mathbf{F}_{avg}\Delta t \tag{3.7.3}$$

$$\mathbf{F}_{avg} = \frac{1}{\Delta t}\int \mathbf{F}\, dt \tag{3.7.4}$$

If there are multiple forces acting on a system there will be multiple integrals of force with respect to time to determine the total (net) impulse $(\Sigma\mathbf{J})$.

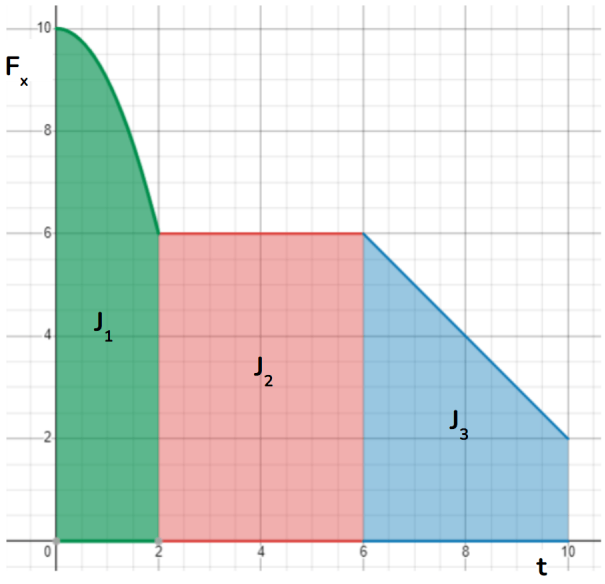
$$\sum \mathbf{J}_{1\to 2} = \sum_i \int_{t_1}^{t_2} \mathbf{F}_i\, dt_i \tag{3.7.5}$$

Similar to before, if the forces act on the object for the same time intervals, it is sometimes easier to add all of the forces together and find impulse from the net force.

$$\sum \mathbf{J}_{1\to 2} = \int_{t_1}^{t_2} \sum \mathbf{F}\, dt \tag{3.7.6}$$

3.7.2 Integrating Piece-wise Forces

Model



$$F = 10x^2 \quad \text{for } 0 < x < 2s \tag{3.7.7}$$

$$F = 6 \quad \text{for } 2s < x < 6s \tag{3.7.8}$$

$$F = 12 - x \quad \text{for } 6s < x < 10s \tag{3.7.9}$$

Explore

1. The Force for $\mathbf{J}_1$ is ... (circle one)
   constant linear quadratic
2. The Force for $\mathbf{J}_2$ is ... (circle one)
   constant linear quadratic
3. The Force for $\mathbf{J}_3$ is ... (circle one)
   constant linear quadratic
4. The total impulse $(\Sigma\mathbf{J})$ is the sum of how many integrals?

Critical Thinking & Concept Invention

1. If A, B, and C are constants, what is:
   - $\int_{t_1}^{t_2} C \ dt$
   - $\int_{t_1}^{t_2} Bt \ dt$
   - $\int_{t_1}^{t_2} At^2 \ dt$
   - $\int_{t_1}^{t_2} (Bt - C) \ dt$
2. Why do you need multiple separate integrals to solve for the total impulse? What determines the amount you need?
3. Construct the equation for the change in momentum of this system $(\Delta\mathbf{p}_{system})$ by adding the equations and bounds of integration that are specific to this problem to the impulse side of the momentum principle, then solve that equation:

$$\Delta\mathbf{p}_{system} = \Sigma\mathbf{J} \tag{3.7.10}$$

$$\Delta\mathbf{p}_{system} = \int + \int + \int \tag{3.7.11}$$

Summary

When forces are constant (not changing with time), you can pull the force vector out of the time integrals.

For each impulse:

$$\mathbf{J}_{1\to2} = \mathbf{F} \int_{t_1}^{t_2} dt \tag{3.7.12}$$

$$\mathbf{J}_{1\to2} = \mathbf{F}\Delta t \qquad \Delta t = t_2 - t_1 \tag{3.7.13}$$

You should always start your Impulse-Momentum Equation with integrals for each impulse. After evaluating the forces, the equation may reduce to this simple product rather than an integral.

$$(1) \quad \Delta\mathbf{p}_{system} = \Sigma\mathbf{J} \tag{3.7.14}$$

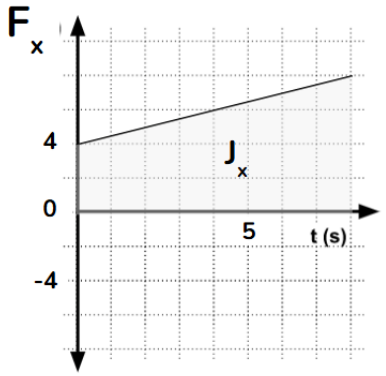
$$(2) \quad \Delta\mathbf{p}_A + \Delta\mathbf{p}_B + \cdots = \int \mathbf{F}_A \, dt + \int \mathbf{F}_B \, dt + \int \mathbf{F}_C \, dt + \cdots \tag{3.7.15}$$

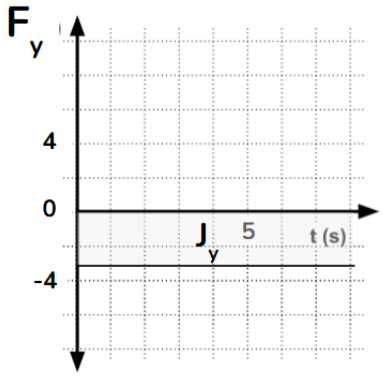
The first line is true for all systems in the universe. The second line is specific to the scenario you are modeling and your choice of system.

3.7.3 Vector Impulse

Model

$$\vec{F} = \begin{pmatrix} \frac{1}{2}t + 4 \\ -3 \\ 0 \end{pmatrix}$$





Explore

1. How many non-zero components (x, y, z) does the Force have?
2. How many Force vs. time graphs are needed to determine the Impulse from that Force?
3. How is J_x represented on a graph? Which graph?
4. How is J_y represented on a graph? Which graph?

Critical Thinking & Concept Invention

1. How can you determine the components of the impulse, J_x and J_y, from the Force that causes that impulse?
2. Identify the following:
   - J_x
   - J_y
   - $\mathbf{J}$
3. Explain how to integrate a vector, such as $\int \mathbf{F}\ dt$. Is the result a vector or a scalar?

Summary

All of our open systems have a right-hand side that is some integral of the external interactions, many of which are vector quantities.

When we take integrals of vectors, we can simply integrate each component of the vector.

$$\vec{F} = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} \qquad \vec{J} = \int \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} dt \qquad = \qquad \begin{bmatrix} \int F_x\, dt \\ \int F_y\, dt \\ \int F_z\, dt \end{bmatrix}$$

If there are multiple external interactions, we will get a sum of vectors that each have integrals of their components.

Apply

A 10,000 kg spaceship has two rockets, Rocket A that applies a force in the $+y$ direction and Rocket B that applies a force in the $-x$ direction. The rocket has an initial velocity of 10 m/s at an angle 30° from the positive x-axis.

Starting at time $t = 0s$, the rockets both turn on. Rocket A applies a constant 20,000 N force. Rocket B applies a force for only 4 seconds given by:

$$F = 40,000 - 10,000t \tag{3.7.16}$$

What is the momentum and velocity of the spaceship after 5 seconds?

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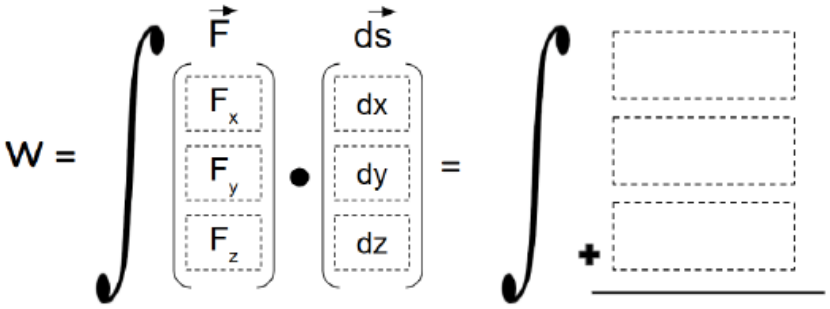
3.8: Integral Work

3.8 Integral Work

3.8.1 Work Integrals

Model

$$W = \int \mathbf{F}_{ext} \cdot d\mathbf{s} \tag{3.8.1}$$



$$W = \int (\qquad + \qquad + \qquad) \tag{3.8.2}$$

$$W = \int \qquad d \quad + \int \qquad d \quad + \int \qquad d \tag{3.8.3}$$

Explore

1. Take the dot product:
   - Multiply across.
   - Add the total inside the parentheses on the equation for work below the dot product model.
2. Fill in the second line with three integral signs by separating the sum of terms into separate integrals dx, dy, and dz.

Critical Thinking & Concept Invention

1. Work involves an integral and a dot product. What happens first: the integral or the dot product?
2. When forces were constant, the Δx, Δy, and Δz terms could be positive or negative quantities. The differentials dx, dy, and dz do not carry signs. How is it possible that Work could be negative in this integral? Identify as many ways as possible.

Summary

Work is the area on a Force vs. position graph for each component of that force. When forces are constant, this area is a rectangle, and the Work is given by:

$$W = \mathbf{F}_{ext} \cdot \Delta \mathbf{s} \tag{3.8.4}$$

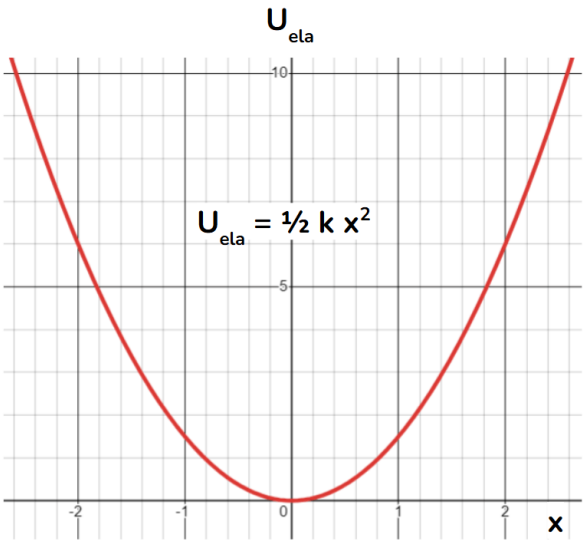
When forces change with position, we need to use an integral to calculate this area. The defining equation for Work is therefore:

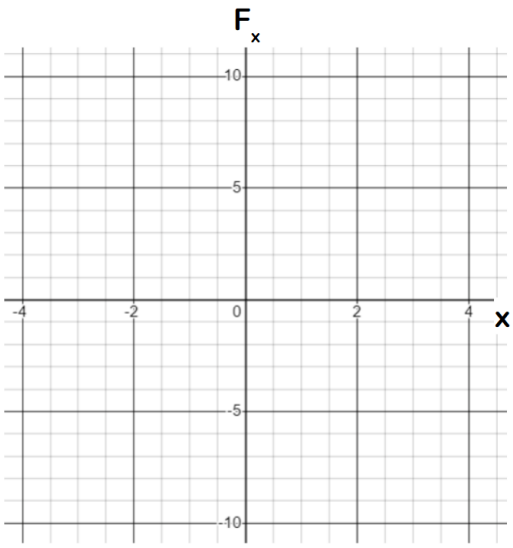
$$W \equiv \int \mathbf{F}_{ext} \cdot d\mathbf{s} \tag{3.8.5}$$

Integrals of dot products are called line integrals. We will only have simple cases in this class where you can evaluate the dot product first to create an equation for work that is a sum of single-variable integrals. The limits of integration for each integral use values for each of the separate coordinates.

3.8.2 Elastic Potential Energy & Force

Model





Explore

Examine the Potential Energy Graph (left).

1. For what values of x (if any) is the value of U_{ela}:
   - positive
   - negative
   - zero
2. For what values of x (if any) is the slope of U_{ela}:
   - positive
   - negative
   - zero

Critical Thinking & Concept Invention

Recall that potential energies relate to work through the equation $W = -\Delta U$ and that $W = \mathbf{F} \cdot \Delta \mathbf{s}$.

1. Construct an equation for Force as it relates to ΔU and Δs.
2. What graphical property (x value, y value, x-intercept, y-intercept, slope, or area) on the Potential Energy Graph does this quantity represent?
3. Re-examine your Exploration questions. For what values of x should the force be:
   - positive
   - negative
   - zero
4. What calculus operation do you need to do to the potential energy equation to get an equation for Force? Perform that operation and pay attention to negative signs.
5. Sketch this function on the Force Graph. Check that it matches where you said the force should be positive, negative, and zero.

Examine your Force Graph.

6. When F is positive:
   - Is the spring elongated (x is positive) or compressed (x is negative)?
   - Does F point in the direction to elongate or compress the spring?
7. When F is negative:
   - Is the spring elongated (x is positive) or compressed (x is negative)?
   - Does F point in the direction to elongate or compress the spring?
8. At larger values of compression or elongation (x), is the magnitude of the Force larger or smaller?

Summary

The Force of a conservative interaction is represented on a Potential Energy Graph as _____.

You can determine the equation for the Force of a conservative interaction by taking the spatial _____ of the Potential Energy Equation.

The equation for the Force of an elastic interaction is:

$$\mathbf{F}_{ela} = _____ \tag{3.8.6}$$

The negative sign in this equation means _____.

Apply

1. **Potential Energy in a Spring**

   We just found that the force of a horizontal spring is given by:

$$\mathbf{F}_{ela} = -k\mathbf{x} \tag{3.8.7}$$

Check this result by integrating force.

$$W_{ela} = \int_a^b -k \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} dx \\ 0 \\ 0 \end{bmatrix} = \tag{3.8.8}$$

$$\Delta U_{ela} = -W_{ela} \tag{3.8.9}$$

$$\Delta U_{ela} = \tag{3.8.10}$$

2. **Surface Gravity**

The potential energy function for gravitational potential energy on the surface of the Earth is:

$$U_g = mgh \tag{3.8.11}$$

Determine the force of gravity using the relationship between Force and Potential Energy. Interpret the sign $(+/-)$ of your result.

Apply (continued)

3. **General Gravity**

Later on we will learn a more general equation for gravity. The potential energy function for gravitational potential energy is:

$$U_{gravity} = -\frac{Gm_1m_2}{r} \tag{3.8.12}$$

Determine the gravitational force using the relationship between Force and Potential Energy. Interpret the sign $(+/-)$ of your result. You can treat r as your coordinate and compute:

$$\frac{d}{dr}U_{gravity} \tag{3.8.13}$$

The vector $\mathbf{r}$ points from one mass to the other.

4. **Electrostatic Force**

In PHYS 4B you will learn about electrostatic forces and electric potential energy. The potential energy function for electric potential energy is:

$$U_{electric} = \frac{kq_1q_2}{r} \tag{3.8.14}$$

Determine the electric force using the relationship between Force and Potential Energy. Interpret the sign $(+/-)$ of your result. You can treat r as your coordinate and compute:

$$\frac{d}{dr}U_{electric} \tag{3.8.15}$$

The vector $\mathbf{r}$ points from one charged object to the other.

3.8.3 Force as Gradient of Potential Energy

Model

The equations below show some symbols that are used in multivariable calculus. We will explore how to interpret these symbols and use them as shorthand for equations in this class that can be solved with single-variable calculus knowledge.

$$F_x = -\frac{\partial U}{\partial x} \tag{3.8.16}$$

$$F_y = -\frac{\partial U}{\partial y} \tag{3.8.17}$$

$$F_z = -\frac{\partial U}{\partial z} \tag{3.8.18}$$

$$\mathbf{F} = -\nabla U \tag{3.8.19}$$

$$\mathbf{F} = \tag{3.8.20}$$

$$U = -\int \mathbf{F} \cdot d\mathbf{s} \tag{3.8.21}$$

$$U = -\int F_x \, dx - \int F_y \, dy - \int F_z \, dz \tag{3.8.22}$$

Explore

1. Are the force components found by taking a derivative or an integral of potential energy (U)?
2. Examine each Force component and the quantities it equals. Are the calculations like coordinates between Force component and differential quantity (for example x and ∂x) or unlike coordinates (for example x and ∂y)?
3. Is the potential energy found by taking a derivative or an integral of Force ($\mathbf{F}$)?
4. What type of product is used for $\mathbf{F}$ and $d\mathbf{s}$?
5. How do you deal with the different components of Force when you calculate potential energy?

Critical Thinking & Concept Invention

1. To the right of the equal sign for $\mathbf{F}$, construct a column vector. Use the equation for each component to identify what should go into the x, y, and z components.
2. Compare the middle relationship to your column vector that you just created. Explain what $-\nabla$ does to a scalar quantity.
3. Recall that potential energy (U) is a scalar quantity. Is the result of ∇U a vector or scalar quantity?
4. Write out the result of the dot product $\mathbf{F} \cdot d\mathbf{s}$.
5. How is it that the integral of a vector quantity ($\mathbf{F}$) results in a scalar quantity (U)?
6. Summarize in your own words: How are Force and potential energy related?

Apply

1. The potential energy of a region of space is given by:

$$U = \left(6.4\frac{J}{m^2}\right)x^2 - \left(4.8\frac{J}{m^2}\right)y^3 \tag{3.8.23}$$

Identify the force as a function of x and y that matches this potential energy function. Report it as a column vector.

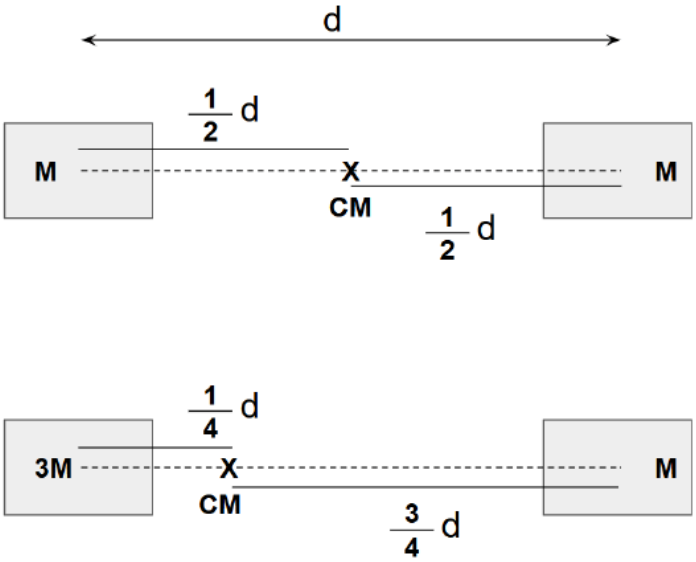
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3.9: Center of Mass

3.9 Center of Mass

3.9.1 Center of Mass

Model



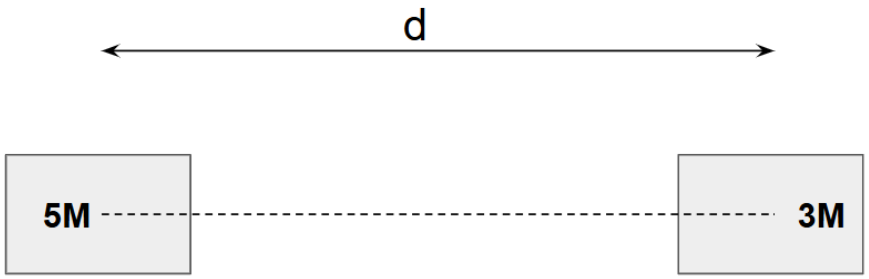
Explore

1. Compare the mass of the object on the left and right for each pair and identify which object the center of mass is closest to.

Pair	M_{left}	M_{right}	**Center of Mass is Closer To...**
Top			
Bottom			

Critical Thinking & Concept Invention

1. What is the general trend for which object the center of mass is closest to?
2. Look at the fractions of the distance to the center of mass for each object. Where do the numbers in the numerator come from?
3. Look at the fractions of the distance to the center of mass for each object. Where do the numbers in the denominator come from?
4. Identify the approximate center of mass for the pair of objects below.



Summary

The center of mass between two objects is located between the objects and is closer to the more massive object. It is weighted by the mass of each object:

$\mathbf{r}_{CM} = \frac{\sum m_i \mathbf{r}_i}{\sum m_i}$

This vector is origin dependent because it represents a vector from a chosen origin. It is often convenient to choose one of the masses as the origin or choose the center of a distribution. A more precise version of this calculation for mass distributions will be explored later.

The denominator is simply the total mass:

$\sum m_i = M_{total}$

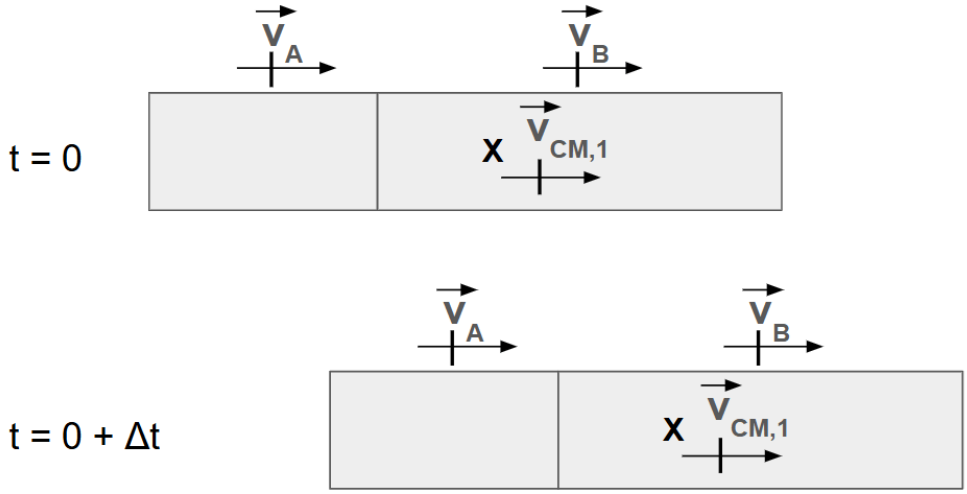
Apply

Use the equation above to calculate the center of mass of the objects in the model. Choose either mass as the origin and set up an appropriate coordinate system.

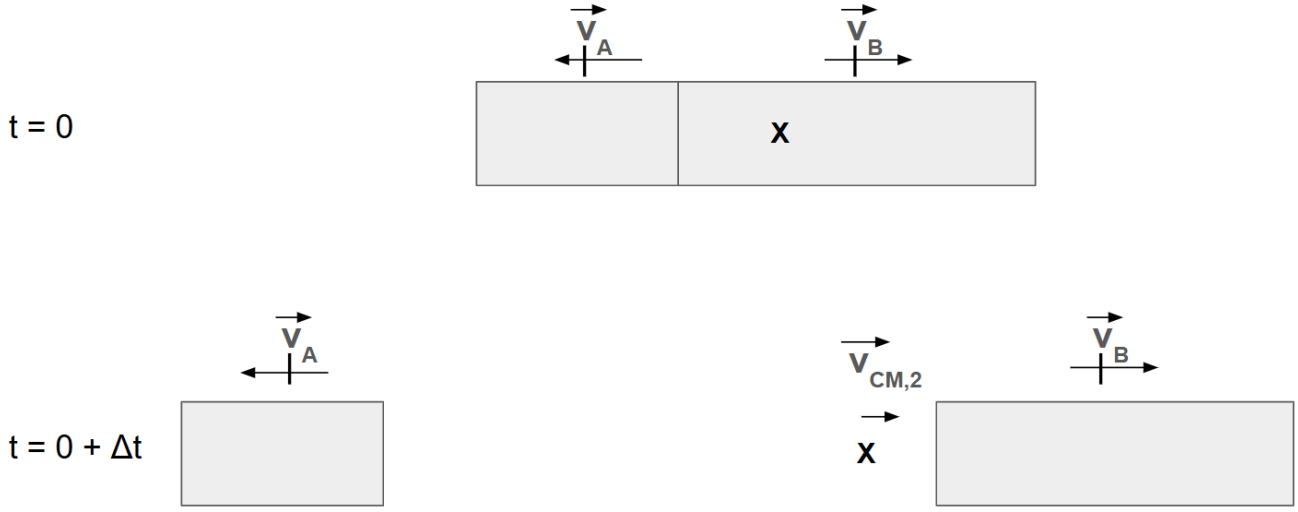
3.9.2 Center of Mass Motion

Model

1)



2)



Explore

1. Compare connected objects and free-moving objects.

	Connected Objects (1)	Free Moving Objects (2)
Is $\mathbf{v}$ the same or different for both objects?		
Is $\mathbf{v}_{CM}$ the same as the velocity of the objects?		

Critical Thinking & Concept Invention

1. Why does the center of mass appear to move with one set of objects but not the other?

For the system with different velocities:

2. Compare the magnitude of v_{CM} to v_B.
3. If $M_A = M_B$, what would v_{CM} be? Explain your thinking.
4. If $M_A > M_B$, what direction would v_{CM} be? Explain your thinking.
5. Is $\sum \mathbf{v}_i$ equal to $\mathbf{v}_{CM}$? Explain why or why not.
6. Use the following relationships to identify an equation for $\mathbf{v}_{CM}$:

$\mathbf{r}_{CM} = \frac{\sum m_i \mathbf{r}_i}{\sum m_i}$

$\mathbf{v}_{CM} = \frac{d}{dt}\mathbf{r}_{CM}$

7. What does the numerator in your equation resemble?

Summary

The velocity of the center of mass is:

$\mathbf{v}_{CM} = \frac{d}{dt}\mathbf{r}_{CM} = \frac{\sum m_i \mathbf{v}_i}{\sum m_i}$

The velocity of the center of mass is the sum of the velocities of all objects in the system weighted by their mass fractions.

The numerator is simply the momentum of the system:

$\sum m_i \mathbf{v}_i = \sum \mathbf{p}_i = \mathbf{p}_{system}$

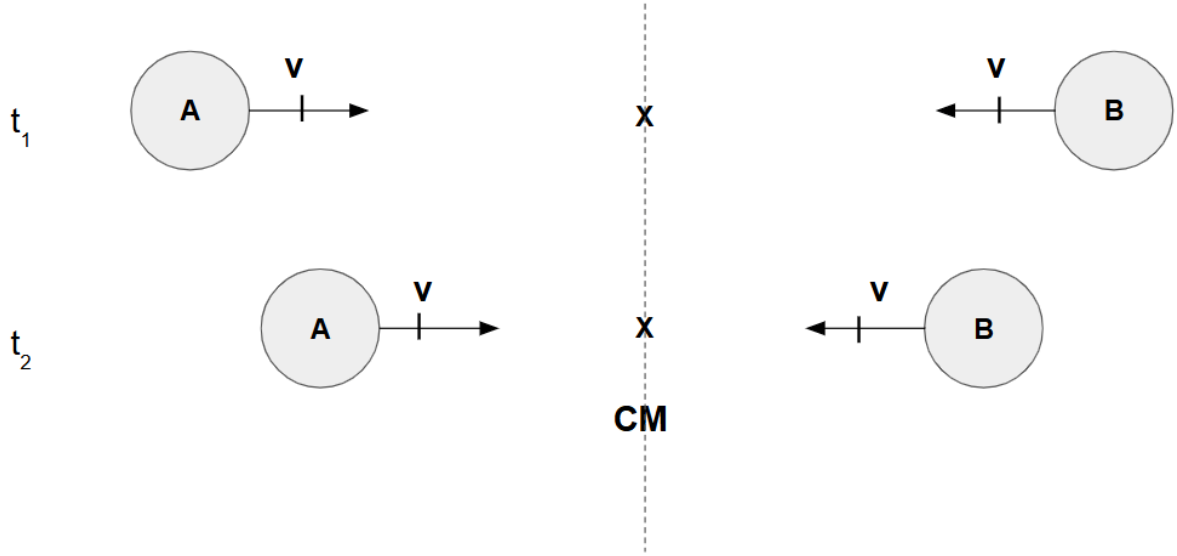
$\mathbf{v}_{CM} = \frac{\mathbf{p}_{system}}{M_{total}}$

$\mathbf{p}_{system} = M_{total}\mathbf{v}_{CM}$

3.9.3 Center of Mass Motion: Momentum and Kinetic Energy

Model

Two objects of equal mass M move toward each other with the same speed v. The lower image represents a later time than the upper image. Assume motion occurs only in the horizontal direction.



Explore

1. Identify the speeds for the objects and their center of mass. Remember that speed is a magnitude and therefore always positive.

v_A	v_B	v_{CM}

2. Determine the kinetic energy of each object. Add them to obtain the kinetic energy of the combined system. Separately determine:

$\frac{1}{2}M_{system}v_{CM}^2$

K_A	K_B	$K_A + K_B$	$\frac{1}{2}M_{system}v_{CM}^2$

Critical Thinking & Concept Invention

1. Does $\frac{1}{2}M_{system}v_{CM}^2$ represent the kinetic energy of the system? Explain why or why not.
2. Identify the momentum of each object, the total system momentum, and the center of mass momentum. Be sure to draw a column vector for vector quantities.

$\mathbf{p}_A$	$\mathbf{p}_B$	$\mathbf{p}_A + \mathbf{p}_B$	$M_{system}\mathbf{v}_{CM}$

3. Does $M_{system}\mathbf{v}_{CM}$ represent the momentum of the system? Explain why or why not.
4. Why might calculations involving $\mathbf{v}_{CM}$ appear to represent momentum but not kinetic energy? Think about how the equations differ.

Summary

The total momentum of a system equals the product of the total mass and the velocity of the center of mass:

$\mathbf{p}_{system} = M_{system}\mathbf{v}_{CM}$

The total kinetic energy of the system is not equal to the kinetic energy associated with the center of mass:

$K_{system} \neq \frac{1}{2}M_{system}v_{CM}^2$

This quantity is generally not useful unless all objects move together with the same velocity or the system contains only a single object.

3.9.4 Center of Mass Equation

Model

The equations below show a transformation of Newton's Second Law:

(1) $\sum \mathbf{F}_{ext} = \frac{d\mathbf{p}_{system}}{dt}$

(2) $\sum \mathbf{F}_{ext} = M_{system}\frac{d\mathbf{v}_{CM}}{dt}$

(3) $\sum \mathbf{F}_{ext}\ dt = M_{system}\ d\mathbf{v}_{CM}$

(4) $\sum \mathbf{F}_{ext} \cdot \frac{d\mathbf{r}_{CM}}{dt}\ dt = M_{system}\ \mathbf{v}_{CM} \cdot d\mathbf{v}_{CM}$

(5) _____

Explore

1. Identify what action was taken to transform the equation:
   - Line 1 to Line 2
   - Line 2 to Line 3
   - Line 3 to Line 4
2. Which lines contain dt?
3. Which lines contain $d\mathbf{r}$?

Critical Thinking & Concept Invention

1. Was the same operation performed on each side of the equation from Step 3 to Step 4? If not, what was different?
2. Is the action used from Line 3 to Line 4 mathematically valid? Explain.
3. Why was a dot product required rather than ordinary multiplication?
4. Evaluate:

$\int_{v_1}^{v_2} v\ dv$

5. Something should cancel on the left side of Line 4. Identify it.

6. Create Line 5 by replacing both sides with your previous results and integrate both sides. Simplify where possible.
7. Examine the resulting equation. Does the left side resemble a familiar quantity? Does the right side resemble a familiar quantity? Identify both.

Summary

This process effectively takes a line integral of Newton's Second Law. By multiplying by velocity, time-based quantities are transformed into position-based quantities. This is useful when position changes are known but elapsed time is not.

The resulting equation is called the **Center of Mass Equation**:

$$\int_{r_1}^{r_2} \sum \mathbf{F}_{ext} \cdot d\mathbf{r}_{CM} = \frac{1}{2} M_{system} v_{CM}^2 \Big|_{v(r_1)}^{v(r_2)}$$

"Not Work" "Not Change in Kinetic Energy"

This equation is another form of Newton's Second Law that is useful for position-based problems. It is not an independent energy equation and should not be mixed with energy methods. Although both sides have units of energy, neither side represents work or energy in the thermodynamic sense.

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3.10: Comparing Center of Mass Equation (CM) and First Law of Thermodynamics (FLT)

Center of Mass Equation and First Law of Thermodynamics Applications

Critical Thinking & Concept Invention

1. Was the same operation performed on each side of the equals sign from step 3 to step 4? If not, what was different?
2. Is the action used to go from line 3 to line 4 mathematically valid? Explain your reasoning.
3. Why did the multiplication action from line 3 to line 4 require a dot product rather than ordinary multiplication?
4. Evaluate:

$$\int_{v_1}^{v_2} v\, dv \tag{3.10.1}$$

   Your answer should be expressed in terms of variables.
5. Something should cancel on the left side in line 4. Cross out the quantity that cancels.
6. Create a line 5 by replacing both sides of the equation with your responses from the previous question, then integrate both sides. Simplify what you can without moving terms across the equals sign.
7. Examine the equation you just wrote. Does the left-hand side look familiar? Does the right-hand side look familiar? Identify what physical quantities each side resembles.

Summary

The process outlined in this model effectively takes a line integral of Newton's Second Law. By multiplying each side by velocity, it becomes possible to transform time-based quantities into position-based quantities. This is useful when the change in position is known but the elapsed time is not known.

The resulting equation is called the **Center of Mass Equation**:

$$\int_{r_1}^{r_2} \sum \mathbf{F}_{ext} \cdot d\mathbf{r}_{CM} = \frac{1}{2} M_{system}\, v_{CM}^2 \Big|_{v(r_1)}^{v(r_2)} \tag{3.10.2}$$

Not Work **Not Change in Kinetic Energy**

This equation is simply another form of Newton's Second Law that works well for position-based problems. It is not an independent energy principle like the First Law of Thermodynamics. Although the units resemble energy units and the mathematics appears similar, the quantities in this equation are neither work nor energy. Do not combine this equation with energy equations.

---

2 FLT and Center of Mass Equation Scenarios

We previously developed the First Law of Thermodynamics (FLT), one of the most important principles in physics:

$$\Delta E_{system} = W + Q \tag{3.10.3}$$

The changes in system energy can include kinetic energy, potential energy, and internal energy changes within the chosen system. The work done on the system is the sum of all external forces integrated over the displacement of those forces. Internal interactions do not do work on the system.

$$\Delta E_{sys} = \sum \Delta K + \sum \Delta U + \sum \Delta E \tag{3.10.4}$$

$$W = \sum_i \int \mathbf{F}_i \cdot d\mathbf{s}_i \tag{3.10.5}$$

We also developed the Center of Mass Equation. Unlike FLT, it is not an energy equation. It focuses on the net external force and the motion of the system's center of mass.

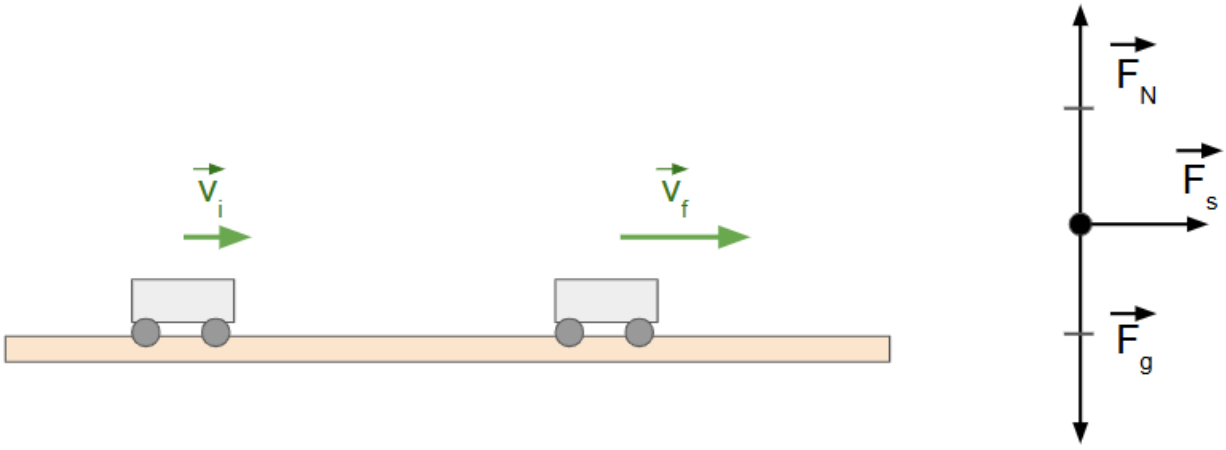
$$\int_{s_1}^{s_2} \sum \mathbf{F}_{ext} \cdot d\mathbf{s}_{CM} = \frac{1}{2} M_{system}\, v_{CM}^2 \Big|_{v(s_1)}^{v(s_2)} \tag{3.10.6}$$

Both equations can be useful, and one may be easier to apply than the other depending on the situation. Assume all forces are constant unless otherwise stated.

3.10.1 Car Speeds Up to the Right on Level Ground

Model

An electric car increases its speed on level ground from 5 m/s to 15 m/s. The model below applies both the First Law of Thermodynamics and the Center of Mass Equation. To simplify the model, assume there is no change in thermal energy.



FLT Model

$$\Delta E_{system} = W \tag{3.10.7}$$

$$\Delta U_{electric} + \Delta K_{trans} = 0 \tag{3.10.8}$$

$$\Delta U_{electric} + \frac{1}{2} M v^2 \Big|_5^{15} = 0 \tag{3.10.9}$$

$$\Delta U_{electric} = -\frac{M(15^2 - 5^2)}{2} \tag{3.10.10}$$

Center of Mass Equation Model

$$\int \sum \mathbf{F} \cdot d\mathbf{s}_{CM} = \frac{1}{2} M_{system} v_{CM}^2 \Big|_5^{15} \tag{3.10.11}$$

$$\int F_s \, ds = \frac{1}{2} M(15^2 - 5^2) \tag{3.10.12}$$

$$F_s = \frac{M(15^2 - 5^2)}{2\Delta s} \tag{3.10.13}$$

Explore

1. Draw a force diagram that includes $\mathbf{F}_g$, $\mathbf{F}_N$, and $\mathbf{F}_s$.
2. How many energy terms change in the FLT equation?
3. How many forces do work? Explain using the definition of work as a dot product.
4. Which energy term from the FLT equation also appears in the CM Equation?
5. How many forces remain in the integral after the dot product?
6. If the mass is known, what quantity can be determined from the FLT equation?
7. If the mass and displacement are known, what quantity can be determined from the CM Equation?

Critical Thinking & Concept Invention

1. In this situation, is

$$\frac{1}{2} M_{system} v_{CM}^2 \Big|_{v(s_1)}^{v(s_2)} \tag{3.10.14}$$

   equal to the change in kinetic energy of the system? Explain.
2. In this situation, is

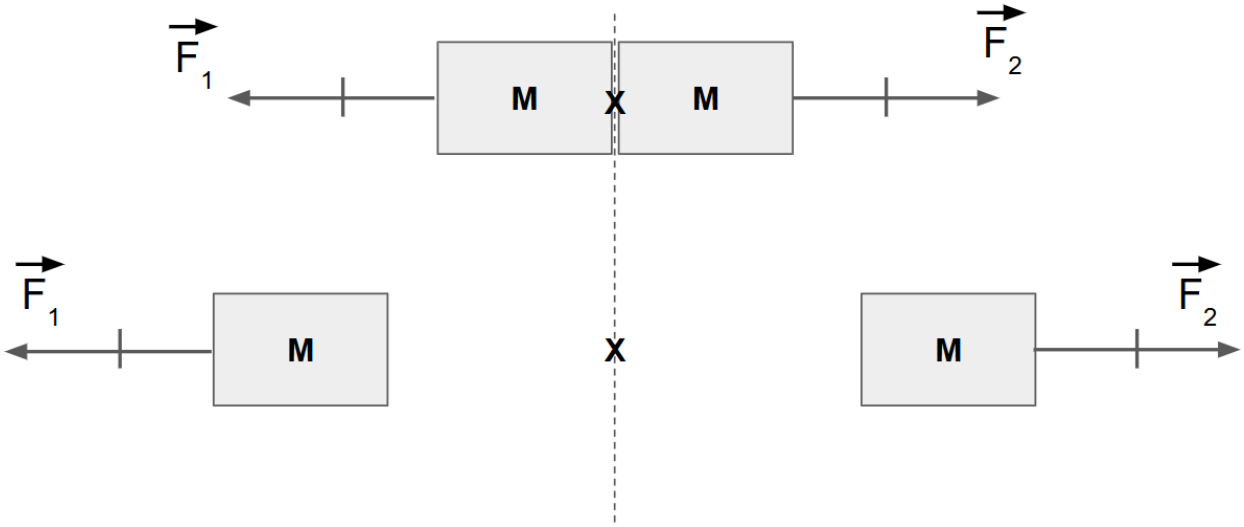
$$\int \mathbf{F}_s \cdot d\mathbf{s} \tag{3.10.15}$$

   also equal to the work done on the system? Explain.
3. What does each equation tell you about the car?

3.10.2 Blocks Pulled Apart

Model

A force of 10 N is applied in the directions shown to two different 0.5 kg blocks. The blocks are initially at rest and move across a frictionless table a distance of 3 m.



Explore

1. What is the net force on the system?
2. What is the initial velocity of the center of mass? Does it change?
3. What is the initial velocity of each individual block? Does it change?

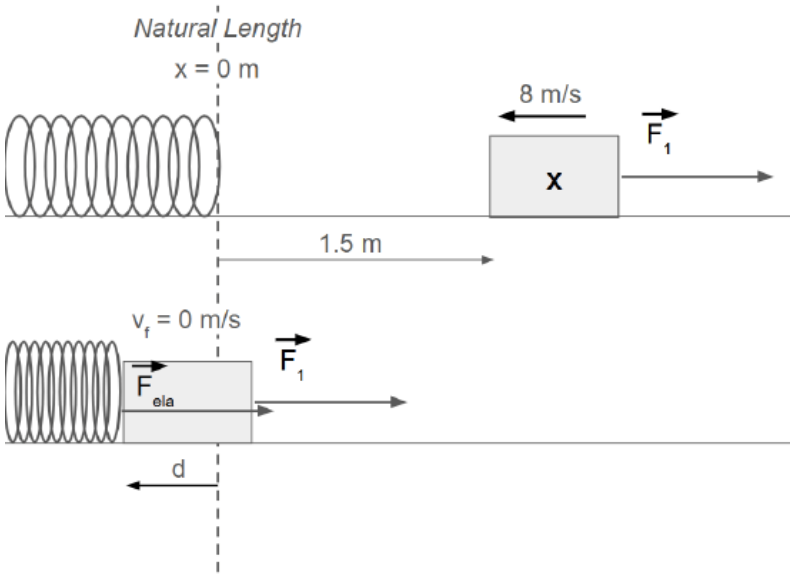
Critical Thinking & Concept Invention

1. Solve both equations:
   - $$\int \sum \mathbf{F} \cdot d\mathbf{s}_{CM} = \frac{1}{2} M_{system} v_{CM}^2 \Big|_{v(s_1)}^{v(s_2)} \tag{3.10.16}$$
   - $$\Delta E_{system} = W \tag{3.10.17}$$
2. Which equation allowed you to calculate something about the system?
3. Why was the other equation not particularly useful in this situation?
4. Summarize when you would use the more useful equation instead of the other.

3.10.3 Collision with a Spring

Model

A 1.5 kg box slides on a surface with a coefficient of friction of 0.4. It begins with a speed of 8 m/s and travels 1.5 m before reaching a spring. Determine how far the spring compresses before the box stops if the spring constant is 24 J/m.



FLT Equation

$$\Delta E_{system} = W \tag{3.10.18}$$

$$\Delta K_{trans} + \Delta E_{thermal} = \int_0^d \mathbf{F}_{ela} \cdot d\mathbf{s} + \int_{1.5}^{d_f} \mathbf{F}_k \cdot d\mathbf{s} \tag{3.10.19}$$

$$\frac{1}{2} M v^2 \Big|_8^0 + \Delta E_{thermal} = \int_0^d (-kx)\, dx + \int_{1.5}^{d_f} \mu_k mg\, dx \tag{3.10.20}$$

Center of Mass Equation

$$\int_{1.5}^{d} \sum \mathbf{F} \cdot d\mathbf{s}_{CM} = \frac{1}{2} M_{system} v_{CM}^2 \Big|_8^0 \tag{3.10.21}$$

$$\int_0^d (-kx)\, dx + \int_{1.5}^{d} \mu_k mg\, dx = \frac{1}{2} M v^2 \Big|_8^0 \tag{3.10.22}$$

Explore

1. Compare the second line of each equation and identify terms unique to each.
2. Circle the unknowns in each equation.
3. For each equation identify:
   - Number of forces
   - What the force integrals equal
   - Number of unknowns

Critical Thinking & Concept Invention

1. Is the variable d in the integral limit positive or negative? Explain.
2. Which equation appears easier to solve?
3. Besides kinetic energy, what energy changes because of friction? Which equation shows this?

4. Why is the work done by friction difficult to calculate?
5. Summarize situations where the CM Equation may be a better choice than FLT.

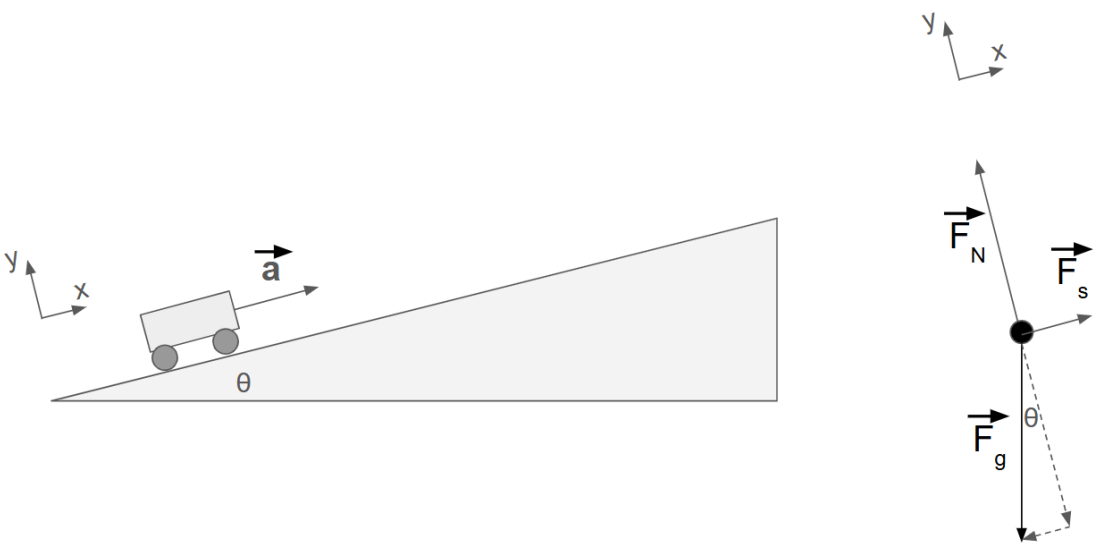
Apply

1. Solve for the final position of the spring using whichever equation is easier.

3.10.4 Driving Up a Hill at a Constant Speed

Model

A 1200 kg car drives up a hill inclined at an angle θ with a constant speed of 30 m/s. The system consists only of the car.



The net force is composed of:

$$\sum \mathbf{F} = \mathbf{F}_g + \mathbf{F}_N + \mathbf{F}_s \tag{3.10.23}$$

$$\begin{bmatrix} -mg\sin(\theta) \\ -mg\cos(\theta) \end{bmatrix} + \begin{bmatrix} 0 \\ mg\cos(\theta) \end{bmatrix} + \begin{bmatrix} F_s \\ 0 \end{bmatrix} \tag{3.10.24}$$

FLT Equation

$$\Delta E_{system} = W \tag{3.10.25}$$

$$\Delta E_{electrical} = \int \mathbf{F}_g \cdot d\mathbf{s} + \int \mathbf{F}_N \cdot d\mathbf{s} + \int \mathbf{F}_s \cdot d\mathbf{s} \tag{3.10.26}$$

$$\Delta E_{electrical} = \int_0^d -mg\sin(\theta)\, dx \tag{3.10.27}$$

Center of Mass Equation

$$\int_{s_1}^{s_2} \sum \mathbf{F} \cdot d\mathbf{s}_{CM} = \frac{1}{2} M_{system} v_{CM}^2 \Big|_{v(s_1)}^{v(s_2)} \tag{3.10.28}$$

$$\int_{s_1}^{s_2} (0) \cdot d\mathbf{s} = \frac{1}{2} M_{system} (0) \tag{3.10.29}$$

$$0 = 0 \tag{3.10.30}$$

Explore

1. What is the chosen system?
2. For each equation identify:
   - Number of forces before the dot product
   - Number of forces after the dot product
   - What the force integrals equal

Critical Thinking & Concept Invention

1. What should $\sum \mathbf{F}$ equal for this situation?
2. Why are there no forces left in the CM Equation?
3. Why is there one force remaining in the FLT equation?
4. Does the FLT equation provide useful information? If so, what?
5. Does the CM Equation provide useful information? If so, what?
6. Under what conditions does the CM Equation reduce to zero?

Apply

1. Assume the car accelerates uphill and the static friction force is: $F_s = 2mg\sin(\theta)$ directed uphill ($+x$). Construct both the FLT and CM equations. Then choose the equation that allows you to determine the speed of the car after traveling 20 m on a hill inclined at 20°.

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4: ANGULAR MOMENTUM

4.1: Angular Momentum Conservation
4.2: Rotating Rigid Bodies
4.3: Torque and Twirl
4.4: Twirl and Angular Work
4.5: Open Angular Momentum Systems

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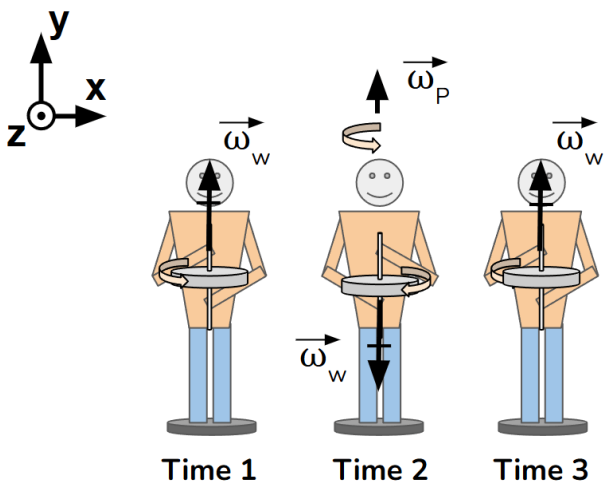
4.1: Angular Momentum Conservation

Angular Momentum Conservation

1.1 Flipping a Bike Wheel

Model

A person stands on a platform that can rotate. The person initially holds a spinning wheel and then flips the wheel upside down. Next to the person is a chart that identifies Angular Momentum ($\mathbf{L}$) at each of the three moments in time.



Angular Momentum

$\vec{L}$	Time 1	Time 2	Time 3
Person	0	↑	0
Wheel	↑	↓	↑
Total (Person + Wheel)	↑	↑↓	↑

$\vec{L}$	L_1	ΔL_{12}	L_2
Person	0		↑
Wheel	↑		↓
Total (Person + Wheel)	↑		↑

$\vec{L}$	L_2	ΔL_{23}	L_3
Person	↑		0
Wheel	↓		↑
Total (Person + Wheel)	↓		↑

Explore

1. Tell a Physical Story about the rotation of objects and their angular velocity vectors. Be sure to comment on the direction of the angular velocity (spin).

- At time 1, the wheel is ___ and the person is ___.
- At time 2, the wheel is ___ and the person is ___.
- The person is spinning (faster/slower) than the wheel.
- At time 3, the wheel is ___ and the person is ___.

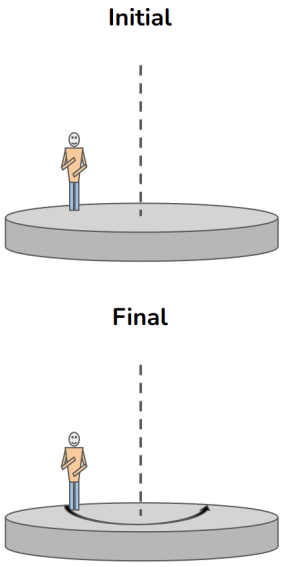
2. Tell an Angular Momentum Story about the same objects. Refer to the L-Chart.
   - At time 1, the wheel ___ and the person ___.
   - At time 2, the wheel ___ and the person ___.
   - The person has (greater/lesser) angular momentum than the wheel.
   - At time 3, the wheel ___ and the person ___.
   - For all three moments in time, the total angular momentum is ___.

Critical Thinking & Concept Invention

1. Is the wheel changing its position? If so, what direction?
2. Is the wheel moving? If so, what direction? Why might this be tricky to identify?
3. Does the wheel have momentum? If so, draw a momentum vector next to the wheel in each image on the diagram.
4. Make a Connection: How is this similar to momentum conservation problems, such as colliding carts?
5. Complete the Change in Angular Momentum columns of the two L-Charts.
6. Use the charts to construct Claim-Evidence-Reasoning responses:
   a. For which systems (Person, Wheel, and Person + Wheel) would you say that angular momentum was always conserved?
   Claim:
   Evidence:
   Reasoning:
   b. For which systems (Person, Wheel, and Person + Wheel) would you say that angular momentum was not always conserved?
   Claim:
   Evidence:
   Reasoning:
   c. Why does the magnitude of the angular momentum for the person have to be greater than the wheel at time 2?
   Claim:
   Evidence:
   Reasoning:

Apply

Jill stands on a merry-go-round. Initially, both Jill and the merry-go-round are at rest. Then, Jill begins to walk around the merry-go-round counterclockwise from above. What will the merry-go-round do? Make a momentum table to justify your response.



	$\vec{L}$	$\vec{L}_i$	$\Delta\vec{L}$	$\vec{L}_f$
Jill				
Merry - go - round				
Total				

Summary

The Angular Momentum Principle is represented by:

$\Delta \mathbf{L}_{system} = \sum \tau$

“The total change in angular momentum of a system” = “The net Twirl acting on the system.”

For closed systems:

$\sum \tau = 0$

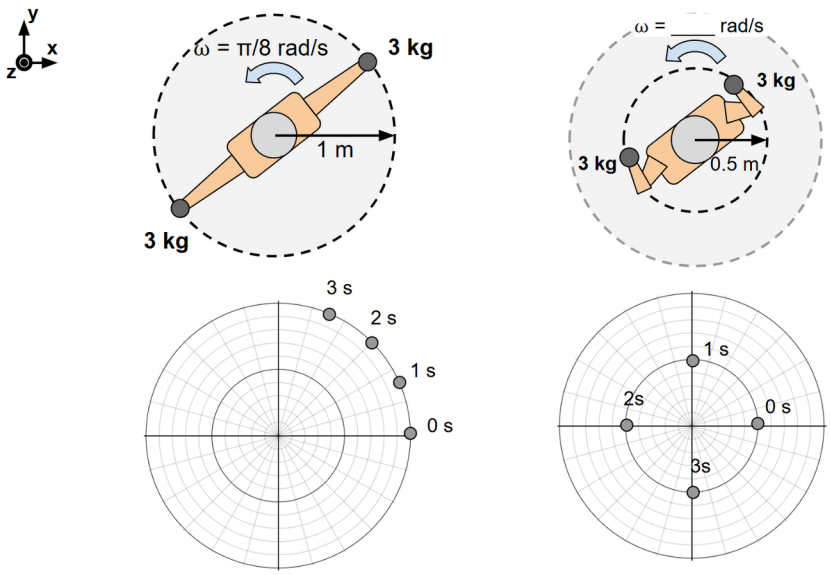
which means angular momentum is conserved.

Angular momentum depends on the chosen origin, pivot point, or rotation axis.

1.2 Angular Momentum Variables and Models

Model

Below is a top-down view of a person who spins on a platform while holding out two weights. The radius changes because the person brings their arms in to half the previous radius. We can consider this a closed system.



Explore

1. Mark up the bottom diagrams (L for Left and R for Right):
   - Draw vectors $\mathbf{r}_0$ and $\mathbf{r}_1$.
   - Label arc lengths Δs_L and Δs_R.
   - Label angle changes $\Delta\theta_L$ and $\Delta\theta_R$.
2. Compare the following quantities.

Quantity	**Left Image (L)**	**Right Image (R)**
Radius (r)		
Angular Speed $\left(\omega = \frac{\Delta\theta}{\Delta t}\right)$		
Linear Speed $\left(v = \frac{\Delta s}{\Delta t}\right)$		

Critical Thinking & Concept Invention

1. Is there an external interaction with the person + weight system? Should angular momentum change?
2. Was either ω or $\mathbf{v}$ conserved? Could $\mathbf{L}$ be based solely on either quantity?
3. Given:
   - $\Delta s = r\Delta\theta$
   - $\Delta A = \frac{1}{2} r\Delta s$

   Determine:

	L	**R**
Arc Length (Δs)		
Area Rate $\left(\frac{\Delta A}{\Delta t}\right)$		

4. What quantity might be the best model for angular momentum?

Summary

Angular motion that best represents angular momentum is related to area swept out per unit time. Like linear momentum ($\mathbf{p}$), angular momentum also depends on mass.

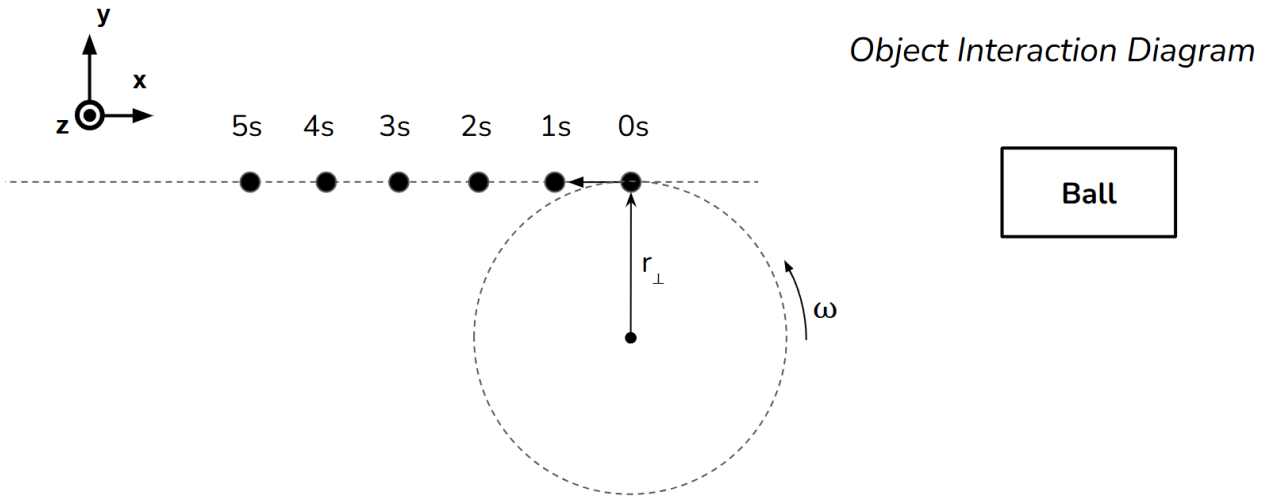
Initial model:

$L =$

1.3 Quantifying Angular Momentum: Moment Arm

Model

This phenomenon involves a ball that was attached to the end of a rope that is initially rotating with angular speed ω. Assume the rope is massless, and therefore cannot have linear or angular momentum. At time t = 0s, the ball is released. Assume there are no external interactions, even in the process of releasing the ball from the rope.



Explore

1. Mark up the diagram
    1. Draw a vector from the origin (center of the circle) to the dots at each second. Label those vectors $\mathbf{r}_1$, (\vec{r}_2\), (\vec{r}_3\), etc.
    2. Highlight the displacement between subsequent dots. Since the time-intervals are one second long, those displacements are also scaled velocity vectors. Draw a vector between dots and label them (\vec{r}_0\), (\vec{r}_1\), (\vec{r}_2\), etc with the subscript matching the initial dot in the pair.
    3. Draw a curve near the origin between each position vector to show the change in angle for each wedge. Label them $\Delta\theta_0 1$, $\Delta\theta_0 2$, $\Delta\theta_0 3$, etc.
2. Identify if each quantity increases, decreases, or remains constant.

r	(4.1.1)

Critical Thinking & Concept Invention

1. Make a claim and provide evidence an reasoning for the following questions:
    1. Should Linear Momentum $\mathbf{p}$ be conserved?
    2. Should Angular Momentum $\mathbf{L}$ be conserved?
2. Construct an equation for wedge area ΔA. Recall the area of a triangle is $A = \frac{1}{2} base \times height$. Consider:
    1. What is the "base" of each triangle?
    2. What is the "height" of each triangle?
3. We said before that $L = 2m\frac{\Delta A}{\Delta t}$. Construct an equation for L in terms of p. Recall that \(v=\frac{\Delta s {\Delta t}\) and $p = mv$

$$L = \tag{4.1.5}$$

Summary

For circular motion:

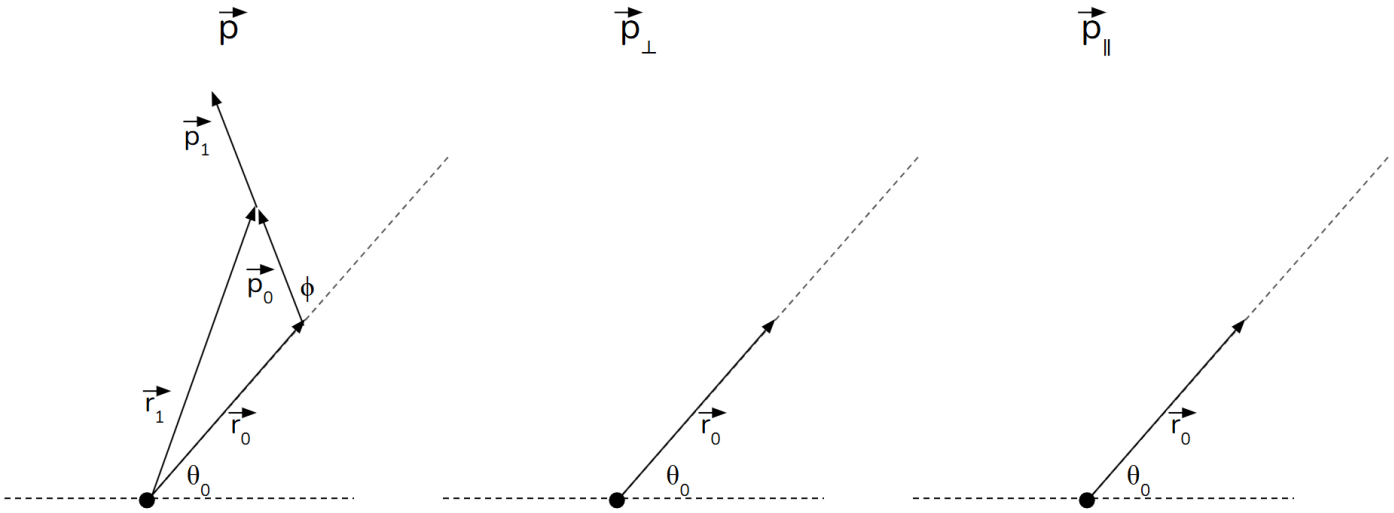
$L = r_{\perp} p$

or

$L = rp \sin\theta$

where θ is the angle between $\mathbf{r}$ and $\mathbf{p}$. Different choices of origin result in different $/vecr$ vectors, so the origin choice affects the rest of the models. Any choice is fine, but some are easier or more useful than others.

1.4 Quantifying Angular Momentum: Parallel and Perpendicular Components



Explore

1. Mark up the model under $\mathbf{p}$ with the following:
   - Draw an arc to show the change in angle $\Delta\theta$ between $\mathbf{r}_1$ and $\mathbf{r}_2$.
   - Turn the vector $\mathbf{p}_0$ into a right triangle where one side is along the dashed extension of $\mathbf{r}_0$ and the other is perpendicular to that line.
   - Label the side of the momentum triangle $\mathbf{p}_{||}$ and $\mathbf{p}_{\perp}$.
2. Mark up the model under $\mathbf{p}_{\perp}$ with the following:
   - Copy your $\mathbf{p}_{\perp}$ vector onto the end of $\mathbf{r}_0$ on the $\mathbf{p}_{\perp}$ model.
   - Draw a position vector to the end of the momentum vector and label it $\mathbf{r}_1$
   - Draw an arc to show the change in angle $\Delta\theta$ between $\mathbf{r}_1$ and $\mathbf{r}_2$ or write $\Delta\theta = 0$.
3. Mark up the model under $\mathbf{p}_{||}$ with the following:
   - Copy your $\mathbf{p}_{||}$ vector onto the end of $\mathbf{r}_0$ on the $\mathbf{p}_{||}$ model.
   - Draw a position vector to the end of the momentum vector and label it $\mathbf{r}_1$
   - Draw an arc to show the change in angle $\Delta\theta$ between $\mathbf{r}_1$ and $\mathbf{r}_2$ or write $\Delta\theta = 0$.

Critical Thinking & Concept Invention

1. Which component of momentum ($\mathbf{p}_{\perp}$ or $\mathbf{p}_{||}$) seems to be most related to **angular motion**?

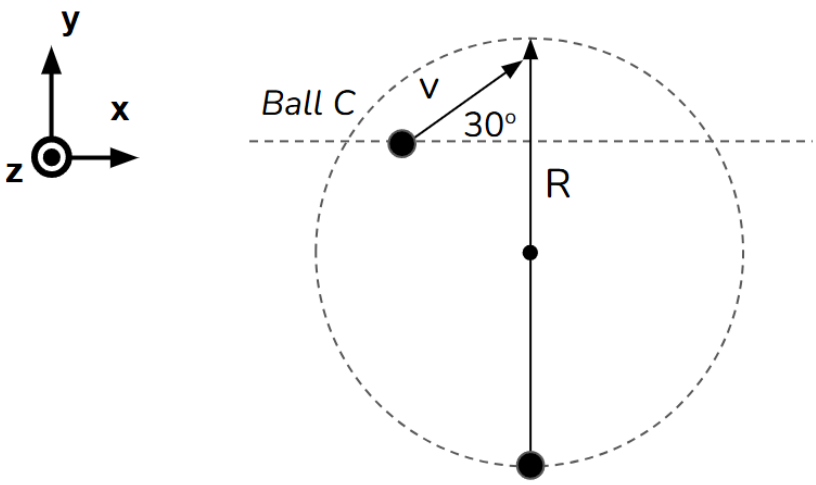
2. Construct equations related each component to $\mathbf{p}$, $\mathbf{r}$, and ϕ.

$$p_{||} =$$
$$p_{\perp} =$$

3. You can turn the equations above into vectors by identify a unit vector for the direction. Add either $\hat{r}$ or $\hat{\theta}$ to the end of the equations above.

Apply

Ball C has mass M and moves towards a massless rod with radius R from a pivot point that it is free to rotate around. Identify the angular momentum of Ball C with respect to this pivot point (in terms of variables given). *Hint: it might help to redraw the figure for the moment that Ball C is at the end of the rod, extend the lines in the direction of* $\mathbf{v}$ *and* $\mathbf{r}$*, and carefully examine all angles.*



Summary

Angular motion is based on the perpendicular components of the position and motion vectors. Any two vectors can be organized into parallel and perpendicular components. **Angular Momentum** is the **perpendicular product** of position and linear momentum:

$$L = r_{\perp}\ p \qquad L = r\ p_{\perp} \qquad L = r\ p\ \sin\phi$$

The angle ϕ is the angle between the vectors $\mathbf{r}$ and $\mathbf{p}$. This **perpendicular product** of vectors is called the **vector cross product**:

$$\mathbf{L} \equiv \mathbf{r} \times \mathbf{p}$$

Angular momentum is defined based on the vector cross product, which will provide a vector from the calculation. Alternatively, you can find the magnitude of the angular momentum vector using the perpendicular components of the vector, then determine the direction of the angular momentum vector is based on the **Right Hand Rule (RHR)**:

Image needed here with Alt Text:
Right-hand rule diagrams showing angular momentum direction from cross product.

Note: These relationships that we have developed so far apply to objects that are modeled as **point particles**. We will learn how to determine the angular momentum of **extended objects** in the next lesson.

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4.2: Rotating Rigid Bodies

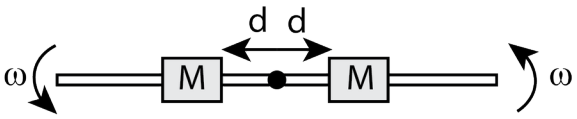
4.2 Rotating Rigid Bodies

4.2.1 Masses on a Rod

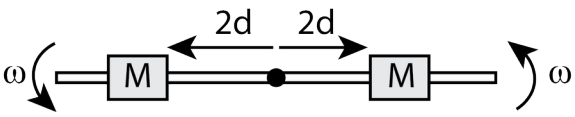
Model

Four massless rods rotate about their center. Each rod has two masses that are fixed at different distances from the center (i.e., from the pivot point).

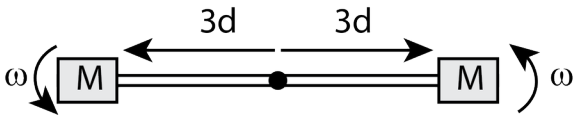
Rod A



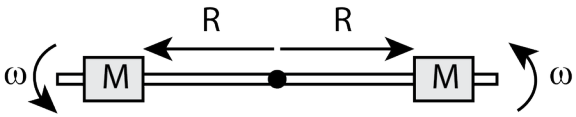
Rod B



Rod C



Rod D



Explore

1. What properties/variables are the same for each rod?
2. What properties/variables are different for each rod?
3. Determine the magnitude of the linear momentum of each mass in terms of the variables given. Draw arrows on the diagrams to show these momentum vectors and scale them to show differences.

Critical Thinking & Concept Invention

1. Determine the angular momentum of each rod in terms of the variables given. Circle the part of each equation that is different. What physical characteristic caused this difference?

   Note: Angular momentum of the rod is the vector sum of the angular momenta of each mass on the rod.
2. Which rod would be more difficult (i.e., require more twirl) to bring the rotation to a stop?
3. How is it possible that three rods with the same mass, rotating at the same rate, can have different angular momenta?

Summary

We previously identified the angular momentum of point particles using:

$\mathbf{L} = \mathbf{r} \times \mathbf{p}$

Extended objects that rotate without changing shape are called **rigid bodies**. For rigid bodies, it is useful to determine angular momentum based on the angular velocity of the object as a whole rather than calculating the angular momentum of each individual piece.

The angular momentum of a rigid body is:

$\mathbf{L} = I\omega$

The quantity I is called the **Moment of Inertia**. It depends on the distribution of mass and the distance of that mass from the rotation axis.

For symmetric objects rotating about a symmetry axis, $\mathbf{L}$ and ω point in the same direction. More complex objects may have angular momentum that is not aligned with angular velocity.

For a point mass:

$I = MR^2$

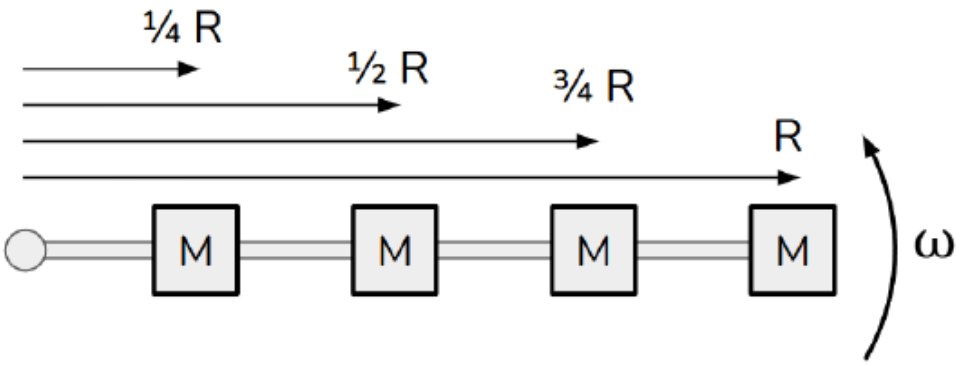
where R is the magnitude of the position vector measured from the rotation axis.

Extended objects have similar expressions with geometric scaling factors.

4.2.2 Adding Up Bits of a Rod

Model

A massless rod of length R_{rod} rotates around its left-most end. The rod has four equal masses fixed along the rod at different distances from the pivot point.



Explore

1. Which angular momentum variables (mass, position, angular velocity) are the same for each mass on the rod?
2. Which angular momentum variables (mass, position, angular velocity) are different for each mass on the rod?
3. What is the total mass of this object?

$$M_{total} = \tag{4.2.1}$$

Critical Thinking & Concept Invention

1. Without doing a calculation, which mass do you think has the least angular momentum and the greatest angular momentum? Explain your reasoning.
2. Without calculating the angular momentum of each mass, what do you expect is the ratio of angular momentum between the two masses identified in the previous question?
3. Apply mathematics to the angular momentum of this object:
   a. Construct an equation for the angular momentum of a single mass. Use the variable $\mathbf{r}$ as a position variable that can apply to any mass.
   b. Calculate the angular momentum of each mass. Do your calculations support your predictions?
   c. Develop an equation for the total angular momentum of the rod and its masses using your previous results. Do not solve the equation yet.
   d. Recall: $\mathbf{L} = I\omega$. Use your previous result to identify an equation for I for this rod and mass system.
   e. Rewrite your equation in terms of total mass M_{total}.

Summary

The moment of inertia of a multiple-mass or extended object is found by adding the moment of inertia of each piece:

$I_{total} = \sum_i M_i R_i^2$

Exact results for common symmetric objects are introduced in the next section. Methods for calculating moments of inertia of continuous bodies will be explored later.

4.2.3 Common Moments of Inertia for Continuous Bodies

Model

Below are common moments of inertia for different objects about the axes shown. The original graphic is from OpenStax University Physics, Chapter 10.4.

Image needed here with Alt Text:
Reference chart showing common rigid-body shapes and their moments of inertia.

Explore

1. Which objects have most of their mass on the outside, away from the pivot point?
2. Which objects have mass both near and away from the pivot point?
3. Examine the moment of inertia equations. Highlight or circle the part that is the same in all equations. Assume $L = R$.
4. List the objects in order from least to greatest moment of inertia. Circle the object identified in Question 1 and box the objects identified in Question 2.

Critical Thinking & Concept Invention

1. How does the location or distribution of mass affect moment of inertia? Use evidence from the previous questions and provide reasoning.
2. Which would have more angular momentum:
   - A wheel with most of its mass concentrated on the rim and light spokes
   - A solid disk wheel of uniform density

   Assume both have the same total mass. Explain your reasoning using equations and patterns identified previously.

Apply

A 70 kg astronaut is motionless in space near the International Space Station. Their partner tosses a 1.5 kg wrench toward them at a speed of 4 m/s. The astronaut catches the wrench 0.4 m to the left of the body's centerline.

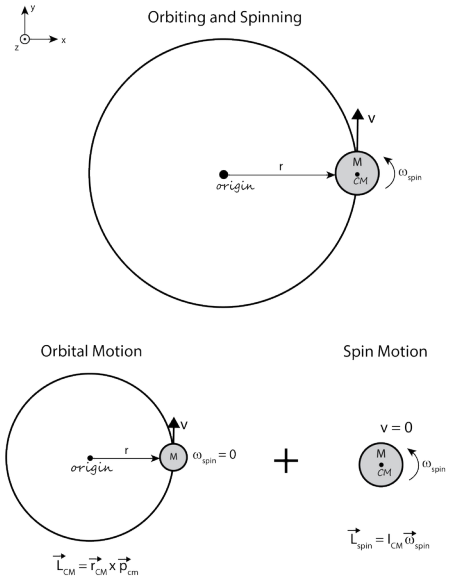
Choose an appropriate shape to estimate the astronaut's moment of inertia and clearly state the assumption.

- Create a linear momentum chart and an angular momentum chart.
- Construct necessary vector equations from both charts.
- Substitute known and assumed values.
- Determine the final linear momentum and final angular momentum of both the astronaut and wrench.

4.2.4 Orbit and Spin

A planet moves in a circular orbit around a star located at the origin. The planet also rotates about its own center of mass (CM).

Model



Explore

1. The motion is separated into how many types of motion? What are their names?
2. For each type of motion:
   - Is linear momentum $\mathbf{p}$ zero or nonzero?
   - Is spin angular velocity ω_{spin} zero or nonzero?
3. For each type of motion, identify whether the rotational reference point is the Origin or the Center of Mass.
4. Use the Right-Hand Rule to determine the direction of angular momentum for each type of motion. Represent directions with vectors or into/out-of-page symbols.

Critical Thinking & Concept Invention

1. In orbital motion, is the planet treated as a point particle or an extended rigid body? How is this shown in the equation for $\mathbf{L}$?
2. In spin motion, is the planet treated as a point particle or an extended rigid body? How is this shown in the equation for $\mathbf{L}$?
3. Do the angular momentum vectors point in the same direction or different directions?
4. How might the total angular momentum be determined for a planet that both orbits and spins? Provide an explanation and propose an equation.

Summary

When an object's center of mass moves with angular momentum about a chosen origin and the object also spins about its center of mass, the total angular momentum is:

$$\mathbf{L}_{object} = \mathbf{L}_{CM} + \mathbf{L}_{spin}$$

$$\mathbf{L}_{object} = \mathbf{r} \times \mathbf{p}_{CM} + I_{CM}\omega_{spin}$$

For $\mathbf{L}_{CM}$, the entire object may be treated as a point particle:

$I = MR_{CM}^2$

For $\mathbf{L}_{spin}$, use the object's moment of inertia about its center of mass:

$I = I_{CM}$

Apply

Your teacher sits motionless on a freely rotating chair while holding a spinning bicycle wheel with arms extended. The teacher then flips the wheel.

Construct diagrams and equations that model the process and explain resulting changes in motion.

- Construct physical scenario diagrams.
- Construct angular and/or linear momentum charts.
- Develop relevant vector equations.
- List assumptions and solve as precisely as possible.

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4.3: Torque and Twirl

4.3 Torque and Twirl

The Angular Momentum Principle is one of the three key dynamics principles used in the study of Mechanics. This principle is represented by the equation:

$$\Delta \mathbf{L}_{\text{system}} = \sum \mathbf{J}_L \tag{4.3.1}$$

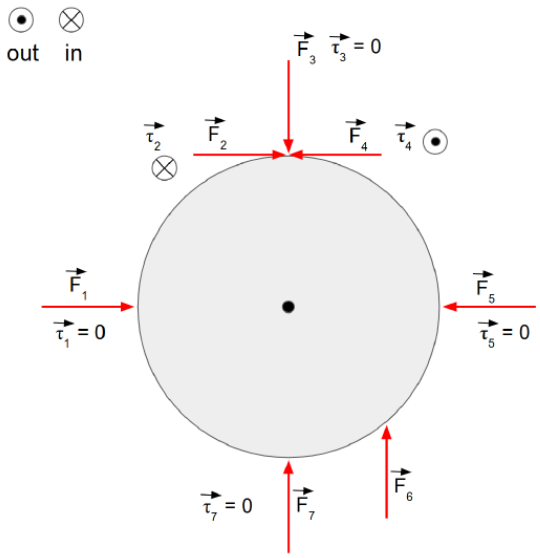
"The total change in angular momentum of a system" = "The net angular impulse acting on the system"

The choice of a system is arbitrary in that any choice can be made and the physics still works. The physics equations will be different (but still accurate) for different choices of system, so it is necessary to identify your chosen system in every problem. We will now begin to explore open angular momentum systems where $\sum \mathbf{J}_L \neq 0$ and angular momentum is not conserved.

Angular Momentum (and other angular quantities) are different from energy and linear momentum in that they depend on the chosen origin, pivot point, or rotation axis.

4.3.1 Torque

Model



Explore

1. Fill in the force/torque story below:

There are _____ forces acting on the wheel. _____ forces do not apply a torque and _____ forces do apply a torque. Of the forces

that apply torque, _____ apply it in a clockwise direction $(-\hat{z})$ and _____ apply torque in the counterclockwise $(+\hat{z})$ direction.

2. Does every torque come from a non-zero force?
3. Examine the force-torque pairs. Does the torque point in the same direction as the force? If not, how does the direction of torque compare to force?

Critical Thinking & Concept Invention

1. In what circumstances does a Force result in a Torque? Pay attention to the direction of those Forces.
2. In what circumstances does a Force not result in a Torque? Pay attention to the direction of those Forces.
3. There is one Force without a torque label. Does it cause a torque? If so, in what direction? Add an appropriate label to the diagram.

Summary

Torque can be thought of as a Rotational Force that can cause an object to change its angular momentum. Torque is given by the equation:

$$\tau \equiv \mathbf{r} \times \mathbf{F} \tag{4.3.2}$$

The magnitude of this expression can be represented as:

$$\tau = r_{\perp} F \tag{4.3.3}$$

or

$$\tau = r F_{\perp} \tag{4.3.4}$$

or

$$\tau = r F \sin(\theta) \tag{4.3.5}$$

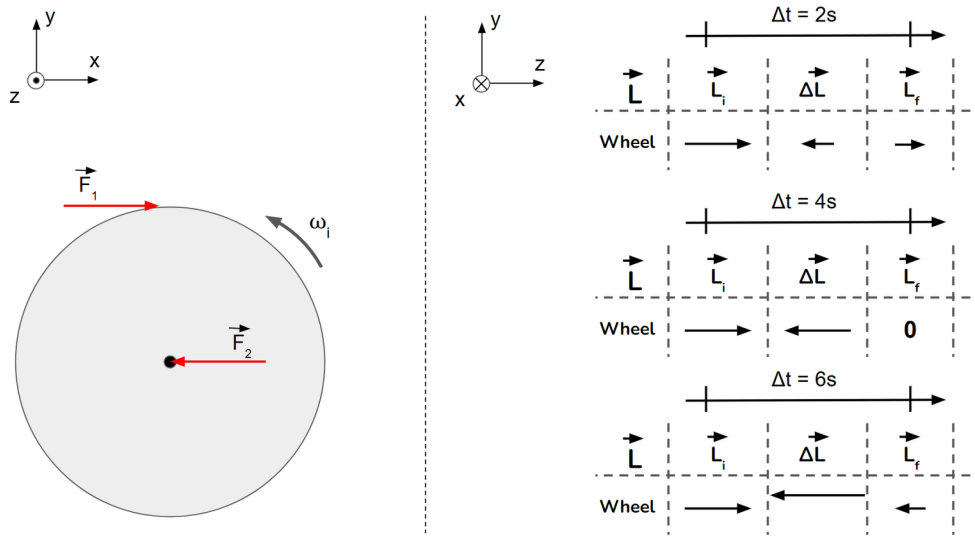
Forces that point in a direction that lines up with (or directly opposite of) the pivot point or rotational axis do not result in a torque. Forces that have a tangential component with respect to the pivot point cause a torque.

The best way to determine this is to draw an extension of the force vector forwards and backwards. If that extension goes through the pivot point, then the force does not cause a torque.

Extended Force Diagrams draw the objects as extended bodies and include force vectors at their point of contact with the object. The force of gravity is drawn from the center of mass of the object.

4.3.2 Angular Impulse and Angular Momentum

Model



Explore

1. How many Forces act on the object? What direction do those forces point?
2. How many Torques act on the object? What direction do those torques point?
3. Is the initial angular velocity of the wheel clockwise or counterclockwise? Use the Right Hand Rule. What direction ($+/-x$, $+/-y$, $+/-z$) is the initial angular velocity vector?
4. Based on your response above, what direction ($+/-x$, $+/-y$, $+/-z$) is the initial angular momentum vector?
5. Compare the direction of

$$\tau \tag{4.3.6}$$

and

$$\mathbf{L}_i \tag{4.3.7}$$

.

Critical Thinking & Concept Invention

1. Is

$$\tau \tag{4.3.8}$$

most related to

$$\mathbf{L}_i \tag{4.3.9}$$

,

$$\Delta\mathbf{L} \tag{4.3.10}$$

, or

$$\mathbf{L}_f \tag{4.3.11}$$

? Hint: Compare the direction of the different vectors.

2. What is different between the angular momentum charts?
3. Construct an equation that relates

$$\Delta\mathbf{L} \tag{4.3.12}$$

and

$$\tau \tag{4.3.13}$$

.

Summary

Torque

$$(\tau) \tag{4.3.14}$$

is the rate that angular momentum

$$(\mathbf{L}) \tag{4.3.15}$$

changes:

$$\tau = \frac{\Delta\mathbf{L}}{\Delta t} \tag{4.3.16}$$

Angular Impulse

$$(\mathbf{J}_L) \tag{4.3.17}$$

is the total amount that the angular momentum of an object changes over a time interval.

Angular Impulse is sometimes called **Twirl** and has a similar equation structure to the Impulse-Momentum Theorem:

$$\Delta\mathbf{L}_{\text{system}} = \sum \mathbf{J}_L \tag{4.3.18}$$

The Angular Impulse on the system depends on both the torque and the time interval over which that torque is applied.

This is represented mathematically as:

$$\mathbf{J}_L = \tau_{\text{avg}}\Delta t \tag{4.3.19}$$

4.3.3 Net Torque

Model

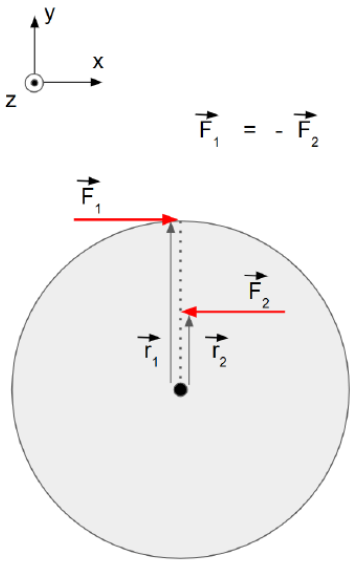
A solid disk of mass M and radius R initially begins at rest. Two forces act on the wheel as shown, one at the edge of the wheel and one halfway between the center of the wheel and the edge.

$$\mathbf{F}_1 \tag{4.3.20}$$

has a magnitude

$$F \tag{4.3.21}$$

.



Explore

1. Compare the following for

   $$\mathbf{F}_1 \tag{4.3.22}$$

   and

   $$\mathbf{F}_2 \tag{4.3.23}$$

   :
   - Magnitude
   - Direction
2. Compare the following for

   $$\mathbf{r}_1 \tag{4.3.24}$$

   and

   $$\mathbf{r}_2 \tag{4.3.25}$$

   :
   - Magnitude
   - Direction
3. For each force, what is the angle between

   $$\mathbf{r} \tag{4.3.26}$$

   and

   $$\mathbf{F} \tag{4.3.27}$$

   ?

Critical Thinking & Concept Invention

1. Assume these forces are constant and are applied for 5 seconds. What will
$$\Delta \mathbf{p}_{\text{wheel}} \tag{4.3.28}$$
be after those 5 seconds? Explain your reasoning.
2. Compare the following for each torque:
   - Magnitude
   - Direction
3. Identify the torque from each Force as a column vector, then add them up to find the net torque. Write in the appropriate mathematical operations $(+, -, \times, \div, =)$.

τ_1 (4.3.29)

4. Assume the forces are constant and applied for 5 seconds. Find
$$\Delta \mathbf{L}_{\text{wheel}} \tag{4.3.32}$$
. Report it as a column vector.
5. Why is there a change in angular momentum but not a change in linear momentum when both relate to the external forces that act on the wheel?

Summary

Where there is more than one torque acting on an object, the change in angular momentum is determined by the net angular impulse (Twirl):

$$\Delta \mathbf{L} = \sum \mathbf{J}_L \tag{4.3.33}$$

The net angular impulse is a sum of the angular impulse from each torque:

$$\sum \mathbf{J}_L = \sum_i \tau_{i,\text{avg}} \Delta t_i \tag{4.3.34}$$

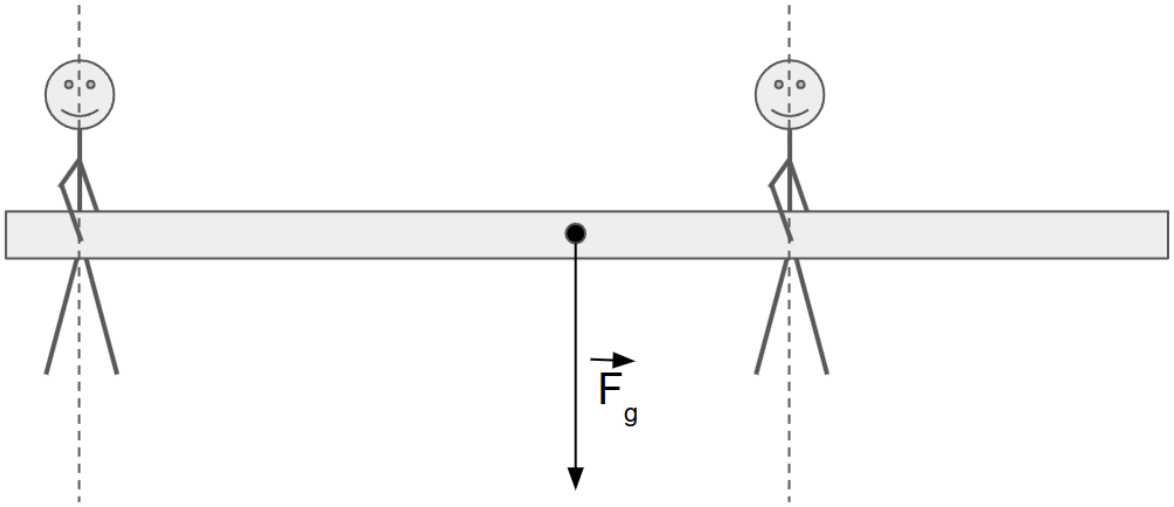
If the torques are all applied over the same time interval, you can take a single integral of the net torque:

$$\sum \mathbf{J}_L = \sum \tau_{\text{avg}} \Delta t \tag{4.3.35}$$

Apply

1. Two construction workers carry a heavy beam of mass M. The forces they apply will align with the dashed line at each worker (so either up or down). The dot on the beam is the middle of the beam. The beam does not change its linear motion
$$(\Delta \mathbf{p} = 0) \tag{4.3.36}$$
or its rotational motion
$$(\Delta \mathbf{L} = 0) \tag{4.3.37}$$
. Your instructor will assign you a specific point to act as the pivot point.

- Draw force vectors from each person and be sure they are consistent with the description of motion.
- Draw the moment arm vectors to each force.
- Determine the direction of the torque from each force and write an equation for the magnitude of each torque.
- Be sure they are consistent with the description of motion.



2. One construction worker carries a heavy beam of mass M that is also held up by a cable at the angle shown. The forces the worker and cable apply will align with the dashed lines at each object. The dot on the beam is the middle of the beam. The beam does not change its linear motion

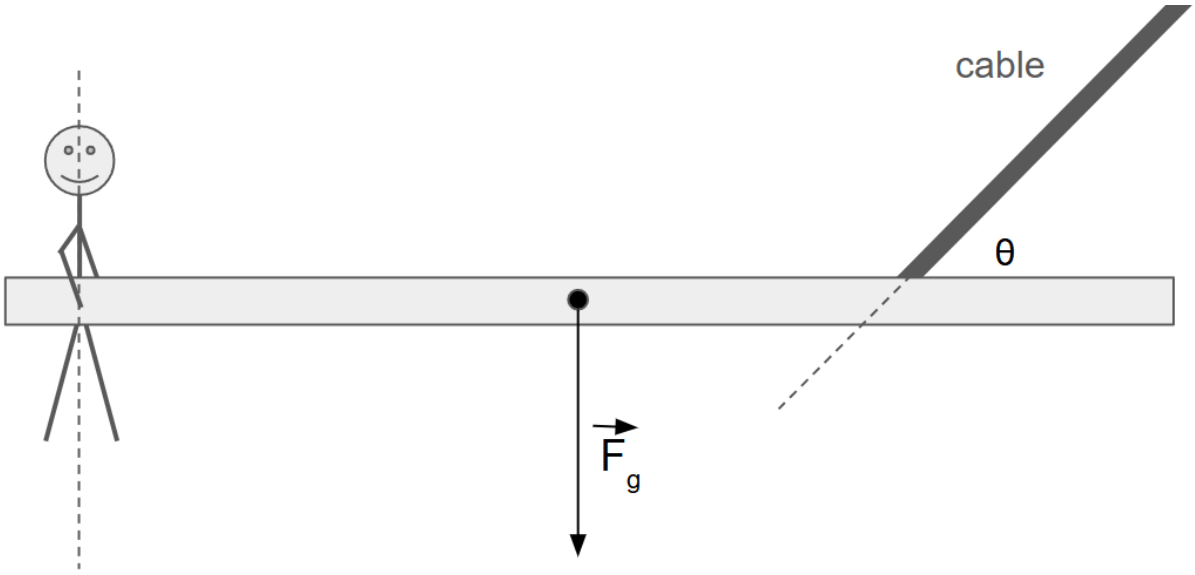
$$(\Delta \mathbf{p} = 0) \tag{4.3.38}$$

or its rotational motion

$$(\Delta \mathbf{L} = 0) \tag{4.3.39}$$

. Your instructor will assign you a specific point to act as the pivot point.

- Draw force vectors from the person and cable.
- Be sure they are consistent with the description of motion.
- Draw the moment arm vectors to each force.
- Determine the direction of the torque from each force and write an equation for the magnitude of each torque.
- Be sure they are consistent with the description of motion.



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4.4: Twirl and Angular Work

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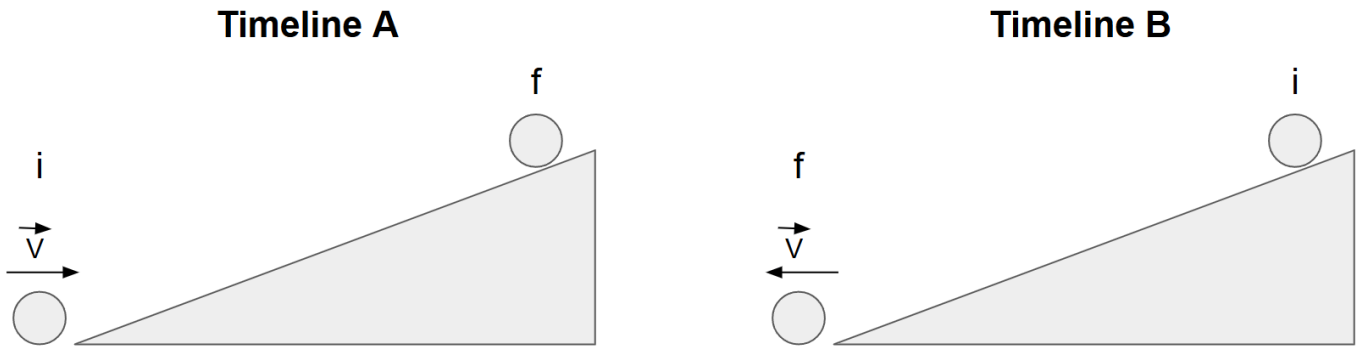
4.5: Open Angular Momentum Systems

4.5 Open Angular Momentum Systems

4.5.1 Torque and Angular Momentum of a Wheel on a Ramp

Model

A wheel is moving and rolling without slipping at the bottom of a ramp. It then rolls up the ramp and slows to a stop. Then, the wheel rolls without slipping back down the ramp.



1. Before analyzing the system in depth, make a prediction about the direction of the frictional Force that acts on the wheel as it goes up the ramp and as it goes down the ramp. Draw Force vectors on the model and explain your thinking here.

Explore

1. On each timeline, identify when (initial / final) the speed and angular speed are zero. Write

$$v = 0 \tag{4.5.1}$$

and

$$\omega = 0 \tag{4.5.2}$$

on the timelines with the appropriate subscript (i or f).
2. On Timeline A (rolling up the ramp):
   - Does the linear speed (v) increase or decrease?
   - Does the angular speed (ω) increase or decrease?
3. On Timeline B (rolling down the ramp):
   - Does the linear speed (v) increase or decrease?
   - Does the angular speed (ω) increase or decrease?

Critical Thinking & Concept Invention

1. Use what you said about speed increasing or decreasing to determine the direction (up or down the ramp) of

$$\Delta \mathbf{p} \tag{4.5.3}$$

for the wheel on each timeline. Use that to identify the direction (up or down the ramp) the net Force

$$\left(\sum \mathbf{F}\right) \tag{4.5.4}$$

must be.

Timeline	$\Delta \mathbf{p}$	$\sum \mathbf{F}$
Timeline A		
Timeline B		

2. Use the right-hand rule to determine the direction $(+/-x, +/-y, +/-z)$ of

$$\omega \tag{4.5.5}$$

when the wheel is moving on each ramp.

Timeline	ω
Timeline A	
Timeline B	

3. Use the right-hand rule and what you said about angular speed increasing or decreasing to determine the direction $(+/-x, +/-y, +/-z)$ of

$$\Delta \mathbf{L} \tag{4.5.6}$$

for each timeline. Use that to identify the direction $(+/-x, +/-y, +/-z)$ the torque

$$\tau \tag{4.5.7}$$

must be to cause that

$$\Delta \mathbf{L}. \tag{4.5.8}$$

Timeline	$\Delta \mathbf{L}$	τ
Timeline A		
Timeline B		

4. Which direction (up or down the ramp) does the Force of friction need to be to create a torque in the directions you just identified?

Timeline	**Direction of Friction**
Timeline A	
Timeline B	

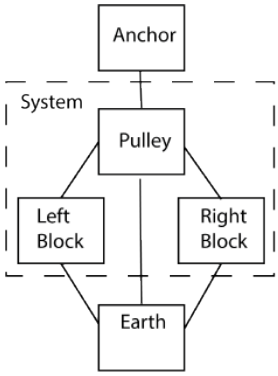
5. Check the forces you drew on the wheel. Do their directions make sense? Explain your thinking and redraw the forces if needed.
6. Re-check what you said about net force

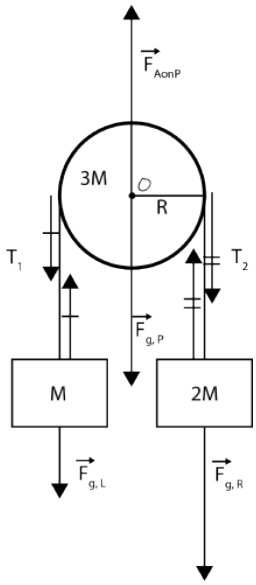
$$\left(\sum \mathbf{F}\right). \tag{4.5.9}$$

Do there need to be additional forces, other than friction acting on the wheel? If so, draw them so that the net Force points in the direction it needs to. Explain your changes and reasoning here.

4.5.2 Torque and Twirl on Systems

Model





Explore

1. Mark up the diagram with the forces:
   - Extend each force forwards and backwards as a dashed line.
   - For non-zero external torques, write a

     $$\tau \tag{4.5.10}$$

     with an appropriate label next to the force. Include a symbol to show the direction (for example, into or out of the page).
2. Tell a system story for the indicated system:
   - There are _____ objects inside the system boundary.
   - This system includes _____.
   - It is an (open / closed) system.
   - There is/are _____ internal interaction(s) and _____ external interaction(s).
   - Of the external interactions, _____ apply a non-zero torque.

Critical Thinking & Concept Invention

1. Compare the internal torques acting on the pulley. Which direction (clockwise or counterclockwise) will the pulley change its motion?
2. Set up a column vector equation for the net external torque acting on the system. Solve it in terms of constants and variables

in this scenario.

3. How many objects in the system might experience a change in angular momentum? Write an equation for the angular momentum of the system:

$$\Delta \mathbf{L}_{\text{system}} = \tag{4.5.11}$$

4. Use the Angular Momentum Principle to model the scenario. Expand both sides of the equation based on your previous responses:

$$\Delta \mathbf{L}_{\text{system}} = \int \sum \tau \, dt \tag{4.5.12}$$

5. Use the relationship between linear and angular speeds

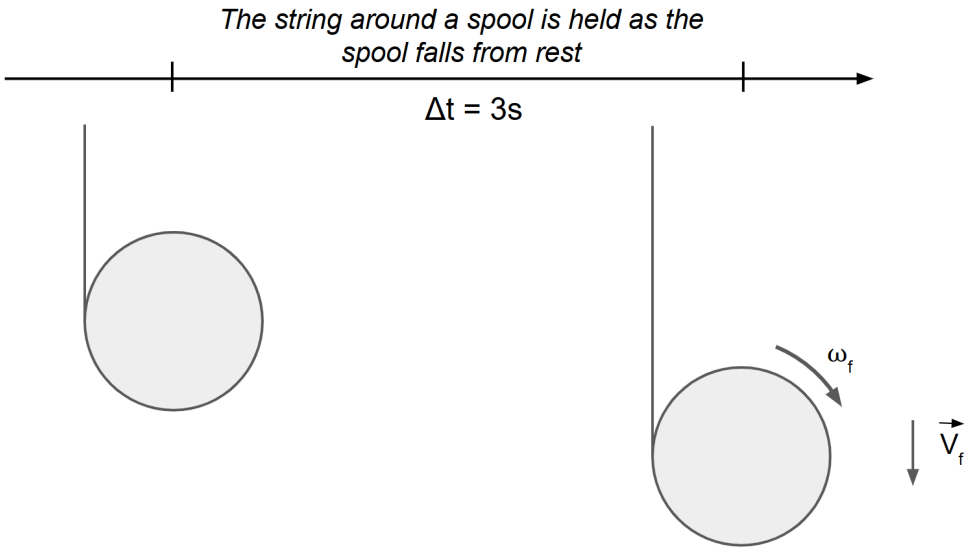
$$v = \omega r \tag{4.5.13}$$

so that you only have v's and not ω's. Solve for the speed of the objects after 3 seconds.

4.5.3 Solving Problems with Multiple Physics Principles

Model

A spool of thread with mass M and radius R can be modeled as a solid cylinder. Find the speed of the spool after it falls for three seconds. You can assume the thread that unwinds has negligible mass and the full mass M is constant for the spool during this time. Analyze just the spool as the system.



Explore

1. Construct an Object Interaction Diagram to the right of the physical scenario diagram and choose just the spool as your system. Circle external interactions as they cross the system boundary.
2. Fill in values (or variables) for speed and angular speed at the start and end of the timeline.
3. Draw a force vector on the physical scenario diagram for any external interaction. Give them relevant labels.

Critical Thinking & Concept Invention

1. Extend each force forwards and backwards as a dashed line. For non-zero external torques, write a

$$\tau \tag{4.5.14}$$

   with an appropriate label next to the force. Include a symbol to show the direction (for example, into or out of the page).
2. Are we set up to solve this system based on momentum (linear and angular) or energy? Explain your thinking.
3. Write a system momentum/energy story for the scenario.
4. Model this storyline and solve for the final speed of the spool:
   - Select the appropriate equations (you will need more than one) to model this story and write that equation in the box.
   - To the right of the box, write a short statement that explains your Problem Solving Approach.
   - Substitute the specifics of this story into the general equation and solve for necessary measurements (for example, speed, time, position) at the end of that phase of the story.
   - Your answer might be a number or an equation with variables.

Equation / Model	Problem Solving Approach

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5: LINEAR AND ANGULAR ACCELERATIONS

5.1: Acceleration Dynamics
5.2: Dynamics of Systems
5.3: Circular Motion
5.4: Angular Acceleration
5.5: Rotational Dynamics
5.6: Power

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5.1: Acceleration Dynamics

Acceleration Dynamics

Recall we learned about **Newton's Three Laws of Motion** in our exploration of Momentum:

(1) In a closed system the momentum of a system does not change:

If $\Sigma \overrightarrow{F}_{ext} = 0$ then $\Delta \overrightarrow{p}_{sys} = 0$ *We can also say* If $\Sigma \overrightarrow{F}_{ext} = 0$ then $\frac{d}{dt} \overrightarrow{p}_{sys} = 0$

(2) In a open system the rate that the momentum of a system changes is equal to the net external force:

$\frac{d}{dt} \overrightarrow{p}_{sys} = \Sigma \overrightarrow{F}_{ext}$

(3) Impulse can be traded between objects. The impulse on one object is equal and opposite to the impulse on the other. Similarly, Forces are an interaction between two objects and each object experiences the same magnitude of force but in an opposite direction

$\overrightarrow{J}_{AB} = -\overrightarrow{J}_{BA}$

$\overrightarrow{F}_{AB} = -\overrightarrow{F}_{BA}$

In this section, we will explore these same laws in terms of accelerations, rather than momenta.

Acceleration

Model

$$(1) \quad \overrightarrow{p}_{sys}(t) = m\ \overrightarrow{v}(t)$$

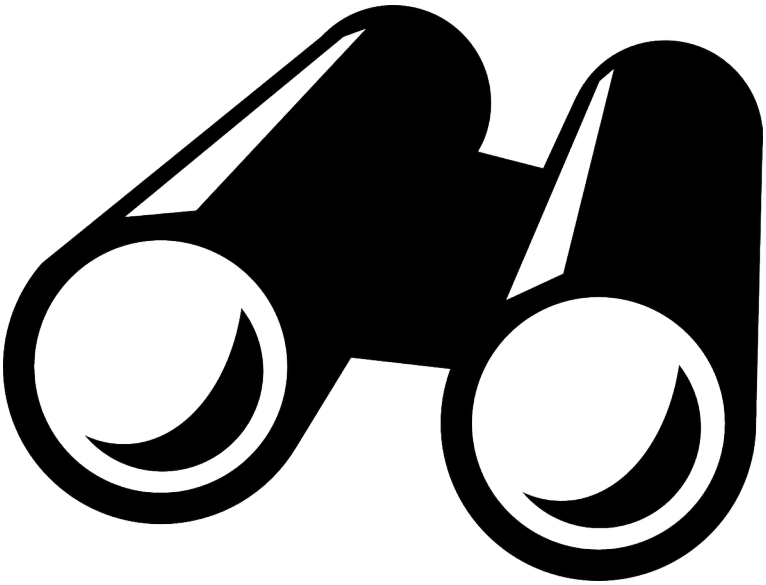
$$(2) \quad \frac{d}{dt} \overrightarrow{p}_{sys}(t) = \frac{d}{dt}[m\ \overrightarrow{v}(t)]$$

If we assume that the mass of the system is constant, then this can be written as...

$$(3) \quad \frac{d}{dt} \overrightarrow{p}_{sys}(t) = m\frac{d}{dt}\ \overrightarrow{v}(t)$$

Which can be rewritten as...

$$(4) \quad = $$



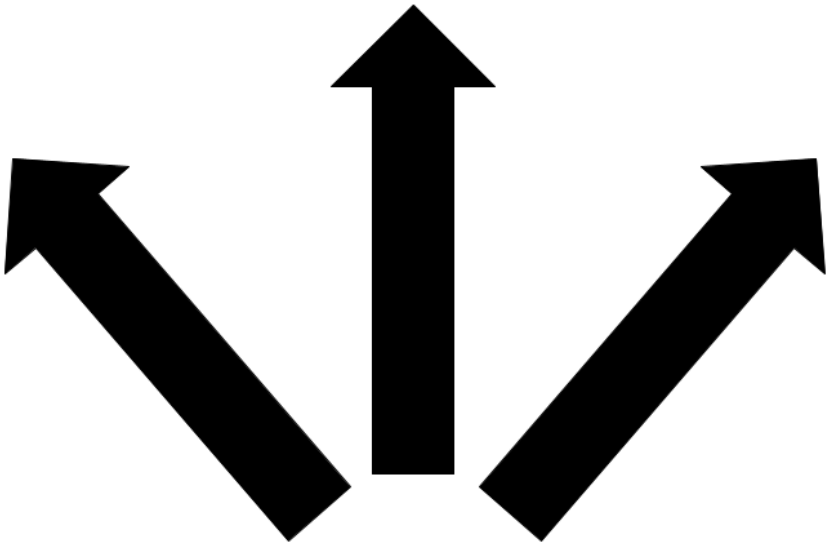
Explore

1. What is different about the first two lines? What did the instructor do to each side of the equation?
2. What happened to mass from line 2 to line 3? Why is that allowed?



Critical Thinking & Concept Invention

1. Annotate each step by describe the action taken and explain why that action is allowed.
2. Examine **Newton's 2nd Law** at the top of the page. What can you write in place of $\frac{d}{dt}\overrightarrow{p}_{sys}(t)$. Right your answer on the **left** side of the equals sign for line 4.
3. What quantity have we learned about previously that is equatl to $\frac{d}{dt}\overrightarrow{v}(t)$? Right your answer on the **right** side of the equals sign for line 4.
4. Copy the final equation here. If this equation **always true**? If not, identify under what circumstances the equation is true. *Hint: Look back over the derivation and search for any assumptions.*

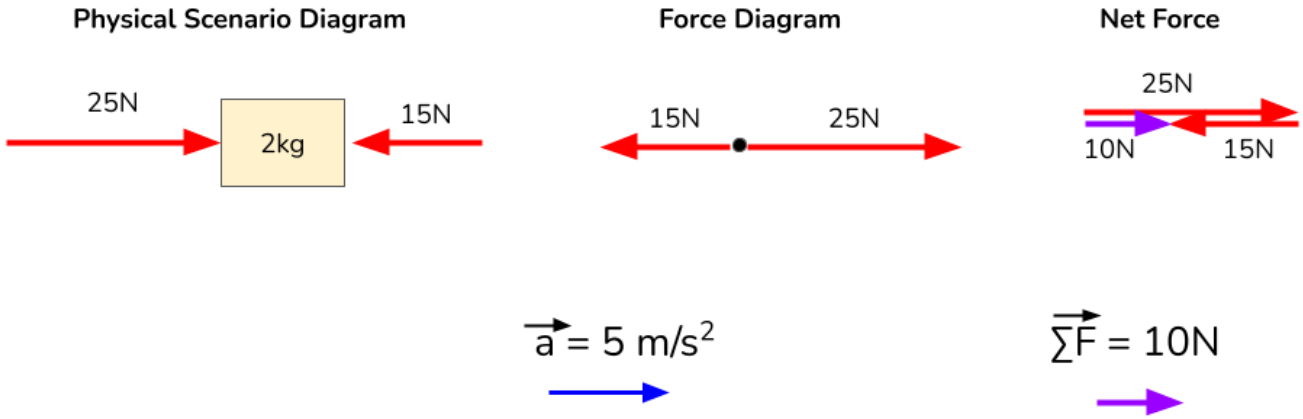


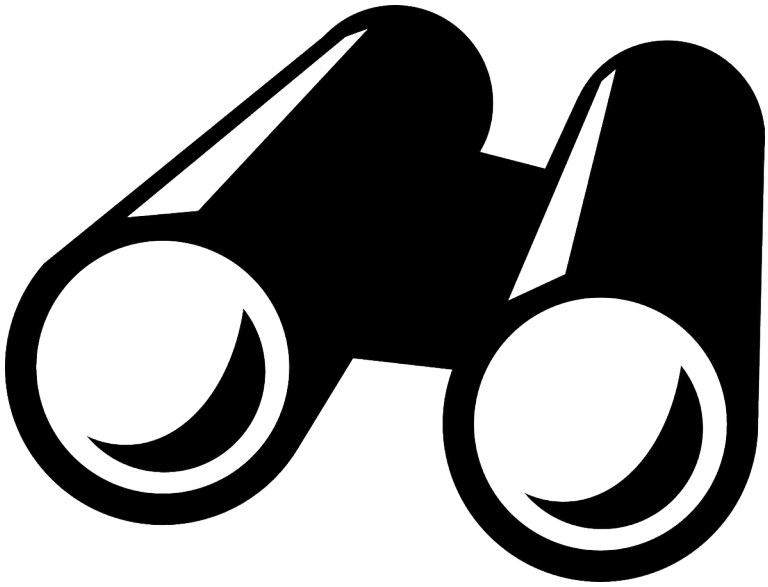
Apply

1. A force with components $F_x = 5N$ and $F_y = 3N$ is applied to a box. Determine the acceleration ($\vec{a}$) of the box if:
    - The mass of the box is 2kg
    - The mass of the box is 6kg
2. A car has an acceleration of $a_x = -2.5\frac{m}{s^2}$ and $a_y = 4.5\frac{m}{s^2}$. Determine the net force ($\Sigma \vec{F}$) of the box if:
    - The mass of the box is 2kg
    - The mass of the box is 6kg

Multiple Forces and Force Diagrams

Model





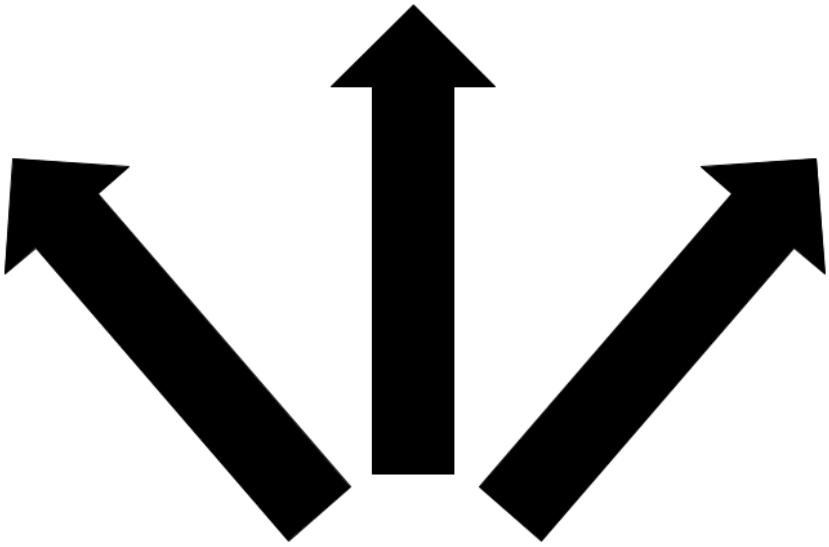
Explore

1. How are force vectors (arrows) arranged on a **Force Diagram**?
2. How are force vectors (arrows) arranged to determine the **net Force**($\Sigma \vec{F}$)?
3. Is the **net Force vector (arrow)**($\Sigma \vec{F}$ drawn on the force diagram?
4. How many forces are applied to the box? List their magnitudes and directions.
5. What is the acceleration ($\vec{a}$) of the box? List the magnitude and direction. $\vec{a}$ = _____ *to the* _____
6. What is the net force ($\Sigma \vec{F}$) on the box? List the magnitude and direction $\Sigma \vec{F}$ = _____ *to the* _____



Critical Thinking & Concept Invention

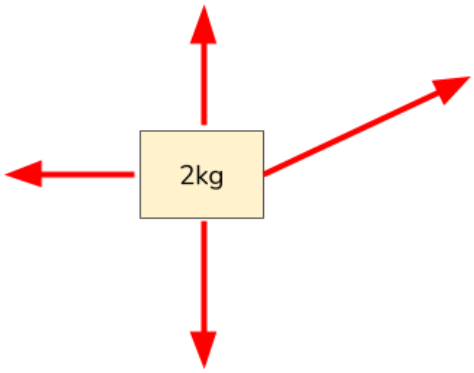
1. How is the net force determined from the other forces?
2. Which force vector is the acceleration based on?



Apply

Make a **Force Diagram** and determine the **Net Force** for this Physical Scenario Diagram by adding the vectors together graphically.

Physical Scenario Diagram | Force Diagram | Net Force



Summary

When there are multiple forces acting on an object, the acceleration is proportional to the **net force**. This gives the equation:

$$\vec{a} = \frac{1}{m}\Sigma \vec{F}_{ext}$$

$$\vec{a} = \frac{1}{m}[\vec{F}_1 + \vec{F}_2 + \vec{F}_3 + ... + \vec{F}_n]$$

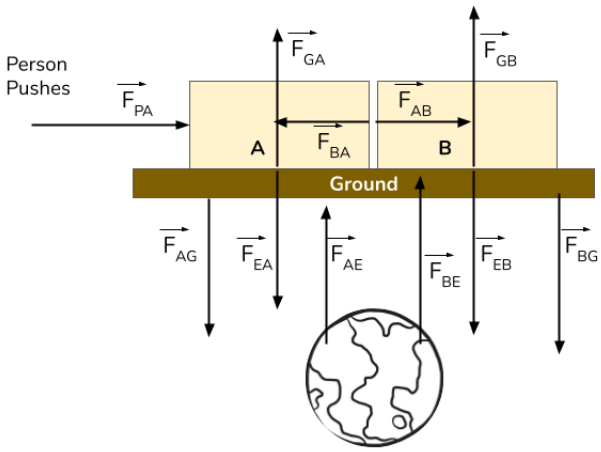
If we know the acceleration and instead want to determine the force, we can rearrange the equation to give us:

$$m\vec{a} = \Sigma \vec{F}_{ext}$$

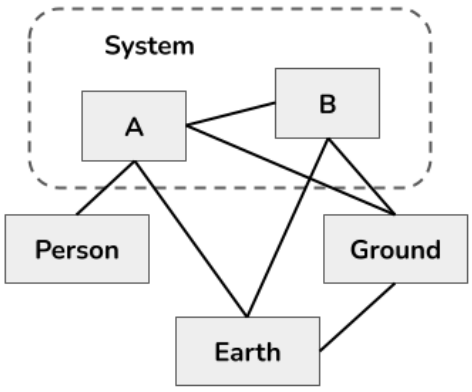
$$m\vec{a} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + ... + \vec{F}_n$$

Model

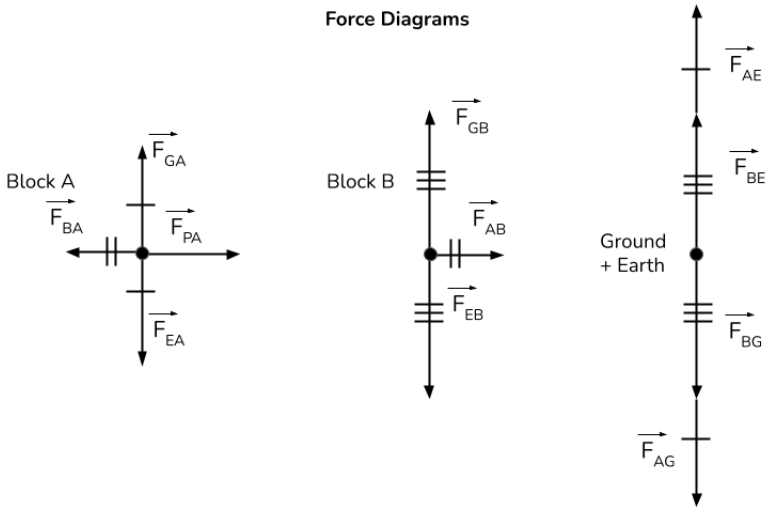
Physical Scenario Diagram

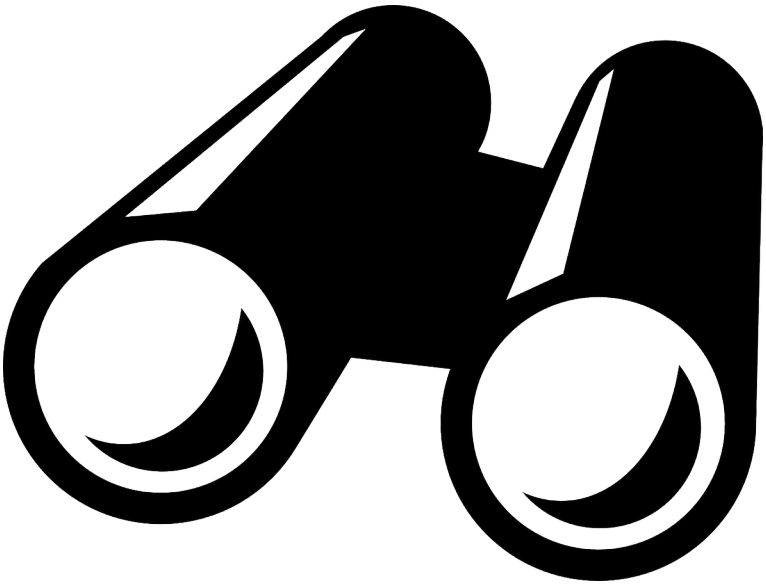


Object Interaction Diagram



Force Diagrams





Explore

1. Identify groups of forces that all have the same magnitude. List them here.
2. Identify forces that have the same letters in the subscripts, but in different order. Draw a dashed line between those forces on the **force diagrams** and label those lines **Third Law Pair**
3. Compare the third law pairs: Do they have the same magnitude? Do they point in the same direction?
4. Circle external interaction lines as they cross the system boundary.
5. How many objects are in the system? List the objects.
6. Examine the force diagrams and make a big dashed box around the force diagrams for the objects that are **a part of the system**.

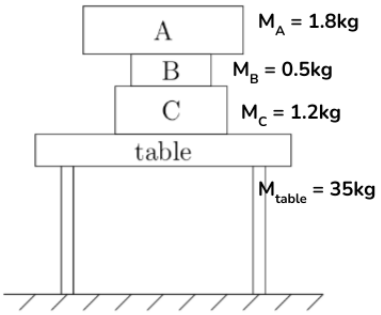


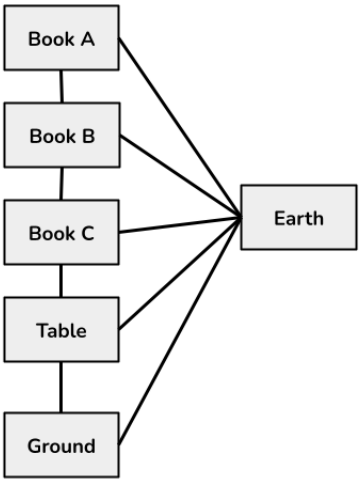
Critical Thinking & Concept Invention

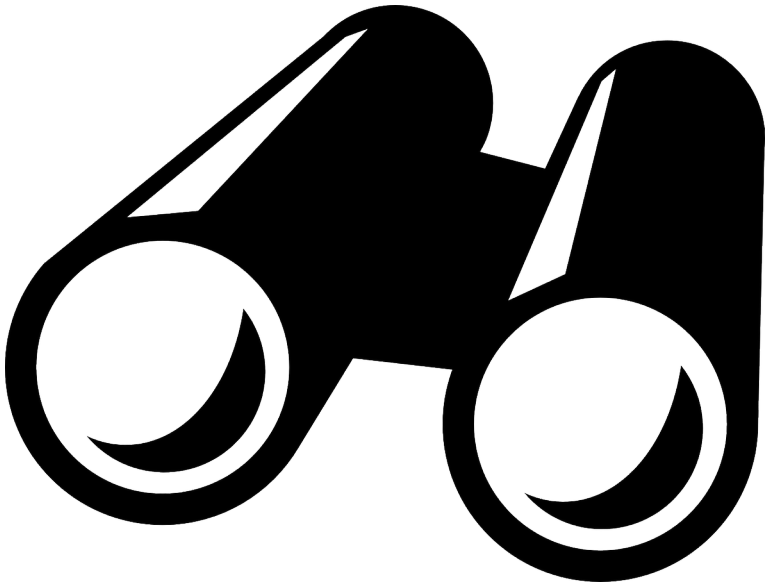
1. Come up with a definition of a **third law pair**.
2. Are all pairs of forces with equal and opposite magnitude third law pairs? Explain why or why not and cite examples from the model.
3. Examine the OID. Do the lines between objects on the OID represent third law pairs? Explain your thinking.
4. Examine the force diagrams that are a part of the **system**. If you added all of them up, what would the **net force** ($\Sigma \vec{F}$) be? Draw just that vector and label it $\Sigma \vec{F}_{sys}$. Explain your reasoning how you determined this vector.

Stacked Books at Rest

Model







Explore

1. How many interactions does each object have? *Book A:* _____ *Book B:* _____ *Book C:* _____ *Table:* _____
2. Are the objects accelerating or is the velocity not changing (ie, $\frac{dv}{dt} = 0$)?
3. Based on your answer, what should the **net force** be on each object?



Critical Thinking & Concept Invention

1. Which interaction (Force) do you have an equation to calculate the force magnitude for?
2. Which object will be easiest to model/calculate the forces for first? Why will that be an easier object to start with?
3. Construct a force diagram for each object. Be sure to identify third law pairs. Do not make a diagram for the Earth.

4. Construct a net force equation for each object. Calculate all forces.

$$m\vec{a} = \Sigma \vec{F}$$

Top Book

$$\boxed{\phantom{m\vec{a}}}_{m\vec{a}} = \boxed{\phantom{\vec{F}}}_{\vec{F}} + \boxed{\phantom{\vec{F}}}_{\vec{F}}$$

Middle Book

$$\boxed{\phantom{m\vec{a}}}_{m\vec{a}} = \boxed{\phantom{\vec{F}}}_{\vec{F}} + \boxed{\phantom{\vec{F}}}_{\vec{F}} + \boxed{\phantom{\vec{F}}}_{\vec{F}}$$

Bottom Book

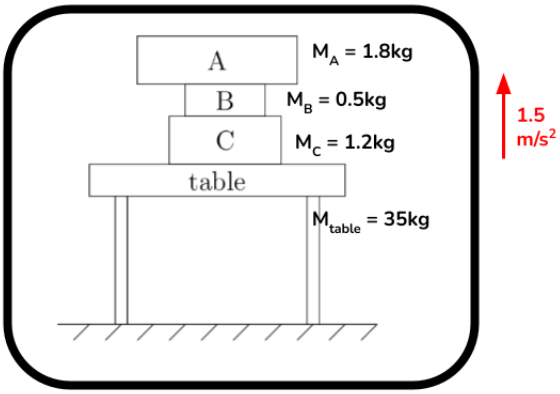
$$\boxed{m\vec{a}} = \boxed{\vec{F}} + \boxed{\vec{F}} + \boxed{\vec{F}}$$

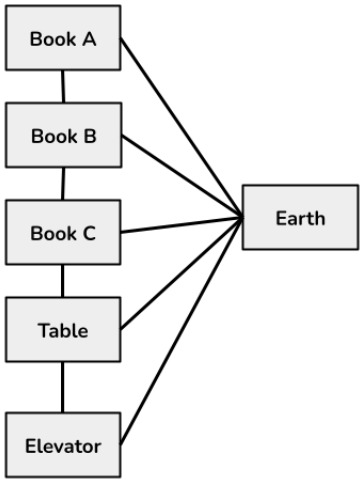
Table

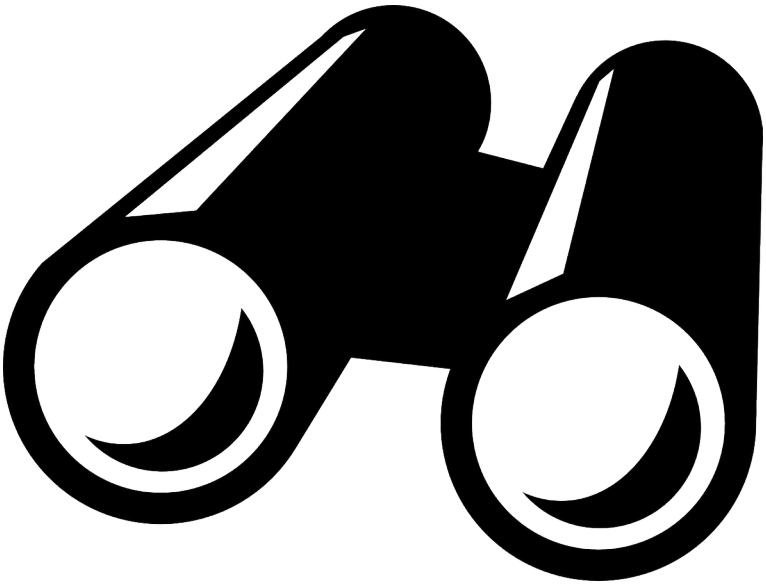
$$\boxed{m\vec{a}} = \boxed{\vec{F}} + \boxed{\vec{F}} + \boxed{\vec{F}}$$

Stacked Books in an Elevator

Model







Explore

1. Are the objects accelerating or is the velocity not changing (ie, $\frac{dv}{dt} = 0$)?

2. Based on your answer, what should the **net force** be on each object? *Book A:* $\Sigma \vec{F} =$ _____ *Book B:* $\Sigma \vec{F} =$ _____ *Book C:* $\Sigma \vec{F} =$ _____ *Table:* $\Sigma \vec{F} =$ _____



Critical Thinking & Concept Invention

1. Which interaction (Force) do you have an equation to calculate the force magnitude for?

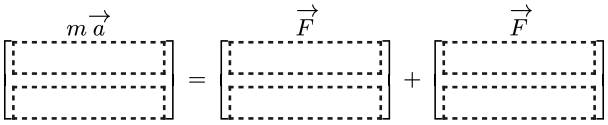
2. Which object will be easiest to model/calculate the forces for first? Why will that be an easier object to start with?

3. Construct a force diagram for each object. Be sure to identify third law pairs. Do not make a diagram for the Earth.

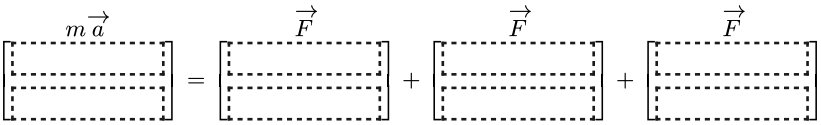
4. Construct a net force equation for each object. Calculate all forces.

$$m\vec{a} = \Sigma \vec{F}$$

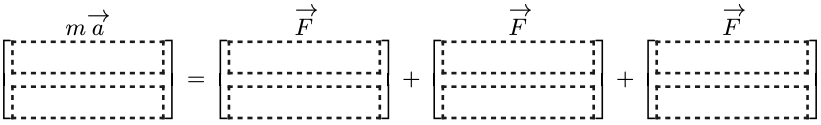
Top Book



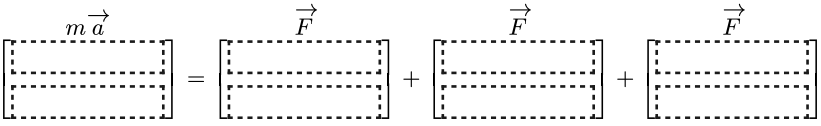
Middle Book



Bottom Book



Table



Summary

Every object follows the Newton's Second Law equation:

$$m\vec{a} = \Sigma \vec{F}$$

When we have multiple objects that interact, we can write a system of equations using this law:

$$m\vec{a}_{object\ 1} = \Sigma \vec{F}_{object\ 1}$$
$$m\vec{a}_{object\ 2} = \Sigma \vec{F}_{object\ 2}$$
$$m\vec{a}_{object\ 2} = \Sigma \vec{F}_{object\ 2}$$
$$\vdots$$

Newton's Third Law shows up across these equations and connects them:

$$\overrightarrow{F}_{1\,on\,2} = -\overrightarrow{F}_{2\,on\,1}$$

By using what we know about the **forces** involved and/or the **accelerations** we can solve for unknown forces or accelerations across these equations.

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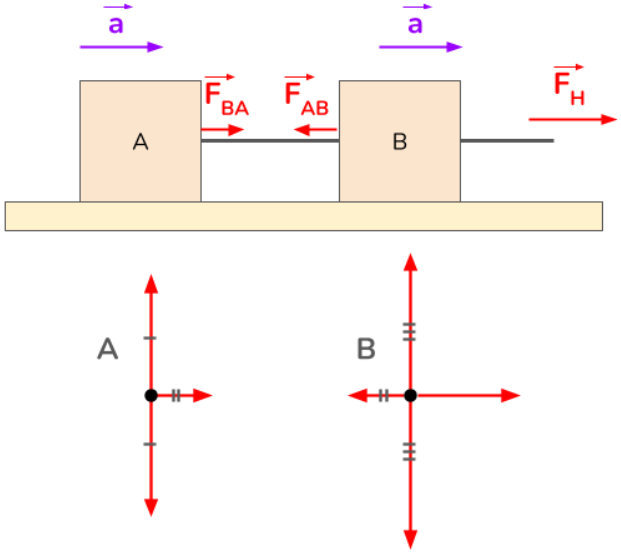
5.2: Dynamics of Systems

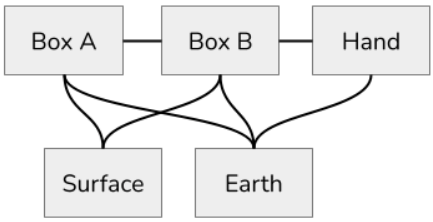
Dynamics of Systems

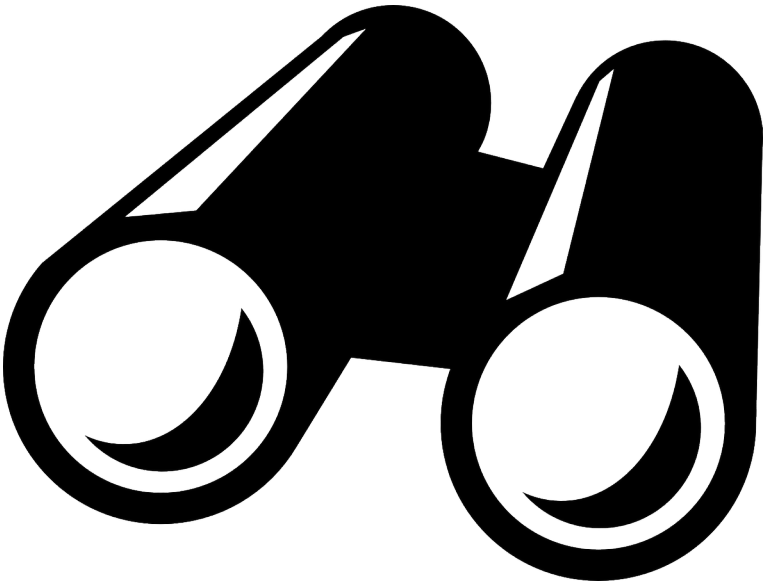
Connected Objects

Model

Two objects are connected by a string and pulled by a person. The forces of gravity and the normal forces are shown on the force diagrams but not on the picture.







Explore

1. Draw a system boundary around Block A and a separate system boundary around Block B
2. Circle external interactions as they cross the system boundary.
3. How many forces act on block A? Label them on the Force Diagram.
4. How many forces act on block B? Label them on the Force Diagram.
5. Do any forces (on an individual object) cancel? If so, cross them out on the diagram.
6. Which direction are the blocks accelerating?



Critical Thinking & Concept Invention

1. If the two blocks are connected, will they have the same acceleration or different acceleration?
2. What direction should the net force be on each block? Explain your reasoning.

3. Construct an acceleration equation using column vectors, then fill in the column vectors with relevant variables (eg, F_{AB} or F_H) and zeros when applicable. You can exclude the forces you crossed out earlier.

$$m\vec{a} = \Sigma\vec{F}$$

Box A

$$\begin{bmatrix} m\vec{a} \\ \\ \\ \end{bmatrix} = \begin{bmatrix} \vec{F} \\ \\ \\ \end{bmatrix}$$

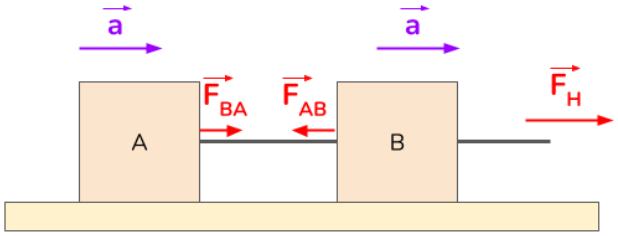
Box B

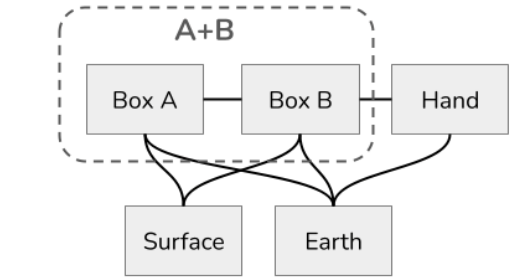
$$\begin{bmatrix} m\vec{a} \\ \\ \\ \end{bmatrix} = \begin{bmatrix} \vec{F} \\ \\ \\ \end{bmatrix} + \begin{bmatrix} \vec{F} \\ \\ \\ \end{bmatrix}$$

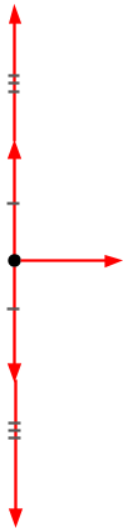
4. Circle unknowns that are not F_H, m_A, or M_B. Which equation is easier to solve for $\vec{a}$? Use that equation to identify what $\vec{a}$ is in terms of other variables in the problem.

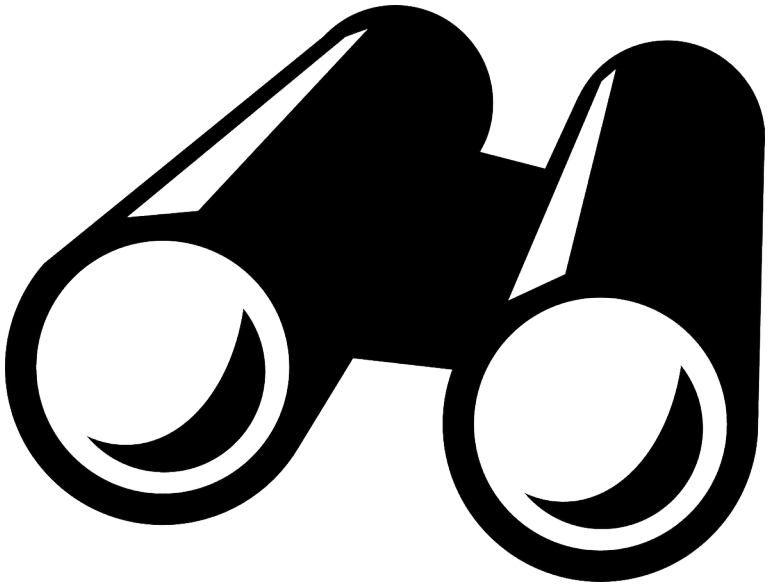
5. Plug that equation for $\vec{a}$ into the other equation. Solve for the force between the blocks (eg, F_{AB}) in terms of the other variables in the problem (eg, in terms of m_A, m_B, and F_H)

Model









Explore

1. What objects are in the system? What objects are in the surroundings?
2. Circle external interactions as they cross the system boundary.
3. How many forces **external to the system boundary** interact with the system?
4. What is the mass of the system (in terms of m_A, m_B)?



Critical Thinking & Concept Invention

1. What should the net force on the system equal? How does it relate to the acceleration of the system?
2. Fill in the column vectors with relevant variables and zeros when applicable. Then solve for the acceleration.

$$m\vec{a} = \Sigma\vec{F}$$

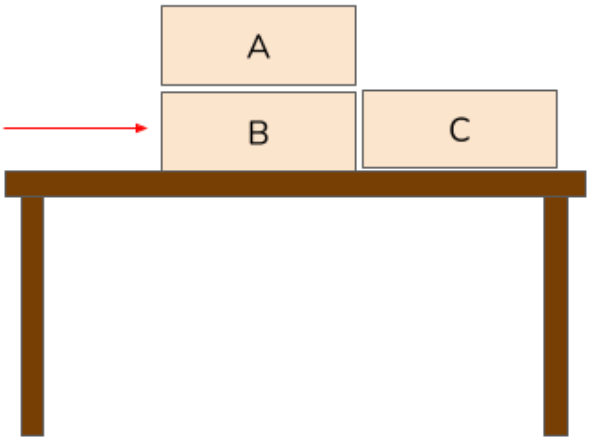
Box A + Box B

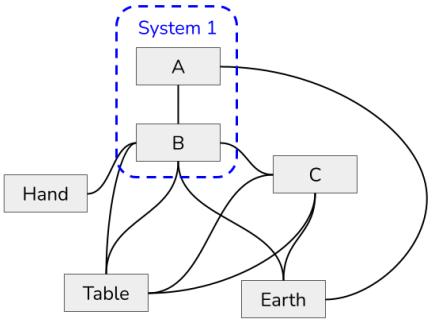
$$\begin{bmatrix} m\vec{a} \\ \\ \end{bmatrix} = \begin{bmatrix} \vec{F} \\ \\ \end{bmatrix}$$

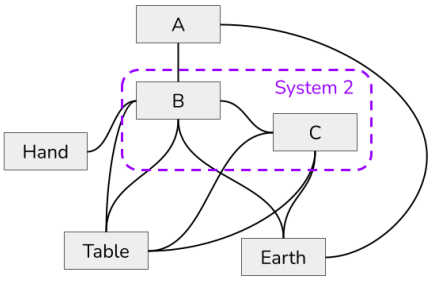
Many Objects

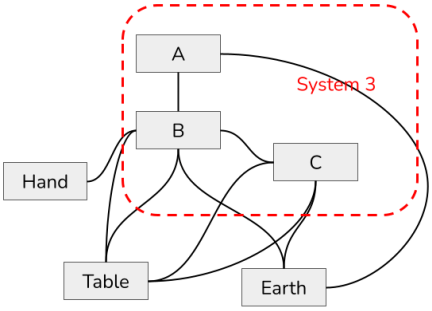
A person pushes on Block B with a force $\vec{F}$ to the right as shown. There is friction between the blocks and between the blocks and the table, such that everything moves at a constant speed to the right.

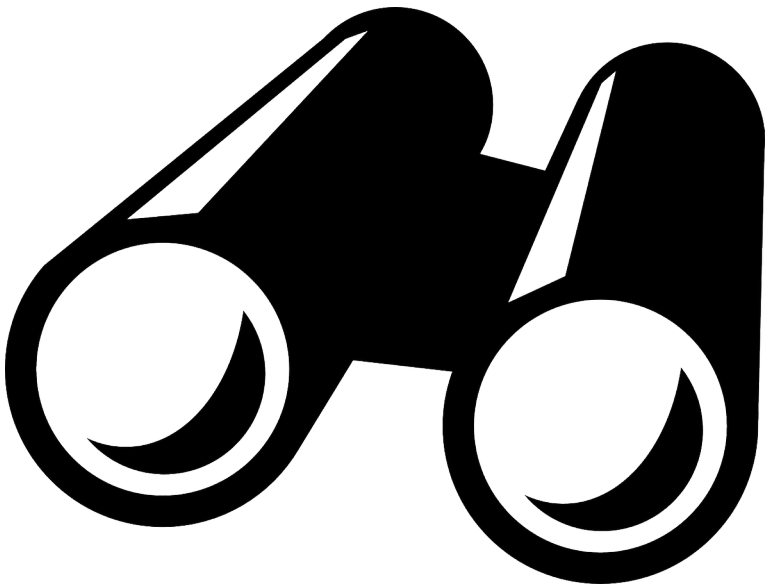
Model











Explore

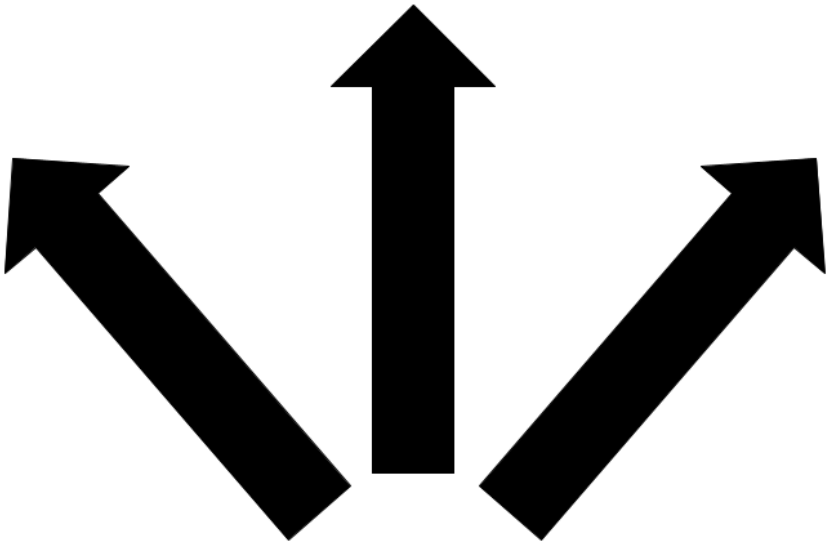
1. On each **Object Interaction Diagram**, circle external interactions as they cross the system boundary.

2. How many interaction lines are drawn between the table and Block B and between the table and Block C? Why does this make sense?
3. How many interaction lines cross the boundary of System 1?
4. How many interaction lines cross the boundary of System 2?
5. How many interaction lines cross the boundary of System 3?



Critical Thinking & Concept Invention

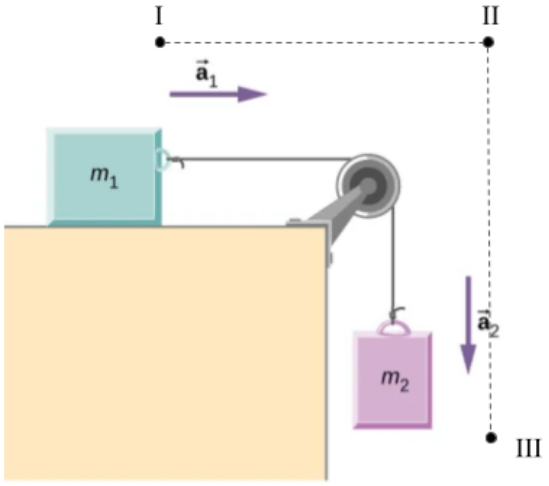
1. What should the net force be on each system? Look back at the description of motion.
2. The **force of gravity** is typically easy to calculate. Identify how many **non-gravitational interactions** cross the boundary of each system.
3. Which system will be easiest to make a force diagram for? Why?
4. Make a force diagram for each **system**. Start with whichever one you think is easiest. You only need to include external forces.
5. Rank the magnitudes of the horizontal forces in your force diagrams.
6. Did you use Newton's 1st/2nd Law when comparing force magnitudes? If so, explain how.
7. Did you use Newton's 3rd Law when comparing force magnitudes? If so, explain how.
8. Are there any forces that appeared on the force diagrams for System 1 or System 2 but not on System 3? Which ones? Why did you not need to include them?

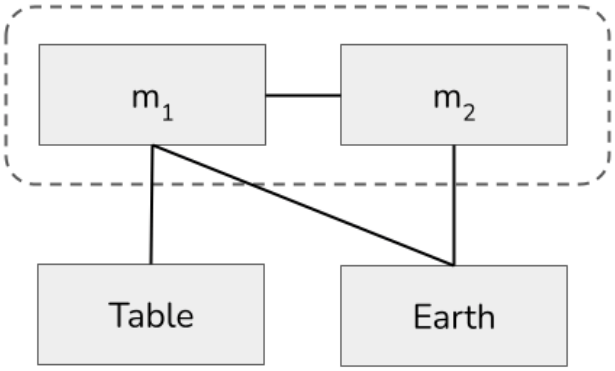


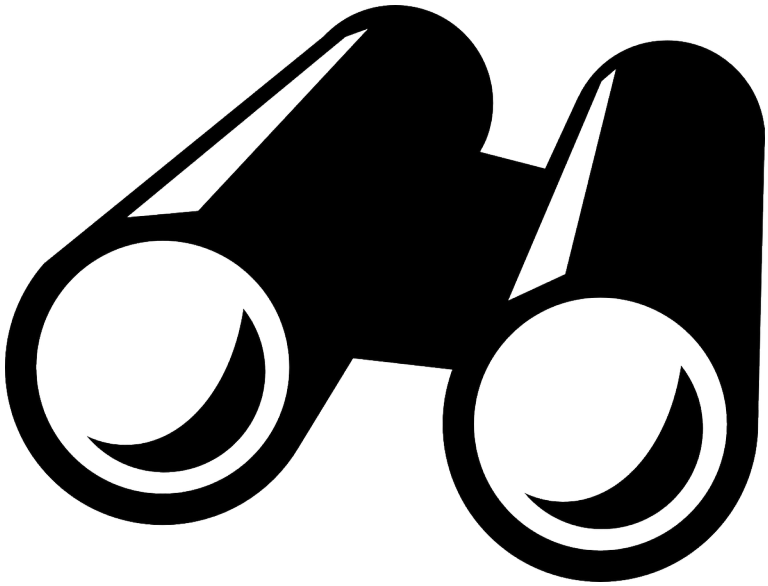
Apply

The mass of each block is 2.5kg and the coefficient of friction between the blocks and the table is 0.2. Determine the magnitude of each force. Recall $F_g = mg$ and $g = 10\frac{m}{s^2}$; $F_k = \mu_k \ F_N$

Model







Explore

1. Without performing any calculation, predict and describe what each object will do as soon as Object B is released. Explain your thinking.

2. Identify whether each object moves towards or away from the labeled points.

	Point I	Point II	Point III
A			
B			

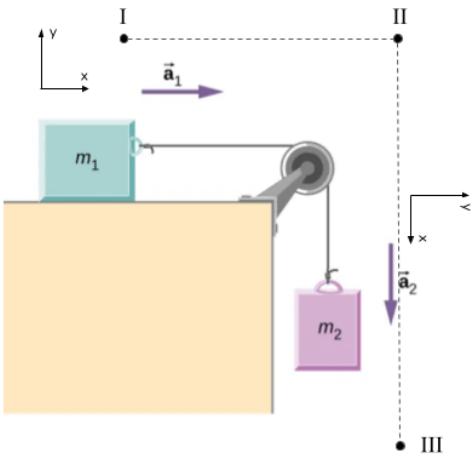


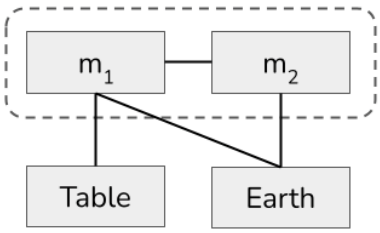
Critical Thinking & Concept Invention

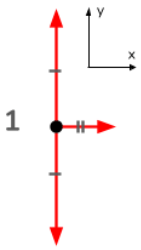
1. Was there a directional response (towards or away from _____) that had the same answer for both blocks? Which one?
2. Could it be accurate to say that both blocks moved along the same direction? Explain your thinking.
3. What would we need to do to identify the same direction for motion when we use column vectors to represent acceleration?

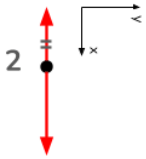
Changing Coordinates

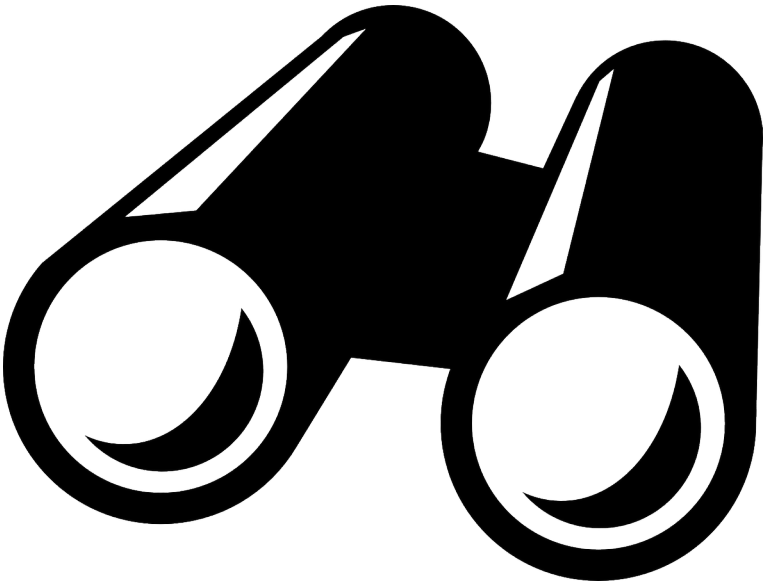
Model











Explore

1. Use the diagram to label the forces (eg, $\vec{F}_{g1}$). Draw a dashed line connecting any **Third Law Pairs**.
2. What direction (up, down, left, right) is the positive x-axis next to **Block A**?
3. What direction (+/- $\hat{x}$ or $\hat{y}$) does **Block A** accelerate?
4. What direction (up, down, left, right) is the positive x-axis next to **Block B**?
5. What direction (+/- $\hat{x}$ or $\hat{y}$) does **Block B** accelerate?



Critical Thinking & Concept Invention

1. Could it be accurate to say that both blocks moved along the same direction? Explain your thinking.
2. How did the coordinate system shift to make the acceleration vectors in the same direction?
3. When do you think it makes sense to make this shift in coordinate system?

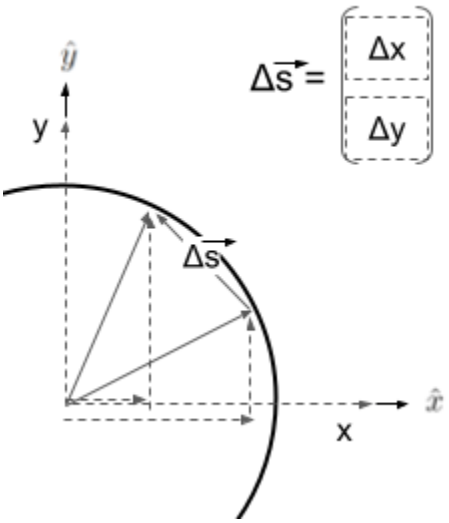
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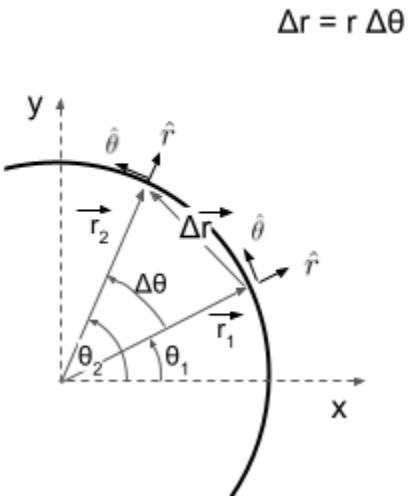
5.3: Circular Motion

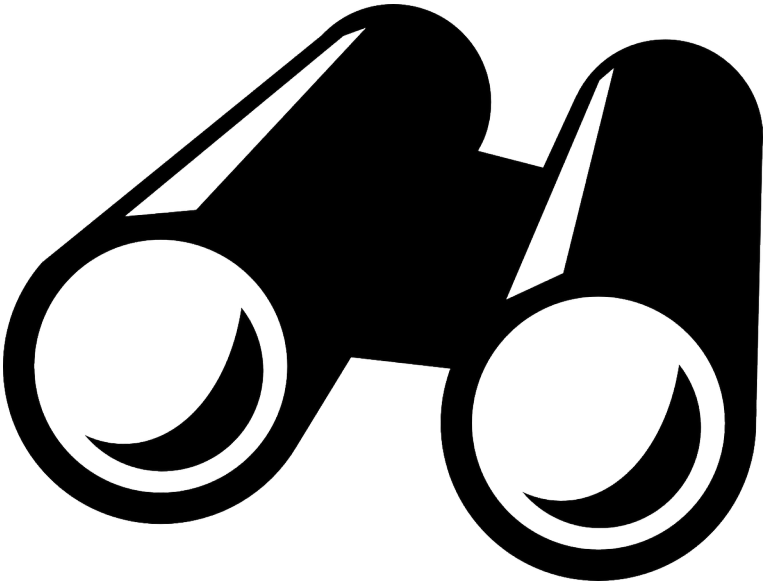
Circular Motion

Cylindrical Coordinates

Model







Explore

1. Identify the direction of each coordinate unit vector: $\hat{x}\ \hat{y}\ \hat{r}\ \hat{\theta}$

2. Which coordinate unit vectors change direction and which are fixed?

3. The vector $\Delta\vec{r}$ points from r_1 to r_2. Is the direction of $\Delta\vec{r}$ in the direction of $\hat{r}$ or $\hat{\theta}$? **Write your answer at the end of the equation for dr on the model and change it to $\Delta\vec{r}$.**

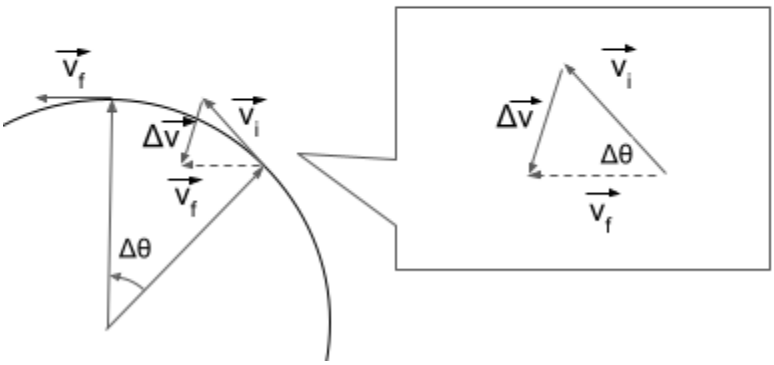


Critical Thinking & Concept Invention

1. Manipulate the equations:
   - Rewrite both equations from the model and divide by Δt
   - If we take the limit as these quantities go to zero, the Δ's become differentials (d's). Rewrite the equations with this limit.
   - Replace the derivative quantities (ie, $\frac{d}{dt}$ _____) with the quantities they represent.
2. The quantities $\frac{d\vec{r}}{dt}$ and $\vec{v}$ are the same thing. Use this to identify a equation for velocity in terms of cylindrical coordinates.

Model

$v_f = v_i$ for constant speed motion

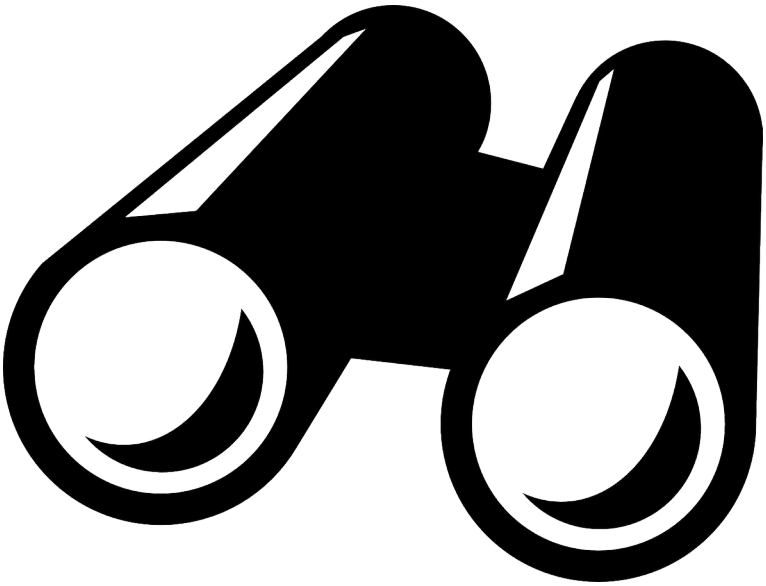


$\Delta\vec{v} = \vec{v}_f - \vec{v}_i$

Arclength = r $\Delta\theta$

$\Delta v \sim v\,\Delta\theta$

For small angles



Explore

Use the model to fill in the blanks in the statements.

Describe the velocity as the object goes around the circle.	*All velocity vectors have the same* _____ *but different* _____

Interpret the angles and how they relate to the direction of the velocity vector	*The change in direction of the velocity vectors can be identified as* Δ_____, *which is also the angle between* _____ *and* _____
Describe the magnitude and direction of the $\Delta\vec{v}$'s.	*The vector* $\Delta\vec{v}$ *points* _____ *and has magnitude* _____



Critical Thinking & Concept Invention

Construct and manipulate the equation below to identify acceleration for **uniform circular motion**.

Identify the relationship between Δv and the other quantities in the model	$\Delta v =$
Divide by Δt on both sides	
Eliminate ω using the relationship $v = \omega r$ or $\omega = \frac{v}{r}$	
Eliminate v using the relationship $v = \omega r$ or $\omega = \frac{v}{r}$	

Identify the direction (+/- $\hat{r}$ or $\hat{\theta}$) of these vector quantities and add it to the end of all of the equations.

Summary

When an object moves in a circle without changing the radius of that circle at a constant speed (v) it has a constant angular velocity ($\vec{\omega}$). The **velocity** ($\vec{v}$) of the object changes because the direction of the velocity vector changes. The **rate** of this change

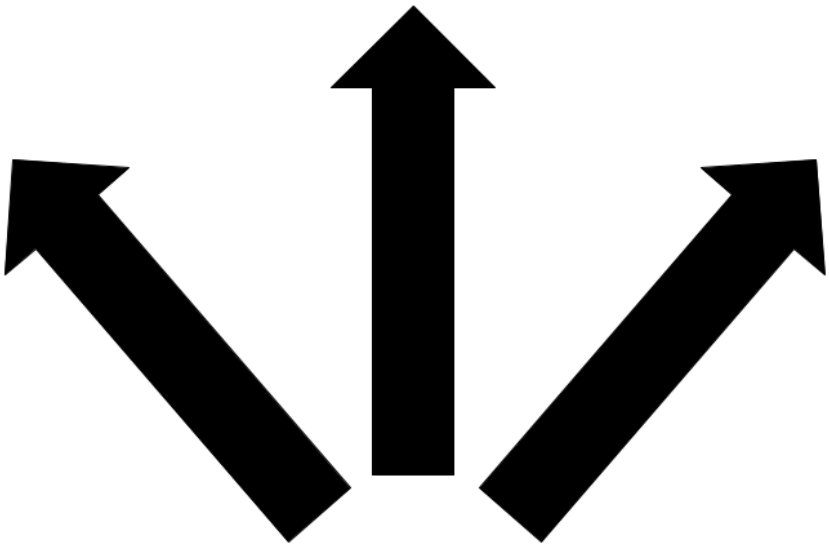
is called the centripetal acceleration ($\vec{a}_c$) and can be represented by the equations:

$$a_c = \frac{dv}{dt} = v\,\omega$$

Using the relationships between speed, angular speed, and radius ($v = \omega\ r$), we can represent this in terms of linear speed (v) or angular speed (ω):

$$a_c = \frac{v^2}{r} \qquad \text{OR} \qquad a_c = \omega^2\, r$$

You can use any of the equations for the magnitude of the centripetal acceleration, based on the information you have for the problem and the question you are trying to answer. In all cases, the direction of the centripetal acceleration is **towards the center of the circle** ($-\hat{r}$).



Apply

A 1200kg car drives around a turn at a speed of 15 m/s. The turn can be approximated as a circle with radius 8 meters.

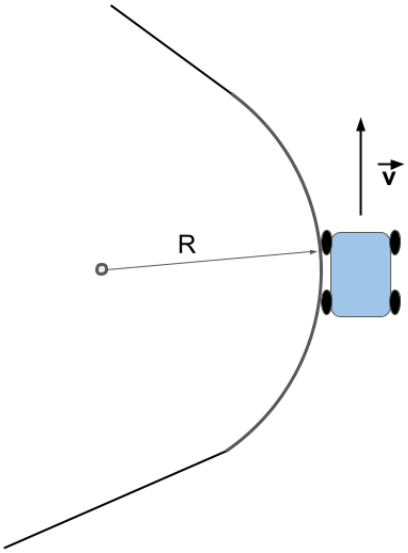
1. What must the centripetal acceleration be to change the direction of the car's velocity but not its speed?
2. If that force is only from friction, what must the frictional force be?
3. If the coefficients of friction between the car tires and the road are $\mu_k = 0.8$ and $\mu_s = 1.2$, will the car make the turn successfully? Explain your thinking.

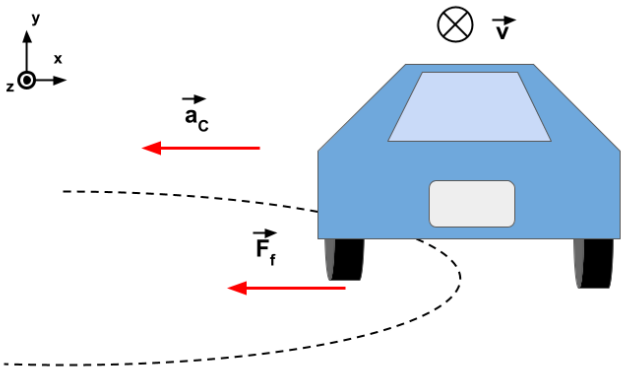
Centripetal Acceleration and Net Force Equations

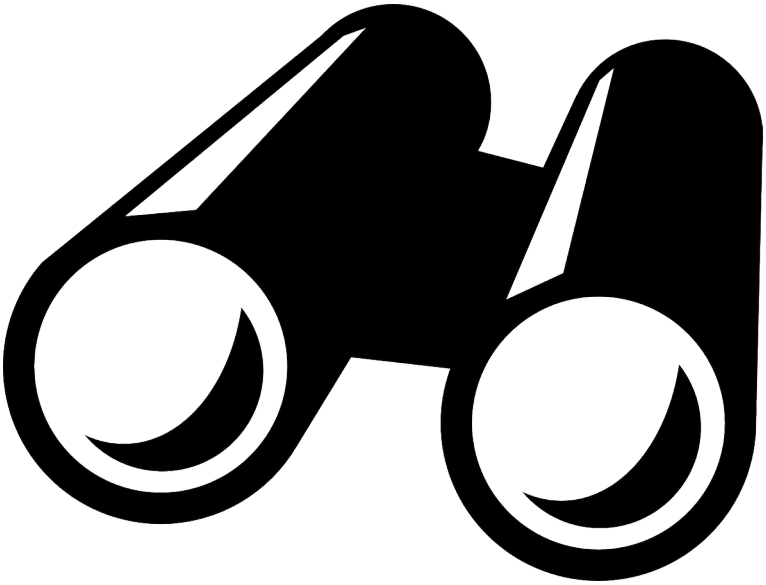
Model

The model below shows a curve in the road that can be modeled as a portion of a circle. The car travels at a constant speed around the turn.

Top View **Rear View**







Explore

1. What direction (+/- $\hat{x}$ or $\hat{y}$) points towards the center of the circle of the car's motion? In the $m\vec{a}$ column vector, put "ma_c" in that component (x or y) with a $+$ or $-$ sign in front.
2. Which force(s) have components that point towards the center of the circle of the car's motion?
3. What other forces might act on the car? Identify the forces and their directions.

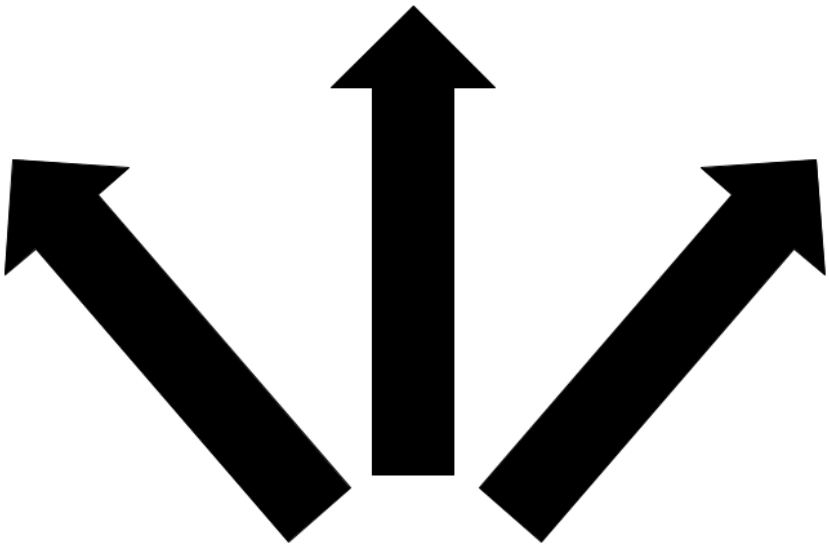


Critical Thinking & Concept Invention

1. What component should be zero for the net force? Fill it in.

2. Finish the net force equation by including variables for friction and the equation (with variables) for gravity. You can replace the variable "a_c" with its equation. Does the mass of the car matter?

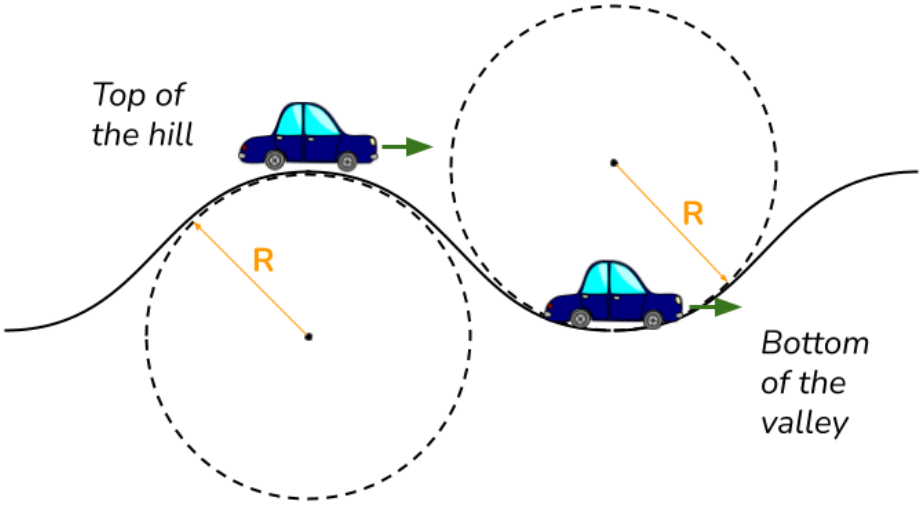
$$\boxed{m\vec{a}} = \boxed{\vec{F}} + \boxed{\vec{F}} + \boxed{\vec{F}}$$



Apply

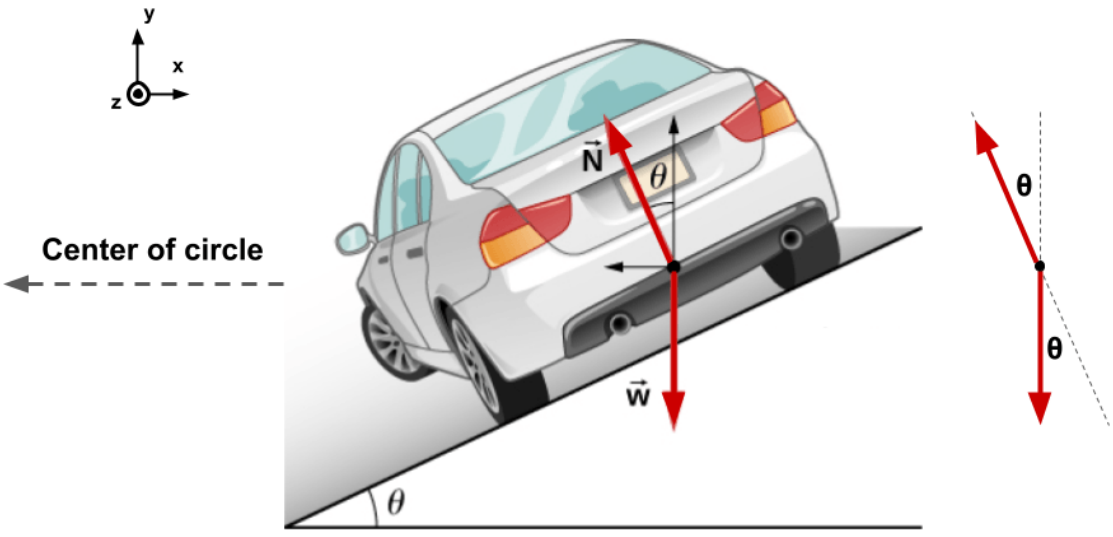
Determine which direction the centripetal acceleration should be based on the "circle" the car travels along. Then draw force

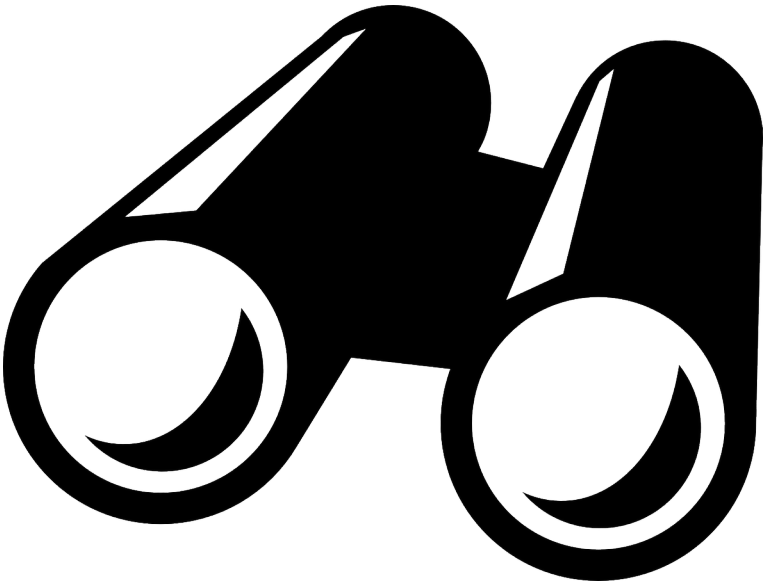
vectors for the **normal force** and **gravitational force** that shows a net force in the necessary direction for the motion of the car. Assume the car drives at a constant speed.



Centripetal Acceleration on Ramps

Model





Explore

1. What forces act on the car?
2. What direction (+/- $\hat{x}$ or $\hat{y}$) points towards the center of the circle of the car's motion? In the $m\vec{a}$ column vector, put "ma_c" in that component (x or y) with a $+$ or $-$ sign in front.
3. Which force(s) have components that point towards the center of the circle of the car's motion?
4. Is that component opposite or adjacent to the angle θ?



Critical Thinking & Concept Invention

1. Finish the net force equation by including the equation (with variables) for gravity and fill in the column vector for the normal force. Does the mass of the car matter?
2. Which force applied the centripetal force in this case?
3. Replace the variable "a_c" with its equation. What must the magnitude of the force with a centripetal component be equal to?
4. As the angle increases, does more or less of that force point in the direction of the centripetal force?
5. Explain as a group: How do banked turns (turns that are ramped at an angle) make it easier for the car to turn?

6. If a roadway is banked at the proper angle, a car can round a corner without any assistance from friction between the tires and the road. Construct a **net force equation** for the motion of this car. Then, find the appropriate banking angle (θ) for a 900-kg car traveling at 20.5 m/s in a turn of radius 85.0 m.

$$\boxed{m\,\vec{a}} = \boxed{\vec{F}} + \boxed{\vec{F}}$$

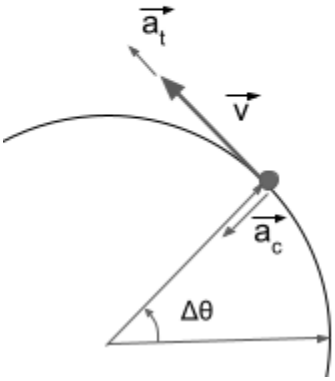
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5.4: Angular Acceleration

General Circular Motion

Model

An object can undergo **circular motion (constant r)** and experience a **change in speed (v and ω are not constant)**. The model below identifies the acceleration components of changes to direction and changes to linear and angular speed.

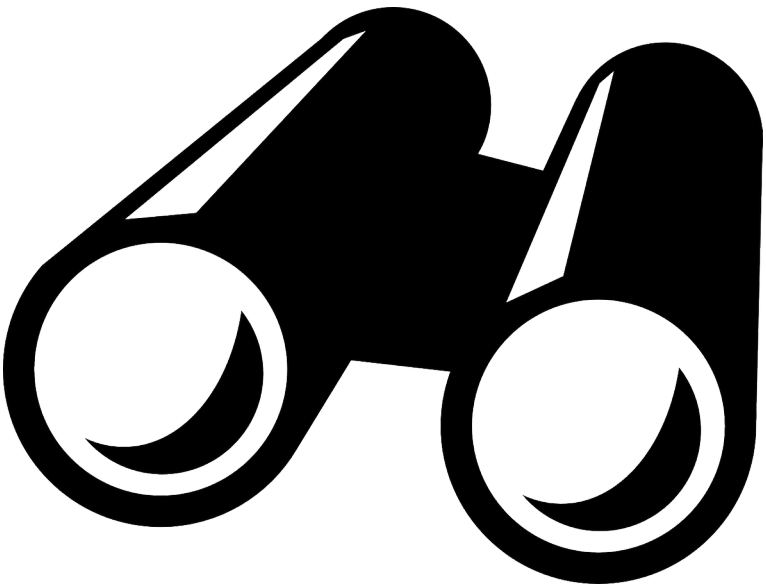


$$v = \omega r$$

$$a_c = v^2 / r$$

$$a_t = dv / dt$$

$$\vec{a} = \vec{a}_c + \vec{a}_t$$



Explore

Use the model to fill in the blanks in the statements.

Describe the **centripetal acceleration** (a_c)	*The magnitude of a_c is based on _____ and _____. The direction is _____*
Describe the **tangential acceleration** (a_t)	*The magnitude of a_t is based on _____ and _____. The direction is _____*
Which acceleration changes the speed and which changes the direction of motion?	_____ changes the speed and _____ changes the direction of motion.



Critical Thinking & Concept Invention

Construct and manipulate the equation below to identify acceleration for **general circular motion**.

Use the relationship $v = \omega r$ to replace the v's in both equations.	$a_c =$ $a_t =$
Assume r is constant. Show how that affects the derivative in the equation for a_t.	$a_t =$
Put it all together: What is the vector equation for $\vec{a}$? Be sure to include directions on each acceleration vector.	$\vec{a} =$ ⬚

Summary

The rate that the angular velocity changes is called **angular acceleration** and uses the variable $\vec{\alpha}$:

$$\vec{\alpha} = \frac{d}{dt}\vec{\omega}$$

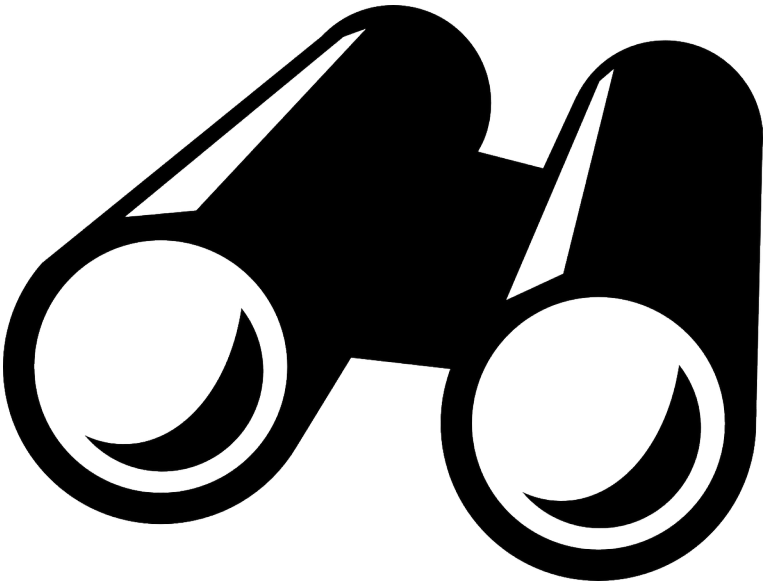
For **rigid objects**, the centripetal acceleration portion is supplied by the internal forces between the different pieces of that rigid object. The angular acceleration is caused by tangential forces, which cause torque.

Center Pointing	Tangential
$a_c = r\ \omega^2$ is **internal**	$a_t = r\ \alpha$ is **external**
$\vec{a}_c$ comes from a center-pointing force so that force does not apply torque.	$a_{cm} = a_t$ so $a_{cm} = r\ \alpha$
	$v_{cm} = v_t$ so $v_{cm} = r\ \omega$

Euler's Second Law

Model

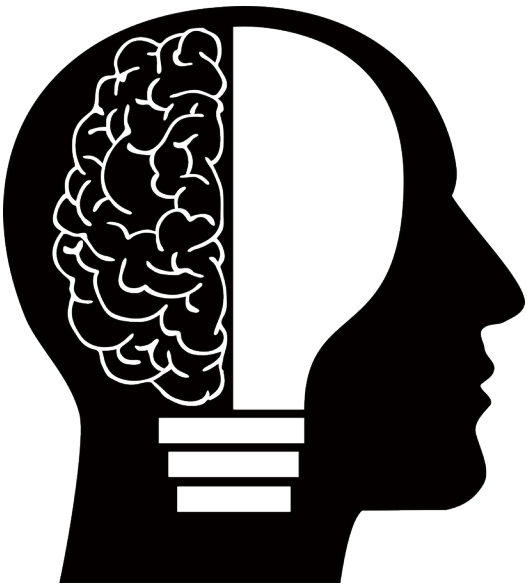
Newton's 2nd Law	Euler's 2nd Law
$\Sigma \vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(m\vec{v})$	$\Sigma \vec{\tau} = \frac{d\vec{L}}{dt} = \frac{d}{dt}(I\vec{\omega})$
$\Sigma \vec{F} = m\frac{d}{dt}(\vec{v}) = m\vec{a}$	$\Sigma \vec{\tau} = I\frac{d}{dt}(\vec{\omega}) = I\vec{\alpha}$



Explore

Explore and markup the model by:

- Identifying and highlight equivalent pairs of quantities that are represented as a derivative or a single variable (eg, $\frac{d\vec{v}}{dt}$ and $\vec{a}$). Use different colors for different pairs.
- Underline the constants that were pulled out of a derivative.



Critical Thinking & Concept Invention

1. Compare the different types of acceleration:

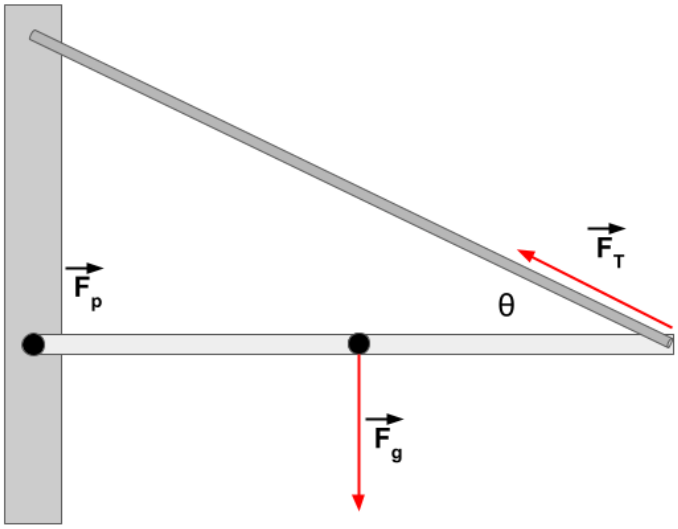
	Linear Acceleration	Angular Acceleration
How does each acceleration vary with an inertial quantity (m or I) determined?	$a \propto$	$\alpha \propto$

	Linear Acceleration	Angular Acceleration
Describe the relationship between inertia and acceleration	*If the **mass** of an object is greater, the **acceleration** will be _____ for the same _____*	*If the **moment of inertia** of an object is greater, the **angular acceleration** will be _____ for the same _____.*
How does each acceleration vary with the net external interaction (Force or Torque)?	$a \propto$	$\alpha \propto$
Describe the relationship between inertia and external interactions	*If the **mass** of an object is greater, I need a _____ **Force** for the same _____.*	*If the **moment of inertia** of an object is greater, I need a _____ **Torque** for the same _____*

Statics

Model

A rod of mass M and length R is connected to a pivot point. The rod can rotate freely around the pivot point. A cable is connected to the rod with an angle θ. The rod remains at rest.

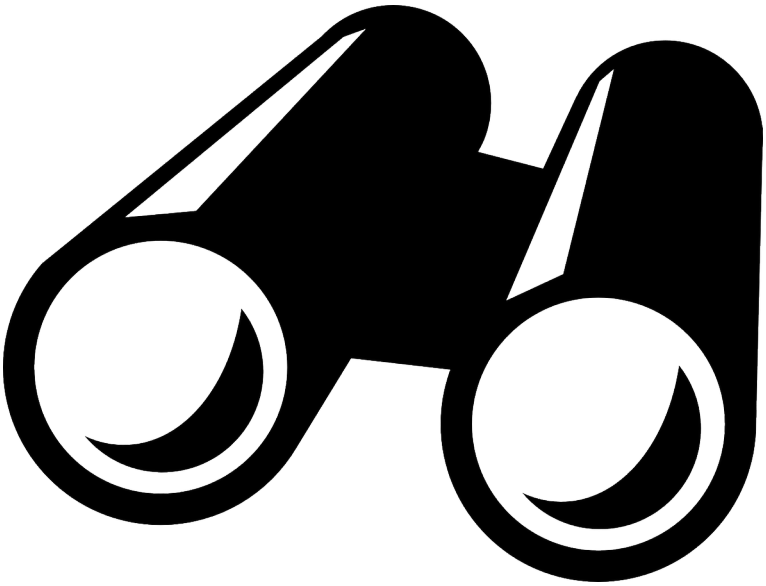


$$m\vec{a} = \Sigma \vec{F}$$

$$\boxed{m\vec{a}} = \boxed{\vec{F}_g} + \boxed{\vec{F}_T} + \boxed{\vec{F}_p}$$

$$I\vec{\alpha} = \Sigma\vec{\tau}$$

$$\begin{bmatrix} \\ \\ \end{bmatrix} I\vec{\alpha} = \begin{bmatrix} \\ \\ \end{bmatrix} \vec{\tau}_g + \begin{bmatrix} \\ \\ \end{bmatrix} \vec{\tau}_T + \begin{bmatrix} \\ \\ \end{bmatrix} \vec{\tau}_p$$



Explore

1. Create an **Object Interaction Diagram** for this scenario with the rod as the system.
2. How many external forces act on the rod?
3. What should the acceleration of the rod be equal to? Why? **Fill in the $m\vec{a}$ column vector based on your response.**
4. Where is the **Force of Gravity ($\vec{F}_g$)** drawn on the rod? What moment arm $\vec{r}$ would this be in a torque calculation?
5. What should the moment arm $\vec{r}$ be for the other two forces?
6. How many forces apply a nonzero torque to the rod? In which direction(s) are those torques? Explain your thinking.
7. What should the angular acceleration of the rod be equal to? Why? **Fill in the $I\vec{\alpha}$ column vector based on your response.**



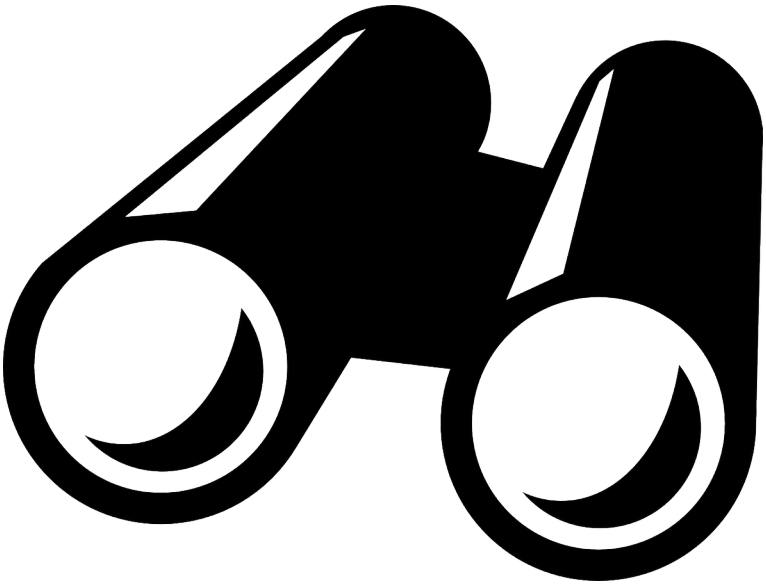
Critical Thinking & Concept Invention

1. Fill in the forces (with variables when needed) into the **Acceleration Equation**.
2. Calculate the torques (with variables when needed) and fill in the **Angular Acceleration Equation**.
3. Solve the set of equations (Acceleration and Angular Acceleration) to identify the magnitude of the force of tension (F_T) in the cable and the full vector force exerted from the pivot point $\vec{F}_p$

Angular Kinematics Relationships

Model

Angular Kinematics	Linear Kinematics
$\vec{\theta}(t) = \vec{\theta}(t)$	$\vec{r}(t) = \vec{r}(t)$
$\vec{\omega}(t) = \frac{d}{dt}\vec{\theta}(t)$	$\vec{v}(t) = \frac{d}{dt}\vec{r}(t)$
$\vec{\alpha}(t) = \frac{d^2}{dt^2}\vec{\theta}(t)$	$\vec{a}(t) = \frac{d^2}{dt^2}\vec{r}(t)$



Explore

Explore and markup the model by:

- Identifying what mathematical operation takes you from one quantity to the next as you go downward.
- Identifying what mathematical operation takes you from one quantity to the next as you go upward.
- Identify each row as one of the following:
    - Description of Motion
    - Description of Change in Motion
    - Description of Position/Orientation, not Motion.



Critical Thinking & Concept Invention

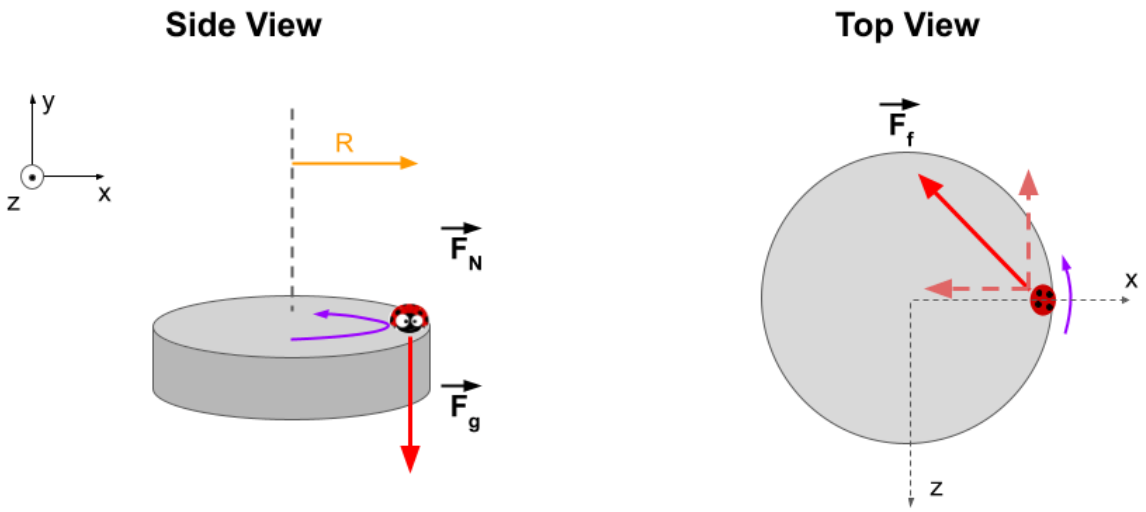
1. Discuss with your group what each quantity represents and identify a clear definition:

Angular	Linear
$\vec{\theta}$	$\vec{r}$
$\vec{\omega}$	$\vec{v}$
$\vec{\alpha}$	$\vec{a}$

Circular Motion with Angular Acceleration

Model

A bug (mass m) is on the rim of a record of radius $15cm$. The record is initially not moving and then begins to spin. Just when the record is turned on, the angular acceleration is constant at $\frac{\pi}{4}\ rad/s^2$. The coefficient of static friction between the bug and the record is $\mu_s = 0.75$.

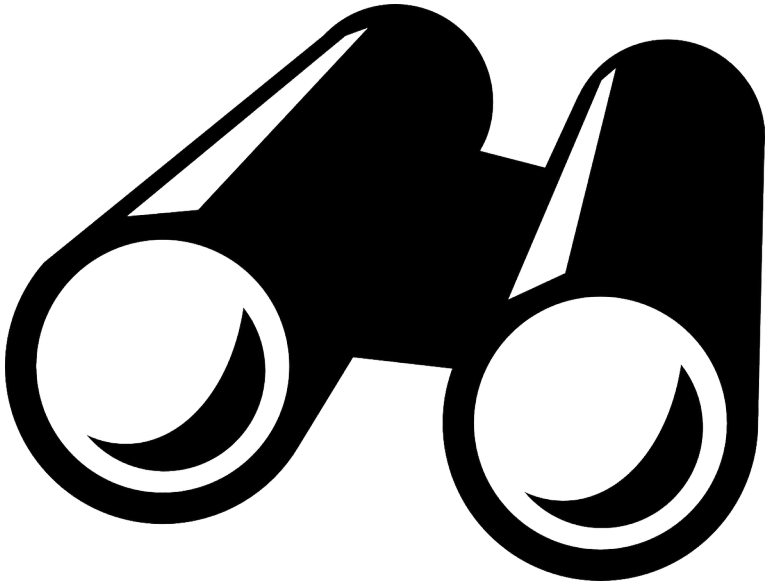


$$m\vec{a} = \Sigma\vec{F}$$

$$\boxed{m\vec{a}} = \boxed{\vec{F}_g} + \boxed{\vec{F}_N} + \boxed{\vec{F}_f}$$

$$I\vec{\alpha} = \Sigma\vec{\tau}$$

$$\boxed{I\vec{\alpha}} = \boxed{\vec{\tau}_g} + \boxed{\vec{\tau}_N} + \boxed{\vec{\tau}_f}$$



Explore

Explore and markup the model by:

- Draw the **Normal Force Vector** next to its label on the side view. If it should be the same length as another vector, put hash marks to illustrate equal magnitudes.
- Examine the Top View Diagram and label the dashed force components as "$\vec{F}_{f,t}$" for the tangential force and "$\vec{F}_{f,c}$" for the centripetal force.
- Based on these center pointing and tangential direction, fill in the components of the $m\vec{a}$ vector with either "$+/-m\ a_c$" or "$+/-m\ a_t$"
- Identify which direction **should be** the positive y-axis on the Top View diagram and add that to the coordinates for that diagram.



Critical Thinking & Concept Invention

1. Fill in the force vectors for the gravitational force and the normal force.
2. Replace a_c and a_t with their equations using angular coordinate (eg, ω's and α's.
3. Identify the friction force column vector based on the other vectors in your equation.
4. Identify the magnitude of this vector (How do you find a magnitude from components)?
5. Remind each other which calculus operation takes you from α to ω. Use this to replace the ω's with α's and another variable that might show up.
6. Identify the maximum force of static friction. At what **time** (did this variable show up in the last step?) will the necessary frictional force (to maintain the acceleration) exceed this value?

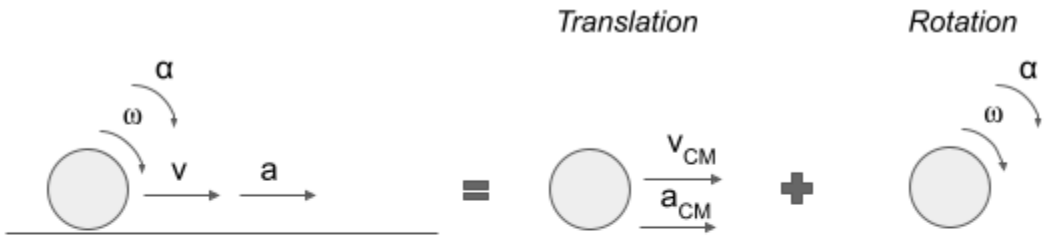
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5.5: Rotational Dynamics

Rotational Dynamics

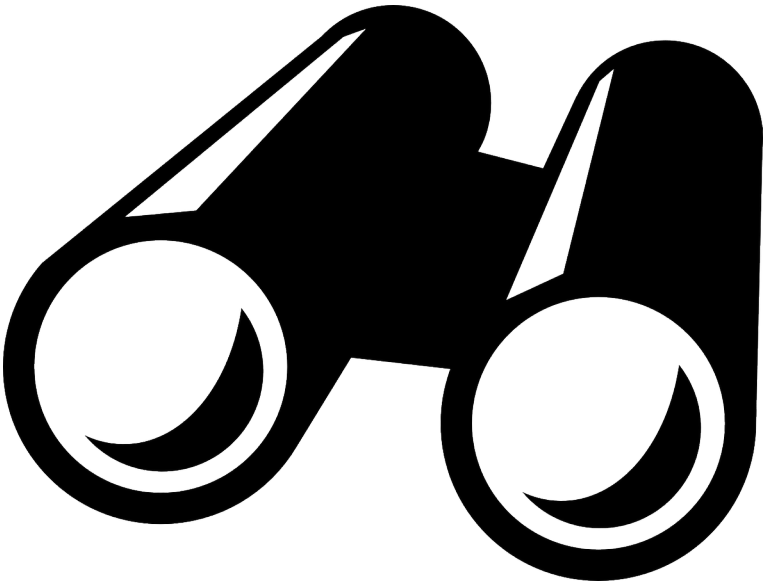
Accelerated Rolling

Model



$\vec{a} = \Sigma F / m$ $\vec{a} = \Sigma \vec{F} / m$ $\vec{\alpha} = \Sigma \vec{\tau} / I$

$\vec{\alpha} = \Sigma \vec{\tau} / I$



Explore

	Translation	Rotation
What is the speed/velocity variable that describes each motion?		
What is the acceleration that changes those speeds/velocities?		
Is the change in that motion based on the net Force or net Torque?		



Critical Thinking & Concept Invention

1. Compare and contrast linear and angular acceleration in the context of this wheel.

	Linear Acceleration	Angular Acceleration
Describe what each quantity represents about the wheel	$\vec{a}$ *represents ...*	$\vec{\alpha}$ *represents ...*

2. Construct equations using a and α:

	Linear Acceleration	Angular Acceleration
Copy / rewrite each equation in terms of a not α	$\begin{bmatrix}\cdots\\\cdots\\\cdots\end{bmatrix} = \frac{1}{m}\begin{bmatrix}\cdots\\\cdots\\\cdots\end{bmatrix}$	$\begin{bmatrix}\cdots\\\cdots\\\cdots\end{bmatrix} = \frac{1}{I}\begin{bmatrix}\cdots\\\cdots\\\cdots\end{bmatrix}$
Copy / rewrite each equation in terms of α not a	$\begin{bmatrix}\cdots\\\cdots\\\cdots\end{bmatrix} = \frac{1}{m}\begin{bmatrix}\cdots\\\cdots\\\cdots\end{bmatrix}$	$\begin{bmatrix}\cdots\\\cdots\\\cdots\end{bmatrix} = \frac{1}{I}\begin{bmatrix}\cdots\\\cdots\\\cdots\end{bmatrix}$

3. Why is it useful to rewrite the equations for both accelerations in terms of a single variable (a or α)?

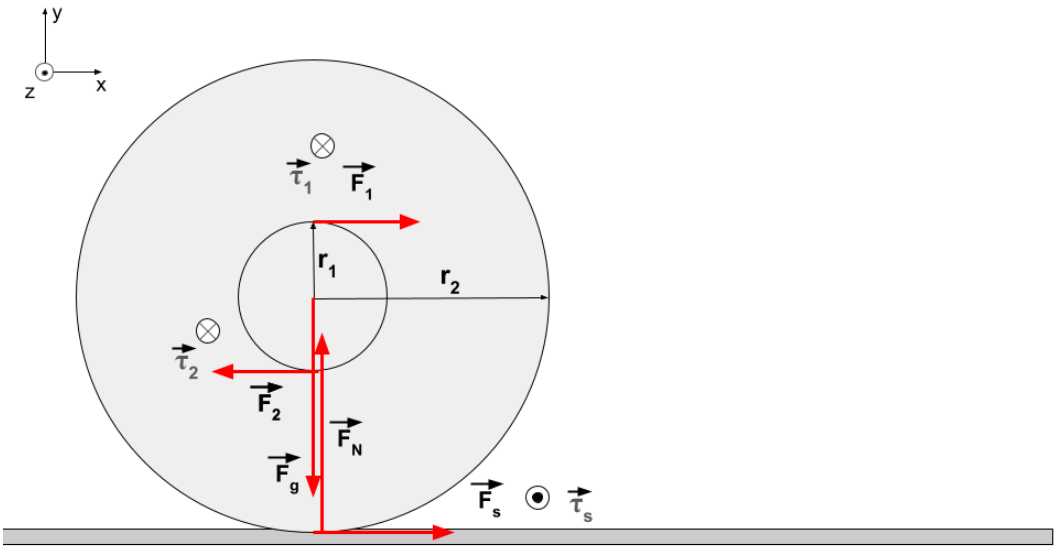
Net Force and Net Torque

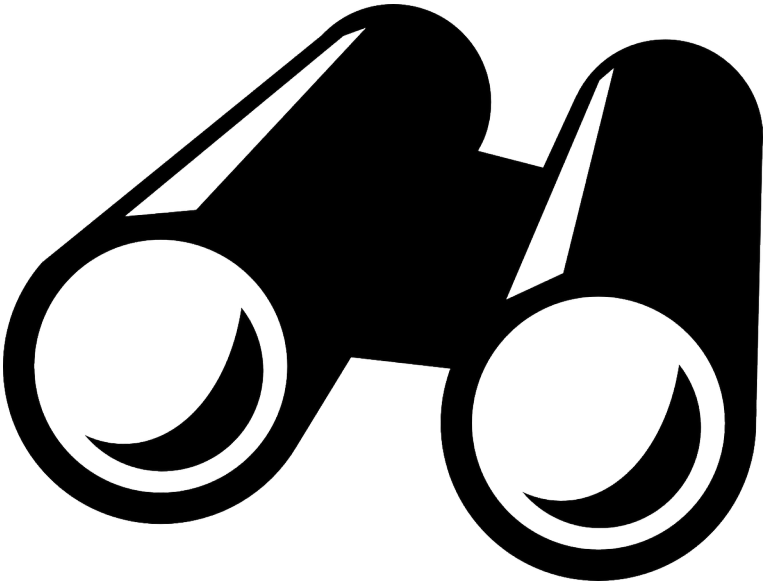
Model

To speed up, a car applies force from the axle (not the very center, but close to the center) to the wheel to make it turn. The road applies force to the wheel (opposing the rotation direction), which propels the car forward. Ideally (so make this assumption) the wheel turns without slipping. To simplify this problem, we will focus just on the wheel and ignore the rest of the car. The rest of the car just adds mass without adding to the moment of inertia of the wheel.

A wheel experiences forces ($\vec{F}_1$ and $\vec{F}_2$) near the axle a distance r from the center with equal magnitudes ($F_1 = F_2 = F_c$) in opposing directions. A pair of equal but opposing forces that apply torque in this manner is called a **couple**. Let's assume the forces from this couple are constant. The friction applies force at the rim of the wheel, R, and might not be constant (so don't make that assumption).

$$F_1 = F_2 = F_c$$





Explore

1. Next to the wheel, create a **Force Diagram** by carefully copying all of the force vectors onto a dot.
2. Add those force vectors together to graphically identify the net force. Which direction (if any) will the wheel accelerate? Does this make sense?
3. Which forces apply torques? In which direction(s)?
4. Which forces do not apply torques? Why not?



Critical Thinking & Concept Invention

1. As a group, identify how many forces act on the wheel. Then, construct a **Net Force Equation**. List the vector variables after the $=$ sign and put empty (for now) column vectors underneath each variable.

$$m\vec{a} = \Sigma\vec{F}$$

$$m\vec{a} \quad \begin{bmatrix} \\ \\ \\ \end{bmatrix} =$$

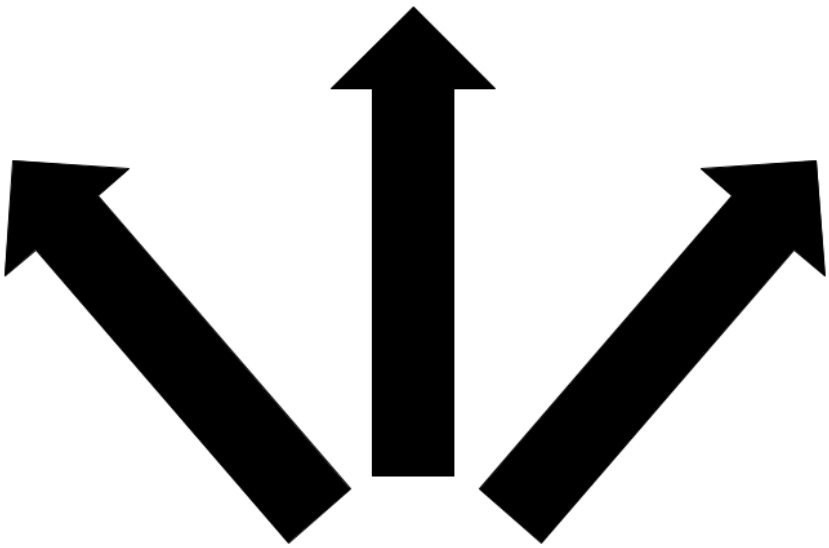
2. As a group, identify how many torques act on the wheel. Then, construct a **Net Torque Equation**. List the vector variables after the $=$ sign and put empty (for now) column vectors underneath each variable.

$$I\vec{\alpha} = \Sigma\vec{\tau}$$

$$I\vec{\alpha} \quad \begin{bmatrix} \\ \\ \\ \end{bmatrix} =$$

3. Solve the system of equations to identify the magnitude of the frictional force F_f based on the force from the couple F_c and the two radii (r and R). Assume the wheel is a **solid disk**.

4. **Summarize this process:** How can you determine unknown forces using the force and torque concepts?



Apply

A 3kg disk with a radius of 0.7m is pulled to the right with a Force of 8N. Assume there is sufficient friction between the disk and the ground that it rolls without slipping.

- Identify all of the forces and torques acting on the disk.
- Identify the linear and angular acceleration of the disk.



$$m\vec{a} = \Sigma\vec{F}$$

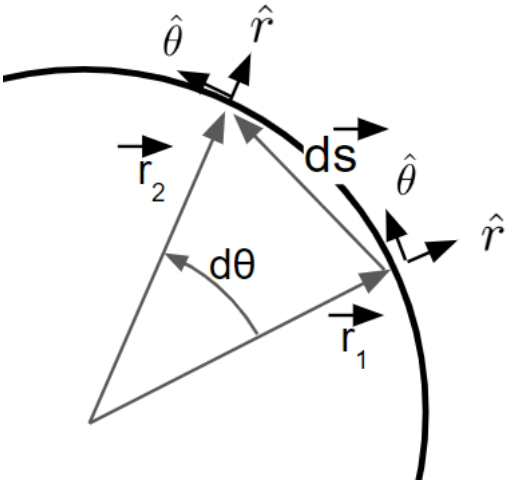
$$\boxed{m\vec{a}} = $$

$$I\vec{\alpha} = \Sigma\vec{\tau}$$

$$\boxed{I\vec{\alpha}} = $$

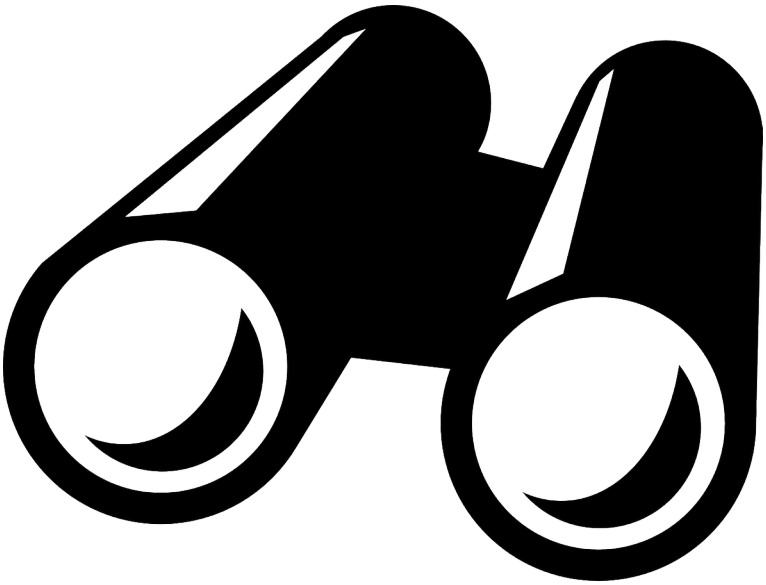
Angular Work

Model



$$d\vec{s} = ds\hat{\theta} \qquad d\vec{s} = \qquad \hat{\theta}$$

$$W = \int_{r1}^{r2} \vec{F} \cdot d\vec{s}$$



Explore

1. What direction, using cylindrical coordinate unit vectors, is the differential displacement $d\ \vec{s}$?
2. Is that displacement vector **radial** or **tangential**?
3. What angle does the displacement vector ($d\ \vec{s}$) make with a radial vector?



Critical Thinking & Concept Invention

What do the differential measurements represent?	$d\ \vec{s}$ represents ... $d\theta$ represents ...
What is $d\vec{s}$ in cylindrical coordinates?	

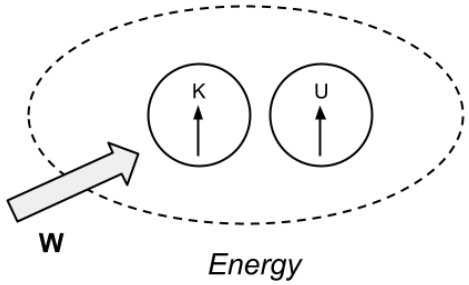
Construct a new **Work integral** using your new version of $d\vec{s}$. Be sure to identify what the new limits of integration should be.	$W = \int F\square$
Examine the integrand (what is in the integral, not including the differential quantity). What other quantity does this resemble?	
Rewrite the **Work integral** with your response for the previous question written in as the integrand.	$W = \int \square$

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5.6: Power

Energy and Power Equations

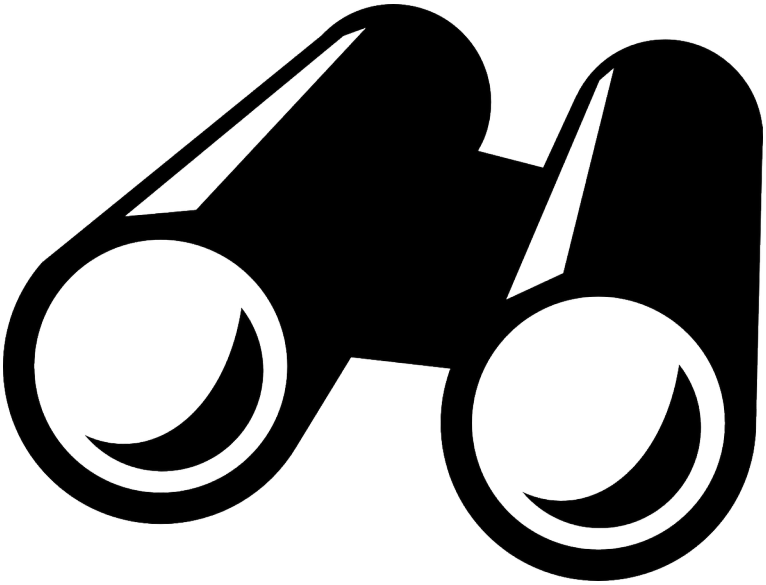
Model



Power

$$P_{avg} = \frac{W}{\Delta t}$$

(1)	$\Delta E_{sys} = W$	$\frac{\Delta E_{sys}}{\Delta t} = \frac{W}{\Delta t}$
(2)	$\Delta K + \Delta U = W$	$\frac{\Delta K}{\Delta t} + \frac{\Delta U}{\Delta t} = P_{avg}$



Explore

1. What operation is performed on the **energy equation** to turn it into a **power equation**?
2. If the typical units for energy are Joules (J), what do you think the units are for power?
3. Do you think the equation in Box 1 is true for all scenarios or just this specific scenario shown by the Energy Interaction

Diagram? Explain your thinking.

4. Do you think the equation in Box 2 is true for all scenarios or just this specific scenario shown by the Energy Interaction Diagram? Explain your thinking.



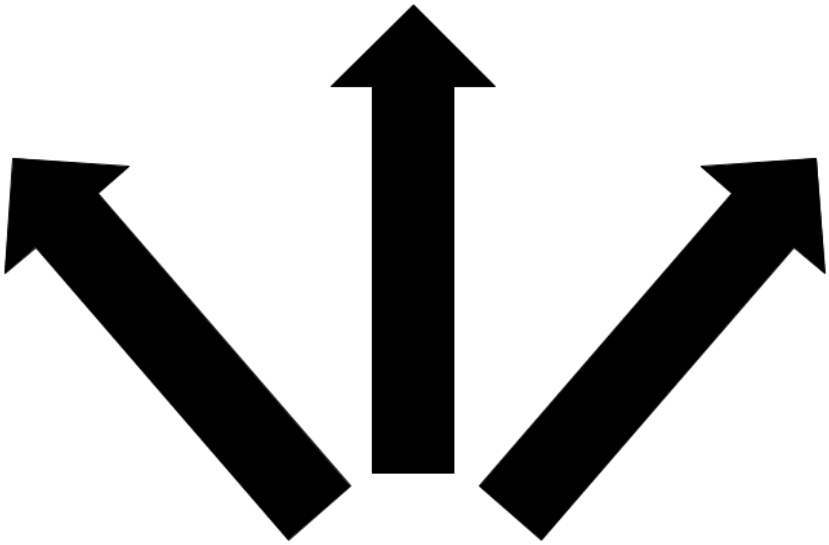
Critical Thinking & Concept Invention

1. Is energy / Work normally associated with **changes in position** or **changes in time**?
2. Which new measurement (spatial or temporal) is incorporated into the power equation?
3. Why do you think the power equation is **average power**? What might we need to do to find **instantaneous power**?

Summary

The rate that work is done, $\frac{dW}{dt}$, is called **Power**. Power is represented by the variable **P**, and is typically measured in units of **Watts (W)** which are Joules per second ($\frac{J}{s}$).

If power is constant, $P = \frac{W}{\Delta t}$. Over any time-interval, $\frac{W}{\Delta t} = P_{avg}$ which means that $W = P_{avg}\Delta t$. This is a way you can use average (or constant) power in an energy equation.



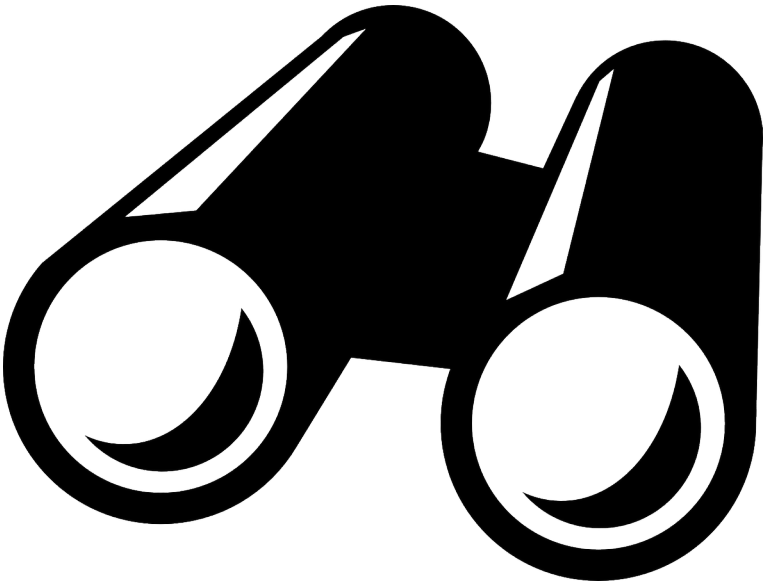
Apply

Find the average power applied to an elevator that lifts a 2500-kg load a height of 35.0 m in 12.0 s and also increases the speed from rest to 4.00 m/s in that time. Construct an energy equation, then turn it into a power equation.

Linear Power

Model

Integral Form	Differential Form	Rate of Change
$W_{ab} = \int_a^b \vec{F} \cdot d\vec{s}$	$dW = \vec{F} \cdot d\vec{s}$	$\frac{dW}{dt} =$ ⬚



Explore

1. What quantities are **the same** and what quantities are **different** in the integral and differential forms of this equation?
2. What operation happens to the left side of the differential form of the equation to turn the equation into a **rate**? Perform that same operation to the right side and record your answer on the model after the equals sign.



Critical Thinking & Concept Invention

What is the differential ratio (fraction) that you just wrote above? What does the quantity represent?	
Rewrite the equation by substituting your response above in place of the differential ratio.	$\frac{dW}{dt} =$ ⬚
Underneath the equation, write what the quantity on each side of the equation represents	

Interpret the sign. Under what conditions is this rate positive or negative?	

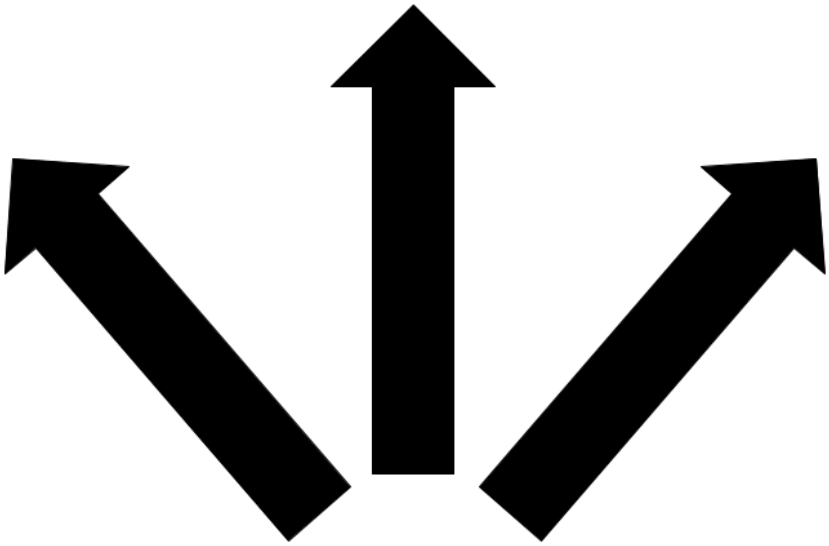
Summary

Work relates to **Force** through a spatial integral. Taking the derivative of that integral, we can relate force and Power.

$$W = \int \vec{F} \cdot d\vec{s}$$

$$P = \vec{F} \cdot \vec{v}$$

The result of this **vector dot product** is a **signed scalar quantity**. It represents ...



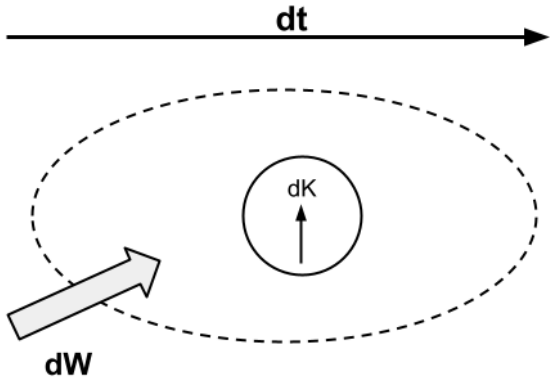
Apply

1. A girl pulls her 15-kg wagon along a flat sidewalk by applying a 10-N force at 37° to the horizontal. Assume that friction is negligible and that the wagon starts from rest.
    - How much work does the girl do on the wagon after pulling it 3 meters?
    - What is the speed of the wagon when it has moved 3 meters from its initial position?
    - What is the instantaneous power the girl applies at the 3 meter mark?
2. A 750kg car is towed by a cable that is powered by an engine. The engine maintains a constant power of 1 horse-power (746 Watts). Assume the car starts at rest ($v(0s) = 0\ m/s$). Identify the following:
    - What is the speed of the car after it is towed for 3 seconds? What is the instantaneous force on the car at the 3 second mark?

- What is the speed of the car after it is towed 8 seconds? What is the instantaneous force on the car at the 8 second mark?
- Identify the speed of the car as a function of time by following the same steps as before with $\Delta t = t$.

Power as a Differential Equation

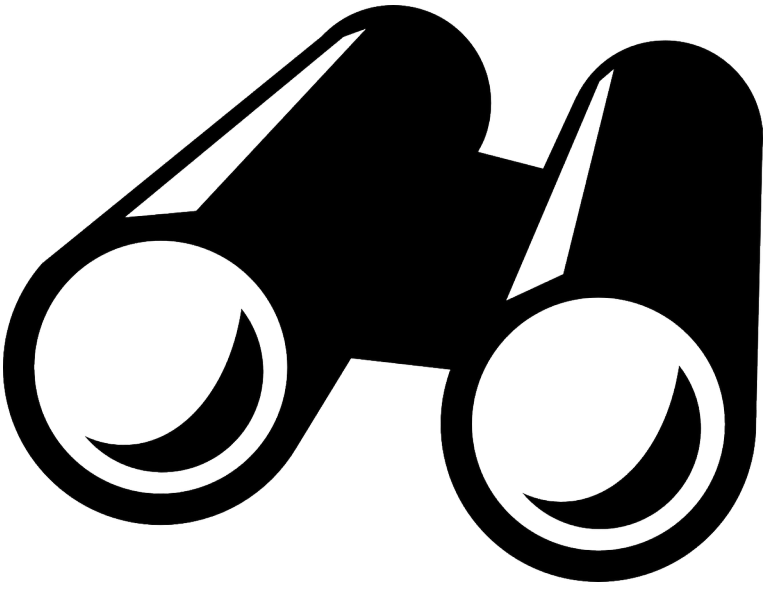
Model



$$dE_{system} = dW$$

$$\frac{d}{dt}K = P(t)$$

$$\frac{d}{dt}(\frac{1}{2}mv^2) = P(t)$$



Explore

1. What is the length of time for this timeline? How is that different than what we typically do for energy processes?
2. What energy is changing in the system?
3. What is different about how these energies are identified?
4. Is there work going into the system? What is different about how this work is labeled?

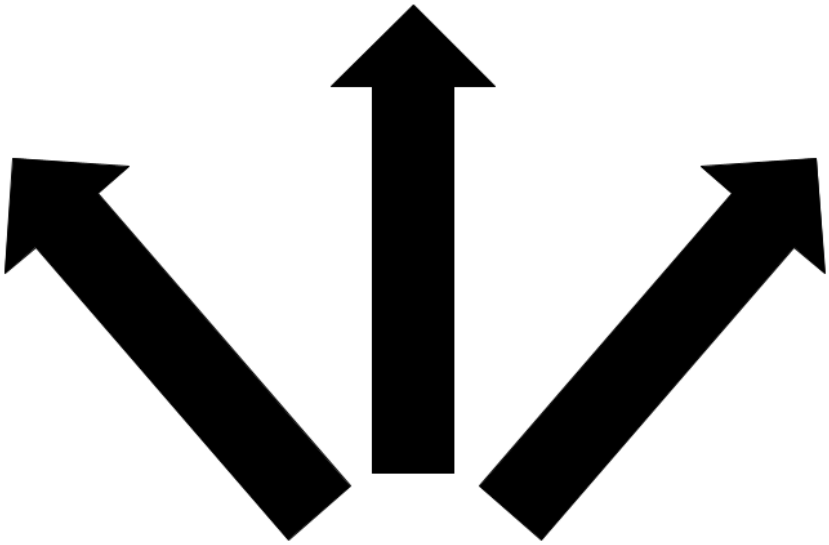


Critical Thinking & Concept Invention

Take the time derivatives (don't forget the chain rule). Assume mass is constant with time.	$\boxed{} = P(t)$
Separate the Variables: Move the dt's to the side of the equation with an explicit time-dependence	
Integrate both sides but only evaluate the integral on the **left hand side**. Use variables to make it a definite integral (eg, v_i to v_f)	
Summarize the **Separation of Variables** process. Identify why it made sense which term has the dt or dv.	

Summary

When power is not constant, we need to use calculus to solve the energy-rate equation. When dealing with different variables, we use the **separation of variables** technique to connect variables with their differential quantities. This lets us take separate integrals for each term.



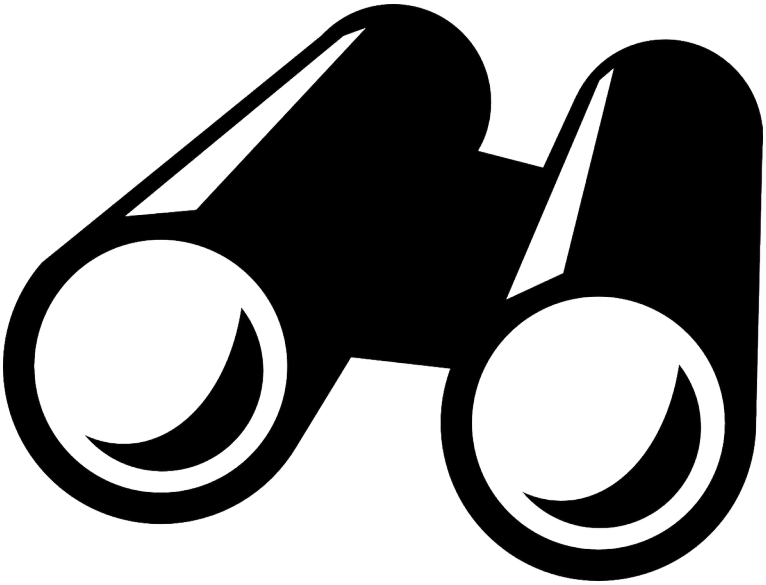
Apply

Let's assume the power is a function of time given by the equation $P = \frac{A}{t}$ with A as a constant. Copy your equation from the critical thinking section, integrate power. Then determine the speed as a function of time (with some constants in the equation).

Angular Power

Model

$$P = \frac{d}{dt}W \qquad dW = \vec{\tau} \cdot \vec{d\theta} \qquad \frac{dW}{dt} = \boxed{}$$



Explore

1. How is this similar to the equation for differential work (dW) for **linear power**? How is each variable shifted to represent angular motion?

2. What operation happens to the left side of the differential form of the equation to turn the equation into a **rate**? Perform that same operation to the right side and record your answer on the model after the equals sign.



Critical Thinking & Concept Invention

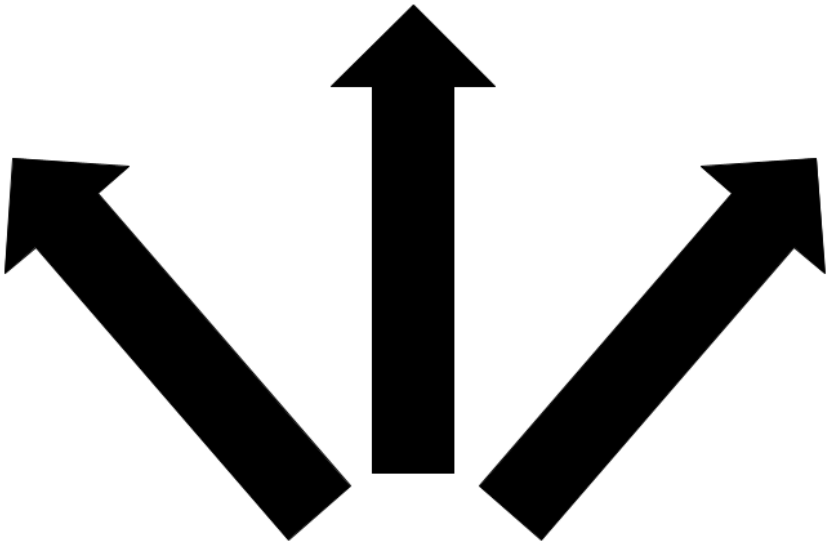
What is the differential ratio (fraction) that you just wrote on the right side of the equals sign in the model above? What does the quantity represent?	$\frac{\square}{\square} =$
Rewrite the equation by substituting your response above in place of the differential ratio.	$P = \square$
Underneath the equation, write what the quantity on each side of the equation represents	
Interpret the sign. Under what conditions is this rate positive or negative?	

Summary

Work relates to **torque** through an angular, spatial integral. Taking the derivative of that integral, we can relate torque and power.

$$W = \int \vec{\tau} \cdot \vec{d\theta}$$

$$P = \vec{\tau} \cdot \vec{\omega}$$



Apply

1. A playground merry-go-round has a radius of 2.2m and mass of 180kg. A child applies a force of 22N at the edge of the merry-go-round in a direction perpendicular to the radius.
    - Draw a picture of the scenario from above and label the Force and radius vectors. Choose and label a coordinate system.
    - Identify the torque on the merry-go-round
    - If the merry-go-round starts at rest, identify the angular velocity of the merry-go-round after pushing for 8 seconds.
    - Identify the power the child must exert at the 8-second mark.
2. A wind turbine rotates at 20 revolutions/min. If its power output is 2.0 MW, what is the torque produced on the turbine from the wind? *Note: You will need to convert to radians per second and watts. The power output of the turbine must equal the power input from the wind.*

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Impulse

1 Impulse

So far we have only explored closed systems so that

$$\sum \mathbf{J} = 0 \tag{1}$$

which means the momentum of the system is conserved. When there are no net external forces acting on a system, the momentum of the system does not change. This principle is known as Newton's 1st Law.

Now we will explore the physics of open systems, which means we need to learn how to determine $\mathbf{J}$ from the interactions that cross system boundaries in open systems.

1.1 Interaction Timeline

Model

Image needed here with Alt Text: Interaction timeline showing momentum vectors and impulse changes over time

Explore

1. Carefully examine the momentum vectors and identify interactions. Draw a vertical dashed line between clock-times when an interaction occurs.
2. For each time-interval where an interaction occurs, draw the change in momentum vector that represents that interaction on the dots in the Impulse row.
3. Find the momentum vector just before the first interaction and put an “i” above it.
4. Find the momentum vector just after the last interaction and put an “f” above it.
5. The Impulse vectors represent the change in momentum. Construct a single vector that represents the total change in momentum for all of the interactions.

Critical Thinking & Concept Invention

1. Use the vectors that relate to the last three Explore questions to fill in the momentum chart for the object.
2. Explain what determined the length of the $\Delta\mathbf{p}$ vector.
3. Under what circumstances could that arrow be longer or shorter? Describe what would need to be different about the scenario and/or the interaction timeline from the model.

Summary

External interactions can apply an Impulse ($\mathbf{J}$) to a system. The total impulse is based on the strength of the interaction and how

long that interaction is applied. For any system, the change in momentum of the system is equal to the total (sum) impulse acting on the system from the surroundings:

$$\Delta \mathbf{p}_{system} = \sum \mathbf{J} \tag{2}$$

1.2 Force

Model

This model represents possible data for the momentum of an object over time. The different lines represent experiments with different amounts of force, such as different fan speeds for a fan cart.

Image needed here with Alt Text: Momentum versus time graph showing different force experiments on an object

Explore

1. What is the x-axis label? What are the units?
2. What is the y-axis label? What are the units?
3. At $t = 2s$, which force resulted in the greatest change in momentum?
4. Which force took longer to change the momentum of the object by an amount Δp?
5. What is the equation for Force (F)? Copy it here.

Critical Thinking & Concept Invention

1. The slope of a line or section of a graph is often written as

   $$\frac{\text{Rise}}{\text{Run}} \tag{3}$$

   and can be written as

   $$\frac{\Delta y}{\Delta x} \tag{4}$$

   or

   $$\frac{dy}{dx}. \tag{5}$$

   Rewrite these equations with y and x replaced by the variables listed on those axes.
2. The area under a line or curve of a graph can be written as

   $$y\Delta x \tag{6}$$

   for a constant function or

   $$\int y\,dx \tag{7}$$

   for a general function. Rewrite these equations with y and x replaced by the variables listed on those axes.
3. Does the equation for force represent a slope or area on this graph? Explain your thinking.
4. Which force is greatest? Least?

5. How does a force, and the amount (magnitude of that force), affect an object's momentum?

Summary

When objects interact, the strength of that interaction is called Force. Force is defined as the rate the momentum of an object changes based on that interaction alone. This definition is represented by the equation:

$$\mathbf{F} \equiv \frac{\Delta \mathbf{p}}{\Delta t} \tag{8}$$

Or more precisely...

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} \tag{9}$$

“Force is the time rate of change of momentum.”

This relationship is known as Newton's Second Law. Force is a vector and points in the same direction as the change in momentum. This might not be the same direction as the actual momentum of the object while the force is applied.

1.3 Impulse

Model

Image needed here with Alt Text: Force versus time graph with shaded impulse regions under the curve

Explore

1. What is the x-axis label? What are the units?
2. What is the y-axis label? What are the units?
3. How many shaded areas are shown? What shapes are those areas?
4. What is the height (symbols/variables) and width (symbols/variables) for each area?
5. In this problem, what are the values for those heights and widths (numbers and units)?
6. What is the equation for Impulse (J)? Copy it here.

Critical Thinking & Concept Invention

1. Does the equation for Impulse represent a slope or area on this graph? Explain your thinking.
2. Compute both areas. Identify J_1, J_2, and $\sum J$.
3. If you know the force acting on an object over time, explain the process to compute the change in momentum (Δp).

Summary

Impulse is a product of Force ($\mathbf{F}$) and the time interval over which that force is applied (Δt). Specifically, Impulse is the area on a Force vs. Time graph. When forces are constant, this area is a rectangle, and the impulse is given by the equation:

$$\mathbf{J} = \mathbf{F}_{ext}\Delta t \tag{10}$$

When forces change with time, we need to use an integral to calculate this area. The general form of Impulse is therefore represented by the equation:

$$\mathbf{J} \equiv \int \mathbf{F}_{ext}\, dt \tag{11}$$

Apply

Car Crash: In a car crash, you and your car change your momentum from whatever initial driving momentum you had to zero in a short amount of time. Let's assume you are traveling 40 miles per hour and the combined mass of you and your car is 1250 kg. At the end of the crash, you and the car are both at rest. Construct Force vs. Time models and determine the forces experienced for each of the following circumstances:

- **Scenario 1: No Brakes**
  You crash and come to a stop in 0.2 seconds.
- **Scenario 2: Brakes**
  You see the hazard in front of you and apply a braking force of 8000 N for 1.5 seconds, then you crash and come to a stop in an additional 0.2 seconds.

Note: The forces you calculated are the total external force on the system of you and the car. You as a person experience much less force.

1.4 Multiple Forces and Multiple Impulses

Model

The timeline below shows different forces that act on an object for intervals of time. We will assume that individual forces do not change and are present for the entire time interval when they are shown.

Image needed here with Alt Text: Timeline showing multiple forces acting during different time intervals

Explore

1. Over what time intervals is each force applied to an object?
   - $\mathbf{F}_1$: _____ to _____
   - $\mathbf{F}_2$: _____ to _____
   - $\mathbf{F}_3$: _____ to _____
2. Fill in the column vectors with the force label to create an equation for the total impulse. Be sure to add +/– signs in front of each one based on axes with $\hat{x}$ to the right.
3. Group the terms in your equation above by factoring the vector portion (Force). Put in the + or – sign in front of each force label in the vectors and fill in the time intervals with Δt_1, Δt_2, and/or Δt_3.
4. Group the terms in your equation above by factoring the time portion. Put in the + or – sign in front of each force label in the

vectors and list all forces for each time interval.

Critical Thinking & Concept Invention

1. Review your equation in Explore Question 4. Do any forces cancel (equal and opposite) during any time intervals? If so, cross them out and then rewrite the equation below.
2. What might be the benefit of grouping according to force, as in Explore Question 3?
3. What might be the benefit of grouping according to time, as in Explore Question 4?
4. Summarize: How do you figure out the net impulse $(\sum \mathbf{J})$ on an object/system when there are multiple forces acting on the system for different time intervals?

Apply

1. Determine the total impulse on the object in the model if $F_1 = F_3 = 8N$ and $F_2 = 6N$. Do this by rewriting the equations from Explore 3 and Explore 4 with all variables filled in. Solve both (the answer should be the same).
2. A fan cart turns on its fan for 6 seconds and experiences a force of 2.5 N toward the right from the interaction between the fan and the air. After traveling for 3 seconds, a gust of wind applies an additional force of 1.8 N toward the left for 1.5 seconds. Write out an equation for the total impulse on the fan cart over the 6 seconds.

1.5 Impulse Between Objects

Model

Image needed here with Alt Text: Three system diagrams showing interacting objects and system boundaries

Explore

Tell a System Story for each system. Be sure to include:

- The number and names of the objects in the system.
- The number of internal interactions.
- The number of external interactions.
- The label of the external interaction(s), if any.

System 1	System 2	System 3

Critical Thinking & Concept Invention

1. Complete the middle column of the momentum chart for System 1. Explain how you completed it here.
2. Use the momentum chart to complete the $\Delta \mathbf{p}$ cell for Systems 2 and 3. How do those two vectors compare?
3. Summarize: When two objects interact, how does the impulse each object experiences compare? Explain why this has to be the case.

Summary

When two objects interact, they exchange equal amounts of impulse with opposite directions. This is given by the equation:

$$\mathbf{J}_{A \text{ on } B} = -\mathbf{J}_{B \text{ on } A} \tag{12}$$

Which is equivalent to...

$$\mathbf{F}_{A \text{ on } B} \Delta t = -\mathbf{F}_{B \text{ on } A} \Delta t \tag{13}$$

Since interactions between objects occur for the same length of time for each object, this leads to:

$$\mathbf{F}_{A \text{ on } B} = -\mathbf{F}_{B \text{ on } A} \tag{14}$$

This is called the Action-Reaction Law and is also known as Newton's Third Law.

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