

PHYS 111: PHYSICS, SYMMETRY, AND CONSERVATION



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This text was compiled on 06/11/2026

TABLE OF CONTENTS

Licensing

1: Conservation and Symmetry

- 1.1: Abstraction and Modeling
 - 1.1.1: About this Text
 - 1.1.2: Modeling in Physics
 - 1.1.3: Units and Standards
 - 1.1.4: Unit Conversion
 - 1.1.5: Dimensional Analysis
 - 1.1.6: Estimates and Fermi Calculations
 - 1.1.E: Exercises
- 1.2: Particles and Interactions
 - 1.2.1: Inertia
 - 1.2.2: Momentum
 - 1.2.3: Force and Impulse
 - 1.2.4: Examples
 - 1.2.5: Particles and Interactions (Exercises)
- 1.3: Vector Analysis
 - 1.3.1: Position Vectors and Components
 - 1.3.2: Vector Algebra in 1 Dimension
 - 1.3.3: Vector Algebra in 2 Dimensions- Graphical
 - 1.3.4: Vector Algebra in Multiple Dimensions- Calculations
 - 1.3.E: Vectors (Exercises)
- 1.4: Systems and The Center of Mass
 - 1.4.1: The Law of Inertia
 - 1.4.2: Extended Systems and Center of Mass
 - 1.4.3: Reference Frame Changes and Relative Motion
 - 1.4.4: Examples
 - 1.4.E: Systems and the Center of Mass Exercises
- 1.5: Conservation of Momentum
 - 1.5.1: Conservation of Linear Momentum
 - 1.5.2: The Problem Solving Framework
 - 1.5.3: Examples
 - 1.5.4: More Examples
 - 1.5.E: Conservation of Momentum (Exercises)
- 1.6: Conservation of Energy- Kinetic and Gravitational
 - 1.6.1: Kinetic Energy
 - 1.6.2: Conservative Interactions
 - 1.6.3: The Inverse-Square Law
 - 1.6.4: Conservation of Energy
 - 1.6.5: Dissipation of Energy and Thermal Energy
 - 1.6.6: Relative Velocity and the Coefficient of Restitution
 - 1.6.7: Examples
 - 1.6.E: Potential Energy and Conservation of Energy (Exercises)
- 1.7: Potential Energy- Graphs and Springs

- 1.7.1: Potential Energy of a System
- 1.7.2: Potential Energy Functions
- 1.7.3: Potential Energy Graphs
- 1.7.4: Examples
- 1.7.E: Potential Energy and Conservation of Energy (Exercises)
- 1.8: Work
 - 1.8.1: Introduction- Work and Impulse
 - 1.8.2: Work on a Single Particle
 - 1.8.3: The "Center of Mass Work"
 - 1.8.4: Examples
 - 1.8.E: Work and Kinetic Energy (Exercises)
- 1.9: Conservation of Angular Momentum I
 - 1.9.1: Angular Momentum
 - 1.9.2: Angular Momentum and Torque
 - 1.9.3: Examples
 - 1.9.E: Angular Momentum (Exercises)
- 1.10: Rotational Energy
 - 1.10.1: Rotational Kinetic Energy, and Moment of Inertia
 - 1.10.2: Rolling Motion
 - 1.10.3: Examples
 - 1.10.E: Fixed-Axis Rotation Introduction (Exercises)
- 1.11: Conservation of Angular Momentum II
 - 1.11.1: The Angular Momentum of a Point and The Cross Product
 - 1.11.2: Torque
 - 1.11.3: Examples
 - 1.11.4: Extended Mass Distributions
 - 1.11.E: Angular Momentum (Exercises)
- 1.12: Thermal Energy
 - 1.12.1: "Lost" Energy and the Discovery of Conservation of Energy
 - 1.12.2: Kinetic Theory of Heat
 - 1.12.3: Thermometers and Temperature Scales
 - 1.12.4: Heat Transfer, Specific Heat, and Calorimetry
 - 1.12.5: Microscopic Origins of Thermal Expansion
 - 1.12.6: Thermal Energy (Exercises)
- 1.13: Other Forms of Energy
 - 1.13.1: Phase Changes
 - 1.13.2: Mechanisms of Heat Transfer
 - 1.13.3: Temperature and Heat (Exercises)
- 1.14: Collisions in Summary
 - 1.14.1: Types of Collisions
 - 1.14.2: Examples
 - 1.14.E: Collisions (Exercises)

2: Forces and Kinematics

- 2.1: N1) Newton's Laws
 - 2.1.1: Forces and Newton's Three Laws
 - 2.1.2: Details on Newton's First Law
 - 2.1.3: Details on Newton's Second Law
 - 2.1.4: Details on Newton's Third Law

- 2.1.5: Free-Body Diagrams
- 2.1.6: Motion on a Circle (Or Part of a Circle)
- 2.1.7: Newton's Laws of Motion (Exercises)
- 2.2: N2) 1 Dimensional Kinematics
 - 2.2.1: Vector Calculus
 - 2.2.2: Position, Displacement, Velocity
 - 2.2.3: Acceleration
 - 2.2.4: Free Fall
 - 2.2.5: The Connection Between Displacement, Velocity, and Acceleration
 - 2.2.6: Examples
 - 2.2.E: Motion Along a Straight Line (Exercises)
- 2.3: N3) 2 Dimensional Kinematics and Projectile Motion
 - 2.3.1: Dealing with Forces in Two Dimensions
 - 2.3.2: Motion in Two Dimensions and Projectile Motion
 - 2.3.3: Inclined Planes
 - 2.3.4: Examples
 - 2.3.E: Projectile Motion (Exercises)
- 2.4: N4) Motion from Forces
 - 2.4.1: Solving Problems with Newton's Laws (Part 1)
 - 2.4.2: Solving Problems with Newton's Laws (Part 2)
 - 2.4.3: Examples
 - 2.4.E: Newton's Laws of Motion (Exercises)
- 2.5: N5) Friction
 - 2.5.1: Friction (Part 1)
 - 2.5.2: Friction (Part 2)
 - 2.5.3: More Examples
 - 2.5.E: Friction (Exercises)
- 2.6: N6) Statics and Springs
 - 2.6.1: Conditions for Static Equilibrium
 - 2.6.2: Springs
 - 2.6.3: Examples
 - 2.6.E: Static Equilibrium and Elasticity (Exercises)
- 2.7: N7) Circular Motion
 - 2.7.1: Banking
 - 2.7.2: Examples
 - 2.7.E: Applications of Newton's Laws (Exercises)
- 2.8: N8) Forces, Energy, and Work
 - 2.8.1: Forces and Potential Energy
 - 2.8.2: Work Done on a System By All the External Forces
 - 2.8.3: Forces Not Derived From a Potential Energy
 - 2.8.4: Examples
 - 2.8.E: Work and Kinetic Energy (Exercises)
- 2.9: N9) Rotational Motion
 - 2.9.1: Rotational Variables
 - 2.9.2: Rotation with Constant Angular Acceleration
 - 2.9.3: Relating Angular and Translational Quantities
 - 2.9.4: Newton's Second Law for Rotation
 - 2.9.5: Examples
 - 2.9.E: Fixed-Axis Rotation Introduction (Exercises)

- 2.10: N10) Simple Harmonic Motion
 - 2.10.1: Introduction- The Physics of Oscillations
 - 2.10.2: Simple Harmonic Motion
 - 2.10.3: Pendulums
 - 2.10.4: In Summary
 - 2.10.5: Examples
 - 2.10.6: Advanced Topics
 - 2.10.7: Simple Harmonic Motion- Exercises

3: Relativity (in progress)

- 3.1: Einstein's Postulates
- 3.2: The Relativity Principle
- 3.3: The Nature of Time
- 3.4: Measuring Lengths and Coincidences
- 3.5: Waves and Aether
- 3.6: (Optional - Includes more math) Derivation of the Lorentz Transformations
- 3.7: Relativistic Velocity Transformation
- 3.8: Doppler Effect for Light
- 3.9: An Invariant Measure of Length
- 3.10: Ultrarelativistic particles
- 3.11: $E=mc^2$
- 3.12: Relativistic Momentum
- 3.13: Relativistic Energy
- 3.14: Time Dialation and Space Contraction Revisited
- 3.15: (Graphical Derivation using World lines) Addition of Velocities
- 3.16: Twin Paradox
- 3.17: Paradoxes
- 3.A: Relativity (Answers)
- 3.E: Relativity (Exercises)

4: Miscellanea

- 4.1: Problem Solving
 - 4.1.1: All models are wrong, some models are useful
- 4.2: Math Review
 - 4.2.1: Introduction
 - 4.2.2: Geometrical Shapes
 - 4.2.3: Triangles
 - 4.2.4: The Rectangular Coordinate Systems and Graphs
 - 4.2.5: Finding Angle Measurements
 - 4.2.6: Parallel and Perpendicular Lines
 - 4.2.7: Solving Linear Equations and Inequalities
 - 4.2.7.1: Solving Linera Equations
 - 4.2.7.2: Solving Inequalities
 - 4.2.7.3: Solving Quadratic Equations
 - 4.2.7.4: Solving a System of Linear Equations
 - 4.2.7.5: Solving a System of Linear Equations with Cramer's Rule
 - 4.2.8: Functions
 - 4.2.8.1: Basic Functions
 - 4.2.8.2: Trigonometric Functions
 - 4.2.8.3: Exponential_and_Logarithmic_Functions

- [4.2.8.4: Properties_of_Logarithms](#)
- [4.2.8.5: Exponential and Logarithmic Models](#)
- [4.2.9: Vectors](#)
 - [4.2.9.1: Review of Trigonometry](#)
 - [4.2.9.2: Right Angle Triangle Trigonometry](#)
 - [4.2.9.3: Scalars and Vectors](#)
 - [4.2.9.4: Coordinate Systems and Components of a Vector](#)
 - [4.2.9.5: Algebra of Vectors](#)
 - [4.2.9.6: Products of Vectors](#)
 - [4.2.9.7: Further Topics](#)
 - [4.2.9.E: Practice](#)
- [4.2.10: Vectors](#)

[Index](#)

[Glossary](#)

[Detailed Licensing](#)

[Detailed Licensing](#)

Licensing

A detailed breakdown of this resource's licensing can be found in [Back Matter/Detailed Licensing](#).

CHAPTER OVERVIEW

1: Conservation and Symmetry

Topic hierarchy

- 1.1: Abstraction and Modeling
 - 1.1.1: About this Text
 - 1.1.2: Modeling in Physics
 - 1.1.3: Units and Standards
 - 1.1.4: Unit Conversion
 - 1.1.5: Dimensional Analysis
 - 1.1.6: Estimates and Fermi Calculations
 - 1.1.E: Exercises
- 1.2: Particles and Interactions
 - 1.2.1: Inertia
 - 1.2.2: Momentum
 - 1.2.3: Force and Impulse
 - 1.2.4: Examples
 - 1.2.5: Particles and Interactions (Exercises)
- 1.3: Vector Analysis
 - 1.3.1: Position Vectors and Components
 - 1.3.2: Vector Algebra in 1 Dimension
 - 1.3.3: Vector Algebra in 2 Dimensions- Graphical
 - 1.3.4: Vector Algebra in Multiple Dimensions- Calculations
 - 1.3.E: Vectors (Exercises)
- 1.4: Systems and The Center of Mass
 - 1.4.1: The Law of Inertia
 - 1.4.2: Extended Systems and Center of Mass
 - 1.4.3: Reference Frame Changes and Relative Motion
 - 1.4.4: Examples
 - 1.4.E: Systems and the Center of Mass Exercises
- 1.5: Conservation of Momentum
 - 1.5.1: Conservation of Linear Momentum
 - 1.5.2: The Problem Solving Framework
 - 1.5.3: Examples
 - 1.5.4: More Examples
 - 1.5.E: Conservation of Momentum (Exercises)
- 1.6: Conservation of Energy- Kinetic and Gravitational
 - 1.6.1: Kinetic Energy
 - 1.6.2: Conservative Interactions
 - 1.6.3: The Inverse-Square Law
 - 1.6.4: Conservation of Energy
 - 1.6.5: Dissipation of Energy and Thermal Energy
 - 1.6.6: Relative Velocity and the Coefficient of Restitution

- 1.6.7: Examples
- 1.6.E: Potential Energy and Conservation of Energy (Exercises)
- 1.7: Potential Energy- Graphs and Springs
 - 1.7.1: Potential Energy of a System
 - 1.7.2: Potential Energy Functions
 - 1.7.3: Potential Energy Graphs
 - 1.7.4: Examples
 - 1.7.E: Potential Energy and Conservation of Energy (Exercises)
- 1.8: Work
 - 1.8.1: Introduction- Work and Impulse
 - 1.8.2: Work on a Single Particle
 - 1.8.3: The "Center of Mass Work"
 - 1.8.4: Examples
 - 1.8.E: Work and Kinetic Energy (Exercises)
- 1.9: Conservation of Angular Momentum I
 - 1.9.1: Angular Momentum
 - 1.9.2: Angular Momentum and Torque
 - 1.9.3: Examples
 - 1.9.E: Angular Momentum (Exercises)
- 1.10: Rotational Energy
 - 1.10.1: Rotational Kinetic Energy, and Moment of Inertia
 - 1.10.2: Rolling Motion
 - 1.10.3: Examples
 - 1.10.E: Fixed-Axis Rotation Introduction (Exercises)
- 1.11: Conservation of Angular Momentum II
 - 1.11.1: The Angular Momentum of a Point and The Cross Product
 - 1.11.2: Torque
 - 1.11.3: Examples
 - 1.11.4: Extended Mass Distributions
 - 1.11.E: Angular Momentum (Exercises)
- 1.12: Thermal Energy
 - 1.12.1: "Lost" Energy and the Discovery of Conservation of Energy
 - 1.12.2: Kinetic Theory of Heat
 - 1.12.3: Thermometers and Temperature Scales
 - 1.12.4: Heat Transfer, Specific Heat, and Calorimetry
 - 1.12.5: Microscopic Origins of Thermal Expansion
 - 1.12.6: Thermal Energy (Exercises)
- 1.13: Other Forms of Energy
 - 1.13.1: Phase Changes
 - 1.13.2: Mechanisms of Heat Transfer
 - 1.13.3: Temperature and Heat (Exercises)
- 1.14: Collisions in Summary
 - 1.14.1: Types of Collisions
 - 1.14.2: Examples
 - 1.14.E: Collisions (Exercises)

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SECTION OVERVIEW

1.1: Abstraction and Modeling

This first section is going to serve two purposes: first, introduce some of the key important concepts that we will need right from the beginning, like units and significant figures. Second, we want to open up the question: "what is physics"? Like many such open-ended questions, there are many ways to answer it. It is not going to be our intention to answer this fully in any sense - we just want to answer it enough so that as a student, you know why this book exists and what you should be getting out of it. To get right to the point: **physics is the study of the abstraction of the physical world.**

So that's a pretty obscure statement, so let's break it down a bit. When we say "abstraction", that's basically a way of replacing all the real physical objects out there that we want to understand (balls, cars, airplanes, etc...objects!) with mathematical representations of those objects. These representations are simple - sometimes they are just points, or maybe box-shaped things with wheels to represent cars, etc. The point of the abstraction is both to simplify the problem (points are easier to deal with than real life planes), and also to make it mathematically precise. "how a car moves" is actually a question far too complicated for a simple introduction to the world of physics to handle. There are literally millions (way more!) of interactions taking place in the motion of a car - things like the engine, the wheels, and the transmission - but even little things like the air molecules hitting the outside of the car and slowing it down. By replacing the entire thing with "a point" or "a box with wheels", we make the problem simple enough for us to actually perform some calculations to understand it's motion.

We should be clear what the kind of mathematical precision we are talking about here as well. We don't mean "calculating the exact value of the speed of the car to 5 decimal places" (we'll see in this chapter that kind of precision is not actually what we are after). We mean something more like "well-defined". For example, if we said "The position of the car is 5 m from the end of the race", we can't actually perform any calculations with that information, because *which part of the car are we talking about?* For sure, you could say "the front", or "the back", or "30 cm from the driver", but now we might need to know even more things about the car - how big is it, or how big the seats are (to determine the position of the driver relative to the finish line). The abstraction avoids all this by the replacement of real objects with simple representations of them. If the car is represented as a point, then "5 m from the end of the race" becomes very well-defined, because it can only refer to one location! Similarly for the representation of the car as a box with wheels - more information is required (dimensions of the box and the wheels?), but it's still far less than what might be required for the real, physical car.

Now that we've covered "abstraction", what does "study of the abstraction" mean? Well, it's often said that physics is "the study of the fundamental interactions" - that's what we mean, but we mean it specifically *within the abstraction*. We aren't actually studying how the fundamental interactions of the real world influence real physical objects - we are representing the world with an abstraction, and studying the abstraction. The force of gravity is actually enormously complicated in the real world (it's everywhere, it acts between each individual molecule and each other individual molecule....you'd be calculating for your entire life!), but in our abstraction, it's just a simple force acting between two points. By proposing simple, fundamental interactions between objects we are hoping to model the real world.

Of course, we claim that physics actually is the study of the real world, but you aren't ever going to see that connection by using solely the abstraction - you have to go into the laboratory and perform experiments to verify that our abstraction is an "accurate and precise" representation of the real world. We aren't going to talk much about that in this text, since it is primarily designed to give you an introduction to the prediction and quantitative aspects of the field - hopefully, your theoretical experience with this text will be coupled with a laboratory experience as well, so that you can see how well physics actually does as describing phenomena and events in the real world.

1.1.1: About this Text

1.1.2: Modeling in Physics

1.1.3: Units and Standards

[1.1.4: Unit Conversion](#)

[1.1.5: Dimensional Analysis](#)

[1.1.6: Estimates and Fermi Calculations](#)

[1.1.E: Exercises](#)

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1.1.1: About this Text

Introduction to Text

This is a remix of a remix of a remix of a text. Each class is taught a little differently and at Gettysburg College, we have been teaching an introductory physics class for majors that follows the general structure of this text for many years. Our teaching has always emphasized "active learning". This means that instead of passively listening to a lecture, students are actively engaged in the class: Answering questions, discussing concepts with each other, and doing active problem solving on the board. Why teach this way? Simply put, there is an ENORMOUS amount of research that shows that this is the most effective way to learn physics. Strangely, often students do not like it quite as much and they even THINK that they are not learning as much, but study after study shows that they simply learn more when using these techniques. (See [Deslauriers et al., PNAS, 2019](#) if you don't believe me.) Since our goal is for all of you to learn as much physics as possible in the short time we have together, these are the techniques we will use.

Below you will see the original introduction to this text. It explains a lot of what was done with this text originally. A lot of this is true for our class as well with a few exceptions. The most prominent one is that we are not using a "studio physics" model. Instead, this class will be more traditional with regular homework, as many examples in the text that I can write, and in-class exams. However, a lot of the things that are said below are true. The main ones are:

1. You MUST do work before you come to class. There will be little-to-no lecture in class. The Professor will assume that you have read the text before coming to class and are familiar with the basic concepts. If you come into class without having learned the basic material beforehand, the class will not be very useful at all.
2. We will try to do approximately one chapter per day or two days. The chapters are purposely very short to allow for this (and for you to be able to read them the night before in an hour or so).
3. We will not be following a "traditional" order for the text. We will start with conservation laws (the most fundamental concepts in physics) and then move onto Newtonian Physics from there. This makes sense for all of the reasons outlined below.

I have tried to update this text to fit better with Gettysburg College's course. This includes adding problems to the end of chapters (largely from OpenStax) and updating chapters, adding chapters/sections, and fixing a few things I see. Hopefully I actually improve the text, but who ever knows?

I want to thank Professor Christopher Duston for adapting the Gea-Banacloche text to the current iteration and, obviously, Gea-Banacloche (and the OpenStax authors) for the original texts.

If you are reading this, please know that I am constantly trying to improve this text. Please contact me ([mpowers at gettysburg dot edu](mailto:mpowers@gettysburg.edu)) with any errors that you find or suggestions for improvements. This is DEFINITELY a work in progress.

Original Introduction:

This text has been written with several goals in mind:

- Break sections into short chunks appropriate for reading before each class meeting, specifically for use in "studio physics" classrooms.
- Present conservation laws before kinematics.
- Acknowledge that students may be co-enrolled in their first calculus course.

Before proceeding with the text, we will motivate and explain each of these goals. Being an open source resource, this text is constantly in a state of improvement. Feel free to write to the editor, Professor Christopher Duston ([dustonc at merrimack dot edu](mailto:dustonc@merrimack.edu)) if you find significant errors that you think should be corrected.

This text has been written to support a so-called "studio" approach to teaching physics. There are many flavors of this pedagogy (for example, [SCALE-UP](#) from North Carolina State, [Studio Physics](#) from Florida State, and [Studio Physics](#) at The Massachusetts Institute of Technology), but an essential element is that class time is significantly devoted to problem solving over lecture time. To support this, it is necessary for students to be exposed to the material at some level before even entering the classroom. There are several ways to achieve that (this is the so-called "[flipped classroom model](#)"), but it is our feeling that student success is maximized by designing this initial exposure around the studio model, rather than using a traditional textbook. A traditional and carefully constructed 1000-page physics textbook is a beautiful thing, but if the material is broken into sections appropriate for the content, rather than the delivery system, there will be a fundamental mismatch between the goals of the students and the goals of the

textbook. So the first goal of this text is to break the content into sections appropriate for single class meetings. At Merrimack College (where this textbook was designed to be used), there are two class meetings per week, and each meeting tackles one section of this text.

This text was also designed to be the initial source of exposure to the content. Another challenge when using a traditional physics textbook in the studio model is that the first thing a novice reads about might be a completely well-thought out and well-motivated description of a physical law, but a student is likely not prepared for that level of description. The process of learning is not linear and hierarchical, but chaotic and varied. Of course, in the end we want students to be experts a la [Bloom's Taxonomy](#), but starting from the bottom and explaining how they get to the top will not service their own internalized development of the material. To that end, the goal of this text is primarily to provide students with enough material to "soak the sponge", so that when they enter the class they are prepared to learn (and make mistakes!), rather than prepared to solve problems on an exam. To achieve this, the material is presented in a streamlined matter, with simplified motivations and only the first few initial steps spelled out. For sure, this means more complicated material is occasionally only briefly touched upon, and in some cases left out completely. However, we would argue that as a fundamental science, more complicated material in the field of physics can always be added in by instructor demand, in a way that their own students can handle it. And, given that anything can be found on the internet, collecting special topics in textbooks seems like something which has already been done by any number of a myriad of authors. If that's what you want, this textbook is not for you!

The content order of this textbook (conservation laws before Newton's laws) is inspired by the wonderful texts of Eric Mazur and Thomas Moore (in fact, the current version of this book blatantly steals the chapter order from Moore's *Six Ideas In Physics*, Units C and N). For more well-thought out motivations, one can consult those admirable texts - for our part, we will simply motivate this in two ways. First, in many ways conservation laws are more fundamental than Newton's laws. Indeed, the argument about the invariance of specific quantities under time and space translations is so fundamental that it borders on the Philosophical nature of the Universe, rather than our particular approach to modeling it. More directly, equations like $\Delta E = 0$ are scalar and discrete, and therefore typically easier for students to deal with than vector expressions that imply continuous change, like $\Sigma \vec{F} = m\vec{a}$. Second, it is our experience that students have already been exposed to Newton's laws and kinematics - and in some cases have built up significant misunderstandings that are difficult to break down. (A trivial example might be the statement that "the force of gravity is negative!", a viewpoint strongly held by many students. This concept is so rife with intellectual inconsistency that I regret even bringing it up...) By starting students with new, consistently correct ideas about modeling and problem solving in the context of conservation laws, we prepare them to tear down their previous knowledge and rebuild it on a more solid foundation.

Finally, this textbook is written with the knowledge that some (many?) students are going to be stepping into their first physics class while simultaneously taking their first calculus class. This breaks the traditional "calculus before physics" mantra that has been a part of our pedagogy for decades. The source of this change will not be covered here, but we are highly motivated to deal with it rather than preach about its evils from the rooftops. Of course, as a textbook that primarily covers mechanics, it's absolutely true that most of what can be found in this textbook is actually directly derivable from simple ideas based in calculus. However, that is also not usually where students coming to the field are, and it would be an injustice to ask them to climb that mountain before demonstrating the joy of understanding basic concepts in the field. More practically, it is our experience that in so-called "calculus-based physics classes", the biggest challenges for students is not the calculus itself, but the vector algebra and analysis which is utilized in nearly no other field as extensively as physics. Therefore, we do not view the loss of a semester of calculus before this class as a significant barrier, and do not start by assuming students have it. By the second half of the book (Newton's laws), it is assumed that students have a grasp of all the basic notations from calculus.

Open Source Resources and the Construction of This Text

From the outset, we must acknowledge that this text fails in all the goals presented above. Writing a text from scratch is a lengthy business, and we have students to teach! However, since the advent of free and open source educational resources, another avenue has been opened - the "collect, evaluate, and modify" approach. Open source solutions have been available for some time ([University Physics](#) by OpenStax is perhaps the most well-known example, as well as [Calculus-Based Physics](#) by Jeffrey Schnick), but these followed the more traditional order of material. However, once Julio Gea-Banacloche published his fantastic [University Physics I: Classical Mechanics](#), the possibility of satisfying our goals via open source publication became realizable. However, rearranging and breaking apart a textbook comes with it's own dangers, and we are squarely in the middle of those dangers with this text, which was constructed in the following way:

1. Rearrange the Gea-Banacloche text to match the specific order of material, which closely follows Thomas Moore's book.
2. Supplement material which is lacking from OpenStax.

3. Write new material to fill in some gaps.

As such, we owe a great deal of gratitude to both Gea-Banacloche and the OpenStax organization, for making such an approach feasible. However, the responsibility is now all on our shoulders, to continue to modify and develop this book so that eventually we can satisfy the lofty pedagogical and practical goals we outlined at the beginning of this section.

Whiteboard Problems

In our studio classroom, over half the time is spent solving problems on whiteboards in groups. During this time, the instructor (and typically one or two learning assistants) are on hand and available to assist if a group gets stuck. Over the years, we've developed many more of these style problems than we will typically use (3-4 per class period, generally), so we've included the extras in this text. At the end of each chapter we will have an "Example" section, and these whiteboard problems will be presented there (along with some good examples from the two main source texts we discussed above). These can be used identically to example problems, or they could be used as actual whiteboard problems for a studio classroom. In the future, we are planning on adding our entire set of whiteboard problems to these sections - this will correspond with a generally shortening of the actual texts of the chapter, to match our goal of being accessible as a pre-class reading assignment.

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1.1.2: Modeling in Physics

Classical mechanics is the branch of physics that deals with the study of the motion of anything (roughly speaking) larger than an atom or a molecule. That is a lot of territory, and the methods and concepts of classical mechanics are at the foundation of any branch of science or engineering that is concerned with the motion of anything from a star to an amoeba—fluids, rocks, animals, planets, and any and all kinds of machines. Moreover, even though the accurate description of processes at the atomic level requires the (formally very different) methods of quantum mechanics, at least three of the basic concepts of classical mechanics that we are going to study this semester, namely, momentum, energy, and angular momentum, carry over into quantum mechanics as well, with the last two playing, in fact, an essential role.

Particles in Classical Mechanics

In the study of motion, the most basic starting point is the concept of the *position* of an object. Clearly, if we want to describe accurately the position of a macroscopic object such as a car, we may need a lot of information, including the precise shape of the car, whether it is turned this way or that way, and so on; however, if all we want to know is how far the car is from Fort Smith or Fayetteville, we do not need any of that: we can just treat the car as a dot, or mathematical point, on the map—which is the way your GPS screen will show it, anyway. When we do this, we say that we are describing the car (or whatever the macroscopic object may be) as a **particle**.

In classical mechanics, an “ideal” particle is an object with no appreciable size—a mathematical point. In one dimension (that is to say, along a straight line), its position can be specified just by giving a single number, the distance from some reference point, as we shall see in a moment (in three dimensions, of course, three numbers are required). In terms of energy (which is perhaps the most important concept in all of physics, and which we will introduce properly in due course), an ideal particle has only one kind of energy, what we will later call *translational kinetic energy*; it cannot have, for instance, rotational kinetic energy (since it has “no shape” for practical purposes), or any form of internal energy (elastic, thermal, etc.), since we assume it is too small to have any internal structure in the first place.

The reason this is a useful concept is not just that we can often treat extended objects as particles in an approximate way (like the car in the example above), but also, and most importantly, that if we want to be more precise in our calculations, *we can always treat an extended object (mathematically) as a collection of “particles.”* The physical properties of the object, such as its energy, momentum, rotational inertia, and so forth, can then be obtained by adding up the corresponding quantities for all the particles making up the object. Not only that, but the interactions between two extended objects can also be calculated by adding up the interactions between all the particles making up the two objects. This is how, once we know the form of the gravitational force between two particles we can use that to calculate the force of gravity between a planet and its satellites, which can be fairly complicated in detail, depending, for instance, on the relative orientation of the planet and the satellite.

The mathematical tool we use to calculate these “sums” is *calculus*—specifically, integration—and you will see many examples of this... in your calculus courses. Calculus I is only lightly used in this text, so we will not make a lot of use of it here, and in any case you would need multidimensional integrals, which are an even more advanced subject, to do these kinds of calculations. But it may be good for you to keep these ideas on the back of your mind. Calculus was, in fact, invented by Sir Isaac Newton precisely for this purpose, and the developments of physics and mathematics have been closely linked together ever since.

Anyway, back to particles, the plan for this semester is as follows: we will start our description of motion by treating every object (even fairly large ones, such as cars) as a “particle,” because we will only be concerned at first with its translational motion and the corresponding energy. Then we will progressively make things more complex: by considering systems of two or more particles, we will start to deal with the *internal energy* of a system. Then we will move to the study of *rigid bodies*, which are another important idealization: extended objects whose parts all move together as the object undergoes a translation or a rotation. This will allow us to introduce the concept of rotational kinetic energy. Eventually we will consider *wave motion*, where different parts of an extended object (or “medium”) move relative to each other. So, you see, there is a logical progression here, with most parts of the course building on top of the previous ones, and energy as one of the main connecting themes. The technical word for the process being described in the preceding paragraphs is **abstraction**; the essence of which is to take the physical world and model it with abstract mathematical quantities. We can then use these quantities to construct physical theories, make predictions, and verify them with experiments. In this text we will primarily be interested in the middle section of this process - *taking physical laws and making predictions about how objects in the real world will behave*.

Aside: The Atomic Perspective

As an aside, it should perhaps be mentioned that the building up of classical mechanics around this concept of ideal particles had nothing to do, initially, with any belief in “atoms,” or an atomic theory of matter. Indeed, for most 18th and 19th century physicists, matter was supposed to be a continuous medium, and its (mental) division into particles was just a mathematical convenience.

The atomic hypothesis became increasingly more plausible as the 19th century wore on, and by the 1920’s, when quantum mechanics came along, physicists had to face a surprising development: matter, it turned out, was indeed made up of “elementary particles,” but these particles could not, in fact, be themselves described by the laws of classical mechanics. One could not, for instance, attribute to them simultaneously well-defined positions and velocities. Yet, in spite of this, most of the conclusions of classical mechanics remain valid for macroscopic objects, because, most of the time, it is OK to (formally) “break up” extended objects into chunks that are small enough to be treated as particles, but large enough that one does not need quantum mechanics to describe their behavior.

Quantum properties were first found to manifest themselves at the macroscopic level when dealing with thermal energy, because at one point it really became necessary to figure out where and how the energy was stored at the truly microscopic (atomic) level. Thus, after centuries of successes, classical mechanics met its first failure with the so-called *problem of the specific heats*, and a completely new physical theory—quantum mechanics—had to be developed in order to deal with the newly-discovered atomic world. But all this, as they say, is another story, and for our very brief dealings with thermal physics—the last chapter in this book—we will just take specific heats as given, that is to say, something you measure (or look up in a table), rather than something you try to calculate from theory.

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1.1.3: Units and Standards

Learning Objectives

- Describe how SI base units are defined.
- Describe how derived units are created from base units.
- Express quantities given in SI units using metric prefixes.

The range of objects and phenomena studied in physics is immense. From the incredibly short lifetime of a nucleus to the age of the universe, from the tiny sizes of subnuclear particles to the vast distance to the edges of the known universe, from the force exerted by a jumping flea to the force between Earth and the Sun, there are enough factors of 10 to challenge the imagination of even the most experienced scientist.

Giving numerical values for physical quantities and equations for physical principles allows us to understand nature much more deeply than qualitative descriptions alone. To comprehend these vast ranges, we must also have accepted units in which to express them. We shall find that even in the potentially mundane discussion of meters, kilograms, and seconds, a profound simplicity of nature appears: all physical quantities can be expressed as combinations of only seven base physical quantities.

We define a **physical quantity** either by specifying how it is measured or by stating how it is calculated from other measurements. For example, we might define distance and time by specifying methods for measuring them, such as using a meter stick and a stopwatch. Then, we could define average speed by stating that it is calculated as the total distance traveled divided by time of travel.

Measurements of physical quantities are expressed in terms of **units**, which are standardized values. For example, the length of a race, which is a physical quantity, can be expressed in units of meters (for sprinters) or kilometers (for distance runners). Without standardized units, it would be extremely difficult for scientists to express and compare measured values in a meaningful way (Figure 1.1.3.1).

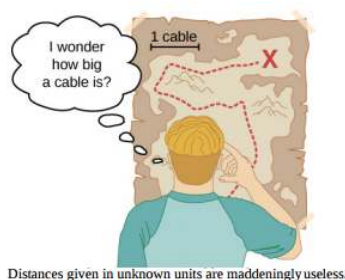


Figure 1.1.3.1: Distances given in unknown units are maddeningly useless.

Two major systems of units are used in the world: **SI units** (for the French **Système International d'Unités**), also known as the **metric system**, and **English units** (also known as the **customary** or **imperial system**). English units were historically used in nations once ruled by the British Empire and are still widely used in the United States. English units may also be referred to as the **foot-pound-second** (fps) system, as opposed to the **centimeter-gram-second** (cgs) system. You may also encounter the term **SAE units**, named after the Society of Automotive Engineers. Products such as fasteners and automotive tools (for example, wrenches) that are measured in inches rather than metric units are referred to as **SAE fasteners** or **SAE wrenches**.

Virtually every other country in the world (except the United States) now uses SI units as the standard. The metric system is also the standard system agreed on by scientists and mathematicians.

SI Units: Base and Derived Units

In any system of units, the units for some physical quantities must be defined through a measurement process. These are called the **base quantities** for that system and their units are the system's **base units**. All other physical quantities can then be expressed as algebraic combinations of the base quantities. Each of these physical quantities is then known as a **derived quantity** and each unit is called a **derived unit**. The choice of base quantities is somewhat arbitrary, as long as they are independent of each other and all other quantities can be derived from them. Typically, the goal is to choose physical quantities that can be measured accurately to a

high precision as the base quantities. The reason for this is simple. Since the derived units can be expressed as algebraic combinations of the base units, they can only be as accurate and precise as the base units from which they are derived.

Based on such considerations, the International Standards Organization recommends using seven base quantities, which form the International System of Quantities (ISQ). These are the base quantities used to define the SI base units. Table 1.1.3.1 lists these seven ISQ base quantities and the corresponding SI base units.

Table 1.1.3.1: ISQ Base Quantities and Their SI Units

ISQ Base Quantity	SI Base Unit
Length	meter (m)
Mass	kilogram (kg)
Time	second (s)
Electrical Current	ampere (A)
Thermodynamic Temperature	kelvin (K)
Amount of Substance	mole (mol)
Luminous Intensity	candela (cd)

You are probably already familiar with some derived quantities that can be formed from the base quantities in Table 1.1.3.1. For example, the geometric concept of area is always calculated as the product of two lengths. Thus, area is a derived quantity that can be expressed in terms of SI base units using square meters ($\text{m} \times \text{m} = \text{m}^2$). Similarly, volume is a derived quantity that can be expressed in cubic meters (m^3). Speed is length per time; so in terms of SI base units, we could measure it in meters per second (m/s). Volume mass density (or just density) is mass per volume, which is expressed in terms of SI base units such as kilograms per cubic meter (kg/m^3). Angles can also be thought of as derived quantities because they can be defined as the ratio of the arc length subtended by two radii of a circle to the radius of the circle. This is how the radian is defined. Depending on your background and interests, you may be able to come up with other derived quantities, such as the mass flow rate (kg/s) or volume flow rate (m^3/s) of a fluid, electric charge ($\text{A} \cdot \text{s}$), mass flux density [$\text{kg}/(\text{m}^2 \cdot \text{s})$], and so on. We will see many more examples throughout this text. For now, the point is that every physical quantity can be derived from the seven base quantities in Table 1.1.3.1, and the units of every physical quantity can be derived from the seven SI base units.

For the most part, we use SI units in this text. Non-SI units are used in a few applications in which they are in very common use, such as the measurement of temperature in degrees Celsius ($^{\circ}\text{C}$), the measurement of fluid volume in liters (L), and the measurement of energies of elementary particles in electron-volts (eV). Whenever non-SI units are discussed, they are tied to SI units through conversions. For example, $1 \text{ L} = 10^{-3} \text{ m}^3$.

Check out a comprehensive source of information on SI units at the National Institute of Standards and Technology (NIST) [Reference on Constants, Units, and Uncertainty](#).

Units of Time, Length, and Mass: The Second, Meter, and Kilogram

The initial chapters in this textmap are concerned with mechanics, fluids, and waves. In these subjects all pertinent physical quantities can be expressed in terms of the base units of length, mass, and time. Therefore, we now turn to a discussion of these three base units, leaving discussion of the others until they are needed later.

The Second

The SI unit for time, the **second** (abbreviated s), has a long history. For many years it was defined as $1/86,400$ of a mean solar day. More recently, a new standard was adopted to gain greater accuracy and to define the second in terms of a nonvarying or constant physical phenomenon (because the solar day is getting longer as a result of the very gradual slowing of Earth's rotation). Cesium atoms can be made to vibrate in a very steady way, and these vibrations can be readily observed and counted. In 1967, the second was redefined as the time required for 9,192,631,770 of these vibrations to occur (Figure 1.1.3.2). Note that this may seem like more precision than you would ever need, but it isn't—GPSs rely on the precision of atomic clocks to be able to give you turn-by-turn directions on the surface of Earth, far from the satellites broadcasting their location.



Figure 1.1.3.2: An atomic clock such as this one uses the vibrations of cesium atoms to keep time to a precision of better than a microsecond per year. The fundamental unit of time, the second, is based on such clocks. This image looks down from the top of an atomic fountain nearly 30 feet tall. (credit: Steve Jurvetson)

The Meter

The SI unit for length is the **meter** (abbreviated m); its definition has also changed over time to become more precise. The meter was first defined in 1791 as 1/10,000,000 of the distance from the equator to the North Pole. This measurement was improved in 1889 by redefining the meter to be the distance between two engraved lines on a platinum–iridium bar now kept near Paris. By 1960, it had become possible to define the meter even more accurately in terms of the wavelength of light, so it was again redefined as 1,650,763.73 wavelengths of orange light emitted by krypton atoms. In 1983, the meter was given its current definition (in part for greater accuracy) as the distance light travels in a vacuum in $1/299,792,458$ of a second (Figure 1.1.3.3). This change came after knowing the speed of light to be exactly 299,792,458 m/s. The length of the meter will change if the speed of light is someday measured with greater accuracy.

