INTRODUCTION TO PARTIAL DIFFERENTIAL EQUATIONS



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Introduction to Partial Differential Equations

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About the Book

CHAPTER OVERVIEW

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1.2: Linear Constant Coefficient Equations
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> *"The profound study of nature is the most fertile source of mathematical discoveries." - Joseph Fourier (1768-1830)*

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1.1: Introduction

We begin our study of partial differential equations with *first order partial differential equations*. Before doing so, we need to define a few terms.

Recall (see the appendix on differential equations) that an n-th order ordinary differential equation is an equation for an unknown function $y(x)$ that expresses a relationship between the unknown function and its first n derivatives. One could write this generally as

$$F(y^{(n)}(x), y^{(n-1)}(x), \ldots, y'(x), y(x), x) = 0. \tag{1.1.1}$$

Here $y^{(n)}(x)$ represents the nth derivative of $y(x)$. Furthermore, and initial value problem consists of the differential equation plus the values of the first $n-1$ derivatives at a particular value of the independent variable, say x_0:

$$y^{(n-1)}(x_0) = y_{n-1}, \quad y^{(n-2)}(x_0) = y_{n-2}, \quad \ldots, \quad y(x_0) = y_0. \tag{1.1.2}$$

If conditions are instead provided at more than one value of the independent variable, then we have a boundary value problem.

If the unknown function is a function of several variables, then the derivatives are partial derivatives and the resulting equation is a partial differential equation. Thus, if $u = u(x, y, \ldots)$, a general partial differential equation might take the form

$$F\left(x, y, \ldots, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \ldots, \frac{\partial^2 u}{\partial x^2}, \ldots\right) = 0. \tag{1.1.3}$$

Since the notation can get cumbersome, there are different ways to write the partial derivatives. First order derivatives could be written as

$$\frac{\partial u}{\partial x}, u_x, \partial_x u, D_x u.$$

$$\frac{\partial^2 u}{\partial x^2}, u_{xx}, \partial_{xx} u, D_x^2 u.$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}, u_{xy}, \partial_{xy} u, D_y D_x u.$$

Note, we are assuming that $u(x, y, \ldots)$has continuous partial derivatives. Then, according to Clairaut's Theorem (Alexis Claude Clairaut, 1713-1765) , mixed partial derivatives are the same.

Examples of some of the partial differential equation treated in this book are shown in Table 2.1.1. However, being that the highest order derivatives in these equation are of second order, these are second order partial differential equations. In this chapter we will focus on first order partial differential equations. Examples are given by

$$\begin{aligned} u_t + u_x &= 0. \\ u_t + uu_x &= 0. \\ u_t + uu_x &= u. \\ 3u_x - 2u_y + u &= x. \end{aligned}$$

For function of two variables, which the above are examples, a general first order partial differential equation for $u = u(x, y)$ is given as

$$F(x, y, u, u_x, u_y) = 0, \quad (x, y) \in D \subset R^2. \tag{1.1.4}$$

This equation is too general. So, restrictions can be placed on the form, leading to a classification of first order equations. A linear first order partial differential equation is of the form

$$a(x, y)u_x + b(x, y)u_y + c(x, y)u = f(x, y). \tag{1.1.5}$$

Note that all of the coefficients are independent of u and its derivatives and each term in linear in u, u_x, or u_y.

We can relax the conditions on the coefficients a bit. Namely, we could assume that the equation is linear only in u_x and u_y. This gives the quasilinear first order partial differential equation in the form

$$a(x,y,u)u_x + b(x,y,u)u_y = f(x,y,u). \tag{1.1.6}$$

Note that the u-term was absorbed by $f(x,y,u)$.

In between these two forms we have the semilinear first order partial differential equation in the form

$$a(x,y)u_x + b(x,y)u_y = f(x,y,u). \tag{1.1.7}$$

Here the left side of the equation is linear in u, u_x and u_y. However, the right hand side can be nonlinear in u.

For the most part, we will introduce the Method of Characteristics for solving quasilinear equations. But, let us first consider the simpler case of linear first order constant coefficient partial differential equations.

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1.2: Linear Constant Coefficient Equations

Let's consider the linear first order constant coefficient partial differential equation

$$au_x + bu_y + cu = f(x,y), \tag{1.2.1}$$

for a, b, and c constants with $a^2 + b^2 > 0$. We will consider how such equations might be solved. We do this by considering two cases, $b = 0$ and $b \neq 0$.

Integrating this equation and solving for $u(x,y)$, we have

$$\begin{aligned} \mu(x)u(x,y) &= \frac{1}{a}\int f(\xi,y)\mu(\xi)d\xi + g(y) \\ e^{\frac{c}{a}x}u(x,y) &= \frac{1}{a}\int f(\xi,y)e^{\frac{c}{a}\xi}d\xi + g(y) \\ u(x,y) &= \frac{1}{a}\int f(\xi,y)e^{\frac{c}{a}(\xi - x)}d\xi + g(y)e^{-\frac{c}{a}x}. \end{aligned} \tag{1.2.2}$$

Here $g(y)$ is an arbitrary function of y.

For the second case, $b \neq 0$, we have to solve the equation

$$au_x + bu_y + cu = f.$$

It would help if we could find a transformation which would eliminate one of the derivative terms reducing this problem to the previous case. That is what we will do.

We first note that

$$\begin{aligned} au_x + bu_y &= (a\mathbf{i} + b\mathbf{j}) \cdot (u_x\mathbf{i} + u_y\mathbf{j}) \\ &= (a\mathbf{i} + b\mathbf{j}) \cdot \nabla u. \end{aligned} \tag{1.2.3}$$

Recall from multivariable calculus that the last term is nothing but a directional derivative of $u(x,y)$ in the direction $a\mathbf{i} + b\mathbf{j}$. [Actually, it is proportional to the directional derivative if $a\mathbf{i} + b\mathbf{j}$ is not a unit vector.]

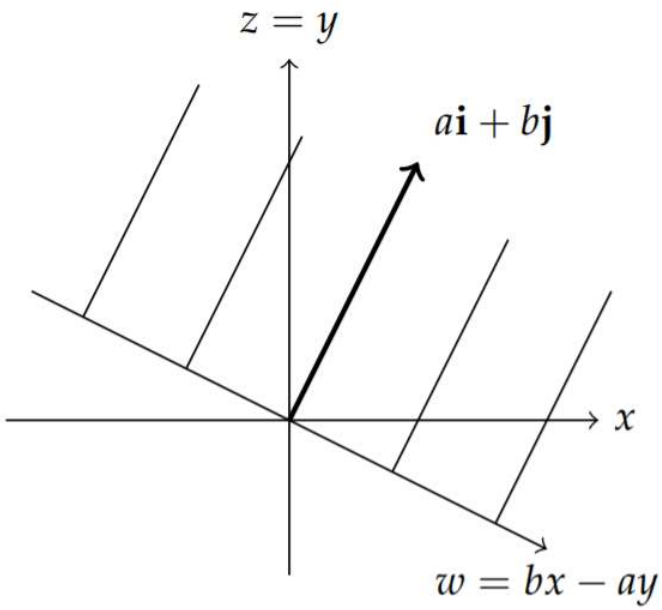


Figure 1.2.1: Coordinate systems for transforming $au_x + bu_y + cu = f$ into $bv_z + cv = f$ using the transformation $w = bx - ay$ and $z = y$.

Therefore, we seek to write the partial differential equation as involving a derivative in the direction $a\mathbf{i} + b\mathbf{j}$ but not in a directional orthogonal to this. In Figure 1.2.1 we depict a new set of coordinates in which the w direction is orthogonal to $a\mathbf{i} + b\mathbf{j}$.

We consider the transformation

$$\begin{aligned} w &= bx - ay, \\ z &= y. \end{aligned} \tag{1.2.4}$$

We first note that this transformation is invertible,

$$\begin{aligned} x &= \frac{1}{b}(w + az), \\ y &= z. \end{aligned} \tag{1.2.5}$$

Next we consider how the derivative terms transform. Let $u(x, y) = v(w, z)$. Then, we have

$$\begin{aligned} au_x + bu_y &= a\frac{\partial}{\partial x}v(w, z) + b\frac{\partial}{\partial y}v(w, z), \\ &= a\left[\frac{\partial v}{\partial w}\frac{\partial w}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right] \\ &\quad + b\left[\frac{\partial v}{\partial w}\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial y}\right] \\ &= a[bv_w + 0 \cdot v_z] + b[-av_w + v_z] \\ &= bv_z. \end{aligned} \tag{1.2.6}$$

Therefore, the partial differential equation becomes

$$bv_z + cv = f\left(\frac{1}{b}(w + az), z\right).$$

This is now in the same form as in the first case and can be solved using an integrating factor.

Example 1.2.1

Find the general solution of the equation $3u_x - 2u_y + u = x$.

Solution

First, we transform the equation into new coordinates.

$$w = bx - ay = -2x - 3y,$$

and $z = y$. The,

$$\begin{aligned} u_x - 2u_y &= 3[-2v_w + 0 \cdot v_z] - 2[-3v_w + v_z] \\ &= -2v_z. \end{aligned} \tag{1.2.7}$$

Using this integrating factor, we can solve the differential equation for $v(w, z)$.

$$\begin{aligned} \frac{\partial}{\partial z}\left(e^{-z/2}v\right) &= \frac{1}{4}(w + 3z)e^{-z/2}, \\ e^{-z/2}v(w, z) &= \frac{1}{4}\int^{z}(w + 3\xi)e^{-\xi/2}d\xi \\ &= -\frac{1}{2}(w + 6 + 3z)e^{-z/2} + c(w) \\ v(w, z) &= -\frac{1}{2}(w + 6 + 3z) + c(w)e^{z/2} \\ u(x, y) &= x - 3 + c(-2x - 3y)e^{y/2}. \end{aligned} \tag{1.2.8}$$

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1.3: Quasilinear Equations - The Method of Characteristics

Geometric Interpretation

We consider the quasilinear partial differential equation in two independent variables,

$$a(x,y,u)u_x + b(x,y,u)u_y - c(x,y,u) = 0. \tag{1.3.1}$$

Let $u = u(x,y)$ be a solution of this equation. Then,

$$f(x,y,u) = u(x,y) - u = 0$$

describes the solution surface, or integral surface.

We recall from multivariable, or vector, calculus that the normal to the integral surface is given by the gradient function,

$$\nabla f = (u_x, u_y, -1).$$

Now consider the vector of coefficients, $\mathbf{v} = (a,b,c)$ and the dot product with the gradient above:

$$\mathbf{v} \cdot \nabla f = au_x + bu_y - c.$$

This is the left hand side of the partial differential equation. Therefore, for the solution surface we have

$$\mathbf{v} \cdot \nabla f = 0,$$

or $\mathbf{v}$ is perpendicular to ∇f. Since ∇f is normal to the surface, $\mathbf{v} = (a,b,c)$ is tangent to the surface. Geometrically, $\mathbf{v}$ defines a direction field, called the characteristic field. These are shown in Figure 1.3.1.

Characteristics

We seek the forms of the characteristic curves such as the one shown in Figure 1.3.1. Recall that one can parametrize space curves,

$$\mathbf{c}(t) = (x(t), y(t), u(t)), \quad t \in [t_1, t_2].$$

The tangent to the curve is then

$$\mathbf{v}(t) = \frac{d\mathbf{c}(t)}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{du}{dt} \right).$$

However, in the last section we saw that $\mathbf{v}(t) = (a,b,c)$ for the partial differential equation $a(x,y,u)u_x + b(x,y,u)u_y - c(x,y,u) = 0$. This gives the parametric form of the characteristic curves as

$$\frac{dx}{dt} = a, \quad \frac{dy}{dt} = b, \quad \frac{du}{dt} = c. \tag{1.3.2}$$

Another form of these equations is found by relating the differentials, dx, dy, du, to the coefficients in the differential equation. Since $x = x(t)$ and $y = y(t)$, we have

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{b}{a}.$$

Similarly, we can show that

$$\frac{du}{dx} = \frac{c}{a}, \quad \frac{du}{dy} = \frac{c}{b}.$$

All of these relations can be summarized in the form

$$dt = \frac{dx}{a} = \frac{dy}{b} = \frac{du}{c}. \tag{1.3.3}$$

How do we use these characteristics to solve quasilinear partial differential equations? Consider the next example.

Example 1.3.1

Find the general solution: $u_x + u_y - u = 0$.

Solution

We first identify $a = 1,\ b = 1,$ and $c = u$. The relations between the differentials is

$$\frac{dx}{1} = \frac{dy}{1} = \frac{du}{u}.$$

We can pair the differentials in three ways:

$$\frac{dy}{dx} = 1, \quad \frac{du}{dx} = u, \quad \frac{du}{dy} = u.$$

Only two of these relations are independent. We focus on the first pair.

The first equation gives the characteristic curves in the xy-plane. This equation is easily solved to give

$$y = x + c_1.$$

The second equation can be solved to give $u = c_2 e^x$.

The goal is to find the general solution to the differential equation. Since $u = u(x, y)$, the integration "constant" is not really a constant, but is constant with respect to x. It is in fact an arbitrary constant function. In fact, we could view it as a function of c_1, the constant of integration in the first equation. Thus, we let $c_2 = G(c_1)$ for G and arbitrary function. Since $c_1 = y - x$, we can write the general solution of the differential equation as

$$u(x, y) = G(y - x)e^x.$$

Example 1.3.2

Solve the advection equation, $u_t + cu_x = 0$, for c a constant, and $u = u(x, t)$, $|x| < \infty$, $t > 0$.

Solution

The characteristic equations are

$$d\tau = \frac{dt}{1} = \frac{dx}{c} = \frac{du}{0} \tag{1.3.4}$$

and the parametric equations are given by

$$\frac{dx}{d\tau} = c, \quad \frac{du}{d\tau} = 0. \tag{1.3.5}$$

These equations imply that

- $u = \text{const.} = c_1$.
- $x = ct + \text{const.} = ct + c_2$.

As before, we can write c_1 as an arbitrary function of c_2. However, before doing so, let's replace c_1 with the variable ξ and then we have that

$$\xi = x - ct, \quad u(x, t) = f(\xi) = f(x - ct)$$

where f is an arbitrary function. Furthermore, we see that $u(x, t) = f(x - ct)$ indicates that the solution is a wave moving in one direction in the shape of the initial function, $f(x)$. This is known as a traveling wave. A typical traveling wave is shown in Figure 1.3.2.

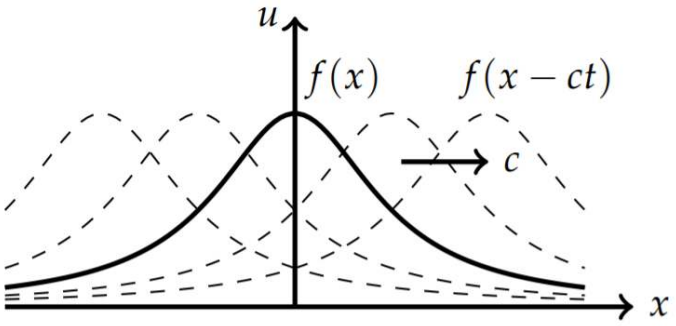


Figure 1.3.2: Depiction of a traveling wave. $u(x,t) = f(x)$ at $t = 0$ travels without changing shape.

Note that since $u = u(x,t)$, we have

$$\begin{aligned} 0 &= u_t + cu_x \\ &= \frac{\partial u}{\partial t} + \frac{dx}{dt}\frac{\partial u}{\partial x} \\ &= \frac{du(x(t),t}{dt}. \end{aligned} \tag{1.3.6}$$

This implies that $u(x,t) =$ constant along the characteristics, $\frac{dx}{dt} = c$.

As with ordinary differential equations, the general solution provides an infinite number of solutions of the differential equation. If we want to pick out a particular solution, we need to specify some side conditions. We investigate this by way of examples.

Example 1.3.3

Find solutions of $u_x + u_y - u = 0$ subject to $u(x,0) = 1$.

Solution

We found the general solution to the partial differential equation as $u(x,y) = G(y-x)e^x$. The side condition tells us that $u = 1$ along $y = 0$. This requires

$$1 = u(x,0) = G(-x)e^x.$$

Thus, $G(-x) = e^{-x}$. Replacing x with $-z$, we find

$$G(z) = e^z.$$

Thus, the side condition has allowed for the determination of the arbitrary function $G(y-x)$. Inserting this function, we have

$$u(x,y) = G(y-x)e^x = e^{y-x}e^x = e^y.$$

Side conditions could be placed on other curves. For the general line, $y = mx + d$, we have $u(x, mx+d) = g(x)$ and for $x = d$, $u(d,y) = g(y)$. As we will see, it is possible that a given side condition may not yield a solution. We will see that conditions have to be given on non-characteristic curves in order to be useful.

Example 1.3.4

Find solutions of $3u_x - 2u_y + u = x$ for

a. $u(x,x) = x$ and
b. $u(x,y) = 0$ on $3y + 2x = 1$.

Solution

Before applying the side condition, we find the general solution of the partial differential equation. Rewriting the differential equation in standard form, we have

$$3u_x - 2u_y = x = u.$$

The characteristic equations are

$$\frac{dx}{3} = \frac{dy}{-2} = \frac{du}{x-u}. \tag{1.3.7}$$

These equations imply that

- $-2dx = 3dy$
  This implies that the characteristic curves (lines) are $2x+3y = c_1$.
- $\frac{du}{dx} = \frac{1}{3}(x-u)$.
  This is a linear first order differential equation, $\frac{du}{dx} + \frac{1}{3}u = \frac{1}{3}x$. It can be solved using the integrating factor,

$$\mu(x) = \exp\left(\frac{1}{3}\int^x d\xi\right) = e^{x/3}.$$

$$\begin{aligned} \frac{d}{dx}\left(ue^{x/3}\right) &= \frac{1}{3}xe^{x/3} \\ ue^{x/3} &= \frac{1}{3}\int^x \xi e^{\xi/3}d\xi + c_2 \\ &= (x-3)e^{x/3} + c_2 \\ u(x,y) &= x - 3 + c_2 e^{-x/3}. \end{aligned} \tag{1.3.8}$$

As before, we write c_2 as an arbitrary function of $c_1 = 2x+3y$. This gives the general solution

$$u(x,y) = x - 3 + G(2x+3y)e^{-x/3}.$$

Note that this is the same answer that we had found in Example 1.1.1

Now we can look at any side conditions and use them to determine particular solutions by picking out specific G's.

a. $u(x,x) = x$
   This states that $u = x$ along the line $y = x$. Inserting this condition into the general solution, we have

$$x = x - 3 + G(5x)e^{-x/3},$$

   or

$$G(5x) = 3e^{x/3}.$$

   Letting $z = 5x$,

$$G(z) = 3e^{z/15}.$$

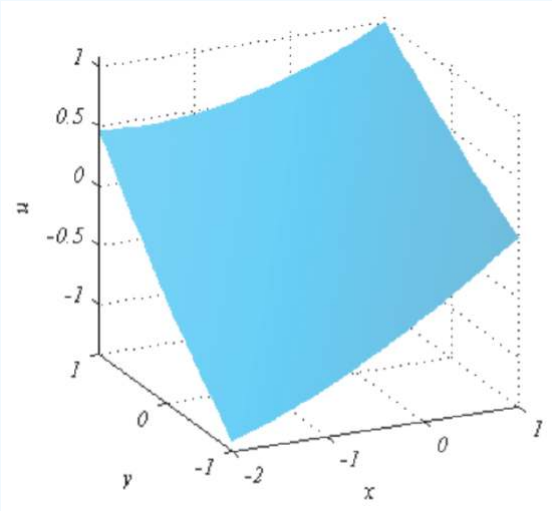


Figure 1.3.3: Integral surface found in Example 1.3.4

   The particular solution satisfying this side condition is

$$\begin{aligned} u(x,y) &= x-3+G(2x+3y)e^{-x/3} \\ &= x-3+3e^{(2x+3y)/15}e^{-x/3} \\ &= x-3+3e^{(y-x)/5}. \end{aligned} \tag{1.3.9}$$

This surface is shown in Figure 1.3.4.

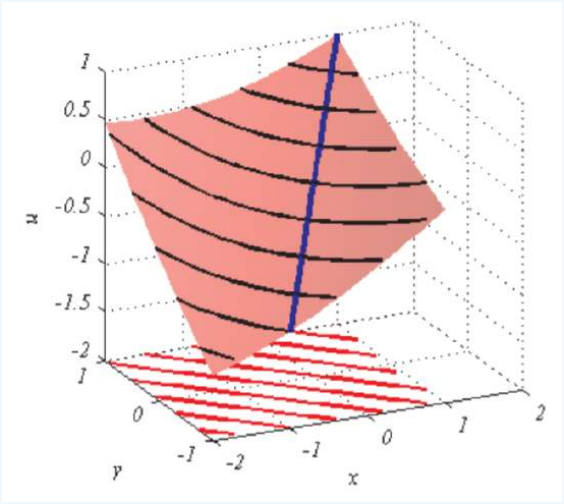


Figure 1.3.4: Integral surface with side condition and characteristics for Example 1.3.4.

In Figure 1.3.4 we superimpose the values of $u(x,y)$ along the characteristic curves. The characteristic curves are the red lines and the images of these curves are the black lines. The side condition is indicated with the blue curve drawn along the surface.

The values of $u(x,y)$ are found from the side condition as follows. For $x=\xi$ on the blue curve, we know that $y=\xi$ and $u(\xi,\xi)=\xi$. Now, the characteristic lines are given by $2x+3y=c_1$. The constant c_1 is found on the blue curve from the point of intersection with one of the black characteristic lines. For $x=y=\xi$, we have $c_1=5\xi$. Then, the equation of the characteristic line, which is red in Figure 1.3.4, is given by $y=\frac{1}{3}(5\xi-2x)$.

Along these lines we need to find $u(x,y)=x-3+c_2e^{-x/3}$. First we have to find c_2. We have on the blue curve, that

$$\begin{aligned} \xi &= u(\xi,\xi) \\ &= \xi-3+c_2e^{-\xi/3}. \end{aligned} \tag{1.3.10}$$

Therefore, $c_2=3e^{\xi/3}$. Inserting this result into the expression for the solution, we have

$$u(x,y)=x-3+e^{(\xi-x)/3}.$$

So, for each ξ, one can draw a family of spacecurves

$$\left(x,\ \frac{1}{3}(5\xi-2x),\ x-3+e^{(\xi-x)/3}\right)$$

yielding the integral surface.

b. $u(x,y)=0$ on $3y+2x=1$.

For this condition, we have

$$0=x-3+G(1)e^{-x/3}.$$

We note that G is not a function in this expression. We only have one value for G. So, we cannot solve for $G(x)$. Geometrically, this side condition corresponds to one of the black curves in Figure 1.3.4.

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1.4: Applications

Conservation Laws

There are many applications of quasilinear equations, especially in fluid dynamics. The **advection equation** is one such example and generalizations of this example to nonlinear equations leads to some interesting problems. These equations fall into a category of equations called conservation laws. We will first discuss one-dimensional (in space) conservations laws and then look at simple examples of nonlinear conservation laws.

Conservation laws are useful in modeling several systems. They can be boiled down to determining the rate of change of some stuff, $Q(t)$, in a region, $a \leq x \leq b$, as depicted in Figure 1.4.1. The simples model is to think of fluid flowing in one dimension, such as water flowing in a stream. Or, it could be the transport of mass, such as a pollutant. One could think of traffic flow down a straight road.

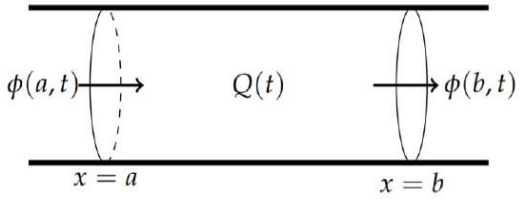


Figure 1.4.1: The rate of change of Q between $x = a$ and $x = b$ depends on the rates of flow through each end.

This is an example of a typical mixing problem. The rate of change of $Q(t)$ is given as

$$\text{the rate of change of } Q = \text{ Rate in } - \text{ Rate Out } + \text{ source term.}$$

Here the "Rate in" is how much is flowing into the region in Figure 1.4.1 from the $x = a$ boundary. Similarly, the "Rate out" is how much is flowing into the region from the $x = b$ boundary. [Of course, this could be the other way, but we can imagine for now that q is flowing from left to right.] We can describe this flow in terms of the flux, $\phi(x,t)$ over the ends of the region. On the left side we have a gain of $\phi(a,t)$ and on the right side of the region there is a loss of $\phi(b,t)$.

The source term would be some other means of adding or removing Q from the region. In terms of fluid flow, there could be a source of fluid inside the region such as a faucet adding more water. Or, there could be a drain letting water escape. We can denote this by the total source over the interval, $\int_a^b f(x,t)\,dx$. Here $f(x,t)$ is the source density.

In summary, the rate of change of $Q(x,t)$ can be written as

$$\frac{dQ}{dt} = \phi(a,t) - \phi(b,t) + \int_a^b f(x,y)dx.$$

We can write this in a slightly different form by noting that $\phi(a,t) - \phi(b,t)$ can be viewed as the evaluation of antiderivatives in the Fundamental Theorem of Calculus. Namely, we can recall that

$$\int_a^b \frac{\partial \phi(x,t)}{\partial x}dx = \phi(b,t) - \phi(a,t).$$

The difference is not exactly in the order that we desire, but it is easy to see that

$$\frac{dQ}{dt} = -\int_a^b \frac{\partial \phi(x,t)}{\partial x}dx + \int_a^b f(x,t)dx. \tag{1.4.1}$$

This is the integral form of the conservation law.

We can rewrite the conservation law in differential form. First, we introduce the density function, $u(x,t)$, so that the total amount of stuff at a given time is

$$Q(t) = \int_a^b u(x,t)dx.$$

Introducing this form into the integral conservation law, we have

$$\frac{d}{dt}\int_a^b u(x,t)dx = -\int_a^b \frac{\partial \phi}{\partial x}dx + \int_a^b f(x,t)dx. \tag{1.4.2}$$

Assuming that a and b are fixed in time and that the integrand is continuous, we can bring the time derivative inside the integrand and collect the three terms into one to find

$$\int_a^b (u_t(x,t)+\phi_x(x,t)-f(x,t))dx = 0, \quad \forall x \in [a,b].$$

We cannot simply set the integrant to zero just because the integral vanishes. However, if this result holds for every region $[a,b]$, then we can conclude the integrand vanishes. So, under that assumption, we have the local conservation law,

$$u_t(x,t)+\phi_x(x,t) = f(x,t). \tag{1.4.3}$$

This partial differential equation is actually an equation in terms of two unknown functions, assuming we know something about the source function. We would like to have a single unknown function. So, we need some additional information. This added information comes from the constitutive relation, a function relating the flux to the density function. Namely, we will assume that we can find the relationship $\phi = \phi(u)$. If so, then we can write

$$\frac{\partial \phi}{\partial x} = \frac{d\phi}{du}\frac{\partial u}{\partial x}$$

or $\phi_x = \phi'(u)u_x$.

Example 1.4.1: Inviscid Burgers' Equation

Find the equation satisfied by $u(x,t)$ for $\phi(u) = \frac{1}{2}u^2$ and $f(x,t) \equiv 0$.

Solution

For this flux function we have $\phi_x = \phi'(u)u_x = uu_x$. The resulting equation is then $u_t + uu_x = 0$. This is the inviscid Burgers' equation. We will later discuss Burgers' equation.

Example 1.4.2: Traffic Flow

This is a simple model of one-dimensional traffic flow. Let $u(x,t)$ be the density of cars. Assume that there is no source term. For example, there is no way for a car to disappear from the flow by turning off the road or falling into a sinkhole. Also, there is no source of additional cars.

Solution

Let $\phi(x,t)$ denote the number of cars per hour passing position x at time t. Note that the units are given by cars/mi times mi/hr. Thus, we can write the flux as $\phi = uv$, where v is the velocity of the carts at position x and time t.

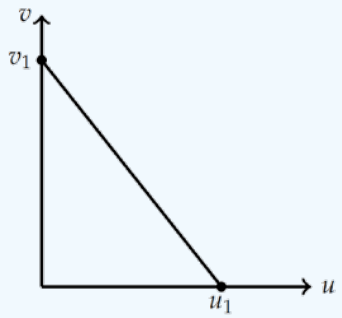


Figure 1.4.2: Car velocity as a function of car density.

We can now write the equation for the car density,

$$\begin{aligned} 0 &= u_t + \phi' u_x \\ &= u_t + v_1\left(1-\frac{2u}{u_1}\right)u_x. \end{aligned} \tag{1.4.4}$$

Nonlinear Advection Equations

In this section we consider equations of the form $u_t + c(u)u_x = 0$. When $c(u)$ is a constant function, we have the advection equation. In the last two examples we have seen cases in which $c(u)$ is not a constant function. We will apply the method of characteristics to these equations. First, we will recall how the method works for the advection equation.

The advection equation is given by $u_t + cu_x = 0$. The characteristic equations are given by

$$\frac{dx}{dt} = c, \quad \frac{du}{dt} = 0.$$

These are easily solved to give the result that

$$u(x,t) = \text{ constant along the lines } x = ct + x_0,$$

where x_0 is an arbitrary constant.

The characteristic lines are shown in Figure 1.4.3. We note that $u(x,t) = u(x_0,0) = f(x_0)$. So, if we know u initially, we can determine what u is at a later time.

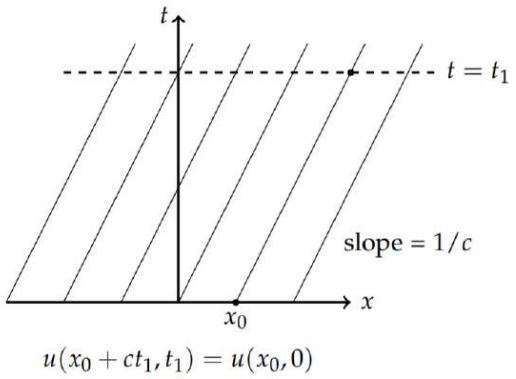


Figure 1.4.3: The characteristics lines the xt-plane.

In Figure 1.4.3 we see that the value of $u(x_0,)$ at $t = 0$ and $x = x_0$ propagates along the characteristic to a point at time $t = t_1$. From $x - ct = x_0$, we can solve for x in terms of t_1 and find that $u(x_0 + ct_1, t_1) = u(x_0, 0)$.

Plots of solutions $u(x,t)$ versus x for specific times give traveling waves as shown in Figure 1.3.1. In Figure 1.4.4 we show how each wave profile for different times are constructed for a given initial condition.

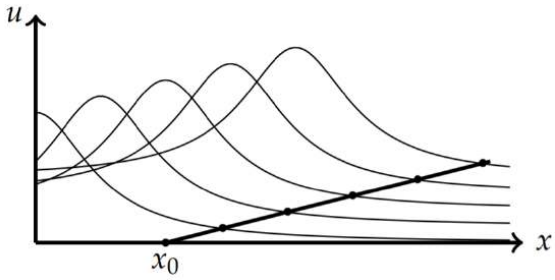


Figure 1.4.4: For each $x = x_0$ at $t = 0$, $u(x_0 + ct, t) = u(x_0, 0)$.

The nonlinear advection equation is given by $u_t + c(u)u_x = 0$, $|x| < \infty$. Let $u(x,0) = u_0(x)$ be the initial profile. The characteristic equations are given by

$$\frac{dx}{dt} = c(u), \quad \frac{du}{dt} = 0.$$

These are solved to give the result that

$$u(x,t) = \text{ constant},$$

along the characteristic curves $x'(t) = c(u)$. The lines passing though $u(x_0,) = u_0(x_0)$ have slope $1/c(u_0(x_0))$.

Example 1.4.3

Solve $u_t + uu_x = 0$, $u(x,0) = e^{-x^2}$.

Solution

For this problem $u =$ constant along

$$\frac{dx}{dt} = u.$$

Since u is constant, this equation can be integrated to yield $x = u(x_0, 0)t + x_0$. Inserting the initial condition, $x = e^{-x_0^2}t + x_0$. Therefore, the solution is

$$u(x,t) = e^{-x_0^2} \text{ along } x = e^{-x_0^2}t + x_0.$$

In Figure 1.4.5 the characteristics a shown. In this case we see that the characteristics intersect. In Figure 1.4.6 we look more specifically at the intersection of the characteristic lines for $x_0 = 0$ and $x_0 = 1$. These are approximately the first lines to intersect; i.e., there are (almost) no intersections at earlier times. At the intersection point the function $u(x,t)$ appears to take on more than one value. For the case shown, the solution wants to take the values $u = 0$ and $u = 1$.

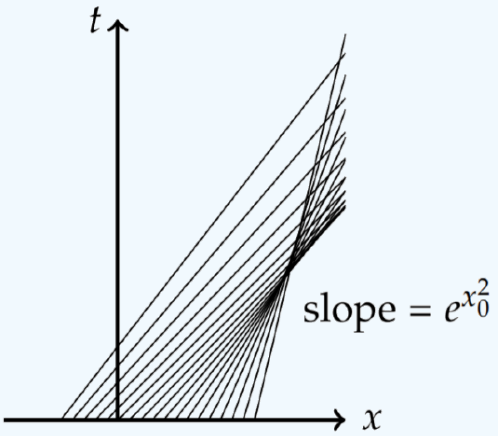


Figure 1.4.5: The characteristics lines the xt-plane for the nonlinear advection equation.

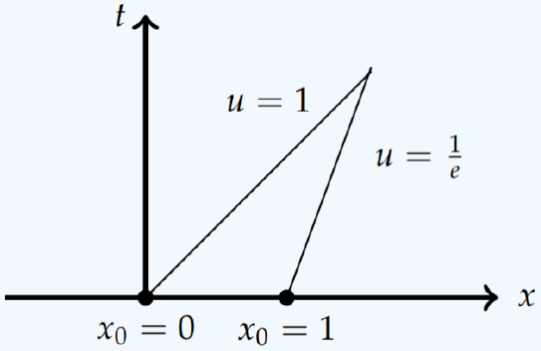


Figure 1.4.6: The characteristics lines for $x_0 = 0, 1$ in the xt-plane for the nonlinear advection equation.

In Figure 1.4.7 we see the development of the solution. This is found using a parametric plot of the points $(x_0 + te^{-x_0^2}, e^{-x_0^2})$ for different times. The initial profile propagates to the right with the higher points traveling faster than the lower points since $x'(t) = u > 0$. Around $t = 1.0$ the wave breaks and becomes multivalued. The time at which the function becomes multivalued is called the breaking time.

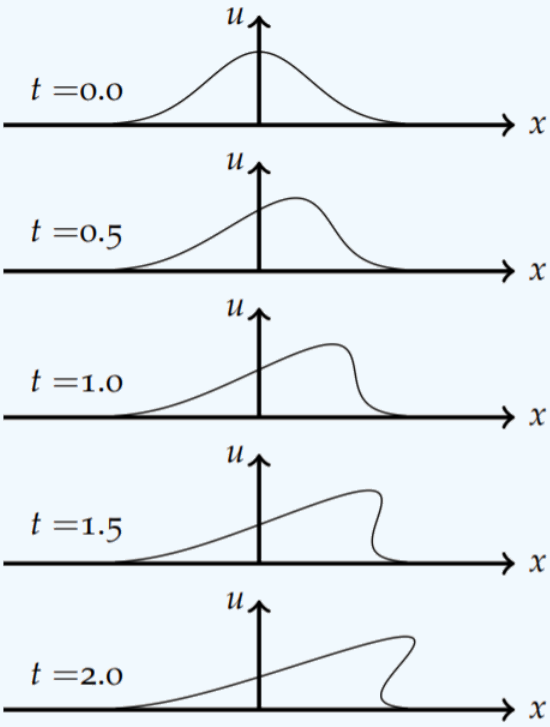


Figure 1.4.7: The development of a gradient catastrophe in Example 1.4.3 leading to a multivalued function.

Breaking Time

In the last example we saw that for nonlinear wave speeds a gradient catastrophe might occur. The first time at which a catastrophe occurs is called the breaking time. We will determine the breaking time for the nonlinear advection equation, $u_t + c(u)u_x = 0$. For the characteristic corresponding to $x_0 = \xi$, the wavespeed is given by

$$F(\xi) = c(u_0(\xi))$$

and the characteristic line is given by

$$x = \xi + tF(\xi).$$

The value of the wave function along this characteristic is

$$\begin{aligned} u(x,t) &= u(\xi + tF(\xi), t) \\ &= . \end{aligned} \tag{1.4.5}$$

Therefore, the solution is

$$u(x,t) = u_0(\xi) \text{ along } x = \xi + tF(\xi).$$

This means that

$$u_x = u_0'(\xi)\xi_x \quad \text{and} \quad u_t = u_0'(\xi)\xi_t.$$

We can determine ξ_x and ξ_t using the characteristic line

$$\xi = x - tF(\xi).$$

Then, we have

$$\begin{aligned}\xi_x &= 1 - tF'(\xi)\xi_x \\ &= \frac{1}{1+tF'(\xi)}. \\ \xi_t &= \frac{\partial}{\partial t}(x - tF(\xi)) \\ &= -F(\xi) - tF'(\xi)\xi_t \\ &= \frac{-F(\xi)}{1+tF'(\xi)}. \end{aligned} \tag{1.4.6}$$

Note that ξ_x and ξ_t are undefined if the denominator in both expressions vanishes, $1+tF'(\xi) = 0$, or at time

$$t = -\frac{1}{F'(\xi)}.$$

The minimum time for this to happen in the breaking time,

$$t_b = \min\left\{-\frac{1}{F'(\xi)}\right\}. \tag{1.4.7}$$

Example 1.4.4

Find the breaking time for $u_t + uu_x = 0$, $u(x,0) = e^{-x^2}$.

Solution

Since $c(u) = u$, we have

$$F(\xi) = c(u_0(\xi)) = e^{-\xi^2}$$

and

$$F'(\xi) = -2\xi e^{-\xi^2}.$$

This goes

$$t = \frac{1}{2\xi e^{-\xi^2}}.$$

We need to find the minimum time. Thus, we set the derivative equal to zero and solve for ξ.

$$\begin{aligned} 0 &= \frac{d}{d\xi}\left(\frac{e^{\xi^2}}{2\xi}\right) \\ &= \left(2 - \frac{1}{\xi^2}\right)\frac{e^{\xi^2}}{2}. \end{aligned} \tag{1.4.8}$$

Thus, the minimum occurs for $2 - \frac{1}{\xi^2} = 0$, or $\xi = 1/\sqrt{2}$. This gives

$$t_b = t\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\frac{2}{\sqrt{2}e^{-1/2}}} = \sqrt{\frac{e}{2}} \approx 1.16. \tag{1.4.9}$$

Shock Waves

Solutions of nonlinear advection equations can become multivalued due to a gradient catastrophe. Namely, the derivatives u_t and u_x become undefined. We would like to extend solutions past the catastrophe. However, this leads to the possibility of discontinuous solutions. Such solutions which may not be differentiable or continuous in the domain are known as weak solutions. In particular, consider the initial value problem

$$u_t + \phi_x = 0, \quad x \in R, \quad t > 0, \quad u(x,0) = u_0(x).$$

Then, $u(x,t)$ is a weak solution of this problem if

$$\int_0^\infty \int_{-\infty}^{\infty} [uv_t + \phi v_x] dx dt + \int_{-\infty}^{\infty} u_0(x) v(x,0) dx = 0$$

for all smooth functions $v \in C^\infty(R \times [0, \infty))$ with compact support, i.e., $v \equiv= 0$ outside some compact subset of the domain.

Effectively, the weak solution that evolves will be a piecewise smooth function with a discontinuity, the shock wave, that propagates with shock speed. It can be shown that the form of the shock will be the discontinuity shown in Figure 1.4.8 such that the areas cut from the solutions will cancel leaving the total area under the solution constant. [See G. B. Whitham's *Linear and Nonlinear Waves*, 1973.] We will consider the discontinuity as shown in Figure 1.4.9.

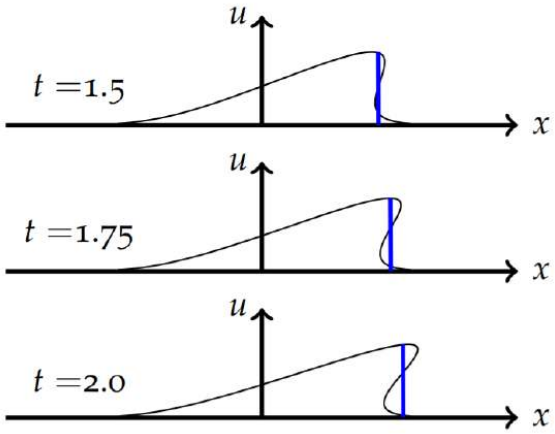


Figure 1.4.8: The shock solution after the breaking time.

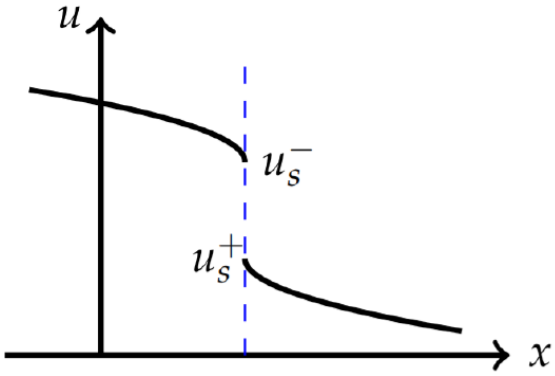


Figure 1.4.9: Depiction of the jump discontinuity at the shock position.

We can find the equation for the shock path by using the integral form of the conservation law,

$$\frac{d}{dt} \int_a^b u(x,t) dx = \phi(a,t) - \phi(b,t).$$

Recall that one can differentiate under the integral if $u(x,t)$ and $u_t(x,t)$ are continuous in x and t in an appropriate subset of the domain. In particular, we will integrate over the interval $[a,b]$ as shown in Figure 1.4.10. The domains on either side of shock path are denoted as R^+ and R^- and the limits of $x(t)$ and $u(x,t)$ as one approaches from the left of the shock are denoted by $x_s^-(t)$ and $u^- = u(x_s^-, t)$. Similarly, the limits of $x(t)$ and $u(x,t)$ as one approaches from the right of the shock are denoted by $x_s^+(t)$ and $u^+ = u(x_s^+, t)$.

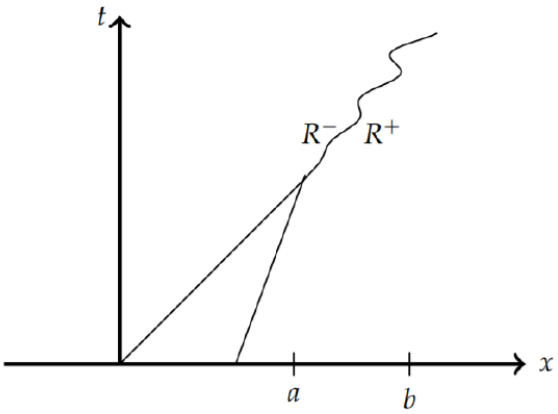


Figure 1.4.10: Domains on either side of shock path are denoted as R^+ and R^-.

We need to be careful in differentiating under the integral,

$$\begin{aligned}\frac{d}{dt}\int_a^b u(x,t)dx &= \frac{d}{dt}\left[\int_a^{x_s^-(t)} u(x,t)dx + \int_{x_s^+(t)}^b u(x,t)dx\right] \\ &= \int_a^{x_s^-(t)} u_t(x,t)dx + \int_{x_s^+(t)}^b u_t(x,t)dx \\ &\quad + u(x_s^-,t)\frac{dx_s^-}{dt} - u(x_s^+,t)\frac{dx_s^+}{dt} \\ &= \phi(a,t) - \phi(b,t). \end{aligned} \tag{1.4.10}$$

Taking the limits $a \to x_s^-$ and $b \to x_s^+$, we have that

$$(u(x_s^-,t) - u(x_s^+,t))\frac{dx_s}{dt} = \phi(x_s^-,t) - \phi(x_s^+,t).$$

Adopting the notation

$$[f] = f(x_s^+) - f(x_s^-),$$

we arrive at the Rankine-Hugonoit jump condition

$$\frac{dx_s}{dt} = \frac{[\phi]}{[u]}. \tag{1.4.11}$$

This gives the equation for the shock path as will be shown in the next example.

Example 1.4.5

Solution

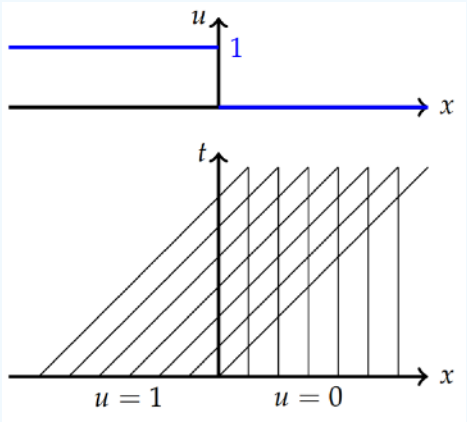


Figure 1.4.11: Initial condition and characteristics for Example 1.4.5.

The characteristics for this partial differential equation are familiar by now. The initial condition and characteristics are shown in Figure 1.4.11. From $x'(t) = u$, there are two possibilities. If $u = 0$, then we have a constant. If $u = 1$ along the characteristics, the we have straight lines of slope one. Therefore, the characteristics are given by

$$x(t) = \begin{cases} x_0, & x > 0, \\ t + x_0, & x < 0. \end{cases}$$

As seen in Figure 1.4.11 the characteristics intersect immediately at $t = 0$. The shock path is found from the Rankine-Hugonoit jump condition. We first note that $\phi(u) = \frac{1}{2}u^2$, since $\phi_x = uu_x$. Then, we have

$$\begin{aligned} \frac{dx_s}{dt} &= \frac{[\phi]}{[u]} \\ &= \frac{\frac{1}{2}u^{+^2} - \frac{1}{2}u^{-^2}}{u^+ - u^-} \\ &= \frac{1}{2}\frac{(u^+ + u^-)(u^+ - u^-)}{u^+ - u^-} \\ &= \frac{1}{2}(u^+ + u^-) \\ &= \frac{1}{2}(0+1) = \frac{1}{2}. \end{aligned} \tag{1.4.12}$$

Now we need only solve the ordinary differential equation $x_s'(t) = \frac{1}{2}$ with initial condition $x_s(0) = 0$. This gives $x_s(t) = \frac{t}{2}$. This line separates the characteristics on the left and right side of the shock solution. The solution is given by

$$u(x,t) = \begin{cases} 1, & x \le t/2, \\ 0, & x > t/2. \end{cases}$$

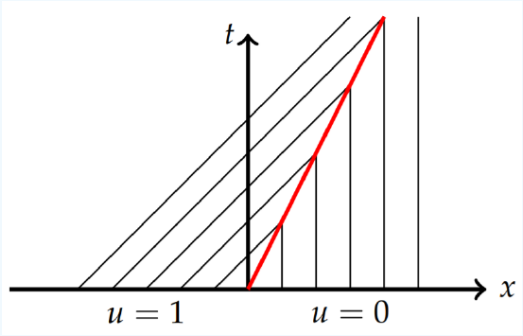


Figure 1.4.12: The characteristic lines end at the shock path (in red). On the left $u = 1$ and on the right $u = 0$.

In Figure 1.4.2 we show the characteristic lines ending at the shock path (in red) with $u = 0$ and on the right and $u = 1$ on the left of the shock path. This is consistent with the solution. One just sees the initial step function moving to the right with speed $1/2$ without changing shape

Rarefaction Waves

Shocks are not the only type of solutions encountered when the velocity is a function of u. There may sometimes be regions where the characteristic lines do not appear. A simple example is the following.

Example 1.4.6

Draw the characteristics for the problem $u_t + uu_x = 0$, $|x| < \infty$, $t > 0$ satisfying the initial condition

$$u(x,0) = \begin{cases} 0, & x \le 0, \\ 1, & x > 0. \end{cases}$$

Solution

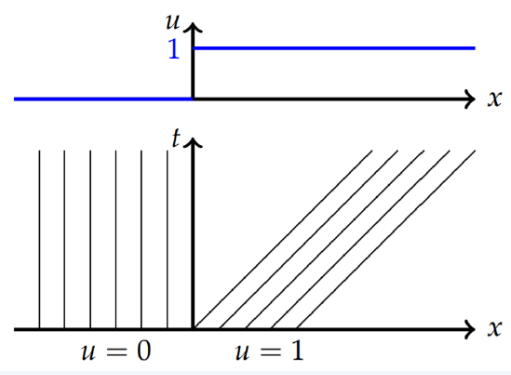


Figure 1.4.13: Initial condition and characteristics for Example 1.4.9.

In this case the solution is zero for negative values of x and positive for positive values of x as shown in Figure 1.4.13. Since the wavespeed is given by u, the $u = 1$ initial values have the waves on the right moving to the right and the values on the left stay fixed. This leads to the characteristics in Figure 1.4.13 showing a region in the xt-plane that has no characteristics. In this section we will discover how to fill in the missing characteristics and, thus, the details about the solution between the $u = 0$ and $u = 1$ values.

As motivation, we consider a smoothed out version of this problem.

Example 1.4.7

Draw the characteristics for the initial condition

$$u(x,0) = \begin{cases} 0, & x \le -\epsilon, \\ \frac{x+\epsilon}{2\epsilon}, & |x| \le \epsilon, \\ 1, & x > \epsilon. \end{cases}$$

Solution

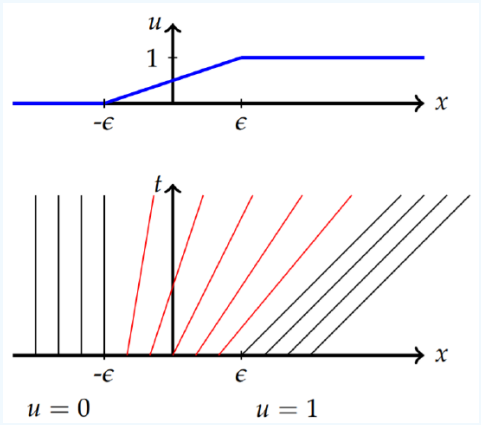


Figure 1.4.14: The function and characteristics for the smoothed step function. Characteristics for rarefaction, or expansion, waves are fan-like characteristics.

The function is shown in the top graph in Figure 1.4.14. The leftmost and rightmost characteristics are the same as the previous example. The only new part is determining the equations of the characteristics for $|x| \le \epsilon$. These are found using the method of characteristics as

$$x = \xi + u_0(\xi)t, \quad u_0(\xi) = \frac{\xi + \epsilon}{2\epsilon}t.$$

These characteristics are drawn in Figure 1.4.14 in red. Note that these lines take on slopes varying from infinite slope to slope one, corresponding to speeds going from zero to one.

Comparing the last two examples, we see that as ϵ approaches zero, the last example converges to the previous example. The characteristics in the region where there were none become a "fan". We can see this as follows. Since $|\xi| < \epsilon$ for the fan region, as ϵ gets small, so does this interval. Let's scale ξ as $\xi = \sigma\epsilon$, $\sigma \in [-1, 1]$. Then,

$$x = \sigma\epsilon + u_0(\sigma\epsilon)t, \quad u_0(\sigma\epsilon) = \frac{\sigma\epsilon + \epsilon}{2\epsilon}t = \frac{1}{2}(\sigma + 1)t.$$

For each $\sigma \in [-1, 1]$ there is a characteristic. Letting $\epsilon \to 0$, we have

$$x = ct, \quad c = \frac{1}{2}(\sigma + 1)t.$$

Thus, we have a family of straight characteristic lines in the xt-plane passing through $(0, 0)$ of the form $x = ct$ for c varying from $c = 0$ to $c = 1$. These are shown as the red lines in Figure 1.4.15.

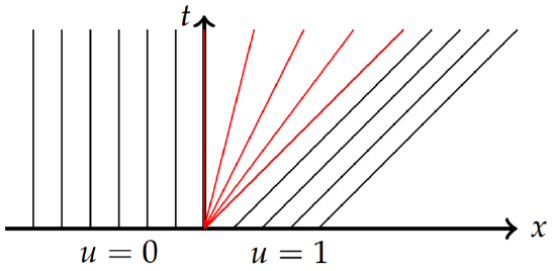


Figure 1.4.15: The characteristics for Example 1.4.9 showing the "fan" characteristics.

The fan characteristics can be written as $x/t =$ constant. So, we can seek to determine these characteristics analytically and in a straight forward manner by seeking solutions of the form $u(x, t) = g\left(\frac{x}{t}\right)$.

Example 1.4.8

Determine solutions of the form $u(x, t) = g\left(\frac{x}{t}\right)$ to $u_t + uu_x = 0$. Inserting this guess into the differential equation, we have

$$\begin{aligned} 0 &= u_t + uu_x \\ &= \frac{1}{t}g'\left(g - \frac{x}{t}\right). \end{aligned} \tag{1.4.13}$$

Solution

Thus, either $g' = 0$ or $g = \frac{x}{t}$. The first case will not work since this gives constant solutions. The second solution is exactly what we had obtained before. Recall that solutions along characteristics give $u(x, t) = \frac{x}{t} =$ constant. The characteristics and solutions for $t = 0, 1, 2$ are shown in Figure 1.4.16. At a specific time one can draw a line (dashed lines in figure) and follow the characteristics back to the $t = 0$ values, $u(\xi, 0)$ in order to construct $u(x, t)$.

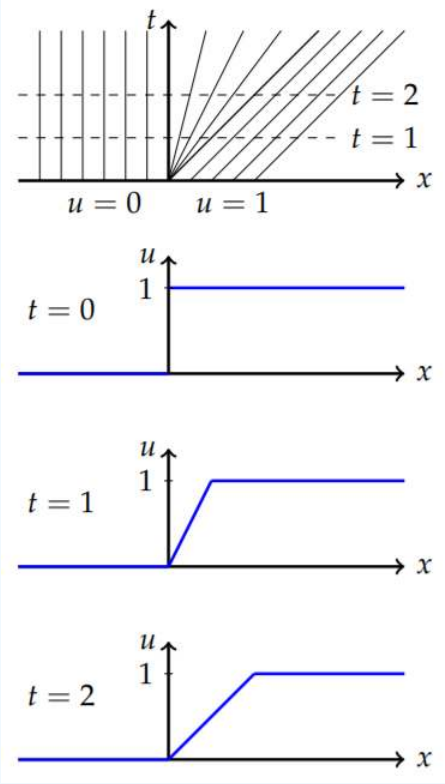


Figure 1.4.16: The characteristics and solutions for $t = 0, 1, 2$ for Example 1.4.9

As a last example, let's investigate a nonlinear model which possesses both shock and rarefaction waves.

Example 1.4.9

Solve the initial value problem $u_t + u^2 u_x = 0$, $|x| < \infty, t > 0$ satisfying the initial condition

$$u(x,0) = \begin{cases} 0, & x \le 0, \\ 1, & 0 < x < 2, \\ 0, & x \ge 2. \end{cases}$$

Solution

The method of characteristics gives

$$\frac{dx}{dt} = u^2, \quad \frac{du}{dt} = 0.$$

Therefore,

$$u(x,t) = u_0(\xi) = \text{ const. along the lines } x(t) = u_0^2(\xi)t + \xi.$$

There are three values of $u_0(\xi)$,

$$u_0(\xi) = \begin{cases} 0, & \xi \le 0, \\ 1, & 0 < \xi < 2, \\ 0, & \xi \ge 2. \end{cases}$$

In Figure 1.4.17we see that there is a rarefaction and a gradient catastrophe.

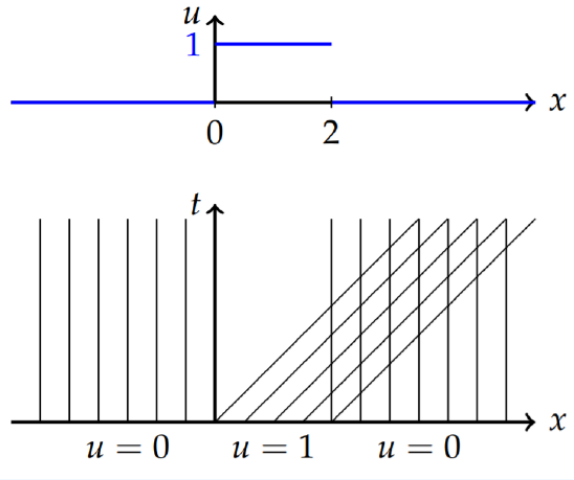


Figure 1.4.17: In this example there occurs a rarefaction and a gradient catastrophe.

Therefore, along the fan characteristics the solutions are $u(x,t) = \sqrt{\frac{x}{t}} = \text{constant}$. These fan characteristics are added in Figure 1.4.18

Next, we turn to the shock path. We see that the first intersection occurs at the point $(x,t) = (2,0)$. The Rankine-Hugonoit condition gives

$$\begin{aligned} \frac{dx_s}{dt} &= \frac{[\phi]}{[u]} \\ &= \frac{\frac{1}{2}u^{+^3} - \frac{1}{3}u^{-^3}}{u^+ - u^-} \\ &= \frac{1}{3}\frac{(u^+ - u^-)(u^{+^2} + u^+u^- + u^{-^2})}{u^+ - u^-} \\ &= \frac{1}{3}(u^{+^2} + u^+u^- + u^{-^2} \\ &= \frac{1}{3}(0+0+1) = \frac{1}{3}. \end{aligned} \tag{1.4.14}$$

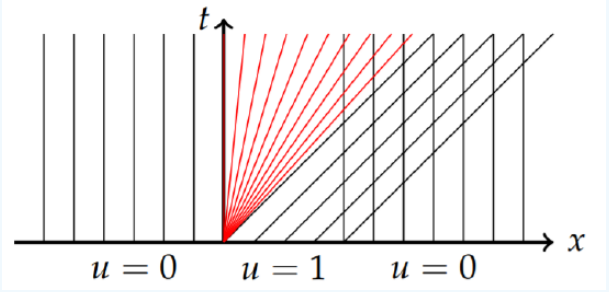


Figure 1.4.18: The fan characteristics are added to the other characteristic lines.

Thus, the shock path is given by $x_s'(t) = \frac{1}{3}$ with initial condition $x_s(0) = 2$. This gives $x_s(t) = \frac{t}{3} + 2$. In Figure 1.4.19 the shock path is shown in red with the fan characteristics and vertical lines meeting the path. Note that the fan lines and vertical lines cross the shock path. This leads to a change in the shock path.

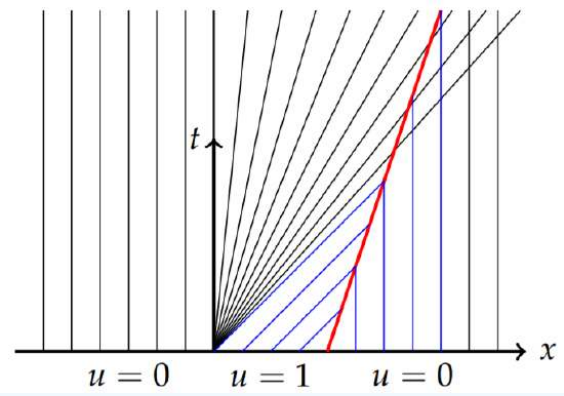


Figure 1.4.19: The shock path is shown in red with the fan characteristics and vertical lines meeting the path.

The new path is found using the Rankine-Hugonoit condition with $u^+ = 0$ and $u^- = \sqrt{\frac{x}{t}}$. Thus,

$$\begin{aligned}
\frac{dx_s}{dt} &= \frac{[\phi]}{[u]} \\
&= \frac{\frac{1}{3}u^{+^3} - \frac{1}{3}u^{-^3}}{u^+ - u^-} \\
&= \frac{1}{3}\frac{(u^+ - u^-)(u^{+^2} + u^+u^- + u^{-^2})}{u^+ - u^-} \\
&= \frac{1}{3}(u^{+^2} + u^+u^- + u^{-^2}) \\
&= \frac{1}{3}(0 + 0 + \sqrt{\frac{x_s}{t}}) = \frac{1}{3}\sqrt{\frac{x_s}{t}}.
\end{aligned} \tag{1.4.15}$$

We need to solve the initial value problem

$$\frac{dx_s}{dt} = \frac{1}{3}\sqrt{\frac{x_s}{t}}, \quad x_s(3) = 3.$$

This can be done using separation of variables. Namely,

$$\int \frac{dx_s}{\sqrt{x_s}} = \frac{1}{3}\frac{t}{\sqrt{t}}.$$

This gives the solution

$$\sqrt{x_s} = \frac{1}{3}\sqrt{t} + c.$$

Since the second shock solution starts at the point $(3, 3)$, we can determine $c = \frac{2}{3}\sqrt{3}$. This gives the shock path as

$$x_s(t) = \left(\frac{1}{3}\sqrt{t} + \frac{2}{3}\sqrt{3}\right)^2.$$

In Figure 1.4.20we show this shock path and the other characteristics ending on the path.

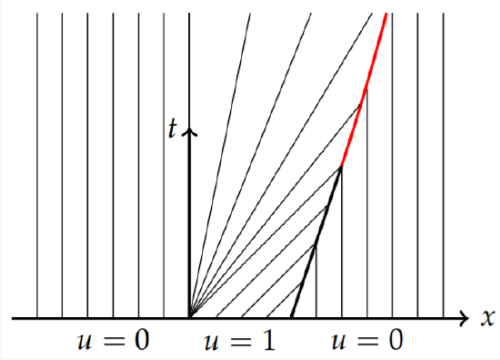


Figure 1.4.20: The second shock path is shown in red with the characteristics shown in all regions.

It is interesting to construct the solution at different times based on the characteristics. For a given time, t, one draws a horizontal line in the xt-plane and reads off the values of $u(x,t)$ using the values at $t=0$ and the rarefaction solutions. This is shown in Figure 1.4.21. The right discontinuity in the initial profile continues as a shock front until $t=3$. At that time the back rarefaction wave has caught up to the shock. After $t=3$, the shock propagates forward slightly slower and the height of the shock begins to decrease. Due to the fact that the partial differential equation is a conservation law, the area under the shock remains constant as it stretches and decays in amplitude.

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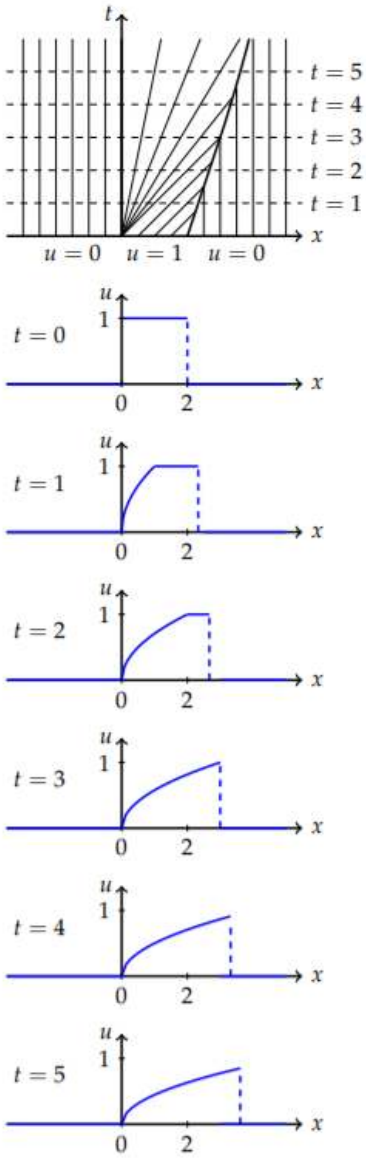


Figure 1.4.21: Solutions for the shock rarefaction example.

Traffic Flow

An interesting application is that of traffic flow. We had already derived the flux function. Let's investigate examples with varying initial conditions that lead to shock or rarefaction waves. As we had seen earlier in modeling traffic flow, we can consider the flux function

$$\phi = uv = v_1 \left(u - \frac{u^2}{u_1} \right),$$

which leads to the conservation law

$$u_t + v_1(1 - \frac{2u}{u_1})u_x = 0.$$

Here $u(x,t)$ represents the density of the traffic and u_1 is the maximum density and v_1 is the initial velocity.

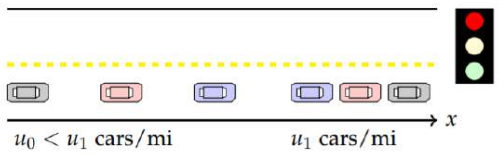


Figure 1.4.22: Cars approaching a red light.

The characteristics for this problem are given by

$$x = c(u(x_0, t))t + x_0,$$

where

$$x(u(x_0, t)) = v_1(1 - \frac{2u(x_0, 0)}{u_1}).$$

Since the initial condition is a piecewise-defined function, we need to consider two cases.

First, for $x \geq 0$, we have

$$c(u(x_0, t)) = c(u_1) = v_1(1 - \frac{2u_1}{u_1}) = -v_1.$$

Therefore, the slopes of the characteristics, $x = -v_1 t + x_0$ are $-1/v_1$.

For $x_0 < 0$, we have

$$c(u(x_0, t)) = c(u_0) = v_1(1 - \frac{2u_0}{u_1}).$$

So, the characteristics are $x = -v_1(1 - \frac{2u_0}{u_1})t + x_0$.

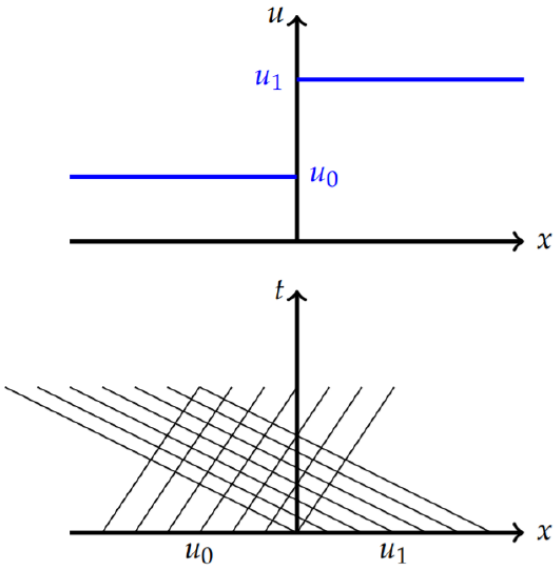


Figure 1.4.23: Initial condition and characteristics for the red light problem.

In Figure 1.4.23 we plot the initial condition and the characteristics for $x < 0$ and $x > 0$. We see that there are crossing characteristics and the begin crossing at $t = 0$. Therefore, the breaking time is $t_b = 0$. We need to find the shock path satisfying $x_s(0) = 0$. The Rankine-Hugonoit conditions give

$$\begin{aligned}\frac{dx_s}{dt} &= \frac{[\phi]}{[u]} \\ &= \frac{\frac{1}{2}u^{+^2} - \frac{1}{2}u^{-^2}}{u^+ - u^-} \\ &= \frac{1}{2}\frac{0 - v_1\frac{u_0^2}{u_1}}{u_1 - u_0} \\ &= -v_1\frac{u_0}{u_1}. \end{aligned} \tag{1.4.16}$$

Thus, the shock path is found as $x_s(t) = -v_1\frac{u_0}{u_1}$.

In Figure 1.4.24 we show the shock path. In the top figure the red line shows the path. In the lower figure the characteristics are stopped on the shock path to give the complete picture of the characteristics. The picture was drawn with $v_1 = 2$ and $u_0/u_1 = 1/3$.

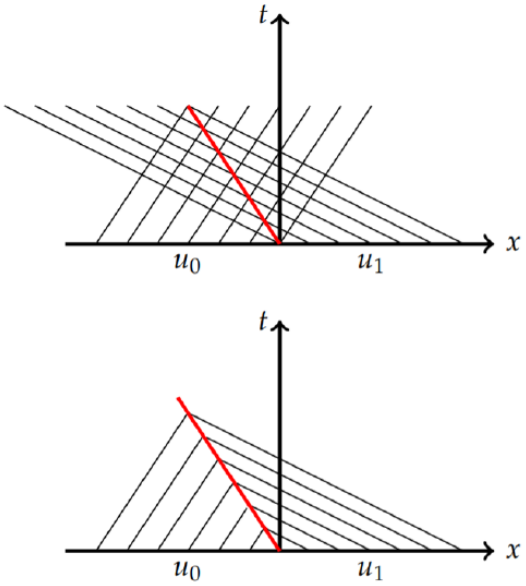


Figure 1.4.24: The addition of the shock path for the red light problem.

The next problem to consider is stopped traffic as the light turns green. The cars in Figure 1.4.25 begin to fan out when the traffic light turns green. In this model the initial condition is given by

$$u(x,0) = \begin{cases} u_1, & x \le 0, \\ 0, & x > 0. \end{cases}$$

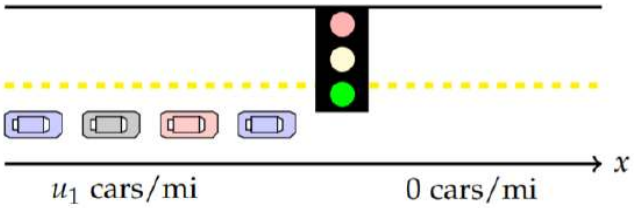


Figure 1.4.25: Cars begin to fan out when the traffic light turns green.

Again,

$$c(u(x_0,t)) = v_1\left(1 - \frac{2u(x_0,0)}{u_1}\right).$$

Inserting the initial values of u into this expression, we obtain constant speeds, $\pm v_1$. The resulting characteristics are given by

$$x(t) = \begin{cases} -v_1 t + x_0, & x \le 0, \\ v_1 t + x_0, & x > 0. \end{cases}$$

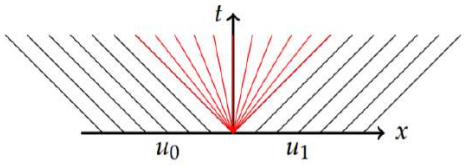


Figure 1.4.26: The characteristics for the green light problem.

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1.5: General First Order PDEs

We have spent time solving quasilinear first order partial differential equations. We now turn to nonlinear first order equations of the form

$$F(x,y,u,u_x,u_y)=0,$$

for $u=u(x,y)$.

If we introduce new variables, $p=u_x$ and $q=u_y$, then the differential equation takes the form

$$F(x,y,u,p,q)=0.$$

Note that for $u(x,t)$ a function with continuous derivatives, we have

$$p_y=u_{xy}=u_{yx}=q_x.$$

We can view $F=0$ as a surface in a five dimensional space. Since the arguments are functions of x and y, we have from the multivariable Chain Rule that

$$\begin{aligned}\frac{dF}{dx} &= F_x+F_u\frac{\partial u}{\partial x}+F_p\frac{\partial p}{\partial x}+F_q\frac{\partial q}{\partial x}\\ 0 &= F_x+pF_u+p_xF_p+p_yF_q.\end{aligned} \tag{1.5.1}$$

Similarly, from $\frac{dF}{dy}=0$ we have that

$$\frac{dx}{F_p}=\frac{dy}{F_q}=-\frac{dq}{F_y+qF_u}.$$

Combining these results we have the Charpit Equations

$$\frac{dx}{F_p}=\frac{dy}{F_q}=\frac{du}{pF_p+qF_q}=-\frac{dp}{F_x+pF_u}=-\frac{dq}{F_y+qF_u}. \tag{1.5.2}$$

These equations can be used to find solutions of nonlinear first order partial differential equations as seen in the following examples.

The Charpit equations

The Charpit equations were named after the French mathematician Paul Charpit Villecourt, who was probably the first to present the method in his thesis the year of his death, 1784. His work was further extended in 1797 by Lagrange and given a geometric explanation by Gaspard Monge (1746-1818) in 1808. This method is often called the Lagrange-Charpit method.

Example 1.5.1

Find the general solution of $u_x^2+yu_y-u=0$.

Solution

First, we introduce $u_x=p$ and $u_y=q$. Then,

$$F(x,y,u,p,q)=p^2+qy-u=0.$$

Next we identify,

$$F_p=2p,\quad F_q=y,\quad F_u=-1,\quad F_x=0,\quad F_y=q.$$

Then,

$$\begin{aligned}pF_p+qF_q &= 2p^2+qy,\\ F_x+pF_u &= -p,\\ F_y+qF_u &= q-q=0.\end{aligned}$$

The Charpit equations are then

$$\frac{dx}{2p} = \frac{dy}{y} = \frac{du}{2p^2+qy} = \frac{dp}{p} = \frac{dq}{0}.$$

The first conclusion is that $q = c_1 =$ constant. So, from the partial differential equation we have $u = p^2 + c_1 y$.

Since $du = pdx + qdy = pdx + c_1 dy$, then

$$du - cdy = \sqrt{u - c_1 y}dx.$$

Therefore,

$$\begin{aligned} \int \frac{d(u-c_1 y)}{\sqrt{u-c_1 y}} &= \int dx \\ \int \frac{z}{\sqrt{z}} &= x + c_2 \\ 2\sqrt{u-c_1 y} &= x + c_2. \end{aligned} \tag{1.5.3}$$

Solving for u, we have

$$u(x,y) = \frac{1}{4}(x+c_2)^2 + c_1 y.$$

This example required a few tricks to implement the solution. Sometimes one needs to find parametric solutions. Also, if an initial condition is given, one needs to find the particular solution. In the next example we show how parametric solutions are found to the initial value problem.

Example 1.5.2

Solve the initial value problem $u_x^2 + u_y + u = 0$, $u(x,0) = x$.

Solution

We consider the parametric form of the Charpit equations,

$$dt = \frac{dx}{F_p} = \frac{dy}{F_q} = \frac{du}{pF_p + qF_q} = -\frac{dp}{F_x + pF_u} = -\frac{dq}{F_y + qF_u}. \tag{1.5.4}$$

This leads to the system of equations

$$\begin{aligned} \frac{dx}{dt} &= F_p = 2p. \\ \frac{dy}{dt} &= F_q = 1. \\ \frac{du}{dt} &= pF_p + qF_q = 2p^2 + q. \\ \frac{dp}{dt} &= -(F_x + pF_u) = -p. \\ \frac{dq}{dt} &= -(F_y + qF_u) = -q. \end{aligned}$$

The second, fourth, and fifth equations can be solved to obtain

$$\begin{aligned} y &= t + c_1. \\ p &= c_2 e^{-t}. \\ q &= c_3 e^{-t}. \end{aligned}$$

Inserting these results into the remaining equations, we have

$$\frac{dx}{dt} = 2c_2 e^{-t}.$$

$$\frac{du}{dt} = 2c_2^2 e^{-2t} + c_3 e^{-t}.$$

These equations can be integrated to find Inserting these results into the remaining equations, we have

$$\begin{aligned} x &= -2c_2 e^{-t} + c_4. \\ u &= -c_2^2 e^{-2t} - c_3 e^{-t} + c_5. \end{aligned}$$

This is a parametric set of equations for $u(x,t)$. Since

$$e^{-t} = \frac{x - c_4}{-2c_2},$$

we have

$$\begin{aligned} u(x,y) &= -c_2^2 e^{-2t} - c_3 e^{-t} + c_5. \\ &= -c_2^2 \left(\frac{x-c_4}{-2c_2} \right)^2 - c_3 \left(\frac{x-c_4}{-2c_2} \right) + c_5 \\ &= \frac{1}{4}(x-c_4)^2 + \frac{c_3}{2c_2}(x-c_4). \end{aligned} \tag{1.5.5}$$

We can use the initial conditions by first parametrizing the conditions. Let $x(s,0) = s$ and $y(s,0) = 0$, Then, $u(s,0) = s$. Since $u(x,0) = x$, $u_x(x,0) = 1$, or $p(s,0) = 1$.

From the partial differential equation, we have $p^2 + q + u = 0$. Therefore,

$$q(s,0) = -p^2(s,0) - u(s,0) = -(1+s).$$

These relations imply that

$$\begin{aligned} y(x,t)|_{t-0} = 0 \Rightarrow c_1 &= 0. \\ p(s,t)|_{t-0} = 1 \Rightarrow c_2 &= 1. \\ q(s,t)|_{t-0} = -(1+s) &= c_3. \end{aligned}$$

So,

$$\begin{aligned} y(s,t) &= t. \\ p(s,t) &= e^{-t}. \\ q(s,t) &= -(1+s)e^{-t}. \end{aligned}$$

The conditions on x and u give

$$\begin{aligned} x(s,t) &= (s+2) - 2e^{-t}, \\ u(s,t) &= (s+1)e^{-t} - e^{-2t}. \end{aligned}$$

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1.6: Modern Nonlinear PDEs

The study of nonlinear partial differential equations is a hot research topic. We will (eventually) describe some examples of important evolution equations and discuss their solutions in the last chapter.

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1.8: Problems

Exercise 1.8.1

Write the following equations in conservation law form, $ut+\varphi_x=0$ by finding the flux function $\varphi(u)$.

a. $u_t+cu_x=0$.
b. $u_t+uu_x-\mu u_{xx}=0$.
c. $u_t+6uu_x+u_{xxx}=0$.
d. $u_t+u^2u_x+u_{xxx}=0$.

Exercise 1.8.2

Consider the Klein-Gordon equation, $u_{tt}-au_{xx}=bu$ for a and b constants. Find traveling wave solutions $u(x,t)=f(x-ct)$.

Exercise 1.8.3

Find the general solution $u(x,y)$ to the following problems.

a. $u_x=0$.
b. $yu_x-xu_y=0$.
c. $2u_x+3u_y=1$.
d. $u_x+u_y=u$.

Exercise 1.8.4

Solve the following problems.

a. $u_x+2u_y=0,\ u(x,0)=\sin x$.
b. $u_t+4u_x=0,\ u(x,0)=\frac{1}{1+x^2}$.
c. $yu_x-xu_y=0,\ u(x,0)=x$.
d. $u_t+xtu_x=0,\ u(x,0)=\sin x$.
e. $yu_x+xu_y=0,\ u(0,y)=e^{-y^2}$.
f. $xu_t-2xtu_x=2tu,\ u(x,0)=x^2$.
g. $(y-u)u_x+(u-x)u_y=x-y,\ u=0$ on $xy=1$.
h. $yu_x+xu_y=xy,\ x,y>0,$ for $u(x,0)=e^{-x^2},\ x>0$ and $u(0,y)=e^{-y^2},\ y>0$.

Exercise 1.8.5

Consider the problem $u_t+uu_x=0,\ |x|<\infty,\ t>0$ satisfying the initial condition $u(x,0)=\frac{1}{1+x^2}$.

a. Find and plot the characteristics.
b. Graphically locate where a gradient catastrophe might occur. Estimate from your plot the breaking time.
c. Analytically determine the breaking time.
d. Plot solutions $u(x,t)$ at times before and after the breaking time.

Exercise 1.8.6

Consider the problem $u_t+u^2u_x=0,\ |x|<\infty,\ t>0$ satisfying the initial condition $u(x,0)=\frac{1}{1+x^2}$.

a. Find and plot the characteristics.
b. Graphically locate where a gradient catastrophe might occur. Estimate from your plot the breaking time.
c. Analytically determine the breaking time.

d. Plot solutions $u(x,t)$ at times before and after the breaking time.

Exercise 1.8.7

a. Find and plot the characteristics.
b. Graphically locate where a gradient catastrophe might occur. Estimate from your plot the breaking time.
c. Analytically determine the breaking time.
d. Find the shock wave solution.

Exercise 1.8.8

Consider the problem $u_t + uu_x = 0,\ |x| < \infty,\ t > 0$ satisfying the initial condition

$$u(x,0) = \begin{cases} 1, & x \le 0, \\ 2, & x > 0. \end{cases}$$

a. Find and plot the characteristics.
b. Graphically locate where a gradient catastrophe might occur. Estimate from your plot the breaking time.
c. Analytically determine the breaking time.
d. Find the shock wave solution.

Exercise 1.8.9

Consider the problem $u_t + uu_x = 0,\ |x| < \infty,\ t > 0$ satisfying the initial condition

$$u(x,0) = \begin{cases} 0, & x \le -1, \\ 2, & |x| < 1, \\ 1, & x > 1. \end{cases}$$

a. Find and plot the characteristics.
b. Graphically locate where a gradient catastrophe might occur. Estimate from your plot the breaking time.
c. Analytically determine the breaking time.
d. Find the shock wave solution.

Exercise 1.8.10

Solve the problem $u_t + uu_x = 0,\ |x| < \infty,\ t > 0$ satisfying the initial condition

$$u(x,0) = \begin{cases} 1, & x \le 0, \\ 1 - \frac{x}{a}, & 0 < x < a, \\ 0, & x \ge a. \end{cases}$$

Exercise 1.8.11

Solve the problem $u_t + uu_x = 0,\ |x| < \infty,\ t > 0$ satisfying the initial condition

$$u(x,0) = \begin{cases} 0, & x \le 0, \\ \frac{x}{a}, & 0 < x < a, \\ 1, & x \ge a. \end{cases}$$

Exercise 1.8.12

Consider the problem $u_t + u^2 u_x = 0,\ |x| < \infty,\ t > 0$ satisfying the initial condition

$$u(x,0)=\begin{cases}2, & x\le 0,\\ 1, & x>0.\end{cases}$$

a. Find and plot the characteristics.
b. Graphically locate where a gradient catastrophe might occur. Estimate from your plot the breaking time.
c. Analytically determine the breaking time.
d. Find the shock wave solution.

? Exercise 1.8.13

Consider the problem $u_t+u^2u_x=0,\ |x|<\infty,\ t>0$ satisfying the initial condition

$$u(x,0)=\begin{cases}1, & x\le 0,\\ 2, & x>0.\end{cases}$$

a. Find and plot the characteristics.
b. Find and plot the fan characteristics.
c. Write out the rarefaction wave solution for all regions of the xt-plane.

? Exercise 1.8.14

Solve the initial-value problem $u_t+uu_x=0,\ |x|<\infty,\ t>0$ satisfying

$$u(x,0)=\begin{cases}1, & x\le 0,\\ 1-x, & 0\le x\le 1,\\ 0, & x\ge 1.\end{cases}$$

? Exercise 1.8.15

Consider the stopped traffic problem in a situation where the maximum car density is 200 cars per mile and the maximum speed is 50 miles per hour. Assume that the cars are arriving at 30 miles per hour. Find the solution of this problem and determine the rate at which the traffic is backing up. How does the answer change if the cars were arriving at 15 miles per hour.

? Exercise 1.8.16

Solve the following nonlinear equations where $p=u_x$ and $q=u_y$.

a. $p^2+q^2=1,\ u(x,x)=x.$
b. $pq=u,\ u(0,y)=y^2.$
c. $p+q=pq,\ u(x,0)=x.$
d. $pq=u^2.$
e. $p^2+qy=u.$

? Exercise 1.8.17

Find the solution of $xp+qy-p^2q-u=0$ in parametric form for the initial conditions at $t=0$:

$$x(t,s)=s,\quad y(t,s)=2,\quad u(t,s)=s+1$$

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CHAPTER OVERVIEW

2: Second Order Partial Differential Equations

> *"Either mathematics is too big for the human mind or the human mind is more than a machine." - Kurt Gödel (1906-1978)*

2.1: Introduction
2.2: Derivation of Generic 1D Equations
2.3: Boundary Value Problems
2.4: Separation of Variables
2.5: Laplace's Equation in 2D
2.6: Classification of Second Order PDEs
2.7: d'Alembert's Solution of the Wave Equation
2.8: Problems

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2.1: Introduction

In this chapter we will introduce several generic second order linear partial differential equations and see how such equations lead naturally to the study of boundary value problems for ordinary differential equations. These generic differential equation occur in one to three spatial dimensions and are all linear differential equations. A list is provided in Table 2.1.1. Here we have introduced the Laplacian operator, $\nabla^2 u = u_{xx} + u_{yy} + u_{zz}$. Depending on the types of boundary conditions imposed and on the geometry of the system (rectangular, cylindrical, spherical, etc.), one encounters many interesting boundary value problems.

Table 2.1.1: List of generic partial differential equations.

Name	2 Vars	3D
Heat Equation	$u_t = ku_{xx}$	$u_t = k\nabla^2 u$
Wave Equation	$u_{tt} = c^2 u_{xx}$	$u_{tt} = c^2 \nabla^2 u$
Laplace's Equation	$u_{xx} + u_{yy} = 0$	$\nabla^2 u = 0$
Poisson's Equation	$u_{xx} + u_{yy} = F(x, y)$	$\nabla^2 u = F(x, y, z)$
Schrödinger's Equation	$iu_t = u_{xx} + F(x, t)u$	$iu_t = \nabla^2 u + F(x, y, z, t)u$

Let's look at the heat equation in one dimension. This could describe the heat conduction in a thin insulated rod of length L. It could also describe the diffusion of pollutant in a long narrow stream, or the flow of traffic down a road. In problems involving diffusion processes, one instead calls this equation the diffusion equation. [See the derivation in Section 2.2.2.]

A typical initial-boundary value problem for the heat equation would be that initially one has a temperature distribution $u(x, 0) = f(x)$. Placing the bar in an ice bath and assuming the heat flow is only through the ends of the bar, one has the boundary conditions $u(0, t) = 0$ and $u(L, t) = 0$. Of course, we are dealing with Celsius temperatures and we assume there is plenty of ice to keep that temperature fixed at each end for all time as seen in Figure 2.1.1. So, the problem one would need to solve is given as [IC = initial condition(s) and BC = boundary conditions.]

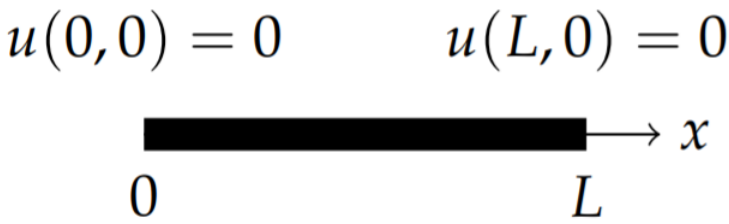


Figure 2.1.1: One dimensional heated rod of length L.

Heat Equation

$$\begin{array}{lll} \text{PDE} & u_t = ku_{xx}, & 0 < t, \quad 0 \le x \le L, \\ \text{IC} & u(x, 0) = f(x), & 0 << L, \\ \text{BC} & u(0, t) = 0, & t > 0, \\ & u(L, t) = 0, & t > 0, \end{array} \tag{2.1.1}$$

Here, k is the heat conduction constant and is determined using properties of the bar.

Another problem that will come up in later discussions is that of the vibrating string. A string of length L is stretched out horizontally with both ends fixed such as a violin string as shown in Figure 2.1.2. Let $u(x, t)$ be the vertical displacement of the string at position x and time t. The motion of the string is governed by the one dimensional wave equation. [See the derivation in Section 2.2.1.] The string might be plucked, giving the string an initial profile, $u(x, 0) = f(x)$, and possibly each point on the string has an initial velocity $u_t(x, 0) = g(x)$. The initial-boundary value problem for this problem is given below.

Wave Equation

$$\begin{array}{lll} \text{PDE} & u_{tt} = c^2 u_{xx} & 0 < t, \quad 0 \le x \le L \\ \text{IC} & u(x,0) = f(x) & 0 < x < L \\ & u_t(x,0) = g(x) & 0 < x < L \\ \text{BC} & u(0,t) = 0 & t > 0 \\ & u(L,t) = 0 & t > 0 \end{array} \tag{2.1.2}$$

In this problem c is the wave speed in the string. It depends on the mass per unit length of the string, μ, and the tension, τ, placed on the string.

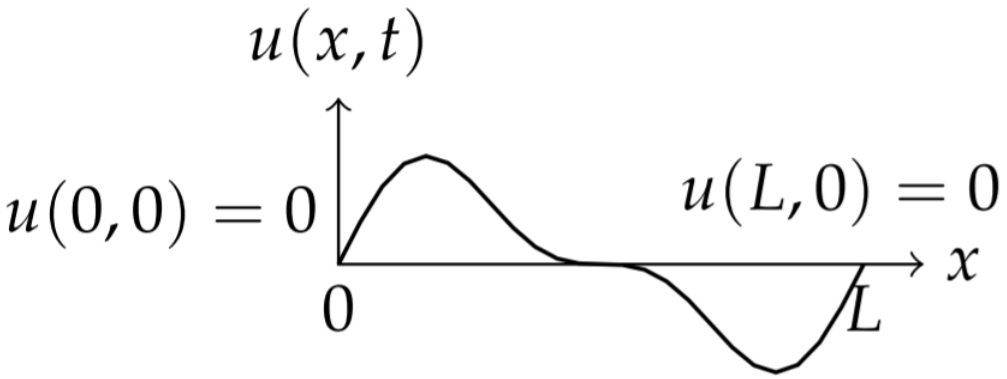


Figure 2.1.2: One dimensional string of length L.

There is a rich history on the study of these and other partial differential equations and much of this involves trying to solve problems in physics. Consider the one dimensional wave motion in the string. Physically, the speed of these waves depends on the tension in the string and its mass density. The frequencies we hear are then related to the string shape, or the allowed wavelengths across the string. We will be interested the harmonics, or pure sinusoidal waves, of the vibrating string and how a general wave on the string can be represented as a sum over such harmonics. This will take us into the field of spectral, or Fourier, analysis. The solution of the heat equation also involves the use of Fourier analysis. However, in this case there are no oscillations in time.

There are many applications that are studied using spectral analysis. At the root of these studies is the belief that continuous waveforms are comprised of a number of harmonics. Such ideas stretch back to the Pythagoreans study of the vibrations of strings, which led to their program of a world of harmony. This idea was carried further by Johannes Kepler (1571-1630) in his harmony of the spheres approach to planetary orbits. In the 1700's others worked on the superposition theory for vibrating waves on a stretched spring, starting with the wave equation and leading to the superposition of right and left traveling waves. This work was carried out by people such as John Wallis (1616-1703), Brook Taylor (1685-1731) and Jean le Rond d'Alembert (1717-1783).

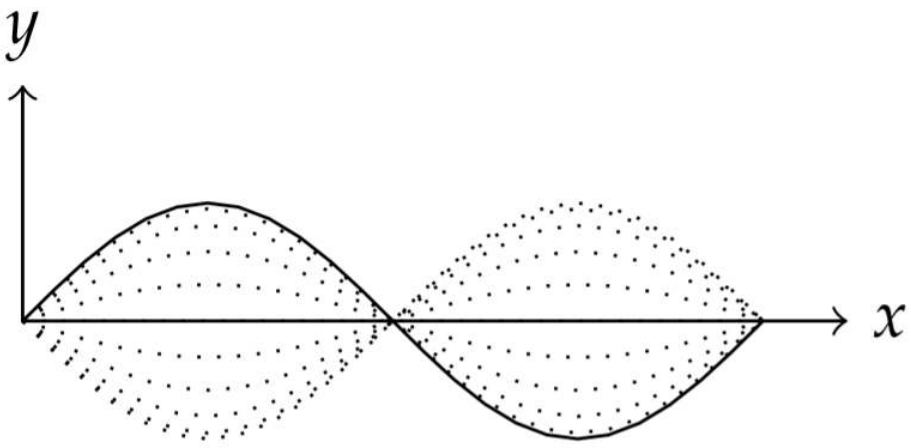


Figure 2.1.3: Plot of the second harmonic of a vibrating string at different times.

In 1742 d'Alembert solved the wave equation

$$c^2 \frac{\partial^2 y}{\partial x^2} - \frac{\partial^2 y}{\partial t^2} = 0,$$

where y is the string height and c is the wave speed. However, this solution led him and others, like Leonhard Euler (1707-1783) and Daniel Bernoulli (1700-1782), to investigate what "functions" could be the solutions of this equation. In fact, this led to a more rigorous approach to the study of analysis by first coming to grips with the concept of a function. For example, in 1749 Euler sought the solution for a plucked string in which case the initial condition $y(x,0) = h(x)$ has a discontinuous derivative! (We will see how this led to important questions in analysis.)

In 1753 Daniel Bernoulli viewed the solutions as a superposition of simple vibrations, or harmonics. Such superpositions amounted to looking at solutions of the form

$$y(x,t) = \sum_k a_k \sin \frac{k\pi x}{L} \cos \frac{k\pi ct}{L},$$

where the string extends over the interval $[0, L]$ with fixed ends at $x = 0$ and $x = L$.

However, the initial profile for such superpositions is given by

$$y(x,0) = \sum_k a_k \sin \frac{k\pi x}{L}.$$

It was determined that many functions could not be represented by a finite number of harmonics, even for the simply plucked string in Figure 2.1.4 given by an initial condition of the form

$$y(x,0) = \begin{cases} Ax, & 0 \le x \le L/2 \\ A(L-x), & L/2 \le x \le L \end{cases}$$

Thus, the solution consists generally of an infinite series of trigonometric functions.

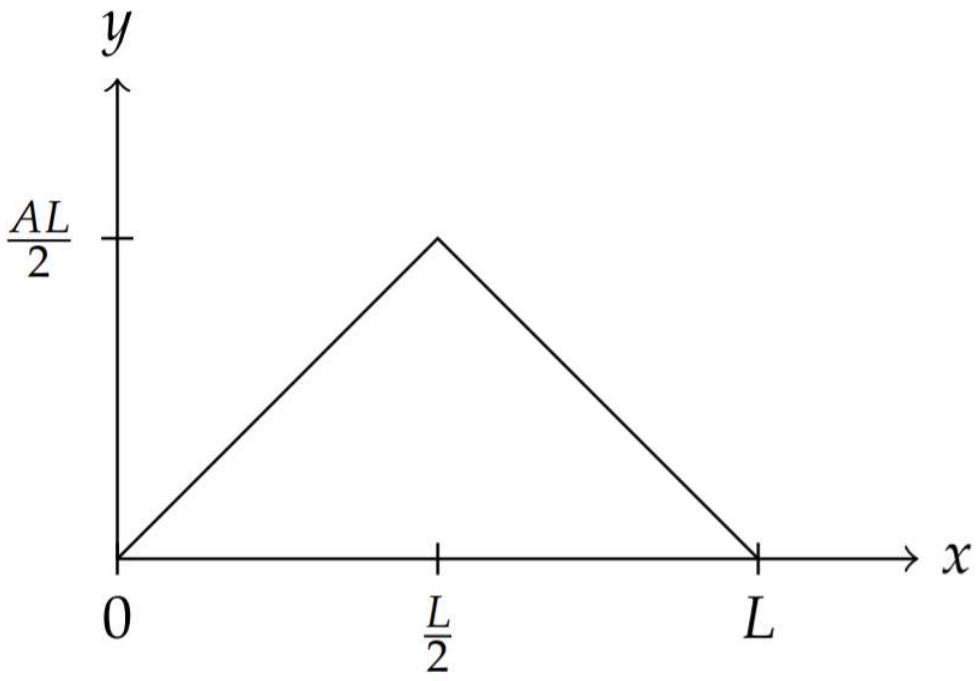


Figure 2.1.4: Plot of an initial condition for a plucked string.

Note

The one dimensional version of the heat equation is a partial differential equation for $u(x,t)$ of the form

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}.$$

Solutions satisfying boundary conditions $u(0,t) = 0$ and $u(L,t) = 0$, are of the form

$$u(x,t) = \sum_{n=0}^{\infty} b_n \sin\frac{n\pi x}{L} e^{-n^2\pi^2 t/L^2}.$$

In this case, setting $u(x,0) = f(x)$, one has to satisfy the condition

$$f(x) = \sum_{n=0}^{\infty} b_n \sin\frac{n\pi x}{L}.$$

This is another example leading to an infinite series of trigonometric functions.

Such series expansions were also of importance in Joseph Fourier's (1768- 1830) solution of the heat equation. The use of Fourier expansions has become an important tool in the solution of linear partial differential equations, such as the wave equation and the heat equation. More generally, using a technique called the Method of Separation of Variables, allowed higher dimensional problems to be reduced to one dimensional boundary value problems. However, these studies led to very important questions, which in turn opened the doors to whole fields of analysis. Some of the problems raised were

1. What functions can be represented as the sum of trigonometric functions?
2. How can a function with discontinuous derivatives be represented by a sum of smooth functions, such as the above sums of trigonometric functions?
3. Do such infinite sums of trigonometric functions actually converge to the functions they represent?

There are many other systems for which it makes sense to interpret the solutions as sums of sinusoids of particular frequencies. For example, we can consider ocean waves. Ocean waves are affected by the gravitational pull of the moon and the sun and other numerous forces. These lead to the tides, which in turn have their own periods of motion. In an analysis of wave heights, one can separate out the tidal components by making use of Fourier analysis.

In the Section 2.4 we describe how to go about solving these equations using the method of separation of variables. We will find that in order to accommodate the initial conditions, we will need to introduce Fourier series before we can complete the problems, which will be the subject of the following chapter. However, we first derive the one-dimensional wave and heat equations.

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2.2: Derivation of Generic 1D Equations

Derivation of Wave Equation for String

The wave equation for a one dimensional string is derived based upon simply looking at Newton's Second Law of Motion for a piece of the string plus a few simple assumptions, such as small amplitude oscillations and constant density.

We begin with $\mathbf{F} = m\mathbf{a}$. The mass of a piece of string of length ds is $m = \rho(x)ds$. From Figure 2.2.1 an incremental length f the string is given by

$$\Delta s^2 = \Delta x^2 + \Delta u^2.$$

The piece of string undergoes an acceleration of $a = \frac{\partial^2 u}{\partial t^2}$.

We will assume that the main force acting on the string is that of tension. Let $T(x,t)$ be the magnitude of the tension acting on the left end of the piece of string. Then, on the right end the tension is $T(x+\Delta x,t)$. At these points the tension makes an angle to the horizontal of $\theta(x,t)$ and $\theta(x+\Delta x,t)$, respectively.

Assuming that there is no horizontal acceleration, the x-component in the second law, $m\mathbf{a} = \mathbf{F}$, for the string element is given by

$$0 = T(x+\Delta x,t)\cos\theta(x+\Delta x,t) - T(x,t)\cos\theta(x,t).$$

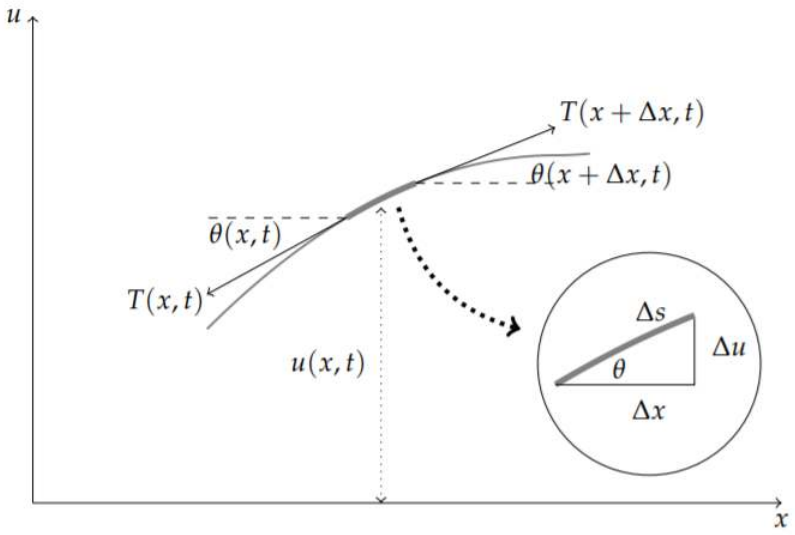


Figure 2.2.1: A small piece of string is under tension.

The vertical component is given by

$$\rho(x)\Delta s\frac{\partial^2 u}{\partial t^2} = T(x+\Delta x,t)\sin\theta(x+\Delta x,t) - T(x,t)\sin\theta(x,t)$$

The length of the piece of string can be written in terms of Δx,

$$\Delta s = \sqrt{\Delta x^2 + \Delta u^2} = \sqrt{1 + \left(\frac{\Delta u}{\Delta x}\right)^2}\,\Delta x.$$

and the right hand sides of the component equation can be expanded about $\Delta x = 0$, to obtain

$$\begin{aligned} T(x+\Delta x,t)\cos\theta(x+\Delta x,t) - T(x,t)\cos\theta(x,t) &\approx \frac{\partial(T\cos\theta)}{\partial x}(x,t)\Delta x \\ T(x+\Delta x,t)\sin\theta(x+\Delta x,t) - T(x,t)\sin\theta(x,t) &\approx \frac{\partial(T\sin\theta)}{\partial x}(x,t)\Delta x. \end{aligned}$$

Furthermore, we note that

$$\tan\theta = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \frac{\partial u}{\partial x}.$$

Now we can divide these component equations by Δx and let $\Delta x \to 0$. This gives the approximations

$$\begin{aligned} 0 &= \frac{T(x+\Delta x,t)\cos\theta(x+\Delta x,t) - T(x,t)\cos\theta(x,t)}{\Delta x} \\ &\approx \frac{\partial (T\cos\theta)}{\partial x}(x,t) \\ \rho(x)\frac{\partial^2 u}{\partial t^2}\frac{\delta s}{\delta s} &= \frac{T(x+\Delta x,t)\sin\theta(x+\Delta x,t) - T(x,t)\sin\theta(x,t)}{\Delta x} \\ \rho(x)\frac{\partial^2 u}{\partial t^2}\sqrt{1+\left(\frac{\partial u}{\partial x}\right)^2} &\approx \frac{\partial (T\sin\theta)}{\partial x}(x,t). \end{aligned} \tag{2.2.1}$$

We will assume a small angle approximation, giving

$$\sin\theta \approx \tan\theta = \frac{\partial u}{\partial x},$$

$\cos\theta \approx 1$, and

$$\sqrt{1+\left(\frac{\partial u}{\partial x}\right)^2} \approx 1.$$

Then, the horizontal component becomes

$$\frac{\partial T(x,t)}{\partial x} = 0.$$

Therefore, the magnitude of the tension $T(x,t) = T(t)$ is at most time dependent.

The vertical component equation is now

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = T(t)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = T(t)\frac{\partial^2 u}{\partial x^2}.$$

Assuming that ρ and T are constant and defining

$$c^2 = \frac{T}{\rho},$$

we obtain the one dimensional wave equation,

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}.$$

Derivation of 1D Heat Equation

Consider a one dimensional rod of length L as shown in Figure 2.2.2. It is heated and allowed to sit. The heat equation is the governing equation which allows us to determine the temperature of the rod at a later time.

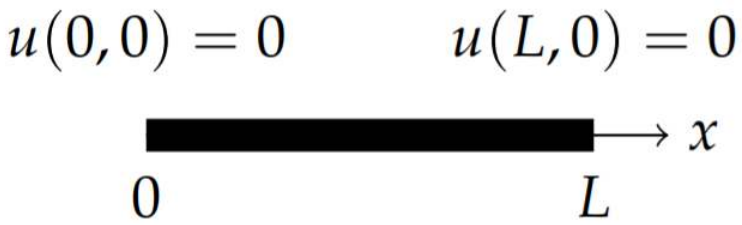


Figure 2.2.2: One dimensional heated rod of length L.

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We begin with some simple thermodynamics. Recall that to raise the temperature of a mass m by ΔT takes thermal energy given by

$$Q = mc\Delta T,$$

assuming the mass does not go through a phase transition. Here c is the specific heat capacity of the substance. So, we will begin with the heat content of the rod as

$$Q = mcT(x,t)$$

and assume that m and c are constant.

We will also need Fourier's law of heat transfer or heat conduction. This law simply states that heat energy flows from warmer to cooler regions and is written in terms of the heat energy flux, $\phi(x,t)$. The heat energy flux, or flux density, gives the rate of energy flow per area. Thus, the amount of heat energy flowing over the left end of the region of cross section A in time Δt is given $\phi(x,t)\,\Delta\, tA$. The units of $\phi(x,t)$ are then $\mathrm{J/s/m^2} = \mathrm{W/m^2}$.

Fourier's law of heat conduction states that the flux density is proportional to the gradient of the temperature,

$$\phi = -K\frac{\partial T}{\partial x}.$$

Here K is the thermal conductivity and the negative sign takes into account the direction of flow from higher to lower temperatures.

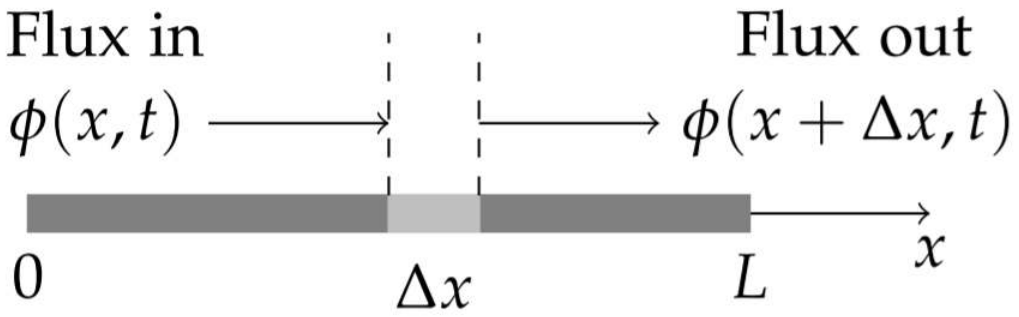


Figure 2.2.3: A one dimensional rod of length L. Heat can flow through increment Δx.

Now we make use of the conservation of energy. Consider a small section of the rod of width Δx as shown in Figure 2.2.3. The rate of change of the energy through this section is due to energy flow through the ends. Namely

$$\text{Rate of change of heat energy } = \text{ Heat in } - \text{ Heat out.}$$

The energy content of the small segment of the rod is given by

$$\Delta Q = (\rho A\,\Delta\, x)cT(x,t+\Delta t) - (\rho A\,\Delta\, x)cT(x,t).$$

The flow rates across the boundaries are given by the flux.

$$(\rho A\,\Delta\, x)cT(x,t+\Delta t) - (\rho A\,\Delta\, x)cT(x,t) = [\phi(x,t) - \phi(x+\Delta x,t)]\,\Delta\, tA.$$

Dividing by Δx and Δt and letting Δx, $\Delta t \to 0$, we obtain

$$\frac{\partial T}{\partial t} = -\frac{1}{c\rho}\frac{\partial \phi}{\partial x}.$$

Using Fourier's law of heat conduction,

$$\frac{\partial T}{\partial t} = \frac{1}{c\rho}\frac{\partial}{\partial x}\left(K\frac{\partial T}{\partial x}\right).$$

Assuming K, c, and ρ are constant, we have the one dimensional heat equation as used in the text:

$$\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2},$$

where $k = \frac{k}{c\rho}$.

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2.3: Boundary Value Problems

You might have only solved initial value problems in your undergraduate differential equations class. For an initial value problem one has to solve a differential equation subject to conditions on the unknown function and its derivatives at one value of the independent variable. For example, for $x = x(t)$ we could have the initial value problem

$$x'' + x = 2, \quad x(0) = 1, \quad x'(0) = 0. \tag{2.3.1}$$

Typically, initial value problems involve time dependent functions and boundary value problems are spatial. So, with an initial value problem one knows how a system evolves in terms of the differential equation and the state of the system at some fixed time. Then one seeks to determine the state of the system at a later time.

Example 2.3.1: Initial Value Problem

Solve the initial value problem, $x'' + 4x = \cos t, \quad x(0) = 1, \quad x'(0) = 0$.

Solution

Note that the conditions are provided at one time, $t = 0$. Thus, this an initial value problem. Recall from your course on differential equations that we need to find the general solution and then apply the initial conditions. Furthermore, this is a nonhomogeneous differential equation, so the solution is a sum of a solution of the homogeneous equation and a particular solution of the nonhomogeneous equation, $x(t) = x_h(t) + x_p(t)$. [See the ordinary differential equations review in the Appendix.]

The solution of $x'' + 4x = 0$ is easily found as

$$x_h(t) = c_1 \cos 2t + c_2 \sin 2t.$$

The particular solution is found using the Method of Undetermined Coefficients. We guess a solution of the form

$$x_p(t) = A \cos t + B \sin t.$$

Differentiating twice, we have

$$x_p''(t) = -(A \cos t + B \sin t).$$

So,

$$x_p'' + 4x_p = -(A \cos t + B \sin t) + 4(A \cos t + B \sin t).$$

Comparing the right hand side of this equation with $\cos t$ in the original problem, we are led to setting $B = 0$ and $A = \frac{1}{3}\cos t$. Thus, the general solution is

$$x(t) = c_1 \cos 2t + c_2 \sin 2t + \frac{1}{3}\cos t.$$

We now apply the initial conditions to find the particular solution. The first condition, $x(0) = 1$, gives

$$1 = c_1 + \frac{1}{3}.$$

Thus, $c_1 = \frac{2}{3}$. Using this value for c_1, the second condition, $x'(0) = 0$, gives $c_2 = 0$. Therefore,

$$x(t) = \frac{1}{3}(2 \cos 2t + \cos t).$$

For boundary values problems, one knows how each point responds to its neighbors, but there are conditions that have to be satisfied at the endpoints. An example would be a horizontal beam supported at the ends, like a bridge. The shape of the beam under the influence of gravity, or other forces, would lead to a differential equation and the boundary conditions at the beam ends would affect the solution of the problem. There are also a variety of other types of boundary conditions. In the case of a beam, one end could be fixed and the other end could be free to move. We will explore the effects of different boundary conditions in our

discussions and exercises. But, we will first solve a simple boundary value problem which is a slight modification of the above problem.

Example 2.3.2: Boundary Value Problem

Solve the boundary value problem, $x'' + x = 2, \quad x(0) = 1, \quad x(1) = 0$.

Solution

Note that the conditions at $t = 0$ and $t = 1$ make this a boundary value problem since the conditions are given at two different points. As with initial value problems, we need to find the general solution and then apply any conditions that we may have. This is a nonhomogeneous differential equation, so the solution is a sum of a solution of the homogeneous equation and a particular solution of the nonhomogeneous equation, $x(t) = x_h(t) + x_p(t)$. The solution of $x'' + x = 0$ is easily found as

$$x_h(t) = c_1 \cos t + c_2 \sin t.$$

The particular solution is found using the Method of Undetermined Coefficients,

$$x_p(t) = 2.$$

Thus, the general solution is

$$x(t) = 2 + c_1 \cos t + c_2 \sin t.$$

We now apply the boundary conditions and see if there are values of c_1 and c_2 that yield a solution to this boundary value problem. The first condition, $x(0) = 0$, gives

$$0 = 2 + c_1.$$

Thus, $c_1 = -2$. Using this value for c_1, the second condition, $x(1) = 1$, gives

$$0 = 2 - 2\cos 1 + c_2 \sin 1.$$

This yields

$$c_2 = \frac{2(\cos 1 - 1)}{\sin 1}.$$

We have found that there is a solution to the boundary value problem and it is given by

$$x(t) = 2\left(1 - \cos t \frac{(\cos 1 - 1)}{\sin 1} \sin t\right).$$

Boundary value problems arise in many physical systems, just as the initial value problems we have seen earlier. We will see in the next sections that boundary value problems for ordinary differential equations often appear in the solutions of partial differential equations. However, there is no guarantee that we will have unique solutions of our boundary value problems as we had found in the example above.

Now that we understand simple boundary value problems for ordinary differential equations, we can turn to initial-boundary value problems for partial differential equations. We will see that a common method for studying these problems is to use the method of separation of variables. In this method the problem of solving partial differential equations is to separate the partial differential equation into several ordinary differential equations of which several are boundary value problems of the sort seen in this section.

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2.4: Separation of Variables

Solving many of the linear partial differential equations presented in the first section can be reduced to solving ordinary differential equations. We will demonstrate this by solving the initial-boundary value problem for the heat equation as given in (2.1.1). We will employ a method typically used in studying linear partial differential equations, called the Method of Separation of Variables. In the next subsections we describe how this method works for the one-dimensional heat equation, one-dimensional wave equation, and the two-dimensional Laplace equation.

Heat Equation

We want to solve the heat equation,

$$u_t = ku_{xx}, \quad 0 < t, \quad 0 \le x \le L.$$

subject to the boundary conditions

$$u(0,t) = 0, \quad u(L,t) = 0, \quad t > 0,$$

and the initial condition

$$u(x,0) = f(x), \quad 0 < x < L.$$

We begin by assuming that u can be written as a product of single variable functions of each independent variable,

$$u(x,t) = X(x)T(t).$$

Substituting this guess into the heat equation, we find that

$$XT' = kX''T.$$

The prime denotes differentiation with respect to the independent variable and we will suppress the independent variable in the following unless needed for emphasis.

Dividing both sides of this result by k and $u = XT$, yields

$$\frac{1}{k}\frac{T'}{T} = \frac{X''}{X}.$$

We have separated the functions of time on one side and space on the other side. The constant k could be on either side of this expression, but we moved it to make later computations simpler.

The only way that a function of t equals a function of x is if the functions are constant functions. Therefore, we set each function equal to a constant, λ : [For example, if $Ae^{ct} = ax^2 + b$ is possible for any x or t, then this is only possible if $a = 0$, $c = 0$ and $b = A$.]

$$\underbrace{\frac{1}{k}\frac{T'}{T}}_{\text{function of } t} = \underbrace{\frac{X''}{X}}_{\text{function of } x} = \underbrace{\lambda.}_{\text{constant}}$$

This leads to two equations:

$$T' = k\lambda T, \tag{2.4.1}$$

$$X'' = \lambda X. \tag{2.4.2}$$

These are ordinary differential equations. The general solutions to these constant coefficient equations are readily found as

$$T(t) = Ae^{k\lambda t}, \tag{2.4.3}$$

$$X(x) = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}. \tag{2.4.4}$$

We need to be a little careful at this point. The aim is to force the final solutions to satisfy both the boundary conditions and initial conditions. Also, we should note that λ is arbitrary and may be positive, zero, or negative. We first look at how the boundary conditions on $u(x,t)$ lead to conditions on $X(x)$.

The first boundary condition is $u(0,t)=0$. This implies that

$$X(0)T(t)=0, \quad \text{for all } t.$$

The only way that this is true is if $X(0)=0$. Similarly, $u(L,t)=0$ for all t implies that $X(L)=0$. So, we have to solve the boundary value problem

$$X''-\lambda X=0, \quad X(0)=0=X(L). \tag{2.4.5}$$

An obvious solution is $X\equiv 0$. However, this implies that $u(x,t)=0$, which is not an interesting solution. We call such solutions, $X\equiv 0$, trivial solutions and will seek nontrivial solution for these problems.

There are three cases to consider, depending on the sign of λ.

$\lambda > 0$

In this case we have the exponential solutions

$$X(x)=c_1e^{\sqrt{\lambda}x}+c_2e^{-\sqrt{\lambda}x.} \tag{2.4.6}$$

For $X(0)=0$, we have

$$0=c_1+c_2.$$

We will take $c_2=-c_1$. Then,

$$X(x)=c_1(e^{\sqrt{\lambda}x}-e^{-\sqrt{\lambda}x})=2c_1\sinh\sqrt{\lambda}x.$$

Applying the second condition, $X(L)=0$ yields

$$c_1\sinh\sqrt{\lambda}L=0.$$

This will be true only if $c_1=0$, since $\lambda>0$. Thus, the only solution in this case is the trivial solution, $X(x)=0$.

$\lambda = 0$

For this case it is easier to set λ to zero in the differential equation. So, $X''=0$. Integrating twice, one finds

$$X(x)=c_1x+c_2.$$

Setting $x=0$, we have $c_2=0$, leaving $X(x)=c_1x$. Setting $x=L$, we find $c_1L=0$. So, $c_1=0$ and we are once again left with a trivial solution.

III. $\lambda < 0$

In this case is would be simpler to write $\lambda=-\mu^2$. Then the differential equation is

$$X''+\mu^2X=0.$$

The general solution is

$$X(x)=c_1\cos\mu x+c_2\sin\mu x.$$

At $x=0$ we get $0=c_1$. This leaves $X(x)=c_2\sin\mu x$. At $x=L$, we find

$$0=c_2\sin\mu L.$$

So, either $c_2=0$ or $\sin\mu L=0$. $c_2=0$ leads to a trivial solution again. But, there are cases when the sine is zero. Namely,

$$\mu L=n\pi, \quad n=1,2,\ldots.$$

Note that $n=0$ is not included since this leads to a trivial solution. Also, negative values of n are redundant, since the sine function is an odd function.

In summary, we can find solutions to the boundary value problem (2.4.5) for particular values of λ. The solutions are

$$X_n(x)=\sin\frac{n\pi x}{L}, \quad n=1,2,3,\ldots$$

for

$$\lambda_n = -\mu_n^2 = -\left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \ldots.$$

We should note that the boundary value problem in Equation (2.4.5) is an eigenvalue problem. We can recast the differential equation as

$$LX = \lambda X,$$

where

$$L = D^2 = \frac{d^2}{dx^2}$$

is a linear differential operator. The solutions, $X_n(x)$, are called eigenfunctions and the λ_n's are the eigenvalues. We will elaborate more on this characterization later in the next chapter.

We have found the product solutions of the heat equation (2.1.1) satisfying the boundary conditions. These are

$$u_n(x,t) = e^{k\lambda_n t} \sin\frac{n\pi x}{L}, \quad n = 1, 2, 3, \ldots. \tag{2.4.7}$$

However, these do not necessarily satisfy the initial condition $u(x,0) = f(x)$. What we do get is

$$u_n(x,0) = \sin\frac{n\pi x}{L}, \quad n = 1, 2, 3, \ldots.$$

So, if the initial condition is in one of these forms, we can pick out the right value for n and we are done.

For other initial conditions, we have to do more work. Note, since the heat equation is linear, the linear combination of the product solutions is also a solution of the heat equation. The general solution satisfying the given boundary conditions is given as

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{k\lambda_n t} \sin\frac{n\pi x}{L}. \tag{2.4.8}$$

The coefficients in the general solution are determined using the initial condition. Namely, setting $t = 0$ in the general solution, we have

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}.$$

So, if we know $f(x)$, can we find the coefficients, b_n? If we can, then we will have the solution to the full initial-boundary value problem.

The expression for $f(x)$ is a Fourier sine series. We will need to digress into the study of Fourier series in order to see how one can find the Fourier series coefficients given $f(x)$. Before proceeding, we will show that this process is not uncommon by applying the Method of Separation of Variables to the wave equation in the next section.

Wave Equation

In this section we will apply the Method of Separation of Variables to the one dimensional wave equation, given by

$$\frac{\partial^2 u}{\partial^2 t} = c^2 \frac{\partial^2 u}{\partial^2 x}, \quad t > 0, \quad 0 \leq x ı L, \tag{2.4.9}$$

subject to the boundary conditions

$$u(0,t) = 0, \quad u(L,t) = 0, \quad t > 0,$$

and the initial conditions

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 < x < L.$$

This problem applies to the propagation of waves on a string of length L with both ends fixed so that they do not move. $u(x,t)$ represents the vertical displacement of the string over time. The derivation of the wave equation assumes that the vertical

displacement is small and the string is uniform. The constant c is the wave speed, given by

$$c = \sqrt{\frac{\tau}{\mu}},$$

where τ is the tension in the string and μ is the mass per unit length. We can understand this in terms of string instruments. The tension can be adjusted to produce different tones and the makeup of the string (nylon or steel, thick or thin) also has an effect. In some cases the mass density is changed simply by using thicker strings. Thus, the thicker strings in a piano produce lower frequency notes.

The u_{tt} term gives the acceleration of a piece of the string. The u_{xx} is the concavity of the string. Thus, for a positive concavity the string is curved upward near the point of interest. Thus, neighboring points tend to pull upward towards the equilibrium position. If the concavity is negative, it would cause a negative acceleration.

The solution of this problem is easily found using separation of variables. We let $u(x,t) = X(x)T(t)$. Then we find

$$XT'' = c^2 X''T,$$

which can be rewritten as

$$\frac{1}{c^2}\frac{T''}{T} = \frac{X''}{X}.$$

Again, we have separated the functions of time on one side and space on the other side. Therefore, we set each function equal to a constant, λ.

$$\underbrace{\frac{1}{c^2}\frac{T''}{T}}_{\text{function of } t} = \underbrace{\frac{X''}{X}}_{\text{function of } x} = \underbrace{\lambda.}_{\text{constant}}$$

This leads to two equations:

$$T'' = c^2 \lambda T, \tag{2.4.10}$$

$$X'' = \lambda X. \tag{2.4.11}$$

As before, we have the boundary conditions on $X(x)$:

$$X(0) = 0, \quad \text{and} \quad X(L) = 0,$$

giving the solutions, as shown in Figure 2.4.1,

$$X_n(x) = \sin\frac{n\pi x}{L}, \quad \lambda_n = -\left(\frac{n\pi}{L}\right)^2.$$

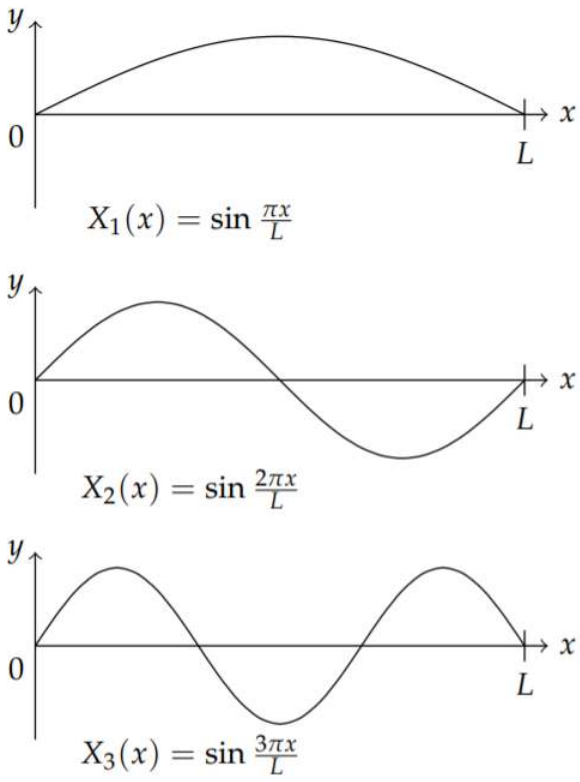


Figure 2.4.1: The first three harmonics of the vibrating string.

The main difference from the solution of the heat equation is the form of the time function. Namely, from Equation (2.4.10) we have to solve

$$T'' + \left(\frac{n\pi c}{L}\right)^2 T = 0. \tag{2.4.12}$$

This equation takes a familiar form. We let

$$\omega_n = \frac{n\pi c}{L},$$

then we have

$$T'' + \omega_n^2 T = 0.$$

This is the differential equation for simple harmonic motion and ω_n is the angular frequency. The solutions are easily found as

$$T(t) = A_n \cos \omega_n t + B_n \sin \omega_n t. \tag{2.4.13}$$

Therefore, we have found that the product solutions of the wave equation take the forms $\sin \frac{n\pi x}{L} \cos \omega_n t$ and $\sin \frac{n\pi x}{L} \sin \omega_n t$. The general solution, a superposition of all product solutions, is given by

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos \frac{n\pi ct}{L} + B_n \sin \frac{n\pi ct}{L} \right] \sin \frac{n\pi x}{L}. \tag{2.4.14}$$

This solution satisfies the wave equation and the boundary conditions. We still need to satisfy the initial conditions. Note that there are two initial conditions, since the wave equation is second order in time.

First, we have $u(x,0) = f(x)$. Thus,

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L}. \tag{2.4.15}$$

In order to obtain the condition on the initial velocity, $u_t(x,0)=g(x)$, we need to differentiate the general solution with respect to t:

$$u_t(x,t)=\sum_{n=1}^{\infty}\frac{n\pi c}{L}\left[-A_n\sin\frac{n\pi ct}{L}+B_n\cos\frac{n\pi ct}{L}\right]\sin\frac{n\pi x}{L}. \tag{2.4.16}$$

Then, we have from the initial velocity

$$g(x)=u_t(x,0)=\sum_{n=1}^{\infty}\frac{n\pi c}{L}B_n\sin\frac{n\pi x}{L}. \tag{2.4.17}$$

So, applying the two initial conditions, we have found that $f(x)$ and $g(x)$, are represented as Fourier sine series. In order to complete the problem we need to determine the coefficients A_n and B_n for $n=1,2,3,\ldots$. Once we have these, we have the complete solution to the wave equation. We had seen similar results for the heat equation. In the next chapter we will find out how to determine these Fourier coefficients for such series of sinusoidal functions.

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2.5: Laplace's Equation in 2D

Another generic partial differential equation is Laplace's equation, $\nabla^2 u = 0$. Laplace's equation arises in many applications. As an example, consider a thin rectangular plate with boundaries set at fixed temperatures. Assume that any temperature changes of the plate are governed by the heat equation, $u_t = k\nabla^2 u$, subject to these boundary conditions. However, after a long period of time the plate may reach thermal equilibrium. If the boundary temperature is zero, then the plate temperature decays to zero across the plate. However, if the boundaries are maintained at a fixed nonzero temperature, which means energy is being put into the system to maintain the boundary conditions, the internal temperature may reach a nonzero equilibrium temperature. Reaching thermal equilibrium means that asymptotically in time the solution becomes time independent. Thus, the equilibrium state is a solution of the time independent heat equation, $\nabla^2 u = 0$.

A second example comes from electrostatics. Letting $\phi(\mathbf{r})$ be the electric potential, one has for a static charge distribution, $\rho(\mathbf{r})$, that the electric field, $\mathbf{E} = \nabla\phi$, satisfies one of Maxwell's equations, $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$. In regions devoid of charge, $\rho(\mathbf{r}) = 0$, the electric potential satisfies Laplace's equation, $\nabla^2\phi = 0$.

As a final example, Laplace's equation appears in two-dimensional fluid flow. For an incompressible flow, $\nabla \cdot \mathbf{v} = 0$. If the flow is irrotational, then $\nabla \times \mathbf{v} = 0$. We can introduce a velocity potential, $\mathbf{v} = \nabla\phi$. Thus, $\nabla \times \mathbf{v}$ vanishes by a vector identity and $\nabla \cdot \mathbf{v} = 0$ implies $\nabla^2\phi = 0$. So, once again we obtain Laplace's equation.

Solutions of Laplace's equation are called *harmonic functions* and we will encounter these in Chapter 8 on complex variables and in Section 2.5 we will apply complex variable techniques to solve the two-dimensional Laplace equation. In this section we use the Method of Separation of Variables to solve simple examples of Laplace's equation in two dimensions. Three dimensional problems will studied in Chapter 6.

Example 2.5.1: Equilibrium Temperature Distribution for a Rectangular Plate

Let's consider Laplace's equation in Cartesian coordinates,

$$u_{xx} + u_{yy} = 0, \quad 0 < x < L, \quad 0 < y < H$$

with the boundary conditions

$$u(0, y) = 0, \quad u(L, y) = 0, \quad u(x, 0) = f(x), \quad u(x, H) = 0.$$

The boundary conditions are shown in Figure 2.5.1.

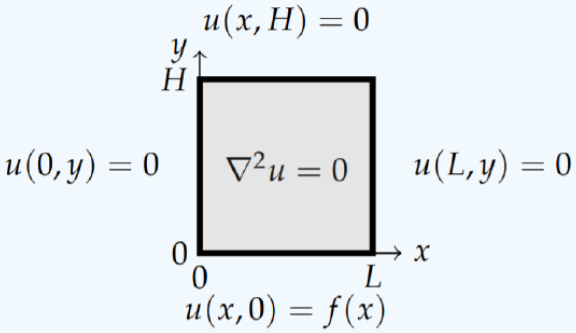


Figure 2.5.1: In this figure we show the domain and boundary conditions for the example of determining the equilibrium temperature distribution for a rectangular plate.

Solution

As with the heat and wave equations, we can solve this problem using the method of separation of variables. Let $u(x, y) = X(x)Y(y)$. Then, Laplace's equation becomes

$$X''Y + XY'' = 0$$

and we can separate the x and y dependent functions and introduce a separation constant, λ,

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda.$$

Thus, we are led to two differential equations,

$$\begin{aligned} X''+\lambda X &= 0, \\ Y''-\lambda Y &= 0. \end{aligned} \tag{2.5.1}$$

The general solution of the equation for $Y(y)$ is given by

$$Y(y) = c_1 e^{\sqrt{\lambda} y} + c_2 e^{-\sqrt{\lambda} y}.$$

The boundary condition $u(x, H) = 0$ implies $Y(H) = 0$. So, we have

$$c_1 e^{\sqrt{\lambda} H} + c_2 e^{-\sqrt{\lambda} H} = 0.$$

Thus,

$$c_2 = -c_1 e^{2\sqrt{\lambda} H}.$$

Inserting this result into the expression for $Y(y)$, we have

$$\begin{aligned} Y(y) &= c_1 e^{\sqrt{\lambda} y} - c_1 e^{2\sqrt{\lambda} H} e^{-\sqrt{\lambda} y} \\ &= c_1 e^{\sqrt{\lambda} H} \left(e^{-\sqrt{\lambda} H} e^{\sqrt{\lambda} y} - e^{\sqrt{\lambda} H} e^{-\sqrt{\lambda} y} \right) \\ &= c_1 e^{\sqrt{\lambda} H} \left(e^{-\sqrt{\lambda}(H-y)} - e^{\sqrt{\lambda}(H-y)} \right) \\ &= -2c_1 e^{\sqrt{\lambda} H} \sinh \sqrt{\lambda}(H-y). \end{aligned} \tag{2.5.2}$$

Since we already know the values of the eigenvalues λ_n from the eigenvalue problem for $X(x)$, we have that the y-dependence is given by

$$Y_n(y) = \sinh \frac{n\pi(H-y)}{L}.$$

So, the product solutions are given by

$$u_n(x, y) = \sin \frac{n\pi x}{L} \sinh \frac{n\pi(H-y)}{L}, \quad n = 1, 2, \ldots.$$

These solutions satisfy Laplace's equation and the three homogeneous boundary conditions and in the problem.

The remaining boundary condition, $u(x, 0) = f(x)$, still needs to be satisfied. Inserting $y = 0$ in the product solutions does not satisfy the boundary condition unless $f(x)$ is proportional to one of the eigenfunctions $X_n(x)$. So, we first write down the general solution as a linear combination of the product solutions,

$$u(x, y) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L} \sinh \frac{n\pi(H-y)}{L}. \tag{2.5.3}$$

Now we apply the boundary condition, $u(x, 0) = f(x)$, to find that

$$f(x) = \sum_{n=1}^{\infty} a_n \sinh \frac{n\pi H}{L} \sin \frac{n\pi x}{L}. \tag{2.5.4}$$

Defining $b_n = a_n \sinh \frac{n\pi H}{L}$, this becomes

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}. \tag{2.5.5}$$

We see that the determination of the unknown coefficients, b_n, is simply done by recognizing that this is a Fourier sine series. We now move on to the study of Fourier series and provide more complete answers in Chapter 6.

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2.6: Classification of Second Order PDEs

We have studied several examples of partial differential equations, the heat equation, the wave equation, and Laplace's equation. These equations are examples of parabolic, hyperbolic, and elliptic equations, respectively. Given a general second order linear partial differential equation, how can we tell what type it is? This is known as the classification of second order PDEs.

Let $u = u(x, y)$. Then, the general form of a linear second order partial differential equation is given by

$$a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy} + d(x,y)u_x + e(x,y)u_y + f(x,y)u = g(x,y). \tag{2.6.1}$$

In this section we will show that this equation can be transformed into one of three types of second order partial differential equations.

Let $x = x(\xi, \eta)$ and $y = y(\xi, \eta)$ be an invertible transformation from coordinates (ξ, η) to coordinates (x, y). Furthermore, let $u(x(\xi,\eta), y(\xi,\eta)) = \mathcal{U}(\xi,\eta)$. How does the partial differential equation (2.6.1) transform?

We first need to transform the derivatives of $u(x, t)$. We have

$$\begin{aligned}
u_x &= \mathcal{U}_\xi \xi_x + \mathcal{U}_\eta \eta_x, \\
u_y &= \mathcal{U}_\xi \xi_y + \mathcal{U}_\eta \eta_y, \\
u_{xx} &= \frac{\partial}{\partial x}(\mathcal{U}_\xi \xi_x + \mathcal{U}_\eta \eta_x), \\
&= \mathcal{U}_{\xi\xi}\xi_x^2 + 2\mathcal{U}_{\xi\eta}\xi_x\eta_x + \mathcal{U}_{\eta\eta}\eta_x^2 + \mathcal{U}_\xi \xi_{xx} + \mathcal{U}_\eta \eta_{xx}, \\
u_{yy} &= \frac{\partial}{\partial y}(\mathcal{U}_\xi \xi_y + \mathcal{U}_\eta \eta_y), \\
&= \mathcal{U}_{\xi\xi}\xi_y^2 + 2\mathcal{U}_{\xi\eta}\xi_y\eta_y + \mathcal{U}_{\eta\eta}\eta_y^2 + \mathcal{U}_\xi \xi_{yy} + \mathcal{U}_\eta \eta_{yy}, \\
u_{xy} &= \frac{\partial}{\partial y}(\mathcal{U}_\xi \xi_x + \mathcal{U}_\eta \eta_x), \\
&= \mathcal{U}_{\xi\xi}\xi_x\xi_y + \mathcal{U}_{\xi\eta}\xi_x\eta_y + \mathcal{U}_{\eta\xi}\xi_y\eta_x + \mathcal{U}_{\eta\eta}\eta_x\eta_y + \mathcal{U}_\xi \xi_{xy} + \mathcal{U}_\eta \eta_{xy}.
\end{aligned} \tag{2.6.2}$$

Inserting these derivatives into Equation (2.6.1), we have

$$\begin{aligned}
g - f\mathcal{U} &= au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y \\
&= a\left(\mathcal{U}_{\xi\xi}\xi_x^2 + 2\mathcal{U}_{\xi\eta}\xi_x\eta_x + \mathcal{U}_{\eta\eta}\eta_x^2 + \mathcal{U}_\xi\xi_{xx} + \mathcal{U}_\eta\eta_{xx}\right) \\
&\quad + 2b(\mathcal{U}_{\xi\xi}\xi_x\xi_y + \mathcal{U}_{\xi\eta}\xi_x\eta_y + \mathcal{U}_{\xi\eta}\xi_y\eta_x \\
&\quad + \mathcal{U}_{\eta\eta}\eta_x\eta + y + \mathcal{U}_\xi\xi_{xy} + \mathcal{U}_\eta\eta_{xy}) \\
&\quad + c\left(\mathcal{U}_{\xi\xi}\xi_y^2 + 2\mathcal{U}_{\xi\eta}\xi_y\eta_y + \mathcal{U}_{\eta\eta}\eta_y^2 + \mathcal{U}_\xi\xi_{yy} + \mathcal{U}_\eta\eta_{yy}\right) \\
&\quad + d(\mathcal{U}_\xi\xi_x + \mathcal{U}_\eta\eta_x) \\
&\quad + e(\mathcal{U}_\xi\xi_y + \mathcal{U}_\eta\eta_y) \\
&= (a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2)\mathcal{U}_{\xi\xi} \\
&\quad + (2a\xi_x\eta_x + 2b\xi_x\eta_y + 2b\xi_y\eta_x + 2c\xi_y\eta_y)\mathcal{U}_{\xi\eta} \\
&\quad + (a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2)\mathcal{U}_{\eta\eta} \\
&\quad + (a\xi_{xx} + 2b\xi_{xy} + c\xi_{yy} + d\xi_x + e\xi_y)\mathcal{U}_\xi \\
&\quad + (a\eta_{xx} + 2b\eta_{xy} + c\eta_{yy} + d\eta_x + e\eta_y)\mathcal{U}_\eta \\
&= A\mathcal{U}_{\xi\xi} + 2B\mathcal{U}_{\xi\eta} + C\mathcal{U}_{\eta\eta} + D\mathcal{U}_\xi + E\mathcal{U}_\eta.
\end{aligned} \tag{2.6.3}$$

Picking the right transformation, we can eliminate some of the second order derivative terms depending on the type of differential equation. This leads to three types: elliptic, hyperbolic, or parabolic.

For example, if transformations can be found to make $A \equiv 0$ and $C \equiv 0$, then the equation reduces

$$\mathcal{U}_{\xi\eta} = \text{lower order terms.}$$

. Such an equation is called **hyperbolic**. A generic example of a hyperbolic equation is the wave equation.

The conditions that $A \equiv 0$ and $C \equiv 0$ give the conditions

$$\begin{aligned} a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 &= 0. \\ a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2 &= 0. \end{aligned} \tag{2.6.4}$$

) We seek ξ and η satisfying these two equations, which are of the same form. Let's assume that $\xi = \xi(x, y)$ is a constant curve in the xy-plane. Furthermore, if this curve is the graph of a function, $y = y(x)$, then

$$\frac{d\xi}{dx} = \xi_x + \frac{dy}{dx}\xi_y = 0.$$

Then

$$\frac{dy}{dx} = -\frac{\xi_x}{\xi_y}.$$

Inserting this expression in $A = 0$, we have

$$\begin{aligned} A &= a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 \\ &= \xi_y^2\left(a\left(\frac{\xi_x}{\xi_y}\right)^2 + 2b\frac{\xi_x}{\xi_y} + c\right) \\ &= \xi_y^2\left(a\left(\frac{dy}{dx}\right)^2 - 2b\frac{dy}{dx} + c\right) = 0. \end{aligned} \tag{2.6.5}$$

This equation is satisfied if $y(x)$ satisfies the differential equation

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - ac}}{a}.$$

So, for $A = 0$, we choose ξ and η to be constant on these characteristic curves.

Example 2.6.1

Show that $u_{xx} - u_{yy} = 0$ is hyperbolic.

Solution

In this case we have $a = 1 = -c$ and $b = 0$. Then,

$$\frac{dy}{dx} = \pm 1.$$

This gives $y(x) = \pm x + c$. So, we choose ξ and η constant on these characteristic curves. Therefore, we let $\xi = x - y$, $\eta = x + y$.

Let's see if this transformation transforms the differential equation into a canonical form. Let $u(x, y) = \mathcal{U}(\xi, \eta)$. Then, the needed derivatives become

$$\begin{aligned} u_x &= \mathcal{U}_\xi\xi_x + \mathcal{U}_\eta\eta_x = \mathcal{U}_\xi + \mathcal{U}_\eta. \\ u_y &= \mathcal{U}_\xi\xi_y + \mathcal{U}_\eta\eta_y = -\mathcal{U}_\xi + \mathcal{U}_\eta. \\ u_{xx} &= \frac{\partial}{\partial x}(\mathcal{U}_\xi + \mathcal{U}_\eta) \\ &= \mathcal{U}_{\xi\xi}\xi_x + \mathcal{U}_{\xi\eta}\eta_x + \mathcal{U}_{\eta\xi}\xi_x + \mathcal{U}_{\eta\eta}\eta_x \\ &= \mathcal{U}_{\xi\xi} + 2\mathcal{U}_{\xi\eta} + \mathcal{U}_{\eta\eta}. \\ u_{yy} &= \frac{\partial}{\partial y}(-\mathcal{U}_\xi + \mathcal{U}_\eta) \\ &= -\mathcal{U}_{\xi\xi}\xi_y - \mathcal{U}_{\xi\eta}\eta_y + \mathcal{U}_{\eta\xi}\xi_y + \mathcal{U}_{\eta\eta}\eta_y \\ &= \mathcal{U}_{\xi\xi} - 2\mathcal{U}_{\xi\eta} + \mathcal{U}_{\eta\eta}. \end{aligned} \tag{2.6.6}$$

Inserting these derivatives into the differential equation, we have

$$0 = u_{xx} - u_{yy} = 4\mathcal{U}_{\xi\eta}.$$

Thus, the transformed equation is $\mathcal{U}_{\xi\eta} = 0$. Thus, showing it is a hyperbolic equation.

We have seen that A and C vanish for $\xi(x,y)$ and $\eta(x,y)$ constant along the characteristics

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - ac}}{a}$$

for second order hyperbolic equations. This is possible when $b^2 - ac > 0$ since this leads to two characteristics.

In general, if we consider the second order operator

$$L[u] = a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy},$$

then this operator can be transformed to the new form

$$L'[\mathcal{U}] = B\mathcal{U}_{\xi\eta}$$

if $b^2 - ac > 0$. An example of a hyperbolic equation is the wave equation, $u_{tt} = u_{xx}$.

When $b^2 - ac = 0$, then there is only one characteristic solution, $\frac{dy}{dx} = \frac{b}{a}$. This is the parabolic case. But, $\frac{dy}{dx} = -\frac{\xi_x}{\xi_y}$. So,

$$\frac{b}{a} = -\frac{\xi_x}{\xi_y},$$

or

$$a\xi_x + b\xi_y = 0.$$

Also, $b^2 - ac = 0$ implies that $c = b^2/a$.

Inserting these expression into coefficient B, we have

$$\begin{aligned} B &= 2a\xi_x\eta_x + 2b\xi_x\eta_y + 2b\xi_y\eta_x + 2c\xi_y\eta_y \\ &= 2(a\xi_x + b\xi_y)\eta_x + 2(b\xi_x + c\xi_y)\eta_y \\ &= 2\frac{b}{a}(a\xi_x + b\xi_y)\eta_y = 0. \end{aligned} \tag{2.6.7}$$

Therefore, in the parabolic case, $A = 0$ and $B = 0$, and $L[u]$ transforms to

$$L'[\mathcal{U}] = C\mathcal{U}_{\eta\eta}$$

when $b^2 - ac = 0$. This is the canonical form for a parabolic operator. An example of a parabolic equation is the heat equation, $u_t = u_{xx}$.

Finally, when $b^2 - ac < 0$, we have the **elliptic** case. In this case we Elliptic case. cannot force $A = 0$ or $C = 0$. However, in this case we can force $B = 0$. As we just showed, we can write

$$B = 2(a\xi_x + b\xi_y)\eta_x + 2(b\xi_x + c\xi_y)\eta_y.$$

Letting $\eta_x = 0$, we can choose ξ to satisfy $b\xi_x + c\xi_y = 0$.

$$A = a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 = a\xi_x^2 - c\xi_y^2 = \frac{ac - b^2}{c}\xi_x^2$$

$$C = a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2 = c\eta_y^2$$

Furthermore, setting $\frac{ac-b^2}{c}\xi_x^2 = c\eta_y^2$, we can make $A = c$ and $L[u]$ transforms to

$$L'[\mathcal{U}] = A[\mathcal{U}_{\xi\xi} + \mathcal{U}_{\eta\eta}]$$

when $b^2 - ac < 0$. This is the canonical form for an elliptic operator. An example of an elliptic equation is Laplace's equation, $u_{xx} + u_{yy} = 0$.

Classification of Second Order PDEs

The second order differential operator

$$L[u] = a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy},$$

can be transformed to one of the following forms:

- $b^2 - ac > 0$. Hyperbolic: $L[u] = B(x,y)u_{xy}$
- $b^2 - ac = 0$. Parabolic: $L[u] = C(x,y)u_{yy}$
- $b^2 - ac < 0$. Elliptic: $L[u] = A(x,y)[u_{xx} + u_{yy}]$

As a final note, the terminology used in this classification is borrowed from the general theory of quadratic equations which are the equations for translated and rotated conics. Recall that the general quadratic equation in two variable takes the form

$$ax^2 + 2bxy + cy^2 + dx + ey + f = 0. \tag{2.6.8}$$

One can complete the squares in x and y to obtain the new form

$$a(x-h)^2 + 2bxy + c(y-k)^2 + f' = 0.$$

So, translating points (x,y) using the transformations $x' = x - h$ and $y' = y - k$, we find the simpler form

$$ax^2 + 2bxy + cy^2 + f = 0.$$

Here we dropped all primes.

We can also introduce transformations to simplify the quadratic terms. Consider a rotation of the coordinate axes by θ,

$$\begin{aligned} x' &= x\cos\theta + y\sin\theta \\ y' &= -x\sin\theta + y\cos\theta, \end{aligned} \tag{2.6.9}$$

or

$$\begin{aligned} x &= x'\cos\theta - y'\sin\theta \\ y &= x'\sin\theta + y'\cos\theta. \end{aligned} \tag{2.6.10}$$

The resulting equation takes the form

$$Ax'^2 + 2Bx'y' + Cy'^2 + D = 0,$$

where

$$\begin{aligned} A &= a\cos^2\theta + 2b\sin\theta\cos\theta + c\sin^2\theta. \\ B &= (c-a)\sin\theta\cos\theta + b(\cos^2\theta - \sin^\theta). \\ C &= a\sin^2\theta - 2b\sin\theta\cos\theta + c\cos^2\theta. \end{aligned} \tag{2.6.11}$$

Furthermore, one can show that $b^2 - ac = B^2 - AC$. From the form $Ax'^2 + 2Bx'y' + Cy'^2 + D = 0$, the resulting quadratic equation takes one of the following forms:

- $b^2 - ac > 0$. Hyperbolic: $Ax^2 - Cy^2 + D = 0$.
- $b^2 - ac = 0$. Parabolic: $Ax^2 + By + D = 0$.
- $b^2 - ac < 0$. Elliptic: $Ax^2 + Cy^2 + D = 0$.

Thus, one can see the connection between the classification of quadratic equations and second order partial differential equations in two independent variables.

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2.7: d'Alembert's Solution of the Wave Equation

A general solution of the one-dimensional wave equation can be found. This solution was first Jean-Baptiste le Rond d'Alembert (1717- 1783) and is referred to as d'Alembert's formula. In this section we will derive d'Alembert's formula and then use it to arrive at solutions to the wave equation on infinite, semi-infinite, and finite intervals.

We consider the wave equation in the form $u_{tt} = c^2 u_{xx}$ and introduce the transformation

$$u(x,t) = U(\xi,\eta), \quad \text{where} \quad \xi = x + ct \quad \text{and} \quad \eta = x - ct.$$

Note that ξ, and η are the characteristics of the wave equation.

We also need to note how derivatives transform. For example

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial U(\xi,\eta)}{\partial x} \\ &= \frac{\partial U(\xi,\eta)}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial U(\xi,\eta)}{\partial \eta}\frac{\partial \eta}{\partial x} \\ &= \frac{\partial U(\xi,\eta)}{\partial \xi} + \frac{\partial U(\xi,\eta)}{\partial \eta}. \end{aligned} \tag{2.7.1}$$

Therefore, as an operator, we have

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}.$$

Similarly, one can show that

$$\frac{\partial}{\partial t} = c\frac{\partial}{\partial \xi} - c\frac{\partial}{\partial \eta}.$$

Using these results, the wave equation becomes

$$\begin{aligned} 0 &= u_{tt} - c^2 u_{xx} \\ &= \left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right) u \\ &= \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) u \\ &= \left(c\frac{\partial}{\partial \xi} - c\frac{\partial}{\partial \eta} + c\frac{\partial}{\partial \xi} + c\frac{\partial}{\partial \eta}\right)\left(c\frac{\partial}{\partial \xi} - c\frac{\partial}{\partial \eta} - c\frac{\partial}{\partial \xi} - c\frac{\partial}{\partial \eta}\right) U \\ &= -4c^2\frac{\partial}{\partial \xi}\frac{\partial}{\partial \eta} U. \end{aligned} \tag{2.7.2}$$

Therefore, the wave equation has transformed into the simpler equation,

$$U_{\eta\xi} = 0.$$

A further integration gives

$$U(\xi,\eta) = \int^{\eta} \Gamma(\eta')d\eta' + F(\xi) = G(\eta) + F(\eta).$$

Therefore, we have as the general solution of the wave equation,

$$u(x,t) = F(x+ct) + G(x-ct), \tag{2.7.3}$$

where F and G are two arbitrary, twice differentiable functions. As t is increased, we see that $F(x+ct)$ gets horizontally shifted to the left and $G(x-ct)$ gets horizontally shifted to the right. As a result, we conclude that the solution of the wave equation can be seen as the sum of left and right traveling waves.

Let's use initial conditions to solve for the unknown functions. We let

$$u(x,0)=f(x), \quad u_t(x,0)=g(x), \quad |x|<\infty.$$

Applying this to the general solution, we have

$$f(x) = F(x)+G(x) \tag{2.7.4}$$
$$g(x) = c[F'(x)-G'x)]. \tag{2.7.5}$$

We need to solve for $F(x)$ and $G(x)$ in terms of $f(x)$ and $g(x)$. Integrating Equation (2.7.5), we have

$$\frac{1}{c}\int_0^x g(s)dx = F(x)-G(x)-F(0)+G(0).$$

Adding this result to Equation (2.7.5), gives

$$F(x)=\frac{1}{2}f(x)+\frac{1}{2c}\int_0^x g(s)ds+\frac{1}{2}[F(0)-G(0)].$$

Subtracting from Equation (2.7.5), gives

$$G(x)=\frac{1}{2}f(x)-\frac{1}{2c}\int_0^x g(s)ds-\frac{1}{2}[F(0)-G(0)].$$

Now we can write out the solution $u(x,t)=F(x+ct)+G(x-ct)$, yielding d'Alembert's solution

$$u(x,t)=\frac{1}{2}[f(x+ct)+f(x-ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds. \tag{2.7.6}$$

When $f(x)$ and $g(x)$ are defined for all $x\in R$, the solution is well-defined. However, there are problems on more restricted domains. In the next examples we will consider the semi-infinite and finite length string problems.In each case we will need to consider the domain of dependence and the domain of influence of specific points. These concepts are shown in Figure 2.7.1. The domain of dependence of point P is red region. The point P depends on the values of u and u_t at points inside the domain. The domain of influence of P is the blue region. The points in the region are influenced by the values of u and u_t at P.

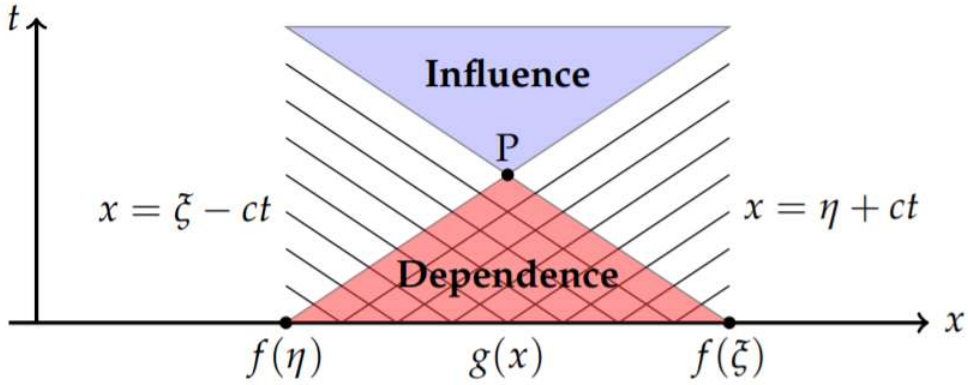


Figure 2.7.1: The domain of dependence of point P is red region. The point P depends on the values of u and u_t at points inside the domain. The domain of influence of P is the blue region. The points in the region are influenced by the values of u and u_t at P.

Example 2.7.1

Use d'Alembert's solution to solve

$$u_{tt}=c^2u_{xx}, \quad u(x,0)=f(x), \quad u_t(x,0)=g(x), \quad 0\le x<\infty.$$

Solution

The d'Alembert solution is not well-defined for this problem because $f(x-ct)$ is not defined for $x-ct<0$ for $c,\ t>0$. There are similar problems for $g(x)$. This can be seen by looking at the characteristics in the xt-plane. In Figure 2.7.2there are characteristics emanating from the points marked by η_0 and ξ_0 that intersect in the domain $x>0$. The point of intersection of the blue lines have a domain of dependence entirely in the region $x,\ t>0$, however the domain of dependence of point P

reaches outside this region. Only characteristics $\xi = x + ct$ reach point P, but characteristics $\eta = x - ct$ do not. But, we need $f(\eta)$ and $g(x)$ for $x < ct$ to form a solution.

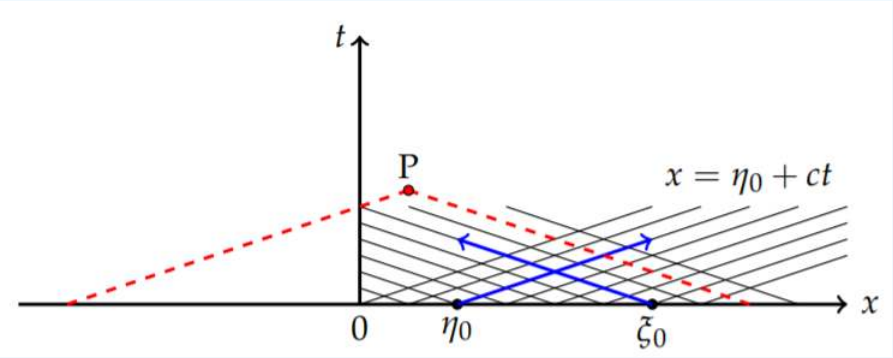


Figure 2.7.2: The characteristics for the semi-infinite string.

This can be remedied if we specified boundary conditions at $x = 0$. For example, Fixed end boundary condition we will assume the end $x = 0$ is fixed,

$$u(0,t) = 0, \quad t \geq 0.$$

Imagine an infinite string with one end (at $x = 0$) tied to a pole.

Since $u(x,t) = F(x+ct) + G(x-ct)$, we have

$$u(0,t) = F(ct) + G(-ct) = 0.$$

Letting $\zeta = -ct$, this gives $G(\zeta) = -F(-\zeta),\ \zeta \leq 0$.

Note that

$$\begin{aligned} G(\zeta) &= \frac{1}{2} f(\zeta) - \frac{1}{2c} \int_0^{\zeta} g(s)ds \\ -F(-\zeta) &= -\frac{1}{2} f(-\zeta) - \frac{1}{2c} \int_0^{-\zeta} g(s)ds \\ &= -\frac{1}{2} f(-\zeta) + \frac{1}{2c} \int_0^{\zeta} g(\sigma)d\sigma \end{aligned} \tag{2.7.7}$$

Comparing the expressions for $G(\zeta)$ and $-F(-\zeta)$, we see that

$$f(\zeta) = -f(-\zeta), \quad g(\zeta) = -g(-\zeta).$$

These relations imply that we can extend the functions into the region $x < 0$ if we make them odd functions, or what are called odd extensions. An example is shown in Figure 2.7.3.

Another type of boundary condition is if the end $x = 0$ is free,

$$u_x(0,t) = 0, \quad t \geq 0.$$

In this case we could have an infinite string tied to a ring and that ring is allowed to slide freely up and down a pole.

One can prove that this leads to

$$f(-\xi) = f(\xi), \quad g(-\xi) = g(\xi).$$

Thus, we can use an even extension of these function to produce solutions.

✔ Example 2.7.2

Solve the initial-boundary value problem

$$
\begin{aligned}
u_{tt} &= c^2 u_{xx}, \quad 0 \le x < \infty, \quad t > 0. \\
u(x,0) &= \begin{cases} x, & 0 \le x \le 1, \\ 2-x, & 1 \le x \le 2, \\ 0, & x > 2, \end{cases} \quad 0 \le x < \infty \\
u_t(x,0) &= 0, \quad 0 \le x < \infty. \\
u(0,t) &= 0, \quad t > 0.
\end{aligned} \tag{2.7.8}
$$

Solution

This is a semi-infinite string with a fixed end. Initially it is plucked to produce a nonzero triangular profile for $0 \le x \le 2$. Since the initial velocity is zero, the general solution is found from d'Alembert's solution,

$$u(x,t) = \frac{1}{2}[f_0(x+ct) + f_0(x-ct)],$$

where $f_0(x)$ is the odd extension of $f(x) = u(x,0)$. In Figure 2.7.3 we show the initial condition and its odd extension. The odd extension is obtained through reflection of $f(x)$ about the origin.

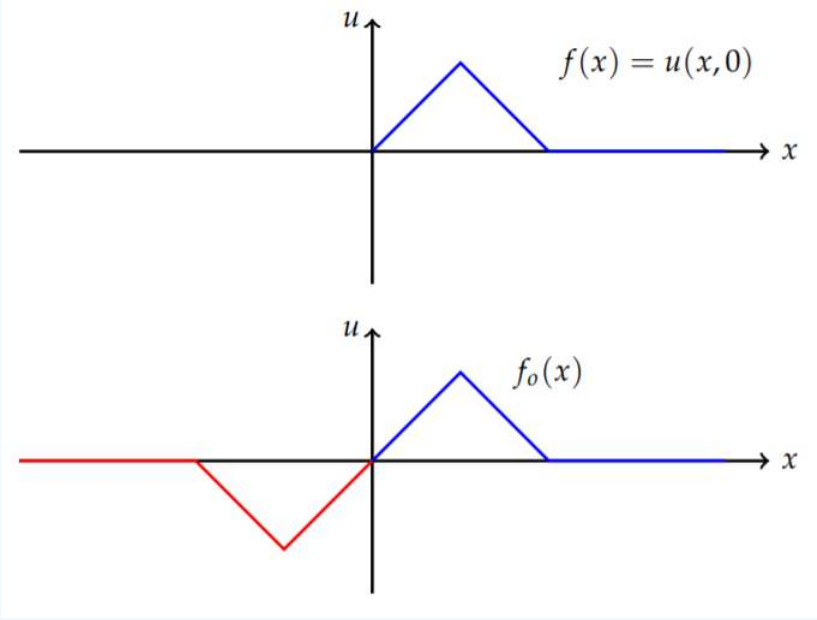


Figure 2.7.3: The initial condition and its odd extension. The odd extension is obtained through reflection of $f(x)$ about the origin.

The next step is to look at the horizontal shifts of $f_0(x)$. Several examples are shown in Figure 2.7.4These show the left and right traveling waves.

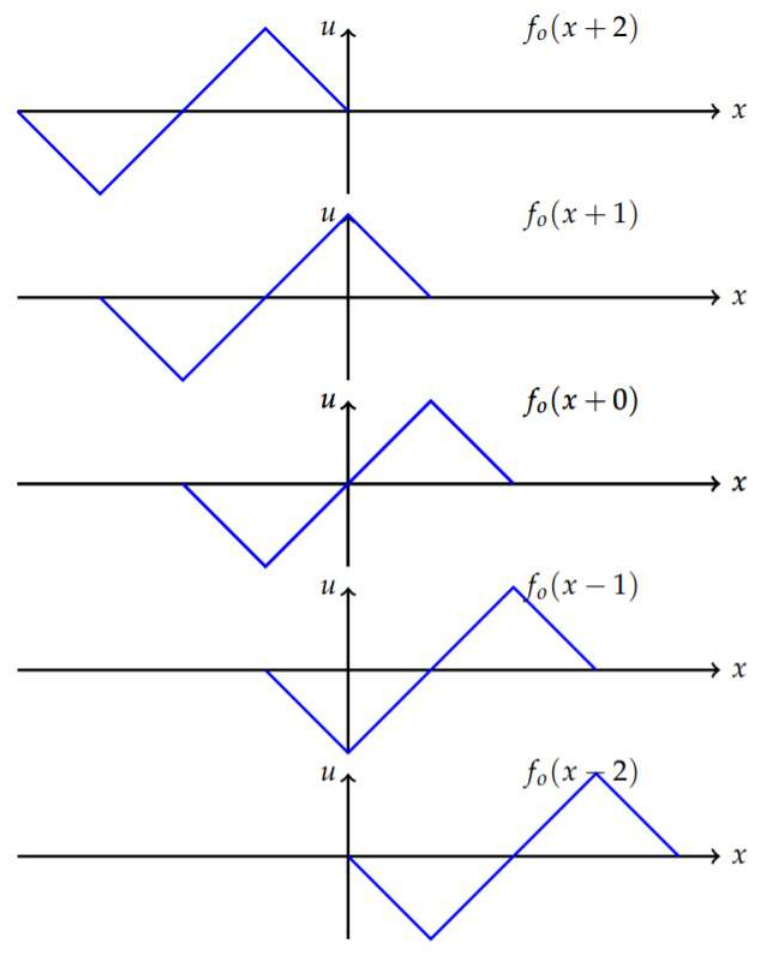


Figure 2.7.4: Examples of $f_0(x+ct)$ and $f_0(x-ct)$.

In Figure $2.7.5$ we show superimposed plots of $f_0(x+ct)$ and $f_0(x-ct)$ for given times. The initial profile in at the bottom. By the time $ct=2$ the full traveling wave has emerged. The solution to the problem emerges on the right side of the figure by averaging each plot.

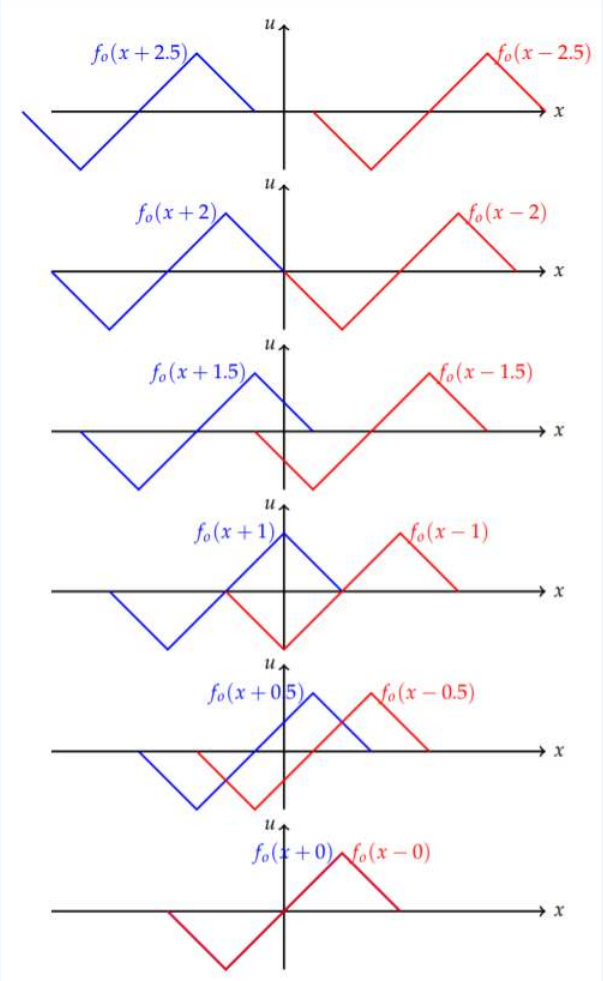


Figure 2.7.5: Superimposed plots of $f_0(x+ct)$ and $f_0(x-ct)$ for given times. The initial profile in at the bottom. By the time $ct=2$ the full traveling wave has emerged.

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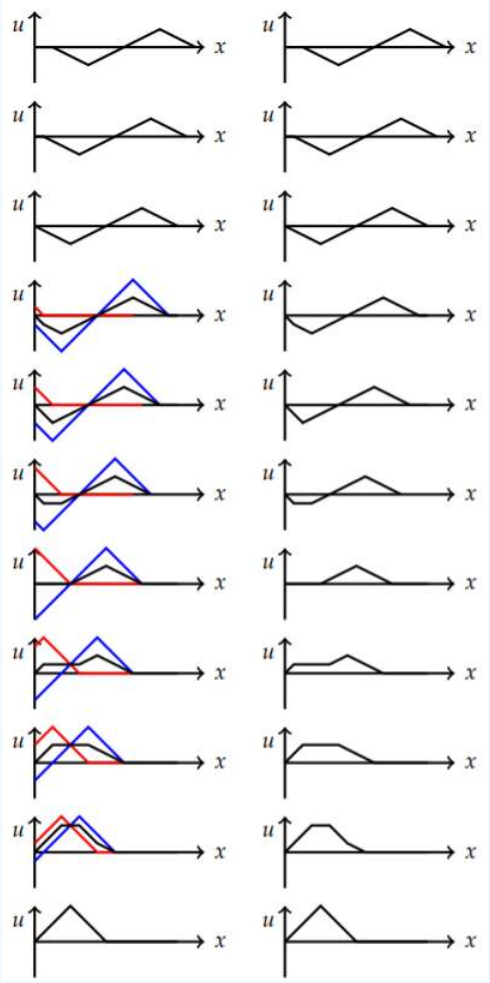


Figure 2.7.6: On the left is a plot of $f(x+ct)$, $f(x-ct)$ from Figure 2.7.5 and the average, $u(x,t)$. On the right the solution alone is shown for $ct=0$ at bottom to $ct=1$ at top for the semi-infinite string problem

Example 2.7.3

Use d'Alembert's solution to solve

$$u_{tt} = c^2 u_{xx}, \quad u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 \le x \le \ell.$$

Solution

The general solution of the wave equation was found in the form

$$u(x,t) = F(x+ct) + G(x-ct).$$

However, for this problem we can only obtain information for values of x and t such that $0 \le x+ct \le \ell$ and $0 \le x-ct \le \ell$. In Figure 2.7.7 the characteristics $x = \xi + ct$ and $x = \eta - ct$ for $0 \le \xi, \eta \le \ell$. The main (gray) triangle, which is the domain of dependence of the point $(\ell, 2, \ell/2c)$, is the only region in which the solution can be found based solely on the initial conditions. As with the previous problem, boundary conditions will need to be given in order to extend the domain of the solution.

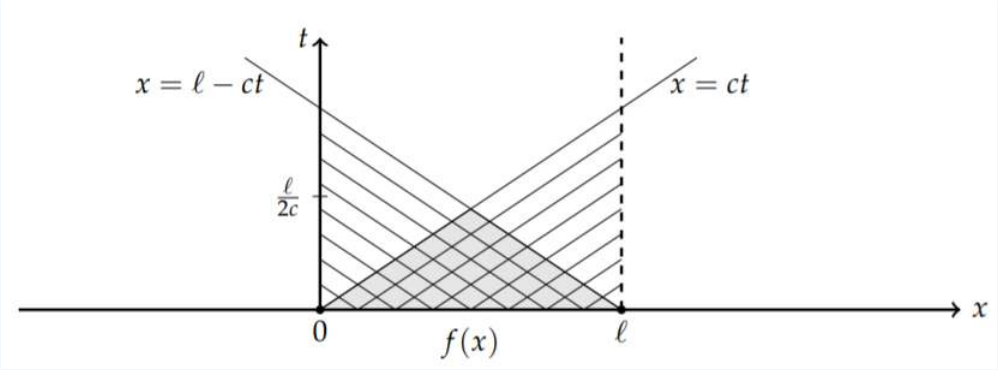


Figure 2.7.7: The characteristics emanating from the interval $0 \le x \le \ell$ for the finite string problem.

In the last example we saw that a fixed boundary at $x = 0$ could be satisfied when $f(x)$ and $g(x)$ are extended as odd functions. In Figure 2.7.8 we indicate how the characteristics are affected by drawing in the new one as red dashed lines. This allows us to now construct solutions based on the initial conditions under the line $x = \ell - ct$ for $0 \le x \le \ell$. The new region for which we can construct solutions from the initial conditions is indicated in gray in Figure 2.7.8.

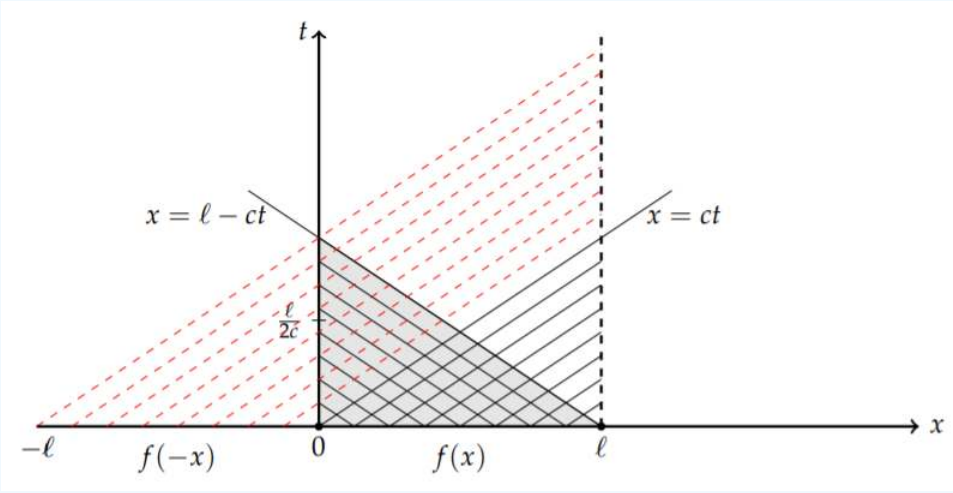


Figure 2.7.8: The red dashed lines are the characteristics from the interval $[-\ell, 0]$ from using the odd extension about $x = 0$.

We can add characteristics on the right by adding a boundary condition at $x = \ell$. Again, we could use fixed $u(\ell, t) = 0$, or free, $u_x(\ell, t) = 0$, boundary conditions. This allows us to now construct solutions based on the initial conditions for $\ell \le x \le 2\ell$.

Let's consider a fixed boundary condition at $x = \ell$. Then, the solution must satisfy

$$u(\ell, t) = F(\ell + ct) + G(\ell - ct) = 0.$$

To see what this means, let $\zeta = \ell + ct$. Then, this condition gives (since $ct = \zeta - \ell$)

$$F(\xi) = -G(2\ell - \xi), \quad \ell \le \xi \le 2\ell.$$

Note that $G(2\ell - \zeta)$ is defined for $0 \le 2\ell - \zeta \le \ell$. Therefore, this is a well-defined extension of the domain of $F(x)$.

Note that

$$\begin{aligned} F(\xi) &= \frac{1}{2} f(\xi) + \frac{1}{2c} \int_0^{\ell} g(s)ds. \\ -G(2\ell - \xi) &= -\frac{1}{2} f(2\ell - \xi) + \frac{1}{2c} \int_0^{2\ell - \xi} g(s)ds \\ &= -\frac{1}{2} f(2\ell - \xi) - \frac{1}{2c} \int_0^{\xi} g(2\ell - \sigma)d\sigma \end{aligned} \tag{2.7.9}$$

Comparing the expressions for $G(\zeta)$ and $-G(2\ell - \zeta)$, we see that

$$f(\xi) = -f(2\ell - \xi), \quad g(\xi) = -g(2\ell - \xi).$$

These relations imply that we can extend the functions into the region $x > \ell$ if we consider an odd extension of $f(x)$ and $g(x)$ about $x = \ell$. This will give the blue dashed characteristics in Figure 2.7.9 and a larger gray region to construct the solution.

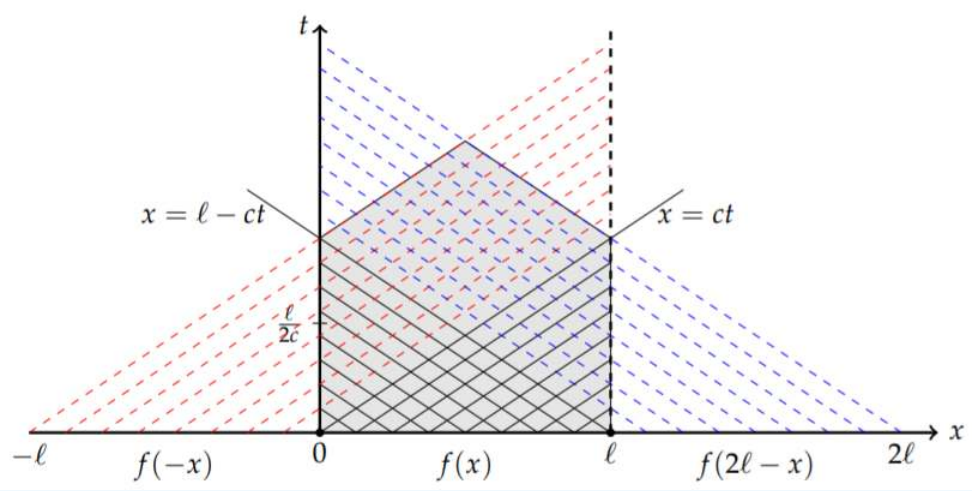


Figure 2.7.9: The red dashed lines are the characteristics from the interval $[-\ell, 0]$ from using the odd extension about $x = 0$ and the blue dashed lines are the characteristics from the interval $[\ell, 2\ell]$ from using the odd extension about $x = \ell$.

So far we have extended $f(x)$ and $g(x)$ to the interval $-\ell \le x \le 2\ell$ in order to determine the solution over a larger xt-domain. For example, the function $f(x)$ has been extended to

$$f_{ext}(x) = \begin{cases} -f(-x), & -\ell < x < 0, \\ f(x), & 0 < x < \ell, \\ -f(2\ell - x), & \ell < x < 2\ell. \end{cases}$$

A similar extension is needed for $g(x)$. Inserting these extended functions into d'Alembert's solution, we can determine $u(x,t)$ in the region indicated in Figure 2.7.9.

Even though the original region has been expanded, we have not determined how to find the solution throughout the entire strip, $[0, \ell] \times [0, \infty)$. This is accomplished by periodically repeating these extended functions with period 2ℓ. This can be shown from the two conditions

$$\begin{aligned} f(x) &= -f(-x), \quad -\ell \le x \le 0, \\ f(x) &= -f(2\ell - x), \quad \ell \le x \le 2\ell. \end{aligned} \tag{2.7.10}$$

Now, consider

$$\begin{aligned} f(x + 2\ell) &= -f(2\ell - (x - 2\ell)) \\ &= -f(-x) \\ &= f(x). \end{aligned} \tag{2.7.11}$$

This shows that $f(x)$ is periodic with period 2ℓ. Since $g(x)$ satisfies the same conditions, then it is as well.

In Figure 2.7.10 we show how the characteristics are extended throughout the domain strip using the periodicity of the extended initial conditions. The characteristics from the interval endpoints zig zag throughout the domain, filling it up. In the next example we show how to construct the odd periodic extension of a specific function.

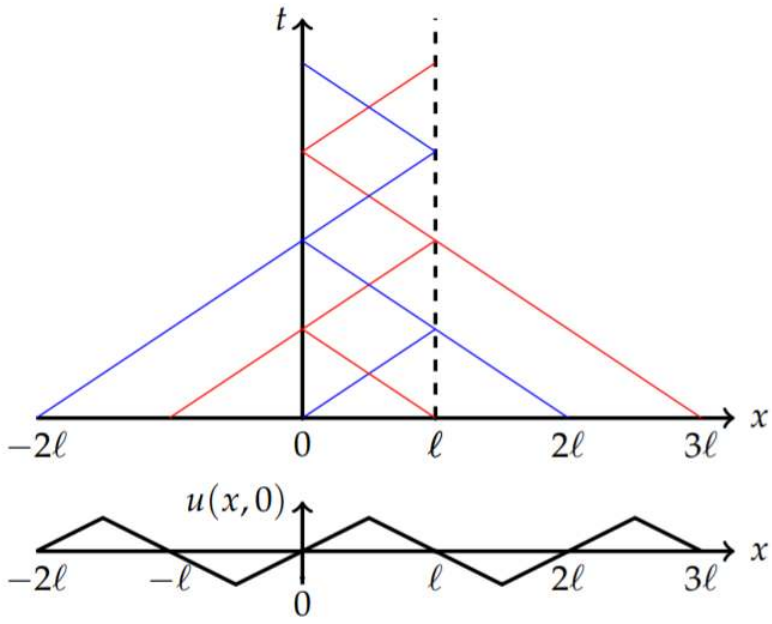


Figure 2.7.10: Extending the characteristics throughout the domain strip.

Example 2.7.4

Construct the periodic extension of the plucked string initial profile given by

$$f(x) = \begin{cases} x, & 0 \le x \le \frac{\ell}{2}, \\ \ell - x, & \frac{\ell}{2} \le x \le \ell, \end{cases}$$

satisfying fixed boundary conditions at $x = 0$ and $x = \ell$.

Solution

We first take the solution and add the odd extension about $x = 0$. Then we add an extension beyond $x = \ell$. This process is shown in Figure 2.7.11.

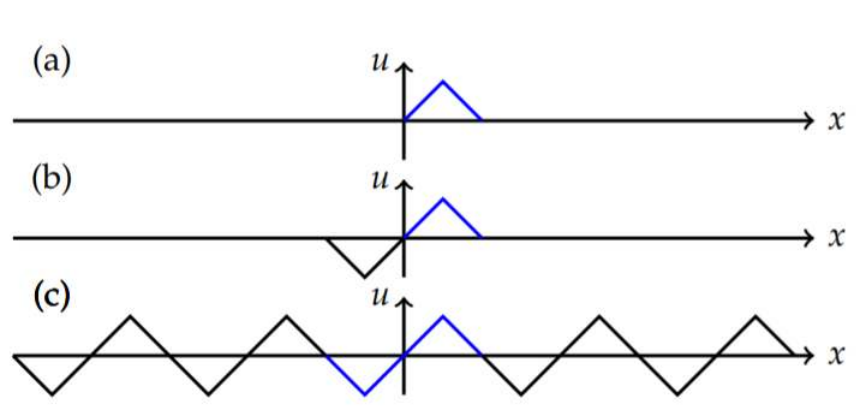


Figure 2.7.11: Construction of odd periodic extension for (a) The initial profile, $f(x)$. (b) Make $f(x)$ an odd function on $[-\ell, \ell]$. (c) Make the odd function periodic with period 2ℓ.

We can use the odd periodic function to construct solutions. In this case we use the result from the last example for obtaining the solution of the problem in which the initial velocity is zero, $u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)]$. Translations of the odd periodic extension are shown in Figure 2.7.12

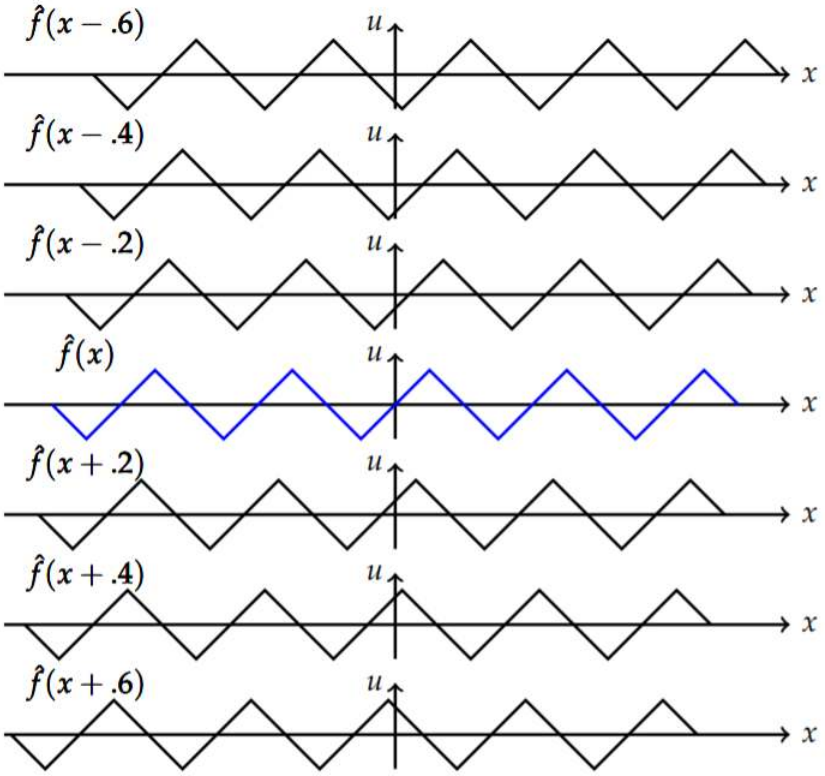


Figure 2.7.12: Translations of the odd periodic extension.

In Figure 2.7.13 we show superimposed plots of $f(x+ct)$ and $f(x-ct)$ for different values of ct. A box is shown inside which the physical wave can be constructed. The solution is an average of these odd periodic extensions within this box. This is displayed in Figure 2.7.14

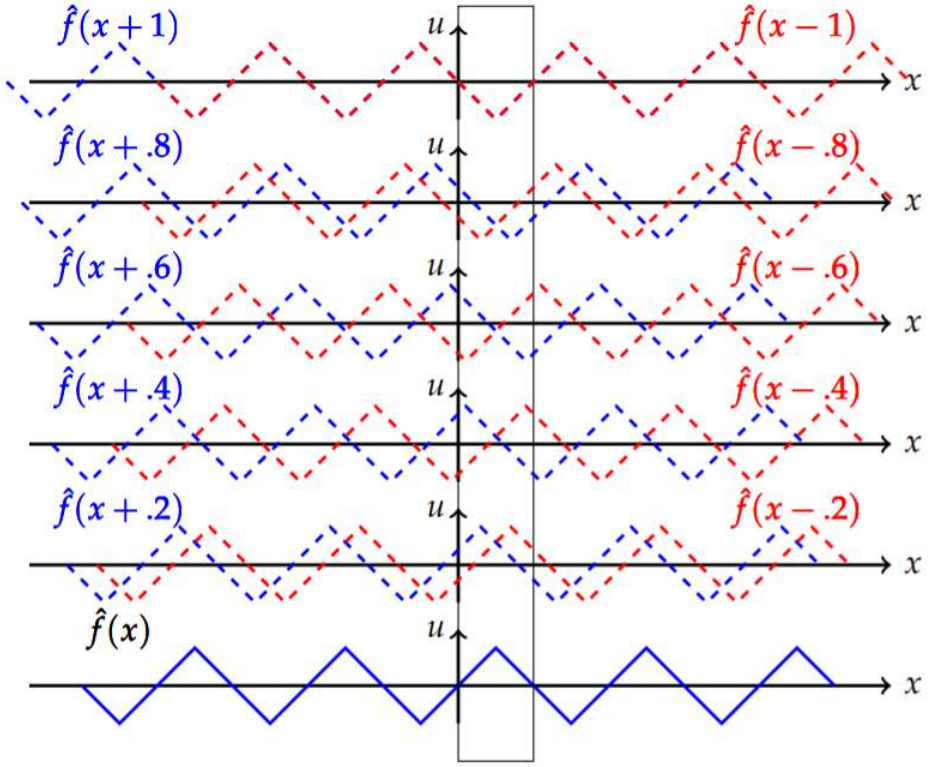


Figure 2.7.13: Superimposed translations of the odd periodic extension.

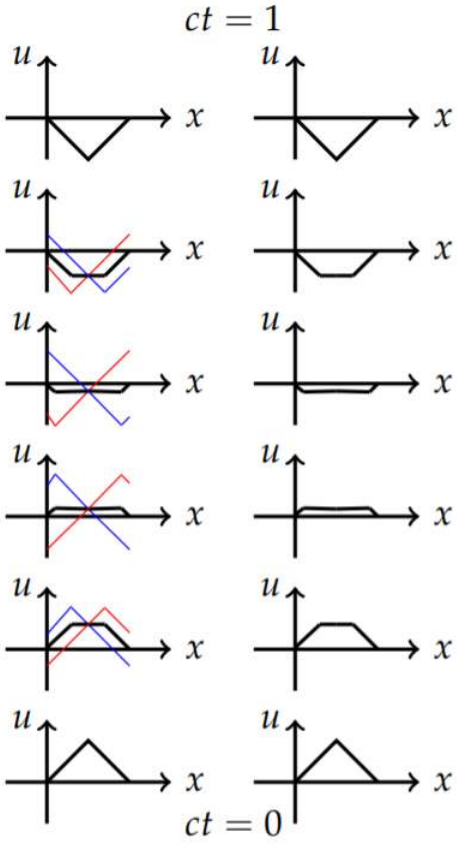


Figure 2.7.14: On the left is a plot of $f(x+ct)$, $f(x-ct)$ from Figure 2.7.13 and the average, $u(x,t)$. On the right the solution alone is shown for $ct = 0$ to $ct = 1$.

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2.8: Problems

Exercise 2.8.1

Solve the following initial value problems.

a. $x'' + x = 0, \quad x(0) = 2, \quad x'(0) = 0$.
b. $y'' + 2y' - 8y = 0, \quad y(0) = 1, \quad y'(0) = 2$.
c. $x^2y'' - 2xy' - 4y = 0, \quad y(1) = 1, \quad y'(1) = 0$.

Exercise 2.8.2

Solve the following boundary value problems directly, when possible.

a. $x'' + x = 2, \quad x(0) = 0, \quad x'(1) = 0$.
b. $y'' + 2y' - 8y = 0, \quad y(0) = 1, \quad y(1) = 0$.
c. $y'' + y = 0, \quad y(0) = 1, \quad y(\pi) = 0$.

Exercise 2.8.3

Consider the boundary value problem for the deflection of a horizontal beam fixed at one end,

$$\frac{d^4y}{dx^4} = C, \quad y(0) = 0, \quad y'(0) = 0, \quad y''(L) = 0, \quad y'''(L) = 0.$$

Solve this problem assuming that C is a constant.

Exercise 2.8.4

Find the product solutions, $u(x,t) = T(t)X(x)$, to the heat equation, $u_t - u_{xx} = 0$, on $[0, \pi]$ satisfying the boundary conditions $u_x(0,t) = 0$ and $u(\pi, t) = 0$.

Exercise 2.8.5

Find the product solutions, $u(x,t) = T(t)X(x)$, to the wave equation $u_{tt} = 2u_{xx}$, on $[0, 2\pi]$ satisfying the boundary conditions $u(0,t) = 0$ and $u_x(2\pi, t) = 0$.

Exercise 2.8.6

Find product solutions, $u(x,t) = X(x)Y(y)$, to Laplace's equation, $u_{xx} + u_{yy} = 0$, on the unit square satisfying the boundary conditions $u(0,y) = 0$, $u(1,y) = g(y)$, $u(x,0) = 0$, and $u(x,1) = 0$.

Exercise 2.8.7

Consider the following boundary value problems. Determine the eigenvalues, λ, and eigenfunctions, $y(x)$ for each problem.

a. $y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(1) = 0$.
b. $y'' - \lambda y = 0, \quad y(-\pi) = 0, \quad y'(\pi) = 0$.
c. $x^2y'' + xy' + \lambda y = 0, \quad y(1) = 0, \quad y(2) = 0$.
d. $(x^2y')' + \lambda y = 0, \quad y(1) = 0, \quad y'(e) = 0$.

Note

In problem d you will not get exact eigenvalues. Show that you obtain a transcendental equation for the eigenvalues in the form $\tan z = 2z$. Find the first three eigenvalues numerically.

Exercise 2.8.8

Classify the following equations as either hyperbolic, parabolic, or elliptic.

a. $u_{yy}+u_{xy}+u_{xx}=0$.
b. $3u_{xx}+2u_{xy}+5u_{yy}=0$.
c. $x^2u_{xx}+2xyu_{xy}+y^2u_{yy}=0$.
d. $y^2u_{xx}+2xyu_{xy}+(x^2+4x^4)u_{yy}=0$.

Exercise 2.8.9

Use d'Alembert's solution to prove

$$f(-\zeta)=f(\zeta), \quad g(-\zeta)=g(\zeta)$$

for the semi-infinite string satisfying the free end condition $u_x(0,t)=0$.

Exercise 2.8.10

Derive a solution similar to d'Alembert's solution for the equation $u_{tt}+2u_{xt}-3u=0$.

Exercise 2.8.11

Construct the appropriate periodic extension of the plucked string initial profile given by

$$f(x)=\begin{cases} x, & 0\le x\le \frac{\ell}{2},\\ \ell-x, & \frac{\ell}{2}\le x\le \ell, \end{cases}$$

satisfying the boundary conditions at $u(0,t)=0$ and $u_x(\ell,t)=0$ for $t>0$.

Exercise 2.8.12

Find and sketch the solution of the problem

$$\begin{aligned} u_{tt} &= u_{xx}, \quad 0\le x\le 1, \quad t>o\\ u(x,0) &= \begin{cases} 0, & 0\le x<\frac{1}{4},\\ 1, & \frac{1}{4}\le x\le\frac{3}{4},\\ 0, & \frac{3}{4}<x\le 1, \end{cases}\\ u_t(x,0) &= 0,\\ u(0,t) &= 0, \quad t>0,\\ u(1,t) &= 0, \quad t>0, \end{aligned}$$

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CHAPTER OVERVIEW

3: Trigonometric Fourier Series

> *"Ordinary language is totally unsuited for expressing what physics really asserts, since the words of everyday life are not sufficiently abstract. Only mathematics and mathematical logic can say as little as the physicist means to say." Bertrand Russell (1872-1970)*

3.1: Introduction to Fourier Series
3.2: Fourier Trigonometric Series
3.3: Fourier Series Over Other Intervals
3.4: Sine and Cosine Series
3.5: Solution of the Heat Equation
3.6: Finite Length Strings
3.7: The Gibbs Phenomenon
3.8: Problems

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3.1: Introduction to Fourier Series

We will now turn to the study of trigonometric series. You have seen that functions have series representations as expansions in powers of x, or $x-a$, in the form of Maclaurin and Taylor series. Recall that the Taylor series expansion is given by

$$f(x)=\sum_{n=0}^{\infty} c_n(x-a)^n,$$

where the expansion coefficients are determined as

$$c_n=\frac{f^{(n)}(a)}{n!}.$$

From the study of the heat equation and wave equation, we have found that there are infinite series expansions over other functions, such as sine functions. We now turn to such expansions and in the next chapter we will find out that expansions over special sets of functions are not uncommon in physics. But, first we turn to Fourier trigonometric series.

We will begin with the study of the Fourier trigonometric series expansion

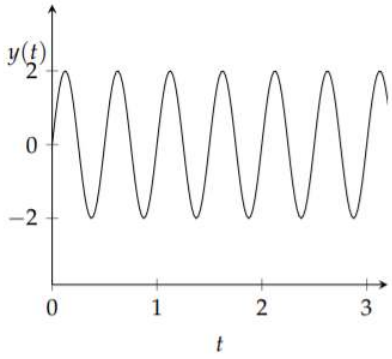
$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}+b_n \sin\frac{n\pi x}{L}.$$

We will find expressions useful for determining the Fourier coefficients $\{a_n, b_n\}$ given a function $f(x)$ defined on $[-L, L]$. We will also see if the resulting infinite series reproduces $f(x)$. However, we first begin with some basic ideas involving simple sums of sinusoidal functions.

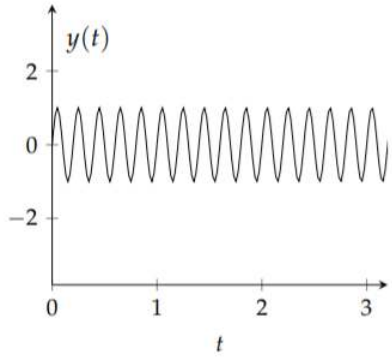
There is a natural appearance of such sums over sinusoidal functions in music. A pure note can be represented as

$$y(t)=A\sin(2\pi f t),$$

where A is the amplitude, f is the frequency in hertz (Hz), and t is time in seconds. The amplitude is related to the volume of the sound. The larger the amplitude, the louder the sound. In Figure 3.1.2 we show plots of two such tones with $f=2$ Hz in the top plot and $f=5$ Hz in the bottom one.



(a) $y(t) = 2\sin(4\pi f t)$



(b) $y(t) = \sin(10\pi f t)$

Figure 3.1.1: Plots of $y(t) = A\sin(2\pi f t)$ on $[0, 5]$ for $f = 2$ Hz and $f = 5$ Hz.

In these plots you should notice the difference due to the amplitudes and the frequencies. You can easily reproduce these plots and others in your favorite plotting utility.

As an aside, you should be cautious when plotting functions, or sampling data. The plots you get might not be what you expect, even for a simple sine function. In Figure 3.1.2 we show four plots of the function $y(t) = 2\sin(4\pi t)$. In the top left you see a proper rendering of this function. However, if you use a different number of points to plot this function, the results may be surprising. In this example we show what happens if you use $N = 200, 100, 101$ points instead of the 201 points used in the first plot. Such disparities are not only possible when plotting functions, but are also present when collecting data. Typically, when you sample a set of data, you only gather a finite amount of information at a fixed rate. This could happen when getting data on ocean wave heights, digitizing music and other audio to put on your computer, or any other process when you attempt to analyze a continuous signal.

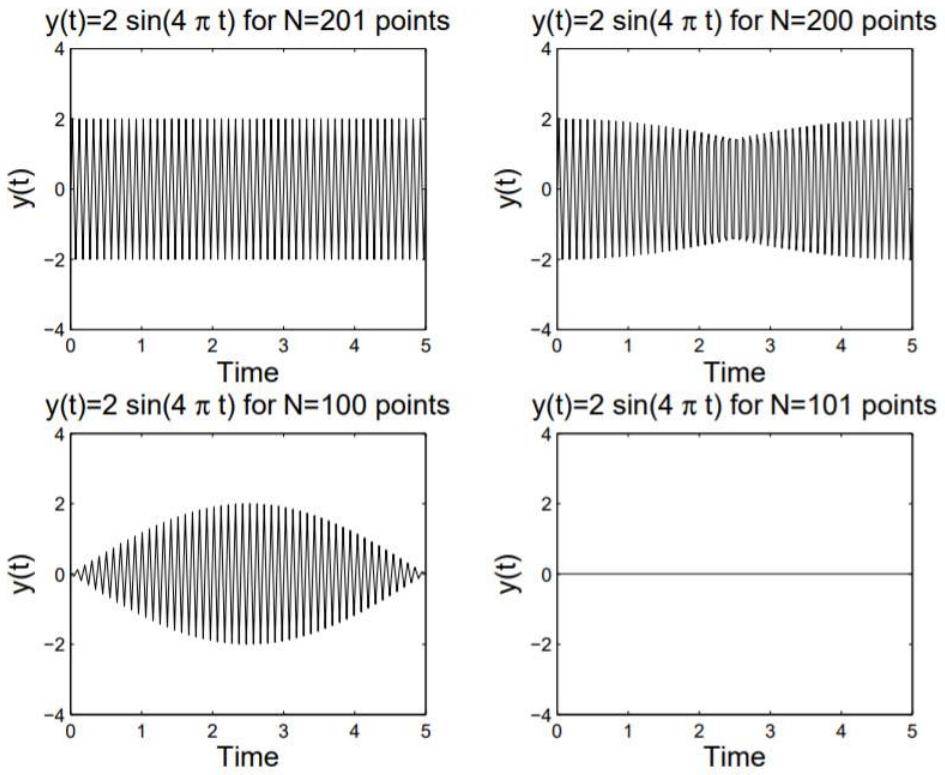
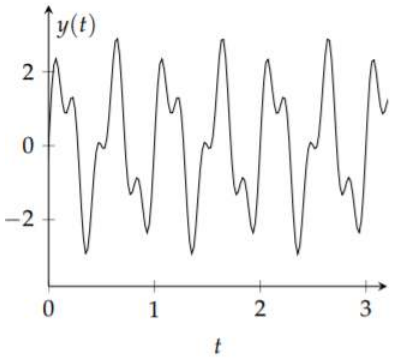
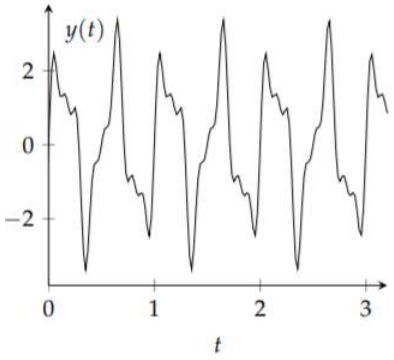


Figure 3.1.2: Problems can occur while plotting. Here we plot the function $y(t) = 2 \sin 4\pi t$ using $N = 201, 200, 100, 101$ points.

Next, we consider what happens when we add several pure tones. After all, most of the sounds that we hear are in fact a combination of pure tones with different amplitudes and frequencies. In Figure 3.1.3 we see what happens when we add several sinusoids. Note that as one adds more and more tones with different characteristics, the resulting signal gets more complicated. However, we still have a function of time. In this chapter we will ask, "Given a function $f(t)$, can we find a set of sinusoidal functions whose sum converges to $f(t)$?"



(a) Sum of signals with frequencies $f = 2$ Hz and $f = 5$ Hz.



(b) Sum of signals with frequencies $f = 2$ Hz, $f = 5$ Hz, and $f = 8$ Hz.

Figure 3.1.3: Superposition of several sinusoids.

Looking at the superpositions in Figure 3.1.3, we see that the sums yield functions that appear to be periodic. This is not to be unexpected. We recall that a periodic function is one in which the function values repeat over the domain of the function. The length of the smallest part of the domain which repeats is called the period. We can define this more precisely: A function is said to be periodic with period T if $f(t+T) = f(t)$ for all t and the smallest such positive number T is called the period.

For example, we consider the functions used in Figure 3.1.3. We began with $y(t) = 2\sin(4\pi t)$. Recall from your first studies of trigonometric functions that one can determine the period by dividing the coefficient of t into 2π to get the period. In this case we have

$$T = \frac{2\pi}{4\pi} = \frac{1}{2}.$$

Looking at the top plot in Figure 3.1.1 we can verify this result. (You can count the full number of cycles in the graph and divide this into the total time to get a more accurate value of the period.)

In general, if $y(t) = A\sin(2\pi f t)$, the period is found as

$$T = \frac{2\pi}{2\pi f} = \frac{1}{f}.$$

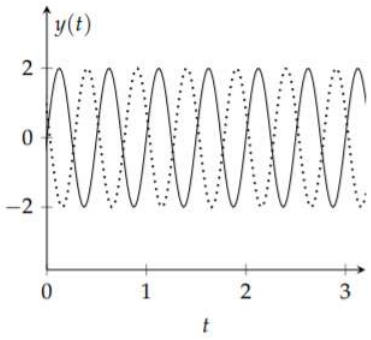
Of course, this result makes sense, as the unit of frequency, the hertz, is also defined as s^{-1}, or cycles per second.

Returning to Figure 3.1.3, the functions $y(t) = 2\sin(4\pi t)$, $y(t) = \sin(10\pi t)$, and $y(t) = 0.5\sin(16\pi t)$ have periods of 0.5s, 0.2s and 0.125s, respectively. Each superposition in Figure 3.1.3 retains a period that is the least common multiple of the periods of the signals added. For both plots, this is $1.0\text{s} = 2(0.5)\text{s} = 5(.2)\text{s} = 8(.125)\text{s}$.

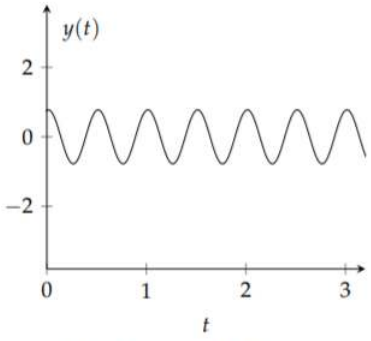
Our goal will be to start with a function and then determine the amplitudes of the simple sinusoids needed to sum to that function. We will see that this might involve an infinite number of such terms. Thus, we will be studying an infinite series of sinusoidal functions.

Secondly, we will find that using just sine functions will not be enough either. This is because we can add sinusoidal functions that do not necessarily peak at the same time. We will consider two signals that originate at different times. This is similar to when your music teacher would make sections of the class sing a song like "Row, Row, Row your Boat" starting at slightly different times.

We can easily add shifted sine functions. In Figure 3.1.4 we show the functions $y(t) = 2\sin(4\pi t)$ and $y(t) = 2\sin(4\pi t + 7\pi/8)$ and their sum. Note that this shifted sine function can be written as $y(t) = 2\sin(4\pi(t + 7/32))$. Thus, this corresponds to a time shift of $-7/32$.



(a) Plot of each function.



(b) Plot of the sum of the functions.

Figure 3.1.4: Plot of the functions $y(t) = 2\sin(4\pi t)$ and $y(t) = 2\sin(4\pi t + 7\pi/8)$ and their sum.

So, we should account for shifted sine functions in the general sum. Of course, we would then need to determine the unknown time shift as well as the amplitudes of the sinusoidal functions that make up the signal, $f(t)$. While this is one approach that some researchers use to analyze signals, there is a more common approach. This results from another reworking of the shifted function.

Note

We should note that the form in the lower plot of Figure 3.4 looks like a simple sinusoidal function for a reason. Let

$$y_1(t) = 2\sin(4\pi t),$$

$$y_2(t) = 2\sin(4\pi t + 7\pi/8).$$

Then,

$$\begin{aligned} y_1 + y_2 &= 2\sin(4\pi t + 7\pi/8) + 2\sin(4\pi t) \\ &= 2[\sin(4\pi t + 7\pi/8) + \sin(4\pi t)] \\ &= 4\cos\frac{7\pi}{16}\sin\left(4\pi t + \frac{7\pi}{16}\right). \end{aligned}$$

[1] Recall the identities (11.2.5)-(11.2.6)

$$\begin{aligned} \sin(x+y) &= \sin x\cos y + \sin y\cos x, \\ \cos(x+y) &= \cos x\cos y - \sin x\sin y. \end{aligned}$$

Consider the general shifted function

$$y(t) = A\sin(2\pi ft + \phi). \tag{3.1.1}$$

Note that $2\pi ft + \phi$ is called the **phase** of the sine function and ϕ is called the **phase shift**. We can use the trigonometric identity (A.17) for the sine of the sum of two angles[1] to obtain

$$\begin{aligned} y(t) &= A\sin(2\pi ft + \phi) \\ &= A\sin(\phi)\cos(2\pi ft) + A\cos(\phi)\sin(2\pi ft). \end{aligned} \tag{3.1.2}$$

We are now in a position to state our goal.

Fourier Analysis

Given a signal $f(t)$, we would like to determine its frequency content by finding out what combinations of sines and cosines of varying frequencies and amplitudes will sum to the given function. This is called Fourier Analysis.

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3.2: Fourier Trigonometric Series

As we have seen in the last section, we are interested in finding representations of functions in terms of sines and cosines. Given a function $f(x)$ we seek a representation in the form

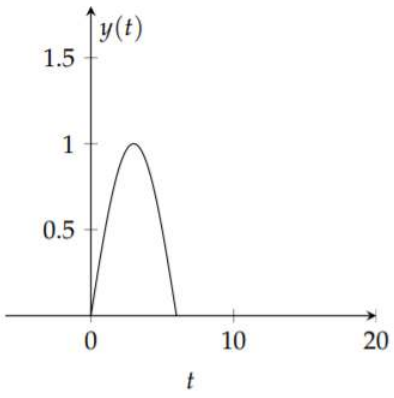
$$f(x)\ \frac{a_0}{2}+\sum_{n=1}^{\infty}[a_n\cos nx+b_n\sin nx]. \tag{3.2.1}$$

Notice that we have opted to drop the references to the time-frequency form of the phase. This will lead to a simpler discussion for now and one can always make the transformation $nx=2\pi f_n t$ when applying these ideas to applications.

The series representation in Equation (3.2.1) is called a **Fourier trigonometric series**. We will simply refer to this as a Fourier series for now. The set of constants $a_0,\ a_n,\ b_n,\ n=1,2,\ldots$ are called the **Fourier coefficient**s. The constant term is chosen in this form to make later computations simpler, though some other authors choose to write the constant term as a_0. Our goal is to find the Fourier series representation given $f(x)$. Having found the Fourier series representation, we will be interested in determining when the Fourier series converges and to what function it converges.

From our discussion in the last section, we see that The Fourier series is periodic. The periods of $\cos nx$ and $\sin nx$ are $\frac{2\pi}{n}$. Thus, the largest period, $T=2\pi$, comes from the $n=1$ terms and the Fourier series has period 2π. This means that the series should be able to represent functions that are periodic of period 2π.

While this appears restrictive, we could also consider functions that are defined over one period. In Figure 3.2.1 we show a function defined on $[0,2\pi]$. In the same figure, we show its periodic extension. These are just copies of the original function shifted by the period and glued together. The extension can now be represented by a Fourier series and restricting the Fourier series to $[0,2\pi]$ will give a representation of the original function. Therefore, we will first consider Fourier series representations of functions defined on this interval. Note that we could just as easily considered functions defined on $[-\pi,\pi]$ or any interval of length 2π. We will consider more general intervals later in the chapter.



(a) Plot of function $f(t)$.

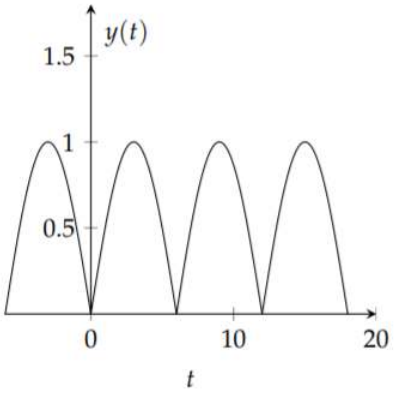


Figure 3.2.1: Plot of the function $f(t)$ defined on $[0,2\pi]$ and its periodic extension.

Theorem 3.2.1: Fourier Coefficients

The Fourier series representation of $f(x)$ defined on $[0, 2\pi]$, when it exists, is given by (3.2.1) with Fourier coefficients

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx, \quad n = 0, 1, 2, \ldots, \\ b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx, \quad n = 1, 2, \ldots \end{aligned} \tag{3.2.2}$$

These expressions for the Fourier coefficients are obtained by considering special integrations of the Fourier series. We will now derive the a_n integrals in (3.2.2).

We begin with the computation of a_0. Integrating the Fourier series term by term in Equation (3.2.1), we have

$$\int_0^{2\pi} f(x)dx = \int_0^{2\pi} \frac{a_0}{2}dx + \int_0^{2\pi}\sum_{n=1}^{\infty}[a_n\cos nx + b_n\sin nx]dx. \tag{3.2.3}$$

Note

Evaluating the integral of an infinite series by integrating term by term depends on the convergence properties of the series.

We will assume that we can integrate the infinite sum term by term. Then we will need to compute

$$\begin{aligned} \int_0^{2\pi} \frac{a_0}{2}dx &= \frac{a_0}{2}(2\pi) = \pi a_0, \\ \int_0^{2\pi} \cos nx\,dx &= \left[\frac{\sin nx}{n}\right]_0^{2\pi} = 0, \\ \int_0^{2\pi} \sin nx\,dx &= \left[\frac{-\cos nx}{n}\right]_0^{2\pi} = 0. \end{aligned} \tag{3.2.4}$$

Note

From these results we see that only one term in the integrated sum does not vanish leaving

$$\int_0^{2\pi} f(x)dx = \pi a_0.$$

This confirms the value for a_0.[2]

Next, we will find the expression for a_n. We multiply the Fourier series (3.2.1) by $\cos mx$ for some positive integer m. This is like multiplying by $\cos 2x$, $\cos 5x$, etc. We are multiplying by all possible $\cos mx$ functions for different integers m all at the same time. We will see that this will allow us to solve for the a_n's.

We find the integrated sum of the series times $\cos mx$ is given by

$$\int_0^{2\pi} f(x)\cos mx\,dx = \int_0^{2\pi} \frac{a_0}{2}\cos mx\,dx + \int_0^{2\pi}\sum_{n=1}^{\infty}[a_n\cos nx + b_n\sin nx]\cos mx\,dx. \tag{3.2.5}$$

Integrating term by term, the right side becomes

$$\int_0^{2\pi} f(x)\cos mx\,dx = \frac{a_0}{2}\int_0^{2\pi}\cos mx\,dx + \sum_{n=1}^{\infty}\left[a_n\int_0^{2\pi}\cos nx\cos mx\,dx + b_n\int_0^{2\pi}\sin nx\cos mx\,dx\right]. \tag{3.2.6}$$

We have already established that $\int_0^{2\pi}\cos mx\,dx = 0$, which implies that the first term vanishes.

Next we need to compute integrals of products of sines and cosines. This requires that we make use of some of the trigonometric identities listed in Chapter 1. For quick reference, we list these here.

Useful Trigonometric Identities

$$\sin(x \pm y) = \sin x \cos y \pm \sin y \cos x \tag{3.2.7}$$
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y \tag{3.2.8}$$
$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \tag{3.2.9}$$
$$\cos^2 x = \frac{1}{2}(1 + \cos 2x) \tag{3.2.10}$$
$$\sin x \sin y = \frac{1}{2}(\cos(x - y) - \cos(x + y)) \tag{3.2.11}$$
$$\cos x \cos y = \frac{1}{2}(\cos(x + y) + \cos(x - y)) \tag{3.2.12}$$
$$\sin x \cos y = \frac{1}{2}(\sin(x + y) + \sin(x - y)) \tag{3.2.13}$$

We first want to evaluate $\int_0^{2\pi} \cos nx \cos mx dx$. We do this by using the product identity (3.2.12). We have

$$\begin{aligned} \int_0^{2\pi} \cos nx \cos mx dx &= \frac{1}{2}\int_0^{2\pi} [\cos(m+n)x + \cos(m-n)x] dx \\ &= \frac{1}{2}\left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right]_0^{2\pi} \\ &= 0. \end{aligned} \tag{3.2.14}$$

There is one caveat when doing such integrals. What if one of the denominators $m \pm n$ vanishes? For this problem $m + n \neq 0$, since both m and n are positive integers. However, it is possible for $m = n$. This means that the vanishing of the integral can only happen when $m \neq n$. So, what can we do about the $m = n$ case? One way is to start from scratch with our integration. (Another way is to compute the limit as n approaches m in our result and use L'Hopital's Rule. Try it!)

For $n = m$ we have to compute $\int_0^{2\pi} \cos^2 mx dx$. This can also be handled using a trigonometric identity. Using the half angle formula, (3.2.10), with $\theta = mx$, we find

$$\begin{aligned} \int_0^{2\pi} \cos^2 mx dx &= \frac{1}{2}\int_0^{2\pi} (1 + \cos 2mx) dx \\ &= \frac{1}{2}\left[x + \frac{1}{2m}\sin 2mx\right]_0^{2\pi} \\ &= \frac{1}{2}(2\pi) = \pi. \end{aligned} \tag{3.2.15}$$

To summarize, we have shown that

$$\int_0^{2\pi} \cos nx \cos mx dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n. \end{cases} \tag{3.2.16}$$

This holds true for $m, n = 0, 1, \ldots$. [Why did we include $m, n = 0$?] When we have such a set of functions, they are said to be an orthogonal set over the integration interval. A set of (real) functions $\{\phi_n(x)\}$ is said to be orthogonal on $[a, b]$ if $\int_a^b \phi_n(x)\phi_m(x)dx = 0$ when $n \neq m$. Furthermore, if we also have that $\int_a^b \phi_n^2(x)dx = 1$, these functions are called orthonormal.

The set of functions $\{cosnx\}_{n=0}^{\infty}$ are orthogonal on $[0, 2\pi]$. Actually, they are orthogonal on any interval of length 2π. We can make them orthonormal by dividing each function by $\sqrt{\pi}$ as indicated by Equation (3.2.15). This is sometimes referred to normalization of the set of functions.

The notion of orthogonality is actually a generalization of the orthogonality of vectors in finite dimensional vector spaces. The integral $\int_a^b f(x)f(x)dx$ is the generalization of the dot product, and is called the scalar product of $f(x)$ and $g(x)$, which are thought of as vectors in an infinite dimensional vector space spanned by a set of orthogonal functions. We will return to these ideas in the next chapter.

Returning to the integrals in equation (3.2.6), we still have to evaluate $\int_0^{2\pi} \sin nx \cos mx dx$. We can use the trigonometric identity involving products of sines and cosines, (3.2.13). Setting $A = nx$ and $B = mx$, we find that

$$\begin{aligned}\int_0^{2\pi} \sin nx \cos mx dx &= \frac{1}{2}\int_0^{2\pi} [\sin(n+m)x + \sin(n-m)x]dx \\ &= \frac{1}{2}\left[\frac{-\cos(n+m)x}{n+m} + \frac{-\cos(n-m)x}{n-m}\right]_0^{2\pi} \\ &= (-1+1)+(-1+1) = 0. \end{aligned} \tag{3.2.17}$$

So,

$$\int_0^{2\pi} \sin nx \cos mx dx = 0. \tag{3.2.18}$$

For these integrals we also should be careful about setting $n = m$. In this special case, we have the integrals

$$\int_0^{2\pi} \sin mx \cos mx dx = \frac{1}{2}\int_0^{2\pi} \sin 2mx dx = \frac{1}{2}\left[\frac{-\cos 2mx}{2m}\right]_0^{2\pi} = 0.$$

Finally, we can finish evaluating the expression in Equation (3.2.6). We have determined that all but one integral vanishes. In that case, $n = m$. This leaves us with

$$\int_0^{2\pi} f(x)\cos mx dx = a_m \pi.$$

Solving for a_m gives

$$a_m = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos mx dx.$$

Since this is true for all $m = 1, 2, \ldots$, we have proven this part of the theorem. The only part left is finding the b_n's This will be left as an exercise for the reader.

We now consider examples of finding Fourier coefficients for given functions. In all of these cases we define $f(x)$ on $[0, 2\pi]$.

Example 3.2.1

$f(x) = 3\cos 2x,\ x \in [0, 2\pi]$.

Solution

We first compute the integrals for the Fourier coefficients.

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_0^{2\pi} 3\cos 2x dx = 0. \\ a_n &= \frac{1}{\pi}\int_0^{2\pi} 3\cos 2x \cos nx dx = 0, \quad n \neq 2. \\ a_2 &= \frac{1}{\pi}\int_0^{2\pi} 3\cos^2 2x dx = 3, \\ b_n &= \frac{1}{\pi}\int_0^{2\pi} 3\cos 2x \sin nx dx = 0, \quad \forall n. \end{aligned} \tag{3.2.19}$$

The integrals for $a_0, a_n,\ n \neq 2$, and b_n are the result of orthogonality. For a_2, the integral can be computed as follows:

$$\begin{aligned} a_2 &= \frac{1}{\pi}\int_0^{2\pi} 3\cos^2 2x\,dx \\ &= \frac{3}{2\pi}\int_0^{2\pi}[1+\cos 4x]dx \\ &= \frac{3}{2\pi}\left[x + \underbrace{\frac{1}{4}\sin 4x}_{\text{This term vanishes!}}\right]_0^{2\pi} = 3. \end{aligned} \tag{3.2.20}$$

Therefore, we have that the only nonvanishing coefficient is $a_2 = 3$. So there is one term and $f(x) = 3\cos 2x$.

Well, we should have known the answer to the last example before doing all of those integrals. If we have a function expressed simply in terms of sums of simple sines and cosines, then it should be easy to write down the Fourier coefficients without much work. This is seen by writing out the Fourier series,

$$\begin{aligned} f(x) &\sim \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n\cos nx + b_n \sin nx]. \\ &= \frac{a_0}{2} + a_1\cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots \end{aligned} \tag{3.2.21}$$

For the last problem, $f(x) = 3\cos 2x$. Comparing this to the expanded Fourier series, one can immediately read off the Fourier coefficients without doing any integration. In the next example we emphasize this point.

Example 3.2.2

$f(x) = \sin^2 x,\ x \in [0, 2\pi]$.

Solution

We could determine the Fourier coefficients by integrating as in the last example. However, it is easier to use trigonometric identities. We know that

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) = \frac{1}{2} - \frac{1}{2}\cos 2x.$$

There are no sine terms, so $b_n = 0,\ n = 1, 2, \ldots$. There is a constant term, implying $a_0/2 = 1/2$. So, $a_0 = 1$. There is a $\cos 2x$ term, corresponding to $n = 2$, so $a_2 = -\frac{1}{2}$. That leaves $a_n = 0$ for $n \neq 0, 2$. So, $a_0 = 1,\ a_2 = -\frac{1}{2}$, and all other Fourier coefficients vanish.

Example 3.2.3

$$f(x) = \begin{cases} 1, & 0 < x < \pi, \\ -1, & \pi < x < 2\pi, \end{cases}$$

Solution

This example will take a little more work. We cannot bypass evaluating any integrals this time. As seen in Figure 3.2.2, this function is discontinuous. So, we will break up any integration into two integrals, one over $[0, \pi]$ and the other over $[\pi, 2\pi]$.

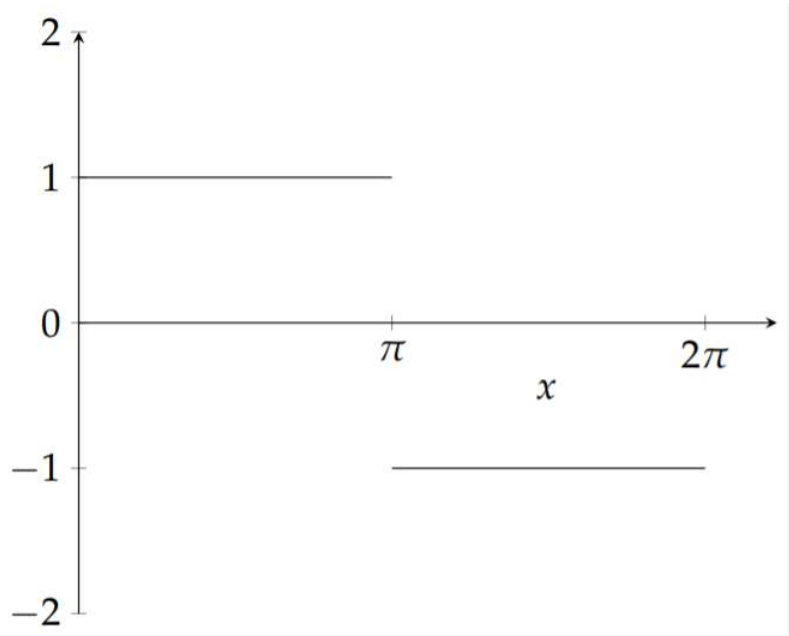


Figure 3.2.2: Plot of discontinuous function in Example 3.2.3.

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_0^{2\pi} f(x)dx \\ &= \frac{1}{\pi}\int_0^{\pi} dx + \frac{1}{\pi}\int_{\pi}^{2\pi}(-1)dx \\ &= \frac{1}{\pi}(\pi) + \frac{1}{\pi}(-2\pi+\pi) = 0. \end{aligned} \tag{3.2.22}$$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx \\ &= \frac{1}{\pi}\left[\int_0^{\pi}\cos nx dx - \int_{\pi}^{2\pi}\cos nx dx\right] \\ &= \frac{1}{\pi}\left[\left(\frac{1}{n}\sin nx\right)_0^{\pi} - \left(\frac{1}{n}\sin nx\right)_{\pi}^{2\pi}\right] \\ &= 0. \end{aligned} \tag{3.2.23}$$

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx \\ &= \frac{1}{\pi}\left[\int_0^{\pi}\sin nx dx - \int_{\pi}^{2\pi}\sin nx dx\right] \\ &= \frac{1}{\pi}\left[\left(-\frac{1}{n}\cos nx\right)_0^{\pi} + \left(\frac{1}{n}\cos nx\right)_{\pi}^{2\pi}\right] \\ &= \frac{1}{\pi}\left[-\frac{1}{n}\cos n\pi + \frac{1}{n} + \frac{1}{n} - \frac{1}{n}\cos n\pi\right] \\ &= \frac{2}{n\pi}(1-\cos n\pi). \end{aligned} \tag{3.2.24}$$

Note

Often we see expressions involving $\cos n\pi = (-1)^n$ and $1 \pm \cos n\pi = 1 \pm (-1)^n$. This is an example showing how to re-index series containing $\cos n\pi$.

We have found the Fourier coefficients for this function. Before inserting them into the Fourier series (3.2.1), we note that $\cos n\pi = (-1)^n$. Therefore,

$$1-\cos n\pi = \begin{cases} 0, & n \text{ even} \\ 2, & n \text{ odd.} \end{cases} \tag{3.2.25}$$

So, half of the b_n's are zero. While we could write the Fourier series representation as

$$f(x) \sim \frac{4}{\pi}\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n}\sin nx,$$

we could let $n = 2k-1$ in order to capture the odd numbers only. The answer can be written as

$$f(x) = \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{\sin(2k-1)x}{2k-1}.$$

Having determined the Fourier representation of a given function, we would like to know if the infinite series can be summed; i.e., does the series converge? Does it converge to $f(x)$? We will discuss this question later in the chapter after we generalize the Fourier series to intervals other than for $x \in [0, 2\pi]$.

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3.3: Fourier Series Over Other Intervals

In many applications we are interested in determining Fourier series representations of functions defined on intervals other than $[0, 2\pi]$. In this section we will determine the form of the series expansion and the Fourier coefficients in these cases.

The most general type of interval is given as $[a, b]$. However, this often is too general. More common intervals are of the form $[-\pi, \pi]$, $[0, L]$,or $[-L/2, L/2]$. The simplest generalization is to the interval $[0, L]$. Such intervals arise often in applications. For example, for the problem of a one dimensional string of length L we set up the axes with the left end at $x = 0$ and the right end at $x = L$. Similarly for the temperature distribution along a one dimensional rod of length L we set the interval to $x \in [0, 2\pi]$. Such problems naturally lead to the study of Fourier series on intervals of length L. We will see later that symmetric intervals, $[-a, a]$, are also useful.

Given an interval $[0, L]$, we could apply a transformation to an interval of length 2π by simply rescaling the interval. Then we could apply this transformation to the Fourier series representation to obtain an equivalent one useful for functions defined on $[0, L]$.

We define $x \in [0, 2\pi]$ and $t \in [0, L]$. A linear transformation relating these intervals is simply $x = \frac{2\pi t}{L}$ as shown in Figure 3.3.1. So, $t = 0$ maps to $x = 0$ and $t = L$ maps to $x = 2\pi$. Furthermore, this transformation maps $f(x)$ to a new function $g(t) = f(x(t))$, which is defined on $[0, L]$. We will determine the Fourier series representation of this function using the representation for $f(x)$ from the last section.

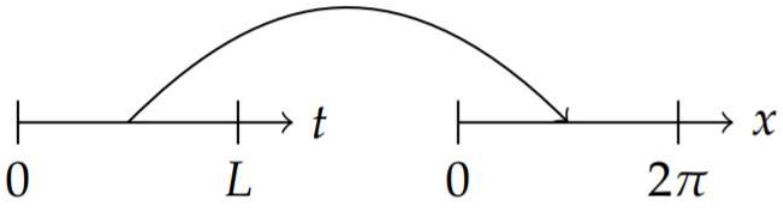


Figure 3.3.1: A sketch of the transformation between intervals $x \in [0, 2\pi]$ and $t \in [0, L]$.

Recall the form of the Fourier representation for $f(x)$ in Equation (3.2.1):

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]. \tag{3.3.1}$$

Inserting the transformation relating x and t, we have

$$g(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{2n\pi t}{L} + b_n \sin \frac{2n\pi t}{L} \right]. \tag{3.3.2}$$

This gives the form of the series expansion for $g(t)$ with $t \in [0, L]$. But, we still need to determine the Fourier coefficients.

Recall, that

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx dx.$$

We need to make a substitution in the integral of $x = \frac{2\pi t}{L}$. We also will need to transform the differential, $dx = \frac{2\pi}{L} dt$. Thus, the resulting form for the Fourier coefficients is

$$a_n = \frac{2}{L} \int_0^{L} g(t) \cos \frac{2n\pi t}{L} dt. \tag{3.3.3}$$

Similarly, we find that

$$b_n = \frac{2}{L} \int_0^{L} g(t) \sin \frac{2n\pi t}{L} dt. \tag{3.3.4}$$

We note first that when $L = 2\pi$ we get back the series representation that we first studied. Also, the period of $\cos \frac{2n\pi t}{L}$ is L/n, which means that the representation for $g(t)$ has a period of L corresponding to $n = 1$.

At the end of this section we present the derivation of the Fourier series representation for a general interval for the interested reader. In Table 3.3.1 we summarize some commonly used Fourier series representations.

Table 3.3.1: Special Fourier Series Representations on Different Intervals

Fourier Series on $[0, L]$	
$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{2n\pi x}{L} + b_n \sin \frac{2n\pi x}{L} \right].$$	(3.3.5)
$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{2n\pi x}{L} dx. \quad n = 0, 1, 2, \ldots,$$ $$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{2n\pi x}{L} dx. \quad n = 1, 2, \ldots.$$	(3.3.6)
Fourier Series on $[-\frac{L}{2}, \frac{L}{2}]$	
$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{2n\pi x}{L} + b_n \sin \frac{2n\pi x}{L} \right].$$	(3.3.7)
$$a_n = \frac{2}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} f(x) \cos \frac{2n\pi x}{L} dx. \quad n = 0, 1, 2, \ldots,$$ $$b_n = \frac{2}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} f(x) \sin \frac{2n\pi x}{L} dx. \quad n = 1, 2, \ldots.$$	(3.3.8)
Fourier Series on $[-\pi, \pi]$	
$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx].$$	(3.3.9)
$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx. \quad n = 0, 1, 2, \ldots,$$ $$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx. \quad n = 1, 2, \ldots.$$	(3.3.10)

At this point we need to remind the reader about the integration of even and odd functions on symmetric intervals.

We first recall that $f(x)$ is an even function if $f(-x) = f(x)$ for all x. One can recognize even functions as they are symmetric with respect to the y-axis as shown in Figure 3.3.2.

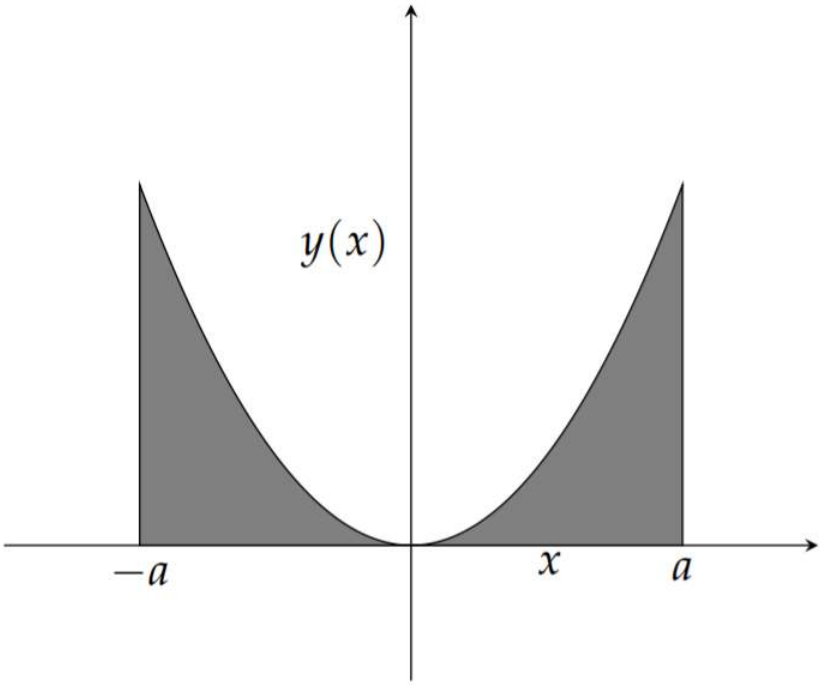


Figure 3.3.2: Area under an even function on a symmetric interval, $[-a, a]$.

If one integrates an even function over a symmetric interval, then one has that

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx. \tag{3.3.11}$$

One can prove this by splitting off the integration over negative values of x, using the substitution $x = -y$, and employing the evenness of $f(x)$. Thus,

$$\begin{aligned}\int_{-a}^{a} f(x)dx &= \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx \\ &= -\int_{a}^{0} f(-y)dy + \int_{0}^{a} f(x)dx \\ &= \int_{0}^{a} f(y)dy + \int_{0}^{a} f(x)dx \\ &= 2\int_{0}^{a} f(x)dx. \end{aligned} \tag{3.3.12}$$

This can be visually verified by looking at Figure 3.3.2.

A similar computation could be done for odd functions. $f(x)$ is an odd function if $f(-x) = -f(x)$ for all x. The graphs of such functions are symmetric with respect to the origin as shown in Figure 3.3.3. If one integrates an odd function over a symmetric interval, then one has that

$$\int_{-a}^{a} f(x)dx = 0. \tag{3.3.13}$$

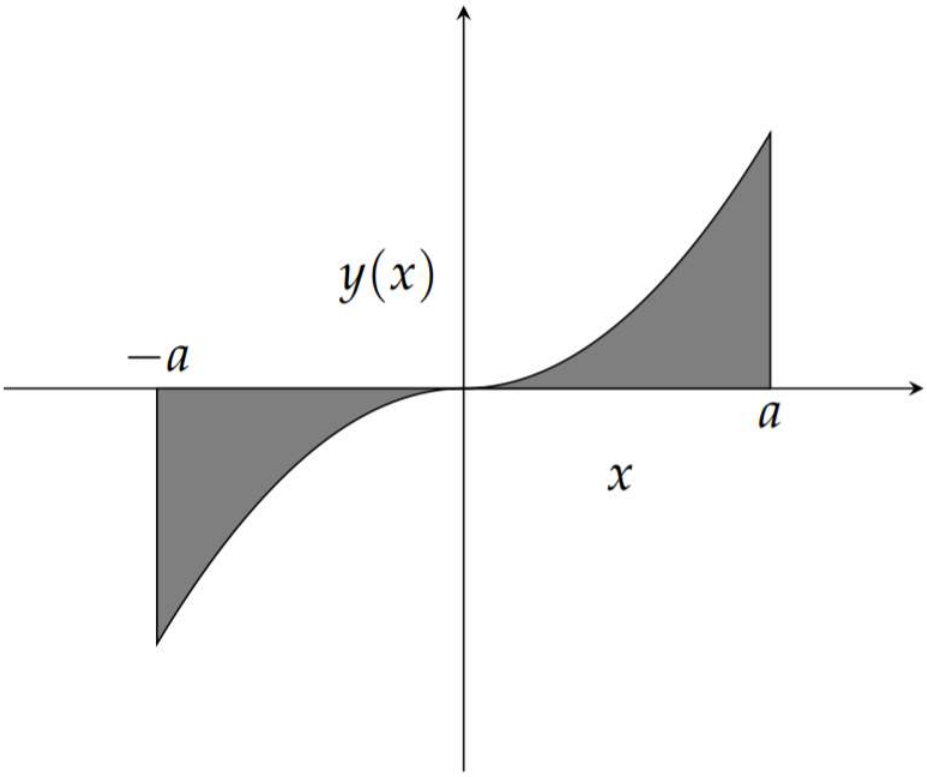


Figure 3.3.3: Area under an odd function on a symmetric interval, $[-a, a]$.

Example 3.3.1

Let $f(x) = |x|$ on $[-\pi, \pi]$. We compute the coefficients, beginning as usual with a_0. We have, using the fact that $|x|$ is an even function,

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} |x| dx \\ &= \frac{2}{\pi}\int_{0}^{\pi} x dx = \pi \end{aligned} \tag{3.3.14}$$

We continue with the computation of the general Fourier coefficients for $f(x) = |x|$ on $[-\pi, \pi]$. We have

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos nx dx = \frac{2}{\pi}\int_{0}^{\pi} x \cos nx dx. \tag{3.3.15}$$

Here we have made use of the fact that $|x| \cos nx$ is an even function.

In order to compute the resulting integral, we need to use integration by parts,

$$\int_a^b u dv = uv|_a^b - \int_a^b v du,$$

by letting $u = x$ and $dv = \cos nx dx$. Thus, $du = dx$ and $v = \int dv = \frac{1}{n}\sin nx$.

Continuing with the computation, we have

$$
\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^\pi x\cos nx\,dx.\\
&= \frac{2}{\pi}\left[\frac{1}{n}x\sin nx\Big|_0^\pi - \frac{1}{n}\int_0^\pi \sin nx\,dx\right]\\
&= -\frac{2}{n\pi}\left[-\frac{1}{n}\cos nx\right]_0^\pi\\
&= -\frac{2}{\pi n^2}(1-(-1)^n). \qquad (3.3.16)
\end{aligned}
$$

Here we have used the fact that $\cos n\pi = (-1)^n$ for any integer n. This leads to a factor $(1-(-1)^n)$. This factor can be simplified as

$$
1-(-1)^n = \begin{cases} 2, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}. \qquad (3.3.17)
$$

So, $a_n = 0$ for n even and $a_n = -\frac{4}{\pi n^2}$ for n odd.

Computing the b_n's is simpler. We note that we have to integrate $|x|\sin nx$ from $x=-\pi$ to π. The integrand is an odd function and this is a symmetric interval. So, the result is that $b_n = 0$ for all n.

Putting this all together, the Fourier series representation of $f(x) = |x|$ on $[-\pi,\pi]$ is given as

$$
f(x) \sim \frac{\pi}{2} - \frac{4}{\pi}\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty}\frac{\cos nx}{n^2}. \qquad (3.3.18)
$$

While this is correct, we can rewrite the sum over only odd n by reindexing. We let $n = 2k-1$ for $k = 1, 2, 3, \ldots$. Then we only get the odd integers. The series can then be written as

$$
f(x) \sim \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos(2k-1)x}{(2k-1)^2}. \qquad (3.3.19)
$$

Throughout our discussion we have referred to such results as Fourier representations. We have not looked at the convergence of these series. Here is an example of an infinite series of functions. What does this series sum to? We show in Figure 3.3.4 the first few partial sums. They appear to be converging to $f(x) = |x|$ fairly quickly.

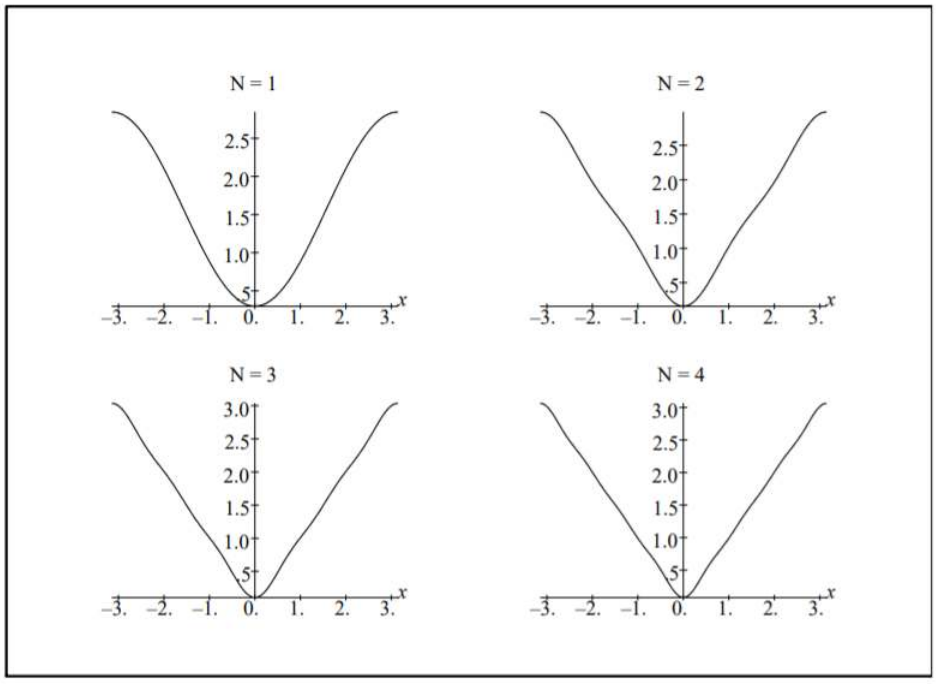


Figure 3.3.4: Plot of the first partial sums of the Fourier series representation for $f(x) = |x|$.

Even though $f(x)$ was defined on $[-\pi, \pi]$ we can still evaluate the Fourier series at values of x outside this interval. In Figure 3.3.5, we see that the representation agrees with $f(x)$ on the interval $[-\pi, \pi]$. Outside this interval we have a periodic extension of $f(x)$ with period 2π.

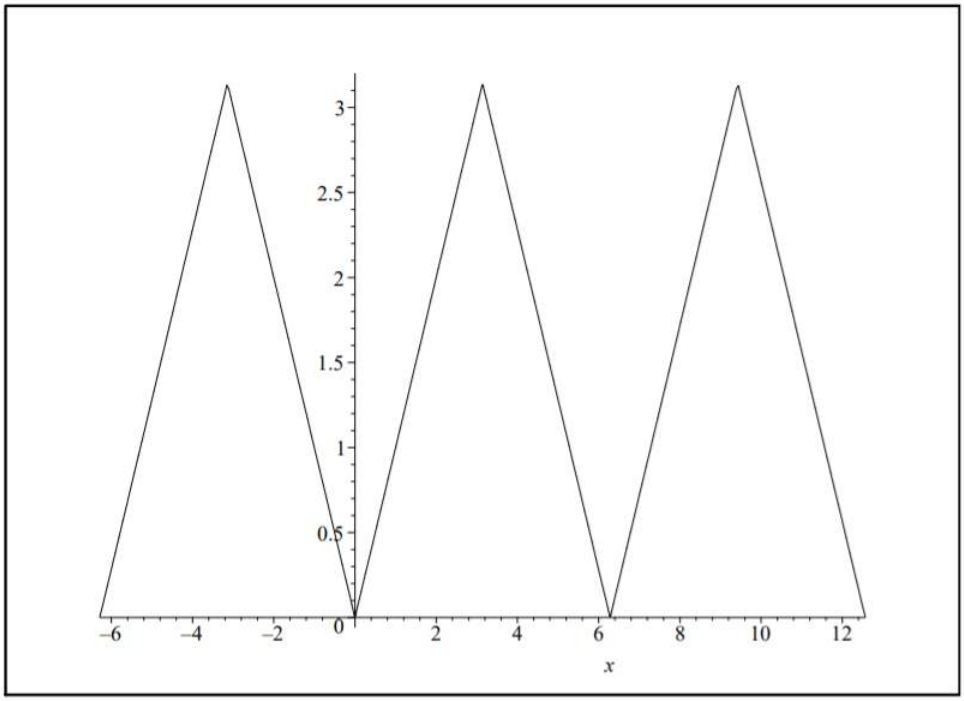


Figure 3.3.5: Plot of the first 10 terms of the Fourier series representation for $f(x) = |x|$ on the interval $[-2\pi, 4\pi]$.

Another example is the Fourier series representation of $f(x) = x$ on $[-\pi, \pi]$ as left for Problem 7. This is determined to be

$$f(x) \sim 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx. \tag{3.3.20}$$

As seen in Figure 3.3.6 we again obtain the periodic extension of the function. In this case we needed many more terms. Also, the vertical parts of the first plot are nonexistent. In the second plot we only plot the points and not the typical connected points that most software packages plot as the default style.

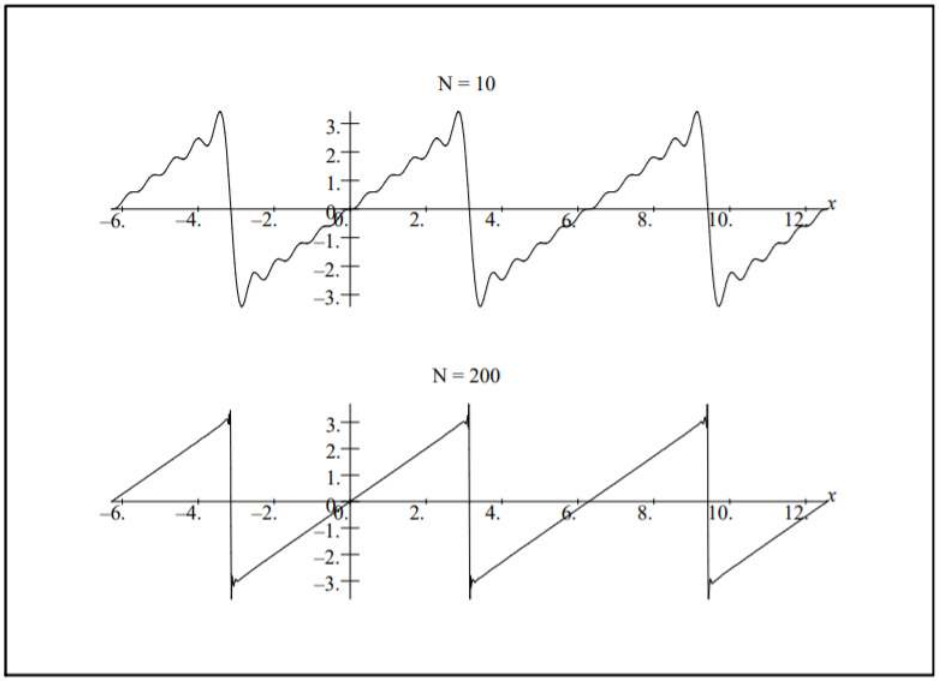


Figure 3.3.6: Plot of the first 10 terms and 200 terms of the Fourier series representation for $f(x) = x$ on the interval $[-2\pi, 4\pi]$.

Example 3.3.2

It is interesting to note that one can use Fourier series to obtain sums of some infinite series. For example, in the last example we found that

$$x \sim 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx.$$

Now, what if we chose $x = \frac{\pi}{2}$? Then, we have

$$\frac{\pi}{2} = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi}{2} = 2\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right].$$

This gives a well known expression for π:

$$\pi = 4\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right].$$

Fourier Series on $[a, b]$

A Fourier series representation is also possible for a general interval, $t \in [a, b]$. As before, we just need to transform this interval to [0, 2π]. Let

$$x = 2\pi \frac{t-a}{b-a}.$$

Inserting this into the Fourier series (3.2.1) representation for $f(x)$ we obtain

$$g(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{2n\pi(t-a)}{b-a} + b_n \sin \frac{2n\pi(t-a)}{b-a}\right]. \tag{3.3.21}$$

Well, this expansion is ugly. It is not like the last example, where the transformation was straightforward. If one were to apply the theory to applications, it might seem to make sense to just shift the data so that $a = 0$ and be done with any complicated expressions. However, some students enjoy the challenge of developing such generalized expressions. So, let's see what is involved.

First, we apply the addition identities for trigonometric functions and rearrange the terms.

$$\begin{aligned} g(t) \sim & \frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n \cos \frac{2n\pi(t-a)}{b-a}+b_n \sin \frac{2n\pi(t-a)}{b-a}\right] \\ = & \frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\left(\cos \frac{2n\pi t}{b-a}\cos \frac{2n\pi a}{b-a}+\sin \frac{2n\pi t}{b-a}\sin \frac{2n\pi a}{b-a}\right)\right. \\ & \left.+b_n\left(\sin \frac{2n\pi t}{b-a}\cos \frac{2n\pi a}{b-a}-\cos \frac{2n\pi t}{b-a}\sin \frac{2n\pi a}{b-a}\right)\right] \\ = & \frac{a_0}{2}+\sum_{n=1}^{\infty}\left[\cos \frac{2n\pi t}{b-a}\left(a_n \cos \frac{2n\pi a}{b-a}-b_n \sin \frac{2n\pi a}{b-a}\right)\right. \\ & \left.+\sin \frac{2n\pi t}{b-a}\left(a_n \sin \frac{2n\pi a}{b-a}+b_n \cos \frac{2n\pi a}{b-a}\right)\right]. \end{aligned} \tag{3.3.22}$$

Defining $A_0 = a_0$ and

$$\begin{aligned} A_n \equiv & a_n \cos \frac{2n\pi a}{b-a}+b_n \sin \frac{2n\pi a}{b-a} \\ B_n \equiv & a_n \sin \frac{2n\pi a}{b-a}+b_n \cos \frac{2n\pi a}{b-a}, \end{aligned} \tag{3.3.23}$$

we arrive at the more desirable form for the Fourier series representation of a function defined on the interval $[a, b]$.

$$g(t) \sim \frac{A_0}{2}+\sum_{n=1}^{\infty}\left[A_n \cos \frac{2n\pi t}{b-a}+B_n \sin \frac{2n\pi t}{b-a}\right]. \tag{3.3.24}$$

We next need to find expressions for the Fourier coefficients. We insert the known expressions for a_n and b_n and rearrange. First, we note that under the transformation $x = 2\pi\frac{t-a}{b-a}$ we have

$$\begin{aligned} a_n = & \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx \\ = & \frac{2}{b-a}\int_a^b g(t)\cos \frac{2n\pi(t-a)}{b-a}dt, \end{aligned} \tag{3.3.25}$$

and

$$\begin{aligned} b_n = & \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx \\ = & \frac{2}{b-a}\int_a^b g(t)\sin \frac{2n\pi(t-a)}{b-a}dt. \end{aligned} \tag{3.3.26}$$

Then, inserting these integrals in A_n, combining integrals and making use of the addition formula for the cosine of the sum of two angles, we obtain

$$\begin{aligned} A_n \equiv & a_n \cos \frac{2n\pi a}{b-a}-b_n \sin \frac{2n\pi a}{b-a} \\ = & \frac{2}{b-a}\int_a^b g(t)\left[\cos \frac{2n\pi(t-a)}{b-a}\cos \frac{2n\pi a}{b-a}-\sin \frac{2n\pi(t-a)}{b-a}\sin \frac{2n\pi a}{b-a}\right]dt \\ = & \frac{2}{b-a}\int_a^b g(t)\cos \frac{2n\pi t}{b-a}dt. \end{aligned} \tag{3.3.27}$$

A similar computation gives

$$B_n = \frac{2}{b-a}\int_a^b g(t)\sin \frac{2n\pi t}{b-a}dt. \tag{3.3.28}$$

Summarizing, we have shown that:

Theorem 3.3.1

The Fourier series representation of $f(x)$ defined on $[a,b]$ when it exists, is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{2n\pi x}{b-a} + b_n \sin \frac{2n\pi x}{b-a} \right]. \tag{3.3.29}$$

with fourier coefficients

$$\begin{aligned} a_n &= \frac{2}{b-a} \int_a^b f(x) \cos \frac{2n\pi x}{b-a} dx. \quad n = 0, 1, 2, \ldots, \\ b_n &= \frac{2}{b-a} \int_a^b f(x) \sin \frac{2n\pi x}{b-a} dx. \quad n = 1, 2, \ldots. \end{aligned} \tag{3.3.30}$$

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3.4: Sine and Cosine Series

In the last two examples ($f(x) = |x|$ and $f(x) = x$ on $[-\pi, \pi]$) we have seen Fourier series representations that contain only sine or cosine terms. As we know, the sine functions are odd functions and thus sum to odd functions. Similarly, cosine functions sum to even functions. Such occurrences happen often in practice. Fourier representations involving just sines are called **sine series** and those involving just cosines (and the constant term) are called **cosine series**.

Another interesting result, based upon these examples, is that the original functions, $|x|$ and x agree on the interval $[0, \pi]$. Note from Figures 3.3.4-3.3.6 that their Fourier series representations do as well. Thus, more than one series can be used to represent functions defined on finite intervals. All they need to do is to agree with the function over that particular interval. Sometimes one of these series is more useful because it has additional properties needed in the given application.

We have made the following observations from the previous examples:

1. There are several trigonometric series representations for a function defined on a finite interval.
2. Odd functions on a symmetric interval are represented by sine series and even functions on a symmetric interval are represented by cosine series.

These two observations are related and are the subject of this section. We begin by defining a function $f(x)$ on interval $[0, L]$. We have seen that the Fourier series representation of this function appears to converge to a periodic extension of the function.

In Figure 3.4.1 we show a function defined on $[0, 1]$. To the right is its periodic extension to the whole real axis. This representation has a period of $L = 1$. The bottom left plot is obtained by first reflecting f about the y axis to make it an even function and then graphing the periodic extension of this new function. Its period will be $2L = 2$. Finally, in the last plot we flip the function about each axis and graph the periodic extension of the new odd function. It will also have a period of $2L = 2$.

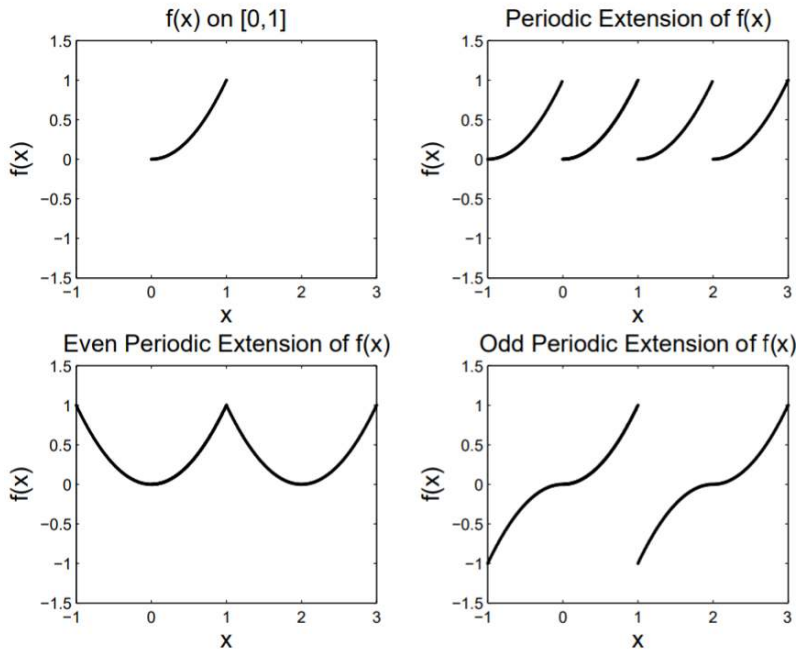


Figure 3.4.1: This is a sketch of a function and its various extensions. The original function $f(x)$ is defined on $[0, 1]$ and graphed in the upper left corner. To its right is the periodic extension, obtained by adding replicas. The two lower plots are obtained by first making the original function even or odd and then creating the periodic extensions of the new function.

In general, we obtain three different periodic representations. In order to distinguish these we will refer to them simply as the periodic, even and odd extensions. Now, starting with $f(x)$ defined on $[0, L]$, we would like to determine the Fourier series representations leading to these extensions. [For easy reference, the results are summarized in Table 3.4.1]

We have already seen from Table 3.3.1 that the periodic extension of $f(x)$, defined on $[0, L]$, is obtained through the Fourier series representation

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{2n\pi x}{L} + b_n \sin \frac{2n\pi x}{L} \right], \tag{3.4.1}$$

where

$$\begin{aligned} a_n &= \frac{2}{L} \int_0^L f(x) \cos \frac{2n\pi x}{L} dx. \quad n = 0, 1, 2, \ldots, \\ b_n &= \frac{2}{L} \int_0^L f(x) \sin \frac{2n\pi x}{L} dx. \quad n = 1, 2, \ldots \end{aligned} \tag{3.4.2}$$

Given $f(x)$ defined on $[0, L]$, the even periodic extension is obtained by simply computing the Fourier series representation for the even function

$$f_e(x) \equiv \begin{cases} f(x), & 0 < x < L, \\ f(-x) & -L < x < 0. \end{cases} \tag{3.4.3}$$

Since $f_e(x)$ is an even function on a symmetric interval $[-L, L]$, we expect that the resulting Fourier series will not contain sine terms. Therefore, the series expansion will be given by [Use the general case in (3.3.29) with $a = -L$ and $b = L$.] :

$$f_e(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}. \tag{3.4.4}$$

with Fourier coefficients

$$a_n = \frac{1}{L} \int_{-L}^{L} f_e(x) \cos \frac{n\pi x}{L} dx. \quad n = 0, 1, 2, \ldots. \tag{3.4.5}$$

However, we can simplify this by noting that the integrand is even and the interval of integration can be replaced by $[0, L]$. On this interval $f_e(x) = f(x)$. So, we have the Cosine Series Representation of $f(x)$ for $x \in [0, L]$ is given as

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}. \tag{3.4.6}$$

where

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx. \quad n = 0, 1, 2, \ldots \tag{3.4.7}$$

Table 3.4.1: Fourier Cosine and Sine Series Representations on $[0, L]$

Fourier Series on $[0, L]$
$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{2n\pi x}{L} + b_n \sin \frac{2n\pi x}{L} \right] \tag{3.4.8}$$
$$\begin{aligned} a_n &= \frac{2}{L} \int_0^L f(x) \cos \frac{2n\pi x}{L} dx. \quad n = 0, 1, 2, \ldots, \\ b_n &= \frac{2}{L} \int_0^L f(x) \sin \frac{2n\pi x}{L} dx. \quad n = 1, 2, \ldots \end{aligned} \tag{3.4.9}$$
Fourier Cosine Series on $[0, L]$
$$f(x) \sim a_0/2 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}. \tag{3.4.10}$$
where

Fourier Series on $[0, L]$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx. \quad n = 0, 1, 2, \ldots \tag{3.4.11}$$

Fourier Sine Series on $[0, L]$

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \tag{3.4.12}$$

where

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx. \quad n = 1, 2, \ldots \tag{3.4.13}$$

Similarly, given $f(x)$ defined on $[0, L]$, the odd periodic extension is obtained by simply computing the Fourier series representation for the odd function

$$f_o(x) \equiv \begin{cases} f(x), & 0 < x < L, \\ -f(-x) & -L < x < 0. \end{cases} \tag{3.4.14}$$

The resulting series expansion leads to defining the Sine Series Representation of $f(x)$ for $x \in [0, L]$ as

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}. \tag{3.4.15}$$

where

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx. \quad n = 1, 2, \ldots. \tag{3.4.16}$$

Example 3.4.1

In Figure 3.4.1 we actually provided plots of the various extensions of the function $f(x) = x^2$ for $x \in [0, 1]$. Let's determine the representations of the periodic, even and odd extensions of this function.

For a change, we will use a CAS (Computer Algebra System) package to do the integrals. In this case we can use Maple. A general code for doing this for the periodic extension is shown in Table 3.4.2.

Example 3.4.2: Periodic Extension - Trigonometric Fourier Series

Using the code in Table 3.4.2, we have that $a_0 = \frac{2}{3}$, $a_n = \frac{1}{n^2\pi^2}$, and $b_n = -\frac{1}{n\pi}$. Thus, the resulting series is given as

$$f(x) \sim \frac{1}{3} + \sum_{n=1}^{\infty}\left[\frac{1}{n^2\pi^2}\cos 2n\pi x - \frac{1}{n\pi}\sin 2n\pi x\right].$$

In Figure 3.4.2 we see the sum of the first 50 terms of this series. Generally, we see that the series seems to be converging to the periodic extension of f. There appear to be some problems with the convergence around integer values of x. We will later see that this is because of the discontinuities in the periodic extension and the resulting overshoot is referred to as the Gibbs phenomenon which is discussed in the last section of this chapter.

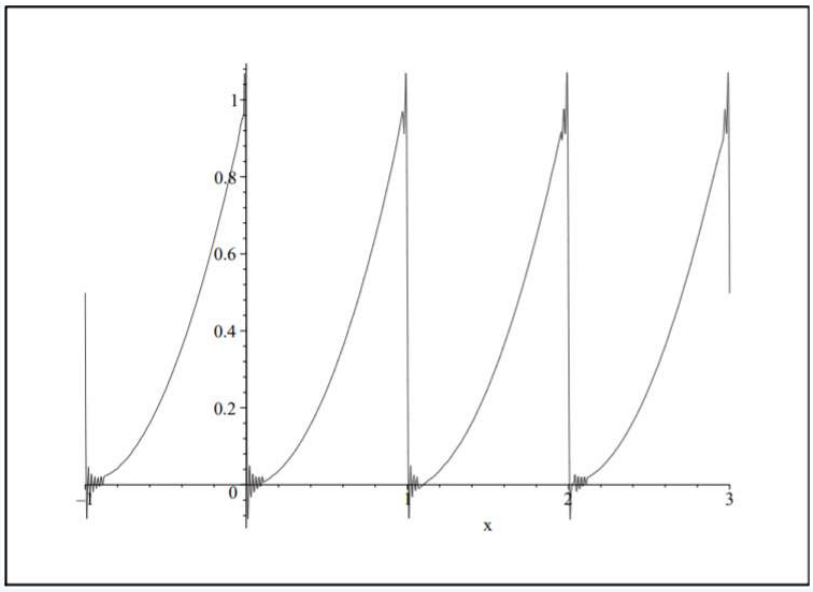


Figure 3.4.2: The periodic extension of $f(x) = x^2$ on $[0, 1]$.

Table 3.4.2: Maple code for computing Fourier coefficients and plotting partial sums of the Fourier series.

```
> restart:
> L:=1:
> f:=x^2:
> assume(n,integer):
>a0:=2/L*int(f,x=0..L);
                                a0 := 2/3
> an:=2/L*int(f*cos(2*n*Pi*x/L),x=0..L);
                                     1
                               an := --------
                                     2    2
> bn:=2/L*int(f*sin(2*n*Pi*x/L),x=0..L);
                                     1
                               bn := --------
                                     n- Pi
> F:=a0/2+sum((1/(k*Pi)^2)*cos(2*k*Pi*x/L)
     -1/(k*Pi)*sin(2*k*Pi*x/L),k=1..50):
> plot(F,x=-1..3,title='Periodic Extension',
      titlefont=[TIMES,ROMAN,14],font=[TIMES,ROMAN,14]);
```

Example 3.4.3: Even Periodic Extension - Cosine Series

In this case we compute $a_0 = \frac{2}{3}$ and $a_n = \frac{4(-1)^n}{n^2\pi^2}$. Therefore, we have

$$f(x) \sim \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\pi x$$

In Figure 3.4.3 we see the sum of the first 50 terms of this series. In this case the convergence seems to be much better than in the periodic extension case. We also see that it is converging to the even extension.

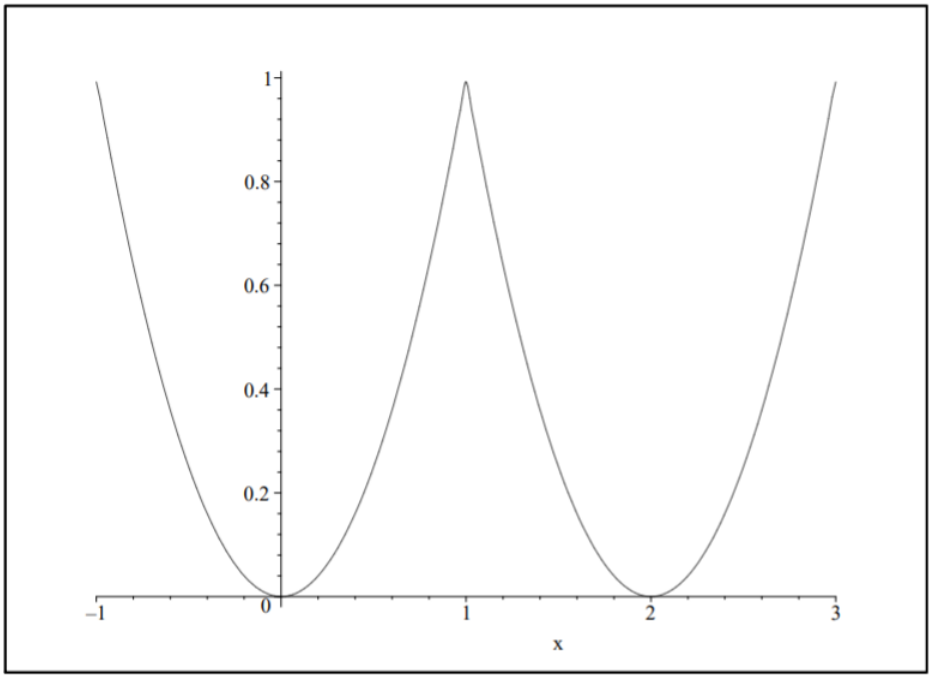


Figure 3.4.3: The even periodic extension of $f(x) = x^2$ on $[0, 1]$

Example 3.4.4: Odd Periodic Extension - Sine Series

Finally, we look at the sine series for this function. We find that

$$b_n = -\frac{2}{n^3\pi^3}\left(n^2\pi^2(-1)^n - 2(-1)^n + 2\right).$$

Therefore,

$$f(x) \sim -\frac{2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}\left(n^2\pi^2(-1)^n - 2(-1)^n + 2\right)\sin n\pi x.$$

Once again we see discontinuities in the extension as seen in Figure 3.4.4. However, we have verified that our sine series appears to be converging to the odd extension as we first sketched in Figure 3.4.1.

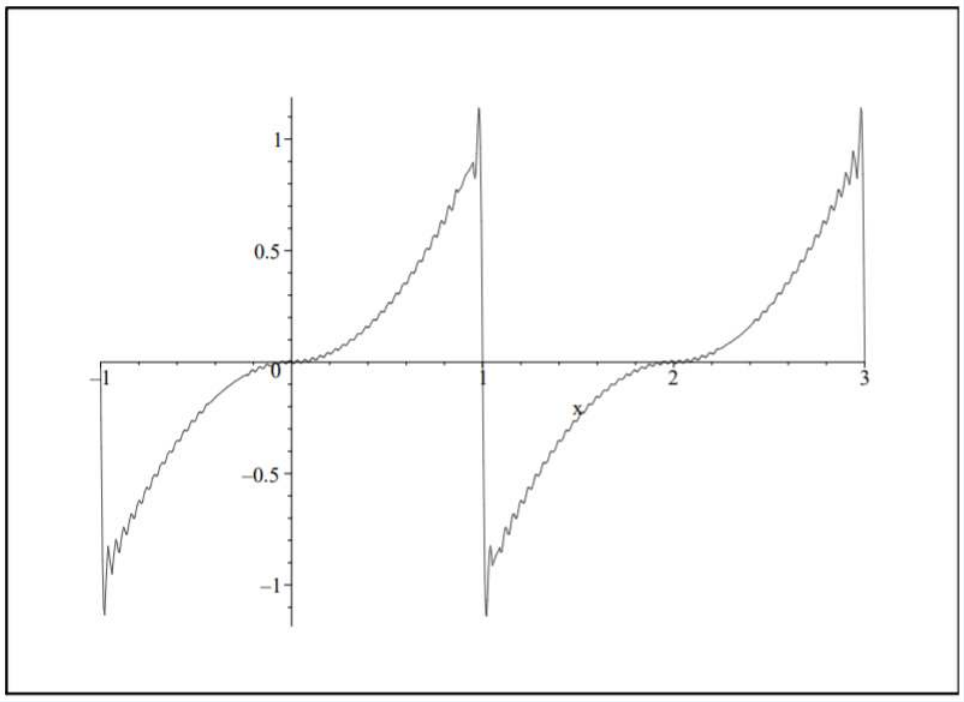


Figure 3.4.4: The odd periodic extension of $f(x) = x^2$ on $[0, 1]$

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3.5: Solution of the Heat Equation

We started this chapter seeking solutions of initial-boundary value problems involving the heat equation and the wave equation. In particular, we found the general solution for the problem of heat flow in a one dimensional rod of length L with fixed zero temperature ends. The problem was given by

$$\begin{array}{lccc} \text{PDE} & u_t = ku_{xx}, & 0<t, & 0\le x\le L, \\ \text{IC} & u(x,0)=f(x), & & 0<x<L, \\ \text{BC} & u(0,t)=0, & & t>0, \\ & u(L,t)=0, & & t>0. \end{array} \tag{3.5.1}$$

We found the solution using separation of variables. This resulted in a sum over various product solutions:

$$u(x,t)=\sum_{n=1}^{\infty} b_n e^{k\lambda_n t}\sin\frac{n\pi x}{L},$$

where

$$\lambda_n = -\left(\frac{n\pi}{L}\right)^2.$$

This equation satisfies the boundary conditions. However, we had only gotten to state initial condition using this solution. Namely,

$$f(x)=u(x,0)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}.$$

We were left with having to determine the constants b_n. Once we know them, we have the solution.

Now we can get the Fourier coefficients when we are given the initial condition, $f(x)$. They are given by

$$b_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx, \quad n=1,2,\ldots.$$

We consider a couple of examples with different initial conditions.

Example 3.5.1

Consider the solution of the heat equation with $f(x)=\sin x$ and $L=\pi$.

In this case the solution takes the form

$$u(x,t)=\sum_{n=1}^{\infty} b_n e^{k\lambda_n t}\sin nx.$$

However, the initial condition takes the form of the first term in the expansion; i.e., the $n=1$ term. So, we need not carry out the integral because we can immediately write $b_1=1$ and $b_n=0, n=2,3,\ldots$. Therefore, the solution consists of just one term,

$$u(x,t)=e^{-kt}\sin x.$$

In Figure 3.5.1 we see that how this solution behaves for $k=1$ and $t\in[0,1]$.

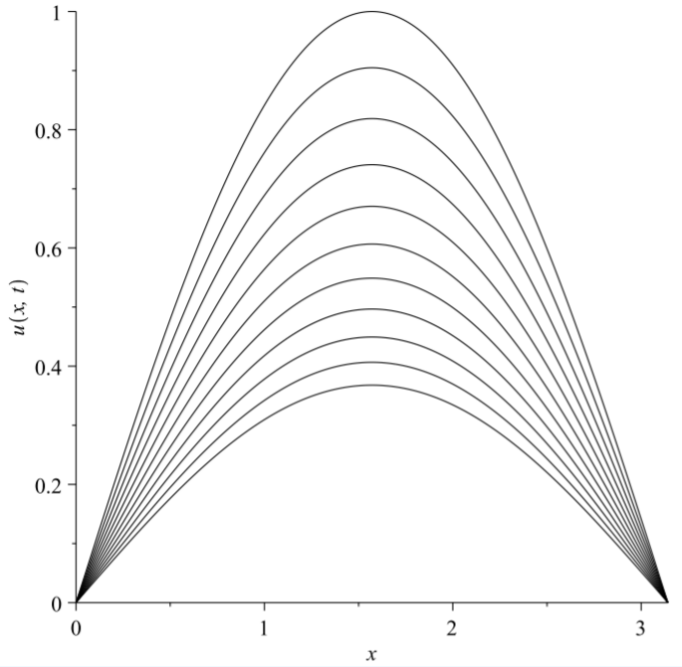


Figure 3.5.1: The evolution of the initial condition $f(x) = \sin x$ for $L = \pi$ and $k = 1$.

Example 3.5.2

Consider solutions of the heat equation with $f(x) = x(1-x)$ and $L = 1$.

This example requires a bit more work. The solution takes the form

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 kt} \sin n\pi x,$$

where

$$b_n = 2\int_0^1 f(x)\sin n\pi x dx.$$

This integral is easily computed using integration by parts

$$\begin{aligned} b_n &= 2\int_0^1 x(1-x)\sin n\pi x dx \\ &= \left[2x(1-x)\left(-\frac{1}{n\pi}\cos n\pi x\right)\right]_0^1 + \frac{2}{n\pi}\int_0^1 (1-2x)\cos n\pi x dx \\ &= -\frac{2}{n^2\pi^2}\left\{[(1-2x)\sin n\pi x]_0^1 + 2\int_0^1 \sin n\pi x dx\right\} \\ &= \frac{4}{n^3\pi^3}[\cos n\pi x]_0^1 \\ &= \frac{4}{n^3\pi^3}(\cos n\pi - 1) \\ &= \begin{cases} 0, & n \text{ even} \\ -\frac{8}{n^3\pi^3} & n \text{ odd} \end{cases} \end{aligned} \tag{3.5.2}$$

So, we have that the solution can be written as

$$u(x,t) = \frac{8}{\pi^3}\sum_{\ell=1}^{\infty} \frac{1}{(2\ell-1)^3} e^{-(2\ell-1)^2\pi^2 kt}\sin(2\ell-1)\pi x.$$

In Figure 3.5.2 we see that how this solution behaves for $k = 1$ and $t \in [0, 1]$. Twenty terms were used. We see that this solution diffuses much faster than in the last example. Most of the terms damp out quickly as the solution asymptotically approaches the first term.

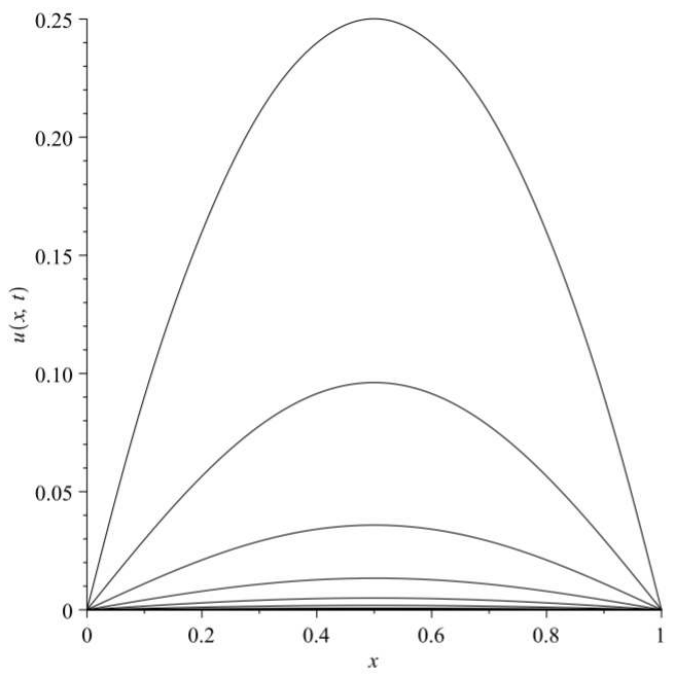


Figure 3.5.2: The evolution of the initial condition $f(x) = x(1 - x)$ for $L = 1$ and $k = 1$.

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3.6: Finite Length Strings

We now return to the physical example of wave propagation in a string. We found that the general solution can be represented as a sum over product solutions. We will restrict our discussion to the special case that the initial velocity is zero and the original profile is given by $u(x,0)=f(x)$. The solution is then

$$u(x,t)=\sum_{n=1}^{\infty}A_n\sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L} \tag{3.6.1}$$

satisfying

$$f(x)=\sum_{n=1}^{\infty}A_n\sin\frac{n\pi x}{L}. \tag{3.6.2}$$

We have seen that the Fourier sine series coefficients are given by

$$A_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx. \tag{3.6.3}$$

We can rewrite this solution in a more compact form. First, we define the wave numbers,

$$k_n=\frac{n\pi}{L},\quad n=1,2,\ldots,$$

and the angular frequencies,

$$\omega_n=ck_n=\frac{n\pi c}{L}.$$

Then, the product solutions take the form

$$\sin k_nx\cos\omega_nt$$

Using trigonometric identities, these products can be written as

$$\sin k_nx\cos\omega_nt=\frac{1}{2}\left[\sin(k_nx+\omega_nt)+\sin(k_nx-\omega_nt)\right].$$

Inserting this expression in the solution, we have

$$u(x,t)=\frac{1}{2}\sum_{n=1}^{\infty}A_n\left[\sin(k_nx+\omega_nt)+\sin(k_nx-\omega_nt)\right]. \tag{3.6.4}$$

Since $\omega_n=ck_n$, we can put this into a more suggestive form:

$$u(x,t)=\frac{1}{2}\left[\sum_{n=1}^{\infty}A_n\sin k_n(x+ct)+\sum_{n=1}^{\infty}A_n\sin k_n(x-ct)\right]. \tag{3.6.5}$$

We see that each sum is simply the sine series for $f(x)$ but evaluated at either $x+ct$ or $x-ct$. Thus, the solution takes the form

$$u(x,t)=\frac{1}{2}[f(x+ct)+f(x-ct)]. \tag{3.6.6}$$

If $t=0$, then we have $u(x,0)=\frac{1}{2}[f(x)+f(x)]=f(x)$. So, the solution satisfies the initial condition. At $t=1$, the sum has a term $f(x-c)$.

Recall from your mathematics classes that this is simply a shifted version of $f(x)$. Namely, it is shifted to the right. For general times, the function is shifted by ct to the right. For larger values of t, this shift is further to the right. The function (wave) shifts to the right with velocity c. Similarly, $f(x+ct)$ is a wave traveling to the left with velocity $-c$.

Thus, the waves on the string consist of waves traveling to the right and to the left. However, the story does not stop here. We have a problem when needing to shift $f(x)$ across the boundaries. The original problem only defines $f(x)$ on $[0,L]$. If we are not careful, we would think that the function leaves the interval leaving nothing left inside. However, we have to recall that our sine

series representation for $f(x)$ has a period of $2L$. So, before we apply this shifting, we need to account for its periodicity. In fact, being a sine series, we really have the odd periodic extension of $f(x)$ being shifted. The details of such analysis would take us too far from our current goal. However, we can illustrate this with a few figures.

We begin by plucking a string of length L. This can be represented by the function

$$f(x) = \begin{cases} \frac{x}{a} & 0 \le x \le a \\ \frac{L-x}{L-a} & a \le x \le L \end{cases} \tag{3.6.7}$$

where the string is pulled up one unit at $x = a$. This is shown in Figure 3.6.1.

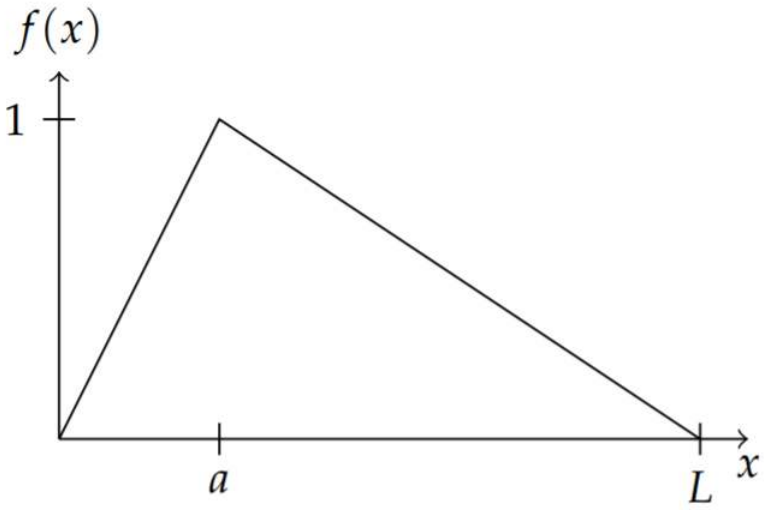


Figure 3.6.1: The initial profile for a string of length one plucked at $x = a$.

Next, we create an odd function by extending the function to a period of $2L$. This is shown in Figure 3.6.2.

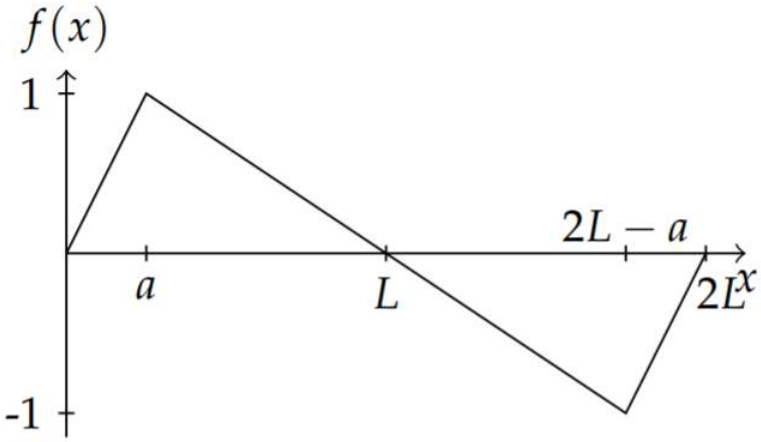


Figure 3.6.2: Odd extension about the right end of a plucked string.

Finally, we construct the periodic extension of this to the entire line. In Figure 3.6.3 we show in the lower part of the figure copies of the periodic extension, one moving to the right and the other moving to the left. (Actually, the copies are $\frac{1}{2}f(x \pm ct)$.) The top plot is the sum of these solutions. The physical string lies in the interval $[0, 1]$. Of course, this is better seen when the solution is animated.

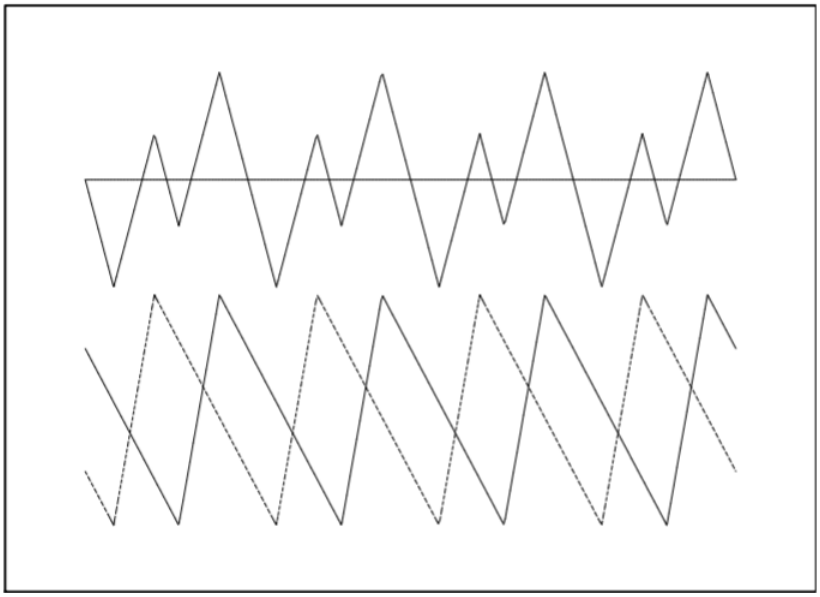


Figure 3.6.3: Summing the odd periodic extensions. The lower plot shows copies of the periodic extension, one moving to the right and the other moving to the left. The upper plot is the sum.

The time evolution for this plucked string is shown for several times in Figure 3.6.4. This results in a wave that appears to reflect from the ends as time increases.

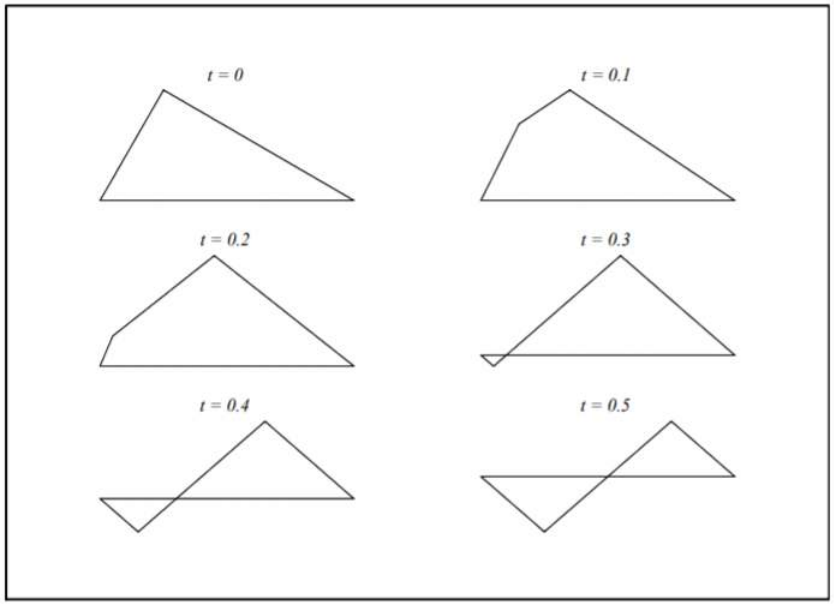


Figure 3.6.4: This Figure shows the plucked string at six successive times.

The relation between the angular frequency and the wave number, $\omega = ck$, is called a dispersion relation. In this case ω depends on k linearly. If one knows the dispersion relation, then one can find the wave speed as $c = \frac{\omega}{k}$. In this case, all of the harmonics travel at the same speed. In cases where they do not, we have nonlinear dispersion, which we will discuss later.

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3.7: The Gibbs Phenomenon

We have seen from the Gibbs Phenomenon when there is a jump discontinuity in the periodic extension of a function, whether the function originally had a discontinuity or developed one due to a mismatch in the values of the endpoints. This can be seen in Figures 3.3.6, 3.4.2 and 3.4.4. The Fourier series has a difficult time converging at the point of discontinuity and these graphs of the Fourier series show a distinct overshoot which does not go away. This is called the Gibbs phenomenon[1] and the amount of overshoot can be computed.

In one of our first examples, Example 3.2.3, we found the Fourier series representation of the piecewise defined function

$$f(x) = \begin{cases} 1, & 0 < x < \pi, \\ -1, & \pi < x < 2\pi, \end{cases} \tag{3.7.1}$$

to be

$$f(x) \sim \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\sin(2k-1)x}{2k-1} \tag{3.7.2}$$

In Figure 3.7.1 we display the sum of the first ten terms. Note the wiggles, overshoots and under shoots. These are seen more when we plot the representation for $x \in [-3\pi, 3\pi]$ as shown in Figure 3.7.2.

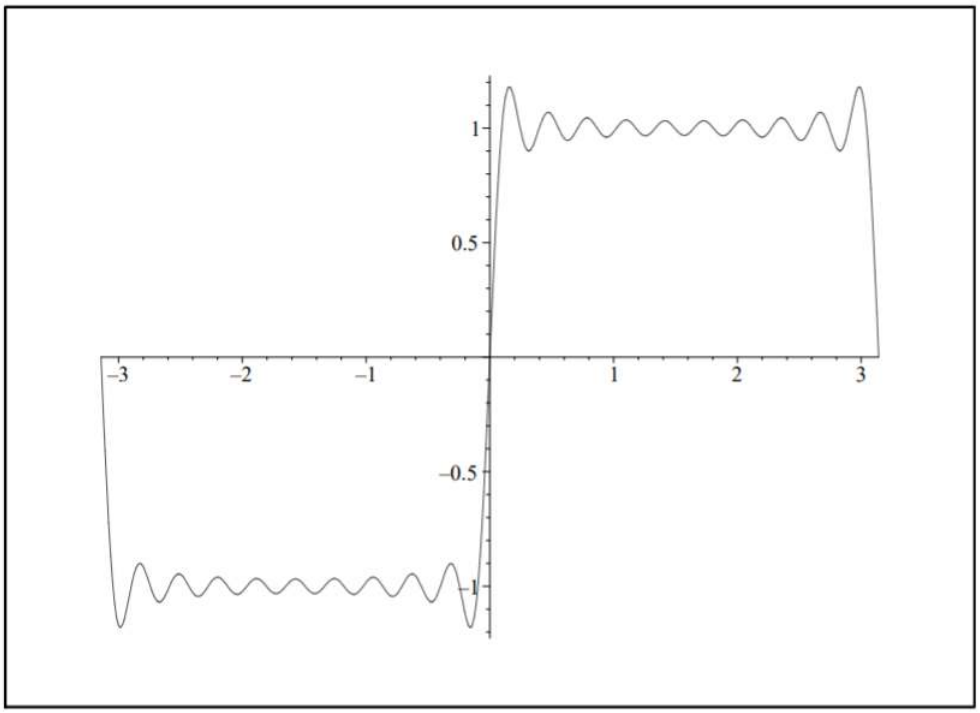


Figure 3.7.1: The Fourier series representation of a step function on $[-\pi, \pi]$ for $N = 10$.

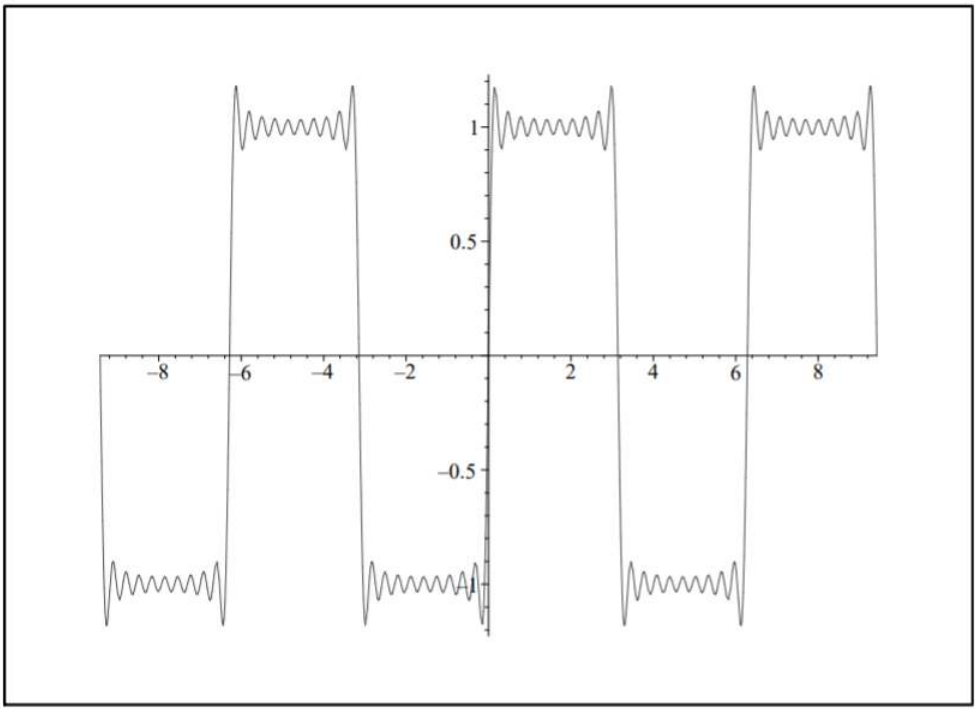


Figure 3.7.2: The Fourier series representation of a step function on $[-\pi, \pi]$ for $N = 10$ plotted on $[-3\pi, 3\pi]$ displaying the periodicity.

We note that the overshoots and undershoots occur at discontinuities in the periodic extension of $f(x)$. These occur whenever $f(x)$ has a discontinuity or if the values of $f(x)$ at the endpoints of the domain do not agree.

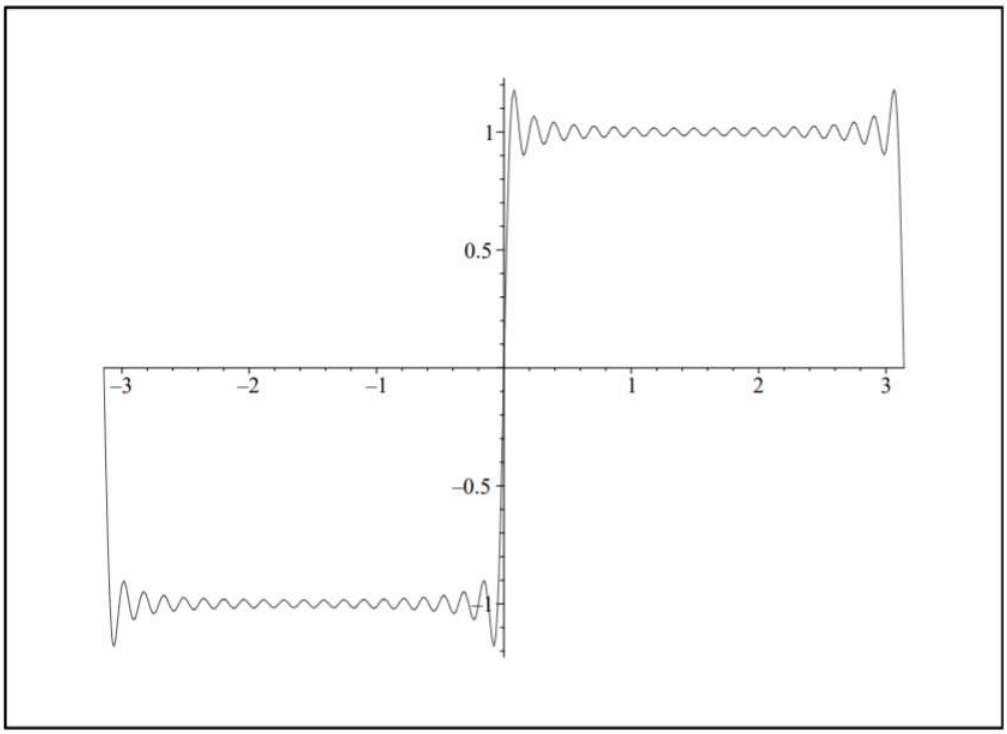


Figure 3.7.3: The Fourier series representation of a step function on $[-\pi, \pi]$ for $N = 20$.

One might expect that we only need to add more terms. In Figure 3.7.3 we show the sum for twenty terms. Note the sum appears to converge better for points far from the discontinuities. But, the overshoots and undershoots are still present. In Figures 3.7.4 and 3.7.5 show magnified plots of the overshoot at $x = 0$ for $N = 100$ and $N = 500$, respectively. We see that the overshoot persists.

The peak is at about the same height, but its location seems to be getting closer to the origin. We will show how one can estimate the size of the overshoot.

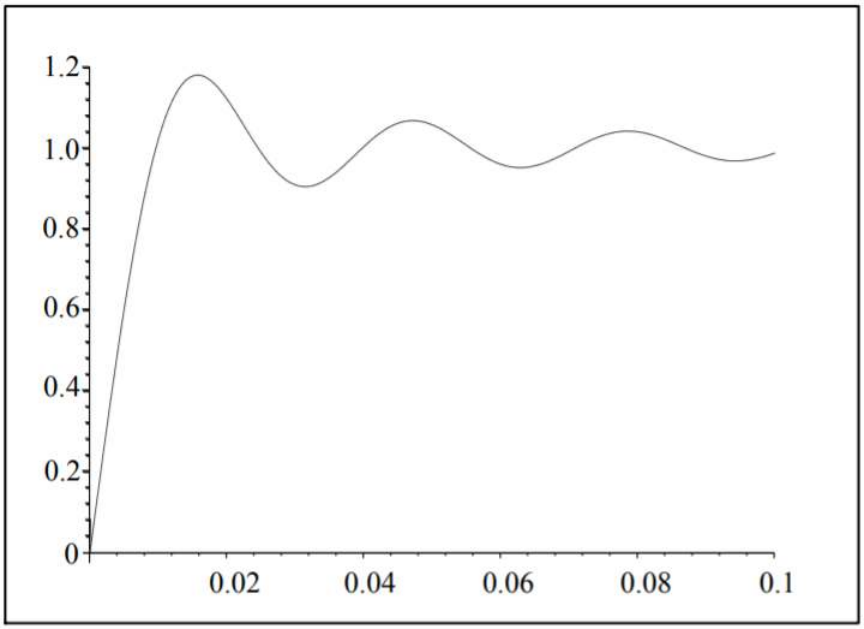


Figure 3.7.4: The Fourier series representation of a step function on $[-\pi, \pi]$ for $N = 100$.

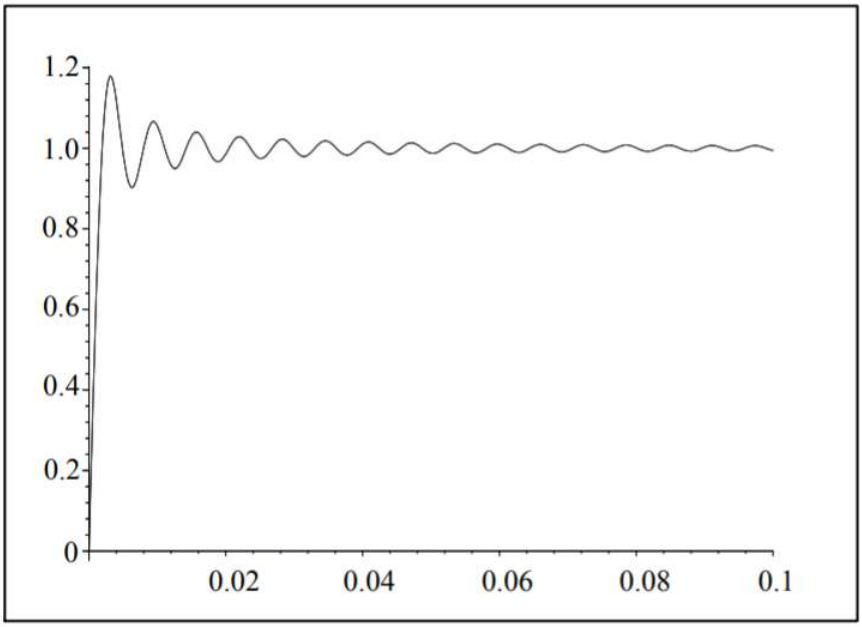


Figure 3.7.5: The Fourier series representation of a step function on $[-\pi, \pi]$ for $N = 500$.

We can study the Gibbs phenomenon by looking at the partial sums of general Fourier trigonometric series for functions $f(x)$ defined on the interval $[-L, L]$. Writing out the partial sums, inserting the Fourier coefficients and rearranging, we have

$$
\begin{aligned}
S_N(x) &= a_0 + \sum_{n=1}^{N} \left[a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right] \\
&= \frac{1}{2L} \int_{-L}^{L} f(y)dy + \sum_{n=1}^{N} \left[\left(\frac{1}{L} \int_{-L}^{L} f(y) \cos \frac{n\pi y}{L} dy \right) \cos \frac{n\pi x}{L} \right. \\
&\quad \left. + \left(\frac{1}{L} \int_{-L}^{L} f(y) \sin \frac{n\pi y}{L} dy. \right) \sin \frac{n\pi x}{L} \right] \\
&= \frac{1}{L} \int_{-L}^{L} \left\{ \frac{1}{2} \right. \\
&\quad \left. + \sum_{n=1}^{N} \left(\cos \frac{n\pi y}{L} \cos \frac{n\pi x}{L} + \sin \frac{n\pi y}{L} \sin \frac{n\pi x}{L} \right) \right\} f(y)dy \\
&= \frac{1}{L} \int_{-L}^{L} \left\{ \frac{1}{2} + \sum_{n=1}^{N} \cos \frac{n\pi (y-x)}{L} \right\} f(y)dy \\
&\equiv \frac{1}{L} \int_{-L}^{L} D_N(y-x) f(y)dy
\end{aligned}
$$

We have defined

$$
D_N(x) = \frac{1}{2} + \sum_{n=1}^{N} \cos \frac{n\pi x}{L},
$$

which is called the N-th Dirichlet kernel.

We now prove

Lemma 3.7.1

The N-th Dirichlet kernel is given by

$$
D_N(x) = \begin{cases} \frac{\sin\left(\left(N+\frac{1}{2}\right)\frac{\pi x}{L}\right)}{2\sin\frac{\pi x}{2L}}, & \sin\frac{\pi x}{2L} \neq 0, \\ N+\frac{1}{2}, & \sin\frac{\pi x}{2L} = 0. \end{cases}
$$

Proof

Let $\theta = \frac{\pi x}{L}$ and multiply $D_N(x)$ by $2\sin\frac{\theta}{2}$ to obtain:

$$
\begin{aligned}
2\sin\frac{\theta}{2} D_N(x) &= 2\sin\frac{\theta}{2}\left[\frac{1}{2} + \cos\theta + \cdots + \cos N\theta\right] \\
&= \sin\frac{\theta}{2} + 2\cos\theta\sin\frac{\theta}{2} + 2\cos 2\theta\sin\frac{\theta}{2} + \cdots + 2\cos N\theta\sin\frac{\theta}{2} \\
&= \sin\frac{\theta}{2} + \left(\sin\frac{3\theta}{2} - \sin\frac{\theta}{2}\right) + \left(\sin\frac{5\theta}{2} - \sin\frac{3\theta}{2}\right) + \cdots \\
&\quad + \left[\sin\left(N+\frac{1}{2}\right)\theta - \sin\left(N-\frac{1}{2}\right)\theta\right] \\
&= \sin\left(N+\frac{1}{2}\right)\theta. \qquad (3.7.3)
\end{aligned}
$$

Thus,

$$
2\sin\frac{\theta}{2} D_N(x) = \sin\left(N+\frac{1}{2}\right)\theta.
$$

If $\sin\frac{\theta}{2} \neq 0$, then

$$D_N(x) = \frac{\sin\left(N+\frac{1}{2}\right)\theta}{2\sin\frac{\theta}{2}}, \quad \theta = \frac{\pi x}{L}.$$

If $\sin\frac{\theta}{2} = 0$, then one needs to apply L'Hospital's Rule as $\theta \to 2m\pi$:

$$\begin{aligned}\lim_{\theta\to 2m\pi} \frac{\sin\left(N+\frac{1}{2}\right)\theta}{2\sin\frac{\theta}{2}} &= \lim_{\theta\to 2m\pi} \frac{\left(N+\frac{1}{2}\right)\cos\left(N+\frac{1}{2}\right)\theta}{\cos\frac{\theta}{2}} \\ &= \frac{\left(N+\frac{1}{2}\right)\cos(2m\pi N + m\pi)}{\cos m\pi} \\ &= \frac{\left(N+\frac{1}{2}\right)(\cos 2m\pi N\cos m\pi - \sin 2m\pi N \sin m\pi)}{\cos m\pi} \\ &= N+\frac{1}{2} \end{aligned} \tag{3.7.4}$$

We further note that $D_N(x)$ is periodic with period $2L$ and is an even function.

So far, we have found that the N th partial sum is given by

$$S_N(x) = \frac{1}{L}\int_{-L}^{L} D_N(y-x)f(y)dy. \tag{3.7.5}$$

Making the substitution $\xi = y - x$, we have

$$\begin{aligned} S_N(x) &= \frac{1}{L}\int_{-L-x}^{L-x} D_N(\xi)f(\xi+x)d\xi \\ &= \frac{1}{L}\int_{-L}^{L} D_N(\xi)f(\xi+x)d\xi. \end{aligned} \tag{3.7.6}$$

In the second integral we have made use of the fact that $f(x)$ and $D_N(x)$ are periodic with period $2L$ and shifted the interval back to $[-L, L]$.

We now write the integral as the sum of two integrals over positive and negative values of ξ and use the fact that $D_N(x)$ is an even function. Then,

$$\begin{aligned} S_N(x) &= \frac{1}{L}\int_{-L}^{0} D_N(\xi)f(\xi+x)d\xi + \frac{1}{L}\int_{0}^{L} D_N(\xi)f(\xi+x)d\xi \\ &= \frac{1}{L}\int_{0}^{L} [f(x-\xi)+f(\xi+x)]D_N(\xi)d\xi. \end{aligned} \tag{3.7.7}$$

We can use this result to study the Gibbs phenomenon whenever it occurs. In particular, we will only concentrate on the earlier example. For this case, we have

$$S_N(x) = \frac{1}{\pi}\int_{0}^{\pi} [f(x-\xi)+f(\xi+x)]D_N(\xi)d\tilde{\xi} \tag{3.7.8}$$

for

$$D_N(x) = \frac{1}{2} + \sum_{n=1}^{N}\cos nx.$$

Also, one can show that

$$f(x-\xi)+f(\xi+x) = \begin{cases} 2, & 0 \le \xi < x, \\ 0, & x \le \xi < \pi - x, \\ -2, & \pi - x \le \xi < \pi. \end{cases}$$

Thus, we have

$$\begin{aligned} S_N(x) &= \frac{2}{\pi}\int_0^x D_N(\xi)d\xi - \frac{2}{\pi}\int_{\pi-x}^{\pi} D_N(\xi)d\xi \\ &= \frac{2}{\pi}\int_0^x D_N(z)dz + \frac{2}{\pi}\int_0^x D_N(\pi - z)dz. \end{aligned} \tag{3.7.9}$$

Here we made the substitution $z = \pi - \xi$ in the second integral.

The Dirichlet kernel for $L = \pi$ is given by

$$D_N(x) = \frac{\sin\left(N + \frac{1}{2}\right)x}{2\sin\frac{x}{2}}.$$

For N large, we have $N + \frac{1}{2} \approx N$, and for small x, we have $\sin\frac{x}{2} \approx \frac{x}{2}$. So, under these assumptions,

$$D_N(x) \approx \frac{\sin Nx}{x}$$

Therefore,

$$S_N(x) \to \frac{2}{\pi}\int_0^x \frac{\sin N\xi}{\xi}d\xi \quad \text{for large } N, \text{ and small } x.$$

If we want to determine the locations of the minima and maxima, where the undershoot and overshoot occur, then we apply the first derivative test for extrema to $S_N(x)$. Thus,

$$\frac{d}{dx}S_N(x) = \frac{2}{\pi}\frac{\sin Nx}{x} = 0.$$

The extrema occur for $Nx = m\pi, m = \pm 1, \pm 2, \ldots$. One can show that there is a maximum at $x = \pi/N$ and a minimum for $x = 2\pi/N$. The value for the overshoot can be computed as

$$\begin{aligned} S_N(\pi/N) &= \frac{2}{\pi}\int_0^{\pi/N} \frac{\sin N\xi}{\xi}d\xi \\ &= \frac{2}{\pi}\int_0^{\pi} \frac{\sin t}{t}dt \\ &= \frac{2}{\pi}\text{Si}(\pi) \\ &= 1.178979744\ldots \end{aligned} \tag{3.7.10}$$

Note that this value is independent of N and is given in terms of the sine integral,

$$\text{Si}(x) \equiv \int_0^x \frac{\sin t}{t}dt.$$

Footnotes

[1] The Gibbs phenomenon was named after Josiah Willard Gibbs (1839-1903) even though it was discovered earlier by the Englishman Henry Wilbraham (1825^- 1883). Wilbraham published a soon forgotten paper about the effect in 1848 . In 1889 Albert Abraham Michelson ($1852-$ 1931), an American physicist,observed an overshoot in his mechanical graphing machine. Shortly afterwards J. Willard Gibbs published papers describing this phenomenon, which was later to be called the Gibbs phenomena. Gibbs was a mathematical physicist and chemist and is considered the father of physical chemistry.

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3.8: Problems

Exercise 3.8.1

Write $y(t) = 3\cos 2t - 4\sin 2t$ in the form $y(t) = A\cos(2\pi ft + \phi)$.

Exercise 3.8.2

Derive the coefficients b_n in Equation (3.2.2).

Exercise 3.8.3

Let $f(x)$ be defined for $x \in [-L, L]$. Parseval's identity is given by

$$\frac{1}{L}\int_{-L}^{L} f^2(x)dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + b_n^2.$$

Assuming the the Fourier series of $f(x)$ converges uniformly in $(-L, L)$, prove Parseval's identity by multiplying the Fourier series representation by $f(x)$ and integrating from $x = -L$ to $x = L$. [In Section 9.6.3 we will encounter Parseval's equality for Fourier transforms which is a continuous version of this identity.]

Exercise 3.8.4

Consider the square wave function

$$f(x) = \begin{cases} 1, & 0 < x < \pi, \\ -1, & \pi < x < 2\pi. \end{cases}$$

a. Find the Fourier series representation of this function and plot the first 50 terms.
b. Apply Parseval's identity in Problem 3 to the result in part a.
c. Use the result of part b to show $\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$.

Exercise 3.8.5

For the following sets of functions: i) show that each is orthogonal on the given interval, and ii) determine the corresponding orthonormal set.

a. $\{\sin 2nx\}, \quad n = 1, 2, 3, \ldots, \quad 0 \le x \le \pi.$
b. $\{\cos n\pi x\}, \quad n = 0, 1, 2, \ldots, \quad 0 \le x \le 2$
c. $\left\{\sin \frac{n\pi x}{L}\right\}, \quad n = 1, 2, 3, \ldots, \quad x \in [-L, L].$

Exercise 3.8.6

Consider $f(x) = 4\sin^3 2x$.

a. Derive the trigonometric identity giving $\sin^3\theta$ in terms of $\sin\theta$ and $\sin 3\theta$ using DeMoivre's Formula.
b. Find the Fourier series of $f(x) = 4\sin^3 2x$ on $[0, 2\pi]$ without computing any integrals.

Exercise 3.8.7

Find the Fourier series of the following:

a. $f(x) = x, x \in [0, 2\pi]$.
b. $f(x) = \frac{x^2}{4}, |x| < \pi$.

c. $f(x) = \begin{cases} \frac{\pi}{2}, & 0 < x < \pi, \\ -\frac{\pi}{2}, & \pi < x < 2\pi. \end{cases}$

Exercise 3.8.8

Find the Fourier Series of each function $f(x)$ of period 2π. For each series, plot the N th partial sum,

$$S_N = \frac{a_0}{2} + \sum_{n=1}^{N} \left[a_n \cos nx + b_n \sin nx \right],$$

for $N = 5, 10, 50$ and describe the convergence (is it fast? what is it converging to, etc.) [Some simple Maple code for computing partial sums is shown in the notes.]

a. $f(x) = x, |x| < \pi.$
b. $f(x) = |x|, |x| < \pi.$
c. $f(x) = \begin{cases} 0, & -\pi < x < 0, \\ 1, & 0 < x < \pi. \end{cases}$

Exercise 3.8.9

Find the Fourier series of $f(x) = x$ on the given interval. Plot the N th partial sums and describe what you see.

a. $0 < x < 2$.
b. $-2 < x < 2$.
c. $1 < x < 2$

Exercise 3.8.10

The result in problem 7 b above gives a Fourier series representation of $\frac{x^2}{4}$. By picking the right value for x and a little arrangement of the series, show that [See Example 3.3.2.]

a. $$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

b. $$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots.$$

Hint: Consider how the series in part a. can be used to do this.

Exercise 3.8.11

Sketch (by hand) the graphs of each of the following functions over four periods. Then sketch the extensions each of the functions as both an even and odd periodic function. Determine the corresponding Fourier sine and cosine series and verify the convergence to the desired function using Maple

a. $f(x) = x^2, 0 < x < 1$.
b. $f(x) = x(2 - x), 0 < x < 2$.
c. $f(x) = \begin{cases} 0, & 0 < x < 1, \\ 1, & 1 < x < 2. \end{cases}$
d. $f(x) = \begin{cases} \pi, & 0 < x < \pi, \\ 2\pi - x, & \pi < x < 2\pi. \end{cases}$

Exercise 3.8.12

Consider the function $f(x) = x, -\pi < x < \pi$.

a. Show that $x = 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n}$.

b.

c. Find the Fourier cosine series of $f(x) = x^2$ on $[0, \pi]$ and compare it to the result in part b.

Exercise 3.8.13

Consider the function $f(x) = x, 0 < x < 2$.

a. Find the Fourier sine series representation of this function and plot the first 50 terms.

b. Find the Fourier cosine series representation of this function and plot the first 50 terms.

c. Apply Parseval's identity in Problem 3 to the result in part b.

d. Use the result of part c to find the sum $\sum_{n=1}^{\infty}\frac{1}{n^4}$.

Exercise 3.8.14

Differentiate the Fourier sine series term by term in Problem 18. Show that the result is not the derivative of $f(x) = x$.

Exercise 3.8.15

Find the general solution to the heat equation, $u_t - u_{xx} = 0$, on $[0, \pi]$ satisfying the boundary conditions $u_x(0, t) = 0$ and $u(\pi, t) = 0$. Determine the solution satisfying the initial condition,

$$u(x, 0) = \begin{cases} x, & 0 \le x \le \frac{\pi}{2}, \\ \pi - x, & \frac{\pi}{2} \le x \le \pi, \end{cases}$$

Exercise 3.8.16

Find the general solution to the wave equation $u_{tt} = 2u_{xx}$, on $[0, 2\pi]$ satisfying the boundary conditions $u(0, t) = 0$ and $u_x(2\pi, t) = 0$. Determine the solution satisfying the initial conditions, $u(x, 0) = x(4\pi - x)$, and $u_t(x, 0) = 0$.

Exercise 3.8.17

Recall the plucked string initial profile example in the last chapter given by

$$f(x) = \begin{cases} x, & 0 \le x \le \frac{\ell}{2}, \\ \ell - x, & \frac{\ell}{2} \le x \le \ell, \end{cases}$$

satisfying fixed boundary conditions at $x = 0$ and $x = \ell$. Find and plot the solutions at $t = 0, .2, \ldots, 1.0$, of $u_{tt} = u_{xx}$, for $u(x, 0) = f(x), u_t(x, 0) = 0$, with $x \in [0, 1]$.

Exercise 3.8.18

Find and plot the solutions at $t = 0, .2, \ldots, 1.0$, of the problem

$$
\begin{aligned}
u_{tt} &= u_{xx}, \quad 0 \le x \le 1, t > 0 \\
u(x,0) &= \begin{cases} 0, & 0 \le x < \frac{1}{4}, \\ 1, & \frac{1}{4} \le x \le \frac{3}{4}, \\ 0, & \frac{3}{4} < x \le 1, \end{cases} \\
u_t(x,0) &= 0, \\
u(0,t) &= 0, \quad t > 0, \\
u(1,t) &= 0, \quad t > 0.
\end{aligned}
$$

? Exercise 3.8.19

Find the solution to Laplace's equation, $u_{xx} + u_{yy} = 0$, on the unit square, $[0,1] \times [0,1]$ satisfying the boundary conditions $u(0,y) = 0, u(1,y) = y(1-y), u(x,0) = 0$, and $u(x,1) = 0$.

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CHAPTER OVERVIEW

4: Sturm-Liouville Boundary Value Problems

We have seen that trigonometric functions and special functions are the solutions of differential equations. These solutions give orthogonal sets of functions which can be used to represent functions in generalized Fourier series expansions. At the same time we would like to generalize the techniques we had first used to solve the heat equation in order to solve more general initial-boundary value problems. Namely, we use separation of variables to separate the given partial differential equation into a set of ordinary differential equations. A subset of those equations provide us with a set of boundary value problems whose eigenfunctions are useful in representing solutions of the partial differential equation. Hopefully, those solutions will form a useful basis in some function space.

A class of problems to which our previous examples belong are the Sturm-Liouville eigenvalue problems. These problems involve self-adjoint (differential) operators which play an important role in the spectral theory of linear operators and the existence of the eigenfunctions needed to solve the interesting physics problems described by the above initial-boundary value problems. In this section we will introduce the Sturm-Liouville eigenvalue problem as a general class of boundary value problems containing the Legendre and Bessel equations and supplying the theory needed to solve a variety of problems.

4.1: Sturm-Liouville Operators

4.2: Properties of Sturm-Liouville Eigenvalue Problems

4.3: The Eigenfunction Expansion Method

4.4: Appendix- The Fredholm Alternative Theorem

4.5: Problems

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4.1: Sturm-Liouville Operators

In physics many problems arise in the form of boundary value problems involving second order ordinary differential equations. For example, we will explore the wave equation and the heat equation in three dimensions. Separating out the time dependence leads to a three dimensional boundary value problem in both cases. Further separation of variables leads to a set of boundary value problems involving second order ordinary differential equations.

In general, we might obtain equations of the form

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x) \tag{4.1.1}$$

subject to boundary conditions. We can write such an equation in operator form by defining the differential operator

$$L = a_2(x)D^2 + a_1(x)D + a_0(x),$$

where $D = d/dx$. Then, Equation (4.1.1) takes the form

$$Ly = f.$$

Recall that we had solved such nonhomogeneous differential equations in Chapter 2. In this section we will show that these equations can be solved using eigenfunction expansions. Namely, we seek solutions to the eigenvalue problem

$$L\phi = \lambda\phi$$

with homogeneous boundary conditions on ϕ and then seek a solution of the nonhomogeneous problem, $Ly = f$, as an expansion over these eigenfunctions. Formally, we let

$$y(x) = \sum_{n=1}^{\infty} c_n \phi_n(x).$$

However, we are not guaranteed a nice set of eigenfunctions. We need an appropriate set to form a basis in the function space. Also, it would be nice to have orthogonality so that we can easily solve for the expansion coefficients.

It turns out that any linear second order differential operator can be turned into an operator that possesses just the right properties (self-adjointedness) to carry out this procedure. The resulting operator is referred to as a **SturmLiouville operator**. We will highlight some of the properties of these operators and see how they are used in applications.

We define the Sturm-Liouville operator as

$$\mathcal{L} = \frac{d}{dx}p(x)\frac{d}{dx} + q(x). \tag{4.1.2}$$

The Sturm-Liouville eigenvalue problem is given by the differential equation

$$\mathcal{L} = -\lambda\sigma(x)y,$$

or

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y + \lambda\sigma(x)y = 0, \tag{4.1.3}$$

for $x \in (a, b), y = y(x)$, plus boundary conditions. The functions $p(x), p'(x)$, $q(x)$ and $\sigma(x)$ are assumed to be continuous on (a, b) and $p(x) > 0, \sigma(x) > 0$ on $[a, b]$. If the interval is finite and these assumptions on the coefficients are true on $[a, b]$, then the problem is said to be a regular Sturm-Liouville problem. Otherwise, it is called a singular Sturm-Liouville problem.

We also need to impose the set of homogeneous boundary conditions

$$\begin{aligned} \alpha_1 y(a) + \beta_1 y'(a) &= 0, \\ \alpha_2 y(b) + \beta_2 y'(b) &= 0. \end{aligned} \tag{4.1.4}$$

The α' s and β' are constants. For different values, one has special types of boundary conditions. For $\beta_i = 0$, we have what are called Dirichlet boundary conditions. Namely, $y(a) = 0$ and $y(b) = 0$. For $\alpha_i = 0$, we have Neumann boundary conditions. In this case, $y'(a) = 0$ and $y'(b) = 0$. In terms of the heat equation example, Dirichlet conditions correspond to maintaining a fixed

temperature at the ends of the rod. The Neumann boundary conditions would correspond to no heat flow across the ends, or insulating conditions, as there would be no temperature gradient at those points. The more general boundary conditions allow for partially insulated boundaries.

Note

- **Dirichlet boundary conditions** - the solution takes fixed values on the boundary. These are named after Gustav Lejeune Dirichlet ($1805-1859$).
- **Neumann boundary conditions** - the derivative of the solution takes fixed values on the boundary. These are named after Carl Neumann (1832-1925).

Another type of boundary condition that is often encountered is the periodic boundary condition. Consider the heated rod that has been bent to form a circle. Then the two end points are physically the same. So, we would expect that the temperature and the temperature gradient should agree at those points. For this case we write $y(a)=y(b)$ and $y'(a)=y'(b)$. Boundary value problems using these conditions have to be handled differently than the above homogeneous conditions. These conditions leads to different types of eigenfunctions and eigenvalues.

As previously mentioned, equations of the form (4.1.1) occur often. We form. now show that any second order linear operator can be put into the form of the Sturm-Liouville operator. In particular, equation (4.1.1) can be put into the form

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right)+q(x)y=F(x). \tag{4.1.5}$$

Another way to phrase this is provided in the theorem:

The proof of this is straight forward as we soon show. Let's first consider the equation (4.1.1) for the case that $a_1(x)=a_2'(x)$. Then, we can write the equation in a form in which the first two terms combine,

$$\begin{aligned} f(x) &= a_2(x)y''+a_1(x)y'+a_0(x)y \\ &= (a_2(x)y')'+a_0(x)y. \end{aligned} \tag{4.1.6}$$

The resulting equation is now in Sturm-Liouville form. We just identify $p(x)=a_2(x)$ and $q(x)=a_0(x)$.

Not all second order differential equations are as simple to convert. Consider the differential equation

$$x^2y''+xy'+2y=0.$$

In this case $a_2(x)=x^2$ and $a_2'(x)=2x\neq a_1(x)$. So, this does not fall into this case. However, we can change the operator in this equation, x^2D+xD, to a Sturm-Liouville operator, $Dp(x)D$ for a $p(x)$ that depends on the coefficients x^2 and x..

In the Sturm Liouville operator the derivative terms are gathered together into one perfect derivative, $Dp(x)D$. This is similar to what we saw in the Chapter 2 when we solved linear first order equations. In that case we sought an integrating factor. We can do the same thing here. We seek a multiplicative function $\mu(x)$ that we can multiply through (4.1.1) so that it can be written in Sturm-Liouville form.

We first divide out the $a_2(x)$, giving

$$y''+\frac{a_1(x)}{a_2(x)}y'+\frac{a_0(x)}{a_2(x)}y=\frac{f(x)}{a_2(x)}.$$

Next, we multiply this differential equation by μ,

$$\mu(x)y''+\mu(x)\frac{a_1(x)}{a_2(x)}y'+\mu(x)\frac{a_0(x)}{a_2(x)}y=\mu(x)\frac{f(x)}{a_2(x)}.$$

The first two terms can now be combined into an exact derivative $(\mu y')'$ if the second coefficient is $\mu'(x)$. Therefore, $\mu(x)$ satisfies a first order, separable differential equation:

$$\frac{d\mu}{dx}=\mu(x)\frac{a_1(x)}{a_2(x)}.$$

This is formally solved to give the sought integrating factor

$$\mu(x) = e^{\int \frac{a_1(x)}{a_2(x)} dx}.$$

Thus, the original equation can be multiplied by factor

$$\frac{\mu(x)}{a_2(x)} = \frac{1}{a_2(x)} e^{\int \frac{a_1(x)}{a_2(x)} dx}$$

to turn it into Sturm-Liouville form.

In summary,

Summary

Equation (4.1.1),

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x), \tag{4.1.7}$$

can be put into the Sturm-Liouville form

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y = F(x), \tag{4.1.8}$$

where

$$\begin{aligned} p(x) &= e^{\int \frac{a_1(x)}{a_2(x)} dx}, \\ q(x) &= p(x)\frac{a_0(x)}{a_2(x)}, \\ F(x) &= p(x)\frac{f(x)}{a_2(x)}. \end{aligned} \tag{4.1.9}$$

Example 4.1.1

Convert $x^2y'' + xy' + 2y = 0$ into Sturm-Liouville form.

Solution

We can multiply this equation by

$$\frac{\mu(x)}{a_2(x)} = \frac{1}{x^2} e^{\int \frac{dx}{x}} = \frac{1}{x'}$$

to put the equation in Sturm-Liouville form:

$$\begin{aligned} 0 &= xy'' + y' + \frac{2}{x}y \\ &= (xy')' + \frac{2}{x}y. \end{aligned} \tag{4.1.10}$$

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4.2: Properties of Sturm-Liouville Eigenvalue Problems

There are several properties that can be proven for the (regular) Sturm-Liouville eigenvalue problem in (4.1.3). However, we will not prove them all here. We will merely list some of the important facts and focus on a few of the properties.

1. The eigenvalues are real, countable, ordered and there is a smallest eigenvalue. Thus, we can write them as $\lambda_1 < \lambda_2 < \dots$. However, there is no largest eigenvalue and $n \to \infty, \lambda_n \to \infty$.
2. For each eigenvalue λ_n there exists an eigenfunction ϕ_n with $n-1$ zeros on (a,b).
3. then the orthogonality of the eigenfunctions can be written in the form

$$\langle \phi_n, \phi_m \rangle = \langle \phi_n, \phi_n \rangle \delta_{nm}, \quad n,m = 1,2,\dots. \tag{4.2.1}$$

4. where

$$c_n = \frac{\langle f, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle}.$$

5. $$\lambda_n = \frac{-p\phi_n \frac{d\phi_n}{dx}\Big|_a^b + \int_a^b \left[p\left(\frac{d\phi_n}{dx}\right)^2 - q\phi_n^2 \right] dx}{\langle \phi_n, \phi_n \rangle}.$$

   This is verified by multiplying the eigenvalue problem

   by ϕ_n and integrating. Solving this result for λ_n, we obtain the Rayleigh quotient. The Rayleigh quotient is useful for getting estimates of eigenvalues and proving some of the other properties.

Example 4.2.1

Verify some of these properties for the eigenvalue problem

$$y'' = -\lambda y, \quad y(0) = y(\pi) = 0.$$

Solution

This is a problem we had seen many times. The eigenfunctions for this eigenvalue problem are $\phi_n(x) = \sin nx$, with eigenvalues $\lambda_n = n^2$ for $n = 1, 2, \dots$. These satisfy the properties listed above.

First of all, the eigenvalues are real, countable and ordered, $1 < 4 < 9 < \dots$. There is no largest eigenvalue and there is a first one.

The eigenfunctions corresponding to each eigenvalue have $n-1$ zeros on $(0, \pi)$. $\phi_n(x) = \sin nx$ for $n = 1, 2, 3, 4$ This is demonstrated for several eigenfunctions in Figure 4.2.1.

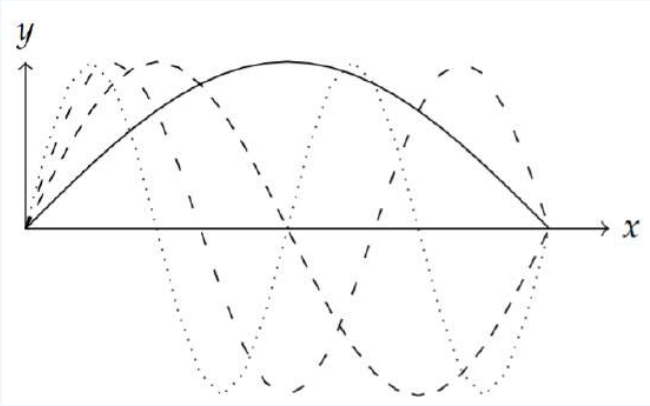


Figure 4.2.1: Plot of the eigenfunctions $\phi_n(x) \sin nx$ for $n = 1, 2, 3, 4$.

We also know that the set $\{\sin nx\}_{n=1}^{\infty}$ is an orthogonal set of basis functions of length

$$||\phi_n|| = \sqrt{\frac{\pi}{2}}.$$

Thus, the Rayleigh quotient can be computed using $p(x)=1$, $q(x)=0$, and the eigenfunctions it is given by

$$R=\frac{-\phi_n\phi_n'|_0^\pi+\int_0^\pi(\phi_n')^2dx}{\frac{\pi}{2}}$$
$$=\frac{2}{\pi}\int_0^\pi(-n^2\cos nx)^2dx=n^2.$$

Therefore, knowing the eigenfunction, the Rayleigh quotient returns the eigenvalues as expected.

Example 4.2.2

We seek the eigenfunctions of the operator found in Example 4.1.1. Namely, we want to solve the eigenvalue problem

$$\mathcal{L}y=(xy')'+\frac{2}{x}y=-\lambda\sigma y \tag{4.2.2}$$

subject to a set of homogeneous boundary conditions. Let's use the boundary conditions

$$y'(1)=0,\quad y'(2)=0.$$

[Note that we do not know $\sigma(x)$ yet, but will choose an appropriate function to obtain solutions.]

Solution

Expanding the derivative, we have

$$xy''+y'+\frac{2}{x}y=-\lambda\sigma y.$$

Multiply through by x to obtain

$$x^2y''+xy'+(2+\lambda x\sigma)y=0.$$

Notice that if we choose $\sigma(x)=x^{-1}$, then this equation can be made a Cauchy-Euler type equation. Thus, we have

$$x^2y''+xy'+(\lambda+2)y=0.$$

The characteristic equation is

$$r^2+\lambda+2=0.$$

For oscillatory solutions, we need $\lambda+2>0$. Thus, the general solution is

$$y(x)=c_1\cos(\sqrt{\lambda+2}\ln|x|)+c_2\sin(\sqrt{\lambda+2}\ln|x|). \tag{4.2.3}$$

Next we apply the boundary conditions. $y'(1)=0$ forces $c_2=0$. This leaves

$$y(x)=c_1\cos(\sqrt{\lambda+2}\ln x).$$

The second condition, $y'(2)=0$, yields

$$\sin(\sqrt{\lambda+2}\ln 2)=0.$$

This will give nontrivial solutions when

$$\sqrt{\lambda+2}\ln 2=n\pi,\quad n=0,1,2,3\ldots.$$

In summary, the eigenfunctions for this eigenvalue problem are

$$y_n(x)=\cos\left(\frac{n\pi}{\ln 2}\ln x\right),\quad 1\le x\le 2$$

and the eigenvalues are $\lambda_n=\left(\frac{n\pi}{\ln 2}\right)^2-2$ for $n=0,1,2,\ldots$

Note

We include the $n=0$ case because $y(x)=$ constant is a solution of the $\lambda=-2$ case. More specifically, in this case the characteristic equation reduces to $r^2=0$. Thus, the general solution of this Cauchy-Euler equation is

$$y(x)=c_1+c_2\ln|x|.$$

Setting $y'(1)=0$, forces $c_2=0$. $y'(2)$ automatically vanishes, leaving the solution in this case as $y(x)=c_1$.

We note that some of the properties listed in the beginning of the section hold for this example. The eigenvalues are seen to be real, countable and ordered. There is a least one, $\lambda_0=-2$. Next, one can find the zeros of each eigenfunction on $[1,2]$. Then the argument of the cosine, $\frac{n\pi}{\ln 2}\ln x$, takes values 0 to $n\pi$ for $x\in[1,2]$. The cosine function has $n-1$ roots on this interval.

Orthogonality can be checked as well. We set up the integral and use the substitution $y=\pi\ln x/\ln 2$. This gives

$$\begin{aligned}\langle y_n,y_m\rangle &= \int_1^2 \cos\Big(\frac{n\pi}{\ln 2}\ln x\Big)\cos\Big(\frac{m\pi}{\ln 2}\ln x\Big)\frac{dx}{x}\\ &= \frac{\ln 2}{\pi}\int_0^\pi \cos ny\cos my dy\\ &= \frac{\ln 2}{2}\delta_{n,m}.\end{aligned} \tag{4.2.4}$$

Adjoint Operators

In the study of the spectral theory of matrices, one learns about the adjoint of the matrix, $A^\dagger$, and the role that self-adjoint, or Hermitian, matrices play in diagonalization. Also, one needs the concept of adjoint to discuss the existence of solutions to the matrix problem $\mathbf{y}=A\mathbf{x}$. In the same spirit, one is interested in the existence of solutions of the operator equation $Lu=f$ and solutions of the corresponding eigenvalue problem. The study of linear operators on a Hilbert space is a generalization of what the reader had seen in a linear algebra course.

Just as one can find a basis of eigenvectors and diagonalize Hermitian, or self-adjoint, matrices (or, real symmetric in the case of real matrices), we will see that the Sturm-Liouville operator is self-adjoint. In this section we will define the domain of an operator and introduce the notion of adjoint operators. In the last section we discuss the role the adjoint plays in the existence of solutions to the operator equation $Lu=f$.

We begin by defining the adjoint of an operator. The adjoint, $L^\dagger$, of operator L satisfies

$$\langle u,Lv\rangle=\left\langle L^+u,v\right\rangle$$

for all v in the domain of L and u in the domain of L^+. Here the domain of a differential operator L is the set of all $u\in L^2_\sigma(a,b)$ satisfying a given set of homogeneous boundary conditions. This is best understood through example.

Example 4.2.3

Find the adjoint of $L=a_2(x)D^2+a_1(x)D+a_0(x)$ for $D=d/dx$.

Solution

In order to find the adjoint, we place the operator inside an integral. Consider the inner product

$$\langle u,Lv\rangle=\int_a^b u\left(a_2v''+a_1v'+a_0v\right)dx.$$

We have to move the operator L from v and determine what operator is acting on u in order to formally preserve the inner product. For a simple operator like $L=\frac{d}{dx}$, this is easily done using integration by parts. For the given operator, we will need to apply several integrations by parts to the individual terms. We consider each derivative term in the integrand separately.

For the a_1v' term, we integrate by parts to find

$$\int_a^b u(x)a_1(x)v'(x)dx = a_1(x)u(x)v(x)|_a^b - \int_a^b (u(x)a_1(x))'v(x)dx. \tag{4.2.5}$$

Now, we consider the a_2v'' term. In this case it will take two integrations by parts:

$$\begin{aligned}\int_a^b u(x)a_2(x)v''(x)dx =& a_2(x)u(x)v'(x)|_a^b - \int_a^b (u(x)a_2(x))'v(x)'dx \\ =& \left[a_2(x)u(x)v'(x) - (a_2(x)u(x))'v(x)\right]\Big|_a^b \\ &+ \int_a^b (u(x)a_2(x))''v(x)dx.\end{aligned} \tag{4.2.6}$$

Combining these results, we obtain

$$\begin{aligned}\langle u, Lv\rangle =& \int_a^b u\left(a_2v'' + a_1v' + a_0v\right)dx \\ =& \left[a_1(x)u(x)v(x) + a_2(x)u(x)v'(x) - (a_2(x)u(x))'v(x)\right]\Big|_a^b \\ &+ \int_a^b \left[(a_2u)'' - (a_1u)' + a_0u\right]vdx.\end{aligned} \tag{4.2.7}$$

Inserting the boundary conditions for v, one has to determine boundary conditions for u such that

$$\left[a_1(x)u(x)v(x) + a_2(x)u(x)v'(x) - (a_2(x)u(x))'v(x)\right]\Big|_a^b = 0.$$

This leaves

$$\langle u, Lv\rangle = \int_a^b \left[(a_2u)'' - (a_1u)' + a_0u\right]vdx \equiv \left\langle L^\dagger u, v\right\rangle.$$

Therefore,

$$L^\dagger = \frac{d^2}{dx^2}a_2(x) - \frac{d}{dx}a_1(x) + a_0(x). \tag{4.2.8}$$

When $L^\dagger = L$, the operator is called formally self-adjoint. When the domain of L is the same as the domain of $L^\dagger$, the term self-adjoint is used. As the domain is important in establishing self-adjointness, we need to do a complete example in which the domain of the adjoint is found.

Example 4.2.4

Determine $L^\dagger$ and its domain for operator $Lu = \frac{du}{dx}$ where u satisfies the boundary conditions $u(0) = 2u(1)$ on $[0, 1]$.

Solution

We need to find the adjoint operator satisfying $\langle v, Lu\rangle = \left\langle L^\dagger v, u\right\rangle$. Therefore, we rewrite the integral

$$\langle v, Lu\rangle\rangle = \int_0^1 v\frac{du}{dx}dx = uv|_0^1 - \int_0^1 u\frac{dv}{dx}dx = \left\langle L^+v, u\right\rangle.$$

From this we have the adjoint problem consisting of an adjoint operator and the associated boundary condition (or, domain of $L^\dagger$.):

1. $L^\dagger = -\dfrac{d}{dx}$.
2. $uv|_0^1 = 0 \Rightarrow 0 = u(1)[v(1) - 2v(0)] \Rightarrow v(1) = 2v(0)$.

Lagrange's and Green's Identities

Before turning to the proofs that the eigenvalues of a Sturm-Liouville problem are real and the associated eigenfunctions orthogonal, we will first need to introduce two important identities. For the Sturm-Liouville operator,

$$\mathcal{L} = \frac{d}{dx}\left(p\frac{d}{dx}\right) + q,$$

we have the two identities:

Definition 4.2.1: Lagrange's Identity

$$u\mathcal{L}v - v\mathcal{L}u = \left[p\left(uv' - vu'\right)\right]'.$$

Definition 4.2.2: Green's Identity

$$\int_a^b (u\mathcal{L}v - v\mathcal{L}u)dx = \left[p\left(uv' - vu'\right)\right]\Big|_a^b.$$

The proof of Lagrange's identity follows by a simple manipulations of the operator:

$$\begin{aligned} u\mathcal{L}v - v\mathcal{L}u &= u\left[\frac{d}{dx}\left(p\frac{dv}{dx}\right) + qv\right] - v\left[\frac{d}{dx}\left(p\frac{du}{dx}\right) + qu\right] \\ &= u\frac{d}{dx}\left(p\frac{dv}{dx}\right) - v\frac{d}{dx}\left(p\frac{du}{dx}\right) \\ &= u\frac{d}{dx}\left(p\frac{dv}{dx}\right) + p\frac{du}{dx}\frac{dv}{dx} - v\frac{d}{dx}\left(p\frac{du}{dx}\right) - p\frac{du}{dx}\frac{dv}{dx} \\ &= \frac{d}{dx}\left[pu\frac{dv}{dx} - pv\frac{du}{dx}\right]. \end{aligned} \tag{4.2.9}$$

Green's identity is simply proven by integrating Lagrange's identity.

Orthogonality and Reality

We are now ready to prove that the eigenvalues of a Sturm-Liouville problem are real and the corresponding eigenfunctions are orthogonal. These are easily established using Green's identity, which in turn is a statement about the Sturm-Liouville operator being self-adjoint.

Example 4.2.5

The eigenvalues of the Sturm-Liouville problem (4.1.3) are real.

Solution

Let $\phi_n(x)$ be a solution of the eigenvalue problem associated with λ_n :

$$\mathcal{L}\phi_n = -\lambda_n\sigma\phi_n.$$

We want to show that Namely, we show that $\bar{\lambda}_n = \lambda_n$, where the bar means complex conjugate. So, we also consider the complex conjugate of this equation,

$$\mathcal{L}\bar{\phi}_n = -\bar{\lambda}_n\sigma\bar{\phi}_n.$$

Now, multiply the first equation by $\bar{\phi}_n$, the second equation by ϕ_n, and then subtract the results. We obtain

$$\bar{\phi}_n\mathcal{L}\phi_n - \phi_n\mathcal{L}\bar{\phi}_n = \left(\bar{\lambda}_n - \lambda_n\right)\sigma\phi_n\bar{\phi}_n.$$

Integrating both sides of this equation, we have

$$\int_a^b \left(\bar{\phi}_n\mathcal{L}\phi_n - \phi_n\mathcal{L}\bar{\phi}_n\right)dx = \left(\bar{\lambda}_n - \lambda_n\right)\int_a^b \sigma\phi_n\bar{\phi}_n dx.$$

We apply Green's identity to the left hand side to find

$$\left[p\left(\bar{\phi}_n\phi_n' - \phi_n\bar{\phi}_n'\right)\right]_a^b = \left(\bar{\lambda}_n - \lambda_n\right)\int_a^b \sigma\phi_n\bar{\phi}_n dx.$$

Using the homogeneous boundary conditions (4.1.4) for a self-adjoint operator, the left side vanishes. This leaves

$$0 = \left(\bar{\lambda}_n - \lambda_n\right)\int_a^b \sigma\|\phi_n\|^2 dx.$$

The integral is nonnegative, so we must have $\bar{\lambda}_n = \lambda_n$. Therefore, the eigenvalues are real.

Example 4.2.6

The eigenfunctions corresponding to different eigenvalues of the Sturm-Liouville problem (4.3) are orthogonal.

Solution

This is proven similar to the last example. Let $\phi_n(x)$ be a solution of the eigenvalue problem associated with λ_n,

$$\mathcal{L}\phi_n = -\lambda_n\sigma\phi_n,$$

and let $\phi_m(x)$ be a solution of the eigenvalue problem associated with $\lambda_m \neq \lambda_n$,

$$\mathcal{L}\phi_m = -\lambda_m\sigma\phi_m,$$

Now, multiply the first equation by ϕ_m and the second equation by ϕ_n. Subtracting these results, we obtain

$$\phi_m\mathcal{L}\phi_n - \phi_n\mathcal{L}\phi_m = \left(\lambda_m - \lambda_n\right)\sigma\phi_n\phi_m$$

Integrating both sides of the equation, using Green's identity, and using the homogeneous boundary conditions, we obtain

$$0 = \left(\lambda_m - \lambda_n\right)\int_a^b \sigma\phi_n\phi_m dx.$$

Since the eigenvalues are distinct, we can divide by $\lambda_m - \lambda_n$, leaving the desired result,

$$\int_a^b \sigma\phi_n\phi_m dx = 0.$$

Therefore, the eigenfunctions are orthogonal with respect to the weight function $\sigma(x)$.

Rayleigh Quotient

The Rayleigh Quotient is useful for getting estimates of eigenvalues and proving some of the other properties associated with Sturm-Liouville eigenvalue problems. The Rayleigh quotient is general and finds applications for both matrix eigenvalue problems as well as self-adjoint operators. For a Hermitian matrix M the Rayleigh quotient is given by

$$R(\mathbf{v}) = \frac{\langle \mathbf{v}, M\mathbf{v}\rangle}{\langle \mathbf{v}, \mathbf{v}\rangle}.$$

One can show that the critical values of the Rayleigh quotient, as a function of $\mathbf{v}$, are the eigenvectors of M and the values of R at these critical values are the corresponding eigenvectors. In particular, minimizing $R(\mathbf{v}$ over the vector space will give the lowest eigenvalue. This leads to the Rayleigh-Ritz method for computing the lowest eigenvalues when the eigenvectors are not known.

This definition can easily be extended to Sturm-Liouville operators,

$$R\left(\phi_n\right) = \frac{\langle \phi_n\mathcal{L}\phi_n\rangle}{\langle \phi_n, \phi_n\rangle}.$$

We begin by multiplying the eigenvalue problem

$$\mathcal{L}\phi_n = -\lambda_n\sigma(x)\phi_n$$

by ϕ_n and integrating. This gives

$$\int_a^b \left[\phi_n \frac{d}{dx}\left(p\frac{d\phi_n}{dx}\right) + q\phi_n^2\right] dx = -\lambda_n \int_a^b \phi_n^2 \sigma dx$$

One can solve the last equation for λ to find

$$\lambda_n = \frac{-\int_a^b \left[\phi_n \frac{d}{dx}\left(p\frac{d\phi_n}{dx}\right) + q\phi_n^2\right] dx}{\int_a^b \phi_n^2 \sigma dx} = R(\phi_n).$$

It appears that we have solved for the eigenvalues and have not needed the machinery we had developed in Chapter 4 for studying boundary value problems. However, we really cannot evaluate this expression when we do not know the eigenfunctions, $\phi_n(x)$ yet. Nevertheless, we will see what we can determine from the Rayleigh quotient.

One can rewrite this result by performing an integration by parts on the first term in the numerator. Namely, pick $u = \phi_n$ and $dv = \frac{d}{dx}\left(p\frac{d\phi_n}{dx}\right) dx$ for the standard integration by parts formula. Then, we have

$$\int_a^b \phi_n \frac{d}{dx}\left(p\frac{d\phi_n}{dx}\right) dx = p\phi_n \frac{d\phi_n}{dx}\Big|_a^b - \int_a^b \left[p\left(\frac{d\phi_n}{dx}\right)^2 - q\phi_n^2\right] dx.$$

Inserting the new formula into the expression for λ, leads to the Rayleigh Quotient

$$\lambda_n = \frac{-p\phi_n \frac{d\phi_n}{dx}\Big|_a^b + \int_a^b \left[p\left(\frac{d\phi_n}{dx}\right)^2 - q\phi_n^2\right] dx}{\int_a^b \phi_n^2 \sigma dx}. \tag{4.2.10}$$

In many applications the sign of the eigenvalue is important. As we had seen in the solution of the heat equation, $T' + k\lambda T = 0$. Since we expect the heat energy to diffuse, the solutions should decay in time. Thus, we would expect $\lambda > 0$. In studying the wave equation, one expects vibrations and these are only possible with the correct sign of the eigenvalue (positive again). Thus, in order to have nonnegative eigenvalues, we see from (4.2.10) that

a. $q(x) \leq 0$, and
b. $-p\phi_n \frac{d\phi_n}{dx}\Big|_a^b \geq 0$.

Furthermore, if λ is a zero eigenvalue, then $q(x) \equiv 0$ and $\alpha_1 = \alpha_2 = 0$ in the homogeneous boundary conditions. This can be seen by setting the numerator equal to zero. Then, $q(x) = 0$ and $\phi_n'(x) = 0$. The second of these conditions inserted into the boundary conditions forces the restriction on the type of boundary conditions.

One of the properties of Sturm-Liouville eigenvalue problems with homogeneous boundary conditions is that the eigenvalues are ordered, $\lambda_1 < \lambda_2 < \ldots$. Thus, there is a smallest eigenvalue. It turns out that for any continuous function, $y(x)$,

$$\lambda_1 = \min_{y(x)} \frac{-py\frac{dy}{dx}\Big|_a^b + \int_a^b \left[p\left(\frac{dy}{dx}\right)^2 - qy^2\right] dx}{\int_a^b y^2 \sigma dx} \tag{4.2.11}$$

and this minimum is obtained when $y(x) = \phi_1(x)$. This result can be used to get estimates of the minimum eigenvalue by using trial functions which are continuous and satisfy the boundary conditions, but do not necessarily satisfy the differential equation.

Example 4.2.7

We have already solved the eigenvalue problem $\phi'' + \lambda\phi = 0$, $\phi(0) = 0, \phi(1) = 0$. In this case, the lowest eigenvalue is $\lambda_1 = \pi^2$. We can pick a nice function satisfying the boundary conditions, say $y(x) = x - x^2$. Inserting this into Equation (4.2.11), we find

$$\lambda_1 \leq \frac{\int_0^1 (1-2x)^2 dx}{\int_0^1 \left(x - x^2\right)^2 dx} = 10.$$

Indeed, $10 \geq \pi^2$.

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4.3: The Eigenfunction Expansion Method

In this section we solve the nonhomogeneous problem $\mathcal{L}y = f$ using expansions over the basis of Sturm-Liouville eigenfunctions. We have seen that Sturm-Liouville eigenvalue problems have the requisite set of orthogonal eigenfunctions. In this section we will apply the eigenfunction expansion method to solve a particular nonhomogeneous boundary value problem.

Recall that one starts with a nonhomogeneous differential equation

$$\mathcal{L}y = f,$$

where $y(x)$ is to satisfy given homogeneous boundary conditions. The method makes use of the eigenfunctions satisfying the eigenvalue problem

$$\mathcal{L}\phi_n = -\lambda_n \sigma \phi_n$$

subject to the given boundary conditions. Then, one assumes that $y(x)$ can be written as an expansion in the eigenfunctions,

$$y(x) = \sum_{n=1}^{\infty} c_n \phi_n(x),$$

and inserts the expansion into the nonhomogeneous equation. This gives

$$f(x) = \mathcal{L}\left(\sum_{n=1}^{\infty} c_n \phi_n(x)\right) = -\sum_{n=1}^{\infty} c_n \lambda_n \sigma(x) \phi_n(x).$$

The expansion coefficients are then found by making use of the orthogonality of the eigenfunctions. Namely, we multiply the last equation by $\phi_m(x)$ and integrate. We obtain

$$\int_a^b f(x)\phi_m(x)dx = -\sum_{n=1}^{\infty} c_n \lambda_n \int_a^b \phi_n(x)\phi_m(x)\sigma(x)dx.$$

Orthogonality yields

$$\int_a^b f(x)\phi_m(x)dx = -c_m \lambda_m \int_a^b \phi_m^2(x)\sigma(x)dx.$$

Solving for c_m, we have

$$c_m = -\frac{\int_a^b f(x)\phi_m(x)dx}{\lambda_m \int_a^b \phi_m^2(x)\sigma(x)dx}.$$

Example 4.3.1

As an example, we consider the solution of the boundary value problem

$$(xy')' + \frac{y}{x} = \frac{1}{x}, \quad x \in [1, e], \tag{4.3.1}$$

$$y(1) = 0 = y(e). \tag{4.3.2}$$

Solution

This equation is already in self-adjoint form. So, we know that the associated Sturm-Liouville eigenvalue problem has an orthogonal set of eigenfunctions. We first determine this set. Namely, we need to solve

$$(x\phi')' + \frac{\phi}{x} = -\lambda\sigma\phi, \quad \phi(1) = 0 = \phi(e). \tag{4.3.3}$$

Rearranging the terms and multiplying by x, we have that

$$x^2\phi'' + x\phi' + (1 + \lambda\sigma x)\phi = 0.$$

This is almost an equation of Cauchy-Euler type. Picking the weight function $\sigma(x) = \frac{1}{x}$, we have

$$x^2\phi'' + x\phi' + (1+\lambda)\phi = 0.$$

This is easily solved. The characteristic equation is

$$r^2 + (1+\lambda) = 0.$$

One obtains nontrivial solutions of the eigenvalue problem satisfying the boundary conditions when $\lambda > -1$. The solutions are

$$\phi_n(x) = A\sin(n\pi \ln x), \quad n = 1, 2, \ldots$$

where $\lambda_n = n^2\pi^2 - 1$.

It is often useful to normalize the eigenfunctions. This means that one chooses A so that the norm of each eigenfunction is one. Thus, we have

$$\begin{aligned} 1 &= \int_1^e \phi_n(x)^2 \sigma(x) dx \\ &= A^2 \int_1^e \sin(n\pi \ln x)\frac{1}{x} dx \\ &= A^2 \int_0^1 \sin(n\pi y) dy = \frac{1}{2}A^2. \end{aligned} \tag{4.3.4}$$

Thus, $A = \sqrt{2}$. Several of these eigenfunctions are show in Figure 4.3.1.

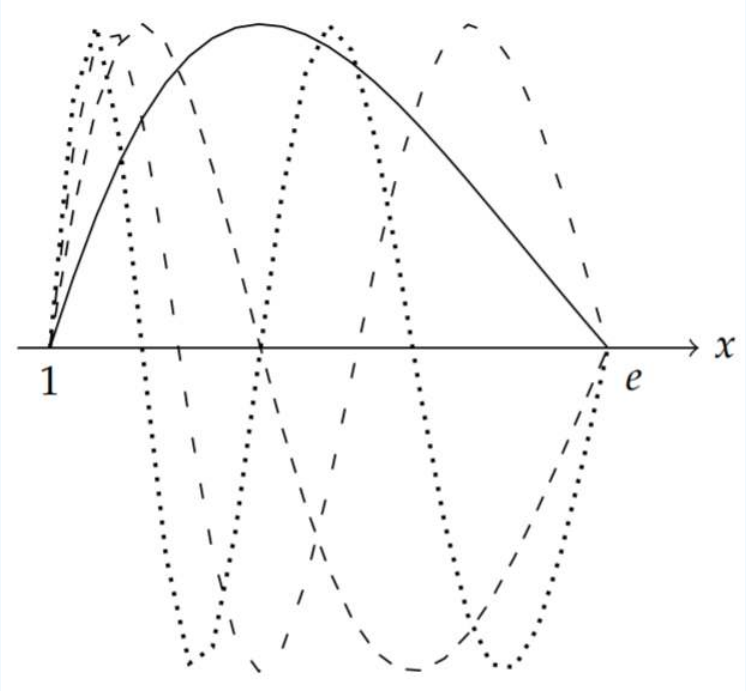


Figure 4.3.1: Plots of the first five eigenfunctions, $y(x) = \sqrt{2}\sin(n\pi \ln x)$.

We now turn towards solving the nonhomogeneous problem, $\mathcal{L}y = \frac{1}{x}$. We first expand the unknown solution in terms of the eigenfunctions,

$$y(x) = \sum_{n=1}^{\infty} c_n \sqrt{2}\sin(n\pi \ln x).$$

Inserting this solution into the differential equation, we have

$$\frac{1}{x} = \mathcal{L}y = -\sum_{n=1}^{\infty} c_n \lambda_n \sqrt{2}\sin(n\pi \ln x)\frac{1}{x}.$$

Next, we make use of orthogonality. Multiplying both sides by the eigenfunction $\phi_m(x) = \sqrt{2}\sin(m\pi \ln x)$ and integrating, gives

$$\lambda_m c_m = \int_1^e \sqrt{2}\sin(m\pi \ln x)\frac{1}{x}dx = \frac{\sqrt{2}}{m\pi}[(-1)^m - 1].$$

Solving for c_m, we have

$$c_m = \frac{\sqrt{2}}{m\pi}\frac{[(-1)^m - 1]}{m^2\pi^2 - 1}.$$

Finally, we insert these coefficients into the expansion for $y(x)$. The solution is then

$$y(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\frac{[(-1)^n - 1]}{n^2\pi^2 - 1}\sin(n\pi \ln(x)).$$

We plot this solution in Figure 4.3.2.

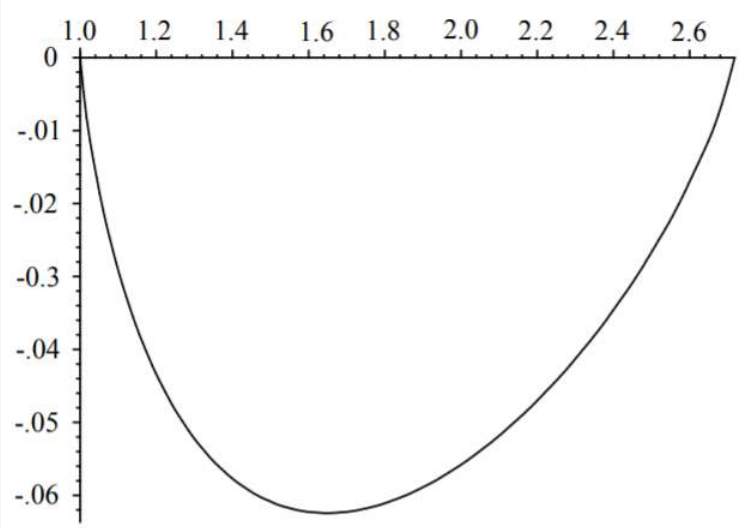


Figure 4.3.2: Plot of the solution in Example 4.3.1.

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4.4: Appendix- The Fredholm Alternative Theorem

Given that $Ly = f$, when can one expect to find a solution? Is it unique? These questions are answered by the Fredholm Alternative Theorem. This theorem occurs in many forms from a statement about solutions to systems of algebraic equations to solutions of boundary value problems and integral equations. The theorem comes in two parts, thus the term "alternative". Either the equation has exactly one solution for all f, or the equation has many solutions for some f 's and none for the rest.

The reader is familiar with the statements of the Fredholm Alternative for the solution of systems of algebraic equations. One seeks solutions of the system $Ax = b$ for A an $n \times m$ matrix. Defining the matrix adjoint, A^* through $\langle Ax, y\rangle = \langle x, A^* y\rangle$ for all $x, y, \in \mathcal{C}^n$, then either

Theorem 4.4.1: First Alternative

The equation $Ax = b$ has a solution if and only if $\langle b, v\rangle = 0$ for all v satisfying $A^* v = 0$.

or

Theorem 4.4.2: Second Alternative

A solution of $Ax = b$, if it exists, is unique if and only if $x = 0$ is the only solution of $Ax = 0$.

The second alternative is more familiar when given in the form: The solution of a nonhomogeneous system of n equations and n unknowns is unique if the only solution to the homogeneous problem is the zero solution. Or, equivalently, A is invertible, or has nonzero determinant.

Proof of Theorem 4.4.2

Proof

We prove the second theorem first. Assume that $Ax = 0$ for $x \neq 0$ and $Ax_0 = b$. Then $A(x_0 + \alpha x) = b$ for all α. Therefore, the solution is not unique. Conversely, if there are two different solutions, x_1 and x_2, satisfying $Ax_1 = b$ and $Ax_2 = b$, then one has a nonzero solution $x = x_1 - x_2$ such that $Ax = A(x_1 - x_2) = 0$.

The proof of the first part of the first theorem is simple. Let $A^* v = 0$ and $Ax_0 = b$. Then we have

$$\langle b, v\rangle = \langle Ax_0, v\rangle = \langle x_0, A^* v\rangle = 0.$$

For the second part we assume that $\langle b, v\rangle = 0$ for all v such that $A^* v = 0$. Write b as the sum of a part that is in the range of A and a part that in the space orthogonal to the range of $A, b = b_R + b_O$. Then, $0 = \langle b_O, Ax\rangle =< A^* b, x >$ for all x. Thus, $A^* b_O$. Since $\langle b, v\rangle = 0$ for all v in the nullspace of A^*, then $\langle b, b_O\rangle = 0$.

Therefore, $\langle b, v\rangle = 0$ implies that

$$0 = \langle b, b_O\rangle = \langle b_R + b_O, b_O\rangle = \langle b_O, b_O\rangle .$$

This means that $b_O = 0$, giving $b = b_R$ is in the range of A. So, $Ax = b$ has a solution.

Example 4.4.1

Determine the allowed forms of $\mathbf{b}$ for a solution of $A\mathbf{x} = \mathbf{b}$ to exist, where

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}.$$

Solution

First note that $A^* = \bar{A}^T$. This is seen by looking at

$$\begin{aligned}\langle A\mathbf{x},\mathbf{y}\rangle &= \langle \mathbf{x}, A^*\mathbf{y}\rangle \\ \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_j\bar{y}_i &= \sum_{j=1}^{n} x_j \sum_{j=1}^{n} a_{ij}\bar{y}_i \\ &= \sum_{j=1}^{n} x_j \sum_{j=1}^{n} \left(\bar{a}^T\right)_{ji} y_i. \end{aligned} \tag{4.4.1}$$

For this example,

$$A^* = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}.$$

We next solve $A^*\mathbf{v} = 0$. This means, $v_1 + 3v_2 = 0$. So, the nullspace of A^* is spanned by $\mathbf{v} = (3,-1)^T$. For a solution of $A\mathbf{x} = \mathbf{b}$ to exist, $\mathbf{b}$ would have to be orthogonal to $\mathbf{v}$. Therefore, a solution exists when

$$\mathbf{b} = \alpha \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

So, what does the Fredholm Alternative say about solutions of boundary value problems? We extend the Fredholm Alternative for linear operators. A more general statement would be

Theorem 4.4.3

If L is a bounded linear operator on a Hilbert space, then $Ly = f$ has a solution if and only if $\langle f, v\rangle = 0$ for every v such that $L^\dagger v = 0$.

The statement for boundary value problems is similar. However, we need to be careful to treat the boundary conditions in our statement. As we have seen, after several integrations by parts we have that

$$\langle \mathcal{L}u, v\rangle = S(u,v) + \left\langle u, \mathcal{L}^+ v\right\rangle,$$

where $S(u,v)$ involves the boundary conditions on u and v. Note that for nonhomogeneous boundary conditions, this term may no longer vanish.

Theorem 4.4.4

The solution of the boundary value problem $\mathcal{L}u = f$ with boundary conditions $Bu = g$ exists if and only if

$$\langle f, v\rangle - S(u,v) = 0$$

for all v satisfying $\mathcal{L}^+ v = 0$ and $B^+ v = 0$.

Example 4.4.2

Consider the problem

$$u'' + u = f(x), \quad u(0) - u(2\pi) = \alpha, u'(0) - u'(2\pi) = \beta.$$

Solution

Only certain values of α and β will lead to solutions. We first note that

$$L = L^+ = \frac{d^2}{dx^2} + 1.$$

Solutions of

$$L^\dagger v = 0, \quad v(0) - v(2\pi) = 0, v'(0) - v'(2\pi) = 0$$

are easily found to be linear combinations of $v = \sin x$ and $v = \cos x$.

Next, one computes

$$\begin{aligned} S(u,v) &= [u'v - uv']_0^{2\pi} \\ &= u'(2\pi)v(2\pi) - u(2\pi)v'(2\pi) - u'(0)v(0) + u(0)v'(0). \end{aligned} \tag{4.4.2}$$

For $v(x) = \sin x$, this yields

$$S(u, \sin x) = -u(2\pi) + u(0) = \alpha.$$

Similarly,

$$S(u, \cos x) = \beta.$$

Using $\langle f, v\rangle - S(u,v) = 0$, this leads to the conditions that we were seeking,

$$\int_0^{2\pi} f(x) \sin x dx = \alpha,$$
$$\int_0^{2\pi} f(x) \cos x dx = \beta.$$

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4.5: Problems

Exercise 4.5.1

Prove the if $u(x)$ and $v(x)$ satisfy the general homogeneous boundary conditions

$$\begin{aligned} \alpha_1 u(a)+\beta_1 u'(a)=0, \\ \alpha_2 u(b)+\beta_2 u'(b)=0 \end{aligned} \tag{4.5.1}$$

at $x=a$ and $x=b$, then

$$p(x)\left[u(x)v'(x)-v(x)u'(x)\right]_{x=a}^{x=b}=0.$$

Exercise 4.5.2

Prove Green's Identity $\int_a^b (u\mathcal{L}v-v\mathcal{L}u)dx=\left[p\left(uv'-vu'\right)\right]\Big|_a^b$ for the general Sturm-Liouville operator $\mathcal{L}$.

Exercise 4.5.3

Find the adjoint operator and its domain for $Lu=u''+4u'-3u, u'(0)+\ 4u(0)=0, u'(1)+4u(1)=0$.

Exercise 4.5.4

Show that a Sturm-Liouville operator with periodic boundary conditions on $[a,b]$ is self-adjoint if and only if $p(a)=p(b)$. [Recall, periodic boundary conditions are given as $u(a)=u(b)$ and $u'(a)=u'(b)$.]

Exercise 4.5.5

The Hermite differential equation is given by $y''-2xy'+\lambda y=0$. Rewrite this equation in self-adjoint form. From the Sturm-Liouville form obtained, verify that the differential operator is self adjoint on $(-\infty,\infty)$. Give the integral form for the orthogonality of the eigenfunctions.

Exercise 4.5.6

Find the eigenvalues and eigenfunctions of the given Sturm-Liouville problems.

a. $y''+\lambda y=0, y'(0)=0=y'(\pi)$.
b. $(xy')'+\frac{\lambda}{x}y=0, y(1)=y\left(e^2\right)=0$.

Exercise 4.5.7

The eigenvalue problem $x^2y''-\lambda xy'+\lambda y=0$ with $y(1)=y(2)=0$ is not a Sturm-Liouville eigenvalue problem. Show that none of the eigenvalues are real by solving this eigenvalue problem.

Exercise 4.5.8

In Example 4.2.7 we found a bound on the lowest eigenvalue for the given eigenvalue problem.

a. Verify the computation in the example.
b. Is this an upper bound on λ_1
c. Use the Rayleigh quotient to obtain a good upper bound for the lowest eigenvalue of the eigenvalue problem:
$\phi''+\left(\lambda-x^2\right)\phi=0$, $\phi(0)=0, \phi'(1)=0$.

Exercise 4.5.9

Use the method of eigenfunction expansions to solve the problems:

a. $y'' = x^2, \quad y(0) = y(1) = 0$.
b. $y'' + 4y = x^2, \quad y'(0) = y'(1) = 0$.

Exercise 4.5.10

Determine the solvability conditions for the nonhomogeneous boundary value problem: $u'' + 4u = f(x), u(0) = \alpha, u'(\pi/4) = \beta$.

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CHAPTER OVERVIEW

5: Non-sinusoidal Harmonics and Special Functions

> *To the pure geometer the radius of curvature is an incidental characteristic-like the grin of the Cheshire cat. To the physicist it is an indispensable characteristic. It would be going too far to say that to the physicist the cat is merely incidental to the grin. Physics is concerned with interrelatedness such as the interrelatedness of cats and grins. In this case the "cat without a grin" and the "grin without a cat" are equally set aside as purely mathematical phantasies.*
>
> *~ Sir Arthur Stanley Eddington (1882-1944)*

In this chapter we provide a glimpse into generalized Fourier series in which the normal modes of oscillation are not sinusoidal. For vibrating strings, we saw that the harmonics were sinusoidal basis functions for a large, infinite dimensional, function space. Now, we will extend these ideas to non-sinusoidal harmonics and explore the underlying structure behind these ideas. In particular, we will explore Legendre polynomials and Bessel functions which will later arise in problems having cylindrical or spherical symmetry.

The background for the study of generalized Fourier series is that of function spaces. We begin by exploring the general context in which one finds oneself when discussing Fourier series and (later) Fourier transforms. We can view the sine and cosine functions in the Fourier trigonometric series representations as basis vectors in an infinite dimensional function space. A given function in that space may then be represented as a linear combination over this infinite basis. With this in mind, we might wonder

- Do we have enough basis vectors for the function space?
- Are the infinite series expansions convergent?
- What functions can be represented by such expansions?

In the context of the boundary value problems which typically appear in physics, one is led to the study of boundary value problems in the form of Sturm-Liouville eigenvalue problems. These lead to an appropriate set of basis vectors for the function space under consideration. We will touch a little on these ideas, leaving some of the deeper results for more advanced We note that the above determination of vector components for finite dimensional spaces is precisely what we had done to compute the Fourier coefficients using trigonometric bases. Reading further, you will see how this works. courses in mathematics. For now, we will turn to function spaces and explore some typical basis functions, many which originated from the study of physical problems. The common basis functions are often referred to as special functions in physics. Examples are the classical orthogonal polynomials (Legendre, Hermite, Laguerre, Tchebychef) and Bessel functions. But first we will introduce function spaces.

5.1: Function Spaces
5.2: Classical Orthogonal Polynomials
5.3: Fourier-Legendre Series
5.4: Gamma Function
5.5: Fourier-Bessel Series
5.6: Appendix- The Least Squares Approximation
5.7: Problems

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5.1: Function Spaces

Earlier we studied finite dimensional vector spaces. Given a set of basis vectors, $\{\mathbf{a}_k\}_{k=1}^n$, in vector space V, we showed that we can expand any vector $\mathbf{v} \in \mathbf{V}$ in terms of this basis, $\mathbf{v} = \sum_{k=1}^n v_k \mathbf{a}_k$. We then spent some time looking at the simple case of extracting the components v_k of the vector. The keys to doing this simply were to have a scalar product and an orthogonal basis set. These are also the key ingredients that we will need in the infinite dimensional case. In fact, we had already done this when we studied Fourier series.

Recall when we found Fourier trigonometric series representations of functions, we started with a function (vector) that we wanted to expand in a set of trigonometric functions (basis) and we sought the Fourier coefficients (components). In this section we will extend our notions from finite dimensional spaces to infinite dimensional spaces and we will develop the needed background in which to think about more general Fourier series expansions. This conceptual framework is very important in other areas in mathematics (such as ordinary and partial differential equations) and physics (such as quantum mechanics and electrodynamics).

We will consider various infinite dimensional function spaces. Functions in these spaces would differ by their properties. For example, we could consider the space of continuous functions on $[0, 1]$, the space of differentiably continuous functions, or the set of functions integrable from a to b. As you will see, there are many types of function spaces. In order to view these spaces as vector spaces, we will need to be able to add functions and multiply them by scalars in such as way that they satisfy the definition of a vector space as defined in Chapter 3.

We will also need a scalar product defined on this space of functions. There are several types of scalar products, or inner products, that we can define. An inner product $\langle , \rangle$ on a real vector space V is a mapping from $V \times V$ into R such that for $u, v, w \in V$ and $\alpha \in R$ one has

1. $\langle v, v \rangle \geq 0$ and $\langle v, v \rangle = 0$ iff $v = 0$.
2. $\langle v, w \rangle = \langle w, v \rangle$.
3. $\langle \alpha v, w \rangle = \alpha \langle v, w \rangle$.
4. $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$.

A real vector space equipped with the above inner product leads to what is called a real inner product space. For complex inner product spaces the above properties hold with the third property replaced with $\langle v, w \rangle = \overline{\langle w, v \rangle}$.

For the time being, we will only deal with real valued functions and, thus, we will need an inner product appropriate for such spaces. One such definition is the following. Let $f(x)$ and $g(x)$ be functions defined on $[a, b]$ and introduce the weight function $\sigma(x) > 0$. Then, we define the inner product, if the integral exists, as

$$\langle f, g \rangle = \int_a^b f(x) g(x) \sigma(x) dx. \tag{5.1.1}$$

Spaces in which $\langle f, f \rangle \langle \infty$ under this inner product are called the space of square integrable functions on (a, b) under weight σ and denoted as $L^2_\sigma(a, b)$. In what follows, we will assume for simplicity that $\sigma(x) = 1$. This is possible to do by using a change of variables.

Now that we have function spaces equipped with an inner product, we seek a basis for the space. For an n-dimensional space we need n basis vectors. For an infinite dimensional space, how many will we need? How do we know when we have enough? We will provide some answers to these questions later.

Let's assume that we have a basis of functions $\{\phi_n(x)\}_{n=1}^\infty$. Given a function $f(x)$, how can we go about finding the components of f in this basis? In other words, let

$$f(x) = \sum_{n=1}^\infty c_n \phi_n(x).$$

How do we find the c_n 's? Does this remind you of Fourier series expansions? Does it remind you of the problem we had earlier for finite dimensional spaces? [You may want to review the discussion at the end of Section ?? as you read the next derivation.]

Formally, we take the inner product of f with each ϕ_j and use the properties of the inner product to find

$$\begin{aligned}\langle \phi_j, f\rangle &= \left\langle \phi_j, \sum_{n=1}^{\infty} c_n \phi_n \right\rangle \\ &= \sum_{n=1}^{\infty} c_n \langle \phi_j, \phi_n\rangle . \end{aligned} \tag{5.1.2}$$

If the basis is an orthogonal basis, then we have

$$\langle \phi_j, \phi_n\rangle = N_j \delta_{jn}, \tag{5.1.3}$$

where δ_{in} is the Kronecker delta. Recall from Chapter 3 that the Kronecker delta is defined as

$$\delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j. \end{cases} \tag{5.1.4}$$

Note

For the generalized Fourier series expansion $f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)$, we have determined the generalized Fourier coefficients to be $c_j = \langle \phi_j, f\rangle / \langle \phi_j, \phi_j\rangle$.

Continuing with the derivation, we have

$$\begin{aligned}\langle \phi_j, f\rangle &= \sum_{n=1}^{\infty} c_n \langle \phi_j, \phi_n\rangle \\ &= \sum_{n=1}^{\infty} c_n N_j \delta_{jn} \end{aligned} \tag{5.1.5}$$

Expanding the sum, we see that the Kronecker delta picks out one nonzero term:

$$\begin{aligned}\langle \phi_j, f\rangle &= c_1 N_j \delta_{j1} + c_2 N_j \delta_{j2} + \ldots + c_j N_j \delta_{jj} + \ldots \\ &= c_j N_j \end{aligned} \tag{5.1.6}$$

So, the expansion coefficients are

$$c_j = \frac{\langle \phi_j, f\rangle}{N_j} = \frac{\langle \phi_j, f\rangle}{\langle \phi_j, \phi_j\rangle} \quad j = 1, 2, \ldots$$

We summarize this important result:

Generalized Basis Expansion

Let $f(x)$ be represented by an expansion over a basis of orthogonal functions, $\{\phi_n(x)\}_{n=1}^{\infty}$,

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x).$$

Then, the expansion coefficients are formally determined as

$$c_n = \frac{\langle \phi_n, f\rangle}{\langle \phi_n, \phi_n\rangle}.$$

This will be referred to as the general Fourier series expansion and the c_j 's are called the Fourier coefficients. Technically, equality only holds when the infinite series converges to the given function on the interval of interest.

Example 5.1.1

Find the coefficients of the Fourier sine series expansion of $f(x)$, given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \quad x \in [-\pi, \pi].$$

Solution

In the last chapter we already established that the set of functions $\phi_n(x) = \sin nx$ for $n = 1, 2, \ldots$ is orthogonal on the interval $[-\pi, \pi]$. Recall that using trigonometric identities, we have for $n \neq m$

$$\langle \phi_n, \phi_m \rangle = \int_{-\pi}^{\pi} \sin nx \sin mx dx = \pi \delta_{nm}. \tag{5.1.7}$$

Therefore, the set $\phi_n(x) = \sin nx$ for $n = 1, 2, \ldots$ is an orthogonal set of functions on the interval $[-\pi, \pi]$.

We determine the expansion coefficients using

$$b_n = \frac{\langle \phi_n, f \rangle}{N_n} = \frac{\langle \phi_n, f \rangle}{\langle \phi_n, \phi_n \rangle} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx.$$

Does this result look familiar?

Just as with vectors in three dimensions, we can normalize these basis functions to arrive at an orthonormal basis. This is simply done by dividing by the length of the vector. Recall that the length of a vector is obtained as $v = \sqrt{\mathbf{v} \cdot \mathbf{v}}$. In the same way, we define the norm of a function by

$$\|f\| = \sqrt{\langle f, f \rangle}.$$

Note, there are many types of norms, but this induced norm will be sufficient.[1]

For this example, the norms of the basis functions are $\|\phi_n\| = \sqrt{\pi}$. Defining $\psi_n(x) = \frac{1}{\sqrt{\pi}} \phi_n(x)$, we can normalize the ϕ_n 's and have obtained an orthonormal basis of functions on $[-\pi, \pi]$.

We can also use the normalized basis to determine the expansion coefficients. In this case we have

$$b_n = \frac{\langle \psi_n, f \rangle}{N_n} = \langle \psi_n, f \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx.$$

Footnotes

[1] The norm defined here is the natural, or induced, norm on the inner product space. Norms are a generalization of the concept of lengths of vectors. Denoting $||\mathbf{v}||$ the norm of $\mathbf{v}$, it needs to satisfy the properties.

1. $||\mathbf{v}|| \geq 0$. $||\mathbf{v}|| = 0$ if and only if $\mathbf{v} = \mathbf{0}$.
2. $||\alpha \mathbf{v}|| = |\alpha| \, ||\mathbf{v}||$.
3. $||\mathbf{u} + \mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||$.

Examples of common norms are

1.
2.
3.

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5.2: Classical Orthogonal Polynomials

There are other basis functions that can be used to develop series representations of functions. In this section we introduce the classical orthogonal polynomials. We begin by noting that the sequence of functions $\{1, x, x^2, \ldots\}$ is a basis of linearly independent functions. In fact, by the Stone-Weierstraß Approximation Theorem [1] this set is a basis of $L^2_\sigma(a,b)$, the space of square integrable functions over the interval $[a,b]$ relative to weight $\sigma(x)$. However, we will show that the sequence of functions $\{1, x, x^2, \ldots\}$ does not provide an orthogonal basis for these spaces. We will then proceed to find an appropriate orthogonal basis of functions.

Theorem 5.2.1: Stone-Weierstraβ Approximation Theorem

Suppose f is a continuous function defined on the interval $[a,b]$. For every $e > 0$, there exists a polynomial function $P(x)$ such that for all $x \in [a,b]$, we have $|f(x) - P(x)| < e$. Therefore, every continuous function defined on $[a,b]$ can be uniformly approximated as closely as we wish by a polynomial function.

We are familiar with being able to expand functions over the basis $\{1, x, x^2, \ldots$ since these expansions are just Maclaurin series representations of the functions about $x = 0$,

$$f(x) \sim \sum_{n=0}^{\infty} c_n x^n.$$

However, this basis is not an orthogonal set of basis functions. One can easily see this by integrating the product of two even, or two odd, basis functions with $\sigma(x) = 1$ and $(a,b) = (-1,1)$. For example,

$$\int_{-1}^{1} x^0 x^2 dx = \frac{2}{3}.$$

Since we have found that orthogonal bases have been useful in determining the coefficients for expansions of given functions, we might ask, "Given a set of linearly independent basis vectors, can one find an orthogonal basis of the given space?" The answer is yes. We recall from introductory linear algebra, which mostly covers finite dimensional vector spaces, that there is a method for carrying this out called the Gram-Schmidt Orthogonalization Process. We will review this process for finite dimensional vectors and then generalize to function spaces.

Let's assume that we have three vectors that span the usual three dimensional space, $\mathbf{R}^3$, given by $\mathbf{a}_1, \mathbf{a}_2$, and $\mathbf{a}_3$ and shown in Figure 5.2.1. We seek an orthogonal basis $\mathbf{e}_1, \mathbf{e}_2$, and $\mathbf{e}_3$, beginning one vector at a time.

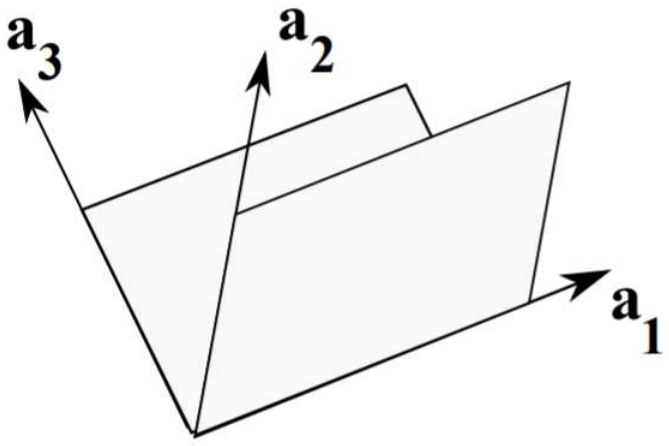


Figure 5.2.1: The basis $\mathbf{a}_1, \mathbf{a}_2$, and $\mathbf{a}_3$, of $\mathbf{R}^3$.

First we take one of the original basis vectors, say $\mathbf{a}_1$, and define

$$\mathbf{e}_1 = \mathbf{a}_1.$$

It is sometimes useful to normalize these basis vectors, denoting such a normalized vector with a "hat":

$$\hat{\mathbf{e}}_1 = \frac{\mathbf{e}_1}{e_1}$$

where $e_1 = \sqrt{\mathbf{e}_1 \cdot \mathbf{e}_1}$.

Next, we want to determine an $\mathbf{e}_2$ that is orthogonal to $\mathbf{e}_1$. We take another element of the original basis, $\mathbf{a}_2$. In Figure 5.2.2 we show the orientation of the vectors. Note that the desired orthogonal vector is $\mathbf{e}_2$. We can now write $\mathbf{a}_2$ as the sum of $\mathbf{e}_2$ and the projection of $\mathbf{a}_2$ on $\mathbf{e}_1$. Denoting this projection by $\mathbf{pr}_1\mathbf{a}_2$, we then have

$$\mathbf{e}_2 = \mathbf{a}_2 - \mathbf{pr}_1\mathbf{a}_2. \tag{5.2.1}$$

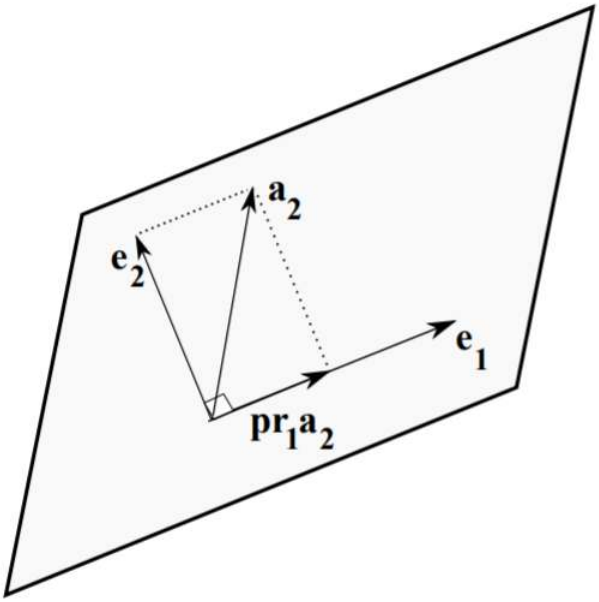


Figure 5.2.2: A plot of the vectors $\mathbf{e}_1, \mathbf{a}_2$, and $\mathbf{e}_2$ needed to find the projection of a_2, on e_1.

Recall the projection of one vector onto another from your vector calculus class.

$$\mathbf{pr}_1\mathbf{a}_2 = \frac{\mathbf{a}_2 \cdot \mathbf{e}_1}{e_1^2}\mathbf{e}_1. \tag{5.2.2}$$

This is easily proven by writing the projection as a vector of length $a_2 \cos\theta$ in direction $\hat{\mathbf{e}}_1$, where θ is the angle between $\mathbf{e}_1$ and $\mathbf{a}_2$. Using the definition of the dot product, $\mathbf{a} \cdot \mathbf{b} = ab\cos\theta$, the projection formula follows.

Combining Equations (5.2.1)-(5.2.2), we find that

$$\mathbf{e}_2 = \mathbf{a}_2 - \frac{\mathbf{a}_2 \cdot \mathbf{e}_1}{e_1^2}\mathbf{e}_1. \tag{5.2.3}$$

It is a simple matter to verify that $\mathbf{e}_2$ is orthogonal to $\mathbf{e}_1$:

$$\begin{aligned} \mathbf{e}_2 \cdot \mathbf{e}_1 &= \mathbf{a}_2 \cdot \mathbf{e}_1 - \frac{\mathbf{a}_2 \cdot \mathbf{e}_1}{e_1^2}\mathbf{e}_1 \cdot \mathbf{e}_1 \\ &= \mathbf{a}_2 \cdot \mathbf{e}_1 - \mathbf{a}_2 \cdot \mathbf{e}_1 = 0. \end{aligned} \tag{5.2.4}$$

Next, we seek a third vector $\mathbf{e}_3$ that is orthogonal to both $\mathbf{e}_1$ and $\mathbf{e}_2$. Pictorially, we can write the given vector $\mathbf{a}_3$ as a combination of vector projections along $\mathbf{e}_1$ and $\mathbf{e}_2$ with the new vector. This is shown in Figure 5.2.3. Thus, we can see that

$$\mathbf{e}_3 = \mathbf{a}_3 - \frac{\mathbf{a}_3 \cdot \mathbf{e}_1}{e_1^2}\mathbf{e}_1 - \frac{\mathbf{a}_3 \cdot \mathbf{e}_2}{e_2^2}\mathbf{e}_2. \tag{5.2.5}$$

Again, it is a simple matter to compute the scalar products with $\mathbf{e}_1$ and $\mathbf{e}_2$ to verify orthogonality.

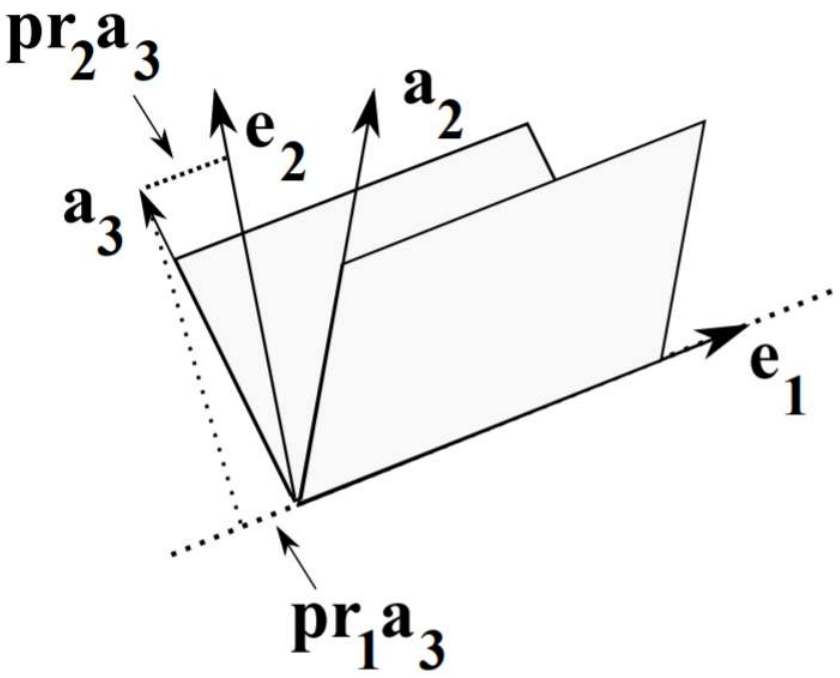


Figure 5.2.3: A plot of vectors for determining $\mathbf{e}_3$.

We can easily generalize this procedure to the N-dimensional case. Let $\mathbf{a}_n, n = 1, \ldots, N$ be a set of linearly independent vectors in $\mathbf{R}^N$. Then, an orthogonal basis can be found by setting $\mathbf{e}_1 = \mathbf{a}_1$ and defining

$$\mathbf{e}_n = \mathbf{a}_n - \sum_{j=1}^{n-1} \frac{\mathbf{a}_n \cdot \mathbf{e}_j}{e_j^2} \mathbf{e}_j, \quad n = 2, 3, \ldots, N \tag{5.2.6}$$

Now, we can generalize this idea to (real) function spaces. Let $f_n(x)$, $n \in N_0 = \{0, 1, 2, \ldots\}$, be a linearly independent sequence of continuous functions defined for $x \in [a, b]$. Then, an orthogonal basis of functions, $\phi_n(x), n \in N_0$ can be found and is given by

$$\phi_0(x) = f_0(x)$$

and

$$\phi_n(x) = f_n(x) - \sum_{j=0}^{n-1} \frac{\langle f_n, \phi_j \rangle}{\|\phi_j\|^2} \phi_j(x), \quad n = 1, 2, \ldots. \tag{5.2.7}$$

Here we are using inner products relative to weight $\sigma(x)$,

$$\langle f, g \rangle = \int_a^b f(x) g(x) \sigma(x) dx. \tag{5.2.8}$$

Note the similarity between the orthogonal basis in (5.2.7) and the expression for the finite dimensional case in Equation (5.2.6).

Example 5.2.1

Apply the Gram-Schmidt Orthogonalization process to the set $f_n(x) = x^n, n \in N_0$, when $x \in (-1, 1)$ and $\sigma(x) = 1$.

Solution

First, we have $\phi_0(x) = f_0(x) = 1$. Note that

$$\int_{-1}^{1} \phi_0^2(x) dx = 2.$$

We could use this result to fix the normalization of the new basis, but we will hold off doing that for now.

Now, we compute the second basis element:

$$\begin{aligned}\phi_1(x) &= f_1(x) - \frac{\langle f_1, \phi_0\rangle}{\|\phi_0\|^2}\phi_0(x)\\ &= x - \frac{\langle x, 1\rangle}{\|1\|^2}1 = x,\end{aligned} \tag{5.2.9}$$

since $\langle x, 1\rangle$ is the integral of an odd function over a symmetric interval.

For $\phi_2(x)$, we have

$$\begin{aligned}\phi_2(x) &= f_2(x) - \frac{\langle f_2, \phi_0\rangle}{\|\phi_0\|^2}\phi_0(x) - \frac{\langle f_2, \phi_1\rangle}{\|\phi_1\|^2}\phi_1(x)\\ &= x^2 - \frac{\langle x^2, 1\rangle}{\|1\|^2}1 - \frac{\langle x^2, x\rangle}{\|x\|^2}x\\ &= x^2 - \frac{\int_{-1}^{1} x^2 dx}{\int_{-1}^{1} dx}\\ &= x^2 - \frac{1}{3}\end{aligned} \tag{5.2.10}$$

So far, we have the orthogonal set $\left\{1, x, x^2 - \frac{1}{3}\right\}$. If one chooses to normalize these by forcing $\phi_n(1) = 1$, then one obtains the classical Legendre polynomials, $P_n(x)$. Thus,

$$P_2(x) = \frac{1}{2}\left(3x^2 - 1\right).$$

Note that this normalization is different than the usual one. In fact, we see the $P_2(x)$ does not have a unit norm,

$$\|P_2\|^2 = \int_{-1}^{1} P_2^2(x)dx = \frac{2}{5}.$$

The set of Legendre[2] polynomials is just one set of classical orthogonal polynomials that can be obtained in this way. Many of these special functions had originally appeared as solutions of important boundary value problems in physics. They all have similar properties and we will just elaborate some of these for the Legendre functions in the next section. Others in this group are shown in Table 5.2.1.

Note

Adrien-Marie Legendre (1752-1833) was a French mathematician who made many contributions to analysis and algebra.

Table 5.2.1: Common classical orthogonal polynomials with the interval and weight function used to define them.

Polynomial	Symbol	Interval	$\sigma(x)$
Hermite	$H_n(x)$	$(-\infty, \infty)$	e^{-x^2}
Laguerre	$L_n^{\alpha}(x)$	$[0, \infty)$	e^{-x}
Legendre	$P_n(x)$	$(-1, 1)$	1
Gegenbauer	$C_n^{\lambda}(x)$	$(-1, 1)$	$\left(1 - x^2\right)^{\lambda - 1/2}$
Tchebychef of the 1st kind	$T_n(x)$	$(-1, 1)$	$\left(1 - x^2\right)^{-1/2}$
Tchebychef of the 2nd kind	$U_n(x)$	$(-1, 1)$	$\left(1 - x^2\right)^{-1/2}$
Jacobi	$P_n^{(v,\mu)}(x)$	$(-1, 1)$	$(1 - x)^v(1 - x)^{\mu}$

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5.3: Fourier-Legendre Series

In the last chapter we saw how useful Fourier series expansions were for solving the heat and wave equations. In Chapter 6 we will investigate partial differential equations in higher dimensions and find that problems with spherical symmetry may lead to the series representations in terms of a basis of Legendre polynomials. For example, we could consider the steady state temperature distribution inside a hemispherical igloo, which takes the form

$$\phi(r,\theta) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta)$$

in spherical coordinates. Evaluating this function at the surface $r = a$ as $\phi(a,\theta) = f(\theta)$, leads to a Fourier-Legendre series expansion of function f:

$$f(\theta) = \sum_{n=0}^{\infty} c_n P_n(\cos\theta),$$

where $c_n = A_n a^n$

In this section we would like to explore Fourier-Legendre series expansions of functions $f(x)$ defined on $(-1,1)$:

$$f(x) \sim \sum_{n=0}^{\infty} c_n P_n(x). \tag{5.3.1}$$

As with Fourier trigonometric series, we can determine the expansion coefficients by multiplying both sides of Equation (5.3.1) by $P_m(x)$ and integrating for $x \in [-1,1]$. Orthogonality gives the usual form for the generalized Fourier coefficients,

$$c_n = \frac{\langle f, P_n\rangle}{\|P_n\|^2}, n = 0, 1, \ldots$$

We will later show that

$$\|P_n\|^2 = \frac{2}{2n+1}.$$

Therefore, the Fourier-Legendre coefficients are

$$c_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)dx. \tag{5.3.2}$$

Properties of Legendre Polynomials

We can do examples of Fourier-Legendre Expansions given just a few facts about Legendre polynomials. The first property that the Legendre polynomials have is the Rodrigues formula:

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left(x^2-1\right)^n, \quad n \in N_0. \tag{5.3.3}$$

From the Rodrigues formula, one can show that $P_n(x)$ is an nth degree polynomial. Also, for n odd, the polynomial is an odd function and for n even, the polynomial is an even function.

Example 5.3.1

Determine $P_2(x)$ from Rodrigues formula:

Solution

$$
\begin{aligned}
P_2(x) &= \frac{1}{2^2 2!}\frac{d^2}{dx^2}\left(x^2-1\right)^2 \\
&= \frac{1}{8}\frac{d^2}{dx^2}\left(x^4-2x^2+1\right) \\
&= \frac{1}{8}\frac{d}{dx}\left(4x^3-4x\right) \\
&= \frac{1}{8}\left(12x^2-4\right) \\
&= \frac{1}{2}\left(3x^2-1\right). \qquad (5.3.4)
\end{aligned}
$$

Note that we get the same result as we found in the last section using orthogonalization.

The first several Legendre polynomials are given in Table 5.3.1. In Figure 5.3.1 we show plots of these Legendre polynomials.

Table 5.3.1: Tabular computation of the Legendre polynomials using the Rodrigues formula.

n	$\left(x^2-1\right)^n$	$\frac{d^n}{dx^n}\left(x^2-1\right)^n$	$\frac{1}{2^n n!}$	$P_n(x)$
O	1	1	1	1
1	x^2-1	$2x$	$\frac{1}{2}$	x
2	x^4-2x^2+1	$12x^2-4$	$\frac{1}{8}$	$\frac{1}{2}\left(3x^2-1\right)$
3	$x^6-3x^4+3x^2-1$	$120x^3-72x$	$\frac{1}{48}$	$\frac{1}{2}\left(5x^3-3x\right)$

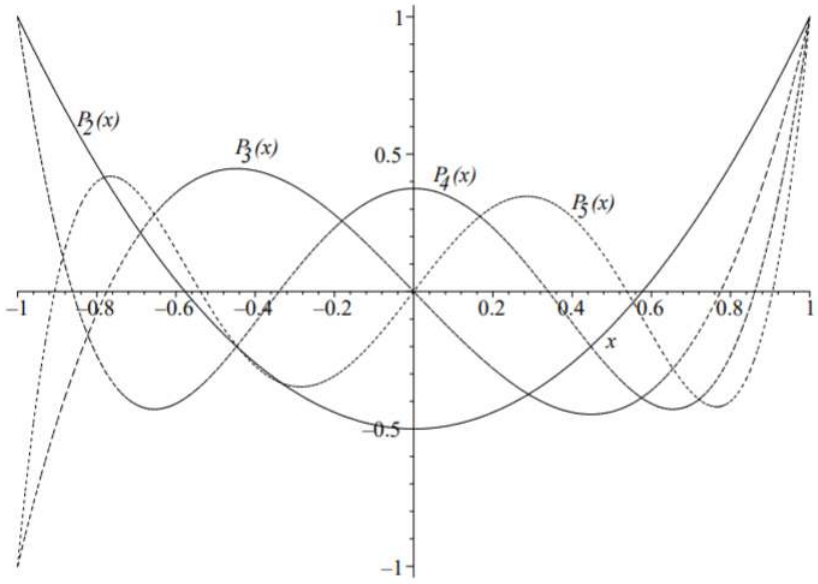


Figure 5.3.1: Plots of the Legendre polynomials $P_2(x), P_3(x), P_4(x)$, and $P_5(x)$.

Note

The first proof of the three term recursion formula is based upon the nature of the Legendre polynomials as an orthogonal basis, while the second proof is derived using generating functions.

All of the classical orthogonal polynomials satisfy a three term recursion formula (or, recurrence relation or formula). In the case of the Legendre

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x), \quad n = 1, 2, \ldots. \qquad (5.3.5)$$

This can also be rewritten by replacing n with $n-1$ as

$$(2n-1)xP_{n-1}(x) = nP_n(x) + (n-1)P_{n-2}(x), \quad n = 1, 2, \ldots \tag{5.3.6}$$

Example 5.3.2

Use the recursion formula to find $P_2(x)$ and $P_3(x)$, given that $P_0(x) = 1$ and $P_1(x) = x$.

Solution

We first begin by inserting $n = 1$ into Equation (5.3.5):

$$2P_2(x) = 3xP_1(x) - P_0(x) = 3x^2 - 1.$$

For $n = 2$, we have

$$\begin{aligned} 3P_3(x) &= 5xP_2(x) - 2P_1(x) \\ &= \frac{5}{2}x\left(3x^2 - 1\right) - 2x \\ &= \frac{1}{2}\left(15x^3 - 9x\right). \end{aligned} \tag{5.3.7}$$

This gives $P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right)$. These expressions agree with the earlier results.

We will prove the three term recursion formula in two ways. First we polynomials, we have use the orthogonality properties of Legendre polynomials and the following lemma.

Lemma 5.3.1

The leading coefficient of x^n in $P_n(x)$ is $\frac{1}{2^n n!}\frac{(2n)!}{n!}$.

Proof

We can prove this using the Rodrigues formula. First, we focus on the leading coefficient of $\left(x^2 - 1\right)^n$, which is x^{2n}. The first derivative of x^{2n} is $2nx^{2n-1}$. The second derivative is $2n(2n-1)x^{2n-2}$. The j th derivative is

$$\frac{d^j x^{2n}}{dx^j} = [2n(2n-1)\ldots(2n-j+1)]x^{2n-j}.$$

Thus, the nth derivative is given by

$$\frac{d^n x^{2n}}{dx^n} = [2n(2n-1)\ldots(n+1)]x^n.$$

This proves that $P_n(x)$ has degree n. The leading coefficient of $P_n(x)$ can now be written as

$$\begin{aligned} \frac{[2n(2n-1)\ldots(n+1)]}{2^n n!} &= \frac{[2n(2n-1)\ldots(n+1)]}{2^n n!}\frac{n(n-1)\ldots 1}{n(n-1)\ldots 1} \\ &= \frac{1}{2^n n!}\frac{(2n)!}{n!} \end{aligned} \tag{5.3.8}$$

Theorem 5.3.1

Legendre polynomials satisfy the three term recursion formula

$$(2n-1)xP_{n-1}(x) = nP_n(x) + (n-1)P_{n-2}(x), \quad n = 1, 2, \ldots. \tag{5.3.9}$$

Proof

In order to prove the three term recursion formula we consider the expression $(2n-1)xP_{n-1}(x) - nP_n(x)$. While each term is a polynomial of degree n, the leading order terms cancel. We need only look at the coefficient of the leading order term first expression. It is

$$\frac{2n-1}{2^{n-1}(n-1)!}\frac{(2n-2)!}{(n-1)!} = \frac{1}{2^{n-1}(n-1)!}\frac{(2n-1)!}{(n-1)!} = \frac{(2n-1)!}{2^{n-1}[(n-1)!]^2}.$$

The coefficient of the leading term for $nP_n(x)$ can be written as

$$n\frac{1}{2^n n!}\frac{(2n)!}{n!} = n\left(\frac{2n}{2n^2}\right)\left(\frac{1}{2^{n-1}(n-1)!}\right)\frac{(2n-1)!}{(n-1)!}\frac{(2n-1)!}{2^{n-1}[(n-1)!]^2}.$$

It is easy to see that the leading order terms in the expression $(2n-1)xP_{n-1}(x) - nP_n(x)$ cancel.

The next terms will be of degree $n-2$. This is because the P_n 's are either even or odd functions, thus only containing even, or odd, powers of x. We conclude that

$$(2n-1)xP_{n-1}(x) - nP_n(x) = \text{ polynomial of degree } n-2.$$

Therefore, since the Legendre polynomials form a basis, we can write this polynomial as a linear combination of Legendre polynomials:

$$(2n-1)xP_{n-1}(x) - nP_n(x) = c_0P_0(x) + c_1P_1(x) + \ldots + c_{n-2}P_{n-2}(x). \tag{5.3.10}$$

Multiplying Equation (5.3.10) by $P_m(x)$ for $m = 0, 1, \ldots, n-3$, integrating from -1 to 1 , and using orthogonality, we obtain

$$0 = c_m\|P_m\|^2, \quad m = 0, 1, \ldots, n-3.$$

[Note: $\int_{-1}^{1} x^k P_n(x)dx = 0$ for $k \le n-1$. Thus, $\int_{-1}^{1} xP_{n-1}(x)P_m(x)dx = 0$ for $m \le n-3$.]

Thus, all of these c_m 's are zero, leaving Equation (5.3.10) as

$$(2n-1)xP_{n-1}(x) - nP_n(x) = c_{n-2}P_{n-2}(x).$$

The final coefficient can be found by using the normalization condition, $P_n(1) = 1$. Thus, $c_{n-2} = (2n-1) - n = n-1$.

Generating Functions The Generating Function for Legendre Polynomials

A second proof of the three term recursion formula can be obtained from the generating function of the Legendre polynomials. Many special functions have such generating functions. In this case it is given by

$$g(x,t) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n, \quad |x| \le 1, |t| < 1. \tag{5.3.11}$$

This generating function occurs often in applications. In particular, it arises in potential theory, such as electromagnetic or gravitational potentials. These potential functions are $\frac{1}{r}$ type functions.

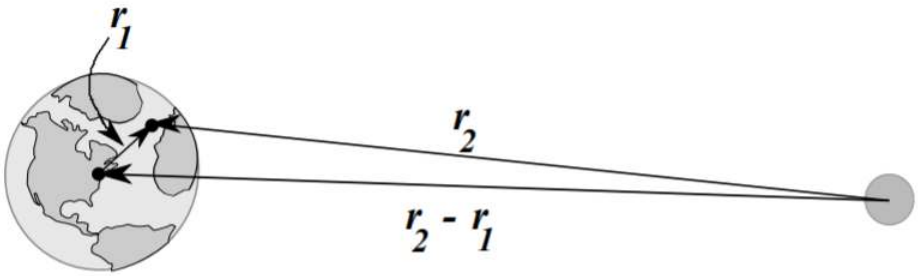


Figure 5.3.2: The position vectors used to describe the tidal force on the Earth due to the moon.

For example, the gravitational potential between the Earth and the moon is proportional to the reciprocal of the magnitude of the difference between their positions relative to some coordinate system. An even better example, would be to place the origin at the center of the Earth and consider the forces on the non-pointlike Earth due to the moon. Consider a piece of the Earth at position r_1 and the moon at position r_2 as shown in Figure 5.3.2. The tidal potential ϕ is proportional to

$$\Phi \propto \frac{1}{|\mathbf{r}_2 - \mathbf{r}_1|} = \frac{1}{\sqrt{(\mathbf{r}_2 - \mathbf{r}_1)\cdot(\mathbf{r}_2 - \mathbf{r}_1)}} = \frac{1}{\sqrt{r_1^2 - 2r_1r_2\cos\theta + r_2^2}}$$

where θ is the angle between $\mathbf{r}_1$ and $\mathbf{r}_2$.

Typically, one of the position vectors is much larger than the other. Let's assume that $r_1 \ll r_2$. Then, one can write

$$\Phi \propto \frac{1}{\sqrt{r_1^2 - 2r_1 r_2 \cos\theta + r_2^2}} = \frac{1}{r_2} \frac{1}{\sqrt{1 - 2\frac{r_1}{r_2}\cos\theta + \left(\frac{r_1}{r_2}\right)^2}}.$$

Now, define $x = \cos\theta$ and $t = \frac{r_1}{r_2}$. We then have that the tidal potential is proportional to the generating function for the Legendre polynomials! So, we can write the tidal potential as

$$\Phi \propto \frac{1}{r_2} \sum_{n=0}^{\infty} P_n(\cos\theta) \left(\frac{r_1}{r_2}\right)^n.$$

The first term in the expansion, $\frac{1}{r_2}$, is the gravitational potential that gives the usual force between the Earth and the moon. [Recall that the gravitational potential for mass m at distance r from M is given by $\Phi = -\frac{GMm}{r}$ and that the force is the gradient of the potential, $\mathbf{F} = -\nabla\Phi \propto \nabla\left(\frac{1}{r}\right)$.] The next terms will give expressions for the tidal effects.

Now that we have some idea as to where this generating function might have originated, we can proceed to use it. First of all, the generating function can be used to obtain special values of the Legendre polynomials.

Example 5.3.3

Evaluate $P_n(0)$ using the generating function. $P_n(0)$ is found by considering $g(0,t)$.

Solution

Setting $x = 0$ in Equation (5.3.11), we have

$$\begin{aligned} g(0,t) &= \frac{1}{\sqrt{1+t^2}} \\ &= \sum_{n=0}^{\infty} P_n(0) t^n \\ &= P_0(0) + P_1(0)t + P_2(0)t^2 + P_3(0)t^3 + \ldots. \end{aligned} \tag{5.3.12}$$

We can use the binomial expansion to find the final answer. Namely, we have

$$\frac{1}{\sqrt{1+t^2}} = 1 - \frac{1}{2}t^2 + \frac{3}{8}t^4 + \ldots.$$

Comparing these expansions, we have the $P_n(0) = 0$ for n odd and for even integers one can show (see Problem 5.7.12) that[1]

$$P_{2n}(0) = (-1)^n \frac{(2n-1)!!}{(2n)!!}, \tag{5.3.13}$$

where $n!!$ is the double factorial,

$$n!! = \begin{cases} n(n-2)\ldots(3)1, & n > 0, \text{ odd}, \\ n(n-2)\ldots(4)2, & n > 0, \text{ even}, \\ 1 & n = 0, -1 \end{cases}.$$

Note

This example can be finished by first proving that

$$(2n)!! = 2^n n!$$

and

$$(2n-1)!! = \frac{(2n)!}{(2n)!!} = \frac{(2n)!}{2^n n!}.$$

Example 5.3.4

Evaluate $P_n(-1)$. This is a simpler problem.

Solution

In this case we have

$$g(-1,t)=\frac{1}{\sqrt{1+2t+t^2}}=\frac{1}{1+t}=1-t+t^2-t^3+\ldots.$$

Therefore, $P_n(-1)=(-1)^n$.

Example 5.3.5

Prove the three term recursion formula,

$$(k+1)P_{k+1}(x)-(2k+1)xP_k(x)+kP_{k-1}(x)=0,\quad k=1,2,\ldots,$$

using the generating function.

Solution

We can also use the generating function to find recurrence relations. To prove the three term recursion (5.3.5) that we introduced above, then we need only differentiate the generating function with respect to t in Equation (5.3.11) and rearrange the result. First note that

$$\frac{\partial g}{\partial t}=\frac{x-t}{\left(1-2xt+t^2\right)^{3/2}}=\frac{x-t}{1-2xt+t^2}g(x,t).$$

Combining this with

$$\frac{\partial g}{\partial t}=\sum_{n=0}^{\infty}nP_n(x)t^{n-1}$$

we have

$$(x-t)g(x,t)=\left(1-2xt+t^2\right)\sum_{n=0}^{\infty}nP_n(x)t^{n-1}.$$

Inserting the series expression for $g(x,t)$ and distributing the sum on the right side, we obtain

$$(x-t)\sum_{n=0}^{\infty}P_n(x)t^n=\sum_{n=0}^{\infty}nP_n(x)t^{n-1}-\sum_{n=0}^{\infty}2nxP_n(x)t^n+\sum_{n=0}^{\infty}nP_n(x)t^{n+1}.$$

Multiplying out the $x-t$ factor and rearranging, leads to three separate sums:

$$\sum_{n=0}^{\infty}nP_n(x)t^{n-1}-\sum_{n=0}^{\infty}(2n+1)xP_n(x)t^n+\sum_{n=0}^{\infty}(n+1)P_n(x)t^{n+1}=0. \tag{5.3.14}$$

Each term contains powers of t that we would like to combine into a single sum. This is done by reindexing. For the first sum, we could use the new index $k=n-1$. Then, the first sum can be written

$$\sum_{n=0}^{\infty}nP_n(x)t^{n-1}=\sum_{k=-1}^{\infty}(k+1)P_{k+1}(x)t^k.$$

Using different indices is just another way of writing out the terms. Note that

$$\sum_{n=0}^{\infty}nP_n(x)t^{n-1}=0+P_1(x)+2P_2(x)t+3P_3(x)t^2+\ldots$$

and

$$\sum_{k=-1}^{\infty}(k+1)P_{k+1}(x)t^k = 0 + P_1(x) + 2P_2(x)t + 3P_3(x)t^2 + \ldots$$

actually give the same sum. The indices are sometimes referred to as dummy indices because they do not show up in the expanded expression and can be replaced with another letter.

If we want to do so, we could now replace all of the k' with n 's. However, we will leave the k' s in the first term and now reindex the next sums in Equation (5.3.14). The second sum just needs the replacement $n = k$ and the last sum we reindex using $k = n+1$. Therefore, Equation (5.3.14) becomes

$$\sum_{k=-1}^{\infty}(k+1)P_{k+1}(x)t^k - \sum_{k=0}^{\infty}(2k+1)xP_k(x)t^k + \sum_{k=1}^{\infty}kP_{k-1}(x)t^k = 0. \tag{5.3.15}$$

We can now combine all of the terms, noting the $k = -1$ term is automatically zero and the $k = 0$ terms give

$$P_1(x) - xP_0(x) = 0. \tag{5.3.16}$$

Of course, we know this already. So, that leaves the $k > 0$ terms:

$$\sum_{k=1}^{\infty}\left[(k+1)P_{k+1}(x) - (2k+1)xP_k(x) + kP_{k-1}(x)\right]t^k = 0. \tag{5.3.17}$$

Since this is true for all t, the coefficients of the t^k 's are zero, or

$$(k+1)P_{k+1}(x) - (2k+1)xP_k(x) + kP_{k-1}(x) = 0, \quad k = 1, 2, \ldots$$

While this is the standard form for the three term recurrence relation, the earlier form is obtained by setting $k = n-1$.

There are other recursion relations which we list in the box below. Equation (5.3.18) was derived using the generating function. Differentiating it with respect to x, we find Equation (5.3.19). Equation (5.3.20) can be proven using the generating function by differentiating $g(x,t)$ with respect to x and rearranging the resulting infinite series just as in this last manipulation. This will be left as Problem 5.7.4. Combining this result with Equation (5.3.18), we can derive Equations (5.3.21)-(5.3.22). Adding and subtracting these equations yields Equations (5.3.23)-(5.3.24).

Table 5.3.2

Recursion Formulae for Legendre Polynomials for $n = 1, 2, \ldots$	
$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$	(5.3.18)
$(n+1)P'_{n+1}(x) = (2n+1)\left[P_n(x) + xP'_n(x)\right] - nP'_{n-1}(x)$	(5.3.19)
$P_n(x) = P'_{n+1}(x) - 2xP'_n(x) + P'_{n-1}(x)$	(5.3.20)
$P'_{n-1}(x) = xP'_n(x) - nP_n(x)$	(5.3.21)
$P'_{n+1}(x) = xP'_n(x) + (n+1)P_n(x)$	(5.3.22)
$P'_{n+1}(x) + P'_{n-1}(x) = 2xP'_n(x) + P_n(x).$	(5.3.23)
$P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x).$	(5.3.24)
$\left(x^2-1\right)P'_n(x) = nxP_n(x) - nP_{n-1}(x)$	(5.3.25)

Finally, Equation (5.3.25) can be obtained using Equations (5.3.21) and (5.3.22). Just multiply Equation (5.3.21) by x,

$$x^2P_n'(x) - nxP_n(x) = xP_{n-1}'(x).$$

Now use Equation (5.3.22), but first replace n with $n-1$ to eliminate the $xP_{n-1}'(x)$ term:

$$x^2P_n'(x) - nxP_n(x) = P_n'(x) - nP_{n-1}(x).$$

Rearranging gives the Equation (5.3.25).

Example 5.3.6

Use the generating function to prove

$$\|P_n\|^2 = \int_{-1}^{1} P_n^2(x)dx = \frac{2}{2n+1}.$$

Solution

Another use of the generating function is to obtain the normalization constant. This can be done by first squaring the generating function in order to get the products $P_n(x)P_m(x)$, and then integrating over x.

Squaring the generating function has to be done with care, as we need to make proper use of the dummy summation index. So, we first write

$$\begin{aligned} \frac{1}{1-2xt+t^2} &= \left[\sum_{n=0}^{\infty} P_n(x)t^n\right]^2 \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} P_n(x)P_m(x)t^{n+m}. \end{aligned} \tag{5.3.26}$$

Integrating from $x=-1$ to $x=1$ and using the orthogonality of the Legendre polynomials, we have

$$\begin{aligned} \int_{-1}^{1}\frac{dx}{1-2xt+t^2} &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} t^{n+m}\int_{-1}^{1} P_n(x)P_m(x)dx \\ &= \sum_{n=0}^{\infty} t^{2n}\int_{-1}^{1} P_n^2(x)dx. \end{aligned} \tag{5.3.27}$$

However, one can show that[2]

$$\int_{-1}^{1}\frac{dx}{1-2xt+t^2} = \frac{1}{t}\ln\left(\frac{1+t}{1-t}\right).$$

Expanding this expression about $t=0$, we obtain[3]

$$\frac{1}{t}\ln\left(\frac{1+t}{1-t}\right) = \sum_{n=0}^{\infty}\frac{2}{2n+1}t^{2n}.$$

Comparing this result with Equation (5.3.27), we find that

$$\|P_n\|^2 = \int_{-1}^{1} P_n^2(x)dx = \frac{2}{2n+1}. \tag{5.3.28}$$

Note

You will need the integral

$$\int\frac{dx}{a+bx} = \frac{1}{b}\ln(a+bx)+C.$$

You will need the series expansion

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$
$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

Differential Equation for Legendre Polynomials

The Legendre Polynomials satisfy a second order linear differential equation. This differential equation occurs naturally in the solution of initialboundary value problems in three dimensions which possess some spherical symmetry. We will see this in the last chapter. There are two approaches we could take in showing that the Legendre polynomials satisfy a particular differential equation. Either we can write down the equations and attempt to solve it, or we could use the above properties to obtain the equation. For now, we will seek the differential equation satisfied by $P_n(x)$ using the above recursion relations.

We begin by differentiating Equation (5.3.25) and using Equation (5.3.21) to simplify:

$$\begin{aligned} \frac{d}{dx}\left(\left(x^2-1\right)P_n'(x)\right) &= nP_n(x) + nxP_n'(x) - nP_{n-1}'(x) \\ &= nP_n(x) + n^2 P_n(x) \\ &= n(n+1)P_n(x) \end{aligned} \tag{5.3.29}$$

Therefore, Legendre polynomials, or Legendre functions of the first kind, are solutions of the differential equation

$$\left(1-x^2\right)y'' - 2xy' + n(n+1)y = 0.$$

As this is a linear second order differential equation, we expect two linearly independent solutions. The second solution, called the Legendre function of the second kind, is given by $Q_n(x)$ and is not well behaved at $x = \pm 1$. For example,

$$Q_0(x) = \frac{1}{2}\ln\frac{1+x}{1-x}.$$

We will not need these for physically interesting examples in this book.

Note

A generalization of the Legendre equation is given by $\left(1-x^2\right)y'' - 2xy' + \left[n(n+1) - \frac{m^2}{1-x^2}\right]y = 0$. Solutions to this equation, $P_n^m(x)$ and $Q_n^m(x)$, are called the associated Legendre functions of the first and second kind.

Fourier-Legendre Series

With these properties of Legendre Functions we are now prepared to compute the expansion coefficients for the Fourier-Legendre series representation of a given function.

Example 5.3.7

Expand $f(x) = x^3$ in a Fourier-Legendre series.

Solution

We simply need to compute

$$c_n = \frac{2n+1}{2}\int_{-1}^{1} x^3 P_n(x)dx. \tag{5.3.30}$$

We first note that

$$\int_{-1}^{1} x^m P_n(x)dx = 0 \quad \text{for } m > n.$$

As a result, we have that $c_n = 0$ for $n > 3$. We could just compute $\int_{-1}^{1} x^3 P_m(x)dx$ for $m = 0, 1, 2, \ldots$ outright by looking up Legendre polynomials. But, note that x^3 is an odd function. *So*, $c_0 = 0$ and $c_2 = 0$.

This leaves us with only two coefficients to compute. We refer to Table 5.3.1 and find that

$$c_1 = \frac{3}{2}\int_{-1}^{1} x^4 dx = \frac{3}{5}$$
$$c_3 = \frac{7}{2}\int_{-1}^{1} x^3\left[\frac{1}{2}\left(5x^3-3x\right)\right]dx = \frac{2}{5}.$$

Thus,

$$x^3 = \frac{3}{5}P_1(x) + \frac{2}{5}P_3(x).$$

Of course, this is simple to check using Table 5.3.1:

$$\frac{3}{5}P_1(x) + \frac{2}{5}P_3(x) = \frac{3}{5}x + \frac{2}{5}\left[\frac{1}{2}\left(5x^3-3x\right)\right] = x^3.$$

We could have obtained this result without doing any integration. Write x^3 as a linear combination of $P_1(x)$ and $P_3(x)$:

$$\begin{aligned} x^3 &= c_1 x + \frac{1}{2}c_2\left(5x^3-3x\right) \\ &= \left(c_1 - \frac{3}{2}c_2\right)x + \frac{5}{2}c_2 x^3. \end{aligned} \tag{5.3.31}$$

Equating coefficients of like terms, we have that $c_2 = \frac{2}{5}$ and $c_1 = \frac{3}{2}c_2 = \frac{3}{5}$.

Note

Oliver Heaviside $(1850-1925)$ was an English mathematician, physicist and engineer who used complex analysis to study circuits and was a co-founder of vector analysis. The Heaviside function is also called the step function.

Example 5.3.8

Expand the Heaviside[3] function in a Fourier-Legendre series.

The Heaviside function is defined as

$$H(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0. \end{cases} \tag{5.3.32}$$

Solution

In this case, we cannot find the expansion coefficients without some integration. We have to compute

$$\begin{aligned} c_n &= \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)dx \\ &= \frac{2n+1}{2}\int_{0}^{1} P_n(x)dx. \end{aligned} \tag{5.3.33}$$

We can make use of identity (5.3.24),

$$P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x), \quad n > 0. \tag{5.3.34}$$

We have for $n > 0$

$$c_n = \frac{1}{2}\int_0^1 [P'_{n+1}(x) - P'_{n-1}(x)]dx = \frac{1}{2}[P_{n-1}(0) - P_{n+1}(0)].$$

For $n = 0$, we have

$$c_0 = \frac{1}{2}\int_0^1 dx = \frac{1}{2}.$$

This leads to the expansion

$$f(x) \sim \frac{1}{2} + \frac{1}{2}\sum_{n=1}^{\infty}[P_{n-1}(0) - P_{n+1}(0)]P_n(x).$$

We still need to evaluate the Fourier-Legendre coefficients

$$c_n = \frac{1}{2}[P_{n-1}(0) - P_{n+1}(0)].$$

Since $P_n(0) = 0$ for n odd, the c_n's vanish for n even. Letting $n = 2k-1$, we re-index the sum, obtaining

$$f(x) \sim \frac{1}{2} + \frac{1}{2}\sum_{n=1}^{\infty}[P_{2k-2}(0) - P_{2k}(0)]P_{2k-1}(x).$$

We can compute the nonzero Fourier coefficients, $c_{2k-1} = \frac{1}{2}[P_{2k-2}(0) - P_{2k}(0)]$, using a result from Problem 12:

$$P_{2k}(0) = (-1)^k \frac{(2k-1)!!}{(2k)!!}. \tag{5.3.35}$$

Namely, we have

$$\begin{aligned} c_{2k-1} &= \frac{1}{2}[P_{2k-2}(0) - P_{2k}(0)] \\ &= \frac{1}{2}\left[(-1)^{k-1}\frac{(2k-3)!!}{(2k-2)!!} - (-1)^k\frac{(2k-1)!!}{(2k)!!}\right] \\ &= -\frac{1}{2}(-1)^k\frac{(2k-3)!!}{(2k-2)!!}\left[1 + \frac{2k-1}{2k}\right] \\ &= -\frac{1}{2}(-1)^k\frac{(2k-3)!!}{(2k-2)!!}\frac{4k-1}{2k}. \end{aligned} \tag{5.3.36}$$

Thus, the Fourier-Legendre series expansion for the Heaviside function is given by

$$f(x) \sim \frac{1}{2} - \frac{1}{2}\sum_{n=1}^{\infty}(-1)^n\frac{(2n-3)!!}{(2n-2)!!}\frac{4n-1}{2n}P_{2n-1}(x). \tag{5.3.37}$$

The sum of the first 21 terms of this series are shown in Figure 5.3.3. We note the slow convergence to the Heaviside function. Also, we see that the Gibbs phenomenon is present due to the jump discontinuity at $x = 0$. [See Section 3.7.]

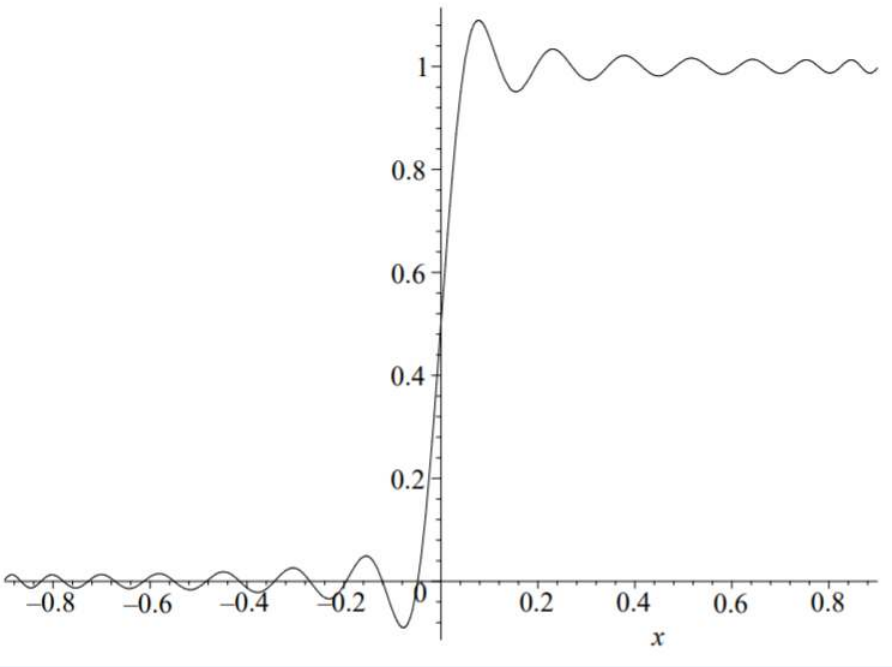


Figure 5.3.3: Sum of first 21 terms for Fourier-Legendre series expansion of Heaviside function.

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5.4: Gamma Function

A function that often occurs in the study of special functions is the **Gamma function**. We will need the Gamma function in the next section on Fourier-Bessel series.

Note

The name and symbol for the Gamma function were first given by Legendre in 1811. However, the search for a generalization of the factorial extends back to the 1720's when Euler provided the first representation of the factorial as an infinite product, later to be modified by others like Gauß, Weierstraß, and Legendre.

For $x >$ we define the Gamma function as

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \quad x > 0. \tag{5.4.1}$$

The Gamma function is a generalization of the factorial function and a plot is shown in Figure 5.4.1. In fact, we have

$$\Gamma(1) = 1$$

and

$$\Gamma(x+1) = x\Gamma(x).$$

The reader can prove this identity by simply performing an integration by parts. (See Problem 5.7.7.) In particular, for integers $n \in \mathbb{Z}^+$, we then have

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-2) = n(n-1)\cdots 2\Gamma(1) = n!.$$

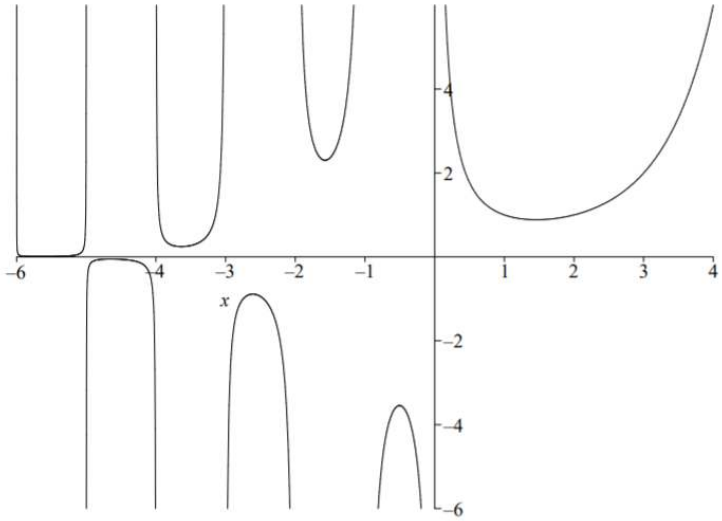


Figure 5.4.1: Plot of the Gamma function.

We can also define the Gamma function for negative, non-integer values of x. We first note that by iteration on $n \in \mathbb{Z}^+$, we have

$$\Gamma(x+n) = (x+n-1)\cdots(x+1)x\Gamma(x), \quad x+n > 0.$$

Solving for $\Gamma(x)$, we then find

$$\Gamma(x) = \frac{\Gamma(x+n)}{(x+n-1)\cdots(x+1)x'}, \quad -n < x < 0$$

Note that the Gamma function is undefined at zero and the negative integers.

Example 5.4.1

We now prove that

$$\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}.$$

Solution

This is done by direct computation of the integral:

$$\Gamma\left(\frac{1}{2}\right)=\int_0^\infty t^{-\frac{1}{2}}e^{-t}dt.$$

Letting $t=z^2$, we have

$$\Gamma\left(\frac{1}{2}\right)=2\int_0^\infty e^{-z^2}dz.$$

Due to the symmetry of the integrand, we obtain the classic integral

$$\Gamma\left(\frac{1}{2}\right)=\int_{-\infty}^\infty e^{-z^2}dz,$$

which can be performed using a standard trick.[1] Consider the integral

$$I=\int_{-\infty}^\infty e^{-x^2}dx.$$

Then,

$$I^2=\int_{-\infty}^\infty e^{-x^2}dx\int_{-\infty}^\infty e^{-y^2}dy.$$

Note that we changed the integration variable. This will allow us to write this product of integrals as a double integral:

$$I^2=\int_{-\infty}^\infty\int_{-\infty}^\infty e^{-\left(x^2+y^2\right)}dxdy.$$

This is an integral over the entire xy-plane. We can transform this Cartesian integration to an integration over polar coordinates. The integral becomes

$$I^2=\int_0^{2\pi}\int_0^\infty e^{-r^2}rdrd\theta$$

This is simple to integrate and we have $I^2=\pi$. So, the final result is found by taking the square root of both sides.

$$\Gamma\left(\frac{1}{2}\right)=I=\sqrt{\pi}.$$

Note

In Example 9.5 we show the more general result:

$$\int_{-\infty}^\infty e^{-\beta y^2}dy=\sqrt{\frac{\pi}{\beta}}$$

In Problem 5.7.12 the reader will prove the identity

$$\Gamma\left(n+\frac{1}{2}\right)=\frac{(2n-1)!!}{2^n}\sqrt{\pi}.$$

Another useful relation, which we only state, is

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}.$$

The are many other important relations, including infinite products, which we will not need at this point. The reader is encouraged to read about these elsewhere. In the meantime, we move on to the discussion of another important special function in physics and mathematics.

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5.5: Fourier-Bessel Series

Bessel functions arise in many problems in physics possessing cylindrical symmetry such as the vibrations of circular drumheads and the radial modes in optical fibers. They also provide us with another orthogonal set of basis functions.

The first occurrence of Bessel functions (zeroth order) was in the work of Daniel Bernoulli on heavy chains (1738). More general Bessel functions were studied by Leonhard Euler in 1781 and in his study of the vibrating membrane in 1764 . Joseph Fourier found them in the study of heat conduction in solid cylinders and Siméon Poisson (1781-1840) in heat conduction of spheres (1823).

> **Note**
>
> Bessel functions have a long history and were named after Friedrich Wilhelm Bessel (1784-1846).

The history of Bessel functions, does not just originate in the study of the wave and heat equations. These solutions originally came up in the study of the Kepler problem, describing planetary motion. According to G. N. Watson in his *Treatise on Bessel Functions*, the formulation and solution of Kepler's Problem was discovered by Joseph-Louis Lagrange (1736-1813), in 1770. Namely, the problem was to express the radial coordinate and what is called the eccentric anomaly, E, as functions of time. Lagrange found expressions for the coefficients in the expansions of r and E in trigonometric functions of time. However, he only computed the first few coefficients. In 1816 Friedrich Wilhelm Bessel (1784-1846) had shown that the coefficients in the expansion for r could be given an integral representation. In 1824 he presented a thorough study of these functions, which are now called Bessel functions.

You might have seen Bessel functions in a course on differential equations as solutions of the differential equation

$$x^2y'' + xy' + \left(x^2 - p^2\right) y = 0. \tag{5.5.1}$$

Solutions to this equation are obtained in the form of series expansions. Namely, one seeks solutions of the form

$$y(x) = \sum_{j=0}^{\infty} a_j x^{j+n}$$

by determining the for the coefficients must take. We will leave this for a homework exercise and simply report the results.

One solution of the differential equation is the Bessel function of the first kind of order p, given as

$$y(x) = J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+p+1)} \left(\frac{x}{2}\right)^{2n+p}. \tag{5.5.2}$$

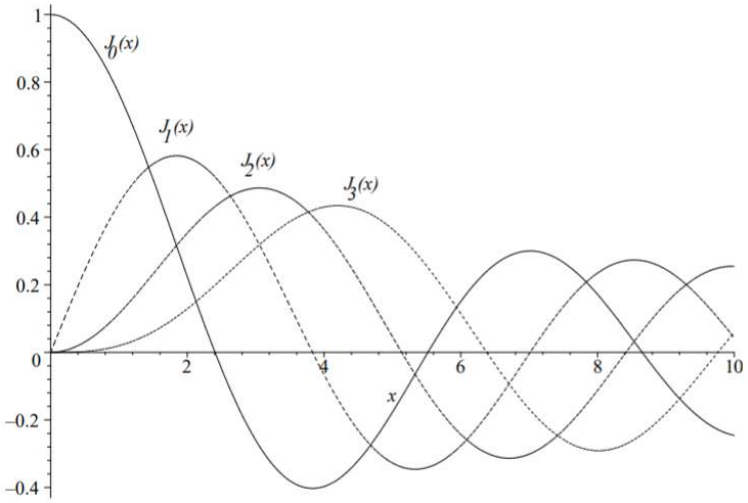


Figure 5.5.1: Plots of the Bessel functions $J_0(x), J_1(x), J_2(x)$, and $J_3(x)$.

In Figure 5.5.1 we display the first few Bessel functions of the first kind of integer order. Note that these functions can be described as decaying oscillatory functions.

A second linearly independent solution is obtained for p not an integer as $J_{-p}(x)$. However, for p an integer, the $\Gamma(n+p+1)$ factor leads to evaluations of the Gamma function at zero, or negative integers, when p is negative. Thus, the above series is not defined in these cases.

Another method for obtaining a second linearly independent solution is through a linear combination of $J_p(x)$ and $J_{-p}(x)$ as

$$N_p(x) = Y_p(x) = \frac{\cos \pi p J_p(x) - J_{-p}(x)}{\sin \pi p}. \tag{5.5.3}$$

These functions are called the Neumann functions, or Bessel functions of the second kind of order p.

In Figure 5.5.2 we display the first few Bessel functions of the second kind of integer order. Note that these functions are also decaying oscillatory functions. However, they are singular at $x = 0$.

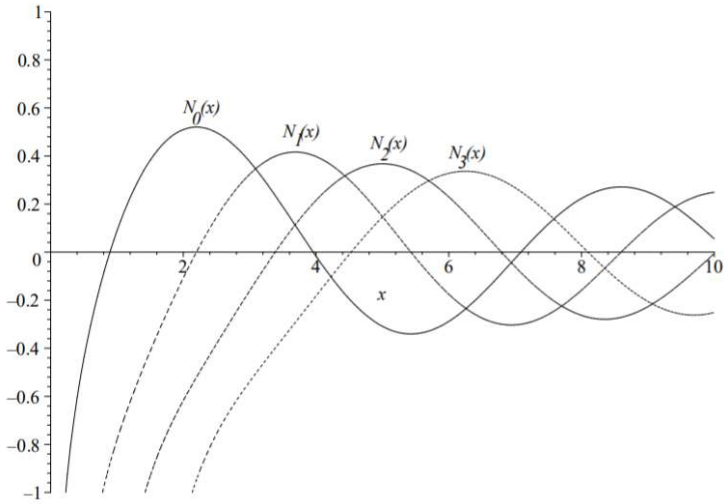


Figure 5.5.2: Plots of the Neumann functions $N_0(x), N_1(x), N_2(x)$, and $N_3(x)$.

In many applications one desires bounded solutions at $x = 0$. These functions do not satisfy this boundary condition. For example, we will later study one standard problem is to describe the oscillations of a circular drumhead. For this problem one solves the two dimensional wave equation using separation of variables in cylindrical coordinates. The radial equation leads to a Bessel equation. The Bessel function solutions describe the radial part of the solution and one does not expect a singular solution at the center of the drum. The amplitude of the oscillation must remain finite. Thus, only Bessel functions of the first kind can be used.

Bessel functions satisfy a variety of properties, which we will only list at this time for Bessel functions of the first kind. The reader will have the opportunity to prove these for homework.

Derivative Identities

These identities follow directly from the manipulation of the series solution.

$$\frac{d}{dx}\left[x^p J_p(x)\right] = x^p J_{p-1}(x). \tag{5.5.4}$$

$$\frac{d}{dx}\left[x^{-p} J_p(x)\right] = -x^{-p} J_{p+1}(x). \tag{5.5.5}$$

Recursion Formulae

The next identities follow from adding, or subtracting, the derivative identities.

$$J_{p-1}(x) + J_{p+1}(x) = \frac{2p}{x} J_p(x). \tag{5.5.6}$$

$$J_{p-1}(x) - J_{p+1}(x) = 2J_p'(x). \tag{5.5.7}$$

Orthogonality

As we will see in the next chapter, one can recast the Bessel equation into an eigenvalue problem whose solutions form an orthogonal basis of functions on $L_x^2(0,a)$. Using Sturm-Liouville theory, one can show that

$$\int_0^a x J_p\left(j_{pn}\frac{x}{a}\right) J_p\left(j_{pm}\frac{x}{a}\right) dx = \frac{a^2}{2}\left[J_{p+1}\left(j_{pn}\right)\right]^2 \delta_{n,m}, \tag{5.5.8}$$

where j_{pn} is the nth root of $J_p(x)$, $J_p\left(j_{pn}\right) = 0, n = 1, 2, \ldots$. A list of some of these roots are provided in Table 5.5.1.

Generating Function

$$e^{x\left(t-\frac{1}{t}\right)/2} = \sum_{n=-\infty}^{\infty} J_n(x) t^n, \quad x > 0, t \neq 0. \tag{5.5.9}$$

Table 5.5.1: The zeros of Bessel Functions, $J_m\left(j_{mn}\right) = 0$.

n	$m=0$	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$
1	2.405	3.832	5.136	6.380	7.588	8.771
2	5.520	7.016	8.417	9.761	**11.065**	**12.339**
3	8.654	**10.173**	**11.620**	**13.015**	**14.373**	**15.700**
4	**11.792**	**13.324**	**14.796**	**16.223**	**17.616**	**18.980**
5	**14.931**	**16.471**	**17.960**	**19.409**	**20.827**	**22.218**
6	**18.071**	**19.616**	**21.117**	**22.583**	**24.019**	**25.430**
7	**21.212**	**22.760**	**24.270**	**25.748**	**27.199**	**28.627**
8	**24.352**	**25.904**	**27.421**	**28.908**	**30.371**	**31.812**
9	**27.493**	**29.047**	**30.569**	**32.065**	**33.537**	**34.989**

Integral Representation

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\sin\theta - n\theta) d\theta, \quad x > 0, n \in \mathrm{Z}. \tag{5.5.10}$$

Fourier-Bessel Series

Since the Bessel functions are an orthogonal set of functions of a SturmLiouville problem, we can expand square integrable functions in this basis. In fact, the Sturm-Liouville problem is given in the form

$$x^2 y'' + xy' + \left(\lambda x^2 - p^2\right) y = 0, \quad x \in [0, a], \tag{5.5.11}$$

satisfying the boundary conditions: $y(x)$ is bounded at $x = 0$ and $y(a) = 0$. The solutions are then of the form $J_p(\sqrt{\lambda}x)$, as can be shown by making the substitution $t = \sqrt{\lambda}x$ in the differential equation. Namely, we let $y(x) = u(t)$ and note that

$$\frac{dy}{dx} = \frac{dt}{dx}\frac{du}{dt} = \sqrt{\lambda}\frac{du}{dt}.$$

Then,

$$t^2 u'' + tu' + \left(t^2 - p^2\right) u = 0,$$

which has a solution $u(t) = J_p(t)$.

Note

In the study of boundary value problems in differential equations, SturmLiouville problems are a bountiful source of basis functions for the space of square integrable functions as will be seen in the next section.

Using Sturm-Liouville theory, one can show that $J_p\left(j_{pn}\frac{x}{a}\right)$ is a basis of eigenfunctions and the resulting *Fourier-Bessel series expansion* of $f(x)$ defined on $x \in [0, a]$ is

$$f(x) = \sum_{n=1}^{\infty} c_n J_p\left(j_{pn}\frac{x}{a}\right), \tag{5.5.12}$$

where the Fourier-Bessel coefficients are found using the orthogonality relation as

$$c_n = \frac{2}{a^2\left[J_{p+1}\left(j_{pn}\right)\right]^2}\int_0^a x f(x) J_p\left(j_{pn}\frac{x}{a}\right) dx. \tag{5.5.13}$$

Example 5.5.1

Expand $f(x) = 1$ for $0 < x < 1$ in a Fourier-Bessel series of the form

$$f(x) = \sum_{n=1}^{\infty} c_n J_0\left(j_{0n}x\right)$$

Solution

We need only compute the Fourier-Bessel coefficients in Equation (5.5.13):

$$c_n = \frac{2}{\left[J_1\left(j_{0n}\right)\right]^2}\int_0^1 x J_0\left(j_{0n}x\right) dx \tag{5.5.14}$$

From the identity

$$\frac{d}{dx}\left[x^p J_p(x)\right] = x^p J_{p-1}(x). \tag{5.5.15}$$

we have

$$\begin{aligned}\int_0^1 x J_0\left(j_{0n}x\right) dx &= \frac{1}{j_{0n}^2}\int_0^{j_{0n}} y J_0(y) dy \\ &= \frac{1}{j_{0n}^2}\int_0^{j_{0n}} \frac{d}{dy}\left[y J_1(y)\right] dy \\ &= \frac{1}{j_{0n}^2}\left[y J_1(y)\right]_0^{j_{0n}} \\ &= \frac{1}{j_{0n}} J_1\left(j_{0n}\right)\end{aligned} \tag{5.5.16}$$

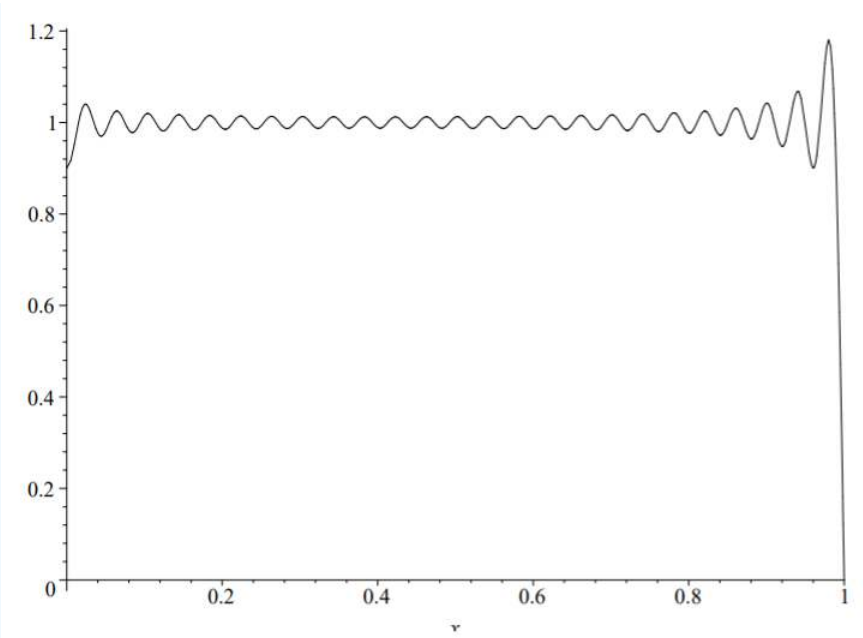


Figure 5.5.3: Plot of the first 50 terms of the Fourier-Bessel series in Equation (5.5.17) for $f(x) = 1$ on $0 < x < 1$.

As a result, the desired Fourier-Bessel expansion is given as

$$1 = 2\sum_{n=1}^{\infty} \frac{J_0\left(j_{0n}x\right)}{j_{0n}J_1\left(j_{0n}\right)}, \quad 0 < x < 1 \tag{5.5.17}$$

In Figure 5.5.3 we show the partial sum for the first fifty terms of this series. Note once again the slow convergence due to the Gibbs phenomenon.

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5.6: Appendix- The Least Squares Approximation

In the first section of this chapter we showed that we can expand functions over an infinite set of basis functions as

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)$$

and that the generalized Fourier coefficients are given by

$$c_n = \frac{\langle \phi_n, f \rangle}{\langle \phi_n, \phi_n \rangle}.$$

In this section we turn to a discussion of approximating $f(x)$ by the partial sums $\sum_{n=1}^{N} c_n \phi_n(x)$ and showing that the Fourier coefficients are the best coefficients minimizing the deviation of the partial sum from $f(x)$. This will lead us to a discussion of the convergence of Fourier series.

More specifically, we set the following goal:

Goal

To find the best approximation of $f(x)$ on $[a, b]$ by $S_N(x) = \sum_{n=1}^{N} c_n \phi_n(x)$ for a set of fixed functions $\phi_n(x)$; i.e., to find the expansion coefficients, c_n, such that $S_N(x)$ approximates $f(x)$ in the least squares sense.

We want to measure the deviation of the finite sum from the given function. Essentially, we want to look at the error made in the approximation. This is done by introducing the mean square deviation:

$$E_N = \int_a^b [f(x) - S_N(x)]^2 \rho(x) dx,$$

where we have introduced the weight function $\rho(x) > 0$. It gives us a sense as to how close the N th partial sum is to $f(x)$.

We want to minimize this deviation by choosing the right c_n 's. We begin by inserting the partial sums and expand the square in the integrand:

$$\begin{aligned} E_N &= \int_a^b [f(x) - S_N(x)]^2 \rho(x) dx \\ &= \int_a^b \left[f(x) - \sum_{n=1}^{N} c_n \phi_n(x) \right]^2 \rho(x) dx \\ &= \int_a^b f^2(x) \rho(x) dx - 2 \int_a^b f(x) \sum_{n=1}^{N} c_n \phi_n(x) \rho(x) dx \\ &\quad + \int_a^b \sum_{n=1}^{N} c_n \phi_n(x) \sum_{m=1}^{N} c_m \phi_m(x) \rho(x) dx \end{aligned} \tag{5.6.1}$$

Looking at the three resulting integrals, we see that the first term is just the inner product of f with itself. The other integrations can be rewritten after interchanging the order of integration and summation. The double sum can be reduced to a single sum using the orthogonality of the ϕ_n 's. Thus, we have

$$\begin{aligned} E_N &= \langle f, f \rangle - 2 \sum_{n=1}^{N} c_n \langle f, \phi_n \rangle + \sum_{n=1}^{N} \sum_{m=1}^{N} c_n c_m \langle \phi_n, \phi_m \rangle \\ &= \langle f, f \rangle - 2 \sum_{n=1}^{N} c_n \langle f, \phi_n \rangle + \sum_{n=1}^{N} c_n^2 \langle \phi_n, \phi_n \rangle . \end{aligned} \tag{5.6.2}$$

We are interested in finding the coefficients, so we will complete the square in c_n. Focusing on the last two terms, we have

$$
\begin{aligned}
&-2\sum_{n=1}^{N} c_n < f, \phi_n > + \sum_{n=1}^{N} c_n^2 < \phi_n, \phi_n > \\
&= \sum_{n=1}^{N} < \phi_n, \phi_n > c_n^2 - 2 < f, \phi_n > c_n \\
&= \sum_{n=1}^{N} < \phi_n, \phi_n > \left[c_n^2 - \frac{2 < f, \phi_n > c_n}{\langle \phi_n, \phi_n \rangle} \right] \\
&= \sum_{n=1}^{N} < \phi_n, \phi_n > \left[\left(c_n - \frac{\langle f, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle} \right)^2 - \left(\frac{\langle f, \phi_n \rangle}{< \phi_n, \phi_n >} \right)^2 \right]. \qquad (5.6.3)
\end{aligned}
$$

To this point we have shown that the mean square deviation is given as

$$
E_N = \langle f, f \rangle + \sum_{n=1}^{N} \langle \phi_n, \phi_n \rangle \left[\left(c_n - \frac{\langle f, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle} \right)^2 - \left(\frac{\langle f, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle} \right)^2 \right].
$$

So, E_N is minimized by choosing

$$
c_n = \frac{\langle f, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle}.
$$

However, these are the Fourier Coefficients. This minimization is often referred to as Minimization in Least Squares Sense.

Inserting the Fourier coefficients into the mean square deviation yields

$$
0 \leq E_N = \langle f, f \rangle - \sum_{n=1}^{N} c_n^2 \langle \phi_n, \phi_n \rangle .
$$

Thus, we obtain Bessel's Inequality:

$$
< f, f > \geq \sum_{n=1}^{N} c_n^2 < \phi_n, \phi_n >.
$$

For convergence, we next let N get large and see if the partial sums converge to the function. In particular, we say that the infinite series converges in the mean if

$$
\int_a^b \left[f(x) - S_N(x) \right]^2 \rho(x) dx \to 0 \text{ as } N \to \infty.
$$

Letting N get large in Bessel's inequality shows that the sum $\sum_{n=1}^{N} c_n^2 < \phi_n, \phi_n >$ converges if

$$
< f, f > = \int_a^b f^2(x) \rho(x) dx < \infty.
$$

The space of all such f is denoted $L_\rho^2(a, b)$, the space of square integrable functions on (a, b) with weight $\rho(x)$.

From the nth term divergence test from calculus we know that $\sum a_n$ converges implies that $a_n \to 0$ as $n \to \infty$. Therefore, in this problem the terms $c_n^2 < \phi_n, \phi_n >$ approach zero as n gets large. This is only possible if the c_n 's go to zero as n gets large. Thus, if $\sum_{n=1}^{N} c_n \phi_n$ converges in the mean to f, then $\int_a^b \left[f(x) - \sum_{n=1}^{N} c_n \phi_n \right]^2 \rho(x) dx$ approaches zero as $N \to \infty$. This implies from the above derivation of Bessel's inequality that

$$
< f, f > - \sum_{n=1}^{N} c_n^2 \left(\phi_n, \phi_n \right) \to 0.
$$

This leads to Parseval's equality:

$$
\langle f, f \rangle = \sum_{n=1}^{\infty} c_n^2 \langle \phi_n, \phi_n \rangle .
$$

Parseval's equality holds if and only if

$$\lim_{N\to\infty}\int_a^b\left(f(x)-\sum_{n=1}^{N}c_n\phi_n(x)\right)^2\rho(x)dx=0.$$

If this is true for every square integrable function in $L^2_\rho(a,b)$, then the set of functions $\{\phi_n(x)\}_{n=1}^{\infty}$ is said to be complete. One can view these functions as an infinite dimensional basis for the space of square integrable functions on (a,b) with weight $\rho(x)>0$.

One can extend the above limit $c_n\to 0$ as $n\to\infty$, by assuming that $\frac{\phi_n(x)}{\|\phi_n\|}$ is uniformly bounded and that $\int_a^b|f(x)|\rho(x)dx<\infty$. This is the RiemannLebesgue Lemma, but will not be proven here.

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5.7: Problems

Exercise 5.7.1

Consider the set of vectors $(-1,1,1),(1,-1,1),(1,1,-1)$

a. Use the Gram-Schmidt process to find an orthonormal basis for R^3 using this set in the given order.
b. What do you get if you do reverse the order of these vectors?

Exercise 5.7.2

Use the Gram-Schmidt process to find the first four orthogonal polynomials satisfying the following:

a. Interval: $(-\infty,\infty)$ Weight Function: e^{-x^2}.
b. Interval: $(0,\infty)$ Weight Function: e^{-x}.

Exercise 5.7.3

Find $P_4(x)$ using

a. The Rodrigues' Formula in Equation (5.3.3).
b. The three term recursion formula in Equation (5.3.5).

Exercise 5.7.4

In Equations (5.3.18)-(5.3.25) we provide several identities for Legendre polynomials. Derive the results in Equations (5.3.19)-(5.3.25) as described in the text. Namely,

a. Differentiating Equation (5.3.18) with respect to x, derive Equation (5.3.19).
b. Derive Equation (5.3.20) by differentiating $g(x,t)$ with respect to x and rearranging the resulting infinite series.
c. Combining the last result with Equation (5.3.18), derive Equations (5.3.21)-(5.3.22).
d. Adding and subtracting Equations (5.3.21)-(5.3.22), obtain Equations (5.3.23)-(5.3.24).
e. Derive Equation (5.3.25) using some of the other identities.

Exercise 5.7.5

Use the recursion relation (5.3.5) to evaluate $\int_{-1}^{1} xP_n(x)P_m(x)dx, n \le m$.

Exercise 5.7.6

Expand the following in a Fourier-Legendre series for $x \in (-1,1)$.

a. $f(x) = x^2$.
b. $f(x) = 5x^4 + 2x^3 - x + 3$.
c. $f(x) = \begin{cases} -1, & -1 < x < 0, \\ 1, & 0 < x < 1. \end{cases}$
d. $f(x) = \begin{cases} x, & -1 < x < 0, \\ 0, & 0 < x < 1. \end{cases}$

Exercise 5.7.7

Use integration by parts to show $\Gamma(x+1) = x\Gamma(x)$.

Exercise 5.7.8

Prove the double factorial identities:

$$(2n)!! = 2^n n!$$

and

$$(2n-1)!! = \frac{(2n)!}{2^n n!}.$$

Exercise 5.7.9

Express the following as Gamma functions. Namely, noting the form $\Gamma(x+1) = \int_0^\infty t^x e^{-t} dt$ and using an appropriate substitution, each expression can be written in terms of a Gamma function.

a. $\int_0^\infty x^{2/3} e^{-x} dx$.
b. $\int_0^\infty x^5 e^{-x^2} dx$
c. $\int_0^1 \left[\ln\left(\frac{1}{x}\right)\right]^n dx$

Exercise 5.7.10

The coefficients C_k^p in the binomial expansion for $(1+x)^p$ are given by

$$C_k^p = \frac{p(p-1)\cdots(p-k+1)}{k!}.$$

a. Write C_k^p in terms of Gamma functions.
b. For $p = 1/2$ use the properties of Gamma functions to write $C_k^{1/2}$ in terms of factorials.
c. Confirm you answer in part b by deriving the Maclaurin series expansion of $(1+x)^{1/2}$.

Exercise 5.7.11

The Hermite polynomials, $H_n(x)$, satisfy the following:

i. $\langle H_n, H_m \rangle = \int_{-\infty}^\infty e^{-x^2} H_n(x) H_m(x) dx = \sqrt{\pi} 2^n n! \delta_{n,m}$.
ii. $H_n'(x) = 2n H_{n-1}(x)$.
iii. $H_{n+1}(x) = 2x H_n(x) - 2n H_{n-1}(x)$.
iv. $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right)$.

Using these, show that

a. $H_n'' - 2x H_n' + 2n H_n = 0$. [Use properties ii. and iii.]
b. $\int_{-\infty}^\infty x e^{-x^2} H_n(x) H_m(x) dx = \sqrt{\pi} 2^{n-1} n! \left[\delta_{m,n-1} + 2(n+1)\delta_{m,n+1}\right]$. [Use properties i. and iii.]
c. $H_n(0) = \begin{cases} 0, & n \text{ odd}, \\ (-1)^m \frac{(2m)!}{m!}, & n = 2m. \end{cases}$ [Let $x = 0$ in iii. and iterate. Note from iv. that $H_0(x) = 1$ and $H_1(x) = 2x$.]

Exercise 5.7.12

In Maple one can type simplify(LegendreP $(2^*n-2, 0)$-LegendreP $(2^*n, 0)$); to find a value for $P_{2n-2}(0) - P_{2n}(0)$. It gives the result in terms of Gamma functions. However, in Example 5.3.8 for Fourier-Legendre series, the value is given in terms of double factorials! So, we have

$$P_{2n-2}(0) - P_{2n}(0) = \frac{\sqrt{\pi}(4n-1)}{2\Gamma(n+1)\Gamma\left(\frac{3}{2}-n\right)} = (-1)^n \frac{(2n-3)!!}{(2n-2)!!} \frac{4n-1}{2n}.$$

You will verify that both results are the same by doing the following:

a. Prove that $P_{2n}(0)=(-1)^n\frac{(2n-1)!!}{(2n)!!}$ using the generating function and a binomial expansion.

b. Prove that $\Gamma\left(n+\frac{1}{2}\right)=\frac{(2n-1)!!}{2^n}\sqrt{\pi}$ using $\Gamma(x)=(x-1)\Gamma(x-1)$ and iteration.

c. Verify the result from Maple that $P_{2n-2}(0)-P_{2n}(0)=\frac{\sqrt{\pi}(4n-1)}{2\Gamma(n+1)\Gamma\left(\frac{3}{2}-n\right)}$.

d. Can either expression for $P_{2n-2}(0)-P_{2n}(0)$ be simplified further?

Exercise 5.7.13

A solution Bessel's equation, $x^2y''+xy'+\left(x^2-n^2\right)y=0$, , can be found using the guess $y(x)=\sum_{j=0}^{\infty}a_jx^{j+n}$. One obtains the recurrence relation $a_j=\frac{-1}{(2n+i)}a_{j-2}$. Show that for $a_0=(n!2^n)^{-1}$ we get the Bessel function of the first kind of order n from the even values $j=2k$:

$$J_n(x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(n+k)!}\left(\frac{x}{2}\right)^{n+2k}.$$

Exercise 5.7.14

Use the infinite series in the last problem to derive the derivative identities (5.5.15) and (5.5.5):

a. $\frac{d}{dx}\left[x^nJ_n(x)\right]=x^nJ_{n-1}(x)$.

b. $\frac{d}{dx}\left[x^{-n}J_n(x)\right]=-x^{-n}J_{n+1}(x)$.

Exercise 5.7.15

Prove the following identities based on those in the last problem.

a. $J_{p-1}(x)+J_{p+1}(x)=\frac{2p}{x}J_p(x)$.

b. $J_{p-1}(x)-J_{p+1}(x)=2J_p'(x)$.

Exercise 5.7.16

Use the derivative identities of Bessel functions, (5.5.15)-(5.5.5), and integration by parts to show that

$$\int x^3J_0(x)dx=x^3J_1(x)-2x^2J_2(x)+C.$$

Exercise 5.7.17

Use the generating function to find $J_n(0)$ and $J_n'(0)$.

Exercise 5.7.18

Bessel functions $J_p(\lambda x)$ are solutions of $x^2y''+xy'+\left(\lambda^2x^2-p^2\right)y=0$. Assume that $x\in(0,1)$ and that $J_p(\lambda)=0$ and $J_p(0)$ is finite.

a. This is the standard Sturm-Liouville form for Bessel's equation.

b. by considering

$$\int_0^1\left[J_p(\mu x)\frac{d}{dx}\left(x\frac{d}{dx}J_p(\lambda x)\right)-J_p(\lambda x)\frac{d}{dx}\left(x\frac{d}{dx}J_p(\mu x)\right)\right]dx.$$

c.

Exercise 5.7.19

We can rewrite Bessel functions, $J_v(x)$, in a form which will allow the order to be non-integer by using the gamma function. You will need the results from Problem 5.7.12b for $\Gamma\left(k+\frac{1}{2}\right)$.

a. Extend the series definition of the Bessel function of the first kind of order v, $J_v(x)$, for $v \geq 0$ by writing the series solution for $y(x)$ in Problem 5.7.13 using the gamma function.
b. Extend the series to $J_{-v}(x)$, for $v \geq 0$. Discuss the resulting series and what happens when v is a positive integer.
c.
d. Use the results in part c with the recursion formula for Bessel functions to obtain a closed form for $J_{3/2}(x)$.

Exercise 5.7.20

In this problem you will derive the expansion

$$x^2 = \frac{c^2}{2} + 4\sum_{j=2}^{\infty} \frac{J_0(\alpha_j x)}{\alpha_j^2 J_0(\alpha_j c)}, \quad 0 < x < c,$$

where the $\alpha_j's$ are the positive roots of $J_1(\alpha c) = 0$, by following the below steps.

a. List the first five values of α for $J_1(\alpha c) = 0$ using the Table 5.5.1 and Figure 5.5.1. [Note: Be careful determining α_1.]
b.
c. Show that $\|J_0(\alpha_j x)\|^2 = \frac{c^2}{2}[J_0(\alpha_j c)]^2, j = 2, 3, \ldots$. (This is the most involved step.) First note from Problem 5.7.18 that $y(x) = J_0(\alpha_j x)$ is a solution of

$$x^2 y'' + xy' + \alpha_j^2 x^2 y = 0.$$

   i. Verify the Sturm-Liouville form of this differential equation: $(xy')' = -\alpha_j^2 xy$.
   ii.
   iii. Noting that $y(x) = J_0(\alpha_j x)$, integrate the left hand side by parts and use the following to simplify the resulting equation.
      1. $J_0'(x) = -J_1(x)$ from Equation (5.5.5).
      2. Equation (5.5.8).
      3. $J_2(\alpha_j c) + J_0(\alpha_j c) = 0$ from Equation (5.5.6).
   iv. Now you should have enough information to complete this part.
d. in order to obtain the desired expansion.

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CHAPTER OVERVIEW

6: Problems in Higher Dimensions

> *"Equations of such complexity as are the equations of the gravitational field can be found only through the discovery of a logically simple mathematical condition that determines the equations completely or at least almost completely."*
>
> *"What I have to say about this book can be found inside this book."*
>
> *~ Albert Einstein (1879-1955)*

In this chapter we will explore several examples of the solution of initial-boundary value problems involving higher spatial dimensions. These are described by higher dimensional partial differential equations, such as the ones presented in Table 2.1.1 in Chapter 2. The spatial domains of the problems span many different geometries, which will necessitate the use of rectangular, polar, cylindrical, or spherical coordinates.

We will solve many of these problems using the method of separation of variables, which we first saw in Chapter 2. Using separation of variables will result in a system of ordinary differential equations for each problem. Adding the boundary conditions, we will need to solve a variety of eigenvalue problems. The product solutions that result will involve trigonometric or some of the special functions that we had encountered in Chapter 5. These methods are used in solving the hydrogen atom and other problems in quantum mechanics and in electrostatic problems in electrodynamics. We will bring to this discussion many of the tools from earlier in this book showing how much of what we have seen can be used to solve some generic partial differential equations which describe oscillation and diffusion type problems.

As we proceed through the examples in this chapter, we will see some common features. For example, the two key equations that we have studied are the heat equation and the wave equation. For higher dimensional problems these take the form

$$u_t = k\nabla^2 u, \tag{6.1}$$
$$u_{tt} = c^2\nabla^2 u. \tag{6.2}$$

We can separate out the time dependence in each equation. Inserting a guess of $u(\mathbf{r},t) = \phi(\mathbf{r})T(t)$ into the heat and wave equations, we obtain

$$T'\phi = kT\nabla^2\phi, \tag{6.3}$$
$$T''\phi = c^2T\nabla^2\phi. \tag{6.4}$$

Note

The Helmholtz equation is named after Hermann Ludwig Ferdinand von Helmholtz (1821-1894). He was both a physician and a physicist and made significant contributions in physiology, optics, acoustics, and electromagnetism.

Dividing each equation by $\phi(\mathbf{r})T(t)$, we can separate the time and space dependence just as we had in Chapter ??. In each case we find that a function of time equals a function of the spatial variables. Thus, these functions must be constant functions. We set these equal to the constant $-\lambda$ and find the respective equations

$$\frac{1}{k}\frac{T'}{T} = \frac{\nabla^2\phi}{\phi} = -\lambda \tag{6.5}$$
$$\frac{1}{c^2}\frac{T''}{T} = \frac{\nabla^2\phi}{\phi} = -\lambda \tag{6.6}$$

The sign of λ is chosen because we expect decaying solutions in time for the heat equation and oscillations in time for the wave equation and will pick $\lambda > 0$.

The respective equations for the temporal functions $T(t)$ are given by

$$T' = -\lambda k T, \tag{6.7}$$
$$T'' + c^2 \lambda T = 0. \tag{6.8}$$

These are easily solved as we had seen in Chapter ??. We have

$$T(t) = T(0) e^{-\lambda k t}, \tag{6.9}$$
$$T(t) = a \cos \omega t + b \sin \omega t, \quad \omega = c\sqrt{\lambda}, \tag{6.10}$$

where $T(0)$, a, and b are integration constants and ω is the angular frequency of vibration.

In both cases the spatial equation is of the same form,

$$\nabla^2 \phi + \lambda \phi = 0. \tag{6.11}$$

This equation is called the Helmholtz equation. For one dimensional problems, which we have already solved, the Helmholtz equation takes the form $\phi'' + \lambda \phi = 0$. We had to impose the boundary conditions and found that there were a discrete set of eigenvalues, λ_n, and associated eigenfunctions, ϕ_n.

In higher dimensional problems we need to further separate out the spatial dependence. We will again use the boundary conditions to find the eigenvalues, λ, and eigenfunctions, $\phi(\mathbf{r})$, for the Helmholtz equation, though the eigenfunctions will be labeled with more than one index. The resulting boundary value problems are often second order ordinary differential equations, which can be set up as Sturm-Liouville problems. We know from Chapter 5 that such problems possess an orthogonal set of eigenfunctions. These can then be used to construct a general solution from the product solutions which may involve elementary, or special, functions, such as Legendre polynomials and Bessel functions.

We will begin our study of higher dimensional problems by considering the vibrations of two dimensional membranes. First we will solve the problem of a vibrating rectangular membrane and then we will turn our attention to a vibrating circular membrane. The rest of the chapter will be devoted to the study of other two and three dimensional problems possessing cylindrical or spherical symmetry.

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6.2: Vibrations of a Kettle Drum
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6.1: Vibrations of Rectangular Membranes

Our first example will be the study of the vibrations of a rectangular membrane. You can think of this as a drumhead with a rectangular cross section as shown in Figure 6.1.1. We stretch the membrane over the drumhead and fasten the material to the boundary of the rectangle. The height of the vibrating membrane is described by its height from equilibrium, $u(x,y,t)$. This problem is a much simpler example of higher dimensional vibrations than that possessed by the oscillating electric and magnetic fields in the last chapter.

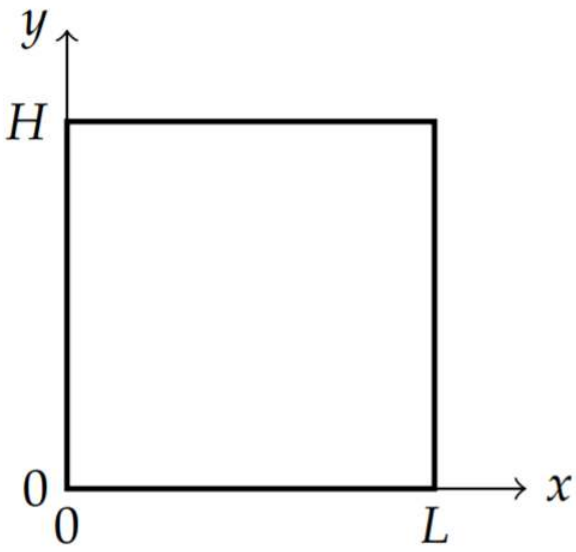


Figure 6.1.1: The rectangular membrane of length L and width H. There are fixed boundary conditions along the edges.

Example 6.1.1: The Vibrating Rectangular Membrane

The problem is given by the two dimensional wave equation in Cartesian coordinates,

$$u_{tt} = c^2\left(u_{xx} + u_{yy}\right), \quad t > 0, 0 < x < L, 0 < y < H, \tag{6.1.1}$$

a set of boundary conditions,

$$\begin{aligned} u(0,y,t) &= 0, \quad u(L,y,t) = 0, \quad t > 0, \quad 0 < y < H, \\ u(x,0,t) &= 0, \quad u(x,H,t) = 0, \quad t > 0, \quad 0 < x < L, \end{aligned} \tag{6.1.2}$$

and a pair of initial conditions (since the equation is second order in time),

$$u(x,y,0) = f(x,y), \quad u_t(x,y,0) = g(x,y). \tag{6.1.3}$$

The first step is to separate the variables: $u(x,y,t) = X(x)Y(y)T(t)$. Inserting the guess, $u(x,y,t)$ into the wave equation, we have

$$X(x)Y(y)T''(t) = c^2\left(X''(x)Y(y)T(t) + X(x)Y''(y)T(t)\right).$$

Dividing by both $u(x,y,t)$ and c^2, we obtain

$$\underbrace{\frac{1}{c^2}\frac{T''}{T}}_{\text{unction of } t} = \underbrace{\frac{X''}{X} + \frac{Y''}{Y}}_{\text{Function of } x \text{ and } y} = -\lambda. \tag{6.1.4}$$

We see that we have a function of t equals a function of x and y. Thus, both expressions are constant. We expect oscillations in time, so we choose the constant λ to be positive, $\lambda > 0$. (Note: As usual, the primes mean differentiation with respect to the specific dependent variable. So, there should be no ambiguity.)

These lead to two equations:

$$T'' + c^2 \lambda T = 0, \tag{6.1.5}$$

and

$$\frac{X''}{X} + \frac{Y''}{Y} = -\lambda. \tag{6.1.6}$$

We note that the spatial equation is just the separated form of Helmholtz's equation with $\phi(x, y) = X(x)Y(y)$.

The first equation is easily solved. We have

$$T(t) = a \cos \omega t + b \sin \omega t, \tag{6.1.7}$$

where

$$\omega = c\sqrt{\lambda}. \tag{6.1.8}$$

This is the angular frequency in terms of the separation constant, or eigenvalue. It leads to the frequency of oscillations for the various harmonics of the vibrating membrane as

$$v = \frac{\omega}{2\pi} = \frac{c}{2\pi}\sqrt{\lambda}. \tag{6.1.9}$$

Once we know λ, we can compute these frequencies.

Next we solve the spatial equation. We need carry out another separation of variables. Rearranging the spatial equation, we have

$$\underbrace{\frac{X''}{X}}_{\text{Function of } x} = \underbrace{-\frac{Y''}{Y} - \lambda}_{\text{Function of } y} = -\mu. \tag{6.1.10}$$

Here we have a function of x equal to a function of y. So, the two expressions are constant, which we indicate with a second separation constant, $-\mu < 0$. We pick the sign in this way because we expect oscillatory solutions for $X(x)$. This leads to two equations:

$$\begin{aligned} X'' + \mu X &= 0, \\ Y'' + (\lambda - \mu)Y &= 0. \end{aligned} \tag{6.1.11}$$

We now impose the boundary conditions. We have $u(0, y, t) = 0$ for all $t > 0$ and $0 < y < H$. This implies that $X(0)Y(y)T(t) = 0$ for all t and y in the domain. This is only true if $X(0) = 0$. Similarly, from the other boundary conditions we find that $X(L) = 0, Y(0) = 0$, and $Y(H) = 0$. We note that homogeneous boundary conditions are important in carrying out this process. Nonhomogeneous boundary conditions could be imposed just like we had in Section 7.3, but we still need the solutions for homogeneous boundary conditions before tackling the more general problems.

In summary, the boundary value problems we need to solve are:

$$\begin{aligned} X'' + \mu X &= 0, \quad X(0) = 0, X(L) = 0. \\ Y'' + (\lambda - \mu)Y &= 0, \quad Y(0) = 0, Y(H) = 0. \end{aligned} \tag{6.1.12}$$

We have seen boundary value problems of these forms in Chapter ??. The solutions of the first eigenvalue problem are

$$X_n(x) = \sin \frac{n\pi x}{L}, \quad \mu_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \ldots$$

The second eigenvalue problem is solved in the same manner. The differences from the first problem are that the "eigenvalue" is $\lambda - \mu$, the independent variable is y, and the interval is $[0, H]$. Thus, we can quickly write down the solutions as

$$Y_m(y) = \sin \frac{m\pi x}{H}, \quad \lambda - \mu_m = \left(\frac{m\pi}{H}\right)^2, \quad m = 1, 2, 3, \ldots$$

At this point we need to be careful about the indexing of the separation constants. So far, we have seen that μ depends on n and that the quantity $\kappa = \lambda - \mu$ depends on m. Solving for λ, we should write $\lambda_{nm} = \mu_n + \kappa_m$, or

$$\lambda_{nm} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2, \quad n, m = 1, 2, \ldots. \tag{6.1.13}$$

Since $\omega = c\sqrt{\lambda}$, we have that the discrete frequencies of the harmonics are given by

$$\omega_{nm} = c\sqrt{\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2}, \quad n, m = 1, 2, \ldots \tag{6.1.14}$$

Note

The harmonics for the vibrating rectangular membrane are given by

$$v_{nm} = \frac{c}{2}\sqrt{\left(\frac{n}{L}\right)^2 + \left(\frac{m}{H}\right)^2}$$

for $n, m = 1, 2, \ldots$

We have successfully carried out the separation of variables for the wave equation for the vibrating rectangular membrane. The product solutions can be written as

$$u_{nm} = (a\cos\omega_{nm}t + b\sin\omega_{nm}t)\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H} \tag{6.1.15}$$

and the most general solution is written as a linear combination of the product solutions,

$$u(x, y, t) = \sum_{n,m}(a_{nm}\cos\omega_{nm}t + b_{nm}\sin\omega_{nm}t)\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}.$$

However, before we carry the general solution any further, we will first concentrate on the two dimensional harmonics of this membrane.

For the vibrating string the nth harmonic corresponds to the function $\sin\frac{n\pi x}{L}$ and several are shown in Figure 6.1.2. The various harmonics correspond to the pure tones supported by the string. These then lead to the corresponding frequencies that one would hear. The actual shapes of the harmonics are sketched by locating the nodes, or places on the string that do not move.

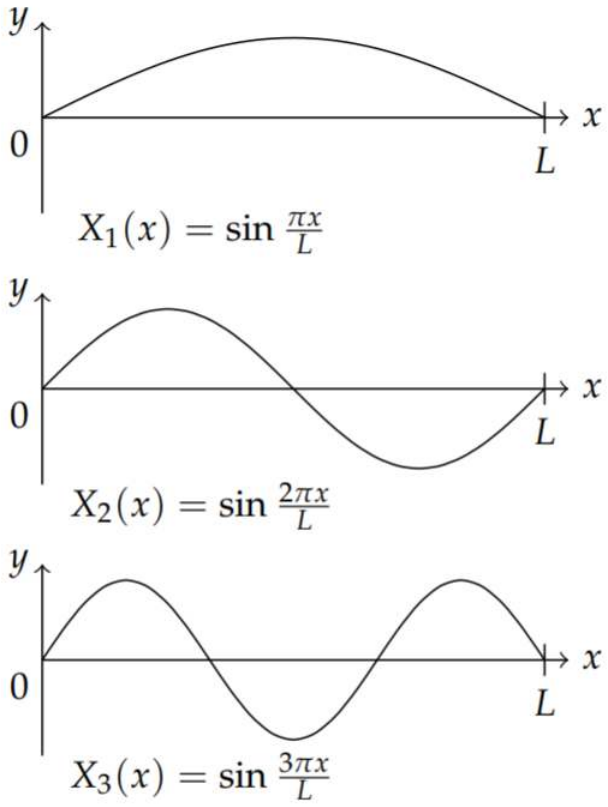


Figure 6.1.2: The first harmonics of the vibrating string

In the same way, we can explore the shapes of the harmonics of the vibrating membrane. These are given by the spacial functions

$$\phi_{nm}(x,y) = \sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}. \tag{6.1.16}$$

Instead of nodes, we will look for the nodal curves, or nodal lines. These are the points (x,y) at which $\phi_{nm}(x,y)=0$. Of course, these depend on the indices, n and m.

For example, when $n=1$ and $m=1$, we have

$$\sin\frac{\pi x}{L}\sin\frac{\pi y}{H} = 0.$$

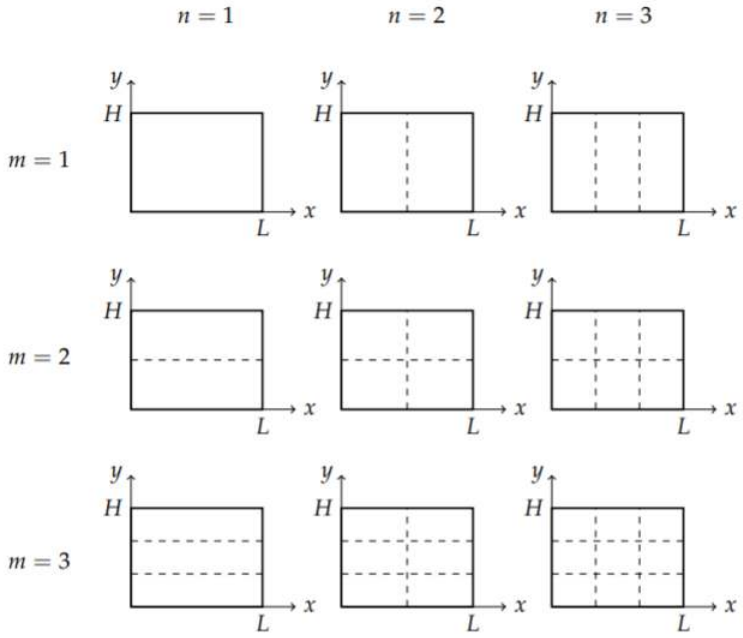


Figure 6.1.3: The first few modes of the vibrating rectangular membrane. The dashed lines show the nodal lines indicating the points that do not move for the particular mode. Compare these the nodal lines to the $_3$D view in Figure 6.1.1.

These are zero when either

$$\sin\frac{\pi x}{L} = 0, \text{ or } \sin\frac{\pi y}{H} = 0.$$

Of course, this can only happen for $x=0, L$ and $y=0, H$. Thus, there are no interior nodal lines.

When $n=2$ and $m=1$, we have $y=0, H$ and

$$\sin\frac{2\pi x}{L} = 0$$

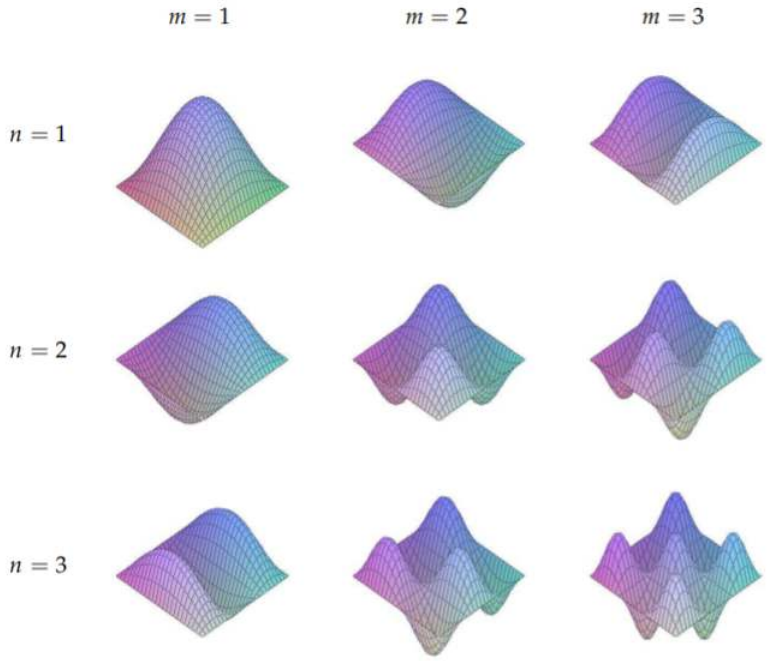
or, $x=0$, $\frac{L}{2}$, L. Thus, there is one interior nodal line at $x=\frac{L}{2}$. These points stay fixed during the oscillation and all other points oscillate on either side of this line. A similar solution shape results for the $(1,2)$-mode; i.e., $n=1$ and $m=2$.

In Figure 6.1.3 we show the nodal lines for several modes for $n,m=1,2,3$ with different columns corresponding to different n-values while the rows are labeled with different m-values. The blocked regions appear to vibrate independently. A better view is the three dimensional view depicted in Figure 6.1.1. The frequencies of vibration are easily computed using the formula for ω_{nm}.

For completeness, we now return to the general solution and apply the initial conditions. The general solution is given by a linear superposition of the product solutions. There are two indices to sum over. Thus, the general solution is

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(a_{nm}\cos\omega_{nm}t + b_{nm}\sin\omega_{nm}t\right)\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H} \tag{6.1.17}$$

Table 6.1.1: A three dimensional view of the vibrating rectangular membrane for the lowest modes. Compare these images with the nodal lines in Figure 6.1.3.



where

$$\omega_{nm} = c\sqrt{\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2}. \tag{6.1.18}$$

The first initial condition is $u(x, y, 0) = f(x, y)$. Setting $t = 0$ in the general solution, we obtain

$$f(x, y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{nm} \sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}. \tag{6.1.19}$$

This is a double Fourier sine series. The goal is to find the unknown coefficients a_{nm}.

The coefficients a_{nm} can be found knowing what we already know about Fourier sine series. We can write the initial condition as the single sum

$$f(x, y) = \sum_{n=1}^{\infty} A_n(y) \sin\frac{n\pi x}{L}, \tag{6.1.20}$$

where

$$A_n(y) = \sum_{m=1}^{\infty} a_{nm} \sin\frac{m\pi y}{H}. \tag{6.1.21}$$

These are two Fourier sine series. Recalling from Chapter ?? that the coefficients of Fourier sine series can be computed as integrals, we have

$$\begin{aligned} A_n(y) &= \frac{2}{L}\int_0^L f(x, y) \sin\frac{n\pi x}{L} dx, \\ a_{nm} &= \frac{2}{H}\int_0^H A_n(y) \sin\frac{m\pi y}{H} dy. \end{aligned} \tag{6.1.22}$$

Inserting the integral for $A_n(y)$ into that for a_{nm}, we have an integral representation for the Fourier coefficients in the double Fourier sine series,

$$a_{nm} = \frac{4}{LH}\int_0^H \int_0^L f(x, y) \sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H} dx dy. \tag{6.1.23}$$

We can carry out the same process for satisfying the second initial condition, $u_t(x,y,0) = g(x,y)$ for the initial velocity of each point. Inserting the general solution into this initial condition, we obtain

$$g(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_{nm}\omega_{nm} \sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}. \tag{6.1.24}$$

Again, we have a double Fourier sine series. But, now we can quickly determine the Fourier coefficients using the above expression for a_{nm} to find that

$$b_{nm} = \frac{4}{\omega_{nm}LH}\int_0^H\int_0^L g(x,y)\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}dxdy. \tag{6.1.25}$$

This completes the full solution of the vibrating rectangular membrane problem. Namely, we have obtained the solution

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(a_{nm}\cos\omega_{nm}t + b_{nm}\sin\omega_{nm}t)\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H} \tag{6.1.26}$$

where

$$a_{nm} = \frac{4}{LH}\int_0^H\int_0^L f(x,y)\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}dxdy, \tag{6.1.27}$$

$$b_{nm} = \frac{4}{\omega_{nm}LH}\int_0^H\int_0^L g(x,y)\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}dxdy, \tag{6.1.28}$$

and the angular frequencies are given by

$$\omega_{nm} = c\sqrt{\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2}. \tag{6.1.29}$$

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6.2: Vibrations of a Kettle Drum

In this section we consider the vibrations of a circular membrane of radius a as shown in Figure 6.2.1. Again we are looking for the harmonics of the vibrating membrane, but with the membrane fixed around the circular boundary given by $x^2+y^2=a^2$. However, expressing the boundary condition in Cartesian coordinates is awkward. Namely, we can only write $u(x,y,t)=0$ for $x^2+y^2=a^2$. It is more natural to use polar coordinates as indicated in Figure 6.2.1. Let the height of the membrane be given by $u=u(r,\theta,t)$ at time t and position (r,θ). Now the boundary condition is given as $u(a,\theta,t)=0$ for all $t>0$ and $\theta\in[0,2\pi]$.

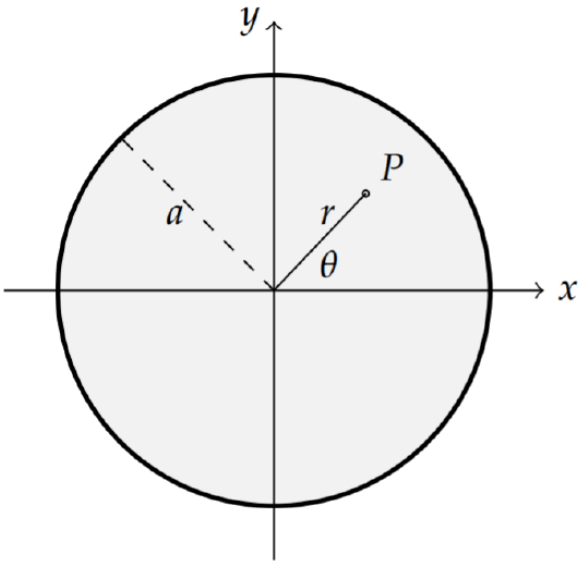


Figure 6.2.1: The circular membrane of radius a. A general point on the membrane is given by the distance from the center, r, and the angle, θ. There are fixed boundary conditions along the edge at $r=a$.

Before solving the initial-boundary value problem, we have to cast the full problem in polar coordinates. This means that we need to rewrite the Laplacian in r and θ. To do so would require that we know how to transform derivatives in x and y into derivatives with respect to r and θ. Using the results from Section ?? on curvilinear coordinates, we know that the Laplacian can be written in polar coordinates. In fact, we could use the results from Problem ?? in Chapter ?? for cylindrical coordinates for functions which are z-independent, $f=f(r,\theta)$. Then, we would have

$$\nabla^2 f=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2}.$$

We can obtain this result using a more direct approach, namely applying the Chain Rule in higher dimensions. First recall the transformations between polar and Cartesian coordinates:

$$x=r\cos\theta,\quad y=r\sin\theta$$

and

$$r=\sqrt{x^2+y^2},\quad \tan\theta=\frac{y}{x}.$$

Now, consider a function $f=f(x(r,\theta),y(r,\theta))=g(r,\theta)$. (Technically, once we transform a given function of Cartesian coordinates we obtain a new function g of the polar coordinates. Many texts do not rigorously distinguish between the two functions.) Thinking of $x=x(r,\theta)$ and $y=y(r,\theta)$, we have from the chain rule for functions of two variables:

$$\begin{aligned}\frac{\partial f}{\partial x}&=\frac{\partial g}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial g}{\partial\theta}\frac{\partial\theta}{\partial x}\\&=\frac{\partial g}{\partial r}\frac{x}{r}-\frac{\partial g}{\partial\theta}\frac{y}{r^2}\\&=\cos\theta\frac{\partial g}{\partial r}-\frac{\sin\theta\partial g}{r}\frac{\partial g}{\partial\theta}.\end{aligned}\tag{6.2.1}$$

Here we have used

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r}$$

and

$$\frac{\partial \theta}{\partial x} = \frac{d}{dx}\left(\tan^{-1}\frac{y}{x}\right) = \frac{-y/x^2}{1+\left(\frac{y}{x}\right)^2} = -\frac{y}{r^2}.$$

Similarly,

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial g}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial g}{\partial \theta}\frac{\partial \theta}{\partial y} \\ &= \frac{\partial g}{\partial r}\frac{y}{r} + \frac{\partial g}{\partial \theta}\frac{x}{r^2} \\ &= \sin\theta\frac{\partial g}{\partial r} + \frac{\cos\theta}{r}\frac{\partial g}{\partial \theta}.\end{aligned} \tag{6.2.2}$$

The 2D Laplacian can now be computed as

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} &= \cos\theta\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial x}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial x}\right) \\ &\quad + \sin\theta\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial y}\right) + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial y}\right) \\ &= \cos\theta\frac{\partial}{\partial r}\left(\cos\theta\frac{\partial g}{\partial r} - \frac{\sin\theta}{r}\frac{\partial g}{\partial \theta}\right) \\ &\quad - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\cos\theta\frac{\partial g}{\partial r} - \frac{\sin\theta}{r}\frac{\partial g}{\partial \theta}\right) \\ &\quad + \sin\theta\frac{\partial}{\partial r}\left(\sin\theta\frac{\partial g}{\partial r} + \frac{\cos\theta}{r}\frac{\partial g}{\partial \theta}\right) \\ &\quad + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial g}{\partial r} + \frac{\cos\theta}{r}\frac{\partial g}{\partial \theta}\right) \\ &= \cos\theta\left(\cos\theta\frac{\partial^2 g}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial g}{\partial \theta} - \frac{\sin\theta}{r}\frac{\partial^2 g}{\partial r \partial \theta}\right) \\ &\quad - \frac{\sin\theta}{r}\left(\cos\theta\frac{\partial^2 g}{\partial \theta \partial r} - \frac{\sin\theta}{r}\frac{\partial^2 g}{\partial \theta^2} - \sin\theta\frac{\partial g}{\partial r} - \frac{\cos\theta}{r}\frac{\partial g}{\partial \theta}\right) \\ &\quad + \sin\theta\left(\sin\theta\frac{\partial^2 g}{\partial r^2} + \frac{\cos\theta}{r}\frac{\partial^2 g}{\partial r \partial \theta} - \frac{\cos\theta}{r^2}\frac{\partial g}{\partial \theta}\right) \\ &\quad + \frac{\cos\theta}{r}\left(\sin\theta\frac{\partial^2 g}{\partial \theta \partial r} + \frac{\cos\theta}{r}\frac{\partial^2 g}{\partial \theta^2} + \cos\theta\frac{\partial g}{\partial r} - \frac{\sin\theta}{r}\frac{\partial g}{\partial \theta}\right) \\ &= \frac{\partial^2 g}{\partial r^2} + \frac{1}{r}\frac{\partial g}{\partial r} + \frac{1}{r^2}\frac{\partial^2 g}{\partial \theta^2} \\ &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial g}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 g}{\partial \theta^2}.\end{aligned} \tag{6.2.3}$$

The last form often occurs in texts because it is in the form of a SturmLiouville operator. Also, it agrees with the result from using the Laplacian written in cylindrical coordinates as given in Problem ?? of Chapter ??.

Now that we have written the Laplacian in polar coordinates we can pose the problem of a vibrating circular membrane.

Example 6.2.1: The Vibrating Circular Membrane

This problem is given by a partial differential equation,[1]

$$u_{tt} = c^2\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right],$$

$$t>0, \quad 0<r<a, \quad -\pi<\theta<\pi, \tag{6.2.4}$$

the boundary condition,

$$u(a,\theta,t)=0, \quad t>0, \quad -\pi<\theta<\pi, \tag{6.2.5}$$

and the initial conditions,

$$\begin{aligned} u(r,\theta,0) &= f(r,\theta), \quad 0<r<a, -\pi<\theta<\pi, \\ u_t(r,\theta,0) &= g(r,\theta),, \quad 0<r<a, -\pi<\theta<\pi. \end{aligned} \tag{6.2.6}$$

Note

Here we state the problem of a vibrating circular membrane. We have chosen $-\pi<\theta<\pi$, but could have just as easily used $0<\theta<2\pi$. The symmetric interval about $\theta=0$ will make the use of boundary conditions simpler.

Now we are ready to solve this problem using separation of variables. As before, we can separate out the time dependence. Let $u(r,\theta,t)=T(t)\phi(r,\theta)$. As usual, $T(t)$ can be written in terms of sines and cosines. This leads to the Helmholtz equation,

$$\nabla^2\phi+\lambda\phi=0.$$

We now separate the Helmholtz equation by letting $\phi(r,\theta)=R(r)\Theta(\theta)$. This gives

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R\Theta}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 R\Theta}{\partial\theta^2}+\lambda R\Theta=0. \tag{6.2.7}$$

Dividing by $u=R\Theta$, as usual, leads to

$$\frac{1}{rR}\frac{d}{dr}\left(r\frac{dR}{dr}\right)+\frac{1}{r^2\Theta}\frac{d^2\Theta}{d\theta^2}+\lambda=0. \tag{6.2.8}$$

The last term is a constant. The first term is a function of r. However, the middle term involves both r and θ. This can be remedied by multiplying the equation by r^2. Rearranging the resulting equation, we can separate out the θ-dependence from the radial dependence. Letting μ be another separation constant, we have

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right)+\lambda r^2=-\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2}=\mu. \tag{6.2.9}$$

This gives us two ordinary differential equations:

$$\begin{aligned} \frac{d^2\Theta}{d\theta^2}+\mu\Theta &= 0, \\ r\frac{d}{dr}\left(r\frac{dR}{dr}\right)+\left(\lambda r^2-\mu\right)R &= 0. \end{aligned} \tag{6.2.10}$$

Let's consider the first of these equations. It should look familiar by now. For $\mu>0$, the general solution is

$$\Theta(\theta)=a\cos\sqrt{\mu}\theta+b\sin\sqrt{\mu}\theta.$$

The next step typically is to apply the boundary conditions in θ. However, when we look at the given boundary conditions in the problem, we do not see anything involving θ. This is a case for which the boundary conditions that are needed are implied and not stated outright.

We can determine the hidden boundary conditions by making some observations. Let's consider the solution corresponding to the endpoints $\theta=\pm\pi$. We note that at these θ-values we are at the same physical point for any $r<a$. So, we would expect the solution to have the same value at $\theta=-\pi$ as it has at $\theta=\pi$. Namely, the solution is continuous at these physical points. Similarly, we expect the slope of the solution to be the same at these points. This can be summarized using the boundary conditions

$$\Theta(\pi)=\Theta(-\pi), \quad \Theta'(\pi)=\Theta'(-\pi).$$

Such boundary conditions are called periodic boundary conditions.

Note

The boundary conditions in θ are periodic boundary conditions.

Let's apply these conditions to the general solution for $\Theta(\theta)$. First, we set $\Theta(\pi) = \Theta(-\pi)$ and use the symmetries of the sine and cosine functions to obtain

$$a\cos\sqrt{\mu}\pi + b\sin\sqrt{\mu}\pi = a\cos\sqrt{\mu}\pi - b\sin\sqrt{\mu}\pi.$$

This implies that

$$\sin\sqrt{\mu}\pi = 0.$$

This can only be true for $\sqrt{\mu} = m$, for $m = 0, 1, 2, 3, \ldots$ Therefore, the eigenfunctions are given by

$$\Theta_m(\theta) = a\cos m\theta + b\sin m\theta, \quad m = 0, 1, 2, 3, \ldots$$

For the other half of the periodic boundary conditions, $\Theta'(\pi) = \Theta'(-\pi)$, we have that

$$-am\sin m\pi + bm\cos m\pi = am\sin m\pi + bm\cos m\pi$$

But, this gives no new information since this equation boils down to $\mathrm{bm} = \mathrm{bm}$.

To summarize what we know at this point, we have found the general solutions to the temporal and angular equations. The product solutions will have various products of $\{\cos\omega t, \sin\omega t\}$ and $\{\cos m\theta, \sin m\theta\}_{m=0}^{\infty}$. We also know that $\mu = m^2$ and $\omega = c\sqrt{\lambda}$.

We still need to solve the radial equation. Inserting $\mu = m^2$, the radial equation has the form

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \left(\lambda r^2 - m^2\right)R = 0. \tag{6.2.11}$$

Expanding the derivative term, we have

$$r^2R''(r) + rR'(r) + \left(\lambda r^2 - m^2\right)R(r) = 0. \tag{6.2.12}$$

The reader should recognize this differential equation from Equation (5.5.11). It is a Bessel equation with bounded solutions $R(r) = J_m(\sqrt{\lambda}r)$.

Recall there are two linearly independent solutions of this second order equation: $J_m(\sqrt{\lambda}r)$, the Bessel function of the first kind of order m, and $N_m(\sqrt{\lambda}r)$, the Bessel function of the second kind of order m, or Neumann functions. Plots of these functions are shown in Figures 5.5.1 and 5.5.2. So, we have the general solution of the radial equation is

$$R(r) = c_1J_m(\sqrt{\lambda}r) + c_2N_m(\sqrt{\lambda}r).$$

Now we are ready to apply the boundary conditions to the radial factor in the product solutions. Looking at the original problem we find only one condition: $u(a, \theta, t) = 0$ for $t > 0$ and $-\pi << \pi$. This implies that $R(a) = 0$. But where is the second condition?

This is another unstated boundary condition. Look again at the plots of the Bessel functions. Notice that the Neumann functions are not well behaved at the origin. Do you expect that the solution will become infinite at the center of the drum? No, the solutions should be finite at the center. So, this observation leads to the second boundary condition. Namely, $|R(0)| < \infty$. This implies that $c_2 = 0$.

Table 5.5.1.

Table 6.2.1: The zeros of Bessel Functions, $J_m(j_{mn}) = 0$.

n	$m = 0$	$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 5$
1	2.405	3.832	5.136	6.380	7.588	8.771
2	5.520	7.016	8.417	9.761	**11.065**	**12.339**
3	8.654	**10.173**	**11.620**	**13.015**	**14.373**	**15.700**
4	**11.792**	**13.324**	**14.796**	**16.223**	**17.616**	**18.980**

n	$m=0$	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$
5	14.931	16.471	17.960	19.409	20.827	22.218
6	18.071	19.616	21.117	22.583	24.019	25.430
7	21.212	22.760	24.270	25.748	27.199	28.627
8	24.352	25.904	27.421	28.908	30.371	31.812
9	27.493	29.047	30.569	32.065	33.537	34.989

Let's denote the nth zero of $J_m(x)$ by j_{mn}. Then, the boundary condition tells us that

$$\sqrt{\lambda}a = j_{mn}, \quad m = 0, 1, \ldots, \quad n = 1, 2, \ldots$$

This gives us the eigenvalues as

$$\lambda_{mn} = \left(\frac{j_{mn}}{a}\right)^2, \quad m = 0, 1, \ldots, \quad n = 1, 2, \ldots.$$

Thus, the radial function satisfying the boundary conditions is

$$R_{mn}(r) = J_m\left(\frac{j_{mn}}{a}r\right).$$

We are finally ready to write out the product solutions for the vibrating circular membrane. They are given by

$$u(r, \theta, t) = \left\{\begin{array}{c}\cos\omega_{mn}t \\ \sin\omega_{mn}t\end{array}\right\}\left\{\begin{array}{c}\cos m\theta \\ \sin m\theta\end{array}\right\} J_m\left(\frac{j_{mn}}{a}r\right). \tag{6.2.13}$$

Here we have indicated choices with the braces, leading to four different types of product solutions. Also, the angular frequency depends on the zeros of the Bessel functions,

$$\omega_{mn} = \frac{j_{mn}}{a}c, \quad m = 0, 1, \ldots, \quad n = 1, 2, \ldots$$

As with the rectangular membrane, we are interested in the shapes of the harmonics. So, we consider the spatial solution $(t = 0)$

$$\phi(r, \theta) = (\cos m\theta) J_m\left(\frac{j_{mn}}{a}r\right).$$

Including the solutions involving $\sin m\theta$ will only rotate these modes. The nodal curves are given by $\phi(r, \theta) = 0$. This can be satisfied if $\cos m\theta = 0$, or $J_m\left(\frac{f_{mn}}{a}r\right) = 0$. The various nodal curves which result are shown in Figure 6.2.2.

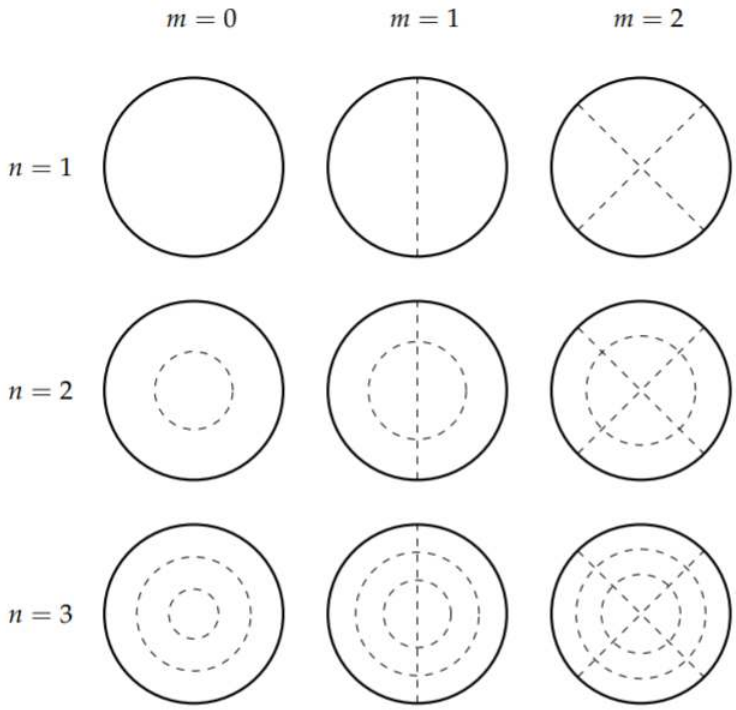


Figure 6.2.2: The first few modes of the vibrating circular membrane. The dashed lines show the nodal lines indicating the points that do not move for the particular mode. Compare these nodal lines with the three dimensional images in Figure 6.1.3.

For the angular part, we easily see that the nodal curves are radial lines, $\theta = \text{const}$. For $m = 0$, there are no solutions, since $\cos m\theta = 1$ for $m = 0$. in Figure 6.2.2 this is seen by the absence of radial lines in the first column.

For $m = 1$, we have $\cos\theta = 0$. This implies that $\theta = \pm\frac{\pi}{2}$. These values give the vertical line as shown in the second column in Figure 6.2.2. For $m = 2, \cos 2\theta = 0$ implies that $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$. This results in the two lines shown in the last column of Figure 6.2.2.

We can also consider the nodal curves defined by the Bessel functions. We seek values of r for which $\frac{j_{mn}}{a}r$ is a zero of the Bessel function and lies in the interval $[0, a]$. Thus, we have

$$\frac{j_{mn}}{a}r = j_{mj}, \quad 1 \le j \le n,$$

or

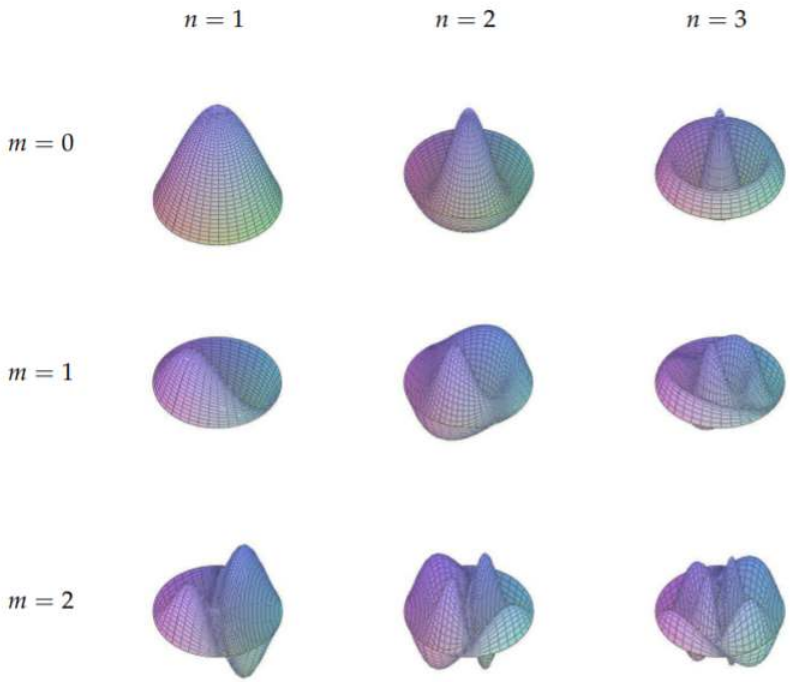
$$r = \frac{j_{mj}}{j_{mn}}a, \quad 1 \le j \le n.$$

These will give circles of these radii with $j_{mj} \le j_{mn}$, or $j \le n$. For $m = 0$ and $n = 1$, there is only one zero and $r = a$. In fact, for all $n = 1$ modes, there is only one zero giving $r = a$. Thus, the first row in Figure 6.2.2 shows no interior nodal circles.

For a three dimensional view, one can look at Figure 6.1.3. Imagine that the various regions are oscillating independently and that the points on the nodal curves are not moving.

We should note that the nodal circles are not evenly spaced and that the radii can be computed relatively easily. For the $n = 2$ modes, we have two circles, $r = a$ and $r = \frac{j_{m1}}{j_{m2}}a$ as shown in the second row of Figure 6.2.2. For

Table 6.2.2: A three dimensional view of the vibrating circular membrane for the lowest modes. Compare these images with the nodal line plots in Figure 6.2.2.



$m = 0$,

$$r = \frac{2.405}{5.520}a \approx 0.4357a$$

for the inner circle. For $m = 1$,

$$r = \frac{3.832}{7.016}a \approx 0.5462a,$$

and for $m = 2$,

$$r = \frac{5.136}{8.417}a \approx 0.6102a.$$

For $n = 3$ we obtain circles of radii

$$r = a, \quad r = \frac{j_{m1}}{j_{m3}}a, \quad \text{and} \quad r = \frac{j_{m2}}{j_{m3}}a.$$

For $m = 0$,

$$r = a, \quad \frac{5.520}{8.654}a \approx 0.6379a, \quad \frac{2.405}{8.654}a \approx 0.2779a.$$

Similarly, for $m = 1$,

$$r = a, \quad \frac{3.832}{10.173}a \approx 0.3767a, \quad \frac{7.016}{10.173}a \approx 0.6897a,$$

and for $m = 2$,

$$r = a, \quad \frac{5.136}{11.620}a \approx 0.4420a, \quad \frac{8.417}{11.620}a \approx 0.7224a.$$

Example 6.2.2: Vibrating Annulus

More complicated vibrations can be dreamt up for this geometry. Consider an annulus in which the drum is formed from two concentric circular cylinders and the membrane is stretch between the two with an annular cross section as shown in Figure

6.2.3. The separation would follow as before except now the boundary conditions are that the membrane is fixed around the two circular boundaries. In this case we cannot toss out the Neumann functions because the origin is not part of the drum head.

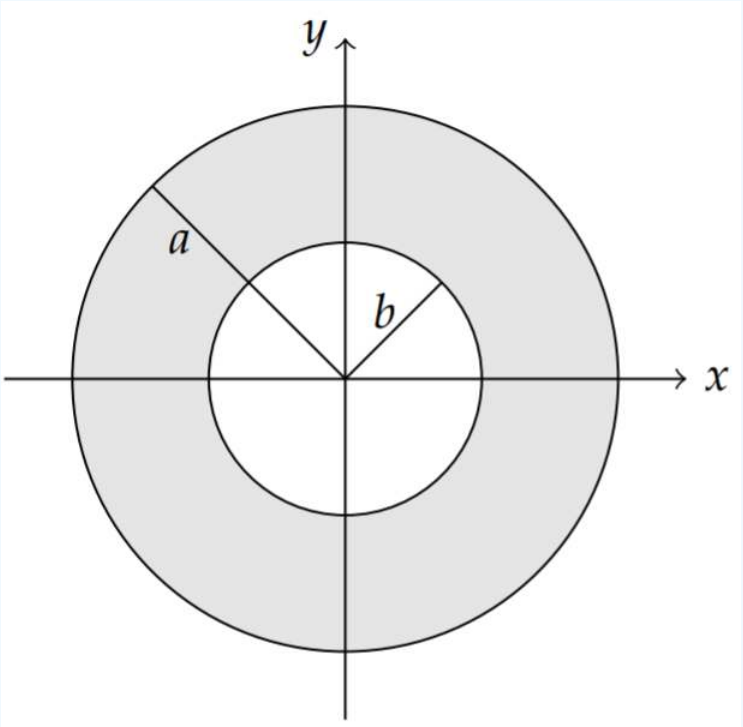


Figure 6.2.3: An annular membrane with radii a and $b > a$. There are fixed boundary conditions along the edges at $r = a$ and $r = b$.

Solution

The domain for this problem is shown in Figure 6.2.3 and the problem is given by the partial differential equation

$$u_{tt} = c^2 \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \right], \tag{6.2.14}$$

$$t > 0, \quad b < r < a, \quad -\pi < \theta < \pi, \tag{6.2.15}$$

the boundary conditions,

$$u(b, \theta, t) = 0, \quad u(a, \theta, t) = 0, \quad t > 0, \quad -\pi < \theta < \pi, \tag{6.2.16}$$

and the initial conditions,

$$\begin{aligned} u(r, \theta, 0) &= f(r, \theta), \quad b < r < a, -\pi < \theta < \pi, \\ u_t(r, \theta, 0) &= g(r, \theta), \quad b < r < a, -\pi < \theta < \pi. \end{aligned} \tag{6.2.17}$$

Since we cannot dispose of the Neumann functions, the product solutions take the form

$$u(r, \theta, t) = \left\{ \begin{array}{c} \cos \omega t \\ \sin \omega t \end{array} \right\} \left\{ \begin{array}{c} \cos m\theta \\ \sin m\theta \end{array} \right\} R_m(r), \tag{6.2.18}$$

where

$$R_m(r) = c_1 J_m(\sqrt{\lambda} r) + c_2 N_m(\sqrt{\lambda} r)$$

and $\omega = c\sqrt{\lambda}, \quad m = 0, 1, \ldots$

For this problem the radial boundary conditions are that the membrane is fixed at $r = a$ and $r = b$. Taking $b < a$, we then have to satisfy the conditions

$$\begin{aligned} R(a) &= c_1 J_m(\sqrt{\lambda} a) + c_2 N_m(\sqrt{\lambda} a) = 0, \\ R(b) &= c_1 J_m(\sqrt{\lambda} b) + c_2 N_m(\sqrt{\lambda} b) = 0. \end{aligned} \tag{6.2.19}$$

This leads to two homogeneous equations for c_1 and c_2. The coefficient determinant of this system has to vanish if there are to be nontrivial solutions. This gives the eigenvalue equation for λ :

$$J_m(\sqrt{\lambda}a)N_m(\sqrt{\lambda}b) - J_m(\sqrt{\lambda}b)N_m(\sqrt{\lambda}a) = 0.$$

There are an infinite number of zeros of the function

$$F(\lambda) = \lambda : J_m(\sqrt{\lambda}a)N_m(\sqrt{\lambda}b) - J_m(\sqrt{\lambda}b)N_m(\sqrt{\lambda}a)$$

In Figure 6.2.4 we show a plot of $F(\lambda)$ for $a = 4, b = 2$ and $m = 0, 1, 2, 3$.

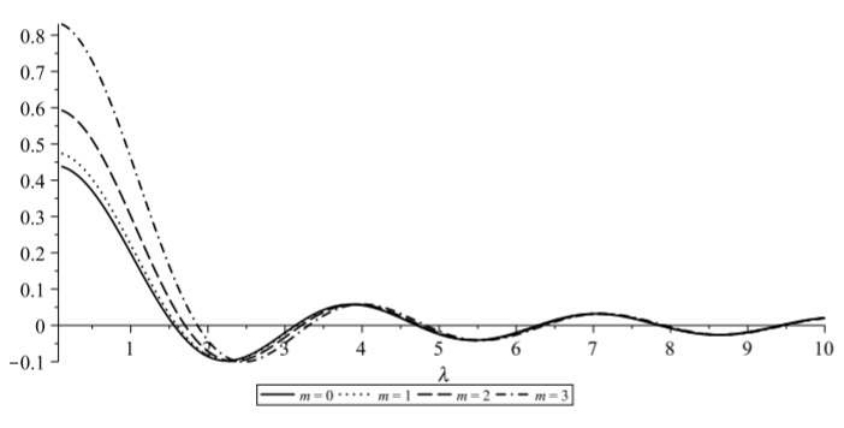


Figure 6.2.4: Plot of the function $F(\lambda) = J_m(\sqrt{\lambda}a)N_m(\sqrt{\lambda}b) - J_m(\sqrt{\lambda}b)N_m(\sqrt{\lambda}a)$ for $a = 4$ and $b = 2$ and $m = 0, 1, 2, 3$.

This eigenvalue equation needs to be solved numerically. Choosing $a = 2$ and $b = 4$, we have for the first few modes

$$\begin{aligned} \sqrt{\lambda_{mn}} &\approx 1.562, 3.137, 4.709, \quad m = 0 \\ &\approx 1.598, 3.156, 4.722, \quad m = 1 \\ &\approx 1.703, 3.214, 4.761, \quad m = 2. \end{aligned} \tag{6.2.20}$$

Note, since $\omega_{mn} = c\sqrt{\lambda_{mn}}$, these numbers essentially give us the frequencies of oscillation.

For these particular roots, we can solve for c_1 and c_2 up to a multiplicative constant. A simple solution is to set

$$c_1 = N_m\left(\sqrt{\lambda_{mn}}b\right), \quad c_2 = J_m\left(\sqrt{\lambda_{mn}}b\right).$$

This leads to the basic modes of vibration,

$$R_{mn}(r)\Theta_m(\theta) = \cos m\theta \left(N_m(\sqrt{\lambda_{mn}}b)J_m(\sqrt{\lambda_{mn}}r) - J_m(\sqrt{\lambda_{mn}}b)N_m(\sqrt{\lambda_{mn}}r)\right),$$

for $m = 0, 1, \ldots$, and $n = 1, 2, \ldots$. In Figure 6.2.1 we show various modes for the particular choice of annular membrane dimensions, $a = 2$ and $b = 4$.

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6.3: Laplace's Equation in 2D

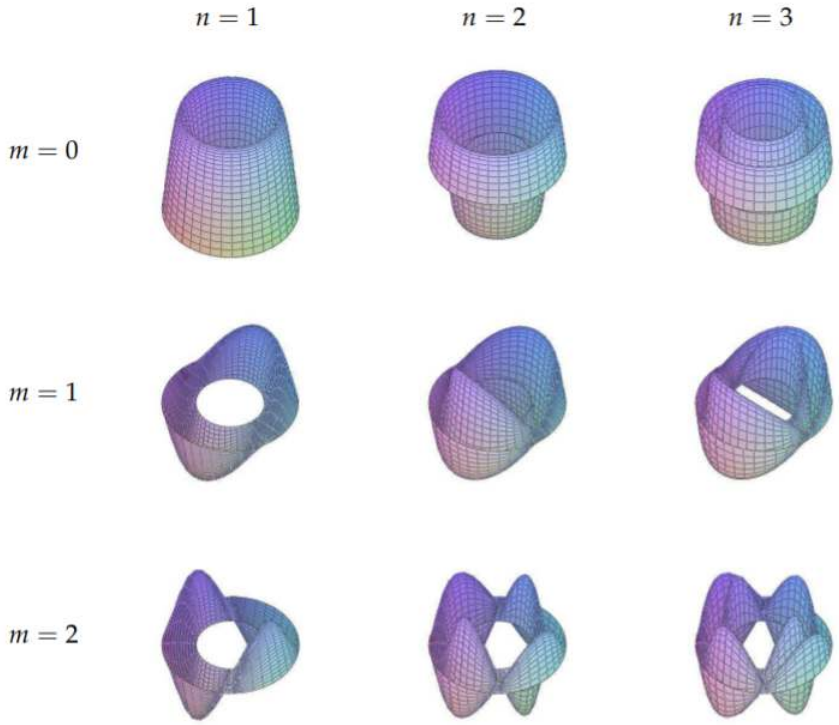
Another of the generic partial differential equations is Laplace's equation, $\nabla^2 u = 0$. This equation first appeared in the chapter on complex variables when we discussed harmonic functions. Another example is the electric potential for electrostatics. As we described Chapter ??, for static electromagnetic fields,

$$\nabla \cdot \mathbf{E} = \rho/\epsilon_0, \quad \mathbf{E} = \nabla\phi.$$

In regions devoid of charge, these equations yield the Laplace equation $\nabla^2\phi = 0$.

Another example comes from studying temperature distributions. Consider a thin rectangular plate with the boundaries set at fixed temperatures. Temperature changes of the plate are governed by the heat equation. The solution of the heat equation subject to these boundary conditions is time dependent. In fact, after a long period of time the plate will reach thermal equilibrium. If the boundary temperature is zero, then the plate temperature decays to zero across the plate. However, if the boundaries are maintained at a fixed nonzero temperature, which means energy is being put into the system to maintain the boundary conditions, the internal temperature may reach a nonzero equilibrium temperature. Reaching thermal equilibrium means that asymptotically in time the solution becomes time independent. Thus, the equilibrium state is a solution of the time independent heat equation, which is another Laplace equation, $\nabla^2 u = 0$.

Table 6.3.1: A three dimensional view of the vibrating annular membrane for the lowest modes.



Note

Thermodynamic equilibrium, $\nabla^2 u = 0$.

Note

Incompressible, irrotational fluid flow, $\nabla^2\phi = 0$, for velocity $\mathbf{v} = \nabla\phi$.

As another example we could look at fluid flow. For an incompressible flow, $\nabla \cdot \mathbf{v} = 0$. If the flow is irrotational, then $\nabla \times \mathbf{v} = 0$. We can introduce a velocity potential, $\mathbf{v} = \nabla\phi$. Thus, $\nabla \times \mathbf{v}$ vanishes by a vector identity and $\nabla \cdot \mathbf{v} = 0$ implies $\nabla^2\phi = 0$. So, once again we obtain Laplace's equation.

In this section we will look at examples of Laplace's equation in two dimensions. The solutions in these examples could be examples from any of the application in the above physical situations and the solutions can be applied appropriately.

Example 6.3.1: Equilibruim Temperature Distribution for a Rectangular Plate

Let's consider Laplace's equation in Cartesian coordinates,

$$u_{xx}+u_{yy}=0, \quad 0<x<L, \quad 0<y<H$$

with the boundary conditions

$$u(0,y)=0, \quad u(L,y)=0, \quad u(x,0)=f(x), \quad u(x,H)=0.$$

The boundary conditions are shown in Figure 6.3.1.

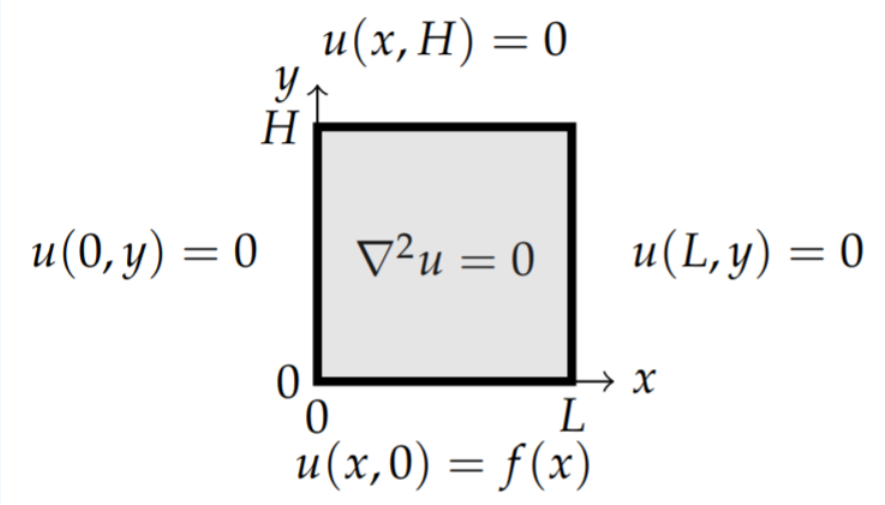


Figure 6.3.1: In this figure we show the domain and boundary conditions for the example of determining the equilibrium temperature for a rectangular plate.

Solution

As with the heat and wave equations, we can solve this problem using the method of separation of variables. Let $u(x,y)=X(x)Y(y)$. Then, Laplace's equation becomes

$$X''Y+XY''=0$$

and we can separate the x and y dependent functions and introduce a separation constant, λ,

$$\frac{X''}{X}=-\frac{Y''}{Y}=-\lambda.$$

Thus, we are led to two differential equations,

$$\begin{aligned} X''+\lambda X&=0 \\ Y''-\lambda Y&=0 \end{aligned} \tag{6.3.1}$$

From the boundary condition $u(0,y)=0, u(L,y)=0$, we have $X(0)=0, X(L)=0$. So, we have the usual eigenvalue problem for $X(x)$,

$$X''+\lambda X=0, \quad X(0)=0, X(L)=0.$$

The solutions to this problem are given by

$$X_n(x)=\sin\frac{n\pi x}{L}, \quad \lambda_n=\left(\frac{n\pi}{L}\right)^2, \quad n=1,2,\ldots$$

The general solution of the equation for $Y(y)$ is given by

$$Y(y) = c_1 e^{\sqrt{\lambda} y} + c_2 e^{-\sqrt{\lambda} y}$$

The boundary condition $u(x, H) = 0$ implies $Y(H) = 0$. So, we have

$$c_1 e^{\sqrt{\lambda} H} + c_2 e^{-\sqrt{\lambda} H} = 0.$$

Thus,

$$c_2 = -c_1 e^{2\sqrt{\lambda} H}.$$

Inserting this result into the expression for $Y(y)$, we have

$$\begin{aligned} Y(y) &= c_1 e^{\sqrt{\lambda} y} - c_1 e^{2\sqrt{\lambda} H} e^{-\sqrt{\lambda} y} \\ &= c_1 e^{\sqrt{\lambda} H} \left(e^{-\sqrt{\lambda} H} e^{\sqrt{\lambda} y} - e^{\sqrt{\lambda} H} e^{-\sqrt{\lambda} y} \right) \\ &= c_1 e^{\sqrt{\lambda} H} \left(e^{-\sqrt{\lambda}(H-y)} - e^{\sqrt{\lambda}(H-y)} \right) \\ &= -2 c_1 e^{\sqrt{\lambda} H} \sinh \sqrt{\lambda}(H-y) \end{aligned} \tag{6.3.2}$$

Note

Having carried out this computation, we can now see that it would be better to guess this form in the future. So, for $Y(H) = 0$, one would guess a solution $Y(y) = \sinh \sqrt{\lambda}(H-y)$ For $Y(0) = 0$, one would guess a solution $Y(y) = \sinh \sqrt{\lambda} y$. Similarly, if $Y'(H) = 0$, one would guess a solution $Y(y) = \cosh \sqrt{\lambda}(H-y)$

Since we already know the values of the eigenvalues λ_n from the eigenvalue problem for $X(x)$, we have that the y-dependence is given by

$$Y_n(y) = \sinh \frac{n\pi(H-y)}{L}.$$

So, the product solutions are given by

$$u_n(x, y) = \sin \frac{n\pi x}{L} \sinh \frac{n\pi(H-y)}{L}, \quad n = 1, 2, \ldots$$

These solutions satisfy Laplace's equation and the three homogeneous boundary conditions and in the problem.

The remaining boundary condition, $u(x, 0) = f(x)$, still needs to be satisfied. Inserting $y = 0$ in the product solutions does not satisfy the boundary condition unless $f(x)$ is proportional to one of the eigenfunctions $X_n(x)$. So, we first write down the general solution as a linear combination of the product solutions,

$$u(x, y) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L} \sinh \frac{n\pi(H-y)}{L}. \tag{6.3.3}$$

Now we apply the boundary condition, $u(x, 0) = f(x)$, to find that

$$f(x) = \sum_{n=1}^{\infty} a_n \sinh \frac{n\pi H}{L} \sin \frac{n\pi x}{L}. \tag{6.3.4}$$

Defining $b_n = a_n \sinh \frac{n\pi H}{L}$, this becomes

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}. \tag{6.3.5}$$

We see that the determination of the unknown coefficients, b_n, is simply done by recognizing that this is a Fourier sine series. The Fourier coefficients are easily found as

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx. \tag{6.3.6}$$

Since $a_n = b_n / \sinh \frac{n\pi H}{L}$, we can finish solving the problem. The solution is

$$u(x,y) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L} \sinh \frac{n\pi (H-y)}{L}, \tag{6.3.7}$$

where

$$a_n = \frac{2}{L \sinh \frac{n\pi H}{L}} \int_0^L f(x) \sin \frac{n\pi x}{L} dx \tag{6.3.8}$$

✔ Example 6.3.2: Equilibrium Temperature Distribution for a Rectangular Plate for General Boundary Conditions

A more general problem is to seek solutions to Laplace's equation in Cartesian coordinates,

$$u_{xx} + u_{yy} = 0, \quad 0 < x < L, 0 < y < H$$

with non-zero boundary conditions on more than one side of the domain,

$$\begin{aligned} u(0,y) &= g_1(y), \quad u(L,y) = g_2(y), \quad 0 < y < H, \\ u(x,0) &= f_1(x), \quad u(x,H) = f_2(x), \quad 0 < x < L. \end{aligned}$$

These boundary conditions are shown in Figure 6.3.2.

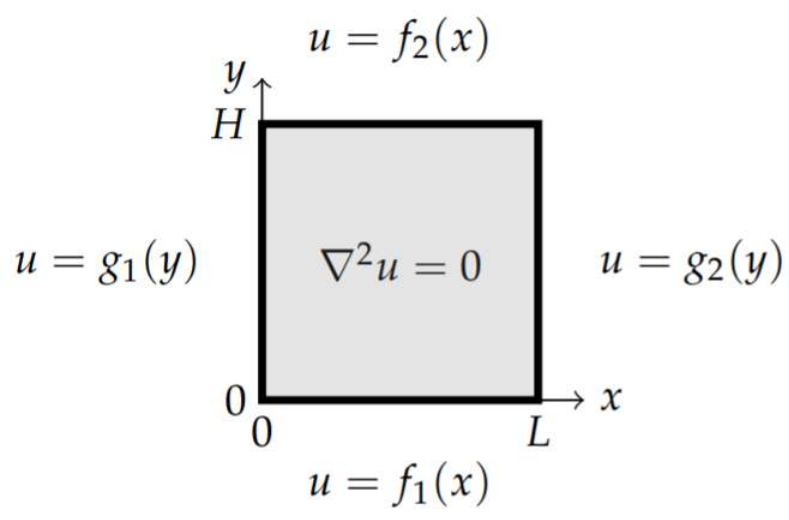


Figure 6.3.2: In this figure we show the domain and general boundary conditions for the example of determining the equilibrium temperature distribution for a rectangular plate.

Solution

The problem with this example is that none of the boundary conditions are homogeneous. This means that the corresponding eigenvalue problems will not have the homogeneous boundary conditions which Sturm-Liouville theory in Section 4 needs. However, we can express this problem in terms of four different problems with nonhomogeneous boundary conditions on only one side of the rectangle.

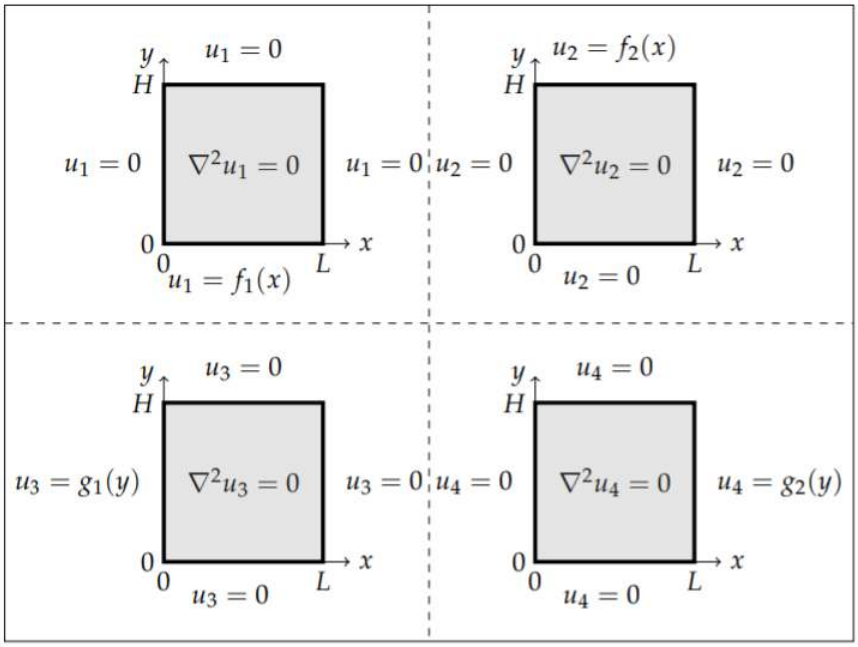


Figure 6.3.3: The general boundary value problem for a rectangular plate can be written as the sum of these four separate problems.

In Figure 6.3.3 we show how the problem can be broken up into four separate problems for functions $u_i(x,y), i = 1, \ldots, 4$. Since the boundary conditions and Laplace's equation are linear, the solution to the general problem is simply the sum of the solutions to these four problems,

$$u(x,y) = u_1(x,y) + u_2(x,y) + u_3(x,y) + u_4(x,y).$$

Then, this solution satisfies Laplace's equation,

$$\nabla^2 u(x,y) = \nabla^2 u_1(x,y) + \nabla^2 u_2(x,y) + \nabla^2 u_3(x,y) + \nabla^2 u_4(x,y) = 0,$$

and the boundary conditions. For example, using the boundary conditions defined in Figure 6.3.3, we have for $y = 0$,

$$u(x,0) = u_1(x,0) + u_2(x,0) + u_3(x,0) + u_4(x,0) = f_1(x).$$

The other boundary conditions can also be shown to hold.

We can solve each of the problems in Figure 6.3.3 quickly based on the solution we obtained in the last example. The solution for $u_1(x,y)$, which satisfies the boundary conditions

$$\begin{aligned} u_1(0,y) = 0, \quad u_1(L,y) = 0, \quad 0 < y < H, \\ u_1(x,0) = f_1(x), \quad u_1(x,H) = 0, \quad 0 < x < L, \end{aligned}$$

is the easiest to write down. It is given by

$$u_1(x,y) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L} \sinh \frac{n\pi (H-y)}{L}. \tag{6.3.9}$$

where

$$a_n = \frac{2}{L \sinh \frac{n\pi H}{L}} \int_0^L f_1(x) \sin \frac{n\pi x}{L} dx \tag{6.3.10}$$

For the boundary conditions

$$\begin{aligned} u_2(0,y) = 0, \quad u_2(L,y) = 0, \quad 0 < y < H, \\ u_2(x,0) = 0, \quad u_2(x,H) = f_2(x), \quad 0 < x < L. \end{aligned}$$

the boundary conditions for $X(x)$ are $X(0)=0$ and $X(L)=0$. So, we get the same form for the eigenvalues and eigenfunctions as before:

$$X_n(x)=\sin\frac{n\pi x}{L}, \quad \lambda_n=\left(\frac{n\pi}{L}\right)^2, n=1,2,\ldots.$$

The remaining homogeneous boundary condition is now $Y(0)=0$. Recalling that the equation satisfied by $Y(y)$ is

$$Y''-\lambda Y=0,$$

we can write the general solution as

$$Y(y)=c_1\cosh\sqrt{\lambda}y+c_2\sinh\sqrt{\lambda}y.$$

Requiring $Y(0)=0$, we have $c_1=0$, or

$$Y(y)=c_2\sinh\sqrt{\lambda}y.$$

Then, the general solution is

$$u_2(x,y)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{L}\sinh\frac{n\pi y}{L}. \tag{6.3.11}$$

We now force the nonhomogeneous boundary condition, $u_2(x,H)=f_2(x)$,

$$f_2(x)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{L}\sinh\frac{n\pi H}{L}. \tag{6.3.12}$$

Once again we have a Fourier sine series. The Fourier coefficients are given by

$$b_n=\frac{2}{L\sinh\frac{n\pi H}{L}}\int_0^L f_2(x)\sin\frac{n\pi x}{L}dx. \tag{6.3.13}$$

Next we turn to the problem with the boundary conditions

$$\begin{aligned} u_3(0,y)=g_1(y), \quad u_3(L,y)=0, \quad 0<y<H,\\ u_3(x,0)=0, \quad u_3(x,H)=0, \quad 0<x<L. \end{aligned}$$

In this case the pair of homogeneous boundary conditions $u_3(x,0)=0, \quad u_3(x,H)=0$ lead to solutions

$$Y_n(y)=\sin\frac{n\pi y}{H}, \quad \lambda_n=-\left(\frac{n\pi}{H}\right)^2, \quad n=1,2\ldots.$$

The condition $u_3(L,0)=0$ gives $X(x)=\sinh\frac{n\pi(L-x)}{H}$.

The general solution satisfying the homogeneous conditions is

$$u_3(x,y)=\sum_{n=1}^{\infty}c_n\sin\frac{n\pi y}{H}\sinh\frac{n\pi(L-x)}{H}. \tag{6.3.14}$$

Applying the nonhomogeneous boundary condition, $u_3(0,y)=g_1(y)$, we obtain the Fourier sine series

$$g_1(y)=\sum_{n=1}^{\infty}c_n\sin\frac{n\pi y}{H}\sinh\frac{n\pi L}{H}. \tag{6.3.15}$$

The Fourier coefficients are found as

$$c_n=\frac{2}{H\sinh\frac{n\pi L}{H}}\int_0^H g_1(y)\sin\frac{n\pi y}{H}dy. \tag{6.3.16}$$

Finally, we can find the solution

$$\begin{aligned} u_4(0,y)=0, \quad u_4(L,y)=g_2(y), \quad 0<y<H,\\ u_4(x,0)=0, \quad u_4(x,H)=0, \quad 0<x<L. \end{aligned}$$

Following the above analysis, we find the general solution

$$u_4(x,y) = \sum_{n=1}^{\infty} d_n \sin\frac{n\pi y}{H}\sinh\frac{n\pi x}{H}. \tag{6.3.17}$$

The nonhomogeneous boundary condition, $u(L,y) = g_2(y)$, is satisfied if

$$g_2(y) = \sum_{n=1}^{\infty} d_n \sin\frac{n\pi y}{H}\sinh\frac{n\pi L}{H}. \tag{6.3.18}$$

The Fourier coefficients, d_n, are given by

$$d_n = \frac{2}{H\sinh\frac{n\pi L}{H}}\int_0^H g_1(y)\sin\frac{n\pi y}{H}dy. \tag{6.3.19}$$

The solution to the general problem is given by the sum of these four solutions.

$$\begin{aligned} u(x,y) = \sum_{n=1}^{\infty}\Bigg[&\left(a_n\sinh\frac{n\pi(H-y)}{L} + b_n\sinh\frac{n\pi y}{L}\right)\sin\frac{n\pi x}{L} \\ &+\left(c_n\sinh\frac{n\pi(L-x)}{H} + d_n\sinh\frac{n\pi x}{H}\right)\sin\frac{n\pi y}{H}\Bigg], \end{aligned} \tag{6.3.20}$$

where the coefficients are given by the above Fourier integrals.

Example 6.3.3: Laplace's Equation on a Disk

We now turn to solving Laplace's equation on a disk of radius a as shown in Figure 6.3.4. Laplace's equation in polar coordinates is given by

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} = 0, \quad 0 < r < a, \quad -\pi < \theta < \pi. \tag{6.3.21}$$

The boundary conditions are given as

$$u(a,\theta) = f(\theta), \quad -\pi < \theta < \pi, \tag{6.3.22}$$

plus periodic boundary conditions in θ.

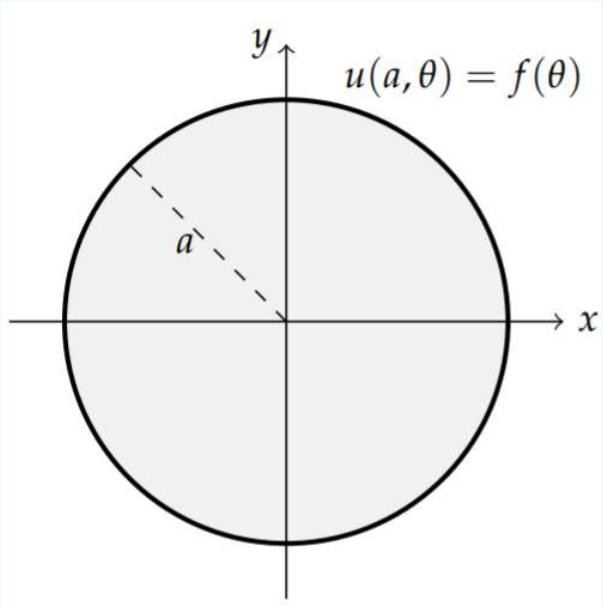


Figure 6.3.4: The disk of radius a with or boundary condition along the edge at $r = a$.

Solution

Separation of variable proceeds as usual. Let $u(r,\theta)=R(r)\Theta(\theta)$. Then

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial(R\Theta)}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2(R\Theta)}{\partial\theta^2}=0, \tag{6.3.23}$$

or

$$\Theta\frac{1}{r}(rR')'+\frac{1}{r^2}R\Theta''=0. \tag{6.3.24}$$

Diving by $u(r,\theta)=R(r)\Theta(\theta)$, multiplying by r^2, and rearranging, we have

$$\frac{r}{R}(rR')'=-\frac{\Theta''}{\Theta}=\lambda. \tag{6.3.25}$$

Since this equation gives a function of r equal to a function of θ, we set the equation equal to a constant. Thus, we have obtained two differential equations, which can be written as

$$r(rR')'-\lambda R=0 \tag{6.3.26}$$

$$\Theta''+\lambda\Theta=0 \tag{6.3.27}$$

We can solve the second equation subject to the periodic boundary conditions in the θ variable. The reader should be able to confirm that

$$\Theta(\theta)=a_n\cos n\theta+b_n\sin n\theta,\quad \lambda=n^2, n=0,1,2,\ldots$$

is the solution. Note that the $n=0$ case just leads to a constant solution.

Inserting $\lambda=n^2$ into the radial equation, we find

$$r^2R''+rR'-n^2R=0.$$

This is a Cauchy-Euler type of ordinary differential equation. Recall that we solve such equations by guessing a solution of the form $R(r)=r^m$. This leads to the characteristic equation $m^2-n^2=0$. Therefore, $m=\pm n$. So,

$$R(r)=c_1r^n+c_2r^{-n}.$$

Since we expect finite solutions at the origin, $r=0$, we can set (. Thus, the general solution is

$$u(r,\theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos n\theta+b_n\sin n\theta)\,r^n. \tag{6.3.28}$$

Note that we have taken the constant term out of the sum and put it into a familiar form.

Now we can impose the remaining boundary condition, $u(a,\theta)=f(\theta)$, or

$$f(\theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos n\theta+b_n\sin n\theta)\,a^n. \tag{6.3.29}$$

This is a Fourier trigonometric series. The Fourier coefficients can be determined using the results from Chapter 4 :

$$a_n=\frac{1}{\pi a^n}\int_{-\pi}^{\pi}f(\theta)\cos n\theta d\theta,\quad n=0,1,\ldots, \tag{6.3.30}$$

$$b_n=\frac{1}{\pi a^n}\int_{-\pi}^{\pi}f(\theta)\sin n\theta d\theta\quad n=1,2\ldots. \tag{6.3.31}$$

Poisson Integral Formula

We can put the solution from the last example in a more compact form by inserting the Fourier coefficients into the general solution. Doing this, we have

$$
\begin{aligned}
u(r,\theta) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos n\theta + b_n \sin n\theta\right) r^n \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\phi) d\phi \\
&\quad + \frac{1}{\pi} \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} [\cos n\phi \cos n\theta + \sin n\phi \sin n\theta] \left(\frac{r}{a}\right)^n f(\phi) d\phi \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} \left[\frac{1}{2} + \sum_{n=1}^{\infty} \cos n(\theta - \phi) \left(\frac{r}{a}\right)^n\right] f(\phi) d\phi. \qquad (6.3.32)
\end{aligned}
$$

The term in the brackets can be summed. We note that

$$
\begin{aligned}
\cos n(\theta - \phi) \left(\frac{r}{a}\right)^n &= \text{Re}\left(e^{in(\theta-\phi)} \left(\frac{r}{a}\right)^n\right) \\
&= \text{Re}\left(\frac{r}{a} e^{i(\theta-\phi)}\right)^n. \qquad (6.3.33)
\end{aligned}
$$

Therefore,

$$
\sum_{n=1}^{\infty} \cos n(\theta - \phi) \left(\frac{r}{a}\right)^n = \text{Re}\left(\sum_{n=1}^{\infty} \left(\frac{r}{a} e^{i(\theta-\phi)}\right)^n\right).
$$

The right hand side of this equation is a geometric series with common ratio of $\frac{r}{a} e^{i(\theta-\phi)}$, which is also the first term of the series. Since $\left|\frac{r}{a} e^{i(\theta-\phi)}\right| = \frac{r}{a} < 1$, the series converges. Summing the series, we obtain

$$
\begin{aligned}
\sum_{n=1}^{\infty} \left(\frac{r}{a} e^{i(\theta-\phi)}\right)^n &= \frac{\frac{r}{a} e^{i(\theta-\phi)}}{1 - \frac{r}{a} e^{i(\theta-\phi)}} \\
&= \frac{re^{i(\theta-\phi)}}{a - re^{i(\theta-\phi)}} \qquad (6.3.34)
\end{aligned}
$$

We need to rewrite this result so that we can easily take the real part. Thus, we multiply and divide by the complex conjugate of the denominator to obtain

$$
\begin{aligned}
\sum_{n=1}^{\infty} \left(\frac{r}{a} e^{i(\theta-\phi)}\right)^n &= \frac{re^{i(\theta-\phi)}}{a - re^{i(\theta-\phi)}} \frac{a - re^{-i(\theta-\phi)}}{a - re^{-i(\theta-\phi)}} \\
&= \frac{are^{-i(\theta-\phi)} - r^2}{a^2 + r^2 - 2ar\cos(\theta - \phi)}. \qquad (6.3.35)
\end{aligned}
$$

The real part of the sum is given as

$$
\text{Re}\left(\sum_{n=1}^{\infty} \left(\frac{r}{a} e^{i(\theta-\phi)}\right)^n\right) = \frac{ar\cos(\theta - \phi) - r^2}{a^2 + r^2 - 2ar\cos(\theta - \phi)}.
$$

Therefore, the factor in the brackets under the integral in Equation (6.3.32) is

$$
\begin{aligned}
\frac{1}{2} + \sum_{n=1}^{\infty} \cos n(\theta - \phi) \left(\frac{r}{a}\right)^n &= \frac{1}{2} + \frac{\arccos(\theta - \phi) - r^2}{a^2 + r^2 - 2\arccos(\theta - \phi)} \\
&= \frac{a^2 - r^2}{2\left(a^2 + r^2 - 2\arccos(\theta - \phi)\right)} \qquad (6.3.36)
\end{aligned}
$$

Thus, we have shown that the solution of Laplace's equation on a disk of radius a with boundary condition $u(a,\theta) = f(\theta)$ can be written in the closed form

$$
u(r,\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{a^2 - r^2}{a^2 + r^2 - 2ar\cos(\theta - \phi)} f(\phi) d\phi. \qquad (6.3.37)
$$

This result is called the Poisson Integral Formula and

$$K(\theta,\phi)=\frac{a^2-r^2}{a^2+r^2-2ar\cos(\theta-\phi)}$$

is called the Poisson kernel.

Example 6.3.4

Evaluate the solution (6.3.37) at the center of the disk.

Solution

We insert $r=0$ into the solution (6.3.37) to obtain

$$u(0,\theta)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(\phi)d\phi.$$

Recalling that the average of a function $g(x)$ on $[a,b]$ is given by

$$g_{\text{ave}}=\frac{1}{b-a}\int_a^b g(x)dx,$$

we see that the value of the solution u at the center of the disk is the average of the boundary values. This is sometimes referred to as the mean value theorem.

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6.4: Three Dimensional Cake Baking

In the rest of the chapter we will extend our studies to three dimensional problems. In this section we will solve the heat equation as we look at examples of baking cakes.

We consider cake batter, which is at room temperature of $T_i = 80^\circ \mathrm{F}$. It is placed into an oven, also at a fixed temperature, $T_b = 350^\circ \mathrm{F}$. For simplicity, we will assume that the thermal conductivity and cake density are constant. Of course, this is not quite true. However, it is an approximation which simplifies the model. We will consider two cases, one in which the cake is a rectangular solid, such as baking it in a $13'' \times 9'' \times 2''$ baking pan. The other case will lead to a cylindrical cake, such as you would obtain from a round cake pan.

Note

This discussion of cake baking is adapted from R. Wilkinson's thesis work. That in turn was inspired by work done by Dr. Olszewski,(2006) From baking a cake to solving the diffusion equation. *American Journal of Physics* $74(6)$.

Assuming that the heat constant k is indeed constant and the temperature is given by $T(\mathbf{r}, t)$, we begin with the heat equation in three dimensions,

$$\frac{\partial T}{\partial t} = k\nabla^2 T. \tag{6.4.1}$$

We will need to specify initial and boundary conditions. Let T_i be the initial batter temperature, $T(x, y, z, 0) = T_i$.

We choose the boundary conditions to be fixed at the oven temperature T_b. However, these boundary conditions are not homogeneous and would lead to problems when carrying out separation of variables. This is easily remedied by subtracting the oven temperature from all temperatures involved and defining $u(\mathbf{r}, t) = T(\mathbf{r}, t) - T_b$. The heat equation then becomes

$$\frac{\partial u}{\partial t} = k\nabla^2 u \tag{6.4.2}$$

with initial condition

$$u(\mathbf{r}, 0) = T_i - T_b.$$

The boundary conditions are now homogeneous. We cannot be any more specific than this until we specify the geometry.

Example 6.4.1: Temperature of a Rectangular Cake

We will consider a rectangular cake with dimensions $0 \le x \le W, 0 \le y \le L$, and $0 \le z \le H$ as show in Figure $6.4.1$. For this problem, we seek solutions of the heat equation plus the conditions

$$\begin{aligned} u(x, y, z, 0) &= T_i - T_b, \\ u(0, y, z, t) = u(W, y, z, t) &= 0, \\ u(x, 0, z, t) = u(x, L, z, t) &= 0, \\ u(x, y, 0, t) = u(x, y, H, t) &= 0. \end{aligned}$$

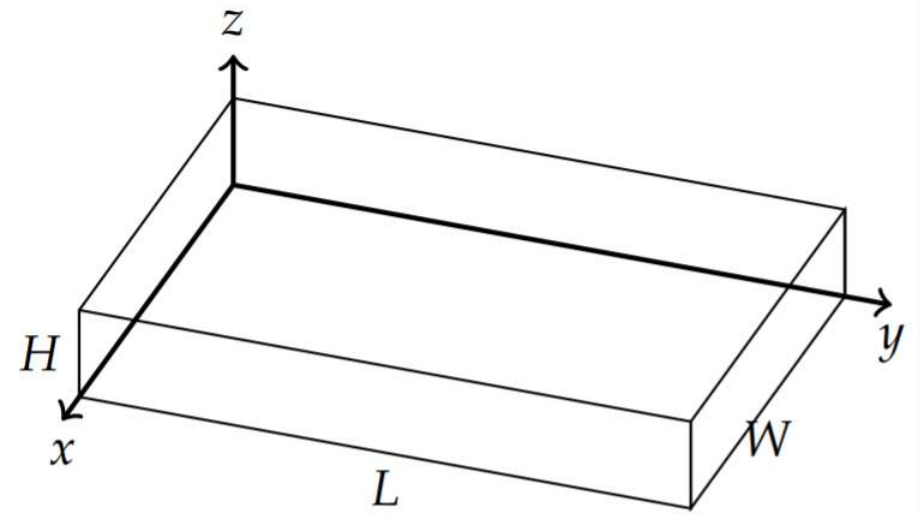


Figure 6.4.1: The dimensions of a rectangular cake.

Solution

Using the method of separation of variables, we seek solutions of the form

$$u(x,y,z,t) = X(x)Y(y)Z(z)G(t). \tag{6.4.3}$$

Substituting this form into the heat equation, we get

$$\frac{1}{k}\frac{G'}{G} = \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z}. \tag{6.4.4}$$

Setting these expressions equal to $-\lambda$, we get

$$\frac{1}{k}\frac{G'}{G} = -\lambda \quad \text{and} \quad \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = -\lambda \tag{6.4.5}$$

Therefore, the equation for $G(t)$ is given by

$$G' + k\lambda G = 0.$$

We further have to separate out the functions of $x, y,$ and z. We anticipate that the homogeneous boundary conditions will lead to oscillatory solutions in these variables. Therefore, we expect separation of variables will lead to the eigenvalue problems

$$\begin{aligned} X'' + \mu^2 X &= 0, & X(0) &= X(W) = 0, \\ Y'' + v^2 Y &= 0, & Y(0) &= Y(L) = 0, \\ Z'' + \kappa^2 Z &= 0, & Z(0) &= Z(H) = 0. \end{aligned} \tag{6.4.6}$$

Noting that

$$\frac{X''}{X} = -\mu^2, \quad \frac{Y''}{Y} = -v^2, \quad \frac{Z''}{Z} = -\kappa^2,$$

we find from the heat equation that the separation constants are related,

$$\lambda^2 = \mu^2 + v^2 + \kappa^2.$$

We could have gotten to this point quicker by writing the first separated equation labeled with the separation constants as

$$\underbrace{\frac{1}{k}\frac{G'}{G}}_{-\lambda} = \underbrace{\frac{X''}{X}}_{-\mu} + \underbrace{\frac{Y''}{Y}}_{-v} + \underbrace{\frac{Z''}{Z}}_{-\kappa}.$$

Then, we can read off the eigenvalues problems and determine that $\lambda^2 = \mu^2 + v^2 + \kappa^2$.

From the boundary conditions, we get product solutions for $u(x,y,z,t)$ in the form

$$u_{mn\ell}(x,y,z,t) = \sin\mu_m x \sin v_n y \sin\kappa_\ell z e^{-\lambda_{mn} kt},$$

for

$$\lambda_{mnl} = \mu_m^2 + v_n^2 + \kappa_\ell^2 = \left(\frac{m\pi}{W}\right)^2 + \left(\frac{n\pi}{L}\right)^2 + \left(\frac{\ell\pi}{H}\right)^2, \quad m,n,\ell = 1,2,\ldots$$

The general solution is a linear combination of all of the product solutions, summed over three different indices,

$$u(x,y,z,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{\ell=1}^{\infty} A_{mnl} \sin\mu_m x \sin v_n y \sin\kappa_\ell z e^{-\lambda_{mn} kt}, \tag{6.4.7}$$

where the A_{mn} 's are arbitrary constants.

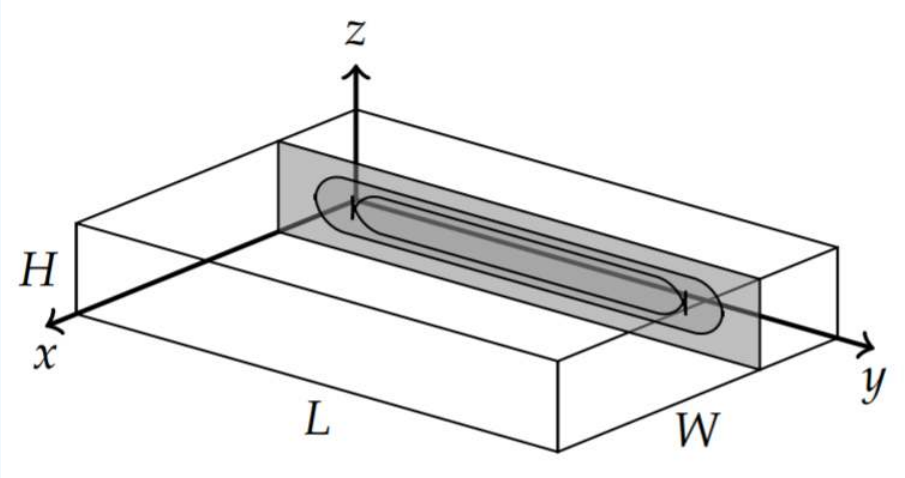


Figure 6.4.2: Rectangular cake showing a vertical slice.

We can use the initial condition $u(x,y,z,0) = T_i - T_b$ to determine the $A_{mn\ell}$'s. We find

$$T_i - T_b = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{\ell=1}^{\infty} A_{mnl} \sin\mu_m x \sin v_n y \sin k_\ell z. \tag{6.4.8}$$

This is a triple Fourier sine series.

We can determine these coefficients in a manner similar to how we handled double Fourier sine series earlier in the chapter. Defining

$$b_m(y,z) = \sum_{n=1}^{\infty}\sum_{\ell=1}^{\infty} A_{mnl} \sin v_n y \sin k_\ell z,$$

we obtain a simple Fourier sine series:

$$T_i - T_b = \sum_{m=1}^{\infty} b_m(y,z) \sin\mu_m x. \tag{6.4.9}$$

The Fourier coefficients can then be found as

$$b_m(y,z) = \frac{2}{W}\int_0^W (T_i - T_b)\sin\mu_m x\, dx.$$

Using the same technique for the remaining sine series and noting that $T_i - T_b$ is constant, we can determine the general coefficients A_{mnl} by carrying out the needed integrations:

$$
\begin{aligned}
A_{mnl} &= \frac{8}{WLH}\int_0^H\int_0^L\int_0^W (T_i - T_b)\sin\mu_m x \sin v_n y \sin k_\ell z dx dy dz \\
&= (T_i - T_b)\frac{8}{\pi^3}\left[\frac{\cos\left(\frac{m\pi x}{W}\right)}{m}\right]_0^W \left[\frac{\cos\left(\frac{n\pi y}{L}\right)}{n}\right]_0^L \left[\frac{\cos\left(\frac{\ell\pi z}{H}\right)}{\ell}\right]_0^H \\
&= (T_i - T_b)\frac{8}{\pi^3}\left[\frac{\cos m\pi - 1}{m}\right]\left[\frac{\cos n\pi - 1}{n}\right]\left[\frac{\cos \ell\pi - 1}{\ell}\right] \\
&= (T_i - T_b)\frac{8}{\pi^3}\begin{cases} 0, & \text{for at least one } m,n,\ell \text{ even,} \\ \left[\frac{-2}{m}\right]\left[\frac{-2}{n}\right]\left[\frac{-2}{\ell}\right], & \text{for } m,n,\ell \text{ all odd.}\end{cases}
\end{aligned}
$$

Since only the odd multiples yield non-zero $A_{mn\ell}$ we let $m = 2m' - 1$, $n = 2n' - 1$, and $\ell = 2\ell' - 1$ for $m', n', \ell' = 1, 2, \ldots$. The expansion coefficients can now be written in the simpler form

$$
A_{mn} = \frac{64(T_b - T_i)}{(2m'-1)(2n'-1)(2\ell'-1)\pi^3}.
$$

Substituting this result into general solution and dropping the primes, we find

$$
u(x,y,z,t) = \frac{64(T_b - T_i)}{\pi^3}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{\sin\mu_m x \sin v_n y \sin k_\ell z e^{-\lambda_{mnt}kt}}{(2m-1)(2n-1)(2\ell-1)},
$$

where

$$
\lambda_{mn\ell} = \left(\frac{(2m-1)\pi}{W}\right)^2 + \left(\frac{(2n-1)\pi}{L}\right)^2 + \left(\frac{(2\ell-1)\pi}{H}\right)^2
$$

for $m, n, \ell = 1, 2, \ldots.$

Recalling that the solution to the physical problem is

$$
T(x,y,z,t) = u(x,y,z,t) + T_b,
$$

we have the final solution is given by

$$
T(x,y,z,t) = T_b + \frac{64\,(T_b - T_i)}{\pi^3}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{\sin\hat{\mu}_m x \sin\hat{v}_n y \sin\hat{\kappa}_\ell z e^{-\hat{\lambda}_{mnt}kt}}{(2m-1)(2n-1)(2\ell-1)}
$$

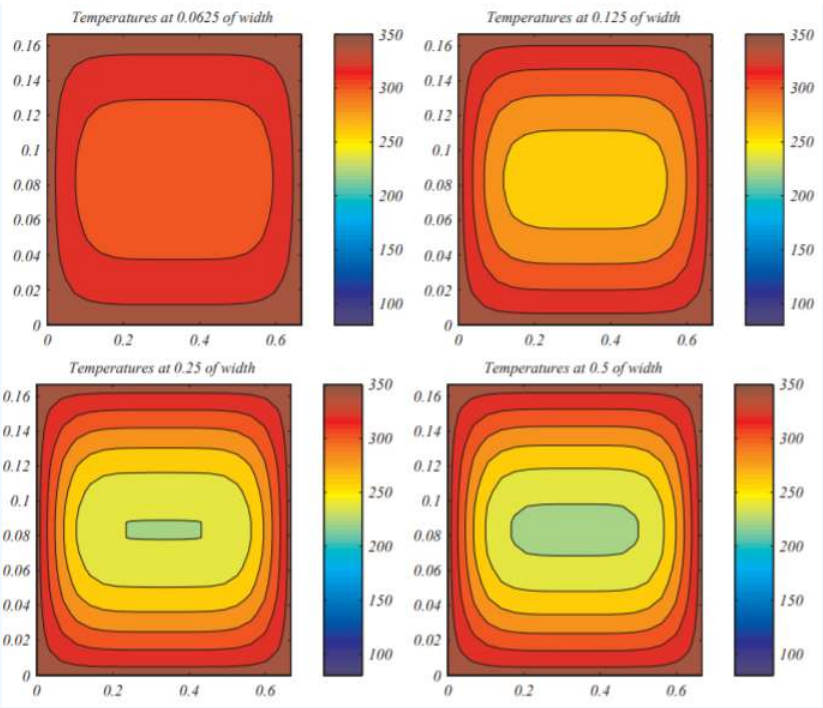


Figure 6.4.3: Temperature evolution for a $13'' \times 9'' \times 2''$ cake shown as vertical slices at the indicated length in feet.

We show some temperature distributions in Figure 6.4.3. Since we cannot capture the entire cake, we show vertical slices such as depicted in Figure 6.4.2. Vertical slices are taken at the positions and times indicated for a $13'' \times 9'' \times 2''$ cake. Obviously, this is not accurate because the cake consistency is changing and this will affect the parameter k. A more realistic model would be to allow $k = k(T(x, y, z, t))$. However, such problems are beyond the simple methods described in this book.

Example 6.4.2: Circular Cakes

In this case the geometry of the cake is cylindrical as show in Figure 6.4.4. Therefore, we need to express the boundary conditions and heat equation in cylindrical coordinates. Also, we will assume that the solution, $u(r, z, t) = T(r, z, t) - T_b$, is independent of θ due to axial symmetry. This gives the heat equation in θ independent cylindrical coordinates as

$$\frac{\partial u}{\partial t} = k\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial z^2}\right), \tag{6.4.10}$$

where $0 \le r \le a$ and $0 \le z \le Z$. The initial condition is

$$u(r, z, 0) = T_i - T_b,$$

and the homogeneous boundary conditions on the side, top, and bottom of the cake are

$$u(a, z, t) = 0,$$
$$u(r, 0, t) = u(r, Z, t) = 0.$$

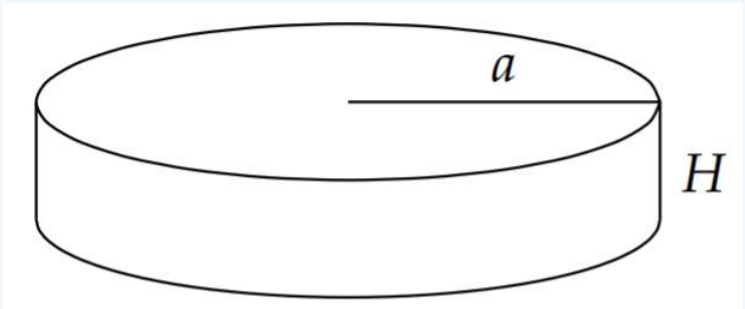


Figure 6.4.4: Geometry for a cylindrical cake.

Solution

Again, we seek solutions of the form $u(r,z,t)=R(r)H(z)G(t)$. Separation of variables leads to

$$\underbrace{\frac{1}{k}\frac{G'}{G}}_{-\lambda}=\underbrace{\frac{1}{rR}\frac{d}{dr}(rR')}_{-\mu^2}+\underbrace{\frac{H''}{H}}_{-v^2}. \tag{6.4.11}$$

Here we have indicated the separation constants, which lead to three ordinary differential equations. These equations and the boundary conditions are

$$\begin{aligned} G'+k\lambda G &= 0, \\ \frac{d}{dr}(rR')+\mu^2 rR &= 0, \quad R(a)=0, \quad R(0) \text{ is finite}, \\ H''+v^2H &= 0, \quad H(0)=H(Z)=0. \end{aligned} \tag{6.4.12}$$

We further note that the separation constants are related by $\lambda=\mu^2+v^2$.

We can easily write down the solutions for $G(t)$ and $H(z)$,

$$G(t)=Ae^{-\lambda kt}$$

and

$$H_n(z)=\sin\frac{n\pi z}{Z}, \quad n=1,2,3,\ldots,$$

where $v=\frac{n\pi}{Z}$. Recalling from the rectangular case that only odd terms arise in the Fourier sine series coefficients for the constant initial condition, we proceed by rewriting $H(z)$ as

$$H_n(z)=\sin\frac{(2n-1)\pi z}{Z}, \quad n=1,2,3,\ldots \tag{6.4.13}$$

with $v=\frac{(2n-1)\pi}{Z}$.

The radial equation can be written in the form

$$r^2R''+rR'+\mu^2r^2R=0.$$

This is a Bessel equation of the first kind of order zero which we had seen in Section 5.5. Therefore, the general solution is a linear combination of Bessel functions of the first and second kind,

$$R(r)=c_1J_0(\mu r)+c_2N_0(\mu r). \tag{6.4.14}$$

Since $R(r)$ is bounded at $r=0$ and $N_0(\mu r)$ is not well behaved at $r=0$, we set $c_2=0$. Up to a constant factor, the solution becomes

$$R(r)=J_0(\mu r). \tag{6.4.15}$$

The boundary condition $R(a)=0$ gives the eigenvalues as

$$\mu_m=\frac{j_{0m}}{a}, \quad m=1,2,3,\ldots,$$

where j_{0m} is the mth roots of the zeroth-order Bessel function, $J_0\left(j_{0m}\right)=0$.

Therefore, we have found the product solutions

$$H_n(z)R_m(r)G(t)=\sin\frac{(2n-1)\pi z}{\mathrm{Z}}J_0\left(\frac{r}{a}j_{0m}\right)e^{-\lambda_{nm}kt}, \tag{6.4.16}$$

where $m=1,2,3,\ldots,n=1,2,\ldots$. Combining the product solutions, the general solution is found as

$$u(r,z,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}A_{nm}\sin\frac{(2n-1)\pi z}{Z}J_0\left(\frac{r}{a}j_{0m}\right)e^{-\lambda_{nm}kt} \tag{6.4.17}$$

with

$$\lambda_{nm} = \left(\frac{(2n-1)\pi}{\mathrm{Z}}\right)^2 + \left(\frac{j_{0m}}{a}\right)^2,$$

for $n, m = 1, 2, 3, \ldots$.

Inserting the solution into the constant initial condition, we have

$$T_i - T_b = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm} \sin\frac{(2n-1)\pi z}{\mathrm{Z}} J_0\left(\frac{r}{a} j_{0m}\right).$$

This is a double Fourier series but it involves a Fourier-Bessel expansion. Writing

$$b_n(r) = \sum_{m=1}^{\infty} A_{nm} J_0\left(\frac{r}{a} j_{0m}\right),$$

the condition becomes

$$T_i - T_b = \sum_{n=1}^{\infty} b_n(r) \sin\frac{(2n-1)\pi z}{Z}.$$

As seen previously, this is a Fourier sine series and the Fourier coefficients are given by

$$\begin{aligned} b_n(r) &= \frac{2}{Z}\int_0^Z (T_i - T_b)\sin\frac{(2n-1)\pi z}{Z} dz \\ &= \frac{2\,(T_i - T_b)}{Z}\left[-\frac{Z}{(2n-1)\pi}\cos\frac{(2n-1)\pi z}{Z}\right]_0^Z \\ &= \frac{4\,(T_i - T_b)}{(2n-1)\pi} \end{aligned}$$

We insert this result into the Fourier-Bessel series,

$$\frac{4\,(T_i - T_b)}{(2n-1)\pi} = \sum_{m=1}^{\infty} A_{nm} J_0\left(\frac{r}{a} j_{0m}\right),$$

and recall from Section 5.5 that we can determine the Fourier coefficients A_{nm} using the Fourier-Bessel series,

$$f(x) = \sum_{n=1}^{\infty} c_n J_p\left(j_{pn}\frac{x}{a}\right), \tag{6.4.18}$$

where the Fourier-Bessel coefficients are found as

$$c_n = \frac{2}{a^2\left[J_{p+1}\left(j_{pn}\right)\right]^2}\int_0^a x f(x) J_p\left(j_{pn}\frac{x}{a}\right) dx. \tag{6.4.19}$$

Comparing these series expansions, we have

$$A_{nm} = \frac{2}{a^2 J_1^2\left(j_{0m}\right)}\frac{4\,(T_i - T_b)}{(2n-1)\pi}\int_0^a J_0\left(\mu_m r\right) r dr. \tag{6.4.20}$$

In order to evaluate $\int_0^a J_0\left(\mu_m r\right) r dr$, we let $y = \mu_m r$ and get

$$\begin{aligned}\int_0^a J_0\left(\mu_m r\right) r dr &= \int_0^{\mu_m a} J_0(y) \frac{y}{\mu_m} \frac{dy}{\mu_m} \\ &= \frac{1}{\mu_m^2} \int_0^{\mu_m a} J_0(y) y dy \\ &= \frac{1}{\mu_m^2} \int_0^{\mu_m a} \frac{d}{dy}\left(y J_1(y)\right) dy \\ &= \frac{1}{\mu_m^2}\left(\mu_m a\right) J_1\left(\mu_m a\right) = \frac{a^2}{j_{0m}} J_1\left(j_{0m}\right). \end{aligned} \tag{6.4.21}$$

Here we have made use of the identity $\frac{d}{dx}(xJ_1(x)) = J_0(x)$ from Section 5.5.

Substituting the result of this integral computation into the expression for A_{nm}, we find

$$A_{nm} = \frac{8\left(T_i - T_b\right)}{(2n-1)\pi} \frac{1}{j_{0m} J_1\left(j_{0m}\right)}.$$

Substituting this result into the original expression for $u(r,z,t)$, gives

$$u(r,z,t) = \frac{8\left(T_i - T_b\right)}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\sin \frac{(2n-1)\pi z}{Z}}{(2n-1)} \frac{J_0\left(\frac{r}{a} j0m\right) e^{-\lambda_{mm} k t}}{j_{0m} J_1\left(j_{0m}\right)}.$$

Therefore, $T(r,z,t)$ is found as

$$T(r,z,t) = T_b + \frac{8\left(T_i - T_b\right)}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\sin \frac{(2n-1)\pi z}{Z}}{(2n-1)} \frac{J_0\left(\frac{r}{a} j_{0m}\right) e^{-\lambda_{nm} k t}}{j_{0m} J_1\left(j_{0m}\right)},$$

where

$$\lambda_{nm} = \left(\frac{(2n-1)\pi}{\mathrm{Z}}\right)^2 + \left(\frac{j_{0m}}{a}\right)^2, \quad n, m = 1, 2, 3, \ldots.$$

We have therefore found the general solution for the three-dimensional heat equation in cylindrical coordinates with constant diffusivity. Similar to the solutions shown in Figure 6.4.3 of the previous section, we show in Figure 6.4.6 the temperature evolution throughout a standard $9''$ round cake pan. These are vertical slices similar to what is depicted in Figure 6.4.5.

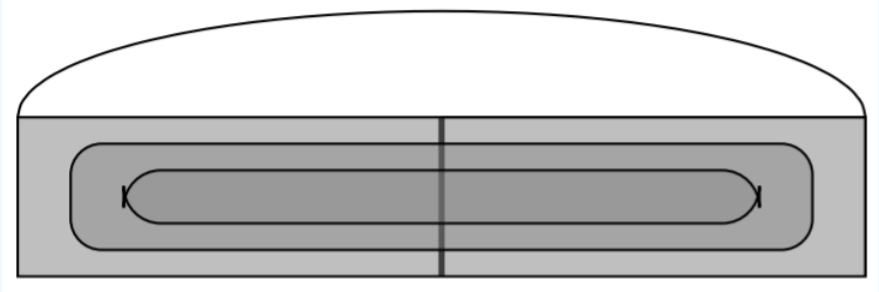


Figure 6.4.5: Depiction of a sideview of a vertical slice of a circular cake.

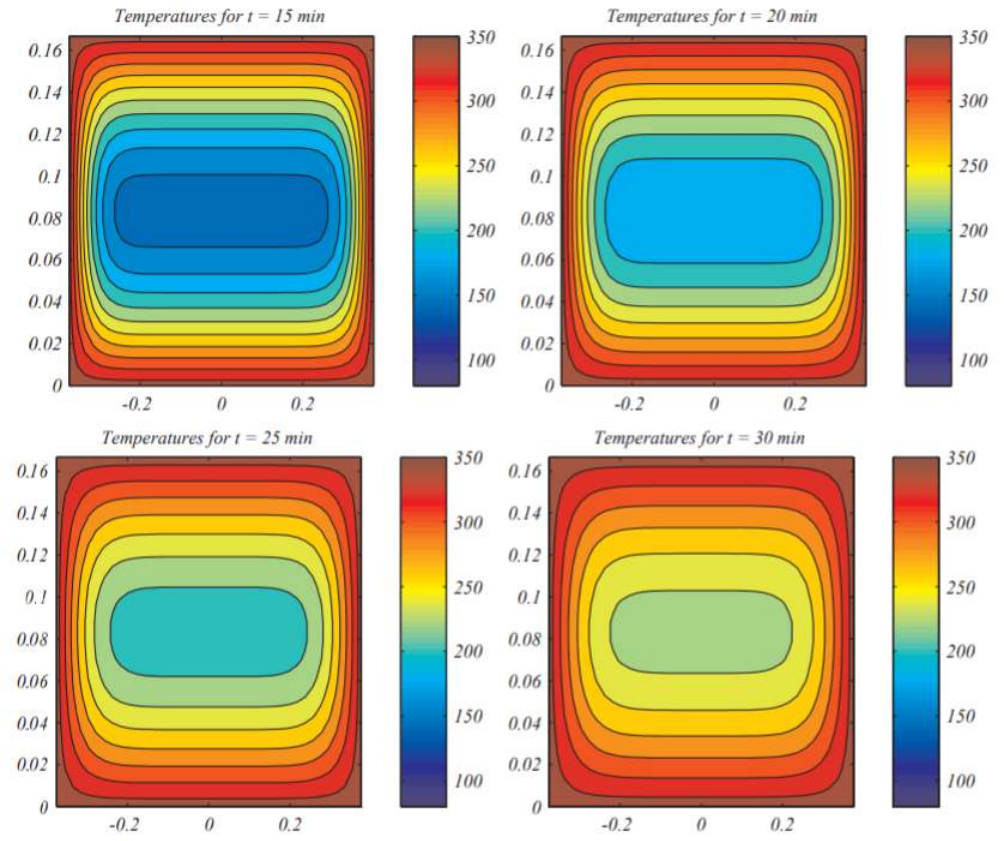


Figure 6.4.6: Temperature evolution for a standard 9″ cake shown as vertical slices through the center.

Again, one could generalize this example to considerations of other types of cakes with cylindrical symmetry. For example, there are muffins, Boston steamed bread which is steamed in tall cylindrical cans. One could also consider an annular pan, such as a bundt cake pan. In fact, such problems extend beyond baking cakes to possible heating molds in manufacturing.

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6.5: Laplace's Equation and Spherical Symmetry

We have seen that Laplace's equation, $\nabla^2 u = 0$, arises in electrostatics as an equation for electric potential outside a charge distribution and it occurs as the equation governing equilibrium temperature distributions. As we had seen in the last chapter, Laplace's equation generally occurs in the study of potential theory, which also includes the study of gravitational and fluid potentials. The equation is named after Pierre-Simon Laplace (1749-1827) who had studied the properties of this equation. Solutions of Laplace's equation are called **harmonic functions**.

Example 6.5.1

Solve Laplace's equation in spherical coordinates.

Solution

We seek solutions of this equation inside a sphere of radius r subject to the boundary condition as shown in Figure 6.5.1. The problem is given by Laplace's equation Laplace's equation in spherical coordinates[1]

$$\frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial u}{\partial \rho}\right) + \frac{1}{\rho^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) + \frac{1}{\rho^2 \sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} = 0, \tag{6.5.1}$$

where $u = u(\rho, \theta, \phi)$.

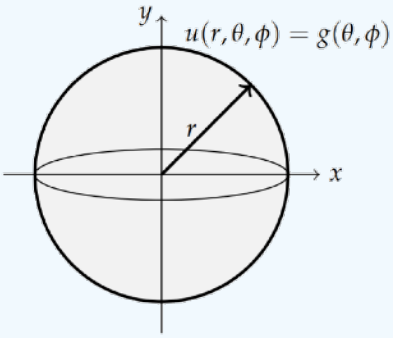


Figure 6.5.1: A sphere of radius r with the boundary condition $u(r, \theta, \phi) = g(\theta, \phi)$.

The boundary conditions are given by

$$u(r, \theta, \phi) = g(\theta, \phi), \quad 0 < \phi < 2\pi, \quad 0 < \theta < \pi,$$

and the periodic boundary conditions

$$u(\rho, \theta, 0) = u(\rho, \theta, 2\pi), \quad u_\phi(\rho, \theta, 0) = u_\phi(\rho, \theta, 2\pi),$$

where $0 < \rho < \infty$, and $0 < \theta < \pi$.

Note

The Laplacian in spherical coordinates is given in Problem ?? in Chapter 8.

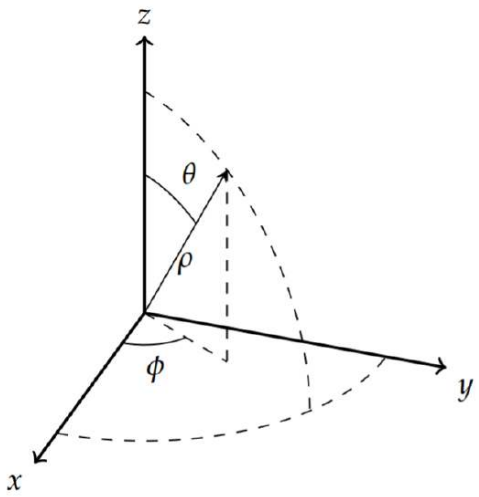


Figure 6.5.2: Definition of spherical coordinates (ρ,θ,ϕ). Note that there are different conventions for labeling spherical coordinates. This labeling is used often in physics.

As before, we perform a separation of variables by seeking product solutions of the form $u(\rho,\theta,\phi)=R(\rho)\Theta(\theta)\Phi(\phi)$. Inserting this form into the Laplace equation, we obtain

$$\frac{\Theta\Phi}{\rho^2}\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right)+\frac{R\Phi}{\rho^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\frac{R\Theta}{\rho^2\sin^2\theta}\frac{d^2\Phi}{d\phi^2}=0. \tag{6.5.2}$$

Multiplying this equation by ρ^2 and dividing by $R\Theta\Phi$, yields

$$\frac{1}{R}\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right)+\frac{1}{\sin\theta\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\frac{1}{\sin^2\theta\Phi}\frac{d^2\Phi}{d\phi^2}=0. \tag{6.5.3}$$

Note that the first term is the only term depending upon ρ. Thus, we can separate out the radial part. However, there is still more work to do on the other two terms, which give the angular dependence. Thus, we have

$$-\frac{1}{R}\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right)=\frac{1}{\sin\theta\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\frac{1}{\sin^2\theta\Phi}\frac{d^2\Phi}{d\phi^2}=-\lambda, \tag{6.5.4}$$

where we have introduced the first separation constant. This leads to two equations:

$$\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right)-\lambda R=0 \tag{6.5.5}$$

and

$$\frac{1}{\sin\theta\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\frac{1}{\sin^2\theta\Phi}\frac{d^2\Phi}{d\phi^2}=-\lambda. \tag{6.5.6}$$

The final separation can be performed by multiplying the last equation by $\sin^2\theta$, rearranging the terms, and introducing a second separation constant:

$$\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\lambda\sin^2\theta=-\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}=\mu. \tag{6.5.7}$$

From this expression we can determine the differential equations satisfied by $\Theta(\theta)$ and $\Phi(\phi)$:

$$\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\left(\lambda\sin^2\theta-\mu\right)\Theta=0 \tag{6.5.8}$$

and

$$\frac{d^2\Phi}{d\phi^2}+\mu\Phi=0. \tag{6.5.9}$$

Note

Equation (6.5.6) is a key equation which occurs when studying problems possessing spherical symmetry. It is an eigenvalue problem for $Y(\theta,\phi)=\Theta(\theta)\Phi(\phi)$, $LY=-\lambda Y$, where

$$L=\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}.$$

The eigenfunctions of this operator are referred to as spherical harmonics.

We now have three ordinary differential equations to solve. These are the radial equation (6.5.5) and the two angular equations (6.5.8)-(6.5.9). We note that all three are in Sturm-Liouville form. We will solve each eigenvalue problem subject to appropriate boundary conditions.

The simplest of these differential equations is Equation (6.5.9) for $\Phi(\phi)$. We have seen equations of this form many times and the general solution is a linear combination of sines and cosines. Furthermore, in this problem $u(\rho,\theta,\phi)$ is periodic in ϕ,

$$u(\rho,\theta,0)=u(\rho,\theta,2\pi),\quad u_\phi(\rho,\theta,0)=u_\phi(\rho,\theta,2\pi).$$

Since these conditions hold for all ρ and θ, we must require that $\Phi(\phi)$ satisfy the periodic boundary conditions

$$\Phi(0)=\Phi(2\pi),\quad \Phi'(0)=\Phi'(2\pi).$$

The eigenfunctions and eigenvalues for Equation (6.5.9) are then found as

$$\Phi(\phi)=\{\cos m\phi,\sin m\phi\},\quad \mu=m^2,\quad m=0,1,\ldots. \tag{6.5.10}$$

Next we turn to solving equation, (6.5.8). We first transform this equation in order to identify the solutions. Let $x=\cos\theta$. Then the derivatives with respect to θ transform as

$$\frac{d}{d\theta}=\frac{dx}{d\theta}\frac{d}{dx}=-\sin\theta\frac{d}{dx}.$$

Letting $y(x)=\Theta(\theta)$ and noting that $\sin^2\theta=1-x^2$, Equation (6.5.8) becomes

$$\frac{d}{dx}\left(\left(1-x^2\right)\frac{dy}{dx}\right)+\left(\lambda-\frac{m^2}{1-x^2}\right)y=0. \tag{6.5.11}$$

We further note that $x\in[-1,1]$, as can be easily confirmed by the reader.

This is a Sturm-Liouville eigenvalue problem. The solutions consist of a set of orthogonal eigenfunctions. For the special case that $m=0$ Equation (6.5.11) becomes

$$\frac{d}{dx}\left(\left(1-x^2\right)\frac{dy}{dx}\right)+\lambda y=0. \tag{6.5.12}$$

In a course in differential equations one learns to seek solutions of this equation in the form

$$y(x)=\sum_{n=0}^{\infty}a_nx^n.$$

This leads to the recursion relation

$$a_{n+2}=\frac{n(n+1)-\lambda}{(n+2)(n+1)}a_n.$$

Setting $n=0$ and seeking a series solution, one finds that the resulting series does not converge for $x=\pm1$. This is remedied by choosing $\lambda=\ell(\ell+1)$ for $\ell=0,1,\ldots$, leading to the differential equation

$$\frac{d}{dx}\left(\left(1-x^2\right)\frac{dy}{dx}\right)+\ell(\ell+1)y=0. \tag{6.5.13}$$

We saw this equation in Chapter 5 in the form

$$\left(1-x^2\right)y''-2xy'+\ell(\ell+1)y=0.$$

The solutions of this differential equation are Legendre polynomials, denoted by $P_\ell(x)$

For the more general case, $m\neq 0$, the differential equation (6.5.11) with $\lambda=\ell(\ell+1)$ becomes

$$\frac{d}{dx}\left(\left(1-x^2\right)\frac{dy}{dx}\right)+\left(\ell(\ell+1)-\frac{m^2}{1-x^2}\right)y=0. \tag{6.5.14}$$

The solutions of this equation are called the associated Legendre functions. The two linearly independent solutions are denoted by $P_\ell^m(x)$ and $Q_\ell^m(x)$. The latter functions are not well behaved at $x=\pm 1$, corresponding to the north and south poles of the original problem. So, we can throw out these solutions in many physical cases, leaving

$$\Theta(\theta)=P_\ell^m(\cos\theta)$$

as the needed solutions. In Table 6.5 we list a few of these.

Table 6.5.1:Associated Legendre Functions, $P_n^m(x)$.

	$P_n^m(x)$	$P_n^m(\cos\theta)$
$P_0^0(x)$	1	1
$P_1^0(x)$	x	$\cos\theta$
$P_1^1(x)$	$-\left(1-x^2\right)^{\frac{1}{2}}$	$-\sin\theta$
$P_2^0(x)$	$\frac{1}{2}\left(3x^2-1\right)$	$\frac{1}{2}\left(3\cos^2\theta-1\right)$
$P_2^1(x)$	$-3x\left(1-x^2\right)^{\frac{1}{2}}$	$-3\cos\theta\sin\theta$
$P_2^2(x)$	$3\left(1-x^2\right)$	$3\sin^2\theta$
$P_3^0(x)$	$\frac{1}{2}\left(5x^3-3x\right)$	$\frac{1}{2}\left(5\cos^3\theta-3\cos\theta\right)$
$P_3^1(x)$	$-\frac{3}{2}\left(5x^2-1\right)\left(1-x^2\right)^{\frac{1}{2}}$	$-\frac{3}{2}\left(5\cos^2\theta-1\right)\sin\theta$
$P_3^2(x)$	$15x\left(1-x^2\right)$	$15\cos\theta\sin^2\theta$
$P_3^3(x)$	$-15\left(1-x^2\right)^{\frac{3}{2}}$	$-15\sin^3\theta$

The associated Legendre functions are related to the Legendre polynomials by[2]

$$P_\ell^m(x)=(-1)^m\left(1-x^2\right)^{m/2}\frac{d^m}{dx^m}P_\ell(x), \tag{6.5.15}$$

for $\ell=0,1,2,\ldots$ and $m=0,1,\ldots,\ell$. We further note that $P_\ell^0(x)=P_\ell(x)$, as one can see in the table. Since $P_\ell(x)$ is a polynomial of degree ℓ, then for $m>\ell$, $\frac{d^m}{dx^m}P_\ell(x)=0$ and $P_\ell^m(x)=0$.

Note

The factor of $(-1)^m$ is known as the Condon-Shortley phase and is useful in quantum mechanics in the treatment of agular momentum. It is sometimes omitted by some

Furthermore, since the differential equation only depends on m^2, $P_\ell^{-m}(x)$ is proportional to $P_\ell^m(x)$. One normalization is given by

$$P_\ell^{-m}(x)=(-1)^m\frac{(\ell-m)!}{(\ell+m)!}P_\ell^m(x).$$

The associated Legendre functions also satisfy the orthogonality condition

$$\int_{-1}^{1}P_\ell^m(x)P_{\ell'}^m(x)dx=\frac{2}{2\ell+1}\frac{(\ell+m)!}{(\ell-m)!}\delta_{\ell\ell'}. \tag{6.5.16}$$

The last differential equation we need to solve is the radial equation. With $\lambda = \ell(\ell+1), \ell = 0, 1, 2, \ldots$, the radial equation (6.5.5) can be written as

$$\rho^2 R'' + 2\rho R' - \ell(\ell+1)R = 0. \tag{6.5.17}$$

The radial equation is a Cauchy-Euler type of equation. So, we can guess the form of the solution to be $R(\rho) = \rho^s$, where s is a yet to be determined constant. Inserting this guess into the radial equation, we obtain the characteristic equation

$$s(s+1) = \ell(\ell+1).$$

Solving for s, we have

$$s = \ell, -(\ell+1).$$

Thus, the general solution of the radial equation is

$$R(\rho) = a\rho^\ell + b\rho^{-(\ell+1)}. \tag{6.5.18}$$

We would normally apply boundary conditions at this point. The boundary condition $u(r, \theta, \phi) = g(\theta, \phi)$ is not a homogeneous boundary condition, so we will need to hold off using it until we have the general solution to the three dimensional problem. However, we do have a hidden condition. Since we are interested in solutions inside the sphere, we need to consider what happens at $\rho = 0$. Note that $\rho^{-(\ell+1)}$ is not defined at the origin. Since the solution is expected to be bounded at the origin, we can set $b = 0$. So, in the current problem we have established that

$$R(\rho) = a\rho^\ell.$$

Note

When seeking solutions outside the sphere, one considers the boundary condition $R(\rho) \to 0$ as $\rho \to \infty$. In this case, $R(\rho) = \rho^{-(\ell+1)}$.

We have carried out the full separation of Laplace's equation in spherical coordinates. The product solutions consist of the forms

$$u(\rho, \theta, \phi) = \rho^\ell P_\ell^m(\cos\theta)\cos m\phi$$

and

$$u(\rho, \theta, \phi) = \rho^\ell P_\ell^m(\cos\theta)\sin m\phi$$

for $\ell = 0, 1, 2, \ldots$ and $m = 0, \pm 1, \ldots, \pm\ell$. These solutions can be combined to give a complex representation of the product solutions as

$$u(\rho, \theta, \phi) = \rho^\ell P_\ell^m(\cos\theta)e^{im\phi}.$$

The general solution is then given as a linear combination of these product can be rewritten as solutions. As there are two indices, we have a double sum:[3]

$$u(\rho, \theta, \phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m}\rho^\ell P_\ell^m(\cos\theta)e^{im\phi}. \tag{6.5.19}$$

Note

While this appears to be a complex-valued solution, it can be rewritten as a sum over real functions. The inner sum contains terms for both $m = k$ and to give a complex representation of the product solutions as $m = -k$. Adding these contributions, we have that

$$a_{\ell k}\rho^\ell P_\ell^k(\cos\theta)e^{ik\phi} + a_{\ell(-k)}\rho^\ell P_\ell^{-k}(\cos\theta)e^{-ik\phi}$$

can be rewritten as

$$\left(A_{\ell k}\cos k\phi + B_{\ell k}\sin k\phi\right)\rho^\ell P_\ell^k(\cos\theta).$$

Example 6.5.2: Laplace's Equation with Azimuthal Symmetry

As a simple example we consider the solution of Laplace's equation in which there is azimuthal symmetry. Let

$$u(r,\theta,\phi) = g(\theta) = 1 - \cos 2\theta.$$

This function is zero at the poles and has a maximum at the equator. So, this could be a crude model of the temperature distribution of the Earth with zero temperature at the poles and a maximum near the equator.

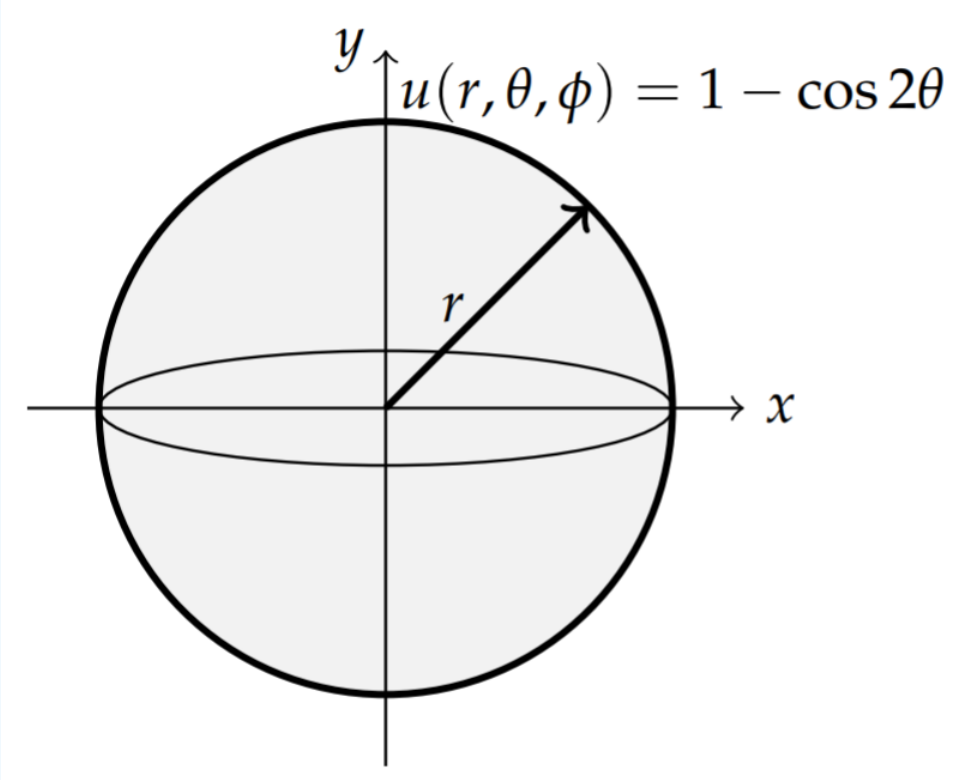


Figure 6.5.3: A sphere of radius r with the boundary condition

$$u(r,\theta,\phi) = 1 - \cos 2\theta.$$

Solution

In problems in which there is no ϕ-dependence, only the $m = 0$ terms of the general solution survives. Thus, we have that

$$u(\rho,\theta,\phi) = \sum_{\ell=0}^{\infty} a_\ell \rho^\ell P_\ell(\cos\theta). \tag{6.5.20}$$

Here we have used the fact that $P_\ell^0(x) = P_\ell(x)$. We just need to determine the unknown expansion coefficients, a_ℓ. Imposing the boundary condition at $\rho = r$, we are lead to

$$g(\theta) = \sum_{\ell=0}^{\infty} a_\ell r^\ell P_\ell(\cos\theta) \tag{6.5.21}$$

This is a Fourier-Legendre series representation of $g(\theta)$. Since the Legendre polynomials are an orthogonal set of eigenfunctions, we can extract the coefficients.

In Chapter 5 we had proven that

$$\int_0^\pi P_n(\cos\theta) P_m(\cos\theta)\sin\theta d\theta = \int_{-1}^1 P_n(x)P_m(x)dx = \frac{2}{2n+1}\delta_{nm}.$$

So, multiplying the expression for $g(\theta)$ by $P_m(\cos\theta)\sin\theta$ and integrating, we obtain the expansion coefficients:

$$a_\ell = \frac{2\ell+1}{2r^\ell}\int_0^\pi g(\theta)P_\ell(\cos\theta)\sin\theta d\theta. \tag{6.5.22}$$

Sometimes it is easier to rewrite $g(\theta)$ as a polynomial in $\cos\theta$ and avoid the integration. For this example we see that

$$\begin{aligned} g(\theta) &= 1-\cos 2\theta \\ &= 2\sin^2\theta \\ &= 2-2\cos^2\theta. \end{aligned} \tag{6.5.23}$$

Thus, setting $x=\cos\theta$ and $G(x)=g(\theta(x))$, we have $G(x)=2-2x^2$.

We seek the form

$$G(x) = c_0P_0(x)+c_1P_1(x)+c_2P_2(x),$$

where $P_0(x)=1, P_1(x)=x$, and $P_2(x)=\frac{1}{2}\left(3x^2-1\right)$. Since $G(x)=2-2x^2$ does not have any x terms, we know that $c_1=0$. So,

$$2-2x^2 = c_0(1)+c_2\frac{1}{2}\left(3x^2-1\right) = c_0-\frac{1}{2}c_2+\frac{3}{2}c_2x^2.$$

By observation we have $c_2=-\frac{4}{3}$ and thus, $c_0=2+\frac{1}{2}c_2=\frac{4}{3}$. Therefore, $G(x)=\frac{4}{3}P_0(x)-\frac{4}{3}P_2(x)$.

We have found the expansion of $g(\theta)$ in terms of Legendre polynomials,

$$g(\theta) = \frac{4}{3}P_0(\cos\theta)-\frac{4}{3}P_2(\cos\theta). \tag{6.5.24}$$

Therefore, the nonzero coefficients in the general solution become

$$a_0=\frac{4}{3}, \quad a_2=\frac{4}{3}\frac{1}{r^2},$$

and the rest of the coefficients are zero. Inserting these into the general solution, we have the final solution

$$\begin{aligned} u(\rho,\theta,\phi) &= \frac{4}{3}P_0(\cos\theta)-\frac{4}{3}\left(\frac{\rho}{r}\right)^2P_2(\cos\theta) \\ &= \frac{4}{3}-\frac{2}{3}\left(\frac{\rho}{r}\right)^2\left(3\cos^2\theta-1\right) \end{aligned} \tag{6.5.25}$$

Spherical Harmonics

The solutions of the angular parts of the problem are often combined into one function of two variables, as problems with spherical symmetry arise often, leaving the main differences between such problems confined to the radial equation. These functions are referred to as spherical harmonics, $Y_{\ell m}(\theta,\phi)$, which are defined with a special normalization as

$$Y_{\ell m}(\theta,\phi) = (-1)^m\sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}P_\ell^m(\cos\theta)e^{im\phi}.$$

These satisfy the simple orthogonality relation

$$\int_0^\pi\int_0^{2\pi} Y_{\ell m}(\theta,\phi)Y^*_{\ell' m'}(\theta,\phi)\sin\theta d\phi d\theta = \delta_{\ell\ell'}\delta_{mm'}.$$

Note

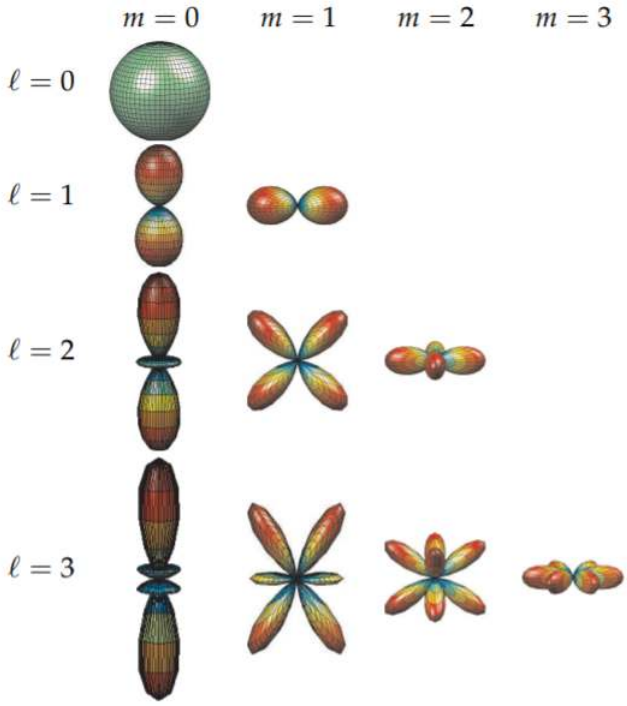
$Y_{\ell m}(\theta,\phi)$, are the spherical harmonics. Spherical harmonics are important in applications from atomic electron configurations to gravitational fields, planetary magnetic fields, and the cosmic microwave background radiation.

As seen earlier in the chapter, the spherical harmonics are eigenfunctions of the eigenvalue problem $LY=-\lambda Y$, where

$$L = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}.$$

This operator appears in many problems in which there is spherical symmetry, such as obtaining the solution of Schrödinger's equation for the hydrogen atom as we will see later. Therefore, it is customary to plot spherical harmonics. Because the $Y_{\ell m}$'s are complex functions, one typically plots either the real part or the modulus squared. One rendition of $|Y_{\ell m}(\theta, \phi)|^2$ is shown in Table 6.5.2 for $\ell, m = 0, 1, 2, 3$.

Table 6.5.2: The first few spherical harmonics, $|Y_{\ell m}(\theta, \phi)|^2$



We could also look for the nodal curves of the spherical harmonics like we had for vibrating membranes. Such surface plots on a sphere are shown in Table 6.5.3. The colors provide for the amplitude of the $|Y_{\ell m}(\theta, \varphi)|^2$. We can match these with the shapes in Table 6.5.2 by coloring the plots with some of the same colors as shown in Table 6.5.3. However, by plotting just the sign of the spherical harmonics, as in Table 6.5.4, we can pick out the nodal curves much easier.

Table 6.5.3: Spherical harmonic contours for $|Y_{\ell m}(\theta, \phi)|^2$.

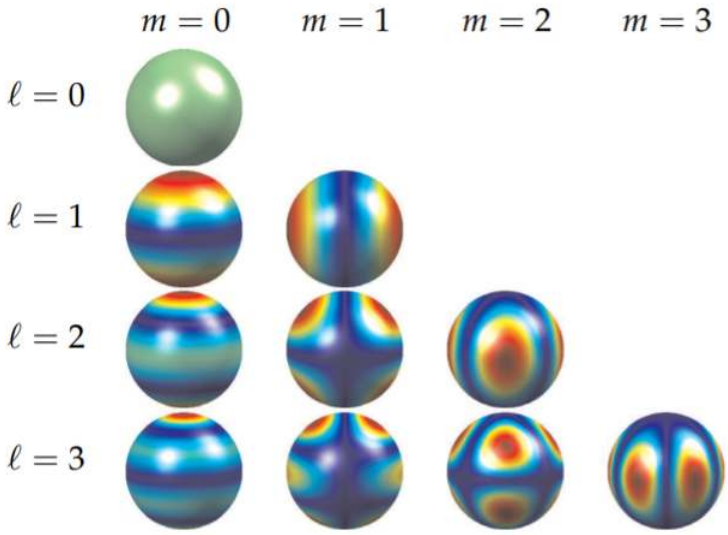
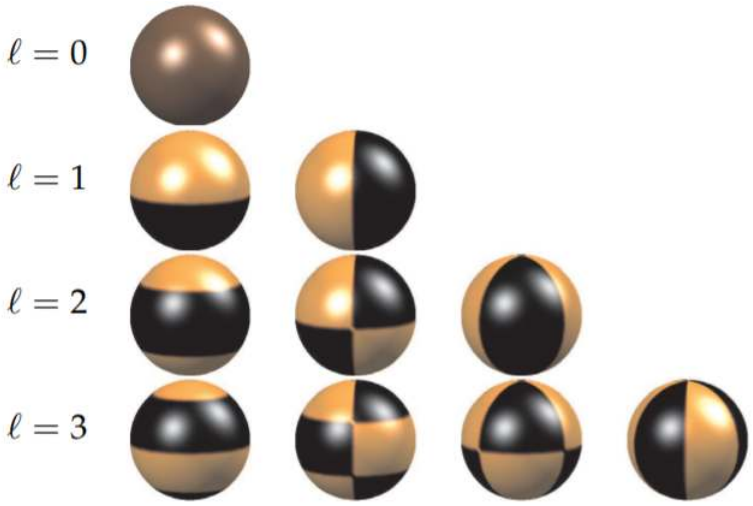


Table 6.5.4: In these figures we show the nodal curves of $|Y_{\ell m}(\theta, \phi)|^2$ Along the first column $(m = 0)$ are the zonal harmonics seen as ℓ horizontal circles. Along the top diagonal $(m = \ell)$ are the sectional harmonics. These look like orange sections formed

from m vertical circles. The remaining harmonics are tesseral harmonics. They look like a checkerboard pattern formed from intersections of $\ell - m$ horizontal circles and m vertical circles.



Spherical, or surface, harmonics can be further grouped into zonal, sectoral, and tesseral harmonics. Zonal harmonics correspond to the $m = 0$ modes. In this case, one seeks nodal curves for which $P_\ell(\cos\theta) = 0$. Solutions of this equation lead to constant θ values such that $\cos\theta$ is a zero of the Legendre polynomial, $P_\ell(x)$. The zonal harmonics correspond to the first column in Table 6.5.4. Since $P_\ell(x)$ is a polynomial of degree ℓ, the zonal harmonics consist of ℓ latitudinal circles.

Sectoral, or meridional, harmonics result for the case that $m = \pm\ell$. For this case, we note that $P_\ell^{\pm\ell}(x) \propto \left(1 - x^2\right)^{m/2}$. This function vanishes for $x = \pm 1$, or $\theta = 0, \pi$. Therefore, the spherical harmonics can only produce nodal curves for $e^{im\phi} = 0$. Thus, one obtains the meridians satisfying the condition $A\cos m\phi + B\sin m\phi = 0$. Solutions of this equation are of the form $\phi =$ constant. These modes can be seen in Table 6.5.4 in the top diagonal and can be described as m circles passing through the poles, or longitudinal circles.

Tesseral harmonics consist of the rest of the modes, which typically look like a checker board glued to the surface of a sphere. Examples can be seen in the pictures of nodal curves, such as Table 6.5.4. Looking in Table 6.5.4 along the diagonals going downward from left to right, one can see the same number of latitudinal circles. In fact, there are $\ell - m$ latitudinal nodal curves in these figures.

In summary, the spherical harmonics have several representations, as show in Tables 6.5.3-6.5.4. Note that there are ℓ nodal lines, m meridional curves, and $\ell - m$ horizontal curves in these figures. The plots in Table 6.5.2 are the typical plots shown in physics for discussion of the wavefunctions of the hydrogen atom. Those in 6.5.3 are useful for describing gravitational or electric potential functions, temperature distributions, or wave modes on a spherical surface. The relationships between these pictures and the nodal curves can be better understood by comparing respective plots. Several modes were separated out in Figures 6.5.4-6.5.9 to make this comparison easier.

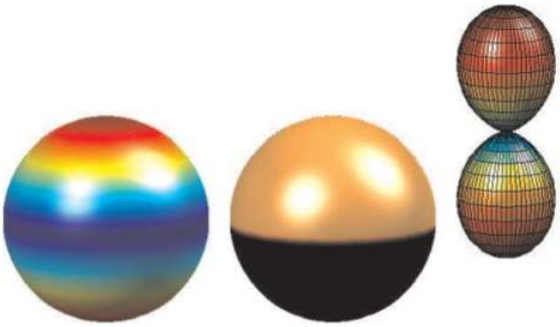


Figure 6.5.4: Zonal harmonics, $\ell = 1$, $m = 0$.

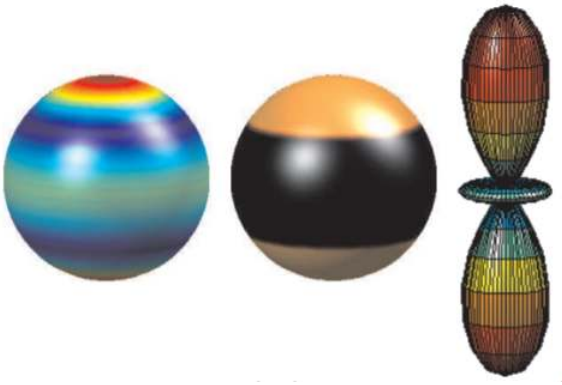


Figure 6.5.5: Zonal harmonics, $\ell = 2, m = 0$.



Figure 6.5.6: Sectoral harmonics, $\ell = 2, m = 2$.

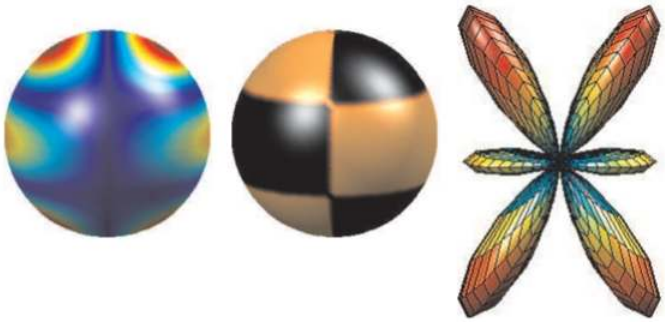


Figure 6.5.7: Tesseral harmonics, $\ell = 3, m = 1$.



Figure 6.5.8: Sectoral harmonics, $\ell = 3, m = 3$.



Figure 6.5.9: Tesseral harmonics, $\ell = 4, m = 3$.

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6.6: Spherically Symmetric Vibrations

Another application of spherical harmonics is a vibrating spherical membrane, such as a balloon. Just as for the two-dimensional membranes encountered earlier, we let $u(\theta,\phi,t)$ represent the vibrations of the surface about a fixed radius obeying the wave equation, $u_{tt}=c^2\nabla^2 u$, and satisfying the initial conditions

$$u(\theta,\phi,0)=f(\theta,\phi), \quad u_t(\theta,\phi,0)=g(\theta,\phi).$$

In spherical coordinates, we have (for $\rho=r=$ constant.)

$$u_{tt}=\frac{c^2}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}\right), \tag{6.6.1}$$

where $u=u(\theta,\phi,t)$.

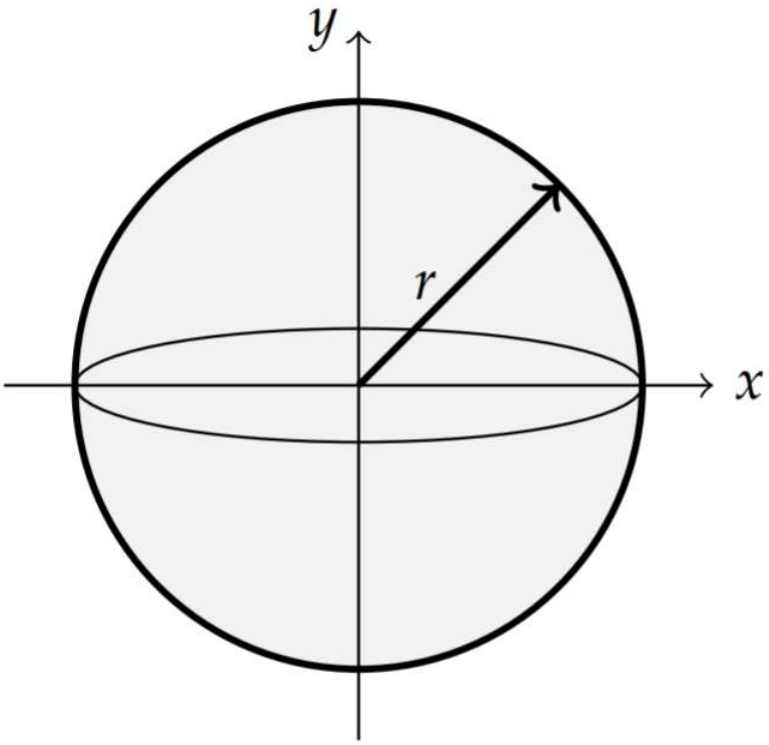


Figure 6.6.1: A vibrating sphere of radius r with the initial conditions

$$u(\theta,\phi,0)=f(\theta,\phi),$$
$$u_t(\theta,\phi,0)=g(\theta,\phi).$$

The boundary conditions are given by the periodic boundary conditions

$$u(\theta,0,t)=u(\theta,2\pi,t), \quad u_\phi(\theta,0,t)=u_\phi(\theta,2\pi,t),$$

where $0<t$, and $0<\theta<\pi$, and that $u=u(\theta,\phi,t)$ should remain bounded.

Noting that the wave equation takes the form

$$u_{tt}=\frac{c^2}{r^2}Lu, \quad \text{where} \quad LY_{\ell m}=-\ell(\ell+1)Y_{\ell m}$$

for the spherical harmonics $Y_{\ell m}(\theta,\phi)=P_\ell^m(\cos\theta)e^{im\phi}$, then we can seek product solutions of the form

$$u_{\ell m}(\theta,\phi,t)=T(t)Y_{\ell m}(\theta,\phi).$$

Inserting this form into the wave equation in spherical coordinates, we find

$$T''Y_{\ell m} = -\frac{c^2}{r^2}T(t)\ell(\ell+1)Y_{\ell m},$$

or

$$T'' + \ell(\ell+1)\frac{c^2}{r^2}T(t)$$

The solutions of this equation are easily found as

$$T(t) = A\cos\omega_\ell t + B\sin\omega_\ell t, \quad \omega_\ell = \sqrt{\ell(\ell+1)}\,\frac{c}{r}.$$

Therefore, the product solutions are given by

$$u_{\ell m}(\theta,\phi,t) = \left[A\cos\omega_\ell t + B\sin\omega_\ell t\right] Y_{\ell m}(\theta,\phi)$$

for $\ell = 0, 1, \ldots, m = -\ell, -\ell+1, \ldots, \ell$.

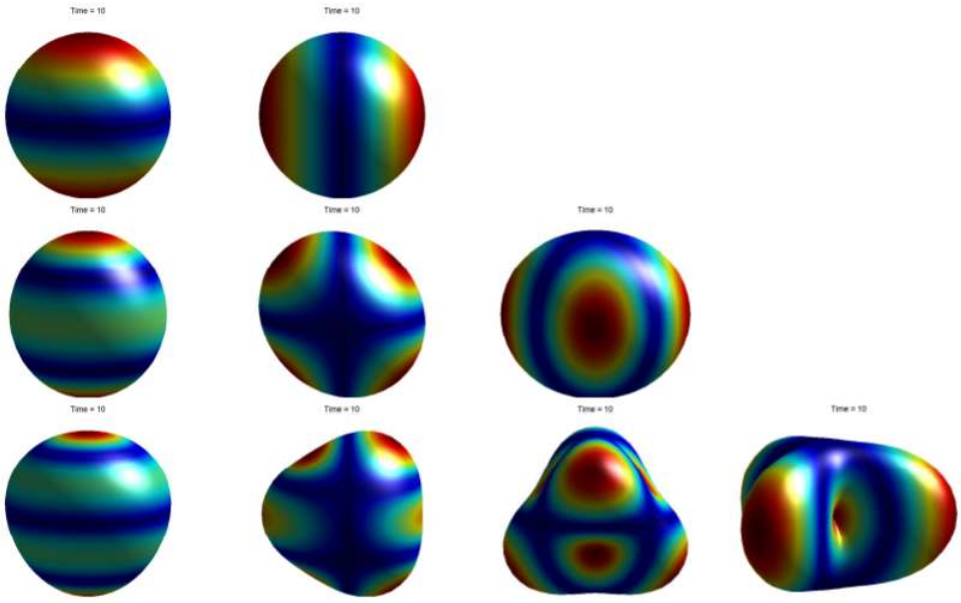


Figure 6.6.2: Modes for a vibrating spherical membrane:

Row 1 :$(1,0),(1,1)$;
Row 2 :$(2,0),(2,1),(2,2)$;
Row 3 :$(3,0),(3,1),(3,2),(3,3)$.

The general solution is found as

$$u(\theta,\phi,t) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left[A_{\ell m}\cos\omega_\ell t + B_{\ell m}\sin\omega_\ell t\right]Y_{\ell m}(\theta,\phi).$$

An interesting problem is to consider hitting the balloon with a velocity impulse while at rest. An example of such a solution is shown in Figure 6.6.3. In this images several modes are excited after the impulse.

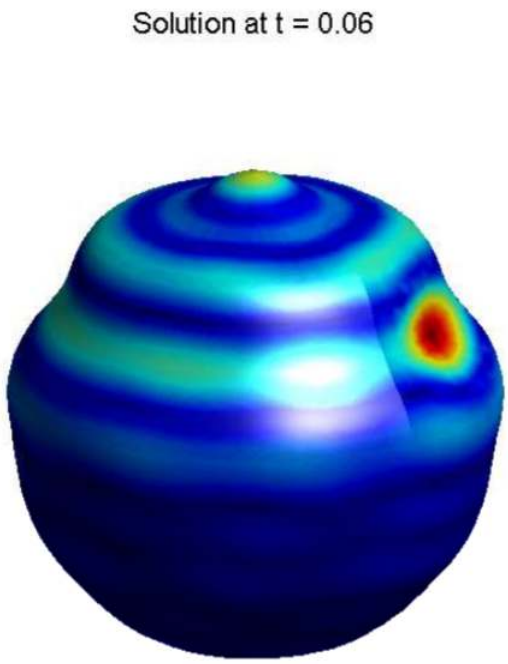


Figure 6.6.3: A moment captured from a simulation of a spherical membrane after hit with a velocity impulse.

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6.7: Baking a Spherical Turkey

During one year as this course was being taught, an instructor returned from the American holiday of Thanksgiving, where it is customary to cook a turkey. Such a turkey is shown in Figure 6.7.1. This reminded the instructor of a typical problem, such as in Weinberger, (1995, p. 92.), where one is given a roast of a certain volume and one is asked to find the time it takes to cook one double the size. In this section, we explore a similar problem for cooking a turkey.



Figure 6.7.1: A 12-lb turkey leaving the oven.

Often during this time of the year, November, articles appear with some scientific evidence as to how to gauge how long it takes to cook a turkey of a given weight. Inevitably it refers to the story, as told in today.slac.stanford.edu/a/2008/11-26.htm today.slac.stanford.edu/a/2008/1126.htm that Pief Panofsky, a former SLAC Director, was determined to find a nonlinear equation for determining cooking times instead of using the rule of thumb of 30 minutes per pound of turkey. He had arrived at the form,

$$t = \frac{W^{2/3}}{1.5}$$

where t is the cooking time and W is the weight of the turkey in pounds. Nowadays, one can go to Wolframalpha.com and enter the question "how long should you cook a turkey" and get results based on a similar formula.

Before turning to the solution of the heat equation for a turkey, let's consider a simpler problem.

Example 6.7.1

If it takes 4 hours to cook a 10 pound turkey in a 350° F oven, then how long would it take to cook a 20 pound turkey at the same conditions?

Solution

In all of our analysis, we will consider a spherical turkey. While the turkey in Figure 6.7.1 is not quite spherical, we are free to approximate the turkey as such. If you prefer, we could imagine a spherical turkey like the one shown in Figure 6.7.2.

This problem is one of scaling. Thinking of the turkey as being spherically symmetric, then the baking follows the heat equation in the form

$$u_t = \frac{k}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right).$$

We can rescale the variables from coordinates (r, t) to (ρ, τ) as $r = \beta\rho$, and $t = \alpha\tau$. Then the derivatives transform as

$$\frac{\partial}{\partial r} = \frac{\partial \rho}{\partial r}\frac{\partial}{\partial \rho} = \frac{1}{\beta}\frac{\partial}{\partial \rho},$$
$$\frac{\partial}{\partial t} = \frac{\partial \tau}{\partial t}\frac{\partial}{\partial \tau} = \frac{1}{\alpha}\frac{\partial}{\partial \tau}. \tag{6.7.1}$$

Inserting these transformations into the heat equation, we have

$$u_\tau = \frac{\alpha}{\beta^2}\frac{k}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial u}{\partial \rho}\right)$$

To keep conditions the same, then we need $\alpha = \beta^2$. So, the transformation that keeps the form of the heat equation the same, or makes it invariant, is $r = \beta\rho$, and $t = \beta^2\tau$. This is also known as a self-similarity transformation.



Figure 6.7.2: The depiction of a spherical turkey.

So, if the radius increases by a factor of β, then the time to cook the turkey (reaching a given temperature, u), would increase by β^2. Returning to the problem, if the weight of the doubles, then the volume doubles, assuming that the density is held constant. However, the volume is proportional to r^3. So, r increases by a factor of $2^{1/3}$. Therefore, the time increases by a factor of $2^{2/3} \approx 1.587$. This give the time for cooking a 20lb turkey as $t = 4\left(2^{2/3}\right) = 2^{8/3} \approx 6.35$ hours.

The previous example shows the power of using similarity transformations to get general information about solutions of differential equations. However, we have focussed on using the method of separation of variables for most of the book so far. We should be able to find a solution to the spherical turkey model using these methods as well. This will be shown in the next example.

Example 6.7.2

Find the temperature, $T(\rho, t)$ inside a spherical turkey, initially at 40°, which is F placed in a 350° F. Assume that the turkey is of constant density and that the surface of the turkey is maintained at the oven temperature. [We will also neglect convection and radaition processes inside the oven.]

Solution

The problem can be formulated as a heat equation problem for $T(\rho, t)$:

$$\begin{aligned} T_t &= \frac{k}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right), \quad 0 < \rho < a, t > 0, \\ T(a,t) &= 350, \quad T(\rho,t) \text{ finite at } \rho = 0, \quad t > 0, \\ T(\rho,0) &= 40. \end{aligned} \tag{6.7.2}$$

We note that the boundary condition is not homogeneous. However, we can fix that by introducing the auxiliary function (the difference between the turkey and oven temperatures) $u(\rho,t)=T(\rho,t)-T_a$, where $T_a=350$. Then, the problem to be solved becomes

$$\begin{aligned} u_t &= \frac{k}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right), \quad 0<\rho<a, t>0, \\ u(a,t) &= 0, \quad u(\rho,t) \quad \text{finite at } \rho=0, \quad t>0, \\ u(\rho,0) &= T_a i - T_a = -310, \end{aligned} \tag{6.7.3}$$

where $T_i=40$.

We can now employ the method of separation of variables. Let $u(\rho,t)=R(\rho)G(t)$. Inserting into the heat equation for u, we have

$$\frac{1}{k}\frac{G'}{G}=\frac{1}{R}\left(R''+\frac{2}{\rho}R'\right)=-\lambda.$$

This give the two ordinary differential equations, the temporal equation,

$$G'=-k\lambda G, \tag{6.7.4}$$

and the radial equation,

$$\rho R''+2R'+\lambda\rho R=0. \tag{6.7.5}$$

The temporal equation is easy to solve,

$$G(t)=G_0e^{-\lambda kt}.$$

However, the radial equation is slightly more difficult. But, making the substitution $R(\rho)=y(\rho)/\rho$, it is readily transformed into a simpler form:[1]

$$y''+\lambda y=0$$

The boundary conditions on $u(\rho,t)=R(\rho)G(t)$ transfer to $R(a)=0$ and $R(\rho)$ finite at the origin. In turn, this means that $y(a)=0$ and $y(\rho)$ has to vanish near the origin. If $y(\rho)$ does not vanish near the origin, then $R(\rho)$ is not finite as $\rho\to 0$.

So, we need to solve the boundary value problem

$$y''+\lambda y=0, \quad y(0)=0, \quad y(a)=0.$$

This gives the well-known set of eigenfunctions

$$y(\rho)=\sin\frac{n\pi\rho}{a}, \quad \lambda_n=\left(\frac{n\pi}{a}\right)^2, \quad n=1,2,3,\ldots.$$

Therefore, we have found

$$R(\rho)=\frac{\sin\frac{n\pi\rho}{a}}{\rho}, \quad \lambda_n=\left(\frac{n\pi}{a}\right)^2, \quad n=1,2,3,\ldots.$$

The general solution to the auxiliary problem is

$$u(\rho,t)=\sum_{n=1}^{\infty}A_n\frac{\sin\frac{n\pi\rho}{a}}{\rho}e^{-(n\pi/a)^2kt}.$$

This gives the general solution for the temperature as

$$T(\rho,t)=T_a+\sum_{n=1}^{\infty}A_n\frac{\sin\frac{n\pi\rho}{a}}{\rho}e^{-(n\pi/a)^2kt}.$$

All that remains is to find the solution satisfying the initial condition, $T(\rho,0=40$. Inserting $t=0$, we have

$$T_i - T_a = \sum_{n=1}^{\infty} A_n \frac{\sin \frac{n\pi\rho}{a}}{\rho}$$

This is almost a Fourier sine series. Multiplying by ρ, we have

$$(T_i - T_a)\,\rho = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi\rho}{a}.$$

Now, we can solve for the coefficients,

$$\begin{aligned} A_n &= \frac{2}{a}\int_0^a (T_i - T_a)\,\rho \sin \frac{n\pi\rho}{a} d\rho \\ &= \frac{2a}{n\pi}(T_i - T_a)\,(-1)^{n+1}. \end{aligned} \tag{6.7.6}$$

This gives the final solution,

$$T(\rho, t) = T_a + \frac{2a\,(T_i - T_a)}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \frac{\sin \frac{n\pi\rho}{a}}{\rho} e^{-(n\pi/a)^2 kt}.$$

For generality, the ambient and initial temperature were left in terms of T_a and T_i, respectively.

Note

The radial equation almost looks familiar when it is multiplied by ρ :

$$\rho^2 R'' + 2\rho R' + \lambda\rho^2 R = 0.$$

If it were not for the ' 2 ', it would be the zeroth order Bessel equation. This is actually the zeroth order spherical Bessel equation. In general, the spherical Bessel functions, $j_n(x)$ and $y_n(x)$, satisfy

$$x^2 y'' + 2xy' + \left[x^2 - n(n+1)\right] y = 0.$$

So, the radial solution of the turkey problem is

$$R(\rho) = j_n(\sqrt{\lambda}\rho) = \frac{\sin \sqrt{\lambda}\rho}{\sqrt{\lambda}\rho}.$$

We further note that

$$j_n(x) = \sqrt{\frac{\pi}{2x}} J_{n+\frac{1}{2}}(x)$$

It is interesting to use the above solution to compare roasting different turkeys. We take the same conditions as above. Let the radius of the spherical turkey be six inches. We will assume that such a turkey takes four hours to cook, i.e., reach a temperature of 180° F. Plotting the solution with 400 terms, one finds that $k \approx 0.000089$. This gives a "baking time" of $t1 = 239.63$. A plot of the temperature at the center point $(\rho = a/2)$ of the bird is in Figure 6.7.3.

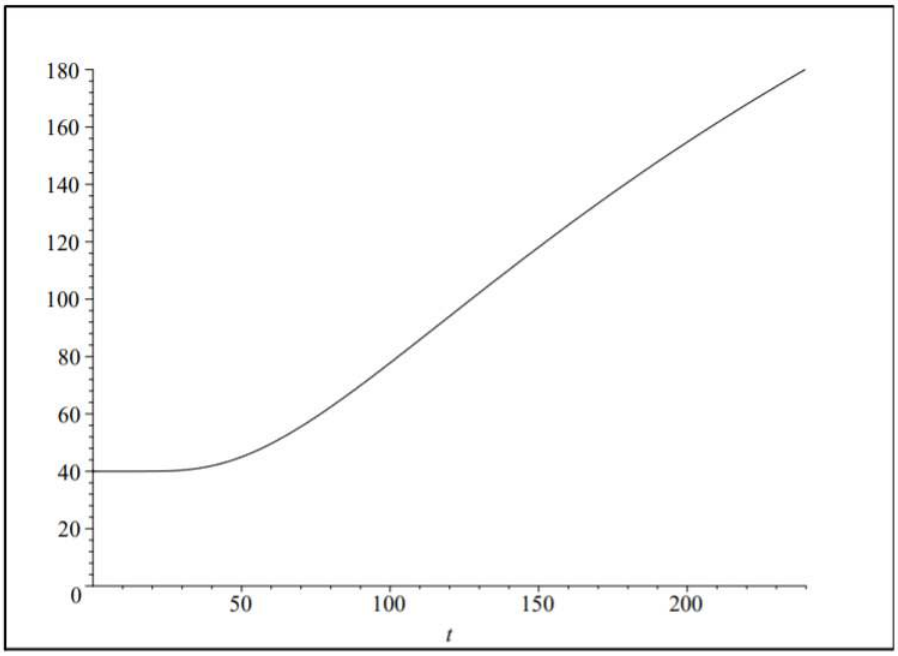


Figure 6.7.3: The temperature at the center of a turkey with radius $a = 0.5\text{ft}$ and $k \approx 0.000089$.

Using the same constants, but increasing the radius of a turkey to $a = 0.5\left(2^{1/3}\right)$ ft, we obtain the temperature plot in Figure 6.7.4. This radius corresponds to doubling the volume of the turkey. Solving for the time at which the center temperature (at $\rho = a/2$) reaches 180°F, we obtained $t2 = 380.38$. Comparing the two temperatures, we find the ratio (using the full computation of the solution in Maple)

$$\frac{t2}{t1} = \frac{380.3813709}{239.6252478} \approx 1.587401054.$$

The compares well to

$$2^{2/3} \approx 1.587401052.$$

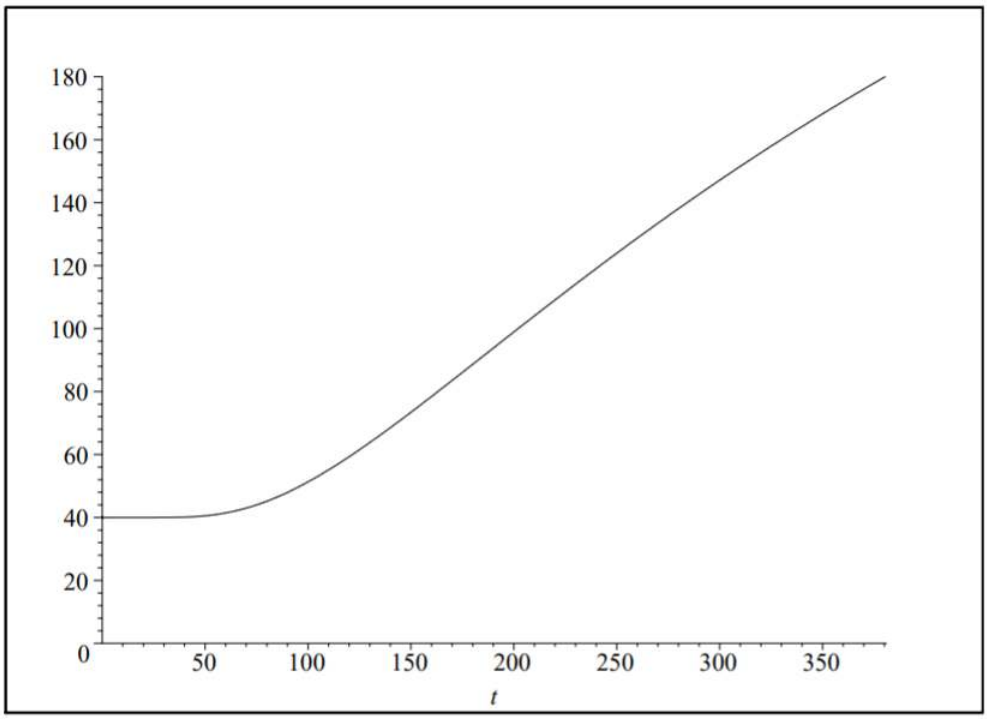


Figure 6.7.4: The temperature at the center of a turkey with radius $a = 0.5(2^{1/3})$ ft and $k \approx 0.000089$.

Of course, the temperature is not quite the center of the spherical turkey. The reader can work out the details for other locations. Perhaps other interesting models would be a spherical shell of turkey with a corse of bread stuffing. Or, one might consider an ellipsoidal geometry.

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6.8: Schrödinger Equation in Spherical Coordinates

Another important eigenvalue problem in physics is the Schrödinger equation. The time-dependent Schrödinger equation is given by

$$i\hbar\frac{\partial\Psi}{\partial t}=-\frac{\hbar^2}{2m}\nabla^2\Psi+V\Psi. \tag{6.8.1}$$

Here $\Psi(\mathbf{r},t)$ is the wave function, which determines the quantum state of a particle of mass m subject to a (time independent) potential, $V(\mathbf{r})$. From Planck's constant, h, one defines $\hbar=\frac{h}{2\pi}$. The probability of finding the particle in an infinitesimal volume, dV, is given by $|\Psi(\mathbf{r},t)|^2dV$, assuming the wave function is normalized,

$$\int_{\text{all space}}|\Psi(\mathbf{r},t)|^2dV=1$$

One can separate out the time dependence by assuming a special form, $\Psi(\mathbf{r},t)=\psi(\mathbf{r})e^{-iEt/\hbar}$, where E is the energy of the particular stationary state solution, or product solution. Inserting this form into the time-dependent equation, one finds that $\psi(\mathbf{r})$ satisfies the time-independent Schrödinger equation,

$$-\frac{\hbar^2}{2m}\nabla^2\psi+V\psi=E\psi. \tag{6.8.2}$$

Assuming that the potential depends only on the distance from the origin, $V=V(\rho)$, we can further separate out the radial part of this solution using spherical coordinates. Recall that the Laplacian in spherical coordinates is given by

$$\nabla^2=\frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial}{\partial\rho}\right)+\frac{1}{\rho^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\rho^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}. \tag{6.8.3}$$

Then, the time-independent Schrödinger equation can be written as

$$\begin{aligned}&-\frac{\hbar^2}{2m}\left[\frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial\psi}{\partial\rho}\right)+\frac{1}{\rho^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)+\frac{1}{\rho^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}\right]\\&=[E-V(\rho)]\psi.\end{aligned} \tag{6.8.4}$$

Let's continue with the separation of variables. Assuming that the wave function takes the form $\psi(\rho,\theta,\phi)=R(\rho)Y(\theta,\phi)$, we obtain

$$\begin{aligned}&-\frac{\hbar^2}{2m}\left[\frac{Y}{\rho^2}\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right)+\frac{R}{\rho^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)+\frac{R}{\rho^2\sin^2\theta}\frac{\partial^2Y}{\partial\phi^2}\right]\\&=RY[E-V(\rho)]\psi.\end{aligned} \tag{6.8.5}$$

Dividing by $\psi=RY$, multiplying by $-\frac{2m\rho^2}{\hbar^2}$, and rearranging, we have

$$\frac{1}{R}\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right)-\frac{2m\rho^2}{\hbar^2}[V(\rho)-E]=-\frac{1}{Y}LY,$$

where

$$L=\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}.$$

We have a function of ρ equal to a function of the angular variables. So, we set each side equal to a constant. We will judiciously write the separation constant as $\ell(\ell+1)$. The resulting equations are then

$$\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right)-\frac{2m\rho^2}{\hbar^2}[V(\rho)-E]R=\ell(\ell+1)R, \tag{6.8.6}$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2Y}{\partial\phi^2}=-\ell(\ell+1)Y. \tag{6.8.7}$$

The second of these equations should look familiar from the last section. This is the equation for spherical harmonics,

$$Y_{\ell m}(\theta,\phi)=\sqrt{\frac{2\ell+1}{2}\frac{(\ell-m)!}{(\ell+m)!}}P_\ell^m e^{im\phi}. \tag{6.8.8}$$

So, any further analysis of the problem depends upon the choice of potential, $V(\rho)$, and the solution of the radial equation. For this, we turn to the determination of the wave function for an electron in orbit about a proton.

Example 6.8.1: The Hydrogen Atom - $\ell=0$ States

Historically, the first test of the Schrödinger equation was the determination of the energy levels in a hydrogen atom. This is modeled by an electron orbiting a proton. The potential energy is provided by the Coulomb potential,

$$V(\rho)=-\frac{e^2}{4\pi\epsilon_0\rho}.$$

Thus, the radial equation becomes

$$\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right)+\frac{2m\rho^2}{\hbar^2}\left[\frac{e^2}{4\pi\epsilon_0\rho}+E\right]R=\ell(\ell+1)R. \tag{6.8.9}$$

Solution

Before looking for solutions, we need to simplify the equation by absorbing some of the constants. One way to do this is to make an appropriate change of variables. Let $\rho=$ ar. Then, by the Chain Rule we have

$$\frac{d}{d\rho}=\frac{dr}{d\rho}\frac{d}{dr}=\frac{1}{a}\frac{d}{dr}.$$

Under this transformation, the radial equation becomes

$$\frac{d}{dr}\left(r^2\frac{du}{dr}\right)+\frac{2ma^2r^2}{\hbar^2}\left[\frac{e^2}{4\pi\epsilon_0 ar}+E\right]u=\ell(\ell+1)u, \tag{6.8.10}$$

where $u(r)=R(\rho)$. Expanding the second term,

$$\frac{2ma^2r^2}{\hbar^2}\left[\frac{e^2}{4\pi\epsilon_0 ar}+E\right]u=\left[\frac{mae^2}{2\pi\epsilon_0\hbar^2}r+\frac{2mEa^2}{\hbar^2}r^2\right]u,$$

we see that we can define

$$a=\frac{2\pi\epsilon_0\hbar^2}{me^2} \tag{6.8.11}$$

$$\epsilon=-\frac{2mEa^2}{\hbar^2} \tag{6.8.12}$$

$$=-\frac{2(2\pi\epsilon_0)^2\hbar^2}{me^4}E. \tag{6.8.13}$$

Using these constants, the radial equation becomes

$$\frac{d}{dr}\left(r^2\frac{du}{dr}\right)+ru-\ell(\ell+1)u=\epsilon r^2u. \tag{6.8.14}$$

Expanding the derivative and dividing by r^2,

$$u''+\frac{2}{r}u'+\frac{1}{r}u-\frac{\ell(\ell+1)}{r^2}u=\epsilon u. \tag{6.8.15}$$

The first two terms in this differential equation came from the Laplacian. The third term came from the Coulomb potential. The fourth term can be thought to contribute to the potential and is attributed to angular momentum. Thus, ℓ is called the angular momentum quantum number. This is an eigenvalue problem for the radial eigenfunctions $u(r)$ and energy eigenvalues ϵ.

The solutions of this equation are determined in a quantum mechanics course. In order to get a feeling for the solutions, we will consider the zero angular momentum case, $\ell=0$:

$$u'' + \frac{2}{r}u' + \frac{1}{r}u = \epsilon u. \tag{6.8.16}$$

Even this equation is one we have not encountered in this book. Let's see if we can find some of the solutions.

First, we consider the behavior of the solutions for large r. For large r the second and third terms on the left hand side of the equation are negligible. So, we have the approximate equation

$$u'' - \epsilon u = 0. \tag{6.8.17}$$

Therefore, the solutions behave like $u(r) = e^{\pm\sqrt{\epsilon}}$ for large r. For bounded solutions, we choose the decaying solution.

This suggests that solutions take the form $u(r) = v(r)e^{-\sqrt{\epsilon}r}$ for some unknown function, $v(r)$. Inserting this guess into Equation (6.8.16), gives an equation for $v(r)$:

$$rv'' + 2(1 - \sqrt{\epsilon}r)v' + (1 - 2\sqrt{\epsilon})v = 0. \tag{6.8.18}$$

Next we seek a series solution to this equation. Let

$$v(r) = \sum_{k=0}^{\infty} c_k r^k.$$

Inserting this series into Equation (6.8.18), we have

$$\sum_{k=1}^{\infty}[k(k-1) + 2k]c_k r^{k-1} + \sum_{k=1}^{\infty}[1 - 2\sqrt{\epsilon}(k+1)]c_k r^k = 0.$$

We can re-index the dummy variable in each sum. Let $k = m$ in the first sum and $k = m - 1$ in the second sum. We then find that

$$\sum_{k=1}^{\infty}\left[m(m+1)c_m + (1 - 2m\sqrt{\epsilon})c_{m-1}\right] r^{m-1} = 0.$$

Since this has to hold for all $m \geq 1$,

$$c_m = \frac{2m\sqrt{\epsilon} - 1}{m(m+1)}c_{m-1}.$$

Further analysis indicates that the resulting series leads to unbounded solutions unless the series terminates. This is only possible if the numerator, $2m\sqrt{\epsilon} - 1$, vanishes for $m = n, n = 1, 2 \ldots$. Thus,

$$\epsilon = \frac{1}{4n^2}.$$

Since ϵ is related to the energy eigenvalue, E, we have

$$E_n = -\frac{me^4}{2(4\pi\epsilon_0)^2\hbar^2 n^2}.$$

Inserting the values for the constants, this gives

$$E_n = -\frac{13.6\text{eV}}{n^2}.$$

This is the well known set of energy levels for the hydrogen atom.

The corresponding eigenfunctions are polynomials, since the infinite series was forced to terminate. We could obtain these polynomials by iterating the recursion equation for the c_m 's. However, we will instead rewrite the radial equation (6.8.18).

Let $x = 2\sqrt{\epsilon}r$ and define $y(x) = v(r)$. Then

$$\frac{d}{dr} = 2\sqrt{\epsilon}\frac{d}{dx}.$$

This gives

$$2\sqrt{\epsilon}xy'' + (2-x)2\sqrt{\epsilon}y' + (1-2\sqrt{\epsilon})y = 0.$$

Rearranging, we have

$$xy'' + (2-x)y' + \frac{1}{2\sqrt{\epsilon}}(1-2\sqrt{\epsilon})y = 0.$$

Noting that $2\sqrt{\epsilon} = \frac{1}{n}$, this equation becomes

$$xy'' + (2-x)y' + (n-1)y = 0. \tag{6.8.19}$$

The resulting equation is well known. It takes the form

$$xy'' + (\alpha + 1 - x)y' + ny = 0. \tag{6.8.20}$$

Solutions of this equation are the associated Laguerre polynomials. The solutions are denoted by $L_n^{\alpha}(x)$. They can be defined in terms of the Laguerre polynomials,

$$L_n(x) = e^x \left(\frac{d}{dx}\right)^n \left(e^{-x}x^n\right).$$

The associated Laguerre polynomials are defined as

$$L_{n-m}^m(x) = (-1)^m \left(\frac{d}{dx}\right)^m L_n(x).$$

Note

The Laguerre polynomials were first encountered in Problem 2 in Chapter 5 as an example of a classical orthogonal polynomial defined on $[0, \infty)$ with weight $w(x) = e^{-x}$. Some of these polynomials are listed in Table 6.8.1 and several Laguerre polynomials are shown in Figure 6.8.1.

Note

The associated Laguerre polynomials are named after the French mathematician Edmond Laguerre (1834-1886). In most derivation in quantum mechanics $a = \frac{a_0}{2}$. where $a_0 = \frac{4\pi\epsilon_0\hbar^2}{me^2}$ is the Bohr radius and $a_0 = 5.2917 \times 10^{-11}$ m.

Comparing Equation (6.8.19) with Equation (6.8.20), we find that $y(x) = L_{n-1}^1(x)$. In summary, we have made the following transformations:

1. $R(\rho) = u(r), \rho = ar$.
2. $u(r) = v(r)e^{-\sqrt{\epsilon}r}$.
3. $v(r) = y(x) = L_{n-1}^1(x), x = 2\sqrt{\epsilon}r$.

Table 6.8.1: Associated Laguerre Functions, $L_n^m(x)$. Note that $L_n^0(x) = L_n(x)$.

	$L_n^m(x)$
$L_0^0(x)$	1
$L_1^0(x)$	$1-x$
$L_2^0(x)$	$\frac{1}{2}\left(x^2-4x+2\right)$
$L_3^0(x)$	$\frac{1}{6}\left(-x^3+9x^2-18x+6\right)$
$L_0^1(x)$	1
$L_1^1(x)$	$2-x$
$L_2^1(x)$	$\frac{1}{2}\left(x^2-6x+6\right)$
$L_3^1(x)$	$\frac{1}{6}\left(-x^3+3x^2-36x+24\right)$

	$L_n^m(x)$
$L_0^2(x)$	1
$L_1^2(x)$	$3-x$
$L_2^2(x)$	$\frac{1}{2}\left(x^2-8x+12\right)$
$L_3^2(x)$	$\frac{1}{12}\left(-2x^3+30x^2-120x+120\right)$

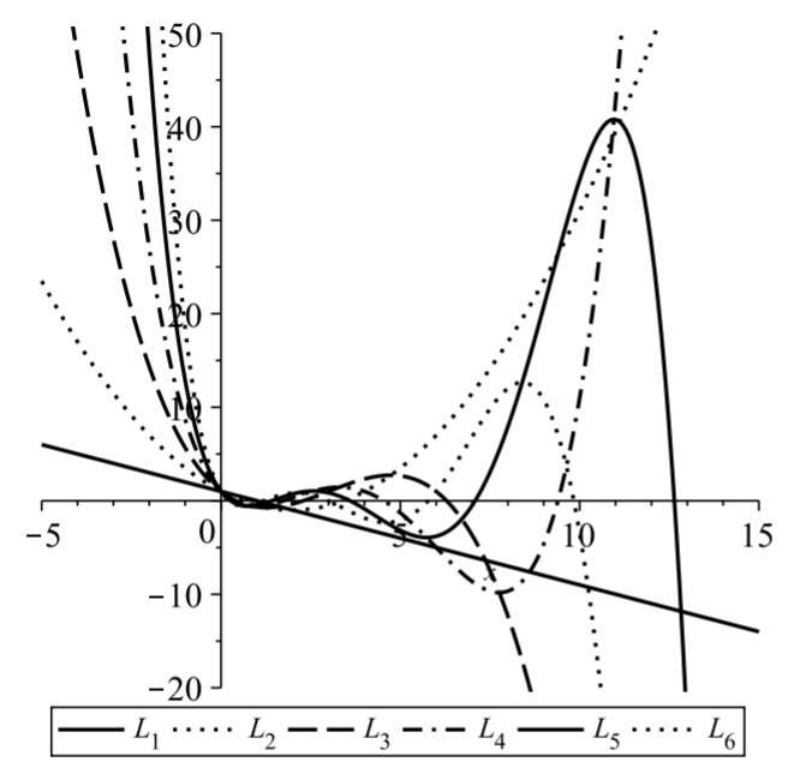


Figure 6.8.1: Plots of the first few Laguerre polynomials.

Therefore,

$$R(\rho)=e^{-\sqrt{\epsilon}\rho/a}L_{n-1}^1(2\sqrt{\epsilon}\rho/a).$$

However, we also found that $2\sqrt{\epsilon}=1/n$. So,

$$R(\rho)=e^{-\rho/2na}L_{n-1}^1(\rho/na).$$

In Figure $6.8.2$ we show a few of these solutions.

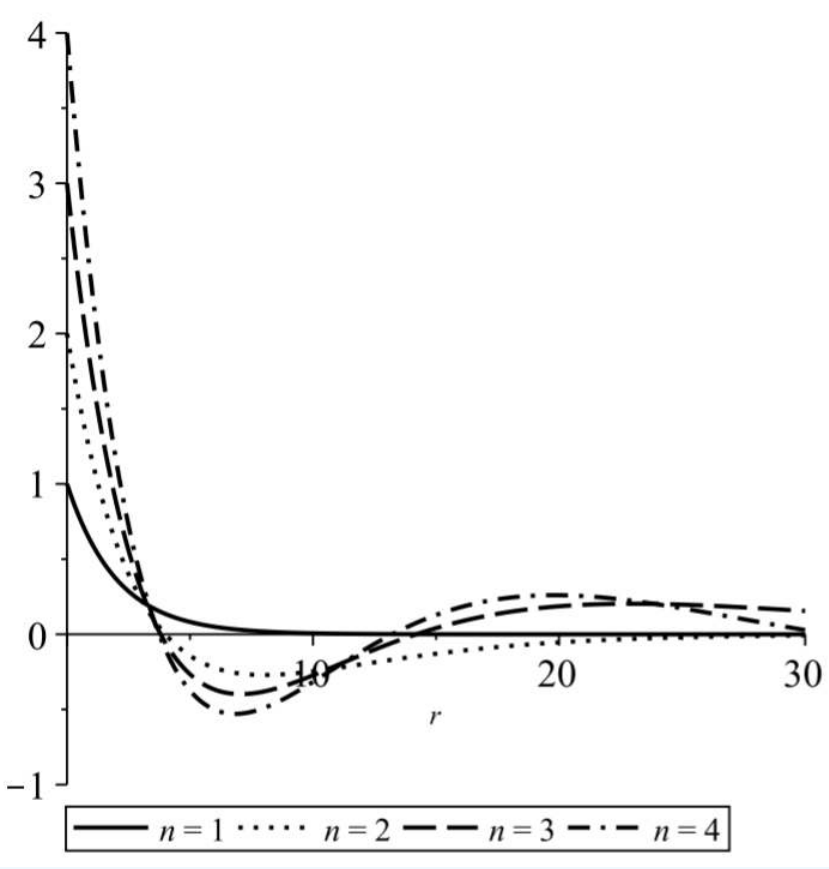


Figure 6.8.2: Plots of $R(\rho)$ for $a=1$ and $n=1,2,3,4$ for the $\ell=0$ states.

Example 6.8.2

Find the $\ell \geq 0$ solutions of the radial equation.

Solution

For the general case, for all $\ell \geq 0$, we need to solve the differential equation

$$u'' + \frac{2}{r}u' + \frac{1}{r}u - \frac{\ell(\ell+1)}{r^2}u = \epsilon u. \tag{6.8.21}$$

Instead of letting $u(r) = v(r)e^{-\sqrt{\epsilon}r}$, we let

$$u(r) = v(r)r^{\ell}e^{-\sqrt{\epsilon}r}.$$

This lead to the differential equation

$$rv'' + 2(\ell + 1 - \sqrt{\epsilon}r)v' + (1 - 2(\ell+1)\sqrt{\epsilon})v = 0. \tag{6.8.22}$$

as before, we let $x = 2\sqrt{\epsilon}$ r to obtain

$$xy'' + 2\left[\ell + 1 - \frac{x}{2}\right]v' + \left[\frac{1}{2\sqrt{\epsilon}} - \ell(\ell+1)\right]v = 0.$$

Noting that $2\sqrt{\epsilon} = 1/n$, we have

$$xy'' + 2[2(\ell+1) - x]v' + (n - \ell(\ell+1))v = 0$$

We see that this is once again in the form of the associate Laguerre equation and the solutions are

$$y(x) = L^{2\ell+1}_{n-\ell-1}(x).$$

So, the solution to the radial equation for the hydrogen atom is given by

$$\begin{aligned} R(\rho) &= r^{\ell} e^{-\sqrt{\epsilon} r} L_{n-\ell-1}^{2\ell+1}(2\sqrt{\epsilon} r) \\ &= \left(\frac{\rho}{2na}\right)^{\ell} e^{-\rho/2na} L_{n-\ell-1}^{2\ell+1}\left(\frac{\rho}{na}\right). \end{aligned} \tag{6.8.23}$$

Interpretations of these solutions will be left for your quantum mechanics course.

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6.9: Curvilinear Coordinates

In order to study solutions of the wave equation, the heat equation, or even Schrödinger's equation in different geometries, we need to see how differential operators, such as the Laplacian, appear in these geometries. The most common coordinate systems arising in physics are polar coordinates, cylindrical coordinates, and spherical coordinates. These reflect the common geometrical symmetries often encountered in physics.

In such systems it is easier to describe boundary conditions and to make use of these symmetries. For example, specifying that the electric potential is 10.0 V on a spherical surface of radius one, we would say $\phi(x,y,z) = 10$ for $x^2 + y^2 + z^2 = 1$. However, if we use spherical coordinates, (r, θ, ϕ), then we would say $\phi(r, \theta, \phi) = 10$ for $r = 1$, or $\phi(1, \theta, \phi) = 10$. This is a much simpler representation of the boundary condition.

However, this simplicity in boundary conditions leads to a more complicated looking partial differential equation in spherical coordinates. In this section we will consider general coordinate systems and how the differential operators are written in the new coordinate systems. This is a more general approach than that taken earlier in the chapter. For a more modern and elegant approach, one can use differential forms.

We begin by introducing the general coordinate transformations between Cartesian coordinates and the more general curvilinear coordinates. Let the Cartesian coordinates be designated by (x_1, x_2, x_3) and the new coordinates by (u_1, u_2, u_3). We will assume that these are related through the transformations

$$\begin{aligned} x_1 &= x_1(u_1, u_2, u_3) \\ x_2 &= x_2(u_1, u_2, u_3) \\ x_3 &= x_3(u_1, u_2, u_3) \end{aligned} \tag{6.9.1}$$

Thus, given the curvilinear coordinates (u_1, u_2, u_3) for a specific point in space, we can determine the Cartesian coordinates, (x_1, x_2, x_3), of that point. We will assume that we can invert this transformation: Given the Cartesian coordinates, one can determine the corresponding curvilinear coordinates.

In the Cartesian system we can assign an orthogonal basis, $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$. As a particle traces out a path in space, one locates its position by the coordinates (x_1, x_2, x_3). Picking x_2 and x_3 constant, the particle lies on the curve $x_1 =$ value of the x_1 coordinate. This line lies in the direction of the basis vector $\mathbf{i}$. We can do the same with the other coordinates and essentially map out a grid in three dimensional space as sown in Figure $6.9.1$. All of the x_i- curves intersect at each point orthogonally and the basis vectors $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ lie along the grid lines and are mutually orthogonal. We would like to mimic this construction for general curvilinear coordinates. Requiring the orthogonality of the resulting basis vectors leads to orthogonal curvilinear coordinates.

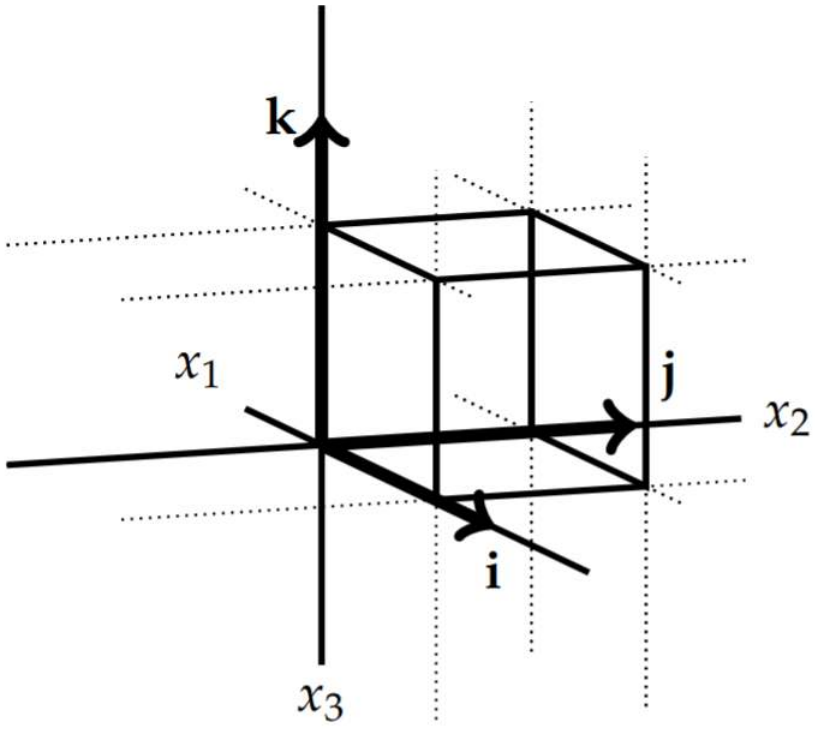


Figure 6.9.1: Plots of x_i-curves forming an orthogonal Cartesian grid.

As for the Cartesian case, we consider u_2 and u_3 constant. This leads to a curve parametrized by $u_1 : \mathbf{r} = x_1(u_1)\mathbf{i} + x_2(u_1)\mathbf{j} + x_3(u_1)\mathbf{k}$. We call this the u_1-curve. Similarly, when u_1 and u_3 are constant we obtain a u_2-curve and for u_1 and u_2 constant we obtain a u_3-curve. We will assume that these curves intersect such that each pair of curves intersect orthogonally as seen in Figure 6.9.2. Furthermore, we will assume that the unit tangent vectors to these curves form a right handed system similar to the $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ systems for Cartesian coordinates. We will denote these as $\{\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2, \hat{\mathbf{u}}_3\}$.

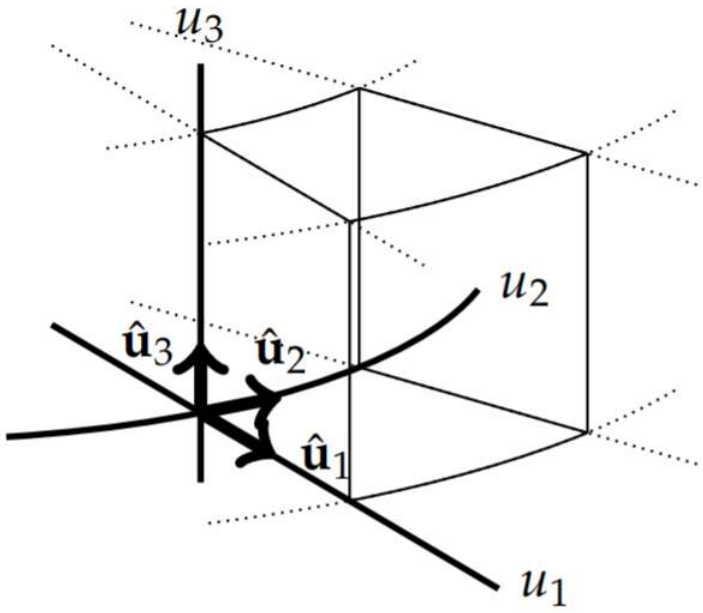


Figure 6.9.2: Plots of general u_i-curves forming an orthogonal grid.

We can determine these tangent vectors from the coordinate transformations. Consider the position vector as a function of the new coordinates,

$$\mathbf{r}(u_1,u_2,u_3)=x_1(u_1,u_2,u_3)\,\mathbf{i}+x_2(u_1,u_2,u_3)\,\mathbf{j}+x_3(u_1,u_2,u_3)\,\mathbf{k}.$$

Then, the infinitesimal change in position is given by

$$d\mathbf{r}=\frac{\partial \mathbf{r}}{\partial u_1}du_1+\frac{\partial \mathbf{r}}{\partial u_2}du_2+\frac{\partial \mathbf{r}}{\partial u_3}du_3=\sum_{i=1}^{3}\frac{\partial \mathbf{r}}{\partial u_i}du_i.$$

We note that the vectors $\frac{\partial \mathbf{r}}{\partial u_i}$ are tangent to the u_i-curves. Thus, we define the unit tangent vectors

$$\hat{\mathbf{u}}_i=\frac{\frac{\partial \mathbf{r}}{\partial u_i}}{\left|\frac{\partial \mathbf{r}}{\partial u_i}\right|}.$$

Solving for the original tangent vector, we have

$$\frac{\partial \mathbf{r}}{\partial u_i}=h_i\hat{\mathbf{u}}_i,$$

where

$$h_i\equiv\left|\frac{\partial \mathbf{r}}{\partial u_i}\right|.$$

The h_i 's are called the scale factors for the transformation. The infinitesimal change in position in the new basis is then given by

$$d\mathbf{r}=\sum_{i=1}^{3}h_iu_i\hat{\mathbf{u}}_i.$$

Note

The scale factors, $h_i\equiv\left|\frac{\partial \mathbf{r}}{\partial u_i}\right|$.

Example 6.9.1

Determine the scale factors for the polar coordinate transformation.

Solution

The transformation for polar coordinates is

$$x=r\cos\theta,\quad y=r\sin\theta.$$

Here we note that $x_1=x, x_2=y, u_1=r$, and $u_2=\theta$. The u_1-curves are curves with $\theta=$ const. Thus, these curves are radial lines. Similarly, the u_2-curves have $r=$ const. These curves are concentric circles about the origin as shown in Figure 6.9.3.

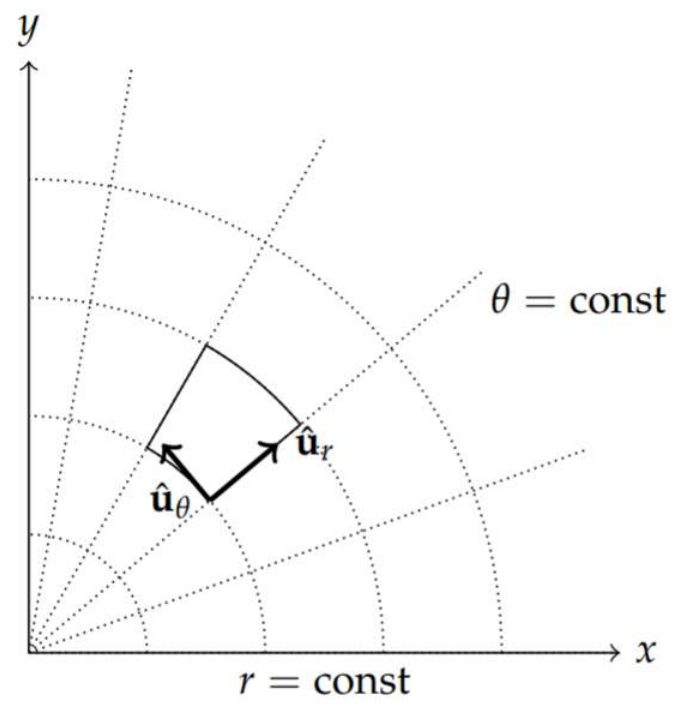


Figure 6.9.3: Plots an orthogonal polar grid.

The unit vectors are easily found. We will denote them by $\hat{\mathbf{u}}_r$ and $\hat{\mathbf{u}}_\theta$. We can determine these unit vectors by first computing $\frac{\partial \mathbf{r}}{\partial u_i}$. Let

$$\mathbf{r} = x(r,\theta)\mathbf{i} + y(r,\theta)\mathbf{j} = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j}.$$

Then,

$$\begin{aligned} \frac{\partial \mathbf{r}}{\partial r} &= \cos\theta\mathbf{i} + \sin\theta\mathbf{j} \\ \frac{\partial \mathbf{r}}{\partial \theta} &= -r\sin\theta\mathbf{i} + r\cos\theta\mathbf{j}. \end{aligned} \tag{6.9.2}$$

The first vector already is a unit vector. So,

$$\hat{\mathbf{u}}_r = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}.$$

The second vector has length r since $|-r\sin\theta\mathbf{i} + r\cos\theta\mathbf{j}| = r$. Dividing $\frac{\partial \mathbf{r}}{\partial \theta}$ by r, we have

$$\hat{\mathbf{u}}_\theta = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}.$$

We can see these vectors are orthogonal $(\hat{\mathbf{u}}_r \cdot \hat{\mathbf{u}}_\theta = 0)$ and form a right hand system. That they form a right hand system can be seen by either drawing the vectors, or computing the cross product,

$$\begin{aligned} (\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) \times (-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}) &= \cos^2\theta\mathbf{i}\times\mathbf{j} - \sin^2\theta\mathbf{j}\times\mathbf{i} \\ &= \mathbf{k}. \end{aligned} \tag{6.9.3}$$

Since

$$\begin{aligned} \frac{\partial \mathbf{r}}{\partial r} &= \hat{\mathbf{u}}_r, \\ \frac{\partial \mathbf{r}}{\partial \theta} &= r\hat{\mathbf{u}}_\theta, \end{aligned}$$

The scale factors are $h_r = 1$ and $h_\theta = r$.

Once we know the scale factors, we have that

$$d\mathbf{r} = \sum_{i=1}^{3} h_i du_i \hat{\mathbf{u}}_i.$$

The infinitesimal arclength is then given by the Euclidean line element

$$ds^2 = d\mathbf{r} \cdot d\mathbf{r} = \sum_{i=1}^{3} h_i^2 du_i^2$$

when the system is orthogonal. The h_i^2 are referred to as the metric coefficients.

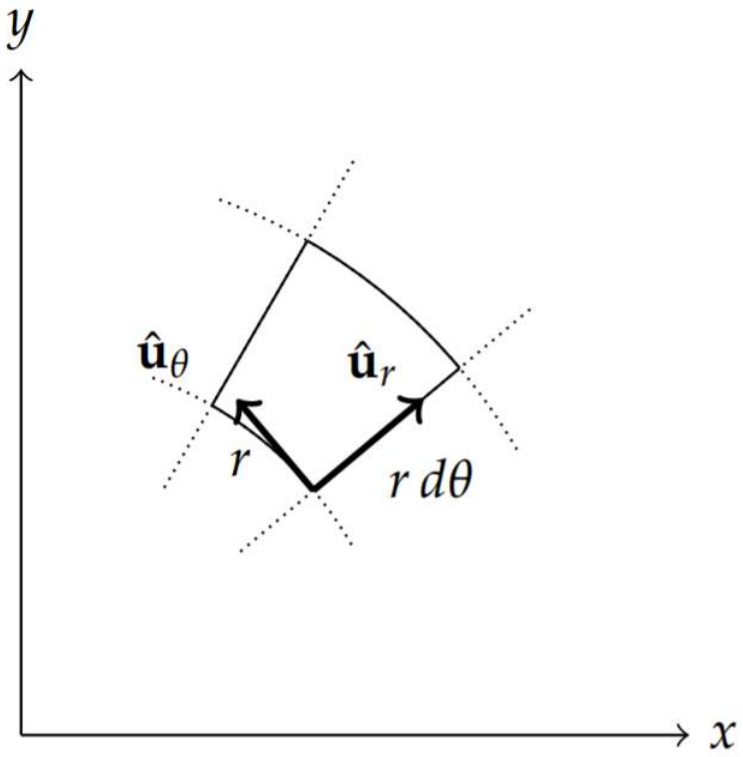


Figure 6.9.4: Infinitesimal area in polar coordinates.

Example 6.9.2

Verify that $d\mathbf{r} = dr\hat{u}_r + rd\theta\hat{\mathbf{u}}_\theta$ directly from $\mathbf{r} = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j}$ and obtain the Euclidean line element for polar coordinates.

Solution

We begin by computing

$$\begin{aligned} d\mathbf{r} &= d(r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j}) \\ &= (\cos\theta\mathbf{i} + \sin\theta\mathbf{j})dr + r(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j})d\theta \\ &= dr\hat{\mathbf{u}}_r + rd\theta\hat{\mathbf{u}}_\theta. \end{aligned} \tag{6.9.4}$$

This agrees with the form $d\mathbf{r} = \sum_{i=1}^{3} h_i du_i \hat{\mathbf{u}}_i$ when the scale factors for polar coordinates are inserted.

The line element is found as

$$\begin{aligned} ds^2 &= d\mathbf{r} \cdot d\mathbf{r} \\ &= (dr\hat{\mathbf{u}}_r + rd\theta\hat{\mathbf{u}}_\theta) \cdot (dr\hat{\mathbf{u}}_r + rd\theta\hat{\mathbf{u}}_\theta) \\ &= dr^2 + r^2 d\theta^2. \end{aligned} \tag{6.9.5}$$

This is the Euclidean line element in polar coordinates.

Also, along the u_i-curves,

$$d\mathbf{r} = h_i du_i \hat{\mathbf{u}}_i, \quad \text{(no summation)}.$$

This can be seen in Figure 6.9.5 by focusing on the u_1 curve. Along this curve, u_2 and u_3 are constant. So, $du_2 = 0$ and $du_3 = 0$. This leaves $d\mathbf{r} = h_1 du_1 \hat{\mathbf{u}}_1$ along the u_1-curve. Similar expressions hold along the other two curves.

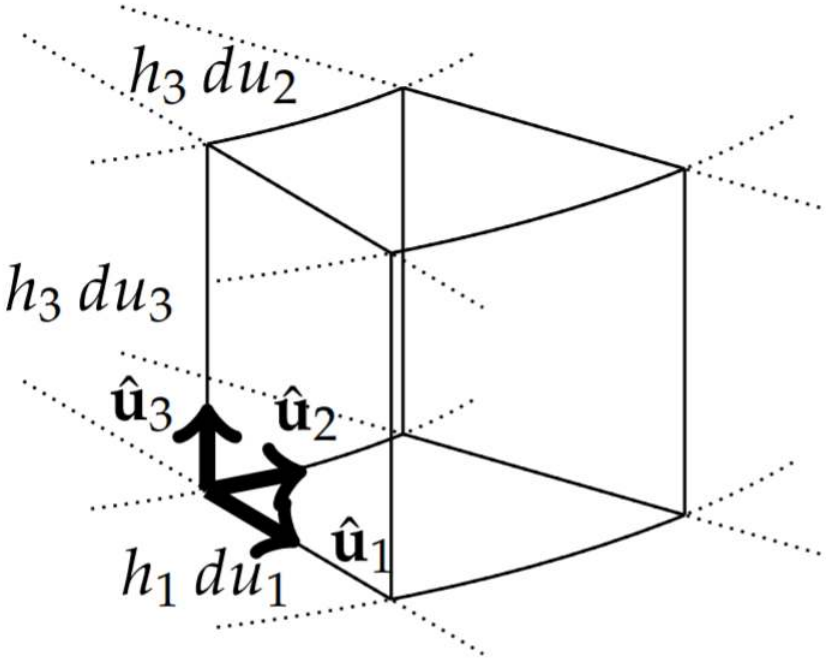


Figure 6.9.5: Infinitesimal volume element with sides of length $h_i du_i$.

We can use this result to investigate infinitesimal volume elements for general coordinate systems as shown in Figure 6.9.5. At a given point (u_1, u_2, u_3) we can construct an infinitesimal parallelepiped of sides $h_i du_i$, $i = 1, 2, 3$. This infinitesimal parallelepiped has a volume of size

$$dV = \left| \frac{\partial \mathbf{r}}{\partial u_1} \cdot \frac{\partial \mathbf{r}}{\partial u_2} \times \frac{\partial \mathbf{r}}{\partial u_3} \right| du_1 du_2 du_3.$$

The triple scalar product can be computed using determinants and the resulting determinant is call the Jacobian, and is given by

$$\begin{aligned} J &= \left| \frac{\partial (x_1, x_2, x_3)}{\partial (u_1, u_2, u_3)} \right| \\ &= \left| \frac{\partial \mathbf{r}}{\partial u_1} \cdot \frac{\partial \mathbf{r}}{\partial u_2} \times \frac{\partial \mathbf{r}}{\partial u_3} \right| \\ &= \begin{vmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_2}{\partial u_1} & \frac{\partial x_3}{\partial u_1} \\ \frac{\partial x_1}{\partial u_2} & \frac{\partial x_2}{\partial u_2} & \frac{\partial x_3}{\partial u_2} \\ \frac{\partial x_1}{\partial u_3} & \frac{\partial x_2}{\partial u_3} & \frac{\partial x_3}{\partial u_3} \end{vmatrix}. \end{aligned} \tag{6.9.6}$$

Therefore, the volume element can be written as

$$dV = J du_1 du_2 du_3 = \left| \frac{\partial (x_1, x_2, x_3)}{\partial (u_1, u_2, u_3)} \right| du_1 du_2 du_3.$$

Example 6.9.3

Determine the volume element for cylindrical coordinates (r, θ, z), given by

$$x = r\cos\theta, \tag{6.9.7}$$
$$y = r\sin\theta, \tag{6.9.8}$$
$$z = z. \tag{6.9.9}$$

Solution

Here, we have $(u_1, u_2, u_3) = (r, \theta, z)$ as displayed in Figure 6.9.6. Then, the Jacobian is given by

$$\begin{aligned} J &= \left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| \\ &= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial z} & \frac{\partial y}{\partial z} & \frac{\partial z}{\partial z} \end{vmatrix} \\ &= \begin{vmatrix} \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} \\ &= r \end{aligned} \tag{6.9.10}$$

Thus, the volume element is given as

$$dV = rdrd\theta dz.$$

This result should be familiar from multivariate calculus.

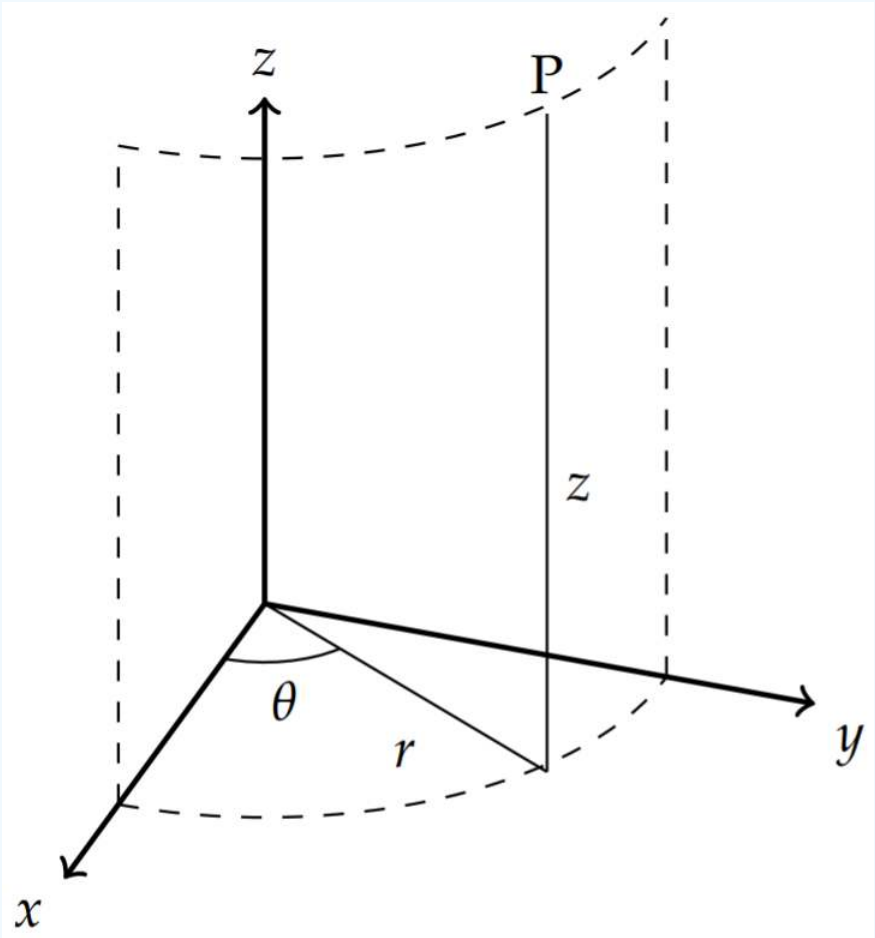


Figure 6.9.6: Cylindrical coordinate system.

Another approach is to consider the geometry of the infinitesimal volume element. The directed edge lengths are given by $d\mathbf{s}_i = h_i du_i \hat{\mathbf{u}}_i$ as seen in Figure 6.9.2. The infinitesimal area element of for the face in direction $\hat{\mathbf{u}}_k$ is found from a simple cross product,

$$d\mathbf{A}_k = d\mathbf{s}_i \times d\mathbf{s}_j = h_i h_j du_i du_j \hat{\mathbf{u}}_i \times \hat{\mathbf{u}}_j.$$

Since these are unit vectors, the areas of the faces of the infinitesimal volumes are $dA_k = h_i h_j du_i du_j$.

The infinitesimal volume is then obtained as

$$dV = |d\mathbf{s}_k \cdot d\mathbf{A}_k| = h_i h_j h_k du_i du_j du_k \left|\hat{\mathbf{u}}_i \cdot (\hat{\mathbf{u}}_k \times \hat{\mathbf{u}}_j)\right|.$$

Thus, $dV = h_1 h_2 h_3 du_1 du_1 du_3$. Of course, this should not be a surprise since

$$J = \left| \frac{\partial \mathbf{r}}{\partial u_1} \cdot \frac{\partial \mathbf{r}}{\partial u_2} \times \frac{\partial \mathbf{r}}{\partial u_3} \right| = |h_1 \hat{\mathbf{u}}_1 \cdot h_2 \hat{\mathbf{u}}_2 \times h_3 \hat{\mathbf{u}}_3| = h_1 h_2 h_3.$$

Example 6.9.4

For polar coordinates, determine the infinitesimal area element.

Solution

In an earlier example, we found the scale factors for polar coordinates as $h_r = 1$ and $h_\theta = r$. Thus, $dA = h_r h_\theta dr d\theta = r dr d\theta$. Also, the last example for cylindrical coordinates will yield similar results if we already know the scales factors without having to compute the Jacobian directly. Furthermore, the area element perpendicular to the z-coordinate gives the polar coordinate system result.

Next we will derive the forms of the gradient, divergence, and curl in curvilinear coordinates using several of the identities in section ??. The results are given here for quick reference.

Note

Gradient, divergence and curl in orthogonal curvilinear coordinates.

$$\begin{aligned}
\nabla\phi &= \sum_{i=1}^{3} \frac{\hat{u}_i}{h_i} \frac{\partial \phi}{\partial u_i} \\
&= \frac{\hat{u}_1}{h_1} \frac{\partial \phi}{\partial u_1} + \frac{\hat{u}_2}{h_2} \frac{\partial \phi}{\partial u_2} + \frac{\hat{u}_3}{h_3} \frac{\partial \phi}{\partial u_3}. && (6.9.11) \\
\nabla \cdot \mathbf{F} &= \frac{1}{h_1 h_2 h_3} \left(\frac{\partial}{\partial u_1}(h_2 h_3 F_1) + \frac{\partial}{\partial u_2}(h_1 h_3 F_2) + \frac{\partial}{\partial u_3}(h_1 h_2 F_3) \right) && (6.9.12) \\
\nabla \times \mathbf{F} &= \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{u}_1 & h_2 \hat{u}_2 & h_3 \hat{u}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ F_1 h_1 & F_2 h_2 & F_3 h_3 \end{vmatrix}. && (6.9.13) \\
\nabla^2 \phi &= \frac{1}{h_1 h_2 h_3} \left(\frac{\partial}{\partial u_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial \phi}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left(\frac{h_1 h_3}{h_2} \frac{\partial \phi}{\partial u_2} \right) \right. && (6.9.14) \\
&\left. + \frac{\partial}{\partial u_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial \phi}{\partial u_3} \right) \right) && (6.9.15)
\end{aligned}$$

Note

Derivation of the gradient form.

We begin the derivations of these formulae by looking at the gradient, $\nabla\phi$, of the scalar function $\phi(u_1, u_2, u_3)$. We recall that the gradient operator appears in the differential change of a scalar function,

$$d\phi = \nabla\phi \cdot d\mathbf{r} = \sum_{i=1}^{3} \frac{\partial \phi}{\partial u_i} du_i.$$

Since

$$d\mathbf{r} = \sum_{i=1}^{3} h_i du_i \hat{\mathbf{u}}_i \tag{6.9.16}$$

we also have that

$$d\phi = \nabla\phi \cdot d\mathbf{r} = \sum_{i=1}^{3} (\nabla\phi)_i h_i du_i.$$

Comparing these two expressions for $d\phi$, we determine that the components of the del operator can be written as

$$(\nabla\phi)_i = \frac{1}{h_i} \frac{\partial \phi}{\partial u_i}$$

and thus the gradient is given by

$$\nabla\phi = \frac{\hat{\mathrm{u}}_1}{h_1} \frac{\partial \phi}{\partial u_1} + \frac{\hat{\mathrm{u}}_2}{h_2} \frac{\partial \phi}{\partial u_2} + \frac{\hat{\mathrm{u}}_3}{h_3} \frac{\partial \phi}{\partial u_3}. \tag{6.9.17}$$

Note

Derivation of the divergence form.

Next we compute the divergence,

$$\nabla \cdot \mathbf{F} = \sum_{i=1}^{3} \nabla \cdot \left(F_i \hat{\mathbf{u}}_i\right).$$

We can do this by computing the individual terms in the sum. We will compute $\nabla \cdot \left(F_1 \hat{\mathbf{u}}_1\right)$.

Using Equation (6.9.17), we have that

$$\nabla u_i = \frac{\hat{\mathbf{u}}_i}{h_i}.$$

Then

$$\nabla u_2 \times \nabla u_3 = \frac{\hat{\mathbf{u}}_2 \times \hat{\mathbf{u}}_3}{h_2 h_3} = \frac{\hat{\mathbf{u}}_1}{h_2 h_3}$$

Solving for $\hat{\mathbf{u}}_1$, gives

$$\hat{\mathbf{u}}_1 = h_2 h_3 \nabla u_2 \times \nabla u_3.$$

Inserting this result into $\nabla \cdot \left(F_1 \hat{\mathbf{u}}_1\right)$ and using the vector identity $2c$ from section ??,

$$\nabla \cdot (f\mathbf{A}) = f \nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla f,$$

we have

$$\begin{aligned} \nabla \cdot \left(F_1 \hat{\mathbf{u}}_1\right) &= \nabla \cdot \left(F_1 h_2 h_3 \nabla u_2 \times \nabla u_3\right) \\ &= \nabla \left(F_1 h_2 h_3\right) \cdot \nabla u_2 \times \nabla u_3 + F_1 h_2 h_3 \nabla \cdot \left(\nabla u_2 \times \nabla u_3\right) \end{aligned} \tag{6.9.18}$$

The second term of this result vanishes by vector identity $3c$,

$$\nabla \cdot (\nabla f \times \nabla g) = 0.$$

Since $\nabla u_2 \times \nabla u_3 = \frac{\hat{1}_1}{h_2 h_3}$, the first term can be evaluated as

$$\nabla \cdot (F_1 \hat{\mathbf{u}}_1) = \nabla (F_1 h_2 h_3) \cdot \frac{\hat{\mathbf{u}}_1}{h_2 h_3} = \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial u_1} (F_1 h_2 h_3) .$$

Similar computations can be carried out for the remaining components, leading to the sought expression for the divergence in curvilinear coordinates:

$$\nabla \cdot \mathbf{F} = \frac{1}{h_1 h_2 h_3} \left(\frac{\partial}{\partial u_1} (h_2 h_3 F_1) + \frac{\partial}{\partial u_2} (h_1 h_3 F_2) + \frac{\partial}{\partial u_3} (h_1 h_2 F_3) \right) . \tag{6.9.19}$$

Example 6.9.5

Write the divergence operator in cylindrical coordinates.

Solution

In this case we have

$$\begin{aligned} \nabla \cdot \mathbf{F} &= \frac{1}{h_r h_\theta h_z} \left(\frac{\partial}{\partial r} (h_\theta h_z F_r) + \frac{\partial}{\partial \theta} (h_r h_z F_\theta) + \frac{\partial}{\partial \theta} (h_r h_\theta F_z) \right) \\ &= \frac{1}{r} \left(\frac{\partial}{\partial r} (r F_r) + \frac{\partial}{\partial \theta} (F_\theta) + \frac{\partial}{\partial \theta} (r F_z) \right) \\ &= \frac{1}{r} \frac{\partial}{\partial r} (r F_r) + \frac{1}{r} \frac{\partial}{\partial \theta} (F_\theta) + \frac{\partial}{\partial \theta} (F_z) \end{aligned} \tag{6.9.20}$$

We now turn to the curl operator. In this case, we need to evaluate

$$\nabla \times \mathbf{F} = \sum_{i=1}^{3} \nabla \times (F_i \hat{\mathbf{u}}_i)$$

Again we focus on one term, $\nabla \times (F_1 \hat{\mathbf{u}}_1)$. Using the vector identity 2e,

$$\nabla \times (f \mathbf{A}) = f \nabla \times \mathbf{A} - \mathbf{A} \times \nabla f,$$

we have

$$\begin{aligned} \nabla \times (F_1 \hat{\mathbf{u}}_1) &= \nabla \times (F_1 h_1 \nabla u_1) \\ &= F_1 h_1 \nabla \times \nabla u_1 - \nabla (F_1 h_1) \times \nabla u_1 \end{aligned} \tag{6.9.21}$$

The curl of the gradient vanishes, leaving

$$\nabla \times (F_1 \hat{\mathbf{u}}_1) = \nabla (F_1 h_1) \times \nabla u_1 .$$

Since $\nabla u_1 = \frac{\hat{u}_1}{h_1}$, we have

$$\begin{aligned} \nabla \times (F_1 \hat{\mathbf{u}}_1) &= \nabla (F_1 h_1) \times \frac{\hat{\mathbf{u}}_1}{h_1} \\ &= \left(\sum_{i=1}^{3} \frac{\hat{u}_i}{h_i} \frac{\partial (F_1 h_1)}{\partial u_i} \right) \times \frac{\hat{\mathbf{u}}_1}{h_1} \\ &= \frac{\hat{\mathbf{u}}_2}{h_3 h_1} \frac{\partial (F_1 h_1)}{\partial u_3} - \frac{\hat{\mathbf{u}}_3}{h_1 h_2} \frac{\partial (F_1 h_1)}{\partial u_2} . \end{aligned} \tag{6.9.22}$$

The other terms can be handled in a similar manner. The overall result is that

$$\begin{aligned} \nabla \times \mathbf{F} = \frac{\hat{\mathbf{u}}_1}{h_2 h_3} \left(\frac{\partial (h_3 F_3)}{\partial u_2} - \frac{\partial (h_2 F_2)}{\partial u_3} \right) + \frac{\hat{\mathbf{u}}_2}{h_1 h_3} \left(\frac{\partial (h_1 F_1)}{\partial u_3} - \frac{\partial (h_3 F_3)}{\partial u_1} \right) \\ + \frac{\hat{\mathbf{u}}_3}{h_1 h_2} \left(\frac{\partial (h_2 F_2)}{\partial u_1} - \frac{\partial (h_1 F_1)}{\partial u_2} \right) \end{aligned} \tag{6.9.23}$$

This can be written more compactly as

$$\nabla \times \mathbf{F} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{\mathbf{u}}_1 & h_2 \hat{\mathbf{u}}_2 & h_3 \hat{\mathbf{u}}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ F_1 h_1 & F_2 h_2 & F_3 h_3 \end{vmatrix} \tag{6.9.24}$$

Example 6.9.6

Write the curl operator in cylindrical coordinates.

Solution

$$\begin{aligned} \nabla \times \mathbf{F} &= \frac{1}{r} \begin{vmatrix} \hat{\mathrm{e}}_r & r\hat{\mathrm{e}}_\theta & \hat{\mathrm{e}}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ F_r & rF_\theta & F_z \end{vmatrix} \\ &= \left(\frac{1}{r} \frac{\partial F_z}{\partial \theta} - \frac{\partial F_\theta}{\partial z} \right) \hat{\mathrm{e}}_r + \left(\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r} \right) \hat{\mathrm{e}}_\theta \\ &\quad + \frac{1}{r} \left(\frac{\partial (rF_\theta)}{\partial r} - \frac{\partial F_r}{\partial \theta} \right) \hat{\mathrm{e}}_z. \end{aligned} \tag{6.9.25}$$

Finally, we turn to the Laplacian. In the next chapter we will solve higher dimensional problems in various geometric settings such as the wave equation, the heat equation, and Laplace's equation. These all involve knowing how to write the Laplacian in different coordinate systems. Since $\nabla^2 \phi = \nabla \cdot \nabla \phi$, we need only combine the results from Equations (6.9.17) and (6.9.19) for the gradient and the divergence in curvilinear coordinates. This is straight forward and gives

$$\begin{aligned} \nabla^2 \phi = \frac{1}{h_1 h_2 h_3} &\left(\frac{\partial}{\partial u_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial \phi}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left(\frac{h_1 h_3}{h_2} \frac{\partial \phi}{\partial u_2} \right) \right. \\ &\left. + \frac{\partial}{\partial u_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial \phi}{\partial u_3} \right) \right). \end{aligned} \tag{6.9.26}$$

The results of rewriting the standard differential operators in cylindrical and spherical coordinates are shown in Problems ?? and ??. In particular, the Laplacians are given as

Definition 6.9.1: Cylindrical Coordinates

$$\nabla^2 f = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}. \tag{6.9.27}$$

Definition 6.9.2: Spherical Coordinates

$$\nabla^2 f = \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left(\rho^2 \frac{\partial f}{\partial \rho} \right) + \frac{1}{\rho^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{\rho^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2}. \tag{6.9.28}$$

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6.10: Problems

Exercise 6.10.1

A rectangular plate $0 \leq x \leq L\,, 0 \leq y \leq H$ with heat diffusivity constant k is insulated on the edges $y = 0, H$ and is kept at constant zero temperature on the other two edges. Assuming an initial temperature of $u(x, y, 0) = f(x, y)$, use separation of variables t find the general solution.

Exercise 6.10.2

Solve the following problem.

$$u_{xx} + u_{yy} + u_{zz} = 0, \quad 0 < x < 2\pi, \quad 0 < y < \pi, \quad 0 < z < 1,$$
$$u(x, y, 0) = \sin x \sin y, \quad u(x, y, z) = 0 \text{ on other faces.}$$

Exercise 6.10.3

Consider Laplace's equation on the unit square, $u_{xx} + u_{yy} = 0, 0 \leq x, y \leq 1$. Let $u(0, y) = 0, u(1, y) = 0$ for $0 < y < 1$ and $u_y(x, 0) = 0$ for $0 < y < 1$. Carry out the needed separation of variables and write down the product solutions satisfying these boundary conditions.

Exercise 6.10.4

Consider a cylinder of height H and radius a.

a. Write down Laplace's Equation for this cylinder in cylindrical coordinates.
b. Carry out the separation of variables and obtain the three ordinary differential equations that result from this problem.
c. What kind of boundary conditions could be satisfied in this problem in the independent variables?

Exercise 6.10.5

Consider a square drum of side s and a circular drum of radius a.

a. Rank the modes corresponding to the first 6 frequencies for each.
b. Write each frequency (in Hz) in terms of the fundamental (i.e., the lowest frequency.)
c. What would the lengths of the sides of the square drum have to be to have the same fundamental frequency? (Assume that $c = 1.0$ for each one.)

Exercise 6.10.6

We presented the full solution of the vibrating rectangular membrane in Equation 6.1.26. Finish the solution to the vibrating circular membrane by writing out a similar full solution.

Exercise 6.10.7

A copper cube 10.0 cm on a side is heated to $100^{\circ}\mathrm{C}$. The block is placed on a surface that is kept at $0^{\circ}\mathrm{C}$. The sides of the block are insulated, so the normal derivatives on the sides are zero. Heat flows from the top of the block to the air governed by the gradient $u_z = -10^{\circ}\mathrm{C/m}$. Determine the temperature of the block at its center after 1.o minutes. Note that the thermal diffusivity is given by $k = \frac{K}{\rho c_p}$, where K is the thermal conductivity, ρ is the density, and c_p is the specific heat capacity.

Exercise 6.10.8

Consider a spherical balloon of radius a. Small deformations on the surface can produce waves on the balloon's surface.

a. Write the wave equation in spherical polar coordinates. (Note: ρ is constant!)
b. Carry out a separation of variables and find the product solutions for this problem.
c. Describe the nodal curves for the first six modes.
d. For each mode determine the frequency of oscillation in Hz assuming $c = 1.0\ \mathrm{m/s}$.

Exercise 6.10.9

Consider a circular cylinder of radius $R = 4.00$ cm and height $H = 20.0$ cm which obeys the steady state heat equation

$$u_{rr} + \frac{1}{r}u_r + u_{zz}.$$

Find the temperature distribution, $u(r, z)$, given that $u(r, 0) = 0^\circ\mathrm{C}$, $u(r, 20) = 20^\circ\mathrm{C}$, and heat is lost through the sides due to Newton's Law of Cooling

$$[u_r + hu]_{r=4} = 0,$$

for $h = 1.0\ \mathrm{cm}^{-1}$.

Exercise 6.10.10

The spherical surface of a homogeneous ball of radius one in maintained at zero temperature. It has an initial temperature distribution $u(\rho, 0) = 100^\circ\mathrm{C}$. Assuming a heat diffusivity constant k, find the temperature throughout the sphere, $u(\rho, \theta, \phi, t)$.

Exercise 6.10.11

Determine the steady state temperature of a spherical ball maintained at the temperature

$$u(x, y, z) = x^2 + 2y^2 + 3z^2, \quad \rho = 1.$$

[Hint - Rewrite the problem in spherical coordinates and use the properties of spherical harmonics.]

Exercise 6.10.12

A hot dog initially at temperature $50^\circ\mathrm{C}$ is put into boiling water at $100^\circ\mathrm{C}$. Assume the hot dog is 12.0 cm long, has a radius of 2.00 cm, and the heat constant is $2.0 \times 10^{-5}\ \mathrm{cm}^2/\mathrm{s}$.

a. Find the general solution for the temperature. [Hint: Solve the heat equation for $u(r, z, t) = T(r, z, t) - 100$, where $T(r, z, t)$ is the temperature of the hot dog.]
b. Indicate how one might proceed with the remaining information in order to determine when the hot dog is cooked; i.e., when the center temperature is $80^\circ\mathrm{C}$.

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CHAPTER OVERVIEW

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7.9: Problems

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7.0: Prelude to Green's Functions and Nonhomogeneous Problems

> *"The young theoretical physicists of a generation or two earlier subscribed to the belief that: If you haven't done something important by age 30, you never will. Obviously, they were unfamiliar with the history of George Green, the miller of Nottingham." - Julian Schwinger (1918-1994)*

The wave equation, heat equation, and Laplace's equation are typical homogeneous partial differential equations. They can be written in the form

$$\mathcal{L}u(x) = 0,$$

where $\mathcal{L}$ is a differential operator. For example, these equations can be written as

$$\begin{aligned} \left(\frac{\partial^2}{\partial t^2} - c^2\nabla^2\right) u &= 0, \\ \left(\frac{\partial}{\partial t} - k\nabla^2\right) u &= 0, \\ \nabla^2 u &= 0. \end{aligned} \tag{7.0.1}$$

In this chapter we will explore solutions of nonhomogeneous partial differential equations,

$$\mathcal{L}u(x) = f(x),$$

by seeking out the so-called Green's function. The history of the Green's function dates back to 1828, when George Green published work in which he sought solutions of Poisson's equation $\nabla^2 u = f$ for the electric potential u defined inside a bounded volume with specified boundary conditions on the surface of the volume. He introduced a function now identified as what Riemann later coined the "Green's function". In this chapter we will derive the initial value Green's function for ordinary differential equations. Later in the chapter we will return to boundary value Green's functions and Green's functions for partial differential equations.

Note

George Green (1793-1841), a British mathematical physicist who had little formal education and worked as a miller and a baker, published *An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism* in which he not only introduced what is now known as Green's function, but he also introduced potential theory and Green's Theorem in his studies of electricity and magnetism. Recently his paper was posted at arXiv.org, arXiv:o807.0088.

As a simple example, consider Poisson's equation,

$$\nabla^2 u(\mathbf{r}) = f(\mathbf{r}).$$

Let Poisson's equation hold inside a region Ω bounded by the surface $\partial\Omega$ as shown in Figure 7.0.1. This is the nonhomogeneous form of Laplace's equation. The nonhomogeneous term, $f(\mathbf{r})$, could represent a heat source in a steady-state problem or a charge distribution (source) in an electrostatic problem.

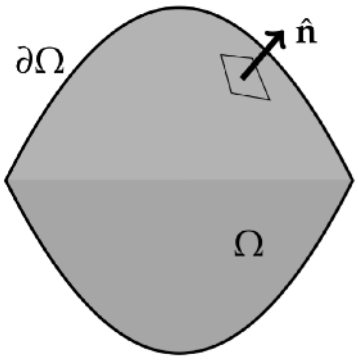


Figure 7.0.1: Let Poisson's equation hold inside region Ω bounded by surface $\partial\Omega$. The Dirac delta function satisfies

$$\delta(\mathbf{r}) = 0, \quad \mathbf{r} \neq \mathbf{0},$$
$$\int_{\Omega} \delta(\mathbf{r}) dV = 1.$$

Now think of the source as a point source in which we are interested in the response of the system to this point source. If the point source is located at a point $\mathbf{r}'$, then the response to the point source could be felt at points r. We will call this response $G(\mathbf{r}, \mathbf{r}')$. The response function would satisfy a point source equation of the form

$$\nabla^2 G(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}').$$

Here $\delta(\mathbf{r} - \mathbf{r}')$ is the Dirac delta function, which we will consider in more detail in Section 9.4. A key property of this generalized function is the sifting property,

$$\int_{\Omega} \delta(\mathbf{r} - \mathbf{r}') f(\mathbf{r}) dV = f(\mathbf{r}').$$

The connection between the Green's function and the solution to Poisson's equation can be found from Green's second identity:

$$\int_{\partial\Omega} [\phi \nabla\psi - \psi \nabla\phi] \cdot \mathbf{n} dS = \int_{\Omega} \left[\phi \nabla^2 \psi - \psi \nabla^2 \phi\right] dV.$$

Letting $\phi = u(\mathbf{r})$ and $\psi = G(\mathbf{r}, \mathbf{r}')$, we have[1]

$$\begin{aligned} &\int_{\partial\Omega} [u(\mathbf{r}) \nabla G(\mathbf{r}, \mathbf{r}') - G(\mathbf{r}, \mathbf{r}') \nabla u(\mathbf{r})] \cdot \mathbf{n} dS \\ &= \int_{\Omega} \left[u(\mathbf{r}) \nabla^2 G(\mathbf{r}, \mathbf{r}') - G(\mathbf{r}, \mathbf{r}') \nabla^2 u(\mathbf{r})\right] dV \\ &= \int_{\Omega} [u(\mathbf{r}) \delta(\mathbf{r} - \mathbf{r}') - G(\mathbf{r}, \mathbf{r}') f(\mathbf{r})] dV \\ &= u(\mathbf{r}') - \int_{\Omega} G(\mathbf{r}, \mathbf{r}') f(\mathbf{r}) dV. \end{aligned} \tag{7.0.2}$$

Solving for $u(\mathbf{r}')$, we have

$$\begin{aligned} u(\mathbf{r}') = &\int_{\Omega} G(\mathbf{r}, \mathbf{r}') f(\mathbf{r}) dV \\ &+ \int_{\partial\Omega} [u(\mathbf{r}) \nabla G(\mathbf{r}, \mathbf{r}') - G(\mathbf{r}, \mathbf{r}') \nabla u(\mathbf{r})] \cdot \mathbf{n} dS. \end{aligned} \tag{7.0.3}$$

If both $u(\mathbf{r})$ and $G(\mathbf{r}, \mathbf{r}')$ satisfied Dirichlet conditions, $u = 0$ on $\partial\Omega$, then the last integral vanishes and we are left with[2]

$$u(\mathbf{r}') = \int_{\Omega} G(\mathbf{r}, \mathbf{r}') f(\mathbf{r}) dV.$$

Note

We note that in the following the volume and surface integrals and differentiation using ∇ are performed using the r-coordinates.

Note

In many applications there is a symmetry,

$$G(\mathbf{r}, \mathbf{r}') = G(\mathbf{r}', \mathbf{r})$$

Then, the result can be written as

$$u(\mathbf{r}) = \int_{\Omega} G(\mathbf{r}, \mathbf{r}') f(\mathbf{r}') \, dV'.$$

So, if we know the Green's function, we can solve the nonhomogeneous differential equation. In fact, we can use the Green's function to solve nonhomogenous boundary value and initial value problems. That is what we will see develop in this chapter as we explore nonhomogeneous problems in more detail. We will begin with the search for Green's functions for ordinary differential equations.

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7.1: Initial Value Green's Functions

In this section we will investigate the solution of initial value problems involving nonhomogeneous differential equations using Green's functions. Our goal is to solve the nonhomogeneous differential equation

$$a(t)y''(t)+b(t)y'(t)+c(t)y(t)=f(t), \tag{7.1.1}$$

subject to the initial conditions

$$y(0)=y_0 \quad y'(0)=v_0.$$

Since we are interested in initial value problems, we will denote the independent variable as a time variable, t.

Equation (7.1.1) can be written compactly as

$$L[y]=f,$$

where L is the differential operator

$$L=a(t)\frac{d^2}{dt^2}+b(t)\frac{d}{dt}+c(t).$$

The solution is formally given by

$$y=L^{-1}[f].$$

The inverse of a differential operator is an integral operator, which we seek to write in the form

$$y(t)=\int G(t,\tau)f(\tau)d\tau.$$

The function $G(t,\tau)$ is referred to as the **kernel of the integral operator** and is called the **Green's function**.

Note

$G(t,\tau)$ is called a Green's function.

In the last section we solved nonhomogeneous equations like Equation (7.1.1) using the Method of Variation of Parameters. Letting,

$$y_p(t)=c_1(t)y_1(t)+c_2(t)y_2(t), \tag{7.1.2}$$

we found that we have to solve the system of equations

\[\begin{align} &c_{1}^{\prime}(t) y_{1}(t)+c_{2}^{\prime}(t) y_{2}(t)=0 .\nonumber \\

4pt] &c_{1}^{\prime}(t) y_{1}^{\prime}(t)+c_{2}^{\prime}(t) y_{2}^{\prime}(t)=\frac{f(t)}{q(t)} .\label{eq:3} \end{align}

This system is easily solved to give

\[\begin{align} &c_{1}^{\prime}(t)=-\frac{f(t) y_{2}(t)}{a(t)\left[y_{1}(t) y_{2}^{\prime}(t)-y_{1}^{\prime}(t) y_{2}(t)\right]}\nonumber \\

4pt] &c_{2}^{\prime}(t)=\frac{f(t) y_{1}(t)}{a(t)\left[y_{1}(t) y_{2}^{\prime}(t)-y_{1}^{\prime}(t) y_{2}(t)\right]} .\label{eq:4} \end{align}

We note that the denominator in these expressions involves the Wronskian of the solutions to the homogeneous problem, which is given by the determinant

\[W\left(y_{1}, y_{2}\right)(t)=\left|\begin{array}{ll} y_{1}(t) & y_{2}(t) \\

4pt] y_{1}^{\prime}(t) & y_{2}^{\prime}(t) \end{array}\right| .\nonumber

When $y_1(t)$ and $y_2(t)$ are linearly independent, then the Wronskian is not zero and we are guaranteed a solution to the above system.

So, after an integration, we find the parameters as

\[\begin{align} &c_{1}(t)=-\int_{t_{0}}^{t} \frac{f(\tau) y_{2}(\tau)}{a(\tau) W(\tau)} d \tau\nonumber \\
4pt] &c_{2}(t)=\int_{t_{1}}^{t} \frac{f(\tau) y_{1}(\tau)}{a(\tau) W(\tau)} d \tau\label{eq:5} \end{align}

where t_0 and t_1 are arbitrary constants to be determined from the initial conditions.

Therefore, the particular solution of Equation (7.1.1) can be written as

\[y_{p}(t)=y_{2}(t) \int_{t_{1}}^{t} \frac{f(\tau) y_{1}(\tau)}{a(\tau) W(\tau)} d \tau-y_{1}(t) \int_{t_{0}}^{t} \frac{f(\tau) y_{2}(\tau)}{a(\tau) W(\tau)} d \tau .\label{eq:6}\]

We begin with the particular solution (Equation (???)) of the nonhomogeneous differential equation Equation (7.1.1). This can be combined with the general solution of the homogeneous problem to give the general solution of the nonhomogeneous differential equation:

$$y_p(t) = c_1y_1(t) + c_2y_2(t) + y_2(t)\int_{t_1}^{t} \frac{f(\tau)y_1(\tau)}{a(\tau)W(\tau)}d\tau - y_1(t)\int_{t_0}^{t} \frac{f(\tau)y_2(\tau)}{a(\tau)W(\tau)}d\tau. \tag{7.1.3}$$

However, an appropriate choice of t_0 and t_1 can be found so that we need not explicitly write out the solution to the homogeneous problem, $c_1y_1(t) + c_2y_2(t)$. However, setting up the solution in this form will allow us to use t_0 and t_1 to determine particular solutions which satisfies certain homogeneous conditions. In particular, we will show that Equation (7.1.3) can be written in the form

$$y(t) = c_1y_1(t) + c_2y_2(t) + \int_0^t G(t,\tau)f(\tau)d\tau, \tag{7.1.4}$$

where the function $G(t,\tau)$ will be identified as the Green's function.

The goal is to develop the Green's function technique to solve the initial value problem

$$a(t)y''(t) + b(t)y'(t) + c(t)y(t) = f(t), \quad y(0) = y_0, \quad y'(0) = v_0. \tag{7.1.5}$$

We first note that we can solve this initial value problem by solving two separate initial value problems. We assume that the solution of the homogeneous problem satisfies the original initial conditions:

$$a(t)y_h''(t) + b(t)y_h'(t) + c(t)y_h(t) = 0, \quad y_h(0) = y_0, \quad y_h'(0) = v_0. \tag{7.1.6}$$

We then assume that the particular solution satisfies the problem

$$a(t)y_p''(t) + b(t)y_p'(t) + c(t)y_p(t) = f(t), \quad y_p(0) = 0, \quad y_p'(0) = 0. \tag{7.1.7}$$

Since the differential equation is linear, then we know that

$$y(t) = y_h(t) + y_p(t)$$

is a solution of the nonhomogeneous equation. Also, this solution satisfies the initial conditions:

\[\begin{aligned} &y(0)=y_{h}(0)+y_{p}(0)=y_{0}+0=y_{0}, \\
4pt] &y^{\prime}(0)=y_{h}^{\prime}(0)+y_{p}^{\prime}(0)=v_{0}+0=v_{0} . \end{aligned} \nonumber\]

Therefore, we need only focus on finding a particular solution that satisfies homogeneous initial conditions. This will be done by finding values for t_0 and t_1 in Equation (???) which satisfy the homogeneous initial conditions, $y_p(0) = 0$ and $y_p'(0) = 0$.

First, we consider $y_p(0) = 0$. We have

$$y_p(0) = y_2(0)\int_{t_1}^{0} \frac{f(\tau)y_1(\tau)}{a(\tau)W(\tau)}d\tau - y_1(0)\int_{t_0}^{0} \frac{f(\tau)y_2(\tau)}{a(\tau)W(\tau)}d\tau. \tag{7.1.8}$$

Here, $y_1(t)$ and $y_2(t)$ are taken to be any solutions of the homogeneous differential equation. Let's assume that $y_1(0) = 0$ and $y_2 \neq (0) = 0$. Then, we have

$$y_p(0) = y_2(0)\int_{t_1}^{0} \frac{f(\tau)y_1(\tau)}{a(\tau)W(\tau)}d\tau \tag{7.1.9}$$

We can force $y_p(0) = 0$ if we set $t_1 = 0$.

Now, we consider $y_p'(0) = 0$. First we differentiate the solution and find that

$$y_p'(t) = y_2'(t) \int_0^t \frac{f(\tau)y_1(\tau)}{a(\tau)W(\tau)}d\tau - y_1'(t) \int_{t_0}^t \frac{f(\tau)y_2(\tau)}{a(\tau)W(\tau)}d\tau, \tag{7.1.10}$$

since the contributions from differentiating the integrals will cancel. Evaluating this result at $t = 0$, we have

$$y_p'(0) = -y_1'(0) \int_{t_0}^0 \frac{f(\tau)y_2(\tau)}{a(\tau)W(\tau)}d\tau. \tag{7.1.11}$$

Assuming that $y_1'(0) \neq 0$, we can set $t_0 = 0$.

Thus, we have found that

\[\begin{align} y_{p}(x) &=y_{2}(t) \int_{0}^{t} \frac{f(\tau) y_{1}(\tau)}{a(\tau) W(\tau)} d \tau-y_{1}(t) \int_{0}^{t} \frac{f(\tau) y_{2}(\tau)}{a(\tau) W(\tau)} d \tau\nonumber \\

4pt] &=\int_{0}^{t}\left[\frac{y_{1}(\tau) y_{2}(t)-y_{1}(t) y_{2}(\tau)}{a(\tau) W(\tau)}\right] f(\tau) d \tau\label{eq:16} \end{align}

This result is in the correct form and we can identify the temporal, or initial value, Green's function. So, the particular solution is given as

$$y_p(t) = \int_0^t G(t,\tau)f(\tau)d\tau, \tag{7.1.12}$$

where the initial value Green's function is defined as

$$G(t,\tau) = \frac{y_1(\tau)y_2(t) - y_1(t)y_2(\tau)}{a(\tau)W(\tau)}.$$

We summarize

Solution of IVP Using the Green's Function

The solution of the initial value problem,

$$a(t)y''(t) + b(t)y'(t) + c(t)y(t) = f(t), \quad y(0) = y_0, \quad y'(0) = v_0,$$

takes the form

$$y(t) = y_h(t) + \int_0^t G(t,\tau)f(\tau)d\tau, \tag{7.1.13}$$

where

$$G(t,\tau) = \frac{y_1(\tau)y_2(t) - y_1(t)y_2(\tau)}{a(\tau)W(\tau)} \tag{7.1.14}$$

is the Green's function and y_1, y_2, y_h are solutions of the homogeneous equation satisfying

$$y_1(0) = 0, y_2(0) \neq 0, y_1'(0) \neq 0, y_2'(0) = 0, y_h(0) = y_0, y_h'(0) = v_0.$$

Example 7.1.1

Solve the forced oscillator problem

$$x'' + x = 2\cos t, \quad x(0) = 4, \quad x'(0) = 0.$$

Solution

We first solve the homogeneous problem with nonhomogeneous initial conditions:

$$x_h'' + x_h = 0, \quad x_h(0) = 4, \quad x_h'(0) = 0.$$

The solution is easily seen to be $x_h(t) = 4\cos t$.

Next, we construct the Green's function. We need two linearly independent solutions, $y_1(x), y_2(x)$, to the homogeneous differential equation satisfying different homogeneous conditions, $y_1(0) = 0$ and $y_2'(0) = 0$. The simplest solutions are $y_1(t) = \sin t$ and $y_2(t) = \cos t$. The Wronskian is found as

$$W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) = -\sin^2 t - \cos^2 t = -1.$$

Since $a(t) = 1$ in this problem, we compute the Green's function,

\[\begin{align} G(t, \tau) &=\frac{y_{1}(\tau) y_{2}(t)-y_{1}(t) y_{2}(\tau)}{a(\tau) W(\tau)}\nonumber \\

\[4pt] &=\sin t \cos \tau-\sin \tau \cos t\nonumber \\

4pt] &=\sin (t-\tau) .\label{eq:20} \end{align}

Note that the Green's function depends on $t - \tau$. While this is useful in some contexts, we will use the expanded form when carrying out the integration.

We can now determine the particular solution of the nonhomogeneous differential equation. We have

\[\begin{align} x_{p}(t)&=\int_{0}^{t} G(t, \tau) f(\tau) d \tau\nonumber \\

\[4pt] &=\int_{0}^{t}(\sin t \cos \tau-\sin \tau \cos t)(2 \cos \tau) d \tau\nonumber \\

\[4pt] &=2 \sin t \int_{0}^{t} \cos ^{2} \tau d \tau-2 \cos t \int_{0}^{t} \sin \tau \cos \tau d \tau\nonumber \\

\[4pt] &=2 \sin t\left[\frac{\tau}{2}+\frac{1}{2} \sin 2 \tau\right]_{0}^{t}-2 \cos t\left[\frac{1}{2} \sin ^{2} \tau\right]_{0}^{t}\nonumber \\

4pt] &=t \sin t .\label{eq:21}\end{align}

Therefore, the solution of the nonhomogeneous problem is the sum of the solution of the homogeneous problem and this particular solution: $x(t) = 4\cos t + t\sin t$.

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7.2: Boundary Value Green's Functions

We solved nonhomogeneous initial value problems in Section 7.1 using a Green's function. In this section we will extend this method to the solution of nonhomogeneous boundary value problems using a boundary value Green's function. Recall that the goal is to solve the nonhomogeneous differential equation

$$L[y] = f, \quad a \le x \le b,$$

where L is a differential operator and $y(x)$ satisfies boundary conditions at $x = a$ and $x = b$.. The solution is formally given by

$$y = L^{-1}[f].$$

The inverse of a differential operator is an integral operator, which we seek to write in the form

$$y(x) = \int_a^b G(x, \xi) f(\xi) d\xi.$$

The function $G(x, \xi)$ is referred to as the kernel of the integral operator and is called the *Green's function*.

We will consider boundary value problems in Sturm-Liouville form,

$$\frac{d}{dx}\left(p(x)\frac{dy(x)}{dx}\right) + q(x)y(x) = f(x), \quad a < x < b, \tag{7.2.1}$$

with fixed values of $y(x)$ at the boundary, $y(a) = 0$ and $y(b) = 0$. However, the general theory works for other forms of homogeneous boundary conditions.

We seek the Green's function by first solving the nonhomogeneous differential equation using the Method of Variation of Parameters. Recall this method from Section B.3.3. We assume a particular solution of the form

$$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x),$$

which is formed from two linearly independent solution of the homogeneous problem, $y_i(x), i = 1, 2$. We had found that the coefficient functions satisfy the equations

$$\begin{aligned} c_1'(x)y_1(x) + c_2'(x)y_2(x) &= 0 \\ c_1'(x)y_1'(x) + c_2'(x)y_2'(x) &= \frac{f(x)}{p(x)}. \end{aligned} \tag{7.2.2}$$

Solving this system, we obtain

$$\begin{aligned} c_1'(x) &= -\frac{fy_2}{pW(y_1, y_2)}, \\ c_1'(x) &= \frac{fy_1}{pW(y_1, y_2)}, \end{aligned}$$

where $W(y_1, y_2) = y_1y_2' - y_1'y_2$ is the Wronskian. Integrating these forms and inserting the results back into the particular solution, we find

$$y(x) = y_2(x)\int_{x_1}^{x} \frac{f(\xi)y_1(\xi)}{p(\xi)W(\xi)}d\xi - y_1(x)\int_{x_0}^{x} \frac{f(\xi)y_2(\xi)}{p(\xi)W(\xi)}d\xi,$$

where x_0 and x_1 are to be determined using the boundary values. In particular, we will seek x_0 and x_1 so that the solution to the boundary value problem can be written as a single integral involving a Green's function. Note that we can absorb the solution to the homogeneous problem, $y_h(x)$, into the integrals with an appropriate choice of limits on the integrals.

We now look to satisfy the conditions $y(a) = 0$ and $y(b) = 0$. First we use solutions of the homogeneous differential equation that satisfy $y_1(a) = 0$, $y_2(b) = 0$ and $y_1(b) \neq 0, y_2(a) \neq 0$. Evaluating $y(x)$ at $x = 0$, we have

$$\begin{aligned} y(a) &= y_2(a)\int_{x_1}^{a}\frac{f(\xi)y_1(\xi)}{p(\xi)W(\xi)}d\xi - y_1(a)\int_{x_0}^{a}\frac{f(\xi)y_2(\xi)}{p(\xi)W(\xi)}d\xi \\ &= y_2(a)\int_{x_1}^{a}\frac{f(\xi)y_1(\xi)}{p(\xi)W(\xi)}d\xi. \end{aligned} \tag{7.2.3}$$

We can satisfy the condition at $x=a$ if we choose $x_1 = a$.

Similarly, at $x=b$ we find that

$$\begin{aligned} y(b) &= y_2(b)\int_{x_1}^{b}\frac{f(\xi)y_1(\xi)}{p(\xi)W(\xi)}d\xi - y_1(b)\int_{x_0}^{b}\frac{f(\xi)y_2(\xi)}{p(\xi)W(\xi)}d\xi \\ &= -y_1(b)\int_{x_0}^{b}\frac{f(\xi)y_2(\xi)}{p(\xi)W(\xi)}d\xi. \end{aligned} \tag{7.2.4}$$

This expression vanishes for $x_0 = b$.

So, we have found that the solution takes the form

$$y(x) = y_2(x)\int_{a}^{x}\frac{f(\xi)y_1(\xi)}{p(\xi)W(\xi)}d\xi - y_1(x)\int_{b}^{x}\frac{f(\xi)y_2(\xi)}{p(\xi)W(\xi)}d\xi. \tag{7.2.5}$$

This solution can be written in a compact form just like we had done for the initial value problem in Section 7.1. We seek a Green's function so that the solution can be written as a single integral. We can move the functions of x under the integral. Also, since $a<x<b$, we can flip the limits in the second integral. This gives

$$y(x) = \int_{a}^{x}\frac{f(\xi)y_1(\xi)y_2(x)}{p(\xi)W(\xi)}d\tilde{\xi} + \int_{x}^{b}\frac{f(\xi)y_1(x)y_2(\xi)}{p(\xi)W(\xi)}d\xi \tag{7.2.6}$$

This result can now be written in a compact form:

Boundary Value Green's Function

The solution of the boundary value problem

$$\begin{aligned} \tfrac{d}{dx}\left(p(x)\tfrac{dy(x)}{dx}\right) + q(x)y(x) = f(x), \quad a<x<b, \\ y(a)=0, \quad y(b)=0. \end{aligned} \tag{7.2.7}$$

takes the form

$$y(x) = \int_{a}^{b} G(x,\xi)f(\xi)d\xi, \tag{7.2.8}$$

where the Green's function is the piecewise defined function

$$G(x,\xi) = \begin{cases} \frac{y_1(\xi)y_2(x)}{pW}, & a\le\xi\le x, \\ \frac{y_1(x)y_2(\xi)}{pW}, & x\le\xi\le b, \end{cases} \tag{7.2.9}$$

where $y_1(x)$ and $y_2(x)$ are solutions of the homogeneous problem satisfying $y_1(a)=0, y_2(b)=0$ and $y_1(b)\neq 0, y_2(a)\neq 0$.

The Green's function satisfies several properties, which we will explore further in the next section. For example, the Green's function satisfies the boundary conditions at $x=a$ and $x=b$. Thus,

$$\begin{aligned} G(a,\xi) &= \frac{y_1(a)y_2(\xi)}{pW} = 0, \\ G(b,\xi) &= \frac{y_1(\xi)y_2(b)}{pW} = 0. \end{aligned}$$

Also, the Green's function is symmetric in its arguments. Interchanging the arguments gives

$$G(\xi,x)=\begin{cases}\frac{y_1(x)y_2(\xi)}{pW}, & a\le x\le\xi,\\ \frac{y_1(\xi)y_2(x)}{pW}, & \xi\le x\le b,\end{cases} \tag{7.2.10}$$

But a careful look at the original form shows that

$$G(x,\xi)=G(\xi,x).$$

We will make use of these properties in the next section to quickly determine the Green's functions for other boundary value problems.

Example 7.2.1

Solve the boundary value problem $y''=x^2,\quad y(0)=0=y(1)$ using the boundary value Green's function.

Solution

We first solve the homogeneous equation, $y''=0$. After two integrations, we have $y(x)=Ax+B$, for A and B constants to be determined.

We need one solution satisfying $y_1(0)=0$ Thus,

$$0=y_1(0)=B.$$

So, we can pick $y_1(x)=x$, since A is arbitrary.

The other solution has to satisfy $y_2(1)=0$. So,

$$0=y_2(1)=A+B.$$

This can be solved for $B=-A$. Again, A is arbitrary and we will choose $A=-1$. Thus, $y_2(x)=1-x$.

For this problem $p(x)=1$. Thus, for $y_1(x)=x$ and $y_2(x)=1-x$,

$$p(x)W(x)=y_1(x)y_2'(x)-y_1'(x)y_2(x)=x(-1)-1(1-x)=-1.$$

Note that $p(x)W(x)$is a constant, as it should be.

Now we construct the Green's function. We have

$$G(x,\xi)=\begin{cases}-\xi(1-x), & 0\le\xi\le x,\\ -x(1-\xi), & x\le\xi\le 1.\end{cases} \tag{7.2.11}$$

Notice the symmetry between the two branches of the Green's function. Also, the Green's function satisfies homogeneous boundary conditions: $G(0,\xi)=0$, from the lower branch, and $G(1,\xi)=0$, from the upper branch.

Finally, we insert the Green's function into the integral form of the solution and evaluate the integral.

$$\begin{aligned}y(x)&=\int_0^1 G(x,\xi)f(\xi)d\xi\\ &=\int_0^1 G(x,\xi)\xi^2 d\xi\\ &=-\int_0^x \xi(1-x)\xi^2 d\xi-\int_x^1 x(1-\xi)\xi^2 d\xi\\ &=-(1-x)\int_0^x \xi^3 d\xi-x\int_x^1(\xi^2-\xi^3)d\xi\\ &=-(1-x)\left[\frac{\xi^4}{4}\right]_0^x-x\left[\frac{\xi^3}{3}-\frac{\xi^4}{4}\right]_x^1\\ &=-\frac{1}{4}(1-x)x^4-\frac{1}{12}x(4-3)+\frac{1}{12}x(4x^3-3x^4)\\ &=\frac{1}{12}(x^4-x).\end{aligned} \tag{7.2.12}$$

Checking the answer, we can easily verify that $y'' = x^2, y(0) = 0$, and $y(1) = 0$.

Properties of Green's Functions

We have noted some properties of Green's functions in the last section. In this section we will elaborate on some of these properties as a tool for quickly constructing Green's functions for boundary value problems. We list five basic properties:

1. **Differential Equation:**

   The boundary value Green's function satisfies the differential equation $\frac{\partial}{\partial x}\left(p(x)\frac{\partial G(x,\tilde{Z})}{\partial x}\right) + q(x)G(x,\xi) = 0, x \neq \xi$ This is easily established. For $x < \xi$ we are on the second branch and $G(x,\xi)$ is proportional to $y_1(x)$. Thus, since $y_1(x)$ is a solution of the homogeneous equation, then so is $G(x,\xi)$. For $x > \xi$ we are on the first branch and $G(x,\xi)$ is proportional to $y_2(x)$. So, once again $G(x,\xi)$ is a solution of the homogeneous problem.
2. **Boundary Conditions:**
   In the example in the last section we had seen that $G(a,\xi) = 0$ and $G(b,\xi) = 0$. For example, for $x = a$ we are on the second branch and $G(x,\xi)$ is proportional to $y_1(x)$. Thus, whatever condition $y_1(x)$ satisfies, $G(x,\xi)$ will satisfy. A similar statement can be made for $x = b$.
3. **Symmetry or Reciprocity:** $G(x\xi) = G(\xi, x)$
   We had shown this reciprocity property in the last section.
4. **Continuity of G at $x = \xi$:** $G(\xi^+,\xi) = G(\xi^-,\xi)$
   Here we define $\xi^{\pm}$ through the limits of a function as x approaches ξ from above or below. In particular,

$$G\left(\xi^+, x\right) = \lim_{x\downarrow\xi} G(x,\xi), \quad x > \xi,$$
$$G\left(\xi^-, x\right) = \lim_{x\uparrow\xi} G(x,\xi), \quad x < \xi.$$

   Setting $x = \xi$ in both branches, we have

$$\frac{y_1(\xi)y_2(\xi)}{pW} = \frac{y_1(\xi)y_2(\xi)}{pW}.$$

   Therefore, we have established the continuity of $G(x,\xi)$ between the two branches at $x = \xi$.
5. $$\frac{\partial G(\xi^+,\xi)}{\partial x} - \frac{\partial G(\xi^-,\xi)}{\partial x} = \frac{1}{p(\xi)}$$

   This case is not as obvious. We first compute the derivatives by noting which branch is involved and then evaluate the derivatives and subtract them. Thus, we have

Here is a summary of the properties of the boundary value Green's function based upon the previous solution.

Properties of the Green's Function

1. $$\frac{\partial}{\partial x}\left(p(x)\frac{\partial G(x,\xi)}{\partial x}\right) + q(x)G(x,\xi) = 0, x \neq \xi$$
2. **Boundary Conditions:** Whatever conditions $y_1(x)$ and $y_2(x)$ satisfy, $G(x,\xi)$ will satisfy.
3. **Symmetry or Reciprocity:** $G(x,\xi) = G(\xi, x)$
4. **Continuity of G at $x = \xi$:** $G(\xi^+,\xi) = G(\xi^-,\xi^-)$
5. $$\frac{\partial G(\xi^+,\xi)}{\partial x} - \frac{\partial G(\xi^-,\xi)}{\partial x} = \frac{1}{p(\xi)}$$

We now show how a knowledge of these properties allows one to quickly construct a Green's function with an example.

Example 7.2.2

Construct the Green's function for the problem

$$y'' + \omega^2 y = f(x), \quad 0 < x < 1,$$
$$y(0) = 0 = y(1),$$

with $\omega \neq 0$.

Solution

I. **Find solutions to the homogeneous equation.**

A general solution to the homogeneous equation is given as

$$y_h(x) = c_1 \sin \omega x + c_2 \cos \omega x.$$

Thus, for $x \neq \xi$

$$G(x, \xi) = c_1(\xi) \sin \omega x + c_2(\xi) \cos \omega x$$

II. **Boundary Conditions.**

First, we have $G(0, \xi) = 0$ for $0 \leq x \leq \xi$. So,

$$G(0, \xi) = c_2(\xi) \cos \omega x = 0$$

So,

$$G(x, \xi) = c_1(\xi) \sin \omega x, \quad 0 \leq x \leq \xi$$

Second, we have $G(1, \xi) = 0$ for $\xi \leq x \leq 1$. So,

$$G(1, \xi) = c_1(\xi) \sin \omega + c_2(\xi) \cos \omega. = 0$$

A solution can be chosen with

$$c_2(\xi) = -c_1(\xi) \tan \omega$$

This gives

$$G(x, \xi) = c_1(\xi) \sin \omega x - c_1(\xi) \tan \omega \cos \omega x$$

This can be simplified by factoring out the $c_1(\xi)$ and placing the remaining terms over a common denominator. The result is

$$\begin{aligned} G(x, \xi) &= \frac{c_1(\xi)}{\cos \omega} [\sin \omega x \cos \omega - \sin \omega \cos \omega x] \\ &= -\frac{c_1(\xi)}{\cos \omega} \sin \omega (1 - x). \end{aligned} \tag{7.2.13}$$

Since the coefficient is arbitrary at this point, as can write the result as

$$G(x, \xi) = d_1(\xi) \sin \omega (1 - x), \quad \xi \leq x \leq 1.$$

We note that we could have started with $y_2(x) = \sin \omega (1 - x)$ as one of the linearly independent solutions of the homogeneous problem in anticipation that $y_2(x)$ satisfies the second boundary condition.

III. **Symmetry or Reciprocity**

We now impose that $G(x, \xi) = G(\xi, x)$. To this point we have that

$$G(x, \xi) = \begin{cases} c_1(\xi) \sin \omega x, & 0 \leq x \leq \xi, \\ d_1(\xi) \sin \omega (1 - x), & \xi \leq x \leq 1. \end{cases}$$

We can make the branches symmetric by picking the right forms for $c_1(\xi)$ and $d_1(\xi)$. We choose $c_1(\xi) = C \sin \omega (1 - \xi)$ and $d_1(\xi) = C \sin \omega \xi$. Then,

$$G(x,\xi) = \begin{cases} C\sin\omega(1-\xi)\sin\omega x, & 0 \le x \le \xi, \\ C\sin\omega(1-x)\sin\omega\xi, & \xi \le x \le 1. \end{cases}$$

Now the Green's function is symmetric and we still have to determine the constant C. We note that we could have gotten to this point using the Method of Variation of Parameters result where $C = \frac{1}{pW}$.

IV. **Continuity of $G(x,\xi)$**

We already have continuity by virtue of the symmetry imposed in the last step.

V. **Jump Discontinuity in $\frac{\partial}{\partial x}G(x,\xi)$.**

We still need to determine C. We can do this using the jump discontinuity in the derivative:

$$\frac{\partial G(\xi^+,\xi)}{\partial x} - \frac{\partial G(\xi^-,\xi)}{\partial x} = \frac{1}{p(\xi)}$$

For this problem $p(x) = 1$. Inserting the Green's function, we have

$$\begin{aligned} 1 &= \frac{\partial G(\xi^+,\xi)}{\partial x} - \frac{\partial G(\xi^-,\xi)}{\partial x} \\ &= \frac{\partial}{\partial x}[C\sin\omega(1-x)\sin\omega\xi]_{x=\xi} - \frac{\partial}{\partial x}[C\sin\omega(1-\xi)\sin\omega x]_{x=\xi} \\ &= -\omega C\cos\omega(1-\xi)\sin\omega\xi - \omega C\sin\omega(1-\xi)\cos\omega\tilde{\xi} \\ &= -\omega C\sin\omega(\xi+1-\xi) \\ &= -\omega C\sin\omega. \end{aligned} \tag{7.2.14}$$

Therefore,

$$C = -\frac{1}{\omega\sin\omega}$$

Finally, we have the Green's function:

$$G(x,\xi) = \begin{cases} -\frac{\sin\omega(1-\xi)\sin\omega x}{\omega\sin\omega}, & 0 \le x \le \xi, \\ -\frac{\sin\omega(1-x)\sin\omega\xi}{\omega\sin\omega}, & \xi \le x \le 1. \end{cases} \tag{7.2.15}$$

It is instructive to compare this result to the Variation of Parameters result.

Example 7.2.3

Use the Method of Variation of Parameters to solve

$$\begin{aligned} y'' + \omega^2 y = f(x), \quad 0 < x < 1, \\ y(0) = 0 = y(1), \quad \omega \neq 0. \end{aligned}$$

Solution

We have the functions $y_1(x) = \sin\omega x$ and $y_2(x) = \sin\omega(1-x)$ as the solutions of the homogeneous equation satisfying $y_1(0) = 0$ and $y_2(1) = 0$. We need to compute pW:

$$\begin{aligned} p(x)W(x) &= y_1(x)y_2'(x) - y_1'(x)y_2(x) \\ &= -\omega\sin\omega x\cos\omega(1-x) - \omega\cos\omega x\sin\omega(1-x) \\ &= -\omega\sin\omega \end{aligned} \tag{7.2.16}$$

Inserting this result into the Variation of Parameters result for the Green's function leads to the same Green's function as above.

Differential Equation for the Green's Function

As we progress in the book we will develop a more general theory of Green's functions for ordinary and partial differential equations. Much of this theory relies on understanding that the Green's function really is the system response function to a point source. This begins with recalling that the boundary value Green's function satisfies a homogeneous differential equation for $x \neq \xi$,

$$\frac{\partial}{\partial x}\left(p(x)\frac{\partial G(x,\xi)}{\partial x}\right)+q(x)G(x,\xi)=0, \quad x\neq\xi \tag{7.2.17}$$

For $x=\xi$, we have seen that the derivative has a jump in its value. This is similar to the step, or Heaviside, function,

$$H(x)=\begin{cases}1, & x>0,\\ 0, & x<0.\end{cases}$$

The function is shown in Figure 7.2.1 and we see that the derivative of the step function is zero everywhere except at the jump, or discontinuity. At the jump, there is an infinite slope, though technically, we have learned that there is no derivative at this point. We will try to remedy this situation by introducing the Dirac delta function,

$$\delta(x)=\frac{d}{dx}H(x).$$

We will show that the Green's function satisfies the differential equation

$$\frac{\partial}{\partial x}\left(p(x)\frac{\partial G(x,\xi)}{\partial x}\right)+q(x)G(x,\xi)=\delta(x-\xi) \tag{7.2.18}$$

However, we will first indicate why this knowledge is useful for the general theory of solving differential equations using Green's functions.

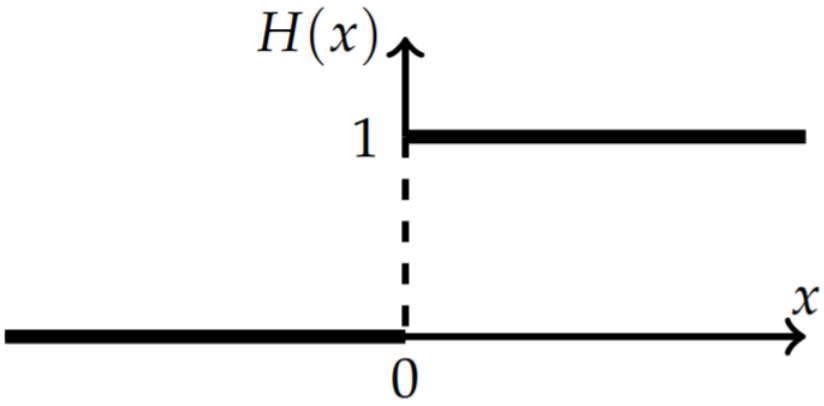


Figure 7.2.1: The Heaviside step function, $H(x)$.

Note

The Dirac delta function is described in more detail in Section 9.4. The key property we will need here is the sifting property,

$$\int_a^b f(x)\delta(x-\xi)dx=f(\xi)$$

for $a<\xi<b$.

As noted, the Green's function satisfies the differential equation

$$\frac{\partial}{\partial x}\left(p(x)\frac{\partial G(x,\xi)}{\partial x}\right)+q(x)G(x,\xi)=\delta(x-\xi) \tag{7.2.19}$$

and satisfies homogeneous conditions. We will use the Green's function to solve the nonhomogeneous equation

$$\frac{d}{dx}\left(p(x)\frac{dy(x)}{dx}\right)+q(x)y(x)=f(x). \tag{7.2.20}$$

These equations can be written in the more compact forms

$$\begin{aligned}\mathcal{L}[y]&=f(x)\\ \mathcal{L}[G]&=\delta(x-\xi).\end{aligned} \tag{7.2.21}$$

Using these equations, we can determine the solution, $y(x)$, in terms of the Green's function. Multiplying the first equation by $G(x,\xi)$, the second equation by $y(x)$, and then subtracting, we have

$$G\mathcal{L}[y]-y\mathcal{L}[G]=f(x)G(x,\xi)-\delta(x-\xi)y(x).$$

Now, integrate both sides from $x=a$ to $x=b$. The left hand side becomes

$$\int_a^b [f(x)G(x,\xi)-\delta(x-\xi)y(x)]dx=\int_a^b f(x)G(x,\xi)dx-y(\xi).$$

Using Green's Identity from Section 4.2.2, the right side is

$$\int_a^b (G\mathcal{L}[y]-y\mathcal{L}[G])dx=\left[p(x)\left(G(x,\xi)y'(x)-y(x)\frac{\partial G}{\partial x}(x,\xi)\right)\right]_{x=a}^{x=b}.$$

Combining these results and rearranging, we obtain

$$\begin{aligned} y(\xi)=&\int_a^b f(x)G(x,\xi)dx\\ &+\left[p(x)\left(y(x)\frac{\partial G}{\partial x}(x,\xi)-G(x,\xi)y'(x)\right)\right]_{x=a}^{x=b}. \end{aligned} \tag{7.2.22}$$

Note

Recall that Green's identity is given by

$$\int_a^b (u\mathcal{L}v-v\mathcal{L}u)dx=\left[p\left(uv'-vu'\right)\right]_a^b$$

The general solution in terms of the boundary value Green's function with corresponding surface terms.

We will refer to the extra terms in the solution,

$$S(b,\tilde{\xi})-S(a,\tilde{\xi})=\left[p(x)\left(y(x)\frac{\partial G}{\partial x}(x,\tilde{\xi})-G(x,\xi)y'(x)\right)\right]_{x=a}^{x=b},$$

as the boundary, or surface, terms. Thus,

$$y(\tilde{\zeta})=\int_a^b f(x)G(x,\xi)dx-[S(b,\tilde{\zeta})-S(a,\xi)].$$

The result in Equation (7.2.22) is the key equation in determining the solution of a nonhomogeneous boundary value problem. The particular set of boundary conditions in the problem will dictate what conditions $G(x,\xi)$ has to satisfy. For example, if we have the boundary conditions $y(a)=0$ and $y(b)=0$, then the boundary terms yield

$$\begin{aligned} y(\xi)=&\int_a^b f(x)G(x,\tilde{})dx-\left[p(b)\left(y(b)\frac{\partial G}{\partial x}(b,\xi)-G(b,\xi)y'(b)\right)\right]\\ &+\left[p(a)\left(y(a)\frac{\partial G}{\partial x}(a,\xi)-G(a,\xi)y'(a)\right)\right]\\ =&\int_a^b f(x)G(x,\xi)dx+p(b)G(b,\xi)y'(b)-p(a)G(a,\xi)y'(a). \end{aligned} \tag{7.2.23}$$

The right hand side will only vanish if $G(x,\xi)$ also satisfies these homogeneous boundary conditions. This then leaves us with the solution

$$y(\xi)=\int_a^b f(x)G(x,\xi)dx$$

We should rewrite this as a function of x. So, we replace ξ with x and x with ξ. This gives

$$y(x) = \int_a^b f(\xi) G(\xi, x) d\xi.$$

However, this is not yet in the desirable form. The arguments of the Green's function are reversed. But, in this case $G(x, \xi)$ is symmetric in its arguments. So, we can simply switch the arguments getting the desired result.

We can now see that the theory works for other boundary conditions. If we had $y'(a) = 0$, then the $y(a)\frac{\partial G}{dx}(a, \xi)$ term in the boundary terms could be made to vanish if we set $\frac{\partial G}{\partial x}(a, \xi) = 0$. So, this confirms that other boundary value problems can be posed besides the one elaborated upon in the chapter so far.

We can even adapt this theory to nonhomogeneous boundary conditions. We first rewrite Equation (7.2.22) as

$$y(x) = \int_a^b G(x, \xi) f(\xi) d\xi - \left[p(\xi) \left(y(\xi) \frac{\partial G}{\partial \xi}(x, \xi) - G(x, \xi) y'(\xi) \right) \right]_{\tilde{z}=a}^{\tilde{\zeta}=b}. \tag{7.2.24}$$

Let's consider the boundary conditions $y(a) = \alpha$ and $y'(b) = \beta$. We also assume that $G(x, \xi)$ satisfies homogeneous boundary conditions,

$$G(a, \xi) = 0, \quad \frac{\partial G}{\partial \xi}(b, \xi) = 0.$$

in both x and ξ since the Green's function is symmetric in its variables. Then, we need only focus on the boundary terms to examine the effect on the solution. We have

$$\begin{aligned} S(b, x) - S(a, x) &= \left[p(b) \left(y(b) \frac{\partial G}{\partial \xi}(x, b) - G(x, b) y'(b) \right) \right] \\ &\quad - \left[p(a) \left(y(a) \frac{\partial G}{\partial \xi}(x, a) - G(x, a) y'(a) \right) \right] \\ &= -\beta p(b) G(x, b) - \alpha p(a) \frac{\partial G}{\partial \xi}(x, a). \end{aligned} \tag{7.2.25}$$

Therefore, we have the solution

$$y(x) = \int_a^b G(x, \xi) f(\xi) d\xi + \beta p(b) G(x, b) + \alpha p(a) \frac{\partial G}{\partial \xi}(x, a). \tag{7.2.26}$$

This solution satisfies the nonhomogeneous boundary conditions.

Note

General solution satisfying the nonhomogeneous boundary conditions $y(a) = \alpha$ and $y'(b) = \beta$. Here the Green's function satisfies homogeneous boundary conditions, $G(a, \xi) = 0, \quad \frac{\partial G}{\partial \xi}(b, \xi) = 0$.

Example 7.2.4

Solve $y'' = x^2, \quad y(0) = 1, y(1) = 2$ using the boundary value Green's function.

Solution

This is a modification of Example 7.2.1. We can use the boundary value Green's function that we found in that problem,

$$G(x, \xi) = \begin{cases} -\tilde{\xi}(1 - x), & 0 \le \xi \le x, \\ -x(1 - \xi), & x \le \xi \le 1 \end{cases} \tag{7.2.27}$$

We insert the Green's function into the general solution (7.2.26) and use the given boundary conditions to obtain

$$\begin{aligned} y(x) &= \int_0^1 G(x,\xi)\tilde{\xi}^2 d\xi - \left[y(\xi)\frac{\partial G}{\partial \xi}(x,\xi) - G(x,\xi)y'(\xi) \right]_{\tilde{\xi}=0}^{\tilde{\zeta}=1} \\ &= \int_0^x (x-1)\xi^3 d\tilde{\xi} + \int_x^1 x(\xi-1)\xi^2 d\tilde{\xi} + y(0)\frac{\partial G}{\partial \xi}(x,0) - y(1)\frac{\partial G}{\partial \xi}(x,1) \\ &= \frac{(x-1)x^4}{4} + \frac{x\left(1-x^4\right)}{4} - \frac{x\left(1-x^3\right)}{3} + (x-1) - 2x \\ &= \frac{x^4}{12} + \frac{35}{12}x - 1. \end{aligned} \tag{7.2.28}$$

Of course, this problem can be solved by direct integration. The general solution is

$$y(x) = \frac{x^4}{12} + c_1 x + c_2.$$

Inserting this solution into each boundary condition yields the same result.

Note

The Green's function satisfies a delta function forced differential equation.

We have seen how the introduction of the Dirac delta function in the differential equation satisfied by the Green's function, Equation (7.2.19), can lead to the solution of boundary value problems. The Dirac delta function also aids in the interpretation of the Green's function. We note that the Green's function is a solution of an equation in which the nonhomogeneous function is $\delta(x-\xi)$. Note that if we multiply the delta function by $f(\xi)$ and integrate, we obtain

$$\int_{-\infty}^{\infty} \delta(x-\xi)f(\xi)d\xi = f(x)$$

We can view the delta function as a unit impulse at $x=\xi$ which can be used to build $f(x)$ as a sum of impulses of different strengths, $f(\xi)$. Thus, the Green's function is the response to the impulse as governed by the differential equation and given boundary conditions.

Note

Derivation of the jump condition for the Green's function.

In particular, the delta function forced equation can be used to derive the jump condition. We begin with the equation in the form

$$\frac{\partial}{\partial x}\left(p(x)\frac{\partial G(x,\xi)}{\partial x} \right) + q(x)G(x,\xi) = \delta(x-\xi). \tag{7.2.29}$$

Now, integrate both sides from $\xi-\epsilon$ to $\xi+\epsilon$ and take the limit as $\epsilon \to 0$. Then,

$$\begin{aligned} \lim_{\epsilon\to 0}\int_{\xi-\epsilon}^{\xi+\epsilon} \left[\frac{\partial}{\partial x}\left(p(x)\frac{\partial G(x,\xi)}{\partial x} \right) + q(x)G(x,\xi) \right] dx &= \lim_{\epsilon\to 0}\int_{\xi-\epsilon}^{\xi+\epsilon} \delta(x-\xi)dx \\ &= 1. \end{aligned} \tag{7.2.30}$$

Since the $q(x)$ term is continuous, the limit as $\epsilon \to 0$ of that term vanishes. Using the Fundamental Theorem of Calculus, we then have

$$\lim_{\epsilon\to 0}\left[p(x)\frac{\partial G(x,\xi)}{\partial x} \right]_{\tilde{\zeta}-\epsilon}^{\tilde{\xi}+\epsilon} = 1. \tag{7.2.31}$$

This is the jump condition that we have been using!

Series Representations of Green's Functions

There are times that it might not be so simple to find the Green's function in the simple closed form that we have seen so far. However, there is a method for determining the Green's functions of Sturm-Liouville boundary value problems in the form of an eigenfunction expansion. We will finish our discussion of Green's functions for ordinary differential equations by showing how one obtains such series representations. (Note that we are really just repeating the steps towards developing eigenfunction expansion which we had seen in Section 4.3.)

We will make use of the complete set of eigenfunctions of the differential operator, $\mathcal{L}$, satisfying the homogeneous boundary conditions:

$$\mathcal{L}\left[\phi_n\right] = -\lambda_n \sigma \phi_n, \quad n = 1, 2, \ldots$$

We want to find the particular solution y satisfying $\mathcal{L}[y] = f$ and homogeneous boundary conditions. We assume that

$$y(x) = \sum_{n=1}^{\infty} a_n \phi_n(x).$$

Inserting this into the differential equation, we obtain

$$\mathcal{L}[y] = \sum_{n=1}^{\infty} a_n \mathcal{L}\left[\phi_n\right] = -\sum_{n=1}^{\infty} \lambda_n a_n \sigma \phi_n = f.$$

This has resulted in the generalized Fourier expansion

$$f(x) = \sum_{n=1}^{\infty} c_n \sigma \phi_n(x)$$

with coefficients

$$c_n = -\lambda_n a_n.$$

We have seen how to compute these coefficients earlier in section 4.3. We multiply both sides by $\phi_k(x)$ and integrate. Using the orthogonality of the eigenfunctions,

$$\int_a^b \phi_n(x)\phi_k(x)\sigma(x)dx = N_k \delta_{nk},$$

one obtains the expansion coefficients (if $\lambda_k \neq 0$)

$$a_k = -\frac{(f, \phi_k)}{N_k \lambda_k}$$

where $(f, \phi_k) \equiv \int_a^b f(x)\phi_k(x)dx$.

As before, we can rearrange the solution to obtain the Green's function. Namely, we have

$$y(x) = \sum_{n=1}^{\infty} \frac{(f, \phi_n)}{-N_n \lambda_n} \phi_n(x) = \int_a^b \underbrace{\sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(\xi)}{-N_n \lambda_n}}_{G(x,\zeta)} f(\xi) d\xi$$

Therefore, we have found the Green's function as an expansion in the eigenfunctions:

$$G(x, \xi) = \sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(\xi)}{-\lambda_n N_n} \tag{7.2.32}$$

We will conclude this discussion with an example. We will solve this problem three different ways in order to summarize the methods we have used in the text.

Example 7.2.5

Solve

$$y'' + 4y = x^2, \quad x \in (0,1), \quad y(0) = y(1) = 0$$

using the Green's function eigenfunction expansion.

Solution

The Green's function for this problem can be constructed fairly quickly for this problem once the eigenvale problem is solved. The eigenvalue problem is

$$\phi''(x) + 4\phi(x) = -\lambda\phi(x),$$

where $\phi(0) = 0$ and $\phi(1) = 0$. The general solution is obtained by rewriting the equation as

$$\phi''(x) + k^2\phi(x) = 0,$$

where

$$k^2 = 4 + \lambda.$$

Solutions satisfying the boundary condition at $x = 0$ are of the form

$$\phi(x) = A\sin kx.$$

Forcing $\phi(1) = 0$ gives

$$0 = A\sin k \Rightarrow k = n\pi, \quad k = 1, 2, 3 \ldots.$$

So, the eigenvalues are

$$\lambda_n = n^2\pi^2 - 4, \quad n = 1, 2, \ldots$$

and the eigenfunctions are

$$\phi_n = \sin n\pi x, \quad n = 1, 2, \ldots$$

We also need the normalization constant, N_n. We have that

$$N_n = \|\phi_n\|^2 = \int_0^1 \sin^2 n\pi x = \frac{1}{2}.$$

We can now construct the Green's function for this problem using Equation (7.2.32).

$$G(x,\xi) = 2\sum_{n=1}^{\infty} \frac{\sin n\pi x \sin n\pi\xi}{(4 - n^2\pi^2)} \tag{7.2.33}$$

Using this Green's function, the solution of the boundary value problem becomes

$$\begin{aligned} y(x) &= \int_0^1 G(x,\xi) f(\xi) d\xi \\ &= \int_0^1 \left(2\sum_{n=1}^{\infty} \frac{\sin n\pi x \sin n\pi\tilde{\xi}}{(4 - n^2\pi^2)} \right) \tilde{\xi}^2 d\xi \\ &= 2\sum_{n=1}^{\infty} \frac{\sin n\pi x}{(4 - n^2\pi^2)} \int_0^1 \xi^2 \sin n\pi\xi d\xi \\ &= 2\sum_{n=1}^{\infty} \frac{\sin n\pi x}{(4 - n^2\pi^2)} \left[\frac{\left(2 - n^2\pi^2\right)(-1)^n - 2}{n^3\pi^3} \right] \end{aligned} \tag{7.2.34}$$

We can compare this solution to the one we would obtain if we did not employ Green's functions directly. The eigenfunction expansion method for solving boundary value problems, which we saw earlier is demonstrated in the next example.

Example 7.2.6

Solve

$$y'' + 4y = x^2, \quad x \in (0,1), \quad y(0) = y(1) = 0$$

using the eigenfunction expansion method.

Solution

We assume that the solution of this problem is in the form

$$y(x) = \sum_{n=1}^{\infty} c_n \phi_n(x).$$

Inserting this solution into the differential equation $\mathcal{L}[y] = x^2$, gives

$$\begin{aligned} x^2 &= \mathcal{L}\left[\sum_{n=1}^{\infty} c_n \sin n\pi x\right] \\ &= \sum_{n=1}^{\infty} c_n \left[\frac{d^2}{dx^2}\sin n\pi x + 4\sin n\pi x\right] \\ &= \sum_{n=1}^{\infty} c_n \left[4 - n^2\pi^2\right] \sin n\pi x \end{aligned} \tag{7.2.35}$$

This is a Fourier sine series expansion of $f(x) = x^2$ on $[0,1]$. Namely,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x.$$

In order to determine the c_n 's in Equation (7.2.35), we will need the Fourier sine series expansion of x^2 on $[0,1]$. Thus, we need to compute

$$\begin{aligned} b_n &= \frac{2}{1}\int_0^1 x^2 \sin n\pi x \\ &= 2\left[\frac{\left(2 - n^2\pi^2\right)(-1)^n - 2}{n^3\pi^3}\right], \quad n = 1, 2, \ldots \end{aligned} \tag{7.2.36}$$

The resulting Fourier sine series is

$$x^2 = 2\sum_{n=1}^{\infty}\left[\frac{\left(2 - n^2\pi^2\right)(-1)^n - 2}{n^3\pi^3}\right]\sin n\pi x.$$

Inserting this expansion in Equation (7.2.35), we find

$$2\sum_{n=1}^{\infty}\left[\frac{\left(2 - n^2\pi^2\right)(-1)^n - 2}{n^3\pi^3}\right]\sin n\pi x = \sum_{n=1}^{\infty} c_n \left[4 - n^2\pi^2\right]\sin n\pi x.$$

Due to the linear independence of the eigenfunctions, we can solve for the unknown coefficients to obtain

$$c_n = 2\frac{\left(2 - n^2\pi^2\right)(-1)^n - 2}{\left(4 - n^2\pi^2\right)n^3\pi^3}.$$

Therefore, the solution using the eigenfunction expansion method is

$$\begin{aligned} y(x) &= \sum_{n=1}^{\infty} c_n \phi_n(x) \\ &= 2\sum_{n=1}^{\infty} \frac{\sin n\pi x}{\left(4 - n^2\pi^2\right)}\left[\frac{\left(2 - n^2\pi^2\right)(-1)^n - 2}{n^3\pi^3}\right]. \end{aligned} \tag{7.2.37}$$

We note that the solution in this example is the same solution as we had obtained using the Green's function obtained in series form in the previous example.

One remaining question is the following: Is there a closed form for the Green's function and the solution to this problem? The answer is yes!

Example 7.2.7

Find the closed form Green's function for the problem

$$y'' + 4y = x^2, \quad x \in (0,1), \quad y(0) = y(1) = 0$$

and use it to obtain a closed form solution to this boundary value problem.

Solution

We note that the differential operator is a special case of the example done in section 7.2. Namely, we pick $\omega = 2$. The Green's function was already found in that section. For this special case, we have

$$G(x,\xi) = \begin{cases} -\frac{\sin 2(1-\xi)\sin 2x}{2\sin 2\sin 2\xi}, & 0 \le x \le \xi, \\ -\frac{\sin 2(1-x)\sin 2}{2\sin 2}, & \xi \le x \le 1. \end{cases} \tag{7.2.38}$$

Using this Green's function, the solution to the boundary value problem is readily computed

$$\begin{aligned} y(x) &= \int_0^1 G(x,\xi) f(\xi) d\xi \\ &= -\int_0^x \frac{\sin 2(1-x)\sin 2\xi}{2\sin 2}\xi^2 d\xi + \int_x^1 \frac{\sin 2(\xi-1)\sin 2x}{2\sin 2}\xi^2 d\xi \\ &= -\frac{1}{4\sin 2}\left[-x^2\sin 2 + (1-\cos^2 x)\sin 2 + \sin x\cos x(1+\cos 2)\right]. \\ &= -\frac{1}{4\sin 2}\left[-x^2\sin 2 + 2\sin^2 x \sin 1\cos 1 + 2\sin x\cos x\cos^2 1)\right]. \\ &= -\frac{1}{8\sin 1\cos 1}\left[-x^2\sin 2 + 2\sin x\cos 1(\sin x\sin 1 + \cos x\cos 1)\right]. \\ &= \frac{x^2}{4} - \frac{\sin x\cos(1-x)}{4\sin 1}. \end{aligned} \tag{7.2.39}$$

In Figure 7.2.2 we show a plot of this solution along with the first five terms of the series solution. The series solution converges quickly to the closed form solution.

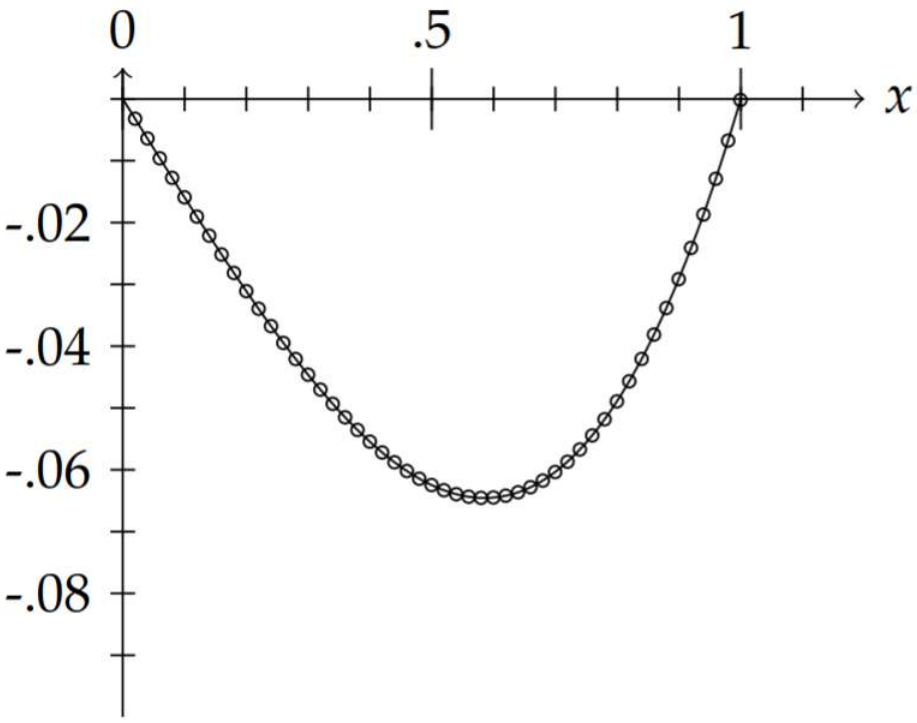


Figure 7.2.2: Plots of the exact solution to Example 7.6 with the first five terms of the series solution.

As one last check, we solve the boundary value problem directly, as we had done in the last chapter.

Example 7.2.8

Solve directly:

$$y'' + 4y = x^2, \quad x \in (0,1), \quad y(0) = y(1) = 0.$$

Solution

The problem has the general solution

$$y(x) = c_1 \cos 2x + c_2 \sin 2x + y_p(x)$$

where y_p is a particular solution of the nonhomogeneous differential equation. Using the Method of Undetermined Coefficients, we assume a solution of the form

$$y_p(x) = Ax^2 + Bx + C.$$

Inserting this guess into the nonhomogeneous equation, we have

$$2A + 4\left(Ax^2 + Bx + C\right) = x^2,$$

Thus, $B = 0, 4A = 1$ and $2A + 4C = 0$. The solution of this system is

$$A = \frac{1}{4}, \quad B = 0, \quad C = -\frac{1}{8}.$$

So, the general solution of the nonhomogeneous differential equation is

$$y(x) = c_1 \cos 2x + c_2 \sin 2x + \frac{x^2}{4} - \frac{1}{8}.$$

We next determine the arbitrary constants using the boundary conditions. We have

$$\begin{aligned} 0 &= y(0) \\ &= c_1 - \frac{1}{8} \\ 0 &= y(1) \\ &= c_1 \cos 2 + c_2 \sin 2 + \frac{1}{8} \end{aligned} \tag{7.2.40}$$

Thus, $c_1 = \frac{1}{8}$ and

$$c_2 = -\frac{\frac{1}{8} + \frac{1}{8}\cos 2}{\sin 2}.$$

Inserting these constants into the solution we find the same solution as before.

$$\begin{aligned} y(x) &= \frac{1}{8}\cos 2x - \left[\frac{\frac{1}{8} + \frac{1}{8}\cos 2}{\sin 2}\right]\sin 2x + \frac{x^2}{4} - \frac{1}{8} \\ &= \frac{(\cos 2x - 1)\sin 2 - \sin 2x(1 + \cos 2)}{8\sin 2} + \frac{x^2}{4} \\ &= \frac{\left(-2\sin^2 x\right) 2\sin 1\cos 1 - \sin 2x\left(2\cos^2 1\right)}{16\sin 1\cos 1} + \frac{x^2}{4} \\ &= -\frac{\left(\sin^2 x\right)\sin 1 + \sin x\cos x(\cos 1)}{4\sin 1} + \frac{x^2}{4} \\ &= \frac{x^2}{4} - \frac{\sin x\cos(1-x)}{4\sin 1}. \end{aligned} \tag{7.2.41}$$

Generalized Green's Function

When solving $Lu = f$ using eigenfunction expansions, there can be a problem when there are zero eigenvalues. Recall from Section 4.3 the solution of this problem is given by

$$\begin{aligned} y(x) &= \sum_{n=1}^{\infty} c_n \phi_n(x), \\ c_n &= -\frac{\int_a^b f(x)\phi_n(x)dx}{\lambda_m \int_a^b \phi_n^2(x)\sigma(x)dx}. \end{aligned} \tag{7.2.42}$$

Here the eigenfunctions, $\phi_n(x)$, satisfy the eigenvalue problem

$$\mathcal{L}\phi_n(x) = -\lambda_n \sigma(x)\phi_n(x), \quad x \in [a,b]$$

subject to given homogeneous boundary conditions.

Note that if $\lambda_m = 0$ for some value of $n = m$, then c_m is undefined. However, if we require

$$(f, \phi_m) = \int_a^b f(x)\phi_n(x)dx = 0,$$

then there is no problem. This is a form of the Fredholm Alternative. Namely, if $\lambda_n = 0$ for some n, then there is no solution unless $f, \phi_m) = 0$; i.e., f is orthogonal to ϕ_n. In this case, a_n will be arbitrary and there are an infinite number of solutions.

Example 7.2.9

$u'' = f(x), u'(0) = 0, u'(L) = 0$.

Solution

The eigenfunctions satisfy $\phi_n'' = -\lambda_n \phi_n(x), \phi_n'(0) = 0, \phi_n'(L) = 0$. There are the usual solutions,

$$\phi_n(x) = \cos\frac{n\pi x}{L}, \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, \ldots.$$

However, when $\lambda_n = 0, \phi_0''(x) = 0$. So, $\phi_0(x) = Ax + B$. The boundary conditions are satisfied if $A = 0$. So, we can take $\phi_0(x) = 1$. Therefore, there exists an eigenfunction corresponding to a zero eigenvalue. Thus, in order to have a solution, we have to require

$$\int_0^L f(x)dx = 0.$$

Example 7.2.10

$u'' + \pi^2 u = \beta + 2x, u(0) = 0, u(1) = 0$.

Solution

In this problem we check to see if there is an eigenfunctions with a zero eigenvalue. The eigenvalue problem is

$$\phi'' + \pi^2\phi = 0, \quad \phi(0) = 0, \quad \phi(1) = 0.$$

A solution satisfying this problem is easily founds as

$$\phi(x) = \sin \pi x.$$

Therefore, there is a zero eigenvalue. For a solution to exist, we need to require

$$\begin{aligned} 0 &= \int_0^1 (\beta + 2x) \sin \pi x dx \\ &= -\frac{\beta}{\pi} \cos \pi x \Big|_0^1 + 2\left[\frac{1}{\pi} x \cos \pi x - \frac{1}{\pi^2} \sin \pi x\right]_0^1 \\ &= -\frac{2}{\pi}(\beta + 1). \end{aligned} \tag{7.2.43}$$

Thus, either $\beta = -1$ or there are no solutions.

Recall the series representation of the Green's function for a Sturm-Liouville problem in Equation (7.2.32),

$$G(x, \zeta) = \sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(\xi)}{-\lambda_n N_n} \tag{7.2.44}$$

We see that if there is a zero eigenvalue, then we also can run into trouble as one of the terms in the series is undefined.

Recall that the Green's function satisfies the differential equation $LG(x, \xi) = \delta(x - \xi), x, \xi \in [a, b]$ and satisfies some appropriate set of boundary conditions. Using the above analysis, if there is a zero eigenvalue, then $L\phi_h(x) = 0$. In order for a solution to exist to the Green's function differential equation, then $f(x) = \delta(x - \xi)$ and we have to require

$$0 = (f, \phi_h) = \int_a^b \phi_h(x)\delta(x - \xi)dx = \phi_h(\xi),$$

for and $\xi \in [a, b]$. Therefore, the Green's function does not exist.

We can fix this problem by introducing a modified Green's function. Let's consider a modified differential equation,

$$LG_M(x, \xi) = \delta(x - \xi) + c\phi_h(x)$$

for some constant c. Now, the orthogonality condition becomes

$$\begin{aligned} 0 = (f, \phi_h) &= \int_a^b \phi_h(x) \left[\delta(x - \xi) + c\phi_h(x)\right] dx \\ &= \phi_h(\xi) + c \int_a^b \phi_h^2(x)dx. \end{aligned} \tag{7.2.45}$$

Thus, we can choose

$$c = -\frac{\phi_h(\xi)}{\int_a^b \phi_h^2(x)dx}$$

Using the modified Green's function, we can obtain solutions to $Lu = f$. We begin with Green's identity from Section 4.2.2, given by

$$\int_a^b (u\mathcal{L}v - v\mathcal{L}u)dx = \left[p\left(uv' - vu'\right)\right]_a^b.$$

Letting $v = G_M$, we have

$$\int_a^b \left(G_M\mathcal{L}[u] - u\mathcal{L}\left[G_M\right]\right)dx = \left[p(x)\left(G_M(x,\xi)u'(x) - u(x)\frac{\partial G_M}{\partial x}(x,\xi)\right)\right]_{x=a}^{x=b}.$$

Applying homogeneous boundary conditions, the right hand side vanishes. Then we have

$$\begin{aligned} 0 &= \int_a^b \left(G_M(x,\xi)\mathcal{L}[u(x)] - u(x)\mathcal{L}\left[G_M(x,\xi)\right]\right)dx \\ &= \int_a^b \left(G_M(x,\xi)f(x) - u(x)\left[\delta(x-\xi) + c\phi_h(x)\right]\right)dx \\ u(\xi) &= \int_a^b G_M(x,\xi)f(x)dx - c\int_a^b u(x)\phi_h(x)dx. \end{aligned} \tag{7.2.46}$$

Noting that $u(x,t) = c_1\phi_h(x) + u_p(x)$, the last integral gives

$$-c\int_a^b u(x)\phi_h(x)dx = \frac{\phi_h(\xi)}{\int_a^b \phi_h^2(x)dx}\int_a^b \phi_h^2(x)dx = c_1\phi_h(\xi).$$

Therefore, the solution can be written as

$$u(x) = \int_a^b f(\xi)G_M(x,\xi)d\tilde{\xi} + c_1\phi_h(x).$$

Here we see that there are an infinite number of solutions when solutions exist.

Example 7.2.11

Use the modified Green's function to solve $u'' + \pi^2 u = 2x - 1$, $u(0) = 0, u(1) = 0$.

Solution

We have already seen that a solution exists for this problem, where we have set $\beta = -1$ in Example 7.2.10

We construct the modified Green's function from the solutions of

$$\phi_n'' + \pi^2\phi_n = -\lambda_n\phi_n, \quad \phi(0) = 0, \quad \phi(1) = 0.$$

The general solutions of this equation are

$$\phi_n(x) = c_1\cos\sqrt{\pi^2 + \lambda_n}\,x + c_2\sin\sqrt{\pi^2 + \lambda_n}\,x.$$

Applying the boundary conditions, we have $c_1 = 0$ and $\sqrt{\pi^2 + \lambda_n} = n\pi$. Thus, the eigenfunctions and eigenvalues are

$$\phi_n(x) = \sin n\pi x, \quad \lambda_n = \left(n^2 - 1\right)\pi^2, \quad n = 1, 2, 3, \ldots.$$

Note that $\lambda_1 = 0$.

The modified Green's function satisfies

$$\frac{d^2}{dx^2}G_M(x,\xi) + \pi^2 G_M(x,\xi) = \delta(x-\xi) + c\phi_h(x),$$

where

$$
\begin{aligned}
c &= -\frac{\phi_1(\xi)}{\int_0^1 \phi_1^2(x)dx} \\
&= -\frac{\sin \pi\xi}{\int_0^1 \sin^2 \pi\xi, dx} \\
&= -2\sin \pi\xi.
\end{aligned} \tag{7.2.47}
$$

We need to solve for $\mathrm{G}_M(x,\xi)$. The modified Green's function satisfies

$$
\frac{d^2}{dx^2}G_M(x,\xi) + \pi^2 G_M(x,\xi) = \delta(x-\xi) - 2\sin\pi\xi\sin\pi x,
$$

and the boundary conditions $G_M(0,\xi) = 0$ and $G_M(1,\xi) = 0$. We assume an eigenfunction expansion,

$$
\mathrm{G}_M(x,\xi) = \sum_{n=1}^{\infty} c_n(\xi)\sin n\pi x.
$$

Then,

$$
\begin{aligned}
\delta(x-\xi) - 2\sin\pi\xi^{\tilde{\sin}\pi x} &= \frac{d^2}{dx^2}G_M(x,\tilde{}) + \pi^2 G_M(x,\tilde{}) \\
&= -\sum_{n=1}^{\infty}\lambda_n c_n(\xi)\sin n\pi x
\end{aligned} \tag{7.2.48}
$$

The coefficients are found as

$$
\begin{aligned}
-\lambda_n c_n &= 2\int_0^1 [\delta(x-\xi) - 2\sin\pi\xi\sin\pi x]\sin n\pi x dx \\
&= 2\sin n\pi\xi - 2\sin\pi\xi\delta_{n1}.
\end{aligned} \tag{7.2.49}
$$

Therefore, $c_1 = 0$ and $c_n = 2\sin n\pi\tilde{\xi}$, for $n > 1$.

We have found the modified Green's function as

$$
G_M(x,\xi) = -2\sum_{n=2}^{\infty}\frac{\sin n\pi x \sin n\pi\tilde{\xi}}{\lambda_n}.
$$

We can use this to find the solution. Namely, we have (for $c_1 = 0$)

$$
\begin{aligned}
u(x) &= \int_0^1 (2\xi - 1)G_M(x,\xi)d\xi \\
&= -2\sum_{n=2}^{\infty}\frac{\sin n\pi x}{\lambda_n}\int_0^1 (2\xi-1)\sin n\pi\xi dx \\
&= -2\sum_{n=2}^{\infty}\frac{\sin n\pi x}{(n^2-1)\pi^2}\left[-\frac{1}{n\pi}(2\xi-1)\cos n\pi\xi + \frac{1}{n^2\pi^2}\sin n\pi\xi\right]_0^1 \\
&= 2\sum_{n=2}^{\infty}\frac{1+\cos n\pi}{n(n^2-1)\pi^3}\sin n\pi x.
\end{aligned} \tag{7.2.50}
$$

We can also solve this problem exactly. The general solution is given by

$$
u(x) = c_1\sin\pi x + c_2\cos\pi x + \frac{2x-1}{\pi^2}.
$$

Imposing the boundary conditions, we obtain

$$
u(x) = c_1\sin\pi x + \frac{1}{\pi^2}\cos\pi x + \frac{2x-1}{\pi^2}.
$$

Notice that there are an infinite number of solutions. Choosing $c_1 = 0$, we have the particular solution

$$u(x) = \frac{1}{\pi^2}\cos \pi x + \frac{2x-1}{\pi^2}.$$

In Figure 7.2.3 we plot this solution and that obtained using the modified Green's function. The result is that they are in complete agreement.

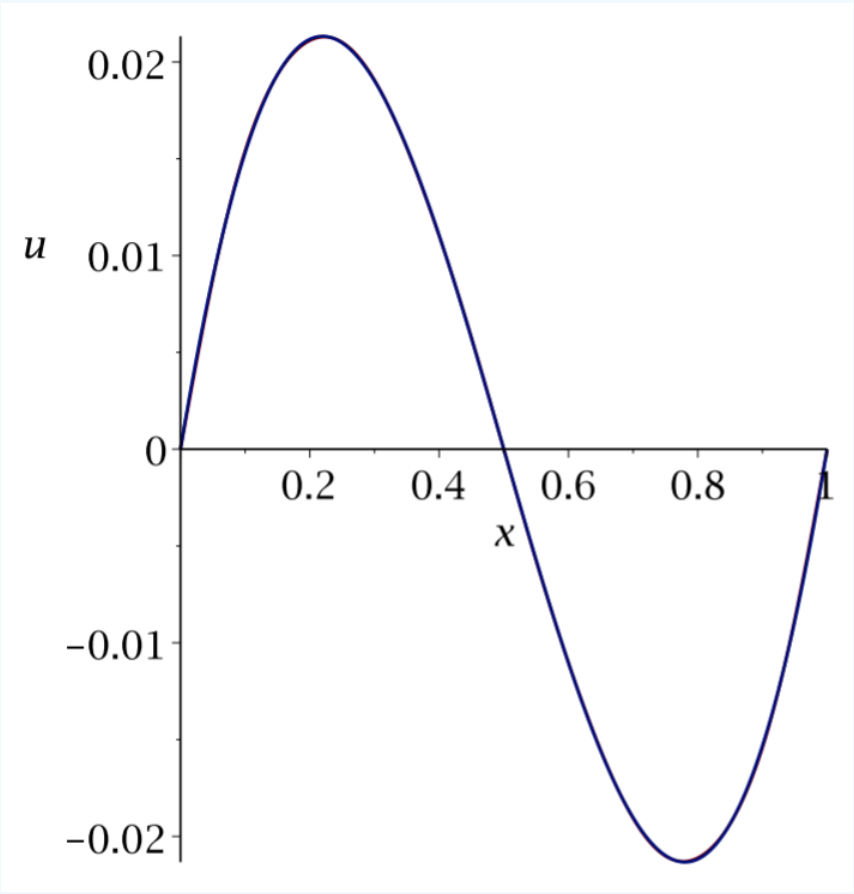


Figure 7.2.3: The solution for Example 7.2.11.

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7.3: The Nonhomogeneous Heat Equation

Boundary value green's functions do not only arise in the solution of nonhomogeneous ordinary differential equations. They are also important in arriving at the solution of nonhomogeneous partial differential equations. In this section we will show that this is the case by turning to the nonhomogeneous heat equation.

Nonhomogeneous Time Independent Boundary Conditions

Consider the nonhomogeneous heat equation with nonhomogeneous boundary conditions:

$$\begin{aligned} u_t - ku_{xx} &= h(x), \quad 0 \le x \le L, \quad t > 0, \\ u(0,t) &= a, \quad u(L,t) = b, \\ u(x,0) &= f(x). \end{aligned} \tag{7.3.1}$$

We are interested in finding a particular solution to this initial-boundary value problem. In fact, we can represent the solution to the general nonhomogeneous heat equation as the sum of two solutions that solve different problems.

First, we let $v(x,t)$ satisfy the homogeneous problem

$$\begin{aligned} v_t - kv_{xx} &= 0, \quad 0 \le x \le L, \quad t > 0, \\ v(0,t) &= 0, \quad v(L,t) = 0, \\ v(x,0) &= g(x), \end{aligned} \tag{7.3.2}$$

which has homogeneous boundary conditions.

Note

The steady state solution, $w(t)$, satisfies a nonhomogeneous differential equation with nonhomogeneous boundary conditions. The transient solution, $v(t)$, satisfies the homogeneous heat equation with homogeneous boundary conditions and satisfies a modified initial condition.

We will also need a steady state solution to the original problem. A steady state solution is one that satisfies $u_t = 0$. Let $w(x)$ be the steady state solution. It satisfies the problem

$$\begin{aligned} -kw_{xx} &= h(x), \quad 0 \le x \le L. \\ w(0,t) &= a, \quad w(L,t) = b. \end{aligned} \tag{7.3.3}$$

Now consider $u(x,t) = w(x) + v(x,t)$, the sum of the steady state solution, $w(x)$, and the transient solution, $v(x,t)$. We first note that $u(x,t)$ satisfies the nonhomogeneous heat equation,

$$\begin{aligned} u_t - ku_{xx} &= (w+v)_t - (w+v)_{xx} \\ &= v_t - kv_{xx} - kw_{xx} \equiv h(x). \end{aligned} \tag{7.3.4}$$

The boundary conditions are also satisfied. Evaluating, $u(x,t)$ at $x = 0$ and $x = L$, we have

$$\begin{aligned} u(0,t) &= w(0) + v(0,t) = a \\ u(L,t) &= w(L) + v(L,t) = b \end{aligned} \tag{7.3.5}$$

Note

The transient solution satisfies

$$v(x,0) = f(x) - w(x).$$

Finally, the initial condition gives

$$u(x,0) = w(x) + v(x,0) = w(x) + g(x).$$

Thus, if we set $g(x) = f(x) - w(x)$, then $u(x,t) = w(x) + v(x,t)$ will be the solution of the nonhomogeneous boundary value problem. We all ready know how to solve the homogeneous problem to obtain $v(x,t)$. So, we only need to find the steady state

solution, $w(x)$.

There are several methods we could use to solve Equation (7.3.3) for the steady state solution. One is the Method of Variation of Parameters, which is closely related to the Green's function method for boundary value problems which we described in the last several sections. However, we will just integrate the differential equation for the steady state solution directly to find the solution. From this solution we will be able to read off the Green's function.

Integrating the steady state equation (7.3.3) once, yields

$$\frac{dw}{dx} = -\frac{1}{k}\int_0^x h(z)dz + A,$$

where we have been careful to include the integration constant, $A = w'(0)$. Integrating again, we obtain

$$w(x) = -\frac{1}{k}\int_0^x \left(\int_0^y h(z)dz\right) dy + Ax + B,$$

where a second integration constant has been introduced. This gives the general solution for Equation (7.3.3).

The boundary conditions can now be used to determine the constants. It is clear that $B = a$ for the condition at $x = 0$ to be satisfied. The second condition gives

$$b = w(L) = -\frac{1}{k}\int_0^L \left(\int_0^y h(z)dz\right) dy + AL + a.$$

Solving for A, we have

$$A = \frac{1}{kL}\int_0^L \left(\int_0^y h(z)dz\right) dy + \frac{b-a}{L}.$$

Inserting the integration constants, the solution of the boundary value problem for the steady state solution is then

$$w(x) = -\frac{1}{k}\int_0^x \left(\int_0^y h(z)dz\right) dy + \frac{x}{kL}\int_0^L \left(\int_0^y h(z)dz\right) dy + \frac{b-a}{L}x + a.$$

This is sufficient for an answer, but it can be written in a more compact form. In fact, we will show that the solution can be written in a way that a Green's function can be identified.

First, we rewrite the double integrals as single integrals. We can do this using integration by parts. Consider integral in the first term of the solution,

$$I = \int_0^x \left(\int_0^y h(z)dz\right) dy.$$

Setting $u = \int_0^y h(z)dz$ and $dv = dy$ in the standard integration by parts formula, we obtain

$$\begin{aligned} I &= \int_0^x \left(\int_0^y h(z)dz\right) dy \\ &= y\int_0^y h(z)dz\Big|_0^x - \int_0^x yh(y)dy \\ &= \int_0^x (x-y)h(y)dy. \end{aligned} \tag{7.3.6}$$

Thus, the double integral has now collapsed to a single integral. Replacing the integral in the solution, the steady state solution becomes

$$w(x) = -\frac{1}{k}\int_0^x (x-y)h(y)dy + \frac{x}{kL}\int_0^L (L-y)h(y)dy + \frac{b-a}{L}x + a.$$

We can make a further simplification by combining these integrals. This can be done if the integration range, $[0, L]$, in the second integral is split into two pieces, $[0, x]$ and $[x, L]$. Writing the second integral as two integrals over these subintervals, we obtain

$$w(x) = -\frac{1}{k}\int_0^x (x-y)h(y)dy + \frac{x}{kL}\int_0^x (L-y)h(y)dy + \frac{x}{kL}\int_x^L (L-y)h(y)dy + \frac{b-a}{L}x + a. \tag{7.3.7}$$

Next, we rewrite the integrands,

$$w(x) = -\frac{1}{k}\int_0^x \frac{L(x-y)}{L}h(y)dy + \frac{1}{k}\int_0^x \frac{x(L-y)}{L}h(y)dy + \frac{1}{k}\int_x^L \frac{x(L-y)}{L}h(y)dy + \frac{b-a}{L}x + a. \tag{7.3.8}$$

It can now be seen how we can combine the first two integrals:

$$w(x) = -\frac{1}{k}\int_0^x \frac{y(L-x)}{L}h(y)dy + \frac{1}{k}\int_x^L \frac{x(L-y)}{L}h(y)dy + \frac{b-a}{L}x + a.$$

The resulting integrals now take on a similar form and this solution can be written compactly as

$$w(x) = -\int_0^L G(x,y)\left[-\frac{1}{k}h(y)\right]dy + \frac{b-a}{L}x + a,$$

where

$$G(x,y) = \begin{cases} \frac{x(L-y)}{L}, & 0 \le x \le y, \\ \frac{y(L-x)}{L}, & y \le x \le L, \end{cases}$$

is the Green's function for this problem.

The full solution to the original problem can be found by adding to this steady state solution a solution of the homogeneous problem,

$$\begin{aligned} u_t - ku_{xx} &= 0, \quad 0 \le x \le L, \quad t > 0, \\ u(0,t) &= 0, \quad u(L,t) = 0, \\ u(x,0) &= f(x) - w(x). \end{aligned} \tag{7.3.9}$$

Example 7.3.1

Solve the nonhomogeneous problem,

$$\begin{aligned} u_t - u_{xx} &= 10, \quad 0 \le x \le 1, \quad t > 0, \\ u(0,t) &= 20, \quad u(1,t) = 0, \\ u(x,0) &= 2x(1-x). \end{aligned} \tag{7.3.10}$$

Solution

In this problem we have a rod initially at a temperature of $u(x,0) = 2x(1-x)$. The ends of the rod are maintained at fixed temperatures and the bar is continually heated at a constant temperature, represented by the source term, 10 .

First, we find the steady state temperature, $w(x)$, satisfying

$$\begin{aligned} -w_{xx} &= 10, \quad 0 \le x \le 1. \\ w(0,t) &= 20, \quad w(1,t) = 0. \end{aligned} \tag{7.3.11}$$

Using the general solution, we have

$$w(x) = \int_0^1 10G(x,y)dy - 20x + 20,$$

where

$$G(x,y) = \begin{cases} x(1-y), & 0 \le x \le y, \\ y(1-x), & y \le x \le 1, \end{cases}$$

we compute the solution

$$\begin{aligned} w(x) &= \int_0^x 10y(1-x)dy + \int_x^1 10x(1-y)dy - 20x + 20 \\ &= 5\left(x - x^2\right) - 20x + 20, \\ &= 20 - 15x - 5x^2. \end{aligned} \tag{7.3.12}$$

Checking this solution, it satisfies both the steady state equation and boundary conditions.

The transient solution satisfies

$$\begin{aligned} v_t - v_{xx} &= 0, \quad 0 \le x \le 1, \quad t > 0, \\ v(0,t) &= 0, \quad v(1,t) = 0, \\ v(x,0) &= x(1-x) - 10. \end{aligned} \tag{7.3.13}$$

Recall, that we have determined the solution of this problem as

$$v(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t} \sin n\pi x,$$

where the Fourier sine coefficients are given in terms of the initial temperature distribution,

$$b_n = 2\int_0^1 [x(1-x) - 10] \sin n\pi x dx, \quad n = 1, 2, \ldots.$$

Therefore, the full solution is

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t} \sin n\pi x + 20 - 15x - 5x^2.$$

Note that for large t, the transient solution tends to zero and we are left with the steady state solution as expected.

Time Dependent Boundary Conditions

In the last section we solved problems with time independent boundary conditions using equilibrium solutions satisfying the steady state heat equation sand nonhomogeneous boundary conditions. When the boundary conditions are time dependent, we can also convert the problem to an auxiliary problem with homogeneous boundary conditions.

Consider the problem

$$\begin{aligned} u_t - ku_{xx} &= h(x), & 0 \le x \le L, t > 0, \\ u(0,t) &= a(t), & u(L,t) = b(t), t > 0, \\ u(x,0) &= f(x), & 0 \le x \le L. \end{aligned} \tag{7.3.14}$$

We define $u(x,t) = v(x,t) + w(x,t)$, where $w(x,t)$ is a modified form of the steady state solution from the last section,

$$w(x,t) = a(t) + \frac{b(t) - a(t)}{L}x.$$

Noting that

$$\begin{aligned} u_t &= v_t + \dot{a} + \frac{\dot{b} - \dot{a}}{L}x, \\ u_{xx} &= v_{xx}, \end{aligned} \tag{7.3.15}$$

we find that $v(x,t)$ is a solution of the problem

$$\begin{aligned} v_t - kv_{xx} &= h(x) - \left[\dot{a}(t) + \frac{\dot{b}(t) - \dot{a}(t)}{L}x\right], \qquad 0 \le x \le L, \quad t > 0, \\ v(0,t) &= 0, \quad v(L,t) = 0, \quad t > 0, \\ v(x,0) &= f(x) - \left[a(0) + \frac{b(0) - a(0)}{L}x\right], \qquad 0 \le x \le L. \end{aligned} \tag{7.3.16}$$

Thus, we have converted the original problem into a nonhomogeneous heat equation with homogeneous boundary conditions and a new source term and new initial condition.

Example 7.3.2

Solve the problem

$$\begin{aligned} u_t - u_{xx} &= x, \quad 0 \le x \le 1, \quad t > 0, \\ u(0,t) &= 2, \quad u(L,t) = t, \quad t > 0 \\ u(x,0) &= 3\sin 2\pi x + 2(1-x), \quad 0 \le x \le 1. \end{aligned} \tag{7.3.17}$$

Solution

We first define

$$u(x,t) = v(x,t) + 2 + (t-2)x.$$

Then, $v(x,t)$ satisfies the problem

$$\begin{aligned} v_t - v_{xx} &= 0, \quad 0 \le x \le 1, \quad t > 0, \\ v(0,t) &= 0, \quad v(L,t) = 0, \quad t > 0, \\ v(x,0) &= 3\sin 2\pi x, \quad 0 \le x \le 1. \end{aligned} \tag{7.3.18}$$

This problem is easily solved. The general solution is given by

$$v(x,t) = \sum_{n=1}^{\infty} b_n \sin n\pi x e^{-n^2\pi^2 t}.$$

We can see that the Fourier coefficients all vanish except for b_2. This gives $v(x,t) = 3\sin 2\pi x e^{-4\pi^2 t}$ and, therefore, we have found the solution

$$u(x,t) = 3\sin 2\pi x e^{-4\pi^2 t} + 2 + (t-2)x.$$

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7.4: Green's Functions for 1D Partial Differential Equations

In Section 7.1 we encountered the initial value green's function for initial value problems for ordinary differential equations. In that case we were able to express the solution of the differential equation $L[y] = f$ in the form

$$y(t) = \int G(t,\tau) f(\tau) d\tau,$$

where the Green's function $G(t,\tau)$ was used to handle the nonhomogeneous term in the differential equation. In a similar spirit, we can introduce Green's functions of different types to handle nonhomogeneous terms, nonhomogeneous boundary conditions, or nonhomogeneous initial conditions. Occasionally, we will stop and rearrange the solutions of different problems and recast the solution and identify the Green's function for the problem.

In this section we will rewrite the solutions of the heat equation and wave equation on a finite interval to obtain an initial value Green;s function. Assuming homogeneous boundary conditions and a homogeneous differential operator, we can write the solution of the heat equation in the form

$$u(x,t) = \int_0^L G\left(x,\xi;t,t_0\right) f(\xi) d\xi.$$

where $u\left(x,t_0\right) = f(x)$, and the solution of the wave equation as

$$u(x,t) = \int_0^L G_c\left(x,\xi,t,t_0\right) f(\xi) d\xi + \int_0^L G_s\left(x,\xi,t,t_0\right) g(\xi) d\xi.$$

where $u\left(x,t_0\right) = f(x)$ and $u_t\left(x,t_0\right) = g(x)$. The functions $G\left(x,\xi;t,t_0\right)$, $G\left(x,\xi;t,t_0\right)$, and $G\left(x,\xi;t,t_0\right)$ are initial value Green's functions and we will need to explore some more methods before we can discuss the properties of these functions. [For example, see Section.]

We will now turn to showing that for the solutions of the one dimensional heat and wave equations with fixed, homogeneous boundary conditions, we can construct the particular Green's functions.

Heat Equation

In Section 3.5 we obtained the solution to the one dimensional heat equation on a finite interval satisfying homogeneous Dirichlet conditions,

$$\begin{aligned} &u_t = k u_{xx}, \quad 0 < t, \quad 0 \le x \le L, \\ &u(x,0) = f(x), \quad 0 < x < L, \\ &u(0,t) = 0, \quad t > 0, \\ &u(L,t) = 0, \quad t > 0. \end{aligned} \tag{7.4.1}$$

The solution we found was the Fourier sine series

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{\lambda_n k t} \sin \frac{n\pi x}{L},$$

where

$$\lambda_n = -\left(\frac{n\pi}{L}\right)^2$$

and the Fourier sine coefficients are given in terms of the initial temperature distribution,

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx, \quad n = 1, 2, \ldots.$$

Inserting the coefficients b_n into the solution, we have

$$u(x,t) = \sum_{n=1}^{\infty} \left(\frac{2}{L} \int_0^L f(\xi) \sin \frac{n\pi \xi}{L} d\xi \right) e^{\lambda_n k t} \sin \frac{n\pi x}{L}.$$

Interchanging the sum and integration, we obtain

$$u(x,t)=\int_0^L\left(\frac{2}{L}\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}\sin\frac{n\pi\tilde{\xi}}{L}e^{\lambda_n kt}\right)f(\xi)d\xi.$$

This solution is of the form

$$u(x,t)=\int_0^L G(x,\xi;t,0)f(\xi)d\xi.$$

Here the function $G(x,\xi;t,0)$ is the initial value Green's function for the heat equation in the form

$$G(x,\xi;t,0)=\frac{2}{L}\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}\sin\frac{n\pi\xi}{L}e^{\lambda_n kt}.$$

which involves a sum over eigenfunctions of the spatial eigenvalue problem, $X_n(x)=\sin\frac{n\pi x}{L}$.

Wave Equation

The solution of the one dimensional wave equation (2.1.2),

$$\begin{aligned}u_{tt}&=c^2u_{xx},\quad 0<t,\quad 0\le x\le L,\\ u(0,t)&=0,\quad u(L,0)=0,\quad t>0,\\ u(x,0)&=f(x),\quad u_t(x,0)=g(x),\quad 0<x<L,\end{aligned}\tag{7.4.2}$$

was found as

$$u(x,t)=\sum_{n=1}^{\infty}\left[A_n\cos\frac{n\pi ct}{L}+B_n\sin\frac{n\pi ct}{L}\right]\sin\frac{n\pi x}{L}.$$

The Fourier coefficients were determined from the initial conditions,

$$\begin{aligned}f(x)&=\sum_{n=1}^{\infty}A_n\sin\frac{n\pi x}{L},\\ g(x)&=\sum_{n=1}^{\infty}\frac{n\pi c}{L}B_n\sin\frac{n\pi x}{L},\end{aligned}\tag{7.4.3}$$

as

$$\begin{aligned}A_n&=\frac{2}{L}\int_0^L f(\xi)\sin\frac{n\pi\xi}{L}d\xi,\\ B_n&=\frac{L}{n\pi c}\frac{2}{L}\int_0^L f(\xi)\sin\frac{n\pi\xi}{L}d\xi.\end{aligned}\tag{7.4.4}$$

Inserting these coefficients into the solution and interchanging integration with summation, we have

$$\begin{aligned}u(x,t)&=\int_0^{\infty}\left[\frac{2}{L}\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}\sin\frac{n\pi\xi}{L}\cos\frac{n\pi ct}{L}\right]f(\xi)d\xi\\ &\quad+\int_0^{\infty}\left[\frac{2}{L}\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}\sin\frac{n\pi\xi}{L}\frac{\sin\frac{n\pi ct}{L}}{n\pi c/L}\right]g(\xi)d\xi\\ &=\int_0^L G_c(x,\xi,t,0)f(\xi)d\xi+\int_0^L G_s(x,\xi,t,0)g(\xi)d\xi.\end{aligned}\tag{7.4.5}$$

In this case, we have defined two Green's functions,

$$\mathrm{G}_c(x,\xi,t,0) = \frac{2}{L}\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}\sin\frac{n\pi\xi}{L}\cos\frac{n\pi ct}{L}$$

$$\mathrm{G}_s(x,\xi,t,0) = \frac{2}{L}\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}\sin\frac{n\pi\tilde{\xi}}{L}\frac{\sin\frac{n\pi ct}{L}}{n\pi c/L} \tag{7.4.6}$$

The first, G_c, provides the response to the initial profile and the second, G_s, to the initial velocity.

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7.5: Green's Functions for the 2D Poisson Equation

In this section we consider the two dimensional Poisson equation with Dirichlet boundary conditions. We consider the problem

$$\begin{aligned} \nabla^2 u &= f, \text{ in } D, \\ u &= g, \text{ on } C, \end{aligned} \tag{7.5.1}$$

for the domain in Figure 7.5.1.

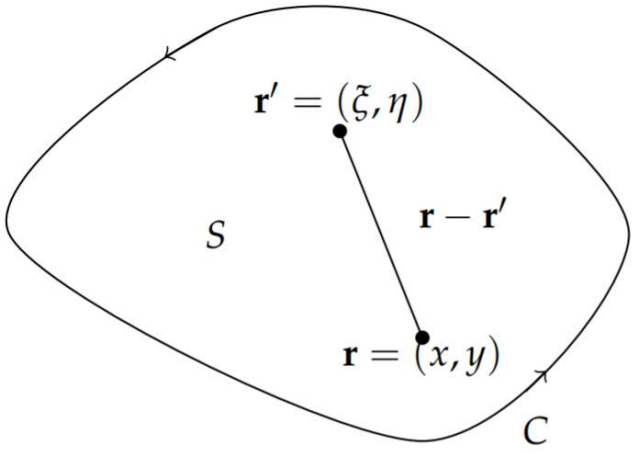


Figure 7.5.1: Domain for solving Poisson's equation.

We seek to solve this problem using a Green's function. As in earlier discussions, the Green's function satisfies the differential equation and homogeneous boundary conditions. The associated problem is given by

$$\begin{aligned} \nabla^2 G = \delta(\xi - x, \eta - y), &\quad \text{in } D, \\ G \equiv 0, &\quad \text{on } C. \end{aligned} \tag{7.5.2}$$

However, we need to be careful as to which variables appear in the differentiation. Many times we just make the adjustment after the derivation of the solution, assuming that the Green's function is symmetric in its arguments. However, this is not always the case and depends on things such as the self-adjointedness of the problem. Thus, we will assume that the Green's function satisfies

$$\nabla^2_{r'} G = \delta(\xi - x, \eta - y),$$

where the notation $\nabla_{r'}$ means differentiation with respect to the variables ξ and η. Thus,

$$\nabla^2_{r'} G = \frac{\partial^2 G}{\partial \tilde{\xi}^2} + \frac{\partial^2 G}{\partial \eta^2}$$

With this notation in mind, we now apply Green's second identity for two dimensions from Problem 8 in Chapter 9. We have

$$\int_D \left(u \nabla^2_{r'} G - G \nabla^2_{r'} u \right) dA' = \int_C \left(u \nabla_{r'} G - G \nabla_{r'} u \right) \cdot d\mathbf{s}'. \tag{7.5.3}$$

Inserting the differential equations, the left hand side of the equation becomes

$$\begin{aligned} &\int_D \left[u \nabla^2_{r'} G - G \nabla^2_{r'} u \right] dA' \\ &= \int_D [u(\xi, \eta) \delta(\xi - x, \eta - y) - G(x, y; \xi, \eta) f(\xi, \eta)] d\xi d\eta \\ &= u(x, y) - \int_D G(x, y; \xi, \eta) f(\xi, \eta) d\xi d\eta. \end{aligned} \tag{7.5.4}$$

Using the boundary conditions, $u(\xi, \eta) = g(\xi, \eta)$ on C and $G(x, y; \xi, \eta) = 0$ on C, the right hand side of the equation becomes

$$\int_C (u\nabla_{r'}G - G\nabla_{r'}u)\cdot d\mathbf{s}' = \int_C g(\xi,\eta)\nabla_{r'}G\cdot d\mathbf{s}'. \tag{7.5.5}$$

Solving for $u(x,y)$, we have the solution written in terms of the Green's function,

$$u(x,y) = \int_D G(x,y;\xi,\eta)f(\xi,\eta)d\xi d\eta + \int_C g(\xi,\eta)\nabla_{r'}G\cdot d\mathbf{s}'.$$

Now we need to find the Green's function. We find the Green's functions for several examples.

Example 7.5.1

Find the two dimensional Green's function for the antisymmetric Poisson equation; that is, we seek solutions that are θ-independent.

Solution

The problem we need to solve in order to find the Green's function involves writing the Laplacian in polar coordinates,

$$v_{rr} + \frac{1}{r}v_r = \delta(r).$$

For $r \neq 0$, this is a Cauchy-Euler type of differential equation. The general solution is $v(r) = A\ln r + B$.

Due to the singularity at $r = 0$, we integrate over a domain in which a small circle of radius ϵ is cut form the plane and apply the two dimensional Divergence Theorem. In particular, we have

$$\begin{aligned} 1 &= \int_{D_\epsilon} \delta(r)dA \\ &= \int_{D_\epsilon} \nabla^2 v dA \\ &= \int_{C_\epsilon} \nabla v \cdot \cdot \cdot d\mathbf{s} \\ &= \int_{C_\epsilon} \frac{\partial v}{\partial r} dS = 2\pi A. \end{aligned} \tag{7.5.6}$$

Therefore, $A = 1/2\pi$. We note that B is arbitrary, so we will take $B = 0$ in the remaining discussion.

Using this solution for a source of the form $\delta(\mathbf{r} - \mathbf{r}')$, we obtain the Green's function for Poisson's equation as

$$G(\mathbf{r},\mathbf{r}') = \frac{1}{2\pi}\ln|\mathbf{r} - \mathbf{r}'|.$$

Example 7.5.2

Find the Green's function for the infinite plane.

Solution

From Figure 7.5.1 we have $|\mathbf{r} - \mathbf{r}'| = \sqrt{(x-\xi)^2 + (y-\eta)^2}$. Therefore, the Green's function from the last example gives

$$G(x,y,\xi,\eta) = \frac{1}{4\pi}\ln\left((\xi - x)^2 + (\eta - y)^2\right).$$

Example 7.5.3

Find the Green's function for the half plane, $\{(x,y) \mid y > 0\}$, using the Method of Images.

Solution

This problem can be solved using the result for the Green's function for the infinite plane. We use the Method of Images to construct a function such that $G = 0$ on the boundary, $y = 0$. Namely, we use the image of the point (x,y) with respect to the x-axis, $(x,-y)$.

Imagine that the Green's function $G(x,y,\xi,\eta)$ represents a point charge at (x,y) and $G(x,y,\xi,\eta)$ provides the electric potential, or response, at (ξ,η). This single charge cannot yield a zero potential along the x-axis $(y=o)$. One needs an additional charge to yield a zero equipotential line. This is shown in Figure 7.5.2.

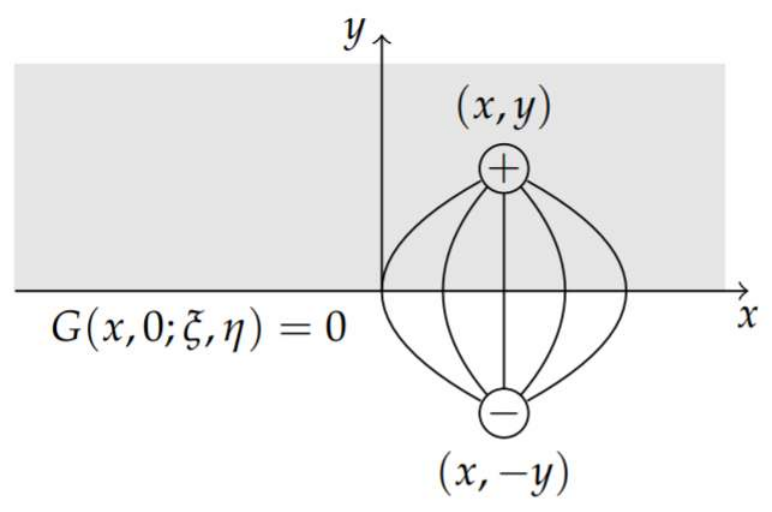


Figure 7.5.2: The Method of Images: The source and image source for the Green's function for the half plane. Imagine two opposite charges forming a dipole. The electric field lines are depicted indicating that the electric potential, or Green's function, is constant along $y=0$

The positive charge has a source of $\delta(\mathbf{r}-\mathbf{r}')$ at $\mathbf{r}=(x,y)$ and the negative charge is represented by the source $-\delta(\mathbf{r}^*-\mathbf{r}')$ at $\mathbf{r}^*=(x,-y)$. We construct the Green's functions at these two points and introduce a negative sign for the negative image source. Thus, we have

$$G(x,y,\xi,\eta)=\frac{1}{4\pi}\ln\left((\xi-x)^2+(\eta-y)^2\right)-\frac{1}{4\pi}\ln\left((\xi-x)^2+(\eta+y)^2\right).$$

These functions satisfy the differential equation and the boundary condition

$$G(x,0,\xi,\eta)=\frac{1}{4\pi}\ln\left((\xi-x)^2+(\eta)^2\right)-\frac{1}{4\pi}\ln\left((\xi-x)^2+(\eta)^2\right).$$

Example 7.5.4

Solve the homogeneous version of the problem; i.e., solve Laplace's equation on the half plane with a specified value on the boundary.

Solution

We want to solve the problem

$$\begin{aligned}\nabla^2 u&=0, \quad \text{in } D,\\ u&=f, \quad \text{on } C,\end{aligned} \tag{7.5.7}$$

This is displayed in Figure 7.5.3.

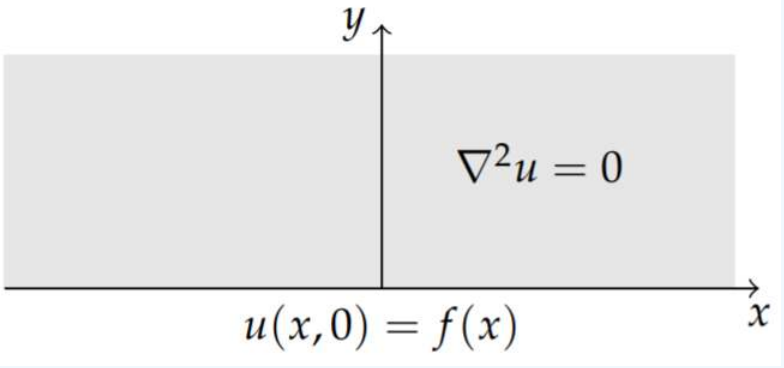


Figure 7.5.3: This is the domain for a semi-infinite slab with boundary value $u(x,0) = f(x)$ and governed by Laplace's equation.

From the previous analysis, the solution takes the form

$$u(x,y) = \int_C f \nabla G \cdot \mathbf{n} ds = \int_C f \frac{\partial G}{\partial n} ds.$$

Since

$$G(x,y,\xi,\eta) = \frac{1}{4\pi}\ln\left((\xi-x)^2+(\eta-y)^2\right) - \frac{1}{4\pi}\ln\left((\xi-x)^2+(\eta+y)^2\right)$$

$$\frac{\partial G}{\partial n} = \left.\frac{\partial G(x,y,\xi,\eta)}{\partial \eta}\right|_{\eta=0} = \frac{1}{\pi}\frac{y}{(\xi-x)^2+y^2}$$

We have arrived at the same surface Green's function as we had found in Example 9.11.2 and the solution is

$$u(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{(x-\xi)^2+y^2} f(\xi) d\xi.$$

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7.6: Method of Eigenfunction Expansions

We have seen that the use of eigenfunction expansions is another technique for finding solutions of differential equations. In this section we will show how we can use eigenfunction expansions to find the solutions to nonhomogeneous partial differential equations. In particular, we will apply this technique to solving nonhomogeneous versions of the heat and wave equations.

Nonhomogeneous Heat Equation

In this section we solve the one dimensional heat equation with a source using an eigenfunction expansion. Consider the problem

$$\begin{aligned} u_t &= ku_{xx} + Q(x,t), \quad 0 < x < L, \quad t > 0 \\ u(0,t) &= 0, \quad u(L,t) = 0, \quad t > 0 \\ u(x,0) &= f(x), \quad 0 < x < L \end{aligned} \tag{7.6.1}$$

The homogeneous version of this problem is given by

$$\begin{aligned} v_t &= kv_{xx}, \quad 0 < x < L, \quad t > 0, \\ v(0,t) &= 0, \quad v(L,t) = 0. \end{aligned} \tag{7.6.2}$$

We know that a separation of variables leads to the eigenvalue problem

$$\phi'' + \lambda\phi = 0, \quad \phi(0) = 0, \quad \phi(L) = 0.$$

The eigenfunctions and eigenvalues are given by

$$\phi_n(x) = \sin\frac{n\pi x}{L}, \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \ldots.$$

We can use these eigenfunctions to obtain a solution of the nonhomogeneous problem (7.6.1). We begin by assuming the solution is given by the eigenfunction expansion

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\phi_n(x). \tag{7.6.3}$$

In general, we assume that $v(x,t)$ and $\phi_n(x)$ satisfy the same boundary conditions and that $v(x,t)$ and $v_x(x,t)$ are continuous functions.Note that the difference between this eigenfunction expansion and that in Section 4.3 is that the expansion coefficients are functions of time.

In order to carry out the full process, we will also need to expand the initial profile, $f(x)$, and the source term, $Q(x,t)$, in the basis of eigenfunctions. Thus, we assume the forms

$$\begin{aligned} f(x) &= u(x,0) \\ &= \sum_{n=1}^{\infty} a_n(0)\phi_n(x), \end{aligned} \tag{7.6.4}$$

$$Q(x,t) = \sum_{n=1}^{\infty} q_n(t)\phi_n(x). \tag{7.6.5}$$

Recalling from Chapter 4, the generalized Fourier coefficients are given by

$$a_n(0) = \frac{\langle f, \phi_n\rangle}{\|\phi_n\|^2} = \frac{1}{\|\phi_n\|^2}\int_0^L f(x)\phi_n(x)dx, \tag{7.6.6}$$

$$q_n(t) = \frac{\langle Q, \phi_n\rangle}{\|\phi_n\|^2} = \frac{1}{\|\phi_n\|^2}\int_0^L Q(x,t)\phi_n(x)dx. \tag{7.6.7}$$

The next step is to insert the expansions (7.6.3) and (7.6.5) into the nonhomogeneous heat equation (7.6.1). We first note that

$$u_t(x,t) = \sum_{n=1}^{\infty} \dot{a}_n(t)\phi_n(x),$$
$$u_{xx}(x,t) = -\sum_{n=1}^{\infty} a_n(t)\lambda_n\phi_n(x). \tag{7.6.8}$$

Inserting these expansions into the heat equation (7.6.1), we have

$$\begin{aligned} u_t &= ku_{xx} + Q(x,t) \\ \sum_{n=1}^{\infty} \dot{a}_n(t)\phi_n(x) &= -k\sum_{n=1}^{\infty} a_n(t)\lambda_n\phi_n(x) + \sum_{n=1}^{\infty} q_n(t)\phi_n(x). \end{aligned} \tag{7.6.9}$$

Collecting like terms, we have

$$\sum_{n=1}^{\infty} \left[\dot{a}_n(t) + k\lambda_n a_n(t) - q_n(t)\right]\phi_n(x) = 0, \quad \forall x \in [0,L].$$

Due to the linear independence of the eigenfunctions, we can conclude that

$$\dot{a}_n(t) + k\lambda_n a_n(t) = q_n(t), \quad n = 1,2,3,\ldots.$$

This is a linear first order ordinary differential equation for the unknown expansion coefficients.

We further note that the initial condition can be used to specify the initial condition for this first order ODE. In particular,

$$f(x) = \sum_{n=1}^{\infty} a_n(0)\phi_n(x).$$

The coefficients can be found as generalized Fourier coefficients in an expansion of $f(x)$ in the basis $\phi_n(x)$. These are given by Equation (7.6.6).

Recall from Appendix B that the solution of a first order ordinary differential equation of the form

$$y'(t) + a(t)y(t) = p(t)$$

is found using the integrating factor

$$\mu(t) = \exp\int^t a(\tau)d\tau.$$

Multiplying the ODE by the integrating factor, one has

$$\frac{d}{dt}\left[y(t)\exp\int^t a(\tau)d\tau\right] = p(t)\exp\int^t a(\tau)d\tau.$$

After integrating, the solution can be found providing the integral is doable.

For the current problem, we have

$$\dot{a}_n(t) + k\lambda_n a_n(t) = q_n(t), \quad n = 1,2,3,\ldots.$$

Then, the integrating factor is

$$\mu(t) = \exp\int^t k\lambda_n d\tau = e^{k\lambda_n t}.$$

Multiplying the differential equation by the integrating factor, we find

$$\begin{aligned} \left[\dot{a}_n(t) + k\lambda_n a_n(t)\right]e^{k\lambda_n t} &= q_n(t)e^{k\lambda_n t} \\ \frac{d}{dt}\left(a_n(t)e^{k\lambda_n t}\right) &= q_n(t)e^{k\lambda_n t}. \end{aligned} \tag{7.6.10}$$

Integrating, we have

$$a_n(t)e^{k\lambda_n t} - a_n(0) = \int_0^t q_n(\tau)e^{k\lambda_n \tau}d\tau,$$

or

$$a_n(t) = a_n(0)e^{-k\lambda_n t} + \int_0^t q_n(\tau)e^{-k\lambda_n(t-\tau)}\,d\tau.$$

Using these coefficients, we can write out the general solution.

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} a_n(t)\phi_n(x) \\ &= \sum_{n=1}^{\infty}\left[a_n(0)e^{-k\lambda_n t} + \int_0^t q_n(\tau)e^{-k\lambda_n(t-\tau)}\,d\tau\right]\phi_n(x). \end{aligned} \tag{7.6.11}$$

We will apply this theory to a more specific problem which not only has a heat source but also has nonhomogeneous boundary conditions.

Example 7.6.1

Solve the following nonhomogeneous heat problem using eigenfunction expansions:

$$\begin{aligned} u_t - u_{xx} &= x + t\sin 3\pi x, \quad 0 \le x \le 1, \quad t > 0 \\ u(0,t) &= 2, \quad u(L,t) = t, \quad t > 0 \\ u(x,0) &= 3\sin 2\pi x + 2(1-x), \quad 0 \le x \le 1 \end{aligned} \tag{7.6.12}$$

Solution

This problem has the same nonhomogeneous boundary conditions as those in Example 7.3.2. Recall that we can define

$$u(x,t) = v(x,t) + 2 + (t-2)x$$

to obtain a new problem for $v(x,t)$. The new problem is

$$\begin{aligned} v_t - v_{xx} &= t\sin 3\pi x, \quad 0 \le x \le 1, \quad t > 0, \\ v(0,t) &= 0, \quad v(L,t) = 0, \quad t > 0, \\ v(x,0) &= 3\sin 2\pi x, \quad 0 \le x \le 1. \end{aligned} \tag{7.6.13}$$

We can now apply the method of eigenfunction expansions to find $v(x,t)$. The eigenfunctions satisfy the homogeneous problem

$$\phi_n'' + \lambda_n\phi_n = 0, \quad \phi_n(0) = 0, \quad \phi_n(1) = 0.$$

The solutions are

$$\phi_n(x) = \sin\frac{n\pi x}{L}, \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \ldots$$

Now, let

$$v(x,t) = \sum_{n=1}^{\infty} a_n(t)\sin n\pi x.$$

Inserting $v(x,t)$ into the PDE, we have

$$\sum_{n=1}^{\infty}\left[\dot{a}_n(t) + n^2\pi^2 a_n(t)\right]\sin n\pi x = t\sin 3\pi x.$$

Due to the linear independence of the eigenfunctions, we can equate the coefficients of the $\sin n\pi x$ terms. This gives

$$\begin{aligned} \dot{a}_n(t) + n^2\pi^2 a_n(t) &= 0, \quad n \neq 3, \\ \dot{a}_3(t) + 9\pi^2 a_3(t) &= t, \quad n = 3. \end{aligned} \tag{7.6.14}$$

This is a system of first order ordinary differential equations. The first set of equations are separable and are easily solved. For $n \neq 3$, we seek solutions of

$$\frac{d}{dt}a_n = -n^2\pi^2 a_n(t).$$

These are given by

$$a_n(t) = a_n(0)e^{-n^2\pi^2 t}, \quad n \neq 3.$$

In the case $n = 3$, we seek solutions of

$$\frac{d}{dt}a_3 + 9\pi^2 a_3(t) = t.$$

The integrating factor for this first order equation is given by

$$\mu(t) = e^{9\pi^2 t}.$$

Multiplying the differential equation by the integrating factor, we have

$$\frac{d}{dt}\left(a_3(t)e^{9\pi^2 t}\right) = te^{9\pi^2 t}.$$

Integrating, we obtain the solution

$$\begin{aligned} a_3(t) &= a_3(0)e^{-9\pi^2 t} + e^{-9\pi^2 t}\int_0^t \tau e^{9\pi^2\tau}d\tau, \\ &= a_3(0)e^{-9\pi^2 t} + e^{-9\pi^2 t}\left[\frac{1}{9\pi^2}\tau e^{9\pi^2\tau} - \frac{1}{(9\pi^2)^2}e^{9\pi^2\tau}\right]_0^t, \\ &= a_3(0)e^{-9\pi^2 t} + \frac{1}{9\pi^2}t - \frac{1}{(9\pi^2)^2}\left[1 - e^{-9\pi^2\tau}\right]. \end{aligned} \tag{7.6.15}$$

Up to this point, we have the solution

$$\begin{aligned} u(x,t) &= v(x,t) + w(x,t) \\ &= \sum_{n=1}^{\infty} a_n(t)\sin n\pi x + 2 + (t-2)x, \end{aligned} \tag{7.6.16}$$

where

$$\begin{aligned} a_n(t) &= a_n(0)e^{-n^2\pi^2 t}, \quad n \neq 3 \\ a_3(t) &= a_3(0)e^{-9\pi^2 t} + \frac{1}{9\pi^2}t - \frac{1}{(9\pi^2)^2}\left[1 - e^{-9\pi^2\tau}\right] \end{aligned} \tag{7.6.17}$$

We still need to find $a_n(0), n = 1, 2, 3, \ldots$.

The initial values of the expansion coefficients are found using the initial condition

$$v(x,0) = 3\sin 2\pi x = \sum_{n=1}^{\infty} a_n(0)\sin n\pi x.$$

It is clear that we have $a_n(0) = 0$ for $n \neq 2$ and $a_2(0) = 3$. Thus, the series for $v(x,t)$ has two nonvanishing coefficients,

$$\begin{aligned} a_2(t) &= 3e^{-4\pi^2 t} \\ a_3(t) &= \frac{1}{9\pi^2}t - \frac{1}{(9\pi^2)^2}\left[1 - e^{-9\pi^2\tau}\right]. \end{aligned} \tag{7.6.18}$$

Therefore, the final solution is given by

$$u(x,t) = 2 + (t-2)x + 3e^{-4\pi^2 t}\sin 2\pi x + \frac{9\pi^2 t - \left(1 - e^{-9\pi^2\tau}\right)}{81\pi^4}\sin 3\pi x.$$

Forced Vibrating Membrane

We now consider the forced vibrating membrane. A two-dimensional membrane is stretched over some domain D. We assume Dirichlet conditions on the boundary, $u = 0$ on ∂D. The forced membrane can be modeled as

$$\begin{aligned} u_{tt} &= c^2\nabla^2 u + Q(\mathbf{r},t), \quad \mathbf{r}\in D, \quad t>0, \\ u(\mathbf{r},t) &= 0, \quad \mathbf{r}\in\partial D, \quad t>0, \\ u(\mathbf{r},0) &= f(\mathbf{r}), \quad u_t(\mathbf{r},0) = g(\mathbf{r}), \quad \mathbf{r}\in D. \end{aligned} \tag{7.6.19}$$

The method of eigenfunction expansions relies on the use of eigenfunctions, $\phi_\alpha(\mathbf{r})$, for $\alpha \in J \subset Z^2$ a set of indices typically of the form (i,j) in some lattice grid of integers. The eigenfunctions satisfy the eigenvalue equation

$$\nabla^2\phi_\alpha(\mathbf{r}) = -\lambda_\alpha\phi_\alpha(\mathbf{r}), \quad \phi_\alpha(\mathbf{r}) = 0, \text{ on } \partial D.$$

We assume that the solution and forcing function can be expanded in the basis of eigenfunctions,

$$\begin{aligned} u(\mathbf{r},t) &= \sum_{\alpha\in J} a_\alpha(t)\phi_\alpha(\mathbf{r}), \\ Q(\mathbf{r},t) &= \sum_{\alpha\in J} q_\alpha(t)\phi_\alpha(\mathbf{r}). \end{aligned} \tag{7.6.20}$$

Inserting this form into the forced wave equation (7.6.19), we have

$$\begin{aligned} u_{tt} &= c^2\nabla^2 u + Q(\mathbf{r},t) \\ \sum_{\alpha\in J}\ddot{a}_\alpha(t)\phi_\alpha(\mathbf{r}) &= -c^2\sum_{\alpha\in J}\lambda_\alpha a_\alpha(t)\phi_\alpha(\mathbf{r}) + \sum_{\alpha\in J} q_\alpha(t)\phi_\alpha(\mathbf{r}) \\ 0 &= \sum_{\alpha\in J}\left[\ddot{a}_\alpha(t) + c^2\lambda_\alpha a_\alpha(t) - q_\alpha(t)\right]\phi_\alpha(\mathbf{r}). \end{aligned} \tag{7.6.21}$$

The linear independence of the eigenfunctions then gives the ordinary differential equation

$$\ddot{a}_\alpha(t) + c^2\lambda_\alpha a_\alpha(t) = q_\alpha(t).$$

We can solve this equation with initial conditions $a_\alpha(0)$ and $\dot{a}_\alpha(0)$ found from

$$\begin{aligned} f(\mathbf{r}) = u(\mathbf{r},0) &= \sum_{\alpha\in J} a_\alpha(0)\phi_\alpha(\mathbf{r}), \\ g(\mathbf{r}) = u_t(\mathbf{r},0) &= \sum_{\alpha\in J}\dot{a}_\alpha(0)\phi_\alpha(\mathbf{r}). \end{aligned} \tag{7.6.22}$$

Example 7.6.2

Periodic Forcing, $Q(\mathbf{r},t) = G(\mathbf{r})\cos\omega t$.

Solution

It is enough to specify $Q(\mathbf{r},t)$ in order to solve for the time dependence of the expansion coefficients. A simple example is the case of periodic forcing, $Q(\mathbf{r},t) = h(\mathbf{r})\cos\omega t$. In this case, we expand Q in the basis of eigenfunctions,

$$\begin{aligned} Q(\mathbf{r},t) &= \sum_{\alpha\in J} q_\alpha(t)\phi_\alpha(\mathbf{r}), \\ G(\mathbf{r})\cos\omega t &= \sum_{\alpha\in J}\gamma_\alpha\cos\omega t\phi_\alpha(\mathbf{r}). \end{aligned} \tag{7.6.23}$$

Inserting these expressions into the forced wave equation (7.6.19), we obtain a system of differential equations for the expansion coefficients,

$$\ddot{a}_\alpha(t) + c^2\lambda_\alpha a_\alpha(t) = \gamma_\alpha\cos\omega t.$$

In order to solve this equation we borrow the methods from a course on ordinary differential equations for solving nonhomogeneous equations. In particular we can use the Method of Undetermined Coefficients as reviewed in Section B.3.1. The solution of these equations are of the form

$$a_\alpha(t) = a_{\alpha h}(t) + a_{\alpha p}(t),$$

where $a_{\alpha h}(t)$ satisfies the homogeneous equation,

$$\ddot{a}_{\alpha h}(t) + c^2\lambda_\alpha a_{\alpha h}(t) = 0, \tag{7.6.24}$$

and $a_{\alpha p}(t)$ is a particular solution of the nonhomogeneous equation,

$$\ddot{a}_{\alpha p}(t) + c^2\lambda_\alpha a_{\alpha p}(t) = \gamma_\alpha \cos\omega t. \tag{7.6.25}$$

The solution of the homogeneous problem (7.6.24) is easily founds as

$$a_{\alpha h}(t) = c_{1\alpha}\cos(\omega_{0\alpha}t) + c_{2\alpha}\sin(\omega_{0\alpha}t),$$

where $\omega_{0\alpha} = c\sqrt{\lambda_\alpha}$.

The particular solution is found by making the guess $a_{\alpha p}(t) = A_\alpha \cos\omega t$. Inserting this guess into Equation (ceqn2), we have

$$\left[-\omega^2 + c^2\lambda_\alpha\right] A_\alpha \cos\omega t = \gamma_\alpha \cos\omega t$$

Solving for A_α, we obtain

$$A_\alpha = \frac{\gamma_\alpha}{-\omega^2 + c^2\lambda_\alpha}, \quad \omega^2 \neq c^2\lambda_\alpha.$$

Then, the general solution is given by

$$a_\alpha(t) = c_{1\alpha}\cos(\omega_{0\alpha}t) + c_{2\alpha}\sin(\omega_{0\alpha}t) + \frac{\gamma_\alpha}{-\omega^2 + c^2\lambda_\alpha}\cos\omega t,$$

where $\omega_{0\alpha} = c\sqrt{\lambda_\alpha}$ and $\omega^2 \neq c^2\lambda_\alpha$

In the case where $\omega^2 = c^2\lambda_\alpha$, we have a resonant solution. This is discussed in Section FO on forced oscillations. In this case the Method of Undetermined Coefficients fails and we need the Modified Method of Undetermined Coefficients. This is because the driving term, $\gamma_\alpha \cos\omega t$, is a solution of the homogeneous problem. So, we make a different guess for the particular solution. We let

$$a_{\alpha p}(t) = t\left(A_\alpha \cos\omega t + B_\alpha \sin\omega t\right).$$

Then, the needed derivatives are

$$\begin{aligned} a_{\alpha p}(t) &= \omega t\left(-A_\alpha \sin\omega t + B_\alpha \cos\omega t\right) + A_\alpha \cos\omega t + B_\alpha \sin\omega t \\ a_{\alpha p}(t) &= -\omega^2 t\left(A_\alpha \cos\omega t + B_\alpha \sin\omega t\right) - 2\omega A_\alpha \sin\omega t + 2\omega B_\alpha \cos\omega t \\ &= -\omega^2 a_{\alpha p}(t) - 2\omega A_\alpha \sin\omega t + 2\omega B_\alpha \cos\omega t \end{aligned} \tag{7.6.26}$$

Inserting this guess into Equation (ceqn 2) and noting that $\omega^2 = c^2\lambda_\alpha$, we have

$$-2\omega A_\alpha \sin\omega t + 2\omega B_\alpha \cos\omega t = \gamma_\alpha \cos\omega t.$$

Therefore, $A_\alpha = 0$ and

$$B_\alpha = \frac{\gamma_\alpha}{2\omega}.$$

So, the particular solution becomes

$$a_{\alpha p}(t) = \frac{\gamma_\alpha}{2\omega} t \sin\omega t.$$

The full general solution is then

$$a_\alpha(t) = c_{1\alpha}\cos(\omega t) + c_{2\alpha}\sin(\omega t) + \frac{\gamma_\alpha}{2\omega} t \sin\omega t,$$

where $\omega = c\sqrt{\lambda_\alpha}$.

We see from this result that the solution tends to grow as t gets large. This is what is called a resonance. Essentially, one is driving the system at its natural frequency for one of the frequencies in the system. A typical plot of such a solution in Figure 7.6.1.

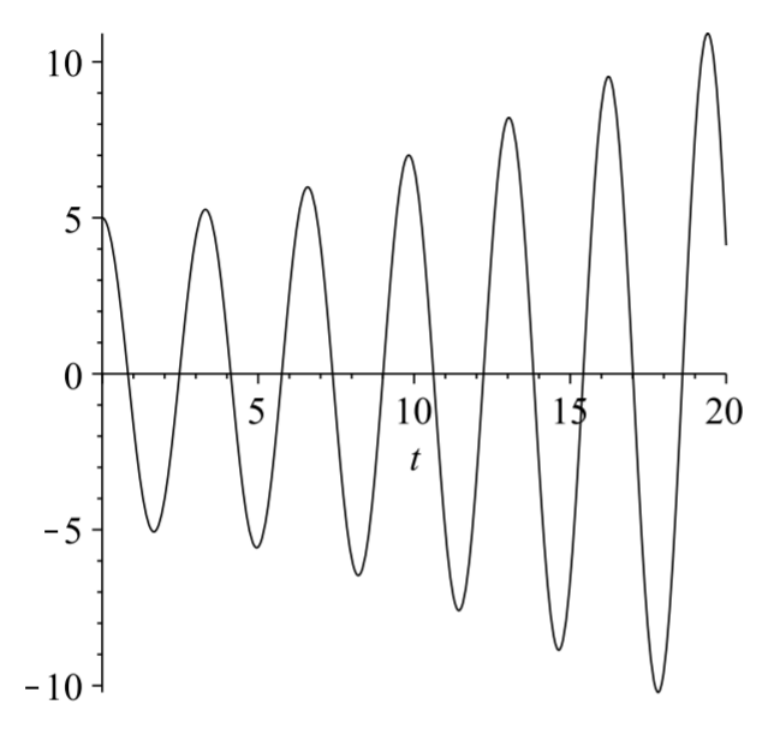


Figure 7.6.1: Plot of a solution showing resonance.

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7.7: Green's Function Solution of Nonhomogeneous Heat Equation

We solved the one dimensional heat equation with a source using an eigenfunction expansion. In this section we rewrite the solution and identify the Green's function form of the solution. Recall that the solution of the nonhomogeneous problem,

$$\begin{aligned} u_t &= ku_{xx} + Q(x,t), \quad 0 < x < L, \quad t > 0, \\ u(0,t) &= 0, \quad u(L,t) = 0, \quad t > 0, \\ u(x,0) &= f(x), \quad 0 < x < L, \end{aligned} \tag{7.7.1}$$

is given by Equation (7.6.11)

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} a_n(t)\phi_n(x) \\ &= \sum_{n=1}^{\infty} \left[a_n(0)e^{-k\lambda_n t} + \int_0^t q_n(\tau)e^{-k\lambda_n(t-\tau)}\,d\tau \right] \phi_n(x) \cdot (7 \cdot 134) \end{aligned} \tag{7.7.2}$$

The generalized Fourier coefficients for $a_n(0)$ and $q_n(t)$ are given by

$$a_n(0) = \frac{1}{\|\phi_n\|^2} \int_0^L f(x)\phi_n(x)dx, \tag{7.7.3}$$

$$q_n(t) = \frac{1}{\|\phi_n\|^2} \int_0^L Q(x,t)\phi_n(x)dx. \tag{7.7.4}$$

The solution in Equation (7.7.2) can be rewritten using the Fourier coefficients in Equations (7.7.3) and (7.7.4).

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} \left[a_n(0)e^{-k\lambda_n t} + \int_0^t q_n(\tau)e^{-k\lambda_n(t-\tau)}\,d\tau \right] \phi_n(x) \\ &= \sum_{n=1}^{\infty} a_n(0)e^{-k\lambda_n t}\phi_n(x) + \int_0^t \sum_{n=1}^{\infty} \left(q_n(\tau)e^{-k\lambda_n(t-\tau)}\phi_n(x) \right) d\tau \\ &= \sum_{n=1}^{\infty} \frac{1}{\|\phi_n\|^2} \left(\int_0^L f(\xi)\phi_n(\xi)d\xi \right) e^{-k\lambda_n t}\phi_n(x) \\ &\quad + \int_0^t \sum_{n=1}^{\infty} \frac{1}{\|\phi_n\|^2} \left(\int_0^L Q(\xi,\tau)\phi_n(\xi)d\xi \right) e^{-k\lambda_n(t-\tau)}\phi_n(x)d\tau \\ &= \int_0^L \left(\sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(\xi)e^{-k\lambda_n t}}{\|\phi_n\|^2} \right) f(\xi)d\xi \\ &\quad + \int_0^t \int_0^L \left(\sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(\xi)e^{-k\lambda_n(t-\tau)}}{\|\phi_n\|^2} \right) Q(\xi,\tau)d\xi d\tau. \end{aligned} \tag{7.7.5}$$

Defining

$$G(x,t;\xi,\tau) = \sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(\xi)e^{-k\lambda_n(t-\tau)}}{\|\phi_n\|^2}$$

we see that the solution can be written in the form

$$u(x,t) = \int_0^L G(x,t;\xi,0)f(\xi)d\xi + \int_0^t \int_0^L G(x,t;\xi,\tau)Q(\xi,\tau)d\xi d\tau.$$

Thus, we see that $G(x,t;\xi,0)$ is the initial value Green's function and $G(x,t;\xi,\tau)$ is the general Green's function for this problem.

Note

The solution can be written in terms of the initial value Green's function, $G(x,t;\xi,0)$, and the general Green's function, $G(x,t;\varepsilon,\tau)$.

The only thing left is to introduce nonhomogeneous boundary conditions into this solution. So, we modify the original problem to the fully nonhomogeneous heat equation:

$$\begin{aligned} u_t &= ku_{xx}+Q(x,t), \quad 0<x<L, \quad t>0 \\ u(0,t) &= \alpha(t), \quad u(L,t)=\beta(t), \quad t>0 \\ u(x,0) &= f(x), \quad 0<x<L \end{aligned} \tag{7.7.6}$$

As before, we begin with the expansion of the solution in the basis of

$$u(x,t)=\sum_{n=1}^{\infty} a_n(t)\phi_n(x)$$

However, due to potential convergence problems, we cannot expect that u_{xx} can be obtained by simply differentiating the series twice and expecting the resulting series to converge to u_{xx}. So, we need to be a little more careful.

We first note that

$$u_t=\sum_{n=1}^{\infty}\dot{a}_n(t)\phi_n(x)=ku_{xx}+Q(x,t).$$

Solving for the expansion coefficients, we have

$$\dot{a}(t)=\frac{\int_0^L\left(ku_{xx}+Q(x,t)\right)\phi_n(x)dx}{\|\phi_n\|^2}.$$

In order to proceed, we need an expression for $\int_a^b u_{xx}\phi_n(x)dx$. We can find this using Green's identity from Section 4.2.2.

We start with

$$\int_a^b(u\mathcal{L}v-v\mathcal{L}u)dx=\left[p\left(uv'-vu'\right)\right]_a^b$$

and let $v=\phi_n$. Then,

$$\begin{aligned} \int_0^L\left(u(x,t)\phi_n''(x)-\phi_n(x)u_{xx}(x,t)\right)dx &= \left[u(x,t)\phi_n'(x)-\phi_n(x)u_x(x,t)\right)]_0^L \\ \int_0^L\left(-\lambda_n u(x,t)+u_{xx}(x,t)\right)\phi_n(x)dx &= \left[u(L,t)\phi_n'(L)-\phi_n(L)u_x(L,t)\right)] \\ &\quad -\left[u(0,t)\phi_n'(0)-\phi_n(0)u_x(0,t)\right)] \\ -\lambda_n a_n\|\phi_n\|^2-\int_0^L u_{xx}(x,t)\phi_n(x)dx &= \beta(t)\phi_n'(L)-\alpha(t)\phi_n'(0). \end{aligned}$$

Thus,

$$\int_0^L u_{xx}(x,t)\phi_n(x)dx=-\lambda_n a_n\|\phi_n\|^2+\alpha(t)\phi_n'(0)-\beta(t)\phi_n'(L).$$

Inserting this result into the equation for $\dot{a}_n(t)$, we have

$$\dot{a}(t)=-k\lambda_n a_n(t)+q_n(t)+k\frac{\alpha(t)\phi_n'(0)-\beta(t)\phi_n'(L)}{\|\phi_n\|^2}.$$

As we had seen before, this first order equation can be solved using the integrating factor

$$\mu(t) = \exp \int^t k\lambda_n d\tau = e^{k\lambda_n t}.$$

Multiplying the differential equation by the integrating factor, we find

$$\begin{aligned} \left[\dot{a}_n(t) + k\lambda_n a_n(t)\right] e^{k\lambda_n t} &= \left[q_n(t) + k\frac{\alpha(t)\phi_n'(0) - \beta(t)\phi_n'(L)}{\|\phi_n\|^2}\right] e^{k\lambda_n t} \\ \frac{d}{dt}\left(a_n(t)e^{k\lambda_n t}\right) &= \left[q_n(t) + k\frac{\alpha(t)\phi_n'(0) - \beta(t)\phi_n'(L)}{\|\phi_n\|^2}\right] e^{k\lambda_n t}. \end{aligned} \tag{7.7.7}$$

Integrating, we have

$$a_n(t)e^{k\lambda_n t} - a_n(0) = \int_0^t \left[q_n(\tau) + k\frac{\alpha(\tau)\phi_n'(0) - \beta(\tau)\phi_n'(L)}{\|\phi_n\|^2}\right] e^{k\lambda_n \tau} d\tau,$$

or

$$a_n(t) = a_n(0)e^{-k\lambda_n t} + \int_0^t \left[q_n(\tau) + k\frac{\alpha(\tau)\phi_n'(0) - \beta(\tau)\phi_n'(L)}{\|\phi_n\|^2}\right] e^{-k\lambda_n (t-\tau)} d\tau.$$

We can now insert these coefficients into the solution and see how to extract the Green's function contributions. Inserting the coefficients, we have

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} a_n(t)\phi_n(x) \\ &= \sum_{n=1}^{\infty} \left[a_n(0)e^{-k\lambda_n t} + \int_0^t q_n(\tau)e^{-k\lambda_n (t-\tau)} d\tau\right] \phi_n(x) \\ &\quad + \sum_{n=1}^{\infty} \left(\int_0^t \left[k\frac{\alpha(\tau)\phi_n'(0) - \beta(\tau)\phi_n'(L)}{\|\phi_n\|^2}\right] e^{-k\lambda_n (t-\tau)} d\tau\right) \phi_n(x) \end{aligned} \tag{7.7.8}$$

Recall that the generalized Fourier coefficients for $a_n(0)$ and $q_n(t)$ are given by

$$a_n(0) = \frac{1}{\|\phi_n\|^2} \int_0^L f(x)\phi_n(x) dx, \tag{7.7.9}$$

$$q_n(t) = \frac{1}{\|\phi_n\|^2} \int_0^L Q(x,t)\phi_n(x) dx. \tag{7.7.10}$$

The solution in Equation (7.7.8) can be rewritten using the Fourier coefficients in Equations (7.7.9) and (7.7.10).

$$
\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty}\left[a_n(0)e^{-k\lambda_n t} + \int_0^t q_n(\tau)e^{-k\lambda_n(t-\tau)}\,d\tau\right]\phi_n(x) \\
&\quad + \sum_{n=1}^{\infty}\left(\int_0^t \left[k\frac{\alpha(\tau)\phi_n'(0) - \beta(\tau)\phi_n'(L)}{\|\phi_n\|^2}\right]e^{-k\lambda_n(t-\tau)}\,d\tau\right)\phi_n(x) \\
&= \sum_{n=1}^{\infty} a_n(0)e^{-k\lambda_n t}\phi_n(x) + \int_0^t \sum_{n=1}^{\infty}\left(q_n(\tau)e^{-k\lambda_n(t-\tau)}\,\phi_n(x)\right)d\tau \\
&\quad + \int_0^t \sum_{n=1}^{\infty}\left(\left[k\frac{\alpha(\tau)\phi_n'(0) - \beta(\tau)\phi_n'(L)}{\|\phi_n\|^2}\right]e^{-k\lambda_n(t-\tau)}\right)\phi_n(x)d\tau \\
&= \sum_{n=1}^{\infty}\frac{1}{\|\phi_n\|^2}\left(\int_0^L f(\xi)\phi_n(\xi)d\xi\right)e^{-k\lambda_n t}\phi_n(x) \\
&\quad + \int_0^t \sum_{n=1}^{\infty}\frac{1}{\|\phi_n\|^2}\left(\int_0^L Q(\xi,\tau)\phi_n(\xi)d\xi\right)e^{-k\lambda_n(t-\tau)}\,\phi_n(x)d\tau \\
&\quad + \int_0^t \sum_{n=1}^{\infty}\left(\left[k\frac{\alpha(\tau)\phi_n'(0) - \beta(\tau)\phi_n'(L)}{\|\phi_n\|^2}\right]e^{-k\lambda_n(t-\tau)}\right)\phi_n(x)d\tau \\
&= \int_0^L\left(\sum_{n=1}^{\infty}\frac{\phi_n(x)\phi_n(\xi)e^{-k\lambda_n t}}{\|\phi_n\|^2}\right)f(\xi)d\xi \\
&\quad + \int_0^t\int_0^L\left(\sum_{n=1}^{\infty}\frac{\phi_n(x)\phi_n(\xi)e^{-k\lambda_n(t-\tau)}}{\|\phi_n\|^2}\right)Q(\xi,\tau)d\xi^2 d\tau. \\
&\quad + k\int_0^t\left(\sum_{n=1}^{\infty}\frac{\phi_n(x)\phi_n'(0)e^{-k\lambda_n(t-\tau)}}{\|\phi_n\|^2}\right)\alpha(\tau)d\tau \\
&\quad - k\int_0^t\left(\sum_{n=1}^{\infty}\frac{\phi_n(x)\phi_n'(L)e^{-k\lambda_n(t-\tau)}}{\|\phi_n\|^2}\right)\beta(\tau)d\tau.
\end{aligned}
\tag{7.7.11}
$$

As before, we can define the general Green's function as

$$
G(x,t;\xi,\tau) = \sum_{n=1}^{\infty}\frac{\phi_n(x)\phi_n(\xi)e^{-k\lambda_n(t-\tau)}}{\|\phi_n\|^2}.
$$

Then, we can write the solution to the fully homogeneous problem as

$$
\begin{aligned}
u(x,t) &= \int_0^t\int_0^L G(x,t;\xi,\tau)Q(\xi,\tau)d\xi d\tau + \int_0^L G(x,t;\xi,0)f(\xi)d\xi \\
&\quad + k\int_0^t\left[\alpha(\tau)\frac{\partial G}{\partial\xi}(x,0;t,\tau) - \beta(\tau)\frac{\partial G}{\partial\xi}(x,L;t,\tau)\right]d\tau.
\end{aligned}
\tag{7.7.12}
$$

The first integral handles the source term, the second integral handles the initial condition, and the third term handles the fixed boundary conditions.

This general form can be deduced from the differential equation for the Green's function and original differential equation by using a more general form of Green's identity. Let the heat equation operator be defined as $\mathcal{L} = \frac{\partial}{\partial t} - k\frac{\partial^2}{\partial x^2}$. The differential equations for $u(x,t)$ and $G(x,t;\xi,\tau)$ for $0 \le x,\xi \le L$ and $t,\tau \ge 0$, are taken to be

$$
\begin{aligned}
\mathcal{L}u(x,t) &= Q(x,t), \\
\mathcal{L}G(x,t;\xi,\tau) &= \delta(x-\xi)\delta(t-\tau).
\end{aligned}
\tag{7.7.13}
$$

Multiplying the first equation by $G(x,t;\xi,\tau)$ and the second by $u(x,t)$, we obtain

$$
\begin{aligned}
G(x,t;\xi,\tau)\mathcal{L}u(x,t) &= G(x,t;\xi,\tau)Q(x,t), \\
u(x,t)\mathcal{L}G(x,t;\xi,\tau) &= \delta(x-\xi)\delta(t-\tau)u(x,t).
\end{aligned}
\tag{7.7.14}
$$

Now, we subtract the equations and integrate with respect to x and t. This gives

$$\begin{aligned}
&\int_0^\infty \int_0^L [G(x,t;\xi,\tau)\mathcal{L}u(x,t) - u(x,t)\mathcal{L}G(x,t;\xi,\tau)]dxdt \\
&= \int_0^\infty \int_0^L [G(x,t;\xi,\tau)Q(x,t) - \delta(x-\xi)\delta(t-\tau)u(x,t)]dxdt \\
&= \int_0^\infty \int_0^L G(x,t;\xi,\tau)Q(x,t)dxdt - u(\xi,\tau).
\end{aligned} \tag{7.7.15}$$

and

$$\begin{aligned}
&\int_0^\infty \int_0^L [G(x,t;\xi,\tau)\mathcal{L}u(x,t) - u(x,t)\mathcal{L}G(x,t;\xi,\tau)]dxdt \\
&= \int_0^L \int_0^\infty [G(x,t;\xi,\tau)u_t - u(x,t)G_t(x,t;\xi,\tau)]\, dtdx \\
&\quad - k\int_0^\infty \int_0^L [G(x,t;\xi,\tau)u_{xx}(x,t) - u(x,t)G_{xx}(x,t;\xi,\tau)]\, dxdt \\
&= \int_0^L \left[G(x,t;\xi,\tau)u_t|_0^\infty - 2\int_0^\infty u(x,t)G_t(x,t;\xi,\tau)dt\right] dx \\
&\quad - k\int_0^\infty \left[G(x,t;\xi,\tau)\frac{\partial u}{\partial x}(x,t) - u(x,t)\frac{\partial G}{\partial x}(x,t;\xi,\tau)\right]_0^L dxdt
\end{aligned} \tag{7.7.16}$$

Equating these two results and solving for $u(\xi,\tau)$, we have

$$\begin{aligned}
u(\xi,\tau) = &\int_0^\infty \int_0^L G(x,t;\xi,\tau)Q(x,t)dxdt \\
&+ k\int_0^\infty \left[G(x,t;\xi,\tau)\frac{\partial u}{\partial x}(x,t) - u(x,t)\frac{\partial G}{\partial x}(x,t;\xi,\tau)\right]_0^L dxdt \\
&+ \int_0^L \left[G(x,0;\xi,\tau)u(x,0) + 2\int_0^\infty u(x,t)G_t(x,t;\xi,\tau)dt\right] dx.
\end{aligned} \tag{7.7.17}$$

Exchanging (ξ,τ) with (x,t) and assuming that the Green's function is symmetric in these arguments, we have

$$\begin{aligned}
u(x,t) = &\int_0^\infty \int_0^L G(x,t;\xi,\tau)Q(\xi,\tau)d\xi d\tau \\
&+ k\int_0^\infty \left[G(x,t;\xi,\tau)\frac{\partial u}{\partial \xi}(\xi,\tau) - u(\xi,\tau)\frac{\partial G}{\partial \xi}(x,t;\xi,\tau)\right]_0^L dxdt \\
&+ \int_0^L G(x,t;\xi,0)u(\xi,0)d\xi + 2\int_0^L \int_0^\infty u(\xi,\tau)G_\tau(x,t;\xi,\tau)d\tau d\xi
\end{aligned} \tag{7.7.18}$$

This result is almost in the desired form except for the last integral. Thus, if

$$\int_0^L \int_0^\infty u(\xi,\tau)G_\tau(x,t;\xi,\tau)d\tau d\xi = 0,$$

then we have

$$\begin{aligned}
u(x,t) = &\int_0^\infty \int_0^L G(x,t;\xi,\tau)Q(\xi,\tau)d\xi d\tau + \int_0^L G(x,t;\xi,0)u(\xi,0)d\xi \\
&+ k\int_0^\infty \left[G(x,t;\xi,\tau)\frac{\partial u}{\partial \xi}(\xi,\tau) - u(\xi,\tau)\frac{\partial G}{\partial \xi}(x,t;\xi,\tau)\right]_0^L dxdt.
\end{aligned} \tag{7.7.19}$$

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7.8: Summary

We have seen throughout the chapter that Green's functions are the solutions of a differential equation representing the effect of a point impulse on either source terms, or initial and boundary conditions. The Green's function is obtained from transform methods or as an eigenfunction expansion. In the text we have occasionally rewritten solutions of differential equations in term's of Green's functions. We will first provide a few of these examples and then present a compilation of Green's Functions for generic partial differential equations.

For example, in section 7.4 we wrote the solution of the one dimensional heat equation as

$$u(x,t) = \int_0^L G(x,\xi;t,0) f(\xi) d\xi$$

where

$$G(x,\xi;t,0) = \frac{2}{L}\sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\sin\frac{n\pi\xi}{L} e^{\lambda_n kt},$$

and the solution of the wave equation as

$$u(x,t) = \int_0^L G_c(x,\xi,t,0) f(\xi) d\xi + \int_0^L G_s(x,\xi,t,0) g(\xi) d\xi,$$

where

$$G_c(x,\xi,t,0) = \frac{2}{L}\sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\sin\frac{n\pi\xi}{L}\cos\frac{n\pi ct}{L},$$

$$G_s(x,\xi,t,0) = \frac{2}{L}\sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\sin\frac{n\pi\xi}{L}\frac{\sin\frac{n\pi ct}{L}}{n\pi c/L}.$$

We note that setting $t = 0$ in $G_c(x,\xi;t,0)$, we obtain

$$\mathrm{G}_c(x,\xi,0,0) = \frac{2}{L}\sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\sin\frac{n\pi\xi}{L}.$$

This is the Fourier sine series representation of the Dirac delta function, $\delta(x-\xi)$. Similarly, if we differentiate $G_s(x,\xi,t,0)$ with repsect to t and set $t = 0$, we once again obtain the Fourier sine series representation of the Dirac delta function.

It is also possible to find closed form expression for Green's functions, which we had done for the heat equation on the infinite interval,

$$u(x,t) = \int_{-\infty}^{\infty} G(x,t;\xi,0) f(\xi) d\xi,$$

where

$$G(x,t;\xi,0) = \frac{e^{-(x-\xi)^2/4t}}{\sqrt{4\pi t}}$$

and for Poisson's equation,

$$\phi(\mathbf{r}) = \int_V G(\mathbf{r},\mathbf{r}') f(\mathbf{r}') d^3r',$$

where the three dimensional Green's function is given by

$$G(\mathbf{r},\mathbf{r}') = \frac{1}{|\mathbf{r}-\mathbf{r}'|}.$$

We can construct Green's functions for other problems which we have seen in the book. For example, the solution of the two dimensional wave equation on a rectangular membrane was found in Equation (6.1.26) as

$$u(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(a_{nm}\cos\omega_{nm}t+b_{nm}\sin\omega_{nm}t\right)\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H},\ (7.153) \tag{7.8.1}$$

where

$$a_{nm}=\frac{4}{LH}\int_0^H\int_0^L f(x,y)\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}dxdy, \tag{7.8.2}$$

$$b_{nm}=\frac{4}{\omega_{nm}LH}\int_0^H\int_0^L g(x,y)\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}dxdy, \tag{7.8.3}$$

where the angular frequencies are given by

$$\omega_{nm}=c\sqrt{\left(\frac{n\pi}{L}\right)^2+\left(\frac{m\pi}{H}\right)^2}. \tag{7.8.4}$$

Rearranging the solution, we have

$$u(x,y,t)=\int_0^H\int_0^L\left[G_c(x,y;\xi,\eta;t,0)f(\xi,\eta)+G_s(x,y;\xi,\eta;t,0)g(\xi,\eta)\right]d\xi d\eta,$$

where

$$G_c(x,y;\xi,\eta;t,0)=\frac{4}{LH}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sin\frac{n\pi x}{L}\sin\frac{n\pi\tilde{\xi}}{L}\sin\frac{m\pi y}{H}\sin\frac{m\pi\eta}{H}\cos\omega_{nm}t$$

and

$$G_s(x,y;\xi,\eta;t,0)=\frac{4}{LH}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sin\frac{n\pi x}{L}\sin\frac{n\pi\xi}{L}\sin\frac{m\pi y}{H}\sin\frac{m\pi\eta}{H}\frac{\sin\omega_{nm}t}{\omega_{nm}}.$$

Once again, we note that setting $t=0$ in $G_c(x,\xi;t,0)$ and setting $t=0$ in $\frac{\partial G_c(x,\zeta;t,0)}{\partial t}$, we obtain a Fourier series representation of the Dirac delta function in two dimensions,

$$\delta(x-\xi)\delta(y-\eta)=\frac{4}{LH}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sin\frac{n\pi x}{L}\sin\frac{n\pi\xi}{L}\sin\frac{m\pi y}{H}\sin\frac{m\pi\eta}{H}.$$

Another example was the solution of the two dimensional Laplace equation on a disk given by Equation (6.3.28). We found that

$$u(r,\theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos n\theta+b_n\sin n\theta\right)r^n. \tag{7.8.5}$$

$$a_n=\frac{1}{\pi a^n}\int_{-\pi}^{\pi}f(\theta)\cos n\theta d\theta,\quad n=0,1,\ldots, \tag{7.8.6}$$

$$b_n=\frac{1}{\pi a^n}\int_{-\pi}^{\pi}f(\theta)\sin n\theta d\theta\quad n=1,2\ldots \tag{7.8.7}$$

We saw that this solution can be written as

$$u(r,\theta)=\int_{-\pi}^{\pi}G(\theta,\phi;r,a)f(\phi)d\phi,$$

where the Green's function could be summed giving the Poisson kernel

$$G(\theta,\phi;r,a)=\frac{1}{2\pi}\frac{a^2-r^2}{a^2+r^2-2ar\cos(\theta-\phi)}.$$

We had also investigated the nonhomogeneous heat equation in section 9.11.4,

$$\begin{aligned} u_t - ku_{xx} &= h(x,t), \quad 0 \le x \le L, \quad t > 0. \\ u(0,t) &= 0, \quad u(L,t) = 0, \quad t > 0, \\ u(x,0) &= f(x), \quad 0 \le x \le. \end{aligned} \tag{7.8.8}$$

We found that the solution of the heat equation is given by

$$u(x,t) = \int_0^L f(\xi) G(x,\xi;t,0) d\xi + \int_0^t \int_0^L h(\xi,\tau) G(x,\xi;t,\tau) d\xi^\tau d\tau,$$

where

$$G(x,\xi;t,\tau) = \frac{2}{L} \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L} \sin\frac{n\pi\tilde{\xi}}{L} e^{-\omega_n^2(t-\tau)}$$

Note that setting $t = \tau$, we again get a Fourier sine series representation of the Dirac delta function.

In general, Green's functions based on eigenfunction expansions over eigenfunctions of Sturm-Liouville eigenvalue problems are a common way to construct Green's functions. For example, surface and initial value Green's functions are constructed in terms of a modification of delta function representations modified by factors which make the Green's function a solution of the given differential equations and a factor taking into account the boundary or initial condition plus a restoration of the delta function when applied to the condition. Examples with an indication of these factors are shown below.

1. $$g(x,y,z;x',y',c) = \sum_{\ell,n} \underbrace{\frac{2}{a}\sin\frac{\ell\pi x}{a}\sin\frac{\ell\pi x'}{a}\frac{2}{b}\sin\frac{n\pi y}{b}\sin\frac{n\pi y'}{b}}_{\delta\text{-function}} [\underbrace{\sinh\gamma_{\ell n} z}_{\text{D.E.}} / \underbrace{\sinh\gamma_{\ell n} c}_{\text{restore }\delta}].$$
2. $$g(r,\phi,\theta;a,\phi',\theta') = \sum_{\ell,m} \underbrace{Y_\ell^{m*}(\psi'\theta') Y_\ell^{m*}(\psi\theta)}_{\delta\text{-function}} [\underbrace{r^\ell}_{\text{D.E.}} / \underbrace{a^\ell}_{\text{restore }\delta}].$$
3. $$g(x,t;x',t_0) = \sum_n \underbrace{\frac{2}{L}\sin\frac{n\pi x}{L}\sin\frac{n\pi x'}{L}}_{\delta-\text{ function}} [\underbrace{e^{-a^2k_n^2 t}}_{\text{D.E.}} / \underbrace{e^{-a^2k_n^2 t_0}}_{\text{restore }\delta}].$$
4.

We can extend this analysis to a more general theory of Green's functions. This theory is based upon Green's Theorems, or identities.

1. This is easily proven starting with the identity

$$\nabla\cdot(\varphi\nabla\chi) = \nabla\varphi\cdot\nabla\chi + \varphi\nabla^2\chi,$$

2. This is proven by interchanging φ and χ in the first theorem and subtracting the two versions of the theorem.

The next step is to let $\varphi = u$ and $\chi = G$. Then,

$$\int_V \left(u\nabla^2 G - G\nabla^2 u\right) dV = \oint_S (u\nabla G - G\nabla u)\cdot\hat{\mathbf{n}} dS.$$

As we had seen earlier for Poisson's equation, inserting the differential equation yields

$$u(x,y) = \int_V G f dV + \oint_S (u\nabla G - G\nabla u)\cdot\hat{\mathbf{n}} dS.$$

If we have the Green's function, we only need to know the source term and boundary conditions in order to obtain the solution to a given problem.

In the next sections we provide a summary of these ideas as applied to some generic partial differential equations.[1]

Note

This is an adaptation of notes from J. Franklin's course on mathematical physics.

Laplace's Equation: $\nabla^2\psi = 0$.

1. **Boundary Conditions**
   a. Dirichlet $-\psi$ is given on the surface.
   b. Neumann $-\hat{\mathbf{n}}\cdot\nabla\psi = \frac{\partial\psi}{\partial n}$ is given on the surface.

Note

Boundary conditions can be Dirichlet on part of the surface and Neumann on part. If they are Neumann on the whole surface, then the Divergence Theorem requires the constraint

$$\int \frac{\partial\psi}{\partial n}\,dS = 0$$

2. **Solution by Surface Green's Function, $g(\vec{\mathbf{r}},\vec{\mathbf{r}}')$.**
   a.
   b. Neumann conditions

$$\nabla^2 g_N\left(\overrightarrow{\mathbf{r}},\overrightarrow{\mathbf{r}}'\right) = 0,$$
$$\frac{\partial g_N}{\partial n}\left(\overrightarrow{\mathbf{r}}_s,\overrightarrow{\mathbf{r}}'_s\right) = \delta^{(2)}\left(\overrightarrow{\mathbf{r}}_s - \overrightarrow{\mathbf{r}}'_s\right),$$
$$\psi(\overrightarrow{\mathbf{r}}) = \int g_N\left(\overrightarrow{\mathbf{r}},\overrightarrow{\mathbf{r}}'_s\right)\frac{\partial\psi}{\partial n}\left(\overrightarrow{\mathbf{r}}'_s\right)dS'.$$

Note

Use of g is readily generalized to any number of dimensions.

Homogeneous Time Dependent Equations

1. **Typical Equations**
   a. Diffusion/Heat Equation $\nabla^2\Psi = \frac{1}{a^2}\frac{\partial}{\partial t}\Psi$.
   b. Schrödinger Equation $-\nabla^2\Psi + U\Psi = i\frac{\partial}{\partial t}\Psi$.
   c. Wave Equation $\nabla^2\Psi = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Psi$.
   d. General form: $\mathcal{D}\Psi = \mathcal{T}\Psi$.
2. **Initial Value Green's Function, $g\left(\overrightarrow{\mathbf{r}},\overrightarrow{\mathbf{r}}';t,t'\right)$.**
   a. **Homogeneous Boundary Conditions**
      i.
$$\Psi(\overrightarrow{\mathbf{r}},t) = \int g\left(\overrightarrow{\mathbf{r}},\overrightarrow{\mathbf{r}}';t,t_0\right)\Psi\left(\mathbf{r}',t_0\right)d^3\mathbf{r}',$$

      where

      $g\left(\mathbf{r}_s\right)$ satisfies homogeneous boundary conditions.

      ii. The first two properties in (a) above hold, but

$$g_c\left(\mathbf{r},\mathbf{r}';t_0,t_0\right) = \delta\left(\mathbf{r}-\mathbf{r}'\right)$$
$$\dot{g}_s\left(\mathbf{r},\mathbf{r}';t_0,t_0\right) = \delta\left(\mathbf{r}-\mathbf{r}'\right)$$

> **Note**
>
> For the diffusion and Schrödinger equations the initial condition is Dirichlet in time. For the wave equation the initial condition is Cauchy, where Ψ and $\dot{\Psi}$ are given.

b. **Inhomogeneous, Time Independent (steady) Boundary Conditions**

i. Solve Laplace's equation, $\nabla^2 \psi_s = 0$, for inhomogeneous B.C.'s

ii.

iii. Then $\Psi(\mathbf{r}, t) = \Psi_t(\mathbf{r}, t) + \psi_s(\mathbf{r})$.

> **Note**
>
> Ψ_t is the transient part and ψ_s is the steady state part.

3. **Time Dependent Boundary Conditions with Homogeneous Initial Conditions**

a. or

$$\Psi(\mathbf{r}, t) = \int_{t_0}^{\infty} \frac{\partial h_N}{\partial n}(\mathbf{r}, \mathbf{r}'_s; t, t') \, \Psi(\mathbf{r}'_s, t') \, dt'$$

b.

c.

> **Note**
>
> For inhomogeneous I.C.,
>
> $$\Psi = \int g \Psi(\mathbf{r}', t_0) + \int dt' h_D \Psi(\mathbf{r}'_s, t') \, d^3\mathbf{r}'.$$

Inhomogeneous Steady State Equation

1.

$$\nabla^2 \psi(\mathbf{r}, t) = f(\mathbf{r}), \quad \psi(\mathbf{r}_s) \quad \text{or} \quad \frac{\partial \psi}{\partial n}(\mathbf{r}_s) \quad \text{given.}$$

a. where ∇' denotes differentiation with respect to r'.

b. Properties of $G(\mathbf{r}, \mathbf{r}')$:

i. $\nabla'^2 G(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}')$.

ii. $G|_s = 0$ or $\frac{\partial G}{\partial n'}\big|_s = 0$.

iii.

c. If there are pure Neumann conditions and S is finite and $\int f d^3\mathbf{r} \neq 0$ by symmetry, then $\vec{n}' \cdot \nabla' G\big|_s \neq 0$ and the Green's function method is much more complicated to solve.

d. or

$$G_N(\mathbf{r}, \mathbf{r}'_s) = -g_N(\mathbf{r}, \mathbf{r}'_s).$$

e. G satisfies a reciprocity property, $G(\mathbf{r}, \mathbf{r}') = G(\mathbf{r}', \mathbf{r})$ for either Dirichlet or Neumann boundary conditions.

f. $G(\mathbf{r}, \mathbf{r}')$ can be considered as a potential at $\mathbf{r}$ due to a point charge $q = -1/4\pi$ at $\mathbf{r}'$, with all surfaces being grounded conductors.

Inhomogeneous, Time Dependent Equations

1. **Diffusion/Heat Flow $\nabla^2 \Psi - \frac{1}{a^2}\dot{\Psi} = f(\mathbf{r}, t)$.**

a.

$$\left[\nabla^2-\frac{1}{a^2}\frac{\partial}{\partial t}\right]G(\mathbf{r},\mathbf{r}';t,t') = \left[\nabla'^2+\frac{1}{a^2}\frac{\partial}{\partial t'}\right]G(\mathbf{r},\mathbf{r}';t,t') = \delta(\mathbf{r}-\mathbf{r}')\,\delta(t-t'). \tag{7.8.9}$$

b. Green's Theorem in 4 dimensions $(\mathbf{r},t)$ yields

$$\Psi(\mathbf{r},t) = \iint_{t_0}^{\infty} G(\mathbf{r},\mathbf{r}';t,t')\,f(\mathbf{r}',t')\,dt'd^3\mathbf{r}' - \frac{1}{a^2}\int G(\mathbf{r},\mathbf{r}';t,t_0)\,\Psi(\mathbf{r}',t_0)\,d^3\mathbf{r}' + \int_{t_0}^{\infty}\int\left[\Psi(\mathbf{r}'_{s'},t)\,\nabla' G_D(\mathbf{r},\mathbf{r}'_{s'}t,t') - G_N(\mathbf{r},\mathbf{r}'_s;t,t')\,\nabla'\Psi(\mathbf{r}'_s,t')\right]\cdot\overrightarrow{d\mathbf{S}}'\,dt'.$$

c. Either $G_D(\mathbf{r}'_s)=0$ or $G_N(\mathbf{r}'_s)=0$ on S at any point $\mathbf{r}'_s$.

d. $\hat{\mathbf{n}}'\cdot\nabla' G_D(\mathbf{r}'_s)=h_D(\mathbf{r}'_s)$, $G_N(\mathbf{r}'_s)=-h_N(\mathbf{r}'_s)$, and $-\frac{1}{a^2}G(\mathbf{r},\mathbf{r}';t,t_0)=g(\mathbf{r},\mathbf{r}';t,t_0)$.

2. **Wave Equation $\nabla^2\Psi-\frac{1}{c^2}\frac{\partial^2\Psi}{\partial^2 t}=f(\mathbf{r},t)$.**

a.
$$\left[\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]G(\mathbf{r},\mathbf{r}';t,t') = \left[\nabla'^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]G(\mathbf{r},\mathbf{r}';t,t') = \delta(\mathbf{r}-\mathbf{r}')\,\delta(t-t'). \tag{7.8.10}$$

b.

c. Cauchy initial conditions are given: $\Psi(t_0)$ and $\Psi\Psi(t_0)$.

d. The wave and diffusion equations satisfy a causality condition $G(t,t')=0, \quad t'>t.$

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7.9: Problems

Exercise 7.9.1

Find the solution of each initial value problem using the appropriate initial value Green's function.

a. $y''-3y'+2y=20e^{-2x},\quad y(0)=0,\quad y'(0)=6$.
b. $y''+y=2\sin 3x,\quad y(0)=5,\quad y'(0)=0$.
c. $y''+y=1+2\cos x,\quad y(0)=2,\quad y'(0)=0$.
d. $x^2y''-2xy'+2y=3x^2-x,\quad y(1)=\pi,\quad y'(1)=0$.

Exercise 7.9.2

Use the initial value Green's function for $x''+x=f(t), x(0)=4, x'(0)=0$, to solve the following problems.

a. $x''+x=5t^2$.
b. $x''+x=2\tan t$.

Exercise 7.9.3

For the problem $y''-k^2y=f(x), y(0)=0, y'(0)=1$,

a. Find the initial value Green's function.
b. Use the Green's function to solve $y''-y=e^{-x}$.
c. Use the Green's function to solve $y''-4y=e^{2x}$.

Exercise 7.9.4

Find and use the initial value Green's function to solve

$$x^2y''+3xy'-15y=x^4e^x,\quad y(1)=1, y'(1)=0.$$

Exercise 7.9.5

Consider the problem $y''=\sin x, y'(0)=0, y(\pi)=0$.

a. Solve by direct integration.
b. Determine the Green's function.
c. Solve the boundary value problem using the Green's function.
d. Change the boundary conditions to $y'(0)=5, y(\pi)=-3$.
   i. Solve by direct integration.
   ii. Solve using the Green's function.

Exercise 7.9.6

Let C be a closed curve and D the enclosed region. Prove the identity

$$\int_C \phi\nabla\phi\cdot\mathbf{n}ds=\int_D\left(\phi\nabla^2\phi+\nabla\phi\cdot\nabla\phi\right)dA.$$

Exercise 7.9.7

Let S be a closed surface and V the enclosed volume. Prove Green's first and second identities, respectively.

a. $\int_S \phi\nabla\psi\cdot\mathbf{n}dS=\int_V\left(\phi\nabla^2\psi+\nabla\phi\cdot\nabla\psi\right)dV$.
b. $\int_S[\phi\nabla\psi-\psi\nabla\phi]\cdot\mathbf{n}dS=\int_V\left(\phi\nabla^2\psi-\psi\nabla^2\phi\right)dV$.

Exercise 7.9.8

Let C be a closed curve and D the enclosed region. Prove Green's identities in two dimensions.

a. First prove

$$\int_D (v\nabla \cdot \mathbf{F} + \mathbf{F} \cdot \nabla v) dA = \int_C (v\mathbf{F}) \cdot d\mathbf{s}$$

b. Let $\mathbf{F} = \nabla u$ and obtain Green's first identity,

$$\int_D \left(v\nabla^2 u + \nabla u \cdot \nabla v\right) dA = \int_C (v\nabla u) \cdot d\mathbf{s}.$$

c.

Exercise 7.9.9

Consider the problem:

$$\frac{\partial^2 G}{\partial x^2} = \delta(x - x_0), \quad \frac{\partial G}{\partial x}(0, x_0) = 0, \quad G(\pi, x_0) = 0.$$

a. Solve by direct integration.
b. Compare this result to the Green's function in part b of the last problem.
c. Verify that G is symmetric in its arguments.

Exercise 7.9.10

Consider the boundary value problem: $y'' - y = x, x \in (0, 1)$, with boundary conditions $y(0) = y(1) = 0$.

a. Find a closed form solution without using Green's functions.
b. Determine the closed form Green's function using the properties of Green's functions. Use this Green's function to obtain a solution of the boundary value problem.
c. Determine a series representation of the Green's function. Use this Green's function to obtain a solution of the boundary value problem.
d. Confirm that all of the solutions obtained give the same results.

Exercise 7.9.11

Rewrite the solution to Problem 15 and identify the initial value Green's function.

Exercise 7.9.12

Rewrite the solution to Problem 16 and identify the initial value Green's functions.

Exercise 7.9.13

Find the Green's function for the homogeneous fixed values on the boundary of the quarter plane $x > 0, y > 0$, for Poisson's equation using the infinite plane Green's function for Poisson's equation. Use the method of images.

Exercise 7.9.14

Find the Green's function for the one dimensional heat equation with boundary conditions $u(0, t) = 0 u_x(L, t), t > 0$.

Exercise 7.9.15

Consider Laplace's equation on the rectangular plate in Figure 6.3.1. Construct the Green's function for this problem.

Exercise 7.9.16

Construct the Green's function for Laplace's equation in the spherical domain in Figure 6.5.1.

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CHAPTER OVERVIEW

8: Complex Representations of Functions

> *"He is not a true man of science who does not bring some sympathy to his studies, and expect to learn something by behavior as well as by application. It is childish to rest in the discovery of mere coincidences, or of partial and extraneous laws. The study of geometry is a petty and idle exercise of the mind, if it is applied to no larger system than the starry one. Mathematics should be mixed not only with physics but with ethics; that is mixed mathematics. The fact which interests us most is the life of the naturalist. The purest science is still biographical."*
>
> *~ Henry David Thoreau (1817-1862)*

8.1: Complex Representations of Waves
8.2: Complex Numbers
8.3: Complex Valued Functions
8.4: Complex Differentiation
8.5: Complex Integration
8.6: Laplace's Equation in 2D, Revisited
8.7: Problems

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8.1: Complex Representations of Waves

We have seen that we can determine the frequency content of a function $f(t)$ defined on an interval $[0,T]$ by looking for the Fourier coefficients in the Fourier series expansion

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{2\pi nt}{T}+b_n\sin\frac{2\pi nt}{T}.$$

The coefficients take forms like

$$a_n=\frac{2}{T}\int_0^T f(t)\cos\frac{2\pi nt}{T}dt.$$

However, trigonometric functions can be written in a complex exponential form. Using Euler's formula, which was obtained using the Maclaurin expansion of e^x in Example 11.7.8,

$$e^{i\theta}=\cos\theta+i\sin\theta,$$

the complex conjugate is found by replacing i with $-i$ to obtain

$$e^{-i\theta}=\cos\theta-i\sin\theta.$$

Adding these expressions, we have

$$2\cos\theta=e^{i\theta}+e^{-i\theta}.$$

Subtracting the exponentials leads to an expression for the sine function. Thus, we have the important result that sines and cosines can be written as complex exponentials:

$$\begin{aligned}\cos\theta&=\frac{e^{i\theta}+e^{-i\theta}}{2}\\ \sin\theta&=\frac{e^{i\theta}-e^{-i\theta}}{2i}\end{aligned}\tag{8.1.1}$$

So, we can write

$$\cos\frac{2\pi nt}{T}=\frac{1}{2}\left(e^{\frac{2\pi int}{T}}+e^{-\frac{2\pi int}{T}}\right).$$

Later we will see that we can use this information to rewrite the series as a sum over complex exponentials in the form

$$f(t)=\sum_{n=-\infty}^{\infty}c_ne^{\frac{2\pi int}{T}},$$

where the Fourier coefficients now take the form

$$c_n=\int_0^T f(t)e^{-\frac{2\pi int}{T}}dt.$$

In fact, when one considers the representation of analogue signals defined over an infinite interval and containing a continuum of frequencies, we will see that Fourier series sums become integrals of complex functions and so do the Fourier coefficients. Thus, we will naturally find ourselves needing to work with functions of complex variables and perform complex integrals.

We can also develop a complex representation for waves. Recall from the discussion in Section 3.6 on finite length strings that a solution to the wave equation was given by

$$u(x,t)=\frac{1}{2}\left[\sum_{n=1}^{\infty}A_n\sin k_n(x+ct)+\sum_{n=1}^{\infty}A_n\sin k_n(x-ct)\right].\tag{8.1.2}$$

We can replace the sines with their complex forms as

$$u(x,t) = \frac{1}{4i}\left[\sum_{n=1}^{\infty} A_n \left(e^{ik_n(x+ct)} - e^{-ik_n(x+ct)}\right) + \sum_{n=1}^{\infty} A_n \left(e^{ik_n(x-ct)} - e^{-ik_n(x-ct)}\right)\right]. \tag{8.1.3}$$

Defining $k_{-n} = -k_n, n = 1, 2, \ldots$, we can rewrite this solution in the form

$$u(x,t) = \sum_{n=-\infty}^{\infty} \left[c_n e^{ik_n(x+ct)} + d_n e^{ik_n(x-ct)}\right]. \tag{8.1.4}$$

Such representations are also possible for waves propagating over the entire real line. In such cases we are not restricted to discrete frequencies and wave numbers. The sum of the harmonics will then be a sum over a continuous range, which means that the sums become integrals. So, we are lead to the complex representation

$$u(x,t) = \int_{-\infty}^{\infty} \left[c(k)e^{ik(x+ct)} + d(k)e^{ik(x-ct)}\right] dk. \tag{8.1.5}$$

The forms $e^{ik(x+ct)}$ and $e^{ik(x-ct)}$ are complex representations of what are called plane waves in one dimension. The integral represents a general wave form consisting of a sum over plane waves. The Fourier coefficients in the representation can be complex valued functions and the evaluation of the integral may be done using methods from complex analysis. We would like to be able to compute such integrals.

With the above ideas in mind, we will now take a tour of complex analysis. We will first review some facts about complex numbers and then introduce complex functions. This will lead us to the calculus of functions of a complex variable, including the differentiation and integration complex functions. This will set up the methods needed to explore Fourier transforms in the next chapter.

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8.2: Complex Numbers

Complex numbers were first introduced in order to solve some simple problems. The history of complex numbers only extends about five hundred years. In essence, it was found that we need to find the roots of equations such as $x^2+1=0$. The solution is $x=\pm\sqrt{-1}$. Due to the usefulness of this concept, which was not realized at first, a special symbol was introduced - the **imaginary unit**, $i=\sqrt{-1}$. In particular, Girolamo Cardano $(1501-1576)$ was one of the first to use square roots of negative numbers when providing solutions of cubic equations. However, complex numbers did not become an important part of mathematics or science until the late seventh and eighteenth centuries after people like Abraham de Moivre ($1667-1754$), the Bernoulli[1] family and Euler took them seriously.

A complex number is a number of the form $z=x+iy$, where x and y are real numbers. x is called the real part of z and y is the imaginary part of z. Examples of such numbers are $3+3i$, $-1i=-i$, $4i$ and 5. Note that $5=5+0i$ and $4i=0+4i$.

Bernoullis

The Bernoullis were a family of Swiss mathematicians spanning three generations. It all started with Jacob Bernoulli $(1654-1705)$ and his brother Johann Bernoulli $(1667-1748)$. Jacob had a son, Nicolaus Bernoulli $(1687-1759)$ and Johann $(1667-1748)$ had three sons, Nicolaus Bernoulli II $(1695-1726)$, Daniel Bernoulli $(1700-1872)$, and Johann Bernoulli II $(1710-1790)$. The last generation consisted of Johann II's sons, Johann Bernoulli III $(1747-1807)$ and Jacob Bernoulli II $(1759-1789)$. Johann, Jacob and Daniel Bernoulli were the most famous of the Bernoulli's. Jacob studied with Leibniz, Johann studied under his older brother and later taught Leonhard Euler and Daniel Bernoulli, who is known for his work in hydrodynamics.

There is a geometric representation of complex numbers in a two dimensional plane, known as the **complex plane** C. This is given by the **Argand diagram** as shown in Figure $8.2.1$. Here we can think of the complex number $z=x+iy$ as a point (x,y) in the z-complex plane or as a vector. The magnitude, or length, of this vector is called the complex modulus of z, denoted by $|z|=\sqrt{x^2+y^2}$. We can also use the geometric picture to develop a polar representation of complex numbers. From Figure $8.2.1$ we can see that in terms of r and θ we have that

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned} \tag{8.2.1}$$

Thus,

$$z=x+iy=r(\cos\theta+i\sin\theta)=re^{i\theta}. \tag{8.2.2}$$

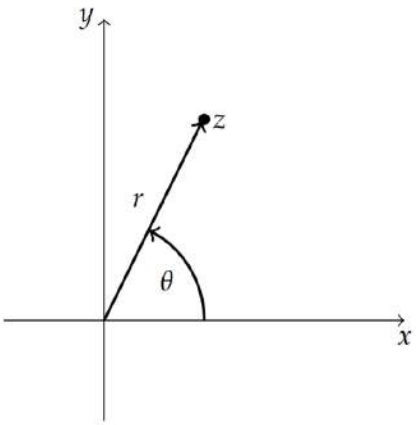


Figure $8.2.1$: The Argand diagram for plotting complex numbers in the complex $z-$ plane.

Note

The complex modulus, $|z|=\sqrt{x^2+y^2}$.

Note

Complex numbers can be represented in rectangular (Cartesian), $z = x + iy$, or polar form, $z = re^{i\theta}$. Here we define the argument, θ, and modulus, $|z| = r$ of complex numbers.

So, given r and θ we have $z = re^{i\theta}$. However, given the Cartesian form, $z = x + iy$, we can also determine the polar form, since

$$\begin{aligned} r &= \sqrt{x^2 + y^2}, \\ \tan\theta &= \frac{y}{x}. \end{aligned} \tag{8.2.3}$$

Note that $r = |z|$.

Locating $1 + i$ in the complex plane, it is possible to immediately determine the polar form from the angle and length of the "complex vector." This is shown in Figure 8.2.2. It is obvious that $\theta = \frac{\pi}{4}$ and $r = \sqrt{2}$.

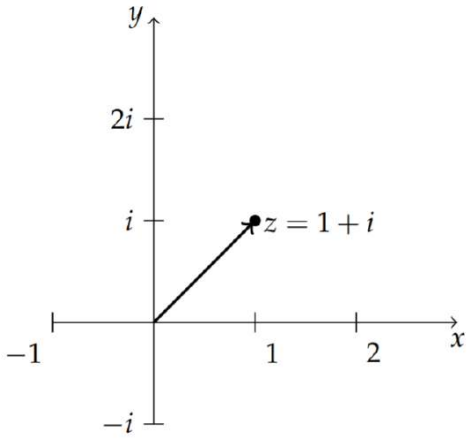


Figure 8.2.2: Locating $1 + i$ in the complex z-plane.

Example 8.2.1

Write $z = 1 + i$ in polar form.

Solution

If one did not see the polar form from the plot in the z-plane, then one could systematically determine the results. First, write $z = 1 + i$ in polar form, $z = re^{i\theta}$, for some r and θ.

Using the above relations between polar and Cartesian representations, we have $r = \sqrt{x^2 + y^2} = \sqrt{2}$ and $\tan\theta = \frac{y}{x} = 1$. This gives $\theta = \frac{\pi}{4}$. So, we have found that

$$1 + i = \sqrt{2}e^{i\pi/4}.$$

We can also define binary operations of addition, subtraction, multiplication and division of complex numbers to produce a new complex number. The addition of two complex numbers is simply done by adding the real and imaginary parts of each number. So,

$$(3 + 2i) + (1 - i) = 4 + i.$$

Subtraction is just as easy,

$$(3 + 2i) - (1 - i) = 2 + 3i.$$

We can multiply two complex numbers just like we multiply any binomials, though we now can use the fact that $i^2 = -1$. For example, we have

$$(3 + 2i)(1 - i) = 3 + 2i - 3i + 2i(-i) = 5 - i.$$

Note

We can easily add, subtract, multiply and divide complex numbers.

We can even divide one complex number into another one and get a complex number as the quotient. Before we do this, we need to introduce the complex conjugate, $\bar{z}$, of a complex number. The complex conjugate of $z = x + iy$, where x and y are real numbers, is given as

$$\bar{z} = x - iy.$$

Note

The complex conjugate of $z = x + iy$, is given as $\bar{z} = x - iy$.

Complex conjugates satisfy the following relations for complex numbers z and w and real number x.

$$\begin{aligned} \overline{z+w} &= \bar{z} + \bar{w} \\ \overline{zw} &= \overline{z}\,\overline{w} \\ \bar{\bar{z}} &= z \\ \bar{x} &= x. \end{aligned} \tag{8.2.4}$$

One consequence is that the complex conjugate of $re^{i\theta}$ is

$$\overline{re^{i\theta}} = \overline{\cos\theta + i\sin\theta} = \cos\theta - i\sin\theta = re^{-i\theta}.$$

Another consequence is that

$$z\bar{z} = re^{i\theta}re^{-i\theta} = r^2.$$

Thus, the product of a complex number with its complex conjugate is a real number. We can also prove this result using the Cartesian form

$$z\bar{z} = (x+iy)(x-iy) = x^2 + y^2 = |z|^2.$$

Now we are in a position to write the quotient of two complex numbers in the standard form of a real plus an imaginary number.

Example 8.2.2

Simplify the expression $z = \frac{3+2i}{1-i}$.

Solution

This simplification is accomplished by multiplying the numerator and denominator of this expression by the complex conjugate of the denominator:

$$z = \frac{3+2i}{1-i} = \frac{3+2i}{1-i}\frac{1+i}{1+i} = \frac{1+5i}{2}.$$

Therefore, the quotient is a complex number and in standard form it is given by $z = \frac{1}{2} + \frac{5}{2}i$.

We can also consider powers of complex numbers. For example,

$$\begin{aligned} (1+i)^2 &= 2i \\ (1+i)^3 &= (1+i)(2i) = 2i - 2. \end{aligned}$$

But, what is $(1+i)^{1/2} = \sqrt{1+i}$?

In general, we want to find the nth root of a complex number. Let $t = z^{1/n}$. To find t in this case is the same as asking for the solution of

$$z = t^n$$

given z. But, this is the root of an nth degree equation, for which we expect n roots. If we write z in polar form, $z = re^{i\theta}$, then we would naively compute

$$\begin{aligned} z^{1/n} &= \left(re^{i\theta}\right)^{1/n} \\ &= r^{1/n}e^{i\theta/n} \\ &= r^{1/n}\left[\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right]. \end{aligned} \tag{8.2.5}$$

For example,

$$(1+i)^{1/2} = \left(\sqrt{2}e^{i\pi/4}\right)^{1/2} = 2^{1/4}e^{i\pi/8}.$$

But this is only one solution. We expected two solutions for $n = 2$.

Note

The function $f(z) = z^{1/n}$ is multivalued.

$$z^{1/n} = r^{1/n}e^{i(\theta+2k\pi)/n}, \quad k = 0, 1, \ldots, n-1$$

The reason we only found one solution is that the polar representation for z is not unique. We note that

$$e^{2k\pi i} = 1, \quad k = 0, \pm 1, \pm 2, \ldots.$$

So, we can rewrite z as $z = re^{i\theta}e^{2k\pi i} = re^{i(\theta+2k\pi)}$. Now, we have that

$$z^{1/n} = r^{1/n}e^{i(\theta+2k\pi)/n}, \quad k = 0, 1, \ldots, n-1$$

Note that these are the only distinct values for the roots. We can see this by considering the case $k = n$. Then, we find that

$$e^{i(\theta+2\pi in)/n} = e^{i\theta/n}e^{2\pi i} = e^{i\theta/n}.$$

So, we have recovered the $n = 0$ value. Similar results can be shown for the other k values larger than n.

Now, we can finish the example we had started.

Example 8.2.3

Determine the square roots of $1+i$, or $\sqrt{1+i}$.

Solution

As we have seen, we first write $1+i$ in polar form, $1+i = \sqrt{2}e^{i\pi/4}$. Then, introduce $e^{2k\pi i} = 1$ and find the roots:

$$\begin{aligned} (1+i)^{1/2} &= \left(\sqrt{2}e^{i\pi/4}e^{2k\pi i}\right)^{1/2}, \quad k = 0, 1, \\ &= 2^{1/4}e^{i(\pi/8+k\pi)}, \quad k = 0, 1, \\ &= 2^{1/4}e^{i\pi/8}, 2^{1/4}e^{9\pi i/8}. \end{aligned} \tag{8.2.6}$$

Note

The nth roots of unity, $\sqrt[n]{1}$.

Finally, what is $\sqrt[n]{1}$? Our first guess would be $\sqrt[n]{1} = 1$. But, we now know that there should be n roots. These roots are called the nth roots of unity. Using the above result with $r = 1$ and $\theta = 0$, we have that

$$\sqrt[n]{1} = \left[\cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n}\right], \quad k = 0, \ldots, n-1.$$

For example, we have

$$\sqrt[3]{1} = \left[\cos\frac{2\pi k}{3} + i\sin\frac{2\pi k}{3}\right], \quad k = 0, 1, 2$$

These three roots can be written out as

$$\sqrt[3]{1} = 1, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i.$$

We can locate these cube roots of unity in the complex plane. In Figure $8.2.3$ we see that these points lie on the unit circle and are at the vertices of an equilateral triangle. In fact, all nth roots of unity lie on the unit circle and are the vertices of a regular n-gon with one vertex at $z = 1$.

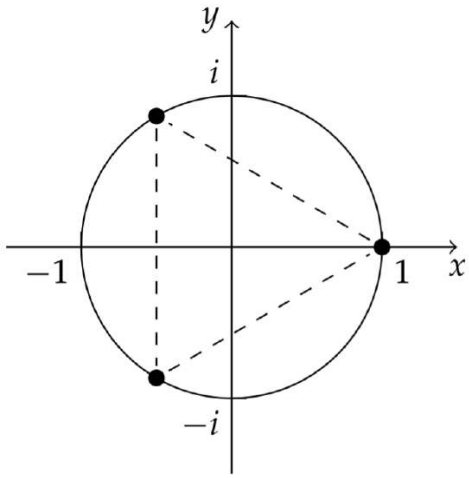


Figure $8.2.3$: Locating the cube unity in the complex z-plane.

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8.3: Complex Valued Functions

We would like to next explore complex functions and the calculus of complex functions. We begin by defining a function that takes complex numbers into complex numbers, $f : C \to C$. It is difficult to visualize such functions. For real functions of one variable, $f : R \to R$, we graph these functions by first drawing two intersecting copies of R and then proceed to map the domain into the range of f.

It would be more difficult to do this for complex functions. Imagine placing together two orthogonal copies of the complex plane, C. One would need a four dimensional space in order to complete the visualization. Instead, typically uses two copies of the complex plane side by side in order to indicate how such functions behave. Over the years there have been several ways to visualize complex functions. We will describe a few of these in this chapter.

We will assume that the domain lies in the z-plane and the image lies in the w-plane. We will then write the complex function as $w = f(z)$. We show these planes in Figure 8.3.1 and the mapping between the planes.

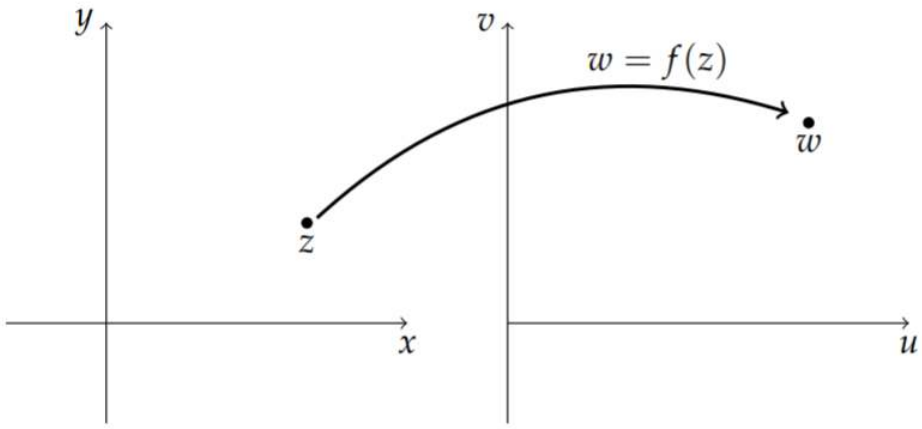


Figure 8.3.1: Defining a complex valued function, $w = f(z)$, on C for $z = x + iy$ and $w = u + iv$

Letting $z = x + iy$ and $w = u + iv$, we can write the real and imaginary parts of $f(z)$:

$$w = f(z) = f(x + iy) = u(x, y) + iv(x, y).$$

We see that one can view this function as a function of z or a function of x and y. Often, we have an interest in writing out the real and imaginary parts of the function, $u(x, y)$ and $v(x, y)$, which are functions of two real variables, x and y. We will look at several functions to determine the real and imaginary parts.

Example 8.3.1

Find the real and imaginary parts of $f(z) = z^2$.

Solution

For example, we can look at the simple function $f(z) = z^2$. It is a simple matter to determine the real and imaginary parts of this function. Namely, we have

$$z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy.$$

Therefore, we have that

$$u(x, y) = x^2 - y^2, \quad v(x, y) = 2xy.$$

In Figure 8.3.2 we show how a grid in the z-plane is mapped by $f(z) = z^2$ into the w-plane. For example, the horizontal line $x = 1$ is mapped to $u(1, y) = 1 - y^2$ and $v(1, y) = 2y$. Eliminating the "parameter" y between these two equations, we have $u = 1 - v^2/4$. This is a parabolic curve. Similarly, the horizontal line $y = 1$ results in the curve $u = v^2/4 - 1$.

If we look at several curves, $x =$ const and $y =$ const, then we get a family of intersecting parabolae, as shown in Figure 8.3.2.

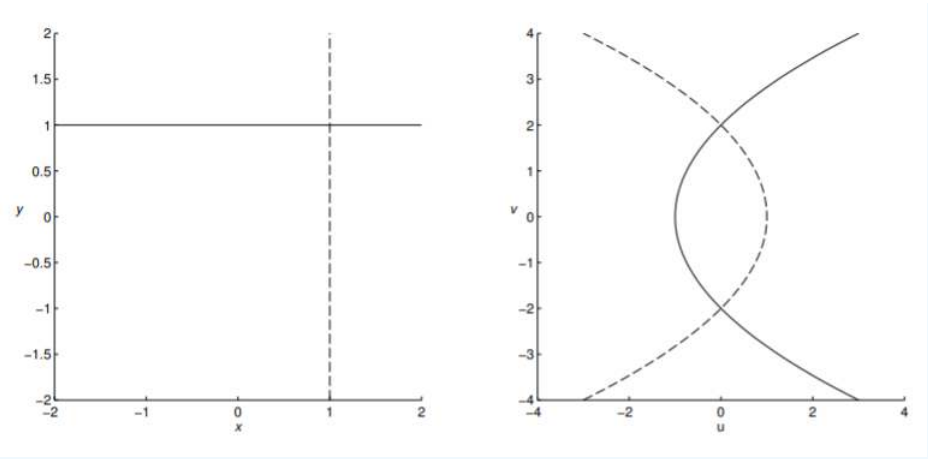


Figure 8.3.2: 2D plot showing how the function $f(z) = z^2$ maps the lines $x = 1$ and $y = 1$ in the z-plane into parabolae in the w-plane.

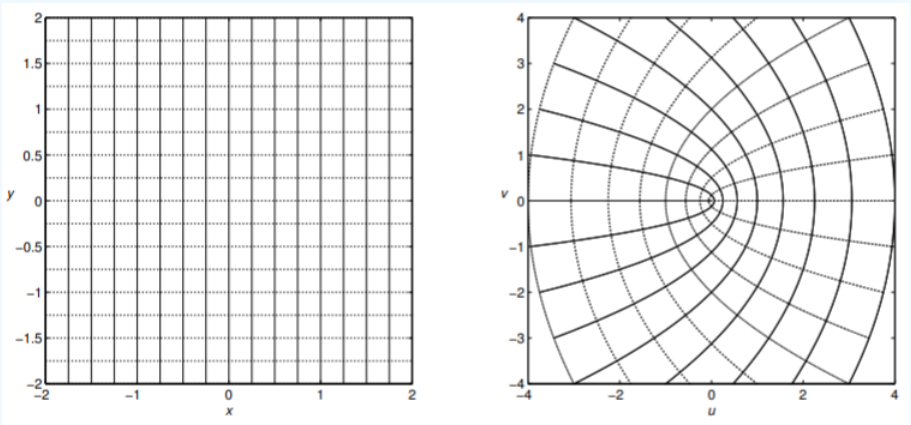


Figure 8.3.3: 2D plot showing how the function $f(z) = z^2$ maps a grid in the z-plane into the w-plane.

Example 8.3.2

Find the real and imaginary parts of $f(z) = e^z$.

Solution

For this case, we make use of Euler's Formula.

$$\begin{aligned} e^z &= e^{x+iy} \\ &= e^x e^{iy} \\ &= e^x(\cos y + i \sin y). \end{aligned} \tag{8.3.1}$$

Thus, $u(x, y) = e^x \cos y$ and $v(x, y) = e^x \sin y$. In Figure 8.3.4 we show how a grid in the z-plane is mapped by $f(z) = e^z$ into the w-plane.

Example 8.3.3

Find the real and imaginary parts of $f(z) = z^{1/2}$.

Solution

We have that

$$z^{1/2} = \sqrt{x^2 + y^2}\,(\cos(\theta + k\pi) + i \sin(\theta + k\pi)), \quad k = 0, 1. \tag{8.3.2}$$

Thus,

$$u = |z|\cos(\theta + k\pi), u = |z|\cos(\theta + k\pi)$$

for $|z| = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}(y/x)$. For each k-value one has a different surface and curves of constant θ give $u/v = c_1$, and curves of constant nonzero complex modulus give concentric circles, $u^2 + v^2 = c_2$, for c_1 and c_2 constants.

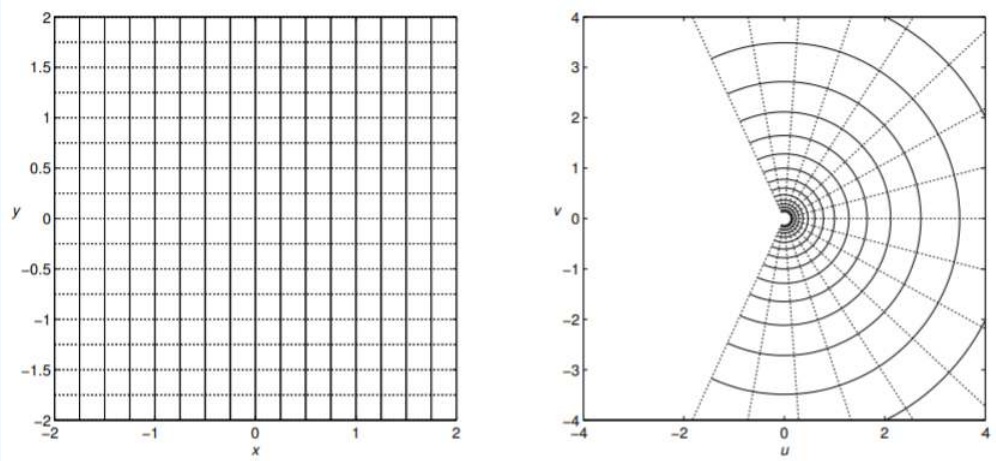


Figure 8.3.4: 2D plot showing how the function $f(z) = e^z$ maps a grid in the z-plane into the w-plane.

Example 8.3.4

Find the real and imaginary parts of $f(z) = \ln z$.

Solution

In this case we make use of the polar form of a complex number, $z = re^{i\theta}$. Our first thought would be to simply compute

$$\ln z = \ln r + i\theta.$$

However, the natural logarithm is multivalued, just like the square root function. Recalling that $e^{2\pi ik} = 1$ for k an integer, we have $z = re^{i(\theta + 2\pi k)}$. Therefore,

$$\ln z = \ln r + i(\theta + 2\pi k), \quad k = \text{ integer}.$$

The natural logarithm is a multivalued function. In fact there are an infinite number of values for a given z. Of course, this contradicts the definition of a function that you were first taught.

Thus, one typically will only report the principal value, $\log z = \ln r + i\theta$, for θ restricted to some interval of length 2π, such as $[0, 2\pi)$. In order to account for the multivaluedness, one introduces a way to extend the complex plane so as to include all of the branches. This is done by assigning a plane to each branch, using (branch) cuts along lines, and then gluing the planes together at the branch cuts to form what is called a Riemann surface. We will not elaborate upon this any further here and refer the interested reader to more advanced texts. Comparing the multivalued logarithm to the principal value logarithm, we have

$$\ln z = \log z + 2n\pi i.$$

We should not that some books use $\log z$ instead of $\ln z$. It should not be confused with the common logarithm.

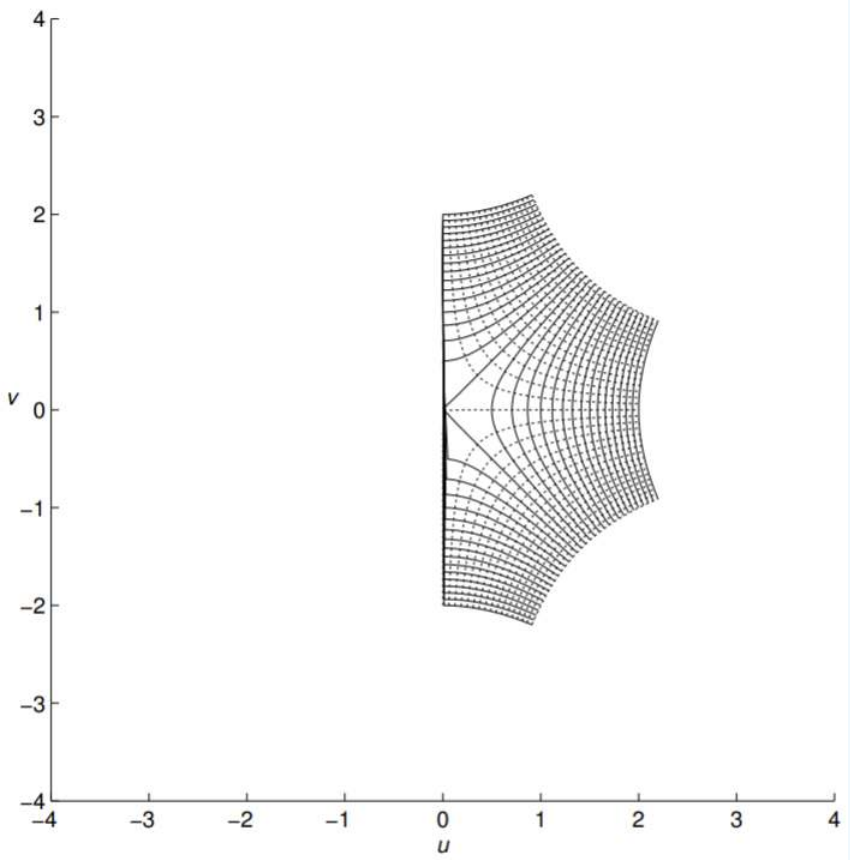


Figure 8.3.5: 2D plot showing how the function $f(z) = \sqrt{z}$ maps a grid in the z-plane into the w-plane.

Complex Domain Coloring

Another method for visualizing complex functions is domain coloring. The idea was described by Frank A. Farris. There are a few approaches to this method. The main idea is that one colors each point of the z-plane (the domain) according to $\arg(z)$ as shown in Figure 8.3.6. The modulus, $|f(z)|$ is then plotted as a surface. Examples are shown for $f(z) = z^2$ in Figure 8.3.7 and $f(z) = 1/z(1-z)$ in Figure 8.3.8.

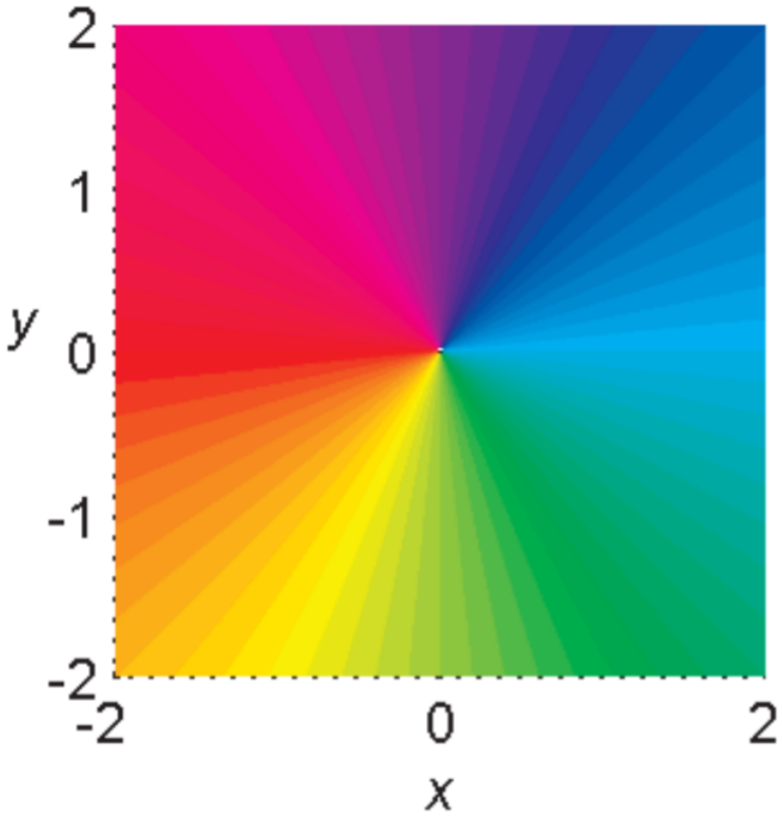


Figure 8.3.6: Domain coloring of the complex z-plane assigning colors $\arg(z)$.

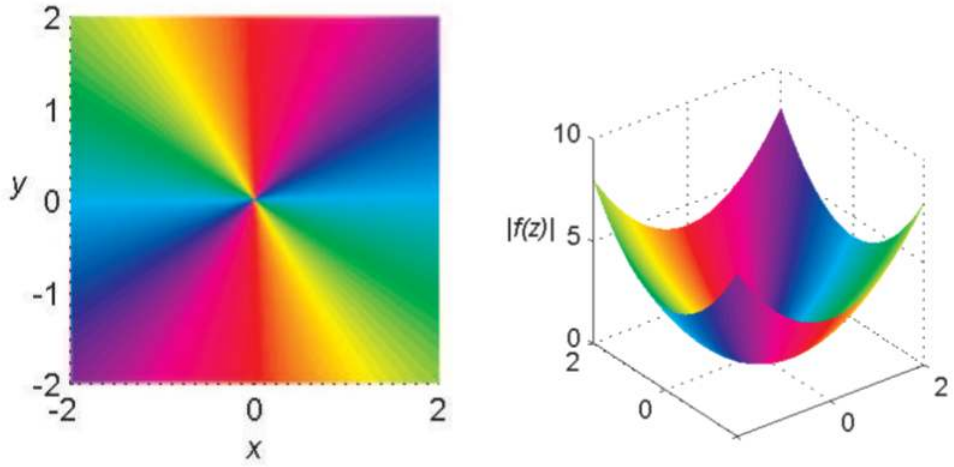


Figure 8.3.7: Domain coloring for $f(z) = z^2$. The left figure shows the phase coloring. The right figure show the colored surface with height $|f(z)|$.

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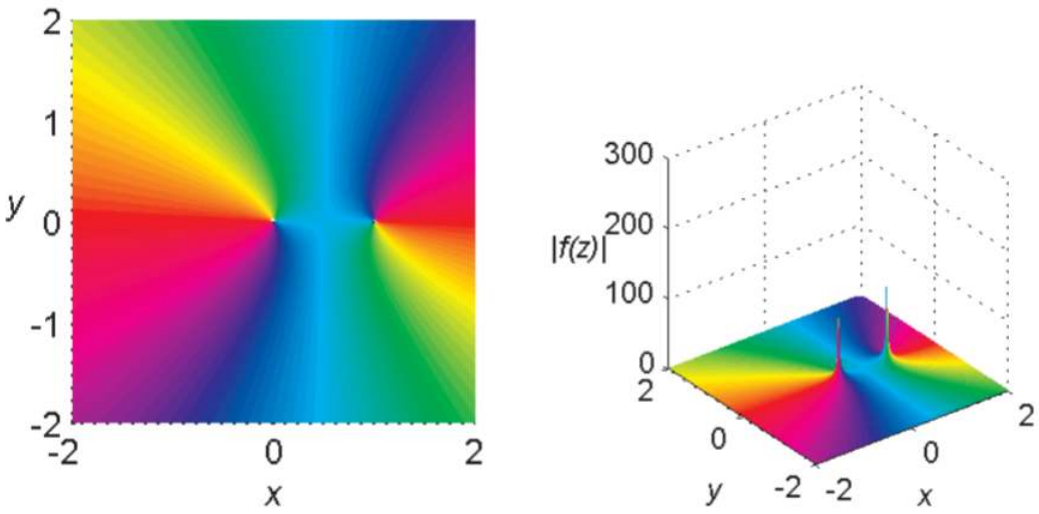


Figure 8.3.8: Domain coloring for $f(z) = 1/z(1-z)$. The left figure shows the phase coloring. The right figure show the colored surface with height $|f(z)|$.

We would like to put all of this information in one plot. We can do this by adjusting the brightness of the colored domain by using the modulus of the function. In the plots that follow we use the fractional part of $\ln|z|$. In Figure 8.3.9 we show the effect for the z-plane using $f(z) = z$. In the figures that follow we look at several other functions. In these plots we have chosen to view the functions in a circular window.

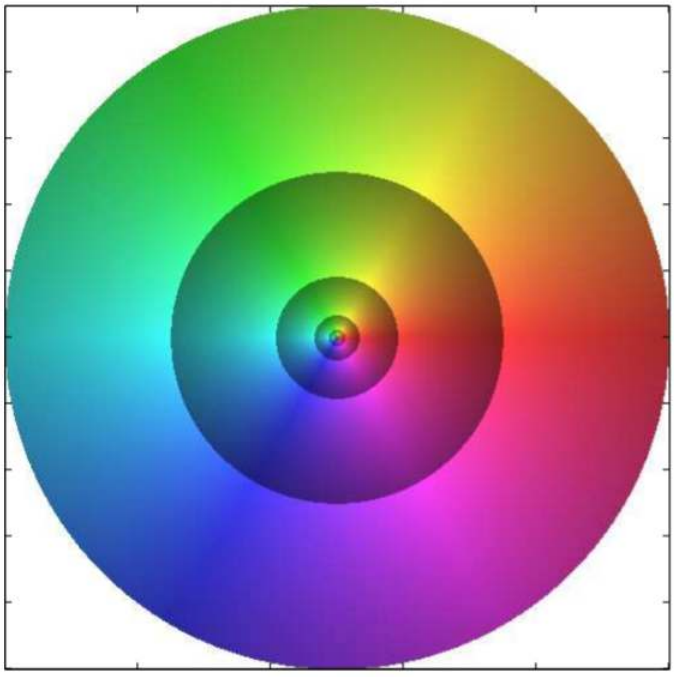


Figure 8.3.9: Domain coloring for the function $f(z) = z$ showing a coloring for $\arg(z)$ and brightness based on $|f(z)|$.

One can see the rich behavior hidden in these figures. As you progress in your reading, especially after the next chapter, you should return to these figures and locate the zeros, poles, branch points and branch cuts. A search online will lead you to other colorings and superposition of the uv grid on these figures.

As a final picture, we look at iteration in the complex plane. Consider the function $f(z) = z^2 - 0.75 - 0.2i$. Interesting figures result when studying the iteration in the complex plane. In Figure 8.3.12 we show $f(z)$ and $f^{20}(z)$, which is the iteration of f twenty times. It leads to an interesting coloring. What happens when one keeps iterating? Such iterations lead to the study of Julia and Mandelbrot sets. In Figure 8.3.13 we show six iterations of $f(z) = (1 - i/2)\sin x$.

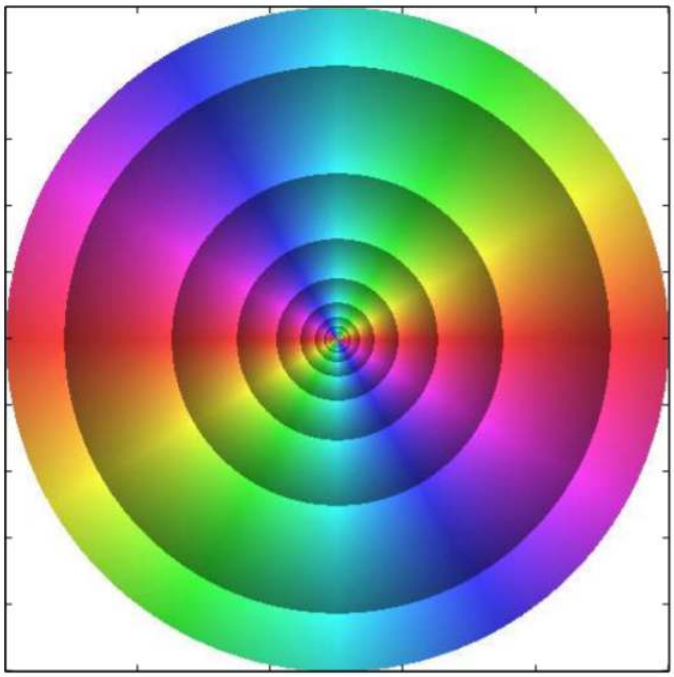


Figure 8.3.10: Domain coloring for the function $f(z) = z^2$.

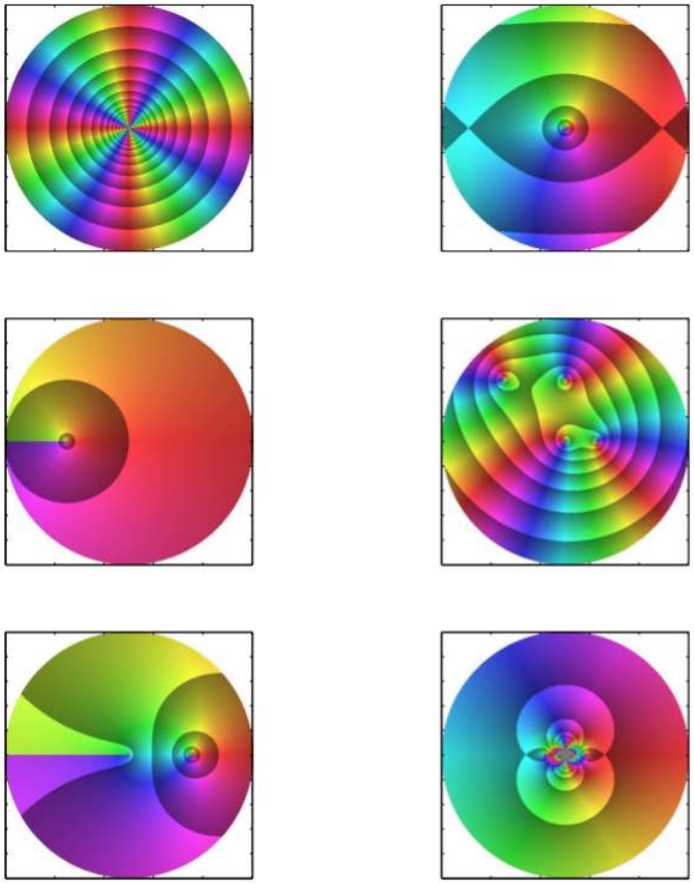


Figure 8.3.11: Domain coloring for several functions. On the top row the domain coloring is shown for $f(z) = z^4$ and $f(z) = \sin z$. On the second row plots for $f(z) = \sqrt{1+z}$ and $f(z) = \overline{z(1/2-z)(z-i)(z-i+1)}$ are shown. In the last row domain colorings for $f(z) = \ln z$ and $f(z) = \sin(1/z)$ are shown.

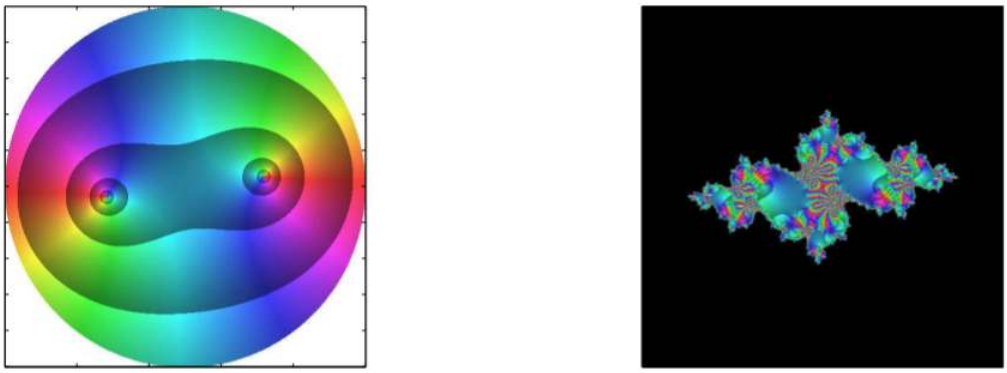


Figure 8.3.12: Domain coloring for $f(z) = z^2 - 0.75 - 0.2i$. The left figure shows the phase coloring. On the right is the plot for $f^{20}(z)$.

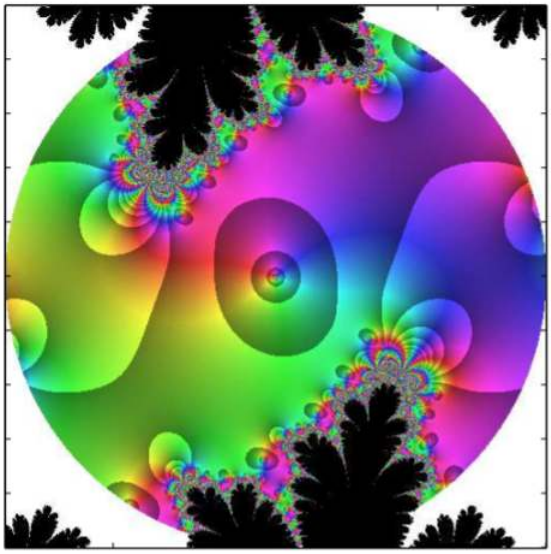


Figure 8.3.13: Domain coloring for six iterations of $f(z) = (1 - i/2)\sin x$

The following code was used in MATLAB to produce these figures.

```
fn = @(x) (1-i/2)*sin(x);
xmin=-2; xmax=2; ymin=-2; ymax=2;
Nx=500;
Ny=500;
x=linspace(xmin,xmax,Nx);
y=linspace(ymin,ymax,Ny);
[X,Y] = meshgrid(x,y); z = complex(X,Y);
tmp=z; for n=1:6
    tmp = fn(tmp);
end Z=tmp;
XX=real(Z);
YY=imag(Z);
R2=max(max(X.^2));
R=max(max(XX.^2+YY.^2));

circle(:,:,1) = X.^2+Y.^2 < R2;
circle(:,:,2)=circle(:,:,1);
```

```
circle(:,:,3)=circle(:,:,1);

addcirc(:,:,1)=circle(:,:,1)==0;
addcirc(:,:,2)=circle(:,:,1)==0;
addcirc(:,:,3)=circle(:,:,1)==0;

warning off MATLAB:divideByZero;
hsvCircle=ones(Nx,Ny,3);
hsvCircle(:,:,1)=atan2(YY,XX)*180/pi+(atan2(YY,XX)*180/pi<0)*360;
hsvCircle(:,:,1)=hsvCircle(:,:,1)/360; lgz=log(XX.^2+YY.^2)/2;
hsvCircle(:,:,2)=0.75; hsvCircle(:,:,3)=1-(lgz-floor(lgz))/2;
hsvCircle(:,:,1) = flipud((hsvCircle(:,:,1)));

hsvCircle(:,:,2) = flipud((hsvCircle(:,:,2)));

hsvCircle(:,:,3) =flipud((hsvCircle(:,:,3)));

rgbCircle=hsv2rgb(hsvCircle);
rgbCircle=rgbCircle.*circle+addcirc;
image(rgbCircle) axis square set(gca,'XTickLabel',{}) set(gca,'YTickLabel',{})
```

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8.4: Complex Differentiation

Next we want to differentiate complex functions. We generalize the definition from single variable calculus,

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}, \tag{8.4.1}$$

provided this limit exists.

The computation of this limit is similar to what one sees in multivariable calculus for limits of real functions of two variables. Letting $z = x + iy$ and $\delta z = \delta x + i\delta y$, then

$$z + \delta x = (x + \delta x) + i(y + \delta y)$$

Letting $\Delta z \to 0$ means that we get closer to z. There are many paths that one can take that will approach z. [See Figure 8.4.1.]

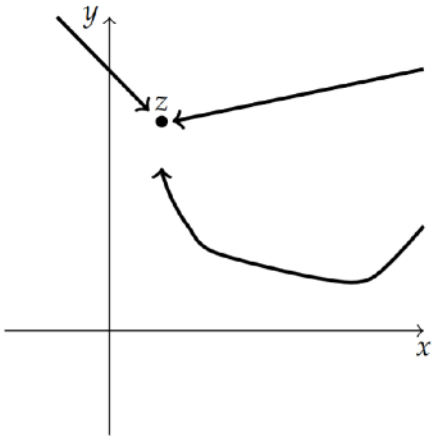


Figure 8.4.1: There are many paths that approach z as $\Delta z \to 0$.

It is sufficient to look at two paths in particular. We first consider the path $y =$ constant. This horizontal path is shown in Figure 8.4.2. For this path, $\Delta z = \Delta x + i\Delta y = \Delta x$, since y does not change along the path. The derivative, if it exists, is then computed as

$$\begin{aligned} f'(z) &= \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \\ &= \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) + iv(x+\Delta x, y) - (u(x,y) + iv(x,y))}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + \lim_{\Delta x \to 0} i\frac{v(x+\Delta x, y) - v(x,y)}{\Delta x} \end{aligned} \tag{8.4.2}$$

The last two limits are easily identified as partial derivatives of real valued functions of two variables. Thus, we have shown that when $f'(z)$ exists,

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \tag{8.4.3}$$

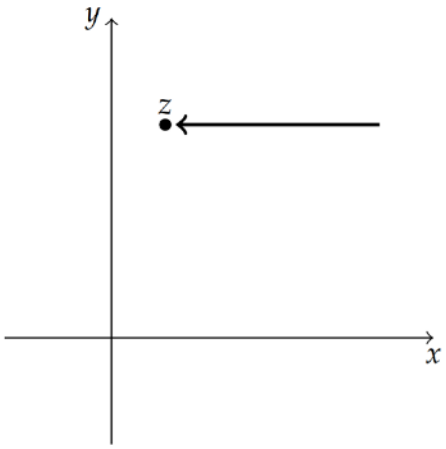


Figure 8.4.2: A path that approaches z with $y =$ constant.

A similar computation can be made if instead we take the vertical path, $x =$ constant, in Figure 8.4.1). In this case $\Delta z = i\Delta y$ and

$$\begin{aligned} f'(z) &= \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \\ &= \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) + iv(x, y+\Delta y) - (u(x,y) + iv(x,y))}{i\Delta y} \\ &= \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y} + \lim_{\Delta y \to 0} \frac{v(x, y+\Delta y) - v(x,y)}{\Delta y} \end{aligned} \tag{8.4.4}$$

Therefore,

$$f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}. \tag{8.4.5}$$

We have found two different expressions for $f'(z)$ by following two different paths to z. If the derivative exists, then these two expressions must be the same. Equating the real and imaginary parts of these expressions, we have

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y} \end{aligned} \tag{8.4.6}$$

These are known as the **Cauchy-Riemann equations**[1].

Note

Augustin-Louis Cauchy $(1789 - 1857)$ was a French mathematician well known for his work in analysis. Georg Friedrich Bernhard Riemann $(1826 - 1866)$ was a German mathematician who made major contributions to geometry and analysis.

Theorem 8.4.1

$f(z)$ is holomorphic (differentiable) if and only if the Cauchy-Riemann equations are satisfied.

Example 8.4.1

$f(z) = z^2$.

Solution

In this case we have already seen that $z^2 = x^2 - y^2 + 2ixy$. Therefore, $u(x,y) = x^2 - y^2$ and $v(x,y) = 2xy$. We first check the Cauchy-Riemann equations.

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y}$$
$$\frac{\partial v}{\partial x} = 2y = -\frac{\partial u}{\partial y}. \tag{8.4.7}$$

Therefore, $f(z) = z^2$ is differentiable.

We can further compute the derivative using either Equation (8.4.3) or Equation (8.4.5). Thus,

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 2x + i(2y) = 2z.$$

This result is not surprising.

Example 8.4.2

$f(z) = \bar{z}$.

Solution

In this case we have $f(z) = x - iy$. Therefore, $u(x, y) = x$ and $v(x, y) = -y$. But, $\frac{\partial u}{\partial x} = 1$ and $\frac{\partial v}{\partial y} = -1$. Thus, the Cauchy-Riemann equations are not satisfied and we conclude the $f(z) = \bar{z}$ is not differentiable.

Note

Harmonic functions satisfy Laplace's equation.

Another consequence of the Cauchy-Riemann equations is that both $u(x, y)$ and $v(x, y)$ are harmonic functions. A real-valued function $u(x, y)$ is harmonic if it satisfies Laplace's equation in 2D, $\nabla^2 u = 0$, or

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Theorem 8.4.1

$f(z) = u(x, y) + iv(x, y)$ is differentiable if and only if u and v are harmonic functions.

This is easily proven using the Cauchy-Riemann equations.

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\frac{\partial u}{\partial x} \\ &= \frac{\partial}{\partial x}\frac{\partial v}{\partial y} \\ &= \frac{\partial}{\partial y}\frac{\partial v}{\partial x} \\ &= -\frac{\partial}{\partial y}\frac{\partial u}{\partial y} \\ &= -\frac{\partial^2 u}{\partial y^2}\end{aligned} \tag{8.4.8}$$

Example 8.4.3

Is $u(x, y) = x^2 + y^2$ harmonic?

Solution

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 + 2 \neq 0.$$

No, it is not.

Example 8.4.4

Is $u(x,y)=x^2-y^2$ harmonic?

Solution

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=2-2=0.$$

Yes, it is.

Given a harmonic function $u(x,y)$, can one find a function, $v(x,y)$, such The harmonic conjugate function. $f(z)=u(x,y)+iv(x,y)$ is differentiable? In this case, v are called the harmonic conjugate of u.

Example 8.4.5

Find the harmonic conjugate of $u(x,y)=x^2-y^2$ and determine $f(z)=u+iv$ such that $u+iv$ is differentiable.

Solution

The Cauchy-Riemann equations tell us the following about the unknown function, $v(x,y)$:

$$\frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}=2y,$$
$$\frac{\partial v}{\partial y}=\frac{\partial u}{\partial x}=2x.$$

We can integrate the first of these equations to obtain

$$v(x,y)=\int 2ydx=2xy+c(y).$$

Here $c(y)$ is an arbitrary function of y. One can check to see that this works by simply differentiating the result with respect to x.

However, the second equation must also hold. So, we differentiate the result with respect to y to find that

$$\frac{\partial v}{\partial y}=2x+c'(y).$$

Since we were supposed to get $2x$, we have that $c'(y)=0$. Thus, $c(y)=k$ is a constant.

We have just shown that we get an infinite number of functions,

$$v(x,y)=2xy+k,$$

such that

$$f(z)=x^2-y^2+i(2xy+k)$$

is differentiable. In fact, for $k=0$ this is nothing other than $f(z)=z^2$.

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8.5: Complex Integration

We have introduced functions of a complex variable. We also established when functions are differentiable as complex functions, or holomorphic. In this chapter we will turn to integration in the complex plane. We will learn how to compute complex path integrals, or contour integrals. We will see that contour integral methods are also useful in the computation of some of the real integrals that we will face when exploring Fourier transforms in the next chapter.

Complex Path Integrals

In this section we will investigate the computation of complex path integrals. Given two points in the complex plane, connected by a path Γ as shown in Figure 8.5.1, we would like to define the integral of $f(z)$ along Γ,

$$\int_\Gamma f(z)dz$$

A natural procedure would be to work in real variables, by writing

$$\int_\Gamma f(z)dz = \int_\Gamma [u(x, y) + iv(x, y)](dx + idy),$$

since $z = x + iy$ and $dz = dx + idy$.

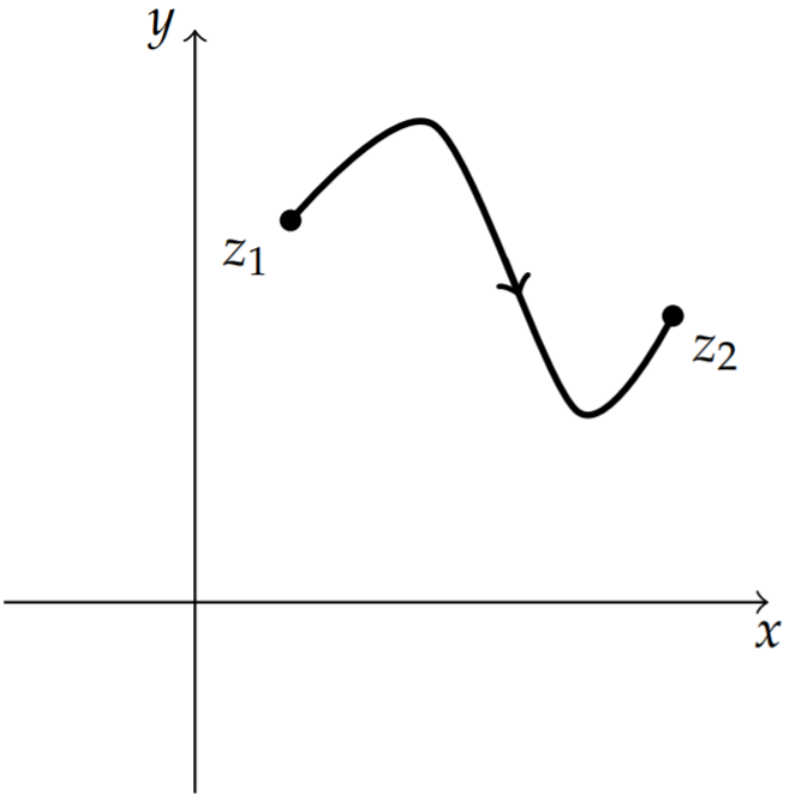


Figure 8.5.1: We would like to integrate a complex function $f(z)$ over the path Γ in the complex plane.

In order to carry out the integration, we then have to find a parametrization of the path and use methods from a multivariate calculus class. Namely, let u and v be continuous in domain D, and Γ a piecewise smooth curve in D. Let $(x(t), y(t))$ be a parametrization of Γ for $t_0 \le t \le t_1$ and $f(z) = u(x, y) + iv(x, y)$ for $z = x + iy$. Then

$$\int_\Gamma f(z)dz = \int_{t_0}^{t_1} [u(x(t), y(t)) + iv(x(t), y(t))]\left(\frac{dx}{dt} + i\frac{dy}{dt}\right)dt.$$

Here we have used

$$dz = dx + idy = \left(\frac{dx}{dt} + i\frac{dy}{dt}\right)dt.$$

Furthermore, a set D is called a domain if it is both open and connected.

Before continuing, we first define open and connected. A set D is connected if and only if for all z_1, and z_2 in D there exists a piecewise smooth curve connecting z_1 to z_2 and lying in D. Otherwise it is called disconnected. Examples are shown in Figure 8.5.2.

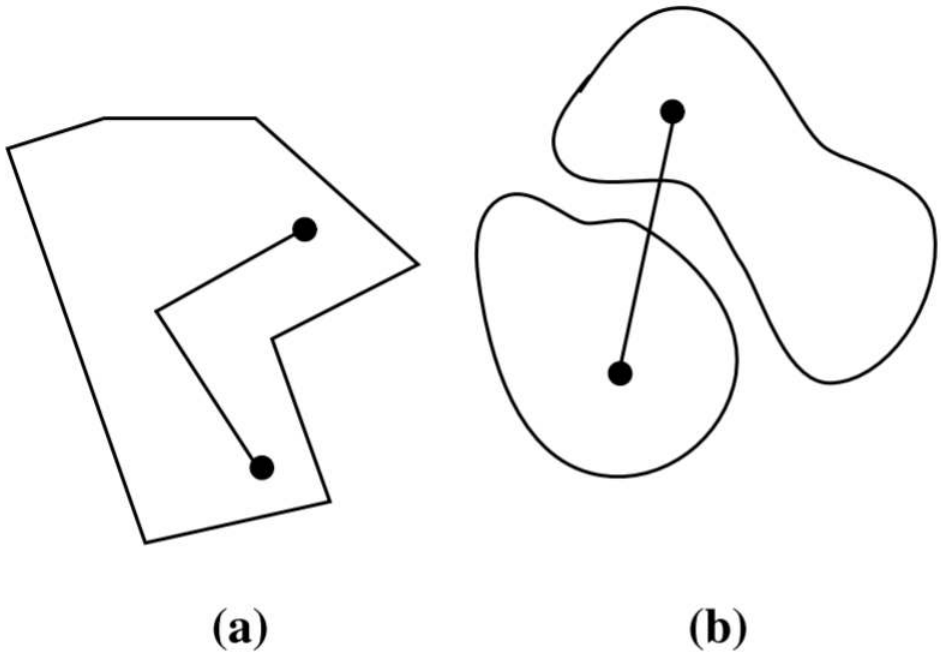


Figure 8.5.2:

A set D is open if and only if for all z_0 in D there exists an open disk $\left|z - z_0\right| < \rho$ in D. In Figure 8.5.3 we show a region with two disks.

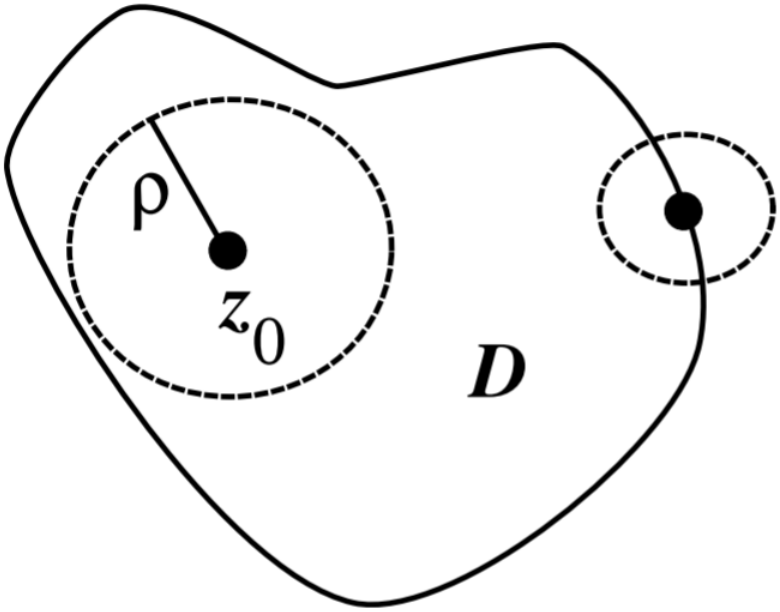


Figure 8.5.3: Locations of open disks inside and on the boundary of a region.

For all points on the interior of the region one can find at least one disk contained entirely in the region. The closer one is to the boundary, the smaller the radii of such disks. However, for a point on the boundary, every such disk would contain points inside and outside the disk. Thus, an open set in the complex plane would not contain any of its boundary points.

We now have a prescription for computing path integrals. Let's see how this works with a couple of examples.

Example 8.5.1

Evaluate $\int_C z^2 dz$, where $C =$ the arc of the unit circle in the first quadrant as shown in Figure 8.5.4.

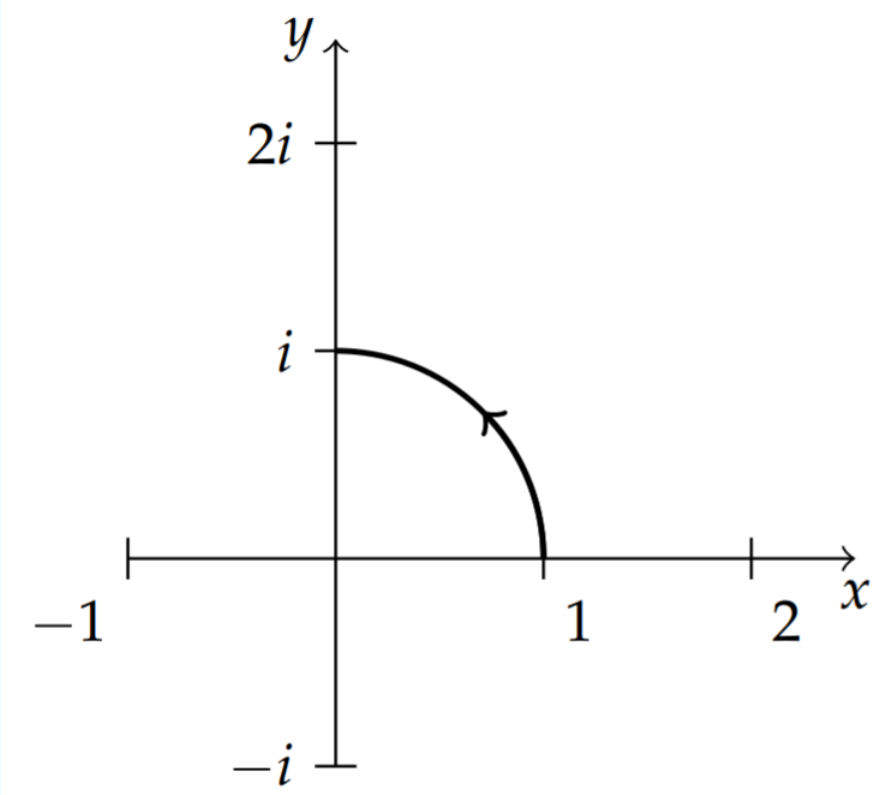


Figure 8.5.4: Contour for Example 8.5.1.

Solution

There are two ways we could carry out the parametrization. First, we note that the standard parametrization of the unit circle is

$$(x(\theta), y(\theta)) = (\cos\theta, \sin\theta), \quad 0 \le \theta \le 2\pi.$$

For a quarter circle in the first quadrant, $0 \le \theta \le \frac{\pi}{2}$, we let $z = \cos\theta + i\sin\theta$. Therefore, $dz = (-\sin\theta + i\cos\theta)d\theta$ and the path integral becomes

$$\int_C z^2 dz = \int_0^{\frac{\pi}{2}} (\cos\theta + i\sin\theta)^2(-\sin\theta + i\cos\theta)d\theta.$$

We can expand the integrand and integrate, having to perform some trigonometric integrations.

$$\int_0^{\frac{\pi}{2}} \left[\sin^3\theta - 3\cos^2\theta\sin\theta + i\left(\cos^3\theta - 3\cos\theta\sin^2\theta\right)\right]d\theta.$$

The reader should work out these trigonometric integrations and confirm the result. For example, you can use

$$\sin^3\theta = \sin\theta\left(1 - \cos^2\theta\right)$$

to write the real part of the integrand as

$$\sin\theta - 4\cos^2\theta\sin\theta.$$

The resulting antiderivative becomes

$$-\cos\theta + \frac{4}{3}\cos^3\theta.$$

The imaginary integrand can be integrated in a similar fashion.

While this integral is doable, there is a simpler procedure. We first note that $z = e^{i\theta}$ on C. S_0, $dz = ie^{i\theta}d\theta$. The integration then becomes

$$\begin{aligned}\int_C z^2 dz &= \int_0^{\frac{\pi}{2}} \left(e^{i\theta}\right)^2 ie^{i\theta}d\theta \\ &= i\int_0^{\frac{\pi}{2}} e^{3i\theta}d\theta \\ &= \frac{ie^{3i\theta}}{3i}\Bigg|_0^{\pi/2} \\ &= -\frac{1+i}{3}\end{aligned}$$

Example 8.5.2

Evaluate $\int_\Gamma z dz$, for the path $\Gamma = \gamma_1 \cup \gamma_2$ shown in Figure 8.5.5.

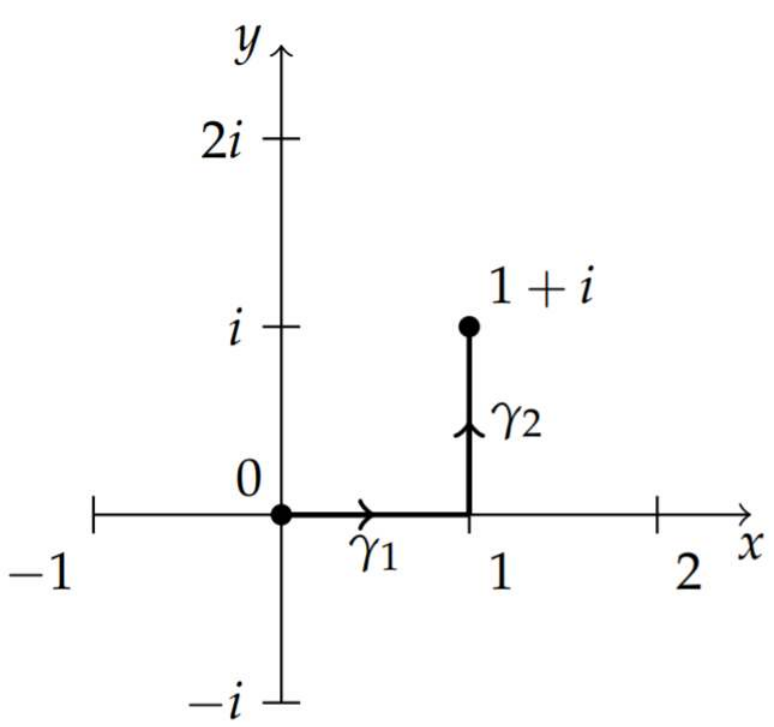


Figure 8.5.5: Contour for Example 8.5.2 with $\Gamma = \gamma_1 \cup \gamma_2$.

Solution

In this problem we have a path that is a piecewise smooth curve. We can compute the path integral by computing the values along the two segments of the path and adding the results. Let the two segments be called γ_1 and γ_2 as shown in Figure 8.5.5 and parametrize each path separately.

Over γ_1 we note that $y = 0$. Thus, $z = x$ for $x \in [0, 1]$. It is natural to take x as the parameter. So, we let $dz = dx$ to find

$$\int_{\gamma_1} z dz = \int_0^1 x dx = \frac{1}{2}.$$

For path γ_2 we have that $z = 1 + \text{ iy}$ for $y \in [0, 1]$ and $dz = idy$. Inserting this parametrization into the integral, the integral becomes

$$\int_{\gamma_2} z dz = \int_0^1 (1 + iy) i dy = i - \frac{1}{2}.$$

Combining the results for the paths γ_1 and γ_2, we have $\int_I z dz = \frac{1}{2} + \left(i - \frac{1}{2}\right) = i$.

Example 8.5.3

Evaluate $\int_{\gamma_3} z dz$, where γ_3, is the path shown in Figure 8.5.6.

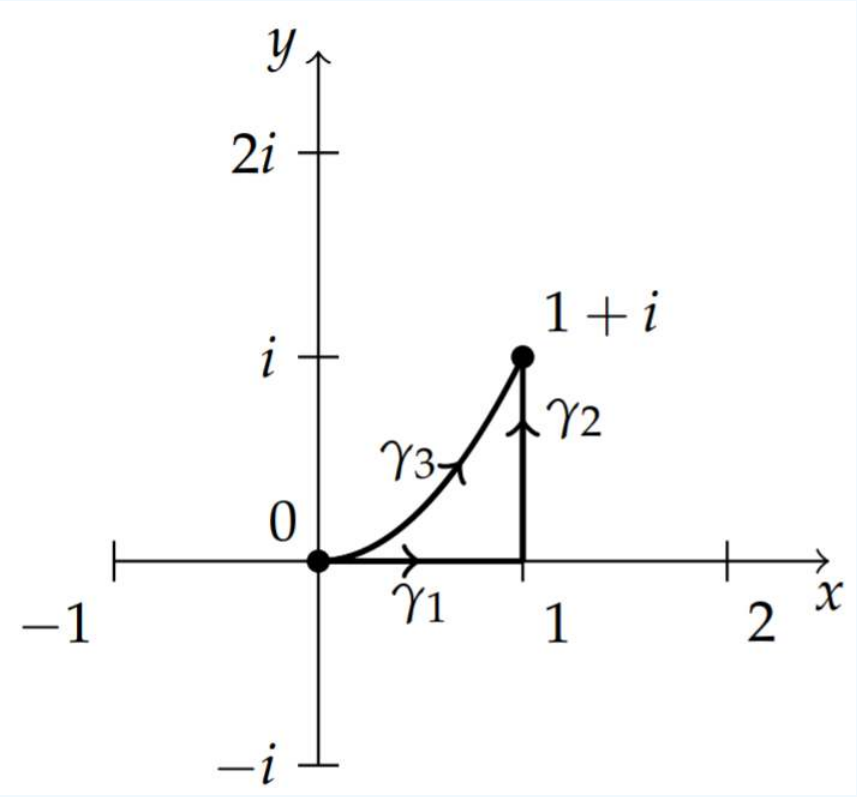


Figure 8.5.6: Contour for Example 8.5.3.

Solution

In this case we take a path from $z = 0$ to $z = 1 + i$ along a different path than in the last example. Let $\gamma_3 = \left\{(x, y) \mid y = x^2, x \in [0, 1]\right\} = \left\{z \mid z = x + ix^2, x \in [0, 1]\right\}$. Then, $dz = (1 + 2ix)dx$.

The integral becomes

$$\int_{\gamma_3} z dz = \int_0^1 \left(x + ix^2\right)(1 + 2ix) dx$$
$$= \int_0^1 \left(x + 3ix^2 - 2x^3\right) dx =$$
$$= \left[\frac{1}{2}x^2 + ix^3 - \frac{1}{2}x^4\right]_0^1 = i$$

In the last case we found the same answer as we had obtained in Example 8.5.3. But we should not take this as a general rule for all complex path integrals. In fact, it is not true that integrating over different paths always yields the same results. However, when this is true, then we refer to this property as path independence. In particular, the integral $\int f(z)dz$ is path independent if

$$\int_{\Gamma_1} f(z)dz = \int_{\Gamma_2} f(z)dz$$

for all paths from z_1 to z_2 as shown in Figure 8.5.7.

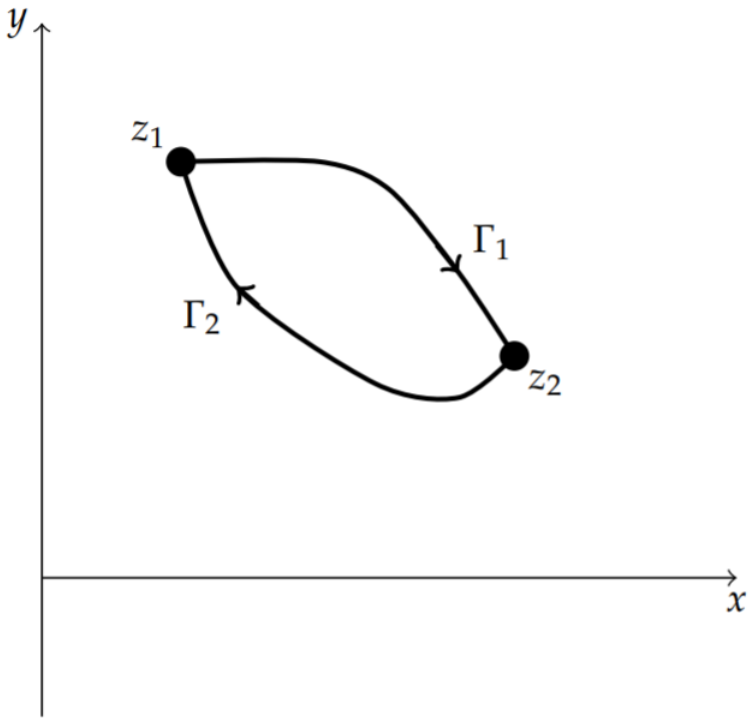


Figure 8.5.7: $\int_{\Gamma_1} f(z)dz = \int_{\Gamma_2} f(z)dz$ for all paths from z_1 to z_2 when the integral of $f(z)$ is path independent.
A simple closed counter.

We can show that if $\int f(z)dz$ is path independent, then the integral of $f(z)$ over all closed loops is zero,

$$\int_{\text{closed loops}} f(z)dz = 0.$$

A common notation for integrating over closed loops is $\oint_C f(z)dz$. But first we have to define what we mean by a closed loop. A simple closed contour is a path satisfying

a. The end point is the same as the beginning point. (This makes the loop closed.)
b. The are no self-intersections. (This makes the loop simple.)

A loop in the shape of a figure eight is closed, but it is not simple.

Now, consider an integral over the closed loop C shown in Figure 8.5.8. We pick two points on the loop breaking it into two contours, C_1 and C_2. Then we make use of the path independence by defining C_2^- to be the path along C_2 but in the opposite

direction. Then,

$$\oint_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$$
$$= \int_{C_1} f(z)dz - \int_{C_2^-} f(z)dz$$

Assuming that the integrals from point 1 to point 2 are path independent, then the integrals over C_1 and C_2^- are equal. Therefore, we have $\oint_C f(z)dz = 0$.

Note

$\oint_C f(z)dz = 0$ if the integral is path independent.

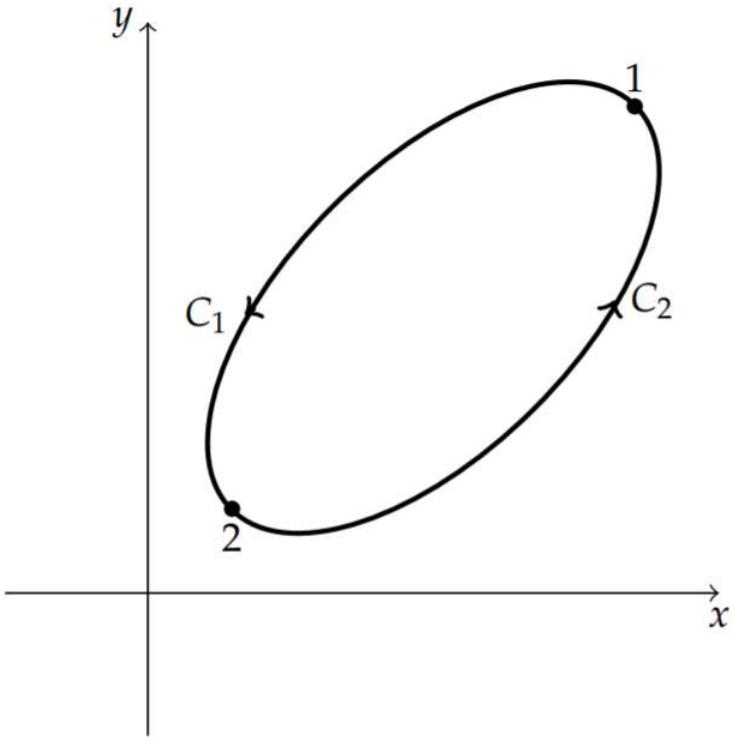


Figure 8.5.8: The integral $\oint_C f(z)dz$ around C is zero if the integral $\int_\Gamma f(z)dz$ is path independent.

Example 8.5.4

Consider the integral $\oint_C zdz$ for C the closed contour shown in Figure 8.5.6 starting at $z = 0$ following path γ_1, then γ_2 and returning to $z = 0$. Based on the earlier examples and the fact that going backwards on γ_3 introduces a negative sign, we have

$$\oint_C zdz = \int_{\gamma_1} zdz + \int_{\gamma_2} zdz - \int_{\gamma_3} zdz = \frac{1}{2} + \left(i - \frac{1}{2}\right) - i = 0.$$

Cauchy's Theorem

Next we want to investigate if we can determine that integrals over simple closed contours vanish without doing all the work of parametrizing the contour. First, we need to establish the direction about which we traverse the contour. We can define the orientation of a curve by referring to the normal of the curve.

Recall that the normal is a perpendicular to the curve. There are two such perpendiculars. The above normal points outward and the other normal points towards the interior of a closed curve. We will define a positively oriented contour as one that is traversed with

the outward normal pointing to the right. As one follows loops, the interior would then be on the left.

Note

A curve with parametrization $(x(t), y(t))$ has a normal $\left(n_x, n_y\right) = \left(-\frac{dx}{dt}, \frac{dy}{dt}\right)$.

We now consider $\oint_C (u + iv)dz$ over a simple closed contour. This can be written in terms of two real integrals in the xy-plane.

$$\begin{aligned}\oint_C (u + iv)dz &= \int_C (u + iv)(dx + idy) \\ &= \int_C udx - vdy + i\int_C vdx + udy.\end{aligned}$$

These integrals in the plane can be evaluated using Green's Theorem in the Plane. Recall this theorem from your last semester of calculus:

Theorem 8.5.1: Green's Theorem in the Plane

Let $P(x, y)$ and $Q(x, y)$ be continuously differentiable functions on and inside the simple closed curve C as shown in Figure 8.5.9. Denoting the enclosed region S, we have

$$\int_C Pdx + Qdy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy$$

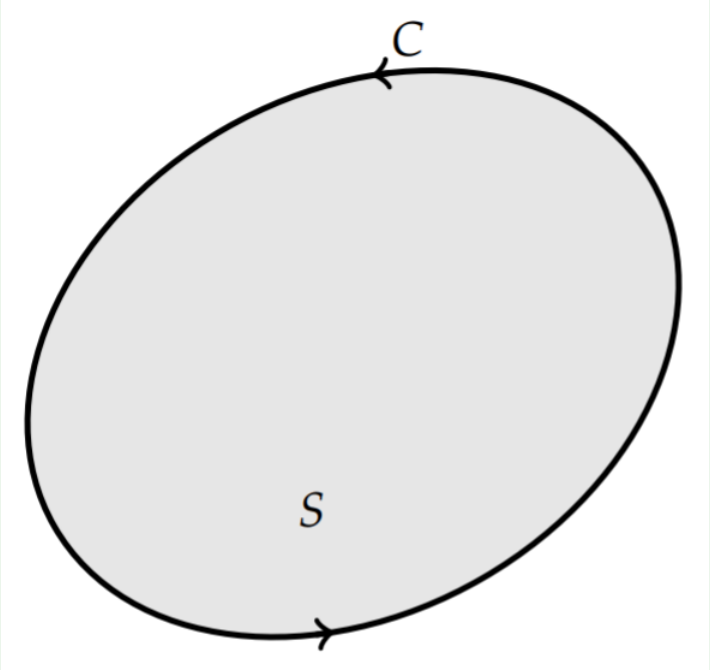


Figure 8.5.9: Region used in Green's Theorem.

Note

Green's Theorem in the Plane is one of the major integral theorems of vector calculus. It was discovered by George Green (1793 − 1841) and published in 1828, about four years before he entered Cambridge as an undergraduate.

Using Green's Theorem to rewrite the first integral in (8.5.5), we have

$$\int_C udx - vdy = \iint_S \left(\frac{-\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dxdy$$

If u and v satisfy the Cauchy-Riemann equations (8.4.6), then the integrand in the double integral vanishes. Therefore,

$$\int_C udx - vdy = 0.$$

In a similar fashion, one can show that

$$\int_C vdx + udy = 0.$$

We have thus proven the following theorem:

Theorem 8.5.2: Cauchy's Theorem

If u and v satisfy the Cauchy-Riemann equations (8.4.6) inside and on the simple closed contour C, then

$$\oint_C (u + iv)dz = 0.$$

Corollary 8.5.1

Corollary $\oint_C f(z)dz = 0$ when f is differentiable in domain D with $C \subset D$.

Either one of these is referred to as **Cauchy's Theorem**.

Example 8.5.5

Evaluate $\oint_{|z-1|=3} z^4 dz$.

Solution

Since $f(z) = z^4$ is differentiable inside the circle $|z - 1| = 3$, this integral vanishes.

We can use Cauchy's Theorem to show that we can deform one contour into another, perhaps simpler, contour.

Note

One can deform contours into simpler ones.

Theorem 8.5.3

If $f(z)$ is holomorphic between two simple closed contours, C and C', then $\oint_C f(z)dz = \oint_{C'} f(z)dz$.

Proof

We consider the two curves C and C' as shown in Figure 8.5.10. Connecting the two contours with contours Γ_1 and Γ_2 (as shown in the figure), C is seen to split into contours C_1 and C_2 and C' into contours C_1' and C_2'. Note that $f(z)$ is differentiable inside the newly formed regions between the curves. Also, the boundaries of these regions are now simple closed curves. Therefore, Cauchy's Theorem tells us that the integrals of $f(z)$ over these regions are zero.

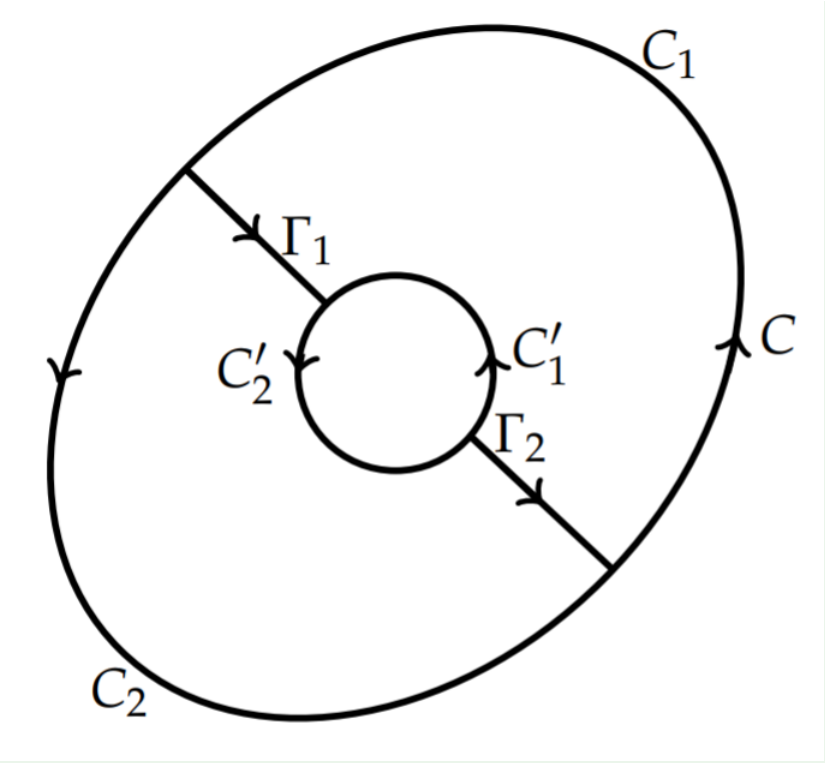


Figure 8.5.10: The contours needed to prove that $\oint_C f(z)dz = \oint_{C'} f(z)dz$ when $f(z)$ is holomorphic between the contours C and C'.

Noting that integrations over contours opposite to the positive orientation are the negative of integrals that are positively oriented, we have from Cauchy's Theorem that

$$\int_{C_1} f(z)dz + \int_{\Gamma_1} f(z)dz - \int_{C_1'} f(z)dz + \int_{\Gamma_2} f(z)dz = 0$$

and

$$\int_{C_2} f(z)dz - \int_{\Gamma_2} f(z)dz - \int_{C_2'} f(z)dz - \int_{\Gamma_1} f(z)dz = 0.$$

In the first integral we have traversed the contours in the following order: C_1, Γ_1, C_1' backwards, and Γ_2. The second integral denotes the integration over the lower region, but going backwards over all contours except for C_2.

Combining these results by adding the two equations above, we have

$$\int_{C_1} f(z)dz + \int_{C_2} f(z)dz - \int_{C_1'} f(z)dz - \int_{C_2'} f(z)dz = 0.$$

Noting that $C = C_1 + C_2$ and $C' = C_1' + C_2'$, we have

$$\oint_C f(z)dz = \oint_{C'} f(z)dz,$$

as was to be proven.

Example 8.5.6

Compute $\oint_R \frac{dz}{z}$ for R the rectangle $[-2, 2] \times [-2i, 2i]$.

Solution

We can compute this integral by looking at four separate integrals over the sides of the rectangle in the complex plane. One simply parametrizes each line segment, perform the integration and sum the four separate results. From the last theorem, we can instead integrate over a simpler contour by deforming the rectangle into a circle as long as $f(z) = \frac{1}{z}$ is differentiable in the region bounded by the rectangle and the circle. So, using the unit circle, as shown in Figure 8.5.11, the integration might be easier to perform.

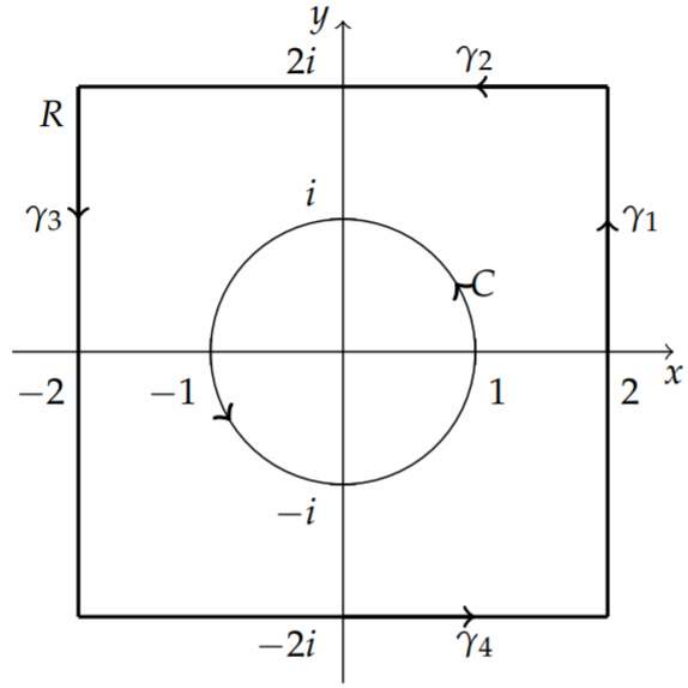


Figure 8.5.11: The contours used to compute $\oint_R \frac{dz}{z}$. Note that to compute the integral around R we can deform the contour to the circle C since $f(z)$ is differentiable in the region between the contours.

More specifically, the last theorem tells us that

$$\oint_R \frac{dz}{z} = \oint_{|z|=1} \frac{dz}{z}$$

The latter integral can be computed using the parametrization $z = e^{i\theta}$ for $\theta \in [0, 2\pi]$. Thus,

$$\begin{aligned} \oint_{|z|=1} \frac{dz}{z} &= \int_0^{2\pi} \frac{ie^{i\theta}d\theta}{e^{i\theta}} \\ &= i\int_0^{2\pi} d\theta = 2\pi i. \end{aligned}$$

Therefore, we have found that $\oint_R \frac{dz}{z} = 2\pi i$ by deforming the original simple closed contour.

For fun, let's do this the long way to see how much effort was saved. We will label the contour as shown in Figure 8.5.12. The lower segment, γ_4 of the square can be simple parametrized by noting that along this segment $z = x - 2i$ for $x \in [-2, 2]$. Then, we have

$$\oint_{\gamma_4} \frac{dz}{z} = \int_{-2}^{2} \frac{dx}{x-2i}$$
$$= \ln|x-2i|\,\Big|_{-2}^{2}$$
$$= \left(\ln(2\sqrt{2}) - \frac{\pi i}{4}\right) - \left(\ln(2\sqrt{2}) - \frac{3\pi i}{4}\right)$$
$$= \frac{\pi i}{2}.$$

We note that the arguments of the logarithms are determined from the angles made by the diagonals provided in Figure 8.5.12.

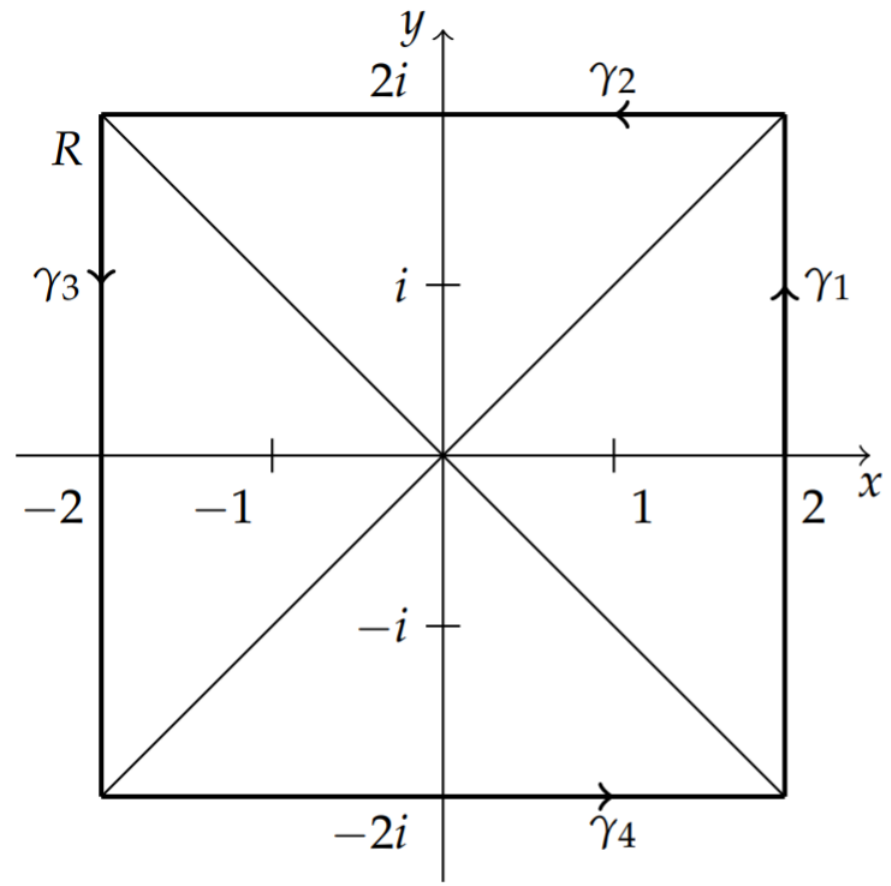


Figure 8.5.12: The contours used to compute $\oint_R \frac{az}{z}$. The added diagonals are for the reader to easily see the arguments used in the evaluation of the limits when integrating over the segments of the square R.

Similarly, the integral along the top segment, $z = x + 2i$, $x \in [-2, 2]$, is computed as

$$\oint_{\gamma_2} \frac{dz}{z} = \int_{2}^{-2} \frac{dx}{x+2i}$$
$$= \ln|x+2i|\,\Big|_{2}^{-2}$$
$$= \left(\ln(2\sqrt{2}) + \frac{3\pi i}{4}\right) - \left(\ln(2\sqrt{2}) + \frac{\pi i}{4}\right)$$
$$= \frac{\pi i}{2}.$$

The integral over the right side, $z = 2 + iy$, $y \in [-2, 2]$, is

$$\oint_{\gamma_1} \frac{dz}{z} = \int_{-2}^{2} \frac{idy}{2+iy}$$
$$= \ln|2+iy|_{-2}^{2}$$
$$= \left(\ln(2\sqrt{2}) + \frac{\pi i}{4}\right) - \left(\ln(2\sqrt{2}) - \frac{\pi i}{4}\right)$$
$$= \frac{\pi i}{2}.$$

Finally, the integral over the left side, $z = -2+iy, y \in [-2,2]$, is

$$\oint_{\gamma_3} \frac{dz}{z} = \int_{2}^{-2} \frac{idy}{-2+iy}$$
$$= \ln|-2+iy|_{-2}^{2}$$
$$= \left(\ln(2\sqrt{2}) + \frac{5\pi i}{4}\right) - \left(\ln(2\sqrt{2}) + \frac{3\pi i}{4}\right)$$
$$= \frac{\pi i}{2}.$$

Therefore, we have that

$$\oint_R \frac{dz}{z} = \int_{\gamma_1} \frac{dz}{z} + \int_{\gamma_2} \frac{dz}{z} + \int_{\gamma_3} \frac{dz}{z} + \int_{\gamma_4} \frac{dz}{z}$$
$$= \frac{\pi i}{2} + \frac{\pi i}{2} + \frac{\pi i}{2} + \frac{\pi i}{2}$$
$$= 4\left(\frac{\pi i}{2}\right) = 2\pi i.$$

This gives the same answer we had found using a simple contour deformation.

The converse of Cauchy's Theorem is not true, namely $\oint_C f(z)dz = 0$ does not always imply that $f(z)$ is differentiable. What we do have is Morera's Theorem(Giacinto Morera, 1856-1909):

Theorem 8.5.4: Moerera's Theorem

Theorem 8.6. Let f be continuous in a domain D. Suppose that for every simple closed contour C in D, $\oint_C f(z)dz = 0$. Then f is differentiable in D.

The proof is a bit more detailed than we need to go into here. However, this theorem is useful in the next section.

Analytic Functions and Cauchy's Integral Formula

In the previous section we saw that Cauchy's Theorem was useful for computing particular integrals without having to parametrize the contours or for deforming contours into simpler contours. The integrand needs to possess certain differentiability properties. In this section, we will generalize the functions that we can integrate slightly so that we can integrate a larger family of complex functions. This will lead us to the Cauchy's Integral Formula, which extends Cauchy's Theorem to functions analytic in an annulus. However, first we need to explore the concept of analytic functions.

A function $f(z)$ is analytic in domain D if for every open disk $|z - z_0| < \rho$ lying in D, $f(z)$ can be represented as a power series in z_0. Namely,

$$f(z) = \sum_{n=0}^{\infty} c_n \left(z - z_0\right)^n.$$

This series converges uniformly and absolutely inside the circle of convergence, $\left|z - z_0\right| < R$, with radius of convergence R. [See the Appendix for a review of convergence.]

Since $f(z)$ can be written as a uniformly convergent power series, we can integrate it term by term over any simple closed contour in D containing z_0. In particular, we have to compute integrals like $\oint_C \left(z - z_0\right)^n dz$. As we will see in the homework exercises, these integrals evaluate to zero for most n. Thus, we can show that for $f(z)$ analytic in D and on any closed contour C lying in D, $\oint_C f(z)dz = 0$. Also, f is a uniformly convergent sum of continuous functions, so $f(z)$ is also continuous. Thus, by Morera's Theorem, we have that $f(z)$ is differentiable if it is analytic. Often terms like analytic, differentiable and holomorphic are used interchangeably, though there is a subtle distinction due to their definitions.

Note

There are various types of complex-valued functions.

A holomorphic function is (complex) differentiable in a neighborhood of every point in its domain.

An analytic function has a convergent Taylor series expansion in a neighborhood of each point in its domain. We see here that analytic functions are holomorphic and vice versa.

If a function is holomorphic throughout the complex plane, then it is called an entire function.

Finally, a function which is holomorphic on all of its domain except at a set of isolated poles (to be defined later), then it is called a meromorphic function.

As examples of series expansions about a given point, we will consider series expansions and regions of convergence for $f(z) = \frac{1}{1+z}$.

Example 8.5.7

Find the series expansion of $f(z) = \frac{1}{1+z}$ about $z_0 = 0$.

Solution

This case is simple. From Chapter 1 we recall that $f(z)$ is the sum of a geometric series for $|z| < 1$. We have

$$f(z) = \frac{1}{1+z} = \sum_{n=0}^{\infty} (-z)^n.$$

Thus, this series expansion converges inside the unit circle ($|z| < 1$) in the complex plane.

Example 8.5.8

Find the series expansion of $f(z) = \frac{1}{1+z}$ about $z_0 = \frac{1}{2}$.

Solution

We now look into an expansion about a different point. We could compute the expansion coefficients using Taylor's formula for the coefficients. However, we can also make use of the formula for geometric series after rearranging the function. We seek an expansion in powers of $z - \frac{1}{2}$. So, we rewrite the function in a form that has is a function of $z - \frac{1}{2}$. Thus,

$$f(z) = \frac{1}{1+z} = \frac{1}{1+\left(z-\frac{1}{2}+\frac{1}{2}\right)} = \frac{1}{\frac{3}{2}+\left(z-\frac{1}{2}\right)}.$$

This is not quite in the form we need. It would be nice if the denominator were of the form of one plus something. [Note: This is similar to what we had seen in Example 11.7.7.] We can get the denominator into such a form by factoring out the $\frac{3}{2}$. Then we would have

$$f(z) = \frac{2}{3}\frac{1}{1+\frac{2}{3}\left(z-\frac{1}{2}\right)}.$$

The second factor now has the form $\frac{1}{1-r}$, which would be the sum of a geometric series with first term $a = 1$ and ratio $r = -\frac{2}{3}\left(z-\frac{1}{2}\right)$ provided that $|r| < 1$. Therefore, we have found that

$$f(z) = \frac{2}{3}\sum_{n=0}^{\infty}\left[-\frac{2}{3}\left(z-\frac{1}{2}\right)\right]^n$$

for

$$\left|-\frac{2}{3}\left(z-\frac{1}{2}\right)\right| < 1.$$

This convergence interval can be rewritten as

$$\left|z-\frac{1}{2}\right| < \frac{3}{2},$$

which is a circle centered at $z = \frac{1}{2}$ with radius $\frac{3}{2}$.

In Figure 8.5.13 we show the regions of convergence for the power series expansions of $f(z) = \frac{1}{1+z}$ about $z = 0$ and $z = \frac{1}{2}$. We note that the first expansion gives that $f(z)$ is at least analytic inside the region $|z| < 1$. The second expansion shows that $f(z)$ is analytic in a larger region, $\left|z-\frac{1}{2}\right| < \frac{3}{2}$. We will see later that there are expansions which converge outside of these regions and that some yield expansions involving negative powers of $z - z_0$.

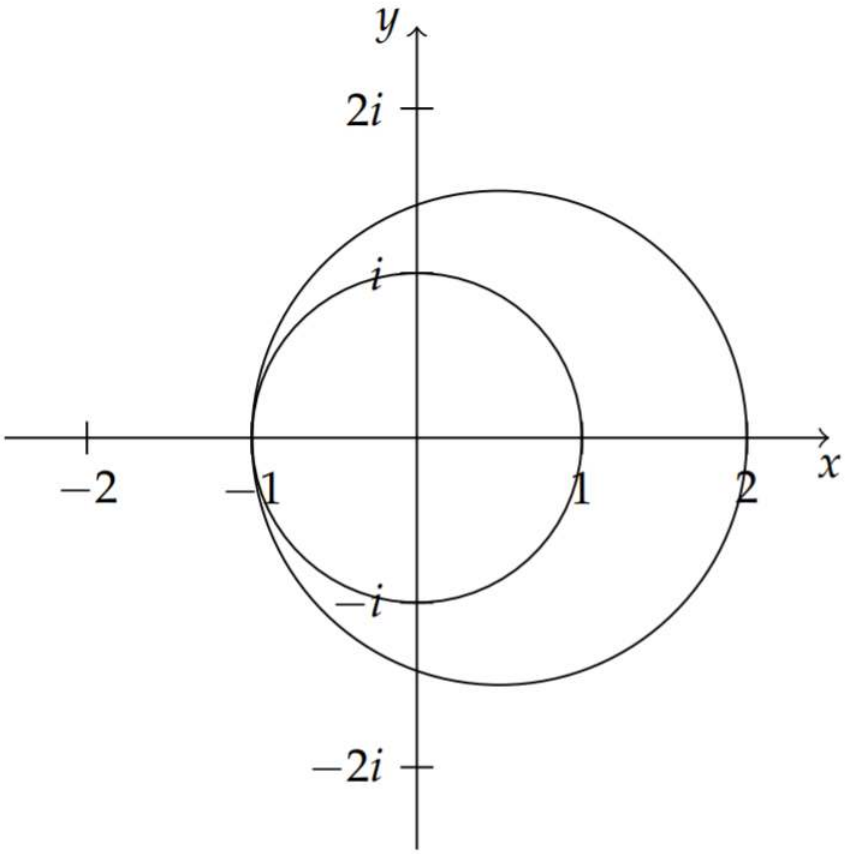


Figure 8.5.13: Regions of convergence for expansions of $f(z) = \frac{1}{1+z}$ about $z = 0$ and $z = \frac{1}{2}$.

We now present the main theorem of this section:

Theorem 8.5.5: Cauchy Integration Formula

Let $f(z)$ be analytic in $\left|z - z_0\right| < \rho$ and let C be the boundary (circle) of this disk. Then,

$$f\left(z_0\right) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}dz.$$

Proof

In order to prove this, we first make use of the analyticity of $f(z)$. We insert the power series expansion of $f(z)$ about z_0 into the integrand. Then we have

$$\begin{aligned}\frac{f(z)}{z - z_0} &= \frac{1}{z - z_0}\left[\sum_{n=0}^{\infty} c_n\left(z - z_0\right)^n\right] \\ &= \frac{1}{z - z_0}\left[c_0 + c_1\left(z - z_0\right) + c_2\left(z - z_0\right)^2 + \ldots\right] \\ &= \frac{c_0}{z - z_0} + \underbrace{c_1 + c_2\left(z - z_0\right) + \ldots}_{\text{analytic function}}\end{aligned}$$

As noted the integrand can be written as

$$\frac{f(z)}{z-z_0} = \frac{c_0}{z-z_0} + h(z),$$

where $h(z)$ is an analytic function, since $h(z)$ is representable as a series expansion about z_0. We have already shown that analytic functions are differentiable, so by Cauchy's Theorem $\oint_C h(z)dz = 0$.

Noting also that $c_0 = f\left(z_0\right)$ is the first term of a Taylor series expansion about $z = z_0$, we have

$$\oint_C \frac{f(z)}{z-z_0}dz = \oint_C \left[\frac{c_0}{z-z_0} + h(z)\right]dz = f\left(z_0\right)\oint_C \frac{1}{z-z_0}dz.$$

We need only compute the integral $\oint_C \frac{1}{z-z_0}dz$ to finish the proof of Cauchy's Integral Formula. This is done by parametrizing the circle, $\left|z-z_0\right| = \rho$, as shown in Figure 8.5.14. This is simply done by letting

$$z - z_0 = \rho e^{i\theta}.$$

(Note that this has the right complex modulus since $\left|e^{i\theta}\right| = 1$. Then $dz = i\rho e^{i\theta}d\theta$. Using this parametrization, we have

$$\oint_C \frac{dz}{z-z_0} = \int_0^{2\pi} \frac{i\rho e^{i\theta}d\theta}{\rho e^{i\theta}} = i\int_0^{2\pi} d\theta = 2\pi i.$$

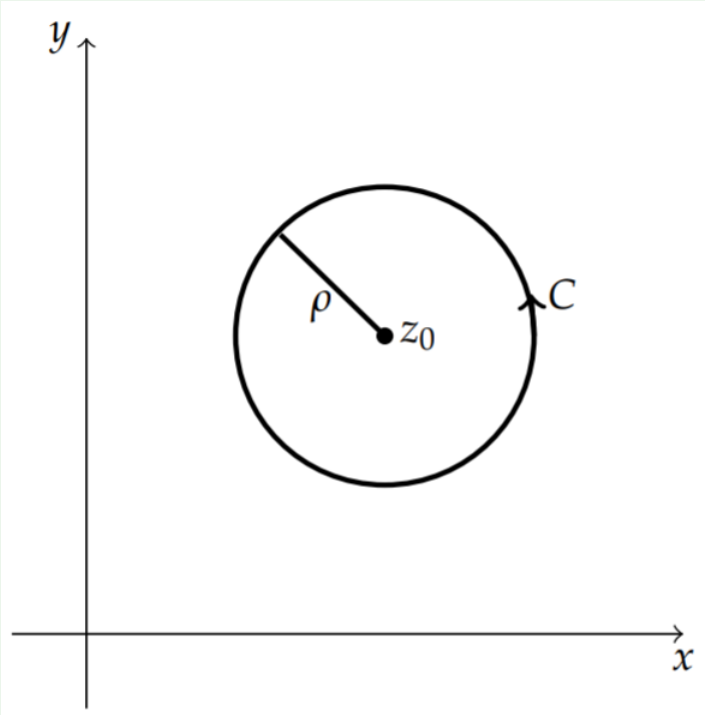


Figure 8.5.14: Circular contour used in proving the Cauchy Integral Formula.

Therefore,

$$\oint_C \frac{f(z)}{z-z_0}dz = f\left(z_0\right)\oint_C \frac{1}{z-z_0}dz = 2\pi i f\left(z_0\right),$$

as was to be shown.

Example 8.5.9

Compute $\oint_{|z|=4} \frac{\cos z}{z^2-6z+5} dz$.

Solution

In order to apply the Cauchy Integral Formula, we need to factor the denominator, $z^2 - 6z + 5 = (z - 1)(z - 5)$. We next locate the zeros of the denominator. In Figure 8.5.15 we show the contour and the points $z = 1$ and $z = 5$. The only point inside the region bounded by the contour is $z = 1$. Therefore, we can apply the Cauchy Integral Formula for $f(z) = \frac{\cos z}{z-5}$ to the integral

$$\int_{|z|=4} \frac{\cos z}{(z-1)(z-5)} dz = \int_{|z|=4} \frac{f(z)}{(z-1)} dz = 2\pi i f(1).$$

Therefore, we have

$$\int_{|z|=4} \frac{\cos z}{(z-1)(z-5)} dz = -\frac{\pi i \cos(1)}{2}.$$

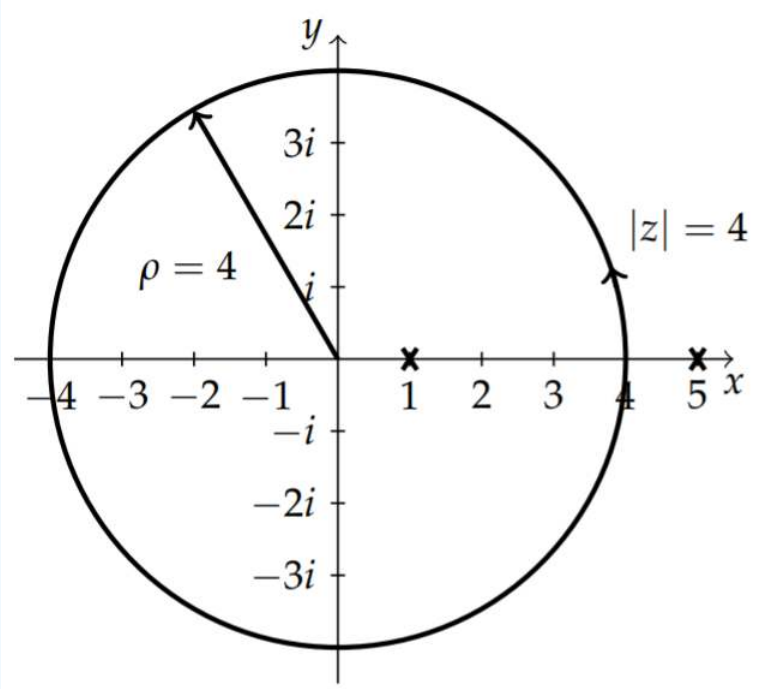


Figure 8.5.15: Circular contour used in computing $\oint_{|z|=4} \frac{\cos z}{z^2-6z+5} dz$.

We have shown that $f(z_0)$ has an integral representation for $f(z)$ analytic in $|z - z_0| < \rho$. In fact, all derivatives of an analytic function have an integral representation. This is given by

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz.$$

This can be proven following a derivation similar to that for the Cauchy Integral Formula. Inserting the Taylor series expansion for $f(z)$ into the integral on the right hand side, we have

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz = \sum_{m=0}^{\infty} c_m \oint_C \frac{(z-z_0)^m}{(z-z_0)^{n+1}}dz$$
$$= \sum_{m=0}^{\infty} c_m \oint_C \frac{dz}{(z-z_0)^{n-m+1}}$$

Picking $k = n - m$, the integrals in the sum can be computed by using the following result:

$$\oint_C \frac{dz}{(z-z_0)^{k+1}} = \begin{cases} 0, & k \neq 0 \\ 2\pi i, & k = 0. \end{cases}$$

The proof is left for the exercises.

The only nonvanishing integrals, $\oint_C \frac{dz}{(z-z_0)^{n-m+1}}$, occur when $k = n - m = 0$, or $m = n$. Therefore, the series of integrals collapses to one term and we have

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz = 2\pi i c_n.$$

We finish the proof by recalling that the coefficients of the Taylor series expansion for $f(z)$ are given by

$$c_n = \frac{f^{(n)}(z_0)}{n!}.$$

Then,

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz = \frac{2\pi i}{n!} f^{(n)}(z_0)$$

and the result follows.

Laurent Series

Until this point we have only talked about series whose terms have nonnegative powers of $z - z_0$. It is possible to have series representations in which there are negative powers. In the last section we investigated expansions of $f(z) = \frac{1}{1+z}$ about $z = 0$ and $z = \frac{1}{2}$. The regions of convergence for each series was shown in Figure 8.5.13. Let us reconsider each of these expansions, but for values of z outside the region of convergence previously found.

Example 8.5.10

$f(z) = \frac{1}{1+z}$ for $|z| > 1$.

Solution

As before, we make use of the geometric series. Since $|z| > 1$, we instead rewrite the function as

$$f(z) = \frac{1}{1+z} = \frac{1}{z}\frac{1}{1+\frac{1}{z}}.$$

We now have the function in a form of the sum of a geometric series with first term $a = 1$ and ratio $r = -\frac{1}{2}$. We note that $|z| > 1$ implies that $|r| < 1$. Thus, we have the geometric series

$$f(z) = \frac{1}{z}\sum_{n=0}^{\infty}\left(-\frac{1}{z}\right)^n.$$

This can be re-indexed[1] as

$$f(z) = \sum_{n=0}^{\infty}(-1)^n z^{-n-1} = \sum_{j=1}^{\infty}(-1)^{j-1}z^{-j}.$$

Note that this series, which converges outside the unit circle, $|z| > 1$, has negative powers of z.

Note

Re-indexing a series is often useful in the geometric series series manipulations. In this case, we have the series

$$\sum_{n=0}^{\infty}(-1)^n z^{-n-1} = z^{-1} - z^{-2} + z^{-3} + \ldots$$

The index is n. You can see that the index does not appear when the sum is expanded showing the terms. The summation index is sometimes refered to as a dummy index for this reason. Reindexing allows one to rewrite the shorthand summation notation while capturing the same terms. In this example, the exponents are $-n-1$. We can simplify the notation by letting $-n-1 = -j$, or $j = n+1$. Noting that $j = 1$ when $n = 0$, we get the sum $\sum_{j=1}^{\infty}(-1)^{j-1}z^{-j}$.

Example 8.5.11

$f(z) = \frac{1}{1+z}$ for $\left|z - \frac{1}{2}\right| > \frac{3}{2}$.

Solution

As before, we express this in a form in which we can use a geometric series expansion. We seek powers of $z - \frac{1}{2}$. So, we add and subtract $\frac{1}{2}$ to the z to obtain:

$$f(z) = \frac{1}{1+z} = \frac{1}{1+\left(z-\frac{1}{2}+\frac{1}{2}\right)} = \frac{1}{\frac{3}{2}+\left(z-\frac{1}{2}\right)}.$$

Instead of factoring out the $\frac{3}{2}$ as we had done in Example 8.5.8, we factor out the $\left(z-\frac{1}{2}\right)$ term. Then, we obtain

$$f(z) = \frac{1}{1+z} = \frac{1}{\left(z-\frac{1}{2}\right)}\frac{1}{\left[1+\frac{3}{2}\left(z-\frac{1}{2}\right)^{-1}\right]}.$$

Now we identify $a = 1$ and $r = -\frac{3}{2}\left(z-\frac{1}{2}\right)^{-1}$. This leads to the series

$$f(z) = \frac{1}{z-\frac{1}{2}} \sum_{n=0}^{\infty} \left(-\frac{3}{2}\left(z-\frac{1}{2}\right)^{-1} \right)^n$$
$$= \sum_{n=0}^{\infty} \left(-\frac{3}{2} \right)^n \left(z-\frac{1}{2}\right)^{-n-1}.$$

This converges for $\left|z-\frac{1}{2}\right| > \frac{3}{2}$ and can also be re-indexed to verify that this series involves negative powers of $z-\frac{1}{2}$.

This leads to the following theorem:

Theorem 8.5.6

Let $f(z)$ be analytic in an annulus, $R_1 < \left|z-z_0\right| < R_2$, with C a positively oriented simple closed curve around z_0 and inside the annulus as shown in Figure 8.5.16. Then,

$$f(z) = \sum_{j=0}^{\infty} a_j \left(z-z_0\right)^j + \sum_{j=1}^{\infty} b_j \left(z-z_0\right)^{-j},$$

with

$$a_j = \frac{1}{2\pi i} \oint_C \frac{f(z)}{\left(z-z_0\right)^{j+1}} dz,$$

and

$$b_j = \frac{1}{2\pi i} \oint_C \frac{f(z)}{\left(z-z_0\right)^{-j+1}} dz.$$

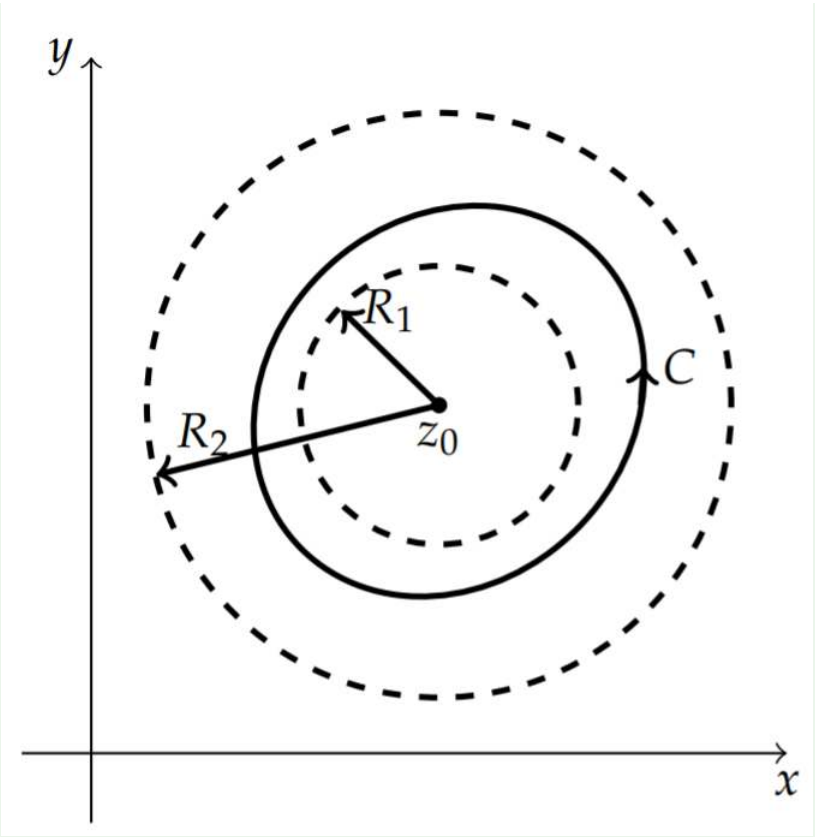


Figure 8.5.16: This figure shows an annulus, $R_1 < \left|z - z_0\right| < R_2$, with C a positively oriented simple closed curve around z_0 and inside the annulus.

The above series can be written in the more compact form

$$f(z) = \sum_{j=-\infty}^{\infty} c_j \left(z - z_0\right)^j.$$

Such a series expansion is called a Laurent series expansion named after its discoverer Pierre Alphonse Laurent (1813-1854).

Example 8.5.12

Expand $f(z) = \frac{1}{(1-z)(2+z)}$ in the annulus $1 < |z| < 2$.

Solution

Using partial fractions, we can write this as

$$f(z) = \frac{1}{3}\left[\frac{1}{1-z} + \frac{1}{2+z}\right].$$

We can expand the first fraction, $\frac{1}{1-z}$, as an analytic function in the region $|z| > 1$ and the second fraction, $\frac{1}{2+z}$, as an analytic function in $|z| < 2$. This is done as follows. First, we write

$$\frac{1}{2+z} = \frac{1}{2\left[1-\left(-\frac{z}{2}\right)\right]} = \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{z}{2}\right)^n.$$

Then, we write

$$\frac{1}{1-z} = -\frac{1}{z\left[1-\frac{1}{z}\right]} = -\frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n}.$$

Therefore, in the common region, $1 < |z| < 2$, we have that

$$\frac{1}{(1-z)(2+z)} = \frac{1}{3}\left[\frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{z}{2}\right)^n - \sum_{n=0}^{\infty}\frac{1}{z^{n+1}}\right]$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{6\left(2^n\right)}z^n + \sum_{n=1}^{\infty}\frac{(-1)}{3}z^{-n}$$

We note that this is not a Taylor series expansion due to the existence of terms with negative powers in the second sum.

Example 8.5.13

Find series representations of $f(z) = \frac{1}{(1-z)(2+z)}$ throughout the complex plane.

Solution

In the last example we found series representations of $f(z) = \frac{1}{(1-z)(2+z)}$ in the annulus $1 < |z| < 2$. However, we can also find expansions which converge for other regions. We first write

$$f(z) = \frac{1}{3}\left[\frac{1}{1-z} + \frac{1}{2+z}\right].$$

We then expand each term separately.

The first fraction, $\frac{1}{1-z}$, can be written as the sum of the geometric series

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \quad |z| < 1.$$

This series converges inside the unit circle. We indicate this by region 1 in Figure 8.5.17.

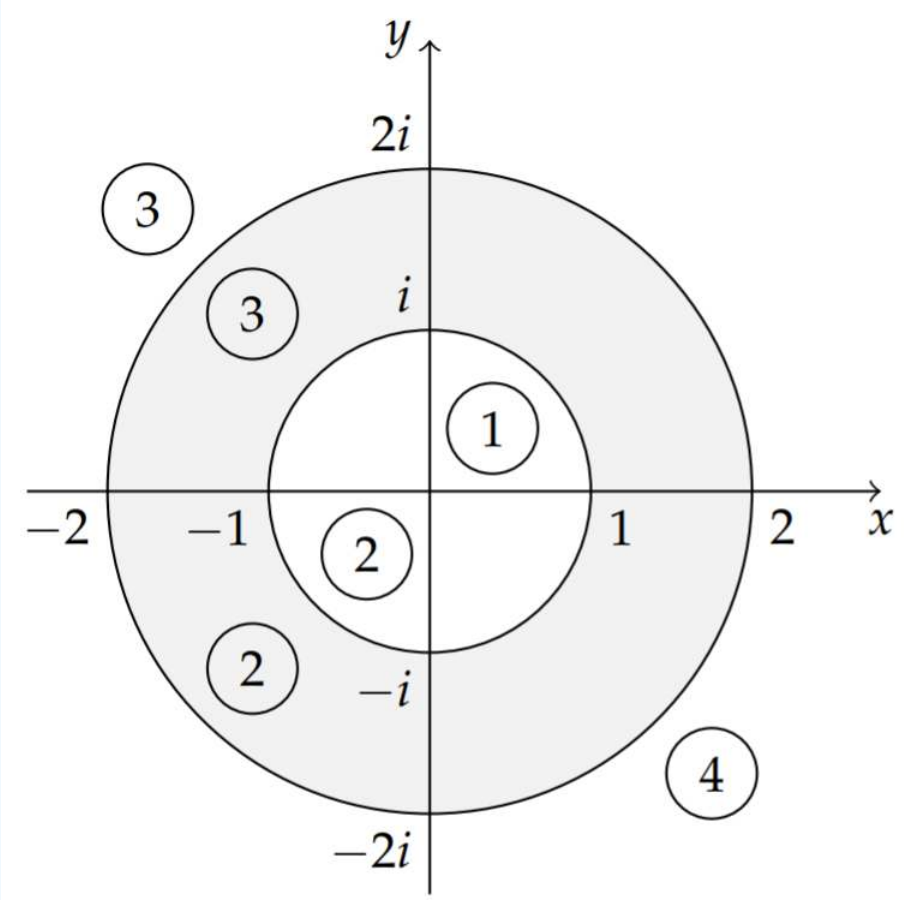


Figure 8.5.17: Regions of convergence for Laurent expansions of $f(z) = \frac{1}{1+z}$.

In the last example, we showed that the second fraction, $\frac{1}{2+z}$, has the series expansion

$$\frac{1}{2+z} = \frac{1}{2\left[1-\left(-\frac{z}{2}\right)\right]} = \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{z}{2}\right)^n.$$

which converges in the circle $|z| < 2$. This is labeled as region 2 in Figure 8.5.17.

Regions 1 and 2 intersect for $|z| < 1$, so, we can combine these two series representations to obtain

$$\frac{1}{(1-z)(2+z)} = \frac{1}{3}\left[\sum_{n=0}^{\infty} z^n + \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{z}{2}\right)^n\right], \quad |z| < 1.$$

In the annulus, $1 < |z| < 2$, we had already seen in the last example that we needed a different expansion for the fraction $\frac{1}{1-z}$. We looked for an expansion in powers of $1/z$ which would converge for large values of z. We had found that

$$\frac{1}{1-z} = -\frac{1}{z\left(1-\frac{1}{z}\right)} = -\frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n}, \quad |z| > 1.$$

This series converges in region 3 in Figure 8.5.17. Combining this series with the one for the second fraction, we obtain a series representation that converges in the overlap of regions 2 and 3 . Thus, in the annulus $1 < |z| < 2$ we have

$$\frac{1}{(1-z)(2+z)} = \frac{1}{3}\left[\frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{z}{2}\right)^n - \sum_{n=0}^{\infty}\frac{1}{z^{n+1}}\right].$$

So far, we have series representations for $|z| < 2$. The only region not covered yet is outside this disk, $|z| > 2$. In in Figure 8.5.17 we see that series 3, which converges in region 3, will converge in the last section of the complex plane. We just need one more series expansion for $1/(2+z)$ for large z. Factoring out a z in the denominator, we can write this as a geometric series with $r = 2/z$,

$$\frac{1}{2+z} = \frac{1}{z\left[\frac{2}{z}+1\right]} = \frac{1}{z}\sum_{n=0}^{\infty}\left(-\frac{2}{z}\right)^n.$$

This series converges for $|z| > 2$. Therefore, it converges in region 4 and the final series representation is

$$\frac{1}{(1-z)(2+z)} = \frac{1}{3}\left[\frac{1}{z}\sum_{n=0}^{\infty}\left(-\frac{2}{z}\right)^n - \sum_{n=0}^{\infty}\frac{1}{z^{n+1}}\right].$$

Singularities and The Residue Theorem

In the last section we found that we could integrate functions satisfying some analyticity properties along contours without using detailed parametrizations around the contours. We can deform contours if the function is analytic in the region between the original and new contour. In this section we will extend our tools for performing contour integrals.

The integrand in the Cauchy Integral Formula was of the form $g(z) = \frac{f(z)}{z-z_0}$, where $f(z)$ is well behaved at z_0. The point $z = z_0$ is called a singularity of $g(z)$, as $g(z)$ is not defined there. More specifically, a singularity of $f(z)$ is a point at which $f(z)$ fails to be analytic.

We can also classify these singularities. Typically these are isolated singularities. As we saw from the proof of the Cauchy Integral Formula, $g(z) = \frac{f(z)}{z-z_0}$ has a Laurent series expansion about $z = z_0$, given by

$$g(z) = \frac{f(z_0)}{z-z_0} + f'(z_0) + \frac{1}{2}f''(z_0)(z-z_0) + \ldots$$

It is the nature of the first term that gives information about the type of singularity that $g(z)$ has. Namely, in order to classify the singularities of $f(z)$, we look at the principal part of the Laurent series of $f(z)$ about $z = z_0$, $\sum_{j=1}^{\infty} b_j (z-z_0)^{-j}$, which consists of the negative powers of $z - z_0$.

There are three types of singularities, removable, poles, and essential singularities. They are defined as follows:

1. If $f(z)$ is bounded near z_0, then z_0 is a removable singularity.

2. If there are a finite number of terms in the principal part of the Laurent series of $f(z)$ about $z = z_0$, then z_0 is called a pole.
3. If there are an infinite number of terms in the principal part of the Laurent series of $f(z)$ about $z = z_0$, then z_0 is called an essential singularity.

Example 8.5.14

$f(z) = \frac{\sin z}{z}$ has a removable singularity at $z = 0$.

Solution

At first it looks like there is a possible singularity at $z = 0$, since the denominator is zero at $z = 0$. However, we know from the first semester of calculus that $\lim_{z \to 0} \frac{\sin z}{z} = 1$. Furthermore, we can expand sinz about $z = 0$ and see that

$$\frac{\sin z}{z} = \frac{1}{z}\left(z - \frac{z^3}{3!} + \ldots\right) = 1 - \frac{z^2}{3!} + \ldots$$

Thus, there are only nonnegative powers in the series expansion. So, $z = 0$ is a removable singularity.

Example 8.5.15

$f(z) = \frac{e^2}{(z-1)^n}$ has poles at $z = 1$ for n a positive integer.

Solution

For $n = 1$ we have $f(z) = \frac{e^z}{z-1}$. This function has a singularity at $z = 1$. The series expansion is found by expanding e^z about $z = 1$:

$$f(z) = \frac{e}{z-1} e^{z-1} = \frac{e}{z-1} + e + \frac{e}{2!}(z-1) + \ldots.$$

Note that the principal part of the Laurent series expansion about $z = 1$ only has one term, $\frac{e}{z-1}$. Therefore, $z = 1$ is a pole. Since the leading term has an exponent of -1, $z = 1$ is called a pole of order one, or a simple pole.

For $n = 2$ we have $f(z) = \frac{e^z}{(z-1)^2}$. The series expansion is found again by expanding e^z about $z = 1$:

$$f(z) = \frac{e}{(z-1)^2} e^{z-1} = \frac{e}{(z-1)^2} + \frac{e}{z-1} + \frac{e}{2!} + \frac{e}{3!}(z-1) + \ldots.$$

Note that the principal part of the Laurent series has two terms involving $(z-1)^{-2}$ and $(z-1)^{-1}$. Since the leading term has an exponent of -2, $z = 1$ is called a pole of order 2, or a double pole.

Example 8.5.16

$f(z) = e^{\frac{1}{2}}$ has an essential singularity at $z = 0$.

Solution

In this case we have the series expansion about $z = 0$ given by

$$f(z) = e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{z}\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n}.$$

We see that there are an infinite number of terms in the principal part of the Laurent series. So, this function has an essential singularity at $z = 0$.

In the above examples we have seen poles of order one (a simple pole) and two (a double pole). In general, we can say that $f(z)$ has a pole of order k at z_0 if and only if $\left(z - z_0\right)^k f(z)$ has a removable singularity at z_0, but $\left(z - z_0\right)^{k-1} f(z)$ for $k > 0$ does not.

Example 8.5.17

Determine the order of the pole of $f(z) = \text{cotzcscz}$ at $z = 0$.

Solution

First we rewrite $f(z)$ in terms of sines and cosines.

$$f(z) = \text{cotzcscz} = \frac{\text{cosz}}{\sin^2 z}.$$

We note that the denominator vanishes at $z = 0$.

How do we know that the pole is not a simple pole? Well, we check to see if $(z - 0)f(z)$ has a removable singularity at $z = 0$:

$$\begin{aligned} \lim_{z \to 0} (z - 0) f(z) &= \lim_{z \to 0} \frac{z\text{cosz}}{\sin^2 z} \\ &= \left(\lim_{z \to 0} \frac{z}{\text{sinz}}\right)\left(\lim_{z \to 0} \frac{\text{cosz}}{\text{sinz}}\right) \\ &= \lim_{z \to 0} \frac{\text{cosz}}{\text{sinz}}. \end{aligned}$$

We see that this limit is undefined. So, now we check to see if $(z - 0)^2 f(z)$ has a removable singularity at $z = 0$:

$$\begin{aligned} \lim_{z \to 0} (z - 0)^2 f(z) &= \lim_{z \to 0} \frac{z^2\text{cosz}}{\sin^2 z} \\ &= \left(\lim_{z \to 0} \frac{z}{\text{sinz}}\right)\left(\lim_{z \to 0} \frac{z\text{cosz}}{\text{sinz}}\right) \\ &= \lim_{z \to 0} \frac{z}{\text{sinz}} \cos(0) = 1. \end{aligned}$$

In this case, we have obtained a finite, nonzero, result. S_0, $z = 0$ is a pole of order 2 .

We could have also relied on series expansions. Expanding both the sine and cosine functions in a Taylor series expansion, we have

$$f(z) = \frac{\text{cosz}}{\sin^2 z} = \frac{1 - \frac{1}{2!} z^2 + \ldots}{\left(z - \frac{1}{3!} z^3 + \ldots\right)^2}.$$

Factoring a z from the expansion in the denominator,

$$f(z) = \frac{1}{z^2}\frac{1 - \frac{1}{2!}z^2 + \ldots}{\left(1 - \frac{1}{3!}z + \ldots\right)^2} = \frac{1}{z^2}\left(1 + O\left(z^2\right)\right),$$

we can see that the leading term will be a $1/z^2$, indicating a pole of order 2.

Note

Integral of a function with a simple pole inside C.

Residues of a function with poles of order k.

We will see how knowledge of the poles of a function can aid in the computation of contour integrals. We now show that if a function, $f(z)$, has a pole of order k, then

$$\oint_C f(z)dz = 2\pi i \text{Res}\left[f(z); z_0\right]$$

where we have defined $\text{Res}\left[f(z); z_0\right]$ as the residue of $f(z)$ at $z = z_0$. In particular, for a pole of order k the residue is given by

Residues - Poles of order k

$$\text{Res}\left[f(z); z_0\right] = \lim_{z \to z_0} \frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}\left[\left(z - z_0\right)^k f(z)\right].$$

Proof

Let $\phi(z) = \left(z - z_0\right)^k f(z)$ be an analytic function. Then $\phi(z)$ has a Taylor series expansion about z_0. As we had seen in the last section, we can write the integral representation of any derivative of ϕ as

$$\phi^{(k-1)}\left(z_0\right) = \frac{(k-1)!}{2\pi i}\oint_C \frac{\phi(z)}{\left(z - z_0\right)^k}dz.$$

Inserting the definition of $\phi(z)$, we then have

$$\phi^{(k-1)}\left(z_0\right) = \frac{(k-1)!}{2\pi i}\oint_C f(z)dz.$$

Solving for the integral, we have the result

$$\oint_C f(z)dz = \frac{2\pi i}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}\left[\left(z - z_0\right)^k f(z)\right]_{z=z_0}$$

$$\equiv 2\pi i \text{Res}\left[f(z); z_0\right]$$

Note

If z_0 is a simple pole, the residue is easily computed as

$$\text{Res}\left[f(z); z_0\right] = \lim_{z \to z_0} \left(z - z_0\right) f(z)$$

In fact, one can show (Problem 18) that for g and h analytic functions at z_0, with $g\left(z_0\right) \neq 0$, $h\left(z_0\right) = 0$, and $h'\left(z_0\right) \neq 0$,

$$\text{Res}\left[\frac{g(z)}{h(z)}; z_0\right] = \frac{g\left(z_0\right)}{h'\left(z_0\right)}$$

Example 8.5.18

Find the residues of $f(z) = \dfrac{z-1}{(z+1)^2\left(z^2+4\right)}$.

Solution

$f(z)$ has poles at $z = -1, z = 2$ i, and $z = -2$ i. The pole at $z = -1$ is a double pole (pole of order 2). The other poles are simple poles. We compute those residues first:

$$\begin{aligned}
\text{Res}[f(z); 2i] &= \lim_{z \to 2i} (z - 2i) \frac{z-1}{(z+1)^2(z+2i)(z-2i)} \\
&= \lim_{z \to 2i} \frac{z-1}{(z+1)^2(z+2i)} \\
&= \frac{2i-1}{(2i+1)^2(4i)} = -\frac{1}{50} - \frac{11}{100} i.
\end{aligned}$$

$$\begin{aligned}
\text{Res}[f(z); -2i] &= \lim_{z \to -2i} (z + 2i) \frac{z-1}{(z+1)^2(z+2i)(z-2i)} \\
&= \lim_{z \to -2i} \frac{z-1}{(z+1)^2(z-2i)} \\
&= \frac{-2i-1}{(-2i+1)^2(-4i)} = -\frac{1}{50} + \frac{11}{100} i.
\end{aligned}$$

For the double pole, we have to do a little more work.

$$\text{Res}[f(z);\ -1] = \lim_{z \to -1} \frac{d}{dz}\left[(z+1)^2 \frac{z-1}{(z+1)^2\left(z^2+4\right)}\right]$$

$$= \lim_{z \to -1} \frac{d}{dz}\left[\frac{z-1}{z^2+4}\right]$$

$$= \lim_{z \to -1} \frac{d}{dz}\left[\frac{z^2+4-2z(z-1)}{\left(z^2+4\right)^2}\right]$$

$$= \lim_{z \to -1} \frac{d}{dz}\left[\frac{-z^2+2z+4}{\left(z^2+4\right)^2}\right]$$

$$= \frac{1}{25}$$

Example 8.5.19

Find the residue of $f(z) = \text{cotz}$ at $z = 0$.

Solution

We write $f(z) = \text{cotz} = \frac{\cos z}{\sin z}$ and note that $z = 0$ is a simple pole. Thus,

$$\text{Res}[\text{cotz}; z = 0] = \lim_{z \to 0} \frac{\text{zcosz}}{\text{sinz}} = \cos(0) = 1.$$

Another way to find the residue of a function $f(z)$ at a singularity z_0 is to look at the Laurent series expansion about the singularity. This is because the residue of $f(z)$ at z_0 is the coefficient of the $\left(z - z_0\right)^{-1}$ term, or $c_{-1} = b_1$.

Note

The residue of $f(z)$ at z_0 is the coefficient of the $(z - z_0)^{-1}$ term, $c_{-1} = b_1$, of the Laurent series expansion about z_0.

Example 8.5.20

Find the residue of $f(z) = \frac{1}{z(3-z)}$ at $z = 0$ using a Laurent series expansion.

Solution

First, we need the Laurent series expansion about $z = 0$ of the form $\sum_{-\infty}^{\infty} c_n z^n$. A partial fraction expansion gives

$$f(z) = \frac{1}{z(3-z)} = \frac{1}{3}\left(\frac{1}{z} + \frac{1}{3-z}\right).$$

The first term is a power of z. The second term needs to be written as a convergent series for small z. This is given by

$$\frac{1}{3-z} = \frac{1}{3(1-z/3)}$$
$$= \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n$$

Thus, we have found

$$f(z) = \frac{1}{3}\left(\frac{1}{z} + \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n\right).$$

The coefficient of z^{-1} can be read off to give $\text{Res}[f(z); z = 0] = \frac{1}{3}$.

Example 8.5.21

Find the residue of $f(z) = z\cos\frac{1}{z}$ at $z = 0$ using a Laurent series expansion.

Solution

In this case $z = 0$ is an essential singularity. The only way to find residues at essential singularities is to use Laurent series. Since

$$\cos z = 1 - \frac{1}{2!}z^2 + \frac{1}{4!}z^4 - \frac{1}{6!}z^6 + \ldots,$$

then we have

$$f(z) = z\left(1 - \frac{1}{2!z^2} + \frac{1}{4!z^4} - \frac{1}{6!z^6} + \ldots\right)$$
$$= z - \frac{1}{2!z} + \frac{1}{4!z^3} - \frac{1}{6!z^5} + \ldots$$

From the second term we have that $\text{Res}[f(z); z = 0] = -\frac{1}{2}$.

We are now ready to use residues in order to evaluate integrals.

Example 8.5.22

Evaluate $\oint_{|z|=1}\frac{dz}{\sin z}$.

Solution

We begin by looking for the singularities of the integrand. These are located at values of z for which $\sin z = 0$. Thus, $z = 0, \pm\pi, \pm 2\pi, \ldots$, are the singularities. However, only $z = 0$ lies inside the contour, as shown in Figure 8.5.18. We note further that $z = 0$ is a simple pole, since

$$\lim_{z\to 0}(z-0)\frac{1}{\sin z} = 1.$$

Therefore, the residue is one and we have

$$\oint_{|z|=1} \frac{dz}{\sin z} = 2\pi i.$$

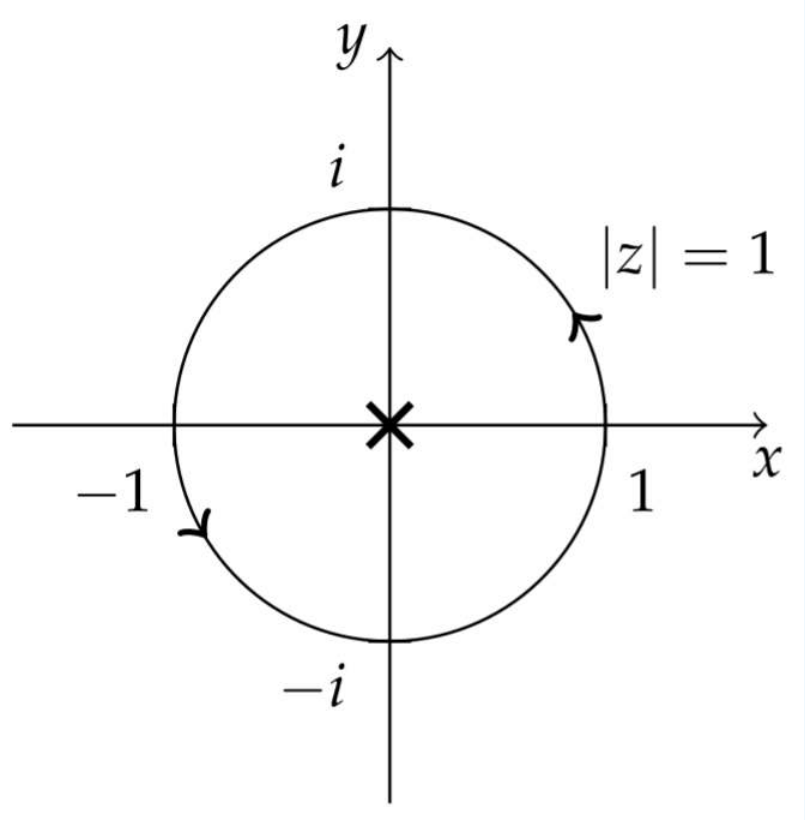


Figure 8.5.18: Contour for computing $\oint_{|z|=1} \frac{dz}{\sin z}$.

In general, we could have several poles of different orders. For example, we will be computing

$$\oint_{|z|=2} \frac{dz}{z^2-1}.$$

The integrand has singularities at $z^2 - 1 = 0$, or $z = \pm 1$. Both poles are inside the contour, as seen in Figure 8.5.20. One could do a partial fraction decomposition and have two integrals with one pole each integral. Then, the result could be found by adding the residues from each pole.

In general, when there are several poles, we can use the Residue Theorem.

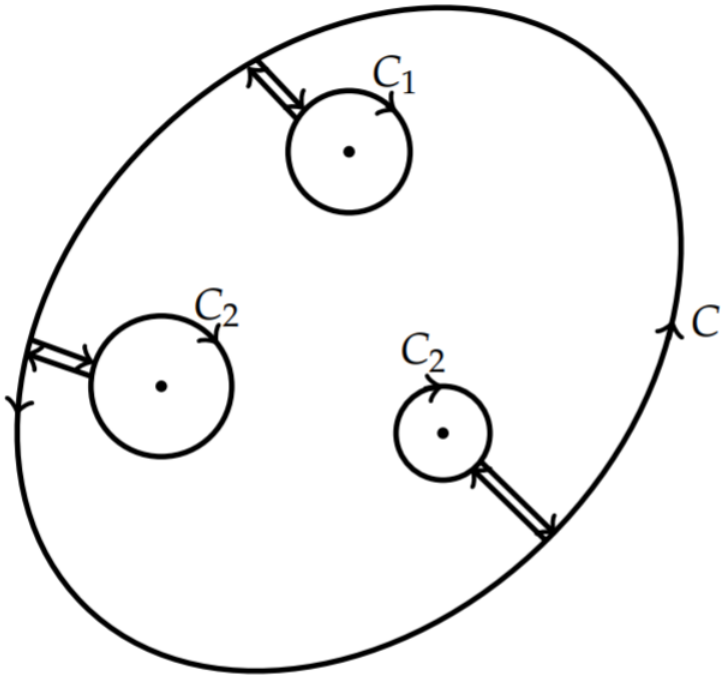


Figure 8.5.19: A depiction of how one cuts out poles to prove that the integral around C is the sum of the integrals around circles with the poles at the center of each.

Theorem 8.5.7: The Residue Theorem

Let $f(z)$ be a function which has poles $z_j, j = 1, \ldots, N$ inside a simple closed contour C and no other singularities in this region. Then,

$$\oint_C f(z)dz = 2\pi i \sum_{j=1}^{N} \text{Res}\left[f(z); z_j\right],$$

where the residues are computed using Equation (8.5.23),

$$\text{Res}\left[f(z); z_0\right] = \lim_{z \to z_0} \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left[\left(z - z_0\right)^k f(z)\right].$$

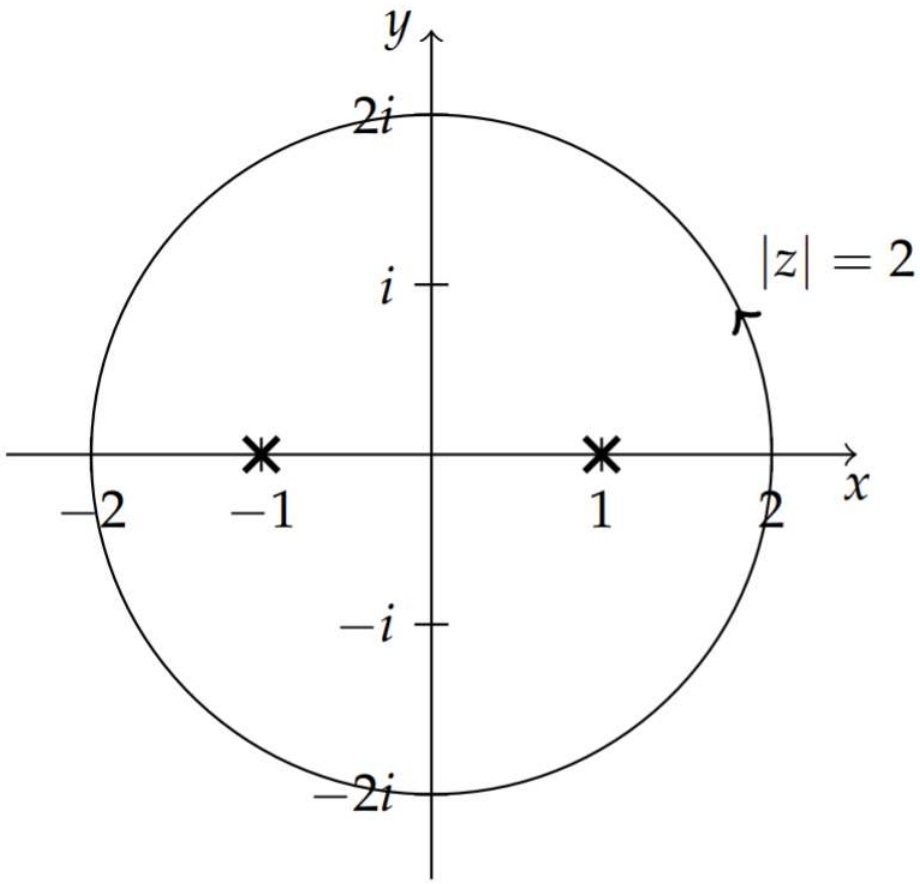


Figure 8.5.20: Contour for computing $\oint_{|z|=2}\frac{dz}{z^2-1}$.

The proof of this theorem is based upon the contours shown in Figure 8.5.19. One constructs a new contour C' by encircling each pole, as show in the figure. Then one connects a path from C to each circle. In the figure two separated paths along the cut are shown only to indicate the direction followed on the cut. The new contour is then obtained by following C and crossing each cut as it is encountered. Then one goes around a circle in the negative sense and returns along the cut to proceed around C. The sum of the contributions to the contour integration involve two integrals for each cut, which will cancel due to the opposing directions. Thus, we are left with

$$\oint_{C'} f(z)dz = \oint_C f(z)dz - \oint_{C_1} f(z)dz - \oint_{C_2} f(z)dz - \oint_{C_3} f(z)dz = 0.$$

Of course, the sum is zero because $f(z)$ is analytic in the enclosed region, since all singularities have been cut out. Solving for $\oint_C f(z)dz$, one has that this integral is the sum of the integrals around the separate poles, which can be evaluated with single residue computations. Thus, the result is that $\oint_C f(z)dz$ is $2\pi i$ times the sum of the residues.

Example 8.5.23

Evaluate $\oint_{|z|=2}\frac{dz}{z^2-1}$.

Solution

We first note that there are two poles in this integral since

$$\frac{1}{z^2-1} = \frac{1}{(z-1)(z+1)}.$$

In Figure 8.5.20 we plot the contour and the two poles, denoted by an "x." Since both poles are inside the contour, we need to compute the residues for each one. They are each simple poles, so we have

$$\begin{aligned}\text{Res}\left[\frac{1}{z^2-1};z=1\right] &= \lim_{z\to 1}(z-1)\frac{1}{z^2-1}\\ &= \lim_{z\to 1}\frac{1}{z+1}=\frac{1}{2},\end{aligned}$$

and

$$\begin{aligned}\text{Res}\left[\frac{1}{z^2-1};z=-1\right] &= \lim_{z\to -1}(z+1)\frac{1}{z^2-1}\\ &= \lim_{z\to -1}\frac{1}{z-1}=-\frac{1}{2}\end{aligned}$$

Then,

$$\oint_{|z|=2}\frac{dz}{z^2-1}=2\pi i\left(\frac{1}{2}-\frac{1}{2}\right)=0.$$

Example 8.5.24

Evaluate $\oint_{|z|=3}\frac{z^2+1}{(z-1)^2(z+2)}dz$.

Solution

In this example there are two poles $z=1,\ -2$ inside the contour. [See Figure 8.5.21.] $z=1$ is a second order pole and $z=-2$ is a simple pole. Therefore, we need to compute the residues at each pole of $f(z)=\frac{z^2+1}{(z-1)^2(z+2)}$:

$$\begin{aligned}\text{Res}[f(z);z=1] &= \lim_{z\to 1}\frac{1}{1!}\frac{d}{dz}\left[(z-1)^2\frac{z^2+1}{(z-1)^2(z+2)}\right]\\ &= \lim_{z\to 1}\left(\frac{z^2+4z-1}{(z+2)^2}\right)\\ &= \frac{4}{9}.\end{aligned}$$

$$\begin{aligned}\text{Res}[f(z);z=-2] &= \lim_{z\to -2}(z+2)\frac{z^2+1}{(z-1)^2(z+2)}\\ &= \lim_{z\to -2}\frac{z^2+1}{(z-1)^2}\\ &= \frac{5}{9}.\end{aligned}$$

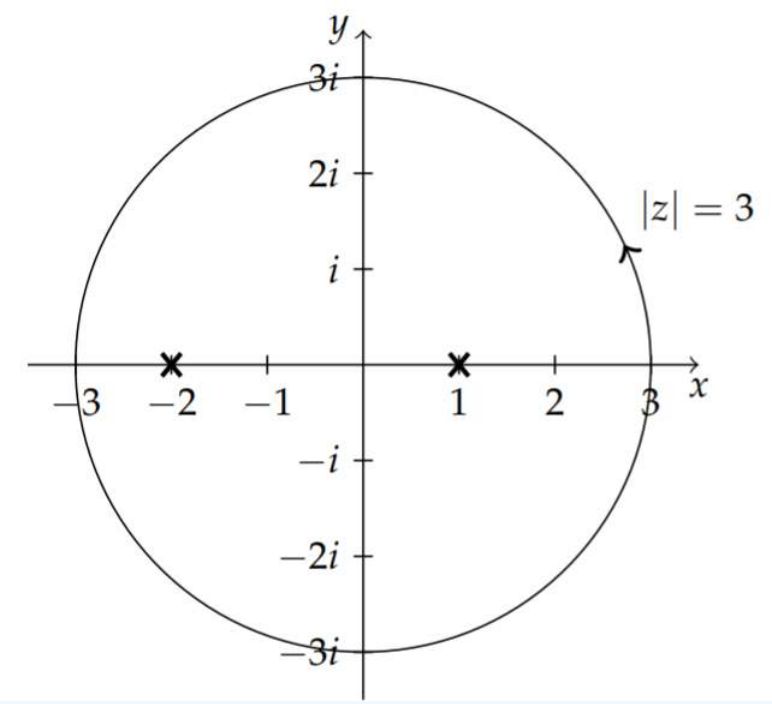


Figure 8.5.21: Contour for computing $\oint_{|z|=3} \frac{z^2+1}{(z-1)^2(z+2)} dz$.

The evaluation of the integral is found by computing $2\pi i$ times the sum of the residues:

$$\oint_{|z|=3} \frac{z^2+1}{(z-1)^2(z+2)} dz = 2\pi i \left(\frac{4}{9} + \frac{5}{9}\right) = 2\pi i.$$

Example 8.5.25

Compute $\oint_{|z|=2} z^3 e^{2/z} dz$.

Solution

In this case, $z = 0$ is an essential singularity and is inside the contour. A Laurent series expansion about $z = 0$ gives

$$\begin{aligned} z^3 e^{2/z} &= z^3 \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{2}{z}\right)^n \\ &= \sum_{n=0}^{\infty} \frac{2^n}{n!} z^{3-n} \\ &= z^3 + \frac{2}{2!} z^2 + \frac{4}{3!} z + \frac{8}{4!} + \frac{16}{5!z} + \ldots \end{aligned}$$

The residue is the coefficient of z^{-1}, or $\text{Res}\left[z^3 e^{2/z}; z = 0\right] = -\frac{2}{15}$. Therefore,

$$\oint_{|z|=2} z^3 e^{2/z} dz = \frac{4}{15}\pi i.$$

Example 8.5.26

Evaluate $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$.

Solution

Here we have a real integral in which there are no signs of complex functions. In fact, we could apply simpler methods from a calculus course to do this integral, attempting to write $1 + \cos\theta = 2\cos^2\frac{\theta}{2}$. However, we do not get very far.

One trick, useful in computing integrals whose integrand is in the form $f(\cos\theta, \sin\theta)$, is to transform the integration to the complex plane through the transformation $z = e^{i\theta}$. Then,

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{1}{2}\left(z + \frac{1}{z}\right),$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = -\frac{i}{2}\left(z - \frac{1}{z}\right).$$

Note

Computation of integrals of functions of sines and cosines, $f(\cos\theta, \sin\theta)$.

Under this transformation, $z = e^{i\theta}$, the integration now takes place around the unit circle in the complex plane. Noting that $dz = ie^{i\theta}d\theta = izd\theta$, we have

$$\begin{aligned}\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} &= \oint_{|z|=1} \frac{\frac{dz}{iz}}{2 + \frac{1}{2}\left(z + \frac{1}{z}\right)} \\ &= -i\oint_{|z|=1} \frac{dz}{2z + \frac{1}{2}\left(z^2+1\right)} \\ &= -2i\oint_{|z|=1} \frac{dz}{z^2+4z+1}.\end{aligned}$$

We can apply the Residue Theorem to the resulting integral. The singularities occur at the roots of $z^2 + 4z + 1 = 0$. Using the quadratic formula, we have the roots $z = -2 \pm \sqrt{3}$.

The location of these poles are shown in Figure 8.5.22. Only $z = -2 + \sqrt{3}$ lies inside the integration contour. We will therefore need the residue of $f(z) = \frac{-2i}{z^2+4z+1}$ at this simple pole:

$$\begin{aligned}
\text{Res}[f(z); z = -2+\sqrt{3}] &= \lim_{z \to -2+\sqrt{3}} (z-(-2+\sqrt{3}))\frac{-2i}{z^2+4z+1} \\
&= -2i \lim_{z \to -2+\sqrt{3}} \frac{z-(-2+\sqrt{3})}{(z-(-2+\sqrt{3}))(z-(-2-\sqrt{3}))} \\
&= -2i \lim_{z \to -2+\sqrt{3}} \frac{1}{z-(-2-\sqrt{3})} \\
&= \frac{-2i}{-2+\sqrt{3}-(-2-\sqrt{3})} \\
&= \frac{-i}{\sqrt{3}} \\
&= \frac{-i\sqrt{3}}{3}.
\end{aligned}$$

Therefore, we have

$$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = -2i \oint_{|z|=1} \frac{dz}{z^2+4z+1} = 2\pi i \left(\frac{-i\sqrt{3}}{3}\right) = \frac{2\pi\sqrt{3}}{3}.$$

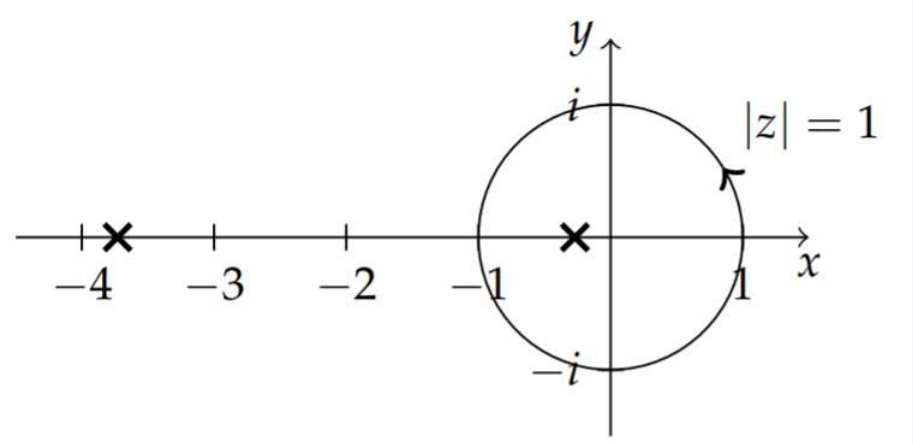


Figure 8.5.22: Contour for computing $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$.

Before moving on to further applications, we note that there is another way to compute the integral in the last example. Karl Theodor Wilhelm Weierstraß (1815-1897) introduced a substitution method for computing integrals involving rational functions of sine and cosine. One makes the substitution $t = \tan\frac{\theta}{2}$ and converts the integrand into a rational function of t. One can show that this substitution implies that

$$\sin\theta = \frac{2t}{1+t^2}, \quad \cos\theta = \frac{1-t^2}{1+t^2},$$

and

$$d\theta = \frac{2dt}{1+t^2}.$$

The details are left for Problem 8 and apply the method. In order to see how it works, we will redo the last problem.

Example 8.5.27

Apply the Weierstraj substitution method to compute $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$.

Solution

$$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \int_{-\infty}^{\infty} \frac{1}{2+\frac{1-t^2}{1+t^2}} \frac{2dt}{1+t^2}$$
$$= 2\int_{-\infty}^{\infty} \frac{dt}{t^2+3}$$
$$= \frac{2}{3}\sqrt{3}\left[\tan^{-1}\left(\frac{\sqrt{3}}{3}t\right)\right]_{-\infty}^{\infty} = \frac{2\pi\sqrt{3}}{3}.$$

Infinite Integrals

Infinite integrals of the Form $\int_{-\infty}^{\infty} f(x)dx$ occur often in physics. They can represent wave packets, wave diffraction, Fourier transforms, and arise in other applications. In this section we will see that such integrals may be computed by extending the integration to a contour in the complex plane.

Recall from your calculus experience that these integrals are improper integrals and the way that one determines if improper integrals exist, or converge, is to carefully compute these integrals using limits such as

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)dx.$$

For example, we evaluate the integral of $f(x) = x$ as

$$\int_{-\infty}^{\infty} xdx = \lim_{R\to\infty} \int_{-R}^{R} xdx = \lim_{R\to\infty} \left(\frac{R^2}{2} - \frac{(-R)^2}{2}\right) = 0.$$

One might also be tempted to carry out this integration by splitting the integration interval, $(-\infty, 0] \cup [0, \infty)$. However, the integrals $\int_0^{\infty} xdx$ and $\int_{-\infty}^{0} xdx$ do not exist. A simple computation confirms this.

$$\int_0^{\infty} xdx = \lim_{R\to\infty} \int_0^{R} xdx = \lim_{R\to\infty} \left(\frac{R^2}{2}\right) = \infty.$$

Therefore,

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_0^{\infty} f(x)dx$$

does not exist while $\lim_{R\to\infty} \int_{-R}^{R} f(x)dx$ does exist. We will be interested in computing the latter type of integral. Such an integral is called the Cauchy Principal Value Integral and is denoted with either a P, PV, or a bar through the integral:

$$P\int_{-\infty}^{\infty} f(x)dx = PV\int_{-\infty}^{\infty} f(x)dx = \text{⨍}_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)dx.$$

If there is a discontinuity in the integral, one can further modify this definition of principal value integral to bypass the singularity. For example, if $f(x)$ is continuous on $a \le x \le b$ and not defined at $x = x_0 \in [a, b]$, then

$$\int_a^b f(x)dx = \lim_{\epsilon \to 0}\left(\int_a^{x_0-\epsilon} f(x)dx + \int_{x_0+\epsilon}^b f(x)dx\right).$$

In our discussions we will be computing integrals over the real line in the Cauchy principal value sense.

Example 8.5.28

Compute $\int_{-1}^{1}\frac{dx}{x^3}$ in the Cauchy Principal Value sense.

Solution

In this case, $f(x) = \frac{1}{x^3}$ is not defined at $x = 0$. So, we have

$$\int_{-1}^{1}\frac{dx}{x^3} = \lim_{\epsilon \to 0}\left(\int_{-1}^{-\epsilon}\frac{dx}{x^3} + \int_{\epsilon}^{1}\frac{dx}{x^3}\right)$$

$$= \lim_{\epsilon \to 0}\left(-\frac{1}{2x^2}\Big|_{-1}^{-\epsilon} - \frac{1}{2x^2}\Big|_{\epsilon}^{1}\right) = 0.$$

Note

Computation of real integrals by embedding the problem in the complex plane.

We now proceed to the evaluation of principal value integrals using complex integration methods. We want to evaluate the integral $\int_{-\infty}^{\infty} f(x)dx$. We will extend this into an integration in the complex plane. We extend $f(x)$ to $f(z)$ and assume that $f(z)$ is analytic in the upper half plane ($\text{Im}(z) > 0$) except at isolated poles. We then consider the integral $\int_{-R}^{R} f(x)dx$ as an integral over the interval $(-R, R)$. We view this interval as a piece of a larger contour C_R obtained by completing the contour with a semicircle Γ_R of radius R extending into the upper half plane as shown in Figure 8.5.23. Note, a similar construction is sometimes needed extending the integration into the lower half plane ($\text{Im}(z) < 0$) as we will later see.

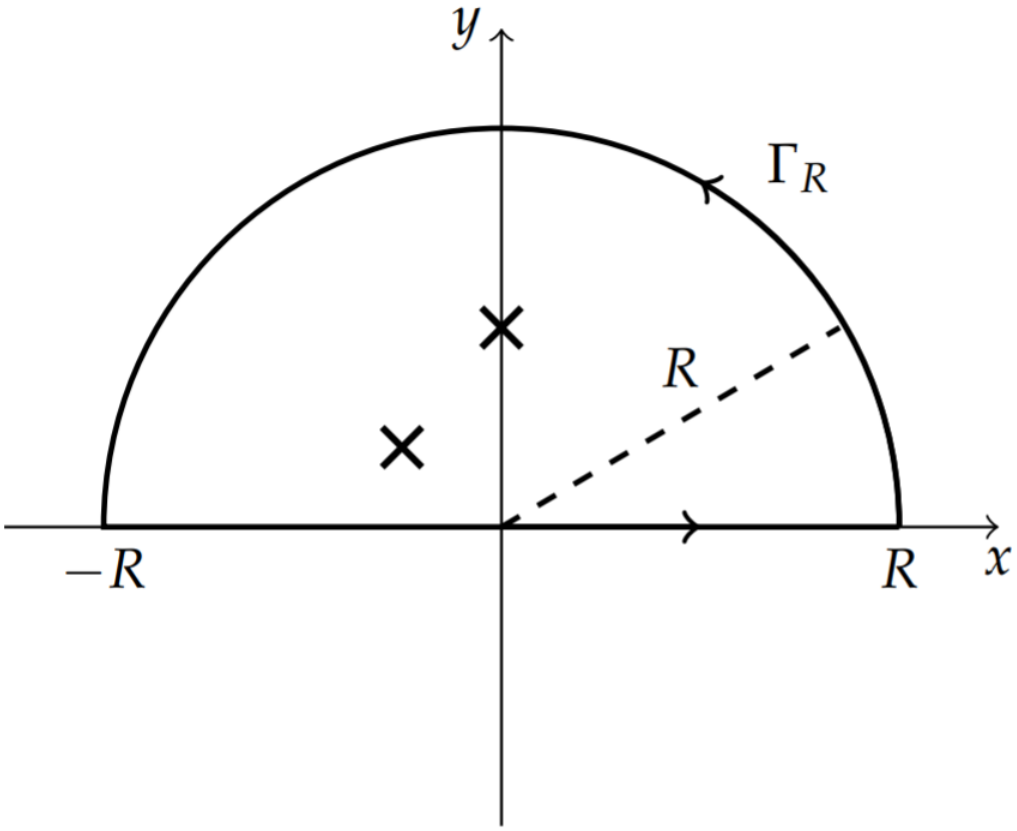


Figure 8.5.23: Contours for computing $P\int_{-\infty}^{\infty} f(x)dx$.

The integral around the entire contour C_R can be computed using the Residue Theorem and is related to integrations over the pieces of the contour by

$$\oint_{C_R} f(z)dz = \int_{\Gamma_R} f(z)dz + \int_{-R}^{R} f(z)dz.$$

Taking the limit $R \to \infty$ and noting that the integral over $(-R, R)$ is the desired integral, we have

$$P\int_{-\infty}^{\infty} f(x)dx = \oint_C f(z)dz - \lim_{R\to\infty}\int_{\Gamma_R} f(z)dz,$$

where we have identified C as the limiting contour as R gets large.

Now the key to carrying out the integration is that the second integral vanishes in the limit. This is true if $R|f(z)| \to 0$ along Γ_R as $R \to \infty$. This can be seen by the following argument. We parametrize the contour Γ_R using $z = Re^{i\theta}$. Then, when $|f(z)| < M(R)$,

$$\begin{aligned}\left|\int_{\Gamma_R} f(z)dz\right| &= \left|\int_0^{2\pi} f\left(Re^{i\theta}\right)Re^{i\theta}d\theta\right| \\ &\le R\int_0^{2\pi}\left|f\left(Re^{i\theta}\right)\right|d\theta \\ &< RM(R)\int_0^{2\pi} d\theta \\ &= 2\pi RM(R).\end{aligned}$$

So, if $\lim_{R\to\infty} RM(R) = 0$, then $\lim_{R\to\infty}\int_{\Gamma_R} f(z)dz = 0$.

We now demonstrate how to use complex integration methods in evaluating integrals over real valued functions.

Example 8.5.29

Evaluate $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$.

Solution

We already know how to do this integral using calculus without complex analysis. We have that

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \lim_{R\to\infty}\left(2\tan^{-1}R\right) = 2\left(\frac{\pi}{2}\right) = \pi.$$

We will apply the methods of this section and confirm this result. The needed contours are shown in Figure 8.5.24 and the poles of the integrand are at $z = \pm i$. We first write the integral over the bounded contour C_R as the sum of an integral from $-R$ to R along the real axis plus the integral over the semicircular arc in the upper half complex plane,

$$\int_{C_R} \frac{dz}{1+z^2} = \int_{-R}^{R} \frac{dx}{1+x^2} + \int_{\Gamma_R} \frac{dz}{1+z^2}.$$

Next, we let R get large.

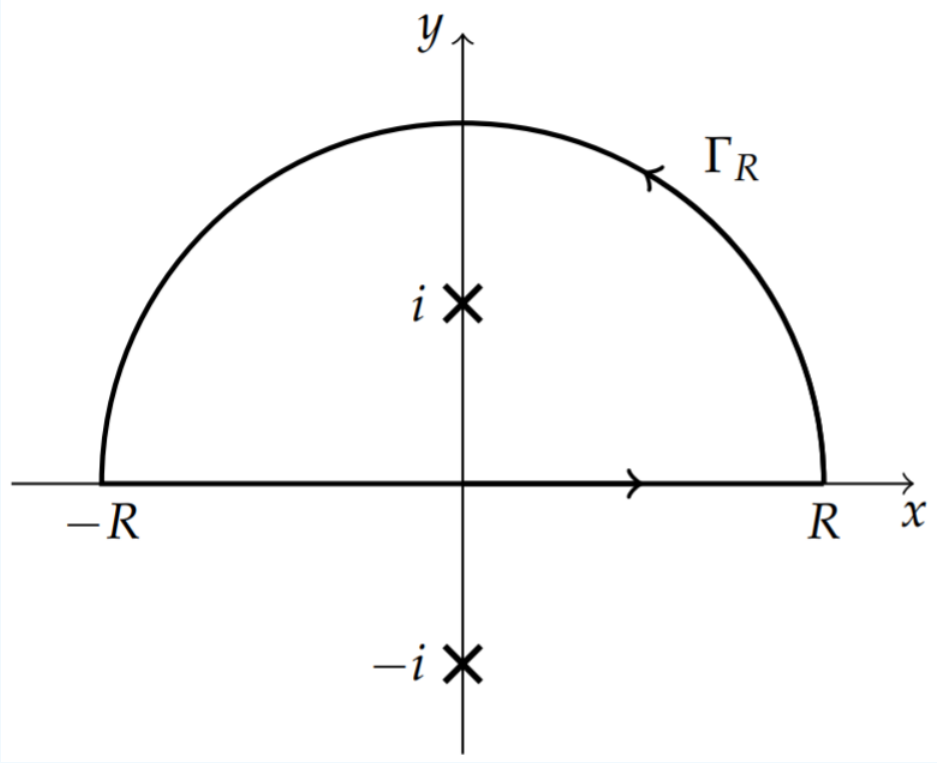


Figure 8.5.24: Contour for computing $P\int_{-\infty}^{\infty} \frac{\sin x}{x} dx$.

We first note that $f(z) = \frac{1}{1+z^2}$ goes to zero fast enough on Γ_R as R gets large.

$$R|f(z)| = \frac{R}{|1+R^2e^{2i\theta}|} = \frac{R}{\sqrt{1+2R^2\cos\theta+R^4}}.$$

Thus, as $R \to \infty$, $R|f(z)| \to 0$ and $C_R \to$ C. So,

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \oint_C \frac{dz}{1+z^2}.$$

We need only compute the residue at the enclosed pole, $z = i$.

$$\text{Res}[f(z); z = i] = \lim_{z \to i} (z - i) \frac{1}{1+z^2} = \lim_{z \to i} \frac{1}{z+i} = \frac{1}{2i}.$$

Then, using the Residue Theorem, we have

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = 2\pi i \left(\frac{1}{2i} \right) = \pi.$$

Example 8.5.30

Evaluate $P\int_{-\infty}^{\infty} \frac{\sin x}{x} dx$.

Solution

For this example the integral is unbounded at $z = 0$. Constructing the contours as before we are faced for the first time with a pole lying on the contour. We cannot ignore this fact. We can proceed with the computation by carefully going around the pole with a small semicircle of radius ϵ, as shown in Figure 8.5.25. Then the principal value integral computation becomes

$$P\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \lim_{\epsilon \to 0, R \to \infty} \left(\int_{-R}^{-\epsilon} \frac{\sin x}{x} dx + \int_{\epsilon}^{R} \frac{\sin x}{x} dx \right).$$

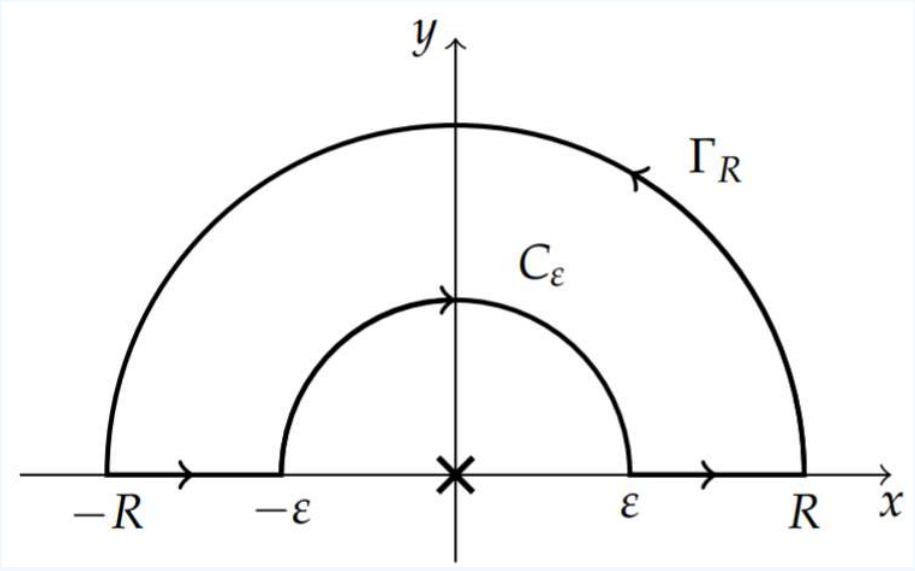


Figure 8.5.25: Contour for computing $P\int_{-\infty}^{\infty} \frac{\sin x}{x} dx$.

We will also need to rewrite the sine function in term of exponentials in this integral. There are two approaches that we could take. First, we could employ the definition of the sine function in terms of complex exponentials. This gives two integrals to compute:

$$P\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \frac{1}{2i} \left(P\int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx - P\int_{-\infty}^{\infty} \frac{e^{-ix}}{x} dx \right).$$

The other approach would be to realize that the sine function is the imaginary part of an exponential, $\text{Im}e^{ix} = \sin x$. Then, we would have

$$P\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \text{Im}\left(P\int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx \right).$$

We first consider $P\int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx$, which is common to both approaches. We use the contour in Figure 8.5.25. Then we have

$$\oint_{\mathcal{C}_R} \frac{e^{iz}}{z} dz = \int_{\Gamma_R} \frac{e^{iz}}{z} dz + \int_{-R}^{-\epsilon} \frac{e^{iz}}{z} dz + \int_{\mathcal{C}_\epsilon} \frac{e^{iz}}{z} dz + \int_{\epsilon}^{R} \frac{e^{iz}}{z} dz.$$

The integral $\oint_{C_R} \frac{e^{iz}}{z} dz$ vanishes since there are no poles enclosed in the contour! The sum of the second and fourth integrals gives the integral we seek as $\epsilon \to 0$ and $R \to \infty$. The integral over Γ_R will vanish as R gets large according to Jordan's Lemma.

Jordan's Lemma give conditions as when integrals over Γ_R will vanish as R gets large. We state a version of Jordan's Lemma here for reference and give a proof is at the end of this chapter.

Lemma 8.5.1: Jordan's Lemma

If $f(z)$ converges uniformly to zero as $z \to \infty$, then

$$\lim_{R \to \infty} \int_{C_R} f(z) e^{ikz} dz = 0$$

where $k > 0$ and C_R is the upper half of the circle $|z| = R$.

A similar result applies for $k < 0$, but one closes the contour in the lower half plane. [See Section 8.5.8 for the proof of Jordan's Lemma.]

The remaining integral around the small semicircular arc has to be done separately. We have

$$\int_{C_\epsilon} \frac{e^{iz}}{z} dz = \int_{\pi}^{0} \frac{\exp\left(i\epsilon e^{i\theta}\right)}{\epsilon e^{i\theta}} i\epsilon e^{i\theta} d\theta = -\int_{0}^{\pi} i \exp\left(i\epsilon e^{i\theta}\right) d\theta$$

Taking the limit as ϵ goes to zero, the integrand goes to i and we have

$$\int_{C_\epsilon} \frac{e^{iz}}{z} dz = -\pi i.$$

So far, we have that

$$P\int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx = -\lim_{\epsilon \to 0} \int_{C_\epsilon} \frac{e^{iz}}{z} dz = \pi i.$$

At this point we can get the answer using the second approach in Equation (8.5.49). Namely,

$$P\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \text{Im}\left(P\int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx \right) = \text{Im}(\pi i) = \pi$$

It is instructive to carry out the first approach in Equation (8.5.48). We will need to compute $P\int_{-\infty}^{\infty}\frac{e^{-ix}}{x}dx$. This is done in a similar to the above computation, being careful with the sign changes due to the orientations of the contours as shown in Figure 8.5.26.

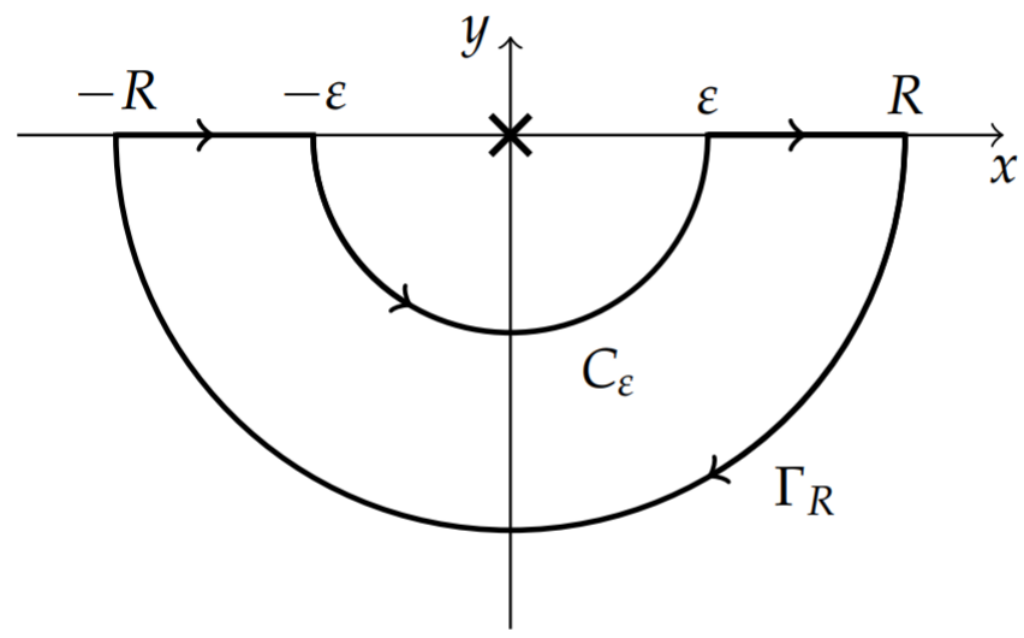


Figure 8.5.26: Contour in the lower half plane for computing $P\int_{-\infty}^{\infty}\frac{e^{-ix}}{x}dx$.

We note that the contour is closed in the lower half plane. This is because $k < 0$ in the application of Jordan's Lemma. One can understand why this is the case from the following observation. Consider the exponential in Jordan's Lemma. Let $z = z_R + iz_I$. Then,

$$e^{ikz} = e^{ik\left(z_R + iz_I\right)} = e^{-kz_I}e^{ikz_R}.$$

As $|z|$ gets large, the second factor just oscillates. The first factor would go to zero if $kz_I > 0$. So, if $k > 0$, we would close the contour in the upper half plane. If $k < 0$, then we would close the contour in the lower half plane. In the current computation, $k = -1$, so we use the lower half plane.

Working out the details, we find the same value for

$$P\int_{-\infty}^{\infty}\frac{e^{-ix}}{x}dx = \pi i.$$

Finally, we can compute the original integral as

$$\begin{aligned} P\int_{-\infty}^{\infty}\frac{\sin x}{x}dx &= \frac{1}{2i}\left(P\int_{-\infty}^{\infty}\frac{e^{ix}}{x}dx - P\int_{-\infty}^{\infty}\frac{e^{-ix}}{x}dx\right) \\ &= \frac{1}{2i}(\pi i + \pi i) \\ &= \pi \end{aligned}$$

This is the same result as we obtained using Equation (8.5.49).

Note

Note that we have not previously done integrals in which a singularity lies on the contour. One can show, as in this example, that points on the contour can be accounted for by using half of a residue (times $2\pi i$). For the semicircle C_e you can verify this. The negative sign comes from going clockwise around the semicircle.

Example 8.5.31

Evaluate $\oint_{|z|=1} \frac{dz}{z^2+1}$.

Solution

In this example there are two simple poles, $z = \pm i$ lying on the contour, as seen in Figure 8.5.27. This problem is similar to Problem 1c, except we will do it using contour integration instead of a parametrization. We bypass the two poles by drawing small semicircles around them. Since the poles are not included in the closed contour, then the Residue Theorem tells us that the integral over the path vanishes. We can write the full integration as a sum over three paths, $C_{\pm}$ for the semicircles and C for the original contour with the poles cut out. Then we take the limit as the semicircle radii go to zero. So,

$$0 = \int_C \frac{dz}{z^2+1} + \int_{C_+} \frac{dz}{z^2+1} + \int_{C_-} \frac{dz}{z^2+1}.$$

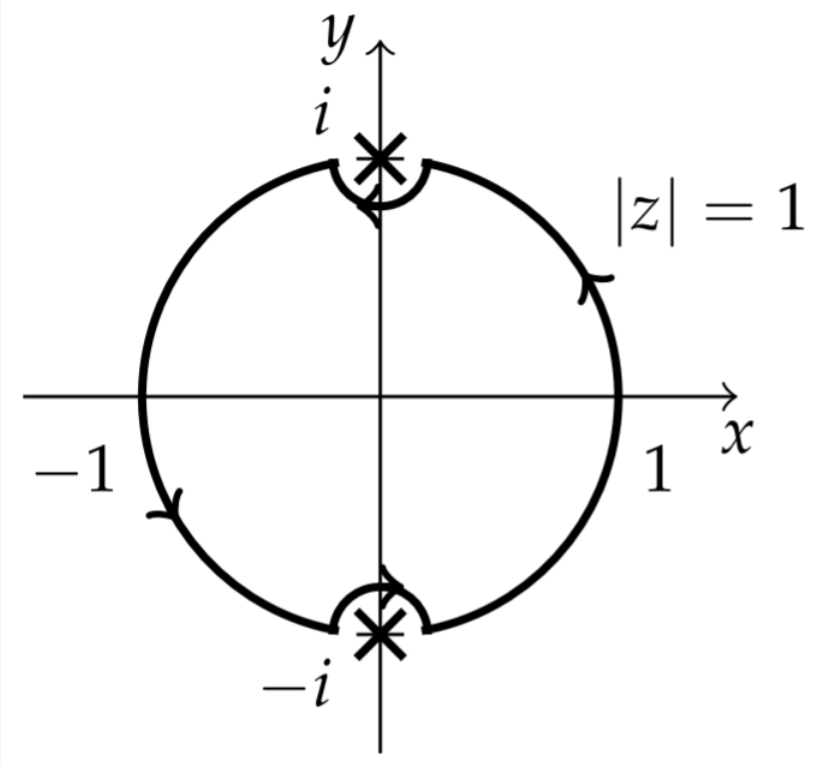


Figure 8.5.27: Example with poles on contour.

The integral over the semicircle around i can be done using the parametrization $z = i + \epsilon e^{i\theta}$. Then $z^2 + 1 = 2i\epsilon e^{i\theta} + \epsilon^2 e^{2i\theta}$. This gives

$$\int_{C_+} \frac{dz}{z^2+1} = \lim_{\epsilon \to 0} \int_0^{-\pi} \frac{i\epsilon e^{i\theta}}{2i\epsilon e^{i\theta} + \epsilon^2 e^{2i\theta}} d\theta = \frac{1}{2}\int_0^{-\pi} d\theta = -\frac{\pi}{2}.$$

As in the last example, we note that this is just πi times the residue, $\text{Res}\left[\frac{1}{z^2+1}; z = i\right] = \frac{1}{21}$. Since the path is traced clockwise, we find the contribution is $-\pi i \text{Res} = -\frac{\pi}{2}$, which is what we obtained above. A Similar computation will give the contribution from $z = -i$ as $\frac{\pi}{2}$. Adding these values gives the total contribution from $C_{\pm}$ as zero. So, the final result is that

$$\oint_{|z|=1} \frac{dz}{z^2+1} = 0.$$

Example 8.5.32

Evaluate $\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx$, for $0 < a < 1$.

Solution

In dealing with integrals involving exponentials or hyperbolic functions it is sometimes useful to use different types of contours. This example is one such case. We will replace x with z and integrate over the contour in Figure 8.5.28. Letting $R \to \infty$, the integral along the real axis is the integral that we desire. The integral along the path for $y = 2\pi$ leads to a multiple of this integral since $z = x + 2\pi i$ along this path. Integration along the vertical paths vanish as $R \to \infty$. This is captured in the following integrals:

$$\begin{aligned}\oint_{C_R} \frac{e^{az}}{1+e^z} dz = &\int_{-R}^{R} \frac{e^{ax}}{1+e^x} dx + \int_0^{2\pi} \frac{e^{a(R+iy)}}{1+e^{R+iy}} dy \\ &+ \int_R^{-R} \frac{e^{a(x+2\pi i)}}{1+e^{x+2\pi i}} dx + \int_{2\pi}^{0} \frac{e^{a(-R+iy)}}{1+e^{-R+iy}} dy\end{aligned}$$

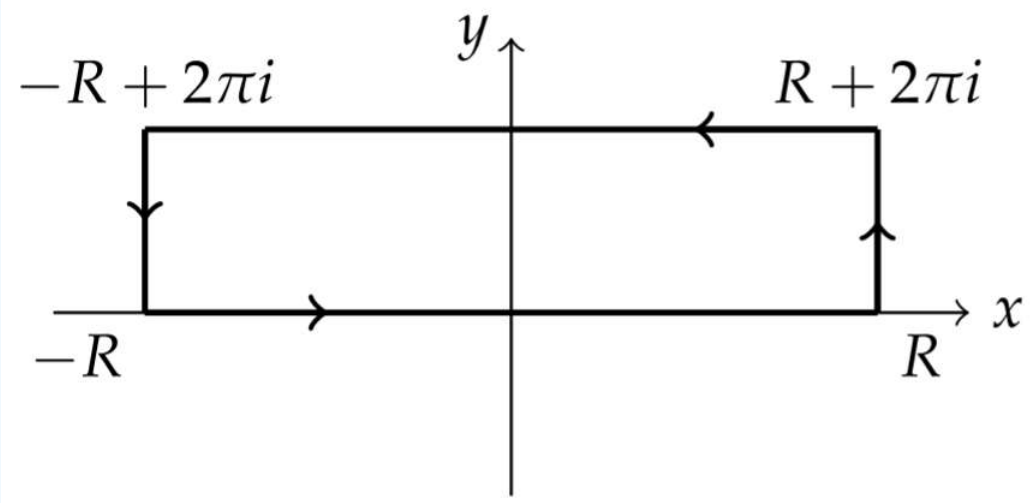


Figure 8.5.28: Example using a rectangular contour.

We can now let $R \to \infty$. For large R the second integral decays as $e^{(a-1)R}$ and the fourth integral decays as e^{-aR}. Thus, we are left with

$$\oint_C \frac{e^{az}}{1+e^z}dz = \lim_{R\to\infty}\left(\int_{-R}^{R}\frac{e^{ax}}{1+e^x}dx - e^{2\pi ia}\int_{-R}^{R}\frac{e^{ax}}{1+e^x}dx\right)$$

$$= \left(1-e^{2\pi ia}\right)\int_{-\infty}^{\infty}\frac{e^{ax}}{1+e^x}dx.$$

We need only evaluate the left contour integral using the Residue Theorem. The poles are found from

$$1+e^z = 0.$$

Within the contour, this is satisfied by $z = i\pi$. So,

$$\text{Res}\left[\frac{e^{az}}{1+e^z}; z = i\pi\right] = \lim_{z\to i\pi}(z-i\pi)\frac{e^{az}}{1+e^z} = -e^{i\pi a}.$$

Applying the Residue Theorem, we have

$$\left(1-e^{2\pi ia}\right)\int_{-\infty}^{\infty}\frac{e^{ax}}{1+e^x}dx = -2\pi i e^{i\pi a}.$$

Therefore, we have found that

$$\int_{-\infty}^{\infty}\frac{e^{ax}}{1+e^x}dx = \frac{-2\pi i e^{i\pi a}}{1-e^{2\pi ia}} = \frac{\pi}{\sin\pi a}, \quad 0 < a < 1.$$

Integration Over Multivalued Functions

We have seen that some complex functions inherently possess multivaluedness; i.e., such "functions" do not evaluate to a single value, but have many values. The key examples were $f(z) = z^{1/n}$ and $f(z) = \ln z$. The nth roots have n distinct values and logarithms have an infinite number of values as determined by the range of the resulting arguments. We mentioned that the way to handle multivaluedness is to assign different branches to these functions, introduce a branch cut and glue them together at the branch cuts to form Riemann surfaces. In this way we can draw continuous paths along the Riemann surfaces as we move from one Riemann sheet to another.

Before we do examples of contour integration involving multivalued functