PRECALCULUS: DESCRIBING RELATIONSHIPS BETWEEN QUANTITIES IN THE WORLD AROUND US



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Precalculus: Describing Relationships Between Quantities in the World Around Us

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Licensing

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1.1: Describing Relationsips Between Two Quantities with Functions

Course Goal:

To describe relationships between two quantities that change and use it to make predications and informed decisions modeling applications.

Focus: How Can We Represent a Relationship Between Two Quantities?

The natural world is full of relationships between quantities that change. Mathematics is the art of making sense of patterns and these relationships. In this setting, we may make sense of the situation by expressing the relationship between the changing quantities through words, through graphs, through data, or through a formula.

Example 1.1.1

If Jaylen throws a ball upward into the air, there is a relationship between time and the height of the ball. To describe this relationship, we first need to define variables to represent each quantity. Let t represent the time in seconds since the ball was thrown and h represent height of the ball in feet.

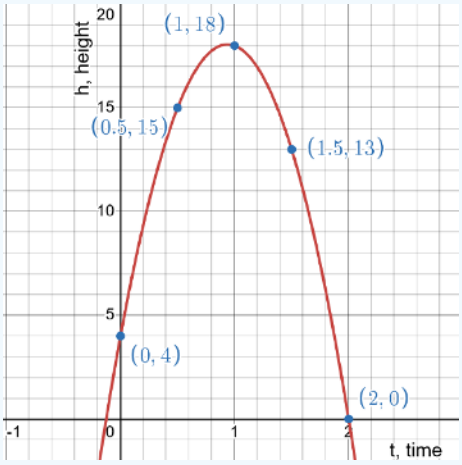
We can represent this relationship with

1. A table

The height of a ball over time since it was thrown

time t	0	0.5	1	1.5	2
height h	4	15	18	13	0

2. A graph



3. A formula or an equation

$$h = -16t^2 + 30t + 4$$

4. With words in an application

" The ball went upward after Jaylen threw before reaching its maximum height, then came dropping back down to the ground."

Notice, the graph allows us to get a quick, general idea of how the height varies relative to time over the flight of the ball. We can summarize that the ball goes upward until about 1 second then comes back down. With the formula, we are unable to quickly see how the height varies over time without evaluating the formula for height at a number of different times. However, if we need detailed information, such as what is the height at 1.5 seconds, the formula allows us to find this information exactly. With the graph, we are only able to estimate the height at 1.5 seconds. Ultimately, if we have a graph, a formula, and a

table, we will be able to more accurately describe the relationship of the height of the ball over time. The ability to describe the relationship with words allows us to communicate the relationship between height and time.

Focus: Are All Relationships Well-Defined Direct Relationships?

Notice in example 1 that the ball is at exactly one height at each point in time. It is impossible for an object to be at two heights at the same time. This is an example of a well defined direct relationship that we call a function. The height is a function of (or depends on) time.

Definition: Function

A **function** is a well-defined direct relationship between two quantities where each input is related to exactly one output.

It is useful to think of a function as a machine, where you put one thing in (the input), the machine does something to it, and then you get one thing out (the output). The output of a function is the dependent quantity. In example 1, the height is the output since the height depends on the time since the ball was thrown. The input of a function is the independent quantity. Typically, the independent quantity is one that can be intentionally manipulated or changed in an experiment. In this example, the time is the independent quantity.

Be aware that it is possible for the ball to be at the same height at two different times. For example, the ball has a height of h = 15 ft at $t \approx 0.5$ and $t \approx 1.4$, In general, one output can be related to two different inputs with a function.

Example 1.1.2

Consider the height of a person, h, over there lifetime.

a. Is height a function of age?

Solution

In this case, the age is the input and the height is the output. It would be correct to say that height is a function of age, since each age uniquely determines a height. For example, on my 16 th birthday, I had exactly one height of 75 inches.

b. Is age a function of height?

Solution

However, age is not a function of height, since one height input might correspond with more than one output age. For example, for an input height of 77 inches, there is more than one output of age since I was 77 inches at the age of 20 and 21.

Example 1.1.3

At a coffee shop, a cup of coffee costs $3. A customer orders a number of cups of coffee.

a. Is total price a function of the number of cups of coffee?

Solution

In this case, the number of cups of coffee is the input and the total price is the output. We could say that total price is a function of the number of items, since each number of cups corresponds to a single total price. For example, four cups of coffee costs a total of $12. In addition, we can build a table as well as develop a formula to further describe this relationship.

The price charged for C cups of coffee

Cups of coffee, C	1	2	3	4	5	C
Price, P	3	6	9	12	15	$P = 3C$

b. Is the number of cups of coffee a function of the total price?

Solution

In this case, the total price is the input and the number of cups of coffee is the output. Yes, the number of items is a function of the total price, since each total cost corresponds to a single number of cups. For example, of total cost of $15 corresponds to five cups of coffee.

? Now You Try: Exercise 1.1.1

Is a percentage earned in a class translated to a letter grade a function?

Answer

Yes it's a function. For example, in our class an 85% would translate to a B.

Tables as Functions

Functions can be represented in many ways: words (as we did in the last few examples), tables of values, graphs, or formulas. Represented as a table, we are presented with a list of input and output values. In some cases, these values represent everything we know about the relationship, while in other cases the table is simply providing us a few select values from a more complete relationship.

Example 1.1.4

Which of these tables define a function (if any)?

Table #1 is a relation between x and y

Input, x	Output, y
1	3
2	4
3	4
4	5

Table #2 is another relation with x and y

Input, x	Output, y
1	3
2	4
2	5
3	5

Solution

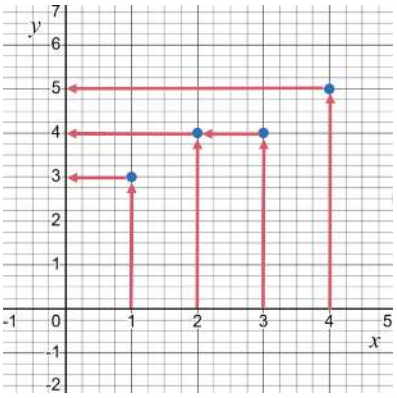
The first table defines a function since each input corresponds to exactly one output. Notice that even though two inputs $x = 2$ and $x = 3$ are related to the same output, the relationship is still a function. The second table does not define a function since the input value of $x = 2$ corresponds with two different output values $y = 4$ and $y = 2$.

Graphs as Functions

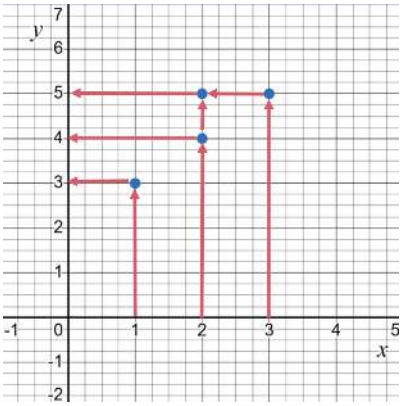
Oftentimes a graph of a relationship can be used to define a function. By convention, graphs are typically created with the input quantity along the horizontal axis and the output quantity along the vertical axis. The most common graph has y on the vertical axis and x on the horizontal axis, and we say y is a function of x.

Plotting the data points for the function described by Table 1 of example 1.1.4 on the graph below, each input on the horizontal axis corresponds a single output on the vertical axis. Moreover, if we draw a vertical line through each input on the horizontal axis, the

vertical line intersects the graph at only one point.



However, plotting the data points for the relation in Table 2 of example 1.1.4 (which was not a function) on the graph below, each input on the horizontal axis does not corresponds a single output on the vertical axis. The input $x = 2$ corresponds to two outputs $y = 4$ and $y = 5$. In this case, when we draw a vertical line $x = 2$, the vertical line intersects the graph at more than one point. This gives us a simple way to identify whether or not a graph represents a function.



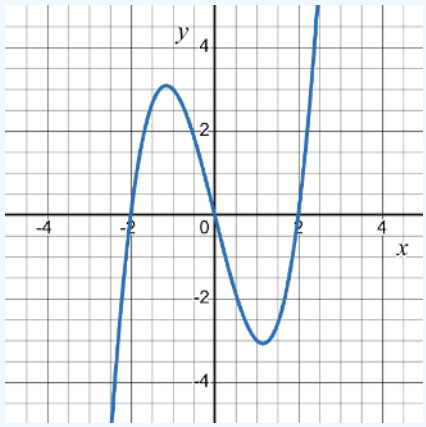
Key Point 1.1.1: The Vertical Line Test

If a vertical line can be drawn that intersects a graph in two or more points, the graph does not represent a function.

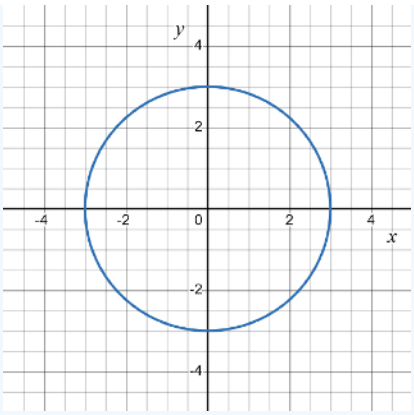
Example 1.1.5

Determine if each graph represents a function.

a. The graph of $y = x^3 - 4x$



b. The graph of the circle of radius 3 centered at the origin, $x^2 + y^2 = 9$



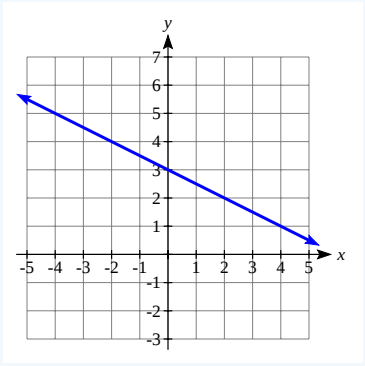
Solution

a. Since each vertical line that can be drawn intersects the graph at only one point, the graph in part (a) represents a functions by the vertical line test.

b. The graph in part b does not represent a function, since it fails the vertical line test. The vertical line $x = 1$ intersects the graph at two points, $y = \sqrt{8} \approx 2.83$ and $y = -\sqrt{8} \approx -2.83$. Therefore, the input $x = 1$ corresponds to two outputs $y = \sqrt{8} \approx 2.83$ and $y = -\sqrt{8} \approx -2.83$.

? Now You Try: Exercise 1.1.2

Determine if the graph of the line represents a function.



Answer

Yes, the graph represents a function since it passes the vertical line test. Each vertical drawn on the graph will intersect the graph at only one point.

Formulas as Functions

When possible, it is very convenient to define relationships using formulas or equations. Notice in example 1.1.5a, the equation can be written as a single formula, $y = x^3 - 4x$, for the output y. In this case, each time we input a value for x, we get a single output for y. For example, when we input $x = 1$, we get the output $y = (1)^3 - 4(1) = -7$. If we input $x = 2$, we get the output $y = (2)^3 - 4(2) = 0$ and so on. Therefore, equations that can be represented as a single formula for the output y represents a function confirming our conclusion from its graph in example 1.1.5a.

However, in example 1.1.5b, the equation $x^2 + y^2 = 9$ can't be written as a single formula for the output y. When we try to solve for y in the equation, we get two formulas for y.

$$x^2 + y^2 = 9$$

Subtracting x^2 from both sides

$$y^2 = 9 - x^2$$

There are two square roots of every positive real number, one positive and one negative, yielding two formulas for y.

$$y = \sqrt{9 - x^2} \text{ and } y = -\sqrt{9 - x^2}$$

An input such as $x = 1$ corresponds to two outputs $y = \sqrt{8}$ and $y = -\sqrt{8}$. Therefore, the equation in example 1.1.5b does not represent a function, confirming our conclusion from its graph in example 1.1.5b.

Key Point 1.1.2

An equation represents a function if it can be written as a single formula for the output y.

Example 1.1.6

Does the equation $3x + 4y = 12$ represent a function for y in terms of x?

Solution

Solving the equation as a single formula for the output y by subtracting $3x$ from both sides

$$4y = -3x + 12$$

Dividing both sides by 4 and simplifying

$$y = \frac{-3x + 12}{4} = \frac{-3x}{4} + \frac{-12}{4} = \frac{3}{4}x - 3$$

Therefore, the equation represents a function.

Now You Try: Exercise 1.1.3

Does the equation $x^2 + y^3 = 1$ represent a function?

Answer

Yes. We can solve the equation as a single formula for the output y where $y = \sqrt[3]{1 - x^2}$.

Focus: Function Notation

If we have two different functions, how can we distinguish between the inputs and outputs of each function? For example, if we have two functions represented by the equations $y = x^2$ and $y = 2x - 1$, which function has the output $y = 3$ when $x = 2$? It's hard to tell without inputting the value $x = 2$ into each equation. When we do, we find that the output $y = 3$ comes from the second equation $y = 2x - 1$.

Our strategy to address this issue is to use a letter or name to represent a function. We could name the function $y = x^2$ with the letter f. For an input, $x = 2$ whose output is $y = 4$ for this function, we write $f(2) = 4$. We say "f of 2 is 4". For an input, $x = 3$ whose output is $y = 9$ for this function, we write $f(3) = 9$. We say "f of 3 is 9". For any input x, we can represent the output of this function with $y = f(x) = x^2$. Note that the parenthesis do not mean multiplication in this context.

We could represent the other function $y = 2x - 1$ with the different letter such as g. For an input, $x = 2$ whose output is $y = 3$ for this function, we write $g(2) = 3$. We say "g of 2 is 3". For any input x, we can represent the output of the second function with $y = g(x) = 2x - 1$.

Function notation allows to distinguish between the inputs and outputs of two or more different functions. To communicate that we are dealing with the input and output of the second function $y = 2x - 1$ for $y = 3$ when $x = 2$, we need only write $g(2) = 3$.

Keypoint 1.1.3

Function notation allows us to distinguish between the inputs and outputs of two different functions.

Be careful

We are using letters in two different contexts. The letters f and g represent two different functions. The letters x and y represent variables or unknown numbers.

Example 1.1.7

A table of values for the functions $y = f(x)$ and $y = g(x)$ is given.

The inputs and outputs of functions f and g

x	0	1	2	3	4
$y = f(x)$	2	3	5	0	3
$y = g(x)$	4	2	0	2	4

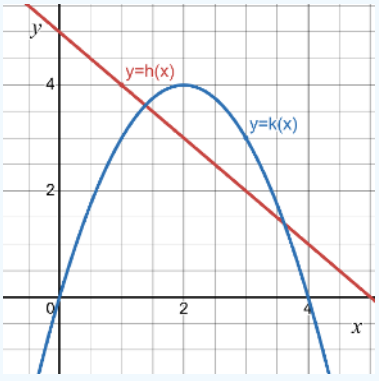
a. Evaluate $f(0)$.
b. Evaluate $f(3)$.
c. Evaluate $g(3)$.
d. Find all values of x where $f(x) = 3$.

Solution

a. $f(0) = 2$ since when $x = 0$, $y = 2$ for function f.
b. $f(3) = 0$ since when $x = 3$, $y = 0$ for function f.
c. $g(3) = 2$ since when $x = 3$, $y = 2$ for function g.
d. When $x = 1$ and $x = 4$, $f(x) = 3$.

Example 1.1.8

The graphs of the functions $y = h(x)$ and $y = k(x)$ are given.



a. Evaluate $h(1)$.
b. Evaluate $k(1)$.
c. Evaluate $k(2)$.
d. Find all values of x where $k(x) = 3$.

Solution

a. $h(1) = 4$ since when $x = 1$, $y = 4$ for function h.
b. $k(1) = 3$ since when $x = 1$, $y = 3$ for function k.
c. $k(2) = 4$ since when $x = 2$, $y = 4$ for function k.

d. When $x = 1$ and $x = 3$, $y = k(x) = 3$.

Example 1.1.9

Given the functions $f(x) = 2x + 1$ and $g(x) = x^2 + 3x$

a. Evaluate $f(2)$
b. Evaluate $g(-3)$
c. Find all values of x where $f(x) = 7$
d. Evaluate $f(4) - f(3)$

Solution

(a) To evaluate $f(2)$, we are given the value of the input $x = 2$ which we can plug into the formula for the function f wherever we see x in the formula and simplify

$$f(2) = 2(2) + 1 = 4 + 1 = 5$$

(b) To evaluate $g(-3)$, we are given the value of the input $x = -3$ which we can plug into the formula for the function g wherever we see x in the formula and simplify

$$g(-3) = (-3)^2 + 3(-3) = 9 + -9 = 0$$

(c) In this case, we are given the output $y = 7$ for the function f. We can set the formula for $y = f(x)$ equal to 1 , and solve for the value the input x that will produce that output

$$f(x) = 7$$

Substituting into the original formula

$$7 = 2x + 1$$

Subtracting 1 from each side

$$6 = 2x$$

Dividing by 2 on both sides

$$3 = x$$

You can always check your answer by using your solution in the original equation to see if your calculated answer is correct.

(d) In this part, we are trying to subtract two outputs where $y_2 = f(4)$ and $y_1 = f(3)$. We need to evaluate each function to find the outputs before we can subtract. Substituting each input into f

$$f(4) - f(3) = (2(4) + 1) - (2(3) + 1)$$

Simplifying

$$= (9) - (7)$$

Then subtracting

$$= 2$$

? Now You Try: Exercise 1.1.4

Given the functions $h(x) = 3x - 2$ and $k(x) = x^2 - 5$

a. Evaluate $h(-2)$
b. Evaluate $k(-2)$
c. Find all values of x where $h(x) = -2$
d. Evaluate $\frac{k(3)-k(1)}{2}$

Answer

(a) $h(-2) = -8$

(b) $k(-2) = -1$

(c) $x = 0$

You can always check your answer by using your solution in the original equation to see if your calculated answer is correct.

(d) $\frac{k(3)-k(1)}{2} = 3$

Focus: Inputting Variables into Functions in Function Notation

As we move forward in using functions to describe relationships between two quantities, we may want to use variables to represent the quantities that we are describing, such as in example 1.1.1 where t represented the time since the ball was thrown and h represented the height of the ball. Just like when we inputted a numerical input into a function in notation, if our input represents another variable or expression, such as $x = a$, we would input the variable a where ever we see the input x.

Given the functions $f(x) = 2x + 1$ and $g(x) = x^2 + 3x$, we can evaluate each with variable inputs. To evaluate $f(a)$, we input the variable a where ever we saw the input x in the formula for function f.

$$f(a) = 2a + 1$$

Similarly, to evaluate $g(a)$, we input the variable a in both places where the input x occurred in the formula for function g.

$$g(a) = a^2 + 3a$$

Now, inputing a box □ into function f, we get

$$f(\square) = 2(\square) + 1$$

If our input is a variable expression such as $a + h$, we can input $a + h$ where ever we see the box □, essentially letting $\square = a + h$. For example,

$$f(a + h) = 2(a + h) + 1$$

Then simplifying, we have

$$f(a + h) = 2a + 2h + 1$$

Example 1.1.10

Given the functions $k(x) = 3x - 4$, $m(x) = x^2 - 5$, and $n(x) = x^2 - 4x + 1$

a. Evaluate $k(a)$
b. Evaluate $m(a)$
c. Evaluate $k(a + h)$
d. Evaluate $m(a + h)$
e. Evaluate $\frac{k(a+h)-k(a)}{h}$
f. Evaluate $\frac{m(a+h)-m(a)}{h}$
g. Evaluate $\frac{m(x)-k(a)}{x-a}$
h. Evaluate $\frac{n(a+h)-n(a)}{h}$

Solution

a. To evaluate $k(a)$, we are given the input $x = a$ which we can plug into the formula for the function k wherever we see the input variable x.

$$k(a) = 3(a) - 4 = 3a - 4$$

b. To evaluate $m(a)$, we are given the input $x = a$ which we can plug into the formula for the function m wherever we see the input variable x.

$$m(a) = (a)^2 - 5 = a^2 - 5$$

c. To evaluate $k(a+h)$, we are given the expression $x = a+h$ for the input which we can plug into the formula for the function k wherever we see the input x.

$$k(a+h) = 3(a+h) - 4$$

Then simplifying we have

$$= 3a + 3h - 4$$

d. To evaluate $m(a+h)$, we are given the expression $x = a+h$ for the input which we can plug into the formula for the function m wherever we see the input x.

$$m(a+h) = (a+h)^2 - 5$$

Then simplifying we have

$$= (a+h)(a+h) - 5 = a^2 + 2ah + h^2 - 5$$

e. In this part, we are trying to subtract two outputs in the numerator where $y_2 = k(a+h)$ and $y_1 = k(a)$, then divide the result by h. We need to evaluate the function with each input to find the outputs before we can subtract. Substituting each input into the function k

$$\frac{k(a+h) - k(a)}{h} = \frac{(3(a+h) - 4) - (3a - 4)}{h}$$

Distributing in the numerator

$$= \frac{(3a + 3h - 4) - 3a + 4}{h}$$

Then combining like terms in the numerator and simplifying

$$= \frac{3h}{h} = 3$$

f. Similar to part (e), we need to evaluate the function with each input to find the outputs before we can subtract in the numerator. Substituting each input into the function m

$$\frac{m(a+h) - m(a)}{h} = \frac{((a+h)^2 - 5) - ((a)^2 - 5)}{h}$$

Squaring $a+h$ and distributing the minus sign in the numerator

$$= \frac{((a+h)(a+h) - 5) - a^2 + 5}{h}$$

$$= \frac{a^2 + 2ah + h^2 - 5 - a^2 + 5}{h}$$

Then combining like terms in the numerator and simplifying

$$= \frac{2ah + h^2}{h}$$

Factoring a common factor of h in the numerator

$$= \frac{h(2a+h)}{h}$$

Then simplifying by canceling a common factor

$$= 2a + h$$

g. Substituting each input into the function m

$$\frac{m(x) - m(a)}{x - a} = \frac{((x)^2 - 5) - ((a)^2 - 5)}{x - a}$$

Subtracting in the numerator

$$= \frac{x^2 - a^2}{x - a}$$

Factoring in the numerator

$$= \frac{(x - a)(x + a)}{x - a}$$

Then simplifying by canceling a common factor

$$= x + a$$

h. Substituting each input into the function n

$$\frac{n(a+h) - n(a)}{h} = \frac{((a+h)^2 - 4(a+h) + 1) - ((a)^2 - 4a + 1)}{h}$$

Performing the operations in the numerator and distributing the minus sign

$$= \frac{((a+h)(a+h) - 4a - 4h + 1) - a^2 + 4a - 1}{h}$$

Then multiplying and combining like terms in the numerator to simplify

$$= \frac{a^2 + 2ah + h^2 - 4h - a^2}{h}$$

Again, combining like terms in the numerator to simplify

$$= \frac{2ah + h^2 - 4h}{h}$$

Factoring a common factor of h in the numerator

$$= \frac{h(2a + h - 4)}{h}$$

Then simplifying by canceling a common factor we have

$$= 2a + h - 4$$

Now You Try: Exercise 1.1.5

Given the functions $r(x) = 5x + 1$, $s(x) = x^2 + 3$, and $t(x) = x^2 + 5x$

a. Evaluate $r(a)$
b. Evaluate $t(a)$
c. Evaluate $s(a + h)$
d. Evaluate $t(a + h))$
e. Evaluate $\frac{r(x)-r(a)}{x-a}$
f. Evaluate $\frac{s(a+h)-s(a)}{h}$
g. Evaluate $\frac{s(x)-s(a)}{x-a}$
h. Evaluate $\frac{t(a+h)-t(a)}{h}$

Answer

a. $r(a) = 5a + 1$
b. $t(a) = a^2 + 5a$
c. $s(a + h) = a^2 + 2ah + h^2 + 3$
d. $t(a + h) = a^2 + 2ah + h^2 + 5a + 5h$
e. $\frac{r(x)-r(a)}{x-a} = 5$
f. $\frac{s(a+h)-s(a)}{h} = 2a + h$

g. $\frac{s(x)-s(a)}{x-a} = x + a$
h. $\frac{t(a+h)-t(a)}{h} = 2a + h + 5$

Focus: Function Notation in Applications

When we work with functions in applications to describe relationships, both the input and output variables in function notation represent real world quantitiessuch as the hight of a ball or the number of infected patients as opposed to abstract unknown numbers. In this situation, it is critical that we identify which quantity is the input and which quantity is the output in order to understand what a statement in function notation is communicating in the application.

Example 1.1.11

The function $P = f(t)$ gives the population of a Clovis, California t years since 2010. The function $P = g(t)$ given the population of Fresno, California t years since 2010 (as reported by The United States Census Bureau on June 1, 2024).

a. Interpret the meaning of $f(10) = 120,674$.
b. Interpret the meaning of $g(10) = 542,159$.

Solution

a. The function f has an input t that represents the years since 2010. The input t = 10 represents 10 years since 2010 or the year 2020. The output of the function f represents the population of Clovis, California. In this example, P = 120,672 represents the population of Clovis. In summary, the statement tells us that the population of Clovis, California was 120,672 in 2020.
b. Similarly, the function g has an input t that represents the years since 2010. The input t = 10 represents 10 years since 2010 or the year 2020. The output of the function g represents the population of Fresno, California. In this example, P = 542,159 represents the population of Fresno. In summary, the statement tells us that the population of Fresno, California was 542,159 in 2020.

✔ Example 1.1.12

Suppose, the function $A(t) = 1.69t + 30.2$ approximates the amount of lumber produced in the United States (in billions of board feet) t years since 2010 (Based on data reported by The United States Department of Agriculture on June 1, 2024).

a. Interpret the meaning of $A(12) = 50.5$.
b. Find and interpret the meaning of $A(5)$.

Solution

We are given the input t = 5, which represents 5 years since 2010 or the year 2015. Since we are given a formula for the amount, we can find the output by evaluating our function at t =5.

$$A(5) = 1.69(5) + 30.2 = 38.7$$

In summary, the statement tells us that the amount of lumber production in the Unites States was 38.7 billion board feet in 2015.

a. Like the previous example, the function has an input t that represents the years since 2010. The input t = 12 represents 12 years since 2010 or the year 2022. The output 50.5 of the function A represents the billions of board feet of lumber produced. In summary, the statement tells us that the amount of lumber produced in the United States in 2022 was 50.5 billion board feet.

? Now You Try: Exercise 1.1.6

Let $C = h(A)$ represent the cost (in US dollars) of painting an area of A square feet of a house.

a. Interpret the meaning of $h(400) = 600$
b. Interpret the meaning of $h(600)$

Answer

a. This function has an input A that represents the square footage of the house to be painted. The input A=400 represents 400 square feet of the house to be painted. The output C=600 of the function h represents the cost. In summary, the statement tells us that the cost of painting an area of 400 square feet is $600.

b. In this case, we are giving the input A = 600 which represents area of the house to be painted in square feet. We are not given the value of the output, so expression represents the unknown cost of painting an area of 600 square feet.

Important Topics of this Section

- Definition of a function
- Input (independent variable)
- Output (dependent variable)
- Function notation
- Descriptive variables
- Functions in words, tables, graphs & formulas
- Vertical line test
- Evaluating a function at a specific input value
- Solving a function given a specific output value

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1.1E: Describing Relationships Between Two Quantities With Functions (Exercises)

Section 1.1 Exercises

1. Select all of the following tables which represent y as a function of x.

Relation #1 between x and y

a.

x	5	10	15
y	3	8	14

Relation #2 between x and y

b.

x	5	10	15
y	3	8	8

Relation #3 between x and y

c.

x	5	10	10
y	3	8	14

2. Select all of the following tables which represent y as a function of x.

Relation #1 between x and y

a.

x	2	6	13
y	3	10	10

Relation #2 between x and y

b.

x	2	6	6
y	3	10	14

Relation #3 between x and y

c.

x	2	6	13
y	3	10	14

3. Select all of the following tables which represent y as a function of x.

Relation #1 between x and y

a.

x	0	3	4	8	3
y	-2	1	6	9	1

Relation #2 between x and y

b.

x	-1	2	5	8	12
y	-4	3	4	7	11

Relation #3 between x and y

c.

x	0	3	3	9	16
y	-5	1	4	8	13

Relation #4 between x and y

d.

x	-1	1	4	9	12
y	-4	2	2	7	13

4. Select all of the following tables which represent y as a function of x.

Relation #1 between x and y

a.

x	-4	3	6	9	12
y	-2	2	4	7	16

Relation #2 between x and y

b.

x	-5	2	2	7	11
y	-3	1	4	9	10

Relation #3 between x and y

c.

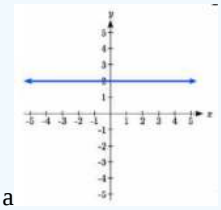
x	-1	1	5	9	1
y	-3	2	4	8	2

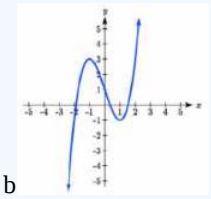
Relation #4 between x and y

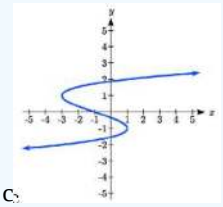
d.

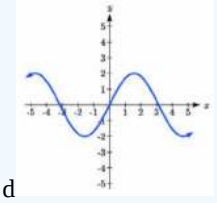
x	-1	3	5	8	14
y	-5	1	1	7	12

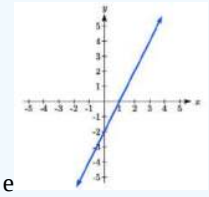
5. Select all of the following graphs which represent y as a function of x.

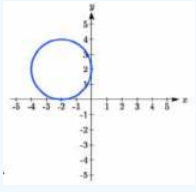
a 

b 

c 

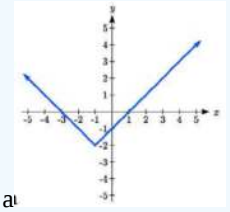
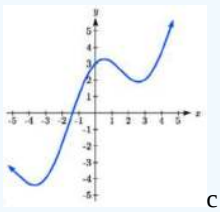
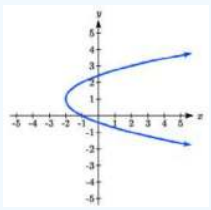
d 

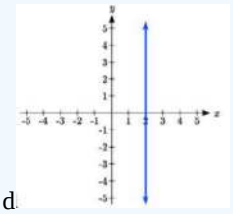
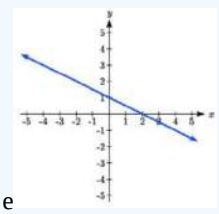
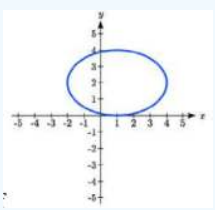
e 

f 

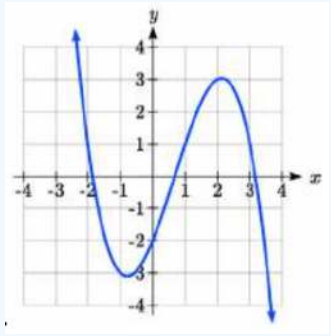
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6. Select all of the following graphs which represent y as a function of x.

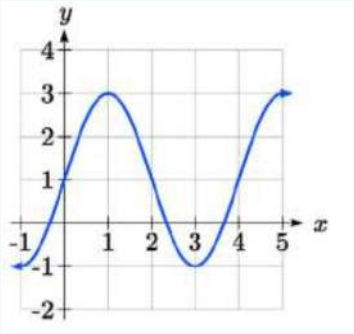
a  b  c 

d  e  f 

7. Is the sales tax on an item in California a function of the price of the item?
8. Is the daily high temperature of Clovis, California a function of the date?
9. Is the date a function of the daily high temperature of Clovis, California?
10. Determine if the following equations represent a function
    a. $5x - 2y = 10$
    b. $3x^2 + 2y^2 = 1$
    c. $3x - \sqrt{y} = 10$
11. Determine if the following equations represent a function
    a. $3x + 5y = 10$
    b. $y^2 = x$
    c. $\dfrac{3x}{y-2} = 3$
12. Given the graph of the function $f(x)$ below (a) Evaluate $f(1)$ (b) Evaluate $f(3)$ (c) Find all values of x where $f(x) = 3$

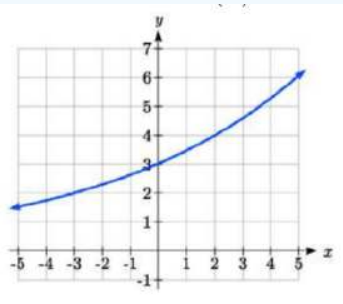


13. Given the graph of the function $g(x)$ below (a) Evaluate $g(1)$ (b) Evaluate $g(3)$ (c) Find all values of x where $g(x) = 3$

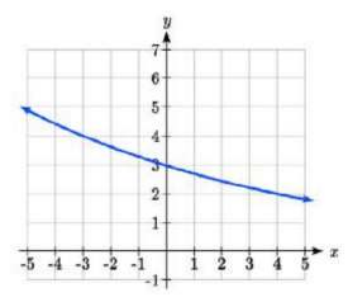


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14. Given the function $g(x)$ graphed here,
    a. Evaluate $g(2)$
    b. Solve $g(x) = 2$



14. Given the function $f(x)$ graphed here,
    a. Evaluate $f(4)$
    b. Solve $f(x) = 4$



15. Based on the table below,
    a. Evaluate $f(3)$ b. Solve $f(x) = 1$

The inputs and outputs of function f

x	0	1	2	3	4	5	6	7	8	9
$f(x)$	74	28	1	53	56	3	36	45	14	47

16. Based on the table below,
    a. Evaluate $f(8)$ b. Solve $f(x) = 1$

The inputs and outputs of function f

x	0	1	2	3	4	5	6	7	8	9
$f(x)$	62	8	7	38	86	73	70	39	75	34

17. Let $f(x) = 4 - 2x$ and $g(x) = 8x^2 - 7x + 3$.
    a. Evaluate $f(2)$
    b. Evaluate $g(2)$
    c. Evaluate $f(-1)$
    d. Evaluate $g(-1)$
    e. Find all values of x where $f(x) = 0$
    f. Evaluate $f(5) - f(3)$
    g. Evaluate $\frac{g(2) - g(1)}{2 - 1}$

18. Let $h(x) = 8 - 3x$ and $k(x) = 6x^2 - 7x + 4$.
    a. Evaluate $h(3)$

b. Evaluate $k(3)$
c. Evaluate $h(-2)$
d. Evaluate $k(-2)$
e. Find all values of x where $h(x) = 0$
f. Evaluate $h(4) + h(1)$
g. Evaluate $\dfrac{k(4) - k(1)}{4 - 1}$

19. Let $f(x) = -x^3 + 2x$ and $g(x) = 3 + \sqrt{x + 3}$
a. Evaluate $f(4)$
b. Evaluate $g(4)$
c. Evaluate $f(-4)$
d. Evaluate $g(-4)$
e. Find all values of x where $g(x) = 0$
f. Evaluate $f(6) - f(1)$
g. Evaluate $\dfrac{g(6) - g(1)}{6 - 1}$

20. Let $m(x) = 5x^4 + x^2$ and $n(x) = 4 - \sqrt[3]{x - 2}$
a. Evaluate $m(3)$
b. Evaluate $n(3)$
c. Evaluate $m(-2)$
d. Evaluate $n(2)$
e. Find all values of x where $n(x) = 0$
f. Evaluate $n(10) + n(1)$
g. Evaluate $\dfrac{m(4) - m(1)}{4 - 1}$

21. Let $f(x) = \dfrac{x - 3}{x + 1}$
a. Evaluate $f(4)$
b. Evaluate $f(0)$
c. Evaluate $f(-1)$
d. Find all values of x where $f(x) = 0$
e. Evaluate $f(4) + f(2)$

22. Let $f(x) = 3x + 2$. Evaluate and simplify each of the following
a. $f(a)$
b. $f(a + h)$
c. $\dfrac{f(x) - f(a)}{x - a}$
d. $\dfrac{f(a + h) - f(a)}{h}$

23. Let $g(x) = x^2 + 7$. Evaluate and simplify each of the following
a. $g(a)$
b. $g(a + h)$
c. $\dfrac{g(x) - g(a)}{x - a}$
d. $\dfrac{g(a + h) - g(a)}{h}$

24. Let $f(x) = x^2 + 5x$. Evaluate and simplify each of the following
a. Evaluate $f(a + h)$
b. Evaluate $\dfrac{f(a + h) - f(a)}{h}$

25. Let $f(x) = 3x^2 - 4x + 2$.
a. Evaluate and simplify $\dfrac{f(a + h) - f(a)}{h}$

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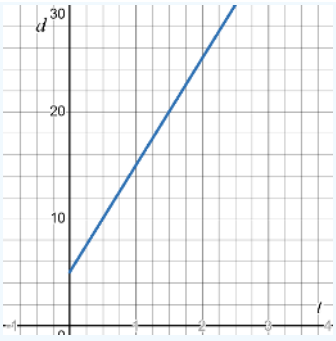
26. Let $f(x) = \frac{1}{x}$. Evaluate and simplify $\frac{f(x) - f(a)}{x - a}$.
27. Let $f(t)$ be the number of ducks in a lake *t* years after 1990. Explain the meaning of each statement:

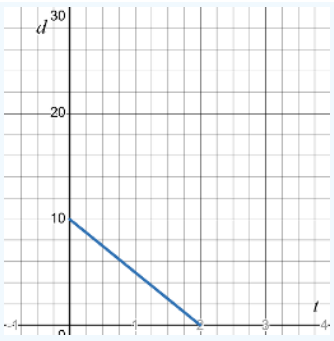
    a. $f(5) = 30$ b. $f(10) = 40$
28. Let $h(t)$ be the height above ground, in feet, of a rocket *t* seconds after launching. Explain the meaning of each statement:
    a. $h(1) = 200$ b. $h(2) = 350$
29. The amount of garbage, *G*, produced by a city with population *p* is given by $G = f(p)$. *G* is measured in tons per week, and p is measured in thousands of people. a. The town of Tola has a population of 40,000 and produces 13 tons of garbage each week. Express this information in terms of the function f.
    b. Explain the meaning of the statement $f(5) = 2$.
30. The number of cubic yards of dirt, *D*, needed to cover a garden with area *a* square feet is given by $D = g(a)$. a. A garden with area 5000 ft ^{2} requires 50 cubic yards of dirt. Express this information in terms of the function *g*.
    b. Explain the meaning of the statement $g(100) = 1$.
31. Dave leaves his office in Padelford Hall on his way to teach in Gould Hall. Below are several different scenarios. In each case, sketch a plausible (reasonable) graph of the function $s = d(t)$ which keeps track of Dave's distance *s* from Padelford Hall at time t. Take distance units to be "feet" and time units to be "minutes." Assume Dave's path to Gould Hall is long a straight line which is 2400 feet long. [UW]

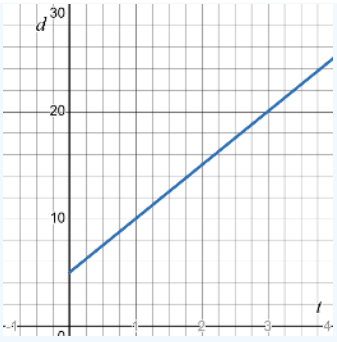


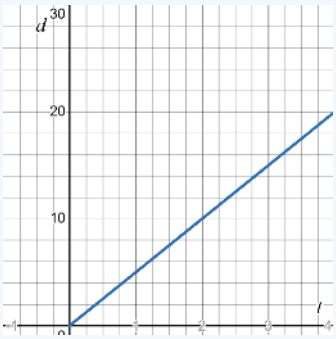
    a. Dave leaves Padelford Hall and walks at a constant spend until he reaches Gould Hall 10 minutes later.
    b. Dave leaves Padelford Hall and walks at a constant speed. It takes him 6 minutes to reach the half-way point. Then he gets confused and stops for 1 minute. He then continues on to Gould Hall at the same constant speed he had when he originally left Padelford Hall.
    c. Dave leaves Padelford Hall and walks at a constant speed. It takes him 6 minutes to reach the half-way point. Then he gets confused and stops for 1 minute to figure out where he is. Dave then continues on to Gould Hall at twice the constant speed he had when he originally left Padelford Hall.
    d. Dave leaves Padelford Hall and walks at a constant speed. It takes him 6 minutes to reach the half-way point. Then he gets confused and stops for 1 minute to figure out where he is. Dave is totally lost, so he simply heads back to his office, walking the same constant speed he had when he originally left Padelford Hall.
    e. Dave leaves Padelford heading for Gould Hall at the same instant Angela leaves Gould Hall heading for Padelford Hall. Both walk at a constant speed, but Angela walks twice as fast as Dave. Indicate a plot of "distance from Padelford" vs. "time" for the both Angela and Dave.
    f. Suppose you want to sketch the graph of a new function $s = g(t)$ that keeps track of Dave's distance s from Gould Hall at time t. How would your graphs change in (a) - (e)?
32. Match each story about a bike ride with the appropriate graph, where d represents the distance from home t hours since the start of the ride.

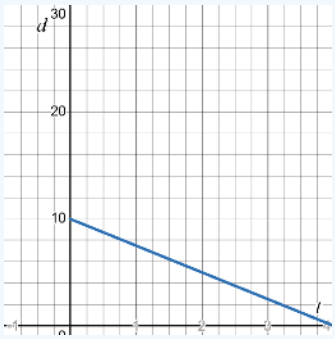
    a. The rider starts from home and rides 5 miles per hour away from home
    b. The rider starts 5 miles from home and rides 5 miles per hour away from home.
    c. The rider starts 5 miles from home and rides 10 miles per hour away from home.
    d. The rider starts 10 miles from home and rides 5 miles per hour towards their home.

I. 

III. 

V. 

II. 

VI. 

Answer

1. Tables (a) and (b) represent y as a function of x because for every value of x there is only one value for y. Table (c) is not a function because for the input $x = 10$, there are two different outputs for y.

3. Tables (a) (b) and (d) represent y as a function of x because for every value of x there is only one value for y. Table (c) is not a function because for the input $x = 3$, there are two different outputs for y. Even though the input $x = 3$ is listed twice in table (a), it is associated with only one output $y = 1$.

5. Graphs (a) (b) (d) and (e) represent y as a function of x because for every value of x there is only one value for y. Graphs (c) and (f) are not functions because they contain points that have more than one output for a given input, or values for x that have 2 or more values for y.

7. Yes, the sales tax on an item in California a function of the price of the item. For each price, there is exactly one tax charged since sales tax is typically a percentage of the price.

9. No, the date a function of the daily high temperature of Clovis, California. Two different days may have the same daily high temperature.

11. (a) Yes, the equation does represent a function since the equation can be written as a single formula for y where $y = \frac{-3}{5}x + 2$.

(b) No, the equation doesn't represent a function since the equation can't be written as a single formula for y where $y = \pm\sqrt{x}$.

(c) (a) Yes, the equation does represent a function since the equation can be written as a single formula for y where $y = \frac{3}{2}x + 2$.

13. (a) $g(1) = 3$ (b) $g(3) = -1$ (c) $x = 1$ and $x = 1$

15. (a) $f(4) = 1$ (b) $x = -2$

17. (a) $f(8) = 75$ (b) There is no solution to $f(x) = 1$

https://math.libretexts.org/@go/page/162260

19. (a) $h(3)-1$ (b) $k(3)=37$ (c) $h(-2)=14$ (d) $k(-2)=42$ (e) $x=\frac{-8}{3}$ (f) $h(4)+h(1)=1$ (g) $\frac{k(4)-k(1)}{4-1}=23$

21. (a) $m(3)=414$ (b) $n(3)=3$ (c) $m(-2)=84$ (d) $n(2)=4$ (e) $x=66$ (f) $n(10)+n(1)=7$ (g) $\frac{m(4)-m(1)}{4-1}=430$

23.(a) $f(a)=3a+2$ (b) $f(a+h)=3a+3h+2$ (c) $\frac{f(x)-f(a)}{x-a}=3$ (d) $\frac{f(a+h)-f(a)}{h}=3$

25. (a) $f(a+h)=a^2+2ah+h^2+5a+5h$ (b) $\frac{g(a+h)-g(a)}{h}=2a+h+5$

27. $\frac{f(x)-f(a)}{x-a}=\frac{-1}{ax}$

29. (a) After 1 second, the height of the rocket is 200 ft. (b) After 2 seconds, the height of the rocket is 350 ft.

31. (a) $g(5000)=50$ (b) A garden with area 100 ft ^{2} requires 1 cubic yards of dirt.

33. (a) Graph II (b) Graph V (c) Graph I (d) Graph III

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1.2: Rates of Change

Focus: Describing How Quantities are Changing

If I traveled 15 miles by foot, are you impressed how fast I went? Ultimately, it depends on how long it took, the change in time. If I went 15 miles in 3 days or 72 hours, I didn't go very fast. I may have been outpaced by a tortoise. If I went 15 miles in 3 hours, that is a decent speed but not breaking any world records. In order to effectively describe the change in distance, we need to compare it to the change in another quantity such as the change in time to give it context. We write the rate of change as a ratio (or fraction) of the changes in the two quantities.

$$\frac{\text{change in distance}}{\text{change in time}} = \frac{15 \text{ miles}}{3 \text{ hours}} = \frac{5 \text{ mi.}}{1 \text{ hr.}}$$

We can simplify this fraction to find an equivalent rate of change. Traveling 15 miles in 3 hours is equivalent to traveling 5 miles per hour for three hours.

If my annual salary increased by $10,000, are you impressed? It depends. If my annual salary increased by $10,000 over the course of my adult lifetime (about 50 - 60 years), my salary hasn't increased that much considering inflation. If my annual salary increased by $10,000 over two years, that is a good raise for many people. In order to effectively describe the change in annual salary, we need to compare it to the change in another quantity such as the change in time in order to give it context. We write the rate of change as a ratio (or fraction) of the change in annual salary to the change in time.

$$\frac{\text{change in annual salary}}{\text{change in time}} = \frac{\$10,000}{2 \text{ years}} = \frac{\$5,000}{1 \text{ yr.}}$$

We can simplify this fraction to find an equivalent rate of change. An increase in annual salary of $10,000 over two years is equivalent to an increase of $5,000 per year over two years.

Key Point: Rate of Change

A **rate of change** is a ratio that compares the change in one quantity, such as y, relative to the change in another quantity, such as x. The rate of change of change is written as

$$\text{rate of change} = \frac{\text{change in output}}{\text{change in input}} = \frac{\text{change in y}}{\text{change in x}} = \frac{\Delta y}{\Delta x}$$

We use the symbol "Δ" to indicate the change in a quantity. For example, to indicate the change in y, we write "Δy". This makes writing a rate of change much simpler.

In the first introductory example, if we let d represent the change in distance and t represent the change in time, then we could write the rate of change as

$$\frac{\Delta d}{\Delta t} = \frac{5 \text{ miles}}{1 \text{ hour}}$$

In the second introductory example, if we let s represent the change in annual salary and t represent the change in time, then we could write the rate of change as

$$\frac{\Delta s}{\Delta t} = \frac{\$5,000}{1 \text{ year}}$$

The rate of change we calculated in the first introductory example was not the rate of change at any one instant in the three hours I was traveling by foot. I could have started faster than 3 miles per hour, then slowed down to a rate slower than 3 miles per hour for the latter portion. I could have traveled at a constant rate of 3 miles per hour. I could have started slower than 3 miles per hour, then went faster than 3 miles per hour for the latter portion. The rate of change we calculated represents t**he average rate of change** of distance relative to time over the three hours we traveled. The rate of change represents the rate we would travel if we traveled at a constant rate of 3 miles per hour for the whole interval.

Example 1.2.1

A medication causes a patient's heart rate to increase by 20 bpm (beats per minute) when an additional 5 mg of the medication is given. Find the average rate of change in heart rate relative to amount of medication taken.

Solution

- Let h represent the change in heart rate and m represent the amount of medication. Then the average rate of change in heart rate relative to amount of medication taken can be written as

$$\frac{\Delta h}{\Delta m} = \frac{\text{change in heart rate}}{\text{change in medication dosage}} = \frac{20 \text{ bpm}}{5 \text{ mg}} = \frac{4 \text{ bpm}}{1 \text{ mg}}$$

  The patient's heart rate is increasing by 4 bpm per mg on average.

Some other examples of rates of change would be quantities like:

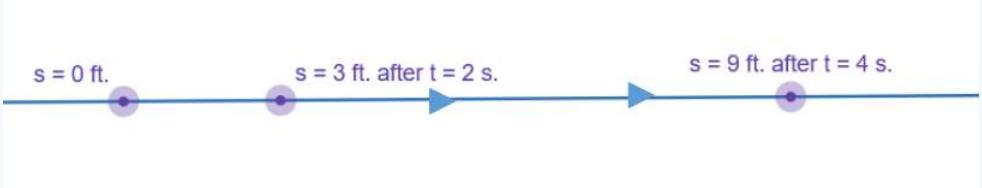
- A population of deer increases by 400 deer per year, $\frac{\Delta D}{\Delta t} = \frac{400 \text{ deer}}{1 \text{ year}}$.
- A barista earns \$9 per hour, $\frac{\Delta M}{\Delta t} = \frac{\$9}{1 \text{ hour}}$.
- A farmer plants 60,000 onions per acre, $\frac{\Delta O}{\Delta a} = \frac{60,000 \text{ onions}}{1 \text{ acre}}$.
- A car can drive 27 miles per gallon, $\frac{\Delta m}{\Delta g} = \frac{27 \text{ miles}}{1 \text{ gallon}}$.
- A population of grey whales decreases by 8 whales per year, $\frac{\Delta W}{\Delta t} = \frac{-8 \text{ whales}}{1 \text{ year}}$.
- The amount of money in your college account decreases by \$4,000 per quarter, $\frac{\Delta m}{\Delta q} \frac{-\$4,000}{1 \text{ year}}$.

Focus: Calculating Rates of Change From Data

In many cases, it may be simpler to measure data from two related quantities (or we may be given data) as opposed to measuring the changes in those two related quantities. Let's explore this situation in the following example.

Example 1.2.2

Let s represent the position of an object moving along a horizontal line after t seconds. The object was at a position of 3 feet after 2 seconts and 9 feet after 4 seconds. Find the average rate of change in position over time.



Solution

The average rate of change in position over time

$$\frac{\text{change in position}}{\text{change in time}} = \frac{\Delta s}{\Delta t} = \frac{s_{end} - s_{start}}{t_{end} - t_{start}} = \frac{9-3}{4-2} = \frac{6 \text{ ft.}}{2 \text{ s.}} = \frac{3 \text{ ft.}}{1 \text{ s.}}$$

The object's position is changing at an average rate of 3 ft. per second from $t = 2$ s to $t = 4$ s.

Notice in this example that we were not given the change in position. Instead, we were given the starting position and ending position. We calculated the change in position by subtracting the ending and starting position.

$$\Delta s = s_{end} - s_{start}$$

Key Point: Average Rate of Change

The **average rate of change** is the ratio of the change in the output y relative to the change in the input x.

$$\text{Average rate of change} = \frac{\text{Change of Output}}{\text{Change of Input}} = \frac{\Delta y}{\Delta x} = \frac{y_{end} - y_{start}}{x_{end} - x_{start}}$$

Example 1.2.3

The value V of a car after t years is given in the table below.

The value of a car after t years

t, years	0	4	6
V, Value (in US dollars)	$40,000	$24,000	$19,000

a. Find the average rate of change in value from year 0 to year 4. Interpret the meaning of the rate of change in this application.
b. Find the average rate of change in value over the interval [4,6]. Interpret the meaning of the rate of change in this application.
c. What do the two rates of change tell you about the decrease in value of the car over time? Would you expect the rate of decrease in value to be bigger or smaller in size over the interval [6,8]?

Solution

a.
$$\text{Average rate of change} = \frac{\Delta V}{\Delta t} = \frac{V_{end} - V_{start}}{t_{end} - t_{start}}$$
$$= \frac{24,000 - 40,000}{4 - 0} = \frac{-\$16,000}{4 \text{ yr.}} = \frac{-\$4,000}{1 \text{ yr.}}$$

The value of the car is decreasing at an average rate of $4,000 per year from year 0 to year 4.

b.
$$\text{Average rate of change} = \frac{\Delta V}{\Delta t} = \frac{V_{end} - V_{start}}{t_{end} - t_{start}} = \frac{19,000 - 24,000}{6 - 4} = \frac{-\$5,000}{2 \text{ yr.}} = \frac{-\$2,500}{1 \text{ yr.}}$$

The value of the car is decreasing at an average rate of $2,500 per year from year 4 to year 6.

c. The rate of decrease in value over time is getting smaller in size over time. On the interval [6,8], the rate of decrease should be smaller in size (or magnitude) than $2,500 per year, such as a decrease of $2,000 per year.

When we have successive intervals of data, such as we did in the previous example, we can find the rate of change on each interval and determine what is happening to the rate of change. Determining whether the rate of increase or decrease is getting bigger or smaller in size (or magnitude) is critical in painting an even clearer and broader picture of the relationship between the two quantities we are working with.

Notice that in the last example the change of output was *negative* since the output value of the function had decreased. Correspondingly, the average rate of change is negative.

Now You Try: Exercise 1.2.1

The number of students at Clovis Community College S during the academic year t beginning in the fall semester is given in the table below

The number of students at Clovis Community College since 2012

t, academic year	2012 (Fall 2012-Spring 2013)	2014 (Fall 2014-2015)	2016 (Fall 2016-2017)
S, number of students	7,691	9,020	10,464

(Unduplicated Student Enrollment as reported by The Clovis Community College Institutional Research Department)

a. Find the average rate of change in the number of students at Clovis Community College from 2012 to 2014. Interpret the meaning of the rate of change in this application.
b. Find the average rate of change in the number of students at Clovis Community College from 2014 to 2016. Interpret the meaning of the rate of change in this application.
c. What do the two rates of change tell you about the increase in the number of students over time? Would you expect the rate of increase in the number of students to to be bigger or smaller in size from 2016 to 2018 based on this data?

Answer

a. The number of students is increasing at an average rate of 665 students per year from 2012 to 2014.
b. The number of students is increasing at an average rate of 722 students per year from 2014 to 2016.
c. The rate of increase in the number of students over time is increasing over time. From 2016-2018, the rate of increase in the number of students should be larger 722 students per year if this trend continued. An increase of 800 students per year would be reasonable based on this data.

Focus: Calculating Rates of Change with Functions

In some cases, we may be given a formula for a function. We would want to calculate the average rate of change on various intervals with the formula in order to describing a relationship with a function.

Example 1.2.4

A ball is dropped from a high building. The distance in feet the ball has dropped after t seconds is given by the function $D(t) = 16t^2$, assuming negligible wind resistance.

a. Find the average rate of change in the distance over time from $t = 1$ second to $t = 2$ second.
b. Find the average rate of change in the distance over time over the interval [2,3].
c. What is happening to the average rate of change in distance over time? Confirm your conclusion by finding the average rate of change in the distance over time over the interval [3,4].

Solution

a. The average rate of change in distance over time on [1,2]

$$\frac{\Delta D}{\Delta t} = \frac{D_{end} - D_{start}}{t_{end} - t_{start}} = \frac{D(2) - D(1)}{2 - 1}$$

Since we are given a formula for the distance, we can evaluate the distance at our ending time $t = 2$ to find the ending distance $D(2)$. Similarly, we can evaulate $D(1)$ to find our starting distance.

$$\frac{\Delta D}{\Delta t} = \frac{16(2)^2 - 16(1)^2}{2 - 1} = \frac{64 - 16}{2 - 1} = \frac{48 \text{ ft}}{1 \text{ s}}$$

The ball is dropping at an average rate of 48 ft. per second from $t = 1$ seconds to $t = 2$ seconds. Note that the average rate of change of distance over time gives us the average speed on the interval.

b. The average rate of change in distance over time on [2,3]

$$\frac{\Delta D}{\Delta t} = \frac{D_{end} - D_{start}}{t_{end} - t_{start}} = \frac{D(3) - D(2)}{3 - 2}$$

As we did in part (a), we can find the ending and starting distances $D(3)$ and $D(2)$, by evaluating the distance at our ending time $t = 3$ and starting time $t = 2$.

$$\frac{\Delta D}{\Delta t} = \frac{16(3)^2 - 16(2)^2}{3 - 2} = \frac{144 - 64}{3 - 2} = \frac{80 \text{ ft}}{1 \text{ s}}$$

The ball is dropping at an average rate of 80 ft. per second from $t = 2$ seconds to $t = 3$ seconds.

c. The average rate of change in distance over time is increasing. Effectively, ball is speeding up as it drops. The speed should be larger that $\frac{80 \text{ ft}}{1 \text{ s}}$ the interval [3,4] . Calculating the average speed on the interval [3,4]

$$\frac{\Delta D}{\Delta t} = \frac{D_{end} - D_{start}}{t_{end} - t_{start}} = \frac{D(4) - D(3)}{4 - 3}$$

$$= \frac{16(4)^2 - 16(3)^2}{4 - 3} = \frac{256 - 144}{4 - 3} = \frac{112 \text{ ft}}{1 \text{ s}}$$

The ball is dropping at an average rate of 112 ft. per second from $t = 3$ seconds to $t = 4$ seconds confirming our conclusion.

We did many repetitive calculations in the last example. Moving forward, we would want to find the rate of change on many successive intervals to describe a relationship between two quantities. To make this process simpler, we can develop a formula for the average rate of change from a starting time $t_{start} = a$ to and ending time $t_{end} = x$. If we do this for the formula in example 1.2.4, we have

$$\text{Average rate of change} = \frac{\Delta D}{\Delta t} = \frac{D_{end} - D_{start}}{t_{end} - t_{start}} = \frac{D(x) - D(a)}{x - a} = \frac{16x^2 - 16a^2}{x - a}$$

Factoring in the numerator

$$\frac{\Delta D}{\Delta t} = \frac{16(x^2 - a^2)}{x - a} = \frac{16(x - a)(x + a)}{x - a}$$

Then simplifying, we have a simple formula for the average rate of change in distance over time.

$$\frac{\Delta D}{\Delta t} = 16(x + a)$$

Notice how much simpler it is to calculate the average rate of change on a specific interval in the following example than in example 1.2.4.

Example 1.2.5

Use the formula for the average rate of change in distance over time for the ball in example 1.2.4 to calculate the average rate of change on the intervals

a. [1,2]
b. [2,3]
c. [3,4]

Solution

a. On the interval [1,2], our starting time $t_{start} = a = 1$ and our ending time $t_{end} = x = 2$. Using our formula for the average rate of change, we have

$$\frac{\Delta D}{\Delta t} = 16(x + a) = 16(2 + 1) = 48$$

b. On the interval [2,3], our starting time $t_{start} = a = 2$ and our ending time $t_{end} = x = 3$. Using our formula for the average rate of change, we have

$$\frac{\Delta D}{\Delta t} = 16(x + a) = 16(3 + 2) = 80$$

c. On the interval [3,4], our starting time $t_{start} = a = 3$ and our ending time $t_{end} = x = 4$. Using our formula for the average rate of change, we have

$$\frac{\Delta D}{\Delta t} = 16(x + a) = 16(4 + 3) = 112$$

In general, for a function $y = f(t)$ we can develop a formula for the average rate of change from a starting value of $t_{start} = a$ and our ending value $t_{end} = x$ in a similar manner.

$$\text{Average rate of change} = \frac{\Delta y}{\Delta t} = \frac{y_{end} - y_{start}}{t_{end} - t_{start}} = \frac{f(x) - f(a)}{x - a}$$

This formula for the average rate of change on the interval $[a, x]$ is called **the difference quotient**.

Key Point: The Difference Quotient as a Formula for the Average Rate of Change

For a function $f(t)$, the difference quotient

$$\frac{\Delta y}{\Delta t} = \frac{f(x) - f(a)}{x - a}$$

is a formula for the average rate of change of f relative to t on the interval [a, x].

If we let h represent the change in t, Δt, then the ending value of t can be represented by $t_{end} = x = a + h$ because if we start at $t_{start} = a$ and increase by $h = \Delta t$ we end at $t_{end} = x = a + h$. The ending value of y can be represented as $y_{end} = f(a + h)$. Therefore, we can represent the formula for the the average rate of change on the interval $[a, x]$ in a second form

$$\text{Average rate of change} = \frac{\Delta y}{\Delta t} = \frac{y_{end} - y_{start}}{h} = \frac{f(a + h) - f(a)}{h}$$

.

Key Point: The Difference Quotient (Form 2) as a Formula for the Average Rate of Change

For a function $f(t)$, the second form of the difference quotient

$$\frac{\Delta y}{\Delta t} = \frac{f(a + h) - f(a)}{h}$$

is another formula for the average rate of change of f relative to t on the interval [a, b] where $h = \Delta t$.

There are advantages and disadvantages to each form of the difference quotient as we will see in the following examples. The first form $\frac{\Delta y}{\Delta t} = \frac{f(x) - f(a)}{x - a}$ is somewhat simpler to evaluate. The second form $\frac{\Delta y}{\Delta t} = \frac{f(a + h) - f(a)}{h}$ is easier to simplify since we only need to factor out a common factor of h in the numerator.

Example 1.2.6

Suppose that the quantity of a product sold q when a price p is charged is given by the function $q = f(p) = 100 - 2p$.

a. Evaluate and simplify the difference quotient $\frac{\Delta q}{\Delta p} = \frac{f(x) - f(a)}{x - a}$ to find a formula for the average rate of change in quantity sold relative to the price on the interval [a,x].

b. Use the difference quotient to find the average rate of change in quantity sold relative to the price on the interval [2.00,2.10]. What does the difference quotient tell you about the rate of change of the quantity sold relative to the price?

Solution

a. The average rate of change in quantity sold relative to the price

$$\frac{\Delta q}{\Delta p} = \frac{f(x) - f(a)}{x - a} = \frac{(100 - 2x) - (100 - 2a)}{x - a}$$

Subtracting in the numerator

$$\frac{\Delta q}{\Delta p} = \frac{-2x + 2a}{x - a}$$

Factoring a common factor of -2 and simplifying

$$\frac{\Delta q}{\Delta p} = \frac{-2(x - a)}{x - a} = \frac{-2 \text{ units}}{\$1}$$

b. For any interval [a,x], the average rate of change is always $\frac{-2 \text{ units}}{\$1}$. The rate of change in quantity sold is decreasing at a constant rate of 2 units per $1 increase in price.

Example 1.2.7

Given the function $y = f(t) = t^2 + 3$

a. Evaluate and simplify the first form of the difference quotient $\frac{\Delta q}{\Delta p} = \frac{f(x) - f(a)}{x - a}$ to find a formula for the average rate of change on the interval [a,x].
b. Use the formula in part (a) to find the average rate of change on the interval [2,5].
c. Use the formula in part (a) to find the average rate of change on the interval [5,9].
d. Evaluate and simplify the second form of the difference quotient $\frac{\Delta q}{\Delta p} = \frac{f(a+h) - f(a)}{h}$ to find a formula for the average rate of change on the interval [a,x] where $h = \Delta t$
e. Use the formula in part (d) to find the average rate of change on the interval [2,5].

Solution

a. The average rate of change

$$\frac{\Delta y}{\Delta t} = \frac{f(x) - f(a)}{x - a} = \frac{(x^2 + 3) - (a^2 + 3)}{x - a}$$

Subtracting in the numerator

$$\frac{\Delta y}{\Delta t} = \frac{x^2 - a^2}{x - a}$$

Then factoring in the numerator and simplifying

$$\frac{\Delta y}{\Delta t} = \frac{(x + a)(x - a)}{x - a} = x + a$$

b. On the interval [2,5], the starting value is $t_{start} = a = 2$ and ending value $t_{end} = x = 5$. Using our formula, the average rate of change is

$$\frac{\Delta y}{\Delta t} = x + a = 5 + 2 = 7$$

c. On the interval [5,9], the starting value is $t_{start} = a = 5$ and ending value $t_{end} = x = 9$. Using our formula, the average rate of change is

$$\frac{\Delta y}{\Delta t} = x + a = 9 + 5 = 14$$

d. The average rate of change

$$\frac{\Delta y}{\Delta t} = \frac{f(a+h) - f(a)}{h} = \frac{((a+h)^2 + 3) - (a^2 + 3)}{h}$$

Squaring $a + h$ and distributing the minus sign in the numerator

$$\frac{\Delta y}{\Delta t} = \frac{a^2 + 2ah + h^2 + 3 - a^2 - 3}{h}$$

Simplifying in the numerator by collecting like terms

$$\frac{\Delta y}{\Delta t} = \frac{2ah + h^2}{h}$$

Factoring our a common factor of h in the numerator and simplifying, we have

$$\frac{\Delta y}{\Delta t} = \frac{h(2a + h)}{h} = 2a + h$$

e. On the interval [2,5], the starting value is $t_{start} = a = 2$, the ending value $t_{end} = x = 5$, and $h = \Delta t \ = 5 - 2 = 3$. Using our formula, the average rate of change is

$$\frac{\Delta y}{\Delta t} = 2a + h = 2(2) + 3 = 5$$

Notice that the average rate of change on this interval was the same using either form of the difference quotient. Although the formulas from the two forms of the difference quotient are different and in terms of different variables, they will calculate the same average rate of change on a specific interval.

Example 1.2.8

A car with an initial velocity of $20\frac{m}{s}$ accelerates with a constant acceleration of $10\frac{m}{s^2}$. The distance the car has traveled after t seconds is given by $s(t) = 5t^2 + 20t$.

a. Evaluate and simplify either form of the difference quotient to find a formula for the average rate of change on the interval [a,x].
b. Use the formula in part (a) to find the average velocity on the interval [5,10].
c. Use the formula in part (a) to find the average velocity on the interval [10,15].

Solution

a. Using the second form of the difference quotient, the average rate of change

$$\frac{\Delta s}{\Delta t} = \frac{f(a+h) - f(a)}{h} = \frac{(5(a+h)^2 + 20(a+h)) - (5a^2 + 20a)}{h}$$

Squaring $a + h$ and distributing in the numerator

$$\frac{\Delta s}{\Delta t} = \frac{5(a^2 + 2ah + h^2) + 20a + 20h - 5a^2 - 20a}{h}$$

Multiplying and simplifying in the numerator by collecting like terms

$$\frac{\Delta s}{\Delta t} = \frac{5a^2 + 10ah + 5h^2 + 20h - 5a^2}{h}$$

Simplifying further in the numerator

$$\frac{\Delta s}{\Delta t} = \frac{10ah + 5h^2 + 20h}{h}$$

Factoring our a common factor of h in the numerator and simplifying, we have

$$\frac{\Delta s}{\Delta t} = \frac{h(10a + 5h + 20)}{h} = 10a + 5h + 20$$

Although it possible to use form one of the difference quotient, it is much harder to factor and simplify than the second form.

b. On the interval [5,10], the starting time is $t_{start} = a = 5$ and the change in time $h = \Delta t \ = 10 - 5 = 5$. Using our formula, the average velocity is

$$\frac{\Delta s}{\Delta t} = 10a + 5h + 20 = 10(5) + 5(5) + 20 = 95\frac{m}{s}$$

c. On the interval [10,15], the starting time is $t_{start} = a = 10$ and the change in time $h = \Delta t \ = 15 - 10 = 5$. Using our formula, the average velocity is

$$\frac{\Delta s}{\Delta t} = 10a + 5h + 20 = 10(10) + 5(5) + 20 = 145\frac{m}{s}$$

Exercise 1.2.2 Now You Try:

Given the functions $y = f(t) = 4t^2 - 1$ and $y = g(t) = 2t^2 + 3t + 1$

a. Evaluate and simplify the first form of the difference quotient $\frac{\Delta y}{\Delta t} = \frac{f(x) - f(a)}{x - a}$ to find a formula for the average rate of change of f on the interval [a,x].

b. Use the formula in part (a) to find the average rate of change on the interval [3,7].

c. Evaluate and simplify the second form of the difference quotient $\frac{\Delta q}{\Delta p} = \frac{g(a+h)-g(a)}{h}$ to find a formula for the average rate of change of g on the interval [a,x] where $h = \Delta t$

d. Use the formula in part (c) to find the average rate of change on the interval [1,4].

Answer

a. The average rate of change is $\frac{\Delta y}{\Delta t} = 4(x+a)$.

b. On the interval [3,7], the average rate of change is $\frac{\Delta y}{\Delta t} = 40$.

c. The average rate of change is $\frac{\Delta y}{\Delta t} = 4a + 2h$.

d. On the interval [1,4], the average rate of change is $\frac{\Delta y}{\Delta t} = 7$.

Important Topics of This Section

- Rate of Change
- Average Rate of Change
- Calculating Average Rate of Change using Function Notation
- The Difference Quotient

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1.2E: Rates of Change (Exercises)

Section 1.2 Exercises

1. The table below gives the annual sales (in millions of dollars) of a product each year from 1998 to 2006.
   a. What was the average rate of change of annual sales between 1998 and 2000?
   b. What was the average rate of change of annual sales between 2000 and 2002?
   c. What was the average rate of change of annual sales between 2002 and 2004?
   d. What was the average rate of change of annual sales between 2004 and 2006?
   e. Describe what is happening to the rate of change in sales over time.

The sales of a product after 1998

year t	1998	1999	2000	2001	2002	2003	2004	2005	2006
sales S	201	219	233	243	249	251	249	243	233

2. The table below gives the population of a town, in thousands each year from 2000 to 2008.
   a. What was the average rate of change of population between 2000 and 2002?
   b. What was the average rate of change of population between 2002 and 2004?
   c. What was the average rate of change of population between 2004 and 2006?
   d. What was the average rate of change of population between 2006 and 2008?
   e. Describe what is happening to the rate of change in population over time.

The population of a town in t years

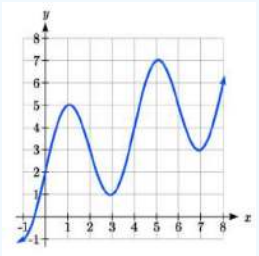
year t	2000	2002	2004	2006	2008
Population P	87	83	77	75	81

3. The value of a company t days since January 1 is given in the table below.
   a. What was the average rate of change in stock value from $t = 0$ to $t = 10$?
   b. What was the average rate of change in stock value from $t = 10$ to $t = 20$?
   c. What was the average rate of change in stock value from $t = 20$ to $t = 40$?
   d. Describe what is happening to the stock price over time

The value of a company since January 1

t days since Jan. 1	0	10	20	30	40
V, Stock Value in $	123	144	161	174	185

4. Based on the graph shown, estimate the average rate of change on each interval
   a. from $x = 1$ to $x = 4$.
   b. from $x = 2$ to $x = 5$.



5. Find the average rate of change of each function on the interval specified of $f(x) = x^2$ on [1, 5].
6. Find the average rate of change of each function on the interval specified.$q(x) = x^3$ on [-4, 2]
7. Find the average rate of change of each function on the interval specified.$g(x) = 3x^3 - 1$ on [-3, 3]

8. Find the average rate of change of each function on the interval specified.$h(x) = 5 - 2x^2$ on [-2, 4]
9. Let $f(x) = 4x^2 - 7$.
    a. Evaluate and simplify the difference quotient $\frac{f(x) - f(a)}{x - a}$
    b. Use the difference quotient to find the average rate of change of f on [1,4].
    c. Use the difference quotient to find the average rate of change of f on [4,8].
10. Let $g(x) = 2x^2 - 9$.
    a. Evaluate and simplify the difference quotient $\frac{g(a + h) - g(a)}{h}$
    b. Use the difference quotient to find the average rate of change of g on [2,5].
    c. Use the difference quotient to find the average rate of change of g on [5,9].
11. Let $k(x) = x^2 + 3x + 4$.
    a. Evaluate and simplify the difference quotient $\frac{k(a + h) - k(a)}{h}$
    b. Use the difference quotient to find the average rate of change of k on [2,5].
    c. Use the difference quotient to find the average rate of change of k on [5,9].
12. Let $f(x) = 4x^2 - 2x + 1$.
    a. Evaluate and simplify the difference quotient $\frac{f(a + h) - f(a)}{h}$
    b. Use the difference quotient to find the average rate of change of f on [-2,1].
    c. Use the difference quotient to find the average rate of change of f on [1,6].
13. Let $g(t) = \frac{1}{t + 4}$.
    1. Evaluate and simplify the difference quotient $\frac{g(x) - g(a)}{x - a}$
    2. Use the difference quotient to find the average rate of change of g on [1,5].
    3. Use the difference quotient to find the average rate of change of g on [5,12].
14. Suppose the profit of a company (in millions) that produces t million units of a product can be modeled by the function $P(t) = 10t - t^2$.
    a. Evaluate and simplify the difference quotient for $P(t)$ to find a formula for the average rate of change in profit relative to production. Use either from of the difference quotient $\frac{P(a + h) - P(a)}{h}$ or $\frac{P(x) - P(a)}{x - a}$.
    b. Use the difference quotient to find the average rate of change in profit if production increased from 1 million units to 2 million units.
    c. Use the difference quotient to find the average rate of change in profit if production increased from 2 million units to 3 million units.
    d. Use the difference quotient to find the average rate of change in profit if production increased from 3 million units to 4 million units.
    e. Use your results in parts (b)-(d) to describe what it happening to the profit with this increased production.-
15. An automobile is driven down a straight highway. It position (in feet) after t seconds can be modeled by the function $s(t) = 4.5t^2$.
    a. Evaluate and simplify the difference quotient for $P(t)$ to find a formula for the average velocity (the average rate of change in position relative to time). Use either from of the difference quotient $\frac{s(a + h) - s(a)}{h}$ or $\frac{s(x) - s(a)}{x - a}$.
    b. Use the difference quotient to find the average rate of change in position on the interval [6,7].
    c. Use the difference quotient to find the average rate of change in position on the interval [6,6.1].
    d. Use the difference quotient to find the average rate of change in position on the interval [6,6.01].
    e. Use the difference quotient to find the average rate of change in position on the interval [6,6.001].
    f. What do your results in parts (b)-(e) tell you about the velocity at the instant $t = 6$ seconds?
16. The pressure of 10 moles of an ideal gas is given by the formula $P = f(V) = \frac{240}{V}$ at the appropriate temperature. Find and interpret each of the following quantities.
    1. $f(8)$

2. $f(8)-f(5)$
3. $\dfrac{f(8)-f(5)}{8-5}$

Answer

1. (a) 16 million dollars per year (b) 8 million dollars per year (c) 0 , no change in sales (d) -8 million dollars per year (e) The increase in annual sales is getting smaller and smaller over time before eventually leveling off and then decreasing.

3. (a) \$2.1 dollars per day (b) \$1.7 dollars per day (c) \$1.3 dollars per day (d) \$1.15 dollars per day (e) The stock value is increasing at a smaller and smaller rate over time.

5. 6

7. 27

9. (a) $\dfrac{\Delta y}{\Delta x}=4(x+a)$ (b) 20 (c) 48

11. (a) $\dfrac{\Delta y}{\Delta x}=2a+h+3$ (b) 10 (c) 17

13. (a) $\dfrac{\Delta y}{\Delta x}=\dfrac{-1}{(x+4)(a+4)}$ (b) $\dfrac{-1}{45}$ (c) $\dfrac{-1}{144}$

15. (a) $\dfrac{\Delta s}{\Delta t}=4.5(x+a)$ (b) 58.5 $\dfrac{\text{ft}}{\text{s}}$ (c) 54.45 $\dfrac{\text{ft}}{\text{s}}$ (d) 54.045 $\dfrac{\text{ft}}{\text{s}}$ (e) 54.0045 $\dfrac{\text{ft}}{\text{s}}$ (f) The velocity at the instant $t=6$ seconds is 54 $\dfrac{\text{ft}}{\text{s}}$.

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1.3: Describing Relationships with Linear Functions

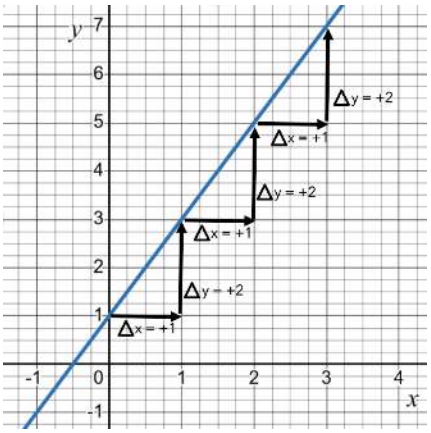
Focus: Describing Relationships with a Constant Rate of Change

Let's explore one of the most basic relationships between quantities where the rate of change is constant. For example, suppose we have a function $y = f(x)$ with a constant rate of change $\frac{\Delta y}{\Delta x} = \frac{2}{1}$ and an initial value of $y = 1$ at $x = 0$. The rate of change tells us that for each increase in the input x of 1 causes an increase in the output y of 2. Describing this relationship with a table, at

The inputs and outputs of function f

x	0	1	2	3	4
$y = f(x)$	1	3	5	7	9

Plotting each of these points we can build a graph this function.



The plotting of this data reveals that the graph of this function with a constant rate of change is a line. We call a graph whose function is a line **a linear function**. Notice how the constant rate of change is related to the graph. Since the value of the input x is on the horizontal axis, the change in x is the horizontal change. Since the value of the output y is on the vertical axis, the change in y is the vertical change. The constant rate of change is the ratio of the vertical change to the horizontal change which graphically represents the **slope** or steepness of the line.

$$\frac{\Delta y}{\Delta x} = \frac{\text{change in y}}{\text{change in x}} = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{\text{rise}}{\text{run}} = \text{slope m}$$

Since the rate of change is constant, the graph will rise at a constant rate for increase in x of 1 confirming that the graph is a line.

Definition: The Slope of a Line

The slope of line is a ratio the vertical change to the horizontal change of the graph of the line.

$$\text{slope m} = \frac{rise}{run} = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{\Delta y}{\Delta x}$$

The slope of a line measures the steepness of the line.

Keypoint 1.3.1

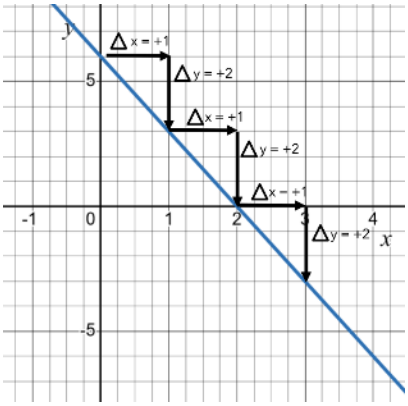
1. A function with constant rate of change is a linear function whose graph is a straight line.
2. The constant rate of change of a linear function is represented graphically by the slope of the line, $m = \frac{\Delta y}{\Delta x}$.

Exploring this concept further with a second function, $y = g(x)$ with a constant rate of change $\frac{\Delta y}{\Delta x} = \frac{-3}{1}$ and an initial value of $y = 6$ at $x = 0$. The rate of change tells us that for each increase in our input x of 1 causes a decrease in our output y of -3. Describing this relationship with a table,

The inputs and outputs of function g

x	0	1	2	3	4
$y = g(x)$	6	3	0	-3	-6

Plotting each of these points we can build a graph this function.



Similarly, we see that the graph of this second function with a constant rate of change is a also line. The constant rate of change $\frac{\Delta y}{\Delta x} = \frac{-3}{1} = m$ describes the slope of the line confirming our conclusion. Notice that this rate of change is negative causing the line to move downward and the value of y to decrease as we move along the line from left to right since the change in y is negative. In contrast, the rate of change in the first introductory example was positive causing the line to move upward and the value of y to increase as we move along the line from left to right since the change in y is positive.

Definition: An Increasing Function

A function $y = f(x)$ is **increasing** on an interval [a,b] if for any two values of x_1 and x_2 in the interval where $x_2 > x_1$, we have $f(x_2) > f(x_1)$,

Definition: An Decreasing Function

A function $y = f(x)$ is **decreasing** on an interval [a,b] if for any two values of x_1 and x_2 in the interval where $x_2 > x_1$, we have $f(x_2) < f(x_1)$,

Key Point 1.3.2

1. If a linear function $y = f(x)$ has a positive slope m, the function is increasing and the line moves upward as we move along the line from left to right.
2. If a linear function $y = f(x)$ has a negative slope m, the function is decreasing and the line moves downward as we move along the line from left to right.

Example 1.3.1

The temperature of the ocean off the coast of California was measured at various depths as shown in the table below (estimated results as reported by NOAA on June 7, 2024).

The temperature at a depth d

d, depth (in feet)	0	5	10	20
T, Temperature (in ◦ F)	60	58	56	52

Determine if the function describing the relationship between temperature and depth is linear. If so, find the slope of the line and interpret its meaning as a rate of change in the context of the application.

Solution

A function is linear if the rate of change is constant. Calculating the rate of change on each interval. On the interval [0,5],

$$m = \frac{\Delta T}{\Delta d} = \frac{58 - 60}{5 - 0} = \frac{-2}{5} = \frac{-0.4^{\circ}\text{F}}{1 \text{ ft}}$$

On the interval [5,10],

$$m = \frac{\Delta T}{\Delta d} = \frac{56 - 58}{10 - 5} = \frac{-2}{5} = \frac{-0.4^{\circ}\text{F}}{1 \text{ ft}}$$

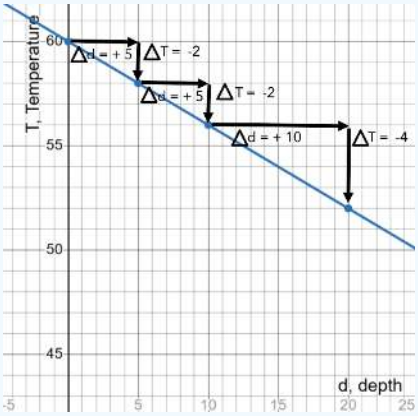
On the interval [10,20],

$$m = \frac{\Delta T}{\Delta d} = \frac{52 - 56}{20 - 10} = \frac{-4}{10} = \frac{-0.4^{\circ}\text{F}}{1 \text{ ft}}$$

The function is linear since the rate of change is constant. In the context of this application, the slope $m = \frac{-0.4^{\circ}\text{F}}{1 \text{ ft}}$ tells us that the temperature is decreasing -0.4 ◦ F per foot in depth.

Although the change in Temperature is different, ΔT = -2 versus ΔT = -4 on the intervals [0,5] and [10,20] respectively, the rate of change is the ratio of the changes in the two quantities.

A graph of the data illustrates that the graph is a line.



Example 1.3.2

The populations of two countries from 1970 - 2020 are given below (as reported by The World Bank on 6/12/24). The country of Australia experienced relatively linear growth in this time period, whereas the country of Afghanistan did not. Which country is Australia? Which country is Afghanistan?

The population of two countries since 1970

t, Year	1970	1980	1990	2000	2010	2020
Country A (in millions)	10.75	12.49	10.69	19.54	28.19	38.94
Country B (in millions)	12.51	14.69	17.07	19.03	22.03	25.65

Solution

The population is linear if the rate of change is constant. Calculating the rates of change on each interval

The rate of change in population of two countries since 1970

Time Interval	[1970,1980]	[1980,1990]	[1990,2000]	[2000,2010]	[2010,2020]
Country A, $\frac{\Delta P}{\Delta t}$ in $\frac{\text{millions}}{\text{year}}$	0.174	-0.18	0.885	0.865	1.075
Country B, $\frac{\Delta P}{\Delta t}$ in $\frac{\text{millions}}{\text{year}}$	0.218	0.238	0.196	0.300	0.362

Country B is experiencing the closest to linear growth since the rate of change only varies slightly from 0.196 to 0.362. The population of Country A increases for a time, then decreases, before increasing again. Therefore, Australia is Country A and Afghanistan is Country B.

? Now You Try: Exercise 1.3.1

The mortality rates for children under 5 years per 1,000 live births from 1995 to 2010 are given in the table below (as reported by The UN Inter-agency Group for Child Mortality Estimation on June 21, 2024)

The mortality rate for children since 1995

Year	1995	2000	2005	2010
mortality rate(in $\frac{\text{deaths}}{\text{yr}}$)	87	76	63	51

Is the mortality rate decreasing at an approximately linear rate?

Answer

Calculating the rates on change on each time interval

The rate of change in the mortality rate for children since 1995

Interval	[1995,2000]	[2000,2005]	[2005,2010]
rate of change in mortality rate per year (in $\frac{\text{deaths per year}}{\text{year}}$)	-2.2	-2.6	-2.4

Yes, the mortality rate is decreasing at an approximately linearly. The rate of decrease in mortality rate is relatively constant varying from 2.2 $\frac{\text{deaths per year}}{\text{year}}$ to 2.6 $\frac{\text{deaths per year}}{\text{year}}$ and then to 2.4 $\frac{\text{deaths per year}}{\text{year}}$ over this time interval .

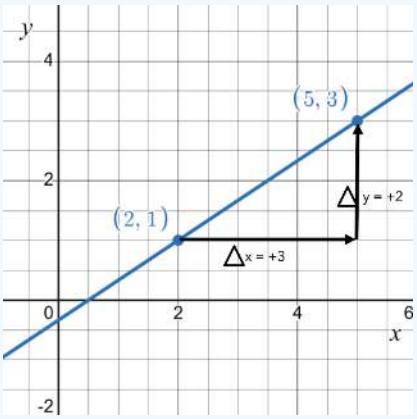
Here is a case where the analysis we or other groups can do to effect positive change in the world by analyzing a relationship. This analysis could then helps make informed decisions on the effectiveness of programs and initiatives which could further reduce infant mortality.

Focus: More on The Slope of a Line and Rates of Change

The slope of a line measures the steepness of a line. For example, if Road A climbed 135 feet over 3900 ft and Road B climbed 120 ft. over 3175 feet at a constant steepness, which is steeper? The slopes of each road allows us to quantify and compare their steepness. The slope of Road A is $m_A = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{135}{3900} \approx 0.0346$ whereas the slope of Road B is $m_B = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{120}{3175} \approx 0.0378$. Road B is steeper. It climbs 0.378 feet every foot of horizontal travel whereas Road A climbs only 0.346 feet every foot of horizontal travel. Although Road A has a greater rise, it rose over a greater horizontal run making the road less steep than Road B.

Example 1.3.3

Find the slope of a line containing the points (2,1) and (5,3).



Solution

The slope $m = \frac{\Delta y}{\Delta x} = \frac{y_{end} - y_{start}}{x_{end} - x_{start}} = \frac{3-1}{5-2} = \frac{2}{3}$

If we know that a function in an application has a constant rate of change, we calculate the slope in the same way from two data points in order to calculate the constant rate of change.

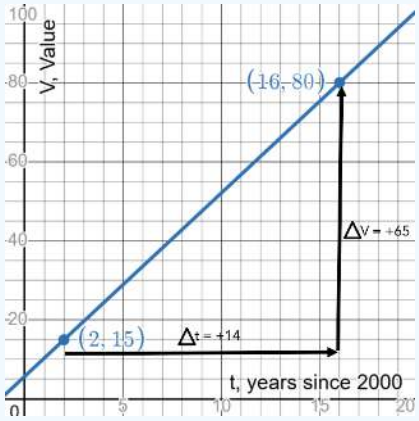
Example 1.3.4

Suppose a company's valuation was $15 million in 2002 and rose to $80 million in 2016. Assume the company's value grew at a constant rate.

a. Graph the data. Then graph a linear function relating the company's value V to the time t in years since 2000.
b. Find the slope of the line.
c. Interpret the slope as a rate of change in this application.

Solution

a. Since we assumed that the company's value grew at a constant rate, the function is linear. The graph is given below.



b. Calculating the slope

$$\text{slope m} = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{\Delta V}{\Delta t} = \frac{80-15}{16-2} = \frac{65}{14} \approx \$4.64 \frac{\text{million}}{\text{year}}$$

c. The value of the company is increasing at a constant rate of \$4.64 million dollars per year.

Example 1.3.5

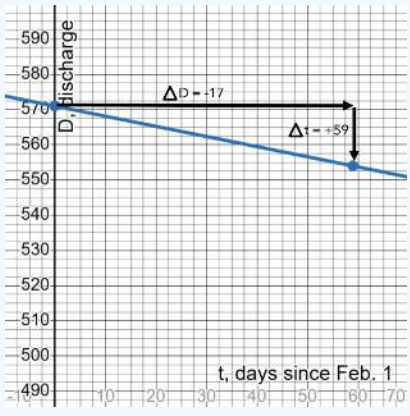
On February 1, 2024, the discharge of the San Joaquin River out of the Millerton lake dam was 571 $\frac{\text{ft}^3}{s}$. On April 1, 2024, the discharge was 554 $\frac{\text{ft}^3}{s}$ (as reported by The United States Geological Survey on June 17, 2024). Assume the rate of decrease in the discharge is constant from April through May.

a. Graph the data. Then graph a linear function relating the discharge D to the time t in days since February 1, 2024.
b. Find the slope of the line.
c. Interpret the slope as a rate of change in this application.

Solution

a. Similar to the previous example, the function is linear since we assumed that the discharge decreased at a constant rate. The graph is given below.

b.



c. Calculating the slope

$$\text{slope m} = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{\Delta D}{\Delta t} = \frac{554-571}{59-0} = \frac{-17}{59} \approx -0.288 \frac{\text{ft.}^3}{\text{day}}$$

d. The discharge out of the dam is decreasing at a rate of 0.288 ft.^3 per day.

In applications, assuming the rate of change is constant is very big assumption and shouldn't be done lightly. In reality, more data such as that of examples 1.3.1 and 1.3.2 or some underlying physical relationship would be needed to justify such a claim. In making projections into the future for certain quantities, it is reasonable to assume the rate of change is constant as one possible projection.

Now You Try: Exercise 1.3.2

Find the slope of a line containing the points (-2,3) and (3,-1).

Answer

The slope $m = \frac{\Delta y}{\Delta x} = \frac{-4}{5}$

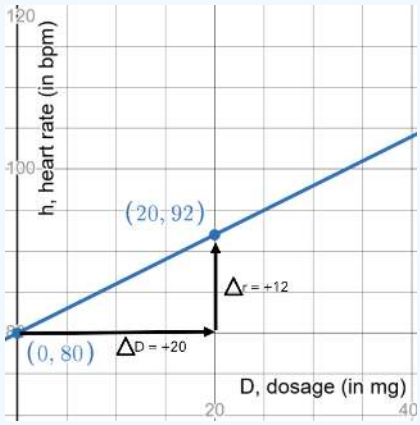
Now You Try: Exercise 1.3.3

Suppose a medication has a side effect that causes a patient's resting heart rate to increase. With no medication, the patient's resting heart rate is 80 bpm (beats per minute). If 20 mg of the medication is taken by the patient, their resting heart rate increases to 92 bpm. Assume that the patient's heart rate increases at a constant rate.

a. Graph the data. Then graph a linear function relating the patient's resting heart rate r to the dosage of the medication d in mg.
b. Find the slope of the line.
c. Interpret the slope as a rate of change in this application.

Answer

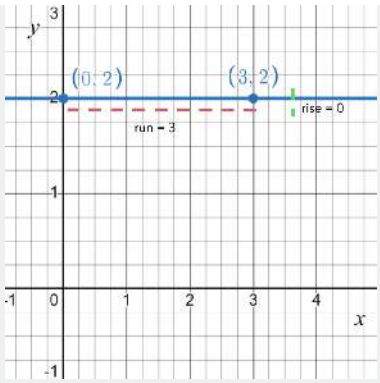
a. The function is linear since we assumed that the heart rate increased at a constant rate. The graph is given below.



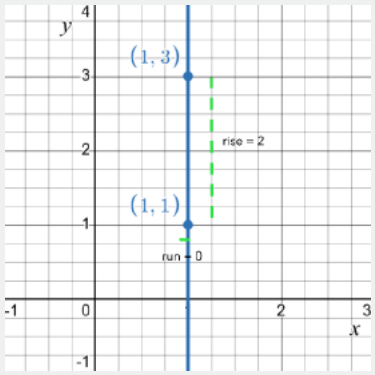
b. The slope $m = 0.6\frac{\text{bpm}}{\text{mg}}$
c. The patient's resting heart rate is increasing at a rate of 0.6 bpm per mg of the medication.

Special Case: Rates of Change and Slopes of Horizontal and Vertical Lines

For a horizontal line, such as y = 2, the slope $m = 0$ since the rise is zero. Calculating the slope from the points (0,2) and (3,2), the slope $m = \frac{\Delta y}{\Delta x} = \frac{2-2}{3-0} = \frac{0}{3} = 0$. Therefore, the average rate of change is zero and there is no change in y.



For a vertical line, such as x = 1, the slope m is undefined since the run is zero. Calculating the slope from the points (1,1) and (1,3) the slope $m = \frac{\Delta y}{\Delta x} = \frac{3-1}{1-1} = \frac{2}{0}$. There is no change in x and so we can't compare it to the change in y. Therefore, there is no rate of change that we can define.



Focus: The Formula for a Linear Function

Looking back at the introductory example of this section where we have a function $y = f(x)$ with a constant rate of change $\frac{\Delta y}{\Delta x} = \frac{2}{1}$ and an initial value of $y = 1$ at $x = 0$. Notice how we can calculate each successive value of the output y with the initial value $b = y_0 = 1$ at $x_0 = 0$ and the slope $m = 2$.

The inputs and outputs of a function f

x	0	1	2	3	4	x
y	1	1+2(1) = 3	1+2(2) = 5	1+2(3) = 7	1+2(4) = 9	1 + 2x = y

For any input x, the output of the function can be calculated with the formula $y = 1 + 2x$ or $y = 2x + 1$. Notice on the graph of this function in the introductory example that the initial value $b = y_0 = 1$ at $x_0 = 0$ is the y-coordinate of the y-intercept of the graph of the line. We summarize this relationship between the formula for a line and its graph with the following key point.

Key Point: The Formula for a Linear Function (Slope-Intercept Form)

The formula for a linear functions can be written in the form

$y = f(x) = mx + b$

where

- m is the slope of the line on the graph and the constant rate of change of the function
- b is the y-coordinate of the y-intercept on the graph and the initial value of the function at x = 0

The slope-intercept formula of a line allows us to relate the parameters of the formula to the features of the graph of the line as well as the rate of change.

Example 1.3.6

Identify the slope and y-intercept of each linear function. Then graph each line.

a. $y = 3x - 2$
b. $3x + 4y = 12$

Solution

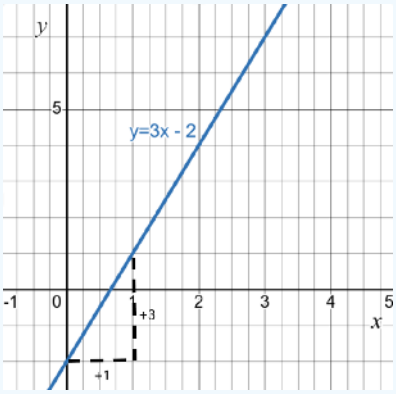
a. Since the linear equation is in slope-intercept form as formula for y, we can identify the slope $m = 3$ as the coefficient of x and the constant $b = -2$ as the y-coordinate of the y-intercept. The graph is shown below left.
b. Notice that this equation is not written as a formula for the output y. Before we can identify the slope and y-intercept, we need to solve for y. Subtracting $3x$ from both sides

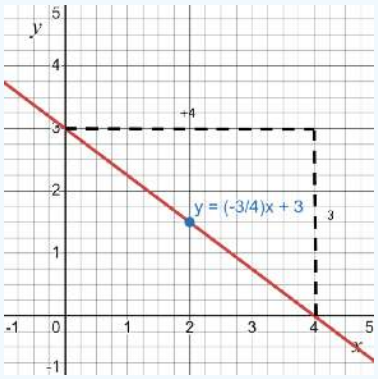
$$4y = -3x + 12$$

Then dividing both sides by 4 and simplifying

$$y = \frac{-3}{4}y + \frac{12}{4} = -\frac{3}{4}y + 3$$

Now that the equation is in slope-intercept form, we can identify the slope $m = -\frac{3}{4}$ as the coefficient of x and the constant $b = 3$ as the y-coordinate of the y-intercept. The graph is shown below right.





Example 1.3.7

Find the equation of a line with the following properties:

a. A slope $m = 3$ that contains the point (0,5)
b. A slope $m = 3$ that contains the point (1,5)

Solution

a. The point (0,5) is a y-intercept since the x-coordinate $x = 0$ and so $b = 5$. Using slope intercept form, the equation of the line is $y = 3x + 5$
b. In contrast to part (a), the point here is not a y-intercept since the x-coordinate $x \neq 0$. However, since the point $x = 1$ and $y = 5$ is on the graph, it must be a solution to the equation for the line $y = mx + b$. Substituting the known values for x, y, and m into the equation, we have

$$5 = (3)(1) + b$$

Solving for b

$$2 = b$$

Therefore, the equation of the line is

$$y = 3x + 2$$

Alternatively, if we are given the slope m and a point (x_1, y_1) that is not the y-intercept of a non-vertical line, then for any other point on the line (x, y) the slope of the line can be calculated as

$$m = \frac{\Delta y}{\Delta x} = \frac{y - y_1}{x - x_1}$$

Multiplying both sides of the equation by $(x - x_1)$, we have

$$m(x - x_1) = y - y_1 \text{ or } y - y_1 = m(x - x_1)$$

This equation can be used to find the equation of a line. It is called **the point-slope form** of the equation of a line.

Key Point: The Formula for a Linear Function (Point-Slope Form)

The equation of a line with slope m that contains the point (x_1, y_1), can be written

$$y - y_1 = m(x - x_1)$$

We could have used the point-slope form of the equation of a line to solve example 1.3.7b. Substituting the slope $m = 3$ and the point $x_1 = 1$ and $y_1 = 5$ into the point-slope equation of the line, we have

$$y - 5 = 3(x - 1)$$

Distributing

$$y - 5 = 3x - 3$$

Then solving for y, we have

$$y = 3x + 2$$

Now You Try: Exercise 1.3.4

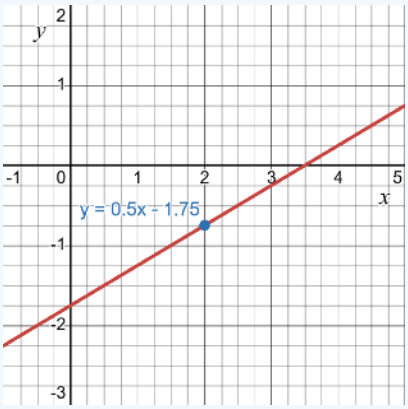
Identify the slope and y-intercept of the linear function $2x - 4y = 7$. Then graph the line.

Answer

Putting the equation in slope-intercept form

$$y = \frac{1}{2}y - \frac{7}{4} = 0.5x - 1.75$$

The slope $m = \frac{1}{2} = 0.5$ and the y-coordinate of the y-intercept is $b = -\frac{7}{4} = -1.75$.



https://math.libretexts.org/@go/page/158862

? Now You Try: Exercise 1.3.5

Find the equation of a line that contains the points (1.2, 3) and (2.4,5).

Answer

The slope

$$m = \frac{\Delta y}{\Delta y} = \frac{5}{3} \approx 1.67$$

Using the point-slope form of the equation of the line,

$$y - 3 = 1.67(x - 1.2)$$

$$y = 1.67x + 1$$

Altervatively, we could have used the slope intercept form of the equation of a line to find the equation of the line similar to how we approached example 1.7.3b.

Focus: Modeling Relationships in Applications with Linear Functions

If we are going to use a linear function to describe a relationship between two quantities, we first need to determine if the rate of change is constant. If so, a linear function is appropriate to model the relationship. If not, we will need another type of function to describe the relationship. In order to fully describe the relationship with a linear function, we need a constant rate of change, a formula, and a graph as well as a description in words to communicate our results. Once we have a formula, we can make informed predictions about the relationship. Understanding how the formula of a linear function is related to the graph allows us to build the formula from the graph itself or information about the graph. Conversely, we can build a graph from the formula.

✓ Example 1.3.8

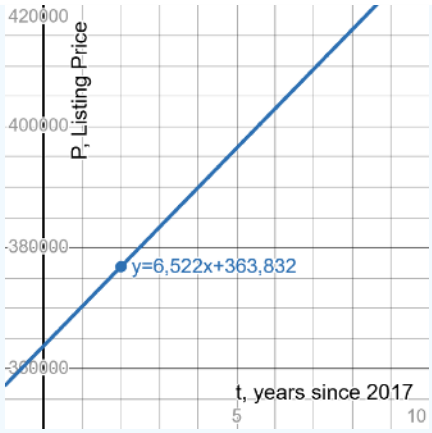
The average listing price of a home in Fresno, California was $363,832 in January 2017. The price increased at an average rate of $6,522 per year from 2017 to 2019 (as reported by The Federal Reserve Bank on June 26,2024). Assume the average listing price continues to increase at this rate.

a. Find a formula for a linear function $P = f(t)$ for the average listing price P in terms of t, the years since 2017. Then graph the function.
b. Predict the average listing price in 2025 if the price continues to increase at this rate.
c. When will the average listing price reach $400,000?

Solution

a. A linear function is appropriate to model this relationship, since we are assuming the rate of change is constant. The initial data point $t = 0$, $P = \$363,832$ is a P-intercept (or y-intercept if $y = P$). So $b = 363,832$. The constant rate of change $6,522 is our slope m. Using the slope-intercept form of the equation of a line, the formula for this linear function is

$$P = f(t) = 6,522t + 363.832$$



b. The year 2025 is $t = 8$ years after 2017. Using our formula for the average listing price

$$P = f(8) = 6,522(8) + 363,832 = 416,008$$

The average listing price of a home in Fresno, California in 2025 will be $416,008 according our model. If we are able to obtain additional data, we could determine the accuracy of the constant rate of change and our prediction.

c. Substituting the given listing price $P = 400,000$ into our formula

$$400,000 = 6,522t + 363,832$$

Solving for t by subtracting 363,832

$$36,168 = 6,522t$$

Dividing by 6,522

$$t = \frac{36,168}{6,522} \approx 5.56$$

The average listing price of a home in Fresno, California will reach $400,000 in mid-2022 if the price keeps increasing at constant rate.

Example 1.3.9

Marcus currently owns 200 songs in his iTunes collection. Every month, he adds 15 new songs. Write a formula for the number of songs, N, in his iTunes collection as a function of the number of months, m. How many songs will he own in a year?

Solution

The initial value for this function is 200, since he currently owns 200 songs, so $N(0) = 200$. The number of songs increases at a constant rate of 15 songs per month which represents the slope of the line on the graph. With this information, we can write the formula:

$$N(m) = 200 + 15m$$

$N(m)$ is an increasing linear function. With this formula we can predict how many songs he will have in 1 year (12 months):

$$N(12) = 200 + 15(12) = 200 + 180 = 380$$

Marcus will have 380 songs in 12 months.

Example 1.3.10

The pressure, P, in pounds per square inch (PSI) on a diver depends upon their depth below the water surface, d, in feet, following the equation $P(d) = 14.696 + 0.434d$. Interpret the parameters of this function in the context of this application.

Solution

The rate of change, or slope, $m = 0.434$ has units $m = \frac{\Delta P}{\Delta d} = \frac{\text{PSI}}{\text{ft}}$. This tells us the pressure on the diver increases by 0.434 PSI for each foot of depth.

The initial value, 14.696, will have the same units as the output, so this tells us that at a depth of 0 feet, the pressure on the diver will be 14.696 PSI.

Example 1.3.11

The cumulative percentage of adults vaccinated with the updated 2023-2024 COVID-19 vaccine was 14.7% in December 2023 and grew to 21.1% in February 2024 (as reported by The CDC vaccines on June 26,2024). Assume the percentage of adults vaccinated increases at a constant rate.

a. Find a formula for a linear function $P = f(t)$ for the cumulative percentage of adults vaccinated in terms of t, the months since December 2023.
b. Interpret the slope as a rate of change.
c. Predict the cumulative percentage of adults vaccinated in June 2024, if the vaccination continues to increase at this rate.

Solution

a. A linear function is appropriate to model this relationship, since we are assuming the rate of change is constant. The initial data corresponds to the point $t = 0$, $P = 14.7$ which is a P-intercept (or y-intercept if $y = P$). So $b = 14.7$. We can use the two data points, $(t, P) = (0, 14.7)$ and $(t, P) = (2, 21.1)$ to calculate the slope of the line.

$$m = \frac{\Delta P}{\Delta t} = \frac{21.1 - 14.7}{2 - 0} = \frac{6.4}{2} = 3.2$$

Using the slope-intercept form of the equation of a line, the formula for this linear function is

$$P = f(t) = 3.2t + 14.7$$

b. Notice the units of the slope as a rate of change

$$m = \frac{\Delta P}{\Delta t} = \frac{}{\text{month}}$$

The slope tells us that the cumulative percentage of adults getting the updated COVID-19 vaccine is increasing at an average rate of 3.2% per month over this period.

c. June 2024 corresponds to $t = 6$ months since December 2023. Using our formula,

$$P = 3.2(6) + 14.7 = 33.9$$

The cumulative percentage of adults vaccinated with the updated 2023-2024 COVID-19 vaccine will 33.9% in June 2024 according our model. If additional data became available that showed the rate of increase began to get smaller, our prediction would be an overestimate.

Now You Try: Exercise 1.3.6

In the winter of 2020, the United States was experiencing the Omicron wave of the COVID pandemic. In the week of October 3, 2020, the number of deaths in the US from COVID-19 was 4,240 per week. Nine weeks later, in the week of December 5, 2020 the number of deaths was 18,545 per week (as reported by The CDC on June 26,2024). At that point in time, no further data was available. In order to predict future weekly COVID death rates moving forward at that time, assume the number of deaths increases at a constant rate.

a. Find a formula for a linear function $D = f(t)$ for the number of deaths from COVID-19 per week in terms of t, the weeks since October, 3 2020.
b. Interpret the slope as a rate of change.
c. Predict the the number of weekly deaths from COVID-19 in the week of February 6, 2021, 19 weeks from October 3, 2020 if the vaccination continues to increase at this rate.

Answer

a. The D-intercept (or y-intercept if $y = D$) is $b = 4,240$. The slope is $m = \frac{\Delta D}{\Delta t} = \frac{14,305}{9} \approx 1,589$. The formula for this linear function is

$$D = f(t) = 1,589t + 4,240$$

b. Notice the units of the slope as a rate of change

$$m = \frac{\Delta D}{\Delta t} = \frac{\text{weekly deaths}}{\text{week}}$$

The slope tells us that the the number of deaths from COVID-19 per week was increasing at an average rate of 1,589 per week over this period.

c. In 19 weeks, the number number of weekly deaths from COVID-19 in the week of February 6, 2021 was 34,431 according our model.

Although the rate of increase in deaths actually increased over this period, it eventually slowed down and the COVID deaths per week peaked during the week of January 9, 2021 at 25,924 before dropping. At that point in time, medical, governmental, business, and other organizations needed informed predictions in order to plan an effective response. The modeling we have done is a basic type of modeling that could have been used as a portion of that response.

Many relationships between two quantities do not have a rate of change that is constant as we saw in the last exercise. Moving forward, we need to explore and analyze other functions to help us describe relationships where the rate of change is not constant. This will encompass our broader goal for the entirety of the remainder of the text.

Important Topics of this Section

- Definition of a linear function
- Formula of a linear function
- Increasing & Decreasing linear functions
- Finding the y-intercept
- Finding the slope/rate of change, *m*
- Modeling in applications with linear functions

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1.3E: Describing Relationships With Linear Functions (Exercises)

Section 1.3 Exercises

1. Which of the following tables could represent a linear function?

The inputs and outputs of function g

x	0	5	10	15
g(x)	5	-10	-25	-40

The inputs and outputs of function h

x	0	5	10	15
h(x)	5	30	105	230

The inputs and outputs of function f

x	0	5	10	15
f(x)	-5	20	45	70

The inputs and outputs of function k

x	5	10	20	25
k(x)	13	28	58	73

2. Which of the following tables could represent a linear function?

The inputs and outputs of function g

x	0	2	4	6
g(x)	6	-19	-44	-69

The inputs and outputs of function h

x	2	4	8	10
h(x)	13	23	43	53

The inputs and outputs of function f

x	2	4	6	8
f(x)	-4	16	36	56

The inputs and outputs of function k

x	0	2	6	8
k(x)	6	31	106	231

3. Suppose a patient's heart rate was measured after different doses of a trial medication was administered as detailed in the table below. Determine if the heart rate is increasing linearly relative to the dosage. Justify your conclusion.

The heart rate of a patient with different doses of a medication

Dosage (in mg)	0	5	10	20
Heart Rate (in bpm)	80	100	115	130

4. Suppose a steel cylindrical steel alloy bar is being pulled in tension along its length with varying loads. The amount of the bar stretches (deflection) is measured. The stress on the steel is the amount of force applied per square unit of cross-sectional area. The strain on the material is the amount of deflection (or stretching) of the material per unit in length. The stress and strain for the steel alloy bar is shown below. Determine if the material stress is increasing linearly relative to the strain. Justify your conclusion.

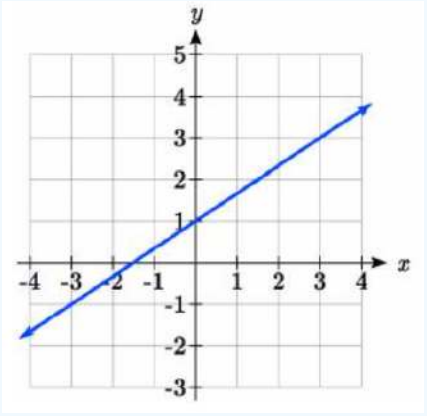
The stress of a steel bar relative to the applied strain

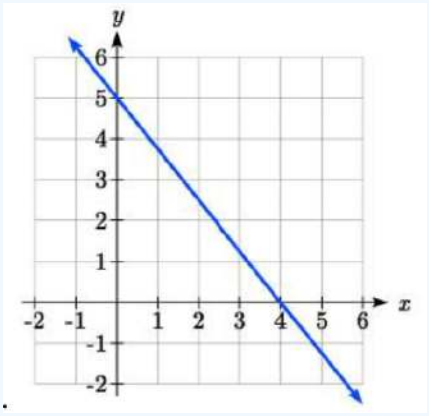
Strain (in inches per inch)	0	5	10	15	20	30
Stress (in pounds per square inch)	0	14229	28508	42737	56916	85374

5. Find the slope of the line that passes through the points (2, 4) and (4, 10).

6. Find the slope of the line that passes through the points (-1, 4) and (5, 2).

7. Find the slope of the lines graphed below.

a) 

b) 

8. The Consumer Price Index (CPI) measure the 12-month percentage change in the price of various items. The 12-month percentage change (or CPI) in Energy in the US was 32.95% in July 2022 and dropped to 17.6% by October 2022 (as reported by The US Bureau of Labor Statistics on July 29, 2024). The CPI was decreasing at a relatively constant rate per month in this time period. Find the slope of the line connecting these data points and interpret it as a rate of change.

9. The 12-month percentage change (or CPI) for all items in the US was 6.2% in October 2021 and increased to 8.6% by May 2022 (as reported by The US Bureau of Labor Statistics on July 29, 2024). The CPI was increasing at a relatively constant rate per month in this time period. Find the slope of the line connecting these data points and interpret it as a rate of change.

10. Determine the slope and y-intercept of the line $f(x) = 4x + 3$. Is the function is increasing or decreasing?

11. Determine the slope and y-intercept of the line $2x - 7y = 1$. Is the function is increasing or decreasing?

12. Determine the slope and y-intercept of the line $3x + 5y = 8$. Is the function is increasing or decreasing?

13. Find the equation of a line with a slope $m = 3$ and y-intercept at (0,-2).

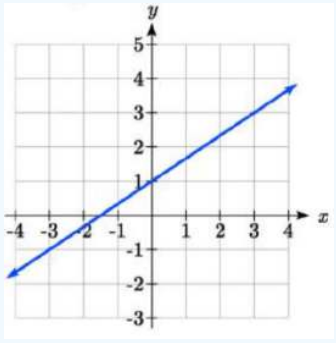
14. Find the equation of a line with a slope $m = 2$ that contains the point (3,-4).

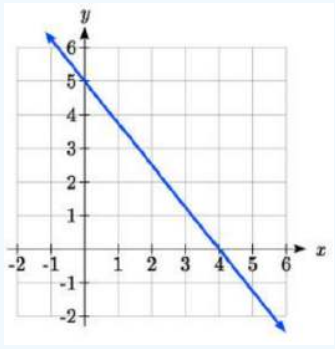
15. Find the equation of a line that contains the points (1,2) and (4,7).

16. Find the equation of the linear function f where $f(-5) = -4$ and $f(5) = 2$.

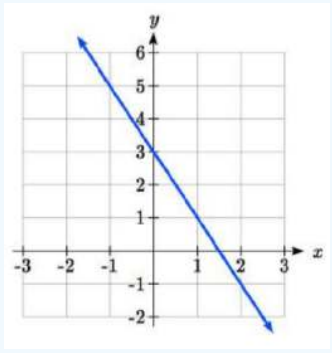
17. Find the equation of the linear function with an x-intercept at (-2, 0) and y-intercept at (0, -3).

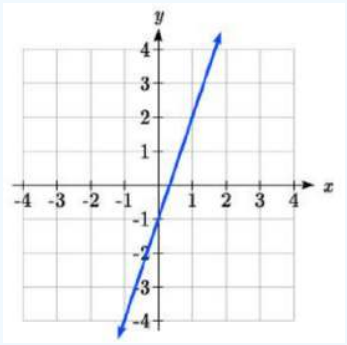
18. Find an equation for the function graphed below.

a) 

b) 

19. Find an equation for the function graphed below.

a) 

b) 

20. Maria is climbing a mountain. Maria's elevation, $E(t)$, in feet after t minutes is given by $E(t) = 1200 + 40t$. Write a complete sentence describing Maria's starting elevation and how it is changing over time.

21. The population of Reedley, California was 24,774 in 2012. Over the next 10 years the population grew by approximately 67 people each year (as reported by The US Census Bureau on July 31, 2024).

a) Write an equation, $P = f(t)$ for the population of Reedley t years since 2010.

b) Predict the population of Reedley in 2025 if this trend continues.

22. A boat is 100 miles away from the marina, sailing directly towards it at 10 miles per hour.

a) Write an equation for the distance of the boat d from the marina after t hours.

b) How long until the boat is 7 miles from the marina?

23. A gym membership with two personal training sessions costs $125, while gym membership with 5 personal training sessions costs $260. Assume the cost increases at a constant rate.

a) Write an equation, $C = f(s)$ for the cost of s training sessions.

b) Interpret the meaning of the slope in this application.

c) Predict the cost of 9 sessions.

24. In the school year beginning in the fall of 2021 there were 13,061 students classified as English Language Learners in the Fresno Unified School District. By the fall 2023 school year, there were 14,432 English Language Learners (as reported by The Ed-Data Education Data Partnership on July 31, 2024). Assume the number of English language learners is increasing at a constant rate.

a) Write an equation, $L = f(t)$ for the number of English language learners in the Fresno Unified School District t years since 2021.

b) Interpret the meaning of the slope in this application.

c) Using your equation, predict the number of English language learners in 2028 if this trend continues.

25. A phone company has a monthly cellular data plan where a customer pays a flat monthly fee and then a certain amount of money per megabyte (MB) of data used on the phone. If a customer uses 20 MB, the monthly cost will be $11.20. If the customer uses 130 MB, the monthly cost will be $17.80.

a) Write a linear equation for the monthly cost $C = f(x)$ of the data plan as a function of x, the number of MB used.

b) Interpret the meaning of the slope in this application.

c) Predict the total monthly cost if 250 MB are used.

26. The world's proven oil reserves was 223.27 billion barrels in 2010. The proven oil reserves grew tp 236.81 billion in 2018(as reported by Our World In Data on July 31, 2024). Assume the proven oil reserves increases at a constant rate in this time period.

a. Find an equation for the proven oil reserves, R, in terms of t, the number of years since 2010.

b. Interpret the meaning of the slope in this application.

c. Predict the proven oil reserves in 2028 if this trend continues.

27. The average hourly wage of a worker in Fresno County was $21.08 in 2015. By 2020, the average hourly wage was $25.08 (as reported by The U.S. Bureau of Labor Statistics on July 24, 2024). Assume the average hourly wage, W, increases at a constant rate relative to time. Let W represent the average hourly wage of a worker in Fresno County t years since 2015.

a) Find a linear function $W = f(t)$ for the average hourly wage of a worker in Fresno County t years since 2015.

b) Interpret the meaning of the slope in the application.

c) Predict the average hourly wage in 2025 if the average hourly wage continues to increase at a constant rate.

28. Suppose in a study on endurance, a person with a heart rate of 100 beats per minute (bpm) consumed 1 liter of oxygen per minute. When the same person's heart rate was raised to 160 bpm, they consumed 2 liters of oxygen per minute. Assume the oxygen consumption increases at a constant rate relative to heart rate.

a. Find a formula for a linear function $C = f(h)$ that models the oxygen consumption C in terms of the heart rate h.

b. Interpret the meaning of the slope in this application.

c. Use your function to predict the oxygen consumption when the heart tate is 198 bpm.

Answer

1. The functions f and g represent linear functions since their rates of change are constant.

3. No, the patient's heart rate is not increasing linearly relative to dosage since the rate of increase is not constant. On the interval [0,5], $\frac{\Delta h}{\Delta d} = 4$ whereas on the interval [5,10], $\frac{\Delta h}{\Delta d} = 3$.

5. $m = 3$

7. (a) $m = \frac{2}{3}$ (b) $m = \frac{-5}{4}$

9. $m \approx 0.343$. The CPI (the 12-month percentage change in price) of all goods increases by 0.343% per month.

11. $m = \frac{2}{7}$. The function is increasing.

13. $y = 3x - 2$

15. $y = \frac{5}{3}x + \frac{1}{3}$

17. $y = \frac{3}{2}x - 3$

19. (a) $y = -2x + 3$ (b) $y = 3x + 1$

21. (a) $P = f(t) = 67t + 24,774$ (b) $P = f(13) = 25,645$

23. (a) $C = f(s) = 45s + 35$ (b) The cost is increasing by \$45 per session. (c) Nine sessions will cost \$440.

25. (a) $C = f(x) = 0.06x + 10$ (b) The cost is increasing by \$0.06 per MB of data. (c) 250 MB of data will cost \$25.

27. (a) $W = f(t) = 0.8t + 21.08$ (b) The average hourly wage is increasing by \$0.80 per yer from 2015 to 2020. (c) The average hourly wage will be \$29.08 in 2025.

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1.4: More on Using Functions to Describe Relationships

Focus: Describing the Graphical Behavior of Functions

If a function describing a relationship between two quantities does not have a constant rate of change, then a linear function is not appropriate to model the relationship. In that case, the rate of increase or decrease is getting larger or smaller in size (or magnitude). The function may change from increasing to decreasing. We need to develop more tools to describe this aspect of the function and explore the graphical behavior of the function.

Consider the function $y = f(x) = x^2 - 4x + 5$. Exploring how the values of the input and output change with a table.

The inputs and corresponding outputs of function f

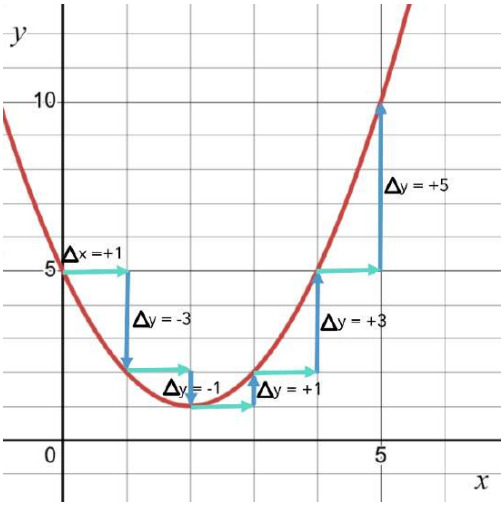
x	0	1	2	3	4	5
y	5	2	1	2	5	10

Notice how the change in the output y changes for each increase in x of one.

The corresponding changes in x and y for function f

Interval	[0,1]	[1,2]	[2,3]	[3,4]	[4,5]
Δx	+1	+1	+1	+1	+1
Δy	-3	-1	1	3	5
$\frac{\Delta y}{\Delta x}$	$\frac{-3}{1} = -3$	$\frac{-1}{1} = -1$	$\frac{1}{1} = 1$	$\frac{3}{1} = 3$	$\frac{5}{1} = 5$

On the interval $[2, \infty)$, a larger value of x gives a larger value of y. This is how we defined an increasing function in section 1.3. On the graph below, the graph moves upward (for a larger y) as we move from left to right (for a larger x) when the function is increasing on this interval. On the interval $(-\infty, 2)$, a larger value of x gives a smaller value of y. This is how we defined an decreasing function in section 1.3. On the graph below, the graph moves downward (for a smaller y) as we move from left to right (for a larger x) when the function is decreasing on this interval. Be aware that the change in x, Δx, does not always have to be $\Delta x = 1$ as shown on the graph to help clarify the different rates of change.

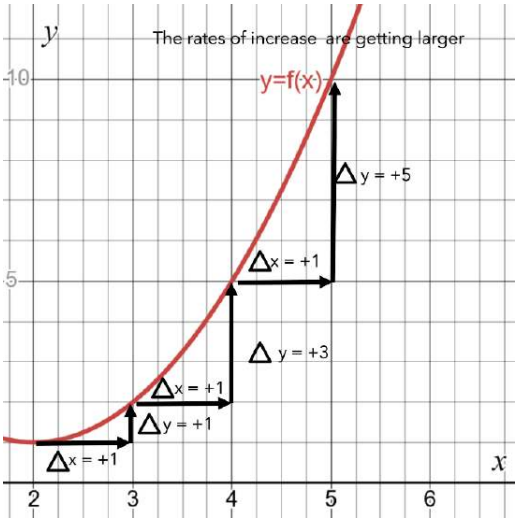


Key Point: Increasing/Decreasing Functions on a Graph

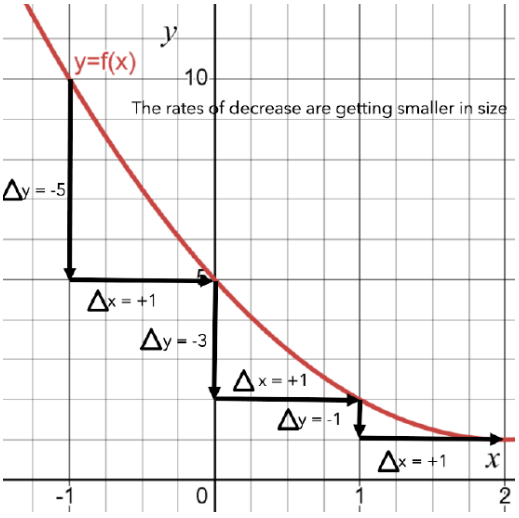
If a function is **increasing** on an interval, the graph moves upward from left to right.

If a function is **decreasing** on an interval, the graph moves upward from left to right.

Moreover, in the introductory example above, notice how the average rates of change on each successive interval get bigger or smaller in size. The rates of increase are getting larger in size on the interval $[2, \infty)$ from $\frac{\Delta y}{\Delta x} = \frac{1}{1}$ on [2,3] to $\frac{\Delta y}{\Delta x} = \frac{3}{1}$ on [3,4] and $\frac{\Delta y}{\Delta x} = \frac{5}{1}$ on [4,5]. Notice how the graph curves upward on the interval $[2, \infty)$ getting steeper and steeper as we move from left to right corresponding to the larger and larger rates of increase as shown on the graph below and the table above. In general, if a function has rates of increase that are getting larger in size over an interval, the shape of that portion of the graph will curve upward from left to right resembling the shape of the graph below.



On the interval $(-\infty, 2)$, the rates of decrease are getting smaller in size, decreasing in size from $\frac{\Delta y}{\Delta x} = \frac{-5}{1}$ on [-1,0] to $\frac{\Delta y}{\Delta x} = \frac{-3}{1}$ on [0,1] and $\frac{\Delta y}{\Delta x} = \frac{-1}{1}$ on [1,2]. The graph moves downward with a smaller and smaller magnitude steepness as we move from left to right along the graph corresponding to the smaller and smaller magnitude rates of decrease as shown in the the graph below and the table above. In general, if a function has rates of decrease that are getting smaller in size over an interval, the shape of that portion of the graph will curve downward from left to right resembling the shape of the graph below.



Consider a second function $y = g(x) = -(x-2)^2 + 5$. Exploring how the values of the input and output change with a table.

The inputs and corresponding outputs of function f

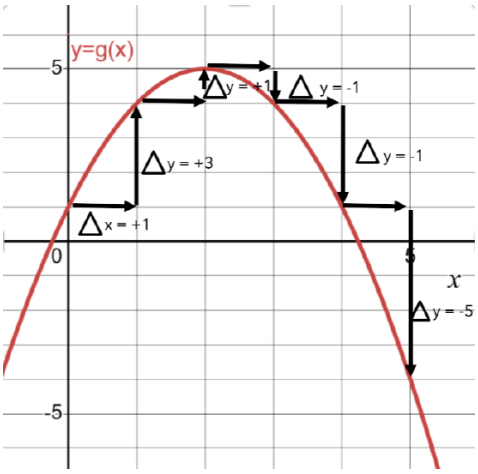
x	0	1	2	3	4	5
y	1	4	5	4	1	-4

Notice how the change in the output y changes for each increase in x of one.

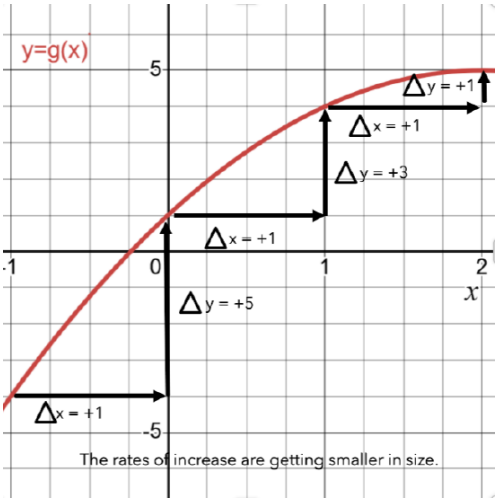
The corresponding changes in x and y for function g

Interval	[0,1]	[1,2]	[2,3]	[3,4]	[4,5]
Δx	+1	+1	+1	+1	+1
Δy	+3	+1	-1	-3	-5
$\frac{\Delta y}{\Delta x}$	$\frac{3}{1}=3$	$\frac{1}{1}=1$	$\frac{-1}{1}=-1$	$\frac{-3}{1}=-3$	$\frac{-5}{1}=-5$

On the interval $(-\infty, 2)$, a larger value of x gives a larger value of y. So the function g is increasing on $(-\infty, 2)$. On the graph below, notice the graph moves upward (for a larger y) as we move from left to right. On the interval $[2, \infty)$, a larger value of x gives a smaller value of y. So the function g is decreasing on $[2, \infty)$. On the graph below, notice the graph moves downward (for a smaller y) as we move from left to right.

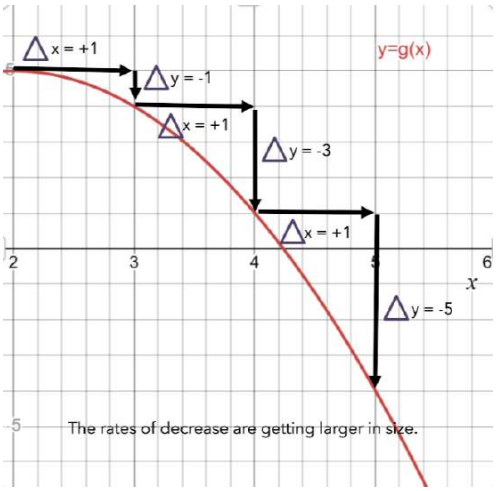


On the interval $(-\infty, 2)$, the rates of increase are getting smaller in size from $\frac{\Delta y}{\Delta x}=\frac{5}{1}$ on [-1,0] to $\frac{\Delta y}{\Delta x}=\frac{3}{1}$ on [0,1] and $\frac{\Delta y}{\Delta x}=\frac{1}{1}$ on [1,2]. Notice how the graph curves upward on the interval $(-\infty, 2)$ getting less and less steep as we move from left to right corresponding to the smaller and smaller rates of increase as shown on the graph below and the table above. In general, if a function has rates of increase that are getting smaller over an interval, the shape of that portion of the graph will curve upward from left to right resembling the shape of the graph below.



On the interval $[2, \infty)$, the rates of decrease are getting larger in size from $\frac{\Delta y}{\Delta x}=\frac{-1}{1}$ on [2,3] to $\frac{\Delta y}{\Delta x}=\frac{-3}{1}$ on [3,4] and $\frac{\Delta y}{\Delta x}=\frac{-5}{1}$ on [4,5]. The graph moves downward with a larger and larger magnitude steepness as we move from left to right along the graph

corresponding to the larger and larger magnitude rates of decrease as shown in the the graph below and the table above. In general, if a function has rates of decrease that are getting larger in size over an interval, the shape of that portion of the graph will curve downward from left to right resembling the shape of the graph below.

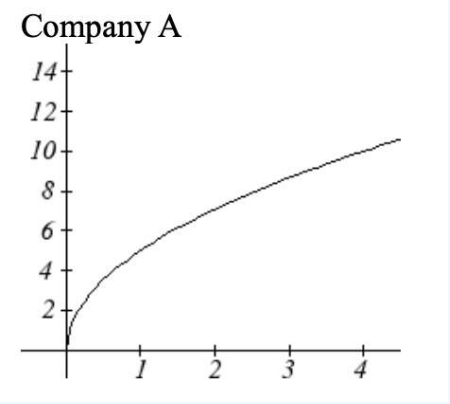


Example 1.4.1

A company's sales over 4 weeks, in thousands of dollars, is shown in the graph below.

a) Are the company's sales increasing or decreasing over the 4 weeks?

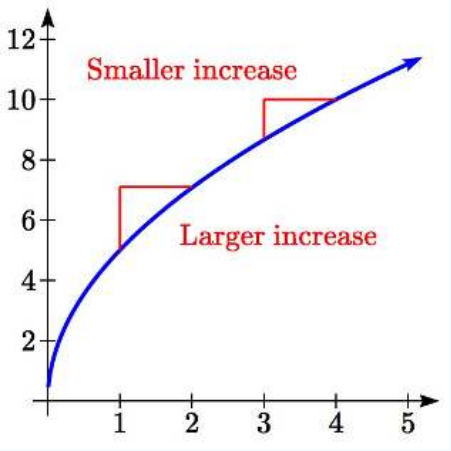
b) Describe what it happening to the rate of change in sales over time. Is the rate of increase or decrease getting larger or smaller in size?



Solution

a) The company's sales are increasing over time.

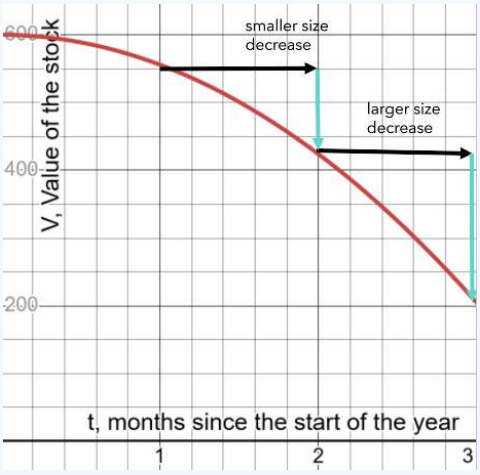
b) The rates of increase in sales in getting smaller as time passes.



Example 1.4.2

The value of the stock of a company reached \$600 per share at the start of the year, then began to decrease. The rate of decrease in value became larger and larger in size over the next few months. Draw a graph that illustrates the decrease in value of the stock over this time.

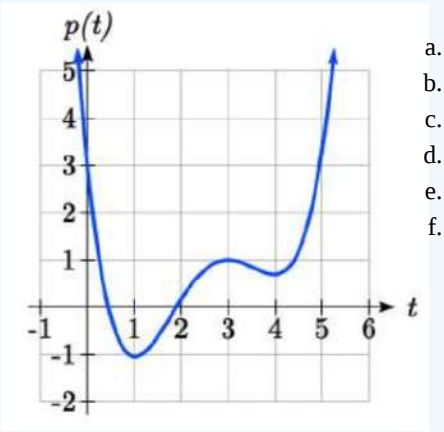
Solution



Example 1.4.3

The graph of the function $p(t)$ is shown below.

a. On what intervals is the function increasing?
b. On what intervals is the function decreasing?
c. On what interval(s) is the rate of increase in f getting larger?
d. On what interval(s) is the rate of increase in f getting smaller?
e. On what interval(s) is the rate of decrease in f getting larger in size?
f. On what interval(s) is the rate of decrease in f getting smaller in size?



Solution

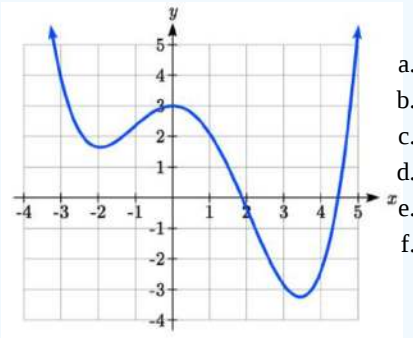
a. The function is increasing on the intervals $(1, 3)$ and $(4, \infty)$.
b. The function is decreasing on the intervals $(-\infty, 1)$ and $(3, 4)$.
c. The rate of increase in f is getting larger on $(1, 2)$ and $(4, \infty)$.
d. The rate of increase in f is getting smaller on $(-2, 3)$.
e. The rate of decrease in f is getting larger in size on $(3, 3.5)$.
f. The rate of decrease in f is getting smaller in size on $(-\infty, 1)$ and $(3.5, 4)$.

Note that all values of x are approximated from the graph.

Now You Try: Exercise 1.4.1

The graph of the function $y = f(x)$ is given below.

a. On what interval(s) is f f increasing?
b. On what interval(s) is f decreasing?
c. On what interval(s) is the rate of increase in f getting larger?
d. On what interval(s) is the rate of increase in f getting smaller?
e. On what interval(s) is the rate of decrease in f getting larger in size?
f. On what interval(s) is the rate of decrease in f getting smaller in size?



Answer

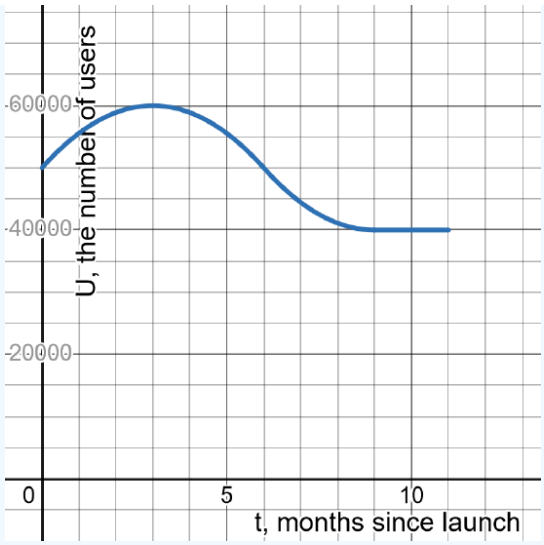
a. The function f increasing $(-2, 0)$ and $(3.5, \infty)$.
b. The function f decreasing $(-\infty, -2)$ and $(0, 3.5)$.
c. The rate of increase in f is getting larger on $(-2, -1)$ and $(3.5, \infty)$.
d. The rate of increase in f is getting smaller on $(-1, 0)$.
e. The rate of decrease in f is getting larger in size on $(0, 2)$.
f. The rate of decrease in f is getting smaller in size on $(-\infty, -2)$ and $(2, 3.5)$.

Note that all values of x are approximated from the graph.

Now You Try: Exercise 1.4.2

At the start of the year, a smartphone app had 50,000 monthly users. The number of users increased at smaller and smaller rate until April 1. After that, the number of users began to decrease. From April 1 to June 1, the rate of decrease in users became larger and larger in size over time. From June 1 to September 1, the rate of decrease in users became smaller and smaller in size over time before leveling off at 40,000 users. Sketch a graph for a function $U = g(t)$ that describes the relationship between the number of users U in t months since the start of the year.

Answer



Focus: A Problem with Using a Function to Describe a Relationship

Consider the function $y = f(x) = \frac{3}{x-2}$. For the input $x = 4$, there is no problem with the corresponding output. The output is a number.

$$y = f(4) = \frac{3}{4-2} = \frac{3}{2}$$

However, for the input $x = 2$, the output is undefined.

$$y = f(2) = \frac{3}{2-2} = \frac{3}{0}$$

If we are attempting to describe a relationship, such as the value of a car or the height of a ball, an undefined value does not make sense in the context of the application. To address this problem, we must specify which inputs of a function give defined outputs. These specified inputs are the only ones we should use for the function. To help us quantify the inputs and defined outputs of a function, we introduce the concept of the domain and range of a function.

Definition: Domain and Range

The **domain** of a function is the set of all inputs of a function that give a defined output.

The **range** of a function is the set of all defined outputs for a given set of inputs.

Example 1.4.4

The function $y = f(x)$ is described by the table below. Find the domain and range of the function.

The inputs and outputs of function f

x		0	1	2	3
$y = f(x)$		5	7	6	5

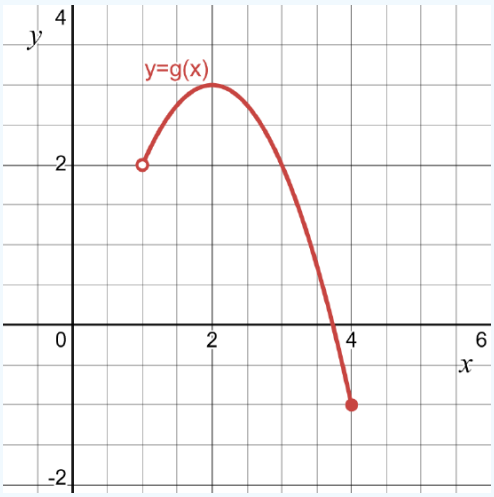
Solution

The domain is a set or list of all the inputs. The domain of f is $\{0,1,2,3\}$. The range is the set of defined outputs for the given domain. The range of f is $\{5,6,7\}$.

Notice that we did not need to list the output of $y = 5$ twice even though it appeared as the output corresponding to two different inputs. Also, the curly brackets $\{\}$ tell us that this a list of the included numbers.

Example 1.4.5

The function $y = g(x)$ is described by the graph below. Find the domain and range of the function.



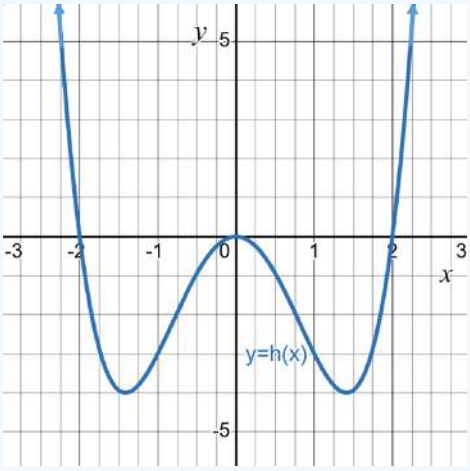
Solution

On a graph, the inputs are listed on the horizontal axis where there are points with defined values of x for all values of x in between 1 and 4. Since there are an infinite number of values of x in this range, we can to use interval notation or an inequality to express this set. The domain of g is (1,4]. We could also use the inequality $1 < x \leq 4$. Notice that we have included the input $x = 4$ corresponding to the closed circle at this endpoint. However, we have excluded the input $x = 2$ corresponding to the open circle at this endpoint.

On a graph, the outputs are listed on the vertical axis where there are defined values of y for all values of y in between -1 and 3. The range of g is [-1,3].

Example 1.4.6

The function $y = h(x)$ is described by the graph below. Find the domain and range of the function.



Solution

Since the graph opens upward on both ends, the graph continues left and right for larger and smaller values of x than those shown on the graph. Therefore, there are points on the graph for all real values of x along the horizontal axis. The domain is $(-\infty, \infty)$.

The smallest output y along the vertical axis is $y = -4$. Since the graph opens upward on both ends, the graph continues upward for larger values of y than those shown on the graph. The range is $(-4, \infty)$.

If we are given a formula for a function, such as $f(x) = x^2 + 3x$, either a domain will be given or one is implied by all the inputs that give a defined output. Let's explore each numerical operation, the corresponding restrictions entailed to get a defined output, and the effect on the domain with a function that involves each operation.

The restrictions on the inputs for each mathematical operation

Operation	Example	Restrictions with the Operation	Explanation	Domain of the Example Function
Addition	$f(x) = x + 2$	none	We can add any number and 2 and get a real number	D is the set of all real numbers. $D = (-\infty, \infty)$
Subtraction	$f(x) = x - 2$	none	We can subtract any number and 2 and get a real number	D is the set of all real numbers. $D = (-\infty, \infty)$
Multiplication	$f(x) = 2x$	none	We can multiply any number and 2 and get a real number	D is the set of all real numbers. $D = (-\infty, \infty)$
Division	$f(x) = \frac{2}{x}$	The denominator $\neq 0$	We can't divide 2 by zero and get a real number	D is the set of all real numbers where $x \neq 0$. $D = (-\infty, 0) \cup (0, \infty)$
Powers	$f(x) = x^2$	none	We can raise any number to a whole number power	D is the set of all real numbers. $D = (-\infty, \infty)$
Roots	$f(x) = \sqrt{x}$	The input of the radical ≥ 0	We can't take the even root of a negative number	$D = [0, \infty)$

At this point, the only functions with restrictions on the input are those involving division and even roots. For a function with multiple operations, we need to identify the operations in the function and corresponding restrictions on the input. For example, the function $f(x) = 2x + 1$ involves multiplication and addition neither of which has a restriction. Therefore, the domain is $(-\infty, \infty)$.

Key Point: The Domain of a Function Given by a Formula

To determine the domain of a function given by a formula identify the restrictions on the input x corresponding to each operation in the formula.

Example 1.4.7

Find the domain of the following functions.

a. $f(x) = x^2 + 3x - 2$
b. $g(x) = \frac{3}{x - 2}$
c. $h(x) = \sqrt{2x - 1}$

Solution

a. Since the function $f(x) = x^2 + 3x - 2$ involves the operations of addition, subtraction, multiplication, and a power, there are no restrictions on the input. Therefore, the domain of f is $(-\infty, \infty)$

b. The function $g(x) = \dfrac{3}{x-2}$ involves a subtraction which has no corresponding restrictions. However, the division has the restriction that the denominator $x - 2 \neq 0$. Solving for x, $x \neq 2$. Therefore, the domain of g is all real numbers where $x \neq 2$ or $(-\infty, 2) \cup (2, \infty)$ in interval notation.

c. The only operation in $h(x) = \sqrt{2x-1}$ that has a restriction is the square root where the input of the radical $2x - 1 \geq 0$. Solving for x by adding 1 to both sides

$$2x \geq 1$$

Then dividing by 2 on both sides

$$x \geq \frac{1}{2}$$

Therefore the domain is $[\frac{1}{2}, \infty)$.

Now You Try: Exercise 1.4.3

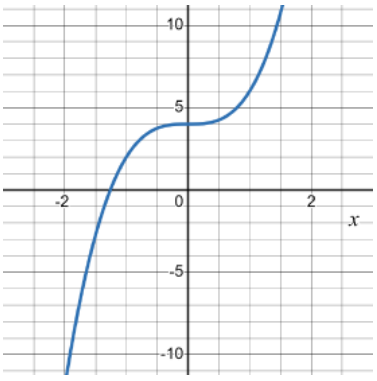
Find the domain of the following functions.

a. $f(x) = 2x^3 + 4$
b. $g(x) = \dfrac{5x+1}{4x-3}$
c. $h(x) = \sqrt{2-7x}$
d. $k(x) = \dfrac{5x+1}{\sqrt{2-7x}}$

Answer

a. The domain of f is $(-\infty, \infty)$
b. The domain of g is all real numbers where $x \neq \frac{3}{4}$ or $(-\infty, \frac{3}{4}) \cup (\frac{3}{4}, \infty)$
c. The domain of $h$is $(-\infty, \frac{2}{7}]$.
d. The domain of k is $(-\infty, \frac{2}{7})$.

Identifying the domain of a function is key to addressing the original problem of undefined outputs by eliminating the inputs whose corresponding outputs are undefined. We could also identify the range if we wanted, but it is usually unnecessary. If we wanted to find the range of the function $y = f(x) = 2x^3 + 4$ exercise 1.4.3a for example, we have two options. We could use the graph to determine the range.



From the graph, the range of outputs y includes all real numbers on the y-axis. Therefore, the domain of f is $(-\infty, \infty)$. The other option is to use f to write a formula for the x if possible, then determine the restrictions on y. Solving for y by subtracting 4 from both sides

$$y-4=2x^3$$

Then dividing by 2

$$\frac{y-4}{2}=x^3$$

Then taking the cube root of both sides

$$\sqrt[3]{\frac{y-4}{2}}=x$$

This is a formula for x in terms of y. None of the operations have a restriction. The division is always by 2. Therefore, y can be any real number and so the range of f is $(-\infty,\infty)$.

Focus: Another Problem with Using Functions to Describe Relationships in Applications

Suppose we used the function $V=f(t)=40000-4000t$ to describe the value V of a car t years after it was purchased. Although we could evaluate $f(-2)$ mathematically, the input $t=-2$ may not make sense in the context of this application. Two years before the car was purchased may not make sense if the car was brand new when purchased. Although the input $t=12$ does make sense in the context of this application as $t=12$ years since it was purchased, the output $V=f(12)=40000-4000(12)=-8000$ does not make sense as the value of a car. The value of a car can't be negative. No one will pay some one else just to have their car (ignoring taxes). A person could argue that the minimum value must be at least a few hundred dollars since it could be sold as scrap metal. Furthermore, it is important to keep in mind the limitations of those models we create. If additional information or data revealed that the linear function that is being used to describe the value of a car is valid for only the first four years, we would need to further restrict the domain accordingly.

Key Point: Domain of an Application

If a function is used to model an application, the domain must be restricted so that

1. the inputs satisfy all mathematical restrictions
2. the inputs make sense as the quantity the input represents in the application
3. the inputs have corresponding outputs that make sense as the quantity the output represents in the application
4. the inputs reflect any other limitations on the quantities the input and the corresponding output represent.

Example 1.4.8

The function $V=f(t)=40000-4000t$ gives the value V of a new car t years after it was purchased years after it was purchased. Find the domain of the function in this application.

Solution

Mathematically, there are no restrictions on the input since the operations in the function are subtraction and multiplication. The input t makes sense as the years since it was purchased when $t\geq 0$ since the car was purchased when it was new. The smallest value of the output V that makes sense as a value of a car is $V=0$ since the value can't be negative. Since the value is decreasing over time, the largest value of the car would be $V=f(0)=40,000-4000(0)=\$40,000$ when the car is brand new. Therefore, the value of the car V must be in the interval $0\leq V\leq 40,000$. However, the domain is a restriction on the input t not the output V. We can use the formula for V to write this double inequality in terms of t,

$$0\leq 40000-4000t\leq 40,000$$

Solving for t by subtracting 40,000 from all sides of the double inequality

$$-40000\leq -4000t\leq 0$$

Then dividing by -4000 on all sides of the double inequality and switching the direction of the inequality signs

$$10\geq t\geq 0$$

The domain of the application is the interval where $0 \leq t \leq 10$ that satisfies both of these restrictions.

Note that we could have also used the graph of $V = f(t)$ to approximate the values of t where $V \geq 0$ if we were unable to solve the double inequality $0 \leq V \leq 40,000$ analytically.

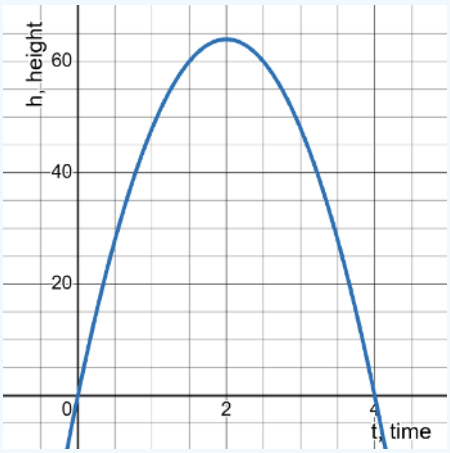
Every function the models a relationship between two quantities in an application should be accompanied by a domain which tells us the range of values for the input for which the model is appropriate and valid.

Example 1.4.9

The height h of a ball thrown upward at $64\frac{ft}{sec}$ from the ground after t seconds is modeled by the function $h = s(t) = -16t^2 + 64t$. Find the domain of the application.

Solution

Mathematically, there are no restrictions on the input since the operations in the function are addition, multiplication, and powers. The input t makes sense as the seconds since it was thrown when $t \geq 0$ since a negative value of t, such as $t = -2$, would before referring to a the time before the ball was thrown. The smallest value of the output h that makes sense as a height is $h = 0$ since the ball can't go below the ground. Therefore, we must have $h \geq 0$. Substituting the formula for h into this inequality, we have $h = -16t^2 + 64t \geq 0$. We will learn later in this course how to solve this type of inequality analytically. In the mean time, we can solve it graphically. On the graph below $h \geq 0$ when $0 \leq t \leq 4$. The domain of the application is the interval where $0 \leq t \leq 4$ that satisfies both of these restrictions on the input t.

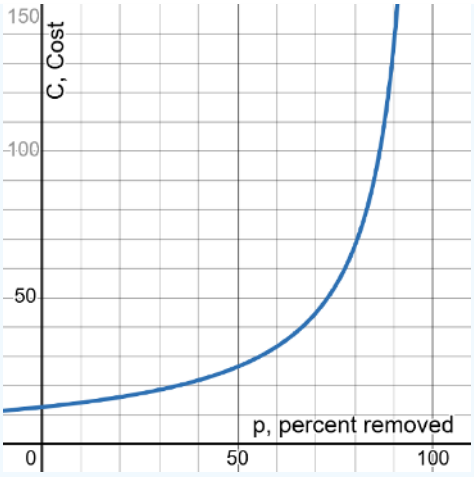


Example 1.4.10

Suppose the cost (in thousands) of removing p percent of a pollutant from an acre foot of water at a waste water treatment facility is approximated with the model $C(p) = \frac{1.25p + 1250}{100 - p}$. Find the domain of the application.

Solution

Mathematically, there is a restriction on the input for the division, the denominator $100 - p \neq 0$. Solving for p, we have $p \neq 100$. The input p makes sense as the percent removed when $p \geq 0$ since it is impossible to remove a negative percentage of the pollutant. The largest percent of the pollutant removed is 100% if we remove it all. Therefore, $0 \leq p \leq 100$. The smallest value of the output C that makes sense is $C = 0$. Without any additional information, we can't identify an upper bound on the the cost. Therefore, we must have $C \geq 0$ for the output C to make sense as a cost. Substituting the formula for C into this inequality, we have $\frac{1.25p + 1250}{100 - p} \geq 0$. This is another type of inequality that we will later learn how to solve analytically in this course. In the mean time, we can solve this inequality graphically. On the graph below, $C \geq 0$ when $p \leq 100$. The domain of the application is the interval where $0 \leq p < 100$ that satisfies all three of these restrictions.



Now You Try: Exercise 1.4.4

Suppose a shoe manufacturer uses test market data to estimate that the number of shoes, N, that they can sell at a particular price p with the function $N = f(p) = 300 - 0.12p$. Find the domain of the application.

Answer

The domain of the application is the interval where $0 \leq p \leq 2,500$.

Mathematically, there are no restrictions. The input p makes sense as a price of shoes when $p \geq 0$ since the price charged can't be negative. The manufacturer can't sell a negative number of shoes, so the output $N \geq 0$. The values of the input that have a corresponding output that satisfy this restriction are $p \leq= 2,500$.

A pair of shoes that costs $2,500 is not realistic in most cases. If we had actual test market data on the prices and number of shoes sold, we should restrict the domain even further to a realistic interval of prices for the shoes.

Important Topics of this Section

- Describing varying rates of change on a graph
- Definition of domain
- Definition of range
- Domain and Range from graphs
- Domain and Range from tables
- Domain from a formula
- Domain of an application

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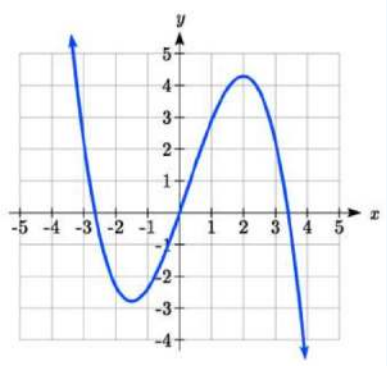
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1.4E: More on Using Functions to Describe Relationships (Exercises)

Section 1.4 Exercises

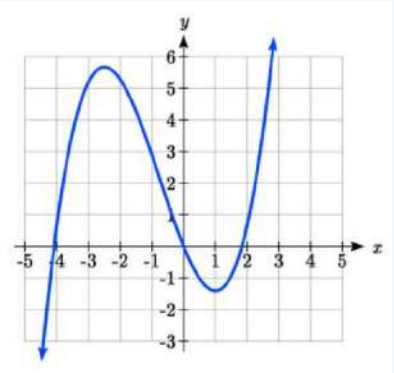
1. For the function graphed below,

   a. Estimate the intervals on which the function is increasing and decreasing.
   b. Determine on which intervals the rate of increase is getting larger in size.
   c. Determine on which intervals the rate of increase is getting smaller in size.
   d. Determine on which intervals the rate of decrease is getting larger in size.
   e. Determine on which intervals the rate of decrease is getting smaller in size.



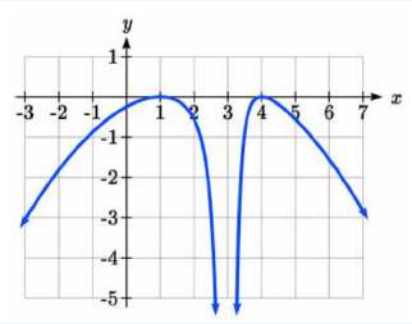
2. For the function graphed below,

   a. Estimate the intervals on which the function is increasing and decreasing.
   b. Determine on which intervals the rate of increase is getting larger in size.
   c. Determine on which intervals the rate of increase is getting smaller in size.
   d. Determine on which intervals the rate of decrease is getting larger in size.
   e. Determine on which intervals the rate of decrease is getting smaller in size.



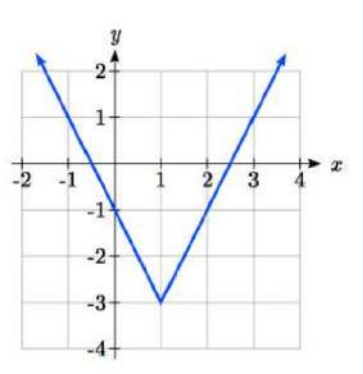
3. For the function graphed below,

   a. Estimate the intervals on which the function is increasing and decreasing.
   b. Determine on which intervals the rate of increase is getting larger in size.
   c. Determine on which intervals the rate of increase is getting smaller in size.
   d. Determine on which intervals the rate of decrease is getting larger in size.
   e. Determine on which intervals the rate of decrease is getting smaller in size.



4. For the function graphed below,

a. Estimate the intervals on which the function is increasing and decreasing.
b. Determine on which intervals the rate of increase is getting larger in size.
c. Determine on which intervals the rate of increase is getting smaller in size.
d. Determine on which intervals the rate of decrease is getting larger in size.
e. Determine on which intervals the rate of decrease is getting smaller in size.



5. For each table below, Determine whether the table represents a function that is increasing or decreasing. Then determine whether the rate of increase or decrease is getting larger or smaller in size.

a.

The inputs and outputs of function f

x	1	2	3	4	5
$f(x)$	2	4	8	16	32

b.

The inputs and outputs of function g

x	1	2	3	4	5
$g(x)$	90	80	75	72	71

c.

The inputs and outputs of function h

x	1	2	3	4	5
$h(x)$	300	290	270	240	200

d.

The inputs and outputs of function k

x	1	2	3	4	5
$k(x)$	0	15	25	32	35

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e.

The inputs and outputs of function L

x	1	2	3	4	5
$L(x)$	-10	-25	-37	-47	-54

f.

The inputs and outputs of function m

x	1	2	3	4	5
$m(x)$	-200	-190	-160	-100	0

g.

The inputs and outputs of function n

x	1	2	3	4	5
$n(x)$	-100	-50	-25	-10	0

h.

The inputs and outputs of function p

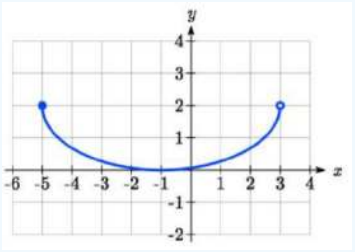
x	1	2	3	4	5
$p(x)$	-50	-100	-200	-400	-900

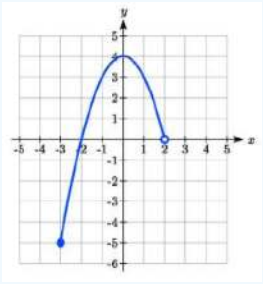
6. A person pours a hot cup of tea into a cup with a temperature of 170 $^\circ F$. The temperature of the cup of tea decreases over time. Initially, the temperature drops quickly then drops more slowly over time. Draw a graph which represents this scenario.

7. A company's monthly revenue is $1.2 million in Jan. 2022. Due to a recent product launch, the company's revenue increasing dramatically at first, then starts to increase more slowly over time as the popularity and newness of the product fades. Draw a graph which represents this scenario.

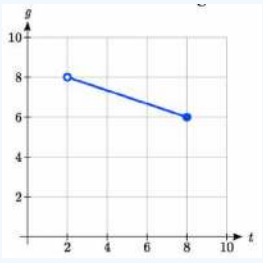
8. Suppose that you are holding your toy submarine 2 ft. under the water. You release it and it begins to ascend at a constant rate. Draw a graph models the depth D of the submarine as a function of time t, stopping once the sub surfaces.

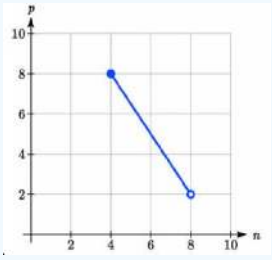
9. Identify the domain and range of each function. Write your result using interval notation or an inequality.

a) 

b) 

10. Identify the domain and range of each function. Write your result using interval notation or an inequality.

a) 

b) 

11. Find the domain of each function

a) $f(x) = 3x + 1$ b) $f(x) = 3\sqrt{x-2}$ c) $f(x) = 3 - \sqrt{6-2x}$ d) $f(x) = \dfrac{9}{x-6}$

12. Find the domain of each function

a) $f(x) = 3x + 1$ b) $f(x) = 5\sqrt{x+3}$ b) $f(x) = 5 - \sqrt{10 - 2x}$ c) 12. $f(x) = \dfrac{6}{x-8}$

13. Find the domain of each function

a) $f(x) = \dfrac{3x+1}{4x+2}$ b) $f(x) = \dfrac{\sqrt{x+4}}{x-4}$ c) $f(x) = \dfrac{x-3}{x^2+9x-22}$

14. Find the domain of each function

a) $f(x) = \dfrac{5x+3}{4x-1}$ b) $f(x) = \dfrac{\sqrt{x+5}}{x-6}$ c) $f(x) = \dfrac{x-8}{x^2+8x-9}$

15. The discharge of the San Joaquin River out of Millerton Lake dam in $\frac{\text{ft}^3}{s}$ t days since February 1,2024 is modeled by the equation $D = f(t) = 571 - 0.288t$ from exercise 1.3.5.

a) Identify the mathematical restrictions on the input t.

b) Identify the restrictions on the input t so that t makes sense as a time since February 1, 2024.

c) Identify the restrictions on the input t so that D makes sense as a discharge from the San Joaquin River.

d) Find the domain of the application.

16. A balloon is filled with 40 in^3 of air. Due to a small hole, the air is released from a balloon at a rate of 1.2 $\frac{\text{in}^3}{s}$. The volume after t seconds is given by the formula $V = f(t) = 40 - 1.2t$.

a) Identify the mathematical restrictions on the input t.

b) Identify the restrictions on the input t so that t makes sense as a time since the air is released from the balloon.

c) Identify the restrictions on the input t so that V makes sense as a volume of air in the balloon.

d) Find the domain of the application.

17. Suppose the revenue (or total sales) of a company (in millions) that produces x thousand units of a product can be modeled by the function $R = R(x) = 300x - 2x^2$.

a) Identify the mathematical restrictions on the input x.

b) Identify the restrictions on the input x so that x makes sense as the thousand of units of a product produced.

c) Identify the restrictions on the input x so that R makes sense as a revenue.

d) Find the domain of the application.

18. An open box is to be made from a 16-inch by 30-inch piece of cardboard by cutting out squares of equal size from the four corners and bending up the sides. Let x represent the side length of the cut out.

a) The function for the length of the side of the box is $L = L(x) = 16 - 2x$. Find the domain of the application.

b) The function for the width of the side of the box is $W = W(x) = 16 - 2x$. Find the domain of the application.

c) The function for the height of the side of the box is $H = H(x) = x$. Find the domain of the application.

d) Find a formula for the volume of the box $V(x)$ in terms of x only. Then find the domain of the application.

Answer

1. a) Increasing on (-1.5,2) and decreasing on $(-\infty, -1.5) \cup (2, \infty)$.

b) (-1.5,0.25) c) (0.25,2) d) $(2, \infty)$ e) $(-\infty, -1.5)$

3. a) Increasing on $(-\infty, 1) \cup (3, 4)$ and decreasing on $(1, 3) \cup (4, \infty)$.

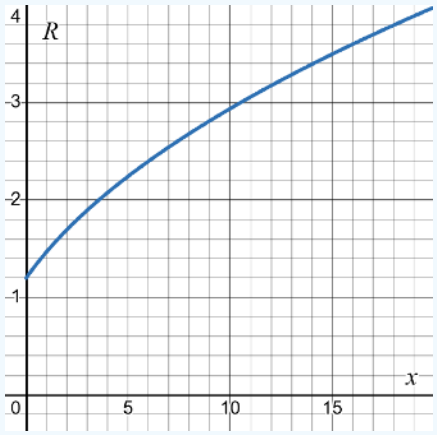
b) (3,4) c) $(-\infty, 1)$ d) nowhere e) $(1, 3) \cup (4, \infty)$

5. a) The function is increasing. The rate of increase is getting larger in size.

b) The function is decreasing. The rate of decrease is getting larger in size.

c) The function is decreasing. The rate of decrease is getting smaller in size.

d) The function is decreasing. The rate of decrease is getting larger in size.

7. 

9. a) D: [-5, 3) R: [0, 2] b) D: [-3, 2) R: [-5, 4]

11. a) $(-\infty, \infty)$ b) $[2, \infty)$ c) $(-\infty, 3]$ d) $(-\infty, 6) \cup (6, \infty)$ or $x \neq 6$

13. a) $(-\infty, -\frac{1}{2}) \cup (-\frac{1}{2}, \infty)$ or $x \neq -\frac{1}{2}$ b) $(-\infty, 4) \cup (4, \infty)$ or $x \neq 4$ c) $(-\infty, -11) \cup (-11, 2) \cup (2, \infty)$ or $x \neq -11$ and $x \neq 2$

15. a) no restrictions b) $[0, \infty)$ c) $(-\infty, 1982]$ d) $[0, 1982]$

17. a) no restrictions b) $[0, \infty)$ c) $[0, 150]$ d) $[0, 150]$

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1.5: Piecewise Defined Functions

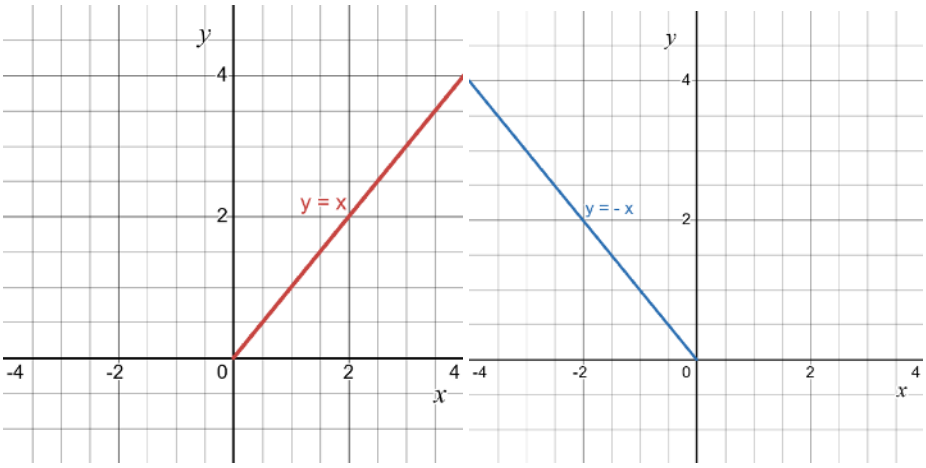
Piecewise Defined Functions

If a function describing a relationship between two quantities does not have a constant rate of change, we can't use a linear function to model the relationship. At this point, it is only linear functions that we have fully developed for use in modeling relationships. We know what the graph looks like (a line), what the formula looks like (y = mx+ b), what the rate of change is, how these are related, and when to use a linear function (the rate of change is constant). Moving forward, our goal is develop more functions in a similar way to describe relationships without a constant rate of change. In this section, we explore piecewise defined functions which are built with pieces of two or more functions on different intervals.

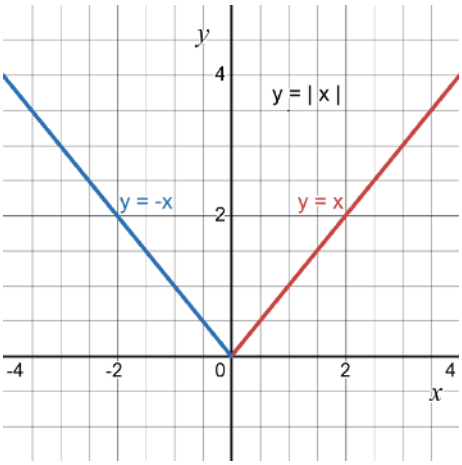
We have already worked with one piecewise defined function, the absolute value function $f(x) = |x|$. For a positive number, such as $x = 3$, the absolute value is the same as the number inputted, $|3| = 3$. For a negative number, such as $x = -2$, the absolute value is the opposite of the number inputted, $|-2| = -(-2) = 2$. There are two different rules or ways of calculating the absolute value of a number. In general, if x is a positive number or zero where $x \geq 0$, the absolute value is the same as the number inputted, $|x| = x$. If x is a negative number where $x < 0$, the absolute value is the opposite of the number inputted, $|x| = -x$. We can write the absolute value function as a piecewise defined function as follows

$$|x| = \begin{cases} x & if \quad x \geq 0 \\ -x & if \quad x < 0 \end{cases}$$

To graph the absolute value function, we need to graph each formula in the piecewise defined function only in the interval where it is defined. The formula $y = x$ is a line with slope $m = 1$ and y-intercept at $y = 0$. We need to graph with line only where it is defined for the absolute value $x \geq 0$. The formula $y = -x$ is a line with slope $m = -1$ and y-intercept at $y = 0$. We need to graph with line only where it is defined for the absolute value $x < 0$. Graphing each piece separately on the interval they are defined on, we have



Putting them on the same graph, we have the graph of the absolute value function below with the different pieces highlighted in different colors.



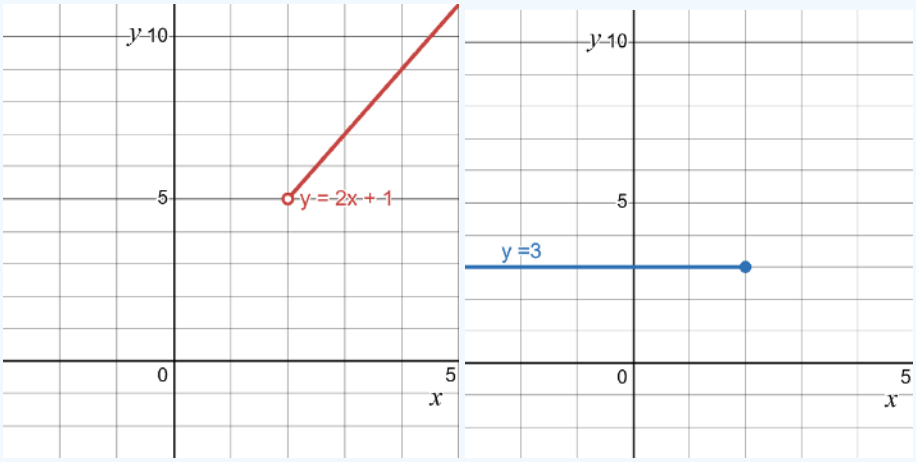
Example 1.5.1

Given the piecewise defined function $g(x) = \begin{cases} 2x+1 & if \quad x > 2 \\ 3 & if \quad x \leq 2 \end{cases}$

a. Evaluate $g(1)$
b. Evaluate $g(2)$
c. Evaluate $g(3)$
d. Evaluate $g(4)$
e. Graph $g(x)$

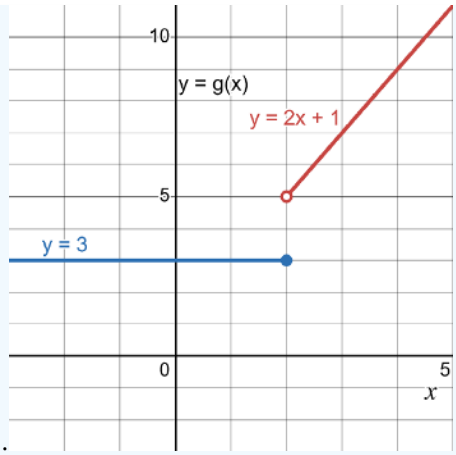
Solution

a. Since the input $x = 1 \leq 2$, we use the second function $y = 3$ to calculate the output. So $g(1) = 3$.
b. Similarly, since the input $x = 2 \leq 2$, we again use the second function $y = 3$ to calculate the output. So $g(2) = 3$.
c. Since the input $x = 3 > 2$, we use the first function $y = 2x + 1$ to calculate the output. So $g(3) = 2(3) + 1 = 7$.
d. Similarly, since the input $x = 4 \geq 2$, we use the first function $y = 2x + 1$ to calculate the output. So $g(4) = 2(4) + 1 = 9$.
e. We need to graph each piece of the function on the interval it is defined on. The formula $y = 2x + 1$ is a line with slope $m = 2$ and y-intercept at $y = 1$. The formula $y = 3$ is a horizontal line with a y-intercept at $y = 3$. Graphing each piece separately on the interval they are defined on, we have



At $x = 2$, we have to differentiate which point we want on the graph. We use a closed circle on the graph of $y = 3$ at the point (2,3) to include this point as part of the graph since the formula prescribes that we use $y = 3$ at $x = 2$. We use an open circle on the graph of $y = 2x + 1$ at the point (2,5) to exclude this point from the graph since the formula prescribes that we don't use $y = 2x + 1$ at $x = 2$.

Putting them on the same graph, we have the graph of $y = g(x)$.

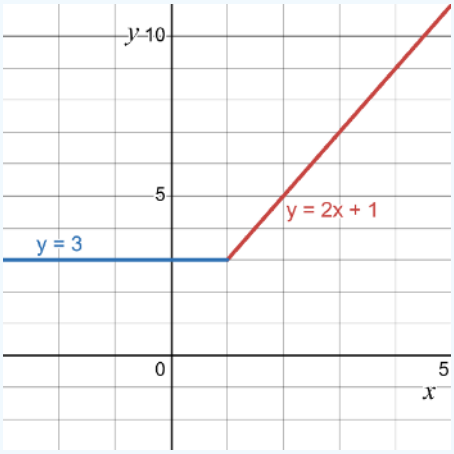


Example 1.5.2

How would the graph in example 1.5.1 change if the function g was changed to $g_{new}(x) = \begin{cases} 2x+1 & if \quad x \geq 1 \\ 3 & if \quad x < 1 \end{cases}$

Solution

The two pieces are the same as g, but the interval where they are defined has change. The first function is defined on $x \geq 1$, while the second function is defined on $x < 1$. We need to adjust the graph accordingly.



Notice that both functions meet at a single point $y = 3$ at $x = 1$ so there is not need to differentiate between points at $x = 1$ like we did in the previous example.

Example 1.5.3

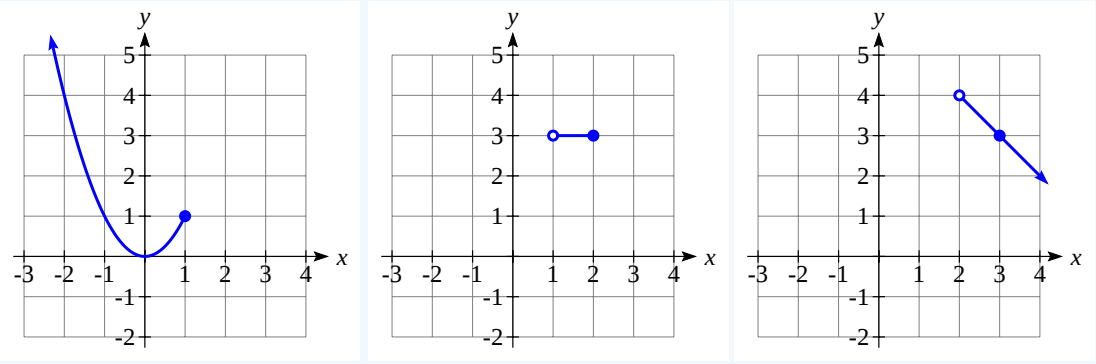
Given the piecewise defined function $f(x) = \begin{cases} x^2 & if \quad x \leq 1 \\ 3 & if \quad 1 < x \leq 2 \\ 6 - x & if \quad x > 2 \end{cases}$

a. Evaluate $f(0)$
b. Evaluate $f(1)$
c. Evaluate $f(2)$
d. Evaluate $f(3)$
e. Graph $f(x)$

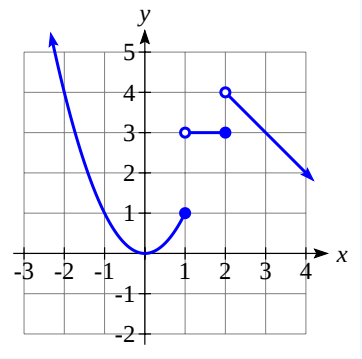
Solution

a. Since the input $x = 0 \leq 1$, we use the first function $y = x^2$ to calculate the output. So $f(0) = 0^2 = 0$.

b. Similarly, since the input $x = 1 \leq 1$, we again use the first function $y = x^2$ to calculate the output. So $f(1) = 1^2 = 1$.
c. Since the input $x = 2 > 1$ and $x = 2 \leq 2$, we use the second function $y = 3$ to calculate the output. So $f(2) = 3$.
d. Similarly, since the input $x = 3 > 2$, we again use the third function $y = 6 - x$ to calculate the output. So $f(3) = 6 - 3 = 3$
e. To graph $f(x)$ we need to graph each piece of the function on the interval it is defined on. The formula $y = x^2$ is a parabola which changes direction at $x = 0$. We can graph this function with points or with a graphing calculator or similar technology. We only need the piece where $x \leq 1$. The formula $y = 3$ is a horizontal line with a y-intercept at $y = 3$. The formula $y = 6 - x$ is a line with slope $m = -1$ and y-intercept at $y = 6$. Graphing each of these separately.



Notice that we used closed circles to indicate the points included in the graph at a place where the formula changes corresponding to a $\leq$ or $\geq$ symbol. We used open circles to indicate the points that are excluded from the graph at a place where the formula changes corresponding to a < or > symbol. Now that we have each piece individually, we combine them onto the same graph below.



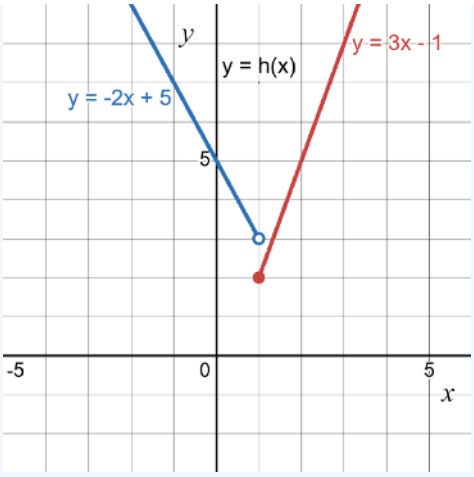
Now You Try: Exercise 1.5.1

Given the piecewise defined function $h(x) = \begin{cases} 3x - 1 & if \quad x \geq 1 \\ -2x + 5 & if \quad x < 1 \end{cases}$

a. Evaluate $h(0)$
b. Evaluate $h(1)$
c. Evaluate $h(2)$
d. Graph $h(x)$

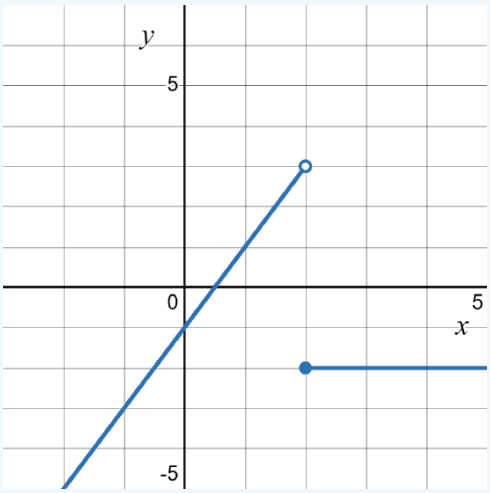
Answer

a. $h(0) = 5$
b. $h(1) = 2$
c. $h(2) = 5$
d.



Now You Try: Exercise 1.5.2

Find a formula for a piecewise defined function $y = f(x)$ whose graph is given below.



Answer

The formula is $f(x) = \begin{cases} 2x - 1 & if \quad x < 2 \\ -2 & if \quad x \geq 2 \end{cases}$.

Focus: Applications of Piecewise Defined Functions

Piecewise defined functions can be used to model relationships between two quantities in applications when there are two are more ways of calculating a particular quantity.

Example 1.5.4

A museum charges \$5 per person for a guided tour with a group of 1 to 9 people, or a fixed \$50 fee for 10 or more people in the group. Write a formula for a piecewise defined function relating the number of people, n, to the cost, C.

Solution

A piecewise defined function is appropriate here, since there are two ways of calculating the cost. For $n \leq 9$, the rate of increase in cost is a constant $\frac{\$5}{1 \text{ person}}$. Therefore, a linear function would be appropriate with the slope $m = 5$. Since the cost of $n = 0$ people is $C = 0$, the linear formula for this piece is $C = 5n$ for values of n under 10. For 10 or more people, the cost is constant $C = 50$.

$$C(n) = \begin{cases} 5n & if \quad 0 < n < 10 \\ 50 & if \quad n \geq 10 \end{cases}$$

Example 1.5.5

A cell phone company uses the function below to determine the cost, C, in dollars for g gigabytes of data transfer.

$$C(g) = \begin{cases} 25 & if \quad 0 < g < 2 \\ 25 + 10(g-2) & if \quad g \geq 2 \end{cases}$$

a. Find the cost of using 1.5 gigabytes of data.
b. Find the cost of using 4 gigabytes of data.

Solution

a. To find the cost of using 1.5 gigabytes of data where we must use the first formula $C(g)$ = \$25 since $g = 1.5 < 2$. So $C(1.5)$ = \$25.
b. To find the cost of using 4 gigabytes of data we must use the second formula $C(g) = 25 + 10(g-2)$ since $g = 4 \geq 2$. So $C(4) = 25 + 10(4-2) = \$45$.

Now You Try: Exercise 1.5.3

At Pierce College during the 2009-2010 school year tuition rates for in-state residents were \$89.50 per credit for the first 10 credits, \$33 per credit for credits 11-18, and for each credit over 18 the rate is \$73 per credit (as reported by Pierce College on August 6, 2010). Write a piecewise defined function for the total tuition, T, at Pierce College during 2009-2010 as a function of the number of credits taken, c. Be sure to consider a reasonable domain for the piecewise defined function

Answer

$$T(c) = \begin{cases} 89.5c & if \quad c \leq 10 \\ 895 + 33(c-10) & if \quad 10 < c \leq 18 \\ 1159 + 73(c-18) & if \quad c > 18 \end{cases}$$ Tuition, T, as a function of credits, c.

A reasonable domain should be whole numbers 0 to (answers may vary), e.g. [0, 23]

Important Topics of this Section

- Using piecewise defined functions
- Graphing with piecewise defined functions
- Modeling with piecewise defined functions

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1.5E: Piecewise-Defined Functions (Exercises)

Section 1.5 Exercises

1. Let $f(x) = \begin{cases} x+1 & if \quad x < 2 \\ 5 & if \quad x \geq 2 \end{cases}$

   a) Evaluate $f(1)$, $f(2)$, $f(3)$, and $f(4)$

   b) Sketch a graph of the piecewise defined function.

2. Let $f(x = \begin{cases} 4 & if \quad x < 0 \\ \sqrt{x} & if \quad x \geq 0 \end{cases}$

   a) Evaluate $f(-1)$, $f(0)$, $f(1)$, and $f(2)$

   b) Sketch a graph of the piecewise defined function.

3. Let $f(x) = \begin{cases} 3x-1 & if \quad x < 1 \\ x+2 & if \quad x \geq 1 \end{cases}$

   a) Evaluate $f(0)$, $f(1)$, $f(2)$, and $f(3)$

   b) Sketch a graph of the piecewise defined function.

4. Let $f(x) = \begin{cases} x^2 & if \quad x < 1 \\ -2x+5 & if \quad x \geq 1 \end{cases}$

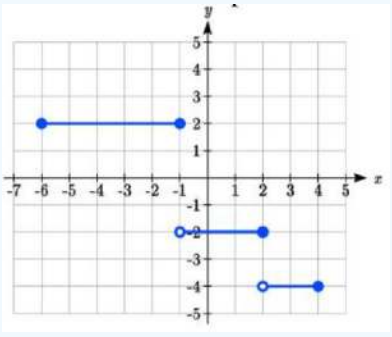
   a) Evaluate $f(0)$, $f(1)$, $f(2)$, and $f(3)$

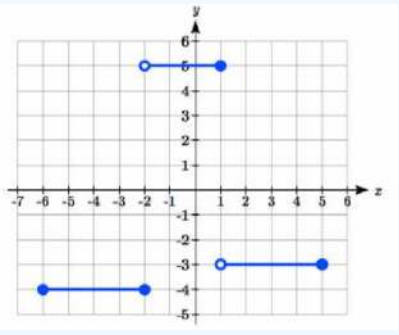
   b) Sketch a graph of the piecewise defined function.

5. Sketch a graph of the piecewise defined function $f(x) = \begin{cases} 3 & if \quad x \leq -2 \\ -x+1 & if \quad -2 < x \leq 1 \\ x+2 & if \quad x > 1 \end{cases}$. Then identify the domain and range of f.

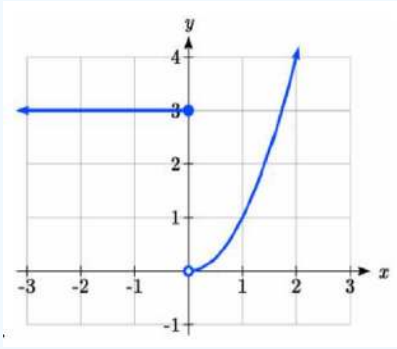
6. Sketch a graph of the piecewise defined function $f(x) = \begin{cases} 2x+1 & if \quad x \leq 0 \\ x+1 & if \quad 0 < x \leq 2 \\ 3 & if \quad x > 2 \end{cases}$ Then identify the domain and range of f.

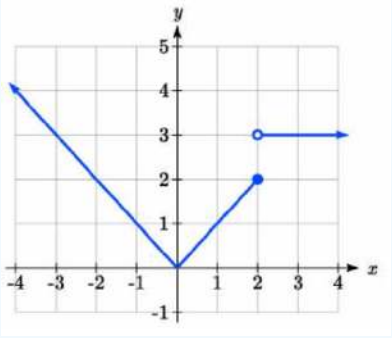
7. Write a formula for the piecewise function graphed below.

a) 

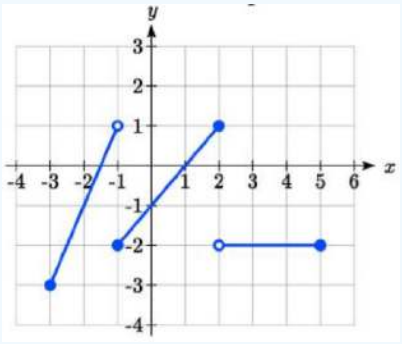
b) 

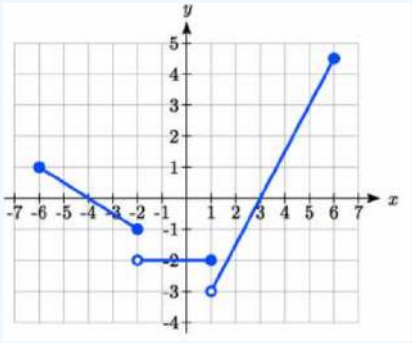
8. Write a formula for the piecewise function graphed below.

a) 

b) 

9. Write a formula for the piecewise function graphed below.

a) 

b) 

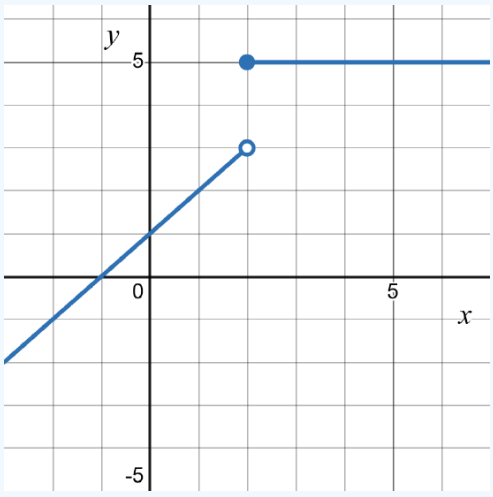
10. Suppose a commercial cloud data plan charges \$5 per year plus \$2.5 per GB for the first 10 GB and then \$1.5 per GB for each additional GB of data. Find a piecewise defined formula for the yearly cost C in terms of the data load D in GB.

11. Suppose a painting company charges \$2.40 per ft^2 to paint a house of up to 2000 ft^2. For houses with more than 2000 ft^2 to paint, they charge \$2.10 per ft^2 for each additional square foot of area above 2000 ft^2. Find a piecewise defined formula for the yearly cost C in terms of the square footage painted A.

12. Suppose an author of a new book receives \$30,000 to write it. For the first 50,000 copies sold, they will make \$1.24 per copy sold. After that, they will make \$1.02 per copy sold. Find a piecewise defined formula for the amount made A in terms of the number of book copies sold b.

Answer

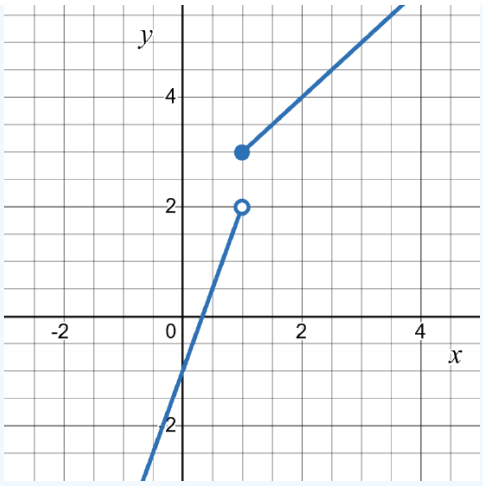
1. a) $f(1) = 2$, $f(2) = 5$, $f(3) = 5$, and $f(4) = 5$

b) 

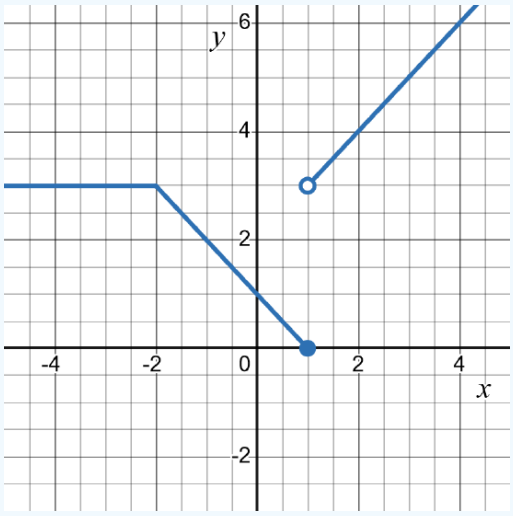
3. a) $f(0) = -1$ $f(1) = 3$ $f(2) = 4$ $f(3) = 4$

https://math.libretexts.org/@go/page/99697

b)



5.



7. a) $f(x) = \begin{cases} 2 & if \quad -6 \leq x \leq -1 \\ -2 & if \quad -1 < x \leq 2 \\ -4 & if \quad 2 < x \leq 4 \end{cases}$

b) $f(x) = \begin{cases} -4 & if \quad -6 \leq x \leq -2 \\ 5 & if \quad -2 < x \leq 1 \\ -3 & if \quad 1 < x \leq 5 \end{cases}$

9. a) $f(x) = \begin{cases} 2x+3 & if \quad -3 \leq x < -1 \\ x-1 & if \quad -1 \leq x \leq 2 \\ -2 & if \quad 2 < x \leq 5 \end{cases}$

11. $C = f(A) = \begin{cases} 2.40A & if \quad 0 \leq A \leq 2000 \\ 2.10A + 600 & if \quad 2000 < A \end{cases}$

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1.6: Combinations and Compositions of Functions

Focus: Combinations of Functions

We need to develop more types of functions to describe relationships. If the rate of change is not constant, we can't use a linear function. If the relationship does not have two or more distinct pieces, we can't use a piecewise defined function. In this section, we will explore techniques to build new functions from old ones using arithmetic operations on the outputs.

Since the outputs of many functions are real numbers, such as $f(1) = 3$ and $g(1) = 5$, arithmetic operations, such as addition, subtraction, multiplication, and division can be performed on those outputs. For example, we can add these outputs for a specific input, such as x = 1,

$$f(1) + g(2) = 3 + 5 = 8$$

We can subtract outputs

$$f(1) - g(1) = 3 - 5 = -2$$

We can multiply outputs

$$f(1) \cdot g(1) = 3 \cdot 5 = 15$$

We can divide outputs

$$\frac{f(1)}{g(1)} = \frac{3}{5} = 0.6$$

If we added the outputs of these two functions for a given set of inputs, we can create a new function. For example, for x = 1,

$$f(1) + g(1) = 3 + 5 = 8$$

For x = 2, if $f(2) = 1$ and $g(2) = 4$, then the sum of outputs

$$f(2) + g(2) = 1 + 4 = 5$$

For x = 3, if $f(3) = 2$ and $g(3) = -1$, then the sum of outputs

$$f(3) + g(3) = 2 + -1 = 1$$

For x = 4, if $f(4) = 6$ and $g(4) = 2$, then the sum of outputs

$$f(4) + g(4) = 6 + 2 = 8$$

For x = 5, if $f(5) = 0$ and $g(5) = -2$, then the sum of outputs

$$f(5) + g(5) = 0 + -2 = -2$$

In general, the formula for this function can be written as $y = f(x) + g(x)$. We call this function **the sum function**. For the function name we use the expression $f + g$ as opposed to a single letter to indicate that it is the function which is the sum of the outputs of f and g.

Definition: Function Arithmetic

Suppose f and g are functions and x is in both the domain of f and the domain of g.

- The **sum function** $f + g$ adds the outputs of f and g and is defined by the formula
$$(f + g)(x) = f(x) + g(x)$$
- The **difference function** $f - g$ subtracts the outputs of f and g and is defined by the formula
$$(f - g)(x) = f(x) - g(x)$$
- The **product function** fg multiplies the outputs of f and g and is defined by the formula
$$(fg)(x) = f(x) \cdot g(x)$$

- The **quotient function** $\frac{f}{g}$ divides the outputs of f and g and is defined by the formula

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \qquad \text{provided } g(x) \neq 0$$

Example 1.6.1

The functions $f(x)$ and $g(x)$ are given in the table below. Use the tables to evaluate the following

a. $(f+g)(3)$
b. $(f \cdot g)(2)$
c. $\left(\frac{f}{g}\right)(1)$
d. $\left(\frac{f}{g}\right)(4)$

The inputs and outputs of functions f and g.

x	1	2	3	4
$f(x)$	4	3	1	-1
$g(x)$	2	5	2	0

Solution

a. Adding the outputs of f and g at $x = 3$ we have

$$(f+g)(3) = f(3) + g(3) = 1 + 2 = 3$$

b. Multiplying the outputs of f and g at $x = 2$ we have

$$f \cdot g)(2) - (f(2))(g(2)) = (3)(5) = 15$$

c. Dividing the outputs of f and g at $x = 1$ we have

$$\left(\frac{f}{g}\right)(1) = \frac{f(1)}{g(1)} = \frac{4}{2} = 2$$

d. Dividing the outputs of f and g at $x = 4$

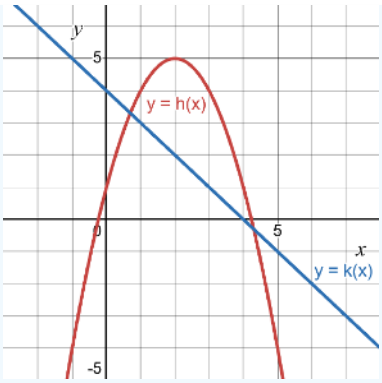
$$\left(\frac{f}{g}\right)(4) = \frac{f(4)}{g(4)} = \frac{-1}{0}$$

is undefined.

Be careful. We can't input values of x into the quotient function that give a zero in the denominator even though they might be valid inputs of f and g in their respective domains.

Example 1.6.2

The functions $h(x)$ and $h(x)$ are given in the table below. Use the tables to evaluate the following



a. $(h-k)(2)$
b. $(h \cdot k)(1)$
c. $\left(\frac{k}{h}\right)(4)$

Solution

a. Subtracting the outputs of f and g at $x=2$ we have

$$(h-k)(2) = h(2) - k(2) = 5 - 2 = 3$$

b. Multiplying the outputs of f and g at $x=1$ we have

$$(h \cdot k)(1) - (h(1))(k(1)) = (4)(3) = 12$$

c. Dividing the outputs of f and g at $x=4$ we have

$$\left(\frac{k}{h}\right)(4) = \frac{k(4)}{h(4)} = \frac{0}{1} = 0$$

Example 1.6.3

Given the functions $f(x) = x - 1$ and $g(x) = x^2 - 1$.

a. Evaluate $(f+g)(2)$
b. Evaluate $\left(\frac{f}{g}\right)(3)$
c. Find a formula for the sum function $(f+g)(x)$ in terms of x.
d. Find a formula for the quotient function $\left(\frac{f}{g}\right)(x)$ in terms of x.

Solution

a. Adding the outputs of f and g at $x=2$, we have

$$(f+g)(2) = f(2) + g(2) = (2-1) + (2^2 - 1) = 1 + 3 = 4$$

b. Dividing the outputs of f and g at $x=3$, we have

$$\left(\frac{f}{g}\right)(3) = \frac{f(3)}{g(3)} = \frac{3-1}{3^2-1} = \frac{2}{8} = \frac{1}{4}$$

c. Adding the outputs of f and g for a general input x and collecting like terms, we have

$$(f+g)(x) = f(x) + g(x) = (x-1) - (x^2-1) = x - 1 - x^2 + 1 = -x^2 + x$$

d. Dividing the outputs of f and $g$for a general input x, we have

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x-1}{x^2-1}$$

Then simplifying

$$= \frac{(x-1)}{(x-1)(x+1)} = \frac{1}{x+1}$$

provided the the input $x \neq 1$ and $x \neq -1$.

In part (c) of the last example, the domain of the sum function is the set of all inputs that are defined inputs of both f and g. Since the f and g are both defined for all real numbers and there is no restriction on adding two numerical outputs, the domain of $(f + g)(x)$ is $(-\infty, \infty)$.

In part (d) of the last example, the domain of the quotient function is the set of all inputs that are defined inputs of both f and g that do not make the denominator zero. Since the f and g are both defined for all real numbers, the only restrictions on the input are those which make the denominator zero, $x \neq 1$ and $x \neq -1$. Therefore, the domain of $\left(\frac{f}{g}\right)$ is all real numbers where $x \neq 1$ and $x \neq -1$. Be careful here. The simplified form of the quotient function by itself is defined at $x = 1$. However, since it is equivalent to the original form of the quotient function, we must have the restriction $x \neq 1$ also.

? Now You Try: Exercise 1.6.1

Given the functions $f(x) = 2x + 1$ and $g(x) = x^2 + 2x$.

a. Evaluate $(f - g)(1)$
b. Evaluate $(f \cdot g)(3)$
c. Find a formula for the difference function $(f - g)(x)$ in terms of x.
d. Find a formula for the product function $(f \cdot g)(x)$ in terms of x.

Answer

a. Subtracting the outputs of f and g at $x = 1$, we have

$$(f - g)(1) = f(1) - g(1) = 0$$

b. Multiplying the outputs of f and g at $x = 3$, we have

$$(f \cdot g)(3) = f(3) \cdot g(3) = 105$$

c. $(f - g)(x) = -x^2 - 1$
d. $(f \cdot g)(x) = 2x^3 + 5x^2 + 2x$

In some cases it may be easier to build a function to describe a relationship in an application using a combination of two simpler functions as shown in the two examples below.

✓ Example 1.6.4

Suppose a manufacturer uses test market data to determine that the price p they can charge for an item is related to the number of items x they can sell by the function $p = 100 - 0.0001x$. Furthermore, the cost of producing the items is given by the function $C(x) = 350,000 + 30x$.

a. Find a formula for the revenue $R(x)$ in terms of the number of items sold.
b. Find a formula for the profit $P(x)$ in terms of the number of items sold.
c. Find the profit from selling and producing 8000 items.

Solution

a. Since revenue typically is regarded as the total sales, we can calculate the revenue from selling x items at a price p as follows

$$R(x) = (\text{price})(\text{number of items sold}) = px = (100 - 0.0001x)x$$

$$R(x) = 100x - 0.0001x^2$$

b. We can calculate the profit from selling x items as a price p using the revenue and cost as follows

$$P(x) = (\text{Revenue}) - (\text{Cost}) = R(x) - C(x) = (100x - 0.0001x^2) - (350,000 + 30x)$$

Then subtracting and simplifying our formula is

$$P(x) = 100x - 0.0001x^2 - 350,000 - 30x$$

$$= -0.0001x^2 + 70x - 350,000$$

c. $P(8000) = -0.0001(8000)^2 + 70(8000) - 350,000 = 203,600$

Notice how much simpler it was to develop the formula for profit from the revenue and cost functions than building the profit function from scratch.

Example 1.6.5

Suppose the population of a country is 2 million and is projected to grow by 4% per year. The population (in millions) t years can be modeled by the function $P(t) = 2(1.04)^t$. The number of people the country's agricultural output can feed is 4 million and is projected to grow by 0.5 million people per year as more land is turned into farmland and agricultural techniques are improved. The number of people the country can feed (in millions) t years can be modeled by the function $N(t) = 4 + 0.5t$. In order to compare the relative prosperity of two countries with regard to their food supply and population, economists build a prosperity index in t years as the ratio of the number of people the country's agricultural output can feed to the population where

$$R(t) = \frac{\text{Number of people fed}}{\text{Population}}$$

a. Find a formula for the prosperity index $R(t)$ in terms of t.
b. Find the prosperity index in 5 years. Interpret the meaning of your result in the context of this application.

Solution

a. The prosperity index is a quotient function

$$R(t) = \frac{N(t)}{P(t)} = \frac{4 + 0.5t}{2(1.04)^t}$$

b. The prosperity in 5 years

$$R(5) = \frac{2(1.04)^5}{4 + 0.5(5)} \approx 2.67$$

Since the prosperity index in five years $R(5) > 1$, the country can feed its population with its agricultural output and export the rest.

Focus: Compositions of Functions

Let's explore another way of building a new function from old ones using a composition of two functions. Since both the inputs and the outputs of many functions are real numbers, we can use the output of one function as the input of another function. For example, with two functions f and g where $g(2) = 5$ and $f(5) = 12$, we can use the output of g at $x = 2$ as the input of f.

$$f(\text{output of g}) = f(g(2)) = f(5) = 12$$

Suppose that $g(1) = 3$ and $f(3) = 7$. Then we can use the output of g at $x = 1$ as the input of f once again.

$$f(\text{output of g}) = f(g(1)) = f(3) = 7$$

If we use the output of g at each value of x in its domain as the input of f, we can create a new function. Suppose the functions f and g are given in the tables below

The inputs and outputs of function f

x	1	2	3	4	5
$g(x)$	3	5	2	1	4

The inputs and outputs of function g

x	1	2	3	4	5
$f(x)$	6	1	7	3	12

Then using the output of g at each value of x in its domain as the input of f

The inputs and outputs of function g and the composition f(g(x))

x	1	2	3	4	5
$g(x)$	3	5	2	1	4
$y = f(g(x))$	$f(3) = 7$	$f(5) = 12$	$f(2) = 1$	$f(1) = 6$	$f(4) = 3$

This gives us a new function that is different from f and g where

The inputs and outputs of the composition

x	1	2	3	4	5
y	7	12	1	6	3

We call a function created in this manner a **composition of functions.**

Definition: Composition of Functions

The **composition of two functions** f and g is a function created using the output of one function g as the input of another function f. We write the composition of functions $f(g(x))$, and read this as "f of g of x" or "f composed with g at x".

We give this function the name $(f \circ g)$ where $(f \circ g)(x) = f(g(x))$.

Be careful, if we do a multistep process in a different order we may get a different result. For example, in the process of putting on your shoes, there three steps done in this order: (1) Put on your socks (2) Put on your shoes (3) Tie your laces. If you do those steps in a different order, such as (1) Put on your shoes (2) Tie your laces (3) Put on your socks, we get a very different and comical result with our socks covering our shoes.

With a composition of functions, the order of composition matters. If we compose the functions in a different order, we may get a different result. If we use the output of the function f as the input of the function g given in the tables above, we create a different function than the first composition $(f \circ g)(x) = f(g(x))$.

The inputs and outputs of function f and the composition g(f(x))

x	1	2	3	4	5
$f(x)$	6	1	7	3	12
$y = g(f(x))$	$g(6)$ is not defined	$g(1) = 3$	$g(7)$ is not defined	$g(3) = 2$	$g(12)$ is not defined

This function we give the name $(g \circ f)$ where $(g \circ f)(x) = g(f(x))$.

Be Careful #1

The order in which you compose two functions matters. If you compose two functions in a different order, you may get a different function.

Also, notice that some of the outputs of f are numbers that are not defined as inputs for the function g. Therefore, we must restrict the domain of $(g \circ f)(x) = g(f(x))$ to the inputs of f where whose outputs $f(x)$ are defined inputs of g. In the example above, the domain of $(f \circ g)$ is D = {1,3}.

Be Careful #2

The domain of the composition $(f \circ g)(x) = f(g(x))$ must be restricted to the inputs of g where the outputs $g(x)$ are defined inputs of f.

Let's build some more functions with compositions.

Example 1.6.6

The functions h and k are given in the tables below.

The inputs and outputs of functions h and k.

x	1	2	3	4
$h(x)$	3	5	2	7
$k(x)$	6	8	3	1

a. Evaluate $(k \circ h)(3)$
b. Evaluate $(h \circ k)(4)$
c. Evaluate $(k \circ h)(4)$
d. Build a table that defines the composition $(k \circ h)(x)$

Solution

a. To evaluate $(k \circ h)(3) = k(h(3))$ we start by evaluate the inside expression $h(3)$ using the table where $h(3) = 2$. We can then use that result as the input to the k function, so

$$k(h(3)) = k(2) = 8$$

b. To evaluate $(h \circ k)(4) = h(k(4))$ we first evaluate the inside expression $k(4)$using the first table $k(4) = 1$. Then using the table for h we can evaluate

$$h(k(4)) = h(1) = 3$$

c. To evaluate $(k \circ h)(4) = k(h(4))$ we start by evaluate the inside expression $h(4)$ using the table where $h(4) = 7$. However 7 is not a defined input for the k function, so $k(h(4))$ is undefined.

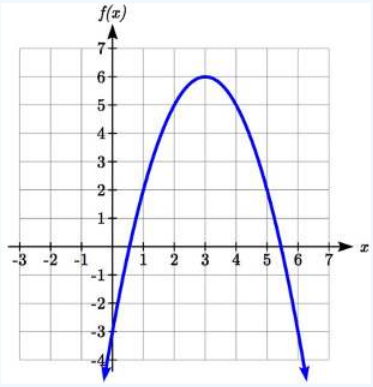
d. Using the output of h as the input of k for each possible

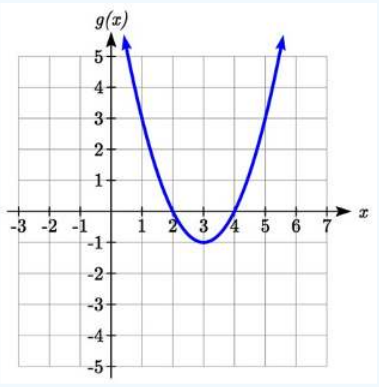
The inputs and outputs for function h and the composition k(h(x))

x	1	2	3	4
$h(x)$	3	5	2	7
$k(h(x))$	3	$k(5)$ is not defined	8	$k(7)$ is not defined

Example 1.6.7

The functions f and g are given in the graphs below.





a. Evaluate $(f \circ g)(1)$
b. Evaluate $(f \circ g)(3)$
c. Evaluate $(g \circ f)(1)$

Solution

a. To evaluate $(f \circ g)(1) = f(g(1))$, we start by using the graph of g to identify $g(1)$, where we have $g(1) = 3$. Using that result as the input to the f function, the graph of f gives us

$$f(g(1)) = f(3) = 6$$

b. To evaluate $(f \circ g)(4) = f(g(4))$, we start by using the graph of g to identify $g(4)$, where we have $g(4) = 0$. Using that result as the input to the f function, the graph of f gives us

$$f(g(4)) = f(0) = -3$$

c. To evaluate $(g \circ f)(4) = g(f(4))$, we start by using the graph of f to identify $f(4)$, where we have $f(4) = 5$. Using that result as the input to the g function, the graph of g gives us

$$g(f(4)) = g(5) = 3$$

Exercise 1.6.2

The functions m and n are given by the set of ordered pairs (x, y) with inputs x and corresponding outputs y below.

m = {(1,2),(2,4),(3,0),(4,1),(5,3)} and n = {(1,4),(2,5),(3,1),(4,2),(5,11)}

a. Evaluate $(m \circ n)(1)$
b. Evaluate $(m \circ n)(2)$
c. Evaluate $(n \circ m)(2)$
d. Build a set of ordered pairs that defines the composition $(n \circ m)$

Answer

a. $(m \circ n)(1) = m(n(1))$ = m(4) = 1
b. $(m \circ n)(2) = m(n(2))$ = m(5) = 3
c. $(n \circ m)(2) = n(m(2))$ = n(4) = 2
d. $(n \circ m)$ = {(1,5),(2,2),(4,4),(5,1)}

Compositions of Functions Using Formulas

We can also build a composition of functions using function given by formulas. When evaluating a composition of functions given by formulas, the concept of working from the inside out remains the same. First, we evaluate the inside function using the input

value provided, then use the resulting output as the input to the outside function

Example 1.6.8

Let $f(x) = 2x + 1$ and $g(x) = x^2 + 3$

a. Evaluate $(f \circ g)(2)$
b. Evaluate $(g \circ f)(1)$
c. Evaluate $(f \circ g)(3)$
d. Find a formula for the composition $f \circ g$ in terms of x.
e. Find a formula for the composition $g \circ f$ in terms of x.

Solution

a. To evaluate $(f \circ g)(2) = (f(g(2))$, we start by evaluating the inside function $g(2)$ using the formula where $g(2) = 2^2 + 3 = 7$. We can then use that result as the input to the outside function f, so

$$f(g(2)) = f(7) = 2(7) + 1 = 15$$

b. To evaluate $(g \circ f)(1) = (g(f(1))$, we start by evaluating the inside function $f(1) = 2(1) + 1 = 3$. Then using this result as the input of the outside function g, so

$$g(f(1)) = g(3) = 3^2 + 3 = 12$$

c. To evaluate $(f \circ g)(3) = (f(g(3))$, we evaluate the inside function $g(3) = 3^2 + 3 = 12$. Then using this result as the input of the outside function f, so

$$f(g(3)) = f(12) = 2(12) + 1 = 25$$

d. To find a formula for the composition $(f \circ g)(x) = (f(g(x))$, we use the output of the inside function $g(x) = x^2 + 3$ as the input of the outside function f, so

$$f(g(x)) = f(x^2 + 3) = 2(x^2 + 3) + 1 = 2x^2 + 6 + 1 = 2x^2 + 7$$

e. To find a formula for the composition $(g \circ f)(x) = (g(f(x))$, we use the output of the inside function $f(x) = 2x + 1$ as the input of the outside function g, so

$$\begin{aligned} g(f(x)) = g(2x + 3) &= (2x + 3)^2 + 3 = (2x + 3)(2x + 3) + 3 \\ &= 4x^2 + 6x + 6x + 9 + 7 = 4x^2 + 12x + 16 \end{aligned}$$

Example 1.6.9

Let $k(x) = x^2$ and $h(x) = \frac{1}{x} - 2x$.

a. Evaluate $(k \circ h)(1)$
b. Find a formula for the composition $k \circ h$ in terms of x.
c. Find a formula for the composition $h \circ k$ in terms of x.

Solution

a. To evaluate $(k \circ h)(1) = (k(h(1))$, we evaluate the inside function $h(1) = \frac{1}{1} - 2(1) = 1 - 2 = -1$. Then using this result as the input of the outside function k, so

$$k(h(1)) = k(-1) = (-1)^2 = 1$$

b. To find a formula for the composition $(k \circ h)(x) = (k(h(x))$, we use the output of the inside function $h(x) = \frac{1}{x} - 2x$ as the input of the outside function k, so

$$k(h(x)) = k(\frac{1}{x} - 2x) = (\frac{1}{x} - 2x)^2$$

c. To find a formula for the composition $(h \circ k)(x) = (h(k(x))$, we use the output of the inside function $k(x) = x^2$ as the input of the outside function h, so

$$h(k(x)) = h(x^2) = \frac{1}{x^2} - 2x^2$$

Example 1.6.10

Let $f(x) = x^3 + 3x$ and $g(x) = \sqrt{x}$.

a. Find a formula for the composition $f \circ g$ in terms of x.
b. Find a formula for the composition $g \circ f$ in terms of x.

Solution

a. Using the output of g as the input of f we have

$$f(g(x) = f(\sqrt{x}) = (\sqrt{x})^3 + 3(\sqrt{x}) = \sqrt{x^3} + 3\sqrt{x}$$

b. Using the output of f as the input of g we have

$$g(f(x) = g(x^3 + 3x) = \sqrt{x^3 + 3x}$$

Now you try: Exercise 1.6.3

Let $f(x) = x^2 - x$ and $g(x) = 3x + 2$, evaluate

a. Evaluate $(f \circ g)(2)$
b. Evaluate $(g \circ f)(1)$
c. Find a formula for the composition $f \circ g$ in terms of x.
d. Find a formula for the composition $g \circ f$ in terms of x.

Answers:

a. $(f \circ g)(2) = (f(g(2)) = f(3(2) + 2) = f(8) = 8^2 - 8 = 56$
b. $(g \circ f)(1) = (g(f(1)) = g(1^2 - 1) = g(0) = 3(0) + 2 = 2$
c. $(f \circ g)(x) = (f(g(x)) = f(3x + 2) = (3x + 2)^2 - (3x + 2) = 9x^2 + 12x + 4 - 3x - 2 = 9x^2 + 9x + 2$
d. $(g \circ f)(x) = (g(f(x)) = g(x^2 - x) = 2(x^2 - x) + 2 = 2x^2 - 2x + 2$

Decomposing Functions

In some cases, it is desirable to view a function as a composition of two simpler functions.

For example, if we let $y = h(x) = \sqrt{x - 3}$. Notice the order of operations done to an input into h:

1. Subtract 3
2. Take the square root.

If we let function g perform the first operation of subtracting three from its input, then $g(x) = x - 3$. If we let function f perform the second operation by taking the square root of its input, then $f(x) = \sqrt{x}$. Then the composition of f and g is our function h

$$y = (f \circ g)(x) = f(g(x)) = f(x - 3) = \sqrt{x - 3}$$

Typically, we refer to the function on the inside of the parenthesis that performs the first set of operations as the "inside function". The inside function would be function g in this example. We refer to the function on the outside of the parenthesis that performs the last set of operations as the "outside function". The outside function would be function f in this example.

Example 1.6.11

Write $f(x) = (5 - 3x)^2$ as the composition of two functions h and g.

Solution

The inside function is represented by the operations that are done first to the input x. In this case, the first operations are those inside the square $5 - 3x$. A formula for the inside function would be

$$h(x) = 5 - 3x$$

Secondly, the outside function is represented by the square which is the operation that is done last in function f. So a formula for the outside function would be

$$g(x) = x^2$$

We can check our answer by recomposing the functions:

$$g(h(x)) = g(5 - 3x) = (5 - 3x)^2$$

Be aware that this is not the only solution to the problem. Another non-trivial decomposition would be to let the inside function be

$$h(x) = -3x$$

and the outside function be

$$g(x) = (5 - x)^2$$

Notice that both compositions do the same three operations on x in the same order.

Example 1.6.12

Write $f(x) = \dfrac{2}{x^2 - 3}$ as the composition of two functions k and m.

Solution

The inside function is represented by the operations that are done first to the input x. The operations in the denominator $x^2 - 3$ are done first. A formula for the inside function would be

$$m(x) = x^2 - 3$$

The next operation is putting that output in the denominator of the fraction. So a formula for the outside function would be

$$k(x) = \frac{2}{x}$$

We can check our answer by recomposing the functions:

$$k(m(x)) = k(x^2 - 3) = \frac{2}{x^2 - 3}$$

Now You Try: Exercise 1.6.4

Write $f(x) = \dfrac{4}{3 - \sqrt{4 + x^2}}$ as the composition of two functions h and g.

Answer

Possible answers:

The inside function is $g(x) = \sqrt{4 + x^2}$

The outside function is $h(x) = \dfrac{4}{3 - x}$

where $f = h \circ g$

Alternatively, we could have

The inside function is $g(x) = 4 + x^2$

The outside function is $h(x) = \dfrac{4}{3 - \sqrt{x}}$

where $f = h \circ g$

This technique of decomposing a function into the composition of two simpler functions is critical for use later in Calculus and calculating the rate of change of a composition of functions.

Applications of Composite Functions

In some cases, it may be easier to build a function modeling an application using a composition of two simpler functions.

Suppose we wanted to calculate how much it costs to heat a house on a particular day of the year. The cost to heat a house depends on the average daily temperature and the average daily temperature depends on the particular day of the year. Notice how we have just defined two relationships: The temperature depends on the day and the cost depends on the temperature.

The first function, $C = g(T)$, gives the cost C of heating a house when the average daily temperature is T degrees Fahrenheit, and the second, $T = f(d)$, gives the average daily temperature on day d of the year in some city. If we wanted to determine the cost of heating the house on the 5 th day of the year, we could form the composition of functions $C = f(g(d))$

$$d \to T \to C$$

We could evaluate function g at $d = 5$ to determine the average daily temperature on the 5 th day of the year. We could then use that temperature as the input to function f to find the cost to heat the house on the 5 th day of the year. In other words, we could evaluate $f(g(5))$.

Example 1.6.13

If a patient has an initial heart rate of 60 bpm and each additional milligram of a medication causes an increase in their heart rate of 0.2 bpm (beats per minute), a formula for the heart rate R in terms of the level of medication L is given by $R = f(L) = 60 + 0.2L$.

If the patient is given an injection of 250 mg of the medication and the level of medication L will decrease by 20% each hour as the body metabolizes the medication, the function $L = g(t) = 250(0.80)^t$ is a formula for the medication level in terms of the hours t since the medication was injected. Find a formula for the heart rate R in terms of the the time t.

Solution

If first use function g to relate the time t to the medication level L, then use the function f to relate the medication level L to the heart rate R,

$$t \to L \to R$$

then we can use the composition $R = f(g(t))$ as a formula for the heart rate R in terms of the the time t. So

$$R = f(g(t)) = f(250(0.80)^t) = 60 + 0.2(250(0.80)^t) = 60 + 50(0.80)^t$$

Note that it is essential when using a composition in an application that the quantity and their corresponding units of the output of the inside function match that of the input to the outside function.

Now You Try: Exercise 1.6.5

A city manager determines that the tax revenue, R, in millions of dollars collected on a population of p thousand people is given by the formula $R = r(p) = 0.03p + \sqrt{p}$, and that the city's population, in thousands, is predicted to follow the formula $P = p(t) = 60 + 2t + 0.3t^2$ where t is measured in years after 2010. Find a formula for the tax revenue as a function of the year.

Answer

$$R = r(p(t)) = r(60 + 2t + 0.3t^2) = 0.03(60 + 2t + 0.3t^2) + \sqrt{60 + 2t + 0.3t^2} = 1.8 + 0.06t + 0.009t^2 + \sqrt{60 + 2t + 0.3t^2}$$

Notice, the composition is this example gives us a single formula which can be used to predict the tax revenue during a given year, without needing to find the intermediary population value.

For example, to predict the tax revenue in 2017, when t = 7 (because t is measured in years after 2010),

$$R(p(7)) = 0.03(60 + 2(7) + 0.3(7)^2) + \sqrt{60 + 2(7) + 0.3(7)^2} \approx 12.079 \text{ million dollars}$$

Important Topics in This Section

- Combinations of Functions
- Compositions of Functions
- Decomposition of Functions

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1.6E: Combinations and Compositions of Functions (Exercises)

SECTION 1.6 EXERCISES

1. Use the table of values to evaluate each expression

The inputs and outputs of functions f and g

x	0	1	2	3	4	5
$f(x)$	7	2	1	0	12	4
$g(x)$	5	3	2	1	4	0

a. $(f+g)(1)$
b. $(f \cdot g)(2)$
c. $(\frac{f}{g})(3)$
d. Write a table of values that defines the function $(f+g)(x)$
e. Write a table of values that defines the function $(f \cdot g)(x)$
f. $(f \circ g)(1)$
g. $(f \circ g)(2)$
h. $(f \circ g)(4)$
i. $(g \circ f)(1)$
j. $(g \circ f)(3)$
k. $(g \circ f)(4)$
l. Write a table of values that defines the function $(f \circ g)(x)$

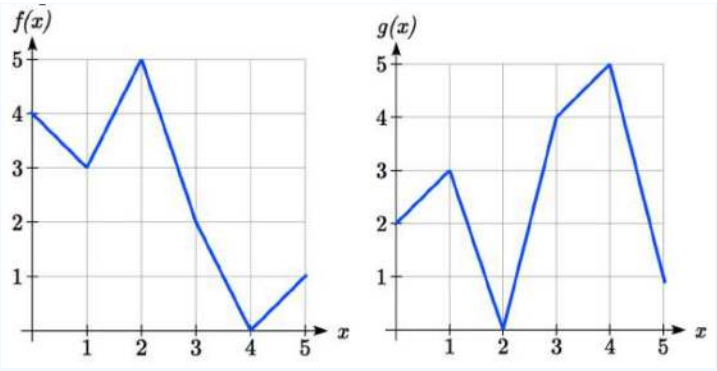
2. Use the table of values to evaluate each expression

The inputs and outputs of functions k and h

x	-2	-1	0	1	2	3
$h(x)$	3	1	0	-1	-2	1
$k(x)$	7	3	2	3	5	-1

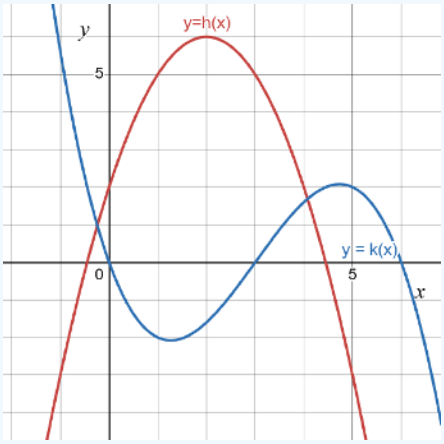
a. $(h-k)(-2)$
b. $(h \cdot k)(1)$
c. $(\frac{h}{k})(3)$
d. $(\frac{k}{h})(0)$
e. Write a table of values that defines the function $(h-k)(x)$
f. Write a table of values that defines the function $(\frac{k}{h})(x)$
g. $(k \circ h)(1)$
h. $(k \circ h)(2)$
i. $(h \circ k)(1)$
j. $(h \circ k)(2)$
k. Write a table of values that defines the function $(h \circ k)(x)$

3. Use the graphs to evaluate the expressions below.



a. $(f+g)(2)$
b. $(f \cdot g)(1)$
c. $(\frac{f}{g})(2)$
d. $(f \circ g)(1)$
e. $(f \circ g)(2)$
f. $(g \circ f)(1)$
g. $(g \circ f)(2)$

4. Use the graphs to evaluate the expressions below.



a. $(h-k)(5)$
b. $(h \cdot k)(1)$
c. $(\frac{h}{k})(3)$
d. $(\frac{k}{h})(0)$
e. $(k \circ h)(1)$
f. $(k \circ h)(2)$
g. $(h \circ k)(0)$
h. $(h \circ k)(1)$

5. Let $f(x) = 4x + 8$ and $g(x) = 7 - x^2$

a. Evaluate $(f+g)(1)$
b. $(f \cdot g)(1)$
c. $(\frac{f}{g})(2)$
d. Find a formula for the sum function $f+g$ in terms of x.
e. Find a formula for the product function $f \cdot g$ in terms of x.
f. Evaluate $(f \circ g)(1)$

g. Evaluate $(g \circ f)(1)$
h. Evaluate $(f \circ g)(3)$
i. Find a formula for the composition $f \circ g$ in terms of x.
j. Find a formula for the composition $g \circ f$ in terms of x.

6. Let $f(x) = \sqrt{x+4}$ and $g(x) = x^2 + 5x$

a. Evaluate $(f - g)(5)$
b. $(f \cdot g)(1)$
c. $(\frac{g}{f})(2)$
d. Find a formula for the function $f - g$ in terms of x.
e. Find a formula for the function $\frac{f}{g}$) in terms of x.
f. Evaluate $(f \circ g)(1)$
g. Evaluate $(g \circ f)(5)$
h. Evaluate $(f \circ g)(3)$
i. Find a formula for the composition $f \circ g$ in terms of x.
j. Find a formula for the composition $g \circ f$ in terms of x.

7. Let $k(x) = \dfrac{1}{x+2}$ and $h(x) = 4x + 3$

a. Evaluate $(k + h)(1)$
b. $(h \cdot k)(1)$
c. $(\frac{k}{h})(-2)$
d. Find a formula for the function $f + g$ in terms of x.
e. Find a formula for the function $f \cdot g$ in terms of x.
f. Evaluate $(k \circ h)(1)$
g. Evaluate $(h \circ k)(-1)$
h. Evaluate $(k \circ h)(3)$
i. Find a formula for the composition $h \circ k$ in terms of x.
j. Find a formula for the composition $k \circ h$ in terms of x.

8. Find functions $f(x)$ and $g(x)$ so the given function can be expressed as the composition of f and g where $h(x) = f(g(x))$.

a. $h(x) = (x+2)^2$
b. $h(x) = \dfrac{3}{x-5}$
c. $h(x) = 3 + \sqrt{x-2}$

9. Find functions $f(x)$ and $g(x)$ so the given function can be expressed as the composition of f and g where $h(x) = f(g(x))$.

a. $h(x) = (x-5)^3$
b. $h(x) = \dfrac{4}{(x+2)^2}$
c. $h(x) = 4 + \sqrt[3]{x+1}$

10. The function $x = D(p) = -2p + 300$ gives the number of items that will be demanded when the price is p. The production cost, $C = f(x) = 5x + 200$ is the cost of producing x items.

a. Find a formula for a function $C = k(p)$ that gives the production cost C when a price p is charged.
b. What is the production cost when a price of $24 is charged.

11. The US Office of Emergency Management calculates risk by multiplying the probability of hazard by the degree of vulnerability. If a natural disaster, such as a hurricane or a wild fire, was expected to impact a certain region, scientists would perform a disaster risk assessment using probability equations to evaluate the potential liability and loss to property and even human life. Suppose the probability of hazard in t years in a particular region is given by $H = f(t) = 0.0012t$. Suppose the degree of vulnerability of the same region is given by $V = g(t) = 0.24(1.013)^t$.

a. Find a formula $R = k(t)$ for the risk R to the region in t years.

b. Use your formula to calculate the risk to the region in 10 years.

12. The radius r, in inches, of a spherical balloon is related to the volume, V, by the function $r = f(V) = \sqrt[3]{\frac{3V}{4\pi}}$. Air is pumped into the balloon, so the volume after t seconds is given by $V = g(t) = 10 + 20t$.

a. Find a formula for a function $r = k(t)$ that gives the radius r after t seconds.
b. Use your formula to find the radius after 20 after 20 seconds.

13. The number of bacteria in a refrigerated food product is given by $N(T) = 23T^2 - 56T + 1$, $3 < T < 33$, where T is the temperature of the food. When the food is removed from the refrigerator, the temperature is given by $T(t) = 5t + 1.5$, where t is the time in hours.

a. Find a formula $N = f(t)$for the number of bacteria in the refrigerated food product in t hours.
b. Find the bacteria count after 4 hours

Answer

1. a) 5 b) 2 c) 0 d) & e) are shown in the tables below

x	0	1	2	3	4	5
$y = (f + g)(x)$	12	5	3	1	16	4

x	0	1	2	3	4	5
$y = (f \cdot g)(x)$	35	6	2	0	48	0

f) 0 g) 1 h) 12 i) 2 j) 5 k) undefined l) shown below in the table

x	0	1	2	3	4	5
$y = f(g(x))$	4	0	1	2	12	7

3. a) 5 b) 9 c) undefined d) 2 e) 4 f) 4 g) 1

5. a) 18 b) 48 c) $\frac{16}{3}$ d) $(f + g)(x) = -x^2 + 4x + 15$ e) $(f \cdot g)(x) = 4x^3 - 8x^2 + 28x + 56$ f) 32 g) -137 h) 0 i) $(f(g(x)) = -4x^2 + 36$ j) $(g(f(x)) = -16x^{-}64x - 57$

9. a) $f(x) = x^3$ and $g(x) = x - 5$ b) $f(x) = \frac{4}{x^2}$ and $g(x) = x + 2$ c) $f(x) = 3 + \sqrt[3]{x}$ and $g(x) = x + 1$

11. $R = k(t) = (0.24(1.013)^t)(0.0012t) = 0.000288(1.013)^t$

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1.7: Inverse Functions

Focus: Understanding Inverse Functions

As we look to build new functions to describe relationships, we explore an important type of functions, inverse functions. On a basic level, inverse functions are functions that undo each other. For example, if f is the function that adds 2 to its input, then the function g that subtracts 2 from its input undoes this. If h is the function that multiplies its input by 2, then the function k that divides 2 from its input undoes this. If m is the function that takes the square root of its input, then the function n that squares its input undoes this.

$$x \underset{g \text{ subtracts } 2}{\overset{f \text{ adds } 2}{\rightleftarrows}} y \qquad x \underset{k \text{ divides by } 2}{\overset{h \text{ multiplies by } 2}{\rightleftarrows}} y \qquad x \underset{n(y)=y^2}{\overset{m(x)=\sqrt{x}}{\rightleftarrows}} y$$

Definition: Inverse Function (Conceptual Definition)

A function g is the inverse of a function f if the operations of function g undo the operations of function f. The inverse of $f(x)$ is typically notated with the function name f^{-1}, which is read "f inverse ". So we write $x = g(y) = f^{-1}(y)$

Be Careful

We are now using negative exponents in two contexts in mathematics. With numbers and variables, an exponent of negative one means we have one factor in the denominator. For example, $3^{-1} = \frac{1}{3^1}$ and $x^{-1} = \frac{1}{x^1}$. In the context of a function, f^{-1} refers to the inverse function of f.

Notice what a function and its inverse do to their inputs and outputs. For the function f which adds two to the input, if we input $x = 3$, we get an output of $y = 5$. For its inverse f^{-1} which subtracts two from its input, if we input $y = 5$, we get an output of $x = 3$. So

$$f(3) = 5 \text{ and } f^{-1}(5) = 3$$

$$x = 3 \underset{x=f^{-1}(y)=y-2}{\overset{y=f(x)=x+2}{\rightleftarrows}} y = 5$$

Similarly, for the function h which multiplies its input by two, if we input $x = 3$, we get an output of $y = 6$. For its inverse h^{-1} which divides its input by two, if we input $y = 6$, we get an output of $x = 3$. So

$$h(3) = 6 \text{ and } h^{-1}(6) = 3.$$

$$x = 3 \underset{x=h^{-1}(y)=\frac{y}{2}}{\overset{y=h(x)=2x}{\rightleftarrows}} y = 6$$

Similarly, for the function m which takes the square root of input, if we input $x = 9$, we get an output of $y = 3$. For its inverse m^{-1} which divides its input by two, if we input $y = 6$, we get an output of $x = 3$. So

$$m(9) = 3 \text{ and } m^{-1}(3) = 9$$

$$x = 9 \underset{x=m^{-1}(y)=y^2}{\overset{y=m(x)=\sqrt{x}}{\rightleftarrows}} y = 3$$

Notice that in each case the inputs and outputs of a function and its inverse are switched. The inputs of f are the outputs of f^{-1} and the outputs of f are the inputs of f^{-1}.

Key Point: 1.7.1

The inputs and outputs of a function f and its inverse f^{-1} are switched. The inputs of f are the outputs of f^{-1} and the outputs of f are the inputs of f^{-1}.

In other words, the domain of f is the range of the inverse f^{-1} and the range of f is the domain of the inverse f^{-1}.

Example 1.7.1

Function $y = f(x)$ is given by the table below. Write a second table that defines the inverse f^{-1}.

The inputs and outputs of function f.

input, x	1	2	3	4
output, $y = f(x)$	2	5	8	3

Solution

the inputs and outputs of the inverse function

input, y	2	5	8	3
output, $x = f^{-1}(y)$	1	2	3	4

Example 1.7.2

The function g is given by the ordered pairs g = {(1,2), (2,7), (3,4), 4,6)}. Write a set of ordered pairs for the inverse function g^{-1}.

Solution

The inputs and outputs of g and g^{-1} are switched. So g^{-1} = {(1,2), (2,7), (3,4), 4,6)}.

Example 1.7.3

The function $P = f(t) = 3000 + 200t$ gives the population of a town in t years. Evaluate and interpret the meaning of the following:

a. $f(20)$
b. $f^{-1}(20,000)$

Solution

Since f relates the years t to the population P, the inverse f^{-1} relates the population P to the years t.

$$t \underset{f^{-1}}{\overset{f}{\rightleftarrows}} P$$

a. For function f the input represents the years t. So t = 20 years . To find the output $P = f(20)$ we substitute for t and

$$P = f(20) = 3000 + 200(20) = 7000$$

In 20 years, the town will have a population of 7000 people.

b. For the inverse function f^{-1} the input represents the population P. So P = 20,000 people. To find the output $t = f^{-1}(20,000)$ we substitute for P into the formula for f

$$20000 = 3000 + 200t$$

Then solving for t

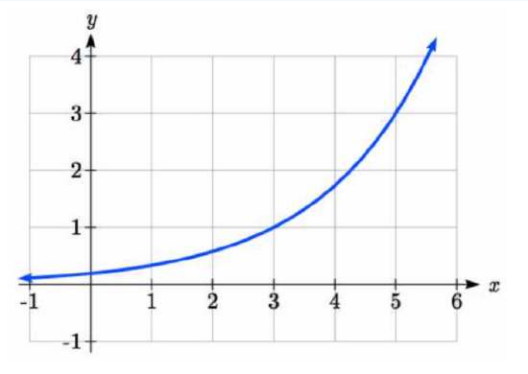
$$17000 = 200t$$

$$t = \frac{17000}{200} = 85$$

In 85 years, the town will have a population of 85,000 people.

Now You Try: Exercise 1.7.1

A function $g(x)$ is given as a graph below. Find $g(3)$ and $g^{-1}(3)$



Answer

$g(3) = 1$ and $g^{-1}(3) = 5$

Now You Try: Exercise 1.7.2

The formula $C = g(F) = \frac{5}{9}(F - 32)$ gives the temperature in Celsius degrees C for a given temperature in Fahrenheit.

Evaluate and interpret the meaning of the following:

a. $g(50)$
b. $g^{-1}(50)$

Answer

1. $C = g(50) = 10$. A temperature of 10 degrees Celsius is equivalent to 50 degrees Fahrenheit.
2. $F = g^{-1}(50) = 122$. A temperature of 122 degrees Fahrenheit is equivalent to 50 degrees Celsius.

Now You Try: Exercise 1.7.3

A function $f(t)$ is given as a table below, showing distance in miles that a car has traveled in t minutes. Find and interpret $f^{-1}(70)$

t (minutes)	30	50	70	90
$f(t)$ (miles)	20	40	60	70

Answer

$f^{-1}(70) = 90$. To drive 70 miles, it took 90 minutes.

Focus: Finding a Formula for the Inverse Function

Since the function f and its inverse f^{-1} switch their inputs and outputs, if $y = f(x)$ where inputting x into f gives the output y, then $x = f^{-1}(y)$ where inputting y into f^{-1} gives the output x. Essentially, the inverse function f^{-1} is a formula for x where x is the original input of function f and y is the original output of function f.

$$x \underset{x=f^{-1}(y)}{\overset{y=f(x)}{\rightleftarrows}} y$$

Key Point: 1.7.2

The formula for the inverse f^{-1} of a function $y = f(x)$ is a formula for x with $x = f^{-1}(y)$ where x is the original input of function f and y is the original output of function f.

So to find a formula for the inverse function f^{-1}, we need only solve for x using the formula for $y = f(x)$.

Example 1.7.4

Let $y = f(x) = 2x + 1$. Find a formula for the inverse f^{-1}.

Solution

The inverse function is a formula for x. Solving for x in the formula for f

$$y = 2x + 1$$

Subtracting one from both sides

$$y - 1 = 2x$$

Then dividing by two on both sides

$$x = \frac{y-1}{2}$$

Therefore, the formula for the inverse is $x = f^{-1}(y) = \frac{y-1}{2}$

Example 1.7.5

Let $y = g(x) = \frac{x-3}{x+1}$. Find a formula for the inverse g^{-1}.

Solution

Again, the inverse function is a formula for x. Solving for x in the formula for g

$$y = \frac{x-3}{x+1}$$

Clearing the fraction by multiplying both sides by $x + 1$ and distributing

$$y(x+1) = x - 3$$

$$xy + y = x - 3$$

Then getting the x terms on one side

$$xy - x = -3 - y$$

Factoring out a common factor and dividing to solve for x

$$x(y-1) = -3 - y$$

$$x = \frac{-3-y}{y-1}$$

Therefore, the formula for the inverse is $x = g^{-1}(y) = \frac{-3-y}{y-1}$

? Now You Try: Exercise 1.7.4

Let $y = f(x) = 5\sqrt[3]{x-1} + 2$. Find a formula for the inverse f^{-1}.

Answer

$x = g^{-1}(y) = (\frac{y-2}{5})^2 + 1$

Focus: Determining whether functions are inverses

Notice what happens when we compose a function and its inverse. For example, if function f adds 2 to its input and its inverse f^{-1} subtracts 2 from its input. If we compose them, applying function f which adds two to an input x and then apply the inverse f^{-1} which subtracts two, we have effectively done nothing to the input x yielding the original input x. So $f^{-1}(f(x)) = x$. Similarly, $f(f^{-1}(y)) = y$.

$$x \xrightarrow[\text{adds 2}]{f(x)=x+2} \xrightarrow[\text{subtracts 2}]{f^{-1}(x+2)=x+2-2} x$$

$$y \xrightarrow[\text{subtracts 2}]{f^{-1}(y)=y-2} \xrightarrow[\text{adds 2}]{f(y-2)=y-2+2} y$$

Key Point: Mathematical Definition of Inverse Functions 1.7.3

Two functions f and g are inverses if and only if their composition gives the original input. That is, $g(f(x)) = x$ and $f(g(x)) = x$

This key point gives us a way to test if two functions are inverses. If their compositions yields the original input, they are inverses. If not, they are not inverses.

Example 1.7.6

Let $f(x) = 2x + 1$, $g(x) = \frac{x-1}{2}$, and $h(x) = \frac{x}{2} - 1$. Determine which of the functions are inverses of one another.

Solution

To test if f and g are inverses, evaluate their composition.

$$f(g(x)) = f(\frac{x-1}{2}) = 2(\frac{x-1}{2}) + 1$$

Simplifying by reducing a common factor of 2 and then adding, we have

$$f(g(x)) = x - 1 + 1 = x$$

Similarly,

$$g(f(x)) = g(2x+1) = \frac{(2x+1)-1}{2}$$

Simplifying in the numerator, then reducing a common factor of 2 we have

$$g(f(x)) = \frac{2x}{2} = x$$

Therefore, f and g are inverses.

To test if f and h are inverses, evaluate their composition.

$$f(h(x)) = f(\frac{x}{2} - 1) = 2(\frac{x}{2} - 1) + 1$$

Distributing 2 and then combining like terms, we have

$$f(h(x)) = \frac{2x}{2} - 2 + 1 = x - 1 \neq x$$

Similarly,

$$h(f(x)) = h(2x + 1) = \frac{2x + 1}{2} - 1$$

Dividing by 2, then simplifying we have

$$h(f(x)) = \frac{2x}{2} + \frac{1}{2} - 1 = x - \frac{1}{2} \neq x$$

Therefore, f and h are not inverses.

Also, g and h are not inverses, since f and g are inverses but f and h are not.

Notice the operations of f and their order in example 6 above:

1. Multiply by 2
2. Add 1

The individual operations of f are undone by the individual operations of g and h. Multiplying by two is undone by dividing by two. Adding 1 is undone by subtracting 1. However, notice that g and h perform their operations in reverse order. The operations of g in order are

1. Subtract 1
2. Divide by 2

Whereas the operations of g in order are

1. Divide by 2
2. Subtract 1

Just because the individual operations of a function may undo the individual operations of another function, the order of multiple operations is important. Notice that the operations of $g = f^{-1}$ undo the operations of f in reverse order.

Consider, the process of putting your shoes:

1. Put on your socks
2. Put on your shoes
3. Tie your laces

Notice the order of the steps undoing putting on your shoes (taking them off) are undoing the operations of putting on your shoes in reverse order.

1. Untie your laces
2. Take off your shoes
3. Take of your socks

In general, if a function f has a single variable term, the inverse of a function f will undo the operations of f in reverse order. However, it is simpler to just test the composition of two functions to see if this gives you the original input x rather than looking at the individual operations. Moreover, this simplistic approach of looking at the operations doesn't apply to functions with multiple variable terms like that in example 1.7.7. However, testing the composition will work.

Example 1.7.7

Let $f(x)=\dfrac{x-3}{x+1}$ and $g(x)=\dfrac{x+3}{1-x}$. Determine if f and g are inverses.

Solution

To test if f and g are inverses, evaluate their composition.

$$f(g(x))=f(\frac{x+3}{1-x})=\frac{\frac{x+3}{1-x}-3}{\frac{x+3}{1-x}+1}$$

Getting a common denominator in the fractions in the numerator and denominator, we have

$$f(g(x))=\frac{\frac{x+3}{1-x}-3(\frac{1-x}{1-x})}{\frac{x+3}{1-x}+\frac{1-x}{1-x}}$$

Distributing and adding the fractions in the numerator and denominator, we have

$$f(g(x))=\frac{\frac{x+3-3+3x}{1-x}}{\frac{x+3+1-x}{1-x}}=\frac{\frac{4x}{1-x}}{\frac{4}{1-x}}$$

To divide we multiply by the reciprocal and then simplifying, we have

$$f(g(x))=(\frac{4x}{1-x})(\frac{1-x}{4})=\frac{4x}{4}=x$$

Similarly, $g(f(x))=x$. Therefore, f and g are inverses.

Now You Try: Exercise 1.7.5

Let $f(x)=4x^3-5$ and $g(x)=\sqrt[3]{\dfrac{x+5}{4}}$. Determine if f and g are inverses.

Answer

Yes, they are inverses since $g(f(x))=x$ and $f(g(x))=x$.

This property of inverse functions is important in fields like cryptography which allows for the secure transmission of important information like passwords. An encryption function f transforms a password into an unreadable or meaningless format (called cipher text). The decryption function is the inverse functions f^{-1} which transforms the cipher text back into the original password. When a person enters their password into a secure website, the website applies the encryption function f to the password before it is sent out via the internet and/or WiFi, ensuring that any third party that intercept the transmission will only see the meaningless cipher text. Once the converted password cipher text is received at the server, the server applies the decryption function f^{-1} to convert the cipher text back into the original password. This ensures that only authorized parties with the correct decryption key can access the original password. For example, if we had a basic encrytion function f where $a\to 1$, $b\to 2$, $c\to 3$, $d\to 4$, and so on, then the decryption function f^{-1} would be $1\to a$, $2\to b$, $3\to c$, $4\to d$, and so on. If we chose a bad password such as "bad", the encryption function f would transform bad $\to$ 123 and the decryption function f^{-1} would transform 123 $\to$ bad. In the field of cryptography, it is essential that an encryption function is designed so the the inverse is not easily identified by a third party.

Focus: Problems with Using Inverse Functions

Consider, the two functions f and g listed in the tables below.

The inputs and outputs of f

x	1	2	3	4
$y = f(x)$	3	4	5	1

The inputs and outputs of g

x	1	2	3	4
$y = g(x)$	2	4	4	5

Notice each is a function since each input is related to exactly one output. However, when we look at their inverse functions described in the tables below where the inverse functions relate each value of y back to the corresponding value of x, we see that f^{-1} is a function but that g^{-1} in not.

The inputs and outputs of the inverse of f

y	3	4	5	1
$x = f^{-1}(y)$	1	2	3	4

Each input of f^{-1} is related to exactly one output. So f^{-1} is a function.

The inputs and outputs of the inverse of g

y	2	4	4	5
$x = g^{-1}(y)$	1	2	3	4

Notice the input $y = 4$ of g^{-1} is related to two outputs $x = 2$ and $x = 3$. So g^{-1} is not a function for these values of x and y.

Problem #1

The inverse $x = f^{-1}(y)$ of a function $y = f(x)$ may not be a function for all values x and y.

Notice with function f each input x was related to a unique output y. So then for the inverse f^{-1} to that unique y back to x ensuring that the inverse f^{-1} was a function. The property of function f is what we call a one-to-one function.

Definition: A One-to-One Function

A function $y = f(x)$ is **a one-to-one function** if each input x was related to a unique output y.

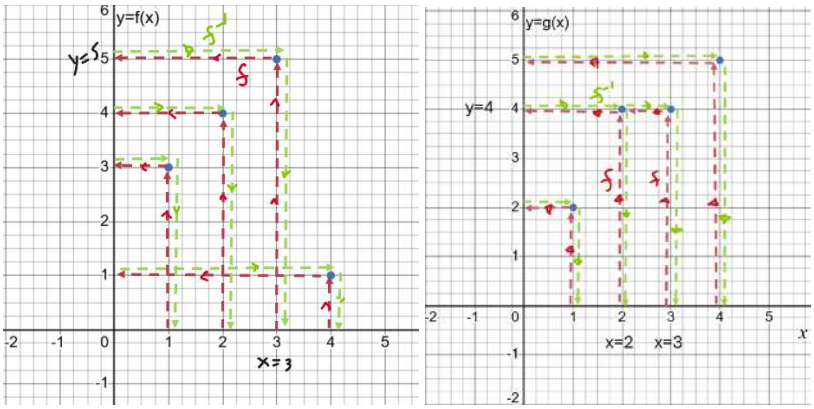
Key Point 1.7.4

If a function $y = f(x)$ is one-to-one on its domain, then its inverse $x = f^{-1}(y)$ is also a function for those values of x in the domain of f.

Notice function g is not one-to-one since the inputs $x = 2$ and $x = 3$ are associated to the same output $y = 4$ and that the inverse g^{-1} is not a function. However, if we restricted the domain of g to $\{x \mid 1,2,4\}$, then the inverse g^{-1} would be a function since the input $y = 4$ of g^{-1} would only be related to a single output $x = 2$. In many cases, it is desirable to have an inverse for a function even though the original function is not one-to-one. In those cases, we can often limit the domain of the original function to an interval or other restricted domain on which the function *is* one-to-one as we did with function g, then find an inverse only on that restricted domain.

In general, determining if a function is one-to-one by looking at each input and determining if its corresponding output is not easily feasible at this level especially with a large or infinite domains. However, we have a much simpler approach using the graph of a

function to determine if the function is one-to-one and if the inverse is a function. Let's look at the graphs of functions f and g from the example above.



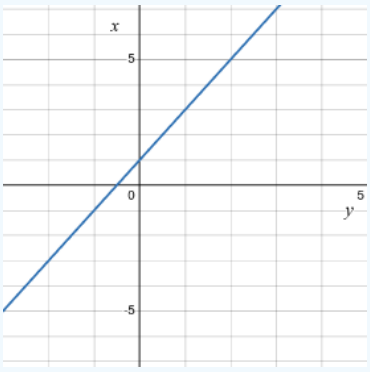
The pair of inputs $x = 2$ and $x = 3$ of g were related to the same output $y = 4$. Therefore, the inverse g^{-1} must relate $y = 4$ back to both $x = 2$ and $x = 3$ as shown in the graph above causing g^{-1} not to be a function and g not to be one-to-one. We can identify this occurring on the graph of g, if we follow the horizontal arrow from $y = 4$ to the two points on the graph, then down to the two inputs. In general, if a horizontal line can be drawn that intersects the graph of a function in two or more points, then the same situation occurs: the inverse must relate a single value of y back to two or more values of x causing the inverse not to be a function. However, for function f every horizontal line intersects the graph at only one point, reflecting the fact that the inverse f^{-1} relates each value of y back to a single value of x, making f a function. This is the idea behind the horizontal line test. It gives us a quick easy was of determining if the inverse of a function is itself a function.

Key Point: The Horizontal Line Test 1.7.5

If a horizontal line can be drawn that intersects the graph of a function f in two or more points, then the inverse f^{-1} is not a function on the specified domain of f.

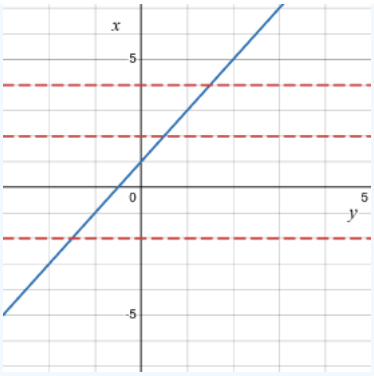
Example 1.7.8

Let $f(x) = 2x + 1$. Use the graph below to determine if the inverse f^{-1} is a function for all real values of x? If not, how can you restrict the domain of f so that the inverse f^{-1} is a function on the restricted domain?



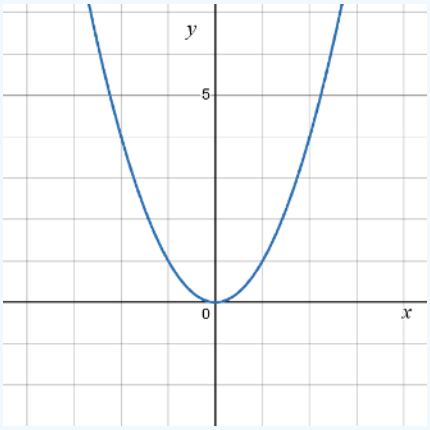
Solution

The inverse f^{-1} is a function since the graph of f passes the horizontal line test. There is no horizontal line that intersects the graph of the linear function f.



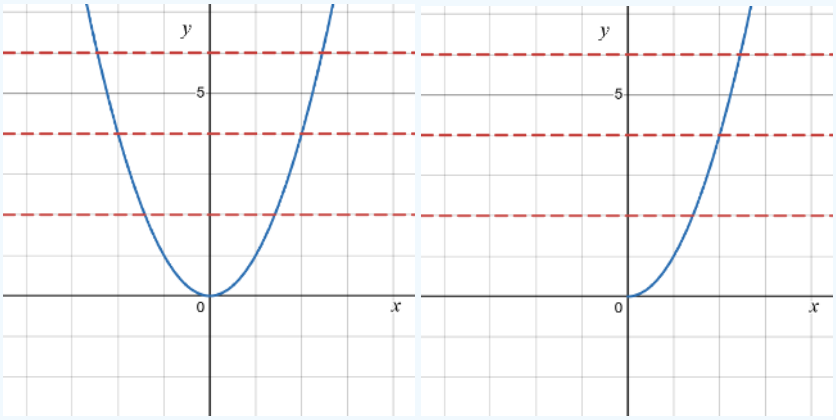
Example 1.7.9

Let $g(x) = x^2$. Use the graph of the parabola below to determine if the inverse g^{-1} is a function for all real values of x? If not, how can you restrict the domain of g so that the inverse g^{-1} is a function on the restricted domain?



Solution

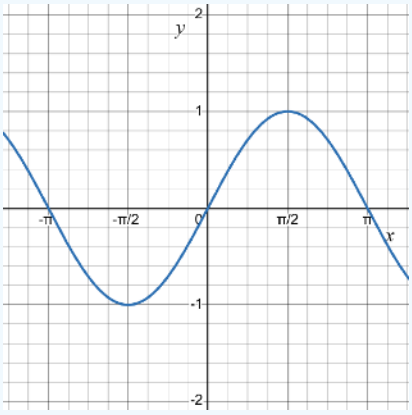
No, the inverse g^{-1} is not a function for all real values of x since it fails the horizontal line test. We can limit the domain of g to $[0, \infty)$ so that that portion of the graph passes the horizontal line test and g^{-1} is a function on the restricted domain of g.



You may have already guessed that since we undo a square with a square root, the inverse of $g(x) = x^2$ on this restricted domain is $g^{-1}(x) = \sqrt{x}$.

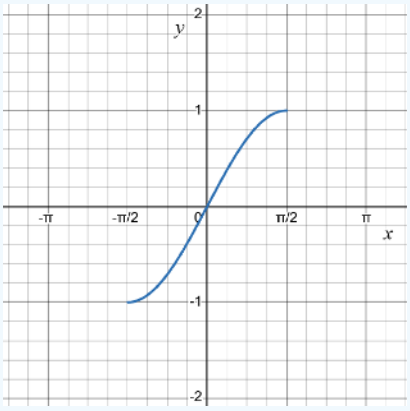
Example 1.7.10

Use the graph of the sine function $y = sin(x)$ from trigonometry below to determine if its inverse sin^{-1} is a function for all real values of x? If not, how can you restrict the domain of $y = sin(x)$ so that the inverse sin^{-1} is a function on the restricted domain?



Solution

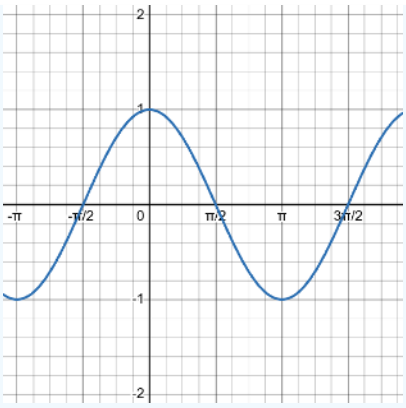
No, sin^{-1} is not a function for all real values of x since it fails the horizontal line test. We can limit the domain of $h(x) = sin(x)$ to $[-\frac{\pi}{2}, \frac{\pi}{2}]$ so that that portion of the graph passes the horizontal line test and sin^{-1} is a function on the restricted domain of sin.



Now You Try: Exercise 1.7.6

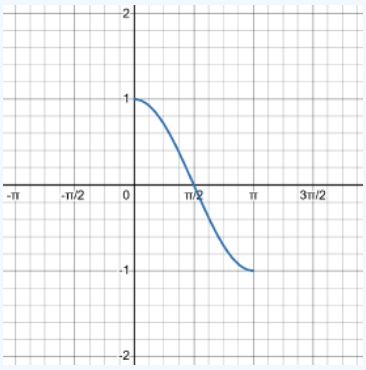
Use the graph the cosine function $y = cos(x)$ from trigonometry below to determine if its inverse cos^{-1} is a function for all real values of x? If not, how can you restrict the domain of $y = cos(x)$ so that the inverse cos^{-1} is a function on the restricted domain?

https://math.libretexts.org/@go/page/99705



Answer

No, cos^{-1} is not a function for all real values of x since the graph of $h(x) = cos(x)$ fails the horizontal line test. We can limit the domain of $h(x) = cos(x)$ to $[0, \pi]$ so that that portion of the graph passes the horizontal line test and cos^{-1} is a function on the restricted domain of cos.



Another problem with inverse functions occurs when using them in the context of applications. In previous examples, we have referred to the inverse of a function $y = f(x) = 2x + 1$ as both $y = g(x) = \frac{x-1}{2}$ and $x = f^{-1}(y) = \frac{y-1}{2}$. In a pure mathematical example where the input quantity and outputs quantity are just numbers without associated units, the expressions $g(3) = 1$ where $x = 3$ and $f^{-1}(3) = 1$ where $y = 3$ are equivalent. However, in the context of an application the units and quantities that the input and output represent have meaning and are distinct from each other. In example 1.7.3 where $P = f(t)$ represents the population of a town in t years, the statements $f(20) = 10,000$ and $f^{-1}(20) = 10,000$ have very distinct meanings. The statement $f(20) = 10,000$ tells us that the population is $P = 10,000$ in $t = 20$ years whereas the statement $f^{-1}(20) = 10,000$ tells us that the population is $P = 20)$ in $t = 10,000$ years. The formula for the inverse $t = f^{-1}(P)$ has an input of P representing the population and the output represents the years from now t. It would render the statement meaningless and loose all connotation to the application if we used other variables, such as $y = f^{-1}(x)$.

Problem #2

When using a function and its inverse function in an application, the variables have a specific meaning and cannot be substituted with other variables.

Example 1.7.11

The formula for converting the Fahrenheit temperature to Celsius temperature is $C = f(F) = \frac{5}{9}(F - 32)$

a. Find and interpret the meaning of $f(92)$
b. Find and interpret the meaning of $f^{-1}(92)$
c. Find a formula for the inverse function f^{-1}

Solution

a. For function f, the input represents the temperature in Fahrenheit degrees and the output represents the Celsius temperature. So we are given $F = 92$ and want $C = f(92)$. Evaluating

$$C = \frac{5}{9}(92 - 32) = \frac{5}{9}(60) = \frac{100}{3} \approx 33.3 \text{ degrees}$$

b. For the inverse function f^{-1} the inputs and outputs are switched to the input represents the temperature in Celcius degrees and the output represents the Fahrenheit temperature. So we are given $C = 92$ and want $F = f^{-1}(92)$. Substituting the value for c and solving

$$92 = \frac{5}{9}(F - 32)$$

$$92 \cdot \frac{9}{5} = F - 32$$

$$92 \cdot \frac{9}{5} + 32 = F$$

$$F = 197.6$$

c. The inverse function $F = f^{-1}(C)$ is a formula for F. Using the formula for f and solving for F, we have

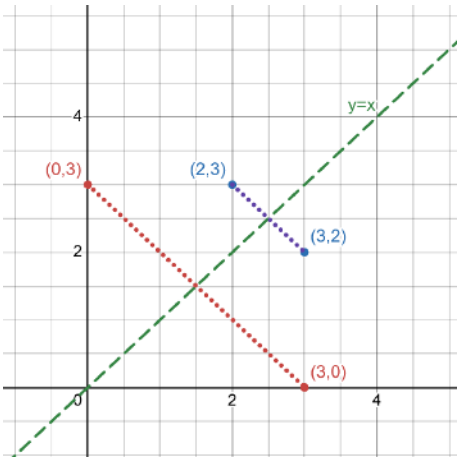
$$C = \frac{5}{9}(F - 32)$$
$$C \cdot \frac{9}{5} = F - 32$$
$$F = \frac{9}{5}C + 32$$

The inverse function is

$$F = h^{-1}(C) = \frac{9}{5}C + 32$$

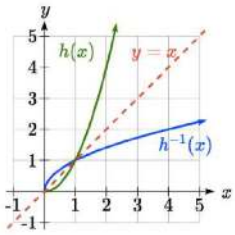
Focus: The Graphs of a Function and its Inverse

In key point 1.7.1, we highlighted that a function and its inverse have their inputs and outputs switched. Let's explore how this relation impacts the graphs of a function and its inverse. For example, if a function f has $f(2) = 3$, then for its inverse $f^{-1}(3) = 2$. If we graph both points on the same axis below, notice that they are reflections of one another about the line $y = x$. Similarly, if $f(0) = 3$, then for its inverse $f^{-1}(3) = 0$. On the graph, the points are once again reflections of one another about the line $y = x$.



If we did this for each point on the graph of a function, the graphs of a function and its inverse are reflections of one another about the line $y = x$.

Notice if we graph the portion of the parabola$y = h(x) = x^2$ on the restricted domain $[0, \infty]$ and its inverse, the square root function, $h^{-1}(x)\sqrt{x}$ on the same axes, we can notice symmetry: the graph of $h^{-1}(x)$ is the graph of h(x) reflected over the line $y = x$.

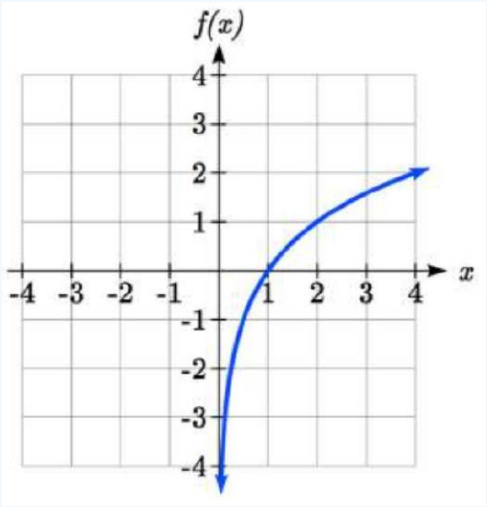


Key Point 1.7.6

The graphs of a function and its inverse are reflections of one another about the line $y = x$.

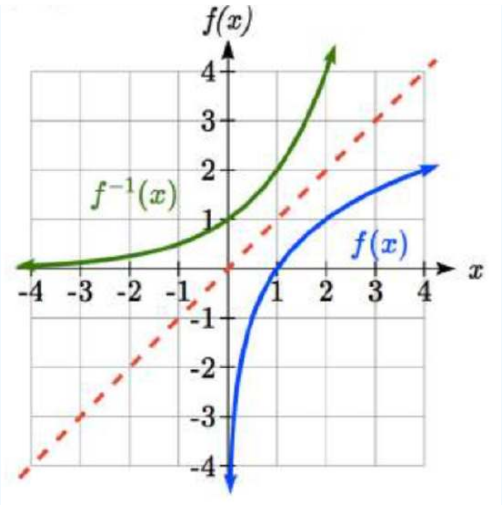
Example 1.7.12

Given the graph of $f(x)$ shown, sketch a graph of $f^{-1}(x)$.



Solution

Since the graph of function f passes the horizontal line test, the inverse is a function. Reflecting this graph of the line $y = x$, the point (1, 0) reflects to (0, 1), and the point (4, 2) reflects to (2, 4). Sketching the inverse on the same axes as the original graph:



Important Topics of this Section

- Definition of an inverse function
- Finding the formula for an inverse function
- Composition of inverse functions yield the original input value
- The inverse of every function may not be a function
- Restricting the domain of functions so the inverse is a function

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1.7E: Inverse Functions (Exercises)

Section 1.7 exercises

1. Assume that the inverse of function f is also a function.

a. If $f(6) = 7$, find $f^{-1}(7)$
b. If $f(3) = 2$, find $f^{-1}(2)$
c. If $f^{-1}(-4) = -8$, find $f(-8)$
d. If $f^{-1}(-2) = -1$, find $f(-1)$

2. If the function f is given in the table below, find a table that defines the inverse function f^{-1}.

The inputs and outputs of function f

x	0	1	2	3	4
$y = f(x)$	7	1	3	5	6

3. Use the table below to find the indicated quantities.

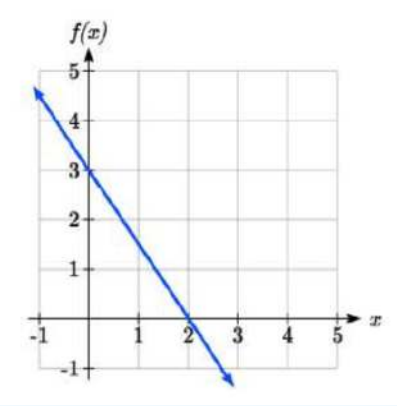
The inputs and outputs of function g

x	0	1	2	3	4	5	6	7	8	9
$g(x)$	8	0	7	4	2	6	5	3	9	1

a. Find $g(0)$
b. Find $g^{-1}(0)$
c. Find all values of x such that $g^{-1}(x) = 7$
d. Write a table of values that defines the inverse function g^{-1}

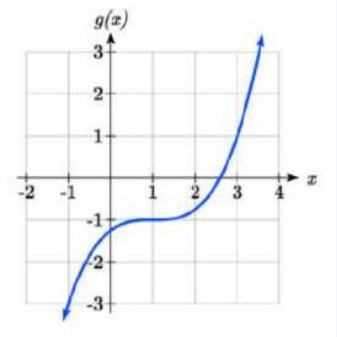
4. The function f is given by the set of ordered pairs f = {(-2,3), (3,1), (1,5), (2,4)}. Find a set of ordered pairs that defines the inverse function f^{-1}.

5. Using the graph of $f(x)$ shown



a. Find $f(0)$
b. Find $f^{-1}(0)$
c. Find $f^{-1}(1)$
d. Find all values of y such that $f^{-1}(y) = 0$

6. Using the graph shown



a. Find $g(1)$
b. Find $g^{-1}(1)$
c. Find $g^{-1}(2)$
d. Find all values of y such that $g^{-1}(y) = 3$

7. Find a formula for the inverse f^{-1} of each function

a. $f(x) = 2x + 3$
b. $f(x) = (x + 7)^2$
c. $f(x) = \sqrt[3]{3x - 7}$

8. Find a formula for the inverse g^{-1} of each function

a. $g(x) = 5x - 8$
b. $g(x) = (3x + 7)^4$
c. $g(x) = 2 + \sqrt{6x - 1}$

9. Find a formula for the inverse f^{-1} of each function

a. $f(x) = \dfrac{x + 1}{x - 2}$
b. $f(x) = \dfrac{3x - 2}{x + 7}$

10. If $f(x) = x^3 - 5$ and $g(x) = \sqrt[3]{x + 5}$, find

a. $f(g(x))$
b. $g(f(x))$
c. What does this tell us about the relationship between $f(x)$ and $g(x)$?

11. If $f(x) = \dfrac{x}{2 + x}$ and $g(x) = \dfrac{2x}{1 - x}$, find

a. $f(g(x))$
b. $g(f(x))$
c. What does this tell us about the relationship between $f(x)$ and $g(x)$?

12. Determine which of the functions are inverses of each other where $f(x) = 5x^2 + 3$, $g(x) = \dfrac{\sqrt{y} - 3}{5}$, and $h(x) = \sqrt{\dfrac{y - 3}{5}}$.

13. The function f is given by the set of ordered pairs f= {(0,1), (1,-2), (2,0), (3,2) (4,1)}.

a. In the inverse f^{-1} a function for all defined values of x?
b. If you answered "Yes, f^{-1} is a function" in part (a), find a new ordered pair that you can add to function f that would make f^{-1} not a function. If you answered "No, f^{-1} is not a function" in part (a), find an ordered pair(s) that you can remove from the function f that would make f^{-1} a function.

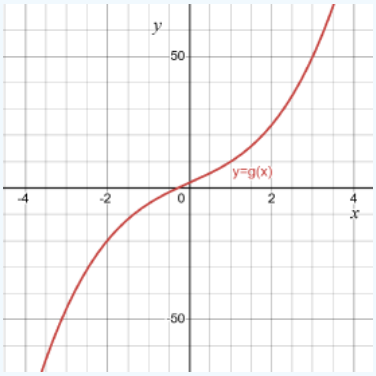
14. The function g is given by the set of ordered pairs g= {(1,2), (3,4), (2,3), (5,1) (6,5)}.

a. In the inverse g^{-1} a function for all defined values of x?

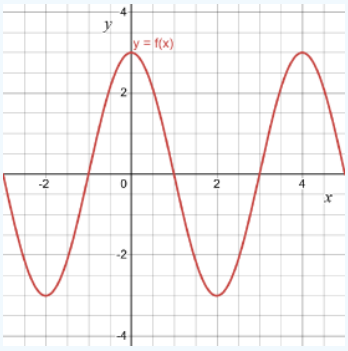
b. If you answered "Yes, g^{-1} is a function" in part (a), find a new ordered pair that you can add to function g that would make g^{-1} not a function. If you answered "No, g^{-1} is not a function" in part (a), find an ordered pair(s) that you can remove from the function g that would make g^{-1} a function.

15. Let $f(x) = x^2(x-2)^2$. Graph $f(x)$ with your graphing utility. Is the inverse f^{-1} a function for all values of x? If not, how can you restrict the domain of f so that f^{-1} is a function.

16. The graph of $g(x) = x^3 + 7x + 2$ is given below. Is the inverse g^{-1} a function for all values of x? If not, how can you restrict the domain of g so that g^{-1} is a function.



17. The graph of $f(x) = 3cos(0.5\pi x)$ is given below. Is the inverse f^{-1} a function for all values of x? If not, how can you restrict the domain of f so that f^{-1} is a function.



18. If $T = f(H)$ represents the time T in minutes it takes to bake a pizza at a temperature H in degrees Fahrenheit. Write a sentence that interpret the meaning of the following statements.

a. $f(400) = 12$
b. $f^{-1}(10) = 450$

19.If $V = g(d)$ represents the volume of water V in cubic feet that will flow through a pipe with interior diameter d in inches in one minute. Write a sentence that interpret the meaning of the following statements.

a. $g(1) = 6.4$
b. $g^{-1}(6.4) = 20.2$

20. The cost (in dollars) of producing x units of a product is given by $C = f(x) = 24x + 700$.

1. Evaluate and interpret $f(4000)$
2. Evaluate and interpret $f^{-1}(4000)$

21. The stopping distance of a car (in feet) traveling at an initial velocity of v mph is given by $D = g(v) = \frac{v^2}{8}$.

1. Evaluate and interpret $g(60)$
2. Evaluate and interpret $g^{-1}(60)$

22. The average value of a solar panel in the US (in dollars per peak Watt) t years since 2012 can estimated with the function $V = f(t) = 0.01(t-9)^2 + 0.34$ using data from The US Energy Information Administration.

1. Evaluate and interpret $f(6)$
2. Evaluate and interpret $f^{-1}(0.87)$

Answer

1. a) 6 c) -4

3. a) 8 b) 1 c) 3 d) shown in the table below

y	8	0	7	4	2	6	5	3	9	1
$x = f^{-1}(y)$	0	1	2	3	4	5	6	7	8	9

5. a) 3 b) 2 c) 165 d) 3

7. a) $x = f^{-1}(y) = \frac{y+3}{2}$ c) $x = f^{-1}(y) = \frac{y^3+7}{3}$

9. a) $x = f^{-1}(y) = \frac{y+1}{y-1}$

11. a) x b) x c) They are inverses.

13. a) f^{-1} is a function b) Adding f^{-1} would make it not a function.

17. a) At 400 degrees, it takes 12 minutes to cook the pizza. b) At 450 degrees, it takes 10 minutes to cook the pizza.

20. a) The cost of producing 4000 units is $96,100. b) The cost of producing 137.5 units is $4,000.

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CHAPTER OVERVIEW

2: Exponential and Logarithmic Functions

2.1: Exponential Growth and Decay

2.1E: Exponential Growth and Decay (Exercises)

2.2: Modeling with Exponential Functions

2.2E: Modeling with Exponential Functions (Exercises)

2.3: More on Compounding Interest and Continuous Exponential Models

2.3E: More on Compounding Interest and Continuous Exponential Models (Exercises)

2.4: Solving Exponential Equations with Logarithms

2.4E: Solving Exponential Equations with Logarithms (Exercises)

2.5: Describing Relationships with Logarithmic Functions

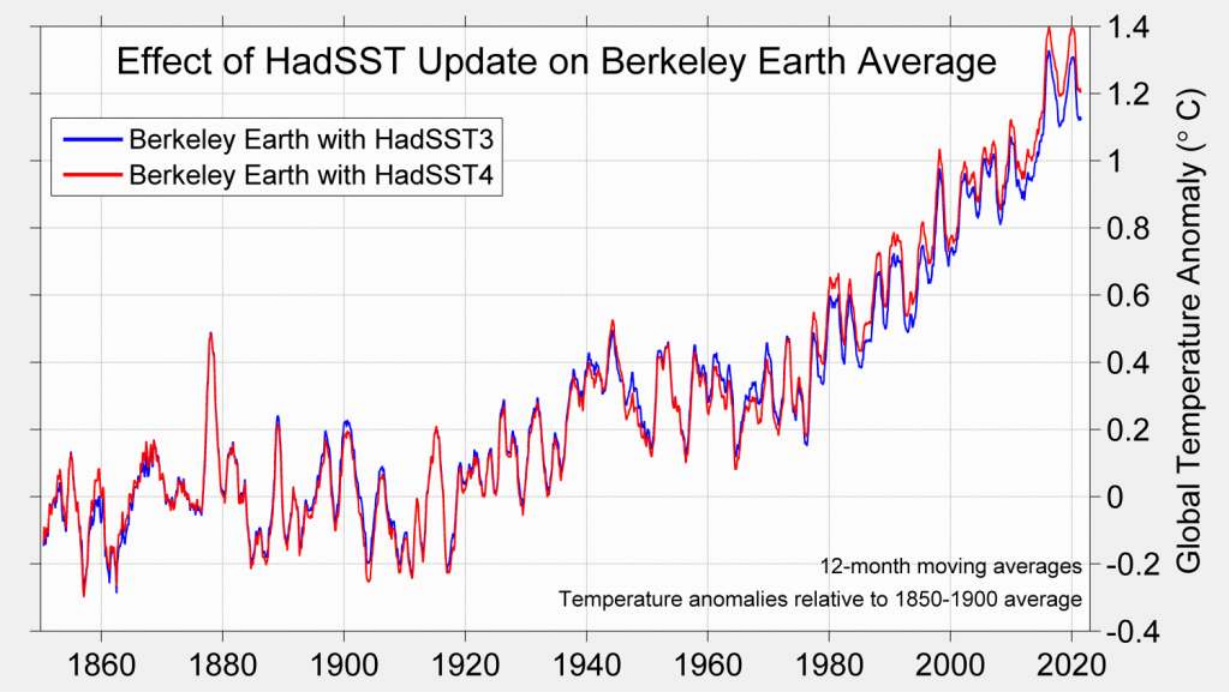
2.5E: Describing Relationships with Logarithmic Functions (Exercises)

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2.1: Exponential Growth and Decay

Motivating Example: Modeling Climate Change

Previously, we modeled the global temperature anomaly relative to time with a linear function. To do so, we assumed that the rate of increase in temperature was constant. As the data shows in the graph below (Figure 3-1)[1], the actual temperature data does not increase at a constant rate. In fact, the global temperature anomalies have increased at an larger and larger rate over time . If we are going to accurately model the the global temperature anomaly over time, we are going to need to develop a new type of function to more accurately describe this type of increase. In this chapter, we will explore modeling relationships with exponential functions that will allow us to do this.



Focus: Understanding Exponential Growth and Decay

Example 3.1.1

Can you fold a piece of paper in half 15 times? Try it. What do you notice?

It gets pretty hard to fold after about 6-7 folds. Why is this? There is a a relationship between the number of folds and the sheets of thickness of our paper is described in the table below.

x = Number of folds	0	1	2	3	4
y = Sheets of thickness	1	2	4	8	16

Do you notice a pattern? For each fold or increase in x of 1, the sheets of thickness, y, is doubled or multiplied by 2. If we fold the paper a fifth time (x= 5), then the sheets of thickness will double again to y = 2(16) = 32. To find the number of sheets of thickness with 15 folds, we could continue to multiply y by 2 for each increase in x by 1 until we get to x= 15 folds. However,

we can calculate this faster if we notice that each output, y, can be calculated from the initial value y = 1 and the number of times we have doubled the sheets of thickness as shown in the table below.

x = Number of folds	0	1	2	3	4	15	x
y = Sheets of thickness	1	$2 = 1(2)^1$	$4 = 1(2)^2$	$8 = 1(2)^3$	$16 = 1(2)^4$	$1(2)^{15} = 32,768$	$1(2)^x$

We can calculate the sheets of thickness of the paper after 15 folds by multiplying 1 by 2 fifteen times or

$$1(2)^{15}$$

to get a paper 32,768 sheets thick. To put this thickness into perspective, two hundred sheets of standard white printer paper is approximately 2 inches thick, or 100 sheets = 1 inch of thickness. The thickness after 15 folds is

$$1(2)^{15} = 32768 \text{ sheets } \cdot \frac{1 \text{ in.}}{100 \text{ sheets}} \cdot \frac{1 \text{ ft .}}{12 \text{ in.}} \approx 27.3 \text{ ft.}$$

This is the thickness of some of the giant sequoia redwood trees in California which no human can fold in half with their hands.

In general, if we fold a piece of paper x times, the sheets of thickness will double x times and can be described by the formula below.

In Summary:

The number of folds of a piece of paper x generates a thickness $y = 1(2)^x$. Notice the parameter in the formula a = 1 represents the initial or starting value. The parameter in the formula b = 2 represents the constant value we are multiplying each output y by for each increase in x by one. We call b = 2 **the growth factor.**

Example 3.1.2

What is the pattern in the data below?

x	0	1	2	3	15
y	8	4	2	1	?

For each increase in x by 1, or $\Delta x = 1$, y is multiplied by $\frac{1}{2}$. To find the value of y when x =15, we could continue to multiply by $\frac{1}{2}$ for each increase in x of 1 until we got to x= 15. However, we can calculate this faster if we notice that each output y, can be calculated from the initial value y = 8 and the number of times we multiply by $\frac{1}{2}$ as show in the table below.

x	0	1	2	3	15	x
y	8	$4 = 8 \cdot (\frac{1}{2})^1$	$2 = 8 \cdot (\frac{1}{2})^2$	$1 = 8 \cdot (\frac{1}{2})^3$	$8 \cdot (\frac{1}{2})^{15} \approx 0.000061035$	$8 \cdot (\frac{1}{2})^x$

In Summary:

The relationship can be described by the formula $y = 8(\frac{1}{2})^x$. Notice the parameter in the formula a = 8 represents the initial value. The parameter in the formula the b = $\frac{1}{2}$ represents the constant value we are multiplying each output by for each increase in x by one. We call the parameter b = $\frac{1}{2}$ the growth factor.

Generalizing further, we arrive at the standard form of the formula for an exponential function.

Key Points:

1) **Exponential growth or decay** by a function is characterized by each increase in the input x by one,($\Delta x = 1$) causes the output y to be multiplied by a constant value b. The constant value b is called **the growth factor**.

2) A function that exhibits exponential growth or decay is called an **exponential function**. The formula for an exponential function can be written in the form

$$f(x) = ab^x \tag{2.1.1}$$

where a is the initial or starting value of the function and b is the growth factor. We limit b to positive values or $b > 0$.

Notice the variable input x of this function is in the exponent and is why we call these types of functions exponential functions.

3) An exponential function $f(x) = ab^x$ increases when the growth factor b > 1 and decreases when 0<b<1.

Example 3.1.3

The population of a county over time and the number of people the county can feed with its agricultural output over time are listed in the table below.

t = Decades from 1960	P = Population of the County	F = Food Supply of the County (in people fed)
0	1,000	5,000
1	3,000	8,000
2	6,000	11,000
3	18,000	14,000
4	54,000	17,000

a) Which quantity is experiencing exponential growth over time? Which quantity is experiencing linear growth over time?

b) Find a formula for the quantity that grows exponentially over time. Then find a formula for the quantity that grows linearly growing over time.

Solution

a) The population of the county is growing exponentially since the population is multiplied by a factor of 3 (the growth factor, b = 3) for each $\Delta t = 1$. The food supply of the county is growing linearly since the food supply increases by 3000, $\Delta F = 3000$, for each $\Delta t = 1$.

b) The population of the county has an initial value a = 1000 and the growth factor b = 3. The exponential formula for the population of the county can be written as

$$P = 1000(3)^t.$$

The food supply of the county has a initial value of a = 5000 and a rate of change $\frac{\Delta F}{\Delta t} = 3000$. The linear formula for the food supply can be written

$$F = 1000 + 3000t.$$

Student Example 3.1.1

Now you try the following example. Determine which functions are exponential. Then write a formula of the form $f(x) = ab^x$ for each exponential function.

x	f(x)	g(x)	h(x)
0	3	54	12
1	12	18	10
2	48	6	8
3	192	2	6
4	768	$\frac{2}{3}$	4

Solution

The functions (\f \) and (\g \) are exponential with a growth factor b = 4 and b = $\frac{1}{3}$, respectively. The function (\h \) is not exponential, since it has a constant rate of change $\frac{\Delta y}{\Delta x} = -2$ and not a constant growth factor.

The formulas for the exponential functions are $f(x) = 3(4)^x$ and $g(x) = 54(\frac{1}{3})^x$.

Focus: Graphing Exponential Functions

Now let's explore the graphs of exponential functions to help us further model relationships with exponential functions.

Graph Example 3.1.1

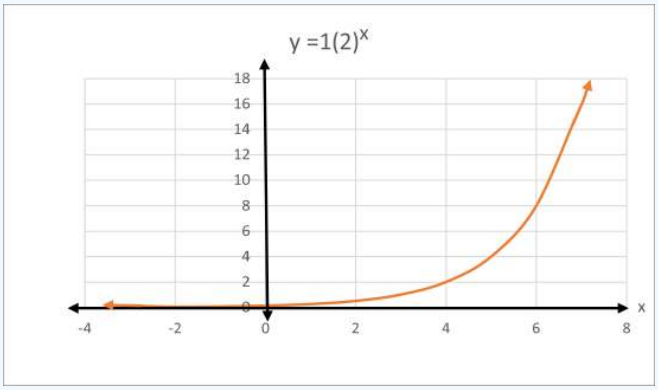
x	0	1	2	3	4
$y = f(x) = 1(2)^x$	1	2	4	8	16

Notice f is increasing for all values of x since we are multiplying by a growth factor b = 2 for each increase in x by 1. Notice the right end behavior of the graph: as x gets larger and larger approaching infinity , y gets larger and larger approaching infinity ($x \to \infty$, $y=f(x) \to \infty$). Also, for negative values of x we have.

x	-1	-2	-3
$y = 1(2)^x$	$1(2)^{-1} = \frac{1}{2^1} = \frac{1}{2}$	$1(2)^{-2} = \frac{1}{2^2} = \frac{1}{4}$	$1(2)^{-3} = \frac{1}{2^3} = \frac{1}{8}$

Notice the left end behavior of the graph: as x becomes larger and larger magnitude negative number approaching negative infinity , y gets smaller and smaller in size approaching zero ($x \to -\infty$, $y=f(x) \to 0$).

Graphing the function, we have



Features of the graph:

- The y-intercept is at y = 1. Equivalently, the y-intercept is the point (0,1).
- The functions is increasing at an increasing rate.
- The right end behavior of the graph: as x $\to \infty$, y=f(x) $\to \infty$.
- The left end behavior of the graph: as x $\to -\infty$, y=f(x) $\to 0$.

Graphs of increasing exponential functions have the same general shape as this graph.

Graph Example 3.1.2

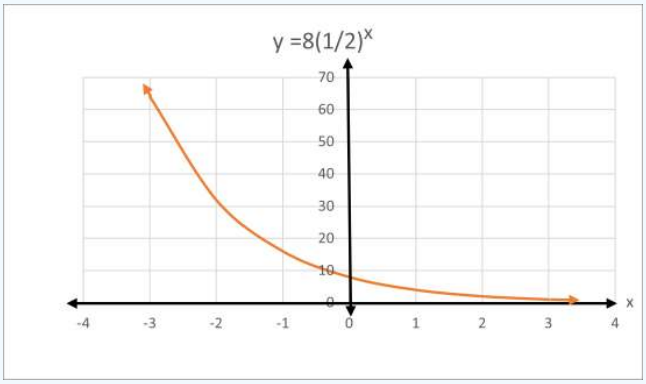
x	0	1	2	3
$y = 8(\frac{1}{2})^x$	8	4	2	1

Notice f is decreasing for all values of x since we are multiplying by the growth factor b = \(\frac{1}{2}) for each increase in x by 1. Also, for negative values of x we have.

x	-1	-2	-3
$y = 8(\frac{1}{2})^x$	$8(\frac{1}{2})^{-1} = 8(2)^1 = 16$	$8(\frac{1}{2})^{-2} = 8(2)^2 = 32$	$8(\frac{1}{2})^{-3} = 8(2)^1 = 64$

Notice the left end behavior of the graph: as x $\to -\infty$, y=f(x) $\to \infty$.

Graphing the function, we have



Features of the graph:

- The y-intercept is at y = 8. Equivalently, the y-intercept is the point (0,8).
- The functions is decreasing. The rate of decrease is getting smaller in size.
- The right end behavior of the graph: as x $\to \infty$, y=f(x) $\to 0$.
- The left end behavior of the graph: as x $\to -\infty$, y=f(x) $\to \infty$.

Graphs of decreasing exponential functions have the same general shape as this graph.

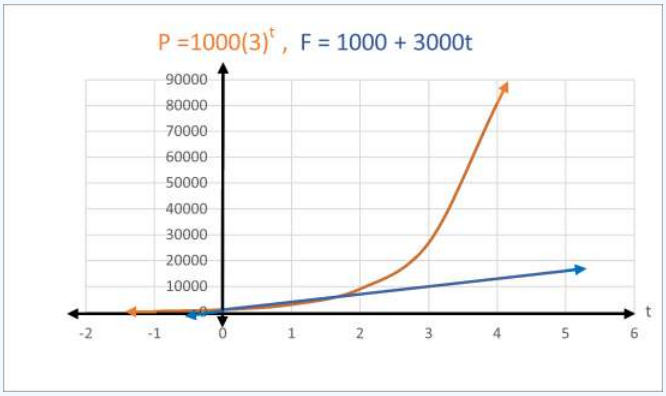
Key Point

The graph of the **exponential function** $f(x) = ab^x$ has a y-intercept at y = a. The function is increasing when the growth factor b >1 and decreasing when 0<b<1.

Graph Example 3.1.3

The population of a county has the formula $P = 1000(3)^t$, where t is the decades from 1960. This is an increasing exponential function since our growth factor b = 3 > 1 with y-intercept at y = 1000. The food supply of the county has a formula $F = 1000 + 3000t$. This is an increasing linear function with slope m = 3000 and y-intercept at y = 1000.

Graphing the both functions, we have



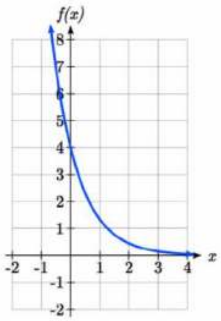
Notice, that in the long run the exponential function increases at a faster rate than the linear function. In general, exponential growth will outpace linear growth in the long run. In the case of our example, this illustrates that an exponentially growing population will outpace its food supply if it grows linearly.

Example 3.1.4

Sketch a graph of $f(x) = 4\left(\frac{1}{3}\right)^x$.

Solution

This graph will have a y-intercept at y=4, and pass through the point $\left(1, \frac{4}{3}\right)$. Since $b < 1$, the function is decreasing.

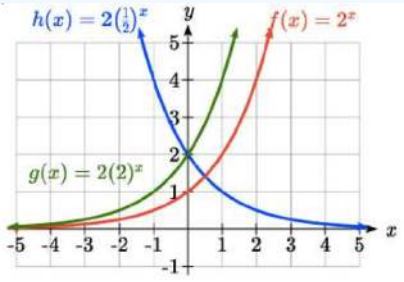


Student Example 3.1.2

Graph the following functions on the same axis:

$f(x) = (2)^x$; $g(x) = 2(2)^x$; $h(x) = 2(1/2)^x$.

Answer



Student Example 3.1.3

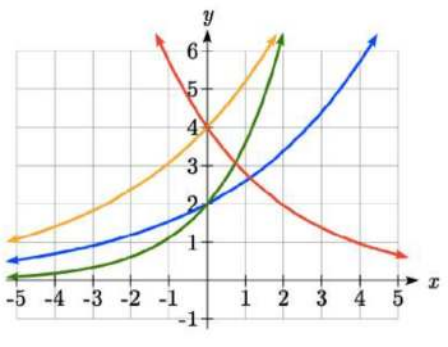
Match each equation with its graph.

$$f(x) = 2(1.3)^x$$

$$g(x) = 2(1.8)^x$$

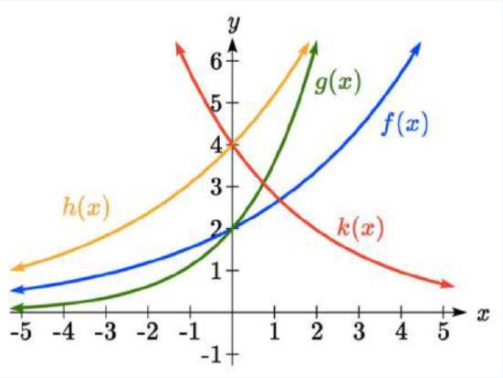
$$h(x) = 4(1.3)^x$$

$$k(x) = 4(0.7)^x$$



Solution

The graph of $k(x)$ is the easiest to identify, since it is the only equation with a growth factor between zero and one, which will produce a decreasing graph. The graph of $h(x)$ can be identified as the only growing exponential function with a y-intercept at y = 4 or (0,4). The graphs of $f(x)$ and $g(x)$ both have a y-intercept intercept at y = 2 or (0,2), but since $g(x)$ has a larger growth factor, we can identify it as the graph increasing faster.



1- the data was reported by berkeleyearth at https://berkeleyearth.org on February 13, 2023.

Important Topics of this Section

- Exponential Growth and Decay
- Growth factor
- The formula for an exponential function
- Graphs of exponential functions

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- **4.2: Graphs of Exponential Functions** by David Lippman & Melonie Rasmussen is licensed CC BY-SA 4.0. Original source: http://www.opentextbookstore.com/details.php?id=30.

2.1E: Exponential Growth and Decay (Exercises)

Section 2.1 Exercises

Part 1: Exponential growth

1. For each table below, determine if the table represents a function that is linear, exponential, or neither? If the function is exponential, find a formula.

a.

x	0	1	2	3
$f(x)$	5	10	20	40

b.

x	0	1	2	3
$g(x)$	7	12	17	22

c.

x	0	1	2	3
$h(x)$	36	12	4	$\frac{4}{3}$

d.

x	1	2	3	5
$k(x)$	50	42	34	26

e.

x	1	2	3	4
$m(x)$	90	81	72.9	65.61

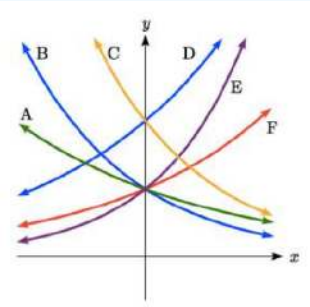
f.

x	2	3	4	5
$n(x)$	3	1	3	7

Part 2: Graphs of exponential functions

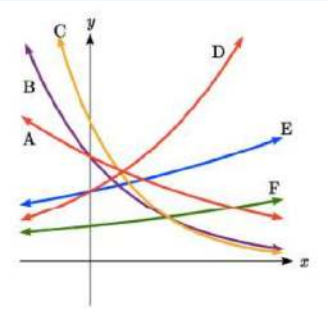
2. Sketch a graph of the function $y = 2(3)^x$ using a table of values with $x = -2, -1, 0, 1, 2, 3, 4$. Label the y-intercept.

3. Match each function with one of the graphs below.



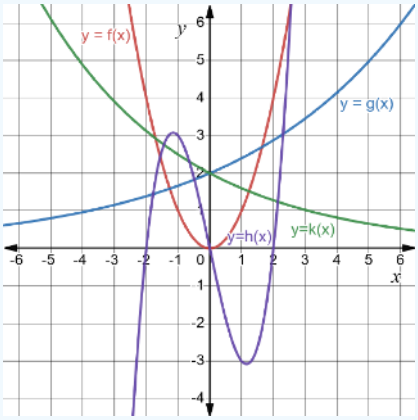
a. $f(x) = 2(0.69)^x$
b. $f(x) = 2(1.28)^x$
c. $f(x) = 2(0.81)^x$
d. $f(x) = 4(1.28)^x$
e. $f(x) = 2(1.59)^x$
f. $f(x) = 4(0.69)^x$

4. If all the graphs to the right have equations with form $f(x) = ab^x$,



a. Which graph has the largest value for b?
b. Which graph has the smallest value for b?
c. Which graph has the largest value for a?
d. Which graph has the smallest value for a?
e. Which of the two graphs has the larger value for b, graph E or F?
f. Which of the two graphs has the smaller value for b, graph A or B?
g. Which of the graphs have a value for the growth factor b where $b > 1$?
h. Which of the graphs have a value for the growth factor b where $0 < b < 1$?

5. Which of the functions graphed below could be exponential? If so, does the growth factor, b, satisfy $b > 1$ or $0 < b < 1$?



6. Without graphing with a graphing calculator or other graphing technology, determine which of the graphs of the two functions will cross in the first quadrant? Explain your reasoning.

a. $f(x) = 8(1.15)^x$ and $f(x) = 3(1.20)^x$

b. $f(x) = 8(1.15)^x$ and $f(x) = 10(1.20)^x$

c. $f(x) = 8(1.15)^x$ and $f(x) = 3(1.12)^x$

d. $f(x) = 6(0.85)^x$ and $f(x) = 3(0.90)^x$

e. $f(x) = 6(0.85)^x$ and $f(x) = 7(0.90)^x$

f. $f(x) = 6(0.85)^x$ and $f(x) = 7(0.83)^x$

Answers to selected exercises:

Part 1:

1. a. Exponential, $f(x) = 5(2)^x$

b. Linear

f. Neither

Part 2:

3. a. Graph B, b. Graph F, c. Graph A, d. Graph D, e. Graph E, f. Graph C

5. The functions y = $g(x)$ and y = $k(x)$ are graphs of exponential functions. For y = $g(x)$, b > 1 since the function is increasing. For y = $k(x)$, we have 0 < b < 1 since the function is decreasing.

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2.2: Modeling with Exponential Functions

Goal: Describing relationships with exponential functions.

Case: The growth factor is given.

Example 2.2.1

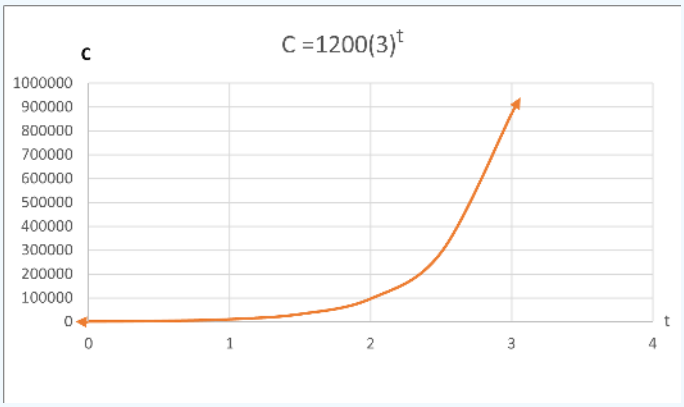
When an outbreak of a contagious virus was first discovered, there were 1200 cases. Each week the number of cases triples.

a) Find a formula for the number of cases, C, in t weeks since the virus was discovered. Then sketch a graph.

b) Predict the number of cases in 3 months if the cases continue to grow in this manner.

Solution

a) The initial value a = 1200 and the growth factor b = 3 since we are multiplying the number of cases by 3 for each week. Using $y = a(b)^x$ or $C = a(b)^t$, our formula is $C = 1200(3)^t$. To sketch the graph, we know the function is increasing since $b = 3 > 1$ and the y-intercept is at y = 1200 since a = 1200.



b) The number of cases in 3 month is $C = 1200(3)^3 = 32,400$.

Case: Functions with a constant percentage rate of change.

Example 2.2.2

Suppose you invest $1000 at a 6% annual interest rate compounded annually.

a) Is this exponential growth?

b) Find a formula for the amount in the account, A, in t years. Then sketch a graph.

c) Predict the amount in the account after 10 years.

Solution

a) After one year, the amount in the account will be

the original amount (100%) + interest (6%) = 106% of the original amount = $1.06(1000)$=$1060.

After the second year, the account will have 106% of the previous years amount of $1060 or

$$1.06(1060) = (1.06)(1.06)(1000) = (1.06)^2(1000) = 1,123.60$$

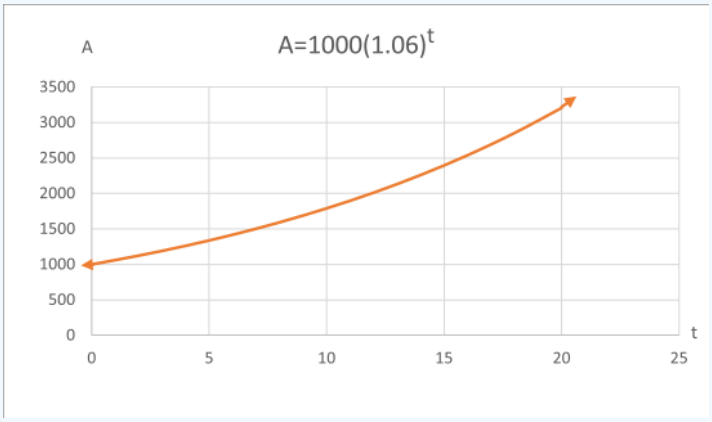
by the same rationale. Each subsequent year, the account will have 106% of the previous years amount = 100% (previous years amount) + 6% (interest). This is exponential growth since we are multiplying the output A by a growth factor b = 1.06 for each increase in x by 1 $(\Delta x = 1)$ as illustrated in the table below.

Years invested, t	0	1	2	3	4
Amount in the account, A	\$1000	$1,000(1.06)^1$ $= \$1,060$	$1,060(1.06)$ $= 1,000(1.06)^2$ $= \$1,123.60$	$1,123.60(1.06)$ $= 1,000(1.06)^3$ $= \$1,191.02$	$1,191.02(1.06)$ $= 1,000(1.06)^4$ $= \$1,262.48$

b) The growth factor is b = 1.06. The initial value is a = \$1000. So the formula for the amount in the account in t years is given by

$$A = 1,000(1.06)^t$$

To sketch the graph, we know the function is increasing since $b = 1.06 > 1$ and the vertical intercept or A-intercept is at A = 1000 since a = 1000.



c) In ten years (t = 10), the amount is predicted to be

$$A = 1,000(1.06)^{10} = \$1,790.85$$

.

Notice that the growth factor b = 1.06 = 1.00 + 0.06 as decimal or b = 106% = 100% + 6%.

Example 2.2.3

A person drinks three cups of coffee containing 390 mg of caffeine. Suppose their body eliminates 13% of the caffeine each hour.

a) Is this exponential decay?

b) Find a formula for the amount in the caffeine, A, in t hours. Then sketch a graph.

c) Predict the amount of caffeine after 6 hours.

Solution

a) After one hour, the amount of caffeine in the body will be

(the original amount (100%)) - (amount eliminated (13%)) = 87% of the original amount = $0.87(390)$= 339.3 mg of caffeine.

After the second year, the amount in the body will have 87% of the previous hour's caffeine level of 339.3 mg or

$$0.87(339.3) = (0.87)(0.87)(390) = 390(0.87)^2$$

by the same rationale. Each subsequent hour, the amount in the body will have

87% of the previous hour's caffeine level = 100% (previous hour's caffeine level) - 13% (amount eliminated).

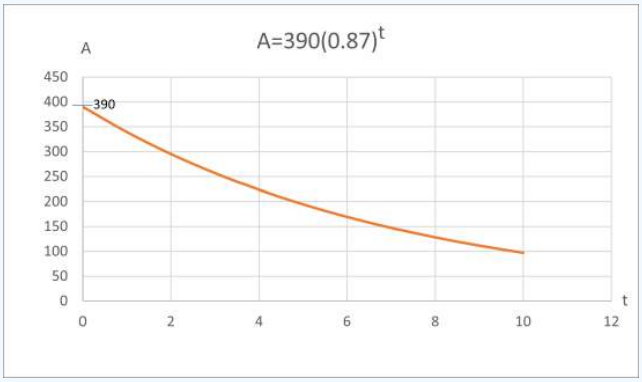
This is exponential decay since we are multiplying the output A by a growth factor b = 0.87 for each increase in x by 1 $(\Delta x = 1)$ as illustrated in the table below.

t hours since the coffee was ingested	0	1	2	3	4
A, the amount of caffeine in the body in mg	390 mg	$390(0.87)^1 = 339.3 \text{ mg}$	$339.3(0.87) = 390(0.87)^2 = 295.19 \text{ mg}$	$295.19(0.87) = 390(0.87)^3 = 256.82 \text{ mg}$	$256.82(0.87) = 390(0.87)^4 = 223.43 \text{ mg}$

b) The growth factor is b = 0.87. The initial value is a = $390 mg. So the formula for the amount in the account in t hours is given by

$$A = 390(0.87)^t.$$

To sketch the graph, we know the function is decreasing since $b = 0.87 < 1$ and the vertical intercept or A-intercept is at A = 390 since a = 390.



c) In six hours (t = 6), the amount of caffeine in the body is predicted to be

$$A = 390(0.87)^6 = 169.11\,mg$$

.

Notice that the growth factor b = 0.87 = 1.00 - 0.13 as decimal or b = 87% = 100% - 13%.

Key Point

A quantity that changes by a constant percentage rate r relative to a another quantity x is experiencing exponential growth (or decay) with a growth factor

b = 100% + r% (as a percent) or b = 1 + r (as a decimal).

Example 2.2.4

India is the second most populous country in the world, with a population in 2008 of about 1.14 billion people. The population is growing by about 1.34% each year (World Bank, World Development Indicators, as reported on http://www.google.com/publicdata, retrieved August 20, 2010).

a) Find a formula to model the population of India, P, as a function of time, t, in years after 2008, if the population continues to grow at this rate.

b) Predict the population of India in 2030 if it continues to grow at this rate.

Solution

a) The growth factor is b = 100% + 1.34% = 101.34% as a percent or b = 1 + 0.0134 = 1.0134 as a decimal. The initial value is a = 1.14 billion people. The formula for the population t years after 2008 is given by

$$P = 1.14(1.0134)^t$$

b) The population in 2030 (t = 22 years after 2008) is predicted to be

$$P = 1.14(1.0134)^{22} = 1.52 \text{ billion people.}$$

Example 2.2.5

$T(q)$ represents the total number of Android smart phone contracts, in thousands, held by a certain Verizon store region q quarters since January 1, 2016, interpret all the parts of the equation $T(2) = 86(1.64)^2 = 231.3056$.

Solution

Interpreting this from the basic exponential form, we know that 86 is our initial value. This means that on Jan. 1, 2016 this region had 86,000 Android smart phone contracts. Since $b = 1 + r = 1.64$, we know that every quarter the number of smart phone contracts grows by 64%. $T(2) = 231.3056$ means that in the 2^{nd} quarter (or at the end of the second quarter) there were approximately 231,306 Android smart phone contracts.

Student Exercise 2.2.1

The number of students at Clovis Community College was 10,092 in the Fall 2014 semester. Over the next few years, the number of students grew by approximately 8% per year (reported by The Clovis Community College Institutional Research Department). Assume the number of students continues to grow at this rate.

a) Write an exponential function for the number of students, S, at Clovis Community College t years after the Fall 2014 if the student population continues to grow in this manner.

b) Predict the number of students at Clovis Community College in the fall 2025 semester.

Answer

a) The growth factor is b = 100% + 8% = 108% as a percent or b = 1 + 0.08 = 1.08 as a decimal. The initial value is a = 10,092 students. The formula for the number of students t years after 2014 is given by

$$S = 10,091(1.08)^t$$

b) The number of students in the Fall 2025 semester (t = 11) is predicted to be

$$S = 10,091(1.08)^{11} = 23,530\, students$$

Student Exercise 2.2.2

Given the three statements below, identify which represent exponential functions.

a. The cost of living allowance for state employees increases salaries by 3.1% each year.
b. State employees can expect a $300 raise each year they work for the state.
c. Tuition costs have increased by 2.8% each year for the last 3 years.

Answer

a & c are exponential functions, they grow by a % not a constant number.

? Student Exercise 2.2.3

Looking at these two equations that represent the balance in two different savings accounts, which account is growing faster, and which account will have a higher balance after 3 years?

- $A(t) = 1000\,(1.05)^t$
- $B(t) = 900\,(1.075)^t$

Answer

$B(t)$ is growing faster (r = 0.075 > 0.05), but after 3 years A(t) still has a higher account balance since

$$A(3) = \$1157.63 > B(3) = \$1,118.07$$

Exponential functions can be used to model quantities that are decreasing at a constant percent rate as we saw in example 3.2.2. Another example of this is radioactive decay, a process in which radioactive isotopes of certain atoms transform to an atom of a different type, causing a percentage decrease of the original material over time.

? Student Exercise 2.2.4

Bismuth-210 is an isotope that radioactively decays each day, meaning a part of the remaining Bismuth-210 transforms into another atom (polonium-210 in this case) each day. If $Q(d)$ represents the amount of Bismuth-210 remaining in d days, interpret the meaning of the formula

$$Q(d) = 100(1 - 0.13)^d = 100(0.87)^d.$$

Answer

The growth factor b = 0.87 = 1 - 0.13 tells use the amount of the isotope is decreasing by 13% percent each hour. Our initial quantity is a = 100 mg of Bismuth-210.

Case: Finding Equations of Exponential Functions from Data

In the previous examples, we were able to write formula for exponential functions since we knew the initial quantity and the growth rate. However, in many cases, we will be able to measure or collect data on exponentially growing (or decaying) quantities. We can use this to construct exponential functions that describe these relationships.

In our original examples in section 3.1, we were given data sets with consecutive values of the input. For example, in the table from student exercise 3.1.1, we are able to easily identify the growth factor b = 4, since b is the value we multiply each output by for each increase in x by 1.

x	0	1	2	3	4
$f(x)$	3	12	48	192	7687

What is the difference in the following table of data? Is the growth factor b = 4 also?

x	0	3	6	9	12
$f(x)$	3	12	48	192	7687

Notice that x is increasing by 3 $(\Delta x = 3)$ instead of by 1. The growth factor is the number we multiply the output by for each increase in x by 1. In this case for $(\Delta x = 3)$, we would have to multiply the output by the growth factor three times. To model the data with an exponential function, we can use the fact that our initial value a = 3 and the point x = 3, y =12 is a solution to our exponential equation described by our function $y = ab^x$.

Substituting in a = 3 and the point (x,y) = (3, 12) gives $12 = 3b^3$.

Solving for a we have,

$$12 = 3b^3$$
$$\frac{12}{3} = b^3$$
$$4 = b^3$$
$$b = \sqrt[3]{4} \approx 1.5974$$

Example 3.2.6

Find a formula for an exponential function passing through the points (0,6) and (5,1).

Solution

The initial value of the function and y-intercept on the graph is at y = 6 so a = 6. Now the point x = 5, y =1 is a solution to our exponential equation described by our function $y = ab^x$.

Substituting in a = 6 and the point (x,y) = (5, 1) gives $1 = 6b^5$.

Solving for a we have,

$$1 = 6b^5$$
$$\frac{1}{6} = b^5$$
$$b = \sqrt[5]{\frac{1}{6}} \approx 0.6988$$

So the formula for the exponential function that models this data is given by $f(x) \approx 6(0.6988)^t$.

In this example, you could also have used $(\frac{1}{6})^{\frac{1}{5}}$ to evaluate the 5^{th} root if your calculator doesn't have an n^{th} root button.

Example 2.2.7

Find a formula for an exponential function passing through the points (1,6) and (5,1).

Solution

What is the difference with this problem and the previous one? We don't know the initial value of the function or y-intercept on the graph since x = 1 when y = 6. However, we can use the same strategy we used with the other point. Since the point x = 5, y =1 is a solution to our exponential equation $y = ab^x$, we can substitute these values into the the equation so that $1 = ab^5$.

This is an equation in terms of two variables. We need a second independent equation in order to solve this system of equations. With the second point (1,6), we can substitute x= 1 and y =6. This gives $6 = ab^1$.

We now solve these as a system of equations.

$$1 = ab^5$$
$$6 = ab^1$$

To do so, we can use a substitution method by solving for the variable a in the second equation $6 = ab^1$. .

So we have,

$$a = \frac{6}{b^1}.$$

Then substituting the expression for a into the second equation $1 = ab^5$:

$$1 = \frac{6}{b^1}(b^5)$$
$$1 = 6b^4$$
$$\frac{1}{6} = b^4$$
$$b = \sqrt[4]{\frac{1}{6}} \approx 0.6988$$

Going back to the equation $a = \frac{6}{b^1}$ lets us find a:

$$a = \frac{6}{0.6389} \approx 8.586$$

Putting this together gives the formula $f(x) \approx 8.586(0.6389)^x$

Notice how much easier it was to find the formula for the exponential function when the initial value a (or y-intercept on the graph) is known. We only need to solve a single equation to find b in example 3.2.6 as opposed to a system of equations in example 3.2.7 when the initial value a is not known. In many applications, it is possible and makes sense to define our variables so that the first data point is the initial value. For example in student example 3.2.1, we defined t as the years after the fall 2014 which made the first data point (the number of students was 10,092 in Fall 2014) an initial value t = 0 and S = 10,092.

Case: Finding Equations of Exponential Functions in Applications

Example 2.2.8

On December 12, 2021, the number of new daily Covid cases in the US was 119,221 (represented as a 7-day average as reported by the Center for Disease Control (CDC) at covid.cdc.gov). By December 30, 2021 , the number of new daily US Covid cases rose to 364,369. Additional data in this time frame show the numbers of cases were increasing at an increasing rate. Assume the number of daily Covid cases is growing exponentially.

a) Find a formula for the number of new daily Covid cases, C, t days since Dec. 12, 2021.

b) Predict the number of new Covid cases on Jan. 10, 2021 if this trend continues.

Solution

a) By defining our input variable to be t days since Dec. 12, 2021, the information listed can be written as two input-output pairs: t = 0, C = 119,221 and t = 18, C = 364,369. Notice that by choosing our input variable to be measured as the days after the first day value provided, we have effectively "given" ourselves the initial value for the function: a = 119,221. This gives us an equation of the form

$$C = f(t) = 119,221b^t$$

Substituting in our second input-output pair allows us to solve for b:

$$364,369 = 119,221b^{18}$$

Dividing by 119,221.

$$b^{18} = \frac{364,369}{119,221} \approx 3.05625$$

Take the 18^{th} root of both sides.

$$b = \sqrt[18]{3.05625} \approx 1.06403$$

This gives us our equation for the number of new daily Covid cases:

$$C = f(t) = 119,221(1.06403)^t$$

Recall that since $b = 1 + r$, we can interpret this to mean that the daily new Covid cases' growth rate is r = 0.06403, and so the daily new Covid cases' is growing by about 6.4303% each day.

b) Jan. 10, 2022 represents t = 29 days since Dec. 12, 2021. Using our formula, the number of new Covid cases (as a 7-day average) on this day is

$$C = f(29) = 119,221(1.06403)^{29} \approx 721,132.$$

Note that the actual number of cases on this day as a 7-day average was 760,924.

When finding equations, the value for b or r will usually have to be rounded to be written easily. To preserve accuracy, it is important to not over-round these values. Typically, you want to be sure to preserve at least 3 or 4 significant digits in the growth rate. For example, if your value for b was 1.064032579, you would want to round this no further than to 1.06403 or 1.0640.

Example 2.2.9

In 1981, the five year average daily maximum temperature in Fresno, County was 67.41 ^{o}F. In 2011, the five year average daily maximum temperature rose to 69.13 ^{o}F. Assume the five year average daily maximum temperature increases exponentially. (Data collected by North America Land Data Assimilation System (NLDAS) and reported on wonder.cdc.gov)

a) Find an exponential model for the five year average daily maximum temperature in Fresno, County, T, in t years since 1981.

b) Use your model to predict the five year average daily maximum temperature in Fresno, County in 2026 assuming the temperatures continue to increase in this manner.

Solution

a) By defining our input variable to be t, years after 1981, the information listed can be written as two input-output pairs: t = 0, T = 67.41 and t = 20, T = 69.13 Notice that by choosing our input variable to be measured as years after the first year value provided, we have effectively "given" ourselves the initial value for the function: a = 67.41. This gives us an equation of the form

$$T = f(t) = 67.41b^{t}$$

Substituting in our second input-output pair allows us to solve for b:

$$69.13 = 67.41b^{20}$$

Divide by 67.41

$$b^{20} = \frac{69.13}{67.41} \approx 1.025516$$

Take the 20^{th} root of both sides.

$$b = \sqrt[20]{1.025516} \approx 1.0012603$$

This gives us our equation for the the five year average daily maximum temperature in Fresno County:

$$T = f(t) = 67.41(1.0012603)^{t}$$

Recall that since $b = 1 + r$, we can interpret this to mean that the five year average daily maximum temperature in Fresno County is growing by about 0.12603% each year.

b) In 2026, t = 45 years since 1981. Substituting this value in for t in our formula

$$T = f(t) = 67.41(1.0012603)^{45} \approx 71.34^{o}F$$

Another important application of exponential functions in physics, chemistry, engineering, and biology is radioactive decay. Radioactive decay is the idea that the amount of radioactive isotopes decreases exponentially over time. One of the common terms associated with radioactive decay is half-life.

Definition: Half Life

The **half-life** of a radioactive isotope is the time it takes for half or 50% of the substance to decay.

Example 2.2.10

Cesium-137 is a radioactive compound that has a half-life of about 30 years. It is a common waste product in the production of nuclear energy. A sample is taken near a nuclear accident that contains 200 mg of cesium-137.

a) Find a formula for the amount of Cesium-137, C, of the sample in t years.

b) Predict the amount of Cesium in the sample in 50 years.

Solution

a) By defining our input variable to be t years since the sample was taken, the initial value is t = 0, C = 200 mg. Notice that by choosing our input variable to be measured as years after the first year value provided, we have effectively "given" ourselves the initial value for the function: a = 200. This gives us an equation of the form

$$C = f(t) = 200b^t$$

The half life can be used to obtain a second data point. In 30 years (t=30), there is half or 50% of the original amount remaining. So C = (0.50)(200) = 100 mg. Substituting this data into out formula allows us to solve for b:

$$100 = 200b^{30}$$

Divide by 200

$$b^{30} = \frac{100}{200} = 0.5$$

Take the 30^{th} root of both sides.

$$b = \sqrt[30]{0.5} \approx 0.97716$$

This gives us our equation for the amount of Cesium:

$$C = f(t) = 200(0.97716)^t$$

Recall that since $b = 1 + r$, so $r = b - 1 = 0.97716 - 1 = -0.0228$ or r = -2.284%. We can interpret this to mean that the amount of cesium is decreasing by about 2.284% each year.

b) In 50 years, t = 50. Substituting this value in for t in our formula

$$T = f(t) = 200(0.97716)^{50} \approx 62.997 \text{ mg}$$

Student Exercise 2.2.5

The number of students at Clovis Community College was 10,092 in the Fall 2014 semester. By the Fall 2019 semester, the number of students as 14,384 (as reported by The Clovis Community College Institutional Research Department). Assume the number of students grows exponentially.

a) Write an exponential function for the number of students, S, at Clovis Community College t years after the Fall 2014 semester, if the student population continues to grow in this way.

b) Predict the number of students at Clovis Community College in the fall 2025 semester.

Answer

a) By defining our input variable to be t, years after the Fall 2014 semester, the information listed can be written as two input-output pairs: t = 0, S = 10092 and t = 5, S = 14,384. Notice that by choosing our input variable to be measured as

years after the first year value provided, we have effectively "given" ourselves the initial value for the function: a = 10,092. This gives us an equation of the form

$$S = f(t) = 10,092b^t$$

Substituting in our second input-output pair allows us to solve for b:

$$14,384 = 10,092b^5$$

Divide by 10,092

$$b^5 = \frac{14,384}{10,092} \approx 1.4253$$

Take the 5^{th} root of both sides.

$$b = \sqrt[5]{1.4253} \approx 1.0734$$

This gives us our equation for the number of students at Clovis Cummunity College:

$$S = f(t) = 10092(1.0734)^t$$

Recall that since $b = 1 + r$, we can interpret this to mean that the of number of students is growing by about 7.34% each year.

b) In fall 2025, t = 11 years since the Fall 2014 semester. Substituting this value in for t in our formula

$$S = f(t) = 10092(1.0734)^{11} \approx 21,997 \text{ students.}$$

Student Exercise 2.2.6

Carbon-14 is a radioactive isotope that is present in organic materials, and is commonly used for dating historical artifacts. Carbon-14 has a half-life of 5730 years.

a) Find a formula for the percentage, P, of the original carbon-14 remaining.

b) If the bone is 10,000 years old, what percentage of the original carbon-14 remains?

Answer

a) Let define t = age of the bone in years. Then, the half-life can be interpretted as an input-output pair:

t = 5,730 years, P = 50%. Although it is not stated explicitly in the problem, we do know a second data point: when the animal died (time t= 0) there is 100% of the original carbon -14 remaining (P=100). This gives us initial value for the function: a = 100. This gives us an equation of the form

$$S = f(t) = 100b^t$$

Substituting in our other input-output pair allows us to solve for b:

$$50 = 100b^{5730}$$

Divide by 100

$$b^{5730} = \frac{50}{100} = 0.5$$

Take the 5730^{th} root of both sides.

$$b = \sqrt[5730]{0.5} \approx 0.999879$$

This gives us our equation for the percentage of the original carbon-14 remaining:

$$P = f(t) = 100(0.999879)^t$$

b) If the bone is 10,000 years old, then t = 10,000. Substituting this value in for t in our formula

$$P = f(t) = 100(0.999879)^{10000} \approx 29.83$$

Important Topics of this Section

- Percent growth
- Exponential functions
- Finding formulas
- Interpreting equations
- Exponential Growth & Decay
- Radioactive decay
- Half life

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2.2E: Modeling with Exponential Functions (Exercises)

Section 2.2 Exercises

Part 1: Modeling with Exponential functions with a constant percentage rate of change or growth factor.

1. On Monday 20 computers are infected with a computer virus. Each day, the number of infected computers triples.

   a. Find an exponential model for the number of infected computers, y, in terms of t, the days since Monday.

   b. Predict the number of infected computers in 2 weeks.

2. What is the growth factor b in each of the following situations of exponential growth or decay?

   a. A customer's power usage increased by 9% per year

   b. The number of subscribers' decreased by 3% per year.

   c. The population of a species of from increased by 2.47% per month?

3. For each of the following situations the percentage rate of change per second and initial value at time t = 0 sec. is given. Find an formula for the quantity y as a function of t. Then predict the amount in 12 seconds.

   a. The initial amount is 520 ft and is increasing by 4% per second.

   b. The initial amount 400 mg and is decreasing by 3% per second.

   c. The initial amount is 20 m^3 and is decreasing by 2.41% per second.

4. The populations, P, of six towns, t years after 2010 are given below.

   i. $P = 12,000(1.07)^t$ ii. $P = 8,000(1.13)^t$

   iii. $P = 9,000(0.94)^t$ iv. $P = 8,500(1.085)^t$

   v. $P = 8,214(0.83)^t$ vi. $P = 8,912(0.997)^t$

   a. Which towns are increasing in size? Which towns are decreasing in size?

   b. Which town is growing the fastest? What is the annual percentage growth rate of that town?

   c. Which town is decreasing in size the fastest? What is the annual percentage decay rate of that town?

   d. Which town had the largest population in 2010?

5. As of January 2, 2022, the rate of people 70 years or older hospitalized for COVID was 13.32 people per 100,000 people in the US population. At that point in time, the number of people hospitalized was increasing by 3.7% per day .[1] Assume the rate of infection continues to increase at this percentage rate.

   a. Find an exponential model for the number of people 70 years or older hospitalized , $H(t)$, in terms of t, the days since January 2, 2022.

   b. Predict the number of people infected on Jan. 22.

6. As of February 4, 2022, the rate of people 70 years or older hospitalized for COVID was 15.52 people per 100,000 people in the US population. At that point in time, the number of people hospitalized was decreasing by 4.475% per day.[2] Assume the rate of infection continues to decrease at this percentage rate.

   a. Find an exponential model for the number of people 70 years or older hospitalized , $H(t)$, in terms of t, the days since February 4, 2022.

   b. Predict the number of people infected on February 23, 2022.

7. Clovis Community College had 10,464 students in the Fall 2016 - Spring 2017 academic year. At the time, growth was projected to be 8% per years based on state funding projections.[3] Assume the student enrollment continues to increase at this percentage rate.

a. Find an exponential model for the number of students, $S(t)$, in terms of the years since the Fall 2016 - Spring 2017.

b. Predict the number of students at Clovis Community College in the Fall 2026 - Spring 2027 semester.

8. The concentration of fine particulate matter pollution (PM2.5) in California was 13.95 $\frac{\mu g}{m^3}$ in 2003. From 2003 to 2011, the concentration decreased by approximately 4.942% per year.[4] Assume the concentration of fine particulate matter pollution continues to decrease at this percentage rate.

a. Find an exponential model for the concentration of fine particulate matter pollution, $c(t)$, in terms of the years since 2003.

b. Predict when the concentration of fine particulate matter pollution in 2025.

9. The fox population in a certain region has an annual growth rate of 9% per year. It is estimated that the population in the year 2010 was 23,900. Estimate the fox population in the year 2018.

10. The amount of area covered by blackberry bushes in a park has been growing by 12% each year. It is estimated that the area covered in 2009 was 4,500 square feet. Estimate the area that will be covered in 2020.

11. A vehicle purchased for $32,500 depreciates at a constant rate of 5% each year. Determine the approximate value of the vehicle 12 years after purchase.

12. A business purchases $125,000 of office furniture which depreciates at a constant rate of 12% each year. Find the residual value of the furniture 6 years after purchase.

Part 2: Modeling with exponential functions from data.

In problems 13- 17, find a formula for an exponential function passing through the two points.

13. $(0,\ 6)\,,\ (3,\ 750)$

14. $(0,\ 3)\,,\ (2,\ 75)$

15. $(0,\ 2000)\,,\ (2,\ 20)$

16. $\left(-1, \frac{3}{2}\right)\,,\ (3,\ 24)$

17. $(3,\ 1)\,,\ (5,\ 4)$

18. A house that was valued at $300,000 in California in January 2018 was valued at $376,000 in January 2021.[5]

a. Find an exponential formula for the value of the house, $v(t)$ t years since January 2018, assuming the value continues at the same percentage rate.

b. Predict the value of the home in January 2025.

c. What was the annual growth rate?

19. A house that was valued at $400,000 in California in January 2019 was valued at $480,000 in January 2022.[6]

a. Find an exponential formula for the value of the house, $v(t)$, t years since January 2019, assuming the value continues to increase at the same percentage rate.

b. Predict the value of the home in January 2025.

c. What was the annual growth rate?

20. A car was valued at $38,000 in the year 2012. The value depreciated to $11,000 by the year 2021. Assume that the car value continues to drop by the same percentage rate.

a. Find an exponential formula for the value of the car, $v(t)$, t years since 2012.

b. Predict the value of the car in 2025.

21. A car was valued at $24,000 in the year 2006. The value depreciated to $15,000 by the year 2009. Assume that the car value continues to drop by the same percentage rate. What was the value in the year 2014?

22. The population of California was 37,253,956 on April 1, 2010. The population grew to 39,538,233 on April 1, 2020. Assume that the population continues to increase by the same percentage rate.[7]

a. Find an exponential formula for the population of California, $P(t)$, t years since April 1, 2010.

b. Predict the population of California on April 1, 2025.

23. A person drinks alcohol at a party. After her last drink, the alcohol level of her blood soon reaches a maximum of 0.28 milligram per milliliter of blood. 50% of the alcohol remained after 2 hours. Assume the blood level decreases exponentially.

a. Find an exponential formula for the alcohol level in her blood, $A(t)$, in t hours.

b. If she waits 3 hours, will her blood alcohol level be below the legal limit of 0.08?

[1,2] The data was reported by the CDC at https://covid.cdc.gov/covid-data-tra...tal-admissions on January 20, 2023.

[3] The data was reported by the Clovis Community College Institutional Research Department at Clovis Community College on August 12, 2018 (https://www.cloviscollege.edu/about/...-research.html) .

[4] The data was reported by the CDC at https://wonder.cdc.gov on February 24, 2023.

[5,6] The data was reported by the U.S. Federal Housing Finance Agency House Price Index Calculator at www.fhfa.gov/DataTools/Tools/Pages/HPI-Calculator.aspx on January 25, 2023.

[7] The data was reported by the U.S. Census Bureau at https://www.census.gov/quickfacts/fa...e/CA/PST045222 on January 26, 2023.

Answer

1. a. $y = 20(3)^t$ b. y = 95,659,380

3. a. $y = 520(1.04)^t$ b. $y = 400(0.97)^t$ c. $y = 20(0.9759)^t$

5. a. $H(t) = 13.32(1.037)^t$ b. 27.55

7. a. $S(t) = 10,464(1.08)^t$ b. 22,591 students

9. The fox population can be modeled by the function $y = 23,900(1.09)^t$. In 2018, there will be 47622 Fox.

11. The vehicle $V = 32,500(0.95)^t$. In 12 years, the value will be $17561.70.

13. $y = 6(5)^x$

15. $y = 2000(0.1)^x$

17. $y = (\frac{1}{8})(2)^x$

19. a. $y = 400,000(1.0627)^t$ b. 576,000 c. The home value is growing by 6.27%

21. The car will be worth $6,853.

23. a. $A(t) = 0.28(0.7071)^t$ b. The blood alcohol level is 0.09899 after 3 hours.

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2.3: More on Compounding Interest and Continuous Exponential Models

Focus: More on Compound Interest

Many types of bank accounts utilize compound interest, such as a bank certificates of deposit (CD), savings accounts, and credit cards. The term $compounding$ comes from the behavior that interest is earned not only on the original value, but on the accumulated value of the account. The time period, or $compounding\ period$, over which interest is calculated may differ by the account. In one CD, interest could be calculated every month (such as 2% interest per month) whereas in another interest could be can calculated every year (such as 12% per year). In order to compare interest rates compounded over different time periods, interest rates are usually expressed as a **nominal rate** or **an annual percentage rate (APR)**. Let's explore an example to understand how this effects how we calculate interest.

Definition: Annual Percentage Rate

The annual percentage rate (APR) of an investment is the percentage of interest gained on the original principal, or amount invested, over a year.

Example 2.3.1

You invest $1,000 into an investment account that pays a 6% APR (annual percentage rate) of interest.

a) If you compounded interest annually (or once per year), how much will be in the account in a year?

b) If you compounded interest quarterly, how much will be in the account in a year?

Solution

a) Since we are starting with $1,000, our initial value $a = 1000$.

Our annual interest rate is 6%, so $r = 0.06$, and our growth factor b = 100% + 6% =106% as a percent or b =1 +0.06 = 1.06 as a decimal. Each time we compound interest, we multiply the previous amount by our growth factor b = 1.06.

So in one year, we will have

$$A(1) = 1000(1.06)^1 = \$1060.$$

b) Since we are compounding quarterly, we are compounding 4 times per year and need to break the annual interest rate into 4 parts. So the interest per quarter is $r = \frac{6}{4}$ = 1.5% as a percent or as a decimal $r = \frac{.06}{4}$ = 0.015.

So our quarterly growth factor b = 100% + 1.5% =101.5% as a percent or b =1 +0.015=1.015 as a decimal. Each time we compound interest, we multiply the previous amount by our growth factor b = 1.015.

So in one year, we will have

$$A(1) = 1000(1.015)^4 = \$1061.36.$$

Can you explain why more interest is gained when we compounded interest quarterly? In the first quarter, we gained $15 in interest since

$$A = 1000(1.015)^1 = \$1015.$$

In the second quarter, we gained interest on the original principal($1000) plus the first month's interest ($15). The interest on the original principal=$1000(.015) = \$15$ and the interest on the first months interest $(15)(.015) = \$0.225$. More interest was gained in the second quarter, $15 + 0.225 = \$15.225$ than the first quarter, since we are gaining interest on previous interest as well as the original principal.

When the interest is compounded more frequently than once a year, the account effectively earns more that the nominal rate or APR. We call this rate the account actually earned the $effective\ annual\ rate$.

Definition: Effective Annual Rate

The **effective annual rate** is the percentage rate r that an account actually increased by over a year incorporating the effects of compounding interest.

In example 3.3.1 part b, the effective annual rate was $r = \frac{61.36}{1000} = 0.06136$ or 6.136%. Notice, that we could have obtained this by using the growth factor. After compounding four times, the account grows by a factor of $b^4 = (1.015)^4 = 1.06136$. As a percent, the account is 106.136% or the original amount after a year. It has grown by 6.136%.

Example 2.3.2

A 529 plan is a college savings plan in which a relative can invest money to pay for a child's later college tuition. The account grows tax free. Lily wants to set up a 529 account for her new granddaughter so that the account grows to $40,000 over 18 years. She believes the account will earn a 4% APR compounded semi-annually (twice a year).

a) How much will Lily need to invest in the account?

b) What is the effective annual rate of her account?

Solution

a) Since the account is earning a 4% APR, the interest per compounding period (semi-annually)is r = $\frac{4}{2}$=2% or as a decimal r = $\frac{0.04}{2} = 0.02$ as a decimal.

Then the growth factor per period (semi-annually) is b = 100% + 2% =102% as a percent or b = 1 +0.02 = 1.02 as a decimal. Each time interest is compounded, we multiply the previous amount by our growth factor b = 1.02. The account will be compounding twice a year for 18 years, so the number of compoundings n = (18)(2) =36.

In this problem, we don't know how much we are starting with, so we will be solving for a, the initial amount. We do know that we want the end amount to be $40,000. We want for the value of a so that $A(18) = 40,000$.

$$40,000 = A(18) = a(1.02)^{36}$$
$$40,000 = a(2.039887)$$
$$a = \frac{40,000}{2.039887} \approx \$19,608.93$$

Lily will need to invest $19,608.93 to have $40,000 in 18 years.

b) In one year, the account will compound twice. So we will multiply our initial amount by our growth factor twice $b^2 = 1.02^2 = 1.0404$. Effectively our account will be 104.04% of the initial value. So the effective annual rate is 4.04%.

Student Exercise 2.3.1

You owe $5000 on a credit card that charges a 12% APR of interest compounded monthly.

a) Find a formula for the amount owed, A, after t years if no payments are made.

b) Predict the amount owed in 5 years.

c) What is the effective annual rate of the credit card account?

Answer

a) Since the account is earning a 12% APR, the interest per month is r = $\frac{12}{12}$=1% as a percent or as a decimal r = $\frac{0.12}{12} = 0.01$ as a decimal. Then the growth factor per month is b = 100% + 1% =101% as a percent or b = 1 +0.01 = 1.01 as a decimal. Each time we compound interest, we multiply the previous amount by our growth factor b = 1.01.

The account will be compounding twelve times a year for t years. So the number of compoundings in t years is n = (12)(t) =12t.

The initial value, or principal, $a = 5000$. So a formula for the amount owed after t years is

$$A = A(t) = 5000(1.01)^{12t}$$

b) In 5 years, t = 5. So the amount in the owed will be

$$A = A(5) = 5000(1.01)^{12(5)} = \$9,083.48$$

c) In one year, the account will compound 12 times. So the initial amount will be multiplied by our growth factor b = 1.01 twelve times and increase by a factor of = $b^{12} = (1.01)^{12} \approx 1.126825$. Effectively, the credit card debt will be 112.6825% of the initial value. So the effective annual rate is 12.6825%.

Generalizing this, we can form a general formula for compound interest. If the APR is written in decimal form as r, and there are k compounding periods per year, then the interest per compounding period will be $\frac{r}{k}$. So the growth factor per period will be b=$(1 + \frac{r}{k})$. Likewise, if we are interested in the value after t years, then there will be $n = kt$ compounding periods in that time.

Compound Interest can be calculated using the formula

$$A(t) = a\left(1 + \frac{r}{k}\right)^{kt} \tag{2.3.1}$$

Where $A(t)$ is the account value

- t is measured in years
- a is the starting amount of the account, often called the principal
- r is the annual percentage rate (APR), also called the nominal rate, written as a decimal
- k is the number of compounding periods in one year

? Student Exercise 2.3.2

Bank A offers an account paying 1.2% compounded quarterly. Bank B offers an account paying 1.1% compounded monthly. Which bank is offering a better rate?

Answer

We can compare these rates using the effective annual percentage rate, the actual percent increase per year.

Bank A has a quarterly interest rate of r = $\frac{0.012}{4} = 0.003$ as a decimal. The growth factor per quarter is $b = 1 + 0.003 = 1.003$. In one year, the account will compound 4 times and grow by a factor of $b^4 = 1.003^4 \approx 1.012085411$. Therefore, the account will be 101.2085411% of the initial value. So the effective annual rate is 1.2085411%.

Bank B has a monthly interest rate of r = $\frac{0.011}{12} = 0.0009166667$ as a decimal. The growth factor per month is $b = 1 + 0.0009166667 = 1.0009166667$. In one year, the account will compound 12 times and grow by a factor of $b^{12} = 1.0009166667^{12} \approx 1.011661873$. Therefore, the account will be 101.1661873% of the initial value. So the effective annual rate is 1.1661873%.

Bank B's monthly compounding is not enough to catch up with Bank A's better APR. Bank A offers a better rate.

Focus: Periodic Exponential Growth versus Continuous Exponential Growth

Compounding interest is an example of *periodic exponential growth*, where the growth occurs at the end of a time period, such as 6% per year or 3% per quarter. Many other exponentially growing (or decaying) quantities grow (or decay) continuously through the time period. For example, the difference in temperature between a hot cup of coffee and its ambient environment decreases at every instant, not just at the end of the time period. Exponential growth (or decay) that occurs continuously through a time period is called *continuous exponential growth (or decay)*. The population growth of a country is another example of continuous exponential growth since the growth in the number of people occurs continuously through the year not just at the end of the year. Be aware that we may communicate the growth of a continuously growing exponential quantity with a periodic growth rate, such as the

population of a country growing by 1.12% per year. The exponential growth formula we have been working with so far, $y = a(b)^x$, is related to the periodic growth rate. For example, if a quantity grows by 1.12% per year, the growth factor would be b = 100% + 1.12% = 101.12% as a percent or b = 1 + 0.0112 = 1.0112 as a decimal. For this reason, we call this the periodic exponential growth (or decay) formula.

Definition: Periodic Exponential Growth Formula

Periodic Exponential Growth can be calculated using the formula

$$f(x) = ab^x \tag{2.3.2}$$

where

a is the starting amount,

r is the periodic growth rate written as a decimal,

and b is the growth factor where b = 1+ r.

How are periodic and continuous exponential growth related? Let's explore what happens to an account with compounded interest, which experiences periodic exponential growth, as we increase the frequency of the compoundings from yearly to quarterly to monthly to daily to hourly to every second as we approach continuous exponential growth. As we saw earlier, the amount we earn increases as we increase the compounding frequency. If we continue to increase the frequency of the compoundings, will the amount we earn keep increasing without bound or eventually level off? The table below shows that the increase from annual to quarterly compounding is larger than the increase from monthly to daily compounding. This might lead us to believe that, although increasing the frequency of compounding will increase our result, there is an upper limit to this process.

To see this, let us examine the value of $1000 invested at 6% interest for 10 years.

Frequency	**Interest per Period, r**	**Growth Factor per Period, b**	**Number of Compoundings in 10 years, n**	**Amount in 10 years** $A = a(b)^n$
Annual	$\frac{0.06}{1} = 0.06$	1+0.06=1.06 (106%)	10(1)=10	$A = 1000(1.06)^{10}$ $= 1790.85$
Quarterly	$\frac{0.06}{4} = 0.015$	1+0.015=1.015	10(4)=40	$A = 1000(1.015)^{40}$ $= 1814.02$
Monthly	$\frac{0.06}{12} = 0.005$	1+ 0.005 = 1.005	10(12)=120	$A = 1000(1.005)^{120}$ $= 1819.40$
Daily	$\frac{0.06}{365} \approx 0.000164384$	1.000164384	10(365)=3,650	A $= 1000(1.000164384)^{3,650}$ $= 1822.03$
Hourly	$\frac{0.06}{8760} \approx 0.0000068493$	1.0000068493	10(8760)=87,600	A $= 1000(1.0000068493)^{87,600}$ $= 1822.11$
Once per minute				1822.12
Once per second				1822.12

These values do indeed appear to be approaching an upper limit of approximately $1822.12 as we approach continuous exponential growth with growth at each instant. In order to describe what this limiting value, we need to introduce a special number, which is so important that it gets represented by its own letter.

Definition: Euler's number: e

e is the letter used to represent the value that $\left(1+\frac{1}{k}\right)^k$ approaches as k gets bigger and bigger.

$$e \approx 2.718282$$

The number is also called the natural base.

e is a number that arises in continuous exponential growth similar to how number π arises when dealing with circles. Notice, that the limiting amount in our interest example can be expressed with e.

$$A = 1000(e)^{(0.06)(10)} \approx 1822.11$$

Notice, the significance of each of the parameters in this equation: the initial value a = 1000, the time t = 10 years, and the percent increase r = 0.06. In this case, r = 0.06 = 6% is *the continuous growth rate* since growth is occurring at each instant.

In general, a continuously growing (or decaying) exponential quantity can be represented by the following formula.

Definition: Continuous Exponential Growth Formula

Continuous Exponential Growth can be calculated using the formula

$$f(t) = ae^{rt} \tag{2.3.3}$$

where

a is the starting amount,

r is the continuous growth rate written as a decimal,

and t is the time.

This type of equation is commonly used when describing quantities that change more or less continuously, like chemical reactions, growth of large populations, and radioactive decay. In general, we can model an exponential quantity with either the periodic or continuous growth formulas, so y = $ab^t = ae^{rt}$ where $b = e^r$. However, we will see that there are some distinct advantages to using the continuous exponential models in some cases in section 3.4.

Example 2.3.3

A virus is detected with 120 people infected initially and is spreading rapidly.

a) If the number of people infected is increasing by 25% per month, find a formula for the number of people infected, I, in t months. Then predict the number infected in 6 months.

b) If the number of people infected is increasing continuously by 25%, find a formula for the number of people infected, I, in t months. Then predict the number infected in 6 months.

Solution

a) In this scenario, we are given the periodic growth rate r = 25% = 0.25. Therefore, we should use the periodic exponential growth formula $I = f(t) = ab^t$, where a = 120 and b = 100% + 25% = 125% as a percent or b = 1.25 as a decimal. So the number of people infected is given by

$$I = 120(1.25)^t$$

The number infected in 6 months is $I = 120(1.25)^6 \approx 458$.

b) In this scenario, we are given the continuous growth rate r = 25% = 0.25. Therefore, we should use the continuous exponential growth formula $I = ae^{rt}$, where a = 120. So the number of people infected is given by

$$I = 120(e)^{0.25t}$$

The number infected in 6 months is $I = 120(e)^{0.25(6)} \approx 538$.

Can you explain why the number infected was more with continuous growth than with periodic growth? It is similar to what happened with compounded interest where we gained interest on previous interest in the continuous model.

Example 2.3.4

Interpret the following: $S(t) = 20e^{0.12t}$ if $S(t)$ represents the growth of a substance in grams, and time is measured in days.

Solution

An initial substance weighing 20g is growing at a continuous rate of 12% per day.

Example 2.3.5

Find the effective annual rate of an account which pays an 8% APR compounded continuously.

Solution

We can interpret the effective annual rate from the growth factor $b = e^r = e^{0.08} \approx 1.08329$. After one year, the account will be 108.329% of the original amount. Therefore the effective annual rate is 8.329%.

Alternatively, after one year the account will have a value of $A = ae^{(0.08)(1)} \approx a(1.08329)$ if an initial value of a is invested. Thus, the amount in one year will be 108.329% of the original amount. Therefore the effective annual rate is 8.329%.

Student Exercise 2.3.3

A person ingests 20 mg of a medication.

a) If the amount of medication in a person's body decreases by 16% per hour, find a formula for the amount of medication, A, in the body after t hours. Then predict the amount of medication after 5 hours.

b) If the amount of medication in a person's body decreases continuously by 16%, find a formula for the amount of medication, A, in the body after t hours. Then predict the amount of medication after 5 hours.

Answer

a) In this scenario, we are given the periodic decay rate r = -16% = -0.16. Therefore, we should use the periodic exponential formula $A = ab^t$, where a = 20 and b = 100% - 16% = 84% as a percent or b = 0.84 as a decimal. So the amount of medication is given by

$$A = 120(0.86)^t$$

The amount of medication in 5 hours is $A = 20(0.86)^5 \approx 9.41$ mg.

b) In this scenario, we are given the continuous decay rate r = -16% = 0.16. Therefore, we should use the continuous exponential growth formula $A = ae^{rt}$, where a = 20. So the amount of medication is given by

$$A = 20(e)^{-0.16t}$$

The amount of medication in 5 hours is $A = 20(e)^{-0.16(5)} \approx 8.99$ mg.

Focus: Some issues with using exponential functions to make predictions

Case: Predict the input when the output is known.

Example 2.3.6

In Example 3.3.1a, the amount in the account after t years was given by $A = 1000(1.06)^t$. How long until there is $5,000 in the account?

Solution

We are given A = 5,000. We want the time t. Substituting we have,

$$5000 = 1000(1.06)^t$$
$$\frac{5000}{1000} = (1.06)^t$$
$$5 = (1.06)^t$$

How can we get the variable t out of the exponent? We need to develop more tools to solve this exponential equation and make predictions like this with exponential models.

Case: Modeling data with a continuous exponential function

Example 2.3.7

Find a continuous exponential model, $y = ae^{rt}$ that contains the points (0,4) and (2,12).

Solution

The initial value is a = 4. To find r, substitute in the other data point t = 2 and y = 12, so

$$12 = 4(e)^{r(2)}$$
$$\frac{12}{3} = e^{2r}$$
$$4 = e^{2r}$$

We run into the same issue as example 5, we need to get the variable out of the exponent.

Important Topics of this Section

- Continuous Exponential Growth
- Compound Interest
- Effective Annual Rate

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2.3E: More on Compounding Interest and Continuous Exponential Models (Exercises)

Section 2.3 Exercise

Part 1: More on compounding interest

1. An investment $4,000 is made in a bank account at an annual percent rate of 7% per year. Predict the amount in the account find the amount in the bank after 9 years if the interest is compounded

a. annually b. quarterly c. monthly

2. A person borrows $20,000 from a bank that charges 12% interest per year that is compounded monthly.

a. Find a formula for the amount owed after t years.

b. Predict the amount owed after 5 years if none of the balance is paid.

c. What is the effective annual yield?

3 . A person invests $100,000 into a fund that pays 8% interest per year that is compounded monthly.

a. Find a formula for the amount in the fund after t years.

b. Predict the amount in the account after 5 years if none of the balance is paid.

c. What is the effective annual yield?

4. An investment grows by 6% per year over a 10 year period. By what percent does it increase over the 10-year period?

5. Find the annual percentage yield (APY) for a savings account with annual percentage rate of 3% compounded

a. quarterly b. Monthly c. daily

6. Which of the following investments is more valuable?

Investment A: Pays 5% APR compounded quarterly

Investment B: Pays 4.9% APR compounded monthly

Investment C: Pays 4.8% APR compounded daily

7. Suppose $5000 is invested into one of five accounts. Let t represent the time in years. Match each formula (A) - (E) with the description of each account (i) - (iv). Some descriptions may match more that once investment. Indicate all investments that match a particular description.

Investment A: $A = 5000(1.16)^t$ Investment B: $A = 5000(1.12)^t$

Investment C: $A = 5000(1.08)^{2t}$ Investment D: $A = 5000(1.01)^{12t}$

Investment E: $A = 5000(1.04)^{4t}$

(i) This investment earned 12% interest per year compounded monthly.

(ii) This investment earned 16% interest per year compounded quarterly.

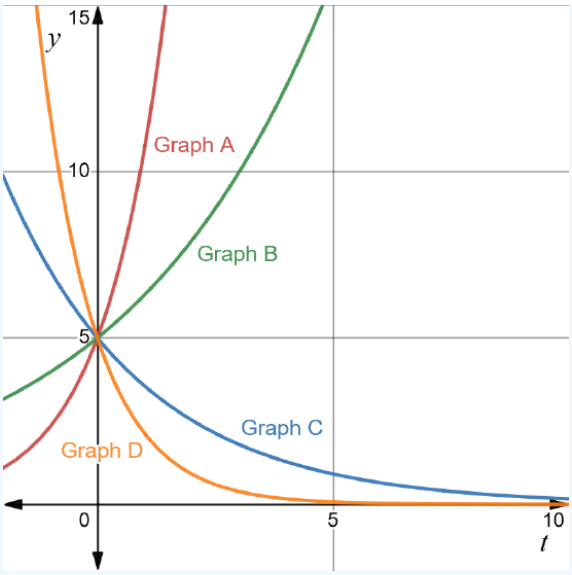
(iii) This investment earned more than 16% interest per year compounded annually.

(iv) This investment earned less than 4% interest per quarter.

Part 2: Continuous exponential growth and decay

8. Without a graphing utility such as a calculator or app, match the functions with a graph

(i) $y = 5e^{0.76t}$ (ii) $y = 5e^{-0.34t}$ (iii) $y = 5e^{0.23t}$ (iv) $y = 5e^{-0.85t}$



9. For each formula below (where t represents the time in years from now), determine

(a) What is the initial value at time t = 0?

(b) Is the quantity increasing or decreasing?

(c) Is the percentage growth rate continuous or not?

(d) What is the percent growth or decay rate?

Formula 1: $y = 20e^{1.27t}$ Formula 2: $y = 3.4(1.27)^t$

Formula 3: $y = 1.29e^{-0.48t}$ Formula 4: $y = 3.4(0.58)^t$

10. The population of a town is currently 12,000 people. In each scenario, find a formula for the population in t years. Then predict the population in 12 years.

(a) Scenario 1: The population is growing continuously at 14% .

(b) Scenario 2: The population is growing periodically by 14% per year.

11. A petri dish contains 1500 bacteria, when an acidic solution is applied. In each scenario, find a formula for the number of bacteria in t hours. Then predict the number of bacteria in 5 hours.

(a) Scenario 1: The number of bacteria decreases periodically by 27% per hour.

(b) Scenario 2: The number of bacteria decreases continuously by 27% per hour.

12. A wildfire has burned 4,257 acres. Assume that the fire continues to expand at a continuous rate of 64% per day.

a. Find a formula for the number of acres burned $A(t)$ in t days.

b. Predict the number of acres burned in two weeks.

c. Use a graphing utility to predict when the number of acres burned will reach 20,000 acres.

13. In an elk habitat, there are currently 12,571 acres contain elk in 2012. Due to environmental and other pressures the number of acres containing elk decreased continuously by 8.9% in the next year.[1] Assume that the acreage containing elk to expand at this percentage rate.

a. Find a formula for the acres containing elk $A(t)$ in t years.

b. Predict the acres containing elk in 10 years.

c. Use a graphing utility to predict when the number of acres containing elk is only 1,000.

---

[1] - The data was reported by the US Fish and Wildlife service at https://www.fws.gov/data/resources on February 2,2023.

Answer

1. a. $7,353.84 b. $7,469.63 c. $7,496.71

3. a. $A = 100,000(1.00667)^{12t}$ b. $148,987.53 c. 8.2996%

5. a. 3.0339% b. 3.0416% c. 3.0453%

7. (i) - D , (ii) - E, (iii) - C & E, (iv) - A & D

9. Formula 1: The initial value a = 20. The function is increasing by 27% per year. The growth is continuous.

Formula 2: The initial value a = 3.4. The function is increasing by 27% per year. The growth is not continuous.

Formula 3: The initial value a = 1.29. The function is decreasing by 48% per year. The growth is continuous.

Formula 4: The initial value a = 3.4. The function is decreasing by 48% per year. The growth is not continuous.

11. a. $A = 1,500(0.73)^t$. After 5 years, there are 4,956 bacteria.

b. $A = 1,500(e)^{0.27t}$. After 5 years, there are 5,786 bacteria.

13. a. $A = 12,571(e)^{-0.089t}$ b. 5,162 acres c. 28.4 years

---

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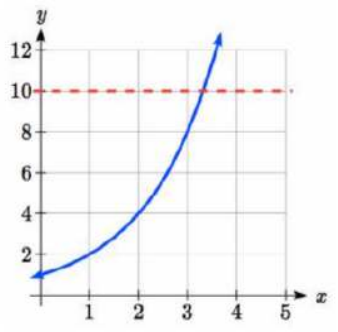
2.4: Solving Exponential Equations with Logarithms

Goal: Solving Exponential Equations

In the previous section, we encountered exponential equations where the variable is in the exponent, such as

$$10 = 2^x.$$

We can solve this equation graphically by graphing the curve $y = 2^x$ and the line $y = 10$ then approximating the point of intersection in the graph below.



The point of intersection is approximately at $x \approx 3.3$. Therefore, the solution to the equation $x \approx 3.3$.

However, in many applications in mathematics, science, and engineering, it would be necessary to have the exact solution or at least a more accurate approximation.

Focus: Solving Exponential Equations Analytically

To solve the equation, $10 = 2^x$, we need to find the exponent of 2. Our strategy to do this is to create a function that tells us the exponent.

Case: Base 10

Definition: The Logarithm Base 10

The logarithm base 10 of x is the exponent of 10 that gives x. We write $\log_{10}(x) = y$ if and only if $10^y = x$.

We read $\log_{10}(x)$ as "log base 10 of x".

The logarithm base 10 is a function whose output is the exponent of 10. In other words, the logarithm base 10 tells us the exponent of 10. Notice each logarithmic statement is equivalent to the corresponding exponential statement. Also, the inputs and outputs are switched with a logarithm and the corresponding exponential statement. Effectively, we have defined the logarithm base 10, $f^{-1}(x) = log_{10}(x)$, as the inverse function of the exponential function base 10, $f(x) = 10^x$. Let's explore this correspondence between exponential and logarithmic form and the definition of a logarithm in the following examples.

Example 2.4.1

Write these exponential equations as logarithmic equations:

a. $10^2 = 100$
b. $10^4 = 10000$
c. $10^{-4} = \frac{1}{10000}$

Solution

a) $10^2 = 100$ is equivalent to $\log_{10}(100) = 2$

b) $10^4 = 10000$ is equivalent to $\log_{10}(10000) = 4$

c) $10^{-4} = \frac{1}{10000}$ is equivalent to $\log_{10}\left(\frac{1}{10000}\right) = -4$

Example 2.4.2

Write these logarithmic equations as exponential equations:

a) $\log_{10}(1000) = 3$

b) $\log_{10}(5) \approx 0.69897$

Solution

a) $\log_{10}(1000) = 3$ is equivalent to $10^3 = 1000$

b) $\log_{10}(5) \approx 0.69897$ is equivalent to $10^{0.69897} \approx 5$

Calculator Note:

On a calculator, the *log* button calculates the $\log_{10}$, the logarithm base 10. Try and use your calculator to calculate a decimal approximation of $\log_{10}(5)$.

Case: Base b, where b > 1 or 0 < b < 1

Definition: logarithm base b

The logarithm base b of x is the exponent of b that gives x. We write $\log_b(x) = y$ if and only if $b^y = x$.

We read $\log_b(x)$ as "log base b of x".

In this case, the logarithm base b is a function whose output is the exponent of b. In other words, the logarithm base b tells us the exponent of b. Effectively, we have defined the logarithm base b, $f^{-1}(x) = log_b(x)$, as the inverse function of the exponential function base b, $f(x) = b^x$.There is a similar relationship between each logarithmic equation and the corresponding exponential equation. Let's further explore this correspondence in the examples below.

Example 2.4.3

Write these exponential equations as logarithmic equations:

a. $2^3 = 8$
b. $3^4 = 81$
c. $e^0 = 1$

Solution

a)$2^3 = 8$ is equivalent to $\log_2(8) = 3$

b) $3^4 = 81$ is equivalent to $\log_3(81) = 4$

c) $e^0 = 1$ is equivalent to $\log_e(1) = 0$

Example 2.4.4

Write these logarithmic equations as exponential equations:

a) $\log_7(49) = 2$

b) $\log_5(625) = 4$

Solution

a) $\log_7(49) = 2$ is equivalent to $7^2 = 49$

b) $\log_5(625) = 4$ is equivalent to $5^4 = 625$

The logarithm base e is a very important logarithmic function. We give it a special name.

Definition: Term

The logarithm base e is also called **the natural logarithm. The natural logarithm** of x is the exponent of e the gives x. We write $\ln(x) = y$ if and only if $e^y = x$.

By writing, $\ln(x) = y$ we mean $\log_e(x) = y$

In example 3.4.3c, we could write $\log_e(1) = 0$ or $\ln(1) = 0$. Both statements mean the same thing.

Calculator Note:

On a calculator, the ln button calculates the natural logarithm or $\log_e$ of a number. For most calculators, log base 10 and the natural log are the only logarithmic functions on the calculator. At this point, we need to develop some tools in order to find decimal approximations of logarithms of other bases.

Now you try some examples converting back and forth between exponential and logarithmic form.

Student Exercise 2.4.1

Write these exponential equations as logarithmic equations:

a. $6^4 = 1296$
b. $10^x = 17$
c. $e^2 \approx 7.389$

Answer

a) $6^4 = 1296$ is equivalent to $\log_6(1296) = 4$

b) $10^x = 17$ is equivalent to $\log_{10}(17) = x$

c) $e^2 \approx 7.389$ is equivalent to $\ln(7.389) \approx 2$

Student Exercise 2.4.2

Write these logarithmic equations as exponential equations:

a) $\log_5(25) = 2$

b) $\log_4(x) = 3$

c) $\ln(5) \approx 1.609$

Answer

a) $\log_5(25) = 2$ is equivalent to $5^2 = 25$

a) $\log_4(x) = 3$ is equivalent to $4^3 = x$

a) $\ln(5) \approx 7.389$ is equivalent to $e^{1.609} = 5$

Notice, in student exercise 3.4.1b, you started with the exponential equation with the variable x in the exponent.

$$10^x = 17$$

By converting to logarithmic form, you have the exponent and solution to the equation.

$$x = log_{10}(17) \approx 1.23$$

You can use a calculator to get a decimal approximation. This highlights a way to solve exponential equations by converting them to logarithmic form.

Example 2.4.5

Solve the exponential equations.

a) $e^x = 12$

b) $5^x = 7$

c) $3(10)^x + 1 = 11$

d) $2(4)^{5x} + 3 = 17$

Solution

a) To solve the equation

$$e^x = 12$$

convert to logarithmic form.

$$\ln(12) = x$$

Using a calculator to find a decimal approximation, we have $x \approx 2.4849$.

b) To solve the equation

$$5^x = 7$$

convert to logarithmic form.

$$\log_5(7) = x$$

With the tools we have developed so far, we can't use a calculator to find a decimal approximation since this logarithm is not base 10 or e.

c) Notice in parts a and b, the equation has the exponential term on one side and a number on the other. In this form, we have defined the equivalent logarthimic form. In the equation

$$3(10)^x + 1 = 11$$

the exponential term $(10)^x$ is not isolated on one side of the equation. First, we need to isolate the exponential term by subtracting 1.

$$3(10)^x = 10$$

Then dividing by 3, we have

$$(10)^x = \frac{10}{3}.$$

Now converting to logarithmic form, we have

$$x = \log_{10}(\frac{10}{3}) \approx 0.5229$$

d) To solve the equation

$$2(4)^{5x} - 3 = 17$$

we need to isolate the exponential term by adding 3.

$$2(4)^{5x} = 20$$

Then dividing by 2, we have

$$(4)^{5x} = \frac{20}{2} = 10.$$

Now converting to logarithmic form, we have

$$\log_4(10) = 5x.$$

Finally, dividing both sides of the equation by 5, we have

$$\frac{\log_4(10)}{5} = x.$$

With the tools we have currently, we can't use a calculator to find a decimal approximation since this logarithm is not base 10 or e.

Now you try to solve a few exponential equations.

? Student Exercise 2.4.3

Solve the exponential equations. Find a decimal approximation if possible with the tools we have up to this point.

a) $10^x = 47$

b) $7(3)^x = 12$

c) $4(e)^{0.012x} + 1 = 15$

Answer

a) To solve the equation

$$10^x = 47$$

convert to logarithmic form.

$$x = \log_{10}(47) \approx 1.672$$

b) To solve the equation

$$7(3)^x = 12$$

we need to isolate the exponential term by dividing by 7.

$$(3)^x = \frac{12}{7}$$

Now converting to logarithmic form, we have

$$\log_3(\frac{12}{7}) = x$$

With the tools we have currently, we can't use a calculator to find a decimal approximation since this logarithm is not base 10 or e.

c) To solve the equation

$$4(e)^{0.012x} + 1 = 15,$$

first we need to isolate the exponential term by subtracting 1.

$$4(e)^{0.012x} = 14$$

Then dividing by 4, we have

$$e^{0.012x} = \frac{14}{4} = \frac{7}{2} = 3.5.$$

Now converting to logarithmic form, we have

$$ln(3.5) = 0.012x.$$

Finally, dividing both sides of the equation by 0.012, we have

$$x = \frac{ln(3.5)}{0.012} \approx 104.39.$$

Focus: Solving Logarithmic Equations

In the students exercise 3.4.2b, you started with the logarithmic equation with the variable x inside the logarithm

$$\log_4(x) = 3.$$

By converting to exponential form, you get the variable out of the logarithm and ultimately the solution to the equation.

$$x = 4^3 = 64$$

This highlights a way to solve logarithmic equations by converting them to exponential form.

Example 2.4.6

Solve the logarithmic equations.

a) $\log_3(x) = 4$

b) $3\log_2(4t) = 20$

Solution

a) To solve the equation

$$\log_3(x) = 4$$

convert to exponential form.

$$x = 3^4 = 81$$

b) Notice in part a, the equation has the logarithmic term on one side and a number on the other. In this form, we have defined the equivalent exponential form. In the equation

$$3\log_2(4t) = 20$$

the logarithmic term $\log_2(4t)$ is not isolated on one side of the equation. First, we need to isolate the logarithmic term by dividing by 3.

$$\log_2(4t) = \frac{20}{3}$$

Then convert to exponential form.

$$2^{\frac{20}{3}} = 4t$$

Finally, to isolate t we divide by 4.

$$t = \frac{2^{\frac{20}{3}}}{4} \approx 25.398$$

Now you try a few exercises solving logarithmic Equations

Student Exercise 2.4.4

Solve the logarithmic equations.

a) $ln(x) = 7$

b) $5\log_6(7t) + 3 = 20$

Answer

a) To solve the equation

$$\ln(x) = 7$$

convert to exponential form.

$$x = e^7 \approx 1096.63$$

b) To solve the equation

$$5\log_6(7t) + 3 = 20$$

we need to isolate the logarithmic term.

$$5\log_6(7t) = 17$$

$$\log_6(7t) = \frac{17}{5} = 3.4$$

Then convert to exponential form.

$$6^{3.4} = 7t$$

Finally, to isolate t we divide by 4.

$$t = \frac{6^{3.4}}{7} \approx 63.18$$

Goal: Finding a decimal approximation of logarithms of other bases than 10 or e

One issue that remains when solving exponential equations is finding a decimal approximation of logarithms of bases other than 10 or e on most calculators. Before we can do this, we need to develop some more properties of logarithms.

Key Point: Properties of Logs

1) The inverse property of logarithms

$$\log_b(b^x) = x \tag{2.4.1}$$

$$b^{\log_b(x)} = x \tag{2.4.2}$$

2) The power property of logarithms

$$\log_b(x^n) = n\log_b(x) \tag{2.4.3}$$

Proof:

For property 3.4.1, the logarithm base b by definition gives us the exponent of b. In this case, the exponent of b is x. Additional, this highlights that the logarithm function base b , $f(x) = \log_b(x)$, and the exponential function base b, $g(x) = b^x$ are inverses. Property 3.4.2 is an additional result of the log and exponential functions being inverses.

For property 3.4.3, we offer a proof:

Let $k = \log_b(x^n)$. Converting to exponential form, we get $b^k = x^n$.

Solving for x by raising both sides of the equation to the $\frac{1}{n}$ power , we have $(b^k)^{\frac{1}{n}} = (x^n)^{\frac{1}{n}}$.

Using the power property of exponents, we have $\left(b^{\frac{k}{n}}\right) = x^{\frac{n}{n}} = x$.

Now converting to logarithmic form $\log_b(x) = \frac{k}{n}$ and solving for k, we have $k = n\log_b(x)$.

Notice that k is equivalent to the expressions on both sides of the equation in property 3.4.3.

Notice that property 3.4.1 has a logarithm and exponential functions of the same base, whereas property 3.4.3 deals with logarithms and exponential functions of the different bases. Let's try a few examples to get comfortable using these properties

Example 2.4.7

Rewrite the following logarithmic expressions using the properties of logarithms.

a) $\log_3(3^7)$

b) $ln(e^x)$

c) $\log_3(5^7)$

d) $ln(2^x)$

Solution

a) Since the logarithmic function and the exponential function have the same base, we can use the inverse property of logarithms 3.4.1. Log base 3 gives us the exponent of 3. So

$$\log_3(3^7) = 7$$

b) Again, the logarithmic function and the exponential function have the same base, so we can us the inverse property of logarithms. Log base e gives us the exponent of e. So

$$ln(e^x) = x$$

c) Notice, the logarithmic function and the exponential function have different bases, so we will need to use the power property of logarithms 3.4.3.

$$\log_3(5^7) = 7\log_3(5)$$

d) Again, the logarithmic function and the exponential function have different bases, so use the power property of logarithms.

$$ln(2^x) = x(ln(2))$$

Example 2.4.8

Evaluate the expressions using the logarithmic properties

a) $\ln(\sqrt{e})$.

b) $2^{\log_2(3)}$

Solution

a) We can rewrite $\ln(\sqrt{e})$ as $\ln(e^{1/2})$. Since ln is a log base e, we can use the inverse property for logs:

$$\ln\left(e^{1/2}\right) = \log_e\left(e^{1/2}\right) = \frac{1}{2}$$

.

b) The logarithmic function and the exponential function have the same base, so the inverse property 3.4.2 applies $2^{\log_2(3)} = 3$.

Notice in example 3.4.8, by using the power property of logarithms on $ln(2^x) = x\,(ln(2))$, we get the variable out of the logarithm. Utilizing this strategy gives us an alternative way to solve an exponential equation from which we can find a decimal approximation no matter what base of logarithm.

Example 2.4.9

Solve $2^x = 3$ in two ways

a) By taking the natural logarithm of both sides of the equation and using the power property of logarithms to solve.

b) Converting to log form.

Solution

a) Strategy #1:

Applying the ln on both sides, we have $ln(2^x) = ln(3)$. Then using the power property of logarithms, we have

$$x\,(ln(2)) = ln(3).$$

Then solving for x we have:

$$x = \frac{ln(3)}{ln(2)} \approx 1.585$$

b) Strategy #2: Convert to log from.

$$x = log_2(3)$$

Previously, we were unable to find a decimal approximation of $log_2(3)$. However, notice the approximation of x with strategy #1 is also and approximation of the logarithm expression we found using strategy #2. Notice, this gives us a way to approximate logs of any base using a calculator.

$$\log_2(3) = \frac{ln(3)}{ln(2)}$$

We can generalize what we have done in example 3.4.9 to approximate logarithms of any base with the following property.

properties of logs: change of base

$$\log_b(a) = \frac{ln(a)}{\ln(b)}$$

Proof

Let $\log_b(a) = x$.

Rewriting as an exponential gives $b^x = a$.

Taking the ln of both sides of this equation gives

$$ln(b^x) = ln(a)$$

Now utilizing the power property for logs on the left side,

$$xln(b) = ln(a)$$

Dividing, we obtain $x = \frac{ln(a)}{ln(b)}$. Replacing our original expression for x,

$$\log_b(a) = \frac{ln(a)}{ln(b)}$$

Notice that the change of base formula can be written with a logarithm of any base on both sides of the equation if we start by taking the logarithm base b of both sides. We have chosen the natural logarithm since it can be approximated on a standard calculator.

Example 2.4.10

Evaluate using the change of base formula.

a) $\log_2(10)$

b) $\log_5(100)$

Solution

a) Writing the logarithm with natural log we have: $\log_2 10 = \dfrac{\ln 10}{\ln 2} \approx 3.3219$

b) $\log_5 100 = \dfrac{\ln 100}{\ln 5} \approx 2.861$

The change of base formula highlights two ways that we can get the solution to an exponential equation like $2^x = 10$ and get a decimal approximation on a standard calculator.

Solving Exponential Equations

Strategy #1

1. Isolate the exponential expressions when possible
2. Convert the equation to logarithm form
3. Use algebra to solve for the variable.
4. Use the change of base formula to find a decimal approximation if needed.

Strategy #2

1. Isolate the exponential expressions when possible
2. Take the natural logarithm of both sides
3. Utilize the exponent property for logarithms to pull the variable out of the exponent
4. Use algebra to solve for the variable.

Example 2.4.11

Solve $2^x = 10$ for x

a) Using strategy #1

b) Using Strategy #2

Solution

a) Converting to logarithmic for:

$$\log_2(10) = x$$

Using the change of base formula, we have

$$x = \frac{\ln(10)}{ln(2)} \approx 3.3219$$

b) Taking the natural logarithm of both sides of the equation:

$$ln\left(2^x\right) = ln(10)$$

Utilizing the exponent property for logs,

$$x\,(ln(2)) = ln(10)$$

Now dividing by ln(2),

$$x = \frac{\ln(10)}{ln(2)} \approx 3.3219$$

Notice that this result matches the result we found using the change of base formula.

Example 2.4.12

Solve $5(1.07)^{3t} = 2$

a) Using strategy #1.

b) Using Strategy #2.

Solution

a) To start, we want to isolate the exponential part of the exponential expression, $(1.07)^{3t}$, on one side of the equation.

$$5(1.07)^{3t} = 2$$

Divide both sides by 5 to isolate the exponential expression.

$$(1.07)^{3t} = \frac{2}{5} = 0.4$$

Convert to log form.

$$\log_{1.07}(0.4) = 3t$$

Solve for t by divide by 3 on both sides.

$$\left(\frac{\log_{1.07}(0.4)}{3}\right)$$

Now use the change of base formula to find a decimal approximation.

$$t = \frac{ln(0.4)}{3ln(1.07)} \approx -4.5143$$

b) To start, we want to isolate the exponential part of the exponential expression, $(1.07)^{3t}$, on one side of the equation as we did in part a.

$$5(1.07)^{3t} = 2$$

Divide both sides by 5 to isolate the exponential

$$(1.07)^{3t} = \frac{2}{5} = 0.4$$

Take the ln of both sides.

$$ln((1.07)^{3t}) = ln(0.4)$$

Use the power property for logs

$$3t\,(ln(1.07)) = ln(0.4)$$

Divide by $3\,(ln(1.07))$ on both sides

$$t = \frac{ln(0.4)}{3ln(1.07)} \approx -4.5143$$

Student Exercise 2.4.5

Evaluate $\log_3(7)$ using the change of base formula.

Answer

Writing the logarithm with natural log we have: $\log_3 7 = \dfrac{\ln 7}{\ln 3} \approx 1.7712$

Student Exercise 2.4.6

Solve $5(0.93)^x = 10$ using any method. Find a decimal approximation.

Answer

Using Strategy 1:

$$5(0.93)^x = 10$$

Divide both sides by 5 to isolate the exponential expression.

$$(0.93)^x = \frac{10}{5} = 2$$

Convert to log form.

$$\log_{0.93}(2) = x$$

Now use the change of base formula to find a decimal approximation.

$$t = \frac{ln(2)}{ln(0.93)} \approx -9.5513$$

Using Strategy 2:

$$5(0.93)^x = 10$$

Divide both sides by 5 to isolate the exponential expression.

$$(0.93)^x = \frac{10}{5} = 2$$

Take the ln of both sides.

$$ln((0.93)^x) = ln(2)$$

Use the power property for logs

$$x\,(ln(0.93)) = ln(2)$$

Divide by $(ln(0.93))$ on both sides

$$t = \frac{ln(2)}{ln(0.93)} \approx -9.5513$$

Focus: More of Modeling With Exponential Functions

In section 3.3, we highlighted two issues when modeling with exponential functions. The first issue arose when making predictions where the output was known and the input was desired. Now that we can solve exponential equations with logarithms, we can make these predictions.

Example 2.4.13

In sec. 3.2, we predicted the population (in billions) of India t years after 2008 by using the function $f(t) = 1.14(1.0134)^t$. Note that we used a periodic exponential model, since the growth factor $b = 1.0134$ =101.34% is easily found from the given percentage growth rate $r = 0.0134$=1.34%. If the population continues following this trend, when will the population reach 2 billion?

Solution

We need to solve for time t so that $f(t) = 2$. Using strategy #2.

$$2 = 1.14(1.0134)^t$$

Divide by 1.14 to isolate the exponential expression

$$\frac{2}{1.14} = 1.0134^t$$

Take the logarithm of both sides of the equation

$$\ln\left(\frac{2}{1.14}\right) = \ln\left(1.0134^t\right)$$

Apply the exponent property on the right side

$$\ln\left(\frac{2}{1.14}\right) = t\ln\left(1.0134\right)$$

Divide both sides by ln(1.0134)

$$t = \frac{\ln\left(\frac{2}{1.14}\right)}{\ln\left(1.0134\right)} \approx 42.23 \text{ years}$$

If this growth rate continues, the model predicts the population of India will reach 2 billion about 42 years after 2008, or approximately in the year 2050.

The second issue arose when trying to find the continuous growth rate r of a continuous exponential model given two data points. Now that we can solve exponential equations with logarithms, we can find continuous exponential models.

Example 2.4.14

A population grows from 100 to 130 in 2 weeks.

a) Find a continuous exponential model for the population P in t weeks.

b) Predict when the population will reach 200.

Solution

a) Measuring t in weeks, we are looking for an equation $P(t) = ae^{rt}$ so that $P(0)$ = 100 and $P(2)$ = 130. Our initial value is a = 100.

Using other data point t = 2 and P = 130,

$$130 = 100e^{r\cdot 2}$$

Divide by 100

$$\frac{130}{100} = e^{2r}$$

Take the natural log of both sides

$$\ln(1.3) = \ln\left(e^{2r}\right)$$

Use the inverse property of logs

$$\ln(1.3) = 2r$$
$$r = \frac{\ln(1.3)}{2} \approx 0.1312$$

This population is growing at a continuous rate of 13.12% per week.

The population in t weeks is given by the function $P(t) = 100e^{0.1312t}$

b) We want the time t when P = 200. Substituting this into our model.

$$P(t) = 100e^{0.1312t}$$

Divide by 100

$$\frac{200}{100} = 2 = e^{0.1312t}$$

Take the natural log of both sides

$$\ln(2) = \ln\left(e^{0.1312t}\right)$$

Use the inverse property of logs

$$\ln(2) = 0.1312t$$
$$t = \frac{\ln(2)}{0.1312} \approx 52.83$$

The population will reach 200 in approximately 53 weeks.

Example 2.4.15

In 1960, the CO_2 emissions was measured at 316.9 ppm (parts per million) at the Mauna Loa Observatory in Hawaii. In 2000, the CO_2 emissions were measured at 369.4 ppm (NOAA - Global Monitoring Laboratory at gml.noaa.gov/ccgg/trends/global.html).

a) Find a continuous exponential model for the CO_2 emissions,C , in t years since 1960.

b) Predict the CO_2 emissions in 2030.

Solution

a) We have two data points given. The first, $C(0)$ = 316.9 gives us our initial value is a = 316.9.

Substituting other data point $C(40)$ = 369.4 into our continuous exponential model, $C(t) = ae^{rt}$, we have

$$369.4 = 316.9e^{r\cdot 40}$$

Divide by 316.9

$$\frac{369.4}{316.9} = e^{r40}$$

$$1.16567 = e^{r40}$$

Take the natural log of both sides

$$\ln(1.16567) = \ln\left(e^{r40}\right)$$

Use the inverse property of logs

$$\ln(1.16567) = 40r$$
$$r = \frac{\ln(1.16567)}{40} \approx 0.0038324$$

The formula for the CO_2 emissions t years since 1960 is given by the function $C(t) = 316.9e^{0.0038324t}$

b) In 2030, t = 70. Substituting this into our model.

$$C(t) = 316.9e^{0.0038324(70)} \approx 414.4ppm$$

In 2030, the CO_2 emissions will be 414.4 ppm.

Newton's Law of Cooling

When a hot object is left in surrounding air that is at a lower temperature, the object's temperature will decrease towards the surrounding air temperature. The difference in temperature between and object and the surrounding air temperature is an exponential function. This is the basis for Newton's Law of Cooling.

Definition: Newton's Law of Cooling

The difference in temperature between an object and the surrounding air , D, is described by the exponential function

$$D(t) = ae^{kt} \tag{2.4.4}$$

Therefore the temperature of an object, T, in surrounding air with temperature T_s will behave according to the formula

$$T(t) = ae^{kt} + T_s \tag{2.4.5}$$

Where

- t is time
- a is a constant determined by the initial temperature of the object
- k is a constant, the continuous rate of cooling of the object

While an equation of the form $T(t) = ab^t + T_s$ could be used, the continuous growth form is more common.

Example 2.4.16

A cheesecake is taken out of the oven with an ideal internal temperature of 165 degrees Fahrenheit, and is placed into a 35 degree refrigerator. After 10 minutes, the cheesecake has cooled to 150 degrees. If you must wait until the cheesecake has cooled to 70 degrees before you eat it, how long will you have to wait?

Solution

Since the surrounding air temperature in the refrigerator is 35 degrees, the cheesecake's temperature will decay exponentially towards 35, following the equation

$$T(t) = ae^{kt} + 35$$

We know the initial temperature was 165, so $T(0) = 165$. Substituting in these values,

$$\begin{aligned} 165 &= ae^{k0} + 35 \\ 165 &= a + 35 \\ a &= 130 \end{aligned}$$

We were given another pair of data, $T(10) = 150$, which we can use to solve for k

$$150 = 130e^{k10} + 35$$

$$115 = 130e^{k10}$$
$$\frac{115}{130} = e^{10k}$$
$$\ln\left(\frac{115}{130}\right) = 10k$$
$$k = \frac{\ln\left(\frac{115}{130}\right)}{10} = -0.0123$$

Together this gives us the equation for cooling:

$$T(t) = 130e^{-0.0123t} + 35$$

Now we can solve for the time it will take for the temperature to cool to 70 degrees.

$$70 = 130e^{-0.0123t} + 35$$

$$35 = 130e^{-0.0123t}$$

$$\frac{35}{130} = e^{-0.0123t}$$

$$\ln\left(\frac{35}{130}\right) = -0.0123t$$

$$t = \frac{\ln\left(\frac{35}{130}\right)}{-0.0123} \approx 106.68$$

It will take about 107 minutes, or one hour and 47 minutes, for the cheesecake to cool. Of course, if you like your cheesecake served chilled, you'd have to wait a bit longer.

Now you try a few examples of modeling with continuous exponential functions.

? Student Exercise 2.4.7

Cancer cells sometimes increase in size exponentially. Suppose a cancerous growth contained 300 cells last month and 360 cells this month.

a) Find a continuous exponential model for the number off cancer cells, C, in t months.

b) Predict how long will it take for the number of cancer cells to double?

Answer

a) We are given two data points: this month, (0, 360), and last month, (-1, 300). The first point gives us our initial value is a = 360.

Substituting other data point $C(-1)$ = 300 into our continuous exponential model, $C(t) = ae^{rt}$, we have

$$300 = 360e^{r\cdot -1}$$

Divide by 360

$$\frac{300}{360} = e^{r(-1)}$$

$$\frac{5}{6} = e^{r(-1)}$$

Take the natural log of both sides

$$\ln(\frac{5}{6}) = \ln\left(e^{r(-1)}\right)$$

Use the inverse property of logs

$$\ln(\frac{5}{6}) = (-1)r$$
$$r = \frac{ln(\frac{5}{6})}{-1} \approx 0.18232$$

The formula for the number of cancer cells in t months is given by the function $C(t) = 360e^{0.18232t}$

b) If the number of cells double, then C = 2(360)=720. Substituting this into our model and solving for t.

$$720 = 360e^{0.18232t}$$

Divide by 360

$$\frac{720}{360} = e^{0.18232t}$$

$$2 = e^{0.18232t}$$

Take the natural log of both sides

$$\ln(2) = \ln\left(e^{0.18232t}\right)$$

Use the inverse property of logs

$$\ln(2) = 0.18232t$$
$$r = \frac{ln(2)}{0.18232} \approx 3.802$$

Exercise 2.4.8

The value V, in billions of dollars, of US imports from China in the 2000 was 100 billion. By 2004, the value rose to 196 billion

a) Find a continuous exponential model for The value V, in billions of dollars, of US imports from China t years after 2000.

b) If this trend continues, predict the value of US imports to China in 2025.

Answer

a) We have two data points given. The first, $V(0)$ = 100 gives us our initial value is a = 100.

Substituting other data point $V(4)$ = 196 into our continuous exponential model, $V(t) = ae^{rt}$, we have

$$196 = 100e^{r\cdot 4}$$

Divide by 100

$$\frac{196}{100} = e^{r4}$$

$$1.96 = e^{r4}$$

Take the natural log of both sides

$$\ln(1.96) = \ln\left(e^{r4}\right)$$

Use the inverse property of logs

$$\ln(1.96) = 4r$$
$$r = \frac{\ln(1.96)}{4} \approx 0.16824$$

The formula for the The value V, in billions of dollars, of US imports from China t years after 2000 is given by the function $V(t) = 100e^{0.16824t}$

b) In 2025, t = 25. Substituting this into our model.

$$V(t) = 100e^{0.16824(25)} \approx 6708.8$$

The value of US imports in 2025 is $6,708.8 billion = 6.7088 trillion.

? Exercise 2.4.9

A pitcher of water at 40 degrees Fahrenheit is placed into a 70 degree room. One hour later the temperature has risen to 45 degrees. How long will it take for the temperature to rise to 60 degrees?

Answer

$T(t) = ae^{kt} + 70$. Substituting (0, 40), we find $a = -30$. Substituting (1, 45), we solve

$$45 = -30e^{k(1)} + 70$$

to get

$$k = ln(\frac{25}{30}) = -0.1823$$

Solving $60 = -30e^{-0.1823t} + 70$ gives

$$t = \frac{ln(1/3)}{-0.1823} = 6.026 \text{ hours}$$

Important Topics of this Section

- The Logarithmic function as the inverse of the exponential function
- Writing logarithmic & exponential expressions
- Properties of logs
- Inverse properties
- Exponential properties
- Change of base
- Common log
- Natural log
- Solving exponential equations
- Newton's law of cooling

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2.4E: Solving Exponential Equations with Logarithms (Exercises)

Section 2.4 Exercises

Part 1: Using logarithmic functions and their properties to solve exponential equations

1. Rewrite each equation in exponential form

   a. $\log_4(q) = m$

   b. $\log_3(t) = k$

   c. $\log_a(b) = c$

   d. $\log_p(z) = u$

   e. $\log(v) = t$

   f. $\ln(w) = n$

2. Rewrite each equation in logarithmic form.

   a. $4^x = y$

   b. $5^y = x$

   c. $c^d = k$

   d. $n^z = L$

   e. $10^a = b$

   f. $e^k = h$

3. Evaluate each expression using your calculator to find a decimal approximation.

   a. $\log(0.04)$

   b. $\log(1045)$

   c. $\ln(15)$

   d. $\ln(0.02)$

4. Find a decimal approximation of each logarithm using the change of base formula and a calculator.

   a. $\log_4(5)$

   b. $\log_8(12)$

   c. $\log_3(\frac{1}{15})$

In exercises 5 - 22, solve each equation for the unknown variable. Then write your solution with a decimal approximation.

5. $5^x = 14$

6. $3^x = 23$

7. $7^x = \frac{1}{15}$

8. $e^{5x} = 17$

9. $3^{4x-5} = 38$

10. $1000(1.03)^t = 5000$

11. $200(1.06)^t = 550$

12. $3(1.04)^{3t} = 8$

13. $2\left(1.08\right)^{4t}=7$

14. $50e^{-0.12t}=10$

15. $10e^{-0.03t}=4$

16. $10-8\left(\frac{1}{2}\right)^{x}=5$

17. $\log_3\left(x\right)=2$

18. $\log_4(x)=3$

19. $5\log_2(x)+4=-3$

20. $3\log_5(x)=-1$

21. $\log\left(x\right)=3$

22. $\ln\left(x\right)=2$

Part 2: Using the properties of logarithms

In exercises 23 - 32, simplify each expression using properties of logarithms. In some cases, it will be helpful to write the input of the logarithm as a power of the base. In other cases, this may not be easily done.

23. $\log_5\left(25\right)$

24. $\log_2\left(8\right)$

25. $\log_3\left(\frac{1}{27}\right)$

26. $\log_6\left(7^3\right))$

27. $\log_6\left(\sqrt{6}\right)$

28. $\log_5\left(\sqrt[3]{5}\right)$

29. $\log\left(10,000\right)$

30. $\log\left(0.001\right)$

31. $\log\left(5^{20}\right)$

32. $\ln\left(e^3\right)$

Part 3: More on Modeling with exponential functions

33. You take 200 milligrams of a headache medicine, and after 4 hours, 120 milligrams remain in your system.

    a. Find a continuous exponential formula that models the milligrams of the medicine remaining in your system after t hours.

    b. If the effects of the medicine wear off when less than 80 milligrams remain, when will you need to take a second dose, assuming the amount of medicine in your system decays exponentially?

34. You go to the doctor and they inject you with 13 milligrams of radioactive dye as part of a test. After 12 minutes, 4.75 milligrams of dye remain in your system.

    a. Find a continuous exponential formula that models The amount of dye in your system after t minutes.

    b. To leave the doctor's office, you must pass through a radiation detector without sounding the alarm. If the detector will sound the alarm whenever more than 2 milligrams of the dye are in your system, how long will your visit to the doctor take, assuming you were given the dye as soon as you arrived and the amount of dye decays exponentially.

35. A ship embarked on a long voyage. At the start of the voyage, there were 500 ants in the cargo hold of the ship. One week into the voyage, there were 800 ants. Suppose the population of ants is an exponential function of time. [UW]

    a. Find a continuous exponential formula for the number of ants, A, after t weeks.

b. How long did it take the population to double?
c. When were there be 10,000 ants on board?

36. The population of Seattle grew from 563,374 in 2000 to 608,660 in 2010. Assume the population continues to grow exponentially at the same rate.

a. Find a continuous exponential model for P, the population of Seattle, t years since 2010.

b. When will the population exceed 1 million people?

37. The median household income (adjusted for inflation) in Seattle grew from $42,948 in 1990 to $45,736 in 2000. Assume it continues to grow exponentially at the same rate.

a. Find a continuous exponential model for I, the median household income in Seattle, t years since 1990.

b. When will median income exceed $70,000?

38. A radioactive substance decays exponentially. A scientist begins with 110 milligrams of a radioactive substance. After 31 hours, 55 mg of the substance remains.

a) Find a continuous exponential formula for the amount of the radioactive

b) How long until only 10 mg remain?

39. The population of Kenya was 39.8 million in 2009 and has been growing by about 2.6% each year. Assume this trend continues

a. Find a formula for P, the population of Kenya t years since 2009.

b. When will the population exceed 60 million?

40. The number of firearm homicide deaths was 4.6 deaths per 100,000 people in 2019. The number of firearm homicide deaths increased by 35% per year from 2019 to 2020.[1] Assume this trend continues.

a. Find a formula for N, the number of firearm homicide deaths, t years since 2019.

b. Predict when the number of firearm homicide deaths will reach 10 deaths per 100,000 people.

41. A bacteria culture initially contains 1500 bacteria and doubles in size every half hour.

a. Find a formula for N, the number of bacteria, in t hours.

b. How long until there are 10,000 bacteria?

42. The half-life of Radium-226 is 1590 years. A sample initially contains 200 mg.

a. Find an exponential formula for the amount in t years.

b. How long until only 10 mg remain?

43. The half-life of Fermium-253 is 3 days. A sample initially contains 100 mg.

a. Find a formula for the amount in t years.

b. how many milligrams will remain after 1 week?

44. A scientist begins with 250 grams of a radioactive substance. After 225 minutes, the sample has decayed to 32 grams. Find the half-life of this substance.

45. A wooden artifact from an archeological dig contains 60 percent of the carbon-14 that is present in living trees. How long ago was the artifact made? (The half-life of carbon-14 is 5730 years.)

46. The count of bacteria in a culture was identified to be 800 after 10 minutes and 1800 after 40 minutes.

a. What was the initial size of the culture?
b. Find a formula for N, the number of bacteria, in t minutes.
c. Find the number of bacteria after 105 minutes.
d. When will the number of bacteria reach 11000?

47. Find the time required for an investment to double in value if invested in an account paying 3% compounded quarterly.

48. A turkey is pulled from the oven when the internal temperature is 165° Fahrenheit, and is allowed to cool in a 75° room. If the temperature of the turkey is 145° after half an hour,

a. What will the temperature be after 50 minutes?
b. How long will it take the turkey to cool to 110°?

49. A cup of coffee is poured at 190° Fahrenheit, and is allowed to cool in a 70° room. If the temperature of the coffee is 170° after half an hour,

a. What will the temperature be after 70 minutes?
b. How long will it take the coffee to cool to 120°\)?

50. As light from the surface penetrates water, its intensity is diminished. In the clear waters of the Caribbean, the intensity is decreased by 15 percent for every 3 meters of depth. Thus, the intensity will have the form of a general exponential function. [UW]

a. If the intensity of light at the water's surface isI_0, find a formula for $I(d)$, the intensity of light at a depth of d meters. Your formula should depend on $I_0$and d.
b. At what depth will the light intensity be decreased to 1% of its surface intensity?

51. The population of fish in a farm-stocked lake after *t* years could be modeled by the equation $P(t) = \dfrac{1000}{1+9e^{-0.6t}}$.

a. Sketch a graph of this equation.
b. What is the initial population of fish?
c. What will the population be after 2 years?
d. How long will it take for the population to reach 900?

---

[1] - The data was reported by the CDC at https://www.cdc.gov/vitalsigns/firea...ths/index.html on February 9, 2023.

Answer

Part 1:

1. a. $4^m = q$ c. $a^c = b$ e. $10^t = v$

2. a. $\log_4(y) = x$ c. $\log_c(k) = d$ e. $\ln(h) = k$

4. a. ≈ 1.161

5. $\dfrac{\ln(14)}{\ln(5)} \approx 1.6397$

7. $\dfrac{\log(\frac{1}{15})}{\log(7)} \approx -1.392$

9. $\dfrac{\frac{\log(38)}{\log(3)}+5}{4} \approx 2.078$

11. $\dfrac{\ln(2.75)}{\ln(1.06)} \approx 17.361$

13. $\dfrac{\ln(\frac{7}{2})}{4\ln(1.08)} \approx 4.069$

15. $\dfrac{\ln(0.4)}{-0.03} \approx 30.543$

17. 9

19. $2^{\frac{-7}{5}} \approx 0.3789$

21. 1,000

Part 2:

23. 2

25. -3

27. $\frac{1}{2}$

29. 4

31. $20\log(5)$

Part 3:

33. a. $A = 200e^{-0.1277t}$ b. You will need to take a second dose after 7.18 hours.

35. a. $A = 500e^{0.47t}$ b. The population will double in 1.47 days. c. In 6.37 days.

37. a. $I = 42,948e^{0.00629t}$ b. The median income will exceed this level in 77.7 years.

39. a. $P = 3.9(1.026)^t$ b. In 16.78 years.

41. a. $N = 1500e^{1.386t}$ b. In 1.369 hours.

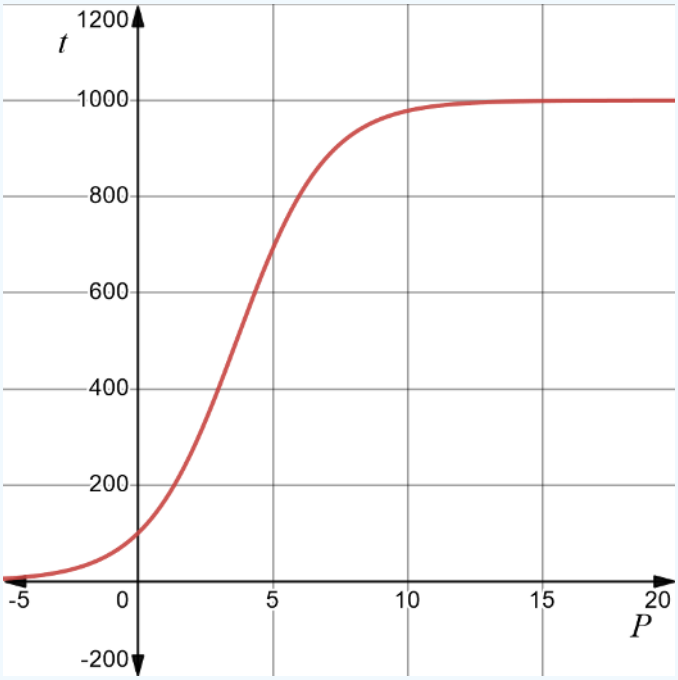
43. a. $A = 100e^{-0.231t}$ b. 19.85 mg

45. It was made 4,222.7 years ago.

47. It will double in value in 23.19 years.

49. a. 148.4 degrees b. In 144 minutes.

51. a.



b. 100 fish c. 269 fish d. 7.324 years

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2.5: Describing Relationships with Logarithmic Functions

Up to this point, we have primarily used logarithmic functions as a tool to solve exponential equations. However, logarithmic functions can be used to describe some relationships on their own, such as sound intensity, soil compaction, earthquakes, or the ph or acidity of a compound. This is our goal for this section. Let's start by graphing logarithmic functions.

Focus: Graphing Logarithmic Functions

First, we will graph the logarithmic function $y = f(x) = \log_2(x)$. We need to find a table of values that represents points on the graph. The relationship is easier to calculate values in exponential form. Converting to exponential form, we have $x = f^{-1}(y) = 2^y$.

The exponential function $x = f^{-1}(y) = 2^y$ produces this table of values.

y	-3	-2	-1	0	1	2	3
$x = f^{-1}(y) = 2^y$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

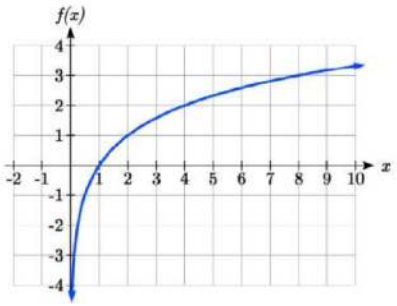
Since the logarithmic function is the inverse of the exponential function, $y = f(x) = \log_2(x)$ produces this table of values, where the inputs and outputs are switched.

x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$y = f(x) = \log_2(x)$	-3	-2	-1	0	1	2	3

In this second table for the values of the logarithm, notice that

1. As the input x increases, the output y increases at a slower and slower rate. In symbolic notation, we write this as $x \to \infty, f(x) \to \infty$.
2. As the input x decreases towards zero, the the output y becomes a larger and larger size negative number approaching negative infinity. In symbolic notation, we write as $x \to 0^+, f(x) \to -\infty$.
3. Since the outputs of an exponential function are only positive values, the inputs of a logarithm can only be positive values. Therefore, the domain of the log function is $(0, \infty)$.
4. Since the inputs of an exponential function can accept all real numbers as inputs, the logarithm can output any real number, so the range is all real numbers or $(-\infty, \infty)$.

Sketching the graph,



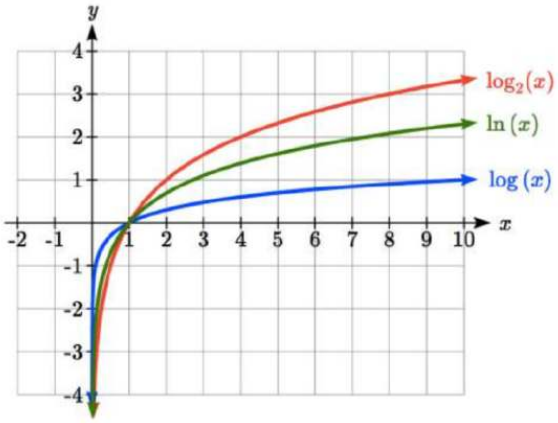
Key Point: The Features of the Graph of a Logarithm

Graphically, in the function $g(x) = \log_b(x)$

- The graph has a horizontal or x-intercept at (1, 0)
- The graph is increasing at a slower and slower rate

- The domain of the function is $x > 0$, or $(0, \infty)$
- The range of the function is all real numbers, or $(-\infty, \infty)$

When sketching a general logarithm with base b, it can be helpful to remember that the graph will pass through the points (1, 0) and (b, 1). To get a feeling for how the base affects the shape of the graph, examine the graphs below.



Notice that the larger the base, the slower the graph grows. For example, the common log graph, while it grows without bound, it does so very slowly. For example, to reach an output of 8, the input must be 100,000,000.

Focus: The Domain of Logarithmic Functions

Another important observation made was the domain of the logarithm. Like the reciprocal and square root functions, the logarithm has a restricted domain which must be considered when finding the domain of a composition involving a log.

Key Point:

The input of a logarithmic function must be positive.

Example 2.5.1

Find the domain of the logarithmic functions

a) $f(x) = \log(5 - 2x)$.

b) $f(x) = ln(x - 5) + 2$.

Solution

a) The logarithm is only defined with the input is positive, so this function will only be defined when $5 - 2x > 0$. Solving this inequality,

$$-2x > -5$$

$$x < \frac{5}{2}$$

The domain of this function is $x < \frac{5}{2}$, or in interval notation, $\left(-\infty, \frac{5}{2}\right)$

b) The logarithm is only defined with the input is positive, so this function will only be defined when $x - 5 > 0$. Solving this inequality,

$$x > 5$$

The domain of this function is $x > 5$, or in interval notation, $(5, \infty)$.

Student Exercise 2.5.1

Find the domain of the function $f(x) = \log_3(3x + 4)$.

Answer

The logarithm is only defined with the input is positive, so this function will only be defined when $3x + 4 > 0$. Solving this inequality,

$$3x > -4$$
$$x > \frac{-4}{3}$$

The domain of this function is $x > \frac{-4}{3}$, or in interval notation, $\left(\frac{-4}{3}, \infty\right)$

Focus: An Application Logarithmic Functions

Case: Soil Compaction

Soil compaction, the relationship between the amount of force F applied to soil and the distance the soil compacts Y, can be described by a logarithmic function. To understand why a logarithmic function is appropriate to describe soil compaction, consider the following scenario. If two people, first a smaller person followed by a larger person, walking in line in the others tracks on deep fresh snow with snowshoes of the same size. If the smaller person with a smaller weight, such as $F_1 = 100lbs$, steps on a fresh snow with their snowshoe, the snow will compact a distance y_1. If the larger person with a larger weight, such as $F_2 = 200lbs$, steps into the same snowshoe track of the smaller person ahead of them, the snow will compact an additional, but smaller distance y_2 corresponding to the increase in force $\Delta F = 100$. In general, as larger and larger force is applied to the snow with the same surface area, the distance compacted increases at a smaller and smaller rate, just like that of a logarithmic function.

In many locations, the soil consists of layers of different types of soil, such as sand, clay, rock, and top soil. Each soil layer compacts differently and would be described by a different logarithmic functions, such as $y_1 = a_1 ln(b_1 F_1 - c_1) + d_1$ for one soil layer and $y_2 = a_2 ln(b_2 F_2 - c_2) + d_2$ for another soil layer where a,b,c, and d are constants. To find the total compaction of all soil layers, we would add two corresponding logarithmic functions. We need to develop additional properties of logarithms in order to perform operations on logarithms like this.

Focus: Operations and Simplifying with Logarithmic Expressions

There are additional properties of logarithms that will allow us to add and subtract logarithmic expressions like the ones mentioned above.

Additional Properties of logarithms

1) The Log Sum Property:

$$\log_b(M) + \log_b(N) = \log_b(MN) \tag{2.5.1}$$

In words, to add two logs of the same base, multiply the inputs in a single logarithm.

2) The Log Difference Property:

$$\log_b(M) - \log_b(N) = \log_b\left(\frac{M}{N}\right) \tag{2.5.2}$$

In words, to subtract two logs of the same base, divide the inputs in a single logarithm.

3) The Power Property of Logarithms:

$$\log_b(x^n) = n\log_b(x) \tag{2.5.3}$$

We offer a proof of Property 3.5.1 to help solidify our new rules and show how they follow from properties you've already seen.

Proof

Let $a = \log_b(M)$ and $c = \log_b(N)$.

Converting each expression to exponential form, $b^a = M$ and $b^c = N$.

Multiplying the expressions on the left side of the two equations by the expressions on the right side of the equations, we have

$$MN = b^a b^c$$

Using the laws of exponents on the expression on the right side, we have

$$MN = b^{a+c}$$

Converting this expression back to logarithmic form we have

$$\log_b(MN) = a + c$$

Replacing a and c with their definition establishes the result

$$\log_b(MN) = \log_b M + \log_b N$$

The proof for the difference property is very similar.

With these properties, we can rewrite expressions involving multiple logs as a single log, or break an expression involving a single log into expressions involving multiple logs.

Example 2.5.2

Simplify the logarithmic expressions by writing them as a single logarithm.

a) $y = \log_3(5) + \log_3(8) - \log_3(2)$

b) $y = 2\log(x) + 3\log(4x^2)$

Solution

a) Using the sum of logs property on the first two terms,

$$\log_3(5) + \log_3(8) = \log_3(5 \cdot 8) = \log_3(40)$$

This reduces our original expression to

$$y = \log_3(40) - \log_3(2)$$

Then using the difference of logs property,

$$y = \log_3(40) - \log_3(2) = \log_3\left(\frac{40}{2}\right) = \log_3(20)$$

b) In this example, we have logarithmic terms with a constant multiples. The Sum Property of logs only applies if we adding two logs without constant multiples. We can remedy this by using the power property of logs on each expression to write

$$y = 2\log(x) + 3\log(4x^2) = \log(x^2) + \log((4x^2)^3)$$

Notice in the second expression, which has two factors in its input, we had to raise the whole input to that power when using the power property of logs. Simplifying the inputs of each logarithm, then utilizing the sum of logs property we have

$$y = \log(x^2) + \log(64x^6) = \log(x^2 \cdot 64x^6)$$

Performing the multiplication in the input of the logarithm we have:

$$y = \log(64x^8)$$

? Student Exercise 2.5.2

Simplify the logarithmic expressions by writing them as a single logarithm.

a) $y = \log_2(4x) + \log_2(8x)$

b) $y = 10 \cdot ln(x) - 2 \cdot ln(3x^4)$

Answer

a) Using the sum of logs property,

$$y = \log_2(4x) + \log_2(8x) = \log_2(4x \cdot 8x) = \log_2(32x^2)$$

b) First, we need to use the power property of logs on each expression to write

$$y = 10 \cdot ln(x) - 2 \cdot ln(3x^4) = ln(x^{10}) - ln((3x^4)^2)$$

Simplifying the inputs of each logarithm, then utilizing the difference of logs property we have

$$y = ln(x^{10}) - ln(9x^8) = ln(\frac{x^{10}}{9x^8}) = ln(\frac{x^2}{9})$$

Notice, in the examples above, we were able to perform the operations inside of the logarithm and ultimately simplify the expression since the expressions involved the same variables. Logarithmic expressions with multiple variables on the other hand may be simplified by writing them as a sum or difference of simpler logarithmic expressions.

✓ Example 2.5.3

Rewrite $y = \ln\left(\frac{x^4y}{z}\right)$ as a sum or difference of logs

Solution

First, noticing we have a quotient of two expressions, we can utilize the difference property of logs to write

$$y = \ln\left(\frac{x^4y}{z}\right) = \ln(x^4y) - \ln(z)$$

Then seeing the product in the first term, we use the sum property

$$y = \ln(x^4y) - \ln(z) = \ln(x^4) + \ln(y) - \ln(z)$$

Finally, we could use the power property on the first term

$$y = \ln(x^4) + \ln(y) - \ln(z) = 4\ln(x) + \ln(y) - \ln(z)$$

Focus: More on Solving Logarithmic Equations

Case: An equation with one log term.

In sec. 3.4, we explored solving logarithmic equations with one log term such as the one in the example below.

✓ Example 2.5.4

$$3\log_2(x) = 12.$$

Solution

We learned that we could solve this type of logarithmic function by isolating the log term.

$$\log_2(x) = \frac{12}{3} = 4$$

Then converting to exponential form and solving if necessary gives us our solution.

$$x = 2^4 = 16$$

Case: An equation with two or more log terms.

If we have two log terms in an equation, such as

$$\log_2(x^2) + \log_2(3x) = 12$$

we can't isolate a single log term. However, we can use the properties of logarithms to write the equation with one log term. Then solve the equation like we did in example 3.5.4.

Example 2.5.5

Solve $\log_2(x) + \log_2(3x) = 5$

Solution

$$\log_2(x) + \log_2(3x) = 5$$

First, we combine the two log terms into one using the sum property of logarithms.

$$\log_2(x \cdot 3x) = 5$$

Simplifying.

$$\log_2(3x^2) = 5$$

Then converting to exponential form

$$3x^2 = 2^5$$

and solving.

$$x^2 = \frac{32}{3}$$

$x = +\sqrt{(\frac{32}{3})} \approx +3.266$ or $x = -\sqrt{(\frac{32}{3})} \approx -3.266$

However, $x \approx -3.266$ is not in the domain of the original logarithmic expressions. We can't input negative numbers into logarithms. We call this an extraneous solution. This equation has no solutions.

Be Careful:

When using the log sum or difference properties in solving logarithmic equations with two or more terms, verify whether or not your solutions are in the domain of the original logarithmic expressions.

Notice that an extraneous solution to a logarithmic equation with two log terms can occur when using the sum or difference properties of logarithms in certain cases. For example, $log(-2)$ and $log(-50)$ are not defined since the inputs of the logarithm are negative. However, when using the log sum property on the $log(-2) + log(-50) = log(-2 \cdot -50) = log(100) = 2$, we end up with an expression on the right side of the equation that is defined even though the original expression was not defined.

Example 2.5.6

Solve $2\log(4x^2) - 3\log(x) = 2$.

Solution

We need a single logarithmic expression on the left side of the equation. First we need to use the power property of logs and simplifying.

$$\log((4x^2)^2) - \log(x^3) = 2$$

$$\log(16x^4) - \log(x^3) = 2$$

Then using the difference property of logs, we can rewrite the left side:

$$\log\left(\frac{16x^4}{x^3}\right) = 2$$

$$\log(16x) = 2$$

Then converting to exponential form

$$10^2 = 16x$$

and solving

$$x = \frac{100}{16} = 6.25$$

Checking this answer in the original equation, we can verify there are no domain issues, and this answer is correct.

Exercise 2.5.7

Solve $\log(x^2 - 4) = 1 + \log(x + 2)$.

Answer

$$\log(x^2 - 4) = 1 + \log(x + 2)$$

Move both logs to one side

$$\log(x^2 - 4) - \log(x + 2) = 1$$

Use the difference property of logs

$$\log\left(\frac{x^2 - 4}{x + 2}\right) = 1$$

Factor

$$\log\left(\frac{(x + 2)(x - 2)}{x + 2}\right) = 1$$

Simplify

$$\log(x - 2) = 1$$

Rewrite as an exponential

$$10^1 = x - 2$$

Add 2 to both sides

$$x = 12$$

Student Exercise 2.5.3

Solve $\ln(3x) + 2 \cdot \ln(x) = 1$.

Answer

$$\ln(3x) + 2\ln(x) = 1$$

Use the power property of exponents to remove the coefficient of the logarithm.

$$\ln(3x) + \ln(x^2) = 1$$

Use the sum of logs property on the right.

$$\ln(3x \cdot x^2) = 1$$

Simplify.

$$\ln(3x^3) = 1$$

. Converting to exponential form.

$$e^1 = 3x^3$$

and solving

$$x = \frac{e}{3}^{\frac{1}{3}} \approx 0.7439$$

Focus: Solving More Complex Exponential Equations

More complex exponential equations can often be solved in more than one way. In the following example, we will solve the same problem in two ways – one using logarithm properties, and the other using exponential properties.

Example 2.5.8a

In 2008, the population of Kenya was approximately 38.8 million, and was growing by 2.64% each year, while the population of Sudan was approximately 41.3 million and growing by 2.24% each year(World Bank, World Development Indicators, as reported on http://www.google.com/publicdata, retrieved August 24, 2010). If these trends continue, when will the population of Kenya match that of Sudan?

Solution

We start by writing an equation for each population in terms of t, the number of years after 2008.

$$Kenya(t) = 38.8(1 + 0.0264)^t$$
$$Sudan(t) = 41.3(1 + 0.0224)^t$$

To find when the populations will be equal, we can set the equations equal

$$38.8(1.0264)^t = 41.3(1.0224)^t$$

For our first approach, we take the log of both sides of the equation.

$$\log\left(38.8(1.0264)^t\right) = \log\left(41.3(1.0224)^t\right)$$

Utilizing the sum property of logs, we can rewrite each side,

$$\log(38.8) + \log\left(1.0264^t\right) = \log(41.3) + \log\left(1.0224^t\right)$$

Then utilizing the exponent property, we can pull the variables out of the exponent

$$\log(38.8) + t\log\left(1.0264\right) = \log(41.3) + t\log\left(1.0224\right)$$

Moving all the terms involving t to one side of the equation and the rest of the terms to the other side,

$$t\log\left(1.0264\right) - t\log\left(1.0224\right) = \log(41.3) - \log(38.8)$$

Factoring out the t on the left,

$$t\left(\log\left(1.0264\right) - \log\left(1.0224\right)\right) = \log(41.3) - \log(38.8)$$

Dividing to solve for t

$$t = \frac{\log(41.3) - \log(38.8)}{\log\left(1.0264\right) - \log\left(1.0224\right)} \approx 15.991$$

It will be 15.991 years until the populations will be equal.

Example 2.5.8b

Solve the problem above by rewriting before taking the log.

Solution

Starting at the equation

$$38.8(1.0264)^t = 41.3(1.0224)^t$$

Divide to move the exponential terms to one side of the equation and the constants to the other side

$$\frac{1.0264^t}{1.0224^t} = \frac{41.3}{38.8}$$

Using exponent rules to group on the left,

$$\left(\frac{1.0264}{1.0224}\right)^t = \frac{41.3}{38.8}$$

Taking the log of both sides

$$\log\left(\left(\frac{1.0264}{1.0224}\right)^t\right) = \log\left(\frac{41.3}{38.8}\right)$$

Utilizing the exponent property on the left,

$$t\log\left(\frac{1.0264}{1.0224}\right) = \log\left(\frac{41.3}{38.8}\right)$$

Dividing gives

$$t = \frac{\log\left(\frac{41.3}{38.8}\right)}{\log\left(\frac{1.0264}{1.0224}\right)} \approx 15.991 \text{ years}$$

While the answer does not immediately appear identical to that produced using the previous method, note that by using the difference property of logs, the answer could be rewritten:

$$t = \frac{\log\left(\frac{41.3}{38.8}\right)}{\log\left(\frac{1.0264}{1.0224}\right)} = \frac{\log(41.3) - \log(38.8)}{\log(1.0264) - \log(1.0224)}$$

Focus: More Applications with Logarithmic Functions

Earthquakes

A logarithmic function is used for earthquakes. This scale is commonly and mistakenly called the Richter Scale, which was a very similar scale succeeded by the MMS.

Moment Magnitude Scale

For an earthquake with seismic moment S, a measurement of earth movement, the MMS value, or magnitude of the earthquake, is

$$M = \frac{2}{3}log(\frac{S}{S_0}) \tag{2.5.4}$$

Where $S_0 = 10^{16}$ is a baseline measure for the seismic moment.

Example 2.5.9

If one earthquake has a MMS magnitude of 6.0, and another has a magnitude of 8.0, how much more powerful (in terms of earth movement) is the second earthquake?

Solution

Since the first earthquake has magnitude 6.0, we can find the amount of earth movement for that quake, which we'll denote S_1. The value of S_0 is not particularity relevant, so we will not replace it with its value.

$$6.0 = \frac{2}{3} log(\frac{S_1}{S_0})$$

$$6.0(\frac{3}{2}) = log(\frac{S_1}{S_0})$$

$$9 = log(\frac{S_1}{S_0})$$

Converting to exponential form

$$\frac{S_1}{S_0} = 10^9$$

$$S_1 = 10^9 S_0$$

This tells us the first earthquake has about 10^9 times more earth movement than the baseline measure.

Doing the same with the second earthquake, S_2, with a magnitude of 8.0,

$$8.0 = \frac{2}{3} log(\frac{S_2}{S_0})$$

$$S_2 = 10^{12} S_0$$

Comparing the earth movement of the second earthquake to the first,

$$\frac{S_2}{S_1} = \frac{10^{12} S_0}{10^9 S_0} = 10^3 = 1000$$

The second value's earth movement is 1000 times as large as the first earthquake.

Example 2.5.10

One earthquake has magnitude of 3.0. If a second earthquake has twice as much earth movement as the first earthquake, find the magnitude of the second quake.

Solution

Since the first quake has magnitude 3.0,

$$3.0 = \frac{2}{3} log(\frac{S}{S_0})$$

Solving for S,

$$3.0 \frac{3}{2} = log(\frac{S}{S_0})$$

$$4.5 = log(\frac{S}{S_0})$$

Converting to exponential form

$$10^{4.5} = \frac{S}{S_0}$$

$$S = 10^{4.5} S_0$$

Since the second earthquake has twice as much earth movement, for the second quake,

$$S = 2 \cdot 10^{4.5} S_0$$

Finding the magnitude,

$$M = \frac{2}{3} log(\frac{2 \cdot 10^{4.5} S_0}{S_0})$$

$$M = \frac{2}{3} log(2 \cdot 10^{4.5}) \approx 3.201$$

The second earthquake with twice as much earth movement will have a magnitude of about 3.2.

Measuring Ph, the acidity or basicity of a liquid.

Example 2.5.11

In chemistry, pH is a measure of the acidity or basicity of a liquid. The pH is related to the concentration of hydrogen ions, $[H^+]$, measured in moles per liter, by the equation

$$pH = -\log\left([H^+]\right) \tag{2.5.5}$$

If a liquid has concentration of 0.0001 moles per liber, determine the pH.

Determine the hydrogen ion concentration of a liquid with pH of 7.

Solution

To answer the first question, we evaluate the expression $-\log(0.0001)$. While we could use our calculators for this, we do not really need them here, since we can use the inverse property of logs:

$$-\log(0.0001) = -\log\left(10^{-4}\right) = -(-4) = 4$$

To answer the second question, we need to solve the equation $7 = -\log([H^+])$. Begin by isolating the logarithm on one side of the equation by multiplying both sides by -1:

$$-7 = \log([H^+])$$

Rewriting into exponential form yields the answer:

$$[H^+] = 10^{-7} = 0.0000001 \text{ moles per liter}$$

Example 2.5.12

If the concentration of hydrogen ions in a liquid is doubled, what is the effect on pH?

Solution

Suppose C is the original concentration of hydrogen ions, and P is the original pH of the liquid, so $P = -\log(C)$. If the concentration is doubled, the new concentration is $2C$. Then the pH of the new liquid is

$$pH = -\log(2C)$$

Using the sum property of logs,

$$pH = -\log(2C) = -(\log(2) + \log(C)) = -\log(2) - \log(C)$$

Since $P = -\log(C)$, the new pH is

$$pH = P - \log(2) = P - 0.301$$

When the concentration of hydrogen ions is doubled, the pH decreases by 0.301.

Important Topics of this Section

- Graphing log functions
- Solving logarithmic functions
- Sum of logs property
- Difference of logs property
- Applications of logarithmic equations

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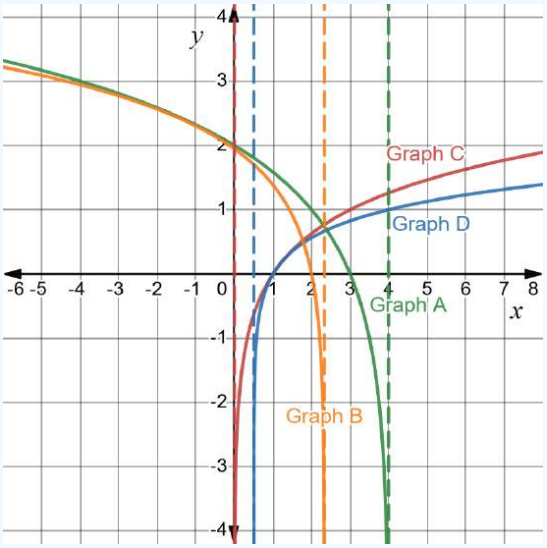
2.5E: Describing Relationships with Logarithmic Functions (Exercises)

Section 2.5 Exercises

Part 1: Working with logarithmic functions

1. For each function find the domain of the logarithmic function. Then match each function with a graph.

a. $f(x) = \log_3(x)$ b. $g(x) = \log_7(2x+1)$ c. $h(x) = ln(7-3x)$ d. $k(x) = \log_2(4-x)$



2. Simplify the logarithmic expressions of a single variable by writing them as a single logarithm.

a. $\log_2(5x) + \log_2(8x^3) - \log_2(2x^2)$

b. $3ln(x^3) + 2ln(x^4)$

c. $5\log_3(5x^5) - 4\log_3(2x^2)$

d. $2\ln(5x^4) + 4\ln(3x^5) - 3\ln(7x^2)$

3. Simplify the logarithmic expressions of multiple variables by writing them as a sum or difference of simpler logarithms.

a. $\log_2(x^3y^4)$

b. $\log_5\left(\frac{z^7}{y^2}\right)$

c. $ln\left(\sqrt{a}\sqrt[3]{b}\right)$

d. $log\left(\frac{xy}{z^2}\right)$

4. Write a sentence explaining the mistake that was made in the simplification below. Then correctly simplify if possible.

$4ln(3) - ln(9)$

To simplify:

Apply the log quotient property $\frac{4ln(3)}{ln(9)}$

Write 9 as a power of 3 $\frac{4ln(3)}{ln(3^2)}$

Apply the power property of logs $\frac{4ln(3)}{2ln(3)}$

Simplify $\frac{4}{2} = 2$

5. Write a sentence explaining the mistake that was made in the simplification below. Then correctly simplify if possible.

$log\,(5) + 2log\,(4) - 3log\,(2)$

To simplify:

Apply the power property of logs $log\,(5) + log\,(4^2) - log\,(2^3)$

Calculate $log\,(5) + log\,(16) - log\,(8)$

Apply the sum & difference properties of logs $log\,(5 + 16 - 8)$

Simplify $log\,(13)$

6. Write a sentence explaining the mistake that was made in the simplification below. Then correctly simplify if possible.

$2\log_2(x^4) + 3\log_2(4x^2)$

To simplify:

Apply the power property of logs $\log_2((x^4)^2) + \log_2(4(x^2)^3)$

Apply the power property of exponents $\log_2(x^8) + \log_2(4x^6)$

Apply the sum property of logs $\log_2(x^8 4x^6)$

Simplify $\log_2(4x^{14})$

7. Solve the logarithmic equations.

a. $\log_2(x) + \log_2(3x) = 5$

b. $ln\,(x^3) - ln\,(x) = 1$

c. $\log_5(x-4) + \log_5(x) = 1$

d. $3\log_4(x) + 5\log_4(3x) = 7$

e. $log\,(x+4) - log\,(x-2) = 1$

8. Write a sentence explaining the mistake that was made in the solving of the equation below. Then correctly solve the equation.

$5e^x = 7$

To solve:

Take the ln of both sides $ln\,(5e^x) = ln\,(7)$

Apply the inverse property of logs and exp. $5x = ln\,(7)$

Apply the sum property of logs $x = \dfrac{ln\,(7)}{5} \approx 0.3892$

9. Write a sentence explaining the mistake that was made in the solving of the equation below. Then correctly solve the equation.

$ln\,(x) + ln\,(5x) = 1$

To solve:

Apply the sum property of logs $ln\,(x + 5x) = 1$

Simplify $ln\,(6x) = 1$

Convert to exponential form $e^1 = 6x$

Isolate x $x = \dfrac{e}{6} \approx 0.4530$

Part 2: Applications

10. The 1906 San Francisco earthquake had a magnitude of 7.9 on the MMS scale. Later there was an earthquake with magnitude 4.7 that caused only minor damage. How many times more intense was the San Francisco earthquake than the second one?

11. The 1906 San Francisco earthquake had a magnitude of 7.9 on the MMS scale. Later there was an earthquake with magnitude 6.5 that caused less damage. How many times more intense was the San Francisco earthquake than the second one?

12. One earthquake has magnitude 3.9 on the MMS scale. If a second earthquake has 750 times as much energy as the first, find the magnitude of the second quake.

13. One earthquake has magnitude 4.8 on the MMS scale. If a second earthquake has 1200 times as much energy as the first, find the magnitude of the second quake.

14. Determine the hydrogen ion concentration of a liquid with pH of 8.

15. Determine the hydrogen ion concentration of an acidic liquid with pH of 4.

16. When will an investment of $10,000 growing by 5% per year be larger in value that an investment of $12,000 that is growing by only 3% per year?

17. A city with a population of 210,000 people is growing by 2.5% per year. Another city with a population of 135,000 is growing by 8% per year. When will the cities have the same population if these trends continue?

Answer

Part 1:

1. a. $(0, \infty)$ Graph C c. $(-\infty, \frac{7}{3})$ Graph B

2 a. $\log_2(20x^2))$ c. $ln\left(\frac{3125x^{17}}{16}\right)$

3. a. $3\log_2(x) + 2\log_2(y)$ c. $\frac{1}{2}ln(a) + \frac{1}{3}(b)$

5. The sum and difference properties of logs were applied incorrectly in the third step. To add log terms of the same base, we multiply the inputs. To subtract log terms of the same base, we divide the inputs. The last two steps should read

$$log\left(\frac{5 \cdot 16}{8}\right) = log(10) = 1$$

7. a. $x = \pm 1$ c. $x = -1, x = 5$ e. $\frac{24}{9}$

9. The sum property of logs was applied incorrectly in the first step. To add log terms of the same base, we multiply the inputs. The first step should read

$$ln(x \cdot 5x) = ln\left(6x^2\right)$$

Part 2:

11. The San Francisco earthquake's earth movement is 125.9 times as large as than the second earthquake.

13. The second quake has a magnitude of 6.85.

15. $[H^+] = 10^{-4} = 0.0001$ moles per liter

17. $x \approx 8.441$

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CHAPTER OVERVIEW

3: Describing Relationships with Quadratic Functions

3.1: Transformations of Functions

3.1E: Transformations of Functions (Exercises)

3.2: Quadratic Functions

3.2E: Describing Relationships with Quadratic Functions (Exercises)

3.3: Complex Zeros

3.3E: Complex Zeros of Quadratics Functions (Exercises)

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3.1: Transformations of Functions

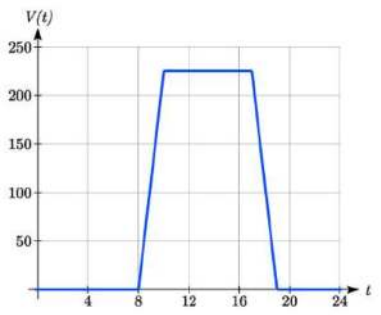
Often when given a problem, we try to model the scenario using mathematics in the form of words, tables, graphs and equations in order to explain or solve it. When building models, it is often helpful to build off of existing formulas or models. Knowing the basic graphs of your tool-kit functions can help you solve problems by being able to model new behavior by adapting something you already know. Unfortunately, the models and existing formulas we know are not always exactly the same as the ones presented in the problems we face.

Fortunately, there are systematic ways to shift, stretch, compress, flip and combine functions to help them become better models for the problems we are trying to solve. We can transform what we already know into what we need, hence the name, "Transformation of functions." When we have a story problem, formula, graph, or table, we can then transform that function in a variety of ways to form new functions.

Shifts

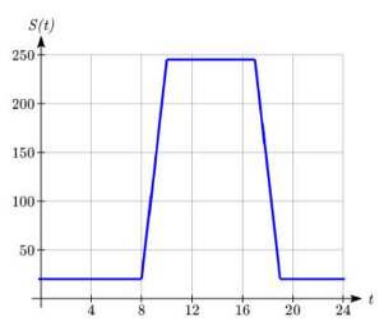
Example 3.1.1

To regulate temperature in a green building, air flow vents near the roof open and close throughout the day to allow warm air to escape. The graph below shows the open vents V (in square feet) throughout the day, t in hours after midnight. During the summer, the facilities staff decides to try to better regulate temperature by increasing the amount of open vents by 20 square feet throughout the day. Sketch a graph of this new function.



Solution

We can sketch a graph of this new function by adding 20 to each of the output values of the original function. This will have the effect of shifting the graph up.



Notice that in the second graph, for each input value, the output value has increased by twenty, so if we call the new function $S(t)$, we could write $S(t) = V(t) + 20$.

Note that this notation tells us that for any value of t, $S(t)$ can be found by evaluating the V function at the same input, then adding twenty to the result.

This defines S as a transformation of the function V, in this case a vertical shift up 20 units.

Notice that with a vertical shift the input values stay the same and only the output values change.

Definition: Vertical shift

Given a function $f(x)$, if we define a new function $g(x)$ as

$$g(x) = f(x) + k \tag{3.1.1}$$

where k is a constant

then $g(x)$ is a **vertical shift** of Functions:Vertical Shifts of the function $f(x)$, where all the output values have been increased by k.

If k is positive, then the graph will shift up

If k is negative, then the graph will shift down

Example 3.1.2

A function $f(x)$ is given as a table below. Create a table for the function $g(x) = f(x) - 3$

x	2	4	6	8
$f(x)$	1	3	7	11

Solution

The formula $g(x) = f(x) - 3$ tells us that we can find the output values of the g function by subtracting 3 from the output values of the f function. For example,

$$f(2) = 1$$

is found from the given table

$$g(x) = f(x) - 3$$

is our given transformation

$$g(2) = f(2) - 3 = 1 - 3 = -2$$

Subtracting 3 from each $f(x)$ value, we can complete a table of values for $g(x)$

x	2	4	6	8
$g(x)$	-2	0	4	8

As with the earlier vertical shift, notice the input values stay the same and only the output values change.

Exercise 3.1.1

The function $h(t) = -4.9t^2 + 30t$ gives the height h of a ball (in meters) thrown upwards from the ground after t seconds. Suppose the ball was instead thrown from the top of a 10 meter building. Relate this new height function $b(t)$ to $h(t)$, then find a formula for $b(t)$.

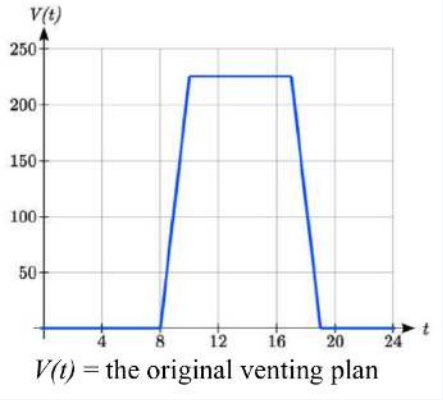
Answer

$$b(t) = h(t) + 10 = -4.9t^2 + 30t + 10$$

The vertical shift is a change to the output, or outside, of the function. We will now look at how changes to input, on the inside of the function, change its graph and meaning.

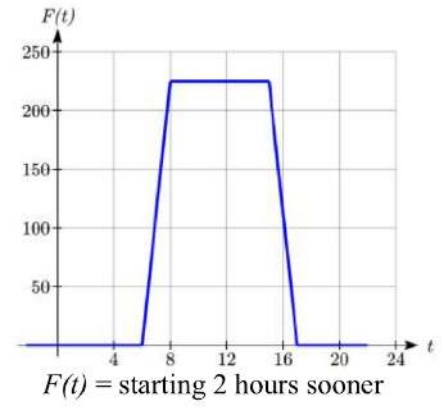
Example 3.1.3

Returning to our building air flow example from the beginning of the section, suppose that in Fall, the facilities staff decides that the original venting plan starts too late, and they want to move the entire venting program to start two hours earlier. Sketch a graph of the new function.



$V(t)$ = the original venting plan

Solution



$F(t)$ = starting 2 hours sooner

In the new graph, which we can call $F(t)$, at each time, the air flow is the same as the original function $V(t)$ was two hours later. For example, in the original function V, the air flow starts to change at 8 am, while for the function $F(t)$ the air flow starts to change at 6am. The comparable function values are $V(8) = F(6)$.

Notice also that the vents first opened to 220 sq. ft. at 10 a.m. under the original plan, while under the new plan the vents reach 220 sq. ft. at 8 a.m., so $V(10) = F(8)$.

In both cases we see that since $F(t)$ starts 2 hours sooner, the same output values are reached when, $F(t) = V(t + 2)$

Note that $V(t + 2)$ had the effect of shifting the graph to the $left$.

Horizontal changes or "inside changes" affect the domain of a function (the input) instead of the range and often seem counterintuitive. The new function $F(t)$ uses the same outputs as $V(t)$, but matches those outputs to inputs two hours earlier than those of $V(t)$. Said another way, we must add 2 hours to the input of V to find the corresponding output for F: $F(t) = V(t + 2)$.

Definition: Horizontal shift

Given a function $f(x)$, if we define a new function $g(x)$ as

$$g(x) = f(x + k)$$

where k is a constant

then $g(x)$ is a **horizontal shift** of the function $f(x)$

If k is positive, then the graph will shift left

If k is negative, then the graph will shift right

Example 3.1.4

A function $f(x)$ is given as a table below. Create a table for the function $g(x) = f(x-3)$.

x	2	4	6	8
$f(x)$	1	3	7	11

Solution

The formula $g(x) = f(x-3)$ tells us that the output values of g are the same as the output value of f with an input value three smaller. For example, we know that $f(2) = 1$. To get the same output from the g function, we will need an input value that is 3 $larger$: We input a value that is three larger for $g(x)$ because the function takes three away before evaluating the function f.

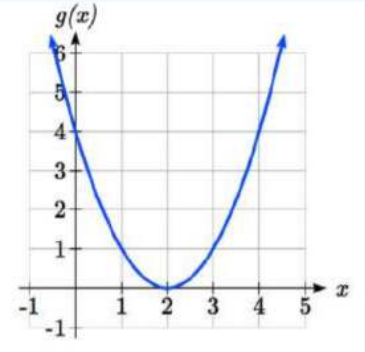
$$g(5) = f(5-3) = f(2) = 1$$

x	5	7	9	11
$g(x)$	1	3	7	11

The result is that the function $g(x)$ has been shifted to the right by 3. Notice the output values for $g(x)$ remain the same as the output values for $f(x)$ in the chart, but the corresponding input values, x, have shifted to the right by 3: 2 shifted to 5, 4 shifted to 7, 6 shifted to 9 and 8 shifted to 11.

Example 3.1.5

The graph shown is a transformation of the toolkit function $f(x) = x^2$. Relate this new function $g(x)$ to $f(x)$, and then find a formula for $g(x)$.



Solution

Notice that the graph looks almost identical in shape to the $f(x) = x^2$ function, but the x values are shifted to the right two units. The vertex used to be at (0, 0) but now the vertex is at (2, 0) . The graph is the basic quadratic function shifted two to the right, so

$$g(x) = f(x-2)$$

Notice how we must input the value $x = 2$, to get the output value $y = 0$; the x values must be two units larger, because of the shift to the right by 2 units.

We can then use the definition of the $f(x)$ function to write a formula for $g(x)$ by evaluating $f(x-2)$:

Since $f(x) = x^2$ and $g(x) = f(x-2)$

$$g(x) = f(x-2) = (x-2)^2$$

If you find yourself having trouble determining whether the shift is +2 or -2, it might help to consider a single point on the graph. For a quadratic, looking at the bottom-most point is convenient. In the original function, $f(0) = 0$. In our shifted function, $g(2) = 0$. To obtain the output value of 0 from the f function, we need to decide whether a +2 or -2 will work to satisfy $g(2) = f(2 \ ? \ 2) = f(0) = 0$. For this to work, we will need to subtract 2 from our input values.

When thinking about horizontal and vertical shifts, it is good to keep in mind that vertical shifts are affecting the output values of the function, while horizontal shifts are affecting the input values of the function.

Example 3.1.6

The function $G(m)$ gives the number of gallons of gas required to drive m miles. Interpret $G(m) + 10$ and $G(m + 10)$.

Solution

$G(m) + 10$ is adding 10 to the output, gallons. This is 10 gallons of gas more than is required to drive m miles. So, this is the gas required to drive m miles, plus another 10 gallons of gas.

$G(m + 10)$ is adding 10 to the input, miles. This is the number of gallons of gas required to drive 10 miles more than m miles.

Exercise 3.1.2

Given the function $f(x) = \sqrt{x}$ graph the original function $f(x)$ and the transformation $g(x) = f(x + 2)$.

a. Is this a horizontal or a vertical change?

b. Which way is the graph shifted and by how many units?

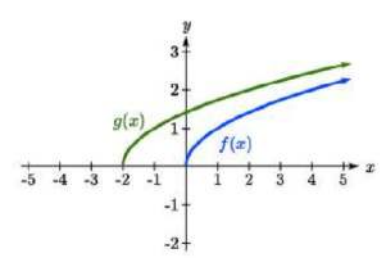
c. Graph $f(x)$ and $g(x)$ on the same axes.

Answer

a. Horizontal shift

b. The function is shifted to the LEFT by 2 units

c. Shown to the right



Now that we have two transformations, we can combine them together.

Remember:

Vertical Shifts are outside changes that affect the output (vertical) axis values shifting the transformed function up or down.

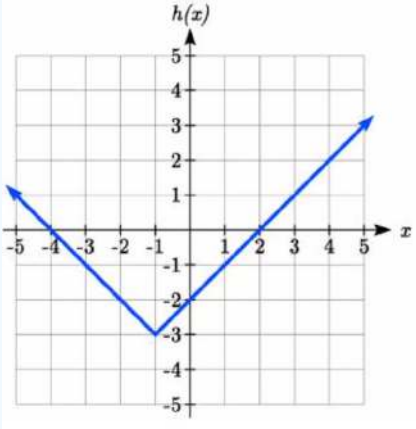
Horizontal Shifts are inside changes that affect the input (horizontal) axis values shifting the transformed function left or right.

Example 3.1.7

Given $f(x) = |x|$, sketch a graph of $h(x) = f(x + 1) - 3$.

Solution

The function f is our toolkit absolute value function. We know that this graph has a V shape, with the point at the origin. The graph of h has transformed f in two ways: $f(x + 1)$ is a change on the inside of the function, giving a horizontal shift left by 1, then the subtraction by 3 in $f(x + 1) - 3$ is a change to the outside of the function, giving a vertical shift down by 3. Transforming the graph gives



Example 3.1.8

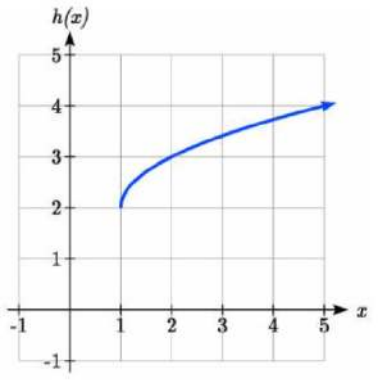
Write a formula for the graph shown, a transformation of the toolkit square root function.

Solution

The graph of the toolkit function starts at the origin, so this graph has been shifted 1 to the right, and up 2. In function notation, we could write that as $h(x) = f(x-1)+2$. Using the formula for the square root function we can write

$h(x) = \sqrt{x-1}+2$

Note that this transformation has changed the domain and range of the function. This new graph has domain $[1, \infty)$ and range $[2, \infty)$.



Reflections

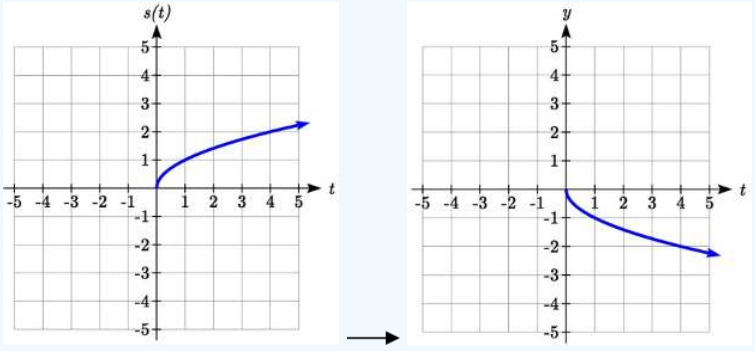
Another transformation that can be applied to a function is a reflection over the horizontal or vertical axis.

Example 3.1.9

Reflect the graph of $s(t) = \sqrt{t}$ both vertically and horizontally.

Solution

Reflecting the graph vertically, each output value will be reflected over the horizontal t axis:



Since each output value is the opposite of the original output value, we can write

$$V(t) = -s(t)$$

$$V(t) = -\sqrt{t}$$

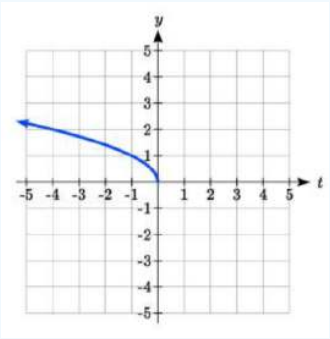
Notice this is an outside change or vertical change that affects the output *s(t)* values so the negative sign belongs outside of the function.

Reflecting horizontally, each input value will be reflected over the vertical axis.

Since each input value is the opposite of the original input value, we can write

$$H(t) = s(-t)$$

$$H(t) = \sqrt{-t}$$



Notice this is an inside change or horizontal change that affects the input values so the negative sign is on the inside of the function.

Note that these transformations can affect the domain and range of the functions. While the original square root function has domain $[0, \infty)$ and range $[0, \infty)$, the vertical reflection gives the $V(t)$ function the range $(-\infty, 0]$, and the horizontal reflection gives the $H(t)$ function the domain $(-\infty, 0]$.

Definition: reflections

Given a function $f(x)$, if we define a new function $g(x)$ as

$$g(x) = -f(x), \tag{3.1.2}$$

then $g(x)$ is a **vertical reflection** of the function $f(x)$, sometimes called a reflection about the x-axis

If we define a new function $g(x)$ as

$$g(x) = f(-x), \tag{3.1.3}$$

then $g(x)$ is a **horizontal reflection** of the function $f(x)$, sometimes called a reflection about the y-axis

Example 3.1.10

A function $f(x)$ is given as a table below. Create a table for the function $g(x) = -f(x)$ and $h(x) = f(-x)$

x	2	4	6	8
$f(x)$	1	3	7	11

Solution

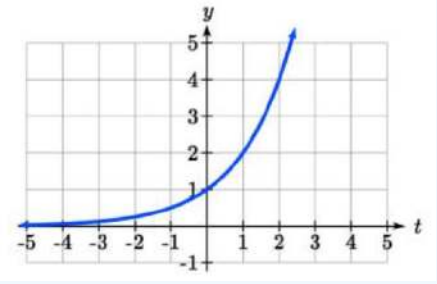
For $g(x)$, this is a vertical reflection, so the x values stay the same and each output value will be the opposite of the original output value

For $h(x)$, this is a horizontal reflection, and each input value will be the opposite of the original input value and the $h(x)$ values stay the same as the $f(x)$ values:

x	-2	-4	-6	-8
$h(x)$	1	3	7	11

Example 3.1.11

A common model for learning has an equation similar to $k(t) = -2^{-t} + 1$, where k is the percentage of mastery that can be achieved after t practice sessions. This is a transformation of the function $f(t) = 2^t$ shown here. Sketch a graph of $k(t)$.



Solution

This equation combines three transformations into one equation.

A horizontal reflection: $f(-t) = 2^{-t}$ combined with

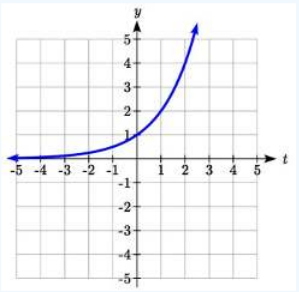
A vertical reflection: $-f(-t) = -2^{-t}$ combined with

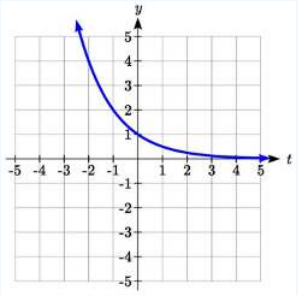
A vertical shift up 1: $-f(-t) + 1 = -2^{-t} + 1$

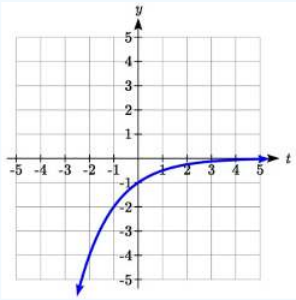
We can sketch a graph by applying these transformations one at a time to the original function:

The original graph Horizontally reflected Then vertically reflected

The Original Graph Horizontally Reflected The Vertically Reflected







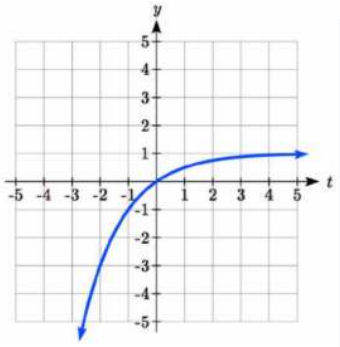
Then, after shifting up 1, we get the final graph.

$$k(t) = -f(-t) + 1 = -2^{-t} + 1$$

Note: As a model for learning, this function would be limited to a domain of $t \geq 0$, with corresponding range $[0, 1)$.

? exercise 3.1.3

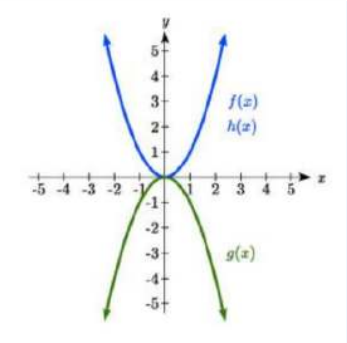
Given the toolkit function $f(x) = x^2$, graph $g(x) = -f(x)$ and $h(x) = f(-x)$. Do you notice anything surprising?



Answer

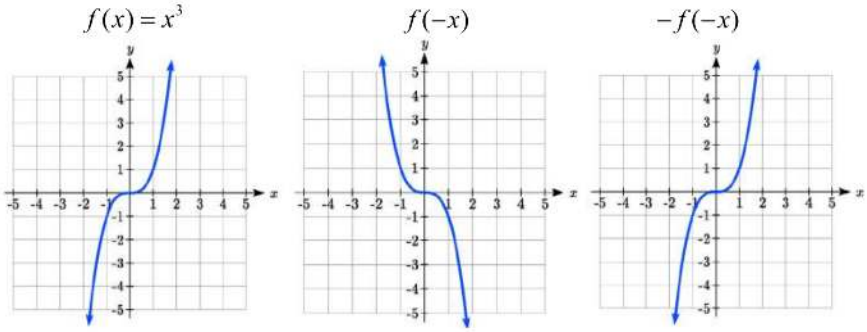
Shown in the graph

Notice: $g(x) = f(-x)$ looks the same as $f(x)$



Some functions exhibit symmetry, in which reflections result in the original graph. For example, reflecting the toolkit functions $f(x) = x^2$ or $f(x) = |x|$ about the *y*-axis will result in the original graph. We call these types of graphs symmetric about the *y*-axis.

Likewise, if the graphs of $f(x) = x^3$ or $f(x) = \frac{1}{x}$ were reflected over both axes, the result would be the original graph:



We call these graphs symmetric about the origin.

Definition: even and odd functions

A function is called an **even function** if

$$f(x) = f(-x) \tag{3.1.4}$$

The graph of an even function is symmetric about the vertical axis

A function is called an **odd function** if

$$f(x) = -f(-x) \tag{3.1.5}$$

The graph of an odd function is symmetric about the origin

Note: A function can be neither even nor odd if it does not exhibit either symmetry. For example, the $f(x) = 2^x$ function is neither even nor odd.

Example 3.1.12

Is the function $f(x) = x^3 + 2x$ even, odd, or neither?

Solution

Without looking at a graph, we can determine this by finding formulas for the reflections, and seeing if they return us to the original function:

$$f(-x) = (-x)^3 + 2(-x) = -x^3 - 2x$$

This does not return us to the original function, so this function is not even.

We can now try also applying a horizontal reflection:

$$-f(-x) = -(-x^3 - 2x) = x^3 + 2x$$

Since $-f(-x) = f(x)$, this is an odd function.

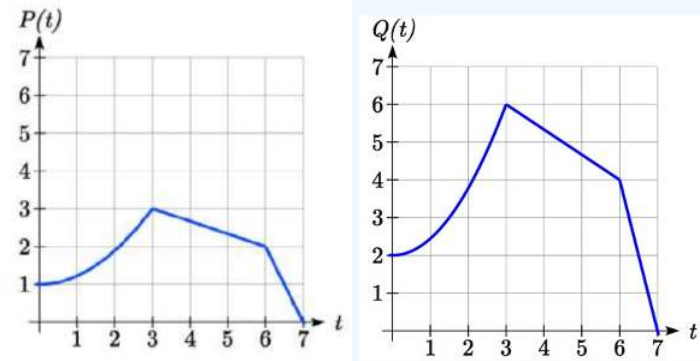
Stretches and Compressions

With shifts, we saw the effect of adding or subtracting to the inputs or outputs of a function. We now explore the effects of multiplying the inputs or outputs.

Remember, we can transform the inside (input values) of a function or we can transform the outside (output values) of a function. Each change has a specific effect that can be seen graphically.

Example 3.1.13

A function $P(t)$ models the growth of a population of fruit flies. The growth is shown in the graph. A scientist is comparing this to another population, Q, that grows the same way, but starts twice as large. Sketch a graph of this population.



Solution

Since the population is always twice as large, the new population's output values are always twice the original function output values. Graphically, this would look like the second graph shown.

Symbolically, $Q(t) = 2P(t)$

This means that for any input t, the value of the Q function is twice the value of the P function. Notice the effect on the graph is a vertical stretching of the graph, where every point doubles its distance from the horizontal axis. The input values, t, stay the same while the output values are twice as large as before.

Definition: Vertical stretch/compression

Given a function $f(x)$, if we define a new function $g(x)$ as

$g(x) = kf(x)$, where k is a constant

then $g(x)$ is a **vertical stretch or compression** of the function $f(x)$.

If $k > 1$, then the graph will be stretched

If $0 < k < 1$, then the graph will be compressed

If $k < 0$, then there will be combination of a vertical stretch or compression with a vertical reflection

Example 3.1.14

A function $f(x)$ is given as a table below. Create a table for the function $g(x) = \frac{1}{2}f(x)$

x	2	4	6	8
$f(x)$	1	3	7	11

Solution

The formula $g(x) = \frac{1}{2}f(x)$ tells us that the output values of g are half of the output values of f with the same inputs. For example, we know that $f(4) = 3$. Then

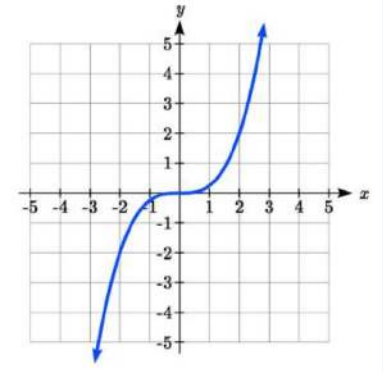
$$g(4) = \frac{1}{2}f(4) = \frac{1}{2}(3) = \frac{3}{2}$$

x	2	4	6	8
$g(x)$	1/2	3/2	7/2	11/2

The result is that the function $g(x)$ has been compressed vertically by 1/2. Each output value has been cut in half, so the graph would now be half the original height.

Example 3.1.15

The graph shown is a transformation of the toolkit function $f(x) = x^3$. Relate this new function $g(x)$ to $f(x)$, then find a formula for $g(x)$.



Solution

When trying to determine a vertical stretch or shift, it is helpful to look for a point on the graph that is relatively clear. In this graph, it appears that $g(2) = 2$. With the basic cubic function at the same input,

$$f(2) = 2^3 = 8$$

Based on that, it appears that the outputs of g are 1/4 the outputs of the function f, since

$$g(2) = \frac{1}{4}f(2)$$

From this we can fairly safely conclude that:

$$g(x) = \frac{1}{4}f(x)$$

We can write a formula for g by using the definition of the function f

$$g(x) = \frac{1}{4}f(x) = \frac{1}{4}x^3$$

Now we consider changes to the inside of a function.

Example 3.1.16

Returning to the fruit fly population we looked at earlier, suppose the scientist is now comparing it to a population that progresses through its lifespan twice as fast as the original population. In other words, this new population, R, will progress in 1 hour the same amount the original population did in 2 hours, and in 2 hours, will progress as much as the original population did in 4 hours. Sketch a graph of this population.

Solution

Symbolically, we could write

$$R(1) = P(2)$$

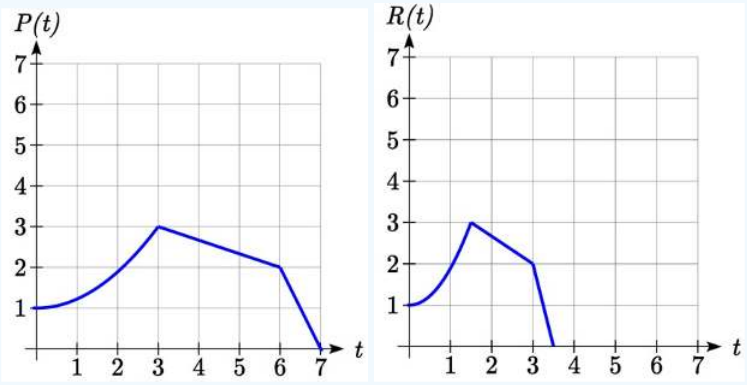
$$R(2) = P(4)$$

and in general,

$$R(t) = P(2t)$$

Graphing this,

Original Population, $P(t)$ Transformed, $R(t)$



Note the effect on the graph is a horizontal compression, where all input values are half their original distance from the vertical axis.

Definition: horizontal stretch/compression

Given a function $f(x)$, if we define a new function $g(x)$ as

$g(x) = f(kx)$, where k is a constant

then $g(x)$ is a **horizontal stretch or compression** of the function $f(x)$.

If $k > 1$, then the graph will be compressed by $1/k$

If $0 < k < 1$, then the graph will be stretched by $1/k$

If $k < 0$, then there will be combination of a horizontal stretch or compression with a horizontal reflection.

Example 3.1.17

A function $f(x)$ is given as a table below. Create a table for the function $g(x) = f(\frac{1}{2}x)$

x	2	4	6	8
$f(x)$	1	3	7	11

Solution

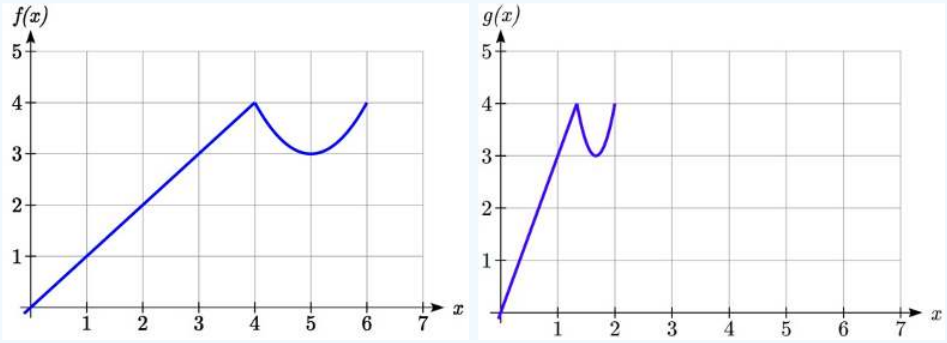
The formula $g(x) = f(\frac{1}{2}x)$ tells us that the output values for g are the same as the output values for the function f at an input half the size. Notice that we don't have enough information to determine $g(2)$ since $g(2) = f(\frac{1}{2} \cdot 2) = f(1)$, and we do not have a value for $f(1)$ in our table. Our input values to g will need to be twice as large to get inputs for f that we can evaluate. For example, we can determine $g(4)$ since $g(4) = f(\frac{1}{2} \cdot 4) = f(2) = 1$.

x	4	8	12	16
$g(x)$	1	3	7	11

Since each input value has been doubled, the result is that the function g(x) has been stretched horizontally by 2.

Example 3.1.18

Two graphs are shown below. Relate the function $g(x)$ to $f(x)$.



Solution

The graph of $g(x)$ looks like the graph of $f(x)$ horizontally compressed. Since $f(x)$ ends at (6,4) and $g(x)$ ends at (2,4) we can see that the x values have been compressed by 1/3, because 6(1/3) = 2. We might also notice that $g(2) = f(6)$, and $g(1) = f(3)$. Either way, we can describe this relationship as $g(x) = f(3x)$. This is a horizontal compression by 1/3.

Notice that the coefficient needed for a horizontal stretch or compression is the *reciprocal* of the stretch or compression. To stretch the graph horizontally by 4, we need a coefficient of 1/4 in our function: $f(\frac{1}{4}x)$. This means the input values must be four times larger to produce the same result, requiring the input to be larger, causing the horizontal stretching.

exercise 3.1.4

Write a formula for the toolkit square root function horizontally stretched by three.

Answer

$$g(x) = f(\frac{1}{3}x$$

so using the square root function we get

$$g(x) = \sqrt{\frac{1}{3}x}$$

It is useful to note that for most toolkit functions, a horizontal stretch or vertical stretch can be represented in other ways. For example, a horizontal compression of the function $f(x) = x^2$ by ½ would result in a new function $g(x) = (2x)^2$, but this can also be written as $g(x) = 4x^2$, a vertical stretch of $f(x)$ by 4. When writing a formula for a transformed toolkit, we only need to find one transformation that would produce the graph.

Combining Transformations

When combining transformations, it is very important to consider the order of the transformations. For example, vertically shifting by 3 and then vertically stretching by 2 does not create the same graph as vertically stretching by 2 then vertically shifting by 3.

When we see an expression like $2f(x) + 3$, which transformation should we start with? The answer here follows nicely from order of operations, for outside transformations. Given the output value of $f(x)$, we first multiply by 2, causing the vertical stretch, then add 3, causing the vertical shift. (Multiplication before Addition)

Definition: combining vertical transformation

When combining vertical transformations written in the form $af(x) + k$,

first vertically stretch by a, then vertically shift by k.

Horizontal transformations are a little trickier to think about. When we write $g(x) = f(2x + 3)$ for example, we have to think about how the inputs to the g function relate to the inputs to the f function. Suppose we know $f(7) = 12$. What input to g would produce that output? In other words, what value of x will allow $g(x) = f(2x + 3) = f(12)$? We would need $2x + 3 = 12$. To solve for x, we would first subtract 3, resulting in horizontal shift, then divide by 2, causing a horizontal compression.

Definition:combining horizontal transformations

When **combining horizontal transformations** written in the form $f(bx + p)$,

first horizontally shift by p, then horizontally stretch by $1/b$.

This format ends up being very difficult to work with, since it is usually much easier to horizontally stretch a graph before shifting. We can work around this by factoring inside the function.

$$f(bx + p) = f(b(x + \frac{p}{b})) \tag{3.1.6}$$

Factoring in this way allows us to horizontally stretch first, then shift horizontally.

Definition: combining horizontal transformations (factored form)

When **combining horizontal transformations** written in the form $f(b(x + h))$,

first horizontally stretch by $1/b$, then horizontally shift by h.

Definition: independence of horizontal and vertical transformations

Horizontal and vertical transformations are independent. It does not matter whether horizontal or vertical transformations are done first.

Example 3.1.19

Given the table of values for the function $f(x)$ below, create a table of values for the function $g(x) = 2f(3x) + 1$

x	6	12	18	24
$f(x)$	10	14	15	17

Solution

There are 3 steps to this transformation and we will work from the inside out. Starting with the horizontal transformations, $f(3x)$ is a horizontal compression by 1/3, which means we multiply each x value by 1/3.

x	2	4	6	8
$f(3x)$	10	14	15	17

Looking now to the vertical transformations, we start with the vertical stretch, which will multiply the output values by 2. We apply this to the previous transformation.

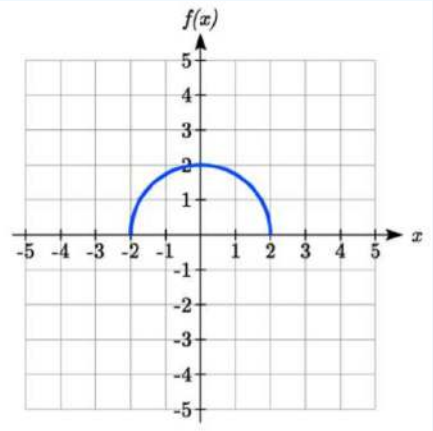
x	2	4	6	8
$2f(3x)$	20	28	30	34

Finally, we can apply the vertical shift, which will add 1 to all the output values.

x	2	4	6	8
$g(x) = 2f(3x) + 1$	21	29	31	35

Example 3.1.20

Using the graph of $f(x)$ below, sketch a graph of $k(x) = f(\frac{1}{2}x + 1) - 3$



Solution

To make things simpler, we'll start by factoring out the inside of the function

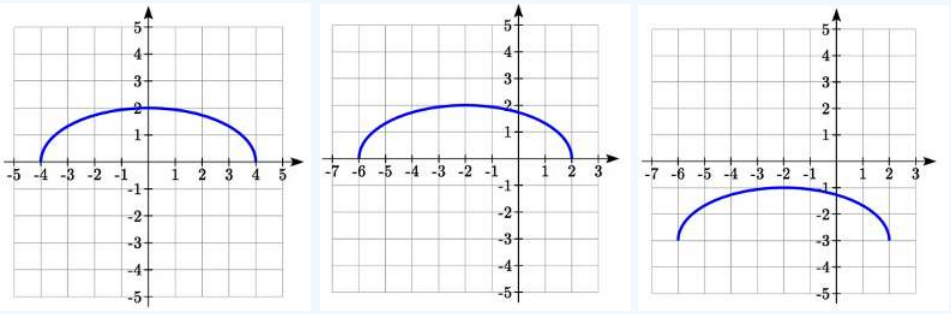
$$f(\frac{1}{2}x + 1) - 3 = f(\frac{1}{2}(x + 2)) - 3$$

By factoring the inside, we can first horizontally stretch by 2, as indicated by the 1/2 on the inside of the function. Remember twice the size of 0 is still 0, so the point (0,2) remains at (0,2) while the point (2,0) will stretch to (4,0).

Next, we horizontally shift left by 2 units, as indicated by the $x + 2$.

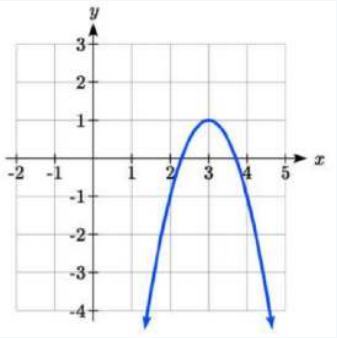
Last, we vertically shift down by 3 to complete our sketch, as indicated by the -3 on the outside of the function.

Horizontal Stretch by 2 Horizontal Shift left by 2 Vertical Shift Down 3



Example 3.1.21

Write an equation for the transformed graph of the quadratic function shown.



Solution

Since this is a quadratic function, first consider what the basic quadratic tool kit function looks like and how this has changed. Observing the graph, we notice several transformations:

The original tool kit function has been flipped over the *x* axis, some kind of stretch or compression has occurred, and we can see a shift to the right 3 units and a shift up 1 unit.

In total there are four operations:

Vertical reflection, requiring a negative sign outside the function

Vertical Stretch *or* Horizontal Compression *

Horizontal Shift Right 3 units, which tells us to put *x*-3 on the inside of the function

Vertical Shift up 1 unit, telling us to add 1 on the outside of the function

* It is unclear from the graph whether it is showing a vertical stretch or a horizontal compression. For the quadratic, it turns out we could represent it either way, so we'll use a vertical stretch. You may be able to determine the vertical stretch by observation.

By observation, the basic tool kit function has a vertex at (0, 0) and symmetrical points at (1, 1) and (-1, 1). These points are one unit up and one unit over from the vertex. The new points on the transformed graph are one unit away horizontally but 2 units away vertically. They have been stretched vertically by two.

Not everyone can see this by simply looking at the graph. If you can, great, but if not, we can solve for it. First, we will write the equation for this graph, with an unknown vertical stretch.

$f(x) = x^2$ The original function

$-f(x) = -x^2$ Vertically reflected

$-af(x) = -ax^2$ Vertically stretched

$-af(x-3) = -a(x-3)^2$ Shifted right 3

$-af(x-3)+1 = -a(x-3)^2+1$ Shifted up 1

We now know our graph is going to have an equation of the form $g(x) = -a(x-3)^2+1$. To find the vertical stretch, we can identify any point on the graph (other than the highest point), such as the point (2,-1), which tells us $g(2) = -1$. Using our general formula, and substituting 2 for x, and -1 for $g(x)$

$$\begin{aligned} -1 &= -a(2-3)^2+1 \\ -1 &= -a+1 \\ -2 &= -a \\ 2 &= a \end{aligned}$$

This tells us that to produce the graph we need a vertical stretch by two.

The function that produces this graph is therefore $g(x) = -2(x-3)^2+1$.

Exercise 3.1.5

Consider the linear function $g(x) = -2x+1$. Describe its transformation in words using the identity tool kit function $f(x) = x$ as a reference.

Answer

The identity tool kit function $f(x) = x$ has been transformed in 3 steps

a. Vertically stretched by 2.

b. Vertically reflected over the x axis.

c. Vertically shifted up by 1 unit.

Example 3.1.22

On what interval(s) is the function $g(x) = \dfrac{-2}{(x-1)^2}+3$ increasing and decreasing?

Solution

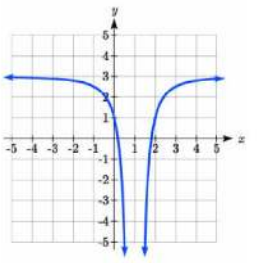
This is a transformation of the toolkit reciprocal squared function, $f(x) = \dfrac{1}{x^2}$:

$-2f(x) = \dfrac{-2}{x^2}$ A vertical flip and vertical stretch by 2

$-2f(x-1) = \dfrac{-2}{(x-1)^2}$ A shift right by 1

$-2f(x-1)+3 = \dfrac{-2}{(x-1)^2}+3$ A shift up by 3

The basic reciprocal squared function is increasing on $(-\infty, 0)$and decreasing on $(0, \infty)$. Because of the vertical flip, the $g(x)$ function will be decreasing on the left and increasing on the right. The horizontal shift right by 1 will also shift these intervals to the right one. From this, we can determine $g(x)$ will be increasing on $(1, \infty)$ and decreasing on $(-\infty, 1)$. We also could graph the transformation to help us determine these intervals.



Exercise 3.1.6

On what interval(s) is the function $h(t) = (t-3)^3 + 2$ concave up and down?

Answer

$h(t)$ is concave down on $(-\infty, 3)$ and concave up on $(3, \infty)$

Important Topics of This Section

Transformations

Vertical Shift (up & down)

Horizontal Shifts (left & right)

Reflections over the vertical & horizontal axis

Even & Odd functions

Vertical Stretches & Compressions

Horizontal Stretches & Compressions

Combinations of Transformation

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3.1E: Transformations of Functions (Exercises)

Section 3.1 Exercises

Part 1: Vertical Stretches, Reflections, and Vertical Shifts

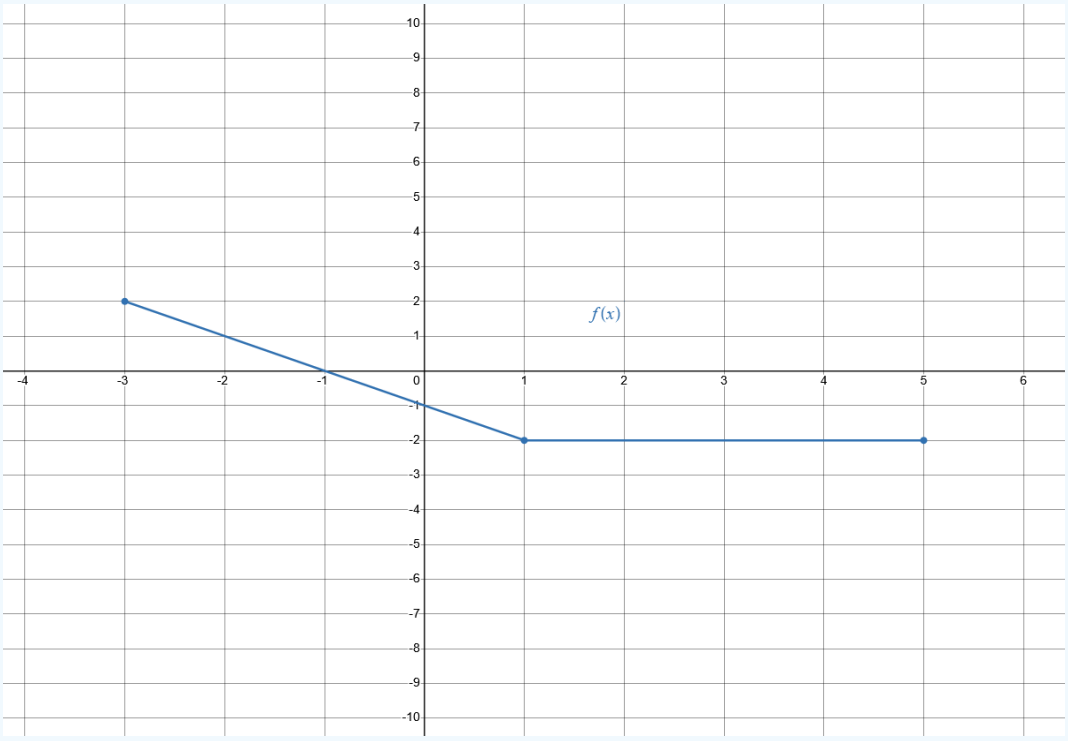
1. Write a function whose graph is the graph of $y = |x|$

   a. Shifted up three units.
   b. Vertically stretched by a factor of 3.
   c. Vertically compressed by a factor of $\frac{1}{3}$, reflected about the x-axis, and then shifted down two units.

2. Write a function whose graph is the graph of $y = x^3$

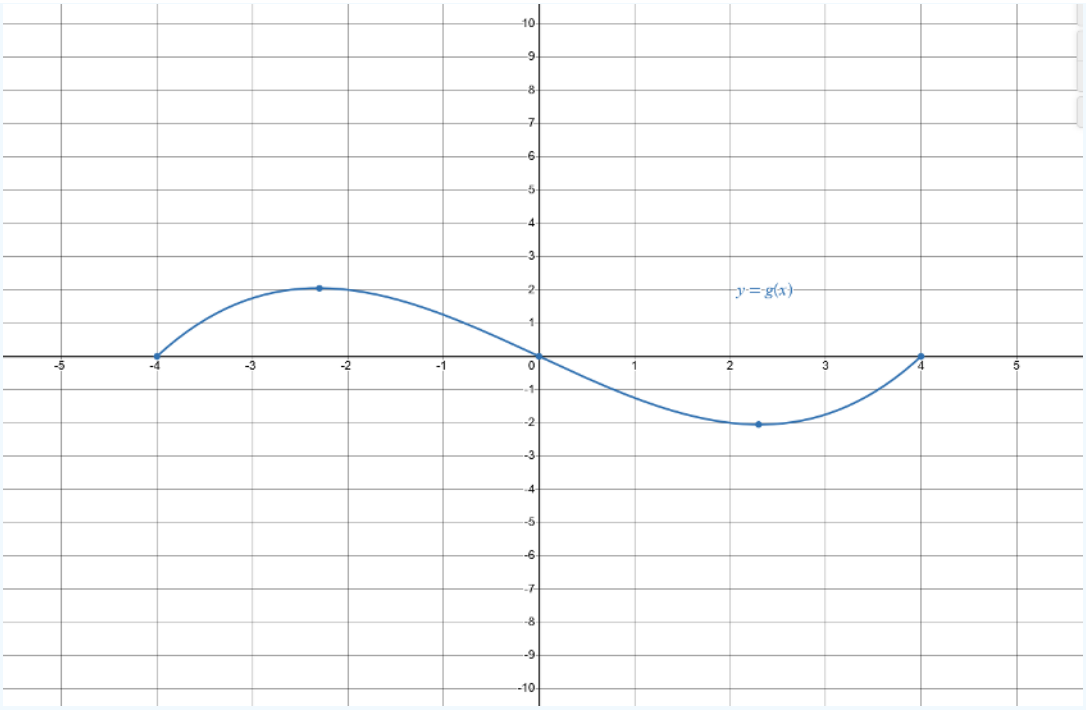
   a. Shifted down two units.
   b. Vertically stretched by a factor of 4.
   c. Vertically compressed by a factor of $\frac{1}{4}$ and reflected about the x-axis.

3. Use the graph of $y = f(x)$ below to sketch the graph of the indicated function with transformations. Show a graph of each transformation if multiple transformations occur and where each indicated point was transformed.



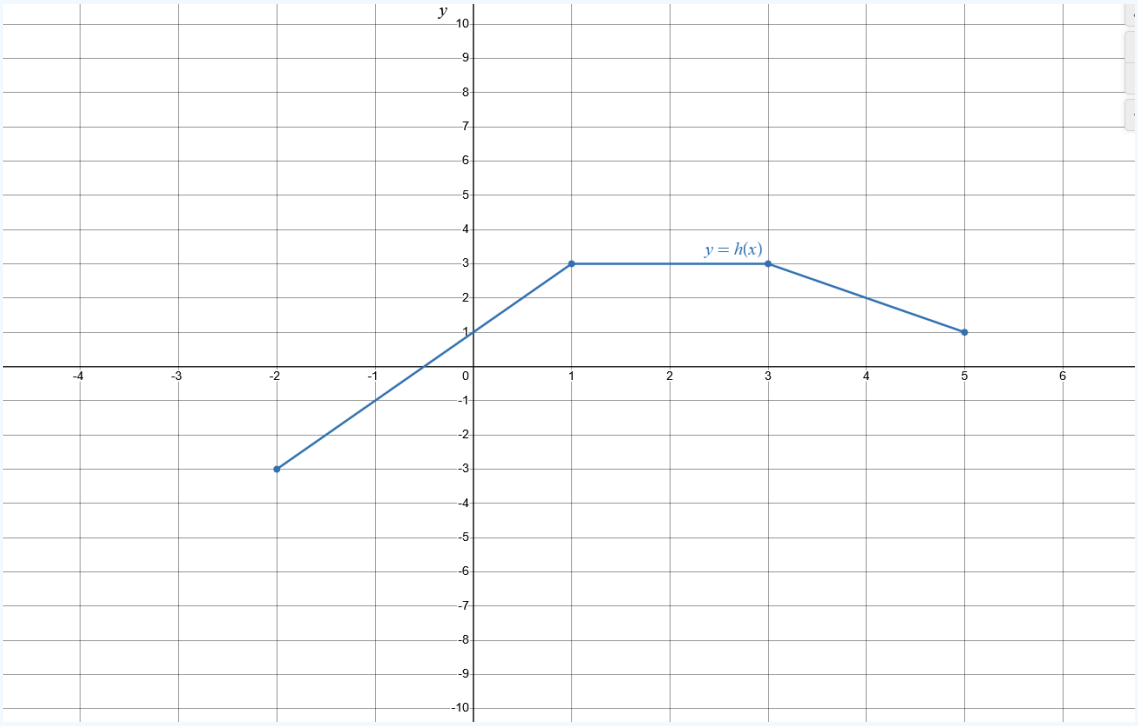
   a. $y = f(x) + 2$
   b. $y = 2f(x)$
   c. $y = -f(x) - 1$
   d. $y = -\frac{1}{2}f(x) + 4$

4. Use the graph of $y = g(x)$ below to sketch the graph of the indicated function with transformations. Show a graph of each transformation if multiple transformations occur and where each indicated point was transformed.



a. $y = g(x) - 3$
b. $y = \frac{1}{2}g(x)$
c. $y = -g(x) + 1$
d. $y = -2g(x) + 3$

5. Use the graph of $y = h(x)$ below to sketch the graph of the indicated function with transformations. Show a graph of each transformation if multiple transformations occur and where each indicated point was transformed.

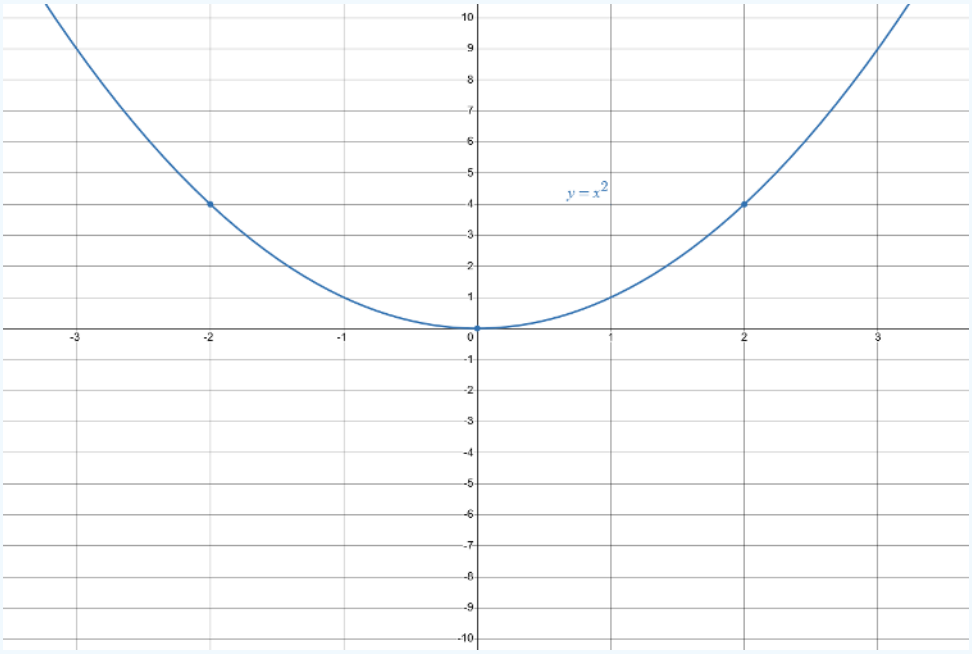


a. $y = h(x) - 1$
b. $y = 3h(x)$

c. $y = -h(x) + 5$

d. $y = -\frac{1}{2}h(x) - 2$

6. Use transformations of the graph of $y = x^2$ below to sketch the graph of the indicated functions. Show a graph of each transformation if multiple transformations occur and where each indicated point was transformed.



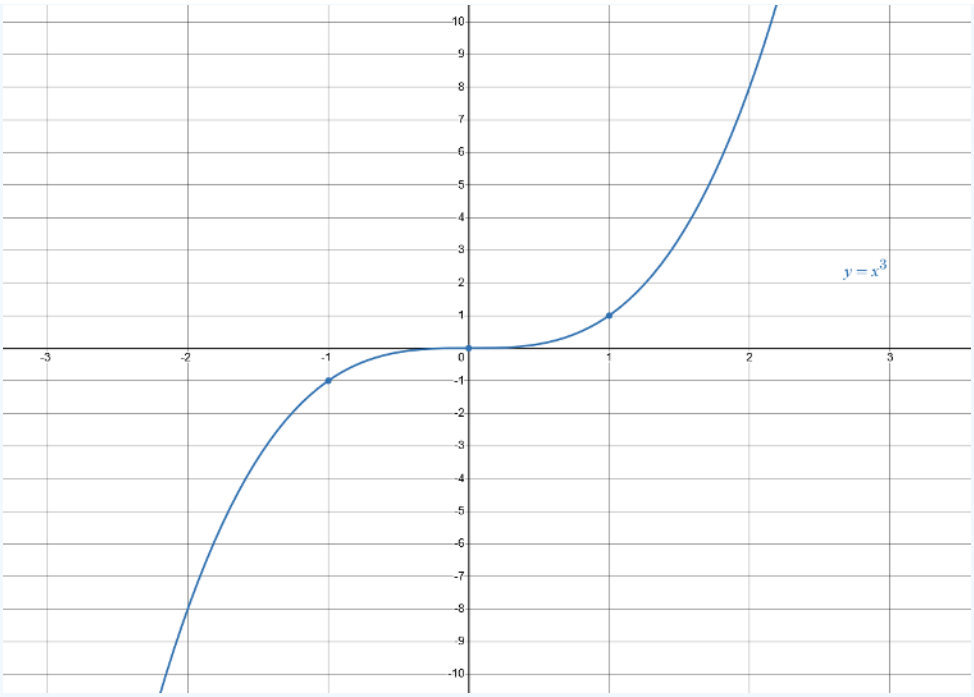
a. $y = x^2 - 2$

b. $y = 3x^2$

c. $y = 2x^2 + 1$

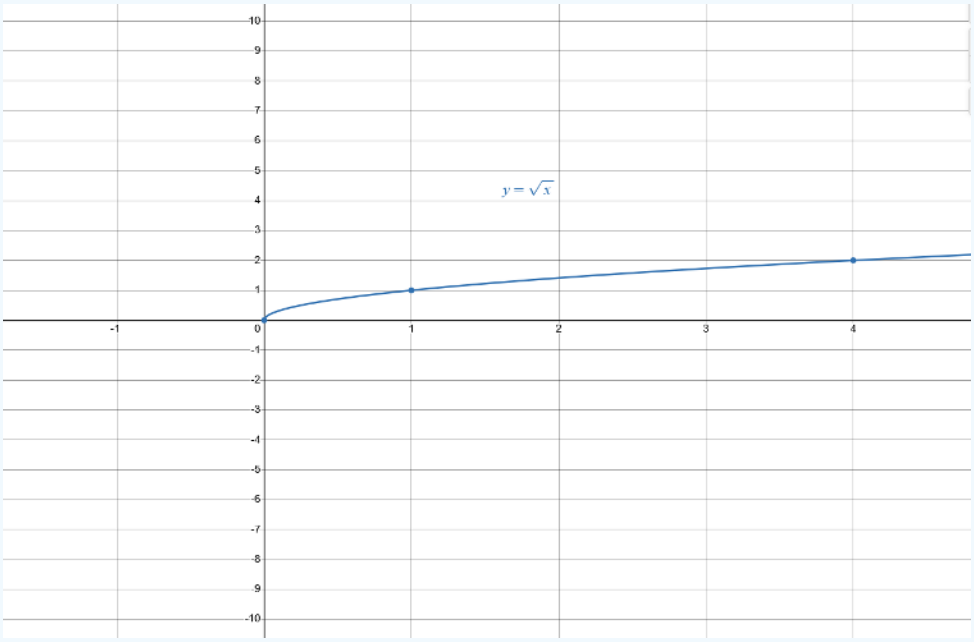
d. $y = -\frac{1}{2}x^2 - 3$

7. Use transformations of the graph of $y = x^3$ below to sketch the graph of the indicated functions. Show a graph of each transformation if multiple transformations occur and where each indicated point was transformed.



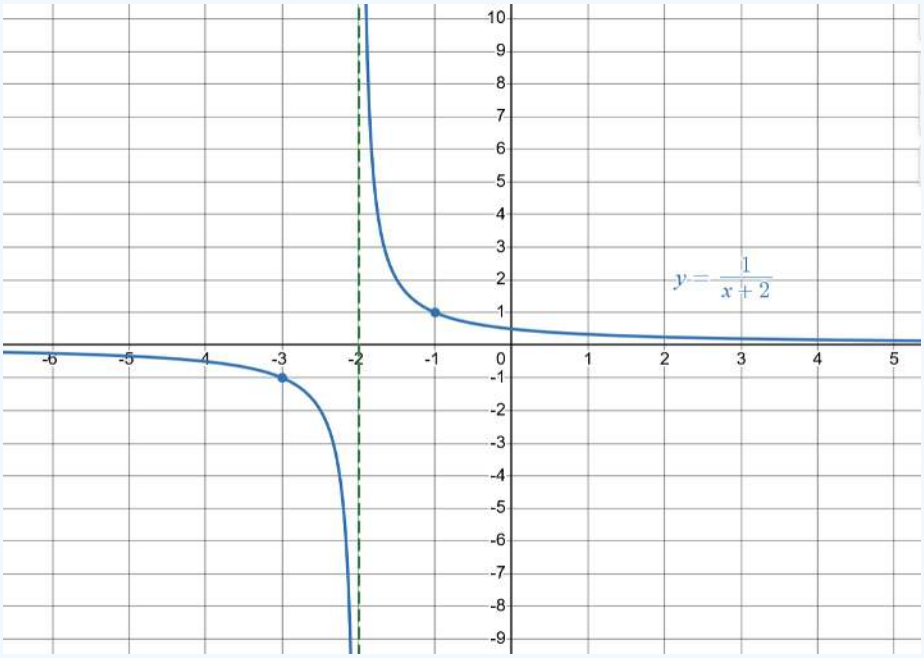
a. $y = x^3 + 4$
b. $y = 4x^3$
c. $y = \frac{1}{2}x^3 - 4$
d. $y = -2\frac{1}{4}x^3 + 2$

8. Use transformations of the graph of $y = \sqrt{x}$ below to sketch the graph of the indicated functions. Show a graph of each transformation if multiple transformations occur and where each indicated point was transformed.



a. $y = \sqrt{x} - 1$
b. $y = 3\sqrt{x}$
c. $y = 2\sqrt{x} + 4$
d. $y = -\frac{1}{2}\sqrt{x} + 2$

9. Use transformations of the graph of $y = \dfrac{1}{x+2}$ below to sketch the graph of the indicated functions. Show a graph of each transformation if multiple transformations occur and where each indicated point was transformed.

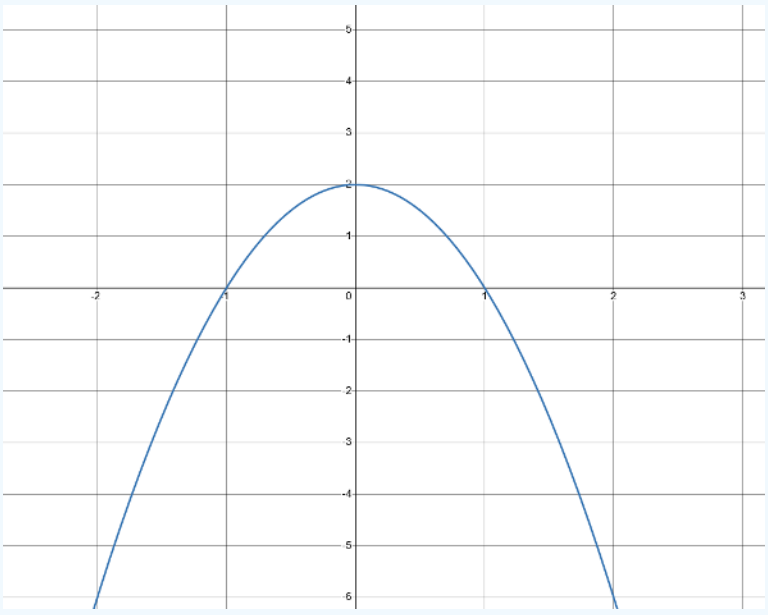


a. $y = \dfrac{1}{x+2} + 2$

b. $y = 2\dfrac{1}{x+2}$

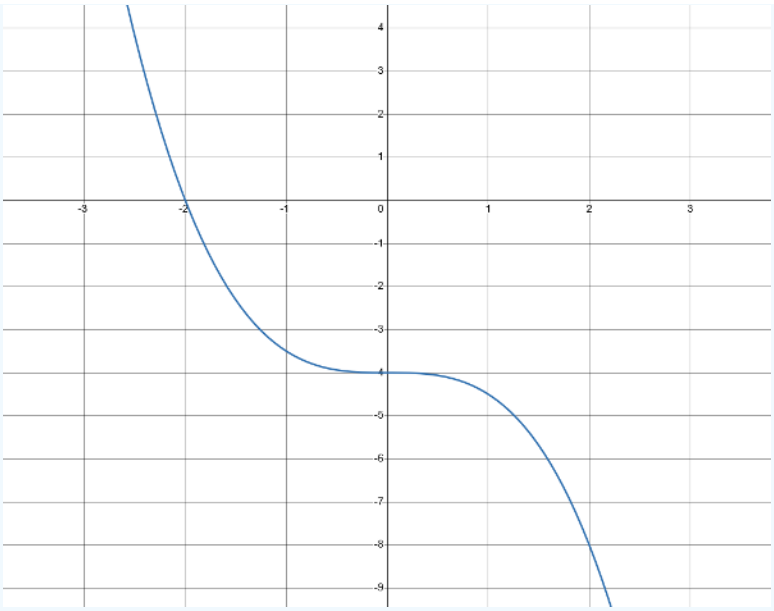
c. $y = -\dfrac{1}{2}\dfrac{1}{x+2} - 1$

10. Find a formula for the function $y = f(x)$ whose graph is given below.

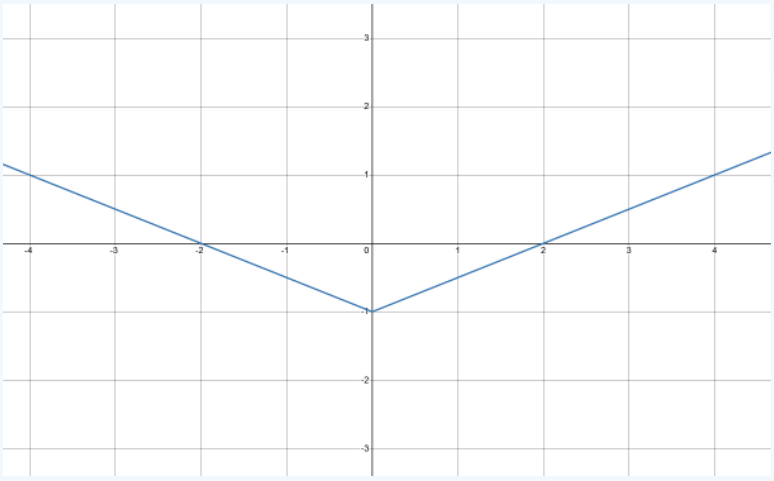


11. Find a formula for the function $y = g(x)$ whose graph is given below.

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12. Find a formula for the function $y = h(x)$ whose graph is given below.



Past 2: Horizontal Shifts

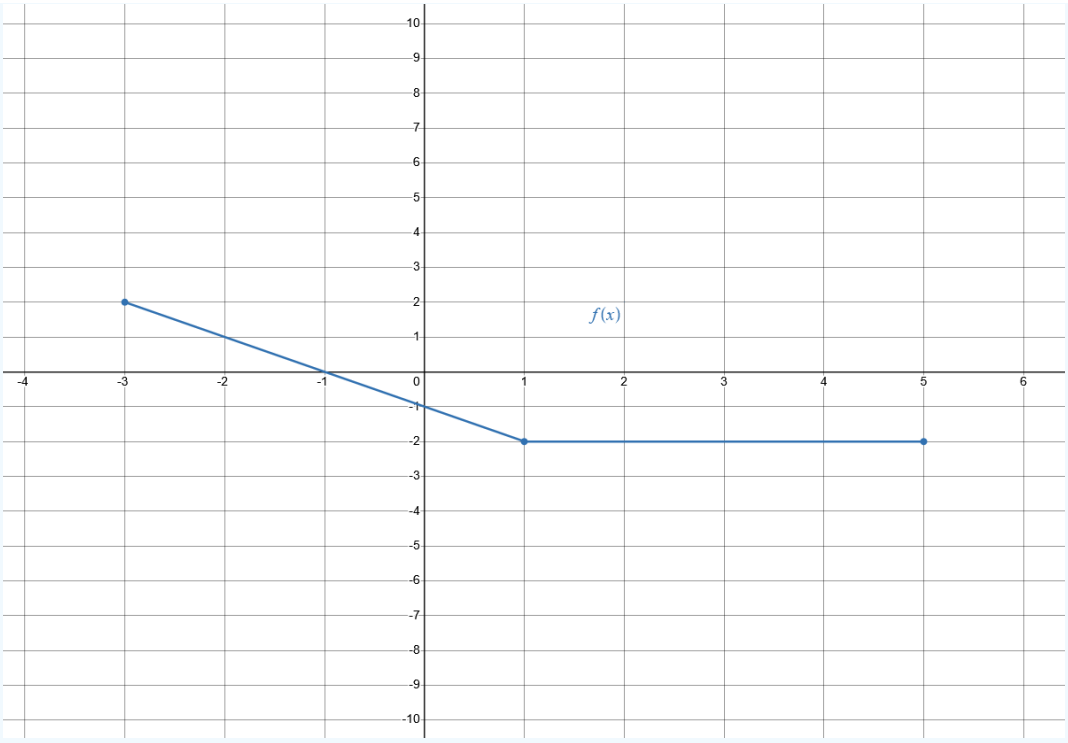
13. Write a function whose graph is the graph of $y = |x|$

    a. Shifted left three units.
    b. Shifted right two units, then vertically stretched by a factor of 2.

14. Write a function whose graph is the graph of $y = x^3$

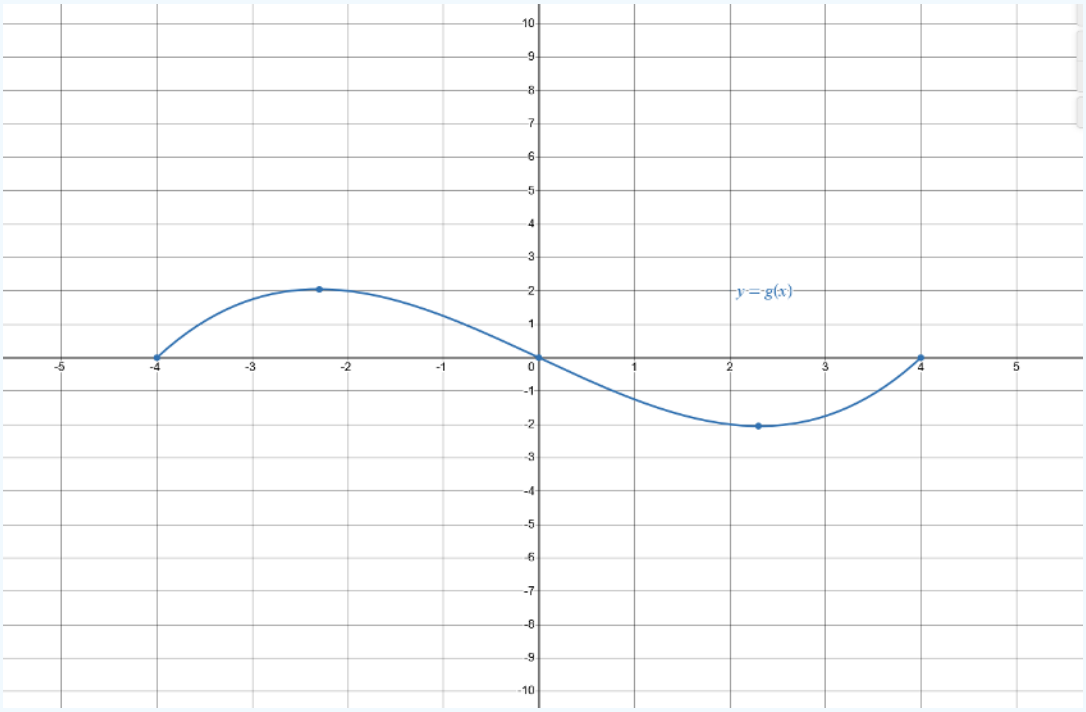
    a. Shifted right one unit.
    b. Shifted left five units, reflected about the y-axis, and shifted up one unit.

15. Use the graph of $y = f(x)$ below to sketch the graph of the indicated function with transformations. Show a graph of each transformation if multiple transformations occur and where each indicated point was transformed.



a. $y = f(x + 2)$
b. $y = 3f(x - 1)$
c. $y = -f(x + 3) + 4$

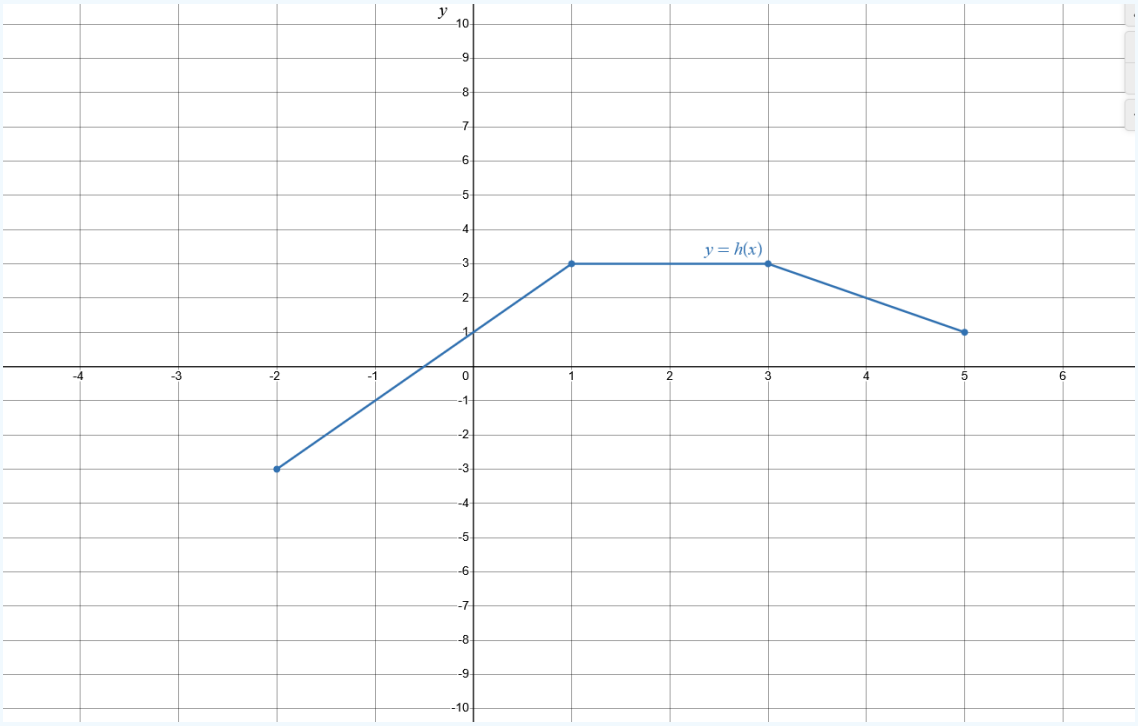
16. Use the graph of $y = g(x)$ below to sketch the graph of the indicated function with transformations. Show a graph of each transformation if multiple transformations occur and where each indicated point was transformed.



a. $y = g(x - 2)$
b. $y = 2g(x + 3)$

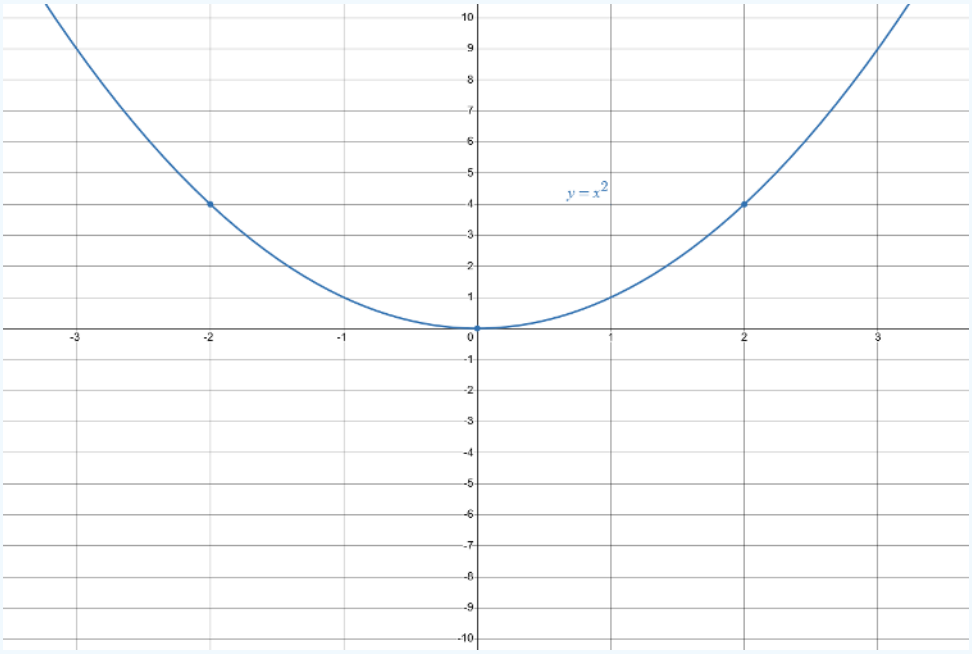
c. $y = -g(x-1)+1$

17. Use the graph of $y = h(x)$ below to sketch the graph of the indicated function with transformations. Show a graph of each transformation if multiple transformations occur and where each indicated point was transformed.



a. $y = h(x+1)$
b. $y = h(x-3)-3$
c. $y = -2h(x+2)$

18. Use transformations of the graph of $y = x^2$ below to sketch the graph of the indicated functions. Show a graph of each transformation if multiple transformations occur and where each indicated point was transformed.

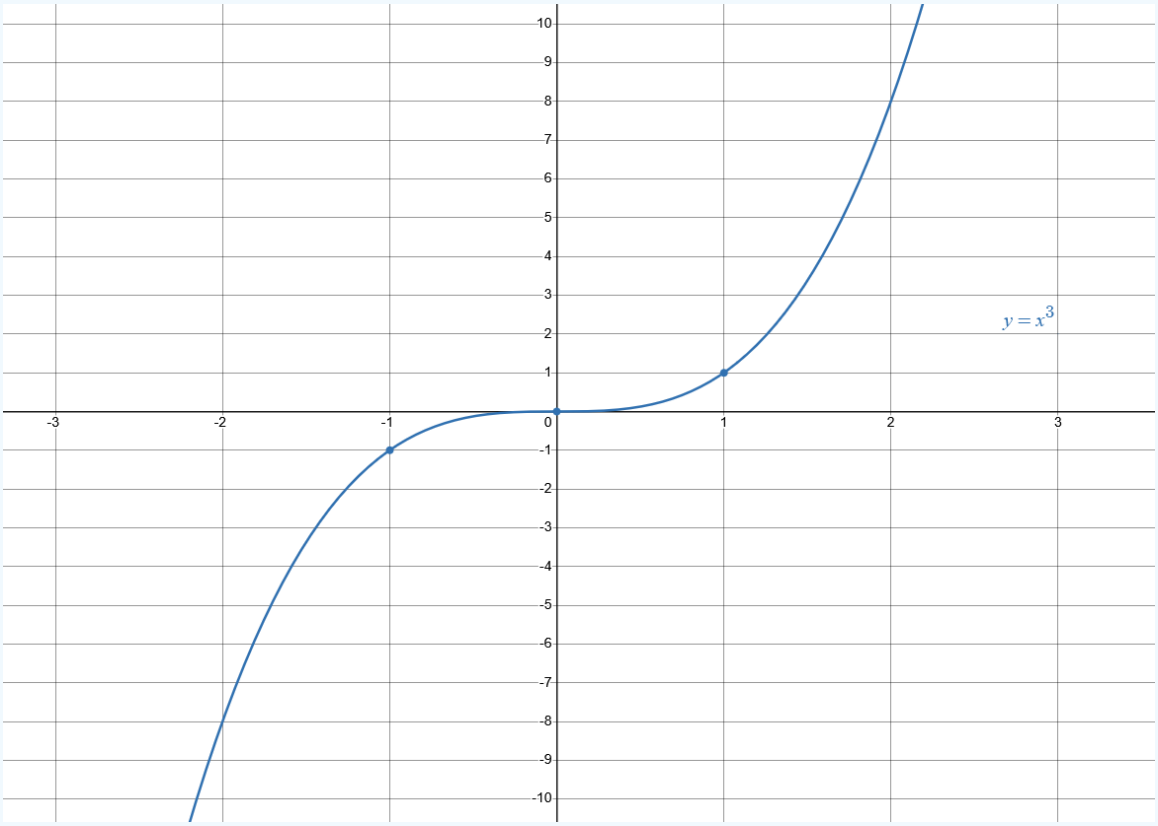


a. $y = (x+4)^2 - 2$

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b. $y = 3(x-1)^2$

19. Use transformations of the graph of $y = x^3$ below to sketch the graph of the indicated functions. Show a graph of each transformation if multiple transformations occur and where each indicated point was transformed.



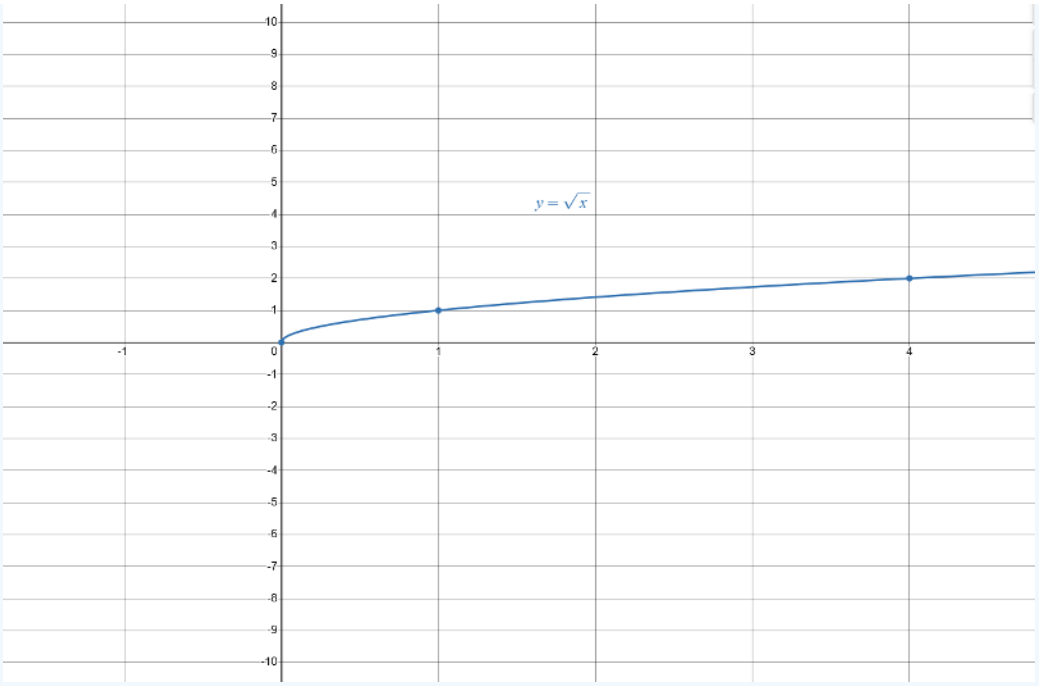
a. $y = (x+3)^3$

b. $y = -(x-1)x^3 + 2$

c. $y = \frac{1}{2}(x+1)^3 - 4$

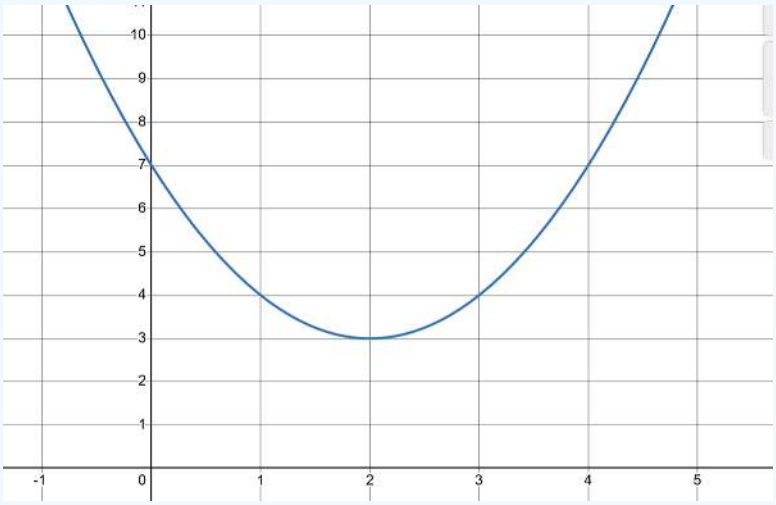
20. Use transformations of the graph of $y = \sqrt{x}$ below to sketch the graph of the indicated functions. Show a graph of each transformation if multiple transformations occur and where each indicated point was transformed.

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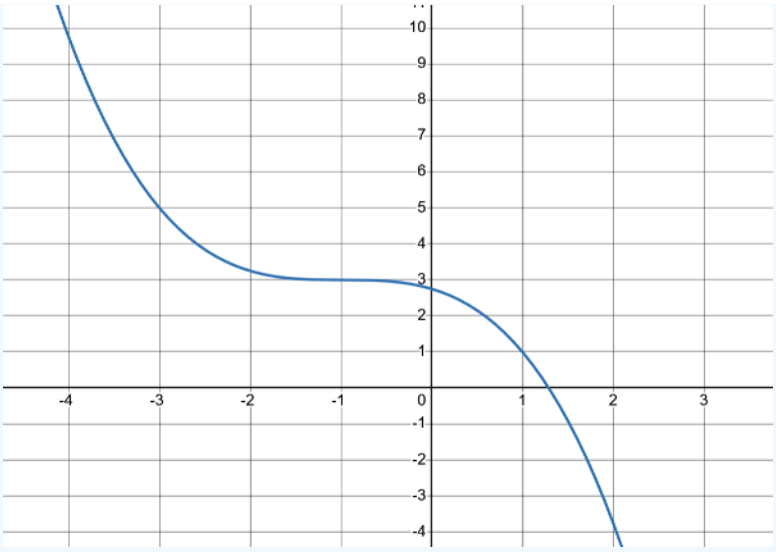
a. $y = \sqrt{x-4})$
b. $y = \sqrt{x+2} - 3$
c. $y = -2\sqrt{x-1}$

21. Find a formula for the function $y = f(x)$ whose graph is given below.

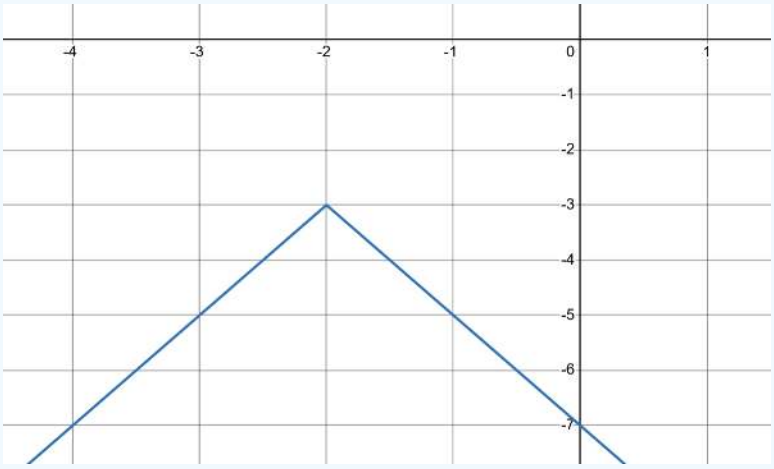


22. Find a formula for the function $y = g(x)$ whose graph is given below.

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23. Find a formula for the function $y = h(x)$ whose graph is given below.



24. A manufacturer used test market data to predict the Revenue R from producing x units of one of their products with the function $R(x) = -1.12x^2 + 200x$.

a. If a new regulatory fee cause the revenue to decrease by \$500, find a formula for the new revenue with this regulation.

b. If a new tariff caused the revenue to decrease 15%, find a formula for the new revenue with this regulation.

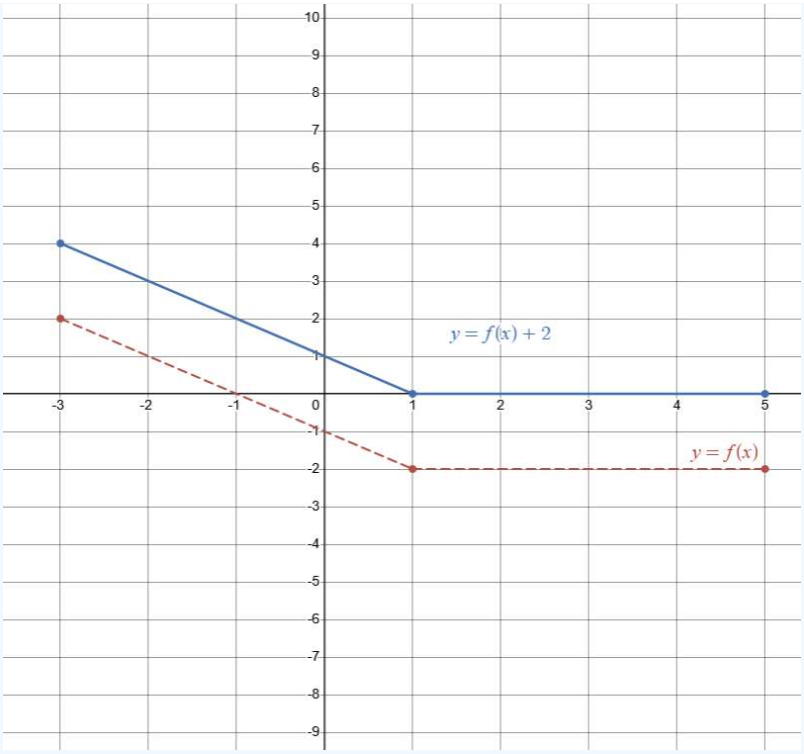
25. The amount of a therapeutic medication prescribed for a condition (in mg) is a function of the weight w in pounds described by the function $R(x) = 2.5\sqrt{x} + 10$. If the scale weighting the patient overestimated the weight by 2.3 pounds, find a formula for the actual amount of the medication that should be prescribed.

Answer

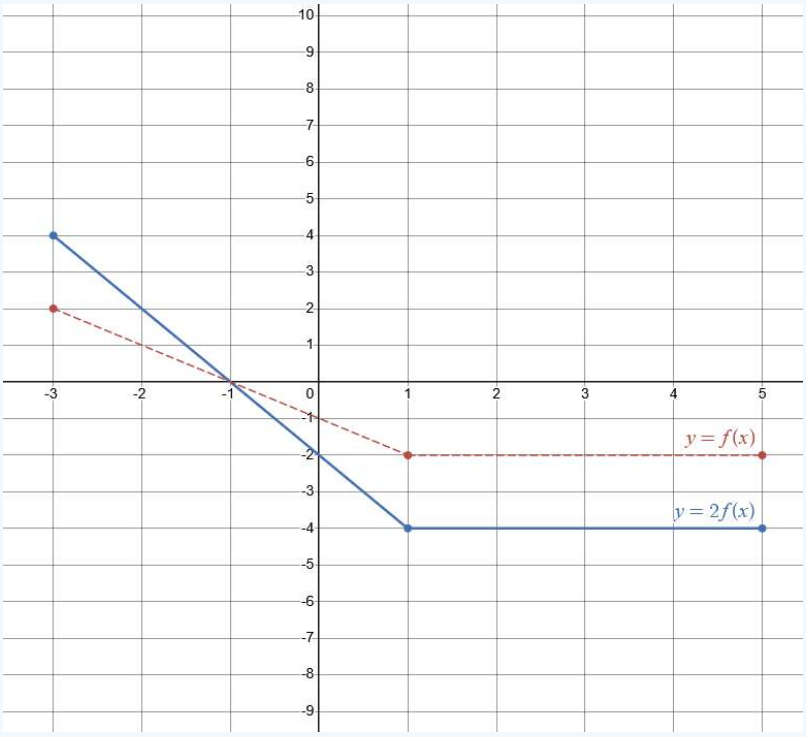
1. a) $y = |x| + 3$ b) $y = 3|x|$ c) $y = -\frac{1}{3}|x| - 2$

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3. a)

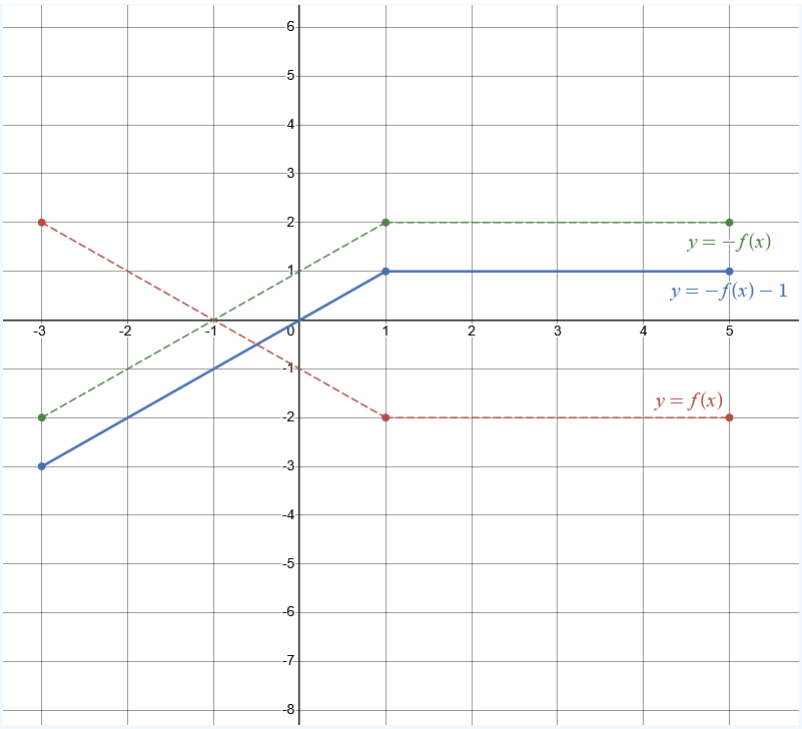


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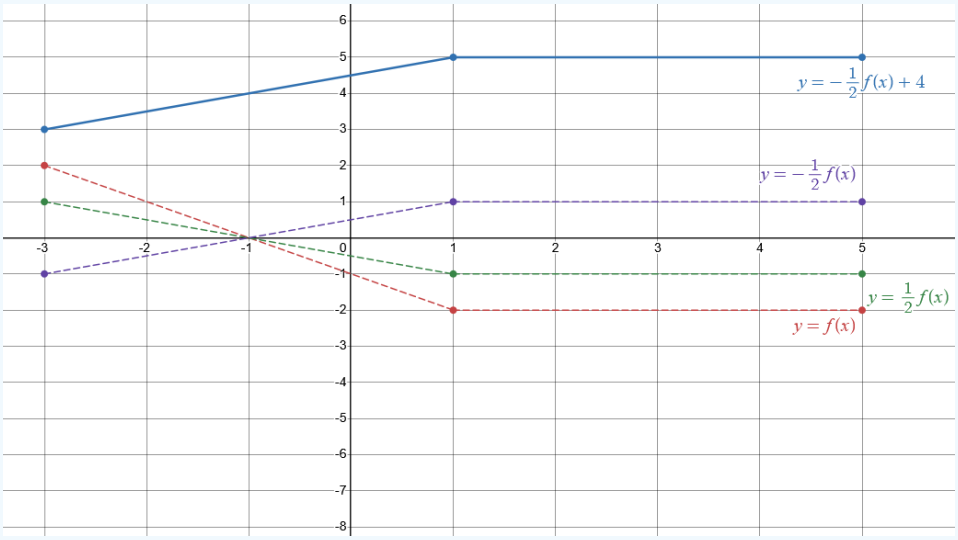


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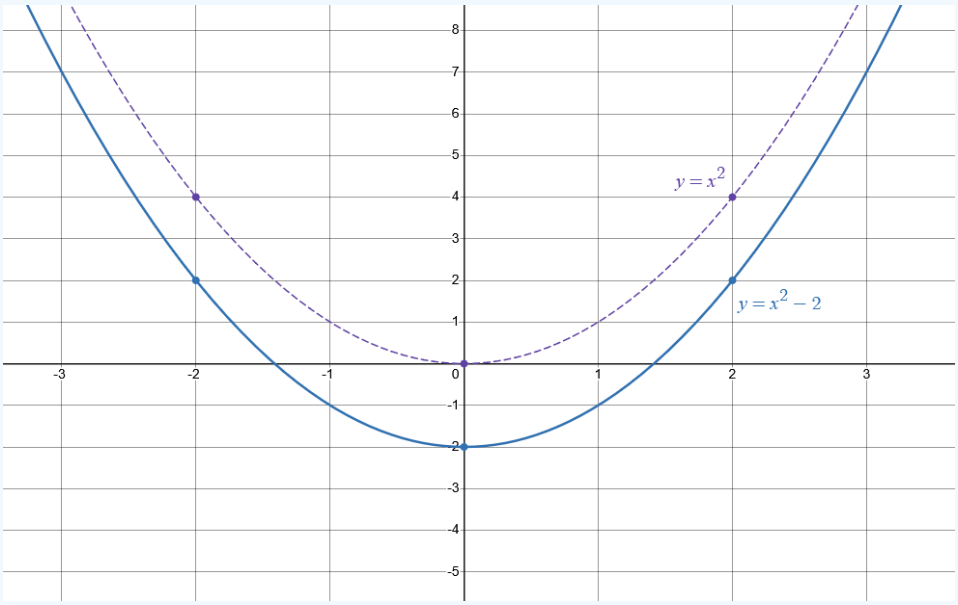


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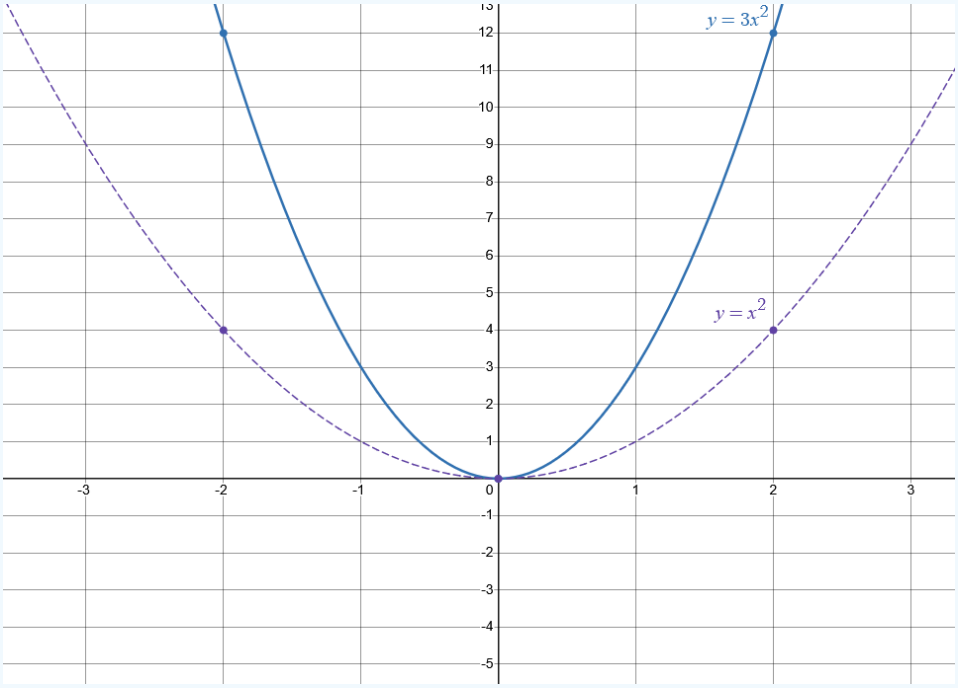


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6. a)

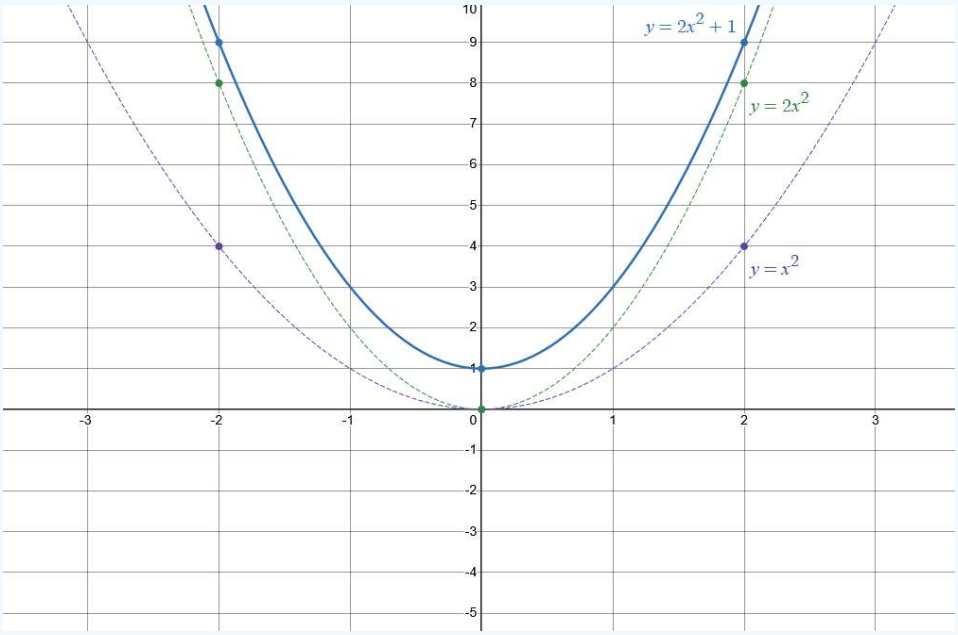


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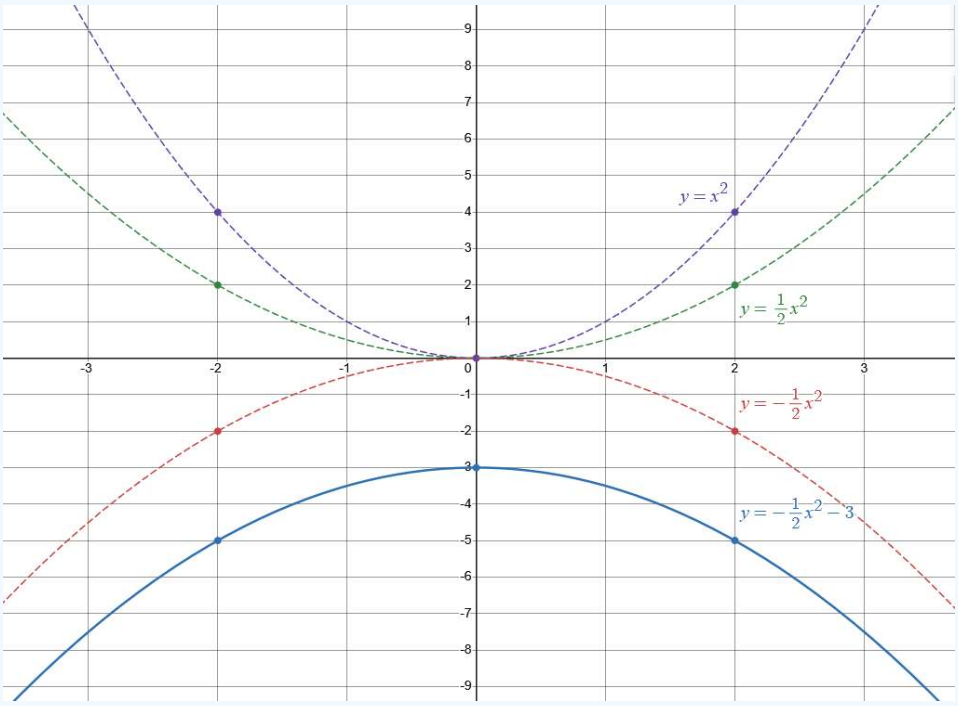


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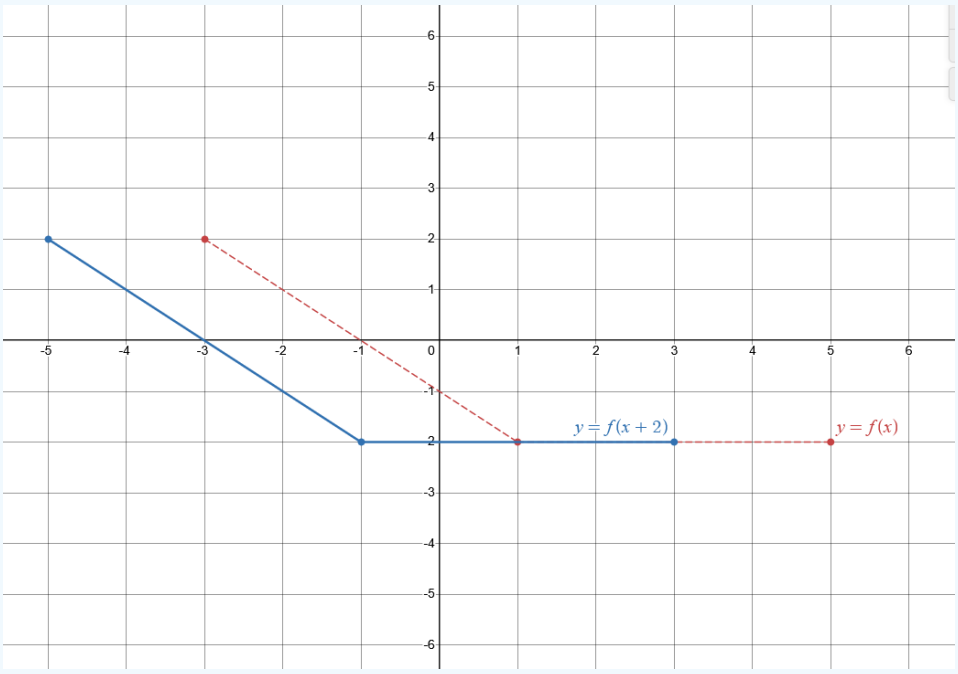


10. $y = -2x^2 + 2$

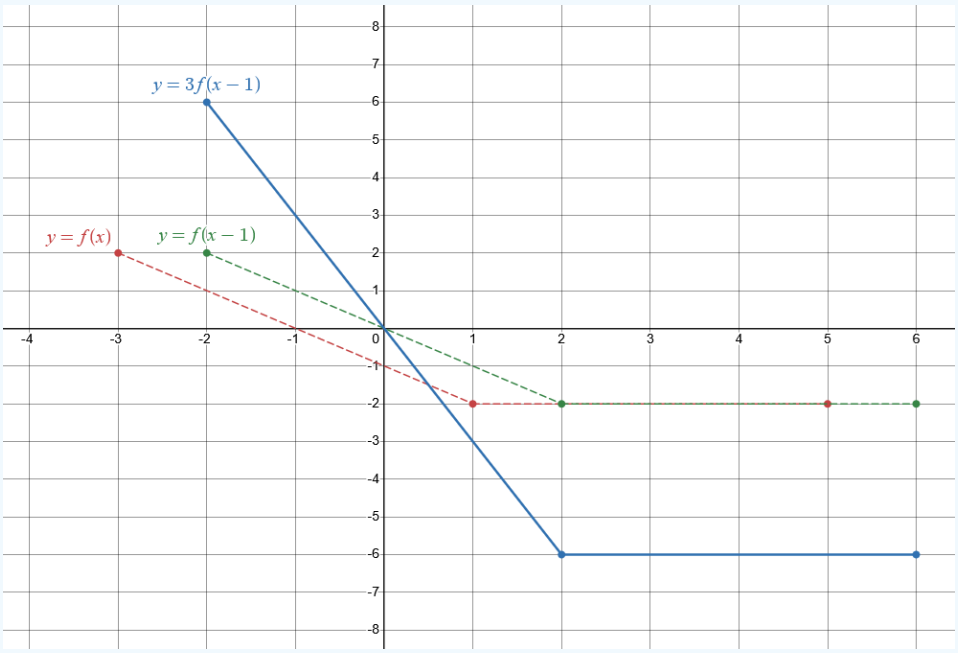
13. a) $y = |x + 3|$ b) $y = 2|x - 2|$

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15. a)

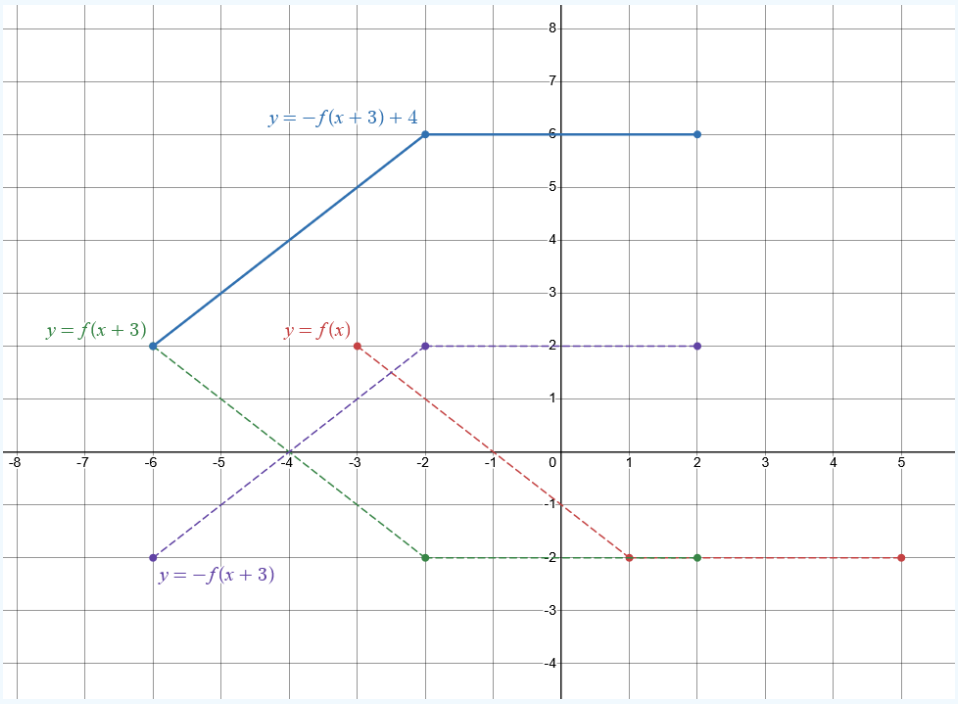


b)



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c)



21. $y = (x-2)^2 + 3$

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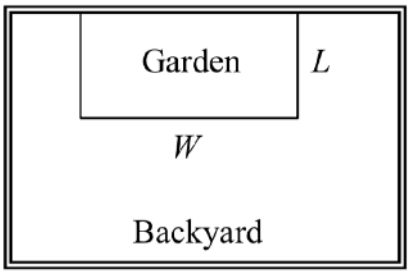
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3.2: Quadratic Functions

In this section, we will explore the family of 2^{nd} degree polynomials, the quadratic functions. While they share many characteristics of polynomials in general, the calculations involved in working with quadratics is typically a little simpler, which makes them a good place to start our exploration of short run behavior. In addition, quadratics commonly arise from problems involving area and projectile motion, providing some interesting applications.

Example 3.2.1

A backyard farmer wants to enclose a rectangular space for a new garden. She has purchased 80 feet of wire fencing to enclose 3 sides, and will put the 4^{th} side against the backyard fence. Find a formula for the area enclosed by the fence if the sides of fencing perpendicular to the existing fence have length L.



Solution

In a scenario like this involving geometry, it is often helpful to draw a picture. It might also be helpful to introduce a temporary variable, W, to represent the side of fencing parallel to the 4^{th} side or backyard fence.

Since we know we only have 80 feet of fence available, we know that $L + W + L = 80$, or more simply, $2L + W = 80$. This allows us to represent the width, W, in terms of L:

$$W = 80 - 2L$$

Now we are ready to write an equation for the area the fence encloses. We know the area of a rectangle is length multiplied by width, so

$$A = LW = L(80 - 2L)$$

$$A(L) = 80L - 2L^2$$

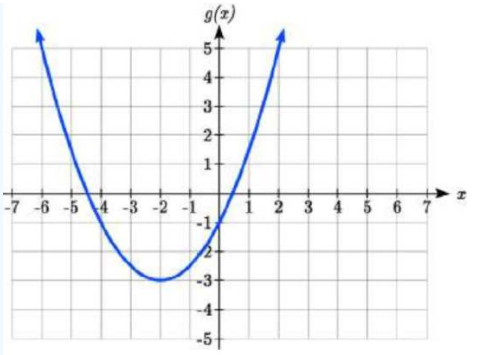
This formula represents the area of the fence in terms of the variable length L.

Short run Behavior: Vertex

We now explore the interesting features of the graphs of quadratics. In addition to intercepts, quadratics have an interesting feature where they change direction, called the **vertex**. You probably noticed that all quadratics are related to transformations of the basic quadratic function $f(x) = x^2$.

Example 3.2.2

Write an equation for the quadratic graphed below as a transformation of $f(x) = x^2$, then expand the formula and simplify terms to write the equation in standard polynomial form.



Solution

We can see the graph is the basic quadratic shifted to the left 2 and down 3, giving a formula in the form $g(x) = a(x+2)^2 - 3$. By plugging in a point that falls on the grid, such as (0,-1), we can solve for the stretch factor:

$$\begin{aligned} -1 &= a(0+2)^2 - 3 \\ 2 &= 4a \\ a &= \frac{1}{2} \end{aligned}$$

Written as a transformation, the equation for this formula is $g(x) = \frac{1}{2}(x+2)^2 - 3$. To write this in standard polynomial form, we can expand the formula and simplify terms:

$$\begin{aligned} g(x) &= \frac{1}{2}(x+2)^2 - 3 \\ g(x) &= \frac{1}{2}(x+2)(x+2) - 3 \\ g(x) &= \frac{1}{2}(x^2+4x+4) - 3 \\ g(x) &= \frac{1}{2}x^2 + 2x + 2 - 3 \\ g(x) &= \frac{1}{2}x^2 + 2x - 1 \end{aligned}$$

Notice that the horizontal and vertical shifts of the basic quadratic determine the location of the vertex of the parabola; the vertex is unaffected by stretches and compressions.

? Exercise 3.2.1

A coordinate grid has been superimposed over the quadratic path of a basketball(From http://blog.mrmeyer.com/?p=4778, © Dan Meyer, CC-BY). Find an equation for the path of the ball. Does he make the basket?



Answer

The path passes through the origin with vertex at (-4, 7).

$$h(x) = -\frac{7}{16}(x+4)^2 + 7$$

To make the shot, $h(-7.5)$ would need to be about 4. $h(-7.5) \approx 1.64$; he doesn't make it.



forms of quadratic function

The **standard form** of a quadratic function is

$$f(x) = ax^2 + bx + c \tag{3.2.1}$$

The **transformation form** of a quadratic function is

$$f(x) = a(x-h)^2 + k \tag{3.2.2}$$

The **vertex** of the quadratic function is located at (h, k), where h and k are the numbers in the transformation form of the function. Because the vertex appears in the transformation form, it is often called the **vertex form**.

In the previous example, we saw that it is possible to rewrite a quadratic function given in transformation form and rewrite it in standard form by expanding the formula. It would be useful to reverse this process since the transformation form reveals the vertex.

Expanding out the general transformation form of a quadratic gives:

$$f(x) = a(x-h)^2 + k = a(x-h)(x-h) + k$$
$$f(x) = a(x^2 - 2xh + h^2) + k = ax^2 - 2ahx + ah^2 + k$$

This should be equal to the standard form of the quadratic:

$$ax^2 - 2ahx + ah^2 + k = ax^2 + bx + c$$

The second degree terms are already equal. For the linear terms to be equal, the coefficients must be equal:

$$-2ah = b \text{ ,so } h = -\frac{b}{2a}$$

This provides us a method to determine the horizontal shift of the quadratic from the standard form. We could likewise set the constant terms equal to find:

$$ah^2 + k = c \text{ ,so } k = c - ah^2 = c - a\left(-\frac{b}{2a}\right)^2 = c - a\frac{b^2}{4a^2} = c - \frac{b^2}{4a}$$

In practice, though, it is usually easier to remember that k is the output value of the function when the input is h, so $k = f(h)$.

finding the vertex of a quadratic

For a quadratic given in standard form, the vertex (h, k) is located at:

$$h = -\frac{b}{2a}, k = f(h) = f\left(\frac{-b}{2a}\right) \tag{3.2.3}$$

Example 3.2.3

Find the vertex of the quadratic $f(x) = 2x^2 - 6x + 7$. Rewrite the quadratic into transformation form (vertex form).

Solution

The horizontal coordinate of the vertex will be at

$$h = -\frac{b}{2a} = -\frac{-6}{2(2)} = \frac{6}{4} = \frac{3}{2}$$

The vertical coordinate of the vertex will be at

$$f(\frac{3}{2}) = 2(\frac{3}{2})^2 - 6(\frac{3}{2}) + 7 = \frac{5}{2}$$

Rewriting into transformation form, the stretch factor will be the same as the a in the original quadratic. Using the vertex to determine the shifts,

$$f(x) = 2(x - \frac{3}{2})^2 + \frac{5}{2}$$

? Exercise 3.2.2

Given the equation $g(x) = 13 + x^2 - 6x$ write the equation in standard form and then in transformation/vertex form.

Answer

$g(x) = x^2 - 6x + 13$ in Standard form (Equation 3.2.1

Finding the vertex, $h = \frac{-(-6)}{2(1)} = 3$.

$$k = g(3) = 3^2 - 6(3) + 13 = 4$$

$g(x) = (x - 3)^2 + 4$ in Transformation form (Equation 3.2.2)

As an alternative to using a formula for finding the vertex, the equation can also be written into vertex form by **completing the square**. This process is most easily explained through example. In most cases, using the formula for finding the vertex will be quicker and easier than completing the square, but completing the square is a useful technique when faced with some other algebraic problems.

✓ Example 3.2.4

Rewrite $f(x) = 2x^2 - 12x + 14$ into vertex form by completing the square.

Solution

We start by factoring the leading coefficient from the quadratic and linear terms.

$$2(x^2 - 6x) + 14$$

Next, we are going to add something inside the parentheses so that the quadratic inside the parentheses becomes a perfect square. In other words, we are looking for values p and q so that $(x^2 - 6x + p) = (x - q)^2$.

Notice that if multiplied out on the right, the middle term would be -2q, so q must be half of the middle term on the left; $q = -3$. In that case, p must be $(-3)^2 = 9$.

$$(x^2 - 6x + 9) = (x - 3)^2$$

Now, we can't just add 9 into the expression – that would change the value of the expression. In fact, adding 9 inside the parentheses actually adds 18 to the expression, since the 2 outside the parentheses will distribute. To keep the expression balanced, we can subtract 18.

$$2(x^2 - 6x + 9) + 14 - 18$$

Simplifying, we are left with vertex form.

$$2(x - 3)^2 - 4$$

In addition to enabling us to more easily graph a quadratic written in standard form, finding the vertex serves another important purpose – it allows us to determine the maximum or minimum value of the function, depending on which way the graph opens.

Example 3.2.5

Returning to our backyard farmer from the beginning of the section, what dimensions should she make her garden to maximize the enclosed area?

Solution

Earlier we determined the area she could enclose with 80 feet of fencing on three sides was given by the equation $A(L) = 80L - 2L^2$. Notice that quadratic has been vertically reflected, since the coefficient on the squared term is negative, so the graph will open downwards, and the vertex will be a maximum value for the area.

In finding the vertex, we take care since the equation is not written in standard polynomial form with decreasing powers. But we know that a is the coefficient on the squared term, so $a = -2$, $b = 80$, and $c = 0$.

Finding the vertex:

$$h = -\frac{80}{2(-2)} = 20, k = A(20) = 80(20) - 2(20)^2 = 800$$

The maximum value of the function is an area of 800 square feet, which occurs when $L = 20$ feet. When the shorter sides are 20 feet, that leaves 40 feet of fencing for the longer side. To maximize the area, she should enclose the garden so the two shorter sides have length 20 feet, and the longer side parallel to the existing fence has length 40 feet.

Example 3.2.6

A local newspaper currently has 84,000 subscribers, at a quarterly charge of $30. Market research has suggested that if they raised the price to $32, they would lose 5,000 subscribers. Assuming that subscriptions are linearly related to the price, what price should the newspaper charge for a quarterly subscription to maximize their revenue?

Solution

Revenue is the amount of money a company brings in. In this case, the revenue can be found by multiplying the charge per subscription times the number of subscribers. We can introduce variables, C for charge per subscription and S for the number subscribers, giving us the equation

$$\text{Revenue} = CS$$

Since the number of subscribers changes with the price, we need to find a relationship between the variables. We know that currently $S = 84,000$ and $C = 30$, and that if they raise the price to $32 they would lose 5,000 subscribers, giving a second pair of values, $C = 32$ and $S = 79,000$. From this we can find a linear equation relating the two quantities. Treating C as the input and S as the output, the equation will have form $S = mC + b$. The slope will be

$$m = \frac{79,000 - 84,000}{32 - 30} = \frac{-5,000}{2} = -2,500$$

This tells us the paper will lose 2,500 subscribers for each dollar they raise the price. We can then solve for the vertical intercept

$$S = -2500C + b$$

Plug in the point $S = 84,000$ and $C = 30$

$$84,000 = -2500(30) + b$$

Solve for b

$$b = 159,000$$

This gives us the linear equation $S = -2,500C + 159,000$ relating cost and subscribers. We now return to our revenue equation.

$$\text{Revenue} = CS$$

Substituting the equation for S from above

$$\text{Revenue} = C(-2,500C + 159,000)$$

Expanding

$$\text{Revenue} = -2,500C^2 + 159,000C$$

We now have a quadratic equation for revenue as a function of the subscription charge. To find the price that will maximize revenue for the newspaper, we can find the vertex:

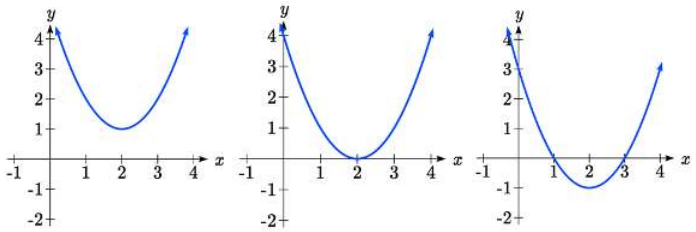
$$h = -\frac{159,000}{2(-2,500)} = 31.8$$

The model tells us that the maximum revenue will occur if the newspaper charges $31.80 for a subscription. To find what the maximum revenue is, we can evaluate the revenue equation:

$$\text{Maximum Revenue} = -2,500(31.8)^2 + 159,000(31.8) = \$2,528,100$$

Short run Behavior: Intercepts

As with any function, we can find the vertical intercepts of a quadratic by evaluating the function at an input of zero, and we can find the horizontal intercepts by solving for when the output will be zero. Notice that depending upon the location of the graph, we might have zero, one, or two horizontal intercepts.



Zero horizontal intercepts one horizontal intercept two horizontal intercepts

Example 3.2.7

Find the vertical and horizontal intercepts of the quadratic $f(x) = 3x^2 + 5x - 2$

Solution

We can find the vertical intercept by evaluating the function at an input of zero:

$$f(0) = 3(0)^2 + 5(0) - 2 = -2$$

Vertical intercept at (0,-2)

For the horizontal intercepts, we solve for when the output will be zero

$$0 = 3x^2 + 5x - 2$$

In this case, the quadratic can be factored easily, providing the simplest method for solution

$$0 = (3x - 1)(x + 2)$$

$$\begin{aligned} 0 &= 3x - 1 \\ x &= \frac{1}{3} \end{aligned} \quad \text{or} \quad \begin{aligned} 0 &= x + 2 \\ x &= -2 \end{aligned}$$

Horizontal intercepts at $\left(\frac{1}{3}, 0\right)$ and (-2,0)

Notice that in the standard form of a quadratic, the constant term c reveals the vertical intercept of the graph.

Example 3.2.8

Find the horizontal intercepts of the quadratic $f(x) = 2x^2 + 4x - 4$

Solution

Again we will solve for when the output will be zero

$$0 = 2x^2 + 4x - 4$$

Since the quadratic is not easily factorable in this case, we solve for the intercepts by first rewriting the quadratic into transformation form.

$$h = -\frac{b}{2a} = -\frac{4}{2(2)} = -1 \quad k = f(-1) = 2(-1)^2 + 4(-1) - 4 = -6$$

$$f(x) = 2(x+1)^2 - 6$$

Now we can solve for when the output will be zero

$$\begin{aligned} 0 &= 2(x+1)^2 - 6 \\ 6 &= 2(x+1)^2 \\ 3 &= (x+1)^2 \\ x+1 &= \pm\sqrt{3} \\ x &= -1 \pm \sqrt{3} \end{aligned}$$

The graph has horizontal intercepts at $(-1-\sqrt{3}, 0)$and $(-1+\sqrt{3}, 0)$

Exercise 3.2.3

In Try it Now problem 2 we found the standard & transformation form for the function $g(x) = 13 + x^2 - 6x$. Now find the Vertical & Horizontal intercepts (if any).

Answer

Vertical intercept at (0, 13), No horizontal intercepts since the vertex is above the x-axis and the graph opens upwards.

The process in the last example is done commonly enough that sometimes people find it easier to solve the problem once in general and remember the formula for the result, rather than repeating the process each time. Based on our previous work we showed that any quadratic in standard form can be written into transformation form as:

$$f(x) = a(x + \frac{b}{2a})^2 + c - \frac{b^2}{4a} \tag{3.2.4}$$

Solving for the horizontal intercepts using this general equation gives:

$$0 = a(x + \frac{b}{2a})^2 + c - \frac{b^2}{4a}$$

start to solve for x by moving the constants to the other side

$$\frac{b^2}{4a} - c = a(x + \frac{b}{2a})^2$$

divide both sides by a

$$\frac{b^2}{4a^2} - \frac{c}{a} = (x + \frac{b}{2a})^2$$

find a common denominator to combine fractions

$$\frac{b^2}{4a^2} - \frac{4ac}{4a^2} = \left(x + \frac{b}{2a}\right)^2$$

combine the fractions on the left side of the equation

$$\frac{b^2 - 4ac}{4a^2} = \left(x + \frac{b}{2a}\right)^2$$

take the square root of both sides

$$\pm\sqrt{\frac{b^2 - 4ac}{4a^2}} = x + \frac{b}{2a}$$

subtract $b/2a$ from both sides

$$-\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = x$$

combining the fractions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Notice that this can yield two different answers for x

Quadratic Formula

For a quadratic function given in standard form $f(x) = ax^2 + bx + c$, the **quadratic formula** gives the horizontal intercepts of the graph of this function.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{3.2.5}$$

Example 3.2.9

A ball is thrown upwards from the top of a 40-foot-tall building at a speed of 80 feet per second. The ball's height above ground can be modeled by the equation $H(t) = -16t^2 + 80t + 40$.

What is the maximum height of the ball?

When does the ball hit the ground?

Solution

To find the maximum height of the ball, we would need to know the vertex of the quadratic.

$$h = -\frac{80}{2(-16)} = \frac{80}{32} = \frac{5}{2}, k = H\left(\frac{5}{2}\right) = -16\left(\frac{5}{2}\right)^2 + 80\left(\frac{5}{2}\right) + 40 = 140$$

The ball reaches a maximum height of 140 feet after 2.5 seconds.

To find when the ball hits the ground, we need to determine when the height is zero – when $H(t) = 0$. While we could do this using the transformation form of the quadratic, we can also use the quadratic formula:

$$t = \frac{-80 \pm \sqrt{80^2 - 4(-16)(40)}}{2(-16)} = \frac{-80 \pm \sqrt{8960}}{-32}$$

Since the square root does not simplify nicely, we can use a calculator to approximate the values of the solutions:

$$t = \frac{-80 - \sqrt{8960}}{-32} \approx 5.458 \text{ or } t = \frac{-80 + \sqrt{8960}}{-32} \approx -0.458$$

The second answer is outside the reasonable domain of our model, so we conclude the ball will hit the ground after about 5.458 seconds.

? Exercise 3.2.4

For these two equations determine if the vertex will be a maximum value or a minimum value.

a. $g(x) = -8x + x^2 + 7$
b. $g(x) = -3(3 - x)^2 + 2$

Answer

a. Vertex is a minimum value, since $a > 0$ and the graph opens upwards

b. Vertex is a maximum value, since $a < 0$ and the graph opens downwards

Important Topics of this Section

- Quadratic functions
- Standard form
- Transformation form/Vertex form
- Vertex as a maximum / Vertex as a minimum
- Short run behavior
- Vertex / Horizontal & Vertical intercepts
- Quadratic formula

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3.2E: Describing Relationships with Quadratic Functions (Exercises)

section 3.2 exercise

Part 1: Quadratic Functions in Vertex Form

1. Let $f(x) = 3(x-2)^2 - 1$

   a. Identify the vertex of the parabola.
   b. Determine the end behavior.
   c. Sketch a graph of the parabola.
   d. Determine where f is increasing.
   e. Determine where f is decreasing.
   f. Identify the minimum value. Where does it occur?

2. Let $g(x) = -2(x+1)^2 + 3$

   a. Identify the vertex of the parabola.
   b. Determine the end behavior.
   c. Sketch a graph of the parabola.
   d. Determine where f is increasing.
   e. Determine where f is decreasing.
   f. Identify the maximum value. Where does it occur?

3. Let $h(x) = 0.5(x-2)^2 - 3$

   a. Identify the vertex of the parabola.
   b. Determine the end behavior.
   c. Sketch a graph of the parabola.
   d. Determine where f is increasing.
   e. Determine where f is decreasing.
   f. Identify the minimum value. Where does it occur?

4. Let $k(x) = -1.2(x+2.1)^2 - 1.5$

   a. Identify the vertex of the parabola.
   b. Determine the end behavior.
   c. Sketch a graph of the parabola.
   d. Determine where f is increasing.
   e. Determine where f is decreasing.
   f. Identify the maximum value. Where does it occur?

5. Let $f(x) = 2(x-1)^2 - 3$

   a. Determine the zeros.
   b. Determine the end behavior.
   c. Sketch a graph of the parabola.
   d. Determine where f is positive.
   e. Determine where f is negative.

6. Let $f(x) = -3(x+2)^2 + 4$

   a. Determine the zeros.
   b. Determine the end behavior.
   c. Sketch a graph of the parabola.
   d. Determine where f is positive.
   e. Determine where f is negative.

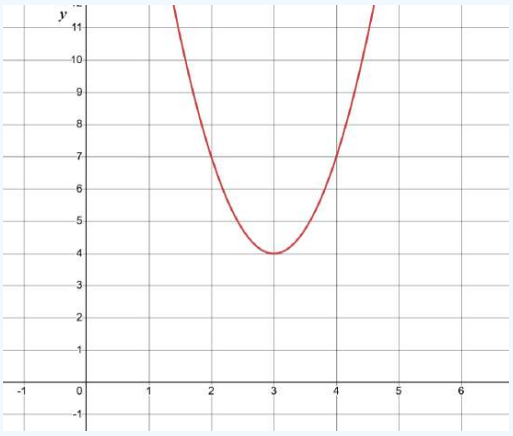
7. Let $f(x) = 0.5(x - 3.2)^2 + 2.5$

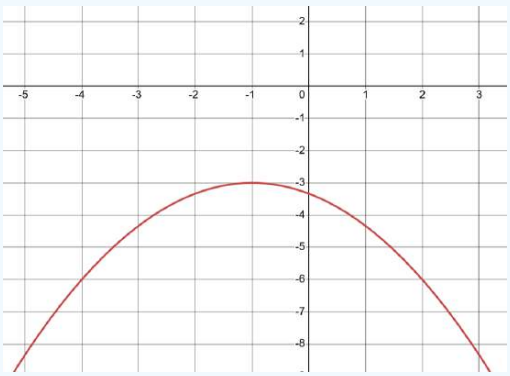
a. Determine the zeros.
b. Determine the end behavior.
c. Sketch a graph of the parabola.
d. Determine where f is positive.
e. Determine where f is negative.

8. Find a formula for the parabola with a

a. Vertex at (4, 0), and y-intercept (0, -4).
b. Vertex at (5, 6), and y-intercept (0, -1).
c. Vertex at (-3, 2) that passes through the point (3, -2).
d. Vertex at (1, -3) that passes through the point (-2, 3).

9. Find a formula for the parabola whose graph is given below

a. 

b. 

10. A ball is kicked into the air. The height of the ball (in feet) after t seconds from where is was kicked is given by $h = f(t) = -16(t - 3)^2 + 50$.

1. What was the maximum height of the ball?
2. When did the ball reach its maximum height?

Part 2: Quadratic functions in Standard Form

11. Let $f(x) = 3x^2 + 6x - 9$

a. Identify the vertex of the parabola.
b. Determine the end behavior.
c. Sketch a graph of the parabola.
d. Determine where f is increasing.
e. Determine where f is decreasing.

f. Identify the minimum value. Where does it occur?

12. Let $f(x) = 2x^2 - 10x + 4$

a. Identify the vertex of the parabola.
b. Determine the end behavior.
c. Sketch a graph of the parabola.
d. Determine where f is increasing.
e. Determine where f is decreasing.
f. Identify the minimum value. Where does it occur?

13. $h(t) = -4t^2 + 6t - 1$

a. Identify the vertex of the parabola.
b. Determine the end behavior.
c. Sketch a graph of the parabola.
d. Determine where h is increasing.
e. Determine where h is decreasing.
f. Identify the maximum value. Where does it occur?

14. $k(x) = 2x^2 + 4x - 15$

a. Identify the vertex of the parabola.
b. Determine the end behavior.
c. Sketch a graph of the parabola.
d. Determine where k is increasing.
e. Determine where k is decreasing.
f. Identify the minimum value. Where does it occur?

15. Let $f(x) = x^2 - 6x + 8$

a. Determine the zeros.
b. Determine the end behavior.
c. Sketch a graph of the parabola.
d. Determine where f is positive.
e. Determine where f is negative.

16. Let $f(x) = 2x^2 - 5x - 3$

a. Determine the zeros.
b. Determine the end behavior.
c. Sketch a graph of the parabola.
d. Determine where f is positive.
e. Determine where f is negative.

17. Let $f(x) = 3.2x^2 - 4.2x - 1.3$

a. Determine the zeros.
b. Determine the end behavior.
c. Sketch a graph of the parabola.
d. Determine where f is positive.
e. Determine where f is negative.

18. Write an equation for a quadratic with the given features

a. Zeros at $x = -3$ and $x = 1$ with a y-intercept (0, 2)
b. Zeros at $x = 2$ and $x = -5$ with a y-intercept (0, 3)
c. Zeros at $x = 2$ and $x = 5$ that passes through the point (-1,4).
d. Zeros at $x = -1.2$ and $x = -2, 3$ that passes through the point (2,5).

19. A rocket is launched in the air. Its height above sea level (in meters) in t seconds is given by $h(t) = -4.9t^2 + 229t + 234$.

a. From what height was the rocket launched?
b. How high above sea level does the rocket reach its peak?
c. Assuming the rocket will splash down in the ocean, at what time does splashdown occur?

20. A ball is thrown in the air from the top of a building. Its height, in meters above ground, as a function of time, in seconds, is given by $h(t) = -4.9t^2 + 24t + 8$.

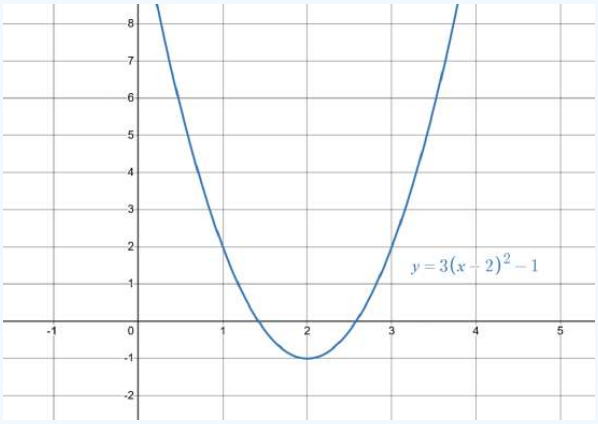
a. From what height was the ball thrown?
b. How high above ground does the ball reach its peak?
c. When does the ball hit the ground?

21. A javelin is thrown in the air. Its height is given by $h(x) = -\frac{1}{20}x^2 + 8x + 6$, where x is the horizontal distance in feet from the point at which the javelin is thrown.

a. How high is the javelin when it was thrown?
b. What is the maximum height of the javelin?
c. How far from the thrower does the javelin strike the ground?

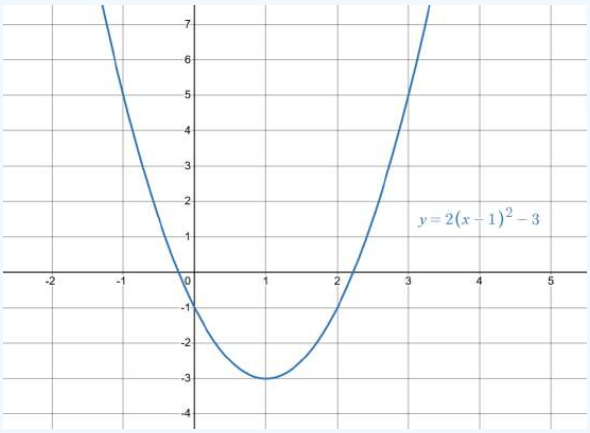
Answer

1. a) Vertex at (2,-1) b) The graph opens up on both ends (as $x \to \pm\infty, y \to \infty$)

c) 

d) f is increasing on $(2, \infty)$ e) f is decreasing on $(-\infty, 2)$ f) The minimum value is $y = -1$ and occurs at $x = 2$

5. a) Zeros at $x = 1 \pm \sqrt{\frac{3}{2}}$ or $x \approx -0.2247$ and $x \approx 2.2247$ b) The graph opens up on both ends (as $x \to \pm\infty, y \to \infty$)

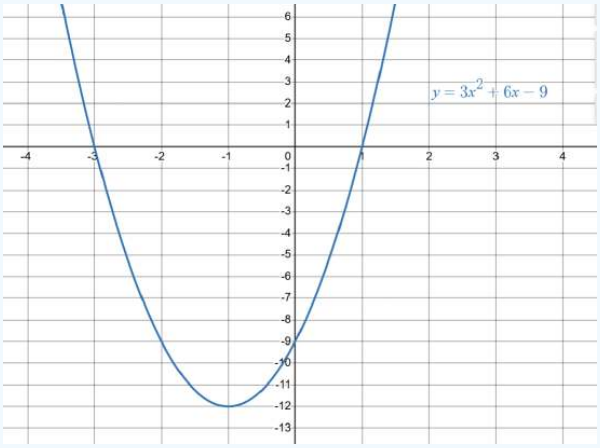
c) 

d) f is positive on $(-\infty, -0.2247) \bigcup (2.2247, \infty)$ e) f is negative on $(-0.2247, 2.2247)$.

8. a) $y = \frac{-1}{4}(x-4)^2$

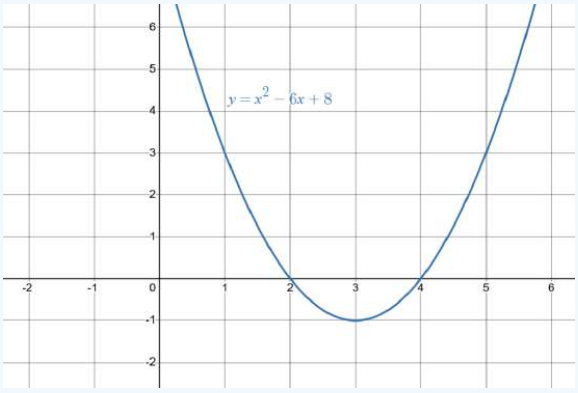
9. a) $y = 3(x-3)^2 + 4$

11. a) Vertex at (-1,-12) b) The graph opens up on both ends (as $x \to \pm\infty, y \to \infty$)

c) 

d) f is increasing on $(-1, \infty)$ e) f is decreasing on $-\infty, -1)$ f) The minimum value is $y = -12$. It occurs at $x = -1$

15. a) The zeros are $x = 2$ and $x = 4$. b) The graph opens up on both ends (as $x \to \pm\infty, y \to \infty$)

c) 

d) f is positive on $(-\infty, 2) \bigcup (4, \infty)$ e) f is negative on $(2, 4)$.

18. a) $y = \frac{-2}{3}(x+3)(x-1)$

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3.3: Complex Zeros

When finding the zeros of polynomials, at some point you're faced with the problem $x^2 = -1$. While there are clearly no real numbers that are solutions to this equation, leaving things there has a certain feel of incompleteness. To address that, we will need utilize the imaginary unit, i.

Definition: Imaginary number i

The most basic complex number is i, defined to be $i = \sqrt{-1}$, commonly called an **imaginary number**. Any real multiple of i is also an imaginary number.

Example 3.3.1

Simplify $\sqrt{-9}$.

Solution

We can separate $\sqrt{-9}$ as $\sqrt{9}\sqrt{-1}$. We can take the square root of 9, and write the square root of -1 as i.

$$\sqrt{-9} = \sqrt{9}\sqrt{-1} = 3i$$

A complex number is the sum of a real number and an imaginary number.

Definition: Complex Numbers

A **complex number** is a number $z = a + bi$, where a and b are real numbers

- a is the real part of the complex number
- b is the imaginary part of the complex number

$$i = \sqrt{-1} \tag{3.3.1}$$

Arithmetic on Complex Numbers

Before we dive into the more complicated uses of complex numbers, let's make sure we remember the basic arithmetic involved. To add or subtract complex numbers, we simply add the like terms, combining the real parts and combining the imaginary parts.

Example 3.3.2

Add $3 - 4i$ and $2 + 5i$.

Solution

Adding $(3 - 4i) + (2 + 5i)$, we add the real parts and the imaginary parts

$$3 + 2 - 4i + 5i$$

$$5 + i$$

Exercise 3.3.1

Subtract $2 + 5i$ from $3 - 4i$.

Answer

$$(3 - 4i) - (2 + 5i) = 1 - 9i$$

We can also multiply and divide complex numbers.

Example 3.3.3

Multiply: $4(2+5i)$.

Solution

To multiply the complex number by a real number, we simply distribute as we would when multiplying polynomials.

$$4(2+5i) = 4 \cdot 2 + 4 \cdot 5i$$
$$= 8 + 20i$$

Example 3.3.4

Divide $\frac{(2+5i)}{(4-i)}$.

Solution

To divide two complex numbers, we have to devise a way to write this as a complex number with a real part and an imaginary part.

We start this process by eliminating the complex number in the denominator. To do this, we multiply the numerator and denominator by a special complex number so that the result in the denominator is a real number. The number we need to multiply by is called the **complex conjugate**, in which the sign of the imaginary part is changed.

Here, 4+*i* is the complex conjugate of 4–*i*. Of course, obeying our algebraic rules, we must multiply by 4+*i* on both the top and bottom.

$$\frac{(2+5i)}{(4-i)} \cdot \frac{(4+i)}{(4+i)}$$

To multiply two complex numbers, we expand the product as we would with polynomials (the process commonly called FOIL – "first outer inner last"). In the numerator:

$$(2+5i)(4+i)$$

Expand

$$= 8 + 20i + 2i + 5i^2$$

Since $i = \sqrt{-1}$, $i^2 = -1$

$$= 8 + 20i + 2i + 5(-1)$$

Simplify

$$= 3 + 22i$$

Following the same process to multiply the denominator

$$(4-i)(4+i)$$

Expand

$$= (16 - 4i + 4i - i^2)$$

Since $i = \sqrt{-1}$, $i^2 = -1$

$$= (16 - (-1))$$

$$= 17$$

Combining this we get $\frac{3+22i}{17} = \frac{3}{17} + \frac{22i}{17}$

Exercise 3.3.2

Multiply $3-4i$ and $2+3i$.

Answer

$$(3-4i)(2+3i) = 18+i$$

In the last example, we used the conjugate of a complex number

Definition: Complex Conjugate

The **conjugate** of a complex number $a+bi$ is the number $a-bi$.

The notation commonly used for conjugation is a bar:

$$\overline{a+bi} = a-bi$$

Complex Zeros of Polynomials

Complex numbers allow us a way to write solutions to quadratic equations that do not have real solutions.

Example 3.3.5

Find the zeros of $f(x) = x^2 - 2x + 5$.

Solution

Using the quadratic formula,

$$x = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(5)}}{2(1)} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i.$$

Exercise 3.3.3

Find the zeros of $f(x) = 2x^2 + 3x + 4$.

Answer

$$x = \frac{-3 \pm \sqrt{(3)^2 - 4(2)(4)}}{2(2)} = \frac{-3 \pm \sqrt{-23}}{4} = \frac{-3 \pm i\sqrt{23}}{4} = \frac{-3}{4} \pm \frac{\sqrt{23}}{4}i$$

Two things are important to note. First, the zeros $1+2i$ and $1-2i$ are complex conjugates. This will always be the case when we find non-real zeros to a quadratic function with real coefficients.

Second, we could write

$$f(x) = x^2 - 2x + 5 = (x - (1+2i))\,(x - (1-2i))$$

if we really wanted to, so the Factor and Remainder Theorems hold.

How do we know if a general polynomial has any complex zeros? We have seen examples of polynomials with no real zeros; can there be polynomials with no zeros at all? The answer to that last question, which comes from the Fundamental Theorem of Algebra, is "No."

A non-constant polynomial f with real or complex coefficients will have at least one real or complex zero.

This theorem is an example of an "existence" theorem in mathematics. It guarantees the existence of at least one zero, but provides no algorithm to use for finding it.

Now suppose we have a polynomial $f(x)$ of degree n. The Fundamental Theorem of Algebra guarantees at least one zero z_1, then the Factor Theorem guarantees that f can be factored as $f(x) = (x - z_1)\, q_1(x)$, where the quotient $q_1(x)$ will be of degree $n - 1$.

If this function is non-constant, than the Fundamental Theorem of Algebra applies to it, and we can find another zero. This can be repeated n times.

If f is a polynomial f with real or complex coefficients with degree $n \geq 1$, then f has exactly n real or complex zeros, counting multiplicities.

If $z_1, z_2, \ldots, z_k$ are the distinct zero of f with multiplicities $m_1, m_2, \ldots, m_k$ respectively, then

$$f(x) = a\,(x - z_1)^{m_1}\,(x - z_2)^{m_2} \cdots (x - z_k)^{m_k} \tag{3.3.2}$$

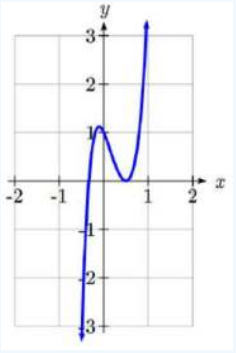
Example 3.3.6

Find all the real and complex zeros of $f(x) = 12x^5 - 20x^4 + 19x^3 - 6x^2 - 2x + 1$.

Solution

Using the Rational Roots Theorem, the possible real rational roots are

$$\left\{\pm\frac{1}{1}, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \pm\frac{1}{6}, \pm\frac{1}{12}\right\}$$



Testing $\frac{1}{2}$,

1/2	12	-20	19	-6	-2	1
	↓	6	-7	6	0	-1
	12	-14	12	0	-2	0

Success! Because the graph bounces at this intercept, it is likely that this zero has multiplicity 2. We can try synthetic division again to test that.

1/2	12	-14	12	0	-2
	↓	6	-4	4	2
	12	-8	8	-4	0

The other real root appears to be $-\frac{1}{3}$ or $-\frac{1}{4}$. Testing $-\frac{1}{3}$,

-1/3	12	-8	8	-4
	↓	-4	4	-4
	12	-12	12	0

Excellent! So far, we have factored the polynomial to

$$f(x) = \left(x - \frac{1}{2}\right)^2 \left(x + \frac{1}{3}\right)(12x^2 - 12x + 12) = 12\left(x - \frac{1}{2}\right)^2 \left(x + \frac{1}{3}\right)(x^2 - x + 1)$$

We can use the quadratic formula to find the two remaining zeros by setting $x^2 - x + 1 = 0$, which are likely complex zeros.

$$x = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)} = \frac{1 \pm \sqrt{-3}}{2} = \frac{1 \pm i\sqrt{3}}{2}$$

The zeros of the function are $x = \frac{1}{2}, -\frac{1}{3}, \frac{1+i\sqrt{3}}{2}, \frac{1-i\sqrt{3}}{2}$. We could write the function fully factored as $f(x) = 12\left(x - \frac{1}{2}\right)^2\left(x + \frac{1}{3}\right)\left(x - \frac{1+i\sqrt{3}}{2}\right)\left(x - \frac{1-i\sqrt{3}}{2}\right)$.

When factoring a polynomial like we did at the end of the last example, we say that it is **factored completely over the complex numbers**, meaning it is impossible to factor the polynomial any further using complex numbers. If we wanted to factor the function over the **real numbers**, we would have stopped at $f(x) = 12\left(x - \frac{1}{2}\right)^2\left(x + \frac{1}{3}\right)(x^2 - x + 1)$. Since the zeros of $x^2 - x + 1$ are nonreal, we call $x^2 - x + 1$ an **irreducible quadratic** meaning it is impossible to break it down any further using real numbers.

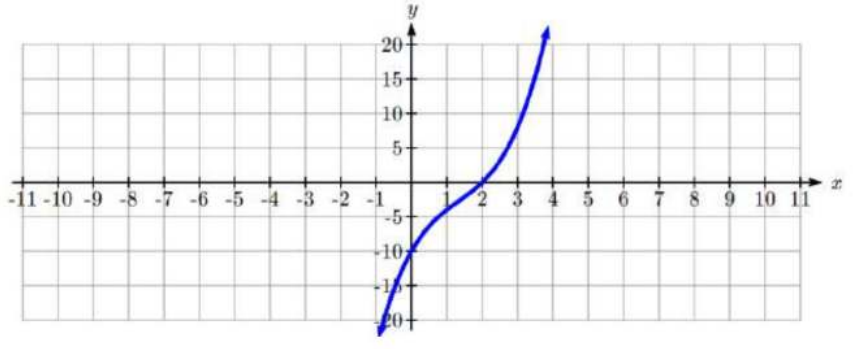
It turns out that a polynomial with real number coefficients can be factored into a product of linear factors corresponding to the real zeros of the function and irreducible quadratic factors which give the nonreal zeros of the function. Consequently, any nonreal zeros will come in conjugate pairs, so if z is a zero of the polynomial, so is $\bar{z}$.

? Exercise 3.3.4

Find the real and complex zeros of $f(x) = x^3 - 4x^2 + 9x - 10$.

Answer

Cauchy's Bound limits us to the interval [-11, 11]. The rational roots theorem gives a list of potential zeros: $\{\pm 1, \pm 2, \pm 5, \pm 10\}$. A quick graph shows that the likely rational root is $x = 2$.



Verifying this,

2	1	-4	9	-10
	↓	2	-4	10
	1	-2	5	0

So $f(x) = (x - 2)(x^2 - 2x + 5)$

Using quadratic formula, we can find the complex roots from the irreducible quadratic.

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(5)}}{2(1)} = \frac{2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

The zeros of this polynomial are $x = 2,\ -1 + 2i,\ -1 - 2i$

Important Topics of This Section

- Complex and Imaginary numbers
- Finding Complex zeros of polynomials

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3.3E: Complex Zeros of Quadratics Functions (Exercises)

? section 3.3 exercise

1. Write each complex expression with the complex number i.

   a. $\sqrt{-9}$
   b. $\sqrt{-16}$
   c. $\sqrt{-51}$
   d. $\sqrt{-24}$

2. Perform each operation and simplify each expression to a single complex number of the form a + bi.

   a. $\sqrt{-6}\sqrt{-24}$
   b. $\sqrt{-81}-\sqrt{-4}$
   c. $3\sqrt{-25}-4\sqrt{-64}$

3. Simplify each expression to a single complex number of the form a + bi.

   a. $\dfrac{4+\sqrt{-9}}{4}$
   b. $\dfrac{2+\sqrt{-12}}{2}$
   c. $\dfrac{4+\sqrt{-20}}{2}$

4. Find the zeros of each quadratic functions. Write complex zeros in the form a + bi.

   a. $f(x)=x^2-4x+13$
   b. $f(x)=x^2-2x+5$
   c. $f(x)=3x^2+2x+10$

5. Perform each operation and simplify each expression to a single complex number of the form a + bi.

   a. $(3+2i)+(5-3i)$
   b. $(-2-4i)+(1+6i)$
   c. $(-5+3i)-(6-i)$
   d. $(2+3i)(4i)$
   e. $(-2+4i)(8)$

6. Perform each operation and simplify each expression to a single complex number of the form a + bi.

   a. $(3+4i)(3-4i)$
   b. $(2+3i)(4-i)$
   c. $(-1+2i)(-2+3i)$
   d. $(4-2i)(4+2i)$

7. The quadratic function $y=f(x)$ has a zero $x=4i$ and the graph contains the point (0,5).

   a. What is the other zero?
   b. Find a formula for the quadratic function $f(x)$ with real coefficients.

8. The quadratic function $y=g(x)$ has a zero $x=-3i$ and the graph contains the point (2,4).

   a. What is the other zero?
   b. Find a formula for the quadratic function $g(x)$ with real coefficients.

9. The quadratic function $y=h(x)$ has a zero $x=2+3i$ and the graph contains the point (3,-1).

   a. What is the other zero?
   b. Find a formula for the quadratic function $h(x)$ with real coefficients.

10. The polynomial function $y=k(x)$ has zeros $x=-4-5i$ and the graph contains the point (-1,2).

a. Are there any other zeros?

b. Find a formula for $k(x)$ with real coefficients. Is $k(x)$a quadratic function?

Answer

1. a) $4i$
2. b) $5i$
4. a) $2 \pm 3i$
5. a) $8 - i$
6. a) 25
7. a) $x = -4i$ is also a zero. b) $f(x) = \frac{5}{16}x^2 + 5$

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CHAPTER OVERVIEW

4: Polynomial and Rational Functions

4.1: An Introduction to The Graphs of Polynomial Functions

4.1E: An Introduction to The Graphs of Polynomial Functions (Exercises)

4.2: More on Finding The Zeros of Polynomial Functions

4.2E: More on Finding The Zeros of Polynomial Functions (Exercises)

4.3: Graphing Polynomial Functions

4.3E: Graphing Polynomial Functions (Exercises)

4.4: Polynomial Inequalities

4.4E: Polynomial Inequalities (Exercises)

4.5: Describing Relationships with Rational Functions

4.5E: Describing Relationships with Rational Functions (Exercises)

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4.1: An Introduction to The Graphs of Polynomial Functions

Focus: Polynomial Functions and Their Graphs

In this chapter, we will explore describing relationships with larger polynomial functions like $f(x) = x^3 + x^2$ or $g(x) = x^4 + 2x^3 + 4x^2 + 2$.

Definition: A Polynomial Function

A **polynomial** is function that can be written as a sum of whole number powers of x such as

$$f(x) = a_n x^n + \cdots + a_2 x^2 + a_1 x + a_0$$

To distinguish between different size polynomials and their graphs, we need to develop some new terminology to describe different polynomials.

Definition: The Degree of a Polynomial Function

The **degree** of the polynomial is the highest power of the variable that occurs in the polynomial.

Example 4.1.1

Identify the degree of each polynomials.

a. $f(x) = 3 + 2x^2 - 4x^3$
b. $g(t) = 5t^5 - 2t^3 + 7t$
c. $h(p) = 6p - p^3 - 2$

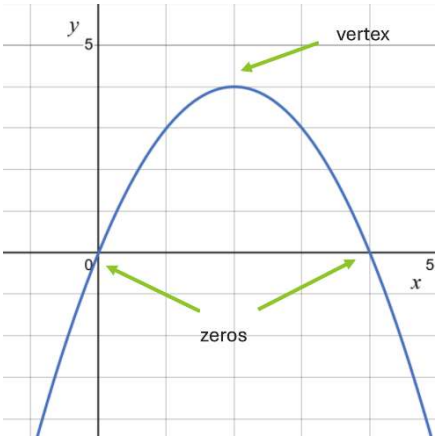
Solution

a. For the function $f(x)$, the degree is 3 since the term with the largest power is $-4x^3$
b. For $g(t)$, the degree is 5 since the term with the largest power is $5t^5$.
c. For $h(p)$, the degree is 3 since the term with the largest power is $-p^3$.

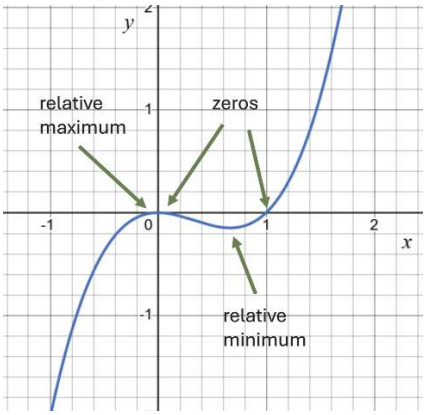
Notice that a linear function, such as $f(x) = 3x + 1$ is a polynomial function of degree one. A quadratic function, such as $g(x) = x^2 + 3x + 1$ is a polynomial function of degree two. Our goal for this chapter is describing relationships with polynomial functions of degree three or larger.

With quadratic functions in chapter three, we identified three key features of the graph of a parabola in order to help us analyze the relationship the quadratic functions were describing. The key features of a quadratic function are

1. The vertex
2. The zeroes
3. The end behavior



The vertex and end behavior allow us to determine where the parabola is increasing or decreasing as well as identify where the maximum or minimum value occurs. The real zeros of a parabola are the x-intercepts which help us determine where the function is positive or negative. With a larger polynomial, we seek to make a similar analysis. On the graph of the larger degree polynomial $f(x) = x^3 - x^2$ below,



the polynomial changes sign from positive to negative (or vice versus) at the real zeros (or x-intercepts) at $x = 0$ and $x = 1$. The graph changes direction at $x = 0$ and $x \approx 0.67$. Unlike a quadratic function, this third degree polynomial changes direction twice. Whereas a quadratic function has only one extrema, this polynomial has two relative extrema at the turning points where the graph changes direction. At the turning point $x = 0$, a relative maximum occurs. By a relative maximum, we mean that in a small interval containing $x = 0$, the function has its largest value in the interval at $x = 0$. At the turning point $x \approx 0.67$, a relative minimum occurs. By a relative minimum, we mean that in a small interval containing $x \approx 0.67$, the function has its smallest value in the interval at $x \approx 0.67$. The relative maximums and minimums of a function are typically referred to as the relative extrema.

Definition: Relative Maximum

The function $y = f(x)$ has a **relative maximum** at x = a if $f(a) \geq f(x)$ for any x in a small interval containing $x = a$.

Definition: Relative Minimum

The function $y = f(x)$ has a **relative minimum** at x = a if $f(a) \leq f(x)$ for any x in a small interval containing $x = a$.

Notice the end behavior of the larger polynomial $f(x)$ above: it opens up on the right (as $x \to +\infty, y \to +\infty$) and down on the left (as $x \to -\infty, y \to -\infty$). These three features of the graph of a polynomial function will help us analyze the relationship it is describing.

Key Point: The Key Features of the Graph of a Polynomial Functions 4.1.1

The key features of the graph of a polynomial function are

1. The zeros
2. The relative extrema
3. The end behavior

In Calculus, we will explore techniques to locate the relative extrema. At this point in Precalculus, we will focus on identifying the zeros and the end behavior. We can use technology, such as a graphing calculator or graphing app, to approximate relative extrema on the graphs of polynomials at this point.

Focus: The Zeros of a Polynomial Function

As with quadratic functions, the zeros of polynomials are the values of x for which the function value $f(x) = 0$. Real zeros are x-intercepts (or intercepts on the horizontal axis) on the graph. The zeros are a significant feature of a polynomial function since it will only change sign from positive to negative (or vice versus) at a real zero. Many of the techniques we used for solving polynomial equations in algebra will help us find the zeros of some polynomials.

Example 4.1.2

Find the zeros of the polynomial functions.

a. $f(x) = x^3 - x^2 - 6x$
b. $f(x) = x^4 + x^3 + x^2$

Solution

1. The zeros occur when $y = f(x) = 0$. So

$$x^3 - x^2 - 6x = 0$$

Factoring out a common factor of x, then factoring the trinomial we have

$$x(x^2 - x - 6) = 0$$

$$x(x - 3)(x + 2) = 0$$

The linear factors correspond to three distinct zeros

$$x = 0, x = 3, \text{and } x = 2$$

2. Letting $y = f(x) = 0$, we have

$$x^4 + x^3 + x^2 = 0$$

Factoring out a common factor of x^2

$$x^2(x^2 + x + 1) = 0$$

Since the trinomial in not easily factorable into linear factors, we can set each factor to zero

$$x^2 = x \cdot x = 0 \text{ and } x^2 + x + 1 = 0$$

In the first equation, we have two factors of x that correspond to the zero $x = 0$. In the second equation, we must use the quadratic equation with $a = 1$, $b = 1$, and $c = 1$ since the equation is not easily factored into linear factors.

$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)}$$

$$x = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm i\sqrt{3}}{2} = \frac{-1}{2} \pm i\frac{\sqrt{3}}{2}$$

Notice in the first polynomials in the previous example, we had three distinct real zeros that corresponding to a single linear factor each. Whereas in the second polynomial, we had a repeated real zero $x = 0$ that corresponds to two of the same linear factor as well as two complex zeros. In general, we could have a distinct real zero(s), a repeated zero(s), or complex conjugate zeros with a larger polynomial function. With quadratic functions, we had either two real zeros or two complex zeros. We need some language to describe the number of time a zero of a polynomial is repeated. We call the number of linear factors $(x - k)$ that corresponding to a zero $x = k$ the multiplicity off the zero.

Definition: The Multiplicity of a Zero

A zero $x = k$ of a polynomial $y = f(x)$ has **multiplicity** n if there are n factors of the corresponding factor $(x - k)$ in the factorization of the polynomial.

Example 4.1.3

Let $f(x) = x^5 + 4x^4 + 4x^3$. Find the zeros and their multiplicity.

Solution

Letting $y = f(x) = 0$, we have

$$x^5 + 4x^4 + 4x^3 = 0$$

Factoring our a common factor of x^3, then factoring the trinomial we have

$$x^3(x^2 + 4x + 4) = 0$$

$$x^3(x + 2)(x + 2) = 0$$

These factors correspond to the zeros $x = 0$ with multiplicity three and $x = -2$ with multiplicity two.

Now you try: Exercise 4.1.1

Let $f(x) = x^6 - 6x^5 + 9x^4$. Find the zeros and their multiplicity.

Answer

$f(x)$ has zeros $x = 0$ with multiplicity four and $x = 3$ with multiplicity two.

Notice when finding the zeros and their multiplicities in each of the preceding examples and exercise, the polynomials had a common factor that when factored out of the polynomial, left a quadratic factor. When we set the quadratic factor to zero, we could solve with the quadratic formula or by factoring. With some larger polynomials, such as $f(x) = x^3 - 13x + 12$, there is no common factor. We can't use the quadratic formula since it is a cubic not a quadratic function. None of the techniques of solving polynomial equations from algebra help us find the zeros in this case. We will develop more tools in the next section to find the zeros of polynomials in this case. As with quadratic functions, real zeros are x-intercepts one the graph. However, we need to determine the effect of the multiplicity of a zero on the graph of a polynomial near that zero.

Building on what we did with quadratic functions, we can use the zeros of a polynomial and their multiplicities to find the formula for a polynomial function.

Example 4.1.4

Find a formula for a polynomial function with zeros $x = 1$ with multiplicity one and $x = -3$ with multiplicity two that passes through the point (0,4).

Solution

Since the polynomial has a zero $x = 1$ with multiplicity one, there is a single factor of $(x - 1)$. Since the polynomial has a zero $x = -3$ with multiplicity two, there are two factors of $(x + 3)$. However, the polynomial $y = f(x) = (x - 1)(x + 3)^2$ passes

through the point (0,-9) since $f(0) = -9$. As with quadratic functions, the polynomial can also have a constant stretch factor a. So

$$f(x) = a(x-1)(x+3)^2$$

To find a, we can substitute the point $x = 0$ and $y = 4$ into the polynomial

$$a(0-1)(0+3)^2 = 4$$

Then solving for a, we have

$$-9a = 4$$

$$a = \frac{-4}{9}$$

Therefore, the formula for the polynomial is $f(x) = \frac{-4}{9}(x-1)(x+3)^2$

For our polynomial in example 4, we could leave our function in factored form as $f(x) = \frac{-4}{9}(x-1)(x+3)^2$ or multiply the factors out as $f(x) = \frac{-4}{9}x^3 + \frac{20}{9}x^2 - \frac{-4}{3}x + 4$. Depending on what we want to do with the formula for the polynomial, either form is useful.

Example 4.1.5

Find a formula for a polynomial function with zeros $x = 0$ with multiplicity three and $x = 2i$ with multiplicity one that passes through the point (1,2).

Solution

Since the polynomial has a zero $x = 0$ with multiplicity three, the polynomial has three factors of x. Since the polynomial has the complex zero $x = 2i$ of multiplicity one, the complex conjugate $x = -2i$ must also be a zero since complex zeros of polynomials appear in conjugate pairs. These complex zeros correspond to the factors $(x - 2i)$ and $(x + 2i)$ in the polynomial. Therefore, the polynomial has the formula

$$f(x) = ax^3(x-2i)(x+2i)$$

Simplifying by multiplying out the complex factors, we have

$$f(x) = ax^3(x^2 - 2ix + 2ix - 4i^2)$$

$$f(x) = ax^3(x^2 + 4)$$

Substituting the given point $x = 1$ and $y = 2$ into the polynomial to find the stretch factor a then solving, we have

$$2 = a1^3(1^2 + 4)$$

$$2 = 5a$$

\[a=\dfrac{2}{5} \nonumber\. Therefore, the polynomial has a formula $f(x) = \frac{2}{5}x^3(x^2 + 4)$

Now You Try: Exercise 4.1.2

Find a formula for a polynomial function with zeros $x = 1$ with multiplicity one, $x = -2$ with multiplicity one, and $x = 3$ with multiplicity two that passes through the point (0,5).

Answer

$f(x) = \frac{-5}{18}(x-1)(x+2)(x-3)^2$

In example 4.1.2, the third degree polynomial $f(x) = x^3 - x^2 - 6x$ had three corresponding linear factors $f(x) = x(x-3)(x+2)$. Each of these three linear factors corresponded to three distinct zeros $x = 0$, $x = 3$, and $x = -2$ each with multiplicity one. In example 4.1.3, the fifth degree polynomial $f(x) = x^5 + 4x^4 + 4x^3$ had five corresponding linear factors $f(x) = x^3(x+2)^2$. The factor x was repeated three times and the factor $(x+2)$ was repeated two times. These corresponded to the zeros $x = 0$ with multiplicity three and the zero $x = -2$ with multiplicity two. Conversely, in example 4.1.3 we were given two distinct zeros of a polynomial: $x = 1$ with multiplicity one and $x = -3$ with multiplicity two. The zero $x = 1$ with multiplicity one corresponds to a single linear factor $(x-1)$ in the polynomial. The zero $x = -3$ with multiplicity two corresponds to two factors of the linear factor $(x+3)$ in the polynomial. So we could write the polynomial as $f(x) = a(x-1)(x+3)^2$. Multiplying the three linear factors, we have the third degree polynomial $f(x) = a(x^3 + 5x^2 + 3x - 9)$ where a is the constant stretch factor. If we have five zeros in a polynomial (some may be repeated zeros with multiplicity two or more), these correspond to five linear factors in the polynomial. When we multiply these linear factors, we have a polynomial of degree five. This demonstrates the relationship between the degree of the polynomial, the number of linear factors, and the number of distinct zeros.

Key Point 4.1.2: The Fundamental Theorem of Algebra

A polynomial of degree n will have at most n distinct real zeros.

Important Topics of this Section

- The Graph of a Polynomial Function
- The Zeros of a Polynomial Function
- The Multiplicity of a Zero
- Degree of a polynomial
- The Fundamental Theorem of Algebra

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4.1E: An Introduction to The Graphs of Polynomial Functions (Exercises)

Section 4.1 Exercises

In # 1- 5 find all the zeros of each polynomial (and state their multiplicity)

1. $f(x) = 2(x-2)(x+3)^3$

2. $f(x) = 4x^6 + 9x^4$

3. $f(x) = 8x^3 + 6x^2 - 27x$

4. $f(x) = 2x^4 + 5x^3 - 3x^2$

5. $f(x) = x^3 + x^2 + x$

6. Find a formula for a polynomial with the zeros $x = -3$, $x = 0$, $x = 1$, and $x = 2$ with multiplicity 1 and whose graph contains the point (3,4).

7. Find a formula for a polynomial with the zero $x = -2$ with multiplicity 2, the zero $x = 0$ with multiplicity 3, and whose graph contains the point (1,-3).

8. Find a formula for a polynomial with the zero $x = 4$ with multiplicity 3, the zero $x = -1$ with multiplicity 1, and whose graph contains the point (0,5).

9. Find a formula for a polynomial with real coefficients that has the zero $x = 3i$ with multiplicity 1, the zero $x = 2$ with multiplicity 2, and whose graph contains the point (1,5).

10. Find a formula for a polynomial with real coefficients that has the zero $x = 2 - i$ with multiplicity 1, the zero $x = 0$ with multiplicity 3, and whose graph contains the point (-2,4).

11. What is the degree of the polynomial $f(x) = 6 + 2x + 1x^5 + x^4$? How many possible distinct zeros can /(f/) have?

12. Determine if the statement is true or false. Justify your conclusion. "A polynomial of degree 7 may have 8 distinct zeros".

13. Determine if the statement is true or false. Justify your conclusion. "A polynomial of degree 7 may have 6 distinct zeros".

Answer

2. The zeros are $x = 0$ with multiplicity 4, $x = \frac{3}{2}$ with multiplicity 1, and $x = -\frac{3}{2}$ with multiplicity 1

7. $y = -\frac{1}{3}x^3(x+2)^2$

9. $y = \frac{1}{2}(x^2+9)(x-2)^2$

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4.2: More on Finding The Zeros of Polynomial Functions

Focus: Finding the Zeros of Larger Polynomial Functions Without a Common Factor

In the previous section, we were able to find the zeros of some polynomial functions which had a common factor like $f(x) = x^3 - x^2 - 6x$ and could be easily factored into linear and/or quadratic factors with tools from algebra. However, other polynomial functions like $g(x) = x^3 - 13x + 12$ do not have a common factor. $g(x)$ not a quadratic function, so we can't use the quadratic formula. In this section, we will explore strategies to find the zeros of larger polynomial functions that don't have a common factor where tools from we know from algebra don't work.

Let's start with a simpler case. What if one zero of a polynomial is known, can we find the others?

Case: One zero is known

Now $x = 1$ is a zero of $g(x) = x^3 - 13x + 12$ since $g(1) = (1)^3 - 13(1) + 12 = 0$. Since $x = 1$ is a zero, we know that $(x - 1)$ is a factor from section 3.3. We know one factor of $g(x)$ but not the other, so

$$g(x) = x^3 - 13x + 12 = (x - 1)(\text{other factor})$$

How can we find the other factor and thus partially factor $g(x)$?

Let's look a simpler version of same question with numbers. Now, 5 is a factor of 105. So

$$5 \cdot ? = 105$$

How can we find the other factor? Divide!

$$? = \frac{105}{5} = 21$$

So 105 factors as

$$105 = 5 \cdot 21$$

which we can further factor 105 as

$$105 = 5 \cdot 7 \cdot 3$$

We can do something similar to find the other factor of our polynomial $g(x) = x^3 - 13x + 12$. We can divide our polynomial by the known factor $(x - 1)$

$$\text{other factor} = \frac{x^3 - 13x + 12}{x - 1}$$

We can divide polynomials in a manner similar to the way we divided numbers with long division.

To divide the numbers $\frac{105}{5}$ with long division,we start by trying to divide the largest digit 1 in the dividend 105 by the divisor 5. Since 1 is smaller that the divisor 5, we included the next largest digit of the dividend and followed these steps

1. We divide the largest digits 10 by 5 $\frac{10}{5} = 2$
2. We multiply this number by the divisor $5 \cdot 2 = 10$
3. Then subtracted this number from the largest digits in the divisor $10 - 10 = 0$

$$\begin{array}{r} 2 \\ \hline 5\overline{)105} \\ -10 \\ \hline 5 \end{array}$$

We then brought down the 5, and repeated the process again with the remaining number, 5, and our divisor 5.

1. We divide the largest remaining digit 5 by the divisor 5, we have $\frac{5}{5} = 1$

2. We multiply this number by the divisor $5 \cdot 1 = 5$
3. Then subtracted this number from the largest digits in the divisor $10 - 10 = 0$

$$\begin{array}{r} 21 \\ \hline 5\overline{)105} \\ -10 \\ \hline 5 \\ -5 \\ \hline 0 \end{array}$$

Since 5 was a factor of 105, we know that 5 should divide into 105 evenly without a remainder. Now we are able to factor partially factor 105 as we did above.

We can use a similar algorithm to divide polynomials with a process called **long division of polynomials**. To divide the polynomials mentioned above to find other factor $\frac{x^3 - 13x + 12}{x - 1}$ where $x - 1$ is our divisor and $x^3 - 13x + 12$ is the dividend.

$$x-1\overline{)\ x^3+0x^2-13x+12}$$

1. Divide the largest term in the dividend x^3 by the largest term in the divisor x. So $\frac{x^3}{x} = x^2$. Put this term in the quotient above the long division symbol.
2. Multiply the term x^2 by the divisor $x - 1$. So $x^2(x - 1) = x^3 - x^2$
3. Then subtract this from the dividend. So $(x^3 - 13x + 12) - (x^3 - x^2) = x^2 - 13x + 12$

$$\begin{array}{r} x^2 \\ x-1\overline{)\ x^3+0x^2-13x+12} \\ \underline{-(x^3-\ x^2)} \\ x^2-13x+12 \end{array}$$

Then we can repeat this process with $x^2 - 13x + 12$. Notice I added a $0x^2$ term to help keep like terms in the same column to assist with the subtraction.

1. Divide the largest term in the result x^2 by the largest term in the divisor x. So $\frac{x^2}{x} = x$. Put this term in the quotient above the long division symbol.
2. Multiply the term x by the divisor $x - 1$. So $x(x - 1) = x^2 - x$
3. Then subtract this from $x^2 - 13x + 12$. So $(x^2 - 13x + 12) - (x^2 - x) = -12x + 12$

$$\begin{array}{r} x^2+x \\ x-1\overline{)\ x^3+0x^2-13x-12} \\ \underline{-(x^3-x^2)} \\ x^2-13x-12 \\ \underline{-(x^2-x)} \\ -12x+12 \end{array}$$

Then we can repeat this process with $-12x + 12$

1. Divide the largest term in the result $-12x$ by the largest term in the divisor x. So $\frac{-12x}{x} = -12$. Put this term in the quotient above the long division symbol.
2. Multiply the term -12 by the divisor $x - 1$. So $-12(x - 1) = -12x + 12$
3. Then subtract this from $-12x + 12$. So $(-12x + 12 - (-12x + 12) = 0$

$$
\begin{array}{r}
x^2+x\quad -12 \\
x-1\overline{)\quad x^3+0x^2-13x-12} \\
\underline{-(x^3-x^2)\qquad\qquad} \\
\qquad x^2-13x-12 \\
\qquad \underline{-(x^2-x)\qquad} \\
\qquad\qquad -12x+12 \\
\qquad\qquad \underline{-(-12x-12)} \\
\qquad\qquad\qquad 0
\end{array}
$$

Like with long division of numbers, we continue this algorithm until the result is smaller in degree that the divisor. Notice that we end up with a remainder of zero. Since we knew that $(x-1)$ was a factor of $x^3-13x+12$, we know that $(x-1)$ should divide into $x^3-13x+12$ evenly without a remainder.

Now we have partially factored $f(x)=x^3-13x+12$ and can factor it further to find the other zeros.

$$f(x)=x^3-13x+12=(x-1)(x^2+x-12)=(x-1)(x+4)(x-3)$$

Therefore $f(x)=x^3-13x+12$ has zeros $x=1$, $x=-4$, and $x=3$ each with multiplicity one.

Example 4.2.1

Let $f(x)=x^3+5x^2+5x+1$. Given that $x=-1$ is a zero of $f(x)$,

a. Use long division of polynomials to divide $f(x)=x^3+5x^2+5x+1$ by the corresponding factor $(x+1)$ of the known zero.
b. Factor $f(x)$ and find the remaining zeros.

Solution

a. Let's start by writing the problem out in long division form

$$x+1\overline{)\quad x^3+5x^2+5x+1}$$

Then begin the algorithm of long division. First, we divide the largest term in the dividend by the largest term in the divisor: $\frac{x^3}{x}=x^2$. I recommend we align it above the same-powered term in the dividend. Then multiply this by the divisor $x^2(x+1)=x^3+x^2$ and then subtract as shown below.

$$
\begin{array}{r}
x^2 \\
x+1\overline{)\quad x^3+5x^2+5x+1} \\
\underline{-(x^3+\ x^2)\qquad\qquad} \\
\qquad 4x^2+5x+1
\end{array}
$$

Then we repeat the algorithm for long division with the result $4x^2+5x+1$. First, we divide the largest term in this polynomial by the largest term in the divisor: $\frac{4x^2}{x}=4x$. Now, multiply this by the divisor $4x(x+1)=4x^2+4x$ and then subtract as shown below.

$$
\begin{array}{r}
x^2+4x \\
x+1\overline{\big)\ x^3+5x^2+5x+1} \\
\underline{-(x^3+\ x^2)} \\
4x^2+5x+1 \\
\underline{-(4x^2+4x)} \\
x+1
\end{array}
$$

Then we repeat the algorithm for long division with the result $x+1$. First, we divide the largest term in this polynomial by the largest term in the divisor: $\frac{x}{x}=1$. Then, multiply this by the divisor $1(x+1)=x+1$ and then subtract as shown below.

$$
\begin{array}{r}
x^2+4x+1 \\
x+1\overline{\big)\ x^3+5x^2+5x+1} \\
\underline{-(x^3+\ x^2)} \\
4x^2+5x+1 \\
\underline{-(4x^2+4x)} \\
x+1 \\
\underline{-(x+1)} \\
0
\end{array}
$$

This tells us x^3+5x^2+5x+1 divided by $x+1$ is x^2+4x+1 with a remainder of zero.

b. We can use our result from the long division to factor x^3+5x^2+5x+1 as

$$f(x)=x^3+5x^2+5x+1=(x+1)(x^2+4x+1)$$

Setting each factor to zero and solving, we have

$$x+1=0 \text{ and } x^2+4x+1=0$$

The first equation gives us the known zero

$$x=-1$$

Using the quadratic formula to solve the second equation since it is not easily factored into linear factors, the other zeros are

$$x=\frac{-4\pm\sqrt{4^2-4(1)(1)}}{2(1)}=\frac{-4\pm\sqrt{12}}{2}=\frac{-4\pm2\sqrt{3}}{2}=\frac{-4}{2}\pm\frac{\sqrt{3}}{2}=-2\pm\frac{\sqrt{3}}{2}$$

So the real zeros (and x-intercepts) are $x=-1$ and $x=-2\pm\frac{\sqrt{3}}{2}$ each with multiplicity one.

Example 4.2.2

Let $f(x)=x^3+4x^2-5x-14$. Given that $x=2$ is a zero of $f(x)$,

a. Use long division of polynomials to divide $f(x)=x^3+4x^2-5x-14$ by the corresponding factor $(x-2)$ of the known zero.
b. Factor $f(x)$ and find the remaining zeros.

Solution

a. Let's start by writing the problem out in long division form

$$x-2\overline{)x^3+4x^2-5x-14}$$

Then begin the algorithm of long division. First, we divide the largest term in the dividend by the largest term in the divisor: $\frac{x^3}{x}=x^2$. Then multiply this by the divisor $x^2(x-2)=x^3-2x^2$ and then subtract as shown below.

$$\begin{array}{r} x^2 \\ x-2\overline{)\ x^3+4x^2-5x-14} \\ \underline{-\left(x^3-2x^2\right)} \\ 6x^2-5x-14 \end{array}$$

Then we repeat the algorithm for long division with the result $6x^2-5x-14$. First, we divide the largest term in this polynomial by the largest term in the divisor: $\frac{6x^2}{x}=6x$. Then multiply this by the divisor $6x(x-2)=6x^2-12x$ and then subtract as shown below.

$$\begin{array}{r} x^2+6x \\ x-2\overline{)\ x^3+4x^2-5x-14} \\ \underline{-\left(x^3-2x^2\right)} \\ 6x^2-5x-14 \\ \underline{-\left(6x^2-12x\right)} \\ 7x-14 \end{array}$$

Then we repeat the algorithm for long division with the result $7x-14$. First, we divide the largest term in this polynomial by the largest term in the divisor: $\frac{7x}{x}=7x$. Then multiply this by the divisor $7(x-2)=7x-14$ and then subtract as shown below.

$$\begin{array}{r} x^2+6x+7 \\ x-2\overline{)\ x^3+4x^2-5x-14} \\ \underline{-\left(x^3-2x^2\right)} \\ 6x^2-5x-14 \\ \underline{-\left(6x^2-12x\right)} \\ 7x-14 \\ \underline{-(7x-14)} \\ 0 \end{array}$$

This tells us $x^3+4x^2-5x-14$ divided by $x-2$ is x^2+6x+7 with a remainder of zero.

b. We can our result from the long division to factor $x^3+4x^2-5x-14$ as

$$f(x)=x^3+4x^2-5x-14=(x-2)(x^2+6x+7)$$

Setting each factor to zero and solving, we have

$$x-2=0 \text{ and } x^2+6x+7=0$$

The first equation gives us the known zero

$$x=2$$

Using the quadratic formula to solve the second equation, the other zeros are

$$x = \frac{-6 \pm \sqrt{6^2 - 4(1)(7)}}{2(1)} = \frac{-6 \pm \sqrt{-16}}{2} = \frac{-6 \pm 4i}{2} = \frac{-6}{2} \pm \frac{4i}{2} = -3 \pm 2i$$

So the only real zero (and x-intercept) is $x = 2$ with multiplicity one.

Example 4.2.3

Given that $x = 3$ is a zero of $f(x) = 2x^3 - 3x^2 - 11x + 6$ find the other zeros.

Solution

Let's start by writing the problem out in long division form

$$x-3\overline{)\ 2x^3-3x^2-11x+6}$$

Then begin the algorithm of long division. First, we divide the largest term in the dividend by the largest term in the divisor: $\frac{2x^3}{x} = 2x^2$. Then multiply this by the divisor $2x^2(x-3) = 2x^3 - 6x^2$ and then subtract as shown below.

$$\begin{array}{r} 2x^2 \\ x-3\overline{)\ 2x^3-3x^2-11x+6} \\ \underline{-(2x^3-6x^2)} \\ 3x^2-11x+6 \end{array}$$

Then we repeat the algorithm for long division with the result $3x^2 - 11x + 6$. First, we divide the largest term in this polynomial by the largest term in the divisor: $\frac{3x^2}{x} = 3x$. Then multiply this by the divisor $3x(x-3) = 3x^2 - 9x$ and then subtract as shown below.

$$\begin{array}{r} 2x^2+3x-2 \\ x-3\overline{)\ 2x^3-3x^2-11x+6} \\ \underline{-(2x^3-6x^2)} \\ 3x^2-11x+6 \\ \underline{-(3x^2-9x)} \\ -2x+6 \end{array}$$

Then we repeat the algorithm for long division with the result $-2x + 6$. First, we divide the largest term in this polynomial by the largest term in the divisor: $\frac{-2x}{x} = -2$. Then multiply this by the divisor $-2(x-3) = -2x + 6$ and then subtract as shown below.

$$\begin{array}{r} 2x^2+3x-2 \\ x-3\overline{)\ 2x^3-3x^2-11x+6} \\ \underline{-(2x^3-6x^2)} \\ 3x^2-11x+6 \\ \underline{-(3x^2-9x)} \\ -2x+6 \\ \underline{-(2x+6)} \\ 0 \end{array}$$

This tells us $2x^3 - 3x^2 - 11x + 6$ divided by $x - 3$ is $2x^2 + 3x - 2$ with a remainder of zero.

We can our result from the long division to factor $f(x)$ as

$$2x^3 - 3x^2 - 11x + 6 = (x-3)(2x^2 + 3x - 2) = (x-3)(2x-1)(x+2)$$

Setting each factor to zero and solving, we have the real zeros (and x-intercepts) are $x = 3$, $x = -2$ and $x = \frac{1}{2}$ each with multiplicity one.

With numbers when we divide a number by a known factor, the remainder is always zero as we saw with 5 and 105 in the introduction of this section:

$$5 \cdot (?) = 105 \text{ and so } \frac{105}{5} = 21$$

with a remainder of zero. In Section 3.3 we learned that if $x = k$ is zero of a polynomial $f(x)$, then $(x - k)$ is a factor of $f(x)$. Therefore, the remainder is zero when $f(x)$ is divided by $(x - k)$ as illustrated by examples 4.2.1 - 4.2.3. This gives us a way to verify if our long division was correct when dividing by the corresponding factor of a zero.

Key Point 4.2.1: The Remainder Theorem Part 1

If a polynomial $y = f(x)$ has a zero $x = k$, then the remainder when $y = f(x)$ is divided by the corresponding factor $(x - k)$ is zero.

Now You Try: Exercise 4.2.1

a. Divide $f(x) = 5x^3 - 6x^2 - 28x + 6$ by $(x + 2)$ using long division.
b. What does the result of your long division tell you about whether $x = 2$ is a zero of $f(x)$?

Answer

a.

$$\begin{array}{r} 5x^2 - 16x + 4 \\ x+2 \overline{)\ 5x^3 - 6x^2 - 28x + 6} \\ \underline{-(5x^3 + 10x^2)} \\ -16x^2 - 28x + 6 \\ \underline{-(-16x^2 - 32x)} \\ 4x + 6 \\ \underline{-(4x + 8)} \\ -2 \end{array}$$

b. $x = 2$ is a not a zero of $f(x) = 5x^3 - 6x^2 - 28x + 6$ since $(x + 2)$ does not divide into it with a remainder of zero.

Now You Try: Exercise 4.2.2

Given that $x = 1$ is a zero of $f(x) = x^3 - 6x^2 + 11x - 6$, use long division of polynomials to find the other zeros.

Answer

$f(x) = x^3 - 6x^2 + 11x - 6$ has zeros $x = 1$, $x = 1$, and $x = 1$ each with multiplicity one since

$$\frac{x^3 - 6x^2 + 11x - 6}{x - 1} = x^2 - 5x + 6$$

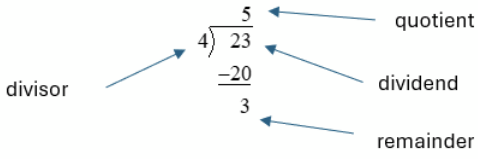
and so

$$x^3 - 6x^2 + 11x - 6 = (x-1)(x^2 - 5x + 6) = (x-1)(x-2)(x-3)$$

The long division is shown below.

$$
\begin{array}{r}
x^2-5x+6 \\
x-1\overline{)\ x^3-6x^2+11x-6} \\
\underline{-(x^3-x^2)} \\
-5x^2+11x-6 \\
\underline{-(-5x^2-5x)} \\
6x+6 \\
\underline{-(6x+6)} \\
0
\end{array}
$$

Exercise 4.3.1 brings up an important question. How should we write our result when the division of two polynomials has a non-zero remainder of zero? Let's look back at what we did when writing as imperfect fraction, such as $\frac{23}{4}$, as the mixed fraction, $5\frac{3}{4}$. When we divided 23 and 4 with long division, we have



This tells us that 4 goes into 23 five times with 3 parts out of 4 left over. So we write

$$\frac{23}{4} = 5\frac{3}{4}$$

In general, we can write the result of a division as

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

We can do the same thing with the two polynomials we divided in exercise 4.3.1. When we divided $f(x) = 5x^3 - 6x^2 - 28x + 6$ by $(x + 2)$, we had a quotient of $5x^2 - 16x + 4$ and remainder $r = -2$. We can write this division as a like a mixed fraction in a similar form

$$\frac{5x^3 - 6x^2 - 28x + 6}{x + 2} = \frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}} = (5x^2 - 16x + 4) + \frac{-2}{x + 2}$$

Although this is not essential in finding the zeros of larger polynomials, writing the division of two polynomials(which we call rational functions) in this way will help us accomplish some of our goals in Calculus with these types of functions.

Furthermore, in exercise 4.3.1 we could have tested whether or not $x = -2$ was a zero of the polynomial $f(x) = 5x^3 - 6x^2 - 28x + 6$ by evaluating the polynomial.

$$f(-2) = 5(-2)^3 - 6(-2)^2 - 28(-2) + 6 = -2$$

When we divided $f(x) = 5x^3 - 6x^2 - 28x + 6$ by $(x + 2)$ with long division of polynomials, there was a remainder of -2. Notice that the remainder was the same as the function value at $x = -2$. This is not a coincidence. When $x = k$ is not a zero of a polynomial, notice what happens when we divide a polynomial f(x) by $(x - k)$.

$$\frac{f(x)}{x - k} = \frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

Multiplying both sides of the equation by $(x - k)$, we have

$$f(x) = \text{dividend} = (\text{quotient})(\text{divisor}) + (\text{remainder}) = (\text{quotient})(x - k) + (\text{remainder})$$

Evaluating f at $x = k$, we have

$$f(k) = (\text{quotient})(k - k) + (\text{remainder}) = 0 + (\text{remainder}) = (\text{remainder})$$

Key Point 4.2.2: The Remainder Theorem Part 2

If $f(x)$ is a polynomial of degree 1 or greater and k is a real number, then when f(x) is divided by $x - k$, the remainder is $r = f(k)$.

Focus: Using Long Division of Polynomials with Complex Zeros

Notice that $x = i$ is a zero of $f(x) = x^3 - 2x^2 + x - 2$ since $f(i) = (i)^3 - 2(i)^2 + (i) - 2 = 0$. So $(x - i)$ is a factor of $f(x)$. At this point, we have a couple options. We could divide $f(x) = x^3 - 2x^2 + x - 2$ by the factor with complex coefficients $(x - i)$ to find the other factor. However, this would introduce complex numbers into our long division making the long division much more challenging. On the other hand, we do know that complex zeros occur in conjugate pairs. Since $x = i$ is a zero of $f(x)$, then so is its complex conjugate $x = -i$. Therefore, both $(x - i)$ and $(x + i)$ are factors of $f(x)$ and so is their product $(x - i)(x + i) = x^2 + 1$. This factor of $f(x)$ has only real coefficients and is much easier to use with long division. To find the other factor of $f(x) = x^3 - 2x^2 + x - 2$, we can divide by $x^2 + 1$ as shown in the long division below.

$$\begin{array}{r} x-2 \\ x^2+1\overline{)\ x^3-2x^2+x-2} \\ \underline{-(x^3\quad\ +x)} \\ -2x^2\ -2 \\ \underline{-(-2x^2\ -2)} \\ 0 \end{array}$$

Then we can use this result to factor $f(x)$ as

$$f(x) = x^3 - 2x^2 + x - 2 = (x^2 + 1)(x - 2)$$

These factors correspond to the zeros $x = i$, $x = -i$, and $x = 2$ each with multiplicity one.

Be Careful:

When looking online, in other textbooks, or prompting and an AI chatbot to divide polynomials, you may encounter a technique called synthetic division. Synthetic division is a technique to streamline polynomial division by a linear factor $(x - k)$. However, synthetic division can not be used to divide a quadratic or larger polynomial. Polynomial long division must be used in this case.

Focus: Finding Zeros of Larger Polynomials When No Zero is Known

With a polynomial function, initially we will not know a zero. In this case, how can we find a zero? Once we find one zero, we can use long division by the corresponding factor to factor the polynomial and find the other zeros as shown earlier in this section.

To help delevop a strategy to finding a zero of a polynomial when none is know, let's explore how we factor a quadratic function in algegra. When trying to find the zeros the quadratic polynomial $f(x) = 6x^2 - x - 35$, notice how we chose the terms of each factor as part of our factoring algorithm:

1. Choose two first terms whose product in the leading term $6x^2$. So the coefficients of the first terms are factors of 6.
2. Choose two last terms whose product is the constant -35.
3. Verify that the product yields the correct middle term.

Using this algorithm to identify the correct terms in each factor and factor the polynomial $f(x)$, we can find the zeros

$$f(x) = 6x^2 - x - 35 = 0$$

$$(3x + 7)(2x - 5) = 0$$

Then setting each factor to zero and solving to find the zeros.

$$3x + 7 = 0 \text{ and } 2x - 5 = 0$$

$$x = \frac{-7}{3} \text{ and } x = \frac{5}{2}$$

Notice each of these zeros are fractions where the numerator is a factor of the constant term -35 and the denominator is a factor of the leading coefficient 6. In other words, each zero is of the form

$$x = \frac{\text{a factor of -35}}{\text{a factor of 6}} = \frac{\text{a factor of the constant term}}{\text{a factor of the leading coefficient}}$$

If another polynomial $g(x)$ had a factor of $(5x-2)$ in addition to the factors of $f(x)$, then

$$g(x) = (5x-2)(3x+7)(2x-5) = 30x^3 - 12x^2 + 2x + 70$$

Notice that each of the first terms of each factor multiply to the leading term $30x^3$. Each of the last terms of each factor multiply to the constant term 70. As we set each factor to zero and solve to find the zeros of $g(x)$

$$5x - 2 = 0 \text{ , } 3x + 7 = 0, \text{ and } 2x - 5 = 0$$

$$x = \frac{2}{5}, x = \frac{-7}{3}, \text{ and } x = \frac{5}{2}$$

Once again, each of these zeros are fractions where the numerator is a factor of the constant term 70 and the denominator is a factor of the leading coefficient 30. In other words, each zero is of the form

$$x = \frac{\text{a factor of 70}}{\text{a factor of 30}} = \frac{\text{a factor of the constant term}}{\text{a factor of the leading coefficient}}$$

We can generalize what we have demonstrated here with the **Rational Zero Theorem**.

Key Point 4.2.3: The Rational Zero Theorem

For a polynomial with integer coefficients

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

if x is a rational zero of f, then x is of the form

$$x = \frac{p}{q} = \frac{\text{a factor of the constant term } a_0}{\text{a factor of the leading coefficient } a_n}$$

where p is a factor of the constant term a_0, and q is a factor of the leading coefficient a_n

The rational zero theorem gives us a list of *possible* integer and rational zeros such as $x = 2$ and $x = \frac{2}{3}$. However, the list may not contain all zeros of a polynomial such as irrational real zeros like $x = \sqrt{2}$ or complex zeros like $x = 2 + 3i$.

Example 4.2.4

Let $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$.

a. Use the Rational Roots Theorem to list all the possible rational zeros of $f(x)$.
b. Find one actual zero.

Solution

a. The possible rational zeros are of the form

$$x = \frac{\text{a factor of the constant term } -3}{\text{a factor of the leading coefficient } 2}$$

The factors of -3 are ± 1 and ± 3. The factors of 2 are ± 1 and ± 2. So the possible rational zeros are

$$x = \frac{\pm 1, \pm 3}{\pm 2, \pm 1}$$

So the Rational Zero Theorem gives the list

$$\left\{\pm\frac{1}{1},\pm\frac{1}{2},\pm\frac{3}{1},\pm\frac{3}{2}\right\} \text{ or } \left\{\pm1,\pm\frac{1}{2},\pm3,\pm\frac{3}{2}\right\}$$

b. Testing at $x=1$, $f(1)=2(1)^4+4(1)^3-(1)^2-6(1)-3=-4\neq 0$. So $x=1$ is not a zero. Testing at $x=3$, $f(3)=2(3)^4+4(3)^3-(3)^2-6(3)-3=240\neq 0$. So $x=3$ is not a zero. Testing at $x=-1$, $f(-1)=2(-1)^4+4(-1)^3-(-1)^2-6(-1)-3=0$. So $x=-1$ is a zero of $f(x)$.

This process could have taken a lot of time to find the actual zero if one there were many more factors of the leading coefficient or the constant term than in example 4.2.4. For this reason, once one rational zero is found, use long division to find the others instead of trying to use the Rational Zero Theorem to find a second zero.

Example 4.2.5

Let $f(x)=x^3-x^2+3x-10$

a. Use the Rational Zero Theorem to list all the possible rational zeros of $f(x)$.
b. Find one actual zero.
c. Use long division of polynomials to find the other zeros of $f(x)$.

Solution

a. The possible rational zeros are of the form

$$x=\frac{\text{factor of the constant term } -10}{\text{factor of the leading coefficient } 1}$$

The factors of -10 are ±1, ±2, ±5, and ±3. The factors of 1 are ±1. So the possible rational zeros are

$$x=\frac{\pm1,\pm3,\pm5,\pm10}{\pm1}$$

So the Rational Zero Theorem gives the list

$$\left\{\pm\frac{1}{1},\pm\frac{2}{1},\pm\frac{5}{1},\pm\frac{10}{1}\right\} \text{ or } \{\pm1,\pm2,\pm5,\pm10\}$$

b. Testing at $x=1$, $f(1)=(1)^3-(1)^2+3(1)-10=-7\neq 0$. So $x=1$ is not a zero. Testing at $x=-1$, $f(-1)=(-1)^3-(-1)^2+3(-1)-10=-15\neq 0$. So $x=-1$ is not a zero. Testing at $x=2$, $f(2)=(2)^3-(2)^2+3(2)-10=0$). So $x=2$ is a zero of $f(x)$.

c. Since $x=2$ is a zero of $f(x)$, $(x-2)$ is a factor of $f(x)$. We can use long division to find the other factor as shown below.

$$\begin{array}{r|l} & x^2+x+5 \\ \hline x-2 & x^3-x^2+3x-10 \\ & \underline{-(x^3-2x^2)} \\ & x^2+3x-10 \\ & \underline{-(x^2-2x)} \\ & 5x-10 \\ & \underline{-(5x-10)} \\ & 0 \end{array}$$

Therefore, we can factor $f(x)$ as

$$x^3-x^2+3x-10=(x-2)(x^2+x+5)$$

Since the quadratic factor is not easily factored into linear factors, we can set each factor to zero

$$x-2=0 \text{ and } x^2+x+5=0$$

The first equation gives us the known zero

$$x=2$$

Using the quadratic formula to solve the second equation, the other zeros are

$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(5)}}{2(1)} = \frac{-1 \pm \sqrt{-19}}{2} = \frac{-1 \pm i\sqrt{19}}{2} = \frac{-1}{2} \pm \frac{i\sqrt{19}}{2}$$

So the only real zero (and x-intercept) is $x = 2$ with multiplicity one.

Notice the complex zeros were not even in our list of possible zeros. This is one down side of the rational zero theorem. It will not give us irrational or complex zeros.

Example 4.2.6

Find the real zeros of $f(x) = 2x^3 - 5x^2 - x + 1$.

Solution

Since no zeros are known, we must use the Rational Zero Theorem to find one, if possible. The possible rational zeros are of the form

$$x = \frac{\text{factor of the constant term } 1}{\text{factor of the leading coefficient } 2} = \frac{\pm 1}{\pm 1, \pm 2}$$

So the possible rational zeros are

$$\left\{\pm\frac{1}{1}, \pm\frac{1}{2}\right\} \text{ ,or } \left\{\pm 1, \pm\frac{1}{2}\right\}$$

Testing at $x = 1$, $f(1) = 2(1)^3 - 5(1)^2 - (1) + 1 = -3 \neq 0$. So $x = 1$ is not a zero. Testing at $x = \frac{1}{2}$, $f(\frac{1}{2}) = 2(\frac{1}{2})^3 - 5(\frac{1}{2})^2 - (\frac{1}{2}) + 1 = -0.5 \neq 0$. So $x = \frac{1}{2}$ is not a zero. Testing at $x = \frac{-1}{2}$, $f(\frac{-1}{2}) = 2(\frac{-1}{2})^3 - 5(\frac{-1}{2})^2 - (\frac{-1}{2}) + 1 = 0$. So $x = \frac{-1}{2}$ is a zero of $f(x)$.

Then $(x + \frac{1}{2})$ is a factor of $f(x)$. We can use long division to find the other factor as shown below.

$$\begin{array}{r} 2x^2 - 6x + 2 \\ x+\frac{1}{2}\overline{)\ 2x^3 - 5x^2 - x + 1} \\ \underline{-(2x^3 + x^2)} \\ -6x^2 - x + 1 \\ \underline{-(-6x^2 - 3x)} \\ 2x + 1 \\ \underline{-(2x + 1)} \\ 0 \end{array}$$

Therefore, we can factor $f(x)$ as

$$x^3 - 5x^2 - x + 1 = \left(x + \frac{1}{2}\right)(2x^2 - 6x + 2)$$

Setting the second factor to zero to find the remaining zeros, then pulling out a common factor

$$2x^2 - 6x + 2 = 2(x^2 - 3x + 1) = 0$$

and use the quadratic equation to solve for the zeros

$$x = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(1)}}{2(1)} = \frac{-3 \pm \sqrt{5}}{2}$$

So the real zeros (and x-intercepts) are $x = \frac{-1}{2}$, $x \approx 2.618$, and $x \approx 0.382$ each with multiplicity one.

? Now You Try: Exercise 4.2.3

List all possible rational zeros of $f(x) = 4x^3 - 10x^2 + 4x + 3$.

Answer

The possible rational zeros are of the form

$$x = \frac{\text{factor of the constant term } 3}{\text{factor of the leading coefficient } 4}$$

So the possible rational zeros are

$$x = \frac{\pm 1, \pm 3}{\pm 1, \pm 2, \pm 4}$$

$$\left\{\pm\frac{1}{1}, \pm\frac{3}{1} =, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{1}{4}, \pm\frac{3}{4}\right\} \text{ or } \left\{\pm 1, \pm 3, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{1}{4}, \pm\frac{3}{4}\right\}$$

? Now You Try: Exercise 4.2.4

Let $f(x) = x^3 + 4x^2 + 4x + 3$

a. Use the Rational Roots Theorem to list all the possible rational zeros of $f(x)$.
b. Find one actual zero.
c. Use long division of polynomials to find the other zeros.

Answer

a. The possible rational zeros are he possible rational zeros are

$$\left\{\pm\frac{1}{1}, \pm\frac{3}{1}\right\} \text{ or } \{\pm 1, \pm 3\}$$

b. Testing at $x = -3$, $f(-3) = (-2)^3 + 4(-3)^2 + 4(-3) + 3 = 0$). So $x = -3$ is a zero of $f(x)$.
c. Since $x = -3$ is a zero of $f(x)$, $(x + 3)$ is a factor of $f(x)$. Using long division to find the other factor, we can factor $f(x)$ as

$$x^3 + 4x^2 + 4x + 3 = (x + 3)(x^2 + x + 1)$$

Using the quadratic formula to find the zeros of the second factor, the other zeros are

$$x = \frac{-1}{2} \pm \frac{i\sqrt{3}}{2}$$

All of the zeros have multiplicity one.

All of the polynomials we have worked with so far have had degree three. Let's explore the case where we want to find the zeros of a larger polynomial like $f(x) = x^4 + x^3 - 3x^2 - 5x - 2$. We would use the rational zero theorem to identify one zero such as $x = 2$, the use long division by the corresponding factor $(x - 2)$ to find the other factor. This would allow us to factor $f(x)$ as

$$f(x) = x^4 + x^3 - 3x^2 - 5x - 2 = (x - 2)(x^3 + 3x^2 + 3x + 1)$$

Notice, the second factor is a cubic polynomial. We need to use the Rational Zero Theorem and long division a second time to factor this factor of $f(x)$ further. We would find a zero such as $x = -1$ and divide by the corresponding factor $(x + 1)$ to factor

$$f(x) = (x - 2)(x^3 + 3x^2 + 3x + 1) = (x - 2)(x + 1)(x^2 + 2x + 1)$$

Then we can factor the quadratic polynomial to fully factor $f(x)$ and find the zeros.

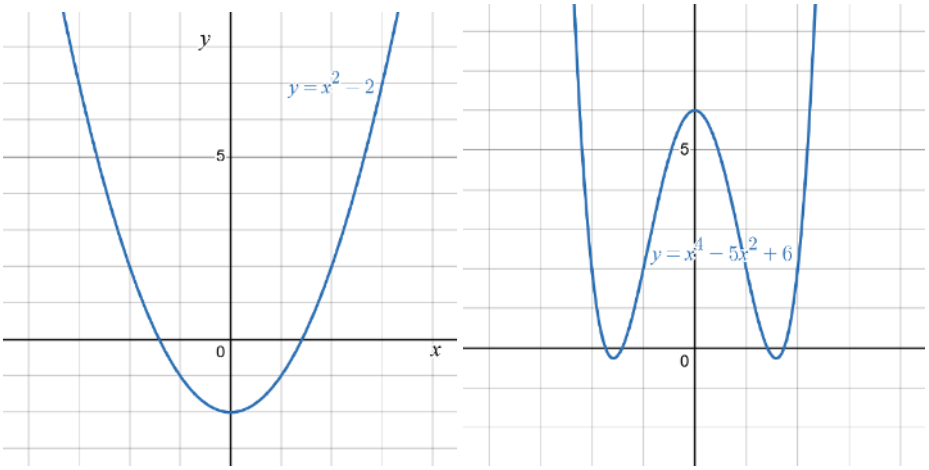
$$f(x) = (x - 2)(x + 1)^3$$

The zeros are $x = 2$ with multiplicity one and $x = -1$ with multiplicity three.

Focus: Approximating Zeros

Although the rational zero theorem is a powerful tool, it allows us to find only the rational zeros that can be written as a fraction of two integers. Some larger polynomials may not have any rational zeros, such as $f(x) = x^4 - 5x^2 + 6 = (x^2 - 2)(x^2 - 3)$ with irrational zeros $x = \pm\sqrt{2}$ and $x = \pm\sqrt{3}$. In this case, the Rational Zero Theorem doesn't help us identify a zero or factor the polynomial. We must turn to approximation methods to approximate the zeros. We will explore the use on one approximation method, The Midpoint Algorithm.

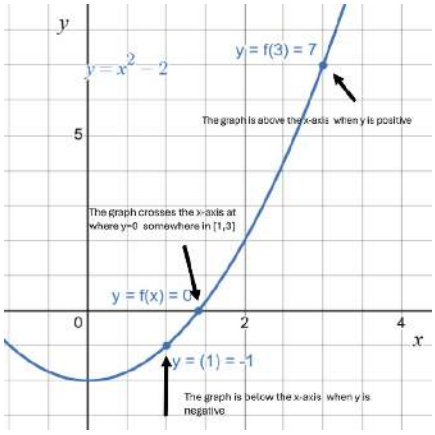
We can utilize an important graphical property of polynomials to approximate a zero. Notice, the graph of a polynomial function, such as $f(x) = x^2 - 2$ or $f(x) = x^4 - 5x^2 + 6$ can drawn with out lifting your pencil for all values of x. We call a function whose graph can be drawn with out lifting your pencil a **continuous function**.



Definition: Continuous Function (Graphical Definition)

A function $y = f(x)$ is continuous at $x = a$ if its graph can be drawn without lifting your pencil near $x = a$.

Consider the polynomial $f(x) = x^2 - 2$. Although we can use techniques from algebra to solve the equation $f(x) = x^2 - 2 = 0$ by isolation to find the irrational zeros $x = \pm\sqrt{2} \approx 1.414$, let's explore how we can approximate the location of one of these zeros of this polynomial. Notice, the polynomial goes from a negative value $y = f(1) = -1$ at $x = 1$ to a positive value $y = f(3) = 7$ at $x = 3$. Since the graph of f is continuous is must cross the x-axis somewhere in the interval $[1, 3]$



We can use this idea to get an even closer approximation of the actual zero using the Midpoint Algorithm on $f(x) = x^2 - 2$ on the interval $[1, 3]$.

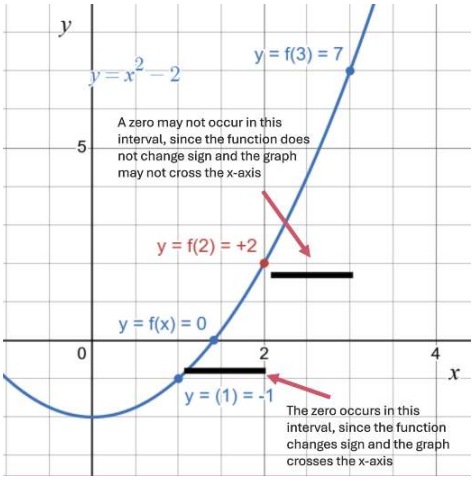
1. First, identify the midpoint of the interval:

$$x_1 = \frac{1+3}{2} = 2$$

2. Evaluate the function at the midpoint:

$$f(x_1) = f(2)^2 - 2 = 2$$

3. The midpoint breaks the interval into two subintervals [1,2] and [2,3]. Compare the function value at each endpoint on each subinterval. A zero lies in the subinterval that changes sign. On the interval [1,2], the function goes from a negative value $y = f(1) = -1$ at $x = 1$ to a positive value $y = f(2) = 7$ at $x = 2$. So the zero must occur in this interval. On the interval [2,3], the function goes from a positive value $y = f(2) = 2$ at $x = 2$ to a positive value $y = f(2) = 7$ at $x = 2$ as shown in the graph below. So a zero may not occur in this interval.



If we repeat the midpoint algorithm with this smaller subinterval [1,2], we will get an even closer approximation than the previous midpoint x_1 as shown in the graph below (left).

1. Identify the midpoint of the interval:

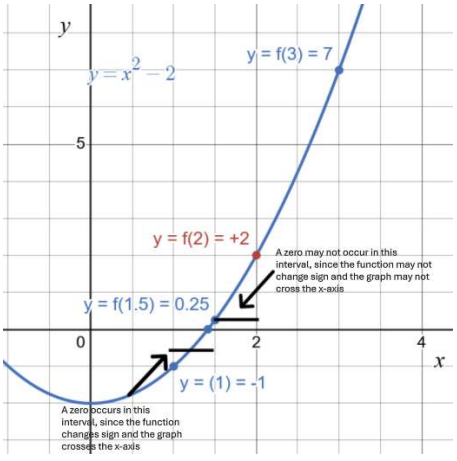
$$x_2 = \frac{1+2}{2} = 1.5$$

2. Evaluate the function at the midpoint:

$$f(x_2) = f(1.5) = (1.5)^2 - 2 = 0.25$$

This is not a zero, but is much closer to out actual zero $x = \sqrt{2} \approx 1.414$

3. The midpoint breaks the interval into two subintervals [1,1.5] and [1.5,2]. Comparing the function value at each endpoint on each subinterval, we have $y = f(1) = -1$, $y = f(1.5) = 0.25$, and $y = f(2) = 7$. So a zero occurs in the interval [1,1.5].



Repeating the midpoint algorithm a third time on [1,1.5]

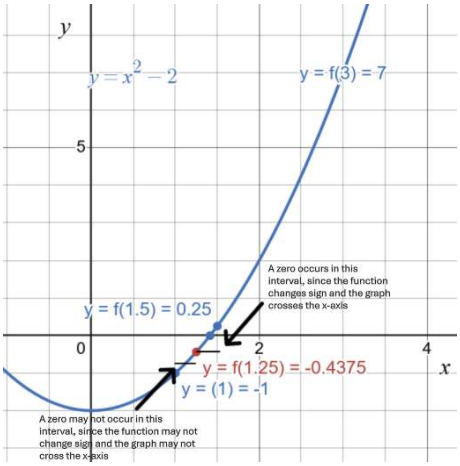
1. Identify the midpoint of the interval:

$$x_3 = \frac{1 + 1.5}{2} = 1.25$$

2. Evaluate the function at the midpoint:

$$f(x_3) = f(1.25) = (1.25)^2 - 2 = -0.4375$$

3. The midpoint breaks the interval into two subintervals [1,1.25] and [1.25,1.5]. Comparing the function value at each endpoint on each subinterval, we have $y = f(1) = -1$, $y = f(1.25) = -0.4375$, and $y = f(1.25) = -0.4375$. So a zero must occur in the interval [1.25,1.5].



Repeating this algorithm on successive smaller subintervals, will give us a better and better approximation of the actual zero as shown in the table below.

Interations of The Midpoint Approximation Method of Finding Zeros

The interval [a,b]	[1.25,1.5]	[1.3751, 1.5]	[1.3751, 1.4375]	[1.4063,1.4375]
The midpoint x_{mid}	$\frac{1.25 + 1.5}{2} \approx 1.3751$	$\frac{1.3751 + 1.5}{2} \approx 1.4375$	$\frac{1.3751 + 1.4375}{2} \approx 1.4063$	$\frac{1.4063 + 1.4375}{2} \approx 1.4219$
The value at the midpoint $y = f(x_{mid})$	$f(1.3751) \approx -0.4375$	$f(1.4375) \approx 0.0664$	$f(1.4063) \approx -0.02246$	$f(1.4219) \approx 0.0217$
Compare the values at each endpoint	$f(1.25) = -0.4375$, $f(1.3751) = -0.1094$ and $f(1.5) = 0.25$	$f(1.3751) = -0.1094$, $f(1.4063) \approx 0.0664$, and $f(1.5) = 0.25$	$f(1.3751) = -0.1094$, $f(1.4063) \approx -0.02246$, and $f(1.4375) \approx 0.0664$	$f(1.4063) \approx -0.02246$, $f(1.4219) \approx 0.0217$, and $f(1.4375) \approx 0.0664$

After our last iteration, the approximated zero $x \approx 1.4219$ is within 0.01 of the actual zero $x = \sqrt{2} \approx 1.4142$. We could continue with this algorithm on the interval [1.4063,1.4219] if we wanted an even better approximation of the zero. This kind of algorithm can be easily utilized with technology and computer programming to implement this algorithm many times in seconds or fractions of a second to get a very accurate approximation.

Strategy for Approximating Zeros: The Midpoint Algorithm

To approximate the zero of a polynomial on an interval [a,b] where the function value differs in sign at the endpoints

1. Identify the midpoint of the interval:

$$x = \frac{a + b}{2}$$

2. Evaluate the function at the midpoint:
3. Compare the function value at each endpoint on each subinterval.

4. Repeat the algorithm on the subinterval where the function value changes sign to find a more accurate approximation of the zero..

Example 4.2.7

Approximate a real zero of the polynomial of $f(x) = 8x^3 + 2x^2 - 1$ on the interval [-1,1] using five iterations of The Midpoint Algorithm.

Solution

Iterations of The Midpoint Algorithm for Finding Zeros.

The interval [a,b]	[-1,1]	[0, 1]	[0, 0.5]	[0.25,0.5]	[0.375,0.5]
The midpoint x_{mid}	$\frac{-1+1}{2} = 0$	$\frac{0+1}{2} \approx 0.5$	$\frac{0+0.5}{2} \approx 0.25$	$\frac{0.25+0.5}{2} = 0.375$	$\frac{0.375+0.5}{2} = 0.1875$
The value at the midpoint $y = f(x_{mid})$	$f(0) = -1$	$f(0.5) = 0.4$	$f(0.25) = -0.75$	$f(0.375) \approx -0.000000042$	$f(0.1875) \approx -0.877$
Compare the values at each endpoint	$f(-1) = -7$, $f(0) = -1$, and $f(1) = 9$	$f(0) = -1$, $f(0.5) = .5$, and $f(1) = 9$	$f(0) = -1$, $f(0.25) = -0.75$, and $f(0.5) = .5$	$f(0.25) = -0.75$, $f(0.375) \approx -0.000000042$, and $f(0.5) = .5$	$f(0.375) \approx -0.000000042$, $f(0.1875) \approx -0.877$, and $f(0.5) = .5$

Although the last iteration of The Midpoint Algorithm gives the approximation of the zero $x \approx 0.1875$, notice that $x = 0.375$ is a better approximation of the zero since $f(0.375) \approx -0.000000042$

Now You Try: Exercise 4.2.5

Approximate a real zero of the polynomial of $f(x) = x^5 - x^4 - 1$ on the interval [1,2] using five iterations of The Midpoint Algorithm.

Although the last iteration of The Midpoint Algorithm below gives the approximation of the zero $x \approx 1.3438$, notice that $x \approx 1.3125$ is a better approximation of the zero since $f(1.3438) \approx 0.1208$.

Answer

Iterations of The Midpoint Algorithm for Finding Zeros of Polynomials.

The interval [a,b]	[1,2]	[1, 1.5]	[1.25, 1.5]	[1.25,1.375]	[1.3125,1.375]
The midpoint x_{mid}	1.5	1.25	1.375	1.3125	1.3438
The value at the midpoint $y = f(x_{mid})$	$f(1.5) \approx 1.53$	$f(1.25) \approx -0.3897$	$f(1.375) \approx 0.3404$	$f(1.3125) \approx -0.0726$	$f(1.3438) \approx 0.1208$
Compare the values at each endpoint	$f(1) = -1$, $f(1.5) \approx 1.53$, and $f(2) = 15$	$f(1) = -1$, $f(1.25) \approx -0.3897$, and $f(1.5) \approx 1.53$	$f(1.25) \approx -0.3897$, $f(1.375) \approx 0.3404$, and $f(1.5) \approx 1.53$	$f(1.25) \approx -0.3897$, $f(1.3125) \approx -0.0726$, and $f(1.375) \approx 0.3404$	$f(1.3125) \approx -0.0726$, $f(1.3438) \approx 0.1208$, and $f(1.375) \approx 0.3404$

Important Topics of this Section

- Finding Zeros of Polynomials
- Long Division of Polynomials
- The Rational Zero Theorem.
- Approximating Zeros of Polynomials

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4.2E: More on Finding The Zeros of Polynomial Functions (Exercises)

Section 4.2 Exercises

In exercises #1-7, use polynomial long division to perform the indicated division. If a non-zero remainder exists, write the result of your division as a mixed fraction.

1. $(4x^2+3x-1) \div (x-3)$

2. $(3x^2-13x-10) \div (x-5)$

3. $(8x^3+27) \div (2x+3)$

4. $(5x^4-3x^3+2x^2-1) \div (x^2+4)$

5. $(9x^3+5) \div (2x-3)$

6. $(x^4-9) \div (x^2+3)$

7. $(-2x^5+3x^4-2x^2) \div (x^2+3)$

In exercises #8-14, given a zero(s) of the polynomial, find the other zeros.

8. $x^3+6x^2+11x+6, \quad x=-1$

9. $x^3-24x^2+192x-512, \; x=8$

10. $3x^3+4x^2-x-2, \quad x=-1$

11. $x^3+2x^2-3x-6, \quad x=-2$

12. $2x^3-3x^2-11x+6, \quad x=\frac{1}{2}$

13. $3x^3+7x^2-4, \quad x=\frac{2}{3}$

14. $x^4-2x^3-11x^2-8x+4, \quad x=5, x=-3$

In exercises 15-16, list all of the possible zeros using the rational zero theorem. Then find one zero.

15. $f(x)=x^4+3x^2-8x+4$

16. $f(x)=3x^5+2x^4-5x^3+x$

In exercises #17-22, use the rational zero theorem and long division of polynomials to find all the zeros.

17. $f(x)=x^3-6x^2+11x-6$

18. $f(x)=x^3-3x-2$

19. $f(x)=x^3-5x^2-4x+20$

20. $f(x)=6x^3+17x^2+x-10$

21. $f(x)=x^4+2x^3-9x^2-2x+8$

22. $f(x)=x^4+2x^3-4x^2-2x+3$

In exercises #23-24, use the given complex zero of the polynomial to find the other zeros.

23. $f(x)=x^4-2x^3-11x^2-8x-60, \quad x=-2i$

24. $f(x)=x^4-x^3+7x^2-9x-18, \quad x=3i$

In exercises #25-28, use the rational zero theorem and long division of polynomials to find all the zeros.

25. $f(x)=x^3-x^2+4x-4$

26. $f(x)=x^3-3x^2+5x-15$

27. $f(x)=x^3+x^2-2$

28. $f(x) = x^4 - 2x^3 - 2x^2 - 2x - 3$

29. Explain the mistake(s) that was(were) made in the following division

$$
\begin{array}{r|l}
x^2+2 & x^3+5x^2-3x+1 \\
\cline{2-2}
 & -x^3 \quad +2x \\
\cline{2-2}
 & 5x^2 - x+1 \\
 & -5x^2 \quad +11 \\
\cline{2-2}
 & -x+11
\end{array}
$$

30. Explain the mistake(s) that was(were) made.

Given that $x = 2$ is a zero of $f(x) = x^3 + 2x^2 - 7x - 2$, find the other zeros.

Solution:

Step 1: Since $x = 2$ is a zero, then $x = -2$ is also a zero

Step 2: Since $x = 2$ and $x = -2$ are zeros, $f(x)$ has factors $(x - 2)$ and $(x + 2)$

Step 3: Therefore $(x - 2)(x + 2) = x^2 - 4$ is also a factor.

Step 4: To find the other factor, we divide $f(x)$ by $x^2 - 4$

$$\frac{x^3 + 2x^2 - 7x - 2}{x^2 - 4} = x + 2 + \frac{-3x + 6}{x^2 - 4}$$

This has a non-zero remainder. The division should have a remainder of zero if $x = 2$ is a zero. What went wrong?

31. Approximate a real zero of the polynomial of $f(x) = x^3 + 3x^2 - 5$ on the interval [0,2] using seven iterations of The Midpoint Algorithm.

32. Approximate a real zero of the polynomial of $f(x) = x^4 - 2x^2 + x - 5$ on the interval [1,3] using seven iterations of The Midpoint Algorithm.

33. Approximate a real zero of the polynomial of $f(x) = 7x^3 - x^2 + 3$ on the interval [-1,0] using seven iterations of The Midpoint Algorithm.

Answer

1. $\frac{4x^2 + 3x - 1}{x - 3} = 4x + 15 + \frac{44}{x - 3}$
5. $\frac{x^4 - 9}{x^2 + 3} = x^2 - 3$
8. The zeros are $x = -1$, $x = -2$, and $x = -3$ all with multiplicity one.
13. The zeros are $x = \frac{2}{3}$, $x = -1$, and $x = -2$ all with multiplicity one.
17. The zeros are $x = 1$, $x = 2$, and $x = 3$ all with multiplicity one.
21. The zeros are $x = -4$, $x = -1$, $x = 1$,and $x = 2$ all with multiplicity one.
23. The zeros are $x = 2i$, $x = -2i$, $x = 5$,and $x = -3$ all with multiplicity one.
25. The zeros are $x = 2i$, $x = -2i$, and $x = 1$ all with multiplicity one.

31.

The interval [a,b]	[0,2]	[1, 2]	[1, 1.5]	[1,1.25]	[1,1.125]	[1.0625,1.125]	[1.09375,1.125]
The midpoint x_{mid}	1	1.5	1.25	1.125	1.0625	1.09375	1.109375
The value at the midpoint $y = f(x_{mid})$	-1	5.125	≈ 1.641	≈ 0.221	≈ -0.414	≈ -0.10269	≈ 0.057461
Compare the values at each endpoint	$f(0) = -5$, $f(1) = -1$, and $f(2) = 15$	$f(1) = -1$, $f(1.5) = 5.125$, and $f(2) = 15$	$f(1) = -1$, $f(1.25) \approx 1.641$, and $f(1.5) = 5.125$	$f(1) = -1$, $f(1.125) \approx 0.221$, and $f(1.25) \approx 1.641$	$f(1) = -1$, $f(1.0625) \approx -0.414$, and $f(1.125) \approx 0.221$	$f(1.0625) \approx -0.414$, $f(1.09375) \approx -0.10269$, and $f(1.125) \approx 0.221$	

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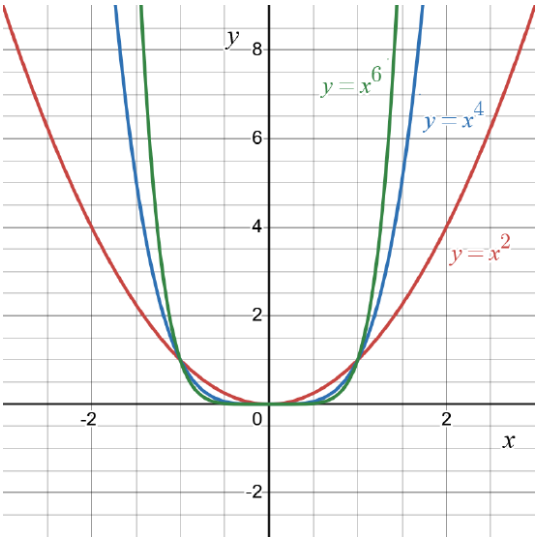
4.3: Graphing Polynomial Functions

In this section, our goal is to graph polynomial functions. As we did with quadratic functions in sections 3.2 and 3.3, we will use the key features of a polynomial to graph it. We learned in sections 4.1 and 4.2 how to find the zeros of a polynomial function. We will also need to determine the end behavior of a larger polynomial in order to graph the polynomial. As well, we need to understand the effect of the multiplicity of real zero on the behavior of the graph near that zero.

Focus: Determining The End Behavior of a Larger Polynomial Functions

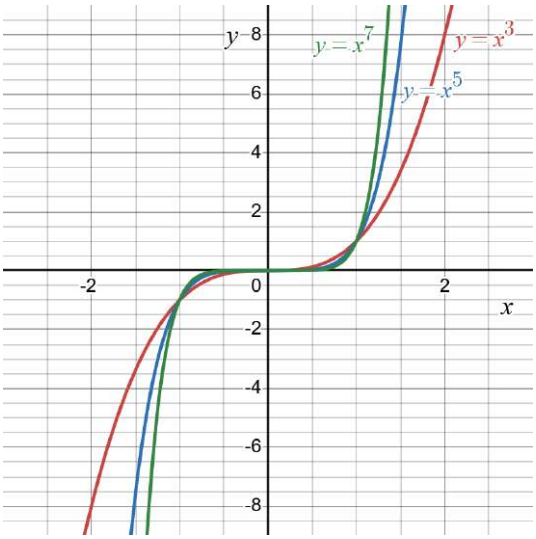
Case: The End Behavior of Powers of x, $y = x^n$

When we graph the even powers $y = x^2$, $y = x^4$, and $y = x^6$ on the same axis below, notice that although the graphs flatten somewhat near the origin, and become steeper away from the origin as the power increases, they all have the same end behavior. All the graphs open up on both ends, just like the parabola $y = x^2$. As we highlighted with quadratic functions in Chapter 3, when we say the graph opens up on the right end, we mean as $x \to \infty$, $y \to \infty$. When we say the graph opens up on the left end, we mean as $x \to -\infty$, $y \to \infty$.



If you graph larger even powers, such as $y = x^8$ and $y = x^{10}$, you will find the same end behavior as $y = x^2$. By extension, we can conclude all even powers of x will have the same end behavior as the parabola $y = x^2$, opening up on both ends (as $x \to \pm\infty$, $y \to \infty$).

When we graph the odd powers $y = x^3$, $y = x^5$, and $y = x^7$ on the same axis below, notice that they all have the same end behavior. The graphs open up on the right end (as $x \to \infty$, $y \to \infty$) and down on the left end (as $x \to -\infty$, $y \to -\infty$)



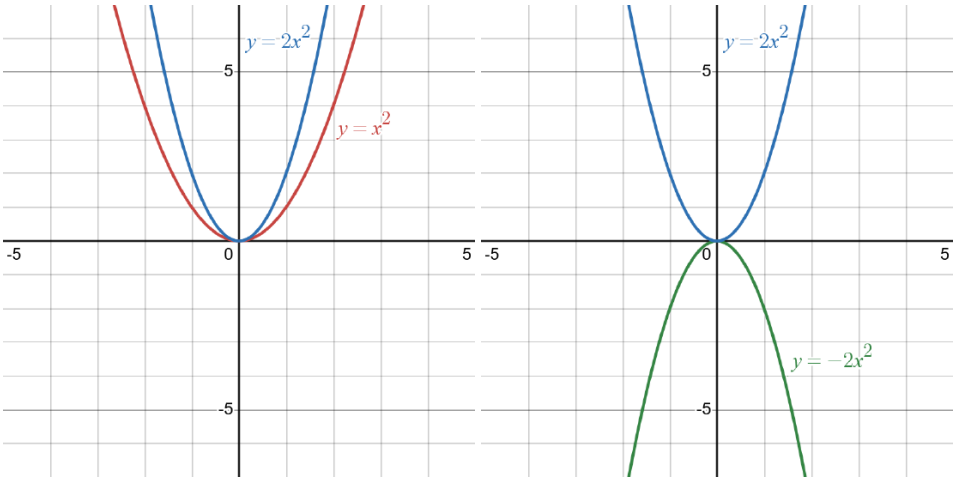
If you graph larger odd powers, such as $y = x^9$ and $y = x^{11}$, you will find the same end behavior as $y = x^3$. By extension, we can conclude all odd powers of x will have the same end behavior as the cubic $y = x^3$, opening up on the right end (as $x \to \infty$, $y \to \infty$) and down on the left end (as $x \to -\infty$, $y \to -\infty$).

Key Point 4.3.1: The End Behavior of Powers of x

For the power functions $y = x^n$, all even powers of x will have the same end behavior as the parabola $y = x^2$, opening up on both ends (as $x \to \pm\infty$, $y \to \infty$). All odd powers of x will have the same end behavior as the cubic $y = x^3$, open up on the right end (as $x \to \infty$, $y \to \infty$) and down on the left end (as $x \to -\infty$, $y \to -\infty$).

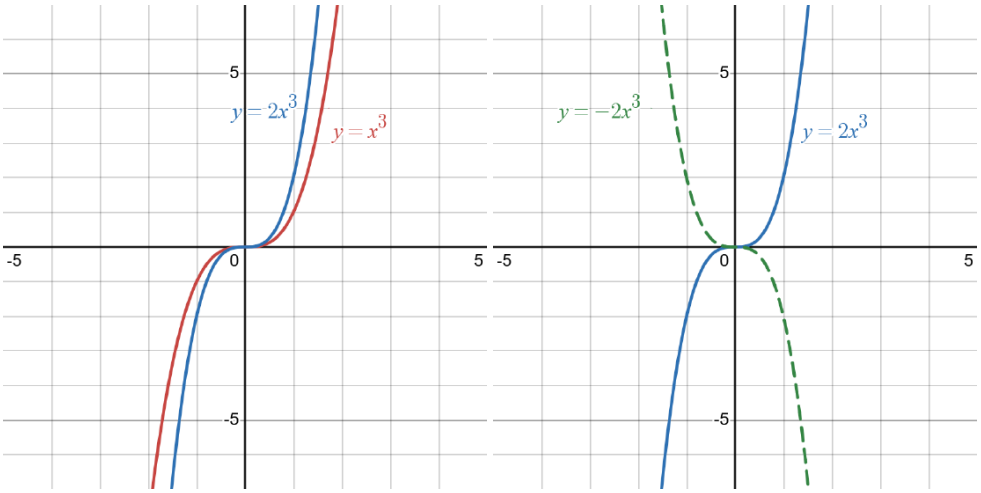
Case: The End Behavior of Power Functions of the Form $y = ax^n$

What is the effect of the stretch factor a on the end behavior? Consider the graphs of $y = x^2$, $y = 2x^2$, and $y = -2x^2$ below with stretch factors $a = 1$, $a = 2$, and $a = -2$ respectively.



Notice with the graph of $y = 2x^2$ in the graph on the left, the stretch factor $a = 2$ causes a vertical stretch by a factor of two on the graph of $y = x^2$. The vertical stretch does not change the end behavior of the graph. The graph opens up on both ends (as $x \to \pm\infty$, $y \to \infty$). Notice with the graph of $y = -2x^2$ on the right with a stretch factor $a = -2$, there is a reflection about the x-axis to the graph of $y = 2x^2$ that causes a change in end behavior. The graph opens of $y = -2x^2$ opens down on both ends (as $x \to \pm\infty$, $y \to -\infty$).

Consider the graphs of $y = x^3$, $y = 2x^3$, and $y = -2x^3$ below with stretch factors $a = 1$, $a = 2$, and $a = -2$ respectively.



Notice with the graph of $y = 2x^3$ in the graph on the left, the stretch factor $a = 2$ causes a vertical stretch by a factor of two on the graph of $y = x^3$. The vertical stretch does not change the end behavior. The graphs both open up on the right end (as $x \to \infty$, $y \to \infty$) and down on the left end (as $x \to -\infty$, $y \to -\infty$). Notice with the graph of $y = -2x^3$ on the right with a stretch factor $a = -2$, there is a reflection about the x-axis to the graph of $y = 2x^3$ that causes a change in end behavior. The graph opens of $y = -2x^3$ opens down on the right end (as $x \to \infty$, $y \to -\infty$) and up on the left end (as $x \to \infty$, $y \to -\infty$). Since all even powers have the same end behavior as $y = x^2$ and all odd powers have the same end behavrion as $y = x^3$, we can conclude that the stretch factor a will only change the end behavior if it causes a reflection about the x-axis.

Key Point 4.3.2: The Effect of The Stretch Factor on The End Behavior

The end behavior of the function $y = ax^n$ is the same end behavior as $y = x^n$ if the stretch factor $a > 0$. The end behavior of the function $y = ax^n$ is the same as the end behavior of a reflection of $y = x^n$ if the stretch factor $a < 0$.

Example 4.3.1

Determine the end behavior of the functions below.

a. $f(x) = 3x^4$
b. $g(x) = -5x^7$

Solution

a. Since the stretch factor of $f(x) = 3x^4$ is positive ($a = 3 > 0$), there is no reflection about the x-axis that will change the end behavior. Therefore, $f(x) = 3x^4$ and $y = x^4$ have the same end behavior. Since the power $n = 4$ is even, $f(x)$ will have the same end behavior as the parabola $y = x^2$ which opens up on both ends (as $x \to \pm\infty$, $y \to \infty$).
b. Since the stretch factor of $g(x) = -5x^7$ is negative $a = -5 < 0$, there is a reflection about the x-axis that will change the end behavior of $y = x^7$. Since the power $n = 7$ is odd, $f(x)$ will have the same end behavior as a reflection of $y = x^3$ which opens down on the right end (as $x \to \infty$, $y \to -\infty$) and up on the left end (as $x \to -\infty$, $y \to +\infty$)

Now You Try: Exercise 4.3.1

Determine the end behavior of the function of $f(x) = -2.3x^6$

Answer

The end behavior of $f(x)$ is the same as a reflection of the parabola $y = x^2$ which opens down on both ends (as $x \to \pm\infty$, $y \to -\infty$).

Case: A Polynomial with Multiple Terms

Polynomials, such as $y = x^3 + x^2$, can have multiple terms. What is the effect of each term on the end behavior? For example, on the left end (as $x \to -\infty$) the cubic term x^3 approaches $-\infty$ but the quadratic term x^2 approaches ∞. What does there sum $y = x^3 + x^2$ approach on the left end? 0? ∞? $-\infty$? 5?

To understand the effect of each term one the end behavior, let's explore the right end behavior of each term of $y = x^3 + x^2$ as well as the whole polynomial itself in the following table as x approaches ∞ .

Understanding the Effect of Each Term of a Polynomial on the End Behavior.

x	x^2	x^3	$x^3 + x^2$	The percent of x^3 as a portion of the total value of the polynomial $x^3 + x^2$
1	1	1	2	$\frac{1}{2} = 0.5$ = 50%
10	100	1,000	1,100	$\frac{1,000}{1,100} \approx 0.9091$ = 90.91%
100	10,000	1,000,000	1,010,000	$\frac{1,000,000}{1,010,000} \approx 0.990099$ = 99.0099%
1000	1,000,000	1,000,000,000	1,001,000,000	$\frac{1,000,000,000}{1,001,000,000} \approx 0.999$ = 99.9%

Notice that as the value of x gets larger and larger approaching infinity on the right end of the graph, the value of the term with the largest power x^3 approaches 100% of the total value of the polynomial. If the x^3 term approaches ∞ on the right end, then the whole polynomial $y = x^3 + x^2$ approaches ∞ on the right end. A similar table with values of x approaching $-\infty$, will show that the value of the term with the largest power x^3 approaches 100% of the total value of the polynomial on the left end of the graph also. Therefore, we can conclude that the end behavior of the term with the largest power is the same as the end behavior of the polynomial. To determine the end behavior of a polynomial, we need only determine the end behavior of the term with the largest power.

Key Point 4.3.3: The End Behavior of a Polynomial

The end behavior of a polynomial

$$f(x) = a_n x^n + \cdots + a_2 x^2 + a_1 x + a_0$$

is the same as the end behavior of the term with the largest power

$$y = a_n x^n$$

Example 4.3.2

Determine the end behavior of the functions below.

a. $f(x) = 2x^5 + 7x^4 - 7x^2$
b. $g(x) = -3x^4 + 3x^2 + 1$

Solution

a. The end behavior of $f(x) = 2x^5 + 7x^4 - 7x^2$ resembles the end behavior of the term of largest power $y = 2x^5$. Since the stretch factor of $y = 2x^5$ is positive ($a = 2 > 0$), there is no reflection about the x-axis that will change the end behavior. Therefore, $y = 2x^5$ and $y = x^5$ have the same end behavior. Since the power $n = 5$ is odd, $f(x)$ will have the same end behavior as $y = x^3$ which opens up on the right end (as $x \to \infty$, $y \to \infty$) and opens down on the left end (as $x \to -\infty$, $y \to -\infty$).

b. The end behavior of $g(x) = -3x^4 + 3x^2 + 1$ resembles the end behavior of the term of largest power $y = -3x^4$. Since the stretch factor of $g(x) = -3x^4$ is negative $a = -3 < 0$, there is a reflection about the x-axis that will change the end behavior of $y = x^4$. Since the power $n = 4$ is even, $g(x)$ will have the same end behavior as a reflection of the parabola $y = x^2$ which opens down on both ends (as $x \to \pm\infty$, $y \to -\infty$).

? Now You Try: Exercise 4.3.2

Determine the end behavior of the function of $f(x) = 5x^4 - 8x^5 + 3x - 4$

Answer

$f(x)$ has the same end behavior as a reflection of $y = x^3$ which opens down on the right end (as $x \to \infty$, $y \to -\infty$) and opens up on the left end (as $x \to -\infty$, $y \to \infty$).

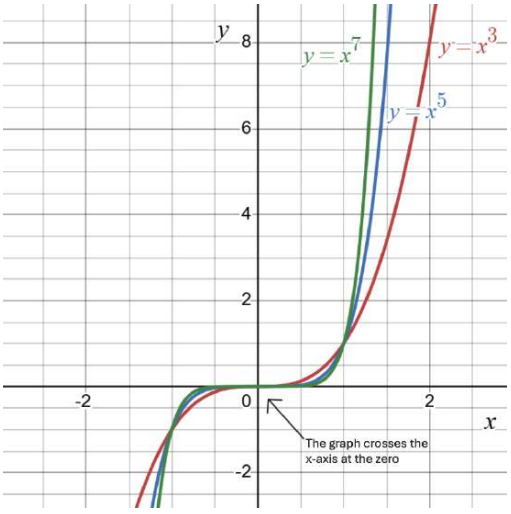
Now that we can determine the end behavior, we have one last facet we need to explore in order to graph a polynomial.

Focus: The Behavior of the Graph of a Polynomial Function Near a Zero

As we have seen in sections 4.1 and 4.2, polynomial functions may have multiple distinct zeros each with a different multiplicity. Each real zero is an x-intercept. How does the multiplicity of a real zero affect the behavior of the graph near the zero? This is an important question we must address in order to graph larger polynomials.

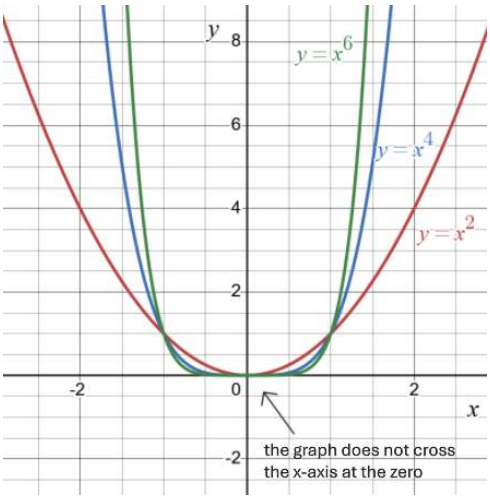
Case: Powers of x with a zero $x = 0$

Let's start by exploring the graphs of the odd powers $y = x^3$, $y = x^5$, and $y = x^7$, near the zero $x = 0$. Notice each odd power has a zero $x = 0$ with odd multiplicity (multiplicity three, five, and seven respectively). In the graph below, notice that each graph crosses the x-axis at the zero.



Try graphing larger odd powers such as $y = x^9$ and $y = x^{11}$ with your graphing utility or app. Notice that they also cross the x-axis at the zero. By extension, we can conclude that all odd powers will cross the x-axis at the zero. In general, the product of an odd number of factors of a positive number is positive whereas the product of an odd number of factors of a negative number is negative. Therefore, an odd power changes sign at the zero $x = 0$ causing the graph to cross the x-axis at $x = 0$.

When we graph the even powers $y = x^2$, $y = x^4$, and $y = x^6$ near the zero $x = 0$, notice that each even power has a zero $x = 0$ with even multiplicity (multiplicity two, four, and six respectively). On the graph below, notice that each even power's graph does not cross the x-axis at the zero. Visually, the graphs of the even powers bounce off the x-axis as you move along the graph from right to left.



Try graphing larger even powers such as $y = x^8$ and $y = x^{10}$ with your graph utility or app. Notice that they also do not cross the x-axis at the zero. By extension, we can conclude that all even powers will not cross the x-axis at the zero. In general, the product of an even number of factors of a positive number is positive and the product of an even number of factors of a negative number is positive also. Therefore, an even power does not change sign at the zero $x = 0$ and the graph does not cross the x-axis at the zero $x = 0$.

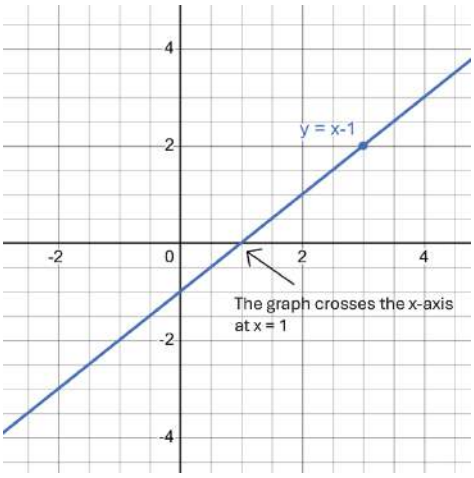
Key Point 4.3.2

For the power function $y = x^n$, odd powers of x cross the x-axis at a zero $x = 0$ whereas even powers of x do not cross the x-axis at the zero $x = 0$.

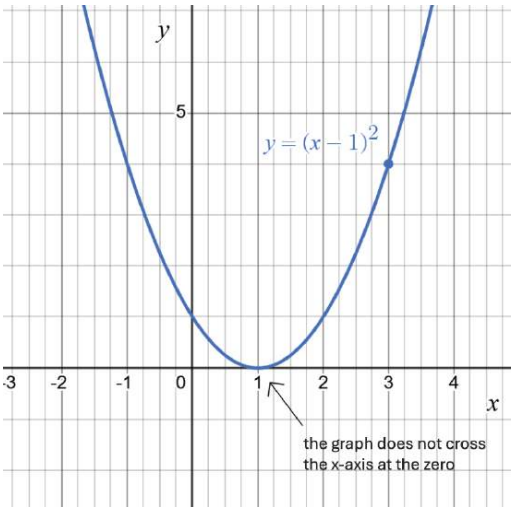
Case: Other zeros with varying multiplicity

Next, let's consider the scenario where we have a zero of $x = 1$. What happens as the multiplicity of the zero increases?

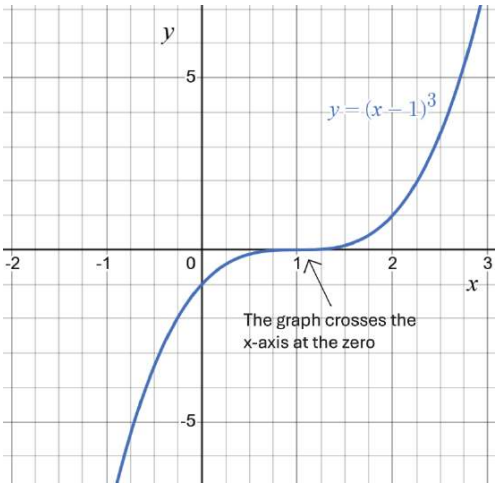
If the zero $x = 1$ has multiplicity one, then a possible formula is $y = x - 1$. Notice, the graph is a line of slope $m = 1$ and y-intercept at $y = -1$ that crosses the x-axis at the zero.



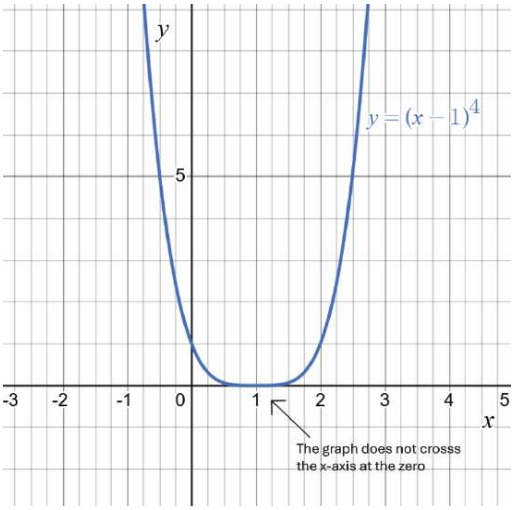
If the zero $x = 1$ has multiplicity two, then a possible formula is $y = (x - 1)^2$ without a vertical stretch or reflection. The graph is a shift right one of the parabola $y = x^2$ which does not cross the x-axis at the zero as shown in the graph below. A vertical stretch or reflection to this graph will not change the fact that the graph will not cross the x-axis at the zero.



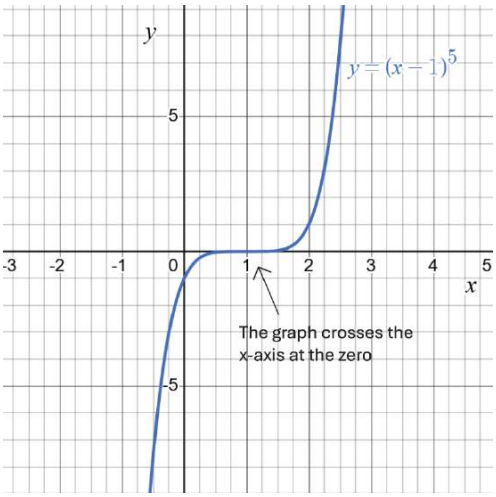
If the zero $x = 1$ has multiplicity three, then a possible formula is $y = (x-1)^3$ without a vertical stretch or reflection. The graph is a shift right one of the cubic $y = x^3$ which crosses the x-axis at the zero as shown in the graph below. Similarly, a vertical stretch or reflection to this graph will not change the fact that the graph will cross the x-axis at the zero.



If the zero $x = 1$ has multiplicity four, then a possible formula is $y = (x-1)^4$. The graph is a shift right one of the even power $y = x^4$ which does not cross the x-axis at the zero as shown in the graph below.



If the zero $x = 1$ has multiplicity five, then a possible formula is $y = (x-1)^5$. The graph is a shift right one of the odd power $y = x^5$. So the graph will cross the x-axis at the zero as shown in the graph below.



In general, a polynomial function with a single real zero with odd multiplicity will have a graph that is a horizontal shift of an odd power which will cross the x-axis at the zero (with or without a vertical stretch or reflection of the graph). A polynomial function with a single real zero with even multiplicity will have a graph that is a horizontal shift of an even power which will not cross the x-axis at the zero.

Key Point 4.3.3

The graph of a polynomial function with a real zero $x = k$ with odd multiplicity will cross the x-axis at the zero. The graph of a polynomial function with a real zero $x = k$ with even multiplicity will not cross the x-axis at the zero.

Case: The effect of other distinct zeros near a zero.

Consider the polynomial $y = x^3(x-1)$ where the zero $x = 1$ has multiplicity one. Notice that as x nears the zero $x = 1$ from above, the factor (x-1) nears 0, the factor x^3 nears 1, so the product $y = x^3(x-1)$ nears $0 \cdot 1 = 0$ as illustrated in the table below.

The effect of mutiple factors on the function value near a zero.

x	2	1.5	1.1	1.001	1.0001
$x-1$	1	0.5	0.1	0.001	0.0001
x^3	8	3.375	1.331	1.003	1.0003
$y = x^3(x-1)$	8	1.686	0.1331	0.001	0.0001

Moreover, the other factor x^3 does not change sign near the zero $x = 1$, but the factor $(x-1)$ does change sign near the zero $x = 1$ since it has odd multiplicity. Therefore, the product $y = x^3(x-1)$ will change sign at the zero. A similar table with values of x approaching $x = 1$ from below will yield a similar result. Therefore, we can conclude the multiplicity of a real zero, determines the behavior of the graph near that real zero. The other factor(s) and corresponding zeros do not effect the behavior of the graph near a specific zero.

Now we are ready to graph polynomials.

Focus: Graphing Polynomials

With the zeros of a polynomial, their multiplicity, and the end behavior, we have enough key features to sketch a graph of a polynomial.

Example 4.3.3

Let $f(x) = x^3 - 4x^2 + 4x$.

a. Identify the zeros and their multiplicities.
b. Determine whether the graph crosses the x-axis at each zero.
c. Determine the end behavior.
d. Use the zeros, their multiplicity, and the end behavior to graph the polynomial $f(x)$.

Solution

a. Factoring the polynomial, we have

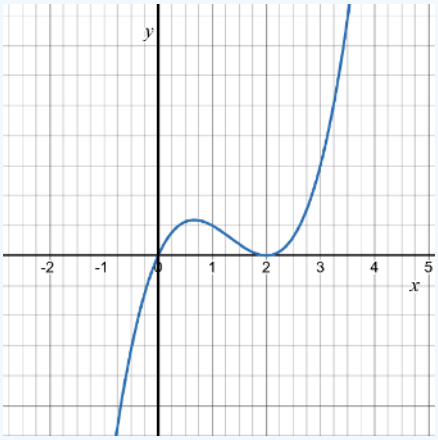
$$f(x) = x^3 - 4x^2 + 4x = x(x^2 - 4x + 4) = x(x-2)(x-2)$$

The factors correspond to the zero $x = 0$ with multiplicity one and the zero $x = 2$ with multiplicity two.

b. Since the zero $x = 0$ has odd multiplicity, the graph will cross the x-axis at $x = 0$. Since the zero $x = 2$ has even multiplicity, the graph will not cross the x-axis at $x = 2$.

c. The end behavior of $f(x)$ resembles the end behavior of the term of largest power $y = x^3$. Therefore, the end behavior resembles that of an odd power with no reflection $(a = 1)$ which opens up on the right (as $x \to \infty$, $y \to \infty$) and down on the left (as $x \to \infty$, $y \to \infty$).

d. From the rightmost zero at $x = 2$, the graph opens upward to the right due to the right end behavior. The graph can't cross the x-axis for any $x > 2$ otherwise their would be a third distinct zero, a contradiction since their are only two real zeros. From the leftmost zero at $x = 0$, the graph opens downward to the left due to the left end behavior. As we move from the left to right along the graph, the graph crosses the x-axis at zero $x = 0$, then must change direction, before dropping down to the x-axis at the other zero $x = 2$. Since this zero has even multiplicity, the graph does not cross the x-axis at this zero.



Example 4.3.4

Let $f(x) = x^4 - x^3 + 6x^2$.

a. Identify the zeros and their multiplicities.
b. Determine whether the graph crosses the x-axis at each zero.
c. Determine the end behavior.
d. Use the zeros, their multiplicity, and the end behavior to graph the polynomial $f(x)$.

Solution

a. Factoring the polynomial, we have

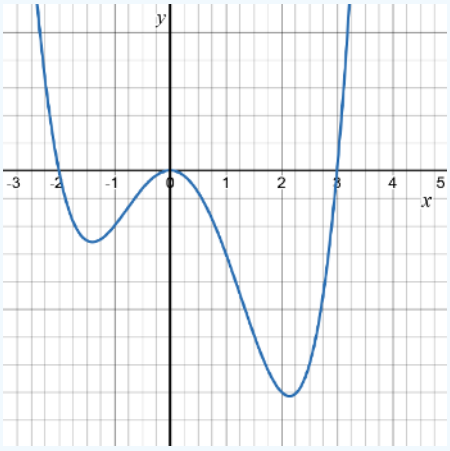
$$f(x) = x^4 - x^3 - 6x^2 = x^2(x^2 - x - 6) = x^2(x-3)(x+2)$$

The factors correspond to the zero are $x = 0$ with multiplicity two, the zero $x = 3$ with multiplicity one, and the zero $x = -2$ with multiplicity one.

b. Since the zero $x = 0$ has even multiplicity, the graph will not cross the x-axis at $x = 0$. Since the zeros $x = 3$ and $x = -2$ have odd multiplicity, the graph will cross the x-axis at these zeros.

c. The end behavior of $f(x)$ resembles the end behavior of the term of largest power $y = x^4$. Therefore, the end behavior resembles that of an even power with no reflection $(a = 1)$ which opens upward on both ends (as $x \to \pm\infty$, $y \to \infty$).

d. From the rightmost zero at $x = 3$, the graph opens upward to the right due to the right end behavior. Using the same rational as example 4.3.3, the graph can't cross the x-axis for any $x > 3$ otherwise their would be a fourth zero, a contradiction since their are only three distinct real zeros. From the leftmost zero at $x = -2$, the graph opens upward to the left due to the left end behavior. As we move from the left to right along the graph, the graph crosses the x-axis at zero $x = -2$, then must change direction, and come up to the x-axis at the next zero $x = 0$. Since the zero $x = 0$ has even multiplicity, the graph does not cross the x-axis at this zero. Continuing rightward along the graph, the graph decreases, then must change direction, and come up to the x-axis at the next zero $x = 3$. The graph crosses the x-axis at the zero $x = 3$.



Example 4.3.5

Let $f(x) = x^3 - 13x + 12$.

a. Identify the zeros and their multiplicities.

b. Determine the end behavior.

c. Use the zeros, their multiplicity, and the end behavior to graph the polynomial $f(x)$.

Solution

a. Since their is no common factor, we must use the rational zero theorem to identify a rational zero if one exists. The possible zeros are $x = \frac{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12}{\pm 1}$. In example 4.2.9, we tested a number of possible zeros for $f(x)$ before we determined that $x = 1$ was a zero since $f(1) = 0$. In example 4.2.1, we divided $f(x) = x^3 - 13x + 12$ by the corresponding factor $(x - 1)$ of the zero $x = 1$ using long division of polynomials. We found that

$$\frac{x^3 - 13x + 12}{x - 1} = x^2 + x - 12$$

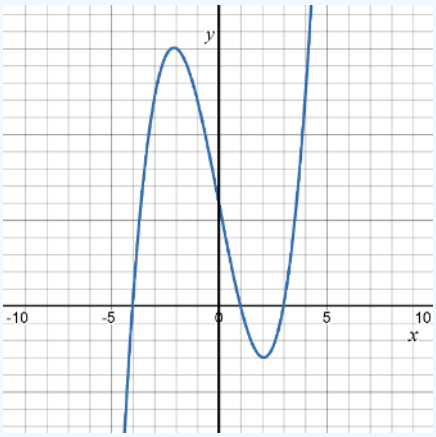
This allowed us to factor $f(x)$ as

$$f(x) = x^3 - 13x + 12 = (x - 1)(x^2 + x - 12) = (x - 1)(x - 3)(x + 4)$$

The factors correspond to zeros of $x = 1$, $x = 3$, and $x = -4$ aeach with multiplicity one. The graph will cross the x-axis at these zeros due to their multiplicities.

b. The end behavior of $f(x)$ resembles the end behavior of the term of largest power $y = x^3$. Therefore, the end behavior resembles that of an odd power with no reflection $(a = 1)$ which opens upward on the right end (as $x \to \infty$, $y \to \infty$) and downward on the left end (as $x \to -\infty$, $y \to -\infty$).

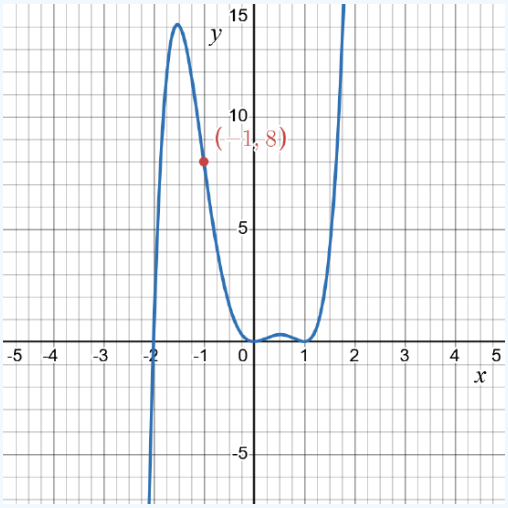
c. From the rightmost zero at $x = 3$, the graph opens upward to the right due to the right end behavior. From the leftmost zero at $x = -4$, the graph opens downward to the left due to the left end behavior. As we move from the left to right along the graph, the graph crosses the x-axis at zero $x = -4$, then must change direction and come back down to the x-axis at the next zero $x = 1$ where it crosses the x-axis. Continuing rightward along the graph, the graph continues to decreas, then must change direction and come up to the x-axis at the next zero at $x = 3$ where it crosses the x-axis.



We can also use the graph of a polynomial to determine its formula, now that we can identify the multiplicity based on the behavior of the graph at each real zero.

Example 4.3.6

Identify a possible formula for the graph of the polynomial given below that passes through the point (1,8).



Solution

The x-intercepts at $x = -2$, $x = 0$, and $x = 1$ are the real zeros of the polynomials. Since the graph crosses the x-axis at the zero $x = -2$, the zero has odd multiplicity and therefore the polynomial has an odd number of factors of the corresponding factor $(x + 2)$. Since the graph does not cross the x-axis at the zero $x = 0$, the zero has even multiplicity and therefore the polynomial has an even number of factors of the corresponding factor (x). Since the graph does not cross the x-axis at the zero $x = 1$, the zero has even multiplicity and the polynomial has an even number of factors of the corresponding factor $(x - 1)$. So a possible formula of minimum degree would be

$$f(x) = ax^2(x+2)(x-1)^2$$

To find the stretch factor a, we can substitute the known point $x = -1$ and $y = 8$ into the polynomial

$$a(-1)^2(-1+2)(-1-1)^2 = 8$$

Then solving for a, we have

$$a4 = 8$$

$$a = 2$$

Therefore, the formula for the polynomial is

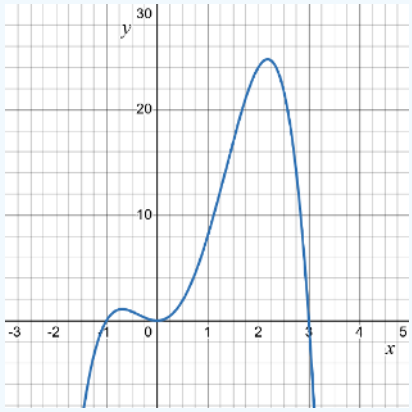
$$f(x) = 2x^2(x+2)(x-1)^2$$

Now You Try: Exercise 4.3.3

Let $f(x) = -2x^4 + 4x^3 + 6x^2$.

a. Identify the zeros and their multiplicities.
b. Determine whether the graph crosses the x-axis at each zero.
c. Determine the end behavior.
d. UUse the zeros, their multiplicity, and the end behavior to graph the polynomial $f(x)$.

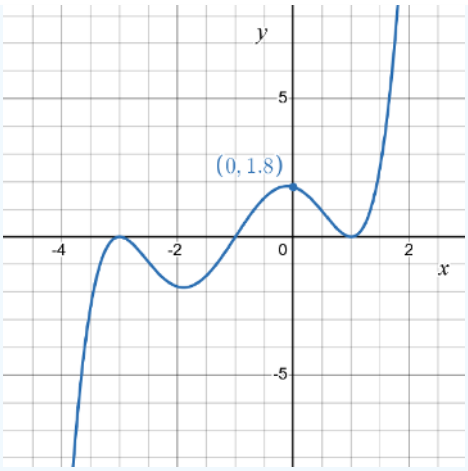
Answer

a. The zeros are $x = 0$ with multiplicity two, $x = 3$ with multiplicity one, and $x = -1$ with multiplicity one.
b. The graph will not cross the x-axis at $x = 0$ since it has even multiplicity. The graph will cross the x-axis at the zeros $x = 3$ and $x = -1$ since they have odd multiplicity.
c. The end behavior resembles that of a reflection of an even power which opens downward on both ends (as $x \to \pm\infty$, $y \to -\infty$).
d. 

Now You Try: Exercise 4.3.4

The graph of a polynomial function $y = f(x)$ containing the point (0,1.8) is given below.

a. Identify each real zero and its smallest possible multiplicity.
b. Write a formula for the polynomial function. Assume the least polynomial is of least possible degree.



Answer

$y = f(x) = 0.2(x+3)^2(x+1)(x-1)^2$

Important Topics of this Section

- The End Behavior of Polynomial Functions
- The Effect of The Multiplicity of A Zero on the Graph
- Graphing Polynomials

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4.3E: Graphing Polynomial Functions (Exercises)

Section 4.3 Exercises

In #1-3, determine the end behavior of each polynomial.

1. $f(x) = -3x^4 + 5x^3 - 2x + 1$

2. $g(x) = 2x^7 + 4x^5 - 2x^3$

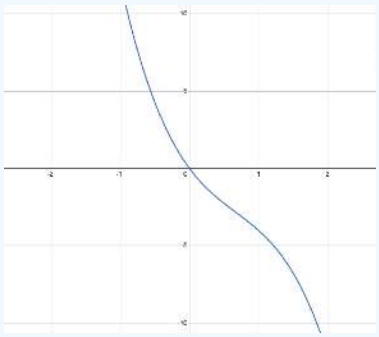
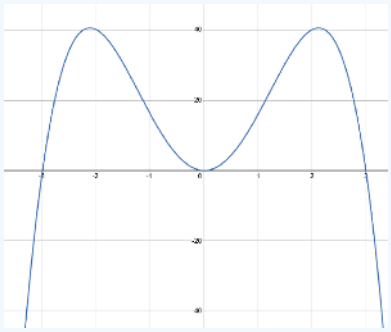
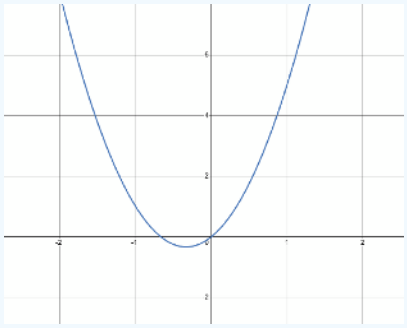
3. $h(x) = -4x^5 - 7x^8 + 5x^3 - 7$

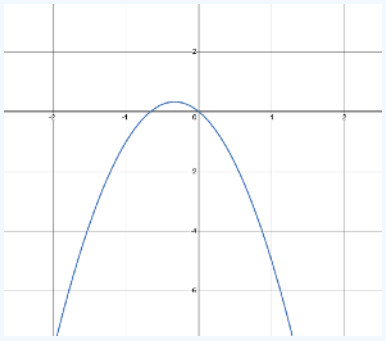
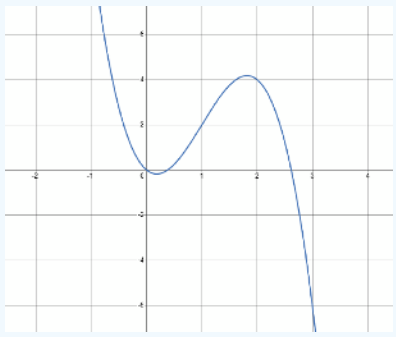
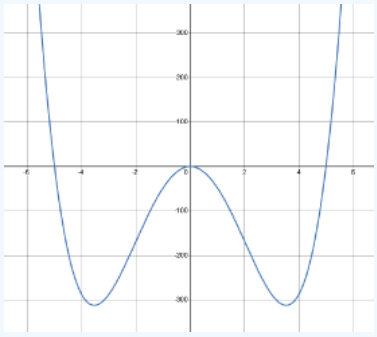
In exercises #4 - 6, match the polynomial function with its graph without using technological assistance.

4. $f(x) = -3x^2 - 2x$

5. $f(x) = 2x^4 - 50x^2$

6. $f(x) = -2x^3 + 4x^2 - 6x$

A.  B.  C. 

D.  E.  F. 

In exercises #7-14, for each polynomial function given: (a) identify each zero and its multiplicity; (b) determine whether the graph crosses the x-axis at each real zero; (c) determine the end behavior; and (d) sketch the graph without using technological assistance.

7. $f(x) = (x+3)^2(x-2) = x^3 + 4x^2 - 3x - 18$

8. $f(x) = -x^3 + 3x^2$

9. $f(x) = -x^3 + x^2 + 2x$

10. $f(x) = x^4 - 4x^2$

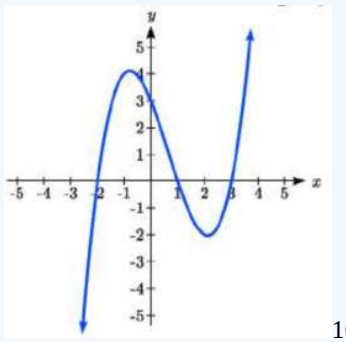
11. $f(x) = -x^4 + x^3 + 6x^2$

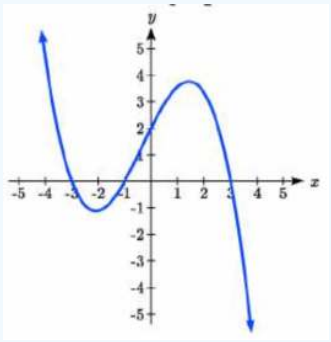
12. $f(x) = x^3 - x^2 - 9x + 9$

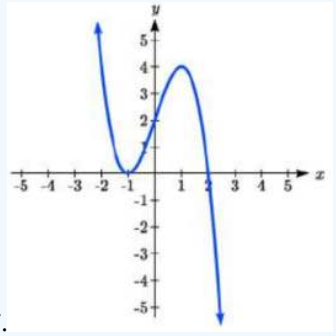
13. $f(x) = x^3 - x^2 - x + 1$

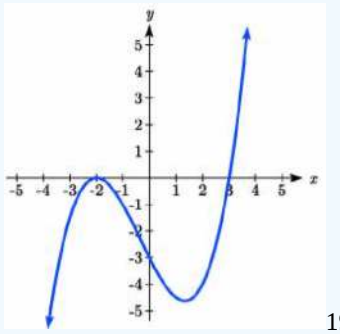
14. $f(x) = x^5 - 4x^4 + 4x^3$

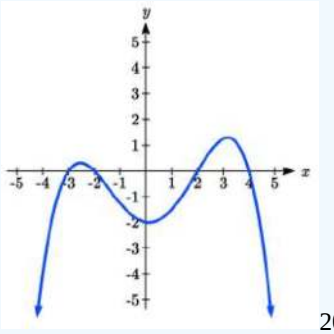
In exercises #15-20, write a formula for each polynomial function graphed. Note that the y-intercept that is given on each graph.

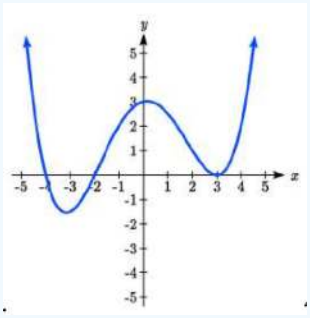
15. 

16. 

17. 

18. 

19. 

20. 

In exercises 21-22, determine whether the statement are true or false. Justify your conclusions.

21. The graph of a polynomial function may not have any x-intercepts.

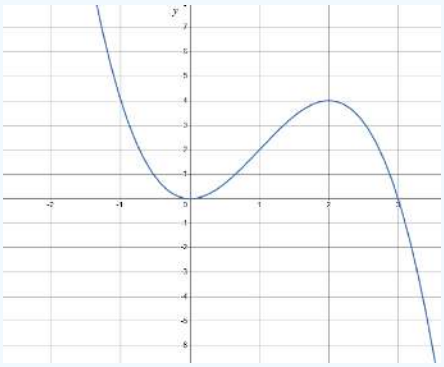
22. The graph of a polynomial function may not have any y-intercepts.

Answer

1. The graph opens down on the right end (as $x \to \infty$, $y \to -\infty$) and up on the left end (as $x \to -\infty$, $y \to \infty$).

4. Graph D

8. (a) The zeros are $x = 0$ with multiplicity two and $x = 3$ with multiplicity one. (b) The graph will cross the x-axis at $x = 3$ but not at $x = 0$. (c) The graph opens down on the right end (as $x \to \infty$, $y \to -\infty$) and up on the left end (as $x \to -\infty$, $y \to \infty$). (d)



15. $f(x) = \frac{1}{2}(x+2)(x-1)(x-3)$

21. True. Since only real zeros of a polynomial are x-intercepts, a polynomial with complex zeros such as $f(x) = x^2 + 4$ will have to x-intercepts.

22. False. The y-intercept of a polynomial occurs at $x = 0$, $y = f(0)$. $y = f(0)$ is always defined for polynomials.

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4.4: Polynomial Inequalities

Solving Polynomial Inequalities

How can we solve inequalities involving polynomials such as $x^3 - 4x^2 + 3x > 0$? Determining where $x^3 - 4x^2 + 3x > 0$ is the same as determining where the function $y = f(x) = x^3 - 4x^2 + 3x$ is positive. Similarly, determining where $x^3 - 4x^2 + 3x < 0$ is the same as determining where the function $y = f(x) = x^3 - 4x^2 + 3x$ is negative. One approach that we can utilize is to use the graph of the polynomial to determine the sign of the polynomial using the zeros and the end behavior. When the graph is above the x-axis, the function is positive. When the graph is below the x-axis, the function is negative. When the graph is on the x-axis, the function value is zero.

Example 4.4.1

Solve $x^3 - 4x^2 + 3x > 0$

Solution

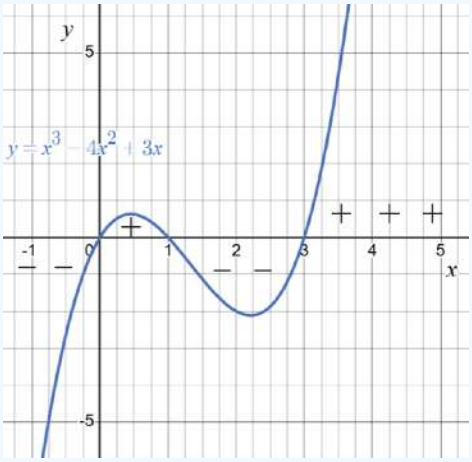
We can solve this inequality by determining where $y = f(x) = x^3 - 4x^2 + 3x$ is positive. Factoring, we have

$$f(x) = x^3 - 4x^2 + 3x = x(x^2 - 4x + 3) = x(x-3)(x-1)$$

So the zeros are

$$x = 0, x = 3, \text{ and } x = 1$$

each with multiplicity one. Therefore, the graph will cross the x-axis at each zero. The end behavior of f is the same as $y = x^3$ which opens up on the right (as $x \to \infty$, $y \to \infty$) and down on the left (as $x \to -\infty$, $y \to -\infty$). Using the zeros and multiplicity, we can graph $y = f(x)$ as shown below.



From the graph, we can see that f is positive on $(0, 1)$ and $(3, \infty)$ when the graph is above the x-axis. These intervals form the solution to our inequality $(0, 1) \cup (3, \infty)$

Example 4.4.2

Solve $x^4 - x^3 - 6x^2 < 0$

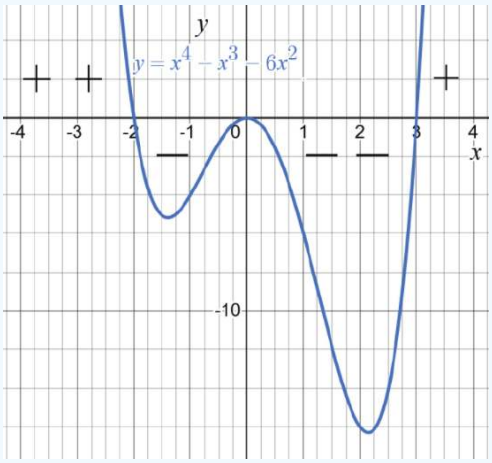
Solution

We can solve this inequality by determining where $y = f(x) = x^4 - x^3 - 6x^2 < 0$ is negative or zero. Factoring, we have

$$f(x) = x^4 - x^3 - 6x^2 = x^2(x^2 - x - 6) = x^2(x-3)(x+2)$$

So the zeros are $x = 3$ with multiplicity one, $x = 0$ with multiplicity two, and $x = -2$ with multiplicity one. Therefore, the graph will cross the x-axis at $x = 3$ and $x = -2$ but will not cross the x-axis at $x = 0$. The end behavior of f is the same as $y = x^4$, an

even power, which opens up on both ends (as $x \to \pm\infty$, $y \to \infty$). Using the zeros and multiplicity, we can graph $y = f(x)$ as shown below.



From the graph, we can see that f is negative on $(-2, 0) \cup (0, 3)$ when the graph is below the x-axis. This interval form the solution to our inequality $(-2, 0) \cup (0, 3)$.

What is different about the inequality below?

$$x^3 + 2x^2 < -5x$$

The expression $-5x$ could be positive or negative depending on the value of x, so we can't make this a question of where a function is positive or negative. However, if we get zero on one side of the inequality by adding $5x$ to both sides, we have

$$x^3 + 2x^2 + 5x < 0$$

This inequality can be solved by determining where $y = f(x) = x^3 + 2x^2 + 5x$ is negative as shown in example 4.4.3 below.

Key Point 4.4.1

A polynomial inequality of the form $f(x) > 0$ or $f(x) < 0$ can be solved by determining the sign of $f(x)$ using the graph.

Example 4.4.3

Solve $x^3 + 2x^2 < -5x$

Solution

First, getting zero on once side of the inequality

$$x^3 + 2x^2 + 5x < 0$$

We can solve this inequality by determining where $y = f(x) = x^3 + 2x^2 + 5x$ is negative. Factoring, we have

$$f(x) = x^3 + 2x^2 + 5x = x(x^2 + 2x + 5)$$

Setting each factor to zero, we have

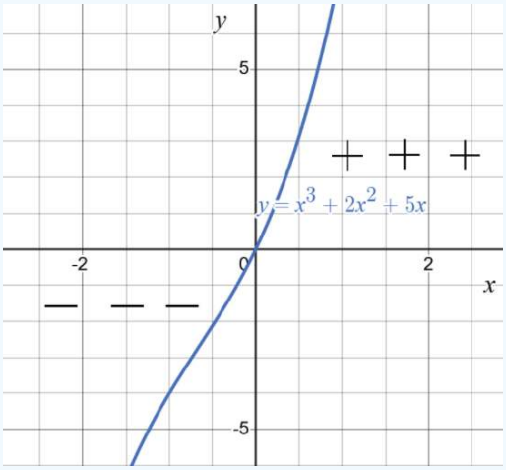
$$x = 0 \text{ and } x^2 + 2x + 5 = 0$$

The first equation gives the zero $x = 0$ with multiplicity one. Using the quadratic formula to solve the second equation with $a = 1$, $b = 2$, and $c = 5$, we have

$$x = \frac{-2 \pm \sqrt{2^2 - 4(1)(5)}}{2(1)}$$

$$x = \frac{-2 \pm \sqrt{-16}}{2} = -1 \pm 2i$$

Since these are a complex zeros, the graph will only cross the x-axis at the real zero $x = 0$. The end behavior of f is the same as $y = x^3$ which opens up on the right (as $x \to \infty$, $y \to \infty$) and down on the left (as $x \to -\infty$, $y \to -\infty$). Using the zeros and multiplicity, we can graph $y = f(x)$ as shown below.



From the graph, we can see that f is negative on $(-\infty, 0)$ when the graph is below the x-axis. This interval form the solution to our inequality $(-\infty, 0)$.

Note that if the inequality in Example 4.4.3 was altered to $x^3 + 2x^2 \leq -5x$, this would change our solution slightly. The inequality is equivalent to $x^3 + 2x^2 + 5x \leq 0$. The solutions are values of x where $f(x) = x^3 + 2x^2 + 5x$ is negative **or zero**. We would have to include the real zero $x = 0$ in our solution in this case. Our solution is $(-\infty, 0]$.

? Now You Try: Exercise 4.4.1

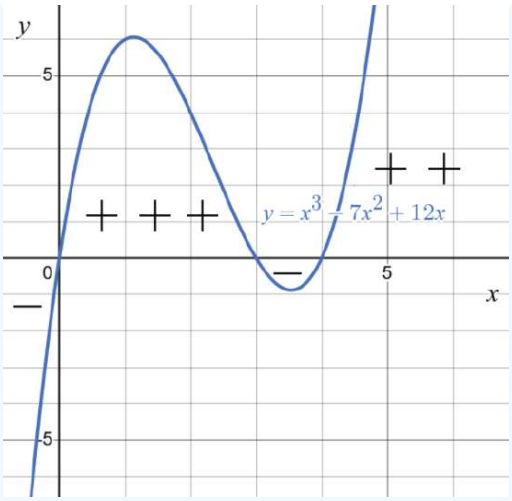
Solve $x^3 - 7x^2 \geq -12x$

Answer

The inequality is equivalent to

$$x^3 - 7x^2 + 12x \geq 0$$

To solve this inequality determine where $y = f(x) = x^3 - 7x^2 + 12x$ is positive, using the zeros $x = 0$, $x = 3$, and $x = 4$ (each with multiplicity one) and the end behavior (resembling that of $y = x^3$). The graph below shows that f is positive or zero on $[3, 0] \cup [4, \infty)$. These intervals are the solution to the inequality.



? Now You Try: Exercise 4.4.2

Explain what is wrong with the following solution to the inequality

$$x^2 - 2x > 0$$

$$x(x-2) > 0$$

$$x > 0 \text{ or } (x-2) > 0$$

$$x > 0 \text{ and } x > 2$$

The solutions to the inequality is $(0, \infty) \cup (2, \infty)$.

Answer

There are two problems with the solution. An error occurs in the third step. Just because two factors multiply to zero in step two $x(x-2) \geq 0$ does not necessarily mean that each factor must be greater that zero or positive as shown in step three where $x \geq 0$ and $(x-2) \geq 0$. The two factors on the left side of the inequality could either be both positive or both negative for the product to be positive. This solution incorrectly reflects how we should solve the equation $x^2 - 2x = 0$ as opposed to an inequality. Factoring

$$x(x-2) = 0$$

Setting each factor to zero

$$x = 0 \text{ or } x - 2 = 0$$

Then solving for x, we have

$$x = 0 \text{ or } x = 2$$

This is the correct way to solve an equality not an inequality.

A second issue occurs in the last step. The values of x where $x > 0$ or $x > 2$ are merely the interval the values of x where $x > 0$. If $x > 2$, then $x > 0$ also. The second inequality is redundant.

Focus: Another Approach to Solving Polynomial Inequalities

Notice on the graph in Example 4.3.2, the function $f(x) = x^4 - x^3 - 6x^2$ only changes sign from positive to negative (or vice versa) at the zeros $x = 3$ and $x = 2$. Although $x = 0$ is also a zero, the graph does not change sign there. Furthermore, the function does not change sign between consecutive zeros since the graph of the polynomial is continuous. If the polynomial did change sign, this would mean there is another zero and that the zeros aren't consecutive, a contradiction. These two concepts are the basis for another

approach to determining the sign of a polynomial function and solving inequalities called **the test point method**. If we pick any value of x in the interval (0,3) between the consecutive zeros $x = 0$ and $x = 3$, we can evaluate the function to determine the sign of f at that point. For example, at $x = 1$, $y = f(1) = 1^4 - 1^3 - 6(1)^2 = -5$ is negative. Since the the polynomial does not change sign between consecutive zeros, the polynomial must be negative on the whole interval (0,3). If we pick any value of x in the interval (-2,0) between the consecutive zeros $x = -2$ and $x = 0$ such as $x = -1$ then evaluate $y = f(-1) = (-1)^4 - (-1)^3 - 6(-1)^2 = -4$, we can make a similar argument to determine that f is also negative on the interval (-2,0). If we repeated this process of evaluating the function at a test point in each interval between consecutive zeros, we could determine the sign of the polynomial f for all values of x. Notice, we do not need the multiplicity of the zero, the end behavior, or the graph itself to determine the sign of the polynomial on each interval.

Example 4.4.4

Solve the inequality $x^4 - 2x^2 > 0$ using the test point method.

Solution

To solve the inequality we need to determine where $f(x) = x^4 - 2x^2$ is positive. First, we need to find the zeros of $y = f(x) = x^4 - 2x^2$ by factoring

$$x^4 - 2x^2 = x^2(x - 2) = 0$$

Then setting each factor to zero

$$x^2 = 0 \text{ and } x^2 - 2 = 0$$

The first equation yields a zero of $x = 0$. Solving the second equation

$$x^2 = 2$$

which yields the zeros

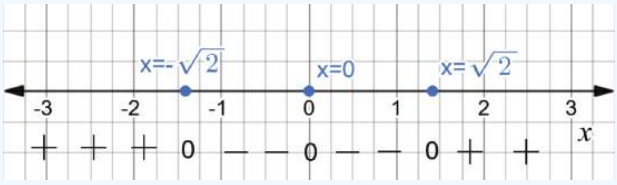
$$x = \pm\sqrt{2} \approx 1.41$$

These zeros break the entire x-axis into four intervals $(-\infty, -\sqrt{2})$, $(-\sqrt{2}, 0)$, $(0, \sqrt{2})$, and $(\sqrt{2}, \infty)$. Choosing a test value in each interval and evaluating the function at each test value in the table below to determine if the function is positive or negative in each interval

Test Points and Their Values in Each Interval

Interval	Test x in interval	f (test value)
$(-\infty, -\sqrt{2})$	-2	8
$(-\sqrt{2}, 0)$	-1	-1
$(0, \sqrt{2})$	1	-1
$(\sqrt{2}, \infty)$	2	8

Therefore, f is positive on $(-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$ and negative on $(-\sqrt{2}, 0) \cup (0, \sqrt{2})$. We can represent the sign on f with a sign chart as shown below.



The solution to the inequality is $(-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$

Example 4.4.5

Solve the inequality $x^2 + x < -2$ using the test point method.

Solution

This inequality is equivalent to the inequality $x^2 + x + 2 < 0$. To solve the inequality we need to determine the zeros of $y = f(x) = x^2 + x + 2$. Using the quadratic formula with $a = 1$, $b = 1$ and $c = 2$, we have

$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(2)}}{2(1)}$$

$$x = \frac{-1 \pm \sqrt{-7}}{2} = \frac{-1 \pm i\sqrt{7}}{2} = \frac{-1}{2} \pm i\frac{\sqrt{7}}{2}$$

So there are no real zeros. Therefore the function never crosses the x-axis and does not change sign. In this case, we need only test one point in the interval $(-\infty, \infty)$.

Test Points and Their Values in Each Interval

Interval	Test x in interval	f (test value)
$(-\infty, \infty)$	0	2

Therefore, f is positive for all values of x (on $(-\infty, -\infty)$. f is never negative. There is no solution to the inequality.

Now you try: Exercise 4.4.3

Solve the inequality $x^3 + 9x^2 + -20x \geq 0$ using the test point method. Build a sign chart for $f(x) = x^3 + 9x^2 + 20x$ with your results.

Answer

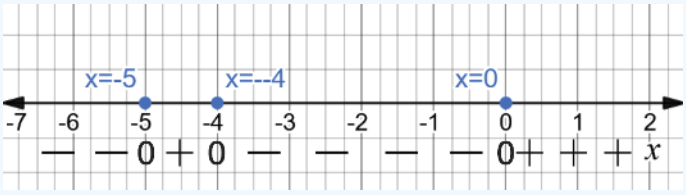
We need to determine where $f(x) = x^3 + 9x^2 + 20x$ is positive or zero. The zeros of $f(x) = x^3 + 9x^2 + 20x$ are $x = 0$, $x = -4$, and $x = -5$. Choosing a test value in each interval and evaluating the function at each test value in the table below

Test Points and Their Values in Each Interval

Interval	Test x in interval	f (test value)
$(-\infty, -5)$	-6	-12
$(-5, -4)$	-4.5	1.125
$(-4, 0)$	-1	-12
$(0, \infty)$	1	30

Therefore, f is positive or zero on $[-5, -4] \cup [0, \infty)$. These intervals are the solutions to the original inequality.

The sign chart for f is given below.



Important Topics of this Section

- Solving Polynomial Inequalities
- The Test Point Method

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4.4E: Polynomial Inequalities (Exercises)

Section 4.4 Exercises

In # 1 - 6, solve each polynomial inequality. Express your result in interval notation when appropriate.

1. $x^2 - 7x + 12 > 0$

2. $x^3 - 4x^2 < 0$

3. $x^3 - 13x + 12 \geq 0$

4. $x^3 - 6x^2 \geq -9x$

5. $x^4 + 4x^3 \leq 21x^2$

6. $x^2 + x > -3$

In #7-9, use the test point method to solve each polynomial inequality. Express your result in interval notation when appropriate.

7. $x^2 - x + 6 > 0$

8. $x^3 - x^2 - 2x - 2 \leq 0$

9. $x^4 \geq 9x^2$

10. Explain the mistake(s) that was(were) made

Solve $x^2 + 2x > 15$

Step 1: $x^2 + 2x - 15 > 0$

Step 2: $(x + 5)(x - 3) > 0$

Step 3: $x + 5 > 0$ and $x - 3 > 0$

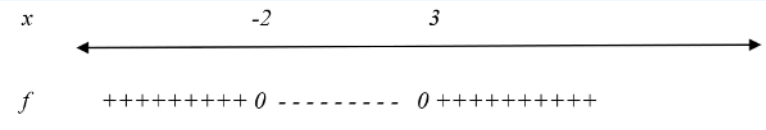
Step 4: $x > -5$ and $x > 3$

Answer

1. $(-\infty, 3) \cup (4, \infty)$

4. $(-\infty, 0] \cup [3, \infty)$

7. $(-\infty, -2) \cup (3, \infty)$



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4.5: Describing Relationships with Rational Functions

In this section, we will explore describing relationships with rational functions. A rational functions is a ratio of two polynomials such as

$$f(x) = \frac{x^2 - 4x + 1}{x^3 + 5x^2 + 3}$$

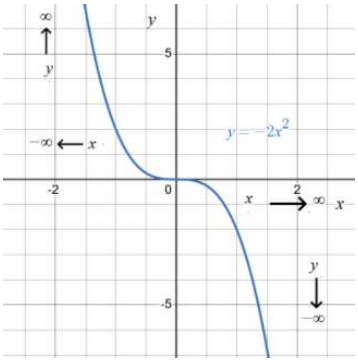
We will approach rational functions in a way similar to how we approached describing relationships with polynomial functions. We will use the key features of the rational function such as the zeros and the end behavior to graph it. We need to identify other key features of the rational function that are needed to draw its graph and describe the relationship between quantities.

Focus: Determining The End Behavior of Rational Functions

For the end behavior, we need to determine what value(s) $y = f(x)$ approaches as x approaches $\pm\infty$. Let's explore a few different cases.

Case #1: A power in the numerator

A power in the numerator such as $y = \frac{-2x^3}{1}$ simplifies to a single power of x, $y = -2x^3$. This is an odd power whose coefficient $a = -2$ causes a reflections about the x-axis. The graph opens down on the right end and up on the left end as shown in the graph below. The end behavior can be describes as $x \to \infty$, $y \to -\infty$ and as $x \to -\infty$, $y \to \infty$.



Case #2: A power in the denominator

Let's examine the end behavior of a power in the denominator such as $f(x) = \frac{1}{x}$ in the table below.

The Right End Behavior

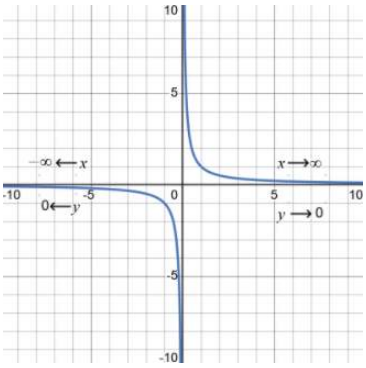
x	1	10	100	1000	$\to \infty$
$y = f(x) = \frac{1}{x}$	$\frac{1}{1} = 1$	$\frac{1}{10} = 0.1$	$\frac{1}{100} = 0.01$	$\frac{1}{1000} = 0.001$	$\to 0$

As x gets larger and larger approaching positive infinity, $y = f(x)$ approaches zero.

The Left End Behavior

x	-1	-10	-100	-1000	$\to -\infty$
$y = f(x) = \frac{1}{x}$	$\frac{1}{-1} = -1$	$\frac{1}{-10} = -0.1$	$\frac{1}{-100} = -0.01$	$\frac{1}{-1000} = -0.001$	$\to 0$

As x approaches negative infinity, $y = f(x)$ approaches zero. The end behavior of $f(x) = \frac{1}{x}$ is shown on the graph below



For a larger power in the denominator, such as $f(x) = \frac{1}{x^2}$, the end behavior is determined in the table below. Notice, as x approaches $\pm\infty$, $y = f(x)$ approaches zero.

The Right End Behavior

x	1	10	100	1000	$\rightarrow \infty$
$y = f(x) = \frac{1}{x^2}$	$\frac{1}{1} = 1$	$\frac{1}{100} = 0.01$	$\frac{1}{10000} = 0.0001$	$\frac{1}{1000000} = 0.000001$	$\rightarrow 0$

The Left End Behavior

x	-1	-10	-100	-1000	$\rightarrow -\infty$
$y = f(x) = \frac{1}{x^2}$	$\frac{1}{1} = -1$	$\frac{1}{100} = 0.01$	$\frac{1}{10000} = 0.0001$	$\frac{1}{1000000} = 0.000001$	$\rightarrow 0$

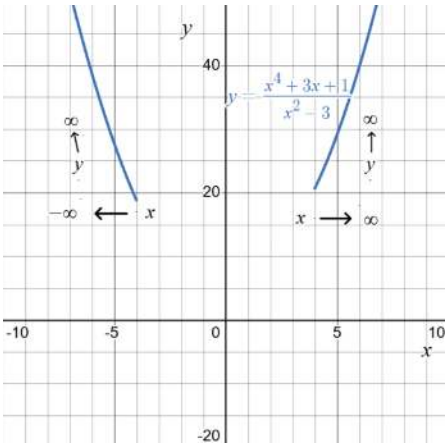
In general, if the denominator of a fraction is getting larger and larger while the numerator is approaching a finite value, the value of the fraction will approach zero. As the whole number power gets larger and larger, the denominator gets larger even faster making the the value of the whole fraction approach zero even faster.

Key Point 4.5.1

A power in the denominator $y = \frac{1}{x^n}$ has end behavior that is described as $x \rightarrow \pm\infty$, $y \rightarrow 0$.

Case #3: A rational function with the degree of the numerator > the degree of the denominator

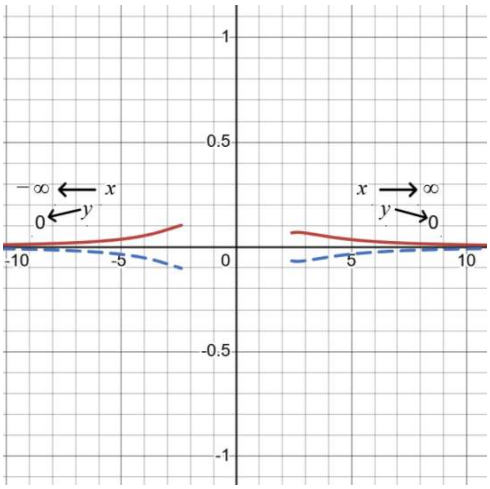
Let $f(x) = \frac{x^4 + 3x + 1}{x^2 - 3}$. The numerator $x^4 + 3x + 1$ is a polynomial whose end behavior is the same as the term with the largest power x^4 as we learned in section 4.3. Similarly the denominator $x^2 - 3$ is a polynomial whose end behavior is the same as the term with the largest power x^2. Therefore, the rational function $f(x) = \frac{x^4 + 3x + 1}{x^2 - 3}$ has the same end behavior as the ratio of leading terms $y = \frac{x^4}{x^2} = x^2$. This is an even power that opens up on both ends (as $x \rightarrow \pm\infty$, $y \rightarrow \infty$). The end behavior is shown in the graph below.



For the end behavior, we are only talking about the left and right ends of the graph. The functions $f(x) = \frac{x^4+3x+1}{x^2-3}$ and $y = x^2$ have very different graphs for smaller values of x.

Case #4: A rational function with the degree of the numerator < the degree of the denominator

Let $f(x) = \frac{x^2-3}{x^4+3x+1}$. The polynomial in the numerator, x^2-3, has the same end behavior is the same as the term with the largest power x^2. The polynomial in the denominatorx^4+3x+1 has the same end behavior is the same as the term with the largest power x^4. Therefore, the rational function $f(x) = \frac{x^2-3}{x^4+3x+1}$ has the same end behavior as the ratio of leading terms $y = \frac{x^2}{x^4} = \frac{1}{x^2}$. This is a power in the denominator that approaches zero on both ends (as $x \to \pm\infty$, $y \to 0$) as shown in case #2. The possible end behaviors are shown in the graph below.

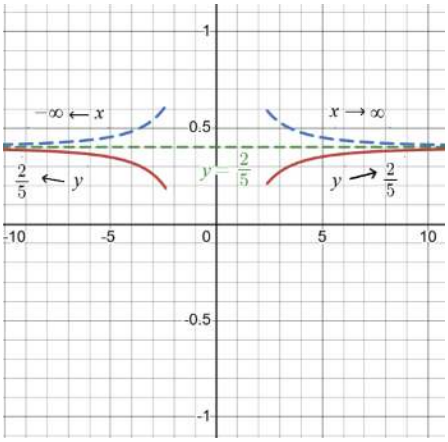


Notice, the graph of $f(x)$ can approach $y = 0$ either from above the x-axis or below the x-axis on the right end as $x \to \infty$. We need more information about the graph to determine from which direction the graph approaches $y = 0$. For example, if we knew that y was positive on the right end, we could deduce that the graph of $f(x)$ approaches $y = 0$ from above the x-axis. Similarly, the graph of $f(x)$ can approach $y = 0$ either from above the x-axis or below the x-axis on the left end as $x \to -\infty$. If we knew that y was also positive on the left end, we could deduce that the graph of $f(x)$ approaches $y = 0$ from above the x-axis.

Case #5: A rational function with the degree of the numerator = the degree of the denominator

Let $f(x) = \frac{2x^3-5x+1}{5x^3+4x}$. The polynomial in the numerator, $2x^3-5x+1$, has the same end behavior is the same as the term with the largest power $2x^3$. The polynomial in the denominator $5x^3+4x$ has the same end behavior is the same as the term with the largest power $5x^3$. Therefore, the rational function $f(x) = \frac{2x^3-5x+1}{5x^3+4x}$ has the same end behavior as the ratio of leading terms $y = \frac{2x^3}{5x^3}$

$= \frac{2}{5}$. On both ends, y approaches $y = \frac{2}{5}$ (as $x \to \pm\infty$, $y \to \frac{2}{5}$). The possible end behaviors are shown in the graph below (in red and blue).



Key Point 4.5.2

The end behavior of a rational function $f(x) = \frac{a_n x^n + \cdots + a_2 x^2 + a_1 x + a_0}{b_m x^m + \cdots + b_2 x^2 + b_1 x + b_0}$ is the same as the end behavior of the ratio of leading terms $y = \frac{a_n x^n}{b_n x^m}$

For the rational function in case #5, $f(x) = \frac{2x^3 - 5x + 1}{5x^3 + 4x}$ where as $x \to \pm\infty$, $y \to \frac{2}{5}$, notice that the graph nears the horizontal line $y = \frac{2}{5}$ shown in green in the graph above. We call this line a **horizontal asymptote**.

Definition: Horizontal Asymptote

A function $y = f(x)$ has a **horizontal asymptote** $y = a$ if as x approaches $\pm\infty$, y approaches a. The graph of $y = f(x)$ nears the line $y = a$ as x approaches $\pm\infty$.

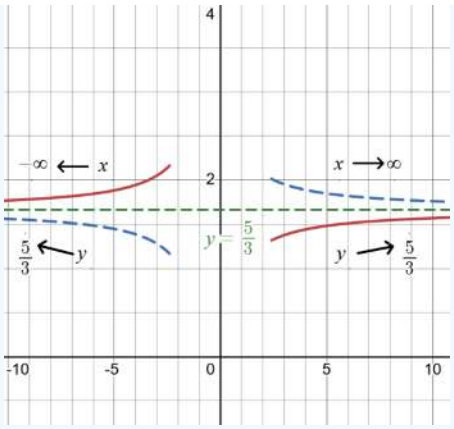
A horizontal asymptote represents a type of end behavior. The rational functions in case #2 and #4 have a horizontal asymptote at $y = 0$ since the graphs near the line $y = 0$ since y approaches 0 as $x \to \pm\infty$. However, be careful. Not all rational functions have a horizontal asymptote for their end behavior such as the rational functions in cases #1 and #3.

Now You Try: Exercise 4.5.1

Let $f(x) = \frac{5x^2 + 2x}{3x^2 + 3x + 1}$. Determine the end behavior. Identify any horizontal asymptotes if they occur. Then sketch a possible graph of the end behavior.

Answer

The rational function $f(x) = \frac{5x^2 + 2x}{3x^2 + 3x + 1}$ has the same end behavior as the ratio of leading terms \9y=\dfrac{5x^2}{3x^2} = \dfrac{5}{3} \). On both ends, y approaches $y = \frac{5}{3}$ (as $x \to \pm\infty$, $y \to \frac{5}{3}$). A horizontal asymptote occurs at $y = 0$. The possible end behavior is shown in the graph below.

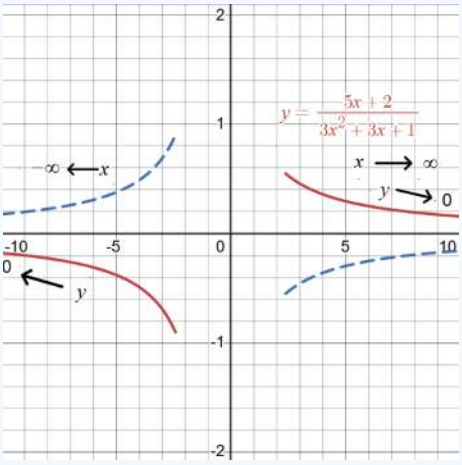


Now You Try: Exercise 4.5.2

Let $f(x) = \dfrac{5x+2}{3x^2+3x+1}$. Determine the end behavior. Identify any horizontal asymptotes if they occur. Then sketch a possible graph of the end behavior of the graph.

Answer

The rational function \[f(x)=\dfrac{5x+2}{3x^2+3x+1} \) has the same end behavior as the ratio of leading terms $y = \dfrac{5x}{3x^2} = \dfrac{5}{3x}$. This is a power in the denominator where y approaches $y = 0$ as $x \to \pm\infty$. A horizontal asymptote occurs at $y = 0$. The possible end behavior is shown in a possible graph below.



Now You Try: Exercise 4.5.3

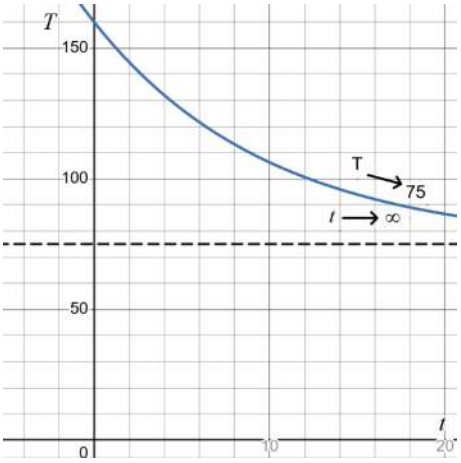
Write a formula for a rational function with no horizontal asymptote.

Answer

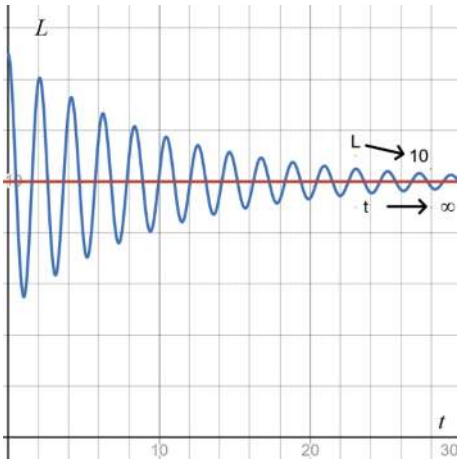
Since the end behavior of a rational function is the same as the ratio of leading terms. If this ratio simplifies to a constant such as $y = k$ or a power of x in the denominator, a horizontal asymptote would occur. Therefore, the ratio must simplify to a power of x in the numerator such as $y = x^3$ which opens up on the right and down on the left. The rational function $\dfrac{x^5+2x^2+1}{x^2+4x}$ has the same end behavior as the ratio of leading terms $\dfrac{x^5}{x^2}$ which simplifies to $y = x^3$

Focus: Horizontal Asymptotes in Applications

Horizontal Asymptotes describe many natural phenomena. For example, if you bring a hot cup of tea with a temperature of 160 $\circ$F into a room with a temperature of 75 $\circ$F. The temperature T of the cup of tea will decrease over time to room temperature (as $t \to \infty, T \to 75$). This describes the right end behavior of the graph with a horizontal asymptote occurring at $T = 75$ as shown in the graph below.



Suppose a mass is attached to a spring with a natural length of 10 inches. If the mass is pulled 5 inches then released , the mass will oscillate back and forth. Over time the amplitude of the oscillations will decrease over time to zero. This describes the right end behavior of the length of the spring L (as $t \to \infty, L \to 0$) with a horizontal asymptote occurring at $L = 10$ as shown in the graph below.



Focus: The Zeros of a Rational Function

When is a fraction zero? Fractions such as $\frac{0}{3}$, $\frac{0}{8}$, and $\frac{0}{2}$ are all equal to zero. A fraction is zero when the numerator is zero and the denominator is not. If the denominator is also zero, such as $\frac{0}{0}$, the fraction is undefined.

Key Point: 4.5.3

The zeros of a rational function occur when the numerator is zero at a defined value.

Example 4.5.1

Find the zeros of the rational functions.

a. $f(x) = \dfrac{x-1}{x-4}$

b. $g(x) = \dfrac{x^2 - x - 6}{x^2 - 4}$

Solution

1. The zeros occur when the numerator $x - 1 = 0$. Solving for x we have $x = 1$. Since the denominator $x - 4 = 1 - 4 = 3 \neq 0$ at $x = 1$, $x = 1$ is a zero of $f(x)$.
2. Setting the numerator to zero and solving by factoring

$$\begin{aligned} x^2 - x - 6 &= 0 \\ (x - 3)(x + 2) &= 0 \\ x - 3 = 0 \text{ and } x &= +2 \\ \text{so } x = 3 \text{ and } x &= -2 \end{aligned}$$

We have possible zeros $x = 3$ and $x = -2$. Since the denominator $x^2 - 4 = 3^2 - 4 = 5 \neq 0$ at $x = 3$, $x = 3$ is a zero of $g(x)$. However, since the denominator $x^2 - 4 = (-2)^2 - 4 = 0$ at $x = -2$, $x = -2$ is not a zero of $g(x)$. g is undefined at $x = -2$.

Focus: The Undefined Values of Rational Functions

Whereas all polynomials are defined for all real values of x, some rational functions may be undefined at certain values of x since division by zero is undefined as we saw in example 4.5.1. The rational function $f(x) = \dfrac{x - 1}{x - 3}$ is undefined when the denominator $x - 3 = 0$. This occurs when $x = 3$.

Key Point 4.5.3

The undefined values of a rational function occur when the denominator is zero.

Example 4.5.2

Identify where the undefined values occur for each function.

a. $f(x) = \dfrac{x + 2}{2x - 3}$

b. $g(x) = \dfrac{x - 1}{x^2 + 5x + 6}$

Solution

The undefined values occur when the denominator of a rational function is zero.

a. Setting the denominator to zero and solving for x

$$\begin{aligned} \text{den.} &= 2x - 3 = 0 \\ 2x &= 3 \\ x &= \frac{3}{2} \end{aligned}$$

So $f(x)$ is undefined at $x = \dfrac{3}{2}$.

b. Setting the denominator to zero and solving by factoring for x

$$\begin{aligned} \text{den.} &= x^2 + 5x + 6 = 0 \\ (x + 2)(x + 3) &= 0 \\ x + 2 = 0 \text{ and } x + 3 &= 0 \\ x = -2 \text{ and } x &= -3 \end{aligned}$$

So $g(x)$ is undefined at $x = -2$ and $x = -3$.

This brings up an important question, what happens on the graph near an undefined value? Let's explore two different cases. A fraction can be simplified or not. A simplified fraction, such as $\dfrac{2}{5}$, has no common factor among the numerator and denominator whereas an unsimplified fraction, such as $\dfrac{4}{6}$, does have a common factor of two among the numerator and denominator.

Case: Simplified Rational Functions

Let's explore behavior of the graph of a simplified rational function $y = f(x) = \frac{1}{x}$ near the undefined value at $x = 0$. Now x can near zero from values above (or greater than) zero or from values below (or less than) zero. Notice what happens to the value of $y = \frac{1}{x}$ as x approaches zero from each side in the table below.

Approaching x=0 from above

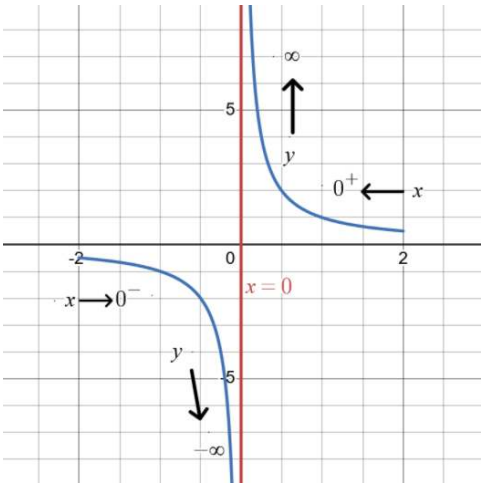
x	1	$\frac{1}{10} = 0.1$	$\frac{1}{100} = 0.01$	$\frac{1}{1000} = 0.001$	$\to 0$
$y = \frac{1}{x}$	1	10	100	1000	$\to \infty$

Approaching x = 0 from below

x	-1	$-\frac{1}{10} = -0.1$	$-\frac{1}{100} = -0.01$	$-\frac{1}{1000} = -0.001$	$\to 0$
$y = \frac{1}{x}$	-1	-10	-100	-1000	$\to -\infty$

Notice as x approaches zero from above, the value of y approaches infinity. Also, as x approaches zero from below, the value of y approaches negative infinity.

The graph of $y = f(x) = \frac{1}{x}$ near the undefined value at $x = 0$ is shown below.



Notice as x approaches zero from above, the value of y approaches infinity, the graph nears the horizontal line $x = 0$ on the top portion of the graph. Also, as x approaches zero from below, the value of y approaches negative infinity, the graph also nears the vertical line $x = 0$ on the bottom portion of the graph. The line $x = 0$ is called **a vertical asymptote** of the function $f(x) = \frac{1}{x}$. The line $x = 0$ is not a part of the graph, but rather describes what the part of the graph nears as x nears zero. The notation $x \to 0^+$ to indicate that x is approaching zero from above. The notation $x \to 0^-$ to indicate that x is approaching zero from below.

Definition:Vertical Asymptote

The function $y = f(x)$ has a **vertical asymptote** at an undefined value $x = a$ if as x approaches a , y approaches ∞ or $-\infty$ from above or below $x = a$. On the graph of $y = f(x)$, the graph nears the vertical line $x = a$.

Let's explore behavior of the graph of another the simplified rational function $y = f(x) = \frac{1}{x^2}$ near the undefined value at $x = 0$ in the tables and graph below.

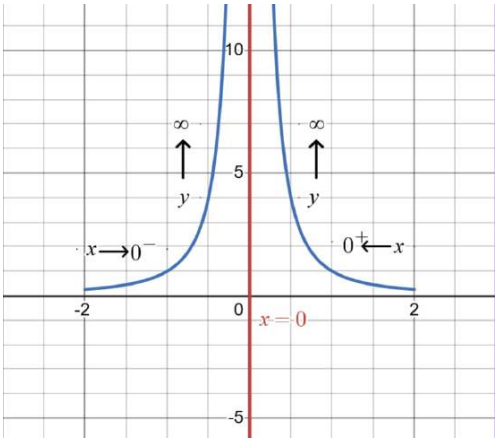
Approaching x=0 from above

x	1	$\frac{1}{10} = 0.1$	$\frac{1}{100} = 0.01$	$\frac{1}{1000} = 0.001$	$\to 0$
$y = \frac{1}{x^2}$	1	100	10,000	1,000,000	$\to \infty$

Approaching x = 0 from below

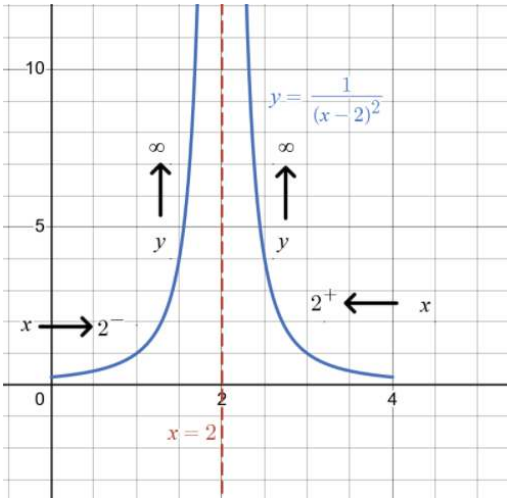
x	-1	$-\frac{1}{10} = -0.1$	$-\frac{1}{100} = -0.01$	$-\frac{1}{1000} = -0.001$	$\to 0$
$y = \frac{1}{x^2}$	-1	-100	-10,000	-1,000,000	$\to -\infty$

Notice as x approaches zero from above and below, the value of y approaches infinity. The graph of $y = f(x) = \frac{1}{x^2}$ near the undefined value at $x = 0$ is shown below.



Notice as x approaches zero from above and below, the value of y approaches infinity, the graph nears the horizontal line $x = 0$ on the top portion of the graph. The graph also approaches the vertical line $x = 0$. The line $x = 0$ is **a vertical asymptote** of $y = \frac{1}{x^2}$. On the graph of the two functions $y = \frac{1}{x}$ and $y = \frac{1}{x^2}$ we see that the graph can either approach ∞ or $-\infty$ on either side of the vertical asymptote depending on the function itself.

If $y = f(x) = \frac{1}{x^2}$, then $y = g(x) = f(x-2) = \frac{1}{(x-2)^2}$ is a horizontal shift of $y = \frac{1}{x^2}$ two units to the right. This horizontal shift shifts the vertical asymptote two units to the right to $x = 2$ where $g(x) = \frac{1}{(x-2)^2}$ is undefined as shown in the graph below.

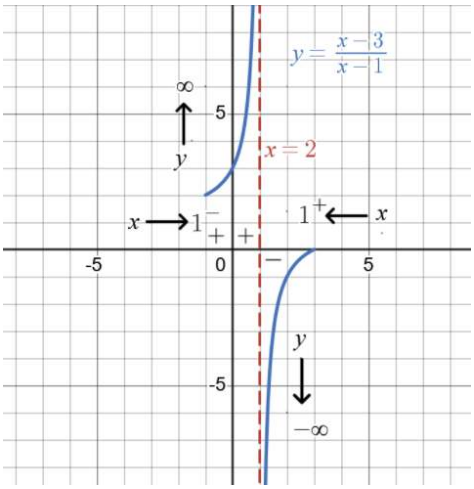


In general, $y = g(x) = f(x-a) = \dfrac{1}{(x-a)^2}$ is a horizontal shift of $y = \dfrac{1}{x^2}$ a units to the right. This horizontal shift shifts the vertical asymptote a units to the right to $x = a$ where $g(x) = \dfrac{1}{(x-a)^2}$ is undefined. In general, if $y = f(x)$ is a simplified rational function and the value of x is nearing an undefined value at $x = a$, then the factor $(x-a)$ in the denominator is nearing zero and so the whole denominator is approaching zero. Furthermore, since the rational function is simplified, $(x-a)$ is not a factor of the numerator and so the numerator is approaching a non-zero value. Therefore, the value of the rational function $y = f(x)$ gets larger and larger in size approaching either positive or negative infinity as the value of x as approaches the undefined value at $x = a$.

Key Point 4.5.4

At an undefined value $x = a$ of a simplified rational function, a vertical asymptote occurs at $x = a$ on the graph.

Although we can identify that a vertical asymptote occurs at an undefined value of a simplified rational function, we don't have enough information to determine which direction the graph approaches on either side of the vertical asymptote. The function value could approach positive or negative infinity as x nears the undefined value. For example, $y = \dfrac{x-3}{x-1}$ is undefined at $x = 1$ since this value of x makes the denominator zero. If we knew that f was positive below the undefined value at $x = 1$ on the interval $(-\infty, 1)$, then we could conclude that the value of y approaches positive infinity as x nears $x = 1$ from below. If we knew that f is negative above $x = 1$ on the interval $(1, 3)$, then we could conclude that the value of y approaches negative infinity as x nears $x = 1$ from above. With this information, we can graph the function near the vertical asymptote as shown below.

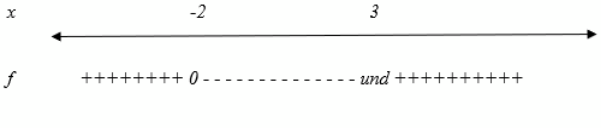


Now we have almost all the tools needed to graph simplified rational functions.

Example 4.5.3

Let $y = \dfrac{x+2}{x-3}$.

a. Identify the undefined value(s). Determine if a vertical asymptote occurs at the undefined value.
b. Find the zero(s).
c. Describe the end behavior.
d. Use part a-c and the sign chart for f to graph the function.



Solution

a. The undefined value(s) occur when the denominator is zero. Setting the denominator to zero and solving

$$x - 3 = 0 \text{ and so } x = 3$$

So f is undefined at $x = 3$. Since f is a simplified rational function, a vertical asymptote occurs at $x = 3$.

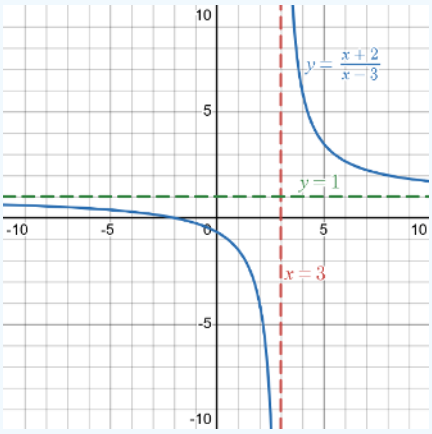
b. The zeros of a simplified rational function occur when the numerator is zero. Setting the numerator to zero and solving

$$x + 2 = 0 \text{ and so } x = -2$$

So f has a zero of $x = -2$ with multiplicity one.

c. The end behavior of f resembles the end behavior of the ratio of leading terms $y = \frac{x}{x} = 1$. So as $x \to \pm\infty$, $y \to 1$. The graph has a horizontal asymptote at $y = 1$.

d.



Example 4.5.4

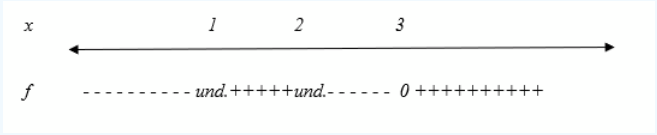
Let $y = \dfrac{x-3}{x^2 - 3x + 2}$.

a. Identify the undefined value(s). Determine if a vertical asymptote occurs at the undefined value.

b. Find the zero(s).

c. Describe the end behavior.

d. Use part a-c and the sign chart for f to graph the function.



Solution

a. The undefined value(s) occur when the denominator is zero. Setting the the denominator to zero and solving by factoring

$$\begin{aligned} \text{den.} &= x^2 - 3x + 2 = 0 \\ (x-2)(x-1) &= 0 \\ x - 2 = 0 &\text{ and } x - 1 = 0 \\ x = 2 &\text{ and } x = 1 \end{aligned}$$

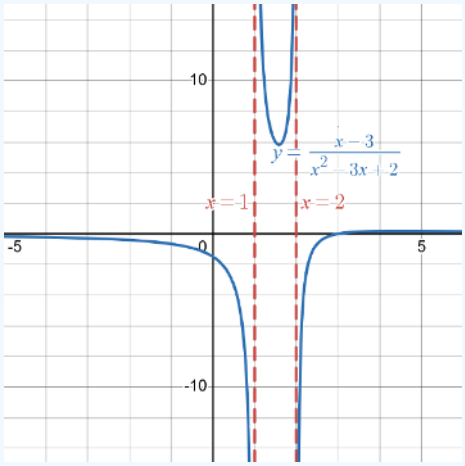
So f is undefined at $x = 2$ and $x = 1$. Since f is a simplified rational function, a vertical asymptote occurs at $x = 2$ and $x = 1$.

b. The zeros of a simplified rational function occur when numerator is zero. Setting the the denominator to zero and solving

$$\begin{aligned} \text{den.} &= x - 3 = 0 \\ x &= 3 \end{aligned}$$

So f has a zero of $x = 3$ with multiplicity one.

c. The end behavior of f resembles the end behavior of the ratio of leading terms $y = \frac{x}{x^2} = \frac{1}{x}$. This is a power in the denominator which approaches $y = 0$ as x approaches $\pm\infty$. The graph has a horizontal asymptote at $y = 0$

d. 

In general, we may not know at first glance whether a rational function is simplified or not. It is critical that we verify that the function is simplified before we determine whether a vertical asymptote occurs at an undefined value. We will explore what happens at undefined values of unsimplified rational functions later in this section.

Now You Try: Exercise 4.5.4

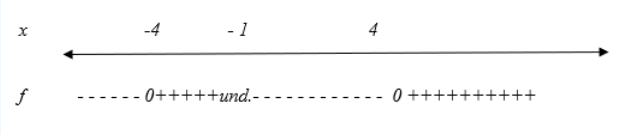
Let $y = \frac{x^2 - 16}{x + 1}$.

a. Identify the undefined value(s). Determine if a vertical asymptote occurs at the undefined value.

b. Find the zero(s).

c. Describe the end behavior.

d. Use part a-c and the sign chart for f to graph the function.



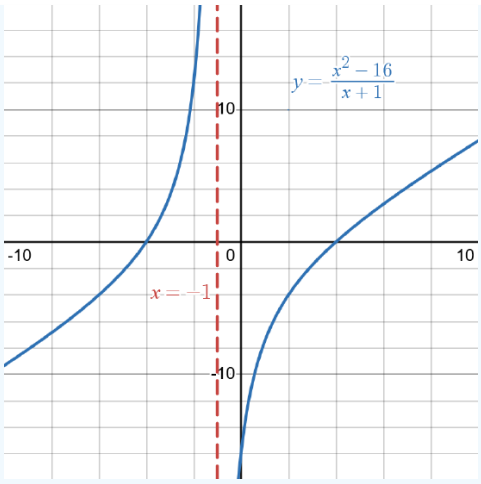
Answer

a. f is undefined at $x = -1$. A vertical asymptote occurs at $x = 1$ since $f(x) = \frac{(x-4)(x-4)}{x+1}$ is a simplified.

b. The zeros are $x = \pm 4$ each with multiplicity one.

c. The end behavior of f resembles the end behavior of $y = x$. This is a odd power which opens up on the right (as $x \to \infty, y \to \infty$) and down on the left (as $x \to -\infty, y \to -\infty$).

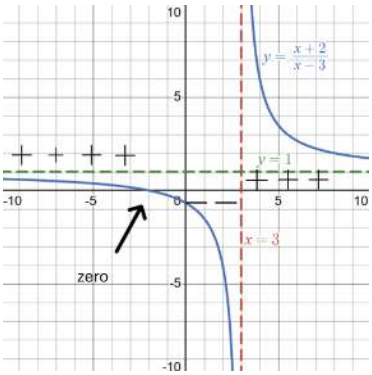
d. The graph has a horizontal asymptote at $y = 0$.

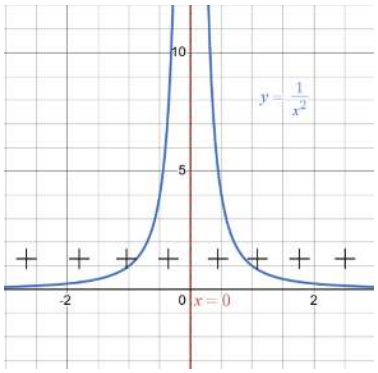
e. 

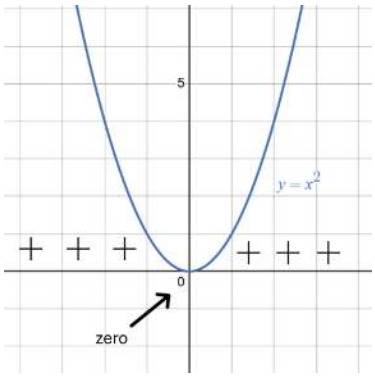
These examples highlight an important piece of graphing rational functions. We need the sign of each interval along with the zeros, undefined values, and end behavior to graph a simplified rational function.

Focus: Determining The Sign of a Rational Function and More Graphing Rational Functions

Unlike polynomials where we could use the graph to determine the sign of the function on different intervals, we need the sign of the rational function (or other information) in order to graph it. How can we determine the sign of each interval for a rational function? We can use an extension of the test point method we used to determine the sign of polynomials. Notice on the graph of the function $f(x) = \dfrac{x+2}{x-3}$ in example 4.5.3 (shown below left), the function changes sign only at the zero $x = -2$ and the undefined value $x = 3$. Whereas the graph of the function $f(x) = \dfrac{1}{x^2}$ in the introductory example (show below middle), the function did not change sign at the undefined value at $x = 0$. The function was positive above and below $x = 0$. We learned in section 4.3 that a polynomial may or may not change sign at a zero. Consider, the graph of $y = x^2$ (below right). The function does not change sign at the zero $x = 0$. Notice between consecutive zeros and/or undefined values the graphs do not change sign.







Key Point 4.5.5

A rational function will only change sign at a zero or an undefined value. Between consecutive zeros and/or undefined values the function does not change signs. It is either strictly increasing or strictly decreasing.

Since the rational function will only change sign at a zero or an undefined value, we can determine the sign of f in an interval between consecutive zeros and/or and undefined value using a test point. If the value of f is positive at the test point, the function is positive on that interval since it can only change sign at the zero or the undefined value. If the value of f is negative at the test point, the function is negative on that interval for the same reason. With this adaption of the test point method, we have a strategy to build a sign chart for f and determine the sign of f on each interval.

Example 4.5.5

Let $f(x) = \dfrac{2x-1}{x-4}$. Determine where f is positive and where f is negative.

Solution

First, we need to find the zeros and undefined values where f can change sign. Setting the denominator to zero and solving to find the undefined value(s)

$$\begin{aligned} \text{den.} &= x - 4 = 0 \\ x &= 4 \end{aligned}$$

So f has an undefined value at $x = 4$. Setting the numerator to zero and solving to find the zero(s)

$$\begin{aligned} \text{den.} &= 2x - 1 = 0 \\ 2x &= 1 \\ x &= \frac{1}{2} \end{aligned}$$

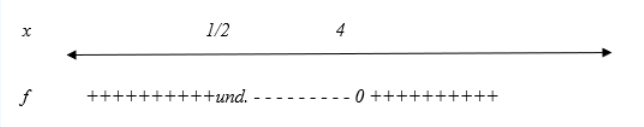
So f has a zero at $x = \dfrac{1}{2}$ since f is defined here.

The zero and undefined value break up the x-axis in to three intervals. Now we need to evaluate f at a test point to determine the sign of each interval.

Building a sign chart for f using the test point method

Interval	$(-\infty, \frac{1}{2})$	$(\frac{1}{2}, 4)$	$(4, \infty)$
Test Point	0	1	5
Value	$\dfrac{2(0)-1}{(0)-4} = \dfrac{1}{4}$	$\dfrac{2(1)-1}{(1)-4} = \dfrac{1}{-3}$	$\dfrac{2(5)-1}{(5)-4} = \dfrac{9}{1} = 9$
Sign of The Interval	positive	negative	positive

So f is positive on $(-\infty, \frac{1}{2}) \cup (4, \infty)$ and negative on the interval $(\frac{1}{2}, 4)$. Although not necessary in this problem, it may be helpful to display this result with a sign chart for f when graphing f.



Example 4.5.6

Let $f(x) = \dfrac{x+1}{x^2-9}$.

a. Identify the undefined value(s). Determine if a vertical asymptote occurs at the undefined value.
b. Find the zero(s).
c. Identify the end behavior.
d. Determine the sign of each interval.
e. Use parts (a)-(d) to graph f.

Solution

a. Setting the denominator to zero and solving to find the undefined value(s)

$$\begin{aligned} \text{den.} &= x^2 - 9 = 0 \\ (x-3)(x+3) &= 0 \\ x - 3 = 0 &\text{ and } x + 3 = 0 \\ x = 3 &\text{ and } x = -3 \end{aligned}$$

Since the rational function $f(x) = \dfrac{x+1}{(x-3)(x+3)}$ is simplified, vertical asymptotes occur at $x = 3$ and $x = 3$.

b. Setting the numerator to zero and solving to find the zero(s)

$$\text{den.} = x + 1 = 0$$
$$x = -1$$

So the zero occurs at $x = -1$.

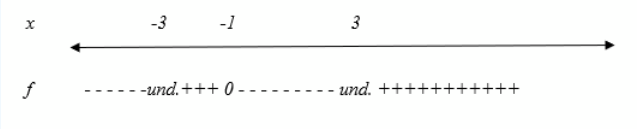
c. The end behavior of f resembles the end behavior of the ratio of leading terms $y = \dfrac{x}{x^2} = \dfrac{1}{x}$. This is a power in the denominator which approaches $y = 0$ as x approaches ∞. Therefore a horizontal asymptote occurs at $y = 0$.

d. The zero and undefined value break up the x-axis in to three intervals. Evaluate f at a test point to determine the sign of each interval.

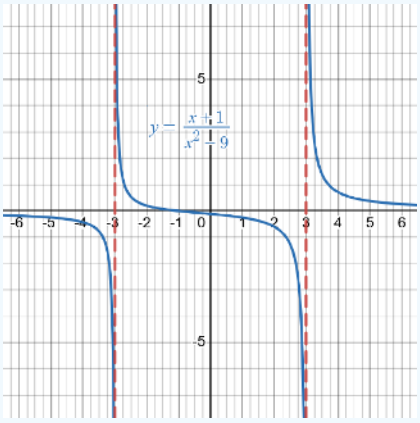
Building a sign chart for f using the test point method

Interval	$(-\infty, -3)$	$(-3, -1)$	$(-1, 3)$	$(3, \infty)$
Test Point	-4	-2	0	4
Value	$\dfrac{(-4)+1}{(-4)^2-9} = \dfrac{-3}{7}$	$\dfrac{(-2)+1}{(-2)^2-9} = \dfrac{1}{5}$	$\dfrac{(0)+1}{(0)^2-9} = \dfrac{1}{-9}$	$\dfrac{(4)+1}{(4)^2-9} = \dfrac{5}{16}$
Sign of The Interval	negative	positive	negative	positive

So f is positive on $(-3, -1) \cup (3, \infty)$ and negative on the interval $(-\infty, -3) \cup (-1, 3)$.



e.

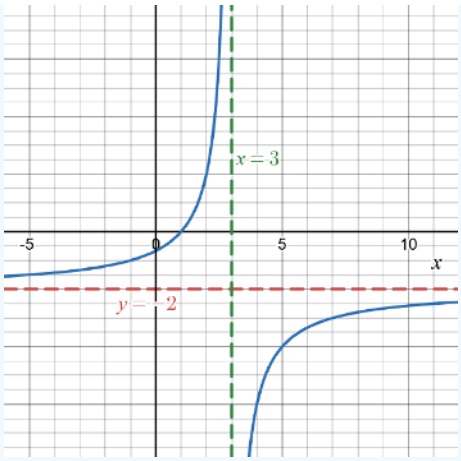


Since we have a sign chart for f in the example above, the multiplicity of the zero is redundant information in determining whether the graph crosses the x-axis or not at the zero as illustrated in example 4.5.6 above. If a zero has odd multiplicity and the graph crosses the x-axis at the zero, the sign changes at the zero. If a zero has even multiplicity and the graph does not cross the x-axis at the zero, the sign does not change at the zero. The multiplicity of a zero will help confirm what the sign chart for f already determines for us, whether the graph crosses the x-axis or not at the zero.

Alternatively if we are given the graph of a rational function or information that would help us graph the function, we can use these same concepts to build a formula for the rational function.

Example 4.5.7

Identify a possible formula for a rational function whose graph is



Solution

Since the rational function has a zero of $x = 1$, $x = 1$ is a zero of the numerator and so $(x - 1)$ is a factor of the numerator. Since there is a vertical asymptote at $x = 3$, $x = 3$ is a zero of the denominator and so$(x - 3)$ is a factor of the denominator. However, the rational function $f(x) = \frac{x-1}{x-3}$ has a horizontal asymptote at $y = 1$ since the end behavior of $y = \frac{(x-1)}{x-3}$ has the same end behavior as $y = \frac{x}{x} = 1$. We can use a stretch factor a to alter end behavior. The end behavior of $y = \frac{a(x-1)}{x-3}$ has the same end behavior as $y = \frac{ax}{x} = a$. Therefore $a = 2$ and so

$$f(x) = \frac{-2(x-1)}{x-3}$$

? Now You Try: Exercise 4.5.5

Let $f(x) = \frac{3x-1}{2x+5}$.

a. Identify the undefined value(s). Determine if a vertical asymptote occurs at the undefined value.
b. Find the zeros
c. Identify the end behavior
d. Determine the sign of each interval
e. Use parts (a)-(d) to graph f

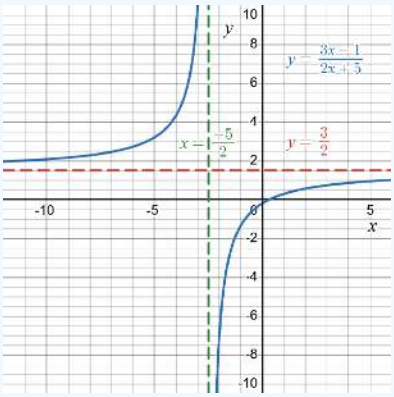
Answer

a. The undefined value occurs at $x = \frac{-5}{2}$ Since the rational function is simplified, a vertical asymptote occurs at $\frac{-5}{2}$.
b. $x = \frac{1}{3}$ is the zero.
c. The end behavior of f resembles the end behavior of $y = \frac{3x}{2x} = \frac{3}{2}$. Therefore $y \to \frac{3}{2}$ as $x \to \infty$. So a horizontal asymptote occurs at $\frac{3}{2}$.
d. Using the test point method, f is positive on $(-\infty, \frac{-5}{2}) \cup (\frac{1}{3}, \infty)$ and negative on the interval $(\frac{-5}{2}, \frac{1}{3})$

Building a sign chart for f using the test point method

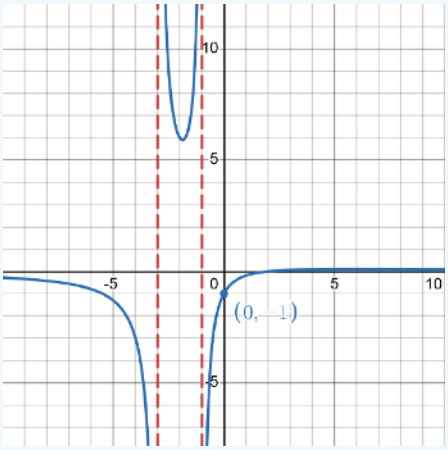
Interval	$(-\infty, \frac{-5}{2})$	$(\frac{-5}{2}, \frac{1}{3})$	$(\frac{1}{3}, \infty)$
Test Point	-3	0	1
Value	10	$\frac{-1}{5}$	$\frac{2}{9}$
Sign of The Interval	positive	negative	positive

x		$-5/2$		$1/3$	
f	$++++++++++$	$und.$	$----------$	0	$++++++++++$



Now You Try: Exercise 4.5.6

Identify a possible formula for a rational function whose graph is



Answer

$f(x) = \frac{1.5(x-2)}{(x+3)(x+1)}$. Notice that in this case, we had to use the y-intercept to find the stretch factor a in $f(x) = \frac{a(x-2)}{(x+3)(x+1)}$ since the end behavior is unchanged by a stretch factor. The end behavior resembles $f(x) = \frac{ax}{x^2} = \frac{a}{x}$. This is a power in the denominator where $y \to 0$ as $x \to \infty$ no matter what value we have for a.

Focus: Applications of Vertical Asymptotes

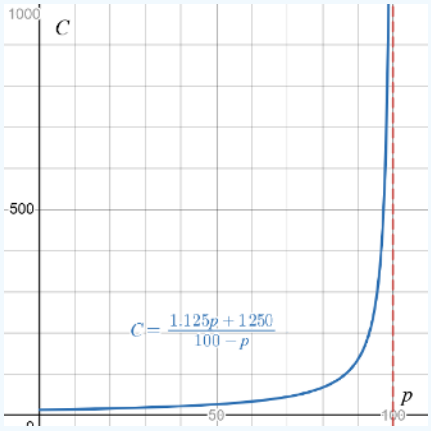
Vertical Asymptotes describe many phenomena in applications in economics, physics, and engineering. Let's explore one.

Example 4.5.8

Suppose the cost of removing p percent of a pollutant from 1000 gallons of water is modeled by the function $C = f(p) = \frac{1.125p + 1250}{100 - p}$ where $0<p<100$. Identify the undefined value(s). Interpret the meaning of the vertical asymptote in terms of cost and the percent of pollutants removed.

Solution

The undefined value occurs at $p = 100$ which makes the denominator $100 - p = 0$. A zero occurs when the numerator $1.125p + 1250 = 0$ Solving for p we have a zero at $p = \frac{-1250}{1.125} \approx -1111.11$ which is not in the domain of f. Therefore the cost function does not change sign in the domain $0<p<100$. Since $f(1) = 12.6$, f is positive on this interval. Therefore, the value of C approaches $+\infty$ as p approaches 100 and the graph nears the vertical asymptote as shown below. Effectively this means that as the percent removed approaches 100%, the cost of removing it gets infinitely large. In this scenario, it is unrealistic that all of the pollutant can be removed given the cost.



Focus: The Undefined Values of Simplified Rational Functions

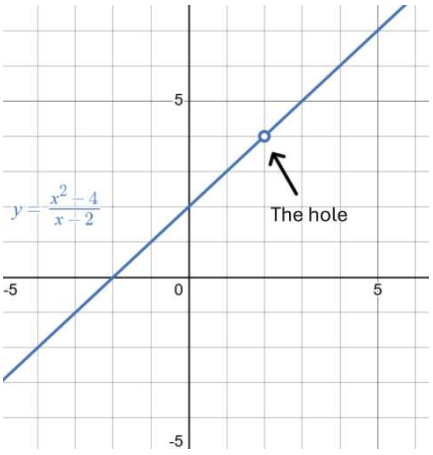
A fraction such as $\frac{4}{6}$ is unsimplified since it contains a common factor of two in the numerator and denominator. It can be simplified as

$$\frac{4}{6} = \frac{2 \cdot 2}{2 \cdot 3} = \frac{2}{3}$$

However, zero can't be a common factor in a fraction since the fraction would be undefined

$$\frac{0 \cdot 2}{0 \cdot 3} = \frac{0}{0}$$

For example, the function $f(x) = \frac{x^2 - 4}{x - 2}$ is undefined at $x = 2$ since this value makes the denominator zero. Notice, the fraction is unsimplified since there is a common factor of $(x - 2)$. We can simplify $f(x) = \frac{x^2 - 4}{x - 2} = \frac{(x + 2)(x - 2)}{x - 2} = x + 2$ as long as the common factor $(x - 2) \neq 0$ which occur when $x \neq 2$. Therefore the functions $y = \frac{x^2 - 4}{x - 2}$ and $y = x + 2$ are equivalent when $x \neq 2$. The graph of $y = x + 2$ is a line with slope $m = 2$ and y-intercept at $y = 2$. Since the function f is not defined at $x = 2$, we need to remove the point on the line at $x = 2$ leaving a hole in the graph below represented by an open dot.



In general, to simplify an unsimplified rational function, the zero corresponding the common factor in the numerator and denominator will need to be removed from the domain of the simplified fraction for the two functions to be equivalent. On the graph of the simplified fraction, the corresponding point will need to be removed from the graph leaving a hole.

Key Point 4.5.6

On the graph of an unsimplified rational function, a hole occurs at the undefined value $x = a$ corresponding to the zero of a common factor $(x - a)$ of the numerator and denominator.

Example 4.5.9

Let $f(x) = \dfrac{x^2 - 1}{x^2 - 3x + 2}$.

a. Identify the undefined value(s). Determine what happens on the graph near each undefined value.

b. Use the zeros, the undefined values, the end behavior and a sign chart for f to graph the function.

Solution

a. Setting the denominator is equal to zero and solving by factoring to find the undefined values

$$x^2 - 3x + 2 = 0$$
$$(x - 2)(x - 1) = 0$$
$$x = 2,\ 1$$

To determine what happens at each undefined value, we need to simplify the fraction

$$f(x) = \frac{x^2 - 1}{x^2 - 3x + 2} = \frac{(x - 1)(x + 1)}{(x - 1)(x - 2)} = \frac{x + 1}{x - 2}$$

provided that $x \neq 2$ and $x \neq 1$. Since the undefined value $x = 1$ corresponding to the common factor $(x - 1)$ in the numerator and denominator, a hole occurs on the graph of $f(x)$ at $x = 1$. Since $x = 2$ is an undefined value of the resulting simplified fraction $y = \dfrac{x + 1}{x - 2}$, a vertical asymptote occurs at $x = 2$.

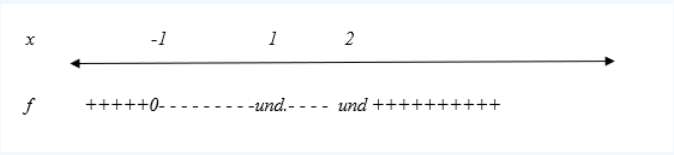
b. Setting the numerator of the simplified fraction is equal to zero and solving to find the zero(s)

$$x + 1 = 0$$
$$x = -1$$

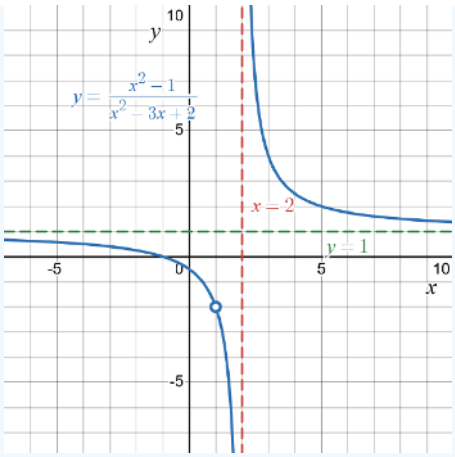
So f has a zero at $x = -1$. The end behavior resembles the end behavior of $y = \dfrac{x}{x} = 1$. Therefore as $x \to \pm\infty$, $y \to 1$. So f has a horizontal asymptote at $y = 1$. Building a sign chart for f using the test point method shown below, we see that f is positive on $(-\infty, -1) \cup (2, \infty)$ and negative on $(-1, 1) \cup (1, 2)$

Building a sign chart for f using the test point method

Interval	$(-\infty, -1)$	$(-1, 1)$	$(1, 2)$	$(2, \infty)$
Test Point	-2	0	1.5	3
Value	$\dfrac{(-2) + 1}{(-2) - 2} = \dfrac{1}{4}$	$\dfrac{(0) + 1}{(0) - 2} = \dfrac{1}{-2}$	$\dfrac{(1.5) + 1}{(1.5) - 2} = -5$	$\dfrac{(-2) + 1}{(-2) - 2} = \dfrac{1}{4}$
Sign of The Interval	positive	negative	negative	positive



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It is possible that a rational function could change sign at an undefined value corresponding to hole on the graph. That is why we tested the function f in example 4.5.9 in the intervals before and after the hole at $x = 1$.

Focus: Solving Rational Inequalities

Solving the rational inequality $\frac{x-3}{x-1} > 0$ is equivalent to finding where $y = \frac{x-3}{x-1}$ is positive. Solving the rational inequality $\frac{x-3}{x-1} < 0$ is equivalent to finding where $y = \frac{x-3}{x-1}$ is negative. On the otherhand, the inequality $\frac{1}{x} > x$ can't be solved in this manner since x and $\frac{1}{x}$ may be either positive or negative depending on the value of x. However, if we get zero on one side of this inequality $\frac{1}{x} - x > 0$, then we can solve this inequality by determining where $y = \frac{1}{x} - x$ is positive.

Key Point 4.5.7

A rational inequality of the form $y = \frac{f(x)}{g(x)} > 0$ or $y = \frac{f(x)}{g(x)} < 0$ can be solved by determining the sign of the rational function.

We can use the sign chart for f using the test point method to determine the sign of f on each interval instead of graphing the entire function.

Example 4.5.10

Solve the inequality $\frac{x+2}{x^2-3x} > 0$

Solution

We can solve this inequality by determining where $y = \frac{x+2}{x^2-3x}$ is positive using a sign chart. Setting the denominator to zero and solving to find the undefined value(s)

$$x^2 - 3x = 0$$
$$x(x-3) = 0$$
$$x = 0 \text{ and } x = 3$$

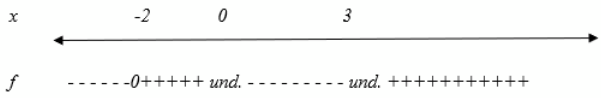
Setting the numerator to zero and solving to find the zero(s)

$$x + 2 = 0$$
$$x = -2$$

Building a sign chart for f using the test point method shown below

A sign chart for function f

Interval	$(-\infty,-2)$	$(-2,0)$	$(0,3)$	$(3,\infty)$
Test Point	-3	-1	1	4
Value	$\frac{(-3)+2}{(-3)^2-3(-3)}=\frac{-1}{18}$	$\frac{(-1)+2}{(-1)^2-3(-1)}=\frac{1}{4}$	$\frac{(1)+2}{(1)^2-3(1)}=\frac{3}{-2}$	$\frac{(4)+2}{(4)^2-3(4)}=\frac{6}{4}=\frac{3}{2}$
Sign of The Interval	negative	positive	negative	positive



Therefore, the solution to the inequality is $(-2,0)\cup(3,\infty)$.

Example 4.5.11

Solve the inequality $\frac{1}{x}\leq x$

Solution

First, getting zero on one side.

$$\frac{1}{x}-x\leq 0$$

We can solve this inequality by determining where $y=\frac{1}{x}-x$ is negative or zero using a sign chart. However, this function is not written as a single fraction that we can analyze as a rational function. Getting a common denominator and adding we have

$$y=\frac{1}{x}-x=\frac{1}{x}-\frac{x^2}{x}=\frac{1-x^2}{x}$$

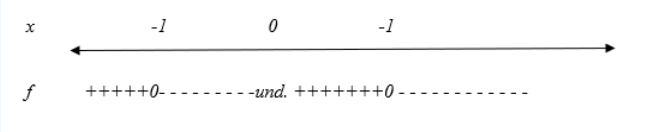
The undefined values occur when the denominator is zero. So x is undefined at $x=0$. The zeros occur when the numerator is zero (and the denominator is not zero)

$$\begin{aligned}1-x^2&=0\\(1-x)(1+x)&=0\\1-x=0 \text{ and } 1+x&=0\\x=1 \text{ and } x&=-1\end{aligned}$$

Building a sign chart for f using the test point method shown below

A sign chart for function f

Interval	$(-\infty,-1)$	$(-1,0)$	$(0,1)$	$(1,\infty)$
Test Point	-2	-0.5	0.5	2
Value	$\frac{1-(-2)^2}{(-2)}=\frac{3}{2}$	$\frac{1-(0.5)^2}{(-0.5)}=\frac{-3}{2}$	$\frac{1-(0.5)^2}{(0.5)}=\frac{3}{2}$	$\frac{1-(2)^2}{(2)}=\frac{-3}{2}$
Sign of The Interval	positive	negative	positive	negative



Therefore, the solution to the inequality is $[-1,0)\cup[1,\infty)$.

? Now You Try: Exercise 4.5.7

Solve the inequality $\frac{x-4}{x+2} \geq 0$

Answer

The solution to the inequality is $(-\infty, -2) \cup [4, \infty)$.

Important Topics of this Section

- The End Behavior of Rational Functions
- Horizontal Asymptotes
- Zeros of Rational Functions
- Vertical Asymptotes
- Rational Functions
- Holes
- Graphing Rational Functions
- Rational Inequalities

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4.5E: Describing Relationships with Rational Functions (Exercises)

Section 4.5 Exercises

Part 1: Simplified Rational Functions

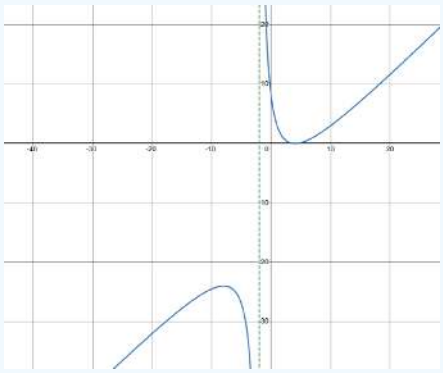
In exercises #1-4, (a) identify the undefined value(s) (b) Find the zero(s); (c) identify the end behavior; (d) match each equation form with one of the graphs without technological assistance.

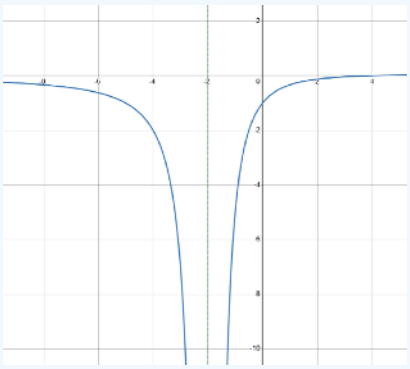
1. $f(x) = \dfrac{x-4}{x+2}$

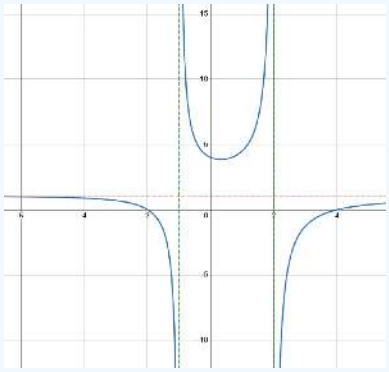
2. $g(x) = \dfrac{x^2-8x+16}{x+2}$

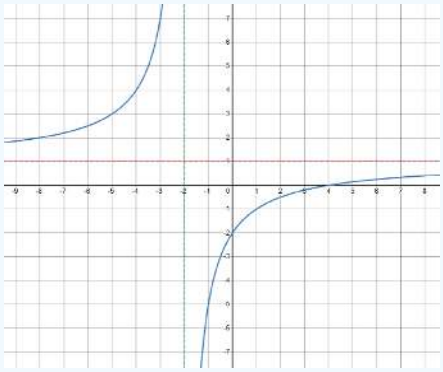
3. $h(x) = \dfrac{x-4}{x^2+4x+4}$

4. $k(x) = \dfrac{x^2-2x-8}{x^2-x-2}$

A. 

B. 

C. 

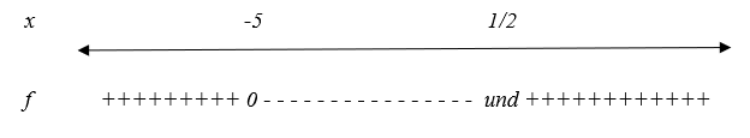
D. 

In exercises #5 -12, for the simplified rational function: (a) identify the undefined value(s) (b) Find the zero(s); (c) identify the end behavior (if a horizontal asymptote occurs, identify the horizontal asymptote); (d) use the given sign chart to graph the function without technological assistance.

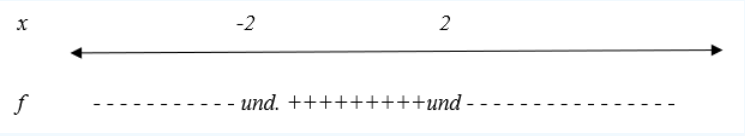
5. $f(x) = \dfrac{x-2}{x+1}$

x	-1	2
f	+++++++++ und. ----------	0 +++++++++++++++++

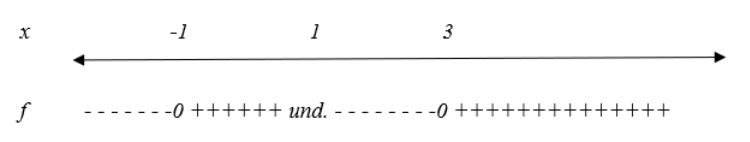
6. $g(x) = \dfrac{x+5}{2x-1}$



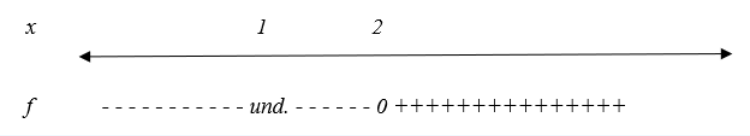
7. $h(x) = \dfrac{-4}{x^2 - 4x + 4}$



8. $k(x) = \dfrac{x^2 - 2x - 3}{x - 1}$



9. $g(x) = \dfrac{x - 2}{x^2 - 2x + 1}$

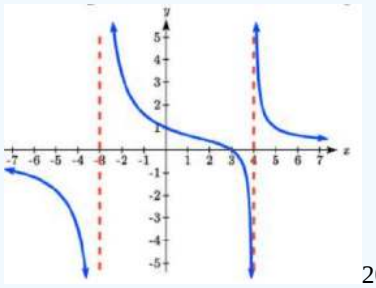


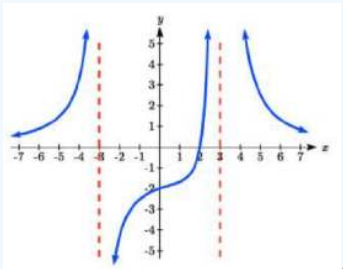
Part 2: General Rational Functions

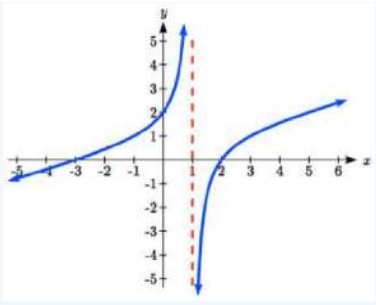
In exercises #10 -19, for the rational function: (a) identify the undefined value(s) and determine whether a vertical asymptote or hole occurs at the undefined value; (b) Find the zero(s); (c) identify the end behavior (if a horizontal asymptote occurs, identify the horizontal asymptote); (d) determine the sign of each interval; and (e) use this information to graph the function without technological assistance.

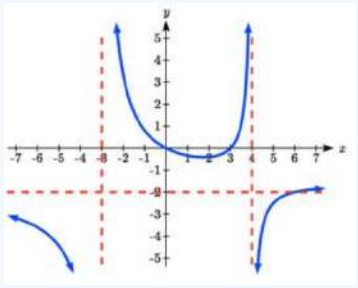
10. $p(x) = \dfrac{2x - 3}{x + 4}$
11. $q(x) = \dfrac{x - 5}{3x - 1}$
12. $s(x) = \dfrac{4}{x^2 - 4x + 4}$
13. $f(x) = \dfrac{x^2 + x - 6}{x^2 - 25}$
14. $g(x) = \dfrac{2x^2 + 7x - 15}{3x^2 - 14x + 15}$
15. $a(x) = \dfrac{x^2 + 2x - 3}{x^2 - 1}$
16. $b(x) = \dfrac{x^2 - x - 6}{x^2 - 4}$
17. $k(x) = \dfrac{2x^2 - 3x - 20}{x - 5}$
18. $f(x) = \dfrac{3x^2 - 14x - 5}{3x^2 + 8x - 16}$
19. $n(x) = \dfrac{3x^2 + 4x - 4}{x^3 - 4x^2}$

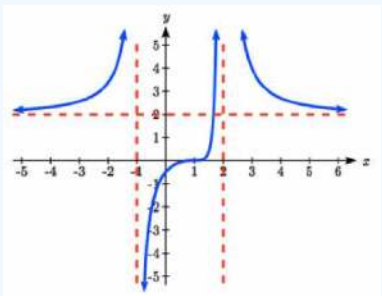
In exercises #19 -24, find a possible formula for a rational function whose graph is given below.

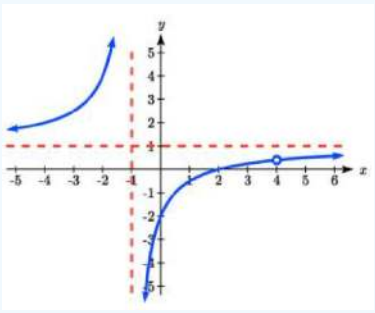
19. 

20. 

21. 

22. 

23. 

24. 

25. A scientist has a beaker containing 20 mL of a solution containing 20% acid. To dilute this, she adds pure water. The concentration in the beaker after adding n mL of water is given by $C(n) = \dfrac{4}{20+n}$

a. Find the concentration if 10 mL of water has been added.

c. How many mL of water must be added to obtain a 4% solution?

d. What is the right end behavior as $n \to \infty$? What is the physical significance of this?

26. A gauss meter is a device for measuring the strength and polarity of magnetic fields. Suppose the following two measurements are made. At a distance of 8 feet from a magnet, the meter reads 2.3. At a distance of 6 feet, the meter reads 4.4. Then the meter reading $m(x)$ can modeled by the rational function \(m(x)=\dfrac{134.9}{x^2-5.33\) where x represents the distance in feet from a magnet.

a. How far must he go for the meter to reach 10? 100?

b. What is the right end behavior as $n \to \infty$? What is the physical significance of this?

27. The speed s it takes to cover a distance of 20 miles is modeled by the function $s = \dfrac{20}{t}$. Find the vertical asymptote. What is the physical significance of the vertical asymptote?

Part 3: Rational Inequalities

In #28 - 31, solve the rational inequalities. Express your result in interval notation.

28. $\dfrac{x+1}{x-3} \geq 0$

29. $\dfrac{x-2}{x^2-4x} < 0$

30. $\dfrac{1}{x-1} > 2$

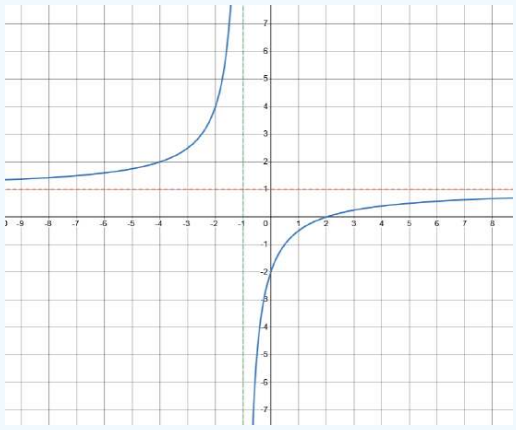
31. $\dfrac{3}{x-2} \leq \dfrac{2}{x}$

Answer

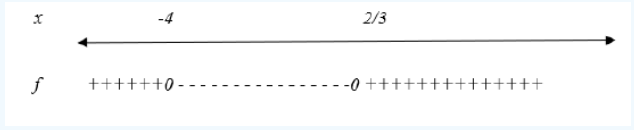
1. (a) $f(x)$ has an undefined value at $x = -2$. A vertical asymptote occurs here. (b) $f(x)$ has a zero at $x = 4$ (c) As $x \to \pm\infty$, $y \to 1$. A horizontal asymptote occurs at $y = 1$. (d) Graph D

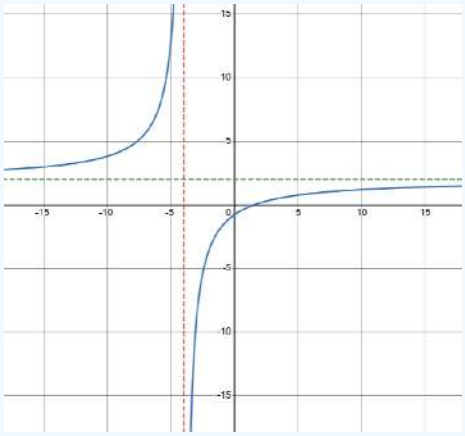
(d)

5. (a) $f(x)$ has an undefined value at $x = -1$. A vertical asymptote occurs here. (b) $f(x)$ has a zero at $x = 2$ (c) As $x \to \pm\infty$, $y \to 1$. A horizontal asymptote occurs at $y = 1$. (d)

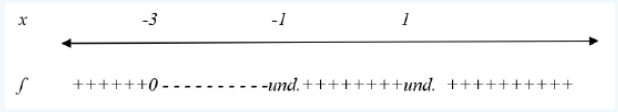
(d) 

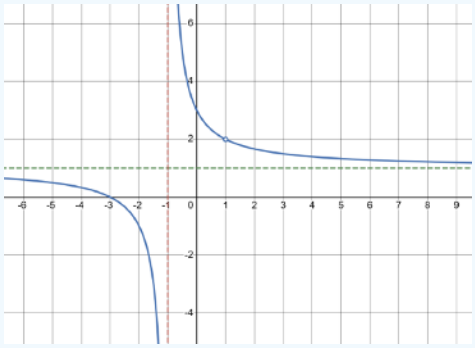
10. (a) $p(x)$ has an undefined value at $x = -4$. A vertical asymptote occurs here since p is simplified. (b) $p(x)$ has a zero at $x = \frac{2}{3}$ (c) As $x \to \pm\infty$, $y \to 2$. A horizontal asymptote occurs at $y = 2$.

(d) 

(e) 

15. (a) $a(x)$ has undefined values at $x = 1$ and $x = -1$. A vertical asymptote occurs at $x = -1$. A hole occurs at $x = 1$. (b) $a(x)$ has a zero at $x = -3$ (c) As $x \to \pm\infty$, $y \to 1$. A horizontal asymptote occurs at $y = 1$.

(d) 

(e) 

19. $q(x) = \dfrac{x-3}{(x+3)(x-4)}$

28. $(-\infty, -1] \cup (3, \infty)$

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CHAPTER OVERVIEW

5: Describing Relationships With Trigonometric Functions

5.1: An Introduction to the Trigonometric Functions

5.1E: An Introduction to the Trigonometric Functions (Exercises)

5.2: Solving Trigonometric Equations with Sine and Cosine

5.2E: Solving Trigonometric Equations with Sine and Cosine (Exercises)

5.3: Graphing The Sine and Cosine Functions

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5.5: Right Triangle Trigonometry

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5.6: Simpliying and Solving Equations with Trigonometric Functions

5.6E: Simpliying and Solving Equations with Trigonometric Functions (Exercises)

5.7: More on Simpliying and Solving Equations with Trigonometric Functions

5.7E: More on Simpliying and Solving Equations with Trigonometric Functions (Exercises)

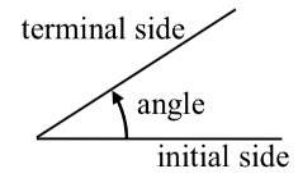
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5.1: An Introduction to the Trigonometric Functions

Because many applications involving circles also involve a rotation of the circle, it is natural to introduce a measure for the rotation, or angle, between two rays (line segments) emanating from the center of a circle. The angle measurement you are most likely familiar with is degrees, so we'll begin there.

Definition: measure of an angle

The **measure of an angle** is a measurement between two intersecting lines, line segments or rays, starting at the **initial side** and ending at the **terminal side**. It is a rotational measure, not a linear measure.



Measuring Angles

Definition: degrees

A **degree** is a measurement of angle. One full rotation around the circle is equal to 360 degrees, so one degree is 1/360 of a circle.

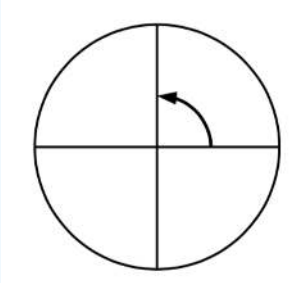
An angle measured in degrees should always include the unit "degrees" after the number, or include the degree symbol $^\circ$. For example, 90 degrees = 90°.

Definition: standard position

When measuring angles on a circle, unless otherwise directed, we measure angles in **standard position**: starting at the positive horizontal axis and with counter-clockwise rotation.

Example 5.1.1

Give the degree measure of the angle shown on the circle.



Solution

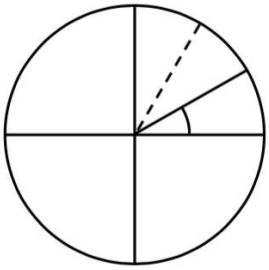
The vertical and horizontal lines divide the circle into quarters. Since one full rotation is 360 degrees=360°, each quarter rotation is 360/4 = 90° or 90 degrees.

Example 5.1.2

Show an angle of 30° on the circle.

Solution

An angle of 30° is 1/3 of 90°, so by dividing a quarter rotation into thirds, we can sketch a line at 30°.



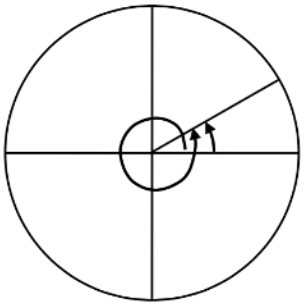
Going Greek

When representing angles using variables, it is traditional to use Greek letters. Here is a list of commonly encountered Greek letters.

θ	$varphi$ or ϕ	α	β	γ
theta	phi	alpha	beta	gamma

Working with Angles in Degrees

Notice that since there are 360 degrees in one rotation, an angle greater than 360 degrees would indicate more than 1 full rotation. Shown on a circle, the resulting direction in which this angle's terminal side points would be the same as for another angle between 0 and 360 degrees. These angles would be called **coterminal**.



coterminal angles

After completing their full rotation based on the given angle, two angles are **coterminal** if they terminate in the same position, so their terminal sides coincide (point in the same direction).

Example 5.1.3

Find an angle θ that is coterminal with 800°, where $0^\circ \leq \theta < 360^\circ$

Solution

Since adding or subtracting a full rotation, 360 degrees, would result in an angle with terminal side pointing in the same direction, we can find coterminal angles by adding or subtracting 360 degrees. An angle of 800 degrees is coterminal with an angle of 800 - 360 = 440 degrees. It would also be coterminal with an angle of 440 - 360 = 80 degrees.

The angle $\theta = 80^\circ$is coterminal with 800°.

By finding the coterminal angle between 0 and 360 degrees, it can be easier to see which direction the terminal side of an angle points in.

Exercise 5.1.1

Find an angle α that is coterminal with 870°, where $0^\circ \leq \alpha < 360^\circ$.

Answer

$$\alpha = 870 - 360 - 360 = 150^\circ$$

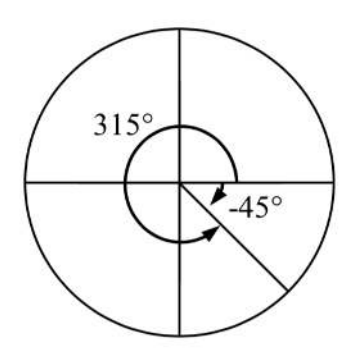
On a number line a positive number is measured to the right and a negative number is measured in the opposite direction (to the left). Similarly a positive angle is measured counterclockwise and a negative angle is measured in the opposite direction

(clockwise).

Example 5.1.4

Show the angle -45°on the circle and find a positive angle α that is coterminal and $0^\circ \leq \alpha < 360^\circ$.

Solution



Since 45 degrees is half of 90 degrees, we can start at the positive horizontal axis and measure clockwise half of a 90 degree angle.

Since we can find coterminal angles by adding or subtracting a full rotation of 360 degrees, we can find a positive coterminal angle here by adding 360 degrees:

$$-45^\circ + 360^\circ = 315^\circ$$

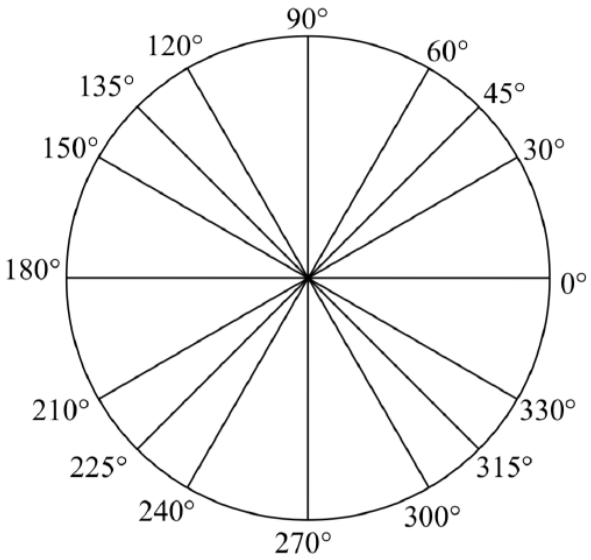
Exercise 5.1.2

Find an angleβ coterminal with -300° where $0^\circ \leq \beta < 360^\circ$.

Answer

$$\beta = -300 + 360 = 60^\circ$$

It can be helpful to have a familiarity with the frequently encountered angles in one rotation of a circle. It is common to encounter multiples of 30, 45, 60, and 90 degrees. These values are shown to the right. Memorizing these angles and understanding their properties will be very useful as we study the properties associated with angles



Angles in Radians

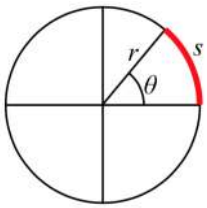
While measuring angles in degrees may be familiar, doing so often complicates matters since the units of measure can get in the way of calculations. For this reason, another measure of angles is commonly used. This measure is based on the distance around a circle.

Definition: arclength

Arclength is the length of an arc, s, along a circle of radius r subtended (drawn out) by an angle θ.

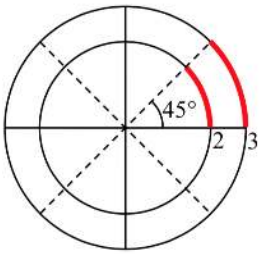
It is the portion of the circumference between the initial and terminal sides of the angle.

The length of the arc around an entire circle is called the circumference of a circle. The circumference of a circle is $C = 2\pi r$. The ratio of the circumference to the radius, produces the constant 2π. Regardless of the radius, this ratio is always the same, just as how the degree measure of an angle is independent of the radius.



To elaborate on this idea, consider two circles, one with radius 2 and one with radius 3. Recall the circumference (perimeter) of a circle is $C = 2\pi r$, where r is the radius of the circle. The smaller circle then has circumference $2\pi(2) = 4\pi$ and the larger has circumference$2\pi(3) = 6\pi$.

Drawing a 45 degree angle on the two circles, we might be interested in the length of the arc of the circle that the angle indicates. In both cases, the 45 degree angle draws out an arc that is $1/8^{th}$ of the full circumference, so for the smaller circle, the arclength = $\frac{1}{8}(4\pi) = \frac{1}{2}\pi$, and for the larger circle, the length of the arc or arclength = $\frac{1}{8}(6\pi) = \frac{3}{4}\pi$.



Notice what happens if we find the *ratio* of the arclength divided by the radius of the circle:

Smaller circle: $\frac{\frac{1}{2}\pi}{2} = \frac{1}{4}\pi$

Larger circle: $\frac{\frac{3}{4}\pi}{3} = \frac{1}{4}\pi$

The ratio is the same regardless of the radius of the circle – it only depends on the angle. This property allows us to define a measure of the angle based on arclength.

radians

The **radian measure** of an angle is the ratio of the length of the circular arc subtended by the angle to the radius of the circle.

In other words, if s is the length of an arc of a circle, and r is the radius of the circle, then

radian measure$= \frac{s}{r}$

If the circle has radius 1, then the radian measure corresponds to the length of the arc.

Because radian measure is the ratio of two lengths, it is a **unitless measure**. It is not necessary to write the label “radians” after a radian measure, and if you see an angle that is not labeled with “degrees” or the degree symbol, you should assume that it is a radian measure.

Considering the most basic case, the unit circle (a circle with radius 1), we know that 1 rotation equals 360 degrees, 360°. We can also track one rotation around a circle by finding the circumference, $C = 2\pi r$, and for the unit circle $C = 2\pi$. These two different ways to rotate around a circle give us a way to convert from degrees to radians.

1 rotation = 360°=2π radians

1/2 rotation = 180° = π radians

1/4 rotation = 90°= $\frac{\pi}{2}$ radians

Example 5.1.5

Find the radian measure of one third of a full rotation.

Solution

For any circle, the arclength along such a rotation would be one third of the circumference, $C = \frac{1}{3}(2\pi r) = \frac{2\pi r}{3}$. The radian measure would be the arclength divided by the radius:

$$\text{Radian measure} = \frac{2\pi r}{3r} = \frac{2\pi}{3} \tag{5.1.1}$$

converting between radians and degrees

1 degree = $\frac{\pi}{180}$ radians

or: to convert from degrees to radians, multiply by $\frac{\pi \text{radians}}{180^\circ}$

1 radian = $\frac{180}{\pi}$ degrees

or: to convert from radians to degrees, multiply by $\frac{180^\circ}{\pi \text{radians}}$

Converting Between Radians and Degrees

Example 5.1.6

Convert $\frac{\pi}{6}$ radians to degrees.

Solution

Since we are given a problem in radians and we want degrees, we multiply by $\frac{180^\circ}{\pi}$.

Remember radians are a unitless measure, so we don't need to write "radians."

$\frac{\pi}{6}$ radians = $\frac{\pi}{6} \cdot \frac{180^\circ}{\pi} = 30$ degrees.

Example 5.1.7

Convert 15 degrees to radians.

Solution

In this example, we start with degrees and want radians so we use the other conversion $\frac{\pi}{180^\circ}$ so that the degree units cancel and we are left with the unitless measure of radians.

15 degrees = $15^\circ \cdot \frac{\pi}{180^\circ} = \frac{\pi}{12}$

Exercise 5.1.3

Convert $\frac{7\pi}{10}$ radians to degrees.

Answer

$$\frac{7\pi}{10} \cdot \frac{180^\circ}{\pi} = 126^\circ$$

Just as we listed all the common angles in degrees on a circle, we should also list the corresponding radian values for the common measures of a circle corresponding to degree multiples of 30, 45, 60, and 90 degrees. As with the degree measurements, it would be advisable to commit these to memory.

We can work with the radian measures of an angle the same way we work with degrees.

Example 5.1.8

Find an angle β that is coterminal with $\frac{19\pi}{4}$, where $0 \leq \beta < 2\pi$.

Solution

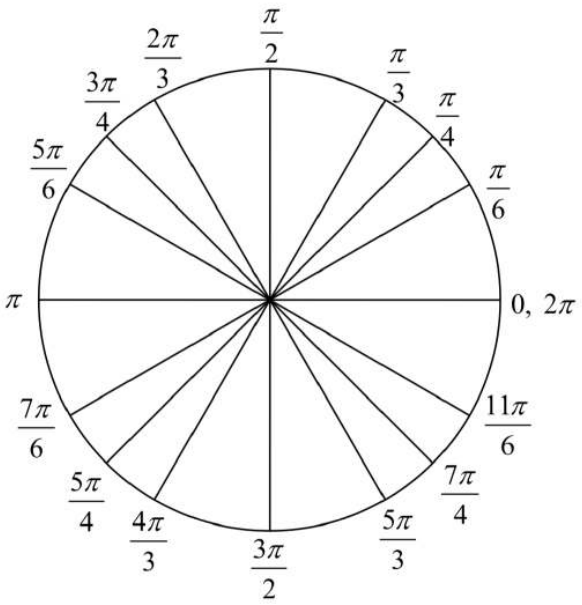
When working in degrees, we found coterminal angles by adding or subtracting 360 degrees, a full rotation. Likewise, in radians, we can find coterminal angles by adding or subtracting full rotations of 2π radians.

$$\frac{19\pi}{4} - 2\pi = \frac{19\pi}{4} - \frac{8\pi}{4} = \frac{11\pi}{4}$$

The angle $\frac{11\pi}{4}$ is coterminal, but not less than 2π, so we subtract another rotation.

$$\frac{11\pi}{4} - 2\pi = \frac{11\pi}{4} - \frac{8\pi}{4} = \frac{3\pi}{4}$$

The angle $\frac{3\pi}{4}$ is coterminal with $\frac{19\pi}{4}$.



? Exercise 5.1.4

Find an angle ϕthat is coterminal with $-\frac{17\pi}{6}$ where $0 \leq \phi < 2\pi$.

Answer

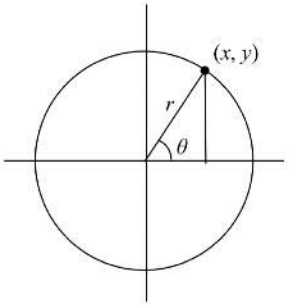
$$-\frac{17\pi}{6} + 2\pi + 2\pi = -\frac{17\pi}{6} + \frac{12\pi}{6} + \frac{12\pi}{6} = \frac{7\pi}{6}$$

Focus: An Introduction to the Trigonometric Functions

While it is convenient to describe the location of a point on a circle using an angle or a distance along the circle, relating this information to the *x* and *y* coordinates and the circle equation we explored in Section 5.1 is an important application of trigonometry.

A distress signal is sent from a sailboat during a storm, but the transmission is unclear and the rescue boat sitting at the marina cannot determine the sailboat's location. Using high powered radar, they determine the distress signal is coming from a distance of 20 miles at an angle of 225 degrees from the marina. How many miles east/west and north/south of the rescue boat is the stranded sailboat?

In a general sense, to investigate this, we begin by drawing a circle centered at the origin with radius r, and marking the point on the circle indicated by some angle θ. This point has coordinates (x, y).



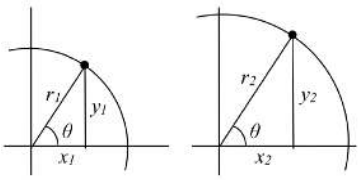
If we drop a line segment vertically down from this point to the *x* axis, we would form a right triangle inside of the circle.

No matter which quadrant our angle θ puts us in we can draw a triangle by dropping a perpendicular line segment to the x axis, keeping in mind that the values of x and y may be positive or negative, depending on the quadrant.

Additionally, if the angle θ puts us on an axis, we simply measure the radius as the x or y with the other value being 0, again ensuring we have appropriate signs on the coordinates based on the quadrant.

Triangles obtained from different radii will all be similar triangles, meaning corresponding sides scale proportionally. While the lengths of the sides may change, as we saw in the last section, the *ratios* of the side lengths will always remain constant for any given angle.

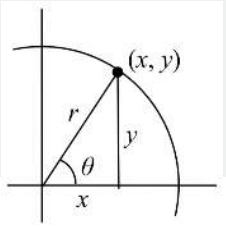
$$\frac{y_1}{r_1} = \frac{y_2}{r_2}$$
$$\frac{x_1}{r_1} = \frac{x_2}{r_2}$$



To be able to refer to these ratios more easily, we will give them names. Since the ratios depend on the angle, we will write them as functions of the angle θ.

Note: sine and cosine

For the point (x, y) on a circle of radius r at an angle of θ, we can define two important functions as the ratios of the sides of the corresponding triangle:



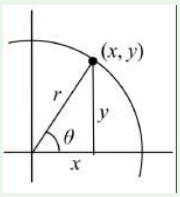
The **sine** function: $\sin(\theta) = \frac{y}{r}$

The **cosine** function: $\cos(\theta) = \frac{x}{r}$

In the previous section, we defined the sine and cosine functions as ratios of the sides of a right triangle in a circle. Since the triangle has 3 sides there are 6 possible combinations of ratios. While the sine and cosine are the two prominent ratios that can be formed, there are four others, and together they define the 6 trigonometric functions.

Definition: Tangent, Secant, Cosecant, and Cotangent Functions

For the point (x, y) on a circle of radius r at an angle of θ, we can define four additional important functions as the ratios of the sides of the corresponding triangle:



- The **tangent** function:

$$\tan(\theta) = \frac{y}{x} \tag{5.1.2}$$

- The **secant** function:

$$\sec(\theta) = \frac{r}{x} \tag{5.1.3}$$

- The **cosecant** function:

$$\csc(\theta) = \frac{r}{y} \tag{5.1.4}$$

- The **cotangent** function:

$$\cot(\theta) = \frac{x}{y} \tag{5.1.5}$$

Geometrically, notice that the definition of tangent corresponds with the slope of the line segment between the origin (0, 0) and the point (x, y). This relationship can be very helpful in thinking about tangent values.

In this chapter, we will explore these functions using both circles and right triangles. In the next chapter, we will take a closer look at the behavior and characteristics of the sine and cosine functions.

Example 5.1.1

The point (3, 4) is on the circle of radius 5 at some angle θ. Find $\cos(\theta)$and $\sin(\theta)$.

Solution

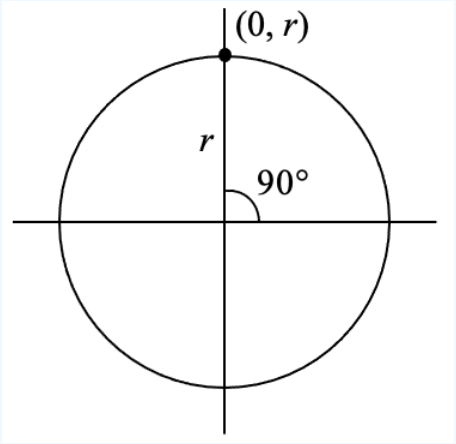
Knowing the radius of the circle and coordinates of the point, we can evaluate the cosine and sine functions as the ratio of the sides.

$$\cos(\theta) = \frac{x}{r} = \frac{3}{5} \sin(\theta) = \frac{y}{r} = \frac{4}{5}$$

There are a few cosine and sine values which we can determine fairly easily because the corresponding point on the circle falls on the x or y axis.

Example 5.1.2

Find $\cos(90^\circ)$ and $\sin(90^\circ)$.



Solution

On any circle, the terminal side of a 90 degree angle points straight up, so the coordinates of the corresponding point on the circle would be (0, *r*). Using our definitions of cosine and sine,

$$\cos(90^\circ) = \frac{x}{r} = \frac{0}{r} = 0$$

$$\sin(90^\circ) = \frac{y}{r} = \frac{r}{r} = 1$$

Exercise 5.1.1

Find cosine and sine of the angle π.

Answer

$$\cos(\pi) = -1 \sin(\pi) = 0$$

Notice that the definitions above can also be stated as:

coordinates of the point on a circle at a given angle

On a circle of radius r at an angle of θ, we can find the **coordinates of the point (x, y)** Circles:Points on a Circle at that angle using

$$x = r\cos(\theta) \tag{5.1.6}$$

$$y = r\sin(\theta) \tag{5.1.7}$$

On a unit circle, a circle with radius 1, $x = \cos(\theta)$ and $y = \sin(\theta)$.

Utilizing the basic equation for a circle centered at the origin, $x^2 + y^2 = r^2$, combined with the relationships above, we can establish a new identity.

$$x^2 + y^2 = r^2$$

substituting the relations above,

$$(r\cos(\theta))^2 + (r\sin(\theta))^2 = r^2$$

simplifying,

$$r^2(\cos(\theta))^2 + r^2(\sin(\theta))^2 = r^2$$

dividing by r^2

$$(\cos(\theta))^2 + (\sin(\theta))^2 = 1$$

or using shorthand notation

$$\cos^2(\theta) + \sin^2(\theta) = 1$$

Here $\cos^2(\theta)$ is a commonly used shorthand notation for $(\cos(\theta))^2$. Be aware that many calculators and computers do not understand the shorthand notation.

In Section 5.1 we related the Pythagorean Theorem $a^2 + b^2 = c^2$ to the basic equation of a circle $x^2 + y^2 = r^2$, which we have now used to arrive at the Pythagorean Identity.

pythagorean indentity

The **Pythagorean Identity**. For any angle θ,

$$\cos^2(\theta) + \sin^2(\theta) = 1$$

One use of this identity is that it helps us to find a cosine value of an angle if we know the sine value of that angle or vice versa. However, since the equation will yield two possible values, we will need to utilize additional knowledge of the angle to help us find the desired value.

Example 5.1.3

If $\sin(\theta) = \frac{3}{7}$ and θ is in the second quadrant, find $\cos(\theta)$.

Solution

Substituting the known value for sine into the Pythagorean identity,

$$\cos^2(\theta) + \sin^2(\theta) = 1$$

$$\cos^2(\theta) + \frac{9}{49} = 1$$

$$\cos^2(\theta) = \frac{40}{49}$$

$$\cos(\theta) = \pm\sqrt{\frac{40}{49}} = \pm\frac{\sqrt{40}}{7} = \pm\frac{2\sqrt{10}}{7}$$

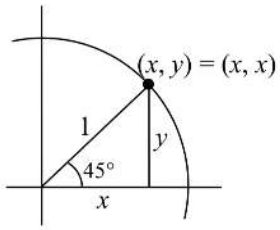
Since the angle is in the second quadrant, we know the x value of the point would be negative, so the cosine value should also be negative. Using this additional information, we can conclude that

$$\cos(\theta) = -\frac{2\sqrt{10}}{7}$$

Values for Sine and Cosine

At this point, you may have noticed that we haven't found any cosine or sine values from angles not on an axis. To do this, we will need to utilize our knowledge of triangles.

First, consider a point on a circle at an angle of 45 degrees, or $\frac{\pi}{4}$. At this angle, the x and y coordinates of the corresponding point on the circle will be equal because 45 degrees divides the first quadrant in half. Since the x and y values will be the same, the sine and cosine values will also be equal. Utilizing the Pythagorean Identity,



$$\cos^2\left(\frac{\pi}{4}\right)+\sin^2\left(\frac{\pi}{4}\right)=1$$

since the sine and cosine are equal, we can substitute sine with cosine

$$\cos^2\left(\frac{\pi}{4}\right)+\cos^2\left(\frac{\pi}{4}\right)=1$$

add like terms

$$2\cos^2\left(\frac{\pi}{4}\right)=1$$

divide

$$\cos^2\left(\frac{\pi}{4}\right)=\frac{1}{2}$$

since the x value is positive, we'll keep the positive root

$$\cos\left(\frac{\pi}{4}\right)=\sqrt{\frac{1}{2}}$$

often this value is written with a rationalized denominator

Remember, to rationalize the denominator we multiply by a term equivalent to 1 to get rid of the radical in the denominator.

$$\cos\left(\frac{\pi}{4}\right)=\sqrt{\frac{1}{2}}\sqrt{\frac{2}{2}}=\sqrt{\frac{2}{4}}=\frac{\sqrt{2}}{2}$$

Since the sine and cosine are equal, $\sin\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}$ as well. The (x, y) coordinates for a point on a circle of radius 1 at an angle of 45 degrees are $\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$.

Example 5.1.4

Find the coordinates of the point on a circle of radius 6 at an angle of $\frac{\pi}{4}$.

Solution

Using our new knowledge that $\sin\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}$ and $\cos\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}$, along with our relationships that stated $x=r\cos(\theta)$ and $y=r\sin(\theta)$, we can find the coordinates of the point desired:

$$x=6\cos\left(\frac{\pi}{4}\right)=6\left(\frac{\sqrt{2}}{2}\right)=3\sqrt{2}$$

$$y=6\sin\left(\frac{\pi}{4}\right)=6\left(\frac{\sqrt{2}}{2}\right)=3\sqrt{2}$$

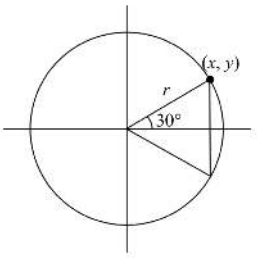
Exercise 5.1.2

Find the coordinates of the point on a circle of radius 3 at an angle of 90°.

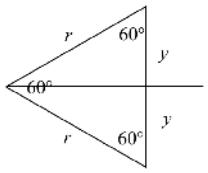
Answer

$$x = 3\cos\left(\frac{\pi}{2}\right) = 3 \cdot 0 = 0$$
$$y = 3\sin\left(\frac{\pi}{2}\right) = 3 \cdot 1 = 3$$

Next, we will find the cosine and sine at an angle of 30 degrees, or $\frac{\pi}{6}$. To do this, we will first draw a triangle inside a circle with one side at an angle of 30 degrees, and another at an angle of -30 degrees. If the resulting two right triangles are combined into one large triangle, notice that all three angles of this larger triangle will be 60 degrees.



Since all the angles are equal, the sides will all be equal as well. The vertical line has length $2y$, and since the sides are all equal we can conclude that $2y = r$, or $y = \frac{r}{2}$. Using this, we can find the sine value:



$$\sin(\frac{\pi}{6}) = \frac{y}{r} = \frac{r/2}{r} = \frac{r}{2} \cdot \frac{1}{r} = \frac{1}{2}$$

Using the Pythagorean Identity, we can find the cosine value:

$$\cos^2\left(\frac{\pi}{6}\right) + \sin^2\left(\frac{\pi}{6}\right) = 1$$

$$\cos^2\left(\frac{\pi}{6}\right) + \left(\frac{1}{2}\right)^2 = 1$$

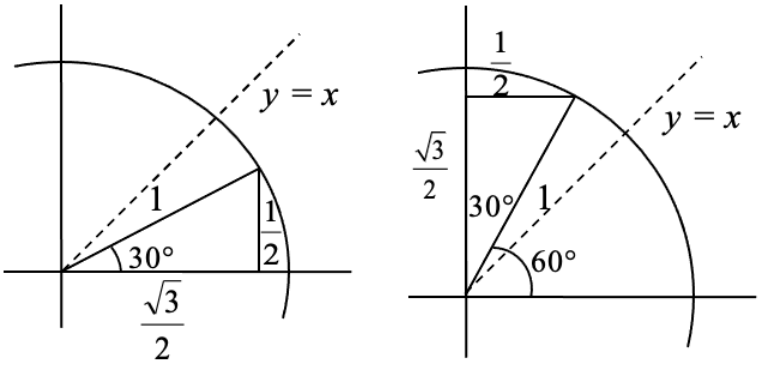
$$\cos^2\left(\frac{\pi}{6}\right) = \frac{3}{4}$$

since the x value is positive, we'll keep the positive root

$$\cos\left(\frac{\pi}{6}\right) = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$

The (x, y) coordinates for the point on a circle of radius 1 at an angle of 30 degrees are $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

By drawing a the triangle inside the unit circle with a 30 degree angle and reflecting it over the line $y = x$, we can find the cosine and sine for 60 degrees, or $\frac{\pi}{3}$, without any additional work.



By this symmetry, we can see the coordinates of the point on the unit circle at an angle of 60 degrees will be $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, giving

$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$ and $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

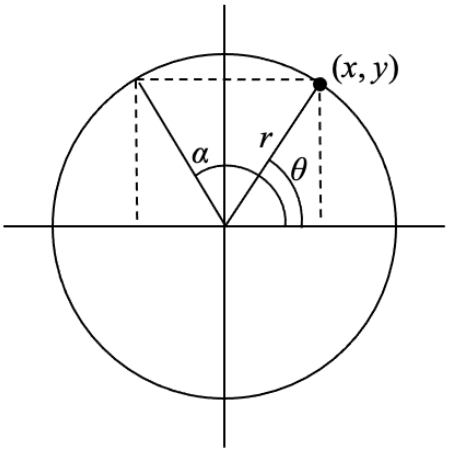
We have now found the cosine and sine values for all the commonly encountered angles in the first quadrant of the unit circle.

Angle	0	$\frac{\pi}{6}$, or 30°	$\frac{\pi}{4}$, or 45°	$\frac{\pi}{3}$, or 60°	$\frac{\pi}{2}$, or 90°
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
Sine	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

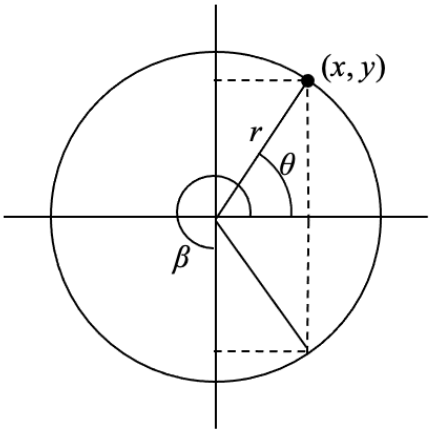
For any given angle in the first quadrant, there will be an angle in another quadrant with the same sine value, and yet another angle in yet another quadrant with the same cosine value. Since the sine value is the y coordinate on the unit circle, the other angle with the same sine will share the same y value, but have the opposite x value. Likewise, the angle with the same cosine will share the same x value, but have the opposite y value.

As shown here, angle α has the same sine value as angle θ; the cosine values would be opposites. The angle β has the same cosine value as the angle θ; the sine values would be opposites.

$\sin(\theta) = \sin(\alpha)$ and $\cos(\theta) = -\cos(\alpha)$



$\sin(\theta) = -\sin(\beta)$ and $\cos(\theta) = \cos(\beta)$



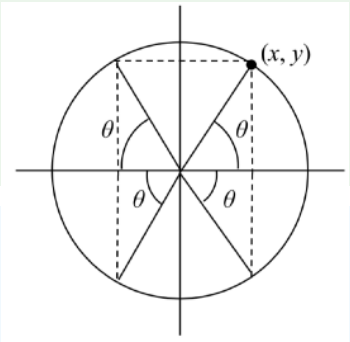
It is important to notice the relationship between the angles. If, from the angle, you measured the smallest angle to the horizontal axis, all would have the same measure in absolute value. We say that all these angles have a **reference angle** of θ.

Definition: reference angle

An angle's **reference angle** is the size of the smallest angle to the horizontal axis.

A reference angle is always an angle between 0 and 90 degrees, or 0 and $\frac{\pi}{2}$ radians.

Angles share the same cosine and sine values as their reference angles, except for signs (positive or negative) which can be determined from the quadrant of the angle.



Example 5.1.5

Find the reference angle of 150 degrees. Use it to find $\cos(150^\circ)$ and $\sin(150^\circ)$.

Solution

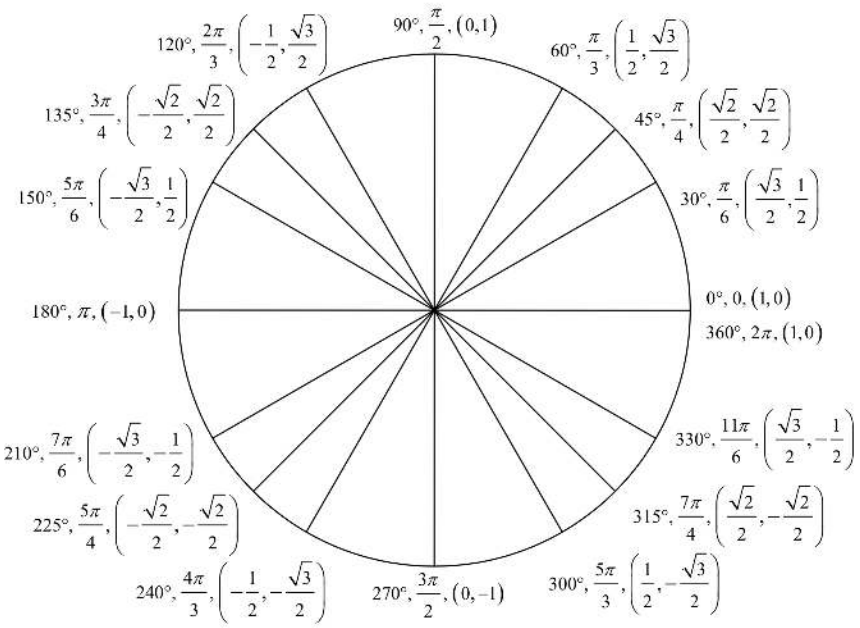
150 degrees is located in the second quadrant. It is 30 degrees short of the horizontal axis at 180 degrees, so the reference angle is 30 degrees.

This tells us that 150 degrees has the same sine and cosine values as 30 degrees, except for sign. We know that $\sin(30^\circ) = \frac{1}{2}$ and $\cos(30^\circ) = \frac{\sqrt{3}}{2}$. Since 150 degrees is in the second quadrant, the x coordinate of the point on the circle would be negative, so the cosine value will be negative. The y coordinate is positive, so the sine value will be positive.

$$\sin(150^\circ) = \frac{1}{2} \text{ and } \cos(150^\circ) = -\frac{\sqrt{3}}{2}$$

The (x, y) coordinates for the point on a unit circle at an angle of 150° are $\left(\frac{-\sqrt{3}}{2}, \frac{1}{2}\right)$.

Using symmetry and reference angles, we can fill in cosine and sine values at the rest of the special angles on the unit circle. Take time to learn the (x, y) coordinates of all the major angles in the first quadrant!



Example 5.1.6

Find the coordinates of the point on a circle of radius 12 at an angle of $\frac{7\pi}{6}$.

Solution

Note that this angle is in the third quadrant, where both *x* and *y* are negative. Keeping this in mind can help you check your signs of the sine and cosine function.

$$x = 12\cos\left(\frac{7\pi}{6}\right) = 12\left(\frac{-\sqrt{3}}{2}\right) = -6\sqrt{3}$$

$$y = 12\sin\left(\frac{7\pi}{6}\right) = 12\left(\frac{-1}{2}\right) = -6$$

The coordinates of the point are $(-6\sqrt{3}, -6)$.

Exercise 5.1.3

Find the coordinates of the point on a circle of radius 5 at an angle of $\frac{5\pi}{3}$.

Answer

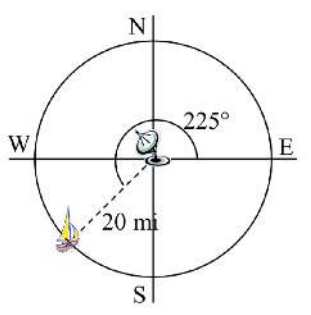
$$\left(5\cos\left(\frac{5\pi}{3}\right), 5\sin\left(\frac{5\pi}{3}\right)\right) = \left(\frac{5}{2}, \frac{-5\sqrt{3}}{2}\right)$$

Example 5.1.7

We now have the tools to return to the sailboat question posed at the beginning of this section.

Solution

A distress signal is sent from a sailboat during a storm, but the transmission is unclear and the rescue boat sitting at the marina cannot determine the sailboat's location. Using high powered radar, they determine the distress signal is coming from a point 20 miles away at an angle of 225 degrees from the marina. How many miles east/west and north/south of the rescue boat is the stranded sailboat?



We can now answer the question by finding the coordinates of the point on a circle with a radius of 20 miles at an angle of 225 degrees.

$$x = 20\cos(225^\circ) = 20\left(\frac{-\sqrt{2}}{2}\right) \approx -14.142 \text{ miles}$$

$$y = 20\sin(225^\circ) = 20\left(\frac{-\sqrt{2}}{2}\right) \approx -14.142 \text{ miles}$$

The sailboat is located 14.142 miles west and 14.142 miles south of the marina.

The special values of sine and cosine in the first quadrant are very useful to know, since knowing them allows you to quickly evaluate the sine and cosine of very common angles without needing to look at a reference or use your calculator. However, scenarios do come up where we need to know the sine and cosine of other angles.

To find the cosine and sine of any other angle, we turn to a computer or calculator. **Be aware**: most calculators can be set into "degree" or "radian" mode, which tells the calculator the units for the input value. When you evaluate "cos(30)" on your calculator, it will evaluate it as the cosine of 30 degrees if the calculator is in degree mode, or the cosine of 30 radians if the calculator is in radian mode. Most computer software with cosine and sine functions only operates in radian mode.

Example 5.1.8

Evaluate the cosine of 20 degrees using a calculator or computer.

Solution

On a calculator that can be put in degree mode, you can evaluate this directly to be approximately 0.939693.

On a computer or calculator without degree mode, you would first need to convert the angle to radians, or equivalently evaluate the expression

$$\cos\left(20 \cdot \frac{\pi}{180}\right)$$

Important Topics of This Section

- Degree measure of angle
- Radian measure of angle
- Conversion between degrees and radians
- Common angles in degrees and radians
- Coterminal angles

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5.1E: An Introduction to the Trigonometric Functions (Exercises)

Section 5.1 Exercises

1. Convert each angle to radian measure.
   a. 120°
   b. 412°
   c. -92°
2. Convert each angle from radians to degrees.
   a. $\frac{5\pi}{6}$
   b. $\frac{11\,\pi}{6}$
3. Find the angle between 0° and 360° that is coterminal to the given angle
   1. 685°
   2. -1400°
   3. $\frac{17\,\pi}{3}$.
   4. $-\frac{7\,\pi}{6}$
4. If the point (-3,6) is a point on the terminal side of the angle θ in standard position, find the exact value of the six trigonometric functions.
5. If the point (5,-12) is a point on the terminal side of the angle θ in standard position, find the exact value of the six trigonometric functions.
6. If $\sin(\theta) = \frac{3}{4}$ and θ is in quadrant II, find the exact value of the remaining trigonometric functions.
7. If $\tan(\theta) = \frac{-5}{12}$ and θ is in quadrant IV, find the exact value of the remaining trigonometric functions.
8. If $\cos(\theta) = -\frac{1}{3}$, and θ is in quadrant II, find the exact value of the remaining trigonometric functions.
9. If $\sin(\theta) = \frac{-1}{5}$ and θ is in quadrant III, find the exact value of the remaining trigonometric functions.
10. If $\tan(\theta) = \frac{12}{5}$, and $0 \le \theta < \frac{\pi}{2}$, find $\sin(\theta)$.
11. If $\cos(\theta) = \frac{-3}{7}$, and $\pi \le \theta < \frac{3\pi}{2}$, find $\tan(\theta)$.
12. If $\sin(\theta) = \frac{4}{9}$, and $\frac{\pi}{2} \le \theta < \pi$, find $\cos(\theta)$.
13. Find the quadrant in which the terminal point determined by t lies if
    a. $\sin(t) < 0$ and $\cos(t) < 0$
    b. $\sin(t) > 0$ and $\cos(t) < 0$
    c. $\tan(t) < 0$ and $\sin(t) > 0$
14. Determine whether the statement is possible or not
    1. $\sin(\theta) = 2.4$
    2. $\cos(\theta) = -0.98$
15. For each of the following angles, identify which quadrant the angle lies in, draw the angle, find the reference angle, and draw the reference angle. Then evaluate the sine and cosine of the angle exactly using the reference angle (without technology).
    a. 225°
    b. 300°
    c. 135°
    d. 210°
16. For each of the following angles, identify which quadrant the angle lies in, draw the angle, find the reference angle, and draw the reference angle. Then evaluate the sine and cosine of the angle exactly using the reference angle (without technology).

a. $\frac{5\pi}{4}$
b. $\frac{7\pi}{6}$
c. $\frac{5\pi}{3}$
d. $\frac{3\pi}{4}$

17. Give exact values for $\sin(\theta)$ and $\cos(\theta)$ for each of these angles using the reference angle.

a. $-\frac{3\pi}{4}$
b. $\frac{23\pi}{6}$
c. $-\frac{\pi}{2}$
d. 5π

18. Find an angle θ with $0 < \theta < 360^\circ$ or $0 < \theta < 2\pi$ that has the same value of sine as:

a. $\frac{\pi}{3}$
b. 80°
c. 140°
d. $\frac{4\pi}{3}$

19. Find an angle θ with $0 < \theta < 360^\circ$ or $0 < \theta < 2\pi$ that has the same cosine value as:

a. $\frac{\pi}{4}$
b. 15°
c. 160°
d. $\frac{7\pi}{6}$

20. Use a calculator to approximate the values of sine, cosine, tangent, secant of the following values:

a. 0.15 radians
b. 4 radians
c. 70°
d. 283°

Answer

1. $\frac{\pi}{3}$

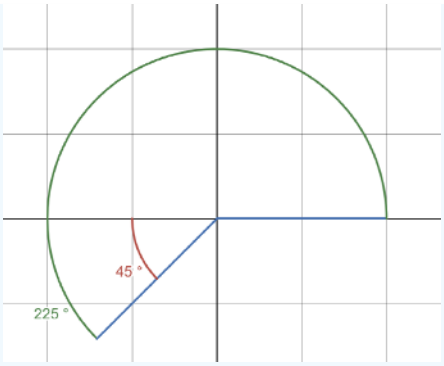
3. (a) 325°

6. $\cos(\theta) = \frac{\sqrt{7}}{4}$, $\tan(\theta) = \frac{3}{\sqrt{7}}$, $\sec(\theta) = \frac{4}{\sqrt{7}}$, $\csc(\theta) = \frac{4}{3}$, and $\cot(\theta) = \frac{\sqrt{7}}{3}$

10. $\sin(\theta) = \frac{12}{13}$

13. Quadrant III

15. (a) The angle lies in Quadrant III, the reference angle is 325°, $\sin(\theta) = \frac{-\sqrt{2}}{2}$, and $\cos(\theta) = \frac{-\sqrt{2}}{2}$



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5.2: Solving Trigonometric Equations with Sine and Cosine

In Section 6.1, we determined the height of a rider on the London Eye Ferris wheel could be determined by the equation

$$h(t) = -65\cos\left(\frac{\pi}{15}t\right) + 70.$$

If we wanted to know length of time during which the rider is more than 100 meters above ground, we would need to solve equations involving trig functions.

Solving using Known Values

In the last chapter, we learned sine and cosine values at commonly encountered angles. We can use these to solve sine and cosine equations involving these common angles.

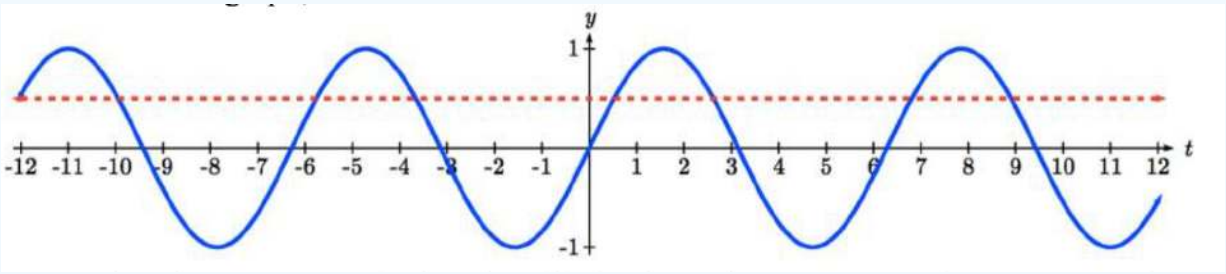
Example 5.2.1

Solve $\sin(t) = \frac{1}{2}$ for all possible values of t.

Solution

Notice this is asking us to identify all angles, t, that have a sine value of $\frac{1}{2}$. While evaluating a function always produces one result, solving for an input can yield multiple solutions. Two solutions should immediately jump to mind from the last chapter: $t = \frac{\pi}{6}$ and $t = \frac{5\pi}{6}$ because they are the common angles on the unit circle with a sin of $\frac{1}{2}$.

Looking at a graph confirms that there are more than these two solutions. While eight are seen on this graph, there are an infinite number of solutions!



Remember that any coterminal angle will also have the same sine value, so any angle coterminal with these our first two solutions is also a solution. Coterminal angles can be found by adding full rotations of 2π, so we can write the full set of solutions:

$t = \frac{\pi}{6} + 2\pi k$ where k is an integer, and $t = \frac{5\pi}{6} + 2\pi k$ where k is an integer.

Example 5.2.2

A circle of radius $5\sqrt{2}$ intersects the line $x = -5$ at two points. Find the angles θ on the interval $0 \le \theta < 2\pi$, where the circle and line intersect.

Solution

The x coordinate of a point on a circle can be found as $x = r\cos(\theta)$, so the x coordinate of points on this circle would be $x = 5\sqrt{2}\cos(\theta)$. To find where the line $x = -5$ intersects the circle, we can solve for where the x value on the circle would be -5.

$$-5 = 5\sqrt{2}\cos(\theta)$$

Isolating the cosine

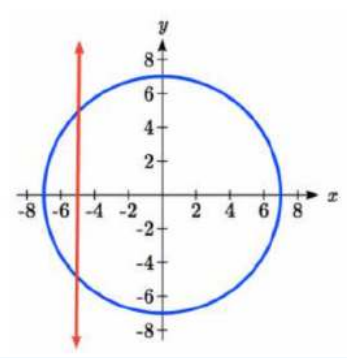
$$\frac{-1}{\sqrt{2}} = \cos(\theta)$$

Recall that $\frac{-1}{\sqrt{2}} = \frac{-\sqrt{2}}{2}$, so we are solving

$$\cos(\theta) = \frac{-\sqrt{2}}{2}$$

We can recognize this as one of our special cosine values from our unit circle, and it corresponds with angles

$$\theta = \frac{3\pi}{4} \text{ and } \theta = \frac{5\pi}{4}$$



Exercise 5.2.1

Solve $\tan(t) = 1$ for all possible values of t.

Answer

From our special angles, we know one answer is $t = \frac{\pi}{4}$. Tangent equations only have one unique solution per cycle or period, so additional solutions can be found by adding multiples of a full period, π.

$$t = \frac{\pi}{4} + \pi k$$

Example 5.2.3

The depth of water at a dock rises and falls with the tide, following the equation $f(t) = 4\sin\left(\frac{\pi}{12}t\right) + 7$, where t is measured in hours after midnight. A boat requires a depth of 9 feet to tie up at the dock. Between what times will the depth be 9 feet?

Solution

To find when the depth is 9 feet, we need to solve $f(t) = 9$.

$$4\sin\left(\frac{\pi}{12}t\right) + 7 = 9$$

Isolating the sine

$$4\sin\left(\frac{\pi}{12}t\right) = 2$$

Dividing by 4

$$\sin\left(\frac{\pi}{12}t\right) = \frac{1}{2}$$

We know $\sin(\theta) = \frac{1}{2}$ when $\theta = \frac{\pi}{6}$ *or* $\theta = \frac{5\pi}{6}$

While we know what angles have a sine value of $\frac{1}{2}$, because of the horizontal stretch/compression it is less clear how to proceed.

To deal with this, we can make a substitution, defining a new temporary variable u to be $u = \frac{\pi}{12}t$, so our equation $\sin\left(\frac{\pi}{12}t\right) = \frac{1}{2}$ becomes

$$\sin(u) = \frac{1}{2}$$

From earlier, we saw the solutions to this equation were

$$u = \frac{\pi}{6} + 2\pi k$$

where k is an integer, and

$$u = \frac{5\pi}{6} + 2\pi k$$

where k is an integer

To undo our substitution, we replace the u in the solutions with $u = \frac{\pi}{12}t$ and solve for t.

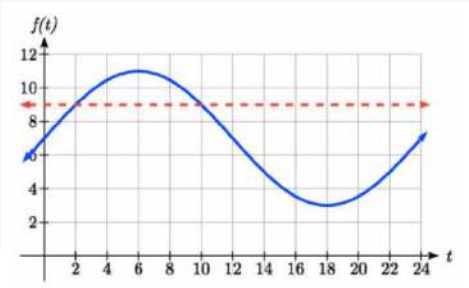
$\frac{\pi}{12}t = \frac{\pi}{6} + 2\pi k$ where k is an integer, and $\frac{\pi}{12}t = \frac{5\pi}{6} + 2\pi k$ where k is an integer.

Dividing by $\pi/12$, we obtain solutions

$t = 2 + 24k$ where k is an integer, and $t = 10 + 24k$ where k is an integer.

The depth will be 9 feet and the boat will be able to approach the dock between 2 am and 10 am.

Notice how in both scenarios, the 24k shows how every 24 hours the cycle will be repeated.



In the previous example, looking back at the original simplified equation $\sin\left(\frac{\pi}{12}t\right) = \frac{1}{2}$, we can use the ratio of the "normal period" to the stretch factor to find the period: $\frac{2\pi}{\left(\frac{\pi}{12}\right)} = 2\pi\left(\frac{12}{\pi}\right) = 24$. Notice that the sine function has a period of 24, which is reflected in the solutions: there were two unique solutions on one full cycle of the sine function, and additional solutions were found by adding multiples of a full period.

? Exercise 5.2.2

Solve $4\sin(5t) - 1 = 1$ for all possible values of t.

Answer

$$4\sin(5t) - 1 = 1$$

$$\sin(5t) = \frac{1}{2}$$

Let $u = 5t$ so this becomes $\sin(u) = \frac{1}{2}$, which has solutions

$$u = \frac{\pi}{6} + 2\pi k, \frac{5\pi}{6} + 2\pi k$$

Solving

$$5t = u = \frac{\pi}{6} + 2\pi k, \frac{5\pi}{6} + 2\pi k$$

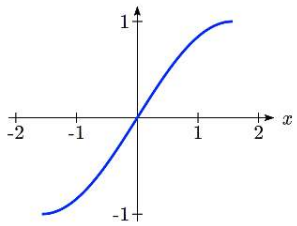
gives the solutions

$$t = \frac{\pi}{30} + \frac{2\pi}{5}k \qquad t = \frac{\pi}{6} + \frac{2\pi}{5}k$$

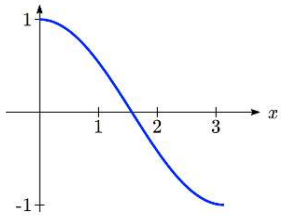
In previous sections, we have evaluated the trigonometric functions at various angles, but at times we need to know what angle would yield a specific sine, cosine, or tangent value. For this, we need inverse functions. Recall that for a one-to-one function, if $f(a) = b$, then an inverse function would satisfy $f^{-1}(b) = a$.

You probably are already recognizing an issue – that the sine, cosine, and tangent functions are not one-to-one functions. To define an inverse of these functions, we will need to restrict the domain of these functions to yield a new function that is one-to-one. We choose a domain for each function that includes the angle zero.

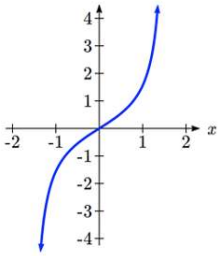
Sine, limited to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



Cosine, limited to $[0, \pi]$



Tangent, limited to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



On these restricted domains, we can define the inverse sine, inverse cosine, and inverse tangent functions.

INVERSE SINE, COSINE, AND TANGENT FUNCTIONS and their inverses

For angles in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, if $\sin(\theta) = a$, then $\sin^{-1}(a) = \theta$

$\sin^{-1}(x)$ has domain [-1, 1] and range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

The $\sin^{-1}(x)$ is sometimes called the **arcsine** function, and notated $\arcsin(a)$.

For angles in the interval $[0, \pi]$, if $\cos(\theta) = a$, then $\cos^{-1}(a) = \theta$

$\cos^{-1}(x)$ has domain [-1, 1] and range $[0, \pi]$

The $\cos^{-1}(x)$ is sometimes called the **arccosine** function, and notated $\arccos(a)$.

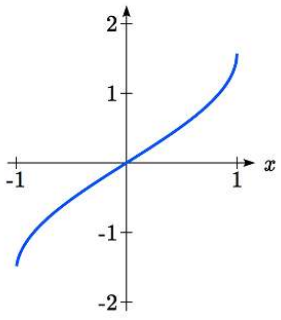
For angles in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, if $\tan(\theta) = a$, then $\tan^{-1}(a) = \theta$

$\tan^{-1}(x)$ has domain of all real numbers and range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

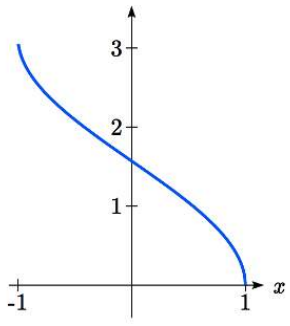
The $\tan^{-1}(x)$ is sometimes called the **arctangent** function, and notated $\arctan(a)$.

The graphs of the inverse functions are shown here:

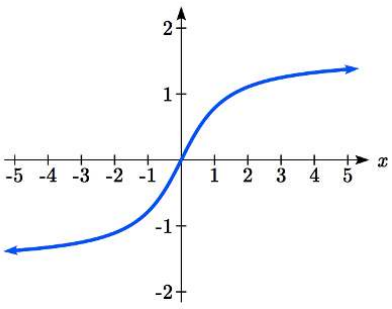
$\sin^{-1}(x)$



$\cos^{-1}(x)$



$\tan^{-1}(x)$



Notice that the output of each of these inverse functions is an *angle*.

Example 5.2.1

Evaluate

a. $\sin^{-1}\left(\frac{1}{2}\right)$

b. $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

c. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

d. $\tan^{-1}(1)$

Solution

a) Evaluating $\sin^{-1}\left(\frac{1}{2}\right)$ is the same as asking what angle would have a sine value of $\frac{1}{2}$. In other words, what angle θ would satisfy $\sin(\theta)=\frac{1}{2}$?

There are multiple angles that would satisfy this relationship, such as $\frac{\pi}{6}$ and $\frac{5\pi}{6}$, but we know we need the angle in the range of $\sin^{-1}(x)$, the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, so the answer will be

$$\sin^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{6}$$

Remember that the inverse is a *function* so for each input, we will get exactly one output.

b) Evaluating $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$, we know that $\frac{5\pi}{4}$ and $\frac{7\pi}{4}$ both have a sine value of $-\frac{\sqrt{2}}{2}$, but neither is in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$. For that, we need the negative angle coterminal with $\frac{7\pi}{4}$.

$$\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)=-\frac{\pi}{4}$$

c) Evaluating $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$, we are looking for an angle in the interval $[0,\pi]$ with a cosine value of $-\frac{\sqrt{3}}{2}$. The angle that satisfies this is

$$\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)=\frac{5\pi}{6}$$

d) Evaluating $\tan^{-1}(1)$, we are looking for an angle in the interval $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ with a tangent value of 1. The correct angle is

$$\tan^{-1}(1)=\frac{\pi}{4}$$

? Exercise 5.2.1

Evaluate

a. $\sin^{-1}(-1)$

b. $\tan^{-1}(-1)$

c. $\cos^{-1}(-1)$

d. $\cos^{-1}\left(\frac{1}{2}\right)$

Answer

a) $-\frac{\pi}{2}$

b) $-\frac{\pi}{4}$

c) π

d) $\frac{\pi}{3}$

Example 5.2.2

Evaluate $\sin^{-1}(0.97)$ using your calculator.

Solution

Since the output of the inverse function is an angle, your calculator will give you a degree value if in degree mode, and a radian value if in radian mode.

In radian mode,

$$\sin^{-1}(0.97) \approx 1.3252$$

In degree mode,

$$\sin^{-1}(0.97) \approx 75.93^{\circ}$$

Exercise

Evaluate $\cos^{-1}(-0.4)$ using your calculator.

Answer

$$1.9823 \text{ or } \text{mtd}$$

In Section 5.5, we worked with trigonometry on a right triangle to solve for the sides of a triangle given one side and an additional angle. Using the inverse trig functions, we can solve for the angles of a right triangle given two sides.

Solving using the Inverse Trig Functions

Not all equations involve the "special" values of the trig functions to we have learned. To find the solutions to these equations, we need to use the inverse trig functions.

Example 5.2.4

Use the inverse sine function to find one solution to $\sin(\theta) = 0.8$.

Solution

Since this is not a known unit circle value, calculating the inverse, $\theta = \sin^{-1}(0.8)$. This requires a calculator and we must approximate a value for this angle. If your calculator is in degree mode, your calculator will give you an angle in degrees as the output. If your calculator is in radian mode, your calculator will give you an angle in radians. In radians, $\theta = \sin^{-1}(0.8) \approx 0.927$, or in degrees, $\theta = \sin^{-1}(0.8) \approx 53.130^{\circ}$.

If you are working with a composed trig function and you are not solving for an angle, you will want to ensure that you are working in radians. In calculus, we will almost always want to work with radians since they are unit-less.

Notice that the inverse trig functions do exactly what you would expect of any function – for each input they give exactly one output. While this is necessary for these to be a function, it means that to find *all* the solutions to an equation like $\sin(\theta) = 0.8$, we need to do more than just evaluate the inverse function.

To find additional solutions, it is good to remember four things:

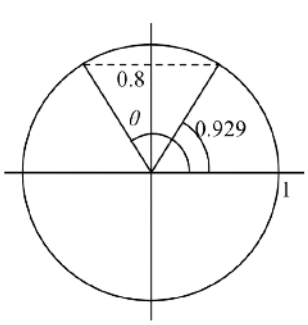
- The sine is the y-value of a point on the unit circle
- The cosine is the x-value of a point on the unit circle
- The tangent is the slope of a line at a given angle
- Other angles with the same $\sin/\cos/\tan$ will have the same reference angle

Example 5.2.5

Find all solutions to $\sin(\theta) = 0.8$.

Solution

We would expect two unique angles on one cycle to have this sine value. In the previous example, we found one solution to be $\theta = \sin^{-1}(0.8) \approx 0.927$. To find the other, we need to answer the question "what other angle has the same sine value as an angle of 0.927?"



We can think of this as finding all the angles where the y-value on the unit circle is 0.8. Drawing a picture of the circle helps how the symmetry.

On a unit circle, we would recognize that the second angle would have the same reference angle and reside in the second quadrant. This second angle would be located at $\theta = \pi - \sin^{-1}(0.8)$, or approximately $\theta \approx \pi - 0.927 = 2.214$.

To find more solutions we recall that angles coterminal with these two would have the same sine value, so we can add full cycles of 2π.

$$\theta = \sin^{-1}(0.8) + 2\pi k$$

and

$$\theta = \pi - \sin^{-1}(0.8) + 2\pi k$$

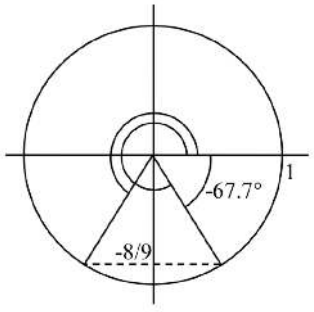
where k is an integer,
or approximately, $\theta = 0.927 + 2\pi k$ and $\theta = 2.214 + 2\pi k$ where k is an integer.

Example 5.2.6

Find all solutions to $\sin(x) = -\frac{8}{9}$ on the interval $0^\circ \le x < 360^\circ$.

Solution

We are looking for the angles with a y-value of -8/9 on the unit circle. Immediately we can see the solutions will be in the third and fourth quadrants.



First, we will turn our calculator to degree mode. Using the inverse, we can find one solution $x = \sin^{-1}\left(-\frac{8}{9}\right) \approx -62.734^\circ$. While this angle satisfies the equation, it does not lie in the domain we are looking for. To find the angles in the desired domain, we start looking for additional solutions.

First, an angle coterminal with -62.734°will have the same sine. By adding a full rotation, we can find an angle in the desired domain with the same sine.

$$x = -62.734^\circ + 360^\circ = 297.266^\circ$$

There is a second angle in the desired domain that lies in the third quadrant. Notice that 62.734° is the reference angle for all solutions, so this second solution would be 62.734° past 180°

$$x = 62.734^\circ + 180^\circ = 242.734^\circ$$

The two solutions on $0^\circ \le x < 360^\circ$ are x = 297.266° and x = 242.734°

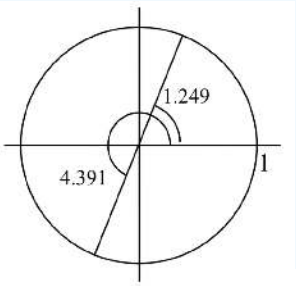
Example 5.2.7

Find all solutions to $\tan(x) = 3$ on $0 \le x < 2\pi$.

Solution

Using the inverse tangent function, we can find one solution $x = \tan^{-1}(3) \approx 1.249$. Unlike the sine and cosine, the tangent function only attains any output value once per cycle, so there is no second solution in any one cycle.

By adding π, a full period of tangent function, we can find a second angle with the same tangent value. Notice this gives another angle where the line has the same slope.



If additional solutions were desired, we could continue to add multiples of π, so all solutions would take on the form $x = 1.249 + k\pi$, however we are only interested in $0 \le x < 2\pi$.

$$x = 1.249 + \pi = 4.391$$

The two solutions on $0 \le x < 2\pi$ are $x = 1.249$ and $x = 4.391$.

Exercise 5.2.3

Find all solutions to $\tan(x) = 0.7$ on $0^\circ \le x < 360^\circ$.

Answer

The first solution is $x = \tan^{-1}(0.7) \approx 34.992^\circ$.

For a standard tangent, the second solution can be found by adding a full period, 180°, giving

$$x = 180^\circ + 34.99^\circ = 214.992^\circ$$

Example 5.2.8

Solve $3\cos(t) + 4 = 2$ for all solutions on one cycle, $0 \le t < 2\pi$

Solution

$$3\cos(t) + 4 = 2$$

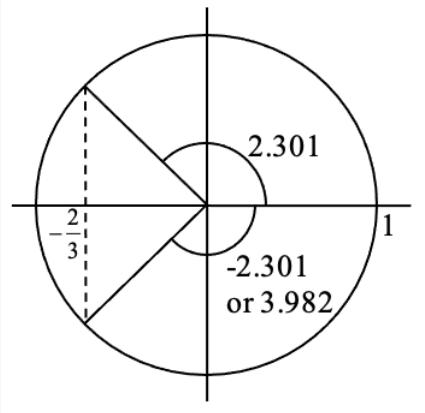
Isolating the cosine

$$3\cos(t) = -2$$

$$\cos(t) = -\frac{2}{3}$$

Using the inverse, we can find one solution

$$t = \cos^{-1}\left(-\frac{2}{3}\right) \approx 2.301$$



We're looking for two angles where the x-coordinate on a unit circle is -2/3. A second angle with the same cosine would be located in the third quadrant. Notice that the location of this angle could be represented as $t = -2.301$. To represent this as a positive angle we could find a coterminal angle by adding a full cycle.

$$t = -2.301 + 2\pi = 3.982$$

The equation has two solutions between 0 and 2π, at $t = 2.301$ and $t = 3.982$.

Example 5.2.9

Solve $\cos(3t) = 0.2$ for all solutions on two cycles, $0 \le t < \frac{4\pi}{3}$.

Solution

As before, with a horizontal compression it can be helpful to make a substitution, $u = 3t$ Making this substitution simplifies the equation to a form we have already solved.

$$\cos(u) = 0.2$$

$$u = \cos^{-1}(0.2) \approx 1.369$$

A second solution on one cycle would be located in the fourth quadrant with the same reference angle.

$$u = 2\pi - 1.369 = 4.914$$

In this case, we need all solutions on two cycles, so we need to find the solutions on the second cycle. We can do this by adding a full rotation to the previous two solutions.

$$u = 1.369 + 2\pi = 7.653$$
$$u = 4.914 + 2\pi = 11.197$$

Undoing the substitution, we obtain our four solutions:

$$3t = 1.369, \text{ so } t = 0.456$$

$$3t = 4.914, \text{ so } t = 1.638$$

$$3t = 7.653, \text{ so } t = 2.551$$

$$3t = 11.197, \text{ so } t = 3.732$$

Example 5.2.10

Solve $3\sin(\pi t) = -2$ for all solutions.

Solution

$$3\sin(\pi t) = -2$$

Isolating the sine

$$\sin(\pi t) = -\frac{2}{3}$$

We make the substitution $u = \pi t$

$$\sin(u) = -\frac{2}{3}$$

Using the inverse, we find one solution

$$u = \sin^{-1}\left(-\frac{2}{3}\right) \approx -0.730$$

This angle is in the fourth quadrant. A second angle with the same sine would be in the third quadrant with 0.730 as a reference angle:

$$u = \pi + 0.730 = 3.871$$

We can write all solutions to the equation $\sin(u) = -\frac{2}{3}$ as

$$u = -0.730 + 2\pi k \text{ or } u = 3.871 + 2\pi k$$

where k is an integer.

Undoing our substitution, we can replace u in our solutions with $u = \pi t$ and solve for t

$$\pi t = -0.730 + 2\pi k \text{ or } \pi t = 3.871 + 2\pi k$$

Divide by π

$$t = -0.232 + 2k \text{ or } t = 1.232 + 2k$$

Exercise 5.2.4

Solve $5\sin\left(\frac{\pi}{2}t\right) + 3 = 0$ for all solutions on one cycle, $0 \le t < 4$.

Answer

$$\sin\left(\frac{\pi}{2}t\right) = -\frac{3}{5}$$

Let $u = \frac{\pi}{2}t$, so this becomes

$$\sin(u) = -\frac{3}{5}$$

Using the inverse, $u = \sin^{-1}\left(-\frac{3}{5}\right) \approx -0.6435$. Since we want positive solutions, we can find the coterminal solution by adding a full cycle: $u = -0.6435 + 2\pi = 5.6397$. Another angle with the same sin would be in the third quadrant with the reference angle 0.6435. $u = \pi + 0.6435 = 3.7851$.

Solving for t,

$$u = \frac{\pi}{2}t = 5.6397$$

, so

$$t = 5.6397\left(\frac{2}{\pi}\right) = 3.5903$$

and

$$u = \frac{\pi}{2}t = 3.7851$$

, so

$$t = 3.7851\left(\frac{2}{\pi}\right) = 2.4097$$

$$t = 2.4097 \text{ or } t = 3.5903$$

SOLVING TRIG EQUATIONS TRIGONOMETRY: SINUSOIDAL FUNCTIONS

1. Isolate the trig function on one side of the equation
2. Make a substitution for the inside of the sine, cosine, or tangent (or other trig function)
3. Use inverse trig functions to find one solution
4. Use symmetries to find a second solution on one cycle (when a second exists)
5. Find additional solutions if needed by adding full periods
6. Undo the substitution

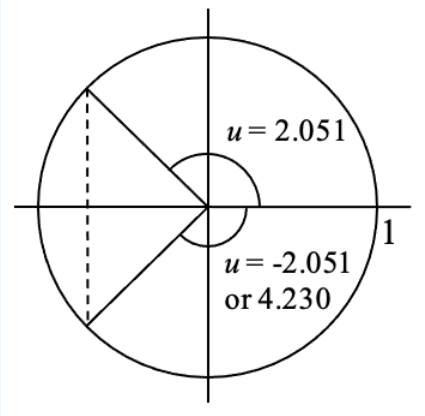
We now can return to the question we began the section with.

Example 5.2.11

The height of a rider on the London Eye Ferris wheel can be determined by the equation $h(t) = -65\cos\left(\frac{\pi}{15}t\right) + 70$. How long is the rider more than 100 meters above ground?

Solution

To find how long the rider is above 100 meters, we first find the times at which the rider is at a height of 100 meters by solving h(t) = 100.



$$100 = -65\cos\left(\frac{\pi}{15}t\right) + 70$$

Isolating the cosine

$$30 = -65\cos\left(\frac{\pi}{15}t\right)$$

$$\frac{30}{-65} = \cos\left(\frac{\pi}{15}t\right)$$

We make the substitution $u = \frac{\pi}{15}t$

$$\frac{30}{-65} = \cos(u)$$

Using the inverse, we find one solution

$$u = \cos^{-1}\left(\frac{30}{-65}\right) \approx 2.051$$

This angle is in the second quadrant. A second angle with the same cosine would be symmetric in the third quadrant. This angle could be represented as u = -2.051, but we need a coterminal positive angle, so we add 2π:

$$u = 2\pi - 2.051 \approx 4.230$$

Now we can undo the substitution to solve for t

$\frac{\pi}{15}t = 2.051$ so $t = 9.793$ minutes after the start of the ride
$\frac{\pi}{15}t = 4.230$ so $t = 20.197$ minutes after the start of the ride

A rider will be at 100 meters after 9.793 minutes, and again after 20.197 minutes. From the behavior of the height graph, we know the rider will be above 100 meters between these times. A rider will be above 100 meters for 20.197 - 9.793 = 10.404 minutes of the ride.

Important Topics of This Section

- Solving trig equations using known values
- Using substitution to solve equations
- Finding answers in one cycle or period vs. finding all possible solutions
- Method for solving trig equations

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5.3: Graphing The Sine and Cosine Functions

The London Eye (London Eye photo by authors, 2010, CC-BY) is a huge Ferris wheel 135 meters (394 feet) tall in London, England, which completes one rotation every 30 minutes. When we look at the behavior of this Ferris wheel it is clear that it completes 1 cycle, or 1 revolution, and then repeats this revolution over and over again.



This is an example of a periodic function, because the Ferris wheel repeats its revolution or one cycle every 30 minutes, and so we say it has a period of 30 minutes.

In this section, we will work to sketch a graph of a rider's height above the ground over time and express this height as a function of time.

Definition: PERIODIC FUNCTIONS

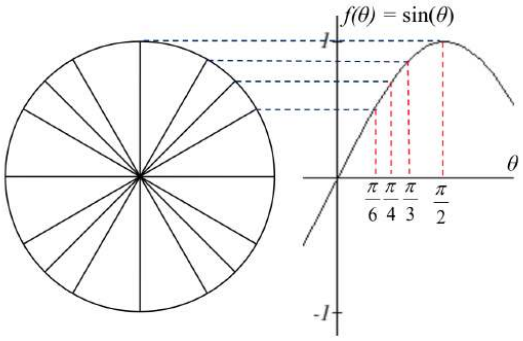
A **periodic function** is a function for which a specific horizontal shift, P, results in the original function: $f(x+P)=f(x)$ for all values of x. When this occurs we call the smallest such horizontal shift with $P>0$ the **period** of the function.

You might immediately guess that there is a connection here to finding points on a circle, since the height above ground would correspond to the y value of a point on the circle. We can determine the y value by using the sine function. To get a better sense of this function's behavior, we can create a table of values we know, and use them to sketch a graph of the sine and cosine functions.

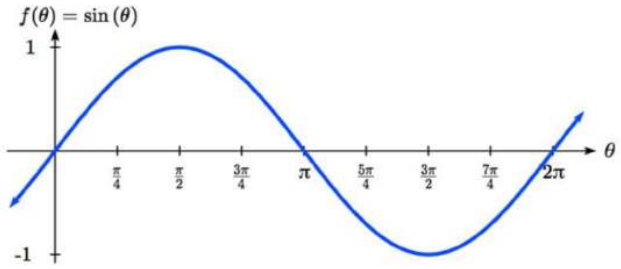
Listing some of the values for sine and cosine on a unit circle,

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1
sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

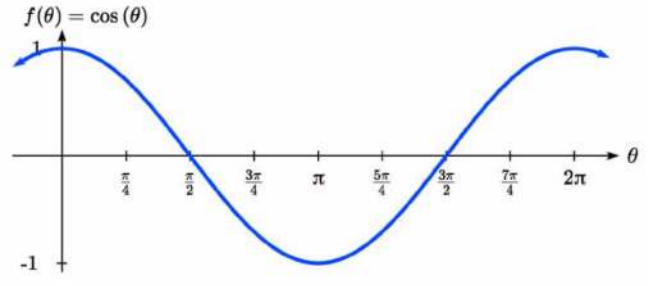
Here you can see how for each angle, we use the y value of the point on the circle to determine the output value of the sine function.



Plotting more points gives the full shape of the sine and cosine functions.



Notice how the sine values are positive between 0 and π, which correspond to the values of sine in quadrants 1 and 2 on the unit circle, and the sine values are negative between π and 2π, corresponding to quadrants 3 and 4.



Like the sine function we can track the value of the cosine function through the 4 quadrants of the unit circle as we place it on a graph.

Both of these functions are defined for all real numbers, since we can evaluate the sine and cosine of any angle. By thinking of sine and cosine as coordinates of points on a unit circle, it becomes clear that the range of both functions must be the interval [-1, 1].

DOMAIN AND RANGE OF SINE AND COSINE

The domain of sine and cosine is all real numbers, $(-\infty, \infty)$.

The range of sine and cosine is the interval [-1, 1].

Both these graphs are called **sinusoidal** graphs.

In both graphs, the shape of the graph begins repeating after 2π. Indeed, since any coterminal angles will have the same sine and cosine values, we could conclude that

$$\sin(\theta + 2\pi) = \sin(\theta)$$

and

$$\cos(\theta + 2\pi) = \cos(\theta)$$

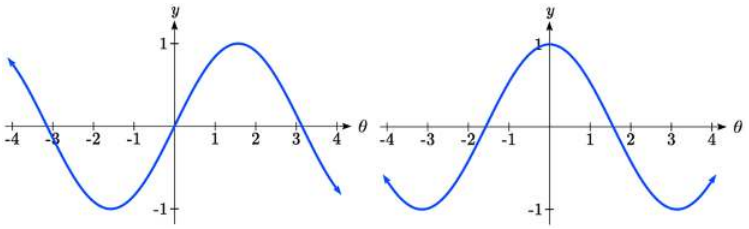
In other words, if you were to shift either graph horizontally by 2π, the resulting shape would be identical to the original function. Sinusoidal functions are a specific type of periodic function.

PERIOD OF SINE AND COSINE

The periods of the sine and cosine functions are both 2π.

Looking at these functions on a domain centered at the vertical axis helps reveal symmetries.

sine cosine



The sine function is symmetric about the origin, the same symmetry the cubic function has, making it an odd function. The cosine function is clearly symmetric about the *y* axis, the same symmetry as the quadratic function, making it an even function.

NEGATIVE ANGLE IDENTITIES

The sine is an odd function, symmetric about the *origin*, so

$$\sin(-\theta) = -\sin(\theta) \tag{5.3.1}$$

The cosine is an even function, symmetric about the *y*-axis, so

$$\cos(-\theta) = \cos(\theta) \tag{5.3.2}$$

These identities can be used, among other purposes, for helping with simplification and proving identities.

You may recall the cofunction identity from last chapter,

$$\sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right)$$

Graphically, this tells us that the sine and cosine graphs are horizontal transformations of each other. We can prove this by using the cofunction identity and the negative angle identity for cosine.

$$\sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right) = \cos\left(-\theta + \frac{\pi}{2}\right) = \cos\left(-\left(\theta - \frac{\pi}{2}\right)\right) = \cos\left(\theta - \frac{\pi}{2}\right) \tag{5.3.3}$$

Now we can clearly see that if we horizontally shift the cosine function to the right by $\pi/2$ we get the sine function.

Remember this shift is not representing the period of the function. It only shows that the cosine and sine function are transformations of each other.

Example 5.3.1

Simplify $\dfrac{\sin(-\theta)}{\tan(\theta)}$.

Solution

We start by using the negative angle identity for sine.

$$\frac{-\sin(\theta)}{\tan(\theta)}$$

Rewriting the tangent

$$\frac{-\sin(\theta)}{\sin(\theta)/\cos(\theta)}$$

Inverting and multiplying

$$-\sin(\theta) \cdot \frac{\cos(\theta)}{\sin(\theta)}$$

Simplifying we get

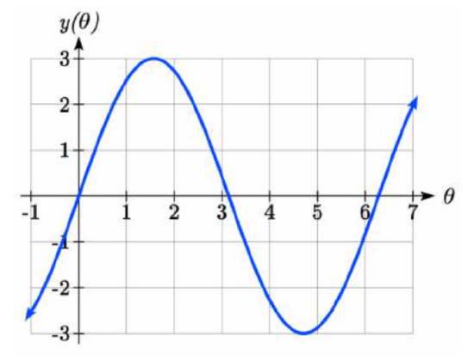
$$-\cos(\theta)$$

Transforming Sine and Cosine

Example 5.3.2

A point rotates around a circle of radius 3. Sketch a graph of the y coordinate of the point.

Solution



Recall that for a point on a circle of radius r, the y coordinate of the point is $y = r\sin(\theta)$, so in this case, we get the equation

$$y(\theta) = 3\sin(\theta)$$

The constant 3 causes a vertical stretch of the y values of the function by a factor of 3.

Notice that the period of the function does not change.

Since the outputs of the graph will now oscillate between -3 and 3, we say that the **amplitude** of the sine wave is 3.

Exercise 5.3.1

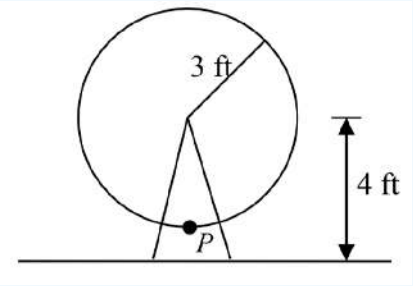
What is the amplitude of the function $f(\theta) = 7\cos(\theta)$? Sketch a graph of this function.

Answer

7

Example 5.3.3

A circle with radius 3 feet is mounted with its center 4 feet off the ground. The point closest to the ground is labeled P. Sketch a graph of the height above ground of the point P as the circle is rotated, then find a function that gives the height in terms of the angle of rotation.



Solution

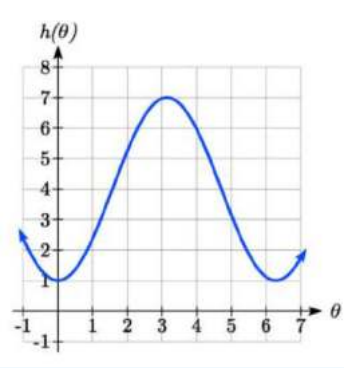
Sketching the height, we note that it will start 1 foot above the ground, then increase up to 7 feet above the ground, and continue to oscillate 3 feet above and below the center value of 4 feet.

Although we could use a transformation of either the sine or cosine function, we start by looking for characteristics that would make one function easier to use than the other.

We decide to use a cosine function because it starts at the highest or lowest value, while a sine function starts at the middle value. A standard cosine starts at the highest value, and this graph starts at the lowest value, so we need to incorporate a vertical reflection.

Second, we see that the graph oscillates 3 above and below the center, while a basic cosine has an amplitude of one, so this graph has been vertically stretched by 3, as in the last example.

Finally, to move the center of the circle up to a height of 4, the graph has been vertically shifted up by 4. Putting these transformations together,



$$h(\theta) = -3\cos(\theta) + 4$$

Definition: MIDLINE

The center value of a sinusoidal function, the value that the function oscillates above and below, is called the **midline** of the function, corresponding to a vertical shift.

The function $f(\theta) = \cos(\theta) + k$ has midline at $y = k$.

Exercise 5.3.2

What is the midline of the function $f(\theta) = 3\cos(\theta) - 4$? Sketch a graph of the function.

Answer

-4

To answer the Ferris wheel problem at the beginning of the section, we need to be able to express our sine and cosine functions at inputs of time. To do so, we will utilize composition. Since the sine function takes an input of an angle, we will look for a function that takes time as an input and outputs an angle. If we can find a suitable $\theta(t)$ function, then we can compose this with our $f(\theta) = \cos(\theta)$ function to obtain a sinusoidal function of time: $f(t) = \cos(\theta(t))$.

Example 5.3.4

A point completes 1 revolution every 2 minutes around a circle of radius 5. Find the x coordinate of the point as a function of time, if it starts at (5, 0).

Solution

Normally, we would express the x coordinate of a point on a unit circle using $x = r\cos(\theta)$, here we write the function $x(\theta) = 5\cos(\theta)$.

The rotation rate of 1 revolution every 2 minutes is an angular velocity. We can use this rate to find a formula for the angle as a function of time. The point begins at an angle of 0. Since the point rotates 1 revolution = 2π radians every 2 minutes, it rotates π radians every minute. After *t* minutes, it will have rotated:

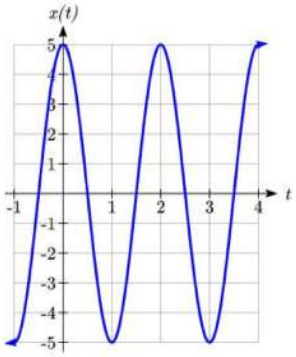
$$\theta(t) = \pi t \text{ radians}$$

Composing this with the cosine function, we obtain a function of time.

$$x(t) = 5\cos(\theta(t)) = 5\cos(\pi t)$$

Notice that this composition has the effect of a horizontal compression, changing the period of the function.

To see how the period relates to the stretch or compression coefficient B in the equation $f(t) = \sin(Bt)$, note that the period will be the time it takes to complete one full revolution of a circle. If a point takes P minutes to complete 1 revolution, then the angular velocity is $\frac{2\pi \text{ radians}}{P \text{ minutes}}$. Then $\theta(t) = \frac{2\pi}{P}t$. Composing with a sine function,



$$f(t) = \sin(\theta(t)) = \sin\left(\frac{2\pi}{P}t\right)$$

From this, we can determine the relationship between the coefficient B and the period:

$$B = \frac{2\pi}{P}$$

Notice that the stretch or compression coefficient B is a ratio of the "normal period of a sinusoidal function" to the "new period." If we know the stretch or compression coefficient B, we can solve for the "new period":

$$P = \frac{2\pi}{B}$$

Summarizing our transformations so far:

TRANSFORMATIONS OF SINE AND COSINE

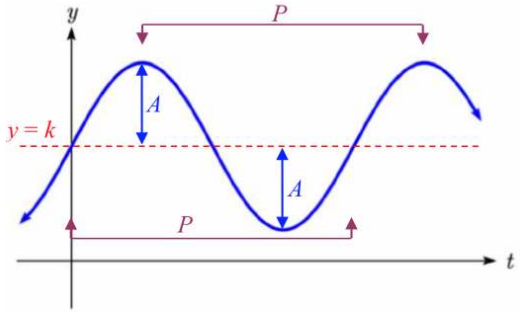
Given an equation in the form

$$f(t) = A\sin(Bt) + k \text{ or } f(t) = A\cos(Bt) + k \tag{5.3.4}$$

A is the vertical stretch, and is the **amplitude** of the function.

B is the horizontal stretch/compression, and is related to the **period,** P, by $P = \frac{2\pi}{B}$.

k is the vertical shift and determines the **midline** of the function.



Example 5.3.5

What is the period of the function $f(t) = \sin\left(\frac{\pi}{6}t\right)$?

Solution

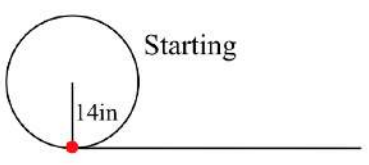
Using the relationship above, the stretch/compression factor is $B = \frac{\pi}{6}$, so the period will be

$$P = \frac{2\pi}{B} = \frac{2\pi}{\frac{\pi}{6}} = 2\pi \cdot \frac{6}{\pi} = 12$$

While it is common to compose sine or cosine with functions involving time, the composition can be done so that the input represents any reasonable quantity.

Example 5.3.6

A bicycle wheel with radius 14 inches has the bottom-most point on the wheel marked in red. The wheel then begins rolling down the street. Write a formula for the height above ground of the red point after the bicycle has travelled x inches.

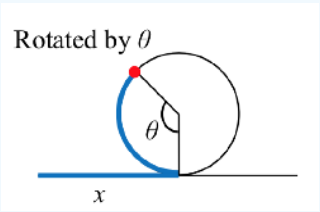


Solution

The height of the point begins at the lowest value, 0, increases to the highest value of 28 inches, and continues to oscillate above and below a center height of 14 inches. In terms of the angle of rotation, θ:

$$h(\theta) = -14\cos(\theta) + 14$$

In this case, x is representing a linear distance the wheel has travelled, corresponding to an arclength along the circle. Since arclength and angle can be related by $s = r\theta$, in this case we can write $x = 14\theta$, which allows us to express the angle in terms of x:



$$\theta(x) = \frac{x}{14}$$

Composing this with our cosine-based function from above,

$$h(x) = h(\theta(x)) = -14\cos\left(\frac{x}{14}\right) + 14 = -14\cos\left(\frac{1}{14}x\right) + 14$$

The period of this function would be $P = \frac{2\pi}{B} = \frac{2\pi}{\frac{1}{14}} = 2\pi \cdot 14 = 28\pi$, the circumference of the circle. This makes sense – the wheel completes one full revolution after the bicycle has travelled a distance equivalent to the circumference of the wheel.

Example 5.3.7

Determine the midline, amplitude, and period of the function $f(t) = 3\sin(2t) + 1$.

Solution

The amplitude is 3

The period is $P = \frac{2\pi}{B} = \frac{2\pi}{2} = \pi$

The midline is at $y = 1$

Amplitude, midline, and period, when combined with vertical flips, allow us to write equations for a variety of sinusoidal situations.

Exercise 5.3.3

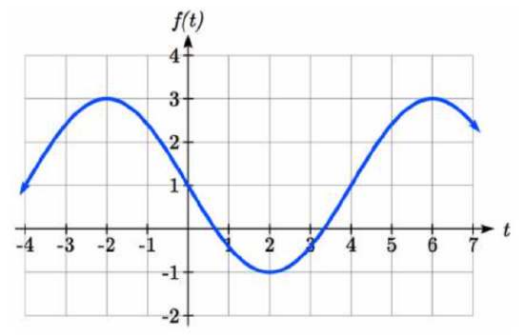
If a sinusoidal function starts on the midline at point (0,3), has an amplitude of 2, and a period of 4, write a formula for the function.

Answer

$$f(x) = 2\sin\left(\frac{\pi}{2}x\right) + 3$$

Example 5.3.8

Find a formula for the sinusoidal function graphed here.



Solution

The graph oscillates from a low of -1 to a high of 3, putting the midline at $y = 1$, halfway between.

The amplitude will be 2, the distance from the midline to the highest value (or lowest value) of the graph.

The period of the graph is 8. We can measure this from the first peak at $x = -2$ to the second at $x = 6$. Since the period is 8, the stretch/compression factor we will use will be

$$B = \frac{2\pi}{P} = \frac{2\pi}{8} = \frac{\pi}{4}$$

At $x = 0$, the graph is at the midline value, which tells us the graph can most easily be represented as a sine function. Since the graph then decreases, this must be a vertical reflection of the sine function. Putting this all together,

$$f(t) = -2\sin\left(\frac{\pi}{4}t\right) + 1$$

With these transformations, we are ready to answer the Ferris wheel problem from the beginning of the section.

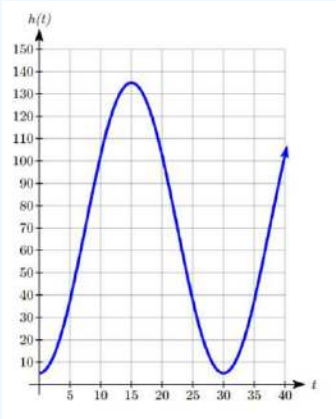
Example 5.3.9

The London Eye is a huge Ferris wheel in London, England, which completes one rotation every 30 minutes. The diameter of the wheel is 120 meters, but the passenger capsules sit outside the wheel. Suppose the diameter at the capsules is 130 meters, and riders board from a platform 5 meters above the ground. Express a rider's height above ground as a function of time in minutes.

Solution

It can often help to sketch a graph of the situation before trying to find the equation.

With a diameter of 130 meters, the wheel has a radius of 65 meters. The height will oscillate with amplitude of 65 meters above and below the center.



Passengers board 5 meters above ground level, so the center of the wheel must be located 65 + 5 = 70 meters above ground level. The midline of the oscillation will be at 70 meters.

The wheel takes 30 minutes to complete 1 revolution, so the height will oscillate with period of 30 minutes.

Lastly, since the rider boards at the lowest point, the height will start at the smallest value and increase, following the shape of a flipped cosine curve.

Putting these together:

Amplitude: 65

Midline: 70

Period: 30, so

$$B = \frac{2\pi}{30} = \frac{\pi}{15}$$

Shape: negative cosine

An equation for the rider's height would be

$$h(t) = -65\cos\left(\frac{\pi}{15}t\right) + 70$$

Exercise 5.3.4

The Ferris wheel at the Puyallup Fair (Photo by photogirl7.1, www.flickr.com/photos/kitkaph...86205/sizes/z/, CC-BY) has a diameter of about 70 feet and takes 3 minutes to complete a full rotation. Passengers board from a platform 10 feet above the ground. Write an equation for a rider's height above ground over time.



Answer

$$h(t) = -35\cos\left(\frac{2\pi}{3}t\right) + 45$$

While these transformations are sufficient to represent many situations, occasionally we encounter a sinusoidal function that does not have a vertical intercept at the lowest point, highest point, or midline. In these cases, we need to use horizontal shifts. Since we are combining horizontal shifts with horizontal stretches, we need to be careful. Recall that when the inside of the function is factored, it reveals the horizontal shift.

HORIZONTAL SHIFTS OF SINE AND COSINE

Given an equation in the form

$$f(t) = A\sin\left(B(t-h)\right) + k \text{ or } f(t) = A\cos\left(B(t-h)\right) + k \tag{5.3.5}$$

h is the horizontal shift of the function

Example 5.3.10

Sketch a graph of $f(t) = 3\sin\left(\frac{\pi}{4}t - \frac{\pi}{4}\right)$.

Solution

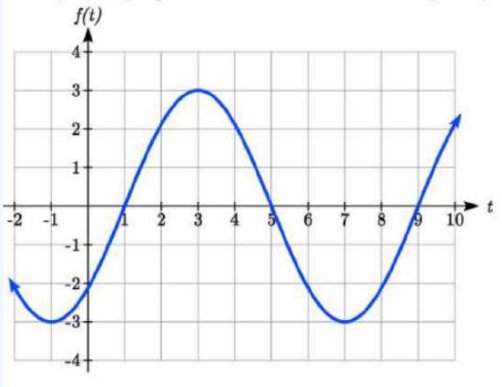
To reveal the horizontal shift, we first need to factor inside the function:

$$f(t) = 3\sin\left(\frac{\pi}{4}(t-1)\right)$$

This graph will have the shape of a sine function, starting at the midline and increasing, with an amplitude of 3. The period of the graph will be

$$P = \frac{2\pi}{B} = \frac{2\pi}{\frac{\pi}{4}} = 2\pi \cdot \frac{4}{\pi} = 8$$

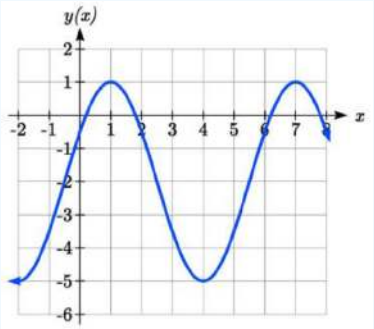
Finally, the graph will be shifted to the right by 1.



In some physics and mathematics books, you will hear the horizontal shift referred to as **phase shift**. In other physics and mathematics books, they would say the phase shift of the equation above is $\frac{\pi}{4}$, the value in the unfactored form. Because of this ambiguity, we will not use the term phase shift any further, and will only talk about the horizontal shift.

Example 5.3.11

Find a formula for the function graphed here.



Solution

With highest value at 1 and lowest value at -5, the midline will be halfway between at -2.

The distance from the midline to the highest or lowest value gives an amplitude of 3.

The period of the graph is 6, which can be measured from the peak at $x = 1$ to the next peak at $x = 7$, or from the distance between the lowest points. This gives

$$B = \frac{2\pi}{P} = \frac{2\pi}{6} = \frac{\pi}{3}$$

For the shape and shift, we have more than one option. We could either write this as:

A cosine shifted 1 to the right

A negative cosine shifted 2 to the left

A sine shifted 1/2 to the left

A negative sine shifted 2.5 to the right

While any of these would be fine, the cosine shifts are easier to work with than the sine shifts in this case, because they involve integer values. Writing these:

$$y(x) = 3\cos\left(\frac{\pi}{3}(x-1)\right) - 2$$

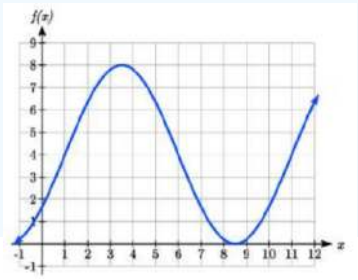
or

$$y(x) = -3\cos\left(\frac{\pi}{3}(x+2)\right) - 2$$

Again, these functions are equivalent, so both yield the same graph.

Exercise 5.3.5

Write a formula for the function graphed here.



Answer

Two possibilities:

$$f(x) = 4\cos\left(\frac{\pi}{5}(x-3.5)\right) + 4 \text{ or } f(x) = 4\sin\left(\frac{\pi}{5}(x-1)\right) + 4$$

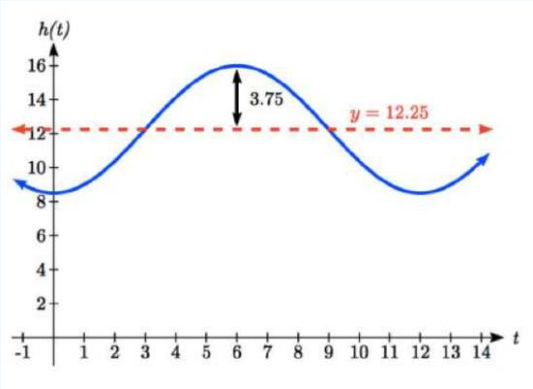
Many modeling situations involve functions that are periodic. Previously we learned that sinusoidal functions are a special type of periodic function. Problems that involve quantities that oscillate can often be modeled by a sine or cosine function and once we create a suitable model for the problem we can use that model to answer various questions.

Example 5.3.4

The hours of daylight in Seattle oscillate from a low of 8.5 hours in January to a high of 16 hours in July (www.mountaineers.org/seattle/...ylightHrs.html). When should you plant a garden if you want to do it during a month where there are 14 hours of daylight?

Solution

To model this, we first note that the hours of daylight oscillate with a period of 12 months. $B = \frac{2\pi}{12} = \frac{\pi}{6}$ corresponds to the horizontal stretch, found by using the ratio of the original period to the new period.



With a low of 8.5 and a high of 16, the midline will be halfway between these values, at

$$\frac{16+8.5}{2} = 12.25.$$

The amplitude will be half the difference between the highest and lowest values:

$$\frac{16-8.5}{2} = 3.75$$

or equivalently the distance from the midline to the high or low value, 16-12.25=3.75.

Letting January be $t = 0$, the graph starts at the lowest value, so it can be modeled as a flipped cosine graph. Putting this together, we get a model:

$$h(t) = -3.75\cos\left(\frac{\pi}{6}t\right) + 12.25$$

$h(t)$ is our model for hours of day light t months after January.

To find when there will be 14 hours of daylight, we solve $h(t) = 14$.

$$14 = -3.75\cos\left(\frac{\pi}{6}t\right) + 12.25$$

Isolating the cosine

$$1.75 = -3.75\cos\left(\frac{\pi}{6}t\right)$$

Subtracting 12.25 and dividing by -3.75

$$-\frac{1.75}{3.75} = \cos\left(\frac{\pi}{6}t\right)$$

Using the inverse

$$\frac{\pi}{6}t = \cos^{-1}\left(-\frac{1.75}{3.75}\right) \approx 2.0563$$

multiplying by the reciprocal

$$t = 2.0563 \cdot \frac{6}{\pi} = 3.927$$

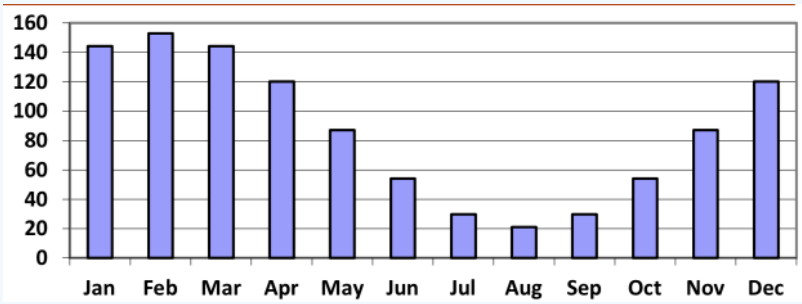
$t = 3.927$ months past January

There will be 14 hours of daylight 3.927 months into the year, or near the end of April.

While there would be a second time in the year when there are 14 hours of daylight, since we are planting a garden, we would want to know the first solution, in spring, so we do not need to find the second solution in this case.

Exercise 5.3.2

The author's monthly gas usage (in therms) is shown here. Find a function to model the data.



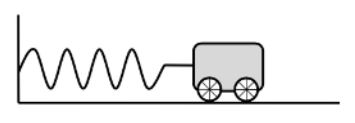
Answer

Approximately

$$G(t) = 66\cos\left(\frac{\pi}{6}(t-1)\right) + 87$$

Example 5.3.6

An object is connected to the wall with a spring that has a natural length of 20 cm. The object is pulled back 8 cm past the natural length and released. The object oscillates 3 times per second. Find an equation for the horizontal position of the object ignoring the effects of friction. How much time during each cycle is the object more than 27 cm from the wall?



Solution

If we use the distance from the wall, x, as the desired output, then the object will oscillate equally on either side of the spring's natural length of 20, putting the midline of the function at 20 cm.

If we release the object 8 cm past the natural length, the amplitude of the oscillation will be 8 cm.

We are beginning at the largest value and so this function can most easily be modeled using a cosine function.

Since the object oscillates 3 times per second, it has a frequency of 3 and the period of one oscillation is 1/3 of second. Using this we find the horizontal compression using the ratios of the periods: $\frac{2\pi}{1/3} = 6\pi$.

Using all this, we can build our model:

$$x(t) = 8\cos(6\pi t) + 20$$

To find when the object is 27 cm from the wall, we can solve $x(t) = 27$

$$27 = 8\cos(6\pi t) + 20$$

Isolating the cosine

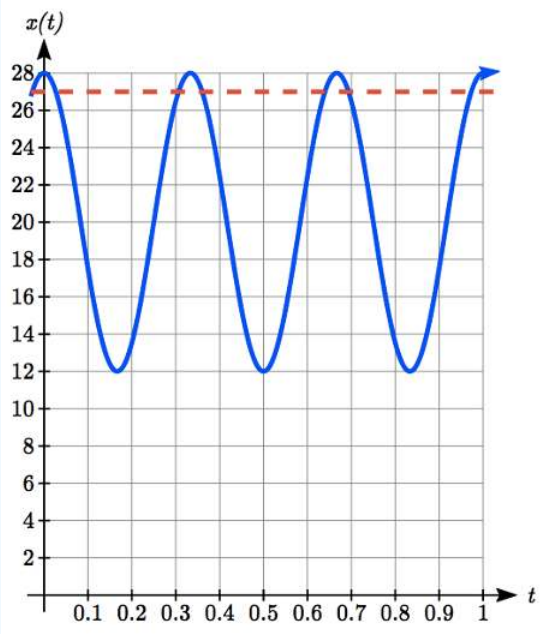
$$7 = 8\cos(6\pi t)$$

$$\frac{7}{8} = \cos(6\pi t)$$

Using the inverse

$$6\pi t = \cos^{-1}\left(\frac{7}{8}\right) \approx 0.505$$

$$t = \frac{0.505}{6\pi} = 0.0268$$

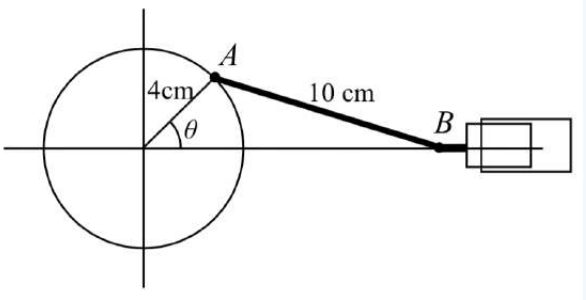


Based on the shape of the graph, we can conclude that the object will spend the first 0.0268 seconds more than 27 cm from the wall. Based on the symmetry of the function, the object will spend another 0.0268 seconds more than 27 cm from the wall at the end of the cycle. Altogether, the object spends 0.0536 seconds each cycle at a distance greater than 27 cm from the wall.

In some problems, we can use trigonometric functions to model behaviors more complicated than the basic sinusoidal function.

Example 5.3.7

A rigid rod with length 10 cm is attached to a circle of radius 4 cm at point A as shown here. The point B is able to freely move along the horizontal axis, driving a piston (For an animation of this situation, see www.mathdemos.org/mathdemos/s...pp/engine1.gif). If the wheel rotates counterclockwise at 5 revolutions per second, find the location of point B as a function of time. When will the point B be 12 cm from the center of the circle?



Solution

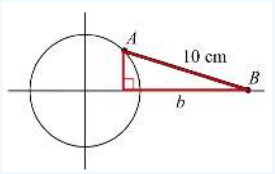
To find the position of point B, we can begin by finding the coordinates of point A. Since it is a point on a circle with radius 4, we can express its coordinates as $(4\cos(\theta), 4\sin(\theta))$, where θ is the angle shown.

The angular velocity is 5 revolutions per second, or equivalently 10π radians per second. After t seconds, the wheel will rotate by $\theta = 10\pi t$ radians. Substituting this, we can find the coordinates of A in terms of t.

$$(4\cos(10\pi t), 4\sin(10\pi t))$$

Notice that this is the same value we would have obtained by observing that the period of the rotation is 1/5 of a second and calculating the stretch/compression factor:

$$\frac{\text{"original"}}{\text{"new"}}\frac{2\pi}{1/5} = 10\pi$$



Now that we have the coordinates of the point A, we can relate this to the point B. By drawing a vertical line segment from A to the horizontal axis, we can form a right triangle. The height of the triangle is the y coordinate of the point A:

$$4\sin(10\pi t)$$

Using the Pythagorean Theorem, we can find the base length of the triangle:

$$(4\sin(10\pi t))^2 + b^2 = 10^2$$

$$b^2 = 100 - 16\sin^2(10\pi t)$$

$$b = \sqrt{100 - 16\sin^2(10\pi t)}$$

Looking at the x coordinate of the point A, we can see that the triangle we drew is shifted to the right of the y axis by $4\cos(10\pi t)$. Combining this offset with the length of the base of the triangle gives the x coordinate of the point B:

$$x(t) = 4\cos(10\pi t) + \sqrt{100 - 16\sin^2(10\pi t)}$$

To solve for when the point B will be 12 cm from the center of the circle, we need to solve $x(t) = 12$.

$$12 = 4\cos(10\pi t) + \sqrt{100 - 16\sin^2(10\pi t)}$$

Isolate the square root

$$12 - 4\cos(10\pi t) = \sqrt{100 - 16\sin^2(10\pi t)}$$

Square both sides

$$(12 - 4\cos(10\pi t))^2 = 100 - 16\sin^2(10\pi t)$$

Expand the left side

$$144 - 96\cos(10\pi t) + 16\cos^2(10\pi t) = 100 - 16\sin^2(10\pi t)$$

Move all terms to the left

$$44 - 96\cos(10\pi t) + 16\cos^2(10\pi t) + 16\sin^2(10\pi t) = 0$$

Factor out 16

$$44 - 96\cos(10\pi t) + 16\left(\cos^2(10\pi t) + \sin^2(10\pi t)\right) = 0$$

At this point, we can utilize the Pythagorean Identity, which tells us that

$$\cos^2(10\pi t) + \sin^2(10\pi t) = 1$$

Using this identity, our equation simplifies to

$$44 - 96\cos(10\pi t) + 16 = 0$$

Combine the constants and move to the right side

$$-96\cos(10\pi t) = -60$$

Divide

$$\cos(10\pi t) = \frac{60}{96}$$

Make a substitution

$$\cos(u) = \frac{60}{96}$$

$$u = \cos^{-1}\left(\frac{60}{96}\right) \approx 0.896$$

By symmetry we can find a second solution

$$u = 2\pi - 0.896 = 5.388$$

Undoing the substitution

$$10\pi t = 0.896, \text{ so } t = 0.0285$$

$$10\pi t = 5.388, \text{ so } t = 0.1715$$

The point B will be 12 cm from the center of the circle 0.0285 seconds after the process begins, 0.1715 seconds after the process begins, and every 1/5 of a second after each of those values.

Important Topics of This Section

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5.3E: Describing Relationships with The Sine and Cosine Functions (Exercises)

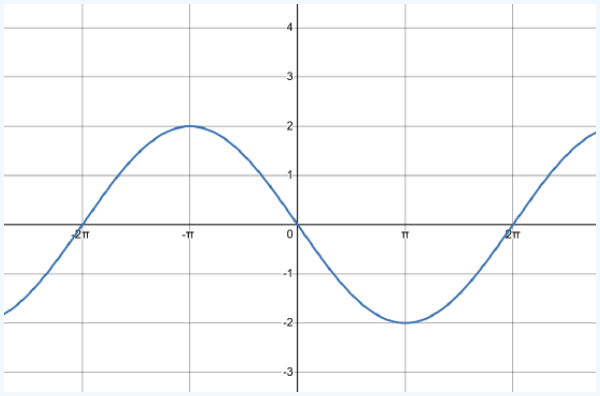
Section 5.3 Exercises

Part 1: Graphs of the form $y = A\sin(B\theta)$ or $y = A\cos(B\theta)$

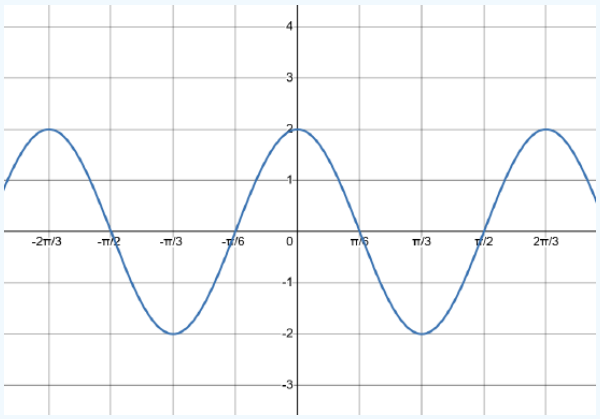
In exercises #1 - 5, match the function with its corresponding graph given below

1. $f(x) = 3\sin(2x)$
2. $g(x) = \sin\left(\frac{1}{2}x\right)$
3. $h(x) = -2\cos\left(\frac{1}{2}x\right)$
4. $k(x) = -2\sin\left(\frac{1}{2}x\right)$
5. $m(x) = 2\cos(3x)$

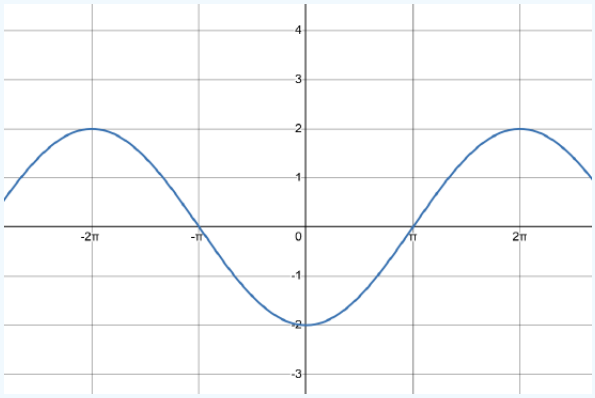
A.



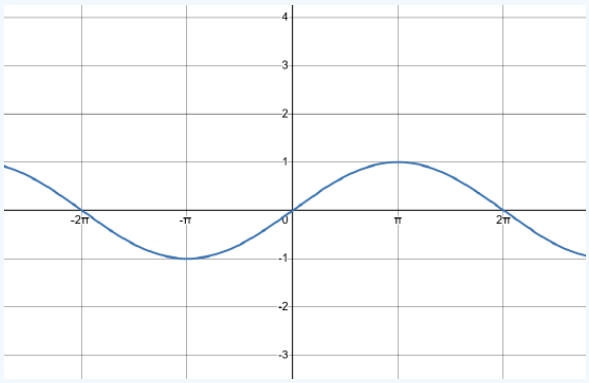
B.



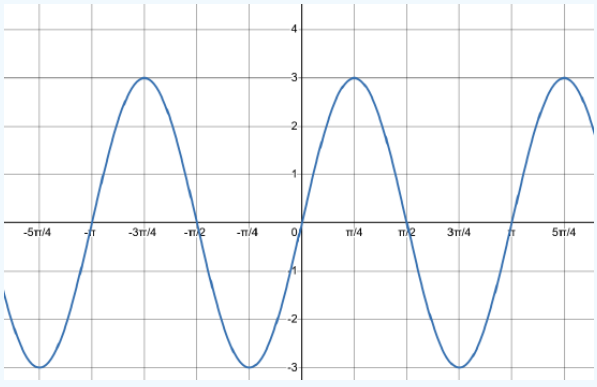
C.



D.



E.



https://math.libretexts.org/@go/page/99748

In exercises #6 - 12, identify the amplitude and period of each function given, then graph two complete cycles with accurate scale on both axis without assistance from technology. Label at least 5 points.

6. $f(x) = 4\cos(3x)$

7. $g(x) = \frac{2}{3}\sin(3x)$

8. $h(x) = 5\cos\left(\frac{\pi}{4}x\right)$

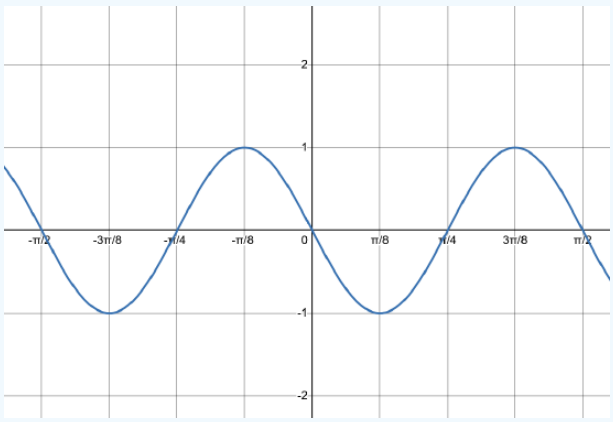
9. $k(x) = -8\cos\left(\frac{1}{3}x\right)$

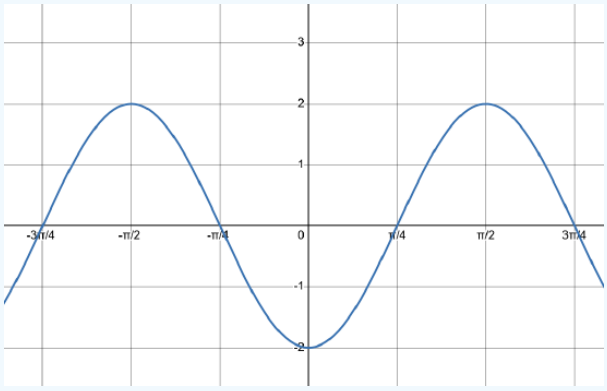
10. $m(x) = -4\sin\left(\frac{\pi}{2}x\right)$

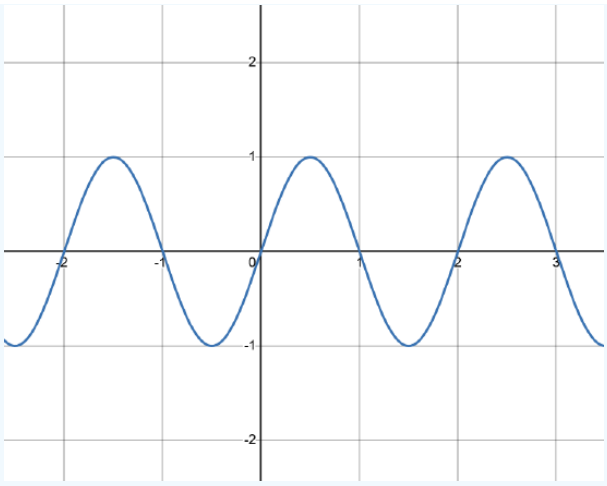
11. $f(x) = 5\cos(6x)$

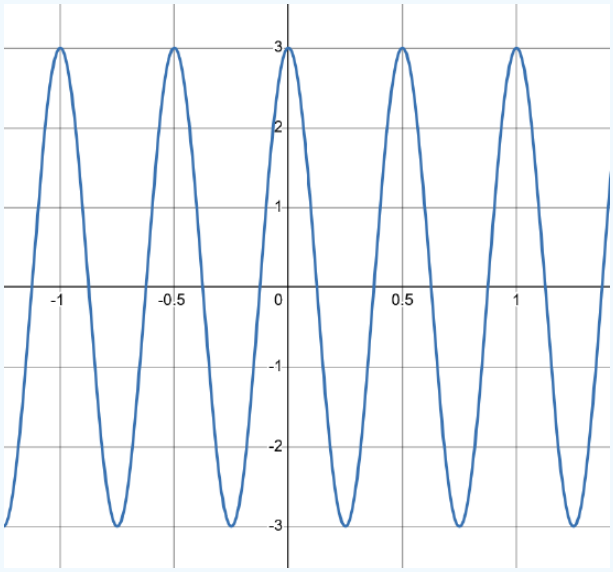
12. $m(x) = -2\sin\left(\frac{2}{5}x\right)$

In exercises #13 - 20, identify the amplitude and period of each wave whose graph is given below, then find a possible formula for the function.

13. 

14. 

15. 

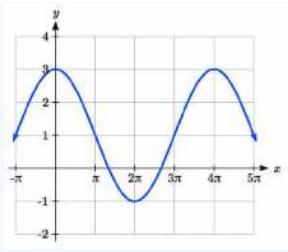
16. 

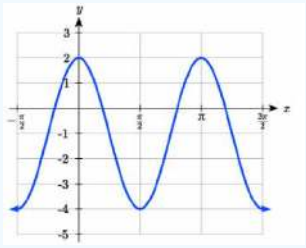
Part 2: Graphs of the form $y = A\sin(B(\theta - h) + D$ **or** $y = A\cos(B(\theta - h) + D$

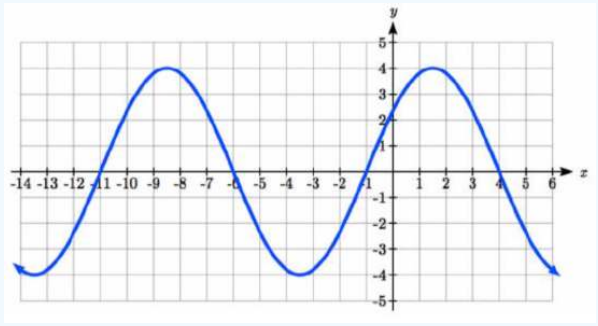
In exercises #17-24, identify the amplitude, period, phase shift, and centerline of each function given, then graph two complete cycles with accurate scale on both axis without assistance from technology. Label at least 5 points.

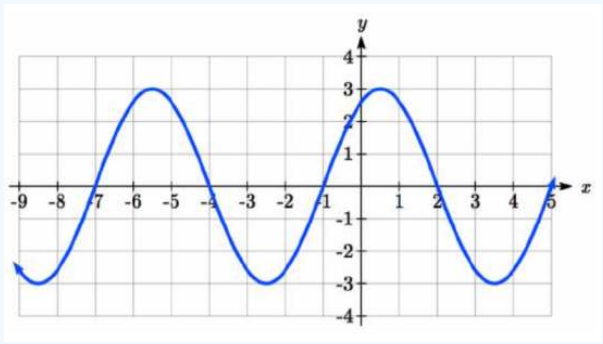
17. $y = 3\sin(x + \frac{\pi}{8})) + 5$
18. $y = -5\cos\left(x - \frac{\pi}{4})\right) - 7$
19. $y = 4\sin\left(\frac{\pi}{2}(x - 3)\right)$
20. $y = 3\cos\left(2(x - \frac{\pi}{3})\right) + 1$
21. $y = 5\sin(5x + 20) - 2$
22. $y = \sin\left(\frac{\pi}{6}x + \pi\right) - 3$
23. $y = 8\cos\left(\frac{7\pi}{6}x + \frac{7\pi}{2}\right) + 6$
24. $y = 2\sin(3x - 21) + 4$

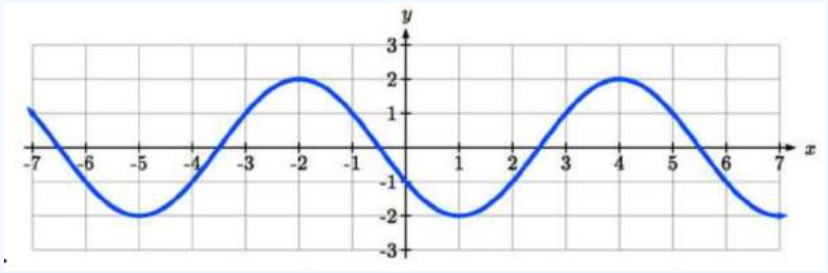
In exercises #25-31, find a formula for each of the functions graphed below.

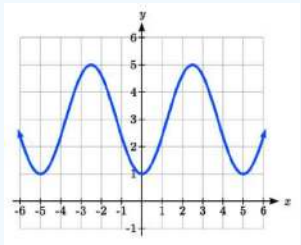
25. 

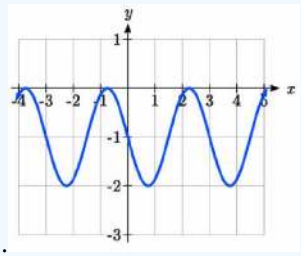
26. 

27. 

28. 

29. 

30. 

31. 

Part 3: Modeling with sine and cosine.

32. The outside temperature is 50 degrees at midnight and the high and low temperature during the day are 57 and 43 degrees, respectively, over the course of two days. Assuming t is the number of hours since midnight, find a function for the temperature, D, in terms of t.

a. Find a formula for a function $T = f(t)$ that models the temperature T in degrees t hours since midnight.

b. Predict when the temperature reaches 52 degrees.

33. The Bay of Fundy in eastern Canada is known for the highest tides in the world. The tides there rise and fall by as much as 50 ft. Assume the tidal cycle takes 11 hours.

a. Find a formula for a function that models the height of the tides in the bay. Assume the low tide corresponds to h=0.
b. Predict when the tide is over 40 feet.

34. An average person has a pulse rate of 72 beats per minute and a blood pressure of 120 over 80. Thus their heart is beating 72 times each minute and her blood pressure is oscillating from a low (diastolic) reading of 80 to a high (systolic) reading of 120 and down again after each beat.

a. Find the period of the person's blood pressure.
b. Sketch a graph that illustrates the blood pressure over 4 seconds.
c. Find a sine or cosine function that models her blood pressure after t seconds.

35. A Ferris wheel is 35 meters in diameter and boarded from a platform that is 3 meters above the ground. The six o'clock position on the Ferris wheel is level with the loading platform. The wheel completes 1 full revolution in 8 minutes. The function $h(t)$ gives your height in meters above the ground *t* minutes after the wheel begins to turn.

a. Find the amplitude, midline, and period of $h\,(t)$.
b. Find a formula for the height function $h\,(t)$.
c. How high are you off the ground after 4 minutes?

36. A herd of elk oscillates from a high of 920 elk to a low of 570 during the year, hitting the lowest value in January ($t = 0$) due to food scarcity or abundance as well as the presence of predators among other factors.

a. Find a formula for the population, P, in terms of the months since January, t.
b. When does the elk herd have a population of 800 or more?

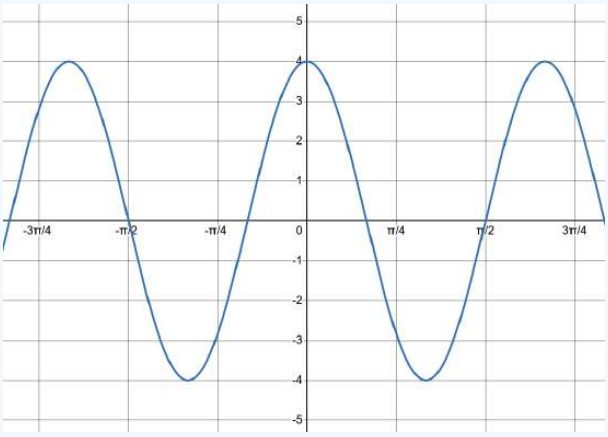
37. The sea ice area around the North Pole fluctuates between about 6 million square kilometers in September to 14 million square kilometers in March.

a. Find a formula for the amount of sea ice around the North Pole over the course of the year.
b. During which months are there less than 9 million square kilometers of sea ice?

Answer

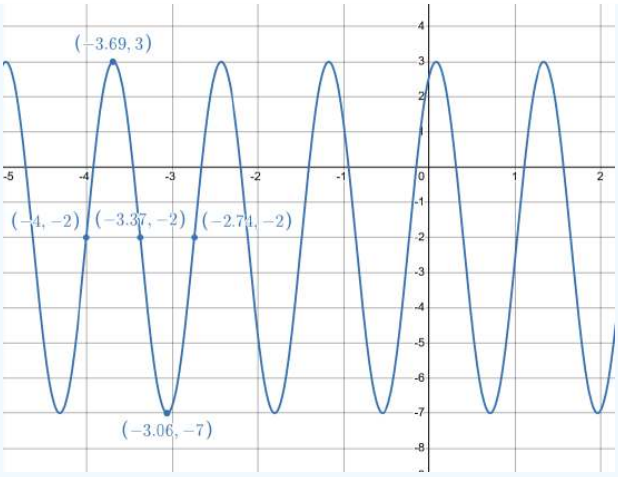
1. Graph E

6. The period is /(P=\dfrac{2 \pi}{3} \) and the amplitude is $A = 4$.



14. $y = 2\cos(2\theta)$

21. The period is /(P=\dfrac{2 \pi}{5}. The amplitude is $A = 4$. The centerline is at $y = -2$. There is no phase shift.



27. $y = 4\sin(\frac{pi}{5}(x+1)$

32. a. $T = -7\sin(\frac{pi}{12}t) + 50$ b. The temperature will reach 52 degrees when $t \approx -1.115$ and $t \approx 13.2$ (just before 11 pm and shortly after 1 pm).

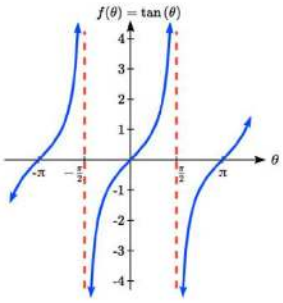
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5.4: Describing Relationships with the Tangent Function

In this section, we will explore the graphs of the other four trigonometric functions. We'll begin with the tangent function. Recall that in Chapter 5 we defined tangent as y/x or sine/cosine, so you can think of the tangent as the slope of a line through the origin making the given angle with the positive x axis.

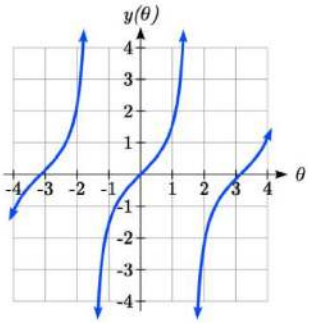
At an angle of 0, the line would be horizontal with a slope of zero. As the angle increases towards $\pi/2$, the slope increases more and more. At an angle of $\pi/2$, the line would be vertical and the slope would be undefined. Immediately past $\pi/2$, the line would have a steep negative slope, giving a large negative tangent value. There is a break in the function at $\pi/2$, where the tangent value jumps from large positive to large negative.



We can use these ideas along with the definition of tangent to sketch a graph. Since tangent is defined as sine/cosine, we can determine that tangent will be zero when sine is zero: at -π, 0, π, and so on. Likewise, tangent will be undefined when cosine is zero: at -$\pi/2$, $\pi/2$, and so on.

The tangent is positive from 0 to $\pi/2$ and π to $3\pi/2$, corresponding to quadrants 1 and 3 of the unit circle.

Using technology, we can obtain a graph of tangent on a standard grid.



Notice that the graph appears to repeat itself. For any angle on the circle, there is a second angle with the same slope and tangent value halfway around the circle, so the graph repeats itself with a period of π; we can see one continuous cycle from - $\pi/2$ to $\pi/2$, before it jumps and repeats itself.

The graph has vertical asymptotes and the tangent is undefined wherever a line at that angle would be vertical: at $\pi/2$, $3\pi/2$, and so on. While the domain of the function is limited in this way, the range of the function is all real numbers.

FEATURES OF THE GRAPH OF TANGENT

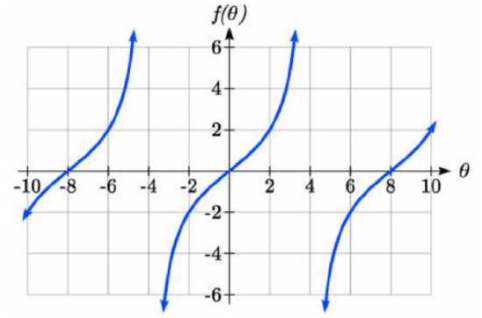
The graph of the tangent function $m(\theta) = \tan(\theta)$

- The **period** of the tangent function is π
- The **domain** of the tangent function is $\theta \neq \frac{\pi}{2} + k\pi$, where k is an integer
- The **range** of the tangent function is all real numbers, $(-\infty, \infty)$

With the tangent function, like the sine and cosine functions, horizontal stretches/compressions are distinct from vertical stretches/compressions. The horizontal stretch can typically be determined from the period of the graph. With tangent graphs, it is often necessary to determine a vertical stretch using a point on the graph.

Example 5.4.1

Find a formula for the function graphed here.



Solution

The graph has the shape of a tangent function, however the period appears to be 8. We can see one full continuous cycle from -4 to 4, suggesting a horizontal stretch. To stretch π to 8, the input values would have to be multiplied by $\frac{8}{\pi}$. Since the constant k in $f(\theta) = a\tan(k\theta)$ is the reciprocal of the horizontal stretch $\frac{8}{\pi}$, the equation must have form

$$f(\theta) = a\tan\left(\frac{\pi}{8}\theta\right).$$

We can also think of this the same way we did with sine and cosine. The period of the tangent function is π but it has been transformed and now it is 8; remember the ratio of the "normal period" to the "new period" is $\frac{\pi}{8}$ and so this becomes the value on the inside of the function that tells us how it was horizontally stretched.

To find the vertical stretch *a*, we can use a point on the graph. Using the point (2, 2)

$$2 = a\tan\left(\frac{\pi}{8}\cdot 2\right) = a\tan\left(\frac{\pi}{4}\right)$$

Since

$$\tan\left(\frac{\pi}{4}\right) = 1, \quad a = 2$$

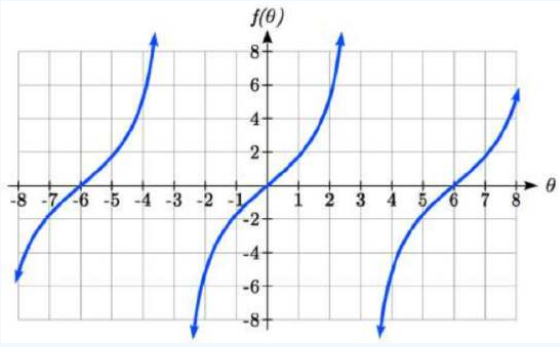
This function would have a formula

$$f(\theta) = 2\tan\left(\frac{\pi}{8}\theta\right)$$

? Exercise 5.4.1

Sketch a graph of $f(\theta) = 3\tan\left(\frac{\pi}{6}\theta\right)$.

Answer



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5.4E: Describing Relationships with the Tangent Function (Exercises)

Section 5.4 Exercises

Part 1: Solving equations with tangent

In exercises #1-4, find all the solutions to the equations.

1. $\tan(\theta) = 0$

2. $\tan(5\theta) = 1$

3. $3\tan(2\theta) + 1 = 0$

4. $4\tan\left(\frac{\theta}{4}\right) - 3 = 0$

In exercises #5 - 6, find all solutions in the interval $-2\pi \leq \theta \leq 2\pi$. Express your results in radians.

5. $\tan(3\theta) = \sqrt{3}$

6. $\tan^2(\theta) - 1 = 0$

Part 2: Graphing with the tangent function

In exercises #7 - 12, find the period and horizontal shift of each of the following functions. Then graph the functions over the indicated intervals.

7. $y = \tan\left(\frac{1}{2}\theta\right)$ on $-2\pi \leq \theta \leq 2\pi$

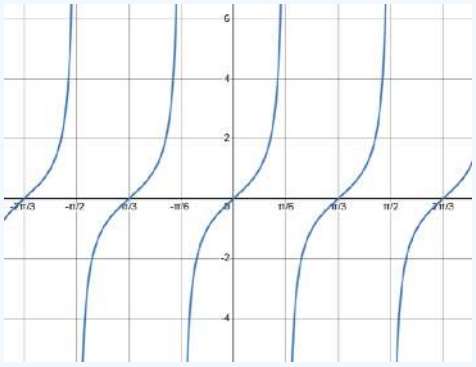
8. $y = 2\tan(3\theta)$ on $-\pi \leq \theta \leq \pi$

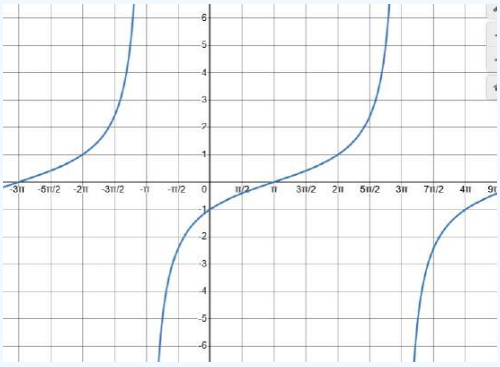
9. $y = 2\tan\left(\frac{1}{3}\theta\right)$ on $-3\pi \leq \theta \leq 3\pi$

10. $y = \tan\left(\theta + \frac{\pi}{4}\right)$ on $-\pi \leq \theta \leq \pi$

11. $y = \tan\left(\frac{\theta}{3} - \frac{\pi}{3}\right)$ on $-\pi \leq \theta \leq \pi$

In exercises #12-13, find a possible formula for each function graphed below.

12. 

13. 

14. Identify the mistake(s) that was(were) made.

Graph $y = \tan(4\theta)$

Step 1: Calculate the period $P = \frac{\pi}{B} = \frac{\pi}{4}$

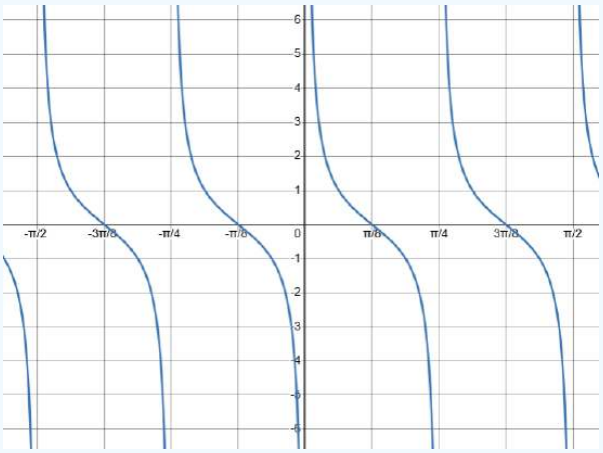
Step 2: Find the x-intercept $4\theta = \frac{\pi}{2}$

$\theta = \frac{\pi}{8}$

Step 3: Find the vertical asymptotes $4\theta = 0$ and $4\theta = \pi$

$\theta = 0$ and $\theta = \frac{\pi}{4}$

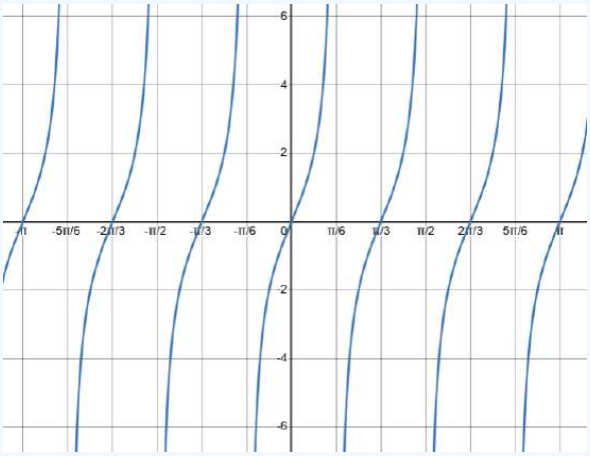
Step 4: Then graph



Answer

2. $\theta == 9^\circ + 36^\circ n$

8. The period is $P = \frac{\pi}{3}$. There is no horizontal shift.



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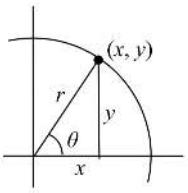
5.5: Right Triangle Trigonometry

In Section 5.3 we were introduced to the sine and cosine function as ratios of the sides of a triangle drawn inside a circle, and spent the rest of that section discussing the role of those functions in finding points on the circle. In this section, we return to the triangle, and explore the applications of the trigonometric functions to right triangles where circles may not be involved.

Recall that we defined sine and cosine as

$$\sin(\theta) = \frac{y}{r}$$

$$\cos(\theta) = \frac{x}{r}$$



Separating the triangle from the circle, we can make equivalent but more general definitions of the sine, cosine, and tangent on a right triangle. On the right triangle, we will label the hypotenuse as well as the side opposite the angle and the side adjacent (next to) the angle.

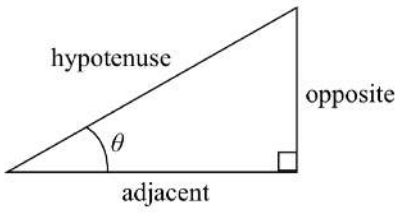
RIGHT TRIANGLE RELATIONSHIPS

Given a right triangle with an angle of θ

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}} \tag{5.5.1}$$

$$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}} \tag{5.5.2}$$

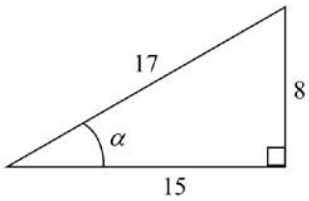
$$\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}} \tag{5.5.3}$$



A common mnemonic for remembering these relationships is SohCahToa, formed from the first letters of "Sine is opposite over hypotenuse, Cosine is adjacent over hypotenuse, Tangent is opposite over adjacent."

Example 5.5.1

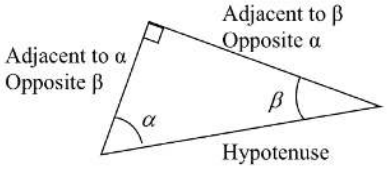
$\alpha\alpha$Given the triangle shown, find the value for $\cos(\alpha)$.



The side adjacent to the angle is 15, and the hypotenuse of the triangle is 17, so

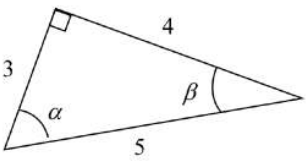
$$\cos(\alpha) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{15}{17}$$

When working with general right triangles, the same rules apply regardless of the orientation of the triangle. In fact, we can evaluate the sine and cosine of either of the two acute angles in the triangle.



Example 5.5.2

Using the triangle shown, evaluate cos(α), sin(α), cos(β), and sin(β),



Solution

$$\cos(\alpha) = \frac{\text{adjacent to } \alpha}{\text{hypotenuse}} = \frac{3}{5}$$

$$\sin(\alpha) = \frac{\text{opposite } \alpha}{\text{hypotenuse}} = \frac{4}{5}$$

$$\cos(\beta) = \frac{\text{adjacent to } \beta}{\text{hypotenuse}} = \frac{4}{5}$$

$$\cos(\beta) = \frac{\text{opposite } \beta}{\text{hypotenuse}} = \frac{3}{5}$$

Exercise 5.5.1

A right triangle is drawn with angle α opposite a side with length 33, angle β opposite a side with length 56, and hypotenuse 65. Find the sine and cosine of α and β

Answer

$$\sin(\alpha) = \frac{33}{65}$$

$$\cos(\alpha) = \frac{56}{65}$$

$$\sin(\beta) = \frac{56}{65}$$

$$\cos(\beta) = \frac{33}{65}$$

You may have noticed that in the above example that $\cos(\alpha) = \sin(\beta)$ and $\cos(\beta) = \sin(\alpha)$. This makes sense since the side opposite α is also adjacent to β. Since the three angles in a triangle need to add to π, or 180 degrees, then the other two angles must add to $\frac{\pi}{2}$, or 90 degrees, so $\beta = \frac{\pi}{2} - \alpha$, and $\alpha = \frac{\pi}{2} - \beta$. Since $\cos(\alpha) = \sin(\beta)$, then $\cos(\alpha) = \sin\left(\frac{\pi}{2} - \alpha\right)$.

COFUNCTION IDENTITIES

The **confunction identities** for sine and cosine are:

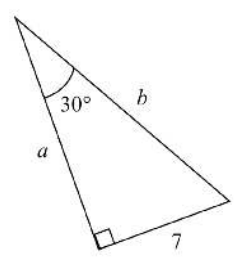
$$\cos(\theta) = \sin(\frac{\pi}{2} - \theta) \tag{5.5.4}$$

$$\sin(\theta) = \cos(\frac{\pi}{2} - \theta) \tag{5.5.5}$$

In the previous examples, we evaluated the sine and cosine on triangles where we knew all three sides of the triangle. Right triangle trigonometry becomes powerful when we start looking at triangles in which we know an angle but don't know all the sides.

Example 5.5.3

Find the unknown sides of the triangle pictured here.



Solution

Since $\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}$. $\sin(30^\circ) = \frac{7}{b}$.

From this, we can solve for the side b.

$$b\sin(30^\circ) = 7$$

$$b = \frac{7}{\sin(30^\circ)}$$

To obtain a value, we can evaluate the sine and simplify

$$b = \frac{7}{1/2} = 14$$

To find the value for side a, we could use the cosine, or simply apply the Pythagorean Theorem:

$$a^2 + 7^2 = b^2$$

$$a^2 + 7^2 = 14^2$$

$$a = \sqrt{147}$$

Notice that if we know at least one of the non-right angles of a right triangle and one side, we can find the rest of the sides and angles.

Exercise 5.5.2

A right triangle has one angle of $\frac{\pi}{3}$ and a hypotenuse of 20. Find the unknown sides and angles of the triangle.

Answer

$$\cos(\frac{\pi}{3}) = \frac{\text{adjacent}}{\text{hypoteuse}} = \frac{\text{Adj}}{20}$$

so,

$$\text{adjacent} = 20\cos(\frac{\pi}{3}) = 20(\frac{1}{2}) = 10$$

$$\sin(\frac{\pi}{3}) = \frac{\text{Opposite}}{\text{hypotenuse}} = \frac{\text{Opp}}{20}$$

so,

$$\text{opposite} = 20\sin(\frac{\pi}{3}) = 20(\frac{\sqrt{3}}{2}) = 10\sqrt{3}$$

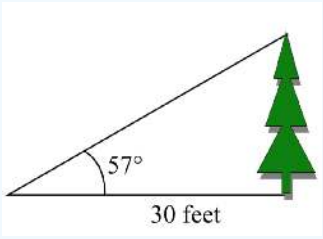
Missing angle = 180 - 90 - 60 = 30 degrees or $\pi/6$.

Example 5.5.4

To find the height of a tree, a person walks to a point 30 feet from the base of the tree, and measures the angle from the ground to the top of the tree to be 57 degrees. Find the height of the tree.

Solution

We can introduce a variable, h, to represent the height of the tree. The two sides of the triangle that are most important to us are the side opposite the angle, the height of the tree we are looking for, and the adjacent side, the side we are told is 30 feet long.



The trigonometric function which relates the side opposite of the angle and the side adjacent to the angle is the tangent.

$$\tan(57^\circ) = \frac{\text{opposite}}{\text{adjacent}} = \frac{h}{30}$$

Solving for h,

$$h = 30\tan(57^\circ)$$

Using technology, we can approximate a value

$$h = 30\tan(57^\circ) \approx 46.2 \text{ feet}$$

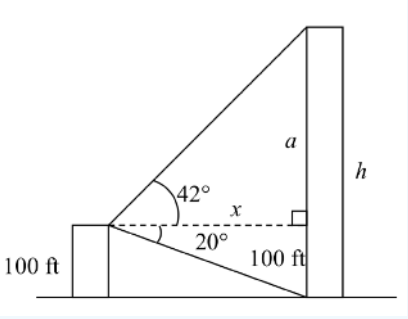
The tree is approximately 46 feet tall.

Example 5.5.5

A person standing on the roof of a 100 foot tall building is looking towards a skyscraper a few blocks away, wondering how tall it is. She measures the angle of declination from the roof of the building to the base of the skyscraper to be 20 degrees and the angle of inclination to the top of the skyscraper to be 42 degrees.

Solution

To approach this problem, it would be good to start with a picture. Although we are interested in the height, h, of the skyscraper, it can be helpful to also label other unknown quantities in the picture – in this case the horizontal distance x between the buildings and a, the height of the skyscraper above the person.



To start solving this problem, notice we have two right triangles. In the top triangle, we know one angle is 42 degrees, but we don't know any of the sides of the triangle, so we don't yet know enough to work with this triangle.

In the lower right triangle, we know one angle is 20 degrees, and we know the vertical height measurement of 100 ft. Since we know these two pieces of information, we can solve for the unknown distance x.

$$\tan(20^\circ) = \frac{\text{opposite}}{\text{adjacent}} = \frac{100}{x}$$

Solving for x

$$x\tan(20^\circ) = 100$$

$$x = \frac{100}{\tan(20^\circ)}$$

Now that we have found the distance x, we know enough information to solve the top right triangle.

$$\tan(42^\circ) = \frac{\text{opposite}}{\text{adjacent}} = \frac{a}{x} = \frac{a}{100/\tan(20^{circ})}$$

$$\tan(42^\circ) = \frac{a\tan(20^\circ)}{100}$$

$$100\tan(42^\circ) = a\tan(20^\circ)$$

$$\frac{\tan(42^\circ)}{\tan(20^\circ)} = a$$

Approximating a value,

$$a = \frac{100\tan(42^\circ)}{\tan(20^\circ)} \approx 247.4 \text{ feet}$$

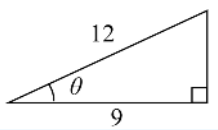
Adding the height of the first building, we determine that the skyscraper is about 347 feet tall.

Important Topics of This Section

- SOH CAH TOA
- Cofunction identities
- Applications with right triangles

Example 5.5.3

Solve the triangle for the angle θ.



Solution

Since we know the hypotenuse and the side adjacent to the angle, it makes sense for us to use the cosine function.

$$\cos(\theta) = \frac{9}{12}$$

Using the definition of the inverse,

$$\theta = \cos^{-1}\left(\frac{9}{12}\right)$$

Evaluating

$$\theta \approx 0.7227, \text{ or about } \text{mtd}$$

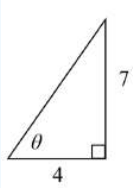
There are times when we need to compose a trigonometric function with an inverse trigonometric function. In these cases, we can find exact values for the resulting expressions

Example 5.5.6

Find an exact value for $\sin\left(\tan^{-1}\left(\frac{7}{4}\right)\right)$.

Solution

While we could use a similar technique as in the last example, we will demonstrate a different technique here. From the inside, we know there is an angle so $\tan(\theta) = \frac{7}{4}$. We can envision this as the opposite and adjacent sides on a right triangle.



Using the Pythagorean Theorem, we can find the hypotenuse of this triangle:

$$4^2 + 7^2 = hypotenuse^2$$

$$hypotenuse = \sqrt{65}$$

Now, we can represent the sine of the angle as opposite side divided by hypotenuse.

$$\sin(\theta) = \frac{7}{\sqrt{65}}$$

This gives us our desired composition

$$\sin\left(\tan^{-1}\left(\frac{7}{4}\right)\right) = \sin(\theta) = \frac{7}{\sqrt{65}}.$$

Solving right triangles for angles

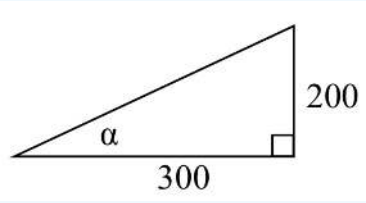
In Section 5.5, we used trigonometry on a right triangle to solve for the sides of a triangle given one side and an additional angle. Using the inverse trig functions, we can solve for the angles of a right triangle given two sides.

Example 5.5.1

An airplane needs to fly to an airfield located 300 miles east and 200 miles north of its current location. At what heading should the airplane fly? In other words, if we ignore air resistance or wind speed, how many degrees north of east should the airplane fly?

Solution

We might begin by drawing a picture and labeling all of the known information.



Drawing a triangle, we see we are looking for the angle α. In this triangle, the side opposite the angle α is 200 miles and the side adjacent is 300 miles. Since we know the values for the opposite and adjacent sides, it makes sense to use the tangent function.

$$\tan(\alpha) = \frac{200}{300}$$

Using the inverse,

$$\alpha = \tan^{-1}\left(\frac{200}{300}\right) \approx 0.588$$

or equivalently about 33.7 degrees.

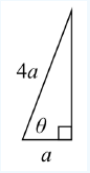
The airplane needs to fly at a heading of 33.7 degrees north of east.

Example 5.5.2

OSHA safety regulations require that the base of a ladder be placed 1 foot from the wall for every 4 feet of ladder length (http://www.osha.gov/SLTC/etools/cons.../4ladders.html). Find the angle such a ladder forms with the ground.

Solution

For any length of ladder, the base needs to be one quarter of the distance the foot of the ladder is away from the wall. Equivalently, if the base is a feet from the wall, the ladder can be $4a$ feet long. Since a is the side adjacent to the angle and $4a$ is the hypotenuse, we use the cosine function.*a*



$$\cos(\theta) = \frac{a}{4a} = \frac{1}{4}$$

Using the inverse

$$\theta = \cos^{-1}\left(\frac{1}{4}\right) \approx 75.52 \text{ degrees}$$

The ladder forms a 75.52 degree angle with the ground.

Exercise 5.5.1

A cable that anchors the center of the London Eye Ferris wheel to the ground must be replaced. The center of the Ferris wheel is 70 meters above the ground and the second anchor on the ground is 23 meters from the base of the wheel. What is the angle from the ground up to the center of the Ferris wheel and how long is the cable?

Answer

Angle of elevation for the cable is 71.81 degrees and the cable is 73.68 m long

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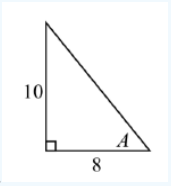
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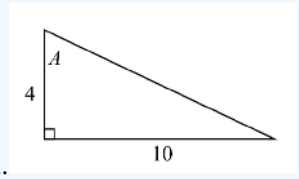
5.5E: Right Triangle Trigonometry (Exercises)

Section 5.5 Exercises

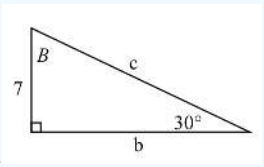
Note: pictures may not be drawn to scale.

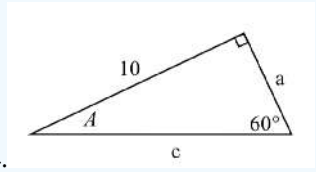
In each of the triangles below, find $\sin(A), \cos(A), \tan(A), \sec(A), \csc(A), \cot(A)$.

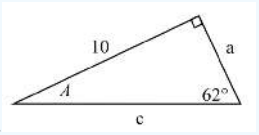
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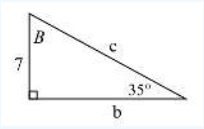
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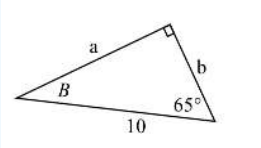
In each of the following triangles, solve for the unknown sides and angles.

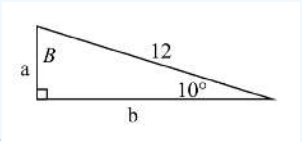
3. 

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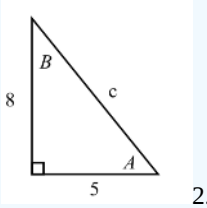
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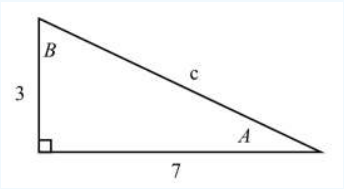
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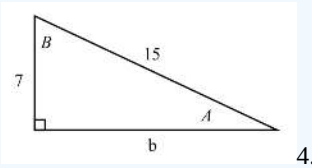
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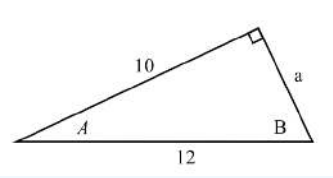
8. 

In each of the following triangles, solve for the unknown side and angles.

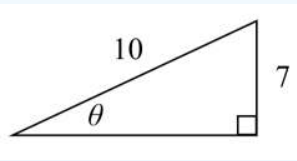
1. 

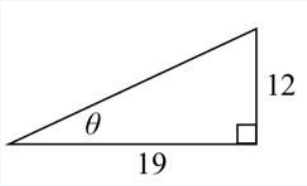
2. 

3. 

4. 

Find the angle θ in degrees.

17. 

18. 

9. A 33-ft ladder leans against a building so that the angle between the ground and the ladder is 80°. How high does the ladder reach up the side of the building?

10. A 23-ft ladder leans against a building so that the angle between the ground and the ladder is 80°. How high does the ladder reach up the side of the building?

11. The angle of elevation to the top of a building in New York is found to be 9 degrees from the ground at a distance of 1 mile from the base of the building. Using this information, find the height of the building.

12. The angle of elevation to the top of a building in Seattle is found to be 2 degrees from the ground at a distance of 2 miles from the base of the building. Using this information, find the height of the building.

13. A radio tower is located 400 feet from a building. From a window in the building, a person determines that the angle of elevation to the top of the tower is 36° and that the angle of depression to the bottom of the tower is 23°. How tall is the tower?

14. A radio tower is located 325 feet from a building. From a window in the building, a person determines that the angle of elevation to the top of the tower is 43° and that the angle of depression to the bottom of the tower is 31°. How tall is the tower?

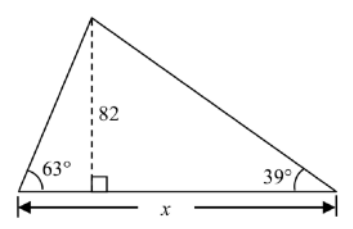
15. A 200 foot tall monument is located in the distance. From a window in a building, a person determines that the angle of elevation to the top of the monument is 15° and that the angle of depression to the bottom of the tower is 2°. How far is the person from the monument?

16. A 400 foot tall monument is located in the distance. From a window in a building, a person determines that the angle of elevation to the top of the monument is 18° and that the angle of depression to the bottom of the tower is 3°. How far is the person from the monument?

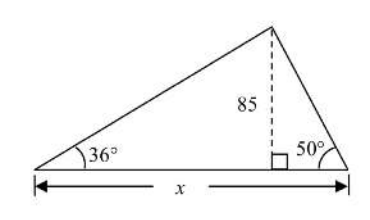
17. There is an antenna on the top of a building. From a location 300 feet from the base of the building, the angle of elevation to the top of the building is measured to be 40°. From the same location, the angle of elevation to the top of the antenna is measured to be 43°. Find the height of the antenna.

18. There is lightning rod on the top of a building. From a location 500 feet from the base of the building, the angle of elevation to the top of the building is measured to be 36°. From the same location, the angle of elevation to the top of the lightning rod is measured to be 38°. Find the height of the lightning rod.

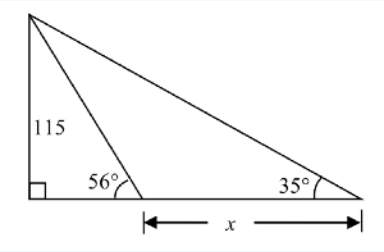
19. Find the length x.



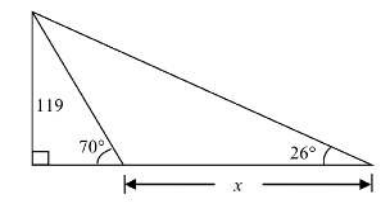
20. Find the length x.



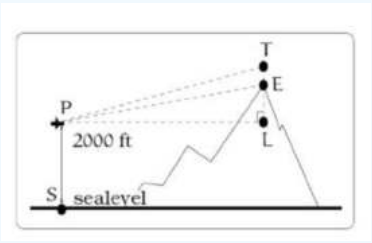
21. Find the length x.



22. Find the length x.

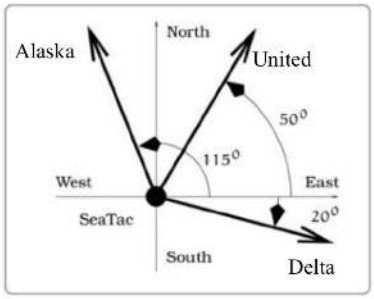


23. A plane is flying 2000 feet above sea level toward a mountain. The pilot observes the top of the mountain to be 18° above the horizontal, then immediately flies the plane at an angle of 20° above horizontal. The airspeed of the plane is 100 mph. After 5 minutes, the plane is directly above the top of the mountain. How high is the plane above the top of the mountain (when it passes over)? What is the height of the mountain? [UW]

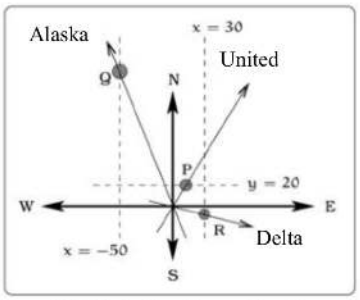


24. Three airplanes depart SeaTac Airport. A United flight is heading in a direction 50° counterclockwise from east, an Alaska flight is heading 115° counterclockwise from east and a Delta flight is heading 20° clockwise from east. [UW]

a. Find the location of the United flight when it is 20 miles north of SeaTac.
b. Find the location of the Alaska flight when it is 50 miles west of SeaTac.
c. Find the location of the Delta flight when it is 30 miles east of SeaTac.

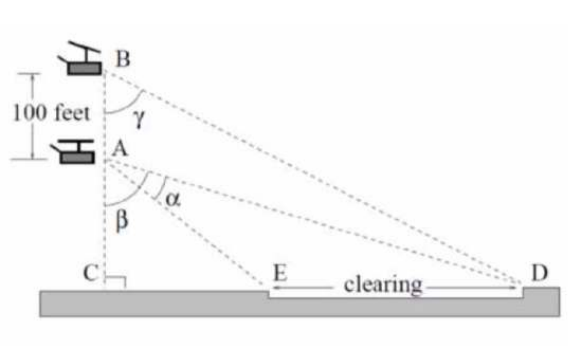


(a) The flight paths of three airplanes

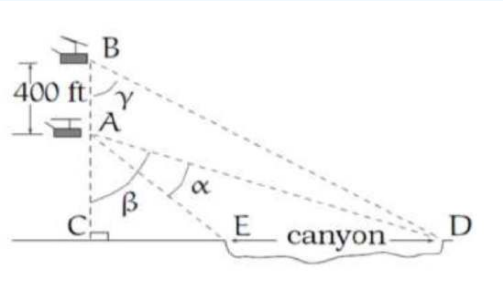


(b) Modeling the paths of each flight

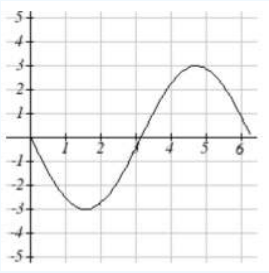
25. The crew of a helicopter needs to land temporarily in a forest and spot a flat piece of ground (a clearing in the forest) as a potential landing site, but are uncertain whether it is wide enough. They make two measurements from A (see picture) finding $\alpha = 25°$ and $\beta = 54°$. They rise vertically 100 feet to B and measure $\gamma = 47°$. Determine the width of the clearing to the nearest foot. [UW]

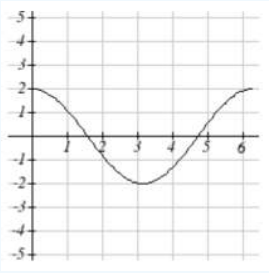


26. A Forest Service helicopter needs to determine the width of a deep canyon. While hovering, they measure the angle $\gamma = 48^\circ$ at position B (see picture), then descend 400 feet to position A and make two measurements: $\alpha = 13^\circ$ (the measure of $\angle EAD$), $\beta = 53^\circ$ (the measure of $\angle CAD$). Determine the width of the canyon to the nearest foot. [UW]



Answer

1. 

3. 

5. Amp: 3. Period = 2. Midline: $y = -4$. $f(t) = 3\sin(\pi t) - 4$

7. Amp: 2. Period = 4π. Midline: $y = 1$. $f(t) = 2\cos(\frac{1}{2}t) + 1$

9. Amp: 2. Period = 5. Midline: $y = 3$. $f(t) = -2\cos(\frac{2\pi}{5}t) + 3$

11. Amp: 3, Period = $\frac{\pi}{4}$, Shift: 4 left, Midline: $y = 5$

13. Amp: 2, Period = $\frac{2\pi}{3}$, Shift: 7 left, Midline: $y = 4$

15. Amp: 1, Period = 12, Shift: 6 left, Midline: $y = -3$

17. $f(x) = 4\sin(\frac{\pi}{5}(x+1))$

19. $f(x) = \cos(\frac{\pi}{5}(x+2))$

21. $D(t) = 50 - 7\sin(\frac{\pi}{12}t)$

23. a. Amp: 12.5. Midline: $y = 13.5$. Period: 10

b. $h(t) = -12.5\cos(\frac{\pi}{5}t) + 13.5$

c. $h(t) = 26$ meters

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5.6: Simpliying and Solving Equations with Trigonometric Functions

You may also notice that the ratios defining the secant, cosecant, and cotangent are the reciprocals of the ratios defining the cosine, sine, and tangent functions, respectively. Additionally, notice that using our results from the last section,

$$\tan(\theta) = \frac{y}{x} = \frac{r\sin(\theta)}{r\cos(\theta)} = \frac{\sin(\theta)}{\cos(\theta)} \tag{5.6.1}$$

Applying this concept to the other trig functions we can state the reciprocal identities.

Identities

The other four trigonometric functions can be related back to the sine and cosine functions using these basic relationships:

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \tag{5.6.2}$$

$$\sec(\theta) = \frac{1}{\cos(\theta)} \tag{5.6.3}$$

$$\csc(\theta) = \frac{1}{\sin(\theta)} \tag{5.6.4}$$

$$\cot(\theta) = \frac{1}{\tan(\theta)} = \frac{\cos(\theta)}{\sin(\theta)} \tag{5.6.5}$$

These relationships are called **identities**. Identities are statements that are true for all values of the input on which they are defined. Identities are usually something that can be derived from definitions and relationships we already know, similar to how the identities above were derived from the circle relationships of the six trig functions. The Pythagorean Identity we learned earlier was derived from the Pythagorean Theorem and the definitions of sine and cosine. We will discuss the role of identities more after an example.

Example 5.6.1

Evaluate $\tan(45^\circ)$ and $\sec\left(\frac{5\pi}{6}\right)$.

Solution

Since we know the sine and cosine values for these angles, it makes sense to relate the tangent and secant values back to the sine and cosine values.

$$\tan(45^\circ) = \frac{\sin(45^\circ)}{\cos(45^\circ)} = \frac{\sqrt{2}/2}{\sqrt{2}/2} = 1$$

Notice this result is consistent with our interpretation of the tangent value as the slope of the line passing through the origin at the given angle: a line at 45 degrees would indeed have a slope of 1.

$$\sec(\frac{5\pi}{6}) = \frac{1}{cos(\frac{5\pi}{6})} = \frac{1}{-\sqrt{3}/2} = \frac{-2}{\sqrt{3}} \quad \text{,which could also be written as} \quad \frac{-2\sqrt{3}}{3}$$

Exercise 5.6.1

Evaluate $\csc\left(\frac{7\pi}{6}\right)$.

Answer

$$\csc(\frac{7\pi}{6}) = \frac{1}{\sin(\frac{7\pi}{6})} = \frac{1}{-1/2} = -2$$

Just as we often need to simplify algebraic expressions, it is often also necessary or helpful to simplify trigonometric expressions. To do so, we utilize the definitions and identities we have established.

Example 5.6.2

Simplify $\frac{\sec(\theta)}{\tan(\theta)}$.

Solution

We can simplify this by rewriting both functions in terms of sine and cosine

$$\frac{\sec(\theta)}{\tan(\theta)} = \frac{1/\cos(\theta)}{\sin(\theta)/\cos(\theta)}$$

To divide the fractions we could invert and multiply

$$= \frac{1}{\cos(\theta)} \frac{\cos(\theta)}{\sin(\theta)}$$

cancelling the cosines,

$$= \frac{1}{\sin(\theta)} = \csc(\theta)$$

simplifying and using the identity

By showing that $\frac{\sec(\theta)}{\tan(\theta)}$ can be simplified to $\csc(\theta)$, we have, in fact, established a new identity: that $\frac{\sec(\theta)}{\tan(\theta)} = \csc(\theta)$.

Occasionally a question may ask you to "prove the identity" or "establish the identity." This is the same idea as when an algebra book asks a question like "show that $(x-1)^2 = x^2 - 2x + 1$." In this type of question, we must show the algebraic manipulations that demonstrate that the left and right side of the equation are in fact equal. You can think of a "prove the identity" problem as a simplification problem where you *know the answer*: you know what the end goal of the simplification should be, and just need to show the steps to get there.

To prove an identity, in most cases you will start with the expression on one side of the identity and manipulate it using algebra and trigonometric identities until you have simplified it to the expression on the other side of the equation. **Do not** treat the identity like an equation to solve – it isn't! The proof is establishing the two expressions are equal, so we must take care to work with one side at a time rather than applying an operation simultaneously to both sides of the equation.

Example 5.6.3

Prove the identity $\frac{1+\cot(\alpha)}{\csc(\alpha)} = \sin(\alpha) + \cos(\alpha)$.

Solution

Since the left side seems a bit more complicated, we will start there and simplify the expression until we obtain the right side. We can use the right side as a guide for what might be good steps to make. In this case, the left side involves a fraction while the right side doesn't, which suggests we should look to see if the fraction can be reduced.

Additionally, since the right side involves sine and cosine and the left does not, it suggests that rewriting the cotangent and cosecant using sine and cosine might be a good idea.

$$\frac{1+\cot(\alpha)}{\csc(\alpha)}$$

Rewriting the cotangent and cosecant

$$= \frac{1 + \frac{\cos(\alpha)}{\sin(\alpha)}}{\frac{1}{\sin(\alpha)}}$$

To divide the fractions, we invert and multiply

$$= \left(1 + \frac{\cos(\alpha)}{\sin(\alpha)}\right) \frac{\sin(\alpha)}{1}$$

Distributing,

$$= 1 \cdot \frac{\sin(\alpha)}{1} + \frac{\cos(\alpha)}{\sin(\alpha)} \cdot \frac{\sin(\alpha)}{1}$$

Simplifying the fractions,

$$= \sin(\alpha) + \cos(\alpha)$$

Establishing the identity.

Notice that in the second step, we could have combined the 1 and $\frac{\cos(\alpha)}{\sin(\alpha)}$ before inverting and multiplying. It is very common when proving or simplifying identities for there to be more than one way to obtain the same result.

We can also utilize identities we have previously learned, like the Pythagorean Identity, while simplifying or proving identities.

Example 5.6.4

Establish the identity $\frac{\cos^2(\theta)}{1 + \sin(\theta)} = 1 - \sin(\theta)$.

Solution

Since the left side of the identity is more complicated, it makes sense to start there. To simplify this, we will have to reduce the fraction, which would require the numerator to have a factor in common with the denominator. Additionally, we notice that the right side only involves sine. Both of these suggest that we need to convert the cosine into something involving sine.

Recall the Pythagorean Identity told us $\cos^2(\theta) + \sin^2(\theta) = 1$. By moving one of the trig functions to the other side, we can establish:

$$\sin^2(\theta) = 1 - \cos^2(\theta) \quad \text{and} \quad \cos^2(\theta) = 1 - \sin^2(\theta)$$

Utilizing this, we now can establish the identity. We start on one side and manipulate:

$$\frac{\cos^2(\theta)}{1 + \sin(\theta)}$$

Utilizing the Pythagorean Identity

$$= \frac{1 - \sin^2(\theta)}{1 + \sin(\theta)}$$

Factoring the numerator

$$= \frac{(1 - \sin(\theta))(1 + \sin(\theta))}{1 + \sin(\theta)}$$

Cancelling the like factors

$$= 1 - \sin(\theta)$$

Establishing the identity

We can also build new identities from previously established identities. For example, if we divide both sides of the Pythagorean Identity by cosine squared (which is allowed since we've already shown the identity is true),

$$\frac{\cos^2(\theta)+\sin^2(\theta)}{\cos^2(\theta)}=\frac{1}{\cos^2(\theta)}$$

Splitting the fraction on the left,

$$\frac{\cos^2(\theta)}{\cos^2(\theta)}+\frac{\sin^2(\theta)}{\cos^2(\theta)}=\frac{1}{\cos^2(\theta)}$$

Simplifying and using the definitions of tan and sec

$$1+\tan^2(\theta)=\sec^2(\theta).$$

? Exercise 5.6.2

Use a similar approach to establish that $\cot^2(\theta)+1=\csc^2(\theta)$.

Answer

$$\frac{\cos^2(\theta)+\sin^2(\theta)}{\sin^2\theta}=1$$

$$\frac{\cos^2(\theta)}{\sin^2(\theta)}+\frac{\sin^2(\theta)}{\sin^2(\theta)}=\frac{1}{\sin^2(\theta)}$$

$$\cot^2(\theta)+1=\csc^2(\theta)$$

Identities: Alternate forms of the Pythagorean Identity

$$1+\tan^2(\theta)=\sec^2(\theta) \tag{5.6.6}$$

$$\cot^2(\theta)+1=\csc^2(\theta) \tag{5.6.7}$$

✓ Example 5.6.5

If $\tan(\theta)=\frac{2}{7}$ and θ is in the 3^{rd} quadrant, find $\cos(\theta)$.

Solution

There are two approaches to this problem, both of which work equally well.

Approach 1

Since $\tan(\theta)=\frac{y}{x}$ and the angle is in the third quadrant, we can imagine a triangle in a circle of some radius so that the point on the circle is (-7, -2), so $\frac{y}{x}=\frac{-2}{-7}=\frac{2}{7}$.

Using the Pythagorean Theorem, we can find the radius of the circle:

$$(-7)^2+(-2)^2=r^2 \quad \text{,so} \quad r=\sqrt{53}$$

Now we can find the cosine value:

$$\cos(\theta)=\frac{x}{r}=\frac{-7}{\sqrt{53}}$$

Approach 2

Using the $1+\tan^2(\theta)=\sec^2(\theta)$ form of the Pythagorean Identity with the known tangent value,

$$1+\tan^2(\theta)=\sec^2(\theta)$$

$$1+\left(\frac{2}{7}\right)^2=\sec^2(\theta)$$

$$\frac{53}{49}=\sec^2(\theta)$$

$$\sec(\theta)=\pm\sqrt{\frac{53}{49}}=\pm\frac{\sqrt{53}}{7}$$

Since the angle is in the third quadrant, the cosine value will be negative so the secant value will also be negative. Keeping the negative result, and using definition of secant,

$$\sec(\theta)=-\frac{\sqrt{53}}{7}$$

$$\frac{1}{\cos(\theta)}=-\frac{\sqrt{53}}{7}$$

Inverting both sides

$$\cos(\theta)=-\frac{7}{\sqrt{53}}=-\frac{7\sqrt{53}}{53}$$

? Exercise 5.6.3

If $\sec(\phi)=-\frac{7}{3}$ and $\frac{\pi}{2}<\phi<\pi$, find $\tan(\phi)$ and $\sin(\phi)$.

Answer

$$\sec(\phi)=-\frac{7}{3}$$

.

By definition, $\frac{1}{\cos(\phi)}=-\frac{7}{3}$, so $\cos(\phi)=-\frac{3}{7}$.

Using Pythagorean Identity with the sec, $1+\tan^2(\phi)=\left(-\frac{7}{3}\right)^2$. Solving gives $\tan(\phi)=\frac{\sqrt{40}}{-3}$. We use the negative square root since an angle in the second quadrant would have a negative tangent.

Using Pythagorean Identity with the cos, $\sin^2(\phi)+\left(-\frac{3}{7}\right)^2=1$. Solving, $\sin(\phi)=\frac{\sqrt{40}}{7}$.

Important Topics of This Section

- Negative Angle identities

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5.6E: Simpliying and Solving Equations with Trigonometric Functions (Exercises)

Section 5.6 Exercises

Part 1: The reciprocal and quotient identities

In exercises # 1 - 10, perform the indicated operation and simplify the expression. Write the expression without fraction and/or as a single trig term if possible.

1. $\cos(t)\tan(t)$
2. $\csc(t)\tan(t)$
3. $\dfrac{\sec(t)}{\csc(t)}$
4. $\dfrac{\cot(t)}{\csc(t)}$
5. $\sin^2(\theta)(\csc^2(\theta)-1)$
6. $\csc(t)(\cos(t)+\sin(t))$
7. $\sec(t)-\tan(t)\sin(t)$
8. $\dfrac{\cos(t)}{\sin(t)\cot(t)}$
9. $\cos(t)(\sec(t)+\csc(t))$
10. $(\sec(\theta)+\csc(\theta))(\cos(\theta)-\sin(\theta))$

In exercises #11 - 17, find all solutions to the equations

11. $\csc(\theta)=4$
12. $3\sec(\theta)+2=5$
13. $4\cot^2(\theta)-1=0$
14. $2\tan(\theta)\cos(\theta)-1=0$
15. $\sin(\theta)-2\csc(\theta)=0$
16. $\sin(\theta)\cos(\theta)=0$
17. $2\sin(\theta)-1=\csc(\theta)$
18. $3\sec^2(\theta)+\tan^2(\theta)=0$

Part 2: Pythagorean Identities

In exercises # 19-25, perform the indicated operation and simplify the expression. Write the expression without fraction if possible.

19. $\cos^2(\theta)+\cos^2(\theta)\tan^2(\theta)$
20. $(1-\cos(\theta))(1+\cos(\theta))$
21. $\sqrt{25-25cos^2(\theta)}$
22. $\dfrac{\sin^2(t)+\cos^2(t)}{\cos^2(t)}$
23. $\dfrac{1-\sin^2(t)}{\sin^2(t)}$
24. $\dfrac{1}{\cos^2(t)}-1$

25. $\dfrac{\cos(t)}{\sec(t)} - \dfrac{\sin(t)}{\csc(t)}$

In exercises #26 - 30, find all solutions to the equation

26. $2\cos^2(\theta) + \sin^2(\theta) - 1 = 0$

27. $3\sec^2(\theta) + \tan^2(\theta) = 1$

28. $2\sin(\theta) - 1 = \csc(\theta)$

Answer

1. $\sin(t)$

6. $1 + \tan(t)$

11. $\theta \approx 14.48^\circ + 360^\circ n$ and $\theta \approx 165.52^\circ + 360^\circ n$ where n is any integer.

16. $\theta = 90^\circ n$ where n is any integer.

19. 1

26. $\theta = 90^\circ + 360^\circ n$ and $\theta \approx 270^\circ + 360^\circ n$ where n is any integer.

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5.7: More on Simpliying and Solving Equations with Trigonometric Functions

Important Topics of This Section

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5.7E: More on Simpliying and Solving Equations with Trigonometric Functions (Exercises)

Section 5.7 Exercises

Part 2: Negative Angles

21. If $\tan(t) = -1.5$, find $\tan(-t)$.

22. If $\cos(t) = 0.23$, find $\cos(-t)$.

23. If $\sin(t) = 0.23$, find $\sin(-t)$.

Simplify each of the following expressions completely.

27. $\cot(-x)\cos(-x) + \sin(-x)$

28. $\cos(-x) + \tan(-x)\sin(-x)$

Part 2: Angle sum identities

Simplify each of the following to an expression involving a single trig function with no fractions.

1. $\sin(180^{circ} + \theta$

2. $\cos(\pi - \theta$

30. $\dfrac{1+\tan^2(b)}{\tan^2(b)} = \csc^2(b)$

31. $\dfrac{\csc^2(x) - \sin^2(x)}{\csc(x) + \sin(x)} = \cos(x)\cot(x)$

33. $\dfrac{\csc^2(\alpha) - 1}{\csc^2(\alpha) - \csc(\alpha)} = 1 + \sin(\alpha)$

36. $2\sec^2(t) = \dfrac{1-\sin(t)}{\cos^2(t)} + \dfrac{1}{1-\sin(t)}$

37. $\dfrac{\sin^4(\gamma) - \cos^4(\gamma)}{\sin(\gamma) - \cos(\gamma)} = \sin(\gamma) + \cos(\gamma)$

23. $\dfrac{1+\cot(t)}{1+\tan(t)}$

38. $\dfrac{(1+\cos(A))(1-\cos(A))}{\sin(A)} = \sin(A)$

Solve the equation

16. $\sin(\theta + 180^{circ}) - sin(\theta) = 1$

Answer

1. $\dfrac{\pi}{4}$
3. $-\dfrac{\pi}{6}$
5. $\dfrac{\pi}{3}$
7. $\dfrac{3\pi}{4}$
9. $\dfrac{\pi}{4}$
11. $-\dfrac{\pi}{3}$
13. 1.9823
15. -0.9273

17. 44.427°
19. $\frac{\pi}{4}$
21. $-\frac{\pi}{6}$
23. $\frac{2\sqrt{10}}{7}$
25. $\frac{1}{\sqrt{17}}$
27. $\frac{\sqrt{25-x^2}}{5}$
29. $\frac{3x}{\sqrt{9x^2+1}}$

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