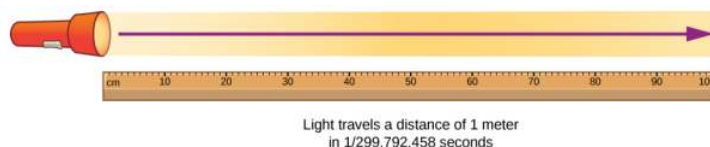


Figure 1.1.3.3: The meter is defined to be the distance light travels in $1/299,792,458$ of a second in a vacuum. Distance traveled is speed multiplied by time.

The Kilogram

The SI unit for mass is the **kilogram** (abbreviated kg); From 1795–2018 it was defined to be the mass of a platinum–iridium cylinder kept with the old meter standard at the International Bureau of Weights and Measures near Paris. However, this cylinder has lost roughly 50 micrograms since it was created. Because this is the standard, this has shifted how we defined a kilogram. Therefore, a new definition was adopted in May 2019 based on the Planck constant and other constants which will never change in value. We will study Planck’s constant in quantum mechanics, which is an area of physics that describes how the smallest pieces of the universe work. The kilogram is measured on a Kibble balance (see 1.1.3.4). When a weight is placed on a Kibble balance, an electrical current is produced that is proportional to Planck’s constant. Since Planck’s constant is defined, the exact current measurements in the balance define the kilogram.



Figure 1.1.3.4: Redefining the SI unit of mass. The U.S. National Institute of Standards and Technology's Kibble balance is a machine that balances the weight of a test mass with the resulting electrical current needed for a force to balance it.

Metric Prefixes

SI units are part of the **metric system**, which is convenient for scientific and engineering calculations because the units are categorized by factors of 10. Table 1.1.3.1 lists the metric prefixes and symbols used to denote various factors of 10 in SI units. For example, a centimeter is one-hundredth of a meter (in symbols, $1 \text{ cm} = 10^{-2} \text{ m}$) and a kilometer is a thousand meters ($1 \text{ km} = 10^3 \text{ m}$). Similarly, a megagram is a million grams ($1 \text{ Mg} = 10^6 \text{ g}$), a nanosecond is a billionth of a second ($1 \text{ ns} = 10^{-9} \text{ s}$), and a terameter is a trillion meters ($1 \text{ Tm} = 10^{12} \text{ m}$).

Table 1.1.3.2: Metric Prefixes for Powers of 10 and Their Symbols

Prefix	Symbol	Meaning	Prefix	Symbol	Meaning
quetta-	Q	10^{30}	quecto-	q	10^{-30}
ronna-	R	10^{27}	ronto-	r	10^{-27}
yotta-	Y	10^{24}	yocto-	y	10^{-24}
zetta-	Z	10^{21}	zepto-	z	10^{-21}
exa-	E	10^{18}	atto-	a	10^{-18}
peta-	P	10^{15}	femto-	f	10^{-15}
tera-	T	10^{12}	pico-	p	10^{-12}
giga-	G	10^9	nano-	n	10^{-9}
mega-	M	10^6	micro-	μ	10^{-6}
kilo-	k	10^3	milli-	m	10^{-3}
hecto-	h	10^2	centi-	c	10^{-2}
deka-	da	10^1	deci-	d	10^{-1}

The only rule when using metric prefixes is that you cannot “double them up.” For example, if you have measurements in petameters ($1 \text{ Pm} = 10^{15} \text{ m}$), it is not proper to talk about megagigameters, although $10^6 \times 10^9 = 10^{15}$. In practice, the only time this

becomes a bit confusing is when discussing masses. As we have seen, the base SI unit of mass is the kilogram (kg), but metric prefixes need to be applied to the gram (g), because we are not allowed to “double-up” prefixes. Thus, a thousand kilograms (10^3 kg) is written as a megagram (1 Mg) since

$$10^3 \text{ kg} = 10^3 \times 10^3 \text{ g} = 10^6 \text{ g} = 1 \text{ Mg}. \quad (1.1.3.1)$$

Incidentally, 10^3 kg is also called a **metric ton**, abbreviated t. This is one of the units outside the SI system considered acceptable for use with SI units.

As we see in the next section, metric systems have the advantage that conversions of units involve only powers of 10. There are 100 cm in 1 m, 1000 m in 1 km, and so on. In nonmetric systems, such as the English system of units, the relationships are not as simple—there are 12 in in 1 ft, 5280 ft in 1 mi, and so on.

Another advantage of metric systems is that the same unit can be used over extremely large ranges of values simply by scaling it with an appropriate metric prefix. The prefix is chosen by the order of magnitude of physical quantities commonly found in the task at hand. For example, distances in meters are suitable in construction, whereas distances in kilometers are appropriate for air travel, and nanometers are convenient in optical design. With the metric system there is no need to invent new units for particular applications. Instead, we rescale the units with which we are already familiar.

✓ Example 1.1.3.1: Using Metric Prefixes

Restate the mass 1.93×10^{13} kg using a metric prefix such that the resulting numerical value is bigger than one but less than 1000.

Strategy

Since we are not allowed to “double-up” prefixes, we first need to restate the mass in grams by replacing the prefix symbol k with a factor of 10^3 (Table 1.1.3.2). Then, we should see which two prefixes in Table 1.1.3.2 are closest to the resulting power of 10 when the number is written in scientific notation. We use whichever of these two prefixes gives us a number between one and 1000.

Solution

Replacing the k in kilogram with a factor of 10^3 , we find that

$$1.93 \times 10^{13} \text{ kg} = 1.93 \times 10^{13} \times 10^3 \text{ g} = 1.93 \times 10^{16} \text{ g}.$$

From Table 1.1.3.2, we see that 10^{16} is between “peta-” (10^{15}) and “exa-” (10^{18}). If we use the “peta-” prefix, then we find that $1.93 \times 10^{16} \text{ g} = 1.93 \times 10^1 \text{ Pg}$, since $16 = 1 + 15$. Alternatively, if we use the “exa-” prefix we find that $1.93 \times 10^{16} \text{ g} = 1.93 \times 10^{-2} \text{ Eg}$, since $16 = -2 + 18$. Because the problem asks for the numerical value between one and 1000, we use the “peta-” prefix and the answer is 19.3 Pg.

Significance

It is easy to make silly arithmetic errors when switching from one prefix to another, so it is always a good idea to check that our final answer matches the number we started with. An easy way to do this is to put both numbers in scientific notation and count powers of 10, including the ones hidden in prefixes. If we did not make a mistake, the powers of 10 should match up. In this problem, we started with 1.93×10^{13} kg, so we have $13 + 3 = 16$ powers of 10. Our final answer in scientific notation is $1.93 \times 10^1 \text{ Pg}$, so we have $1 + 15 = 16$ powers of 10. So, everything checks out.

If this mass arose from a calculation, we would also want to check to determine whether a mass this large makes any sense in the context of the problem. For this, Figure 1.4 might be helpful.

? Exercises 1.1.3.1

Restate 4.79×10^5 kg using a metric prefix such that the resulting number is bigger than one but less than 1000.

Answer

Most of the metric prefixes are spaced a factor of 1000 apart (centi-, deci-, deka- and hecto- notwithstanding). So we can drop a factor of 10^3 in order to take a step up to the next metric prefix.

$$10^5 \text{ kg} = 10^{5-3} \text{ Mg} = 10^2 \text{ Mg}$$

So an appropriate choice of prefix would give us 479 megagrams

Getting a feeling for sizes

It is useful when solving problems (and in general) to know the approximate value that your final answer should be. To know this, one needs to know the approximate lengths, masses, and times that exist in our everyday lives. Here is a list to get you started.

Table 1.1.3.3: Approximate Values of Length, Mass, and Time

Lengths in meters		Masses in kilograms		Times in seconds	
10^{-18}	Present experimental limit to smallest observable detail	10^{-30}	Mass of an electron	10^{-23}	Time for light to cross a proton
10^{-15}	Diameter of a proton	10^{-27}	Mass of a hydrogen atom	10^{-22}	Mean life of an extremely unstable nucleus
10^{-14}	Diameter of a uranium nucleus	10^{-15}	Mass of a bacterium	10^{-15}	Time for one oscillation of visible light
10^{-10}	Diameter of a hydrogen atom	10^{-5}	Mass of a mosquito	10^{-13}	Time for one vibration of an atom in a solid
10^{-8}	Thickness of membranes in cells of living organisms	10^{-2}	Mass of a hummingbird	10^{-8}	Time for one oscillation of an FM radio wave
10^{-6}	Wavelength of visible light	10^0	Mass of a liter of water (about a quart)	10^{-3}	Duration of a nerve impulse
10^{-3}	Size of a grain of sand	10^2	Mass of a person	10^0	Time for one heartbeat
10^0	Height of a 4-year-old child	10^3	Mass of a car	10^5	One day
10^2	Length of a football field	10^8	Mass of a large ship	10^6	About two weeks
10^4	Greatest ocean depth	10^{12}	Mass of a large iceberg	10^9	~32 years
10^7	Diameter of the Earth	10^{15}	Mass of the nucleus of a comet	10^{11}	Recorded history
10^{11}	Distance from the Earth to the Sun	10^{23}	Mass of the Moon	10^{17}	Age of the Earth
10^{16}	Distance traveled by light in 1 year (a light year)	10^{25}	Mass of the Earth	10^{18}	Age of the universe
10^{21}	Diameter of the Milky Way galaxy	10^{30}	Mass of the Sun		

Lengths in meters		Masses in kilograms		Times in seconds	
10^{22}	Distance from the Earth to the nearest large galaxy (Andromeda)	10^{42}	Mass of the Milky Way galaxy (current upper limit)		
10^{26}	Distance from the Earth to the edges of the known universe	10^{53}	Mass of the known universe (current upper limit)		

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1.1.4: Unit Conversion

Learning Objectives

- Use conversion factors to express the value of a given quantity in different units.

It is often necessary to convert from one unit to another. For example, if you are reading a European cookbook, some quantities may be expressed in units of liters and you need to convert them to cups. Or perhaps you are reading walking directions from one location to another and you are interested in how many miles you will be walking. In this case, you may need to convert units of feet or meters to miles.

Let's consider a simple example of how to convert units. Suppose we want to convert 80 m to kilometers. The first thing to do is to list the units you have and the units to which you want to convert. In this case, we have units in meters and we want to convert to kilometers. Next, we need to determine a conversion factor relating meters to kilometers. A **conversion factor** is a ratio that expresses how many of one unit are equal to another unit. For example, there are 12 in. in 1 ft, 1609 m in 1 mi, 100 cm in 1 m, 60 s in 1 min, and so on. Refer to [Appendix B](#) for a more complete list of conversion factors. In this case, we know that there are 1000 m in 1 km. Now we can set up our unit conversion. We write the units we have and then multiply them by the conversion factor so the units cancel out, as shown:

$$80 \cancel{m} \times \frac{1 \text{ km}}{1000 \cancel{m}} = 0.080 \text{ km.} \quad (1.1.4.1)$$

Note that the unwanted meter unit cancels, leaving only the desired kilometer unit. You can use this method to convert between any type of unit. Now, the conversion of 80 m to kilometers is simply the use of a metric prefix, as we saw in the preceding section, so we can get the same answer just as easily by noting that

$$80 \text{ m} = 8.0 \times 10^1 \text{ m} = 8.0 \times 10^{-2} \text{ km} = 0.080 \text{ km}, \quad (1.1.4.2)$$

since “kilo-” means 10^3 and $1 = -2 + 3$. However, using conversion factors is handy when converting between units that are not metric or when converting between derived units, as the following examples illustrate.

✓ Example 1.1.4.1: Converting Nonmetric Units to Metric

The distance from the university to home is 10 mi and it usually takes 20 min to drive this distance. Calculate the average speed in meters per second (m/s). (**Note:** Average speed is distance traveled divided by time of travel.)

Strategy

First we calculate the average speed using the given units, then we can get the average speed into the desired units by picking the correct conversion factors and multiplying by them. The correct conversion factors are those that cancel the unwanted units and leave the desired units in their place. In this case, we want to convert miles to meters, so we need to know the fact that there are 1609 m in 1 mi. We also want to convert minutes to seconds, so we use the conversion of 60 s in 1 min.

Solution

- Calculate average speed. Average speed is distance traveled divided by time of travel. (Take this definition as a given for now. Average speed and other motion concepts are covered in later chapters.) In equation form,

$$\text{Average speed} = \frac{\text{Distance}}{\text{Time}} \quad (1.1.4.3)$$

- Substitute the given values for distance and time:

$$\text{Average speed} = \frac{10 \text{ mi}}{20 \text{ min}} = 0.50 \frac{\text{mi}}{\text{min}}.$$

- Convert miles per minute to meters per second by multiplying by the conversion factor that cancels miles and leave meters, and also by the conversion factor that cancels minutes and leave seconds:

$$0.50 \frac{\cancel{\text{mile}}}{\cancel{\text{min}}} \times \frac{1609 \text{ m}}{1 \cancel{\text{mile}}} \times \frac{1 \cancel{\text{min}}}{60 \text{ s}} = \frac{(0.50)(1609)}{60} \text{ m/s} = 13 \text{ m/s}.$$

Significance

Check the answer in the following ways:

1. Be sure the units in the unit conversion cancel correctly. If the unit conversion factor was written upside down, the units do not cancel correctly in the equation. We see the “miles” in the numerator in 0.50 mi/min cancels the “mile” in the denominator in the first conversion factor. Also, the “min” in the denominator in 0.50 mi/min cancels the “min” in the numerator in the second conversion factor.
2. Check that the units of the final answer are the desired units. The problem asked us to solve for average speed in units of meters per second and, after the cancelations, the only units left are a meter (m) in the numerator and a second (s) in the denominator, so we have indeed obtained these units.

? Exercise 1.1.4.1

Light travels about 9 Pm in a year. Given that a year is about 3×10^7 s, what is the speed of light in meters per second?

Answer

We are given the distance light travels (9 Pm) in an amount of time (3×10^7 s). A speed has dimensions of length over time, so we will divide the distance by the time.

$$v = (9 \text{ Pm}) / (3 \times 10^7 \text{ s}) = 3 \times 10^{-7} \text{ Pm/s}$$

We weren't asked for petameters per second (Pm = 10^{15} m), so we should replace the prefix with an order of magnitude and get our result in m/s.

$$v = 3 \times 10^{15-7} \text{ m/s} = 3 \times 10^8 \text{ m/s}$$

✓ Example 1.1.4.2: Converting between Metric Units

The density of iron is 7.86 g/cm^3 under standard conditions. Convert this to kg/m^3 .

Strategy

We need to convert grams to kilograms and cubic centimeters to cubic meters. The conversion factors we need are $1 \text{ kg} = 10^3 \text{ g}$ and $1 \text{ cm} = 10^{-2} \text{ m}$. However, we are dealing with cubic centimeters ($\text{cm}^3 = \text{cm} \times \text{cm} \times \text{cm}$), so we have to use the second conversion factor three times (that is, we need to cube it). The idea is still to multiply by the conversion factors in such a way that they cancel the units we want to get rid of and introduce the units we want to keep.

Solution

$$7.86 \frac{\cancel{\text{g}}}{\cancel{\text{cm}^3}} \times \frac{\text{kg}}{10^3 \cancel{\text{g}}} \times \left(\frac{\cancel{\text{cm}}}{10^{-2} \text{ m}} \right)^3 = \frac{7.86}{(10^3)(10^{-6})} \text{ kg/m}^3 = 7.86 \times 10^3 \text{ kg/m}^3$$

Significance

Remember, it's always important to check the answer.

1. Be sure to cancel the units in the unit conversion correctly. We see that the gram (“g”) in the numerator in 7.86 g/cm^3 cancels the “g” in the denominator in the first conversion factor. Also, the three factors of “cm” in the denominator in 7.86 g/cm^3 cancel with the three factors of “cm” in the numerator that we get by cubing the second conversion factor.
2. Check that the units of the final answer are the desired units. The problem asked for us to convert to kilograms per cubic meter. After the cancelations just described, we see the only units we have left are “kg” in the numerator and three factors of “m” in the denominator (that is, one factor of “m” cubed, or “m³”). Therefore, the units on the final answer are correct.

? Exercise 1.1.4.2

We know from Figure 1.4 that the diameter of Earth is on the order of 10^7 m, so the order of magnitude of its surface area is 10^{14} m^2 . What is that in square kilometers (that is, km^2)? (Try doing this both by converting 10^7 m to km and then squaring it and then by converting 10^{14} m^2 directly to square kilometers. You should get the same answer both ways.)

Answer

This space intentionally (and temporarily) left blank

Unit conversions may not seem very interesting, but not doing them can be costly. One famous example of this situation was seen with the **Mars Climate Orbiter**. This probe was launched by NASA on December 11, 1998. On September 23, 1999, while attempting to guide the probe into its planned orbit around Mars, NASA lost contact with it. Subsequent investigations showed a piece of software called SM_FORCES (or “small forces”) was recording thruster performance data in the English units of pound-seconds ($\text{lb} \cdot \text{s}$). However, other pieces of software that used these values for course corrections expected them to be recorded in the SI units of newton-seconds ($\text{N} \cdot \text{s}$), as dictated in the software interface protocols. This error caused the probe to follow a very different trajectory from what NASA thought it was following, which most likely caused the probe either to burn up in the Martian atmosphere or to shoot out into space. This failure to pay attention to unit conversions cost hundreds of millions of dollars, not to mention all the time invested by the scientists and engineers who worked on the project.

? Exercise 1.1.4.3

Given that 1 lb (pound) is 4.45 N, were the numbers being output by SM_FORCES too big or too small?

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1.1.5: Dimensional Analysis

Learning Objectives

- Find the dimensions of a mathematical expression involving physical quantities.
- Determine whether an equation involving physical quantities is dimensionally consistent.

The **dimension** of any physical quantity expresses its dependence on the base quantities as a product of symbols (or powers of symbols) representing the base quantities. Table 1.1.5.1 lists the base quantities and the symbols used for their dimension. For example, a measurement of length is said to have dimension L or L^1 , a measurement of mass has dimension M or M^1 , and a measurement of time has dimension T or T^1 . Like units, dimensions obey the rules of algebra. Thus, area is the product of two lengths and so has dimension L^2 , or length squared. Similarly, volume is the product of three lengths and has dimension L^3 , or length cubed. Speed has dimension length over time, L/T or LT^{-1} . Volumetric mass density has dimension M/L^3 or ML^{-3} , or mass over length cubed. In general, the dimension of any physical quantity can be written as

$$L^a M^b T^c I^d \Theta^e N^f J^g \quad (1.1.5.1)$$

for some powers a, b, c, d, e, f, and g. We can write the dimensions of a length in this form with a = 1 and the remaining six powers all set equal to zero:

$$L^1 = L^1 M^0 T^0 I^0 \Theta^0 N^0 J^0. \quad (1.1.5.2)$$

Any quantity with a dimension that can be written so that all seven powers are zero (that is, its dimension is $L^0 M^0 T^0 I^0 \Theta^0 N^0 J^0$) is called **dimensionless** (or sometimes “of dimension 1,” because anything raised to the zero power is one). Physicists often call dimensionless quantities **pure numbers**.

Table 1.1.5.1: Base Quantities and Their Dimensions

Base Quantity	Symbol for Dimension
Length	L
Mass	M
Time	T
Current	I
Thermodynamic Temperature	Θ
Amount of Substance	N
Luminous Intensity	J

Physicists often use square brackets around the symbol for a physical quantity to represent the dimensions of that quantity. For example, if r is the radius of a cylinder and h is its height, then we write $[r] = L$ and $[h] = L$ to indicate the dimensions of the radius and height are both those of length, or L. Similarly, if we use the symbol A for the surface area of a cylinder and V for its volume, then $[A] = L^2$ and $[V] = L^3$. If we use the symbol m for the mass of the cylinder and ρ for the density of the material from which the cylinder is made, then $[m] = M$ and $[\rho] = ML^{-3}$.

The importance of the concept of dimension arises from the fact that any mathematical equation relating physical quantities must be **dimensionally consistent**, which means the equation must obey the following rules:

- Every term in an expression must have the same dimensions; it does not make sense to add or subtract quantities of differing dimension (think of the old saying: “You can’t add apples and oranges”). In particular, the expressions on each side of the equality in an equation must have the same dimensions.
 - If $A + B = C$, then we need $[A+B] = [C]$ and also $[A] = [B]$. The terms being added together need to agree for that operation to make sense. We need the left and right hand side of the equation to agree in order for the equality to hold.
 - If $A \times B = C$, then the dimensions of C must be the same as those of A and B combined together

- The arguments of any of the standard mathematical functions such as trigonometric functions (such as sine and cosine), logarithms, or exponential functions that appear in the equation must be dimensionless. These functions require pure numbers as inputs and give pure numbers as outputs.
 - For example, a common way to write a sine function is $f(x) = \sin(2\pi \frac{x}{L} + \phi)$. The entire argument must be dimensionless (angles, whether expressed in degrees or radians, are inherently pure numbers as they represent a fraction of a circle). In this case, the quantities that are not already dimensionless have been collected together into a dimensionless fraction: $[x] = [L]$, so $[x/L] = 1$.

If either of these rules is violated, an equation is not dimensionally consistent and cannot possibly be a correct statement of physical law. This simple fact can be used to check for typos or algebra mistakes, to help remember the various laws of physics, and even to suggest the form that new laws of physics might take.

✓ Example 1.1.5.1: Using Dimensions to Remember an Equation

Suppose we need the formula for the area of a circle for some computation. Like many people who learned geometry too long ago to recall with any certainty, two expressions may pop into our mind when we think of circles: πr^2 and $2\pi r$. One expression is the circumference of a circle of radius r and the other is its area. But which is which?

Strategy

One natural strategy is to look it up, but this could take time to find information from a reputable source. Besides, even if we think the source is reputable, we shouldn't trust everything we read. It is nice to have a way to double-check just by thinking about it. Also, we might be in a situation in which we cannot look things up (such as during a test). Thus, the strategy is to find the dimensions of both expressions by making use of the fact that dimensions follow the rules of algebra. If either expression does not have the same dimensions as area, then it cannot possibly be the correct equation for the area of a circle.

Solution

We know the dimension of area is L^2 . Now, the dimension of the expression πr^2 is

$$[\pi r^2] = [\pi] \cdot [r]^2 = 1 \cdot L^2 = L^2, \quad (1.1.5.3)$$

since the constant π is a pure number and the radius r is a length. Therefore, πr^2 has the dimension of area. Similarly, the dimension of the expression $2\pi r$ is

$$[2\pi r] = [2] \cdot [\pi] \cdot [r] = 1 \cdot 1 \cdot L = L, \quad (1.1.5.4)$$

since the constants 2 and π are both dimensionless and the radius r is a length. We see that $2\pi r$ has the dimension of length, which means it cannot possibly be an area.

We rule out $2\pi r$ because it is not dimensionally consistent with being an area. We see that πr^2 is dimensionally consistent with being an area, so if we have to choose between these two expressions, πr^2 is the one to choose.

Significance

This may seem like kind of a silly example, but the ideas are very general. As long as we know the dimensions of the individual physical quantities that appear in an equation, we can check to see whether the equation is dimensionally consistent. On the other hand, knowing that true equations are dimensionally consistent, we can match expressions from our imperfect memories to the quantities for which they might be expressions. Doing this will not help us remember dimensionless factors that appear in the equations (for example, if you had accidentally conflated the two expressions from the example into $2\pi r^2$, then dimensional analysis is no help), but it does help us remember the correct basic form of equations.

? Exercise 1.1.5.1

Suppose we want the formula for the volume of a sphere. The two expressions commonly mentioned in elementary discussions of spheres are $4\pi r^2$ and $\frac{4}{3}\pi r^3$. One is the volume of a sphere of radius r and the other is its surface area. Which one is the volume?

Answer

The dimensions of area and volume are $[A] = L^2$ and $[V] = L^3$. If one of these is area and the other is volume then one of them must have one additional dimension of length than the other. As long as $[r] = L$, then $[4\pi r^2] = L^2$ and must be the surface area.

✓ Example 1.1.5.2: Checking Equations for Dimensional Consistency

Consider the physical quantities s , v , a , and t with dimensions $[s] = L$, $[v] = LT^{-1}$, $[a] = LT^{-2}$, and $[t] = T$. Determine whether each of the following equations is dimensionally consistent:

- $s = vt + 0.5at^2$;
- $s = vt^2 + 0.5at$; and
- $v = \sin\left(\frac{at^2}{s}\right)$.

Strategy

By the definition of dimensional consistency, we need to check that each term in a given equation has the same dimensions as the other terms in that equation and that the arguments of any standard mathematical functions are dimensionless.

Solution

- There are no trigonometric, logarithmic, or exponential functions to worry about in this equation, so we need only look at the dimensions of each term appearing in the equation. There are three terms, one in the left expression and two in the expression on the right, so we look at each in turn:

$$[s] = L \quad (1.1.5.5)$$

$$[vt] = [v] \cdot [t] = LT^{-1} \cdot T = LT^0 = L \quad (1.1.5.6)$$

$$[0.5at^2] = [a] \cdot [t]^2 = LT^{-2} \cdot T^2 = LT^0 = L. \quad (1.1.5.7)$$

- Again, there are no trigonometric, exponential, or logarithmic functions, so we only need to look at the dimensions of each of the three terms appearing in the equation:

$$[s] = L \quad (1.1.5.8)$$

$$[vt^2] = [v] \cdot [t]^2 = LT^{-1} \cdot T^2 = LT \quad (1.1.5.9)$$

$$[at] = [a] \cdot [t] = LT^{-2} \cdot T = LT^{-1}. \quad (1.1.5.10)$$

None of the three terms has the same dimension as any other, so this is about as far from being dimensionally consistent as you can get. The technical term for an equation like this is **nonsense**.

- This equation has a trigonometric function in it, so first we should check that the argument of the sine function is dimensionless:

$$\left[\frac{at^2}{s}\right] = \frac{[a] \cdot [t]^2}{[s]} = \frac{LT^{-2} \cdot T^2}{L} = \frac{L}{L} = 1. \quad (1.1.5.11)$$

The argument is dimensionless. So far, so good. Now we need to check the dimensions of each of the two terms (that is, the left expression and the right expression) in the equation:

$$[v] = LT^{-1} \quad (1.1.5.12)$$

$$\left[\sin\left(\frac{at^2}{s}\right)\right] = 1. \quad (1.1.5.13)$$

The two terms have different dimensions—meaning, the equation is not dimensionally consistent. This equation is another example of “nonsense.”

Significance

If we are trusting people, these types of dimensional checks might seem unnecessary. But, rest assured, any textbook on a quantitative subject such as physics (including this one) almost certainly contains some equations with typos. Checking equations routinely by dimensional analysis save us the embarrassment of using an incorrect equation. Also, checking the

dimensions of an equation we obtain through algebraic manipulation is a great way to make sure we did not make a mistake (or to spot a mistake, if we made one).

? Exercise 1.1.5.2

Is the equation $v = v_i + at + \frac{ax}{v_i}$ dimensionally consistent? Use $[x] = L$, $[v] = [v_i] = LT^{-1}$, and $[a] = LT^{-2}$

Answer

There are several terms, and each of them are added together. As a result we need each term to have the same dimensions. We also need the dimensions on the left hand side of the equation to be the same as the dimensions on the right hand side.

The left hand side is simple, so we will start there and use that to define our "target" dimensions for the terms on the right to be $[v] = LT^{-1}$. Each term on the right must agree, so we can go term by term

$$[v_i] = [v] = L T^{-1} \checkmark$$

$$[a t] = [a][t] = (L T^{-2})(T) = L T^{-1} \checkmark$$

$$\left[\frac{ax}{v_i}\right] = [a][x]/[v_i] = (L T^{-2})(L)(L^{-1} T^1) = L^{1+1-1} T^{-2+1} = L T^{-1} \checkmark$$

Each term has the appropriate dimensions, so the equation is consistent. As for how to interpret it, that is a separate matter.

One further point that needs to be mentioned is the effect of the operations of calculus on dimensions. We have seen that dimensions obey the rules of algebra, just like units, but what happens when we take the derivative of one physical quantity with respect to another or integrate a physical quantity over another? The derivative of a function is just the slope of the line tangent to its graph and slopes are ratios, so for physical quantities v and t , we have that the dimension of the derivative of v with respect to t is just the ratio of the dimension of v over that of t :

$$\left[\frac{dv}{dt}\right] = \frac{[v]}{[t]}. \quad (1.1.5.14)$$

Similarly, since integrals are just sums of products, the dimension of the integral of v with respect to t is simply the dimension of v times the dimension of t :

$$\left[\int v dt\right] = [v] \cdot [t]. \quad (1.1.5.15)$$

By the same reasoning, analogous rules hold for the units of physical quantities derived from other quantities by integration or differentiation.

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1.1.6: Estimates and Fermi Calculations

Learning Objectives

- Estimate the values of physical quantities.

On many occasions, physicists, other scientists, and engineers need to make estimates for a particular quantity. Other terms sometimes used are **guesstimates**, **order-of-magnitude approximations**, **back-of-the-envelope calculations**, or **Fermi calculations**. (The physicist Enrico Fermi mentioned earlier was famous for his ability to estimate various kinds of data with surprising precision.) Will that piece of equipment fit in the back of the car or do we need to rent a truck? How long will this download take? About how large a current will there be in this circuit when it is turned on? How many houses could a proposed power plant actually power if it is built? Note that estimating does not mean guessing a number or a formula at random. Rather, **estimation** means using prior experience and sound physical reasoning to arrive at a rough idea of a quantity's value. Because the process of determining a reliable approximation usually involves the identification of correct physical principles and a good guess about the relevant variables, estimating is very useful in developing physical intuition. Estimates also allow us perform “sanity checks” on calculations or policy proposals by helping us rule out certain scenarios or unrealistic numbers. They allow us to challenge others (as well as ourselves) in our efforts to learn truths about the world.

Many estimates are based on formulas in which the input quantities are known only to a limited precision. As you develop physics problem-solving skills (which are applicable to a wide variety of fields), you also will develop skills at estimating. You develop these skills by thinking more quantitatively and by being willing to take risks. As with any skill, experience helps. Familiarity with dimensions (see [Table 1.5.1](#)) and units (see [Table 1.3.1](#) and [Table 1.3.2](#)), and the scales of base quantities (see [Figure 1.2.3](#)) also helps.

To make some progress in estimating, you need to have some definite ideas about how variables may be related. The following strategies may help you in practicing the art of estimation:

- **Get big lengths from smaller lengths.** When estimating lengths, remember that anything can be a ruler. Thus, imagine breaking a big thing into smaller things, estimate the length of one of the smaller things, and multiply to get the length of the big thing. For example, to estimate the height of a building, first count how many floors it has. Then, estimate how big a single floor is by imagining how many people would have to stand on each other's shoulders to reach the ceiling. Last, estimate the height of a person. The product of these three estimates is your estimate of the height of the building. It helps to have memorized a few length scales relevant to the sorts of problems you find yourself solving. For example, knowing some of the length scales in [Figure 1.2.3](#) might come in handy. Sometimes it also helps to do this in reverse—that is, to estimate the length of a small thing, imagine a bunch of them making up a bigger thing. For example, to estimate the thickness of a sheet of paper, estimate the thickness of a stack of paper and then divide by the number of pages in the stack. These same strategies of breaking big things into smaller things or aggregating smaller things into a bigger thing can sometimes be used to estimate other physical quantities, such as masses and times.
- **Get areas and volumes from lengths.** When dealing with an area or a volume of a complex object, introduce a simple model of the object such as a sphere or a box. Then, estimate the linear dimensions (such as the radius of the sphere or the length, width, and height of the box) first, and use your estimates to obtain the volume or area from standard geometric formulas. If you happen to have an estimate of an object's area or volume, you can also do the reverse; that is, use standard geometric formulas to get an estimate of its linear dimensions.
- **Get masses from volumes and densities.** When estimating masses of objects, it can help first to estimate its volume and then to estimate its mass from a rough estimate of its average density (recall, density has dimension mass over length cubed, so mass is density times volume). For this, it helps to remember that the density of air is around 1 kg/m^3 , the density of water is 10^3 kg/m^3 , and the densest everyday solids max out at around 10^4 kg/m^3 . Asking yourself whether an object floats or sinks in either air or water gets you a ballpark estimate of its density. You can also do this the other way around; if you have an estimate of an object's mass and its density, you can use them to get an estimate of its volume.
- **If all else fails, bound it.** For physical quantities for which you do not have a lot of intuition, sometimes the best you can do is think something like: Well, it must be bigger than this and smaller than that. For example, suppose you need to estimate the mass of a moose. Maybe you have a lot of experience with moose and know their average mass offhand. If so, great. But for most people, the best they can do is to think something like: It must be bigger than a person (of order 10^2 kg) and less than a car

(of order 10^3 kg). If you need a single number for a subsequent calculation, you can take the geometric mean of the upper and lower bound—that is, you multiply them together and then take the square root. For the moose mass example, this would be

$$(10^2 \times 10^3)^{0.5} = 10^{2.5} = 10^{0.5} \times 10^2 \approx 3 \times 10^2 \text{ kg.} \quad (1.1.6.1)$$

The tighter the bounds, the better. Also, no rules are unbreakable when it comes to estimation. If you think the value of the quantity is likely to be closer to the upper bound than the lower bound, then you may want to bump up your estimate from the geometric mean by an order or two of magnitude.

- **One “sig. fig.” is fine.** There is no need to go beyond one significant figure when doing calculations to obtain an estimate. In most cases, the order of magnitude is good enough. The goal is just to get in the ballpark figure, so keep the arithmetic as simple as possible.
- **Ask yourself: Does this make any sense?** Last, check to see whether your answer is reasonable. How does it compare with the values of other quantities with the same dimensions that you already know or can look up easily? If you get some wacky answer (for example, if you estimate the mass of the Atlantic Ocean to be bigger than the mass of Earth, or some time span to be longer than the age of the universe), first check to see whether your units are correct. Then, check for arithmetic errors. Then, rethink the logic you used to arrive at your answer. If everything checks out, you may have just proved that some slick new idea is actually bogus.

✓ Example 1.1.6.1: Mass of Earth's Oceans

Estimate the total mass of the oceans on Earth.

Strategy

We know the density of water is about 10^3 kg/m^3 , so we start with the advice to “get masses from densities and volumes.” Thus, we need to estimate the volume of the planet’s oceans. Using the advice to “get areas and volumes from lengths,” we can estimate the volume of the oceans as surface area times average depth, or $V = AD$. We know the diameter of Earth from Figure 1.4 and we know that most of Earth’s surface is covered in water, so we can estimate the surface area of the oceans as being roughly equal to the surface area of the planet. By following the advice to “get areas and volumes from lengths” again, we can approximate Earth as a sphere and use the formula for the surface area of a sphere of diameter d —that is, $A = \pi d^2$, to estimate the surface area of the oceans. Now we just need to estimate the average depth of the oceans. For this, we use the advice: “If all else fails, bound it.” We happen to know the deepest points in the ocean are around 10 km and that it is not uncommon for the ocean to be deeper than 1 km, so we take the average depth to be around $(10^3 \times 10^4)^{0.5} \approx 3 \times 10^3 \text{ m}$. Now we just need to put it all together, heeding the advice that “one ‘sig. fig.’ is fine.”

Solution

We estimate the surface area of Earth (and hence the surface area of Earth’s oceans) to be roughly

$$A = \pi d^2 = \pi (10^7 \text{ m})^2 \approx 3 \times 10^{14} \text{ m}^2. \quad (1.1.6.2)$$

Next, using our average depth estimate of $D = 3 \times 10^3 \text{ m}$, which was obtained by bounding, we estimate the volume of Earth’s oceans to be

$$V = AD = (3 \times 10^{14} \text{ m}^2) (3 \times 10^3 \text{ m}) = 9 \times 10^{17} \text{ m}^3. \quad (1.1.6.3)$$

Last, we estimate the mass of the world’s oceans to be

$$M = \rho V = (10^3 \text{ kg/m}^3) (9 \times 10^{17} \text{ m}^3) = 9 \times 10^{20} \text{ kg.} \quad (1.1.6.4)$$

Thus, we estimate that the order of magnitude of the mass of the planet’s oceans is 10^{21} kg .

Significance

To verify our answer to the best of our ability, we first need to answer the question: Does this make any sense? From Figure 1.4, we see the mass of Earth’s atmosphere is on the order of 10^{19} kg and the mass of Earth is on the order of 10^{25} kg . It is reassuring that our estimate of 10^{21} kg for the mass of Earth’s oceans falls somewhere between these two. So, yes, it does seem to make sense. It just so happens that we did a search on the Web for “mass of oceans” and the top search results all said $1.4 \times 10^{21} \text{ kg}$, which is the same order of magnitude as our estimate. Now, rather than having to trust blindly whoever first put that

number up on a website (most of the other sites probably just copied it from them, after all), we can have a little more confidence in it.

? Exercise 1.1.6.1

Figure 1.4 says the mass of the atmosphere is 10^{19} kg. Assuming the density of the atmosphere is 1 kg/m^3 , estimate the height of Earth's atmosphere. Do you think your answer is an underestimate or an overestimate? Explain why.

How many piano tuners are there in New York City? How many leaves are on that tree? If you are studying photosynthesis or thinking of writing a smartphone app for piano tuners, then the answers to these questions might be of great interest to you. Otherwise, you probably couldn't care less what the answers are. However, these are exactly the sorts of estimation problems that people in various tech industries have been asking potential employees to evaluate their quantitative reasoning skills. If building physical intuition and evaluating quantitative claims do not seem like sufficient reasons for you to practice estimation problems, how about the fact that being good at them just might land you a high-paying job?

📌 PhET Simulation: Estimation

For practice estimating relative lengths, areas, and volumes, check out this PhET simulation, titled "[Estimation](#)."

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1.1.E: Exercises

Conceptual Questions

1.2 Modeling in Physics

1. What is physics?
2. Some have described physics as a “search for simplicity.” Explain why this might be an appropriate description.
3. If two different theories describe experimental observations equally well, can one be said to be more valid than the other (assuming both use accepted rules of logic)?
4. What determines the validity of a theory?
5. Certain criteria must be satisfied if a measurement or observation is to be believed. Will the criteria necessarily be as strict for an expected result as for an unexpected result?
6. Can the validity of a model be limited or must it be universally valid? How does this compare with the required validity of a theory or a law?

1.3 Units and Standards

7. Identify some advantages of metric units.
8. What are the SI base units of length, mass, and time?
9. What is the difference between a base unit and a derived unit? (b) What is the difference between a base quantity and a derived quantity? (c) What is the difference between a base quantity and a base unit?
10. For each of the following scenarios, refer to Figure 1.4 and Table 1.2 to determine which metric prefix on the meter is most appropriate for each of the following scenarios. (a) You want to tabulate the mean distance from the Sun for each planet in the solar system. (b) You want to compare the sizes of some common viruses to design a mechanical filter capable of blocking the pathogenic ones. (c) You want to list the diameters of all the elements on the periodic table. (d) You want to list the distances to all the stars that have now received any radio broadcasts sent from Earth 10 years ago.

Problems

1.3 Units and Standards

For the remaining questions, you need to use Figure 1.4 to obtain the necessary orders of magnitude of lengths, masses, and times.

1. Find the order of magnitude of the following physical quantities.
 - a. The mass of Earth’s atmosphere: 5.1×10^{18} kg;
 - b. The mass of the Moon’s atmosphere: 25,000 kg;
 - c. The mass of Earth’s hydrosphere: 1.4×10^{21} kg;
 - d. The mass of Earth: 5.97×10^{24} kg;
 - e. The mass of the Moon: 7.34×10^{22} kg;
 - f. The Earth–Moon distance (semi-major axis): 3.84×10^8 m;
 - g. The mean Earth–Sun distance: 1.5×10^{11} m;
 - h. The equatorial radius of Earth: 6.38×10^6 m;
 - i. The mass of an electron: 9.11×10^{-31} kg;
 - j. The mass of a proton: 1.67×10^{-27} kg;
 - k. The mass of the Sun: 1.99×10^{30} kg.
2. Use the orders of magnitude you found in the previous problem to answer the following questions to within an order of magnitude.
 - a. How many electrons would it take to equal the mass of a proton?
 - b. How many Earths would it take to equal the mass of the Sun?
 - c. How many Earth–Moon distances would it take to cover the distance from Earth to the Sun?
 - d. How many Moon atmospheres would it take to equal the mass of Earth’s atmosphere?
 - e. How many moons would it take to equal the mass of Earth?
 - f. How many protons would it take to equal the mass of the Sun?
3. Roughly how many heartbeats are there in a lifetime?
4. A generation is about one-third of a lifetime. Approximately how many generations have passed since the year 0 AD?

5. Roughly how many times longer than the mean life of an extremely unstable atomic nucleus is the lifetime of a human?
6. Calculate the approximate number of atoms in a bacterium. Assume the average mass of an atom in the bacterium is 10 times the mass of a proton.
7. (a) Calculate the number of cells in a hummingbird assuming the mass of an average cell is 10 times the mass of a bacterium.
(b) Making the same assumption, how many cells are there in a human?
8. Assuming one nerve impulse must end before another can begin, what is the maximum firing rate of a nerve in impulses per second?
9. About how many floating-point operations can a supercomputer perform each year?
10. Roughly how many floating-point operations can a supercomputer perform in a human lifetime?
11. The following times are given using metric prefixes on the base SI unit of time: the second. Rewrite them in scientific notation without the prefix. For example, 47 Ts would be rewritten as 4.7×10^{13} s.
 - a. 980 Ps;
 - b. 980 fs;
 - c. 17 ns;
 - d. 577 μ s.
12. The following times are given in seconds. Use metric prefixes to rewrite them so the numerical value is greater than one but less than 1000. For example, 7.9×10^{-2} s could be written as either 7.9 cs or 79 ms.
 - a. 9.57×10^5 s;
 - b. 0.045 s;
 - c. 5.5×10^{-7} s;
 - d. 3.16×10^7 s.
13. The following lengths are given using metric prefixes on the base SI unit of length: the meter. Rewrite them in scientific notation without the prefix. For example, 4.2 Pm would be rewritten as 4.2×10^{15} m.
 - a. 89 Tm;
 - b. 89 μ m;
 - c. 711 mm;
 - d. 0.45 μ m.
14. The following lengths are given in meters. Use metric prefixes to rewrite them so the numerical value is bigger than one but less than 1000. For example, 7.9×10^{-2} m could be written either as 7.9 cm or 79 mm.
 - a. 7.59×10^7 m;
 - b. 0.0074 m;
 - c. 8.8×10^{-11} m;
 - d. 1.63×10^{13} m.
15. The following masses are written using metric prefixes on the gram. Rewrite them in scientific notation in terms of the SI base unit of mass: the kilogram. For example, 40 Mg would be written as 4×10^4 kg.
 - a. 23 mg;
 - b. 320 Tg;
 - c. 42 ng;
 - d. 7 g;
 - e. 9 Pg.
16. The following masses are given in kilograms. Use metric prefixes on the gram to rewrite them so the numerical value is bigger than one but less than 1000. For example, 7×10^{-4} kg could be written as 70 cg or 700 mg.
 - a. 3.8×10^{-5} kg;
 - b. 2.3×10^{17} kg;
 - c. 2.4×10^{-11} kg;
 - d. 8×10^{15} kg;
 - e. 4.2×10^{-3} kg.

1.4 Unit Conversion

17. The volume of Earth is on the order of 10^{21} m^3 . (a) What is this in cubic kilometers (km^3)? (b) What is it in cubic miles (mi^3)? (c) What is it in cubic centimeters (cm^3)?
18. The speed limit on some interstate highways is roughly 100 km/h. (a) What is this in meters per second? (b) How many miles per hour is this?
19. A car is traveling at a speed of 33 m/s. (a) What is its speed in kilometers per hour? (b) Is it exceeding the 90 km/h speed limit?
20. In SI units, speeds are measured in meters per second (m/s). But, depending on where you live, you're probably more comfortable of thinking of speeds in terms of either kilometers per hour (km/h) or miles per hour (mi/h). In this problem, you will see that 1 m/s is roughly 4 km/h or 2 mi/h, which is handy to use when developing your physical intuition. More precisely, show that (a) $1.0 \text{ m/s} = 3.6 \text{ km/h}$ and (b) $1.0 \text{ m/s} = 2.2 \text{ mi/h}$.
21. American football is played on a 100-yd-long field, excluding the end zones. How long is the field in meters? (Assume that 1 m = 3.281 ft.)
22. Soccer fields vary in size. A large soccer field is 115 m long and 85.0 m wide. What is its area in square feet? (Assume that 1 m = 3.281 ft.)
23. What is the height in meters of a person who is 6 ft 1.0 in. tall?
24. Mount Everest, at 29,028 ft, is the tallest mountain on Earth. What is its height in kilometers? (Assume that 1 m = 3.281 ft.)
25. The speed of sound is measured to be 342 m/s on a certain day. What is this measurement in kilometers per hour?
26. Tectonic plates are large segments of Earth's crust that move slowly. Suppose one such plate has an average speed of 4.0 cm/yr. (a) What distance does it move in 1.0 s at this speed? (b) What is its speed in kilometers per million years?
27. The average distance between Earth and the Sun is $1.5 \times 10^{11} \text{ m}$. (a) Calculate the average speed of Earth in its orbit (assumed to be circular) in meters per second. (b) What is this speed in miles per hour?
28. The density of nuclear matter is about 10^{18} kg/m^3 . Given that 1 mL is equal in volume to cm^3 , what is the density of nuclear matter in megagrams per microliter (that is, $\text{Mg}/\mu\text{L}$)?
29. The density of aluminum is 2.7 g/cm^3 . What is the density in kilograms per cubic meter?
30. A commonly used unit of mass in the English system is the pound-mass, abbreviated lbm, where $1 \text{ lbm} = 0.454 \text{ kg}$. What is the density of water in pound-mass per cubic foot?
31. A furlong is 220 yd. A fortnight is 2 weeks. Convert a speed of one furlong per fortnight to millimeters per second.
32. It takes 2π radians (rad) to get around a circle, which is the same as 360° . How many radians are in 1° ?
33. Light travels a distance of about $3 \times 10^8 \text{ m/s}$. A light-minute is the distance light travels in 1 min. If the Sun is $1.5 \times 10^{11} \text{ m}$ from Earth, how far away is it in lightminutes?
34. A light-nanosecond is the distance light travels in 1 ns. Convert 1 ft to light-nanoseconds.
35. An electron has a mass of $9.11 \times 10^{-31} \text{ kg}$. A proton has a mass of $1.67 \times 10^{-27} \text{ kg}$. What is the mass of a proton in electron-masses?
36. A fluid ounce is about 30 mL. What is the volume of a 12 fl-oz can of soda pop in cubic meters?

1.5 Dimensional Analysis

37. A student is trying to remember some formulas from geometry. In what follows, assume A is area, V is volume, and all other variables are lengths. Determine which formulas are dimensionally consistent. (a) $V = \pi r^2 h$; (b) $A = 2\pi r^2 + 2\pi r h$; (c) $V = 0.5bh$; (d) $V = \pi d^2$; (e) $V = \frac{\pi d^3}{6}$
38. Consider the physical quantities s, v, a, and t with dimensions $[s] = L$, $[v] = LT^{-1}$, $[a] = LT^{-2}$, and $[t] = T$. Determine whether each of the following equations is dimensionally consistent. (a) $v^2 = 2as$; (b) $s = vt^2 + 0.5at^2$; (c) $v = s/t$; (d) $a = v/t$.
39. Consider the physical quantities m, s, v, a, and t with dimensions $[m] = M$, $[s] = L$, $[v] = LT^{-1}$, $[a] = LT^{-2}$, and $[t] = T$. Assuming each of the following equations is dimensionally consistent, find the dimension of the quantity on the left-hand side of the equation: (a) $F = ma$; (b) $K = 0.5mv^2$; (c) $p = mv$; (d) $W = mas$; (e) $L = mvr$.
40. Suppose quantity s is a length and quantity t is a time. Suppose the quantities v and a are defined by $v = ds/dt$ and $a = dv/dt$. (a) What is the dimension of v? (b) What is the dimension of the quantity a? What are the dimensions of (c) $\int v dt$, (d) $\int a dt$, and (e) da/dt ?
41. Suppose $[V] = L^3$, $[p] = ML^{-3}$, and $[t] = T$. (a) What is the dimension of $\int \rho dV$? (b) What is the dimension of dV/dt ? (c) What is the dimension of $\rho(dV/dt)$?
42. The arc length formula says the length s of arc subtended by angle Θ in a circle of radius r is given by the equation $s = r\Theta$. What are the dimensions of (a) s, (b) r, and (c) Θ ?

1.6 Estimates and Fermi Calculations

43. Assuming the human body is made primarily of water, estimate the volume of a person.
44. Assuming the human body is primarily made of water, estimate the number of molecules in it. (Note that water has a molecular mass of 18 g/mol and there are roughly 10^{24} atoms in a mole.)
45. Estimate the mass of air in a classroom.
46. Estimate the number of molecules that make up Earth, assuming an average molecular mass of 30 g/mol. (Note there are on the order of 10^{24} objects per mole.)
47. Estimate the surface area of a person.
48. Roughly how many solar systems would it take to tile the disk of the Milky Way?
49. (a) Estimate the density of the Moon. (b) Estimate the diameter of the Moon. (c) Given that the Moon subtends at an angle of about half a degree in the sky, estimate its distance from Earth.
50. The average density of the Sun is on the order 10^3 kg/m^3 . (a) Estimate the diameter of the Sun. (b) Given that the Sun subtends at an angle of about half a degree in the sky, estimate its distance from Earth. 64. Estimate the mass of a virus.
51. A floating-point operation is a single arithmetic operation such as addition, subtraction, multiplication, or division. (a) Estimate the maximum number of floating-point operations a human being could possibly perform in a lifetime. (b) How long would it take a supercomputer to perform that many floating-point operations?
52. Calculate the approximate mass and volume (both in SI units) of ketchup that your college uses every year (assume water's density of $1,000 \text{ kg/m}^3$)? Be sure to state all assumptions and be explicit in your calculations.

Challenge Problems

53. The first atomic bomb was detonated on July 16, 1945, at the Trinity test site about 200 mi south of Los Alamos. In 1947, the U.S. government declassified a film reel of the explosion. From this film reel, British physicist G. I. Taylor was able to determine the rate at which the radius of the fireball from the blast grew. Using dimensional analysis, he was then able to deduce the amount of energy released in the explosion, which was a closely guarded secret at the time. Because of this, Taylor did not publish his results until 1950. This problem challenges you to recreate this famous calculation.
 - a. Using keen physical insight developed from years of experience, Taylor decided the radius r of the fireball should depend only on time since the explosion, t , the density of the air, ρ , and the energy of the initial explosion, E . Thus, he made the educated guess that $r = kE^a \rho^b t^c$ for some dimensionless constant k and some unknown exponents a , b , and c . Given that $[E] = \text{ML}^2\text{T}^{-2}$, determine the values of the exponents necessary to make this equation dimensionally consistent. (Hint: Notice the equation implies that $k = rE^{-a}\rho^{-b}t^{-c}$ and that $[k] = 1$.)
 - b. By analyzing data from high-energy conventional explosives, Taylor found the formula he derived seemed to be valid as long as the constant k had the value 1.03. From the film reel, he was able to determine many values of r and the corresponding values of t . For example, he found that after 25.0 ms, the fireball had a radius of 130.0 m. Use these values, along with an average air density of 1.25 kg/m^3 , to calculate the initial energy release of the Trinity detonation in joules (J). (Hint: To get energy in joules, you need to make sure all the numbers you substitute in are expressed in terms of SI base units.) (c) The energy released in large explosions is often cited in units of “tons of TNT” (abbreviated “t TNT”), where 1 t TNT is about 4.2 GJ. Convert your answer to (b) into kilotons of TNT (that is, kt TNT). Compare your answer with the quick-and-dirty estimate of 10 kt TNT made by physicist Enrico Fermi shortly after witnessing the explosion from what was thought to be a safe distance. (Reportedly, Fermi made his estimate by dropping some shredded bits of paper right before the remnants of the shock wave hit him and looked to see how far they were carried by it.)

Contributors and Attributions

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SECTION OVERVIEW

1.2: Particles and Interactions

1.2.1: Inertia

1.2.2: Momentum

1.2.3: Force and Impulse

1.2.4: Examples

1.2.5: Particles and Interactions (Exercises)

In this chapter we will study particles and interactions. We will make our first step into the abstraction of the real world by **replacing any objects we want to describe with points**. As we talked about in the last chapter, this is to make things not only 1) simpler, but more importantly 2) mathematical well-defined. The way this works with points is dead simple - take a real object, made up of many (essentially an infinite number of) points, and pick one of it's point to represent the entire thing. This is now well-defined because we can now talk about the position $\vec{r}$ of the object as being exactly the position of the points. And it's far simpler, since we are only dealing with a single point instead of innumerable many.

So after we have a bunch of points, what then? Well, we want to understand the **interactions between the points** (remember, that was one of the ways we described what physics is - the study of these interactions). While we might be tempted to call these interactions "forces", in many cases the forces acting on a particular object are relatively complicated. So first, we are actually going to choose the simplest possible interaction between two objects we can consider - a collision. This is simply when one object comes into contact with another, changing the motion of both.

So what can happen in such a collision? Well, the position $\vec{r}$ and the velocity $\vec{v}$ of either object can change - and that's kind of it, since our objects are just points! The points might have masses, for sure, but for now we are going to assume these masses don't change. (Consider a two-car collision - for sure, some mass is moved back and forth if the collision is bad enough, but generally the two cars keep all their points within each other and don't exchange them.) But how does (for example) the velocity change when the objects interact? It turns out that the velocity is actually not the thing that tells us what happens when two objects collide, it's actually the product of the mass and the velocity, the momentum $\vec{p} = m\vec{v}$. It's pretty intuitive that both the mass and the velocity have to play a role here, since a heavy object hitting a light object is going to be different from two light objects hitting each other. This brings us to our first real principle of physics we are going to study:

Principle of Momentum

Objects interact by transferring momentum

Although this statement seems quite trivial, it actually allows us to perform our first calculations. Consider a collision between two points, each of mass 1 kg, and let's say one of the objects is at rest, while the other is moving towards the first at a speed of 10 m/s. That means the first object has a momentum of 10 kg m/s, while the second has zero (since it's not moving). After they collide, let's say the second object moved away at a speed of 2 m/s. That means it gained 2 kg m/s of momentum...and based on our principle of momentum transfer, the first object must have lost that same 2 kg m/s, leaving it with 8 kg m/s. But that also means we know how fast the object is moving after the collision - since it's mass is 1 kg, it's moving at 8 m/s! This simple example gives us all the essential concepts we are going to be studying not just for this chapter, but for the entire first half of the text.

A brief final note - we didn't really consider anything about the directions that the two objects moved in that example, because it was pretty clear they were moving in a straight line. However, some quantities in physics carry direction, like velocity $\vec{v}$ and position $\vec{r}$ - what about momentum? Well, we defined momentum with a particular formula, $m\vec{v}$. Velocity carries direction, but mass does not, so the momentum will be in the same direction as the velocity. This makes perfect sense with our momentum

transfer principle above as well - an object with a particular momentum $\vec{p}$ in a particular direction will transfer that same momentum to the other object, in the same direction.

This chapter is dedicated to a conceptual understanding of this principle - not that there are no calculations, but we are going to be a little careless with things like vectors and directions. We first want to be sure we get some of the conceptual ideas surrounding momentum transfer, and in later chapters we will add in the mathematical formalism required to handle the directions - and all kinds of other interesting collisions!

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1.2.1: Inertia

In everyday language, we speak of something or someone “having a large inertia” to mean, essentially, that they are very difficult to set in motion. We do know, from experience, that lighter objects are easier to set in motion than heavier objects, but most of us probably have an intuition that gravity (the force that pulls an object towards the earth and hence determines its weight) is not involved in an essential way here. Imagine, for instance, the difference between slapping a volleyball and a bowling ball. It is not hard to believe that the latter would hurt as much if we did it while floating in free fall in the space station (in a state of effective “weightlessness”) as if we did it right here on the surface of the earth. In other words, it is not (necessarily) how heavy something feels, but just how *massive* it is.

But just what is this “massiveness” quality that we associate intuitively with a large inertia? Is there a way (other than resorting to the weight again) to assign to it a numerical value?

1.2.1.1: Relative Inertia and Collisions

One possible way to determine the *relative* inertias of two objects, conceptually, at least, is to try to use one of them to set the other one in motion. Most of us are familiar with what happens when two identical objects (presumably, therefore, having the same inertia) collide: if the collision is head-on (so the motion, before and after, is confined to a straight line), they basically exchange velocities. For instance, a billiard ball hitting another one will stop dead and the second one will set off with the same speed as the first one. The toy sometimes called “Newton’s balls” or “Newton’s cradle” also shows this effect. Intuitively, we understand that what it takes to stop the first ball is exactly the same as it would take to set the second one in motion with the same velocity.

But what if the objects colliding have different inertias? We expect that the change in their velocities as a result of the collision will be different: the velocity of the object with the largest inertia will not change very much, and conversely, the change in the velocity of the object with the smallest inertia will be comparatively larger. A velocity vs. time graph for the two objects might look somewhat like the one sketched in Figure 1.2.1.1.

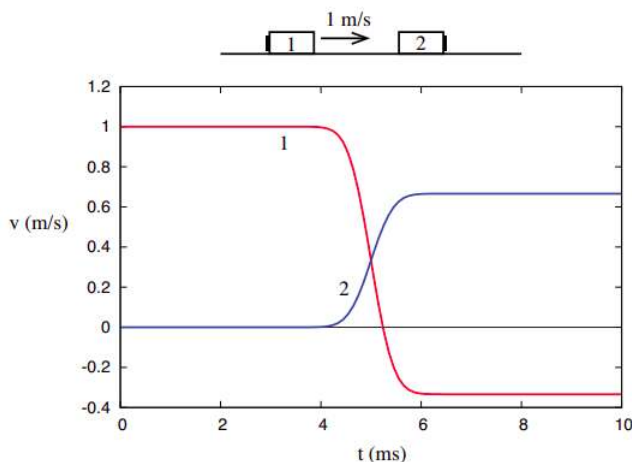


Figure 1.2.1.1: An example of a velocity vs. time graph for a collision of two objects with different inertias.

In this picture, object 1, initially moving with velocity $\vec{v}_{1i} = 1 \text{ m/s}$, collides with object 2, initially at rest. After the collision, which here is assumed to take a millisecond or so, object 1 actually bounces back, so its final velocity is $\vec{v}_{1f} = -1/3 \text{ m/s}$, whereas object 2 ends up moving to the right with velocity $\vec{v}_{2f} = 2/3 \text{ m/s}$. So the change in the velocity of object 1 is $\Delta\vec{v}_1 = \vec{v}_{1f} - \vec{v}_{1i} = -4/3 \text{ m/s}$, whereas for object 2 we have $\Delta\vec{v}_2 = \vec{v}_{2f} - \vec{v}_{2i} = 2/3 \text{ m/s}$. (Note: The small arrows above the velocity variables are simply indications that velocity is a “vector”, which means it has a magnitude and a direction. We will discuss this more later, but for now just know that is what it stands for.)

It is tempting to use this ratio, $\Delta\vec{v}_1/\Delta\vec{v}_2$, as a measure of the *relative inertia* of the two objects, only we’d want to use it upside down and with the opposite sign: that is, since $\Delta\vec{v}_2/\Delta\vec{v}_1 = -1/2$ we would say that object 2 has *twice* the inertia of object 1. (Actually, it makes more sense to take the absolute value of each, but this will work for now.) But then we have to ask: is this a reliable, repeatable measure? Will it work for any kind of collision (within reason, of course: we clearly need to stay in one dimension, and eliminate external influences such as friction), and for any initial velocity?

To begin with, we have reason to expect that it does not matter whether we shoot object 1 towards object 2 or object 2 towards object 1, because (as we will learn later) *only relative motion is detectable*, and the relative motion is the same in both cases. Consider, for instance, what the collision in Figure 1.2.1.1 appears like to a hypothetical observer moving along with object 1, at 1 m/s. To him, object 1 appears to be at rest, and it is object 2 that is coming towards him, with a velocity of -1 m/s. To see what the outcome of the collision looks like to him, just add the same -1 m/s to the final velocities we obtained before: object 1 will end up moving at $v_{1f} = -4/3$ m/s, and object 2 would move at $v_{2f} = -1/3$ m/s, and we would have a situation like the one shown in Figure 1.2.1.2, where both curves have simply been shifted down by 1 m/s:

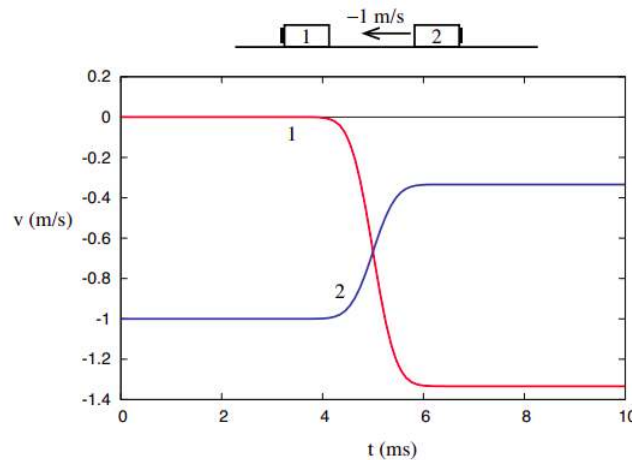


Figure 1.2.1.2: Another example (really the same collision as in Figure 1.2.1.1, only as seen by an observer initially moving to the right at 1 m/s).

But then, this is exactly what we should expect to find also in our laboratory if we actually did send the second object at 1 m/s towards the first one sitting at rest. All the individual velocities have changed relative to Figure 1.2.1.1, but the *velocity changes*, $\Delta \vec{v}_1$ and $\Delta \vec{v}_2$, are clearly still the same, and therefore so is our (tentative) measure of the objects' relative inertia.

Clearly, the same argument can be used to conclude that the same result will be obtained when both objects are initially moving towards each other, as long as their *relative velocity* is the same as in these examples, namely, 1 m/s. However, unless we do the experiments we cannot really predict what will happen if we increase (or decrease) their relative velocity. In fact, we could imagine smashing the two objects at very high speed, so they might even become seriously mangled in the process. Yet, experimentally (and this is not at all an obvious result!), we would still find the same value of $-1/2$ for the ratio $\Delta \vec{v}_2 / \Delta \vec{v}_1$, at least as long as the collision is not so violent that the objects actually break up into pieces.

Perhaps the most surprising result of our experiments would be the following: imagine that the objects have a “sticky” side (for instance, the small black rectangles shown in the pictures could be strips of Velcro), and we turn them around so that when they collide they will end up stuck to each other. In this case (which, as we shall see later, is termed a **completely inelastic** collision), the v -vs- t graph might look like Figure 1.2.1.3 below.

Now the two objects end up moving together to the right, fairly slowly: $v_{1f} = v_{2f} = 1/3$ m/s. The velocity changes are $\Delta \vec{v}_1 = -2/3$ m/s and $\Delta \vec{v}_2 = 1/3$ m/s, both of which are different from what they were before, in Figs. 1.2.1.1 and 1.2.1.2: yet, the ratio $\Delta \vec{v}_2 / \Delta \vec{v}_1$ is still equal to $-1/2$, just as in all the previous cases.

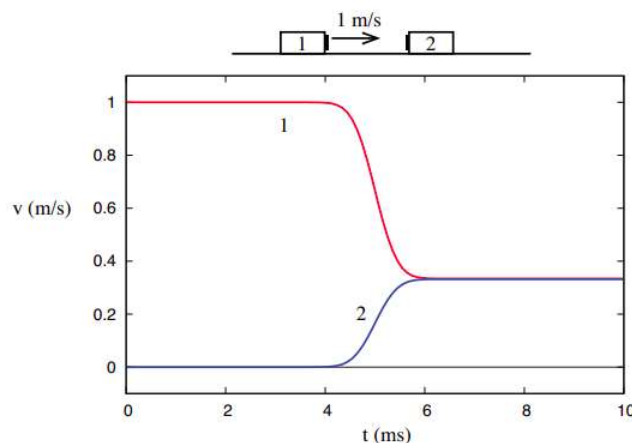


Figure 1.2.1.3 became stuck together when they collided.

1.2.1.2: Inertial Mass: Definition and Properties

At this point, it would seem reasonable to assume that this ratio, $\Delta v_2/\Delta v_1$, is, in fact, telling us something about an *intrinsic* property of the two objects, what we have called above their “relative inertia.” It is easy, then, to see how one could assign a value to the inertia of any object (at least, conceptually): choose a “standard” object, and decide, arbitrarily, that its inertia will have the numerical value of 1, in whichever units you choose for it (these units will turn out, in fact, to be kilograms, as you will see in a minute). Then, to determine the inertia of another object, which we will label with the subscript 1, just arrange a one-dimensional collision between object 1 and the standard, under the right conditions (basically, no net external forces), measure the velocity changes Δv_1 and Δv_s , and take the quantity $-\Delta v_s/\Delta v_1$ as the numerical value of the ratio of the inertia of object 1 to the inertia of the standard object. In symbols, using the letter m to represent an object’s inertia,

$$\frac{m_1}{m_s} = -\frac{\Delta v_s}{\Delta v_1} \quad (1.2.1.1)$$

But, since $m_s = 1$ by definition, this gives us directly the numerical value of m_1 .

The reason we use the letter m is, as you must have guessed, because, in fact, the inertia defined in this way turns out to be identical to what we have traditionally called “mass.” More precisely, the quantity defined this way is an object’s *inertial mass*. The remarkable fact that the force of gravity between two objects turns out to be proportional to their inertial masses, allows us to determine the inertial mass of an object by the more traditional procedure of simply weighing it, rather than elaborately staging a collision between it and the standard kilogram on an ice-hockey rink. But, in principle, we could conceive of the existence of two different quantities that should be called “inertial mass” and “gravitational mass,” and the identity (or more precisely, the—so far as we know—exact proportionality) of the two is a rather mysterious experimental fact¹.

In any case, by the way we have constructed it, the inertial mass, defined as in Equation (1.2.1.1), does capture, in a quantitative way, the concept that we were trying to express at the beginning of the chapter: namely, how difficult it may be to set an object in motion. In principle, however, other experiments would need to be conducted to make sure that it does have the properties we have traditionally associated with the concept of mass. For instance, suppose we join together two objects of mass m . Is the mass of the resulting object $2m$? Collision experiments would, indeed, show this to be the case with great accuracy in the macroscopic world (with which we are concerned this semester), but this is a good example of how you cannot take anything for granted: at the

microscopic level, it is again a fact that the inertial mass of an atomic nucleus is a little *less* than the sum of the masses of all its constituent protons and neutrons².

Probably the last thing that would need to be checked is that *the ratio of inertias is independent of the standard*. Suppose that we have two objects, to which we have assigned masses m_1 and m_2 by arranging for each to collide with the “standard object” independently. If we now arrange for a collision between objects 1 and 2 directly, will we actually find that the ratio of their velocity changes is given by the ratio of the separately determined masses m_1 and m_2 ? We certainly would need that to be the case, in order for the concept of inertia to be truly useful; but again, we should not assume anything until we have tested it! Fortunately, the tests would indeed reveal that, in every case, the expected relationship holds³

$$-\frac{\Delta \vec{v}_2}{\Delta \vec{v}_1} = \frac{m_1}{m_2}. \quad (1.2.1.2)$$

At this point, we are not just in possession of a useful definition of inertia, but also of a veritable *law of nature*, as I will explain next.

¹This fact, elevated to the category of a principle by Einstein (the *equivalence principle*) is the starting point of the general theory of relativity.

²And this is not just an unimportant bit of trivia: all of nuclear power depends on this small difference.

³Equation 1.2.1.2 actually is found to hold also at the microscopic (or *quantum*) level, although there we prefer to state the result by saying that conservation of momentum holds (see the following section).

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1.2.2: Momentum

For an object of (inertial) mass m moving, in one dimension, with velocity $\vec{v}$, we define its *momentum* as

$$\vec{p} = m\vec{v} \quad (1.2.2.1)$$

(the choice of the letter $\vec{p}$ for momentum is apparently related to the Latin word “impetus”).

We can think of momentum as a sort of extension of the concept of inertia, from an object at rest to an object in motion. When we speak of an object’s inertia, we typically think about what it may take to get it moving; when we speak of its momentum, we typically think of that it may take to stop it (or perhaps deflect it). So, both the inertial mass m and the velocity $\vec{v}$ are involved in the definition.

We may also observe that what looks like inertia in some reference frame may look like momentum in another. For instance, if you are driving in a car towing a trailer behind you, the trailer has only a large amount of inertia, but no momentum, relative to you, because its velocity relative to you is zero; however, the trailer definitely has a large amount of momentum (by virtue of both its inertial mass and its velocity) relative to somebody standing by the side of the road.

1.2.2.1: Conservation of Momentum; Isolated Systems

For a system of objects, we treat the momentum as an *additive* quantity. So, if two colliding objects, of masses m_1 and m_2 , have initial velocities $\vec{v}_{1i}$ and $\vec{v}_{2i}$, we say that the total initial momentum of the system is $\vec{p}_i = m_1\vec{v}_{1i} + m_2\vec{v}_{2i}$, and similarly if the final velocities are $\vec{v}_{1f}$ and $\vec{v}_{2f}$, the total final momentum will be $\vec{p}_f = m_1\vec{v}_{1f} + m_2\vec{v}_{2f}$.

We then assert that *the total momentum of the system is not changed by the collision*. Mathematically, this means

$$\vec{p}_i = \vec{p}_f \quad (1.2.2.2)$$

or

$$m_1\vec{v}_{1i} + m_2\vec{v}_{2i} = m_1\vec{v}_{1f} + m_2\vec{v}_{2f}. \quad (1.2.2.3)$$

But this last equation, in fact, follows directly from Equation (2.1.2): to see this, move all the quantities in Equation (1.2.2.3) having to do with object 1 to one side of the equal sign, and those having to do with object 2 to the other side. You then get

$$\begin{aligned} m_1(\vec{v}_{1i} - \vec{v}_{1f}) &= m_2(\vec{v}_{2f} - \vec{v}_{2i}) \\ -m_1\Delta\vec{v}_1 &= m_2\Delta\vec{v}_2 \end{aligned} \quad (1.2.2.4)$$

which is just another way to write Equation (2.1.2). Hence, the result (2.1.2) ensures the conservation of the total momentum of a system of any two interacting objects (“particles”), regardless of the form the interaction takes, as long as there are no external forces acting on them.

Momentum conservation is one of the most important principles in all of physics, so let me take a little time to explain how we got here and elaborate on this result. First, as I just mentioned, we have been more or less implicitly assuming that the two interacting objects form an *isolated* system, by which we mean that, throughout, they interact with nothing other than each other. (Equivalently, there are no external forces acting on them.)

It is pretty much impossible to set up a system so that it is *really* isolated in this strict sense; instead, in practice, we settle for making sure that the external forces on the two objects *cancel out*. This is what happens on the air tracks with which you will be doing experiments this semester: gravity is acting on the carts, but that force is balanced out by the upwards push of the air from the track. A system on which there is no *net* external force is as good as isolated for practical purposes, and we will refer to it as such. (It is harder, of course, to completely eliminate friction and drag forces, so we just have to settle for approximately isolated systems in practice.)

Secondly, we have assumed so far that the motion of the two objects is restricted to a straight line—one dimension. In fact, momentum is a *vector* quantity (just like velocity is), which is why we have been writing the previous equations with the small arrows over them.

Conservation of momentum, in general, holds as a vector equation for any isolated system in three dimensions:

$$\vec{p}_i = \vec{p}_f. \quad (1.2.2.5)$$

What this means, in turn, is that each separate component (x , y and z) of the momentum will be separately conserved (so Equation (1.2.2.5) is equivalent to three scalar equations, in three dimensions).

Thirdly, although this may not be immediately obvious, for an isolated system of two colliding objects the momentum is truly conserved throughout the whole collision process. It is not just a matter of comparing the initial and final velocities: at any of the times shown in Figures 2.1.1, 2.1.2, or 2.1.3, if we were to measure v_1 and v_2 and compute $m_1v_1 + m_2v_2$, we would obtain the same result. In other words, the total momentum of an isolated system is *constant*: it has the same value at all times.

Finally, all these examples have involved interactions between only two particles. Can we really generalize this to conclude that the total momentum of an isolated system of any number of particles is constant, even when all the particles may be interacting with each other simultaneously? Here, again, the experimental evidence is overwhelmingly in favor of this hypothesis⁴, but much of our confidence on its validity comes in fact from a consideration of the nature of the internal interactions themselves. It is a mathematical fact that all of the interactions so far known to physics have the property of conserving momentum, whether acting individually or simultaneously. No experiments have ever suggested the existence of an interaction that does not have this property.

⁴For an important piece of indirect evidence, just consider that any extended object is in reality a collection of interacting particles, and the experiments establishing conservation of momentum almost always involve such extended objects. See the following section for further thoughts on this matter.

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1.2.3: Force and Impulse

We have already seen how we can think of interacting objects as objects that transfer momentum between each other. We've written this transfer of momentum as $\Delta\vec{p}$, and we've discussed how this momentum cannot be lost (a phenomena known as **conservation**); if one object gains $\Delta\vec{p}$, the object that it's interacting with must have lost that same amount of momentum. For a two-object system, that means precisely that $\Delta\vec{p}_1 = -\Delta\vec{p}_2$.

Fundamentally, you can imagine momentum as a "currency" that is constantly being exchanged between objects. When two objects interact, one object can gain momentum only by the other losing momentum. There is no way to create "new" momentum. (Unfortunately, in my experience, this is also true with money. I have not yet found a way to increase the amount I have without getting it from some other person or entity.)

We often give a name to the momentum that is exchanged between two objects. It is called "impulse". If object A interacts with object B, A might have a change in momentum, $\Delta\vec{p}_A$ that it has received from object B. This value is the "impulse" that object A has received. We know from conservation of momentum (and our currency analogy) that B will also receive an impulse, $\Delta\vec{p}_B$, that is equal to $-\Delta\vec{p}_A$. ($\Delta\vec{p}_B = -\Delta\vec{p}_A$).

But of course we know that this is not the only way to describe interacting objects - in fact, perhaps the most intuitive way to describe two objects interacting is actually using a **force between them**. We are going to talk a lot more about forces in the second half of the this book, but right now we just want to acknowledge that if interactions can be described with either forces or momentum transfer, there must be some relationship between these two quantities. In fact, the relationship can be made concrete once you know the **time period** Δt over which the interaction occurs.

Specifically, if you have a change in momentum $\Delta\vec{p}$ (also called an "impulse") resulting from an interaction that happens over a time period Δt , we can associate a force with this interaction via

$$\vec{F}_{ave} = \frac{\Delta\vec{p}}{\Delta t} \quad (1.2.3.1)$$

Notice that this definition only really works for an *average* force $\vec{F}_{ave}$, since we are talking about the changes in time being possibly relatively large. If the change in time is very small (in the sense of an infinitesimal dt from calculus), we can actually write

$$\vec{F} = \frac{d\vec{p}}{dt} \quad (1.2.3.2)$$

for the *instantaneous* force, which is the usual notion of force we are used to (this relationship can also be derived from Newton's second law).

All of the above is a perfectly fine description of a force that does not depend on time, which is often what we are dealing with. However, if a force does depend on time (think of throwing a baseball, or a car crash - the force over the time period of those interactions may not be constant at all), we can still find the impulse delivered by performing an integration,

$$\vec{J} = \int \vec{F} dt. \quad (1.2.3.3)$$

1.2.3.1: Gravity

We will be looking at many different forces in future chapters, but for now we will just learn one of the most important forces, the force of gravity. Actually, we should be a little more specific. We will learn about a special force which is the force of gravity on (and near) the surface of the Earth. You are probably aware of this force. It is the reason you fall when you trip, why you shouldn't hold your ice cream cone upside down, and (at least partially) why baseball is an interesting sport. It is the force that pulls things down. In fact, I often tell my students, gravity is actually how we know which direction *is* down. "Down" is generally agreed upon as the direction that gravity pulls things.

Close to earth, the force of gravity has a very simple form:



$$\vec{F}_g = m\vec{g} \quad (1.2.3.4)$$

Where $\vec{g} = 9.8\text{m/s}^2$ is the acceleration due to gravity and points "down" (generally towards the center of the Earth). The S.I. unit for force is a Newton (N).

So what does this mean in terms of momentum and impulse? Let's take a 1 kg weight that is being acted on only by the force of gravity. Using our Equation 1.2.3.4, we know that the magnitude of force it experiences will be $(1\text{ kg})(9.8\text{m/s}) = 9.8\text{ N}$. We know from our Equation 1.2.3.1 that in every second (1 s), we will have a change of momentum (impulse) of:

$$|\Delta\vec{p}| = |\vec{F}|\Delta t = (9.8\text{ N})(1\text{ s}) = 9.8\text{kg} \cdot \text{m/s} \quad (1.2.3.5)$$

where I have used absolute value signs around our momentum and force to show that I am just interested in the magnitude (not direction). Besides, we already know that the impulse from the Earth will pull us *down*.

Since we knew from the beginning that we have a 1 kg weight, we can also calculate our change in speed in one second:

$$|\Delta\vec{v}| = |\Delta\vec{p}|/m = 9.8\text{ m/s} \quad (1.2.3.6)$$

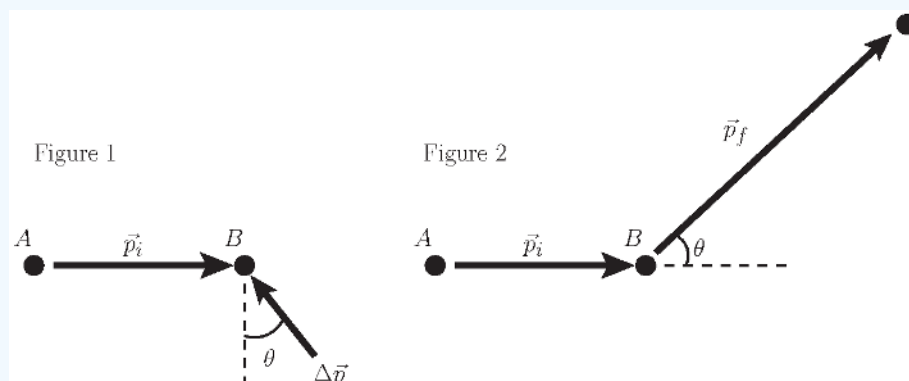
which, we will find later, we could have discovered directly just by knowing the acceleration due to gravity.

Hopefully, this gives some idea of how momentum, change in momentum, impulse, and force are all related.

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1.2.4: Examples

✓ Practice Problem 1.2.4.1: Hockey Puck Scale Drawings



Consider a hockey puck of mass $1/6$ kg, traveling between points A and B in a straight line at a speed of 10.0 m/s. At point B, it gets hit by a hockey stick which delivers an impulse of $\Delta \vec{p}$ to the puck. Using meter sticks and protractors, make scale drawings of this process on the whiteboards, and answer the following two questions:

- In figure 1, the amount of impulse delivered is $\Delta \vec{p} = 30$ kg m/s at an angle of 30° , as shown.
 - What is the final momentum (magnitude and direction) of the hockey puck, $\vec{p}_f$?
- In figure 2, we measure the final momentum $\vec{p}_f$ to be 100 kg m/s, moving at an angle of 40° , as shown.
 - What was the impulse transferred to the hockey puck, $\Delta \vec{p}$?
 - At what angle did the impulse act on the puck?

? Practice Problem 1.2.4.2: Momentum Cart Transfer, Part II

Suppose a cart with mass m is moving to the right at a speed of v_0 . It collides with a cart of mass $2m$, which is initially at rest. After the collision, the first cart rebounds to the left with a speed of $\frac{1}{3}v_0$.

- How much impulse did the first cart deliver to the second? Use v_f as the final speed of the second cart.
- How much impulse did the second cart deliver to the first?
- Assuming that these two impulse transfers are the same, determine the final speed v_f in terms of the initial speed v_0 .

? Practice Problem 1.2.4.3: Bouncing Ball

I drop a golf ball of mass $m = 46$ g onto the floor, and right before it hits it has a velocity of $\vec{v}_{yi} = -2.2$ m/s.

- What is the momentum of the ball right before it hits the ground?
- If the speed of the ball after it bounces, v_{yf} , is 75% of the speed right before it hits, what was the impulse the floor delivered to the ball?
- If the ball interacts with the floor for a time period Δt , find an expression for the ratio between the average force the floor acted on the ball with to the weight of the ball mg , in terms of v_{yi} , v_{yf} , g , and Δt .
- Calculate this ratio using $\Delta t = 150$ ms and the other parameters of the problem.

✓ Example 1.2.4.1: Reading a Collision Graph

The graph shows a collision between two carts (possibly equipped with magnets so that they repel each other before they actually touch) on an air track. The inertia (mass) of cart 1 is 1 kg. Note: this is a *position vs. time* graph!

- What are the initial velocities of the carts?
- What are the final velocities of the carts?
- What is the mass of the second cart?

- d. Does the air track appear to be level? Why? (Hint: does the graph show any evidence of acceleration, for either cart, outside of the collision region?)
- e. At the collision time, is the change in velocity (Δv) of the first cart positive or negative? How about the second cart? (Justify your answers.)
- f. For the system consisting of the two carts, what is its initial (total) momentum? What is its final momentum?
- g. Imagine now that one of the magnets is reversed, so when the carts collide they stick to each other. What would then be the final momentum of the system? What would be its final velocity?

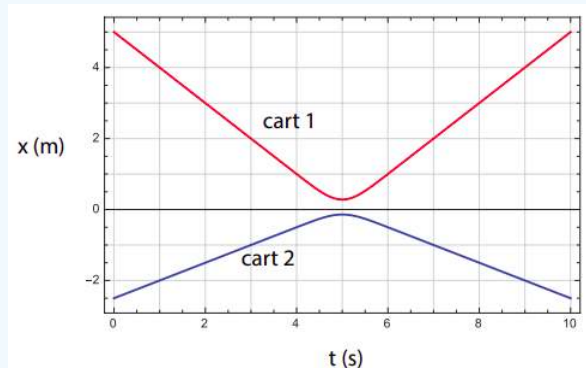


Figure 1.2.4.1: A collision between two carts

Solution

- (a) All the velocities are to be calculated by picking an easy straight part of each curve and calculating

$$v = \frac{\Delta x}{\Delta t}$$

for suitable intervals. In this way one gets

$$\begin{aligned} v_{1i} &= -1 \frac{\text{m}}{\text{s}} \\ v_{2i} &= 0.5 \frac{\text{m}}{\text{s}} \end{aligned} \quad (1.2.4.1)$$

- (b) Similarly, one gets

$$\begin{aligned} v_{1f} &= 1 \frac{\text{m}}{\text{s}} \\ v_{2f} &= -0.5 \frac{\text{m}}{\text{s}} \end{aligned} \quad (1.2.4.2)$$

- (c) Use this equation, or equivalent (conservation of momentum is OK)

$$\begin{aligned} \frac{m_2}{m_1} &= -\frac{\Delta v_1}{\Delta v_2} \\ \frac{m_2}{m_1} &= -\frac{1 - (-1)}{-0.5 - 0.5} = 2 \end{aligned} \quad (1.2.4.3)$$

so the mass of the second cart is 2 kg.

- (d) Yes, the track appears to be level because the carts do not show any evidence of acceleration outside of the collision region (the position vs. time curves are straight lines outside of the region approximately given by $4.5 \text{ s} < t < 5.5 \text{ s}$).

- (e) The change in velocity of the first cart is positive. You can see this either graphically (the curve is like a parabola that opens upwards, i.e., concave), or algebraically (the cart's velocity increases, going from -1 m/s to 1 m/s)

Similarly, the change in velocity of the second cart is negative. The curve is like a parabola that opens downwards, i.e., convex; or, algebraically, the cart's velocity decreases, going from 0.5 m/s to -0.5 m/s .

- (f) The initial momentum of the system is

$$p_i = m_1 v_{1i} + m_2 v_{2i} = (1 \text{ kg}) \times \left(-1 \frac{\text{m}}{\text{s}}\right) + (2 \text{ kg}) \times \left(0.5 \frac{\text{m}}{\text{s}}\right) = 0$$

The final momentum is

$$p_f = m_1 v_{1f} + m_2 v_{2f} = (1 \text{ kg}) \times \left(1 \frac{\text{m}}{\text{s}}\right) + (2 \text{ kg}) \times \left(-0.5 \frac{\text{m}}{\text{s}}\right) = 0$$

You could also just say that the final momentum should be the same as the initial momentum, since the system appears to be isolated.

(g) The momentum should be conserved in this case as well, so $p_f = 0$. The velocity would be

$$v_f = \frac{p_f}{m_1 + m_2} = 0.$$

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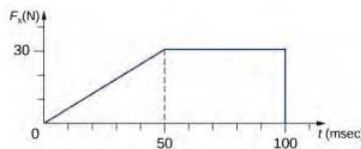
1.2.5: Particles and Interactions (Exercises)

1.2.5.1: Conceptual Questions

1. An object that has a small mass and an object that has a large mass have the same momentum. Which object is moving faster?
2. An object that has a small mass and an object that has a large mass are moving with the same speed. Which mass has the largest momentum?
3. Is it possible for a small force to produce a larger impulse on a given object than a large force? Explain.
4. Why is a 10-m fall onto concrete far more dangerous than a 10-m fall onto water?
5. What external force is responsible for changing the momentum of a car moving along a horizontal road?
6. A piece of putty and a tennis ball with the same mass are thrown against a wall with the same velocity. Which object experiences a greater force from the wall or are the forces equal? Explain.

1.2.5.2: Problems

1. (a) Calculate the momentum of a 2000-kg elephant charging a hunter at a speed of 7.50 m/s. (b) Compare the elephant's momentum with the momentum of a 0.0400-kg tranquilizer dart fired at a speed of 600 m/s. (c) What is the momentum of the 90.0-kg hunter running at 7.40 m/s after missing the elephant?
2. A skater of mass 40 kg is carrying a box of mass 5 kg. The skater has a speed of 5 m/s with respect to the floor and is gliding without any friction on a smooth surface. (a) Find the momentum of the box with respect to the floor. (b) Find the momentum of the box with respect to the floor after she puts the box down on the frictionless skating surface.
3. A car of mass 2000 kg is moving with a constant velocity of 10 m/s due east. What is the momentum of the car?
4. The mass of Earth is 5.97×10^{24} kg and its orbital radius is an average of 1.50×10^{11} m. Calculate the magnitude of its average linear momentum.
5. If a rainstorm drops 1 cm of rain over an area of 10 km^2 in the period of 1 hour, what is the momentum of the rain that falls in one second? Assume the terminal velocity of a raindrop is 10 m/s.
6. What is the average momentum of an avalanche that moves a 40-cm-thick layer of snow over an area of 100 m by 500 m over a distance of 1 km down a hill in 5.5 s? Assume a density of 350 kg/m^3 for the snow.
7. What is the average momentum of a 70.0-kg sprinter who runs the 100-m dash in 9.65 s?
8. A tennis ball of mass 0.06 kg is thrown at a wall with a speed of 15 m/s. The ball bounces back from the wall with a speed of 12 m/s. What is the change in momentum and impulse of the ball due to the collision with the wall?
9. A 5 kg cannonball is shot out of a cannon at 400 m/s on the Gettysburg Battlefield. The cannon has a mass of 1,200 kg.
 1. What is the impulse that was delivered to the cannonball by the cannon?
 2. What is the impulse that is delivered to the cannon?
 3. What is the final speed of the cannon?
10. One hazard of space travel is debris left by previous missions. There are several thousand objects orbiting Earth that are large enough to be detected by radar, but there are far greater numbers of very small objects, such as flakes of paint. Calculate the force exerted by a 0.100-mg chip of paint that strikes and sticks to a spacecraft window at a relative speed of 4.00×10^3 m/s, given the collision lasts 6.00×10^{-8} s.
11. Calculate the final speed of a 110-kg rugby player who is initially running at 8.00 m/s but collides head-on with a padded goalpost and experiences a backward force of 1.76×10^4 N for 5.50×10^{-2} s.
12. Water from a fire hose is directed horizontally against a wall at a rate of 50.0 kg/s and a speed of 42.0 m/s. Calculate the force exerted on the wall, assuming the water's horizontal momentum is reduced to zero.
13. A 300 g scoop of ice cream fall from a cone. It falls for 2 seconds.
 1. What was the impulse that was delivered to the ice cream cone by the Earth?
 2. What was the impulse that was delivered to the Earth by the ice cream cone?
 3. If the ice cream started from rest, how fast is the ice cream going now?
 4. If the Earth started from rest, how fast is it moving towards the ice cream now?
14. The x-component of a force on a 46-g golf ball by a 7-iron versus time is plotted in the following figure:
 - a. Find the x-component of the impulse during the intervals (i) [0, 50 ms], and (ii) [50 ms, 100 ms].
 - b. Find the change in the x-component of the momentum during the intervals (iii) [0, 50 ms], and (iv) [50 ms, 100 ms].



1.2.5.3: Additional Problems

- Which has a larger magnitude of momentum: a 3000-kg elephant moving at 40 km/h or a 60-kg cheetah moving at 112 km/h?
- A driver applies the brakes and reduces the speed of her car by 20%, without changing the direction in which the car is moving. By how much does the car's momentum change?
- You friend claims that momentum is mass multiplied by velocity, so things with more mass have more momentum. Do you agree? Explain.
- Dropping a glass on a cement floor is more likely to break the glass than if it is dropped from the same height on a grass lawn. Explain in terms of the impulse.
- Your 1500-kg sports car accelerates from 0 to 30 m/s in 10 s. What average force is exerted on it during this acceleration?
- A 5.0-g egg falls from a 90-cm-high counter onto the floor and breaks. What impulse is exerted by the floor on the egg?
- You are coasting on your 10-kg bicycle at 15 m/s and a 5.0-g bug splatters on your helmet. The bug was initially moving at 2.0 m/s in the same direction as you. If your mass is 60 kg, (a) what is the initial momentum of you plus your bicycle? (b) What is the initial momentum of the bug? (c) What is your change in velocity due to the collision with the bug? (d) What would the change in velocity have been if the bug were traveling in the opposite direction?
- A 100-kg astronaut finds himself separated from his spaceship by 10 m and moving away from the spaceship at 0.1 m/s. To get back to the spaceship, he throws a 10-kg tool bag away from the spaceship at 5.0 m/s. How long will he take to return to the spaceship?
- You friend wonders how a rocket continues to climb into the sky once it is sufficiently high above the surface of Earth so that its expelled gasses no longer push on the surface. How do you respond?
- To increase the acceleration of a rocket, should you throw rocks out of the front window of the rocket or out of the back window?

Contributors and Attributions

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SECTION OVERVIEW

1.3: Vector Analysis

1.3.1: Position Vectors and Components

1.3.2: Vector Algebra in 1 Dimension

1.3.3: Vector Algebra in 2 Dimensions- Graphical

1.3.4: Vector Algebra in Multiple Dimensions- Calculations

1.3.E: Vectors (Exercises)

The language of physics is mathematics. This is not just a thing we tell ourselves, but is a basic fact that separates the physical sciences from the social sciences. Although we may speak and write about physics using a spoken and written language (English, in this case), this is just for descriptive convenience. Anytime we actually want to do anything with physics, we have to cast it into mathematics. Part of this is a desire for precision - if I say "the acceleration due to gravity is 9.81 m/s^2 ", I know exactly how precise that number is (typically we assume the uncertainty is in the last decimal point, so that implies $\pm 0.01 \text{ m/s}^2$ on that number). However, it's also a necessity, since no amount of words can actually be used to exactly describe any physical law, since words don't inherently have precise meaning. Take even the most basic of physical laws - Newton's first law "An object in motion remains in motion until acted on by an external force." What is "motion"? What is "being acted on"? What is "force"? No amount of language is going to answer these questions, only mathematics is going to give us careful definitions of these concepts¹.

So, once we are convinced that we need to use mathematics to do physics, we know that we need to understand how to associate physics quantities ("mass", "speed", "force") with mathematical variables. In some cases, this is very easy - for example, the temperature of something is just a number, call it T . Or the mass of something is a number m . Single numbers are called **scalars**. Sometimes these numbers have constraints on them that come from reality - for example, mass must be positive, and temperature has a minimum value (absolute zero). But there are also quantities which need more than just a single variable, or additional conditions on that variable - they actually must carry more information. The most important of these are **vectors**, which carry both a number (the *magnitude*, or length of the vector), and a *direction*.

The length of a vector is a pretty straightforward idea - vectors are arrows, and the size of the arrow is the length. But direction is a little bit more confusing, because it can be specified in several different ways. For example, I can be driving at a speed of 32 mph, in the direction "north". Or maybe in the direction "towards you", or "90° from East". Because of this uncertainty, we will often choose to represent vectors *using components* - that is, an xy -Cartesian coordinate system. Of course, this coordinate system is something we created out of thin air to help us solve the problem, so which coordinate system we pick should not influence our calculation in the slightest.

For a concrete example, consider a vector of length 5, that is pointed 45° above the horizontal. We can choose to represent this vector in a number of ways. In a coordinate system fixed to the horizontal, the vector has components

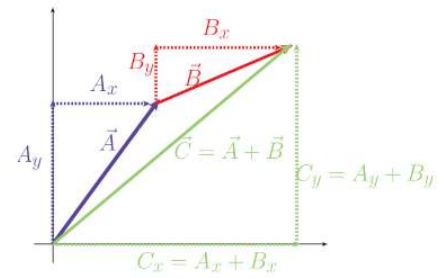
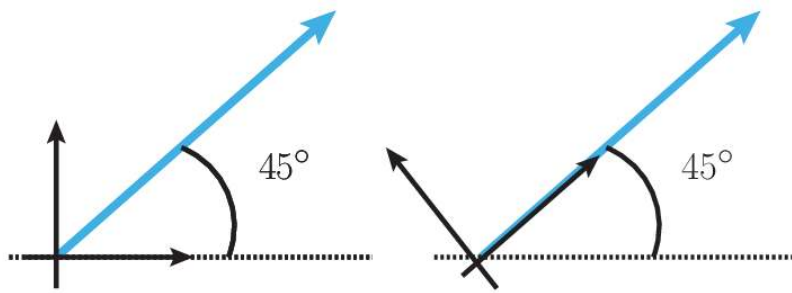
$$v_x = 5 \cos(45) \simeq 3.53, v_y = 5 \sin(45) \simeq 3.53. \quad (1.3.1)$$

Alternatively, if you pick a coordinate system with the x -axis along the vector itself, the components would be

$$v_x = 5, v_y = 0. \quad (1.3.2)$$

In some cases one of these might be preferable to the other, but there is something important here: although the coordinate representation changed, the magnitude and direction did not. No choice of coordinate system can change the magnitude and direction of a physical variable like a vector, because coordinates are just helping tools to do physics.

There is one last point to make about vectors in coordinate systems, which is summed up by the last figure: performing mathematics like addition on vectors means the same mathematics applies to the components. This makes working with components very convenient - we don't actually need to go around making pictures of arrows and measuring them, we can just use the components and get the results algebraically.



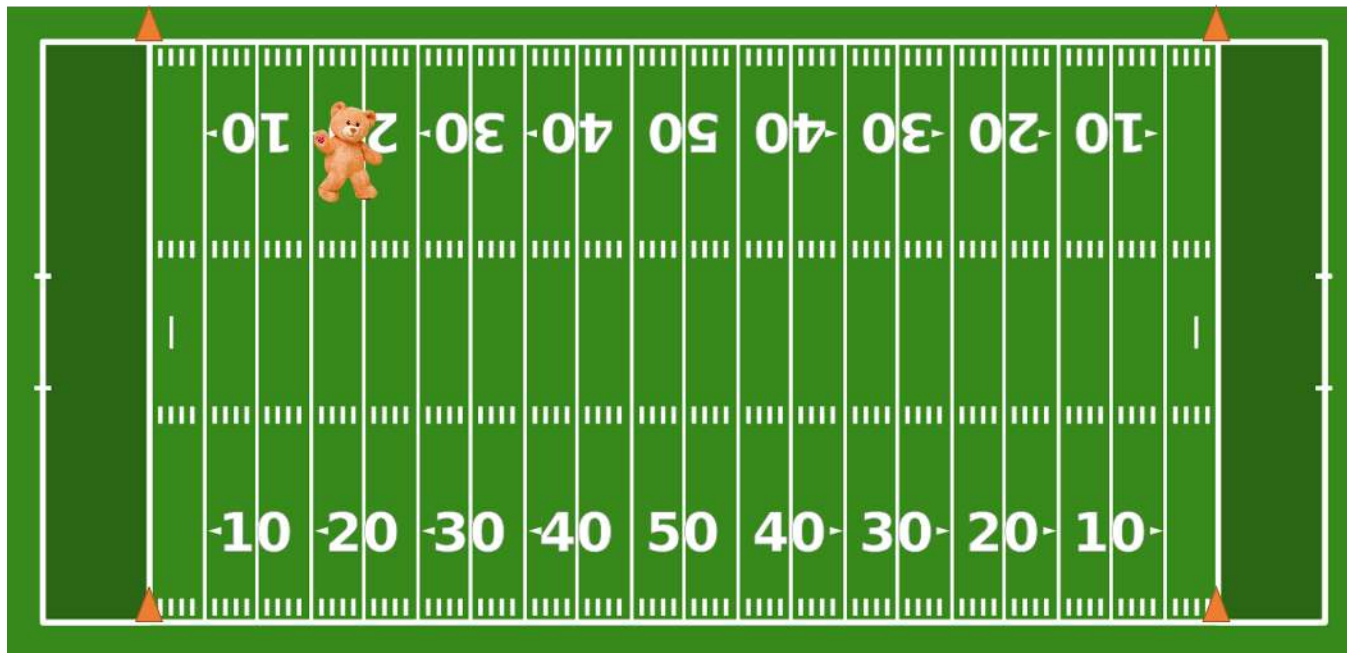
¹If you're interested, the mathematical statement of Newton's first law is something like " $\vec{a} = 0 \supset \Sigma \vec{F}_{ext} = 0$ ", using the symbol $\supset$ for "if".

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1.3.1: Position Vectors and Components

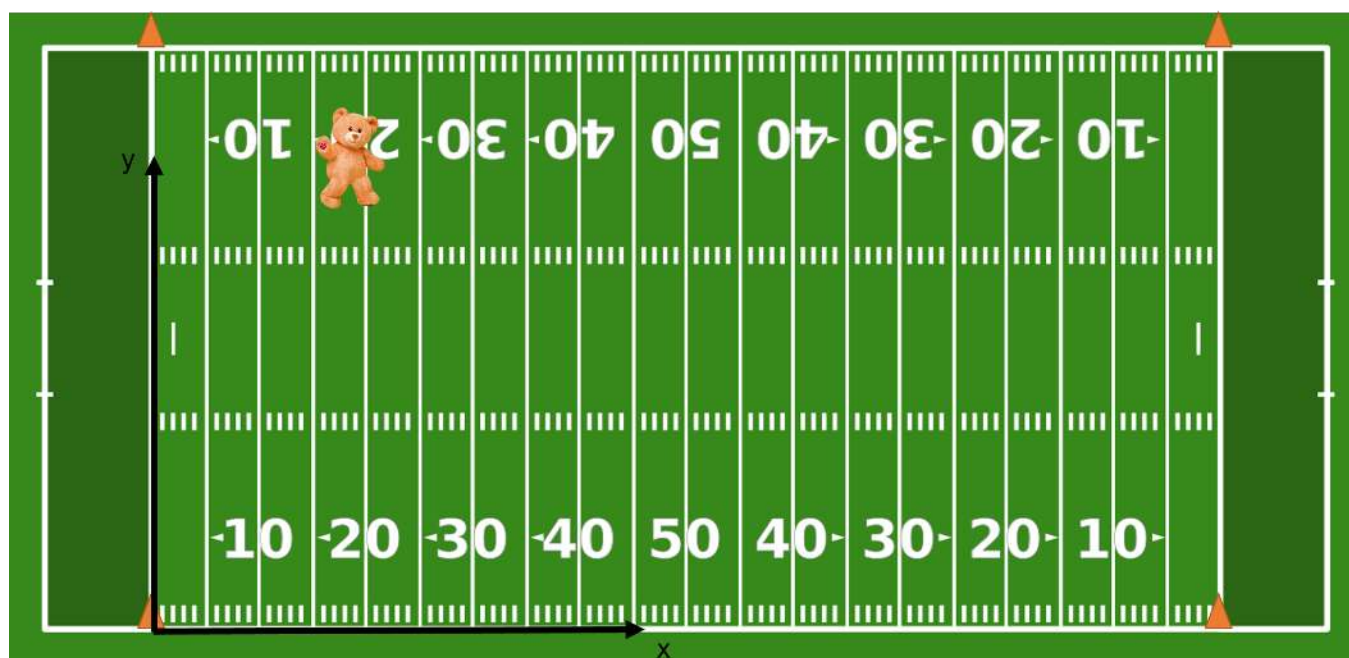
Vectors

As we saw at the beginning of the chapter, vectors are a simple way of representing a value that has both a magnitude and a direction. One of the simplest things we can represent using vectors is the position of an object. Imagine you have a teddy bear somewhere on a football field.

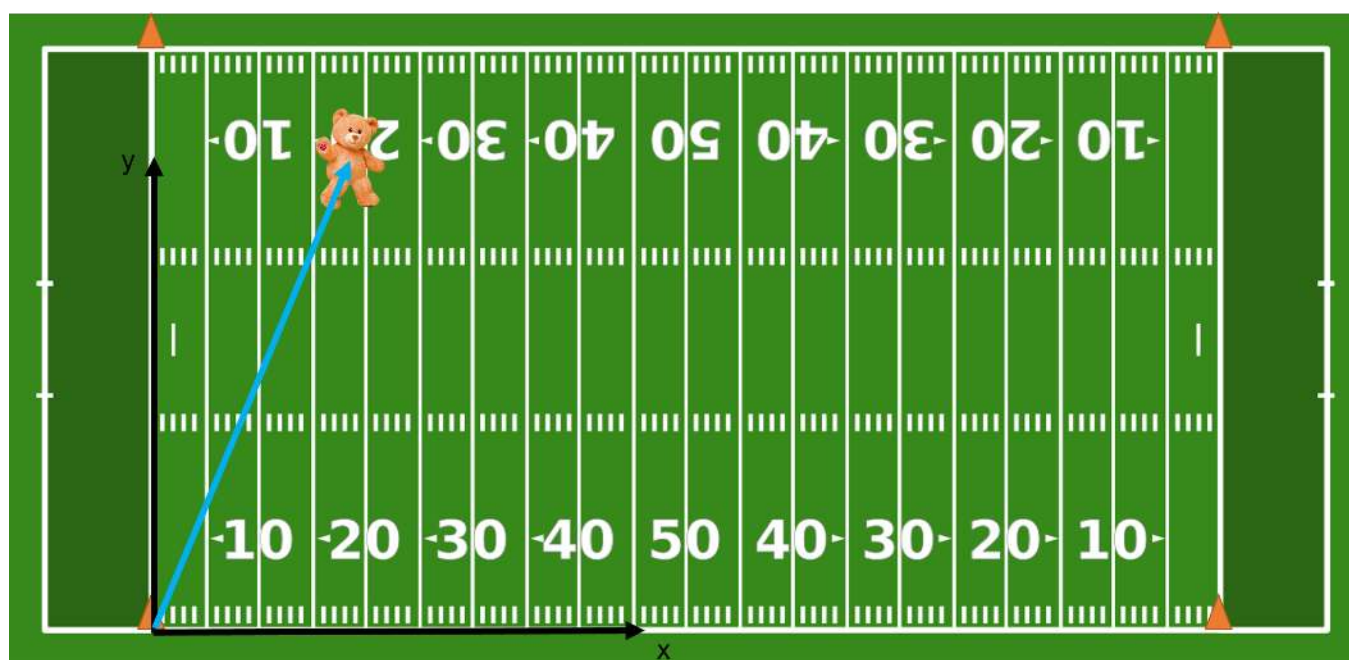


Now, imagine you have forgotten it and you need to tell someone where to find it in the middle of the night.

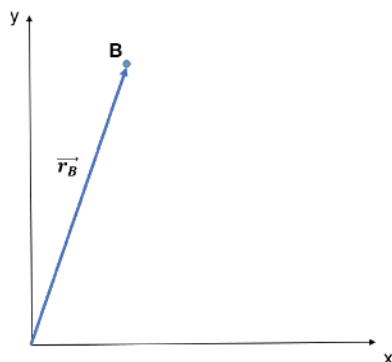
First, you have to set up a coordinate system. This means you need to tell someone what direction your "x" values will go and what direction your "y" values will go. What we can do is say that going the length of the field will be our "x" direction and going across the field will be our "y" direction. We also need to say where we are going to be measuring from, in other words where do we start from. This is called our origin. We will set this in the bottom left corner at the orange cone. When we have done all of this, we will have the following.



Now that we have this all set up, we can draw a vector that points directly to the teddy bear!

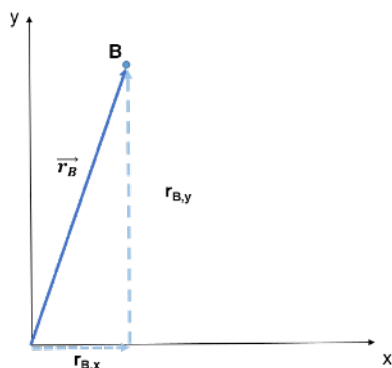


Great, but this is getting pretty hard to see. Let's remove all of the distracting parts and just show the representations of everything:



where I've labeled the point the teddy bear is at "B" for Bear! The vector I will call r_B . For some reason, we often use the letter "r" for vectors that point to positions and use subscripts as "name tags" when we need to be more descriptive.

Great! Now we can see everything and start do do some math! First, the best way of representing a vector is by its components. You can think of components as telling someone how much to move in each direction to get to the bear. For instance, you may say that they should move 17 yards in the x-direction and 35 yards in the y-direction. (*Apologies for using silly units like "yards" instead of nice units like "meters". That's American football for you! I promise not to do it again.*) These measurements are called the x-component and y-component, respectively. You would say that $r_{B,x} = 17$ yards and $r_{B,y} = 35$ yards. We can show these components on the graph.

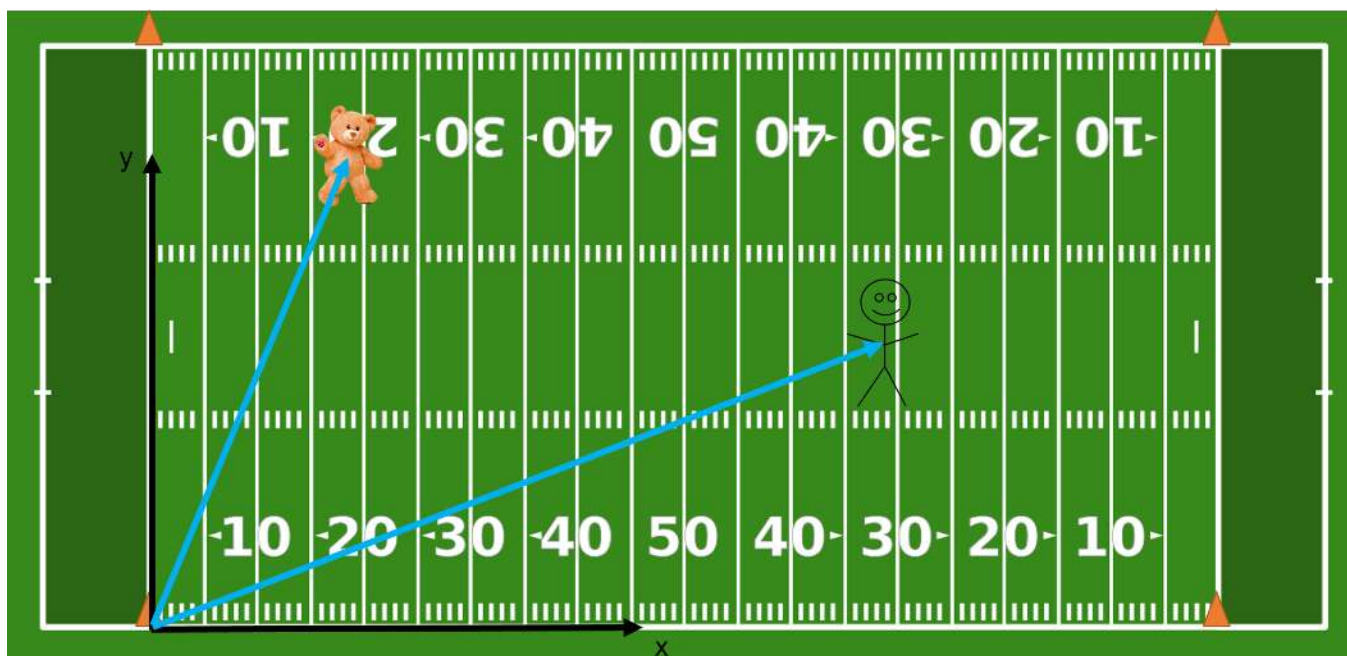


However, this will get cumbersome very quickly and I hate the double subscripts (although they are necessary sometimes). This representation will also be a real pain when we need to start adding and subtracting vectors. So, we will introduce a new representation of vectors called **column vector** notation. Here is that same vector, r_B represented in column vector form:

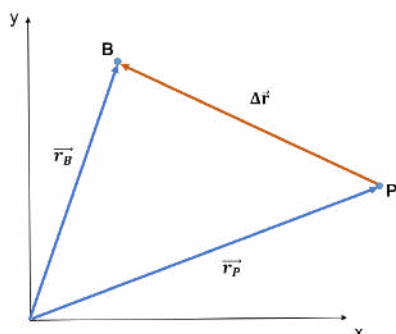
$$\vec{r}_B = \begin{bmatrix} r_{B,x} \\ r_{B,y} \end{bmatrix} = \begin{bmatrix} 17 \text{ yards} \\ 35 \text{ yards} \end{bmatrix} \quad (1.3.1.1)$$

All three of these representations of a vector are equally valid ways of representing and interacting with a vector. Each has their own strengths and weaknesses that make them useful in certain situations. For example, certain geometry problems are more easily solved with a compass, straight edge and steady hand than with algebraic manipulations.

This is all looking great. But, what if the person getting your teddy bear is not starting at the origin? What do we do then?



Well, of course now we need to construct our second position vector. To tell the person where to go, we also need a third vector. This will be called a **displacement vector**. This is just a vector that points from one point to another. However, unlike our position vectors, a displacement vector can start anywhere. We'll have our start at the person and point to the teddy bear.



As you can see, $\Delta \vec{r}$ is the vector that points between point P and point B. This is where the person, P, would need to walk to get to the bear, B. How do we find this mathematically? Well, first we need to know the coordinates of point P. Let's say $\vec{r}_P = \begin{bmatrix} 68 \text{ yards} \\ 16 \text{ yards} \end{bmatrix}$. Now, to get the displacement vector, $\Delta \vec{r}$, we simply subtract the components of vector r_P from the components of r_B .

$$\Delta \vec{r} = \begin{bmatrix} r_B, x \\ r_B, y \end{bmatrix} - \begin{bmatrix} r_P, x \\ r_P, y \end{bmatrix} \quad (1.3.1.2)$$

$$\Delta \vec{r} = \begin{bmatrix} 17 \text{ yards} \\ 35 \text{ yards} \end{bmatrix} - \begin{bmatrix} 68 \text{ yards} \\ 16 \text{ yards} \end{bmatrix} \quad (1.3.1.3)$$

$$\Delta \vec{r} = \begin{bmatrix} -51 \text{ yards} \\ 19 \text{ yards} \end{bmatrix} \quad (1.3.1.4)$$

So, this is the vector that says where the person should walk to get to the teddy bear from where they are currently standing.

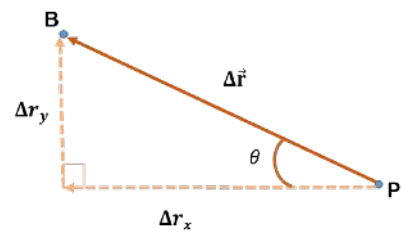
We may want to express this vector in different ways. If we want to know how far the person needs to walk, we just use the Pythagorean theorem. This is just the **magnitude** of the vector. We normally indicate this with an absolute value sign around the vector, $|\Delta \vec{r}|$.

$$|\Delta \vec{r}| = \sqrt{\Delta r_x^2 + \Delta r_y^2} = \sqrt{(-51 \text{ yards})^2 + (19 \text{ yards})^2} = 54.4 \text{ yards} \quad (1.3.1.5)$$

And we can find the angle by using $\sin$, $\cos$, or $\tan$ functions. I will use $\tan$, as it does not rely on me getting the magnitude of $|\Delta \vec{r}|$ correct.

$$\theta = \tan^{-1} \frac{\Delta r_y}{\Delta r_x} = \tan^{-1} \frac{19}{51} = 20.4^\circ \quad (1.3.1.6)$$

So now, we can just tell our helper to walk 54.4 yards at an angle 20.4° above the x-axis.



Vectors in 3 Dimensions

The last thing we should do is extend this into three dimensions. After all, we live in a (spatially) 3D world. The good news is that this is easy! Just add one more component (the z-component) to your column vector! For a 3-dimensional column vector, $\vec{A}$, we will have:

$$\vec{A} = \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} \quad (1.3.1.7)$$

This is pretty straightforward. Luckily, it is rare that we encounter a 3-dimensional problem, so often you will be setting one of these components to zero.

In the next section, we will learn how to do math with vectors in one dimension and the two dimensions.

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1.3.2: Vector Algebra in 1 Dimension

Learning Objectives

- Explain the effect of multiplying a vector quantity by a scalar.
- Describe how one-dimensional vector quantities are added or subtracted.
- Explain the geometric construction for the addition or subtraction of vectors in a plane.
- Distinguish between a vector equation and a scalar equation.

Suppose your friend walks from the campsite at A to the fishing pond at B and then walks back: from the fishing pond at B to the campsite at A. The magnitude of the displacement vector $\vec{D}_{AB}$ from A to B is the same as the magnitude of the displacement vector $\vec{D}_{BA}$ from B to A (it equals 6 km in both cases), so we can write $\vec{D}_{AB} = \vec{D}_{BA}$. However, vector $\vec{D}_{AB}$ is not equal to vector $\vec{D}_{BA}$ because these two vectors have different directions: $\vec{D}_{AB} \neq \vec{D}_{BA}$. In Figure 2.3, vector $\vec{D}_{BA}$ would be represented by a vector with an origin at point B and an end at point A, indicating vector $\vec{D}_{BA}$ points to the southwest, which is exactly 180° opposite to the direction of vector $\vec{D}_{AB}$. We say that vector $\vec{D}_{BA}$ is **antiparallel** to vector $\vec{D}_{AB}$ and write $\vec{D}_{AB} = -\vec{D}_{BA}$, where the minus sign indicates the antiparallel direction.

Two vectors that have identical directions are said to be **parallel vectors**—meaning, they are **parallel** to each other. Two parallel vectors $\vec{A}$ and $\vec{B}$ are equal, denoted by $\vec{A} = \vec{B}$, if and only if they have equal magnitudes $|\vec{A}| = |\vec{B}|$. Two vectors with directions perpendicular to each other are said to be **orthogonal vectors**. These relations between vectors are illustrated in Figure 1.3.2.4.

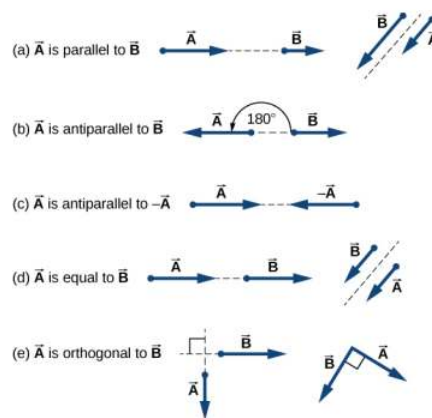


Figure 1.3.2.4: Various relations between two vectors $\vec{A}$ and $\vec{B}$. (a) $\vec{A} \neq \vec{B}$ because $A \neq B$. (b) $\vec{A} \neq \vec{B}$ because they are not parallel and $A \neq B$. (c) $\vec{A} \neq -\vec{A}$ because they have different directions (even though $|\vec{A}| = |-\vec{A}| = A$). (d) $\vec{A} = \vec{B}$ because they are parallel and have identical magnitudes $A = B$. (e) $\vec{A} \neq \vec{B}$ because they have different directions (are not parallel); here, their directions differ by 90° —meaning, they are orthogonal.

? Exercise 2.1

Two motorboats named **Alice** and **Bob** are moving on a lake. Given the information about their velocity vectors in each of the following situations, indicate whether their velocity vectors are equal or otherwise.

- Alice** moves north at 6 knots and **Bob** moves west at 6 knots.
- Alice** moves west at 6 knots and **Bob** moves west at 3 knots.
- Alice** moves northeast at 6 knots and **Bob** moves south at 3 knots.
- Alice** moves northeast at 6 knots and **Bob** moves southwest at 6 knots.
- Alice** moves northeast at 2 knots and **Bob** moves closer to the shore northeast at 2 knots.

Algebra of Vectors in One Dimension

Vectors can be multiplied by scalars, added to other vectors, or subtracted from other vectors. We can illustrate these vector concepts using an example of the fishing trip seen in Figure 1.3.2.5.

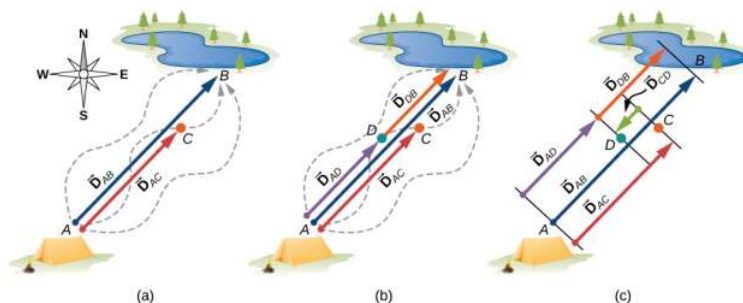


Figure 1.3.2.5: Displacement vectors for a fishing trip. (a) Stopping to rest at point C while walking from camp (point A) to the pond (point B). (b) Going back for the dropped tackle box (point D). (c) Finishing up at the fishing pond.

Suppose your friend departs from point A (the campsite) and walks in the direction to point B (the fishing pond), but, along the way, stops to rest at some point C located three-quarters of the distance between A and B, beginning from point A (Figure 1.3.2.5a). What is his displacement vector $\vec{D}_{AC}$ when he reaches point C? We know that if he walks all the way to B, his displacement vector relative to A is $\vec{D}_{AB}$, which has magnitude $D_{AB} = 6$ km and a direction of northeasterly. If he walks only a 0.75 fraction of the total distance, maintaining the northeasterly direction, at point C he must be $0.75 D_{AB} = 4.5$ km away from the campsite at A. So, his displacement vector at the rest point C has magnitude $D_{AC} = 4.5$ km $= 0.75 D_{AB}$ and is parallel to the displacement vector $\vec{D}_{AB}$. All of this can be stated succinctly in the form of the following **vector equation**:

$$\vec{D}_{AC} = 0.75 \vec{D}_{AB}.$$

In a vector equation, both sides of the equation are vectors. The previous equation is an example of a vector multiplied by a positive scalar (number) $\alpha = 0.75$. The result, $\vec{D}_{AC}$, of such a multiplication is a new vector with a direction parallel to the direction of the original vector $\vec{D}_{AB}$. In general, when a vector $\vec{D}_A$ is multiplied by a positive scalar α , the result is a new vector $\vec{D}_B$ that is parallel to $\vec{D}_A$:

$$\vec{B} = \alpha \vec{A} \quad (1.3.2.1)$$

The magnitude $|\vec{B}|$ of this new vector is obtained by multiplying the magnitude $|\vec{A}|$ of the original vector, as expressed by the **scalar equation**:

$$B = |\alpha| A. \quad (1.3.2.2)$$

In a scalar equation, both sides of the equation are numbers. Equation 1.3.2.2 is a scalar equation because the magnitudes of vectors are scalar quantities (and positive numbers). If the scalar α is **negative** in the vector equation Equation 1.3.2.1, then the magnitude $|\vec{B}|$ of the new vector is still given by Equation 1.3.2.2, but the direction of the new vector $\vec{B}$ is **antiparallel** to the direction of $\vec{A}$. These principles are illustrated in Figure 1.3.2.6a by two examples where the length of vector $\vec{A}$ is 1.5 units. When $\alpha = 2$, the new vector $\vec{B} = 2\vec{A}$ has length $B = 2A = 3.0$ units (twice as long as the original vector) and is parallel to the original vector. When $\alpha = -2$, the new vector $\vec{C} = -2\vec{A}$ has length $C = |-2| A = 3.0$ units (twice as long as the original vector) and is antiparallel to the original vector.

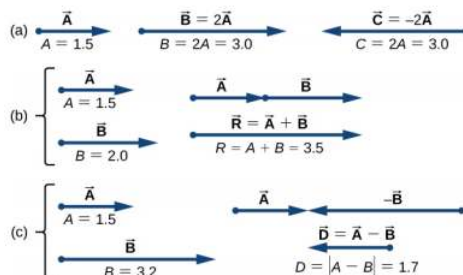


Figure 1.3.2.6: Algebra of vectors in one dimension. (a) Multiplication by a scalar. (b) Addition of two vectors ($\vec{R}$ is called the resultant of vectors $(\vec{A}$ and $(\vec{B})$). (c) Subtraction of two vectors ($\vec{D}$ is the difference of vectors $(\vec{A}$ and $(\vec{B})$).

Now suppose your fishing buddy departs from point A (the campsite), walking in the direction to point B (the fishing hole), but he realizes he lost his tackle box when he stopped to rest at point C (located three-quarters of the distance between A and B, beginning from point A). So, he turns back and retraces his steps in the direction toward the campsite and finds the box lying on the path at some point D only 1.2 km away from point C (see Figure 1.3.2.5b). What is his displacement vector $\vec{D}_{AD}$ when he finds the box at

point D? What is his displacement vector $\vec{D}_{DB}$ from point D to the hole? We have already established that at rest point C his displacement vector is $\vec{D}_{AC} = 0.75 \vec{D}_{AB}$. Starting at point C, he walks southwest (toward the campsite), which means his new displacement vector $\vec{D}_{CD}$ from point C to point D is antiparallel to $\vec{D}_{AB}$. Its magnitude $|\vec{D}_{CD}|$ is $D_{CD} = 1.2 \text{ km} = 0.2 D_{AB}$, so his second displacement vector is $\vec{D}_{CD} = -0.2 \vec{D}_{AB}$. His total displacement $\vec{D}_{AD}$ relative to the campsite is the vector sum of the two displacement vectors: vector $\vec{D}_{AC}$ (from the campsite to the rest point) and vector $\vec{D}_{CD}$ (from the rest point to the point where he finds his box):

$$\vec{D}_{AD} = \vec{D}_{AC} + \vec{D}_{CD}. \quad (1.3.2.3)$$

The vector sum of two (or more vectors) is called the **resultant vector** or, for short, the **resultant**. When the vectors on the right-hand-side of Equation 1.3.2.3 are known, we can find the resultant $\vec{D}_{AD}$ as follows:

$$\vec{D}_{AD} = \vec{D}_{AC} + \vec{D}_{CD} = 0.75 \vec{D}_{AB} - 0.2 \vec{D}_{AB} = (0.75 - 0.2) \vec{D}_{AB} = 0.55 \vec{D}_{AB}. \quad (1.3.2.4)$$

When your friend finally reaches the pond at B, his displacement vector $\vec{D}_{AB}$ from point A is the vector sum of his displacement vector $\vec{D}_{AD}$ from point A to point D and his displacement vector $\vec{D}_{DB}$ from point D to the fishing hole: $\vec{D}_{AB} = \vec{D}_{AD} + \vec{D}_{DB}$ (see Figure 1.3.2.5c). This means his displacement vector $\vec{D}_{DB}$ is the difference of two vectors:

$$\vec{D}_{DB} = \vec{D}_{AB} - \vec{D}_{AD} = \vec{D}_{AB} + (-\vec{D}_{AD}). \quad (1.3.2.5)$$

Notice that a difference of two vectors is nothing more than a vector sum of two vectors because the second term in Equation 1.3.2.5 is vector $-\vec{D}_{AD}$ (which is antiparallel to $\vec{D}_{AD}$). When we substitute Equation 1.3.2.4 into Equation 1.3.2.5, we obtain the second displacement vector:

$$\vec{D}_{DB} = \vec{D}_{AB} - \vec{D}_{AD} = \vec{D}_{AB} - 0.55 \vec{D}_{AB} = (1.0 - 0.55) \vec{D}_{AB} = 0.45 \vec{D}_{AB}. \quad (1.3.2.6)$$

This result means your friend walked $D_{DB} = 0.45 D_{AB} = 0.45(6.0 \text{ km}) = 2.7 \text{ km}$ from the point where he finds his tackle box to the fishing hole.

When vectors $\vec{A}$ and $\vec{B}$ lie along a line (that is, in one dimension), such as in the camping example, their resultant $\vec{R} = \vec{A} + \vec{B}$ and their difference $\vec{D} = \vec{A} - \vec{B}$ both lie along the same direction. We can illustrate the addition or subtraction of vectors by drawing the corresponding vectors to scale in one dimension, as shown in Figure 1.3.2.6.

To illustrate the resultant when $\vec{A}$ and $\vec{B}$ are two parallel vectors, we draw them along one line by placing the origin of one vector at the end of the other vector in head-to-tail fashion (see Figure (\PageIndex{6b})). The magnitude of this resultant is the sum of their magnitudes: $R = A + B$. The direction of the resultant is parallel to both vectors. When vector $\vec{A}$ is antiparallel to vector $\vec{B}$, we draw them along one line in either head-to-head fashion (Figure (\PageIndex{6c})) or tail-to-tail fashion. The magnitude of the vector difference, then, is the **absolute value** $D = |A - B|$ of the difference of their magnitudes. The direction of the difference vector $\vec{D}$ is parallel to the direction of the longer vector.

In general, in one dimension—as well as in higher dimensions, such as in a plane or in space—we can add any number of vectors and we can do so in any order because the addition of vectors is **commutative**,

$$\vec{A} + \vec{B} = \vec{B} + \vec{A}. \quad (1.3.2.7)$$

and **associative**,

$$(\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C}). \quad (1.3.2.8)$$

Moreover, multiplication by a scalar is **distributive**:

$$\alpha_1 \vec{A} + \alpha_2 \vec{A} = (\alpha_1 + \alpha_2) \vec{A}. \quad (1.3.2.9)$$

We used the distributive property in Equation 1.3.2.4 and Equation 1.3.2.6.

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1.3.3: Vector Algebra in 2 Dimensions- Graphical

Algebra of Vectors in Two Dimensions

When vectors lie in a plane—that is, when they are in two dimensions—they can be multiplied by scalars, added to other vectors, or subtracted from other vectors in accordance with the general laws expressed by [Equation 2.2.1](#), [Equation 2.2.2](#), [Equation 2.2.7](#), and [Equation 2.2.8](#). However, the addition rule for two vectors in a plane becomes more complicated than the rule for vector addition in one dimension. We have to use the laws of geometry to construct resultant vectors, followed by trigonometry to find vector magnitudes and directions. This geometric approach is commonly used in navigation (Figure 1.3.3.2). In this section, we need to have at hand two rulers, a triangle, a protractor, a pencil, and an eraser for drawing vectors to scale by geometric constructions.



Figure 1.3.3.2: In navigation, the laws of geometry are used to draw resultant displacements on nautical maps.

For a geometric construction of the sum of two vectors in a plane, we follow the **parallelogram rule**. Suppose two vectors $\vec{A}$ and $\vec{B}$ are at the arbitrary positions shown in Figure 1.3.3.3. Translate either one of them in parallel to the beginning of the other vector, so that after the translation, both vectors have their origins at the same point. Now, at the end of vector $\vec{A}$ we draw a line parallel to vector $\vec{B}$ and at the end of vector $\vec{B}$ we draw a line parallel to vector $\vec{A}$ (the dashed lines in Figure 1.3.3.3). In this way, we obtain a parallelogram. From the origin of the two vectors we draw a diagonal that is the resultant $\vec{R}$ of the two vectors: $\vec{R} = \vec{A} + \vec{B}$ (Figure 1.3.3.3a). The other diagonal of this parallelogram is the vector difference of the two vectors $\vec{D} = \vec{A} - \vec{B}$, as shown in Figure 1.3.3.3b. Notice that the end of the difference vector is placed at the end of vector $\vec{A}$.

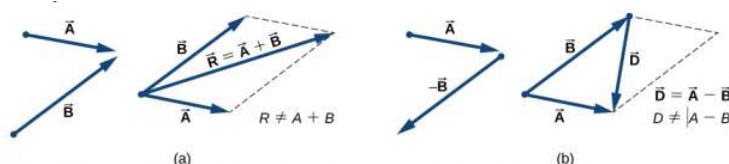


Figure 1.3.3.3: The parallelogram rule for the addition of two vectors. Make the parallel translation of each vector to a point where their origins (marked by the dot) coincide and construct a parallelogram with two sides on the vectors and the other two sides (indicated by dashed lines) parallel to the vectors. (a) Draw the resultant vector $\vec{R}$ along the diagonal of the parallelogram from the common point to the opposite corner. Length R of the resultant vector is not equal to the sum of the magnitudes of the two vectors. (b) Draw the difference vector $\vec{D} = \vec{A} - \vec{B}$ along the diagonal connecting the ends of the vectors. Place the origin of vector $\vec{D}$ at the end of vector $\vec{B}$ and the end (arrowhead) of vector $\vec{D}$ at the end of vector $\vec{A}$. Length D of the difference vector is not equal to the difference of magnitudes of the two vectors.

It follows from the parallelogram rule that neither the magnitude of the resultant vector nor the magnitude of the difference vector can be expressed as a simple sum or difference of magnitudes A and B , because the length of a diagonal cannot be expressed as a simple sum of side lengths. When using a geometric construction to find magnitudes $|\vec{R}|$ and $|\vec{D}|$, we have to use trigonometry laws for triangles, which may lead to complicated algebra. There are two ways to circumvent this algebraic complexity. One way is to use the method of components, which we examine in the next section. The other way is to draw the vectors to scale, as is done in navigation, and read approximate vector lengths and angles (directions) from the graphs. In this section we examine the second approach.

If we need to add three or more vectors, we repeat the parallelogram rule for the pairs of vectors until we find the resultant of all of the resultants. For three vectors, for example, we first find the resultant of vector 1 and vector 2, and then we find the resultant of

this resultant and vector 3. The order in which we select the pairs of vectors does not matter because the operation of vector addition is commutative and associative (see [Equation 2.2.7](#) and [Equation 2.2.8](#)). Before we state a general rule that follows from repetitive applications of the parallelogram rule, let's look at the following example.

Suppose you plan a vacation trip in Florida. Departing from Tallahassee, the state capital, you plan to visit your uncle Joe in Jacksonville, see your cousin Vinny in Daytona Beach, stop for a little fun in Orlando, see a circus performance in Tampa, and visit the University of Florida in Gainesville. Your route may be represented by five displacement vectors $\vec{A}$, $\vec{B}$, $\vec{C}$, $\vec{D}$, and $\vec{E}$, which are indicated by the red vectors in Figure 1.3.3.4. What is your total displacement when you reach Gainesville? The total displacement is the vector sum of all five displacement vectors, which may be found by using the parallelogram rule four times. Alternatively, recall that the displacement vector has its beginning at the initial position (Tallahassee) and its end at the final position (Gainesville), so the total displacement vector can be drawn directly as an arrow connecting Tallahassee with Gainesville (see the green vector in Figure 1.3.3.4). When we use the parallelogram rule four times, the resultant $\vec{R}$ we obtain is exactly this green vector connecting Tallahassee with Gainesville: $\vec{R} = \vec{A} + \vec{B} + \vec{C} + \vec{D} + \vec{E}$.

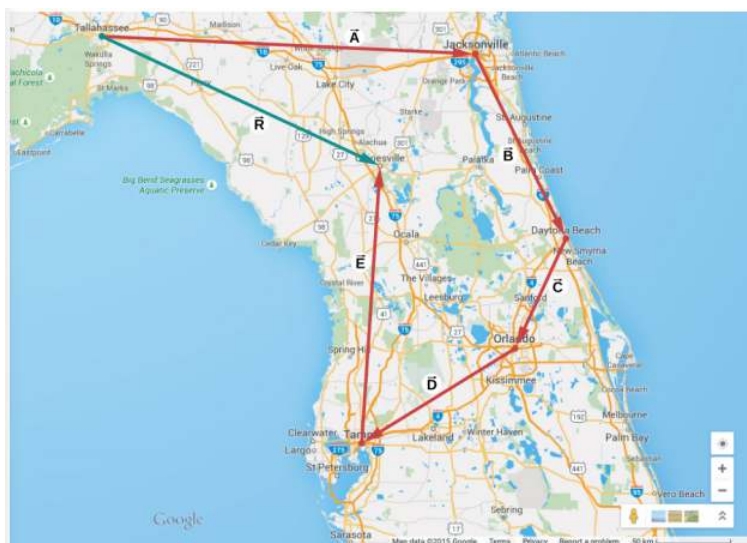


Figure 1.3.3.4: When we use the parallelogram rule four times, we obtain the resultant vector $\vec{R} = \vec{A} + \vec{B} + \vec{C} + \vec{D} + \vec{E}$, which is the green vector connecting Tallahassee with Gainesville.

Drawing the resultant vector of many vectors can be generalized by using the following tail-to-head geometric construction. Suppose we want to draw the resultant vector $\vec{R}$ of four vectors $\vec{A}$, $\vec{B}$, $\vec{C}$, and $\vec{D}$ (Figure 1.3.3.5a). We select any one of the vectors as the first vector and make a parallel translation of a second vector to a position where the origin ("tail") of the second vector coincides with the end ("head") of the first vector. Then, we select a third vector and make a parallel translation of the third vector to a position where the origin of the third vector coincides with the end of the second vector. We repeat this procedure until all the vectors are in a head-to-tail arrangement like the one shown in Figure 1.3.3.5. We draw the resultant vector $\vec{R}$ by connecting the origin ("tail") of the first vector with the end ("head") of the last vector. The end of the resultant vector is at the end of the last vector. Because the addition of vectors is associative and commutative, we obtain the same resultant vector regardless of which vector we choose to be first, second, third, or fourth in this construction.

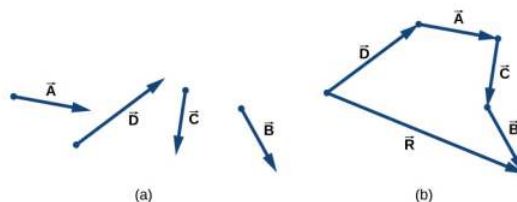


Figure 1.3.3.5: Tail-to-head method for drawing the resultant vector $\vec{R} = \vec{A} + \vec{B} + \vec{C} + \vec{D}$. (a) Four vectors of different magnitudes and directions. (b) Vectors in (a) are translated to new positions where the origin ("tail") of one vector is at the end ("head") of another vector. The resultant vector is drawn from the origin ("tail") of the first vector to the end ("head") of the last vector in this arrangement.

✓ Example 1.3.3.2: Geometric Construction of the Resultant

The three displacement vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$ in Figure 1.3.3.6 are specified by their magnitudes $A = 10.0$, $B = 7.0$, and $C = 8.0$, respectively, and by their respective direction angles with the horizontal direction $\alpha = 35^\circ$, $\beta = -110^\circ$, and $\gamma = 30^\circ$. The physical units of the magnitudes are centimeters. Choose a convenient scale and use a ruler and a protractor to find the following vector sums: (a) $\vec{R} = \vec{A} + \vec{B}$, (b) $\vec{D} = \vec{A} - \vec{B}$, and (c) $\vec{S} = \vec{A} - 3\vec{B} + \vec{C}$.

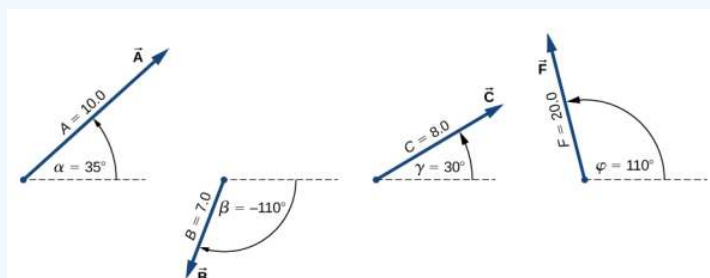


Figure 1.3.3.6: Vectors used in Example 1.3.3.2 and in the Exercise feature that follows.

Strategy

In geometric construction, to find a vector means to find its magnitude and its direction angle with the horizontal direction. The strategy is to draw to scale the vectors that appear on the right-hand side of the equation and construct the resultant vector. Then, use a ruler and a protractor to read the magnitude of the resultant and the direction angle. For parts (a) and (b) we use the parallelogram rule. For (c) we use the tail-to-head method.

Solution

For parts (a) and (b), we attach the origin of vector $\vec{B}$ to the origin of vector $\vec{A}$, as shown in Figure 1.3.3.7, and construct a parallelogram. The shorter diagonal of this parallelogram is the sum $\vec{A} + \vec{B}$. The longer of the diagonals is the difference $\vec{A} - \vec{B}$. We use a ruler to measure the lengths of the diagonals, and a protractor to measure the angles with the horizontal. For the resultant $\vec{R}$, we obtain $R = 5.8$ cm and $\theta_R \approx 0^\circ$. For the difference $\vec{D}$, we obtain $D = 16.2$ cm and $\theta_D = 49.3^\circ$, which are shown in Figure 1.3.3.7.

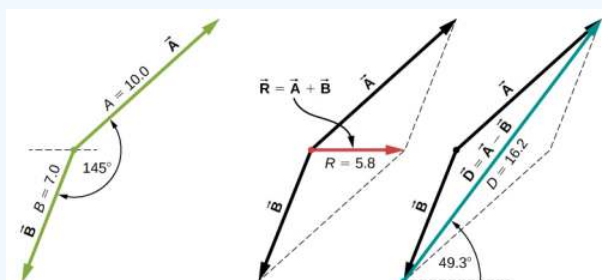


Figure 1.3.3.7: Using the parallelogram rule to solve (a) (finding the resultant, red) and (b) (finding the difference, blue).

For (c), we can start with vector $-3\vec{B}$ and draw the remaining vectors tail-to-head as shown in Figure 1.3.3.8. In vector addition, the order in which we draw the vectors is unimportant, but drawing the vectors to scale is very important. Next, we draw vector $\vec{S}$ from the origin of the first vector to the end of the last vector and place the arrowhead at the end of $\vec{S}$. We use a ruler to measure the length of $\vec{S}$, and find that its magnitude is $S = 36.9$ cm. We use a protractor and find that its direction angle is $\theta_S = 52.9^\circ$. This solution is shown in Figure 1.3.3.8.

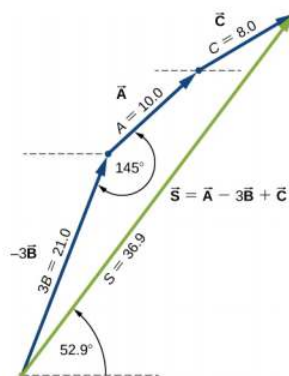


Figure 1.3.3.8: Using the tail-to-head method to solve (c) (finding vector $\vec{S}$, green).

? Exercise 2.3

Using the three displacement vectors $\vec{A}$, $\vec{B}$, and $\vec{F}$ in Figure 1.3.3.6, choose a convenient scale, and use a ruler and a protractor to find vector $\vec{G}$ given by the vector equation $\vec{G} = \vec{A} + 2\vec{B} - \vec{F}$.

📌 Simulation

Observe the addition of vectors in a plane by visiting this [vector calculator](#) and this [PhET simulation](#).

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1.3.4: Vector Algebra in Multiple Dimensions- Calculations

Adding Vectors

Luckily, adding vectors is very easy when we use our column vector notation. We simply add the components together, which is the same as just adding across each row of the matrix. Let's say we have two vectors, $\vec{A}$ and $\vec{B}$, and we want to add them together to create a new vector, $\vec{C} = \vec{A} + \vec{B}$. We can easily find the components of the new vector.

$$\vec{C} = \vec{A} + \vec{B} = \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} + \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} A_x + B_x \\ A_y + B_y \\ A_z + B_z \end{bmatrix} \quad (1.3.4.1)$$

✓ Example 1.3.4.1

Vector $\vec{A}$ has components $A_x = 3$ m, $A_y = 7$ m, $A_z = -4$ m. Vector $\vec{B}$ has components $B_x = 5$ m, $B_y = -2$ m, $B_z = 8$ m. What is $\vec{C} = \vec{A} + \vec{B}$?

Solution

$$\vec{C} = \vec{A} + \vec{B} = \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} + \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} 3 \text{ m} + 5 \text{ m} \\ 7 \text{ m} + (-2) \text{ m} \\ -4 \text{ m} + 8 \text{ m} \end{bmatrix} \quad (1.3.4.2)$$

$$\vec{C} = \begin{bmatrix} 8 \text{ m} \\ 5 \text{ m} \\ 4 \text{ m} \end{bmatrix} \quad (1.3.4.3)$$

For multiplication, we simply multiply each component. So:

$$3\vec{A} = \begin{bmatrix} 3A_x \\ 3A_y \\ 3A_z \end{bmatrix} \quad (1.3.4.4)$$

Subtraction is just the same as adding the negative of the matrix that is being subtracted.

You cannot multiply two vectors together in a normal way. We will cover different types of vector multiplication later in this book.

Finally, you cannot divide a vector by another vector.

Let's look at a little more realistic example.

✓ Example 1.3.4.2 (Adapted from OpenStax)

During a takeoff of IAI Heron, its position with respect to a control tower is 100 m above the ground, 300 m to the east, and 200 m to the north. One minute later, its position is 250 m above the ground, 1200 m to the east, and 2100 m to the north. What is the drone's displacement vector with respect to the control tower? What is the magnitude of its displacement vector?



Figure The drone IAI Heron in flight. (credit: SSgt Reynaldo Ramon, USAF)

Solution

We are given two position vectors. Let's call them $\vec{r}_i$ and $\vec{r}_f$. Let us call east +x, north +y, and "above the ground" +z. Then these will be our two position vectors:

$$\vec{r}_i = \begin{bmatrix} 300 \text{ m} \\ 200 \text{ m} \\ 100 \text{ m} \end{bmatrix} \text{ and } \vec{r}_f = \begin{bmatrix} 1200 \text{ m} \\ 2100 \text{ m} \\ 250 \text{ m} \end{bmatrix} \quad (1.3.4.5)$$

To find the displacement vector of the drone with respect to the control tower, we need to subtract the initial position $\vec{r}_i$ of the drone from the final position, $\vec{r}_f$.

$$\Delta\vec{r} = \vec{r}_f - \vec{r}_i = \begin{bmatrix} 1200 \text{ m} \\ 2100 \text{ m} \\ 250 \text{ m} \end{bmatrix} - \begin{bmatrix} 300 \text{ m} \\ 200 \text{ m} \\ 100 \text{ m} \end{bmatrix} \quad (1.3.4.6)$$

$$\Delta\vec{r} = \begin{bmatrix} 900 \text{ m} \\ 1900 \text{ m} \\ 150 \text{ m} \end{bmatrix} \quad (1.3.4.7)$$

Finally, if we want the magnitude of the displacement, we should use our 3-dimensional Pythagorean theorem:

$$|\Delta\vec{r}| = \sqrt{(900 \text{ m})^2 + (1900 \text{ m})^2 + (150 \text{ m})^2} \quad (1.3.4.8)$$

$$|\Delta\vec{r}| = 2108 \text{ m} \quad (1.3.4.9)$$

,
which sounds reasonable!

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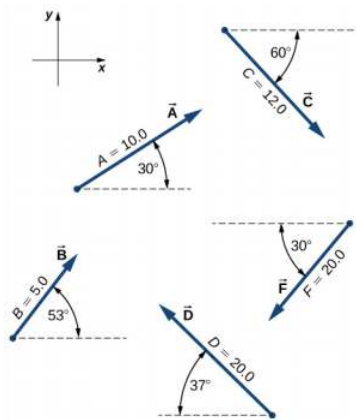
1.3.E: Vectors (Exercises)

Conceptual Questions

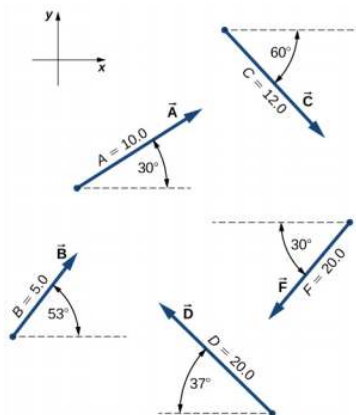
1. A weather forecast states the temperature is predicted to be -5°C the following day. Is this temperature a vector or a scalar quantity? Explain.
2. Which of the following is a vector: a person's height, the altitude on Mt. Everest, the velocity of a fly, the age of Earth, the boiling point of water, the cost of a book, Earth's population, or the acceleration of gravity?
3. Give a specific example of a vector, stating its magnitude, units, and direction.
4. What do vectors and scalars have in common? How do they differ?
5. Suppose you add two vectors $\vec{A}$ and $\vec{B}$. What relative direction between them produces the resultant with the greatest magnitude? What is the maximum magnitude? What relative direction between them produces the resultant with the smallest magnitude? What is the minimum magnitude?
6. Is it possible to add a scalar quantity to a vector quantity?
7. Is it possible for two vectors of different magnitudes to add to zero? Is it possible for three vectors of different magnitudes to add to zero? Explain.
8. Does the odometer in an automobile indicate a scalar or a vector quantity?
9. When a 10,000-m runner competing on a 400-m track crosses the finish line, what is the runner's net displacement? Can this displacement be zero? Explain.
10. A vector has zero magnitude. Is it necessary to specify its direction? Explain.
11. Can a magnitude of a vector be negative?
12. Can the magnitude of a particle's displacement be greater than the distance traveled?
13. If two vectors are equal, what can you say about their components? What can you say about their magnitudes? What can you say about their directions?
14. If three vectors sum up to zero, what geometric condition do they satisfy?
15. Give an example of a nonzero vector that has a component of zero.
16. Explain why a vector cannot have a component greater than its own magnitude.
17. If two vectors are equal, what can you say about their components?
18. If vectors $\vec{A}$ and $\vec{B}$ are orthogonal, what is the component of $\vec{B}$ along the direction of $\vec{A}$? What is the component of $\vec{A}$ along the direction of $\vec{B}$?
19. If one of the two components of a vector is not zero, can the magnitude of the other vector component of this vector be zero?
20. If two vectors have the same magnitude, do their components have to be the same?

Problems

21. A scuba diver makes a slow descent into the depths of the ocean. His vertical position with respect to a boat on the surface changes several times. He makes the first stop 9.0 m from the boat but has a problem with equalizing the pressure, so he ascends 3.0 m and then continues descending for another 12.0 m to the second stop. From there, he ascends 4 m and then descends for 18.0 m, ascends again for 7 m and descends again for 24.0 m, where he makes a stop, waiting for his buddy. Assuming the positive direction up to the surface, express his net vertical displacement vector in terms of the unit vector. What is his distance to the boat?
22. Suppose you walk 18.0 m straight west and then 25.0 m straight north. How far are you from your starting point and what is the compass direction of a line connecting your starting point to your final position? Write your displacement as a column vector and compute its magnitude.
23. For the vectors given in the following figure, find the components of each lettered vector and find the following vectors:
 - a. $\vec{A} + \vec{B}$,
 - b. $\vec{C} + \vec{B}$,
 - c. $\vec{D} + \vec{F}$,
 - d. $\vec{A} - \vec{B}$,
 - e. $\vec{D} - \vec{F}$,
 - f. $\vec{A} + 2\vec{F}$,
 - g. $\vec{A} - 4\vec{D} + 2\vec{F}$.



28. A delivery man starts at the post office, drives 40 km north, then 20 km west, then 60 km northeast, and finally 50 km north to stop for lunch. What is the delivery man's displacement magnitude and direction from the post office?
29. An adventurous dog strays from home, runs three blocks east, two blocks north, one block east, one block north, and two blocks west. Assuming that each block is about 100 m, how far from home and in what direction is the dog?
30. In an attempt to escape a desert island, a castaway builds a raft and sets out to sea. The wind shifts a great deal during the day and he is blown along the following directions: 2.50 km and 45.0° north of west, then 4.70 km and 60.0° south of east, then 1.30 km and 25.0° south of west, then 5.10 km straight east, then 1.70 km and 5.00° east of north, then 7.20 km and 55.0° south of west, and finally 2.80 km and 10.0° north of east. Use a graphical method to find the castaway's final position relative to the island.
31. A small plane flies 40.0 km in a direction 60° north of east and then flies 30.0 km in a direction 15° north of east. Find the total distance the plane covers from the starting point and the direction of the path to the final position.
32. A surveyor measures the distance across a river that flows straight north by the following method. Starting directly across from a tree on the opposite bank, the surveyor walks 100 m along the river to establish a baseline. She then sights across to the tree and reads that the angle from the baseline to the tree is 35° . How wide is the river?
33. The magnitudes of two displacement vectors are $A = 20$ m and $B = 6$ m. What are the largest and the smallest values of the magnitude of the resultant $\vec{R} = \vec{A} + \vec{B}$?
34. Assuming the +x-axis is horizontal and points to the right, resolve the vectors given in the following figure to their scalar components and express them in vector component form.



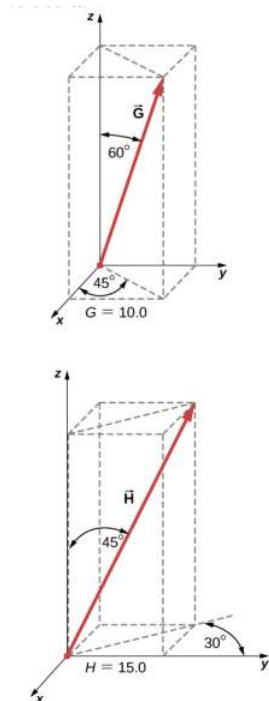
35. Suppose you walk 18.0 m straight west and then 25.0 m straight north. How far are you from your starting point? What is your displacement vector? What is the direction of your displacement? Assume the +x-axis is to the east.
36. You drive 7.50 km in a straight line in a direction 15° east of north. (a) Find the distances you would have to drive straight east and then straight north to arrive at the same point. (b) Show that you still arrive at the same point if the east and north legs are reversed in order. Assume the +x-axis is to the east.

37. A sledge is being pulled by two horses on a flat terrain. The net force on the sledge can be expressed in the Cartesian coordinate system as vector $\vec{F} = \begin{bmatrix} -2980.0 \\ 8200.0 \\ 0 \end{bmatrix}$ N. Find the magnitude and direction of the pull.
38. A fly enters through an open window and zooms around the room. In a Cartesian coordinate system with three axes along three edges of the room, the fly changes its position from point b(4.0 m, 1.5 m, 2.5 m) to point e(1.0 m, 4.5 m, 0.5 m). Find the scalar components of the fly's displacement vector and express its displacement vector in vector component form. What is its magnitude?
39. For vectors $\vec{B} = \begin{bmatrix} 1 \\ -4 \\ 0 \end{bmatrix}$ and $\vec{A} = \begin{bmatrix} -3 \\ -2 \\ 0 \end{bmatrix}$ (a) $\vec{A} + \vec{B}$ and its magnitude and direction angle, and (b) $\vec{A} - \vec{B}$ and its magnitude and direction angle.
40. A particle undergoes three consecutive displacements given by vectors $\vec{D}_1 = \begin{bmatrix} 3.0 \text{ mm} \\ -4.0 \text{ mm} \\ -2.0 \text{ mm} \end{bmatrix}$, $\vec{D}_2 = \begin{bmatrix} 4.0 \text{ mm} \\ -7.0 \text{ mm} \\ 4.0 \text{ mm} \end{bmatrix}$, and $\vec{D}_3 = \begin{bmatrix} -7.0 \text{ mm} \\ 4.0 \text{ mm} \\ 1.0 \text{ mm} \end{bmatrix}$. (a) Find the resultant displacement vector of the particle. (b) What is the magnitude of the resultant displacement? (c) If all displacements were along one line, how far would the particle travel?
41. Given two displacement vectors $\vec{A} = \begin{bmatrix} 3.0 \text{ m} \\ -4.0 \text{ m} \\ 4.0 \text{ m} \end{bmatrix}$ and $\vec{B} = \begin{bmatrix} 2.0 \text{ m} \\ 3.0 \text{ m} \\ -7.0 \text{ m} \end{bmatrix}$, find the displacements and their magnitudes for (a) $\vec{C} = \vec{A} + \vec{B}$ and (b) $\vec{D} = 2\vec{A} - \vec{B}$.
42. A small plane flies 40.0 km in a direction 60° north of east and then flies 30.0 km in a direction 15° north of east. Use the analytical method to find the total distance the plane covers from the starting point, and the geographic direction of its displacement vector. What is its displacement vector?
43. In an attempt to escape a desert island, a castaway builds a raft and sets out to sea. The wind shifts a great deal during the day, and she is blown along the following straight lines: 2.50 km and 45.0° north of west, then 4.70 km and 60.0° south of east, then 1.30 km and 25.0° south of west, then 5.10 km due east, then 1.70 km and 5.00° east of north, then 7.20 km and 55.0° south of west, and finally 2.80 km and 10.0° north of east. Use the analytical method to find the resultant vector of all her displacement vectors. What is its magnitude and direction?
44. A barge is pulled by the two tugboats shown in the following figure. One tugboat pulls on the barge with a force of magnitude 4000 units of force at 15° above the line AB (see the figure and the other tugboat pulls on the barge with a force of magnitude 5000 units of force at 12° below the line AB. Resolve the pulling forces to their scalar components and find the components of the resultant force pulling on the barge. What is the magnitude of the resultant pull? What is its direction relative to the line AB?
45. In the control tower at a regional airport, an air traffic controller monitors two aircraft as their positions change with respect to the control tower. One plane is a cargo carrier Boeing 747 and the other plane is a Douglas DC-3. The Boeing is at an altitude of 2500 m, climbing at 10° above the horizontal, and moving 30° north of west. The DC-3 is at an altitude of 3000 m, climbing at 5° above the horizontal, and cruising directly west. (a) Find the position vectors of the planes relative to the control tower. (b) What is the distance between the planes at the moment the air traffic controller makes a note about their positions?

Additional Problems

46. You fly 32.0 km in a straight line in still air in the direction 35.0° south of west. (a) Find the distances you would have to fly due south and then due west to arrive at the same point. (b) Find the distances you would have to fly first in a direction 45.0° south of west and then in a direction 45.0° west of north. Note these are the components of the displacement along a different set of axes—namely, the one rotated by 45° with respect to the axes in (a).
47. An air traffic controller notices two signals from two planes on the radar monitor. One plane is at altitude 800 m and in a 19.2-km horizontal distance to the tower in a direction 25° south of west. The second plane is at altitude 1100 m and its horizontal distance is 17.6 km and 20° south of west. What is the distance between these planes?
48. Show that when $\vec{A} + \vec{B} = \vec{C}$, then $C^2 = A^2 + B^2 + 2AB \cos \varphi$, where φ is the angle between vectors $\vec{A}$ and $\vec{B}$.
49. Four force vectors each have the same magnitude f . What is the largest magnitude the resultant force vector may have when these forces are added? What is the smallest magnitude of the resultant? Make a graph of both situations.

50. A stubborn dog is being walked on a leash by its owner. At one point, the dog encounters an interesting scent at some spot on the ground and wants to explore it in detail, but the owner gets impatient and pulls on the leash with force $\vec{F}$ $= \begin{bmatrix} 98.0 \text{ N} \\ 132.0 \text{ N} \\ 32.0 \text{ N} \end{bmatrix}$ along the leash. (a) What is the magnitude of the pulling force? (b) What angle does the leash make with the vertical?
51. If the velocity vector of a polar bear is $\vec{u} = \begin{bmatrix} -18.0 \text{ km/h} \\ -13.0 \text{ km/h} \\ 0.0 \text{ km/h} \end{bmatrix}$, how fast and in what geographic direction is it heading? Here, the positive x and y axes are directions to geographic east and north, respectively.
52. Find the scalar components of three-dimensional vectors $\vec{G}$ and $\vec{H}$ in the following figure and write the vectors in column vector form.



55. A diver explores a shallow reef off the coast of Belize. She initially swims 90.0 m north, makes a turn to the east and continues for 200.0 m, then follows a big grouper for 80.0 m in the direction 30° north of east. In the meantime, a local current displaces her by 150.0 m south. Assuming the current is no longer present, in what direction and how far should she now swim to come back to the point where she started?
56. A force vector $\vec{A}$ has x - and y -components, respectively, of -8.80 units of force and 15.00 units of force. The x - and y -components of force vector $\vec{B}$ are, respectively, 13.20 units of force and -6.60 units of force. Find the components of force vector $\vec{C}$ that satisfies the vector equation $\vec{A} - \vec{B} + 3\vec{C} = 0$.

Challenge Problems

57. Vector $\vec{B}$ is 5.0 cm long and vector $\vec{A}$ is 4.0 cm long. Find the angle between these two vectors when $|\vec{A} + \vec{B}| = 3.0$ cm and $|\vec{A} - \vec{B}| = 3.0$ cm.

Contributors

Samuel J. Ling (Truman State University), Jeff Sanny (Loyola Marymount University), and Bill Moebs with many contributing authors. This work is licensed by OpenStax University Physics under a [Creative Commons Attribution License \(by 4.0\)](https://creativecommons.org/licenses/by/4.0/).

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SECTION OVERVIEW

1.4: Systems and The Center of Mass

1.4.1: The Law of Inertia

1.4.2: Extended Systems and Center of Mass

1.4.3: Reference Frame Changes and Relative Motion

1.4.4: Examples

1.4.E: Systems and the Center of Mass Exercises

There are two basic concepts we are going to cover in this chapter: what a *system* is, and *the center of mass*.

A system is a very generic word ("the system of the Earth and the Moon", "a fridge is a closed system", "my computer system") that we are going to make precise so that we can use it to describe our abstraction of the real world. Simply put, a **system is a collection of objects**. When we turn our objects into points, we are going to describe specific collections of those points as the systems. For a given problem, there may be multiple possible systems. For a simple example, consider the collision of two blocks sliding along the ground. We replace the blocks with points, and say "the system is the two blocks." Well that's a fine system if we're going to ignore friction, but if the blocks are experiencing friction between themselves and the ground, we had better include the ground in our system as well. So maybe system A is the blocks, but system B is the blocks *and* the ground, and if we want to include the effects of friction, we have to use system B. A simple idea, but really important for us as we start to think about how we model interactions, and which interactions are present in any given physical problem.

The center of mass, in comparison, is a definite mathematical concept. It's the answer to "when we replace our objects with points, what point do we actually use?" We use the center of mass because it's the balance point of the system, and therefore satisfies some important properties that random other points in our system don't satisfy. For example, consider the motion of a runner. If we replaced the runner with a point, and used the tip of their fingers as the point, we would have a very hard time confirming any of calculations, because a real runners fingers are flying back and forth as they run! However, if we described their motion using their center of mass (a point inside their chest, at the center of their body), their motion would essentially be just a straight line - much easier to understand and verify.

The specific mathematical definition of the center of mass is

$$\vec{r}_{CM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \dots}{m_1 + m_2 + \dots} = \frac{\sum m_i\vec{r}_i}{\sum m_i} \quad (1.4.1)$$

In this formula, each m_i is the mass of an object, and each $\vec{r}_i$ is the position of the mass. If we were going to use this definition to find the center of mass of the runner, we would break up their body into a bunch of little masses and find all their vector positions - it would take a long time¹! In this course, we will generally do things more like "find the center of mass between two objects of mass m_1 and m_2 , separated by a distance d ". So what are $\vec{r}_1$ and $\vec{r}_2$ in this situation? They are position vectors, so you first have to decide where the origin of your system is - where you are measuring the position relative to. It's often easy to pick one of the masses as the origin, say the first one, so that $r_1 = 0$, $r_2 = d$, and the center of mass is

$$r_{CM} = \frac{m_1 \cdot 0 + m_2 \cdot d}{m_1 + m_2}. \quad (1.4.2)$$

Notice that I got rid of the vector signs in this example - we will generally be more careful and keep track of both x- and y-directions separately, but here it's easy to see that the center of mass will be in a line between the two masses. If our numbers were $m_1 = 1$ kg, $m_2 = 10$ kg, and $d = 1$ m, the answer is $r_{CM} \approx 0.91$ m. This makes sense as "the balance point", because it's much closer to the larger mass.

¹Of course, if we really wanted to do that calculation, we would probably use an integral!

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1.4.1: The Law of Inertia

There is something funny about motion with constant velocity: it is *indistinguishable from rest*. Of course, you can usually tell whether you are moving *relative to something else*. But if you are enjoying a smooth airplane ride, without looking out the window, you have no idea how fast you are moving, or even, indeed (if the flight is exceptionally smooth) whether you are moving at all.

If I were to drop something from such an airplane, I know from experience that it would fall on a straight line—relative to me, that is. If it falls from my hand it will land at my feet, just as if we were all at rest. But we are *not* at rest. In the half second or so it takes for the object to fall, the airplane has moved forward 111 meters relative to the ground. Yet the (hypothetical) object I drop does not land 300 feet behind me! It moves forward with me as it falls, even though I am not touching it. *It keeps its initial forward velocity*, even though it is no longer in contact with me or anything connected to the airplane.

This remarkable observation is one of the most fundamental principles of physics, which we call the **law of inertia**. It can be stated as follows: in the absence of any external influence (or *force*) acting on it, an object at rest will stay at rest, while an object that is already moving with some velocity will keep that same velocity (speed and direction of motion)—at least until it is, in fact, acted upon by some force.

Please let that sink in for a moment, before we start backtracking, which we have to do now on several accounts. First, we used repeatedly the term “force,” but we have not defined it properly. What if we just said that forces are precisely any “external influences” that may cause a change in the velocity of an object? That will work until it is time to explore the concept in more detail, a few chapters from now.

Next, I need to draw your attention to the fact that the object I (hypothetically) dropped did not actually keep its *total* initial velocity: it only kept its initial *forward* velocity. In the downward direction, it was speeding up from the moment it left my hand, as would any other falling object (and as we shall see later in this chapter). But this actually makes sense in a certain way: there was no forward force, so the forward velocity remained constant; there was, however, a vertical force acting all along (the force of gravity), and so the object did speed up in that direction. This observation is, in fact, telling us something profound about the world’s geometry: namely, that forces and velocities are *vectors*, and laws such as the law of inertia will typically apply to the vector as a whole, as well as to each component separately (that is to say, each dimension of space). This anticipates, in fact, the way we will deal, later on, with motion in two or more dimensions; but we do not need to worry about that for a few chapters still.

Finally, it is worth spending a moment reflecting on how radically the law of inertia seems to contradict our intuition about the way the world works. What it seems to be telling us is that, if we throw or push an object, it should continue to move forever with the same speed and in the same direction with which it set out—something that we know is certainly not true. But what’s happening in “real life” is that, just because we have left something alone, it doesn’t mean the world has left it alone. After we lose contact with the object, all sorts of other forces will continue to act on it. A ball we throw, for instance, will experience air resistance or drag, and that will slow it down. An object sliding on a surface will experience friction, and that will slow it down too. Perhaps the closest thing to the law of inertia in action that you may get to see is a hockey puck sliding on the ice: it is remarkable (perhaps even a bit frightening) to see how little it slows down, but even so the ice does exert a (very small) frictional force that would bring the puck to a stop eventually.

This is why, historically, the law of inertia was not discovered until people started developing an appreciation for frictional forces, and the way they are constantly acting all around us to oppose the relative motion of any objects trying to slide past each other.

This mention of relative motion, in a way, brings us full circle. Yes, relative motion is certainly detectable, and for objects in contact it actually results in the occurrence of forces of the frictional, or drag, variety. But *absolute* (that is, without reference to anything external) motion with constant velocity is fundamentally undetectable. And in view of the law of inertia, it makes sense: if no force is required to keep me moving with constant velocity, it follows that as long as I am moving with constant velocity I should not be feeling any net force acting on me; nor would any other detection apparatus I might be carrying with me.

This is the next very interesting fact about the physical world that we are about to discover: forces cause accelerations, or changes in velocity, but they do so in different degrees for different objects; and, moreover, the ultimate change in velocity *takes time*. The first part of this statement has to do with the concept of *inertial mass*, to be introduced in the next chapter; the second part we are going to explore right now, after a brief detour to define *inertial reference frames*.

Inertial Reference Frames

The example we just gave you of what happens when a plane in flight experiences turbulence points to an important phenomenon, namely, that there may be times where the law of inertia may not *seem* to apply in a certain reference frame. By this we mean that an object left at rest, like the water in a cup, may suddenly start to move—relative to the reference frame coordinates—even though nothing and nobody is acting on it. More dramatically still, if a car comes to a sudden stop, the passengers may be “projected forward”—they were initially at rest relative to the car frame, but now they find themselves moving forward (always in the car reference frame), to the point that, if they are not wearing seat belts, they may end up hitting the dashboard, or the seat in front of them.

Again, nobody has pushed on them, and in fact what we can see in this case, from outside the car, is nothing but the law of inertia at work: the passengers were just keeping their initial velocity, when the car suddenly slowed down under and around them. So there is nothing wrong with the law of inertia, but *there is a problem with the reference frame*: if one wants to describe the motion of objects in a reference frame like a plane being shaken up or a car that is speeding up or slowing down, we need to allow for the fact that objects may move—always relative to that frame—in an *apparent* violation of the law of inertia.

The way we deal with this in physics is by introducing the very important concept of an *inertial reference frame*, by which we mean a reference frame in which all objects will, at all times, be observed to move (or not move) in a way fully consistent with the law of inertia. In other words, the law of inertia has to hold *when we use that frame's own coordinates to calculate the objects' velocities*. This, of course, is what we always do instinctively: when I am on a plane I locate the various objects around me relative to the plane frame itself, not relative to the distant ground.

To ascertain whether a frame is inertial or not, we start by checking to see if the description of motion using that frame's coordinates obeys the law of inertia: does an object left at rest on the counter in the laboratory stay at rest? If set in motion, does it move with constant velocity on a straight line? The Earth's surface, as it turns out, is *not* quite a perfect inertial reference frame, but it is good enough that it made it possible for us to discover the law of inertia in the first place!

What spoils the inertial-ness of an Earth-bound reference frame is the Earth's rotation, which, as we shall see later, is an example of *accelerated motion*. In fact, if you think about the grossly non-inertial frames I have introduced above—the bouncy plane, the braking car—they all have this in common: that their velocities are changing; they are *not* moving with constant speed on a straight line.

So, once you have found an inertial reference frame, to decide whether another one is inertial or not is simple: if it is moving with constant velocity (relative to the first, inertial frame), then it is itself inertial; if not, it is not. We will show you how this works, formally, in a little bit ([Chapter 15](#)), after we get around to properly introducing the concept of acceleration.

It is a fundamental principle of physics that *the laws of physics take the same form in all inertial reference frames*. The law of inertia is, of course, an example of such a law. Since all inertial frames are moving with constant velocity relative to each other, this is another way to say that absolute motion is undetectable, and all motion is ultimately relative. Accordingly, this principle is known as the **principle of relativity**.

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1.4.2: Extended Systems and Center of Mass

Consider a collection of particles with masses $m_1, m_2, \dots$, and located, at some given instant, at positions $x_1, x_2, \dots$. (We are still, for the time being, considering only motion in one dimension, but all these results generalize easily to three dimensions.) The **center of mass** of such a system is a mathematical point whose position coordinate is given by

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2 + \dots}{m_1 + m_2 + \dots}. \quad (1.4.2.1)$$

Clearly, the denominator of (1.4.2.1) is just the total mass of the system, which we may just denote by M . If all the particles have the same mass, the center of mass will be somehow “in the middle” of all of them; otherwise, it will tend to be closer to the more massive particle(s). The “particles” in question could be spread apart, or they could literally be the “parts” into which we choose to subdivide, for computational purposes, a single extended object.

If the particles are in motion, the position of the center of mass, x_{cm} , will in general change with time. For a solid object, where all the parts are moving together, the displacement of the center of mass will just be the same as the displacement of any part of the object. In the most general case, we will have (by subtracting x_{cmi} from x_{cmf})

$$\Delta x_{cm} = \frac{1}{M} (m_1 \Delta x_1 + m_2 \Delta x_2 + \dots). \quad (1.4.2.2)$$

Dividing Equation (1.4.2.2) by Δt and taking the limit as $\Delta t \rightarrow 0$, we get the instantaneous velocity of the center of mass:

$$v_{x,cm} = \frac{1}{M} (m_1 v_{x,1} + m_2 v_{x,2} + \dots). \quad (1.4.2.3)$$

But this is just

$$v_{x,cm} = \frac{p_{x,sys}}{M}. \quad (1.4.2.4)$$

where $p_{x,sys} = m_1 v_{x,1} + m_2 v_{x,2} + \dots$ is the total momentum of the system along the x-axis.

Center of Mass in 3 dimensions

So far, we have only been dealing with one dimension, and in particular the x-dimension. However, there is nothing special about the x direction. We could simply call the direction we were calling “x” the “y” or “z” direction and show that everything above holds for those dimensions as well. We can combine all of these results into one great equation with our column vector notation:

$$\vec{r}_{cm} = \frac{1}{M} (m_1 \vec{r}_1 + m_2 \vec{r}_2 + \dots) \quad (1.4.2.5)$$

or in column vector notation:

$$\begin{bmatrix} x_{cm} \\ y_{cm} \\ z_{cm} \end{bmatrix} = \frac{1}{M} \left(m_1 \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + m_2 \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} + \dots \right) \quad (1.4.2.6)$$

Similarly, our velocity and momentum can be generalized to three dimensions simply by getting rid of the “x” subscript and putting a vector symbol over the variables:

$$\vec{v}_{cm} = \frac{1}{M} (m_1 \vec{v}_1 + m_2 \vec{v}_2 + \dots) \quad (1.4.2.7)$$

$$\vec{p}_{sys} = M \vec{v}_{cm} \quad (1.4.2.8)$$

Center of Mass Motion for an Isolated System

Equation (1.4.2.4) is a very interesting result. Since the total momentum of an isolated system is constant, it tells us that the center of mass of an isolated system of particles moves at constant velocity, regardless of the relative motion of the particles themselves or their possible interactions with each other. As indicated above, this generalizes straightforwardly to more than one dimension, so we can write $\vec{v}_{cm} = \vec{p}_{sys}/M$. Thus, we can say that, for an isolated system in space, not only the speed, but also the direction of motion of its center of mass does not change with time.

Clearly this result is a sort of generalization of the law of inertia. For a solid, extended object, it does, in fact, provide us with the precise form that the law of inertia must take: in the absence of external forces, *the center of mass* will just move on a straight line with constant velocity, whereas the object itself may be moving in any way that does not affect the center of mass trajectory. Specifically, the most general motion of an isolated rigid body is a straight line motion of its center of mass at constant speed, combined with a possible rotation of the object as a whole around the center of mass.

For a system that consists of separate parts, on the other hand, the center of mass is generally just a point in space, which may or may not coincide at any time with the position of any of the parts, but which will nonetheless move at constant velocity as long as the system is isolated. This is illustrated by Figure 1.4.2.1, where the position vs. time curves have been drawn for the colliding objects of Figure 2.1.1. I have assumed that object 1 starts out at $x_{1i} = -5$ mm at $t = 0$, and object 2 starts at $x_{2i} = 0$ at $t = 0$. Because object 2 has twice the inertia of object 1, the position of the center of mass, as given by Equation (1.4.2.1), will always be

$$x_{cm} = x_1/3 + 2x_2/3$$

that is to say, the center of mass will always be in between objects 1 and 2, and its distance from object 2 will always be half its distance to object 1:

$$\begin{aligned} |x_{cm} - x_1| &= \frac{2}{3} |x_1 - x_2| \\ |x_{cm} - x_2| &= \frac{1}{3} |x_1 - x_2| \end{aligned} \quad (1.4.2.9)$$

Figure 1.4.2.1 shows that this simple prescription does result in motion with constant velocity for the center of mass (the green straight line), even though the x -vs- t curves of the two objects themselves look relatively complicated. (Please check it out! Take a ruler to Figure 1.4.2.1 and verify that the center of mass is, at every instant, where it is supposed to be.)

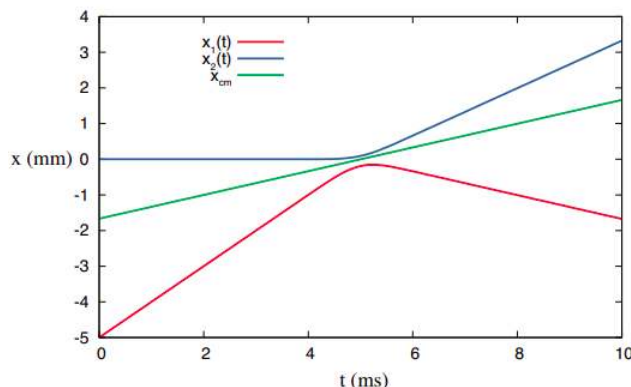


Figure 1.4.2.1. The green line shows the position of the center of mass as a function of time.

The concept of center of mass gives us an important way to simplify the description of the motion of potentially complicated systems. We will make good use of it in forthcoming chapters.

A very nice demonstration of the motion of the center of mass in two-body one-dimensional collisions can be found at https://phet.colorado.edu/sims/collision-lab/collision-lab_en.html (you need to check the “center of mass” box to see it).

Recoil and Rocket Propulsion

As we have just seen, you cannot alter the motion of your center of mass without relying on some external force—which is to say, some kind of external support. This is actually something you may have experienced when you are resting on a very slippery surface and you just cannot “get a grip” on it. There is, however, one way to circumvent this problem which, in fact, relies on conservation of momentum itself: if you are carrying something with you, and can throw it away from you at high speed, you will recoil as a result of that. If you can keep throwing things, you (with your store of as yet unthrown things) will speed up a little more every time. This is, in essence, the principle behind rocket propulsion.

Mathematically, consider two objects, of masses m_1 and m_2 , initially at rest, so their total momentum is zero. If mass 1 is thrown away from mass 2 with a speed v_{1f} , then, by conservation of momentum (always assuming the system is isolated) we must have

$$0 = m_1 v_{1f} + m_2 v_{2f} \quad (1.4.2.10)$$

and therefore $v_{2f} = -m_1 v_{1f} / m_2$. This is how a rocket moves forward, by constantly expelling mass (the hot exhaust gas) backwards at a high velocity. Note that, even though both objects move, the center of mass of the whole system does *not* (in the absence of any external force), as discussed above.

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1.4.3: Reference Frame Changes and Relative Motion

Everything up to this point assumes that we are using a fixed, previously agreed upon **reference frame**. Basically, this is just an origin and a set of axes along which to measure our coordinates.

There are, however, a number of situations in physics that call for the use of different reference frames, and, more importantly, that require us to *convert* various physical quantities from one reference frame to another. For instance, imagine you are on a boat on a river, rowing downstream. You are moving with a certain velocity relative to the water around you, but the water itself is flowing with a different velocity relative to the shore, and your actual velocity relative to the shore is the sum of those two quantities. Ships generally have to do this kind of calculation all the time, as do airplanes: the “airspeed” is the speed of a plane relative to the air around it, but that air is usually moving at a substantial speed relative to the earth.

The way we deal with all these situations is by introducing two reference frames, which here I am going to call A and B. One of them, say A, is “at rest” relative to the earth, and the other one is “at rest” relative to something else—which means, really, moving along with that something else. (For instance, a reference frame at rest “relative to the river” would be a frame that’s moving along with the river water, like a piece of driftwood that you could measure your progress relative to.)

In any case, graphically, this will look as in Figure 1.4.3.1, which we have drawn for the two-dimensional case:

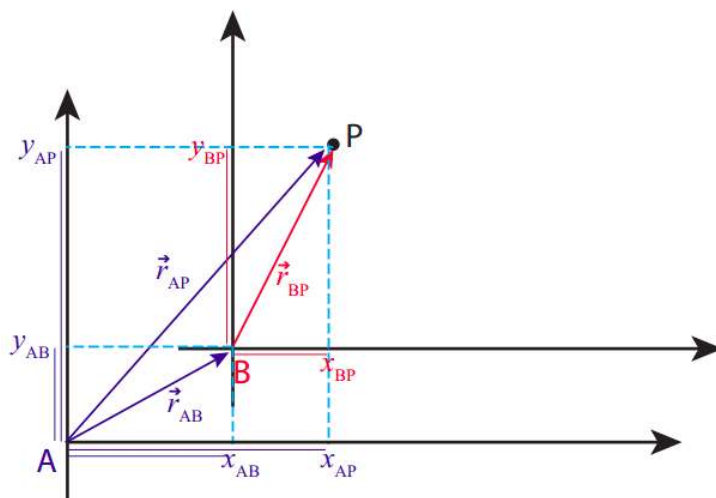


Figure 1.4.3.1: Position vectors and coordinates of a point P in two different reference frames, A and B.

In the reference frame A, the point P has position coordinates (x_{AP}, y_{AP}) . Likewise, in the reference frame B, its coordinates are (x_{BP}, y_{BP}) . As you can see, the notation chosen is such that every coordinate in A will have an “A” as a first subscript, while the second subscript indicates the object to which it refers, and similarly for coordinates in B.

The coordinates (x_{AB}, y_{AB}) are special: they are the coordinates, in the reference frame A, of the origin of reference frame B. This is enough to fully locate the frame B in A, as long as the frames are not rotated relative to each other.

The thin colored lines I have drawn along the axes in Figure 1.4.3.1 are intended to make it clear that the following equations hold:

$$\begin{aligned} x_{AP} &= x_{AB} + x_{BP} \\ y_{AP} &= y_{AB} + y_{BP} \end{aligned} \quad (1.4.3.1)$$

Although the figure is drawn for the easy case where all these quantities are positive, you should be able to convince yourself that Eqs. (1.4.3.1) hold also when one or more of the coordinates have negative values.

All these coordinates are also the components of the respective position vectors, shown in the figure and color-coded by reference frame (so, for instance, $\vec{r}_{AP}$ is the position vector of P in the frame A), so the equations (1.4.3.1) can be written more compactly as the single vector equation

$$\vec{r}_{AP} = \vec{r}_{AB} + \vec{r}_{BP}. \quad (1.4.3.2)$$

From all this you can see how to add vectors: algebraically, you just add their components separately, as in Eqs. (1.4.3.1); graphically, you draw them so the tip of one vector coincides with the tail of the other (we call this “tip-to-tail”), and then draw the sum vector from the tail of the first one to the tip of the other one.

Of course, I showed you already how to *subtract* vectors with Figure 1.2.3: again, algebraically, you just subtract the corresponding coordinates, whereas graphically you draw them with a common origin, and then draw the vector from the tip of the vector you are subtracting to the tip of the other one. If you read the previous paragraph again, you can see that Figure 1.2.3 can equally well be used to show that $\Delta \vec{r} = \vec{r}_f - \vec{r}_i$, as to show that $\vec{r}_f = \vec{r}_i + \Delta \vec{r}$.

In a similar way, you can see graphically from Figure 1.4.3.1 (or algebraically from Equation (1.4.3.2)) that the position vector of P in the frame B is given by $\vec{r}_{BP} = \vec{r}_{AP} - \vec{r}_{AB}$. The last term in this expression can be written in a different way, as follows. If I follow the convention I have introduced above, the quantity x_{BA} (with the order of the subscripts reversed) would be the x coordinate of the origin of frame A in frame B, and algebraically that would be equal to $-x_{AB}$, and similarly $y_{BA} = -y_{AB}$. Hence the vector equality $\vec{r}_{AB} = -\vec{r}_{BA}$ holds. Then,

$$\vec{r}_{BP} = \vec{r}_{AP} - \vec{r}_{AB} = \vec{r}_{AP} + \vec{r}_{BA}. \quad (1.4.3.3)$$

This is, in a way, the “inverse” of Equation (1.4.3.2): it tells us how to get the position of P in the frame B if we know its position in the frame A.

Let's show next how all this extends to displacements and velocities. Suppose the point P indicates the position of a particle at the time t . Over a time interval Δt , both the position of the particle and the relative position of the two reference frames may change. We can add yet another subscript, i or f , (for initial and final) to the coordinates, and write, for example,

$$\begin{aligned} x_{AP,i} &= x_{AB,i} + x_{BP,i} \\ x_{AP,f} &= x_{AB,f} + x_{BP,f} \end{aligned} \quad (1.4.3.4)$$

Subtracting these equations gives us the corresponding displacements:

$$\Delta x_{AP} = \Delta x_{AB} + \Delta x_{BP}. \quad (1.4.3.5)$$

Dividing Equation (1.4.3.5) by Δt we get the average velocities¹, and then taking the limit $\Delta t \rightarrow 0$ we get the instantaneous velocities. This applies in the same way to the y coordinates, and the result is the vector equation

$$\vec{v}_{AP} = \vec{v}_{BP} + \vec{v}_{AB}. \quad (1.4.3.6)$$

We have rearranged the terms on the right-hand side to (hopefully) make it easier to visualize what's going on. In words: the velocity of the particle P relative to (or *measured in*) frame A is equal to the (vector) sum of the velocity of the particle as measured in frame B, plus the velocity of frame B relative to frame A.

The result (1.4.3.6) is just what we would have expected from the examples mentioned at the beginning of this section, like rowing in a river or an airplane flying in the wind. For instance, for the airplane $\vec{v}_{BP}$ could be its “airspeed” (only it has to be a vector, so it would be more like its “airvelocity”: that is, its velocity relative to the air around it), and $\vec{v}_{AB}$ would be the velocity of the air relative to the earth (the wind velocity, at the location of the airplane). In other words, A represents the earth frame of reference and B the air, or wind, frame of reference. Then, $\vec{v}_{AP}$ would be the “true” velocity of the airplane relative to the earth. You can see how it would be important to add these quantities as vectors, in general, by considering what happens when you fly in a cross wind, or try to row across a river, as in Figure 1.4.3.2 below.

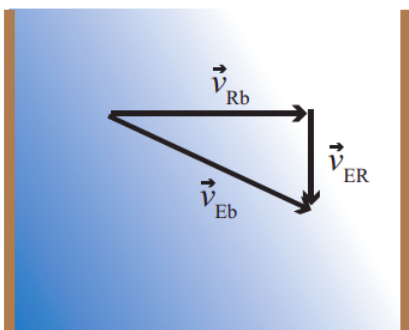


Figure 1.4.3.2: Rowing across a river. If you head “straight across” the river (with velocity vector $\vec{v}_{Rb}$ in the moving frame of the river, which is flowing with velocity $\vec{v}_{ER}$ in the frame of the earth), your actual velocity relative to the shore will be the vector $\vec{v}_{Eb}$. This is an instance of Equation (1.4.3.6), with frame A being E (the earth), frame B being R (the river), and “b” (for “boat”) standing for the point P we are tracking.

As you can see from this couple of examples, Equation (1.4.3.6) is often useful as it is written, but sometimes the information we have is given to us in a different way: for instance, we could be given the velocity of the object in frame A ($\vec{v}_{AP}$), and the velocity of frame B as seen in frame A ($\vec{v}_{AB}$), and told to calculate the velocity of the object as seen in frame B. This can be easily accomplished if we note that the vector $\vec{v}_{AB}$ is equal to $-\vec{v}_{BA}$; that is to say, the velocity of frame B as seen from frame A is just the opposite of the velocity of frame A as seen from frame B. Hence, Equation (1.4.3.6) can be rewritten as

$$\vec{v}_{AP} = \vec{v}_{BP} - \vec{v}_{BA}. \quad (1.4.3.7)$$

For most of the next few chapters we are going to be considering only motion in one dimension, and so we will write Equation (1.4.3.6) (or (1.4.3.7)) without the vector symbols, and it will be understood that v refers to the component of the vector $\vec{v}$ along the coordinate axis of interest.

A quantity that will be particularly important later on is the *relative velocity* of two objects, which we could label 1 and 2. The velocity of object 2 relative to object 1 is, by definition, the velocity which an observer moving along with 1 would measure for object 2. So it is just a simple frame change: let the earth frame be frame E and the frame moving with object 1 be frame 1, then the velocity we want is v_{12} (“velocity of object 2 in frame 1”). If we make the change $A \rightarrow 1$, $B \rightarrow E$, and $P \rightarrow 2$ in Equation (1.4.3.7), we get

$$v_{12} = v_{E2} - v_{E1}. \quad (1.4.3.8)$$

In other words, the velocity of 2 relative to 1 is just the velocity of 2 minus the velocity of 1. This is again a familiar effect: if you are driving down the highway at 50 miles per hour, and the car in front of you is driving at 55, then its velocity relative to you is 5 mph, which is the rate at which it is moving away from you (in the forward direction, assumed to be the positive one).

It is important to realize that all these velocities are *real* velocities, each in its own reference frame. Something may be said to be truly moving at some velocity in one reference frame, and just as truly moving with a different velocity in a different reference frame. I will have a lot more to say about this in the next chapter, but in the meantime you can reflect on the fact that, if a car moving at 55 mph collides with another one moving at 50 mph in the same direction, the damage will be basically the same as if the first car had been moving at 5 mph and the second one had been at rest. For practical purposes, where you are concerned, another car’s velocity relative to yours is that car’s “real” velocity.

Resources

A good app for practicing how to add vectors (and how to break them up into components, magnitude and direction, etc.) may be found here: <https://phet.colorado.edu/en/simulation/vector-addition>.

Perhaps the most dramatic demonstration of how Equation (1.4.3.6) works in the real world is in this episode of *Mythbusters*: <https://www.youtube.com/watch?v=BLu118nhzc>. (If this link does not work, do a search for “Mythbusters cancel momentum.”) They shoot a ball from the bed of a truck, with a velocity (relative to the truck) of 60 mph backwards, while the truck is moving forward at 60 mph. I think the result is worth watching.

A very old, but also very good, educational video about different frames of reference is this one: <https://www.youtube.com/watch?v=sS17fCom0Ns>. You should try to watch at least part of it. Many things will be relevant to later parts of the course, including projectile motion.

¹ We have made a very natural assumption, that the time interval Δt is the same for observers tracking the particle’s motion in frames A and B, respectively (where each observer is understood to be moving along with his or her frame). This, however, turns out to be *not* true when any of the velocities involved is close to the speed of light, and so the simple addition of velocities formula (1.4.3.6) does not hold in Einstein’s relativity theory.

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1.4.4: Examples

? Whiteboard Problem 1.4.4.1

Two blocks, of mass m and $3m$, are compressed on either side of a spring and tied together with a rope. They are sitting at rest on a frictionless surface, as shown in the figure.

1. The rope breaks and the larger block flies away from the smaller one at a speed of 2.00 m/s (relative to the surface they are on). If $m=0.450$ kg, what is the speed of the smaller block?
2. Now retie the rope and perform the experiment again, but slide the blocks along the surface with an initial speed of 3.5 m/s to the right. The rope breaks and the spring acts in the same way on the blocks. What is the speed of the smaller block?
3. For both parts (a) and (b): What is the speed of the center of mass of the system both before and after the rope breaks?

You can find the velocity of the center of mass without knowing where the center of mass is. However, if you want to find where it is, take both blocks to be 10cm wide and made of a uniform density material.

✓ Example 1.4.4.2: Center of Mass of the Earth-Moon System

Using data from text appendix, determine how far the center of mass of the Earth-moon system is from the center of Earth. Compare this distance to the radius of Earth, and comment on the result. Ignore the other objects in the solar system.

Strategy

We get the masses and separation distance of the Earth and moon, impose a coordinate system, and use Equation 4.3.1 with just $N = 2$ objects. We use a subscript “e” to refer to Earth, and subscript “m” to refer to the moon.

Solution

Define the origin of the coordinate system as the center of Earth. Then, with just two objects, Equation 4.3.1 becomes

$$R = \frac{m_e r_e + m_m r_m}{m_e + m_m}. \quad (1.4.4.1)$$

We can find the values of the distances and masses from the Internet,

$$m_e = 5.97 \times 10^{24} \text{ kg} \quad (1.4.4.2)$$

$$m_m = 7.36 \times 10^{22} \text{ kg} \quad (1.4.4.3)$$

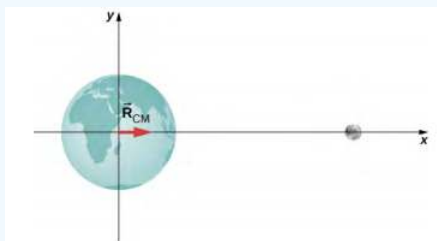
$$r_m = 3.82 \times 10^5 \text{ m}. \quad (1.4.4.4)$$

We defined the center of Earth as the origin, so $r_e = 0$ m. Inserting these into the equation for R gives

$$\begin{aligned} R &= \frac{(5.97 \times 10^{24} \text{ kg})(0 \text{ m}) + (7.36 \times 10^{22} \text{ kg})(3.82 \times 10^8 \text{ m})}{(5.98 \times 10^{24} \text{ kg}) + (7.36 \times 10^{22} \text{ kg})} \\ &= 4.64 \times 10^6 \text{ m}. \end{aligned} \quad (1.4.4.5)$$

Significance

The radius of Earth is 6.37×10^6 m, so the center of mass of the Earth-moon system is $(6.37 - 4.64) \times 10^6 \text{ m} = 1.73 \times 10^6 \text{ m} = 1730 \text{ km}$ (roughly 1080 miles) **below** the surface of Earth. The location of the center of mass is shown (not to scale).



? Exercise 1.4.4.3

Suppose we included the sun in the system. Approximately where would the center of mass of the Earth-moon-sun system be located? (Feel free to actually calculate it.)

✓ Example 1.4.4.4: Center of Mass of a Salt Crystal

Figure 1.4.4.3 shows a single crystal of sodium chloride—ordinary table salt. The sodium and chloride ions form a single unit, NaCl. When multiple NaCl units group together, they form a cubic lattice. The smallest possible cube (called the unit cell) consists of four sodium ions and four chloride ions, alternating. The length of one edge of this cube (i.e., the bond length) is 2.36×10^{-10} m. Find the location of the center of mass of the unit cell. Specify it either by its coordinates ($r_{CM,x}$, $r_{CM,y}$, $r_{CM,z}$), or by r_{CM} and two angles.

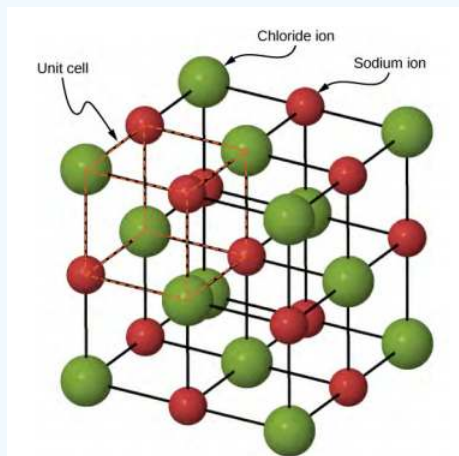


Figure 1.4.4.3: A drawing of a sodium chloride (NaCl) crystal.

Strategy

We can look up all the ion masses. If we impose a coordinate system on the unit cell, this will give us the positions of the ions. We can then apply Equation 4.3.1 in each direction (along with the Pythagorean theorem).

Solution

Define the origin to be at the location of the chloride ion at the bottom left of the unit cell. Figure 1.4.4.4 shows the coordinate system.

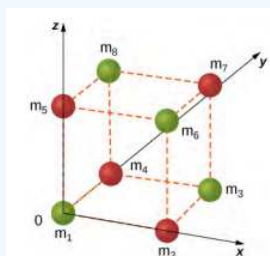


Figure 1.4.4.4: A single unit cell of a NaCl crystal.

There are eight ions in this crystal, so $N = 8$:

$$\vec{r}_{CM} = \frac{1}{M} \sum_{j=1}^8 m_j \vec{r}_j. \quad (1.4.4.6)$$

The mass of each of the chloride ions is

$$35.453u \times \frac{1.660 \times 10^{-27} \text{ kg}}{u} = 5.885 \times 10^{-26} \text{ kg} \quad (1.4.4.7)$$

so we have

$$m_1 = m_3 = m_6 = m_8 = 5.885 \times 10^{-26} \text{ kg}. \quad (1.4.4.8)$$

For the sodium ions,

$$m_2 = m_4 = m_5 = m_7 = 3.816 \times 10^{-26} \text{ kg}. \quad (1.4.4.9)$$

The total mass of the unit cell is therefore

$$M = (4)(5.885 \times 10^{-26} \text{ kg}) + (4)(3.816 \times 10^{-26} \text{ kg}) = 3.880 \times 10^{-25} \text{ kg}. \quad (1.4.4.10)$$

From the geometry, the locations are

$$\begin{aligned} \vec{r}_1 &= 0 \\ \vec{r}_2 &= (2.36 \times 10^{-10} \text{ m})\hat{i} \\ \vec{r}_3 &= r_{3x}\hat{i} + r_{3y}\hat{j} = (2.36 \times 10^{-10} \text{ m})\hat{i} + (2.36 \times 10^{-10} \text{ m})\hat{j} \\ \vec{r}_4 &= (2.36 \times 10^{-10} \text{ m})\hat{j} \\ \vec{r}_5 &= (2.36 \times 10^{-10} \text{ m})\hat{k} \\ \vec{r}_6 &= r_{6x}\hat{i} + r_{6z}\hat{k} = (2.36 \times 10^{-10} \text{ m})\hat{i} + (2.36 \times 10^{-10} \text{ m})\hat{k} \\ \vec{r}_7 &= r_{7x}\hat{i} + r_{7y}\hat{j} + r_{7z}\hat{k} = (2.36 \times 10^{-10} \text{ m})\hat{i} + (2.36 \times 10^{-10} \text{ m})\hat{j} + (2.36 \times 10^{-10} \text{ m})\hat{k} \\ \vec{r}_8 &= r_{8y}\hat{j} + r_{8z}\hat{k} = (2.36 \times 10^{-10} \text{ m})\hat{j} + (2.36 \times 10^{-10} \text{ m})\hat{k}. \end{aligned} \quad (1.4.4.11)$$

Substituting:

$$\begin{aligned} |\vec{r}_{CM,x}| &= \sqrt{r_{CM,x}^2 + r_{CM,y}^2 + r_{CM,z}^2} \\ &= \frac{1}{M} \sum_{j=1}^8 m_j (r_x)_j \\ &= \frac{1}{M} (m_1 r_{1x} + m_2 r_{2x} + m_3 r_{3x} + m_4 r_{4x} + m_5 r_{5x} + m_6 r_{6x} + m_7 r_{7x} + m_8 r_{8x}) \\ &= \frac{1}{3.8804 \times 10^{-25} \text{ kg}} \left[(5.885 \times 10^{-26} \text{ kg})(0 \text{ m}) + (3.816 \times 10^{-26} \text{ kg})(2.36 \times 10^{-10} \text{ m}) \right. \\ &\quad + (5.885 \times 10^{-26} \text{ kg})(2.36 \times 10^{-10} \text{ m}) + (3.816 \times 10^{-26} \text{ kg})(2.36 \times 10^{-10} \text{ m}) + 0 + 0 \\ &\quad \left. + (3.816 \times 10^{-26} \text{ kg})(2.36 \times 10^{-10} \text{ m}) + 0 \right] \\ &= 1.18 \times 10^{-10} \text{ m}. \end{aligned} \quad (1.4.4.12)$$

Similar calculations give $r_{CM,y} = r_{CM,z} = 1.18 \times 10^{-10} \text{ m}$ (you could argue that this must be true, by symmetry, but it's a good idea to check).

Significance

As it turns out, it was not really necessary to convert the mass from atomic mass units (u) to kilograms, since the units divide out when calculating r_{CM} anyway.

To express r_{CM} in terms of magnitude and direction, first apply the three-dimensional Pythagorean theorem to the vector components:

$$\begin{aligned} r_{CM} &= \sqrt{r_{CM,x}^2 + r_{CM,y}^2 + r_{CM,z}^2} \\ &= (1.18 \times 10^{-10} \text{ m})\sqrt{3} \\ &= 2.044 \times 10^{-10} \text{ m}. \end{aligned} \quad (1.4.4.13)$$

Since this is a three-dimensional problem, it takes two angles to specify the direction of $\vec{r}_{CM}$. Let ϕ be the angle in the x,y-plane, measured from the +x-axis, counterclockwise as viewed from above; then:

$$\phi = \tan^{-1} \left(\frac{r_{CM,y}}{r_{CM,x}} \right) = 45^\circ. \quad (1.4.4.14)$$

Let θ be the angle in the y,z-plane, measured downward from the +z-axis; this is (not surprisingly):

$$\theta = \tan^{-1} \left(\frac{R_z}{R_y} \right) = 45^\circ. \quad (1.4.4.15)$$

Thus, the center of mass is at the geometric center of the unit cell. Again, you could argue this on the basis of symmetry

? Exercise 1.4.4.5

Suppose you have a macroscopic salt crystal (that is, a crystal that is large enough to be visible with your unaided eye). It is made up of a **huge** number of unit cells. Is the center of mass of this crystal necessarily at the geometric center of the crystal?

Two crucial concepts come out of these examples:

1. As with all problems, you must define your coordinate system and origin. For center-of-mass calculations, it often makes sense to choose your origin to be located at one of the masses of your system. That choice automatically defines its distance in Equation 4.3.1 to be zero. However, you must still include the mass of the object at your origin in your calculation of M , the total mass Equation 4.3.1. In the Earth-moon system example, this means including the mass of Earth. If you hadn't, you'd have ended up with the center of mass of the system being at the center of the moon, which is clearly wrong.
2. In the second example (the salt crystal), notice that there is no mass at all at the location of the center of mass. This is an example of what we stated above, that there does not have to be any actual mass at the center of mass of an object.

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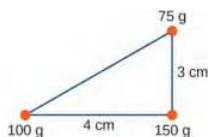
1.4.E: Systems and the Center of Mass Exercises

Conceptual Questions

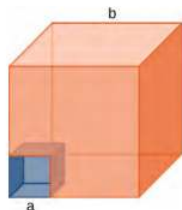
1. Suppose a fireworks shell explodes, breaking into three large pieces for which air resistance is negligible. How does the explosion affect the motion of the center of mass? How would it be affected if the pieces experienced significantly more air resistance than the intact shell?
2. It is possible for the velocity of a rocket to be greater than the exhaust velocity of the gases it ejects. When that is the case, the gas velocity and gas momentum are in the same direction as that of the rocket. How is the rocket still able to obtain thrust by ejecting the gases?

Problems

3. Three point masses are placed at the corners of a triangle as shown in the figure below. Find the center of mass of the three-mass system.



4. Two particles of masses m_1 and m_2 separated by a horizontal distance D are released from the same height h at the same time. Find the vertical position of the center of mass of these two particles at a time before the two particles strike the ground. Assume no air resistance.
5. Two particles of masses m_1 and m_2 separated by a horizontal distance D are let go from the same height h at different times. Particle 1 starts at $t = 0$, and particle 2 is let go at $t = T$. Find the vertical position of the center of mass at a time before the first particle strikes the ground. Assume no air resistance.
6. A cube of side a is cut out of another cube of side b as shown in the figure below. Find the location of the center of mass of the structure. (**Hint:** Think of the missing part as a negative mass overlapping a positive mass.)



7. A 5.00-kg squid initially at rest ejects 0.250 kg of fluid with a velocity of 10.0 m/s. (a) What is the recoil velocity of the squid if the ejection is done in 0.100 s and there is a 5.00-N frictional force opposing the squid's movement? (b) How much energy is lost to work done against friction?
8. A rocket takes off from Earth and reaches a speed of 100 m/s in 10.0 s. If the exhaust speed is 1500 m/s and the mass of fuel burned is 100 kg, what was the initial mass of the rocket?
9. Repeat the preceding problem but for a rocket that takes off from a space station, where there is no gravity other than the negligible gravity due to the space station.
10. How much fuel would be needed for a 1000-kg rocket (this is its mass with no fuel) to take off from Earth and reach 1000 m/s in 30 s? The exhaust speed is 1000 m/s.
11. What exhaust speed is required to accelerate a rocket in deep space from 800 m/s to 1000 m/s in 5.0 s if the total rocket mass is 1200 kg and the rocket only has 50 kg of fuel left?
12. **Unreasonable Results** Squids have been reported to jump from the ocean and travel 30.0 m (measured horizontally) before re-entering the water. (a) Calculate the initial speed of the squid if it leaves the water at an angle of 20.0° , assuming negligible lift from the air and negligible air resistance. (b) The squid propels itself by squirting water. What fraction of its mass would it have to eject in order to achieve the speed found in the previous part? The water is ejected at 12.0 m/s; gravitational force and friction are neglected. (c) What is unreasonable about the results? (d) Which premise is unreasonable, or which premises are inconsistent?

Additional Problems

13. If the entire population of Earth were transferred to the Moon, how far would the center of mass of the Earth-Moon-population system move? Assume the population is 7 billion, the average human has a mass of 65 kg, and that the population is evenly distributed over both the Earth and the Moon. The mass of the Earth is 5.97×10^{24} kg and that of the Moon is 7.34×10^{22} kg. The radius of the Moon's orbit is about 3.84×10^5 m.
14. Two friends are in small boats. The mass of the first boat plus the person is 300 kg; the mass for the second boat plus the person is 400 kg. They have a rope between the two boats that is 30 feet long. If both friends pull slowly on the rope, how far will the lighter person move?
15. You friend wonders how a rocket continues to climb into the sky once it is sufficiently high above the surface of Earth so that its expelled gasses no longer push on the surface. How do you respond?
16. To increase the acceleration of a rocket, should you throw rocks out of the front window of the rocket or out of the back window?

Contributors and Attributions

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SECTION OVERVIEW

1.5: Conservation of Momentum

1.5.1: Conservation of Linear Momentum

1.5.2: The Problem Solving Framework

1.5.3: Examples

1.5.4: More Examples

1.5.E: Conservation of Momentum (Exercises)

In this chapter we are going to combine what we've done in Chapter 2 and Chapter 3, and bring us to the **law of conservation of momentum**, which is the first real "rule" of nature that we are going to study in this book. In Chapter 2 we used this law in the form "interactions transfer momentum". That is core concept, but it turns out that really implementing this is challenging without some mathematical tools - primarily that of vectors. Now that we have a better understanding (and some practice) doing vector analysis, we are ready to state the full law of conservation of momentum: *the momentum of an isolated system does not change in time*. This is a seemingly simple statement, but it turns out to be extraordinarily powerful, because you can use it to make predictions about the systems you are studying - the final velocity of an object in a collision, for instance.

Of course, to make quantitative predictions, we need cast this law into mathematics, which is the following:

Conservation of Momentum

$$\Delta \vec{p} = 0 \Leftrightarrow \vec{p}_f - \vec{p}_i = 0 \Leftrightarrow \vec{p}_f = \vec{p}_i \quad (1.5.1)$$

(You should convince yourself that all three of those statements say exactly the same thing, using the definition of the delta and basic algebra!)

We should take note that this is a vector equation, which means there are actually three parts to it: $\Delta p_x = 0$, $\Delta p_y = 0$, and $\Delta p_z = 0$. These three equations are independent, in the sense that they are always all true at the same time, but aren't directly related.

Let's also just briefly note how we should understand this mathematical statement in conjunction with our old "interactions transfer momentum" concept. If a system is isolated, there are no interactions between the system and the outside world, so there is no momentum transfer into or out of the system. That's exactly the same information contained in the equation above, since that says the change in momentum of an isolated system is zero.

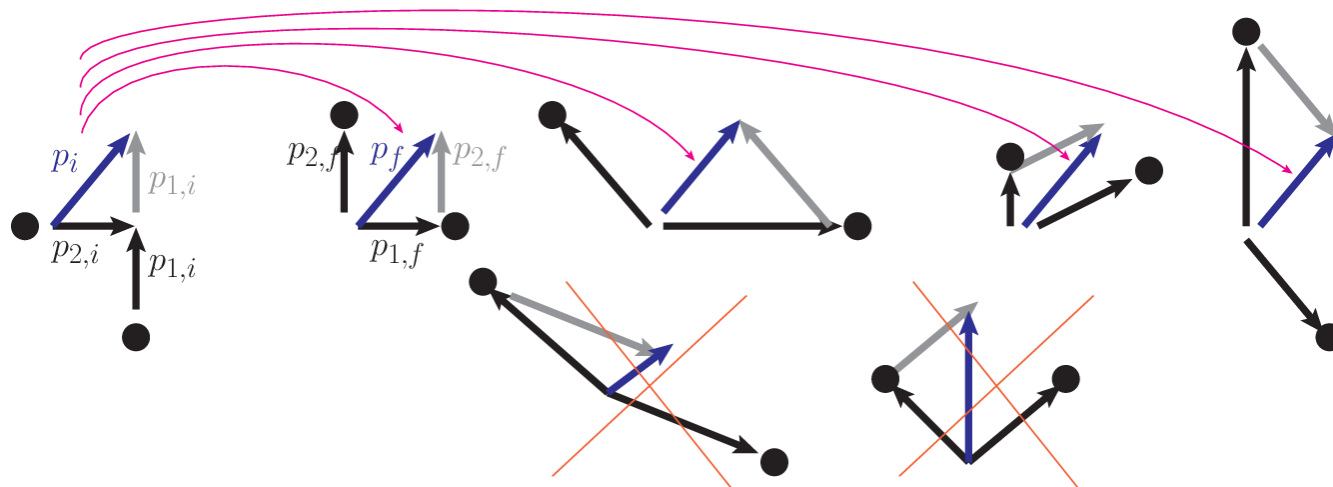
Let's try some examples. Say you have a hockey puck (mass 0.5 kg) sliding on ice at a speed of 10 m/s, and it collides head on into another puck moving at 14 m/s. After the collision, the second puck moves away at 6 m/s, and we can use our laws to determine the final speed of the first. The equation says "final equal initial", so let's try to calculate the initial. The first puck has a momentum of 5 kg m/s, the second has 7 kg m/s - but it's moving the opposite direction, so the initial momentum in the x-direction is $p_{i,x} = 5 \text{ kg m/s} - 7 \text{ kg m/s} = -2 \text{ kg m/s}$. We can also try to calculate the final momentum - the second puck has 3 kg m/s, in the positive direction, but we don't know the final speed of the first. However, using the equation above we can write

$$-2 \text{ kg m/s} = (0.5 \text{ kg})v_{1,f} + 3 \text{ kg m/s} \quad (1.5.2)$$

This equation can be solved for the unknown variable to get $v_{1,f} = -10 \text{ m/s}$, which is negative because apparently the first puck bounces off the second and moves backward after the collision.

That example was pretty simple, since the collision only took place in a single direction (we choose x). However, our conservation of momentum law tells us that the vector (magnitude and direction) of the momentum must remain unchanged during a collision. The figure below illustrates that for the collision of two objects of similar masses (note the grey arrows are just helping us to make

sure we add the vectors head to tail). The final momentum must be equal to the initial, and even though some of the collisions below look totally reasonable, they are prohibited by the law of conservation of momentum.



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1.5.1: Conservation of Linear Momentum

Learning Objectives

- Explain the meaning of “conservation of momentum”
- Correctly identify if a system is, or is not, closed
- Define a system whose momentum is conserved
- Mathematically express conservation of momentum for a given system
- Calculate an unknown quantity using conservation of momentum

Recall what we learned about momentum transfer in [Chapter 2](#): when objects interact with each other (like in a collision), they do so by *transferring momentum between each other*. What's more, this momentum is conserved - in that chapter, we expressed this conservation by noting the amount that one object lost was the same that the other object gained. Mathematically, we could write this as

$$\Delta \vec{p}_1 = -\Delta \vec{p}_2, \quad (1.5.1.1)$$

for two objects. If this collision happens over some time period Δt , we can write this change as rate of change,

$$\frac{\Delta \vec{p}_1}{\Delta t} = -\frac{\Delta \vec{p}_2}{\Delta t}. \quad (1.5.1.2)$$

So now one object gains or loses momentum at the same rate that the other loses or gains it. Now performing some simple manipulations on this expression:

$$\frac{\Delta \vec{p}_1}{\Delta t} = -\frac{\Delta \vec{p}_2}{\Delta t} \rightarrow \frac{\Delta \vec{p}_1}{\Delta t} + \frac{\Delta \vec{p}_2}{\Delta t} = 0 \rightarrow \frac{\Delta \vec{p}_1 + \Delta \vec{p}_2}{\Delta t} = 0. \quad (1.5.1.3)$$

Using the definition of Δ , the top of this expression can be written as

$$\Delta \vec{p}_1 + \Delta \vec{p}_2 = (\vec{p}_{1f} - \vec{p}_{1i}) + (\vec{p}_{2f} - \vec{p}_{2i}) = (\vec{p}_{1f} + \vec{p}_{2f}) - (\vec{p}_{1i} + \vec{p}_{2i}). \quad (1.5.1.4)$$

Now looking at this expression, if we now define the total momentum of the system to be $\vec{P}_{sys} = \vec{p}_1 + \vec{p}_2$, we can see that what we just wrote was

$$\vec{P}_{sys,f} - \vec{P}_{sys,i}, \quad (1.5.1.5)$$

and combined with equation [1.5.1.3](#), we see this gives us

$$\vec{P}_{sys,f} - \vec{P}_{sys,i} = 0 \rightarrow \Delta \vec{P}_{sys} = 0. \quad (1.5.1.6)$$

Or, in other words, the **total momentum of the system does not change in time**. This is the best statement of conservation of momentum we have gotten so far, and is the best one to remember going forward. It does not matter "when" the initial and final states happen, all that matters is the amount of momentum does not change between those initial and final states.

Since momentum is a vector, both the magnitude and direction of this momentum must be conserved, as shown in Figure 1.5.1.1, the total momentum of the system before and after the collision remains the same, in both magnitude and direction.

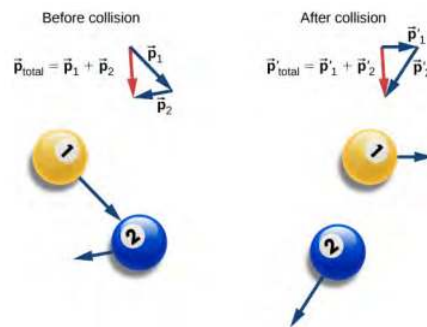


Figure 1.5.1.1: Before the collision, the two billiard balls travel with momenta $\vec{p}_1$ and $\vec{p}_2$. The total momentum of the system is the sum of these, as shown by the red vector labeled $\vec{p}_{total}$ on the left. After the collision, the two billiard balls travel with different momenta $\vec{p}'_1$ and $\vec{p}'_2$. The total momentum, however, has not changed, as shown by the red vector arrow $\vec{p}'_{total}$ on the right.

Generalizing this result to N objects, we obtain

$$\vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \cdots + \vec{p}_N = \text{constant} \quad (1.5.1.7)$$

$$\sum_{j=1}^N \vec{p}_j = \text{constant}. \quad (1.5.1.8)$$

Equation 1.5.1.8 is the definition of the total (or net) momentum of a system of N interacting objects, along with the statement that the total momentum of a system of objects is constant in time—that is, *momentum is conserved*.

Conservation Laws

If the value of a physical quantity is constant in time, we say that the quantity is conserved.

Requirements for Momentum Conservation

There is a complication, however. A system must meet two requirements for its momentum to be conserved:

1. **The mass of the system must remain constant during the interaction.** As the objects interact (apply forces on each other), they may transfer mass from one to another; but any mass one object gains is balanced by the loss of that mass from another. The total mass of the system of objects, therefore, remains unchanged as time passes:

$$\left[\frac{dm}{dt} \right]_{\text{system}} = 0. \quad (1.5.1.9)$$

2. **The net external force on the system must be zero.** As the objects collide, or explode, and move around, they exert forces on each other. However, all of these forces are internal to the system, and thus each of these internal forces is balanced by another internal force that is equal in magnitude and opposite in sign. As a result, the change in momentum caused by each internal force is cancelled by another momentum change that is equal in magnitude and opposite in direction. Therefore, internal forces cannot change the total momentum of a system because the changes sum to zero. However, if there is some external force that acts on all of the objects (gravity, for example, or friction), then this force changes the momentum of the system as a whole; that is to say, the momentum of the system is changed by the external force. Thus, for the momentum of the system to be conserved, we must have

$$\vec{F}_{ext} = \vec{0}. \quad (1.5.1.10)$$

A system of objects that meets these two requirements is said to be a **closed system** (also called an isolated system). Another way to state equation 1.5.1.6 is

Law of Conservation of Momentum

The total momentum of a closed system is conserved:

$$\sum_{j=1}^N \vec{p}_j = \text{constant}. \quad (1.5.1.11)$$

This statement is called the **Law of Conservation of Momentum**. Along with the conservation of energy, it is one of the foundations upon which all of physics stands. All our experimental evidence supports this statement: from the motions of galactic clusters to the quarks that make up the proton and the neutron, and at every scale in between. In a **closed system**, the **total momentum never changes**.

Note that there absolutely **can** be external forces acting on the system; but for the system's momentum to remain constant, these external forces have to cancel, so that the **net** external force is zero. Billiard balls on a table all have a weight force acting on them, but the weights are balanced (canceled) by the normal forces, so there is no net force.

The Meaning of 'System'

A **system** (mechanical) is the collection of objects in whose motion (kinematics and dynamics) you are interested. If you are analyzing the bounce of a ball on the ground, you are probably only interested in the motion of the ball, and not of Earth; thus, the ball is your system. If you are analyzing a car crash, the two cars together compose your system (Figure 1.5.1.2).

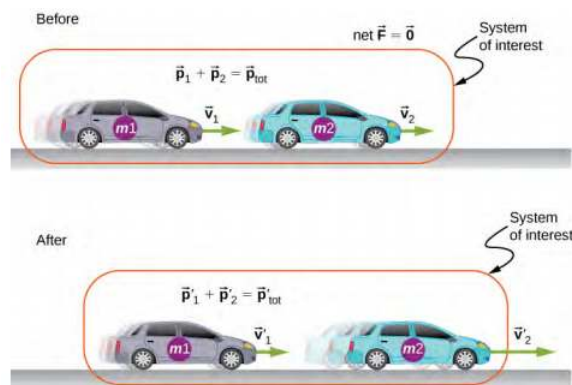


Figure 1.5.1.2: The two cars together form the system that is to be analyzed. It is important to remember that the contents (the mass) of the system do not change before, during, or after the objects in the system interact.

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1.5.2: The Problem Solving Framework

Solving problems in physics can be a daunting task. Some solutions appear to be insights from the great beyond, or algebraic tricks that you'd never be able to reproduce on your own. However, the great thing about physics is that the only thing we have to work with are the basic laws of the universe, and there are actually not many of those. For example, one could argue that in this textbook the only laws of the universe we are actually dealing with are the following:

- Conservation of Momentum
- Conservation of Angular Momentum
- Conservation of Energy

Such a short list of things to learn! If all of mechanics is covered by that list, why does physics appear to be so hard? Well, that's because we are often applying these laws in situations which are new to us - if they weren't new, we could simply look up the answer, and that doesn't demonstrate any understanding of the physical world. To this end, we'd like to present the **problem solving framework**, which can tackle difficult (and easy!) problems we come across when trying to apply the physical laws we listed above.

The problem-solving framework is a 4-step process to get you from a problem statement to a solution, and is as follows¹:

1. **Translate:** You might think about this step as "list the knowns and unknowns": Take the words on the page and translate them into symbols, as appropriate. For example, "A train of mass 45 kg is traveling at a speed of 67 m/s", means that you can call the variable m the mass, v the speed, and they have particular values as part of the given information. This translation is necessary both because the language of physics is mathematical, and also it will help us organize our solution. This step also includes drawing the situation presented in the problem, and is a critical step that is often skipped by students. *A drawing will always help your thinking* - this is maybe even more true for a bad drawing, because it will be a sign that you do not yet understand the set up of the problem! Walk by the office of any physicist in the world and you will see sketches on a blackboard of whatever they were last working on.
2. **Model:** Decide what physical law you want to use to solve the problem. Sometimes, this is simply picking an equation to use ("this problem is asking us to find the center of mass, so I'm going to use the center of mass formula..."), but more often this step means picking one of the physical laws from the list above to use to solve the problem.
3. **Solve:** Solve! Perform whatever mathematical manipulations need to be done to find the quantity that you can use to answer the question. Note that in many cases, this can be the longest and most difficult step, but *it also contains no physics!* The physics is done already in step 2, now we are just using math.
4. **Check:** It's critical to check your answer - at least to make sure it *could* be physically reasonable. Sometimes this is just "I found that this car is traveling at 800 mph, is that reasonable?" (no, you did something wrong!) A better way to do this step would be to pick an alternative solution to verify you get the same answer. The classic way to do this is to use Newton's laws if you solved it first with a conservation law, or vice versa. But since this is physics, we are always calculating something about the real world, and we should always be able to see if our calculations roughly match our expectations.

¹This framework was inspired by Unit C of Thomas Moore's textbook series *Six Ideas That Shaped Physics*.

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1.5.3: Examples

? Whiteboard Problem 1.5.3.1: System Definitions

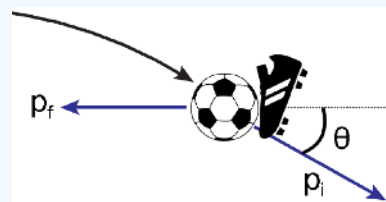
For each of the following sets of objects, define *two systems each*; one that is isolated and one that is not.

1. The Earth, Sun, and Moon.
2. A football flying through the air, during the time after it is thrown and before it is caught. (you may ignore air resistance here)
3. Two colliding carts, ignoring friction and air resistance.
4. Two colliding carts, ignore air resistance but including the friction between the ground and the carts.

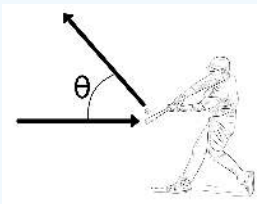
? Whiteboard Problem 1.5.3.2: Impulsive Shooter

A soccer player receives a pass in the air from a teammate and immediately fires the ball along the ground into the goal. The initial speed of the soccer ball is 15.0 m/s at an angle of 40.0° below the horizontal. The mass of the ball is 435 g.

If this is David Beckham, he can get the shot moving with a speed of 30.0 m/s, moving straight along the ground. What impulse (magnitude and direction) did his foot transfer to the soccer ball?



? Whiteboard Problem 1.5.3.3: Impulsive Batter



A baseball player is hitting a baseball as shown in the figure. The baseball has a mass of 145 g, and is traveling at the batter at a speed of 35 m/s (that's an 80-mph fastball). The baseball player can deliver an impulse of 5.5 kg m/s to the baseball when they hit it.

1. The baseball player hits the ball at an angle of 15° , as shown in the figure. How much impulse does the baseball player give to the ball in the x- and y-directions?
2. What is the final speed of the ball?
3. How much impulse was transferred to the player (through the bat) when they hit the ball?

✓ Example 1.5.3.4: Collision, Center of Mass, and Recoil¹

An 80-kg hockey player (call him player 1), moving at 3 m/s to the right, collides with a 90-kg player (player 2) who was moving at 2 m/s to the left. For a brief moment, they are stuck sliding together as they grab at each other.

- a. What is their joint velocity as they slide together?
- b. What was the velocity of their center of mass before and after the collision?

Solution

(a) First we must determine whether the hockey players are isolated. We can consider them isolated from a couple of standpoints.

First of all, they are functionally isolated. This is because if they are on ice, there is very little friction and the only external forces on the hockey players are gravity and the normal force, which lead to a net zero force. (I assume this is why we are using hockey players instead of football/soccer/rugby/etc. players.)

Secondly, they are temporarily isolated due to the collision. In the collision, the collision forces are (briefly) larger than all other forces acting on the players. This is why we can do problems of football players, etc. as long as we are careful to look only right before and right after the collision. In any case, they are isolated, so we can proceed.

Call the initial velocities $\vec{v}_{1i}$ and $\vec{v}_{2i}$, the joint final velocity $\vec{v}_f$. Also, call the "right" direction +x.

Then conservation of momentum reads:

$$m_1 \begin{bmatrix} v_{1i,x} \\ 0 \end{bmatrix} + m_2 \begin{bmatrix} v_{2i,x} \\ 0 \end{bmatrix} = (m_1 + m_2) \begin{bmatrix} v_{f,x} \\ v_{f,y} \end{bmatrix} \quad (1.5.3.1)$$

$$m_1 \begin{bmatrix} 0 \\ 0 \end{bmatrix} + m_2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} = (m_1 + m_2) \begin{bmatrix} v_{f,y} \\ v_{f,z} \end{bmatrix}$$

Our y and z component equations simply tell us that those velocities are zero throughout:

$$0 + 0 = (m_1 + m_2) v_{f,y} \Rightarrow v_{f,y} = 0 \text{ and similarly } v_{f,z} = 0 \quad (1.5.3.2)$$

As for our x component (the interesting one):

$$m_1 v_{1i,x} + m_2 v_{2i,x} = (m_1 + m_2) v_{f,x}. \quad (1.5.3.3)$$

Solving for the final velocity, we get

$$v_{f,x} = \frac{m_1 v_{1i,x} + m_2 v_{2i,x}}{m_1 + m_2}. \quad (1.5.3.4)$$

Substituting the values given, we get

$$v_{f,x} = \frac{80 \times 3 - 90 \times 2}{170} = 0.353 \frac{\text{m}}{\text{s}}. \quad (1.5.3.5)$$

(b) According to Equation (3.3.3), the velocity of the center of mass, $v_{cm,x}$, is just the same as what we just calculated (Equation 1.5.3.4) above). This makes sense: after the collision, if the players are moving together, their system's center of mass has to be moving with them. Also, if the system is isolated, the center of mass velocity should be the same before and after the collision. So the answer is $v_{cm,x} = v_{f,x} = 0.353 \text{ m/s}$.

¹For a variation of this problem that studies the relative velocity of this system with respect to another frame, check out Example 3.5.2 in [University Physics 1 - Classical Mechanics](#), by Gea-Banacloche.

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1.5.4: More Examples

? Problem-Solving Strategy: Conservation of Momentum

Using conservation of momentum requires four basic steps. The first step is crucial:

1. Identify a closed system (total mass is constant, no net external force acts on the system).
2. Write down an expression representing the total momentum of the system before the “event” (explosion or collision).
3. Write down an expression representing the total momentum of the system after the “event.”
4. Set these two expressions equal to each other, and solve this equation for the desired quantity

✓ Example 1.5.4.1: Colliding Carts

Two carts in a physics lab roll on a level track, with negligible friction. These carts have small magnets at their ends, so that when they collide, they stick together (Figure 1.5.4.1). The first cart has a mass of 675 grams and is rolling at 0.75 m/s to the right; the second has a mass of 500 grams and is rolling at 1.33 m/s, also to the right. After the collision, what is the velocity of the two joined carts?



Figure 1.5.4.1: Two lab carts collide and stick together after the collision.

Strategy

We have a collision. We’re given masses and initial velocities; we’re asked for the final velocity. This all suggests using conservation of momentum as a method of solution. However, we can only use it if we have a closed system. So we need to be sure that the system we choose has no net external force on it, and that its mass is not changed by the collision.

Defining the system to be the two carts meets the requirements for a closed system: The combined mass of the two carts certainly doesn’t change, and while the carts definitely exert forces on each other, those forces are internal to the system, so they do not change the momentum of the system as a whole. In the vertical direction, the weights of the carts are canceled by the normal forces on the carts from the track.

Solution

Conservation of momentum is

$$\vec{p}_f = \vec{p}_i.$$

Define the direction of their initial velocity vectors to be the +x-direction. The initial momentum is then

$$\vec{p}_i = m_1 \begin{bmatrix} v_{1,x} \\ 0 \\ 0 \end{bmatrix} + m_2 \begin{bmatrix} v_{2,x} \\ 0 \\ 0 \end{bmatrix}.$$

The final momentum of the now-linked carts is

$$\vec{p}_f = (m_1 + m_2) \begin{bmatrix} v_{f,x} \\ v_{f,y} \\ v_{f,z} \end{bmatrix}.$$

Once again, our y and z equations are not interesting. We can concentrate on the x direction. Using the x equation:

$$(m_1 + m_2)\vec{v}_{f,x} = m_1 v_{1,x} + m_2 v_{2,x}$$

$$\vec{v}_f = \left(\frac{m_1 v_{1,x} + m_2 v_{2,x}}{m_1 + m_2} \right).$$

Substituting the given numbers:

$$\vec{v}_{f,x} = \left[\frac{(0.675 \text{ kg})(0.75 \text{ m/s}) + (0.5 \text{ kg})(1.33 \text{ m/s})}{1.175 \text{ kg}} \right] \hat{i} \\ = (0.997 \text{ m/s}).$$

Significance

The principles that apply here to two laboratory carts apply identically to all objects of whatever type or size. Even for photons, the concepts of momentum and conservation of momentum are still crucially important even at that scale. (Since they are massless, the momentum of a photon is defined very differently from the momentum of ordinary objects. You will learn about this when you study quantum physics.)

? Exercise 1.5.4.2

Suppose the second, smaller cart had been initially moving to the left. What would the sign of the final velocity have been in this case?

✓ Example 1.5.4.3: Ice Hockey 1

Two hockey pucks of identical mass are on a flat, horizontal ice hockey rink. The red puck is motionless; the blue puck is moving at 2.5 m/s to the left (Figure 1.5.4.3). It collides with the motionless red puck. The pucks have a mass of 15 g. After the collision, the red puck is moving at 2.5 m/s, to the left. What is the final velocity of the blue puck?

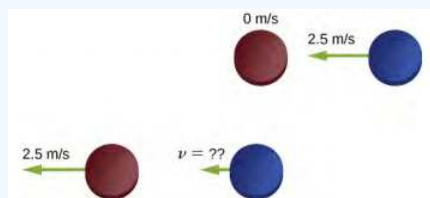


Figure 1.5.4.3: Two identical hockey pucks colliding. The top diagram shows the pucks the instant before the collision, and the bottom diagram show the pucks the instant after the collision. The net external force is zero.

Strategy

We're told that we have two colliding objects, we're told the masses and initial velocities, and one final velocity; we're asked for both final velocities. Conservation of momentum seems like a good strategy. Define the system to be the two pucks; there's no friction, so we have a closed system.

Before you look at the solution, what do you think the answer will be?

The blue puck final velocity will be:

- zero
- 2.5 m/s to the left
- 2.5 m/s to the right
- 1.25 m/s to the left
- 1.25 m/s to the right
- something else

Solution

Define the +x-direction to point to the right. Conservation of momentum then reads

$$\vec{p}_f = \vec{p}_i \\ m \begin{bmatrix} v_{r,f,x} \\ v_{r,f,y} \\ v_{r,f,z} \end{bmatrix} + m \begin{bmatrix} v_{b,f,x} \\ v_{b,f,y} \\ v_{b,f,z} \end{bmatrix} = m \begin{bmatrix} v_{r,i,x} \\ 0 \\ 0 \end{bmatrix} + m \begin{bmatrix} v_{b,i,x} \\ 0 \\ 0 \end{bmatrix}$$

As in the other examples, only the x direction is of interest. Before the collision, the momentum of the system is entirely and only in the blue puck. Thus,

$$mv_{r_{f,x}} + mv_{b_{f,x}} = -mv_{b_{i,x}}$$

$$v_{r_{f,x}} + v_{b_{f,x}} = -v_{b_{i,x}}.$$

(Remember that the masses of the pucks are equal.) Substituting numbers:

$$\begin{aligned} -(2.5 \text{ m/s}) + v_{b_{f,x}} &= -(2.5 \text{ m/s}) \\ v_{b_{f,x}} &= 0. \end{aligned}$$

Significance

Evidently, the two pucks simply exchanged momentum. The blue puck transferred all of its momentum to the red puck. In fact, this is what happens in similar collision where $m_1 = m_2$.

? Exercise 1.5.4.4

Even if there were some friction on the ice, it is still possible to use conservation of momentum to solve this problem, but you would need to impose an additional condition on the problem. What is that additional condition?

✓ Example 1.5.4.5: Philae

On November 12, 2014, the European Space Agency successfully landed a probe named **Philae** on Comet 67P/Churyumov/Gerasimenko (Figure 1.5.4.4). During the landing, however, the probe actually landed three times, because it bounced twice. Let's calculate how much the comet's speed changed as a result of the first bounce.



Figure 1.5.4.4: An artist's rendering of Philae landing on a comet. (credit: modification of work by "DLR German Aerospace Center"/Flickr)

Let's define upward to be the $+y$ -direction, perpendicular to the surface of the comet, and $y = 0$ to be at the surface of the comet. Here's what we know:

- The mass of Comet 67P: $M_c = 1.0 \times 10^{13} \text{ kg}$
- The acceleration due to the comet's gravity: $\vec{a} = -(5.0 \times 10^{-3} \text{ m/s}^2)$ in y direction
- **Philae's** mass: $M_p = 96 \text{ kg}$
- Initial touchdown speed: $\vec{v}_1 = -(1.0 \text{ m/s})$ in y direction
- Initial upward speed due to first bounce: $\vec{v}_2 = (0.38 \text{ m/s})$ in y direction.
- Landing impact time: $\Delta t = 1.3 \text{ s}$

Strategy

We're asked for how much the comet's speed changed, but we don't know much about the comet, beyond its mass and the acceleration its gravity causes. However, we are told that the **Philae** lander collides with (lands on) the comet, and bounces off of it. A collision suggests momentum as a strategy for solving this problem.

If we define a system that consists of both **Philae** and Comet 67P, then there is no net external force on this system, and thus the momentum of this system is conserved. (We'll neglect the gravitational force of the sun.) Thus, if we calculate the change of momentum of the lander, we automatically have the change of momentum of the comet. Also, the comet's change of velocity is directly related to its change of momentum as a result of the lander "colliding" with it.

Solution

Let $\vec{p}_1$ be **Philae's** momentum at the moment just before touchdown, and $\vec{p}_2$ be its momentum just after the first bounce. Then its momentum just before landing was

$$\begin{aligned}\vec{p}_1 &= M_p \vec{v}_1 = \begin{bmatrix} 0 \\ (96 \text{ kg})(-1.0 \text{ m/s}) \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ -(96 \text{ kg} \cdot \text{m/s}) \\ 0 \end{bmatrix}\end{aligned}$$

and just after was

$$\vec{p}_2 = M_p \vec{v}_2 = \begin{bmatrix} 0 \\ (96 \text{ kg})(+0.38 \text{ m/s}) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ (36.5 \text{ kg} \cdot \text{m/s}) \\ 0 \end{bmatrix}.$$

Therefore, the lander's change of momentum during the first bounce is

$$\begin{aligned}\Delta \vec{p} &= \vec{p}_2 - \vec{p}_1 \\ &= \begin{bmatrix} 0 \\ (36.5 \text{ kg} \cdot \text{m/s}) \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ -(96 \text{ kg} \cdot \text{m/s}) \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ (133 \text{ kg} \cdot \text{m/s}) \\ 0 \end{bmatrix}\end{aligned}$$

Notice how important it is to include the negative sign of the initial momentum.

Now for the comet. Since momentum of the system must be conserved, the comet's momentum changed by exactly the negative of this:

$$\Delta \vec{p}_c = -\Delta \vec{p} = \begin{bmatrix} 0 \\ (-133 \text{ kg} \cdot \text{m/s}) \\ 0 \end{bmatrix}.$$

Therefore, its change of velocity (entirely in the y) direction is

$$\Delta v_{c,y} = \frac{\Delta p_{c,y}}{M_c} = \frac{-(133 \text{ kg} \cdot \text{m/s})}{1.0 \times 10^{13} \text{ kg}} = -(1.33 \times 10^{-11} \text{ m/s}).$$

Significance

This is a very small change in velocity, about a thousandth of a billionth of a meter per second. Crucially, however, it is **not** zero.

? Exercise 1.5.4.6

The changes of momentum for **Philae** and for Comet 67/P were equal (in magnitude). Were the impulses experienced by **Philae** and the comet equal? How about the forces? How about the changes of kinetic energies?

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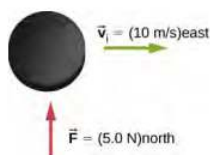
1.5.E: Conservation of Momentum (Exercises)

Conceptual Questions

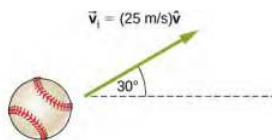
- Under what circumstances is momentum conserved?
- Can momentum be conserved for a system if there are external forces acting on the system? If so, under what conditions? If not, why not?
- Explain in terms of momentum and Newton's laws how a car's air resistance is due in part to the fact that it pushes air in its direction of motion.
- Can objects in a system have momentum while the momentum of the system is zero? Explain your answer.
- A sprinter accelerates out of the starting blocks. Can you consider him as a closed system? Explain.

Problems

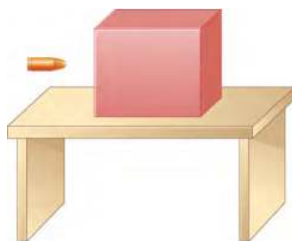
- A hockey puck of mass 150 g is sliding due east on a frictionless table with a speed of 10 m/s. Suddenly, a constant force of magnitude 5 N and direction due north is applied to the puck for 1.5 s. Find the north and east components of the momentum at the end of the 1.5-s interval.



- A ball of mass 250 g is thrown with an initial velocity of 25 m/s at an angle of 30° with the horizontal direction. Ignore air resistance. What is the momentum of the ball after 0.2 s? (Do this problem by finding the components of the momentum first, and then constructing the magnitude and direction of the momentum vector from the components.)



- Train cars are coupled together by being bumped into one another. Suppose two loaded train cars are moving toward one another, the first having a mass of $1.50 \times 10^5 \text{ kg}$ and a velocity of (0.30 m/s) in the x-direction, and the second having a mass of $1.10 \times 10^5 \text{ kg}$ and a velocity of $-(0.12 \text{ m/s})$ in the x-direction. What is their final velocity?
- The figure below shows a bullet of mass 200 g traveling horizontally towards the east with speed 400 m/s, which strikes a block of mass 1.5 kg that is initially at rest on a frictionless table. After striking the block, the bullet is embedded in the block and the block and the bullet move together as one unit. (a) What is the magnitude and direction of the velocity of the block/bullet combination immediately after the impact? (b) What is the magnitude and direction of the impulse by the block on the bullet? (c) What is the magnitude and direction of the impulse from the bullet on the block? (d) If it took 3 ms for the bullet to change the speed from 400 m/s to the final speed after impact, what is the average force between the block and the bullet during this time?



- Explain why a cannon recoils when it fires a shell.

11. Two figure skaters are coasting in the same direction, with the leading skater moving at 5.5 m/s and the trailing skating moving at 6.2 m/s. When the trailing skater catches up with the leading skater, he picks her up without applying any horizontal forces on his skates. If the trailing skater is 50% heavier than the 50-kg leading skater, what is their speed after he picks her up?
12. A 2000-kg railway freight car coasts at 4.4 m/s underneath a grain terminal, which dumps grain directly down into the freight car. If the speed of the loaded freight car must not go below 3.0 m/s, what is the maximum mass of grain that it can accept?
13. A 90.0-kg ice hockey player hits a 0.150-kg puck, giving the puck a velocity of 45.0 m/s. If both are initially at rest and if the ice is frictionless, how far does the player recoil in the time it takes the puck to reach the goal 15.0 m away?
14. A 100-g firecracker is launched vertically into the air and explodes into two pieces at the peak of its trajectory. If a 72-g piece is projected horizontally to the left at 20 m/s, what is the speed and direction of the other piece?
15. You are standing on a very slippery icy surface and throw a 1-kg football horizontally at a speed of 6.7 m/s. What is your velocity when you release the football? Assume your mass is 65 kg.
16. A 35-kg child sleds down a hill and then coasts along the flat section at the bottom, where a second 35-kg child jumps on the sled as it passes by her. If the speed of the sled is 3.5 m/s before the second child jumps on, what is its speed after she jumps on?
17. Two hockey players approach each other head on, each traveling at the same speed v_i . They collide and get tangled together, falling down and moving off at a speed $\frac{v_i}{5}$. What is the ratio of their masses?
18. A load of gravel is dumped straight down into a 30 000-kg freight car coasting at 2.2 m/s on a straight section of a railroad. If the freight car's speed after receiving the gravel is 1.5 m/s, what mass of gravel did it receive?
19. Two carts on a straight track collide head on. The first cart was moving at 3.6 m/s in the positive x direction and the second was moving at 2.4 m/s in the opposite direction. After the collision, the second car continues moving in its initial direction of motion at 0.24 m/s. If the mass of the second car is 5.0 times that of the first, what is the final velocity of the first car?
20. A 90-kg football player jumps vertically into the air to catch a 0.50-kg football that is thrown essentially horizontally at him at 17 m/s. What is his horizontal speed after catching the ball?
21. Three skydivers are plummeting earthward. They are initially holding onto each other, but then push apart. Two skydivers of mass 70 and 80 kg gain horizontal velocities of 1.2 m/s north and 1.4 m/s southeast, respectively. What is the horizontal velocity of the third skydiver, whose mass is 55 kg?
22. A kitten (3.5 kg) is running 5.0 m/s to the east and tackles a puppy (6.0 kg) who is running 2.0 m/s to the south. The kitten holds onto the puppy after the collision. Write down both the initial momentum of the kitten and the initial momentum of the puppy in column vector form. What is the velocity (magnitude and direction) of the kitten and puppy both travel after the collision?
23. A soccer ball (0.4 kg) is kicked with a speed of 20 m/s to the north. It hits a stationary basketball (0.6 kg) and bounces back with a speed of 15 m/s to the south. The basketball moves to the north after the collision. Write down both the initial momentum of the soccer ball and the initial momentum of the basketball in column vector form. What is the velocity (magnitude and direction) of the basketball after the collision?
24. A snowball (0.2 kg) is thrown with a speed of 10 m/s to the west. It collides with another snowball (0.3 kg) that is moving with a speed of 5 m/s to the north. The snowballs stick together after the collision and move as one mass. Write down both the initial momentum of the first snowball and the initial momentum of the second snowball in column vector form. What is the velocity (magnitude and direction) of the combined snowballs after the collision?
25. A baseball player (80 kg) is running with a speed of 6 m/s to the east to catch a fly ball. He collides with another player (70 kg) who is running with a speed of 5 m/s to the north. They both catch the ball (0.15 kg) and hold onto it after the collision. Write down both the initial momentum of the first player and the initial momentum of the second player in column vector form. What is the velocity (magnitude and direction) of the players and the ball after the collision?

Contributors and Attributions

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SECTION OVERVIEW

1.6: Conservation of Energy- Kinetic and Gravitational

1.6.1: Kinetic Energy

1.6.2: Conservative Interactions

1.6.3: The Inverse-Square Law

1.6.4: Conservation of Energy

1.6.5: Dissipation of Energy and Thermal Energy

1.6.6: Relative Velocity and the Coefficient of Restitution

1.6.7: Examples

1.6.E: Potential Energy and Conservation of Energy (Exercises)

Starting in this chapter we are going to move away from our first basic principle of physics (conservation of momentum), and onto **conservation of energy**. As another conservation law, the essentials are the same - there is a quantity in the system that is not changing, and by keeping track of that quantity, we can learn things about unknown aspects of the system. The things we are looking for will be similar (final speeds, for example), but since the quantity we are keeping track of is different (energy instead of momentum), the details will be different.

One major difference is that unlike momentum, *energy takes many different forms*. Momentum was only ever mass times velocity. But energy can take several different forms, and most notably can be stored without motion - something momentum cannot do¹. Here we will quickly outline two different forms that energy can take.

The first is kinetic energy, or energy of motion. This is the energy stored in a moving object, and all moving objects have it. The nice thing about kinetic energy is that there is a simple formula to describe it (kind of like for momentum!):

$$K = \frac{1}{2}mv^2. \quad (1.6.1)$$

Here m is the mass of the object, and v is the speed. Although this shares some of the same intuition as momentum ("more mass and speed, more energy"), it's important to recognize one difference - unlike momentum, *kinetic energy is a scalar and does not have direction*. So it doesn't matter what direction $\vec{v}$ is pointed in, what matters is the square of the magnitude, $|\vec{v}|^2$.

The second form of energy we want to immediately introduce is potential energy, or *energy of interaction*. This new concept describes how objects interact, and allows us to study interactions at a more fundamental level than momentum. Objects interact by storing and moving energy around to different parts of the system. For example, when you stretch a spring, you are transferring energy from your arms into the spring, which stays there until you let the spring return to its original length. Every interaction has a different way of storing energy, although sometimes we may not know how to easily describe it (friction is an example of this).

Our first example of potential energy will be gravitational potential energy near the surface of the Earth. You already know about this - when you drop something it falls! The interaction of gravity allowed you (or someone else, whoever lifted the object in the first place) to store energy in the system, and you can release it by dropping it. In this case, the energy turns into the kind we discussed before, kinetic energy. The formula for gravitational potential energy near the Earth is

$$U_g = mgh, \quad (1.6.2)$$

where m is the mass of the object, $g = 9.81 \text{ m/s}^2$ is the acceleration due to gravity, and h is the height above the reference point (often taken to be the ground).

We are going to spend a lot of time understanding these two kinds of energy, but let's start by considering the simple example of dropping an object we just mentioned. Assuming the object is not moving before you drop it, the initial energy is completely made up of U_g . As the object falls, it converts the energy from U_g into K , increasing the speed v while the height h decreases. That allows us to construct the formula,

$$mgh = \frac{1}{2}mv^2 \rightarrow v = \sqrt{2gh}, \quad (1.6.3)$$

which tells us the speed of the object when it falls from a height h . That's already a new result for us, but we should also point out that the only thing that changes how fast the object is moving is how high it's dropped from; the mass doesn't matter, and also we have no knowledge of the direction of motion (ok, yes we know in this case it's "downwards", but generally energy cannot tell us that!). This is typical for equations in energy problems - they are often pretty simple, but we have to be careful to know exactly what they can and can't tell us.

¹We have to be a little careful here - momentum can be stored in certain fields, like the electromagnetic and gravitational fields. However, understanding how that works takes a greater knowledge of the fundamental interactions than we can give here!

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1.6.1: Kinetic Energy

For a long time in the development of classical mechanics, physicists were aware of the existence of two different quantities that one could define for an object of inertia m and velocity v . One was the momentum, mv , and the other was something proportional to mv^2 . Despite their obvious similarities, these two quantities exhibited different properties and seemed to be capturing different aspects of motion.

When things got finally sorted out, in the second half of the 19th century, the quantity $\frac{1}{2}mv^2$ came to be recognized as a form of *energy*—itself perhaps the most important concept in all of physics. *Kinetic energy*, as this quantity is called, may be the most obvious and intuitively understandable kind of energy, and so it is a good place to start our study of the subject.

We will use the letter K to denote kinetic energy, and, since it is a form of energy, we will express it in the units especially named for this purpose, which is to say joules (J). 1 joule is $1 \text{ kg}\cdot\text{m}^2/\text{s}^2$. In the definition

$$K = \frac{1}{2}mv^2 \quad (1.6.1.1)$$

the letter v is meant to represent the *magnitude* of the velocity vector, that is to say, the *speed* of the particle. Hence, unlike momentum, *kinetic energy is not a vector, but a scalar*: there is no sense of direction associated with it. In three dimensions, one could write

$$K = \frac{1}{2}m(v_x^2 + v_y^2 + v_z^2) \quad (1.6.1.2)$$

There is, therefore, some amount of kinetic energy associated with each component of the velocity vector, but in the end they are all added together in a lump sum.

For a system of particles, we will treat kinetic energy as an additive quantity, just like we did for momentum, so the total kinetic energy of a system will just be the sum of the kinetic energies of all the particles making up the system. Note that, unlike momentum, this is a scalar (not a vector) sum, and most importantly, that kinetic energy is, by definition, always positive, so there can be no question of a “cancellation” of one particle’s kinetic energy by another, again unlike what happened with momentum. Two objects of equal mass moving with equal speeds in opposite directions have a total momentum of zero, but their total kinetic energy is definitely nonzero. Basically, the kinetic energy of a system can never be zero as long as there is any kind of motion going on in the system.

Kinetic Energy in Collisions

To gain some further insights into the concept of kinetic energy, and the ways in which it is different from momentum, it is useful to look at it in the same setting in which we “discovered” momentum, namely, one-dimensional collisions in an isolated system. If we look again at the collision represented in Figure 2.1.1 of Chapter 2, reproduced below,

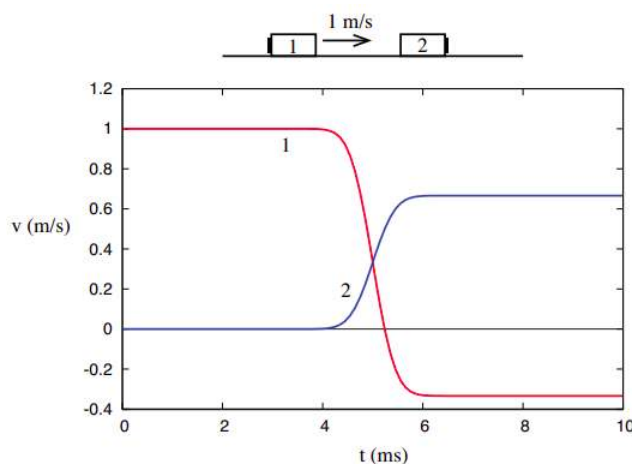


Figure 8.1.1

we can use the definition (1.6.1.1) to calculate the initial and final values of K for each object, and for the system as a whole. Remember we found that, for this particular system, $m_2 = 2m_1$, so we can just set $m_1 = 1 \text{ kg}$ and $m_2 = 2 \text{ kg}$, for simplicity. The

initial and final velocities are $v_{1i} = 1 \text{ m/s}$, $v_{2i} = 0$, $v_{1f} = -1/3 \text{ m/s}$, $v_{2f} = 2/3 \text{ m/s}$, and so the kinetic energies are

$$K_{1i} = \frac{1}{2} \text{ J}, K_{2i} = 0; \quad K_{1f} = \frac{1}{18} \text{ J}, K_{2f} = \frac{4}{9} \text{ J}.$$

Note that $1/18 + 4/9 = 9/18 = 1/2$, and so

$$K_{sys,i} = K_{1i} + K_{2i} = \frac{1}{2} \text{ J} = K_{1f} + K_{2f} = K_{sys,f}.$$

In words, we find that, in this collision, the final value of the total kinetic energy is the same as its initial value, and so it looks like we have “discovered” *another* conserved quantity (besides momentum) for this system.

This belief may be reinforced if we look next at the collision depicted in Figure 2.1.2, again reproduced below. In this collision, the second object is now moving towards the first, which is stationary.

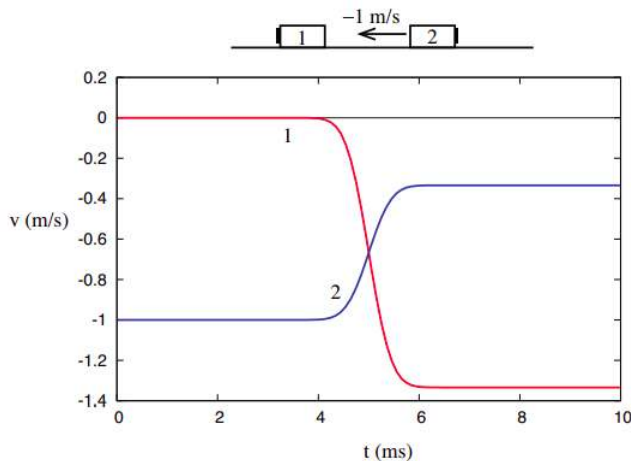


Figure 8.1.2.

The corresponding kinetic energies are, accordingly, $K_{1i} = 0$, $K_{2i} = 1 \text{ J}$, $K_{1f} = \frac{8}{9} \text{ J}$, $K_{2f} = \frac{1}{9} \text{ J}$. These are all different from the values we had in the previous example, but note that once again the total kinetic energy after the collision equals the total kinetic energy before—namely, 1 J in this case.

Things are, however, very different when we consider the third collision example shown in Chapter 2, namely, the one where the two objects are stuck together after the collision.

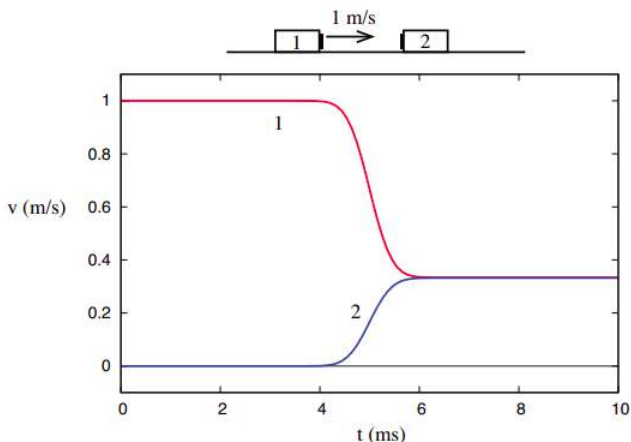


Figure 8.1.3

Their joint final velocity, consistent with conservation of momentum, is $v_{1f} = v_{2f} = 1/3 \text{ m/s}$. Since the system starts as in Figure 1.6.1.1, its kinetic energy is initially $K_{sys,i} = \frac{1}{2} \text{ J}$, but after the collision we have only

$$K_{sys,f} = \frac{1}{2} (3 \text{ kg}) \left(\frac{1}{3} \frac{\text{m}}{\text{s}} \right)^2 = \frac{1}{6} \text{ J}.$$

What this shows, however, is that unlike the total momentum of a system, which is completely unaffected by internal interactions, the total kinetic energy does depend on the details of the interaction, and thus conveys some information about its nature. We can then refine our study of collisions to distinguish two kinds: the ones where the initial kinetic energy is recovered after the collision, which we will call **elastic**, and the ones where it is not, which we call **inelastic**. A special case of inelastic collision is the one called *totally inelastic*, where the two objects end up stuck together, as in Figure 1.6.1.3. As we shall see later, the kinetic energy “deficit” is largest in that case.

Since whatever ultimately happens depends on the details and the nature of the interaction, we will be led to distinguish between “conservative” interactions, where kinetic energy is reversibly stored as some other form of energy somewhere, and “dissipative” interactions, where the energy conversion is, at least in part, irreversible. Clearly, elastic collisions are associated with conservative interactions and inelastic collisions are associated with dissipative interactions. This preliminary classification of interactions will have to be reviewed a little more carefully, however, in the next chapter.

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1.6.2: Conservative Interactions

Let's summarize the physical concepts and principles we have encountered so far in our study of classical mechanics. We have “discovered” one important quantity, the inertia or inertial mass of an object, and introduced two different quantities based on that concept, the momentum $m\vec{v}$ and the kinetic energy $\frac{1}{2}mv^2$. We found that these quantities have different but equally intriguing properties. The total momentum of a system is insensitive to the interactions between the parts that make up the system, and therefore it stays constant in the absence of external influences (a more general statement of the law of inertia, the first important principle we encountered). The total kinetic energy, on the other hand, changes while any sort of interaction is taking place, but in some cases it may actually return to its original value afterwards. This chapter deals with interactions from an energy point of view, whereas later chapters will deal with them from a force point of view.

In the previous chapter I suggested that what was going on in an elastic collision could be interpreted, or described (perhaps in a figurative way) more or less as follows: as the objects come together, the total kinetic energy goes down, but it is as if it was being temporarily stored away somewhere, and as the objects separate, that “stored energy” is fully recovered as kinetic energy. Whether this does happen or not in any particular collision (that is, whether the collision is elastic or not) depends, as we have seen, on the kind of interaction (“bouncy” or “sticky,” for instance) that takes place between the objects.

We are going to take the above description literally, and use the name *conservative interaction* for any interaction that can “store and restore” kinetic energy in this way. The “stored energy” itself—which is *not* actually kinetic energy while it remains stored, since it is not given by the value of $\frac{1}{2}mv^2$ at that time—we are going to call *potential energy*. Thus, conservative interactions will be those that have a “potential energy” associated with them, and vice-versa.

Potential Energy

Perhaps the simplest and clearest example of the storage and recovery of kinetic energy is what happens when you throw an object straight upwards, as it rises and eventually falls back down. The object leaves your hand with some kinetic energy; as it rises it slows down, so its kinetic energy goes down, down... all the way down to zero, eventually, as it momentarily stops at the top of its rise. Then it comes down, and its kinetic energy starts to increase again, until eventually, as it comes back to your hand, it has very nearly the same kinetic energy it started out with (exactly the same, actually, if you neglect air resistance).

The interaction responsible for this change in the object's kinetic energy is, of course, the gravitational interaction between it and the Earth, so we are going to say that the “missing” kinetic energy is temporarily stored as *gravitational potential energy* of the system formed by the Earth and the object. This potential energy takes a simple mathematical form,

$$U_g = mgy, \quad (1.6.2.1)$$

where y is the height of the object above the Earth. We have carefully called this height the coordinate y to emphasize that it can be negative if y is negative. It turns out that with this definition of potential energy, the total energy in freefall is

$$K + U^G = \text{constant}. \quad (1.6.2.2)$$

This is a statement of conservation of energy under the gravitational interaction. For any interaction that has a potential energy associated with it, the quantity $K + U$ is called the (total) *mechanical energy*.

Figure 1.6.2.1 shows how the kinetic and potential energies of an object thrown straight up change with time (don't worry where this plot comes from yet, you will learn about that when we study projectile motion in detail). I have arbitrarily assumed that the object has a mass of 1 kg and an initial velocity of 2 m/s, and it is thrown from an initial height of 0.5 m above the ground. Note how the change in potential energy exactly mirrors the change in kinetic energy (so $\Delta U^G = -\Delta K$, mathematically), and the total mechanical energy remains equal to its initial value of 6.9 J throughout.

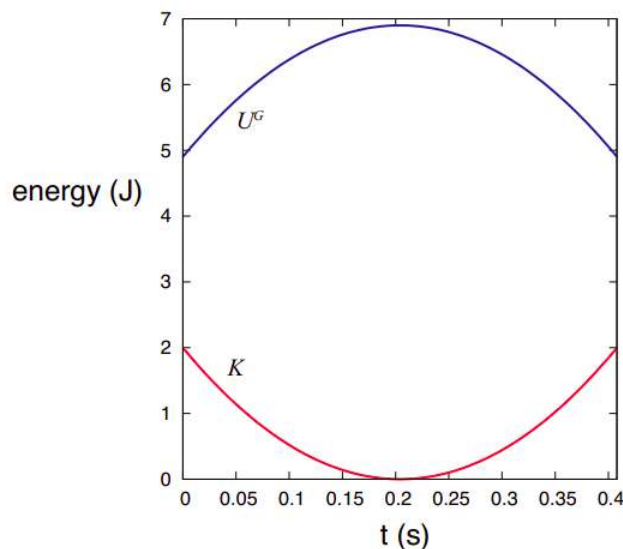


Figure 1.6.2.1: Potential and kinetic energy as a function of time for a system consisting of the earth and a 1-kg object sent upwards with $v_i = 2$ m/s from a height of 0.5 m.

There is something about potential energy that probably needs to be mentioned at this point. Because I have chosen to launch the object from 0.5 m above the ground, and I have chosen to measure y from the ground, I started out with a potential energy of $mgy_i = 4.9$ J. This makes sense, in a way: it tells you that if you simply dropped the object from this height, it would have picked up an amount of kinetic energy equal to 4.9 J by the time it reached the ground. But, actually, where I choose the vertical origin of coordinates is arbitrary. I could start measuring y from any height I wanted to—for instance, taking the initial height of my hand to correspond to $y = 0$. This would shift the blue curve in Figure 1.6.2.1 down by 4.9 J, but it would not change any of the physics. The only important thing I really want the potential energy for is to calculate the kinetic energy the object will lose or gain *as it moves from one height to another*, and for that only *changes* in potential energy matter. I can always add or subtract any (constant) number to or from U , and it will still be true that $\Delta K = -\Delta U$.

Closed Systems

Today, physics is pretty much founded on the belief that the energy of a closed system (defined as one that does not exchange energy with its surroundings—more on this in a minute) is always *conserved*: that is, internal processes and interactions will only cause energy to be “converted” from one form into another, but the total, after all the forms of energy available to the system have been carefully accounted for, will not change. This belief is based on countless experiments, on the one hand, and, on the other, on the fact that all the fundamental interactions that we are aware of do conserve a system’s total energy.

Of course, recognizing whether a system is “closed” or not depends on having first a complete catalogue of all the ways in which energy can be stored and exchanged—to make sure that there is, in fact, no exchange of energy going on with the surroundings. Note, incidentally, that a “closed” system is not necessarily the same thing as an “isolated” system: the former relates to the total energy, the latter to the total momentum. A parked car getting hotter in the sun is not a closed system (it is absorbing energy all the time) but, as far as its total momentum is concerned, it is certainly fair to call it “isolated.” Hopefully all these concepts will be further clarified when we introduce the additional auxiliary concepts of force, work, and heat.

For a closed system, we can state the principle of conservation of energy (somewhat symbolically) in the form

$$K + U + E_{\text{source}} + E_{\text{diss}} = \text{constant} \quad (1.6.2.3)$$

where K is the total, macroscopic, kinetic energy; U the sum of all the applicable potential energies associated with the system’s *internal* interactions; E_{source} is any kind of internal energy (such as chemical energy) that is *not* described by a potential energy function, but can increase the system’s mechanical energy; and E_{diss} stands for the contents of the “dissipated energy reservoir”—typically thermal energy. As with the potential energy U , the absolute value of E_{source} and E_{diss} does not (usually) really matter: all we are interested in is how much they change in the course of the process under consideration.

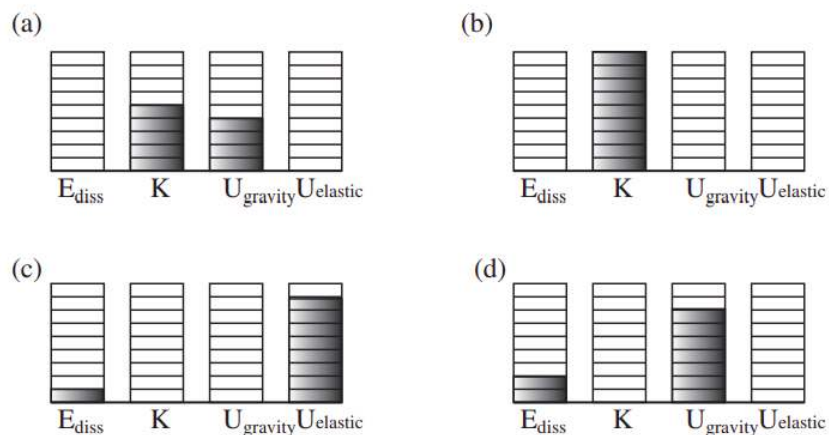


Figure 1.6.2.1: Energy bar diagrams for a system formed by the earth and a ball thrown downwards. (a) As the ball leaves the hand. (b) Just before it hits the ground. (c) During the collision, at the time of maximum compression. (d) At the top of the first bounce. The total number of energy “units” is the same in all the diagrams, as required by the principle of conservation of energy.

Figure 1.6.2.1 above is an example of this kind of “energy accounting” for a ball bouncing on the ground. If the ball is thrown down, the system formed by the ball and the earth initially has both gravitational potential energy, and kinetic energy (diagram (a)). So all the kinetic energy that we have is the kinetic energy of the ball. As the ball falls, gravitational potential energy is being converted into kinetic energy, and the ball speeds up. As it is about to hit the ground (diagram (b)), the potential energy is zero and the kinetic energy is maximum. During the collision with the ground, all the kinetic energy is temporarily converted into other forms of energy, which are essentially elastic energy of deformation (like the energy in a spring) and some thermal energy (diagram (c)). When it bounces back, its kinetic energy will only be a fraction e^2 of what it had before the collision (where e is the coefficient of restitution). This kinetic energy is all converted into gravitational potential energy as the ball reaches the top of its bounce (diagram (d)). Note there is more dissipated energy in diagram (d) than in (c); this is because I have assumed that dissipation of energy takes place both during the compression and the subsequent expansion of the ball.

Note that in this closed system, some energy was transferred to heat, but the total energy stayed the same, and was therefore conserved. The real problem appears if you can't actually *keep track of* that energy that gets transferred to heat, then the system would appear *not to* conserve energy. Mostly, we will assume no transfer to heat in our problems, but we should acknowledge this is not a problem with physics, it is rather a problem with our ability to keep track of all the sources of energy in a system. As you move through the field of physics, you will learn how we keep track of more of these sources of energy, and the problems you solve will become better and better approximations to “the real world”.

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1.6.3: The Inverse-Square Law

Up to this point, all we have said about gravity is that, near the surface of the Earth, the gravitational force exerted by the Earth on an object of mass m is $F^G = mg$, which corresponds to a potential energy of interaction $U_g = mgh$. This is, indeed, a pretty good approximation, but it does not really tell you anything about what the gravitational force is where other objects or distances are involved.

The first comprehensive theory of gravity, formulated by Isaac Newton in the late 17th century, postulates that any two “particles” with masses m_1 and m_2 will exert an attractive force (a “pull”) on each other, whose magnitude is proportional to the product of the masses, and inversely proportional to the square of the distance between them. Mathematically, we write

$$F_{12}^G = \frac{Gm_1m_2}{r_{12}^2}. \quad (1.6.3.1)$$

Here, r_{12} is just the magnitude of the position vector of particle 2 relative to particle 1 (so r_{12} is, indeed, the distance between the two particles), and G is a constant, known as “Newton’s constant” or the *gravitational constant*, which at the time of Newton still had not been determined experimentally. You can see from Equation (1.6.3.1) that G is simply the magnitude, in newtons, of the attractive force between two 1-kg masses a distance of 1 m apart. This turns out to have the ridiculously small value $G = 6.674 \times 10^{-11} \text{ N m}^2/\text{kg}^2$ (or, as is more commonly written, $\text{m}^3\text{kg}^{-1}\text{s}^{-2}$). It was first measured by Henry Cavendish in 1798, in what was, without a doubt, an experimental tour de force for that time. As you can see, gravity as a force between any two ordinary objects is absolutely insignificant, and it takes the mass of a planet to make it into something you can feel.

There is a potential energy corresponding to this force, *a la* mgh :

$$U_G = -\frac{Gm_1m_2}{r_{12}} \quad (1.6.3.2)$$

The variables are the same as in the force equation (notice the *negative derivative of the potential gives you the force* - that's a universal rule, which we will explore later). These two equations together constitute what is often called Universal Gravity, because it's the law of gravity that applies to any two objects in the universe. This should be contrasted with the more familiar $F_g = mg$ and $U_g = mgh$, which only apply near the surface of the Earth.

Since U_G should apply everywhere, it should be true that we can derive U_g from U_G , so let's show how that is done. The situation we want to understand is the gravitational interaction near the Earth - in fact, very near the Earth, so that we can write the height of the object from the surface h is much smaller than the radius of the Earth, $h \ll R_E$. Setting the two masses in equation (1.6.3.2) to the Earth M_E and the mass of the object in question m , we can write

$$U_G = -\frac{GM_Em}{R_E + h} = -\frac{GM_Em}{R_E(1 + h/R_E)} \quad (1.6.3.3)$$

Notice in going to either side of the equation sign, I have simply pulled out R_E from the expression in the denominator, those two expressions are exactly equal. But now we have that quantity h/R_E , and we've already said $h \ll R_E$, which means h/R_E is a very small value. Recall from your calculus class that for a small value x , you can write

$$\frac{1}{1+x} \approx 1 - x + x^2 - x^3 + \dots \quad (1.6.3.4)$$

(This is called a [Taylor Expansion](#), and you can check our work on [Wolfram Alpha](#).) We have to be careful to write this expression as an approximation now, since it's not exact, but it should be a good approximation for things that actually are close to the surface of the Earth - perhaps as high as Everest, for example.

Anyway, let's truncate the series after two terms (keep more if you want it to be more exact!) and write

$$U_G \approx -\frac{GM_Em}{R_E} \left(1 - \frac{h}{R_E}\right) = -\frac{GM_Em}{R_E} + \frac{GM_Em}{R_E^2}h. \quad (1.6.3.5)$$

Notice in this last expression, the first term $-GM_Em/R_E$ is a particular constant which stays the same for a particular planet (Earth) and object of mass m - let's call that C . The second term is similar, but depends linearly on the height h above the surface of the Earth. In fact, if we define a new constant $g = GM_E/R_E^2$, we can write that second term as mgh , and we have found the relationship between the two different kinds of gravitational potential energy:

$$U_G \approx C + U \quad (1.6.3.6)$$

$$U_G \approx C + U_g. \quad (1.6.3.6)$$

That's quite satisfying, but what is going on with the constant? To understand that, we can go back to something we discussed last chapter - for closed systems that do not dissipate energy, we can write $\Delta K = -\Delta U$. Notice here that's it's just the *change in energy that matters, and not the total value*. For example, if I wanted to find the change in Universal Gravitational energy I could write

$$\Delta U_G = U_{G,f} - U_{G,i} \approx (C + U_{g,f}) - (C + U_{g,i}) = U_{g,f} - U_{g,i} = \Delta U_g. \quad (1.6.3.7)$$

In other words, if we consider the change in energy, the constant doesn't matter - in fact, we usually just set this constant to be 0 in all of our problems. The conceptual understanding of that is essentially that it does not matter where we set our $h = 0$ point to be - if we drop an object 10 m, we have to get the same answer for the final speed if we consider the $h = 0$ point to be either at the bottom of the motion or the top.

It's important to keep in mind this approximation technique - it's a very powerful tool that is used all over the place in the physics. Sometimes it's to make our work easier, while having essentially no impact on the outcome of calculations, like in this case. In other cases, the exact answer is so difficult to calculate that we are forced to make approximations to get an answer at all. One example of this which is just beyond what we will study is [air drag](#), another example which is far beyond what we will study is the use of [Feynman diagrams](#) in quantum field theory.

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1.6.4: Conservation of Energy

Today, physics is pretty much founded on the belief that the energy of a closed system (defined as one that does not exchange energy with its surroundings—more on this in a minute) is always *conserved*: that is, internal processes and interactions will only cause energy to be “converted” from one form into another, but the total, after all the forms of energy available to the system have been carefully accounted for, will not change. This belief is based on countless experiments, on the one hand, and, on the other, on the fact that all the fundamental interactions that we are aware of do conserve a system’s total energy.

Of course, recognizing whether a system is “closed” or not depends on having first a complete catalogue of all the ways in which energy can be stored and exchanged—to make sure that there is, in fact, no exchange of energy going on with the surroundings. Note, incidentally, that a “closed” system is not necessarily the same thing as an “isolated” system: the former relates to the total energy, the latter to the total momentum. A parked car getting hotter in the sun is not a closed system (it is absorbing energy all the time) but, as far as its total momentum is concerned, it is certainly fair to call it “isolated.” (And as you keep this in mind, make sure you also do not mistake “isolated” for “insulated”!) Hopefully all these concepts will be further clarified when we introduce the additional auxiliary concepts of force, work, and heat (although the latter will not come until the end of the semester).

For a closed system, we can state the principle of conservation of energy (somewhat symbolically) in the form

$$K + U + E_{\text{source}} + E_{\text{diss}} = \text{constant} \quad (1.6.4.1)$$

where K is the total, macroscopic, kinetic energy; U the sum of all the applicable potential energies associated with the system’s *internal* interactions; E_{source} is any kind of internal energy (such as chemical energy) that is *not* described by a potential energy function, but can increase the system’s mechanical energy; and E_{diss} stands for the contents of the “dissipated energy reservoir”—typically thermal energy. As with the potential energy U , the absolute value of E_{source} and E_{diss} does not (usually) really matter: all we are interested in is how much they change in the course of the process under consideration.

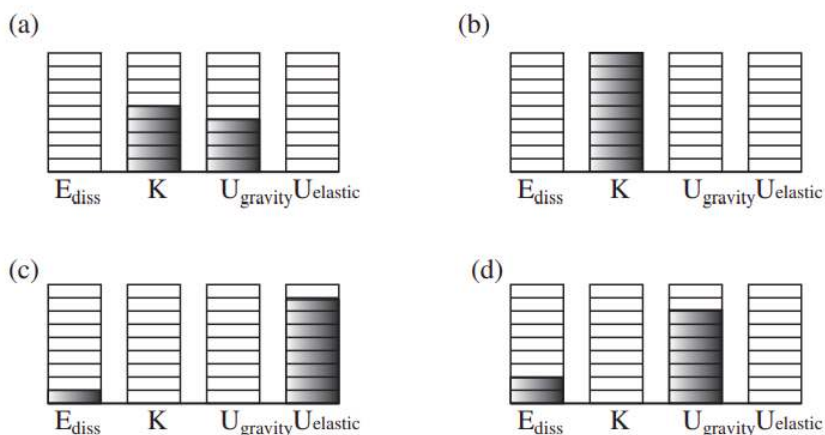


Figure 1.6.4.1: Energy bar diagrams for a system formed by the earth and a ball thrown downwards. (a) As the ball leaves the hand. (b) Just before it hits the ground. (c) During the collision, at the time of maximum compression. (d) At the top of the first bounce. The total number of energy “units” is the same in all the diagrams, as required by the principle of conservation of energy. From the diagrams you can tell that the coefficient of restitution $e = \sqrt{7/9}$.

Figure 1.6.4.1 above is an example of this kind of “energy accounting” for a ball bouncing on the ground. If the ball is thrown down, the system formed by the ball and the earth initially has both gravitational potential energy, and kinetic energy (diagram (a)). Note that we would normally need to worry about the Earth moving as well, but since the Earth is so massive, its movement is essentially zero.

As the ball falls, gravitational potential energy is being converted into kinetic energy, and the ball speeds up. As it is about to hit the ground (diagram (b)), the potential energy is zero and the kinetic energy is maximum. During the collision with the ground, all the kinetic energy is temporarily converted into other forms of energy, which are essentially elastic energy of deformation (like the energy in a spring) and some thermal energy (diagram (c)). When it bounces back, its kinetic energy will only be a fraction e^2 of what it had before the collision (where e is the coefficient of restitution). This kinetic energy is all converted into gravitational potential energy as the ball reaches the top of its bounce (diagram (d)). Note there is more dissipated energy in diagram (d) than in

(c); this is because I have assumed that dissipation of energy takes place both during the compression and the subsequent expansion of the ball.

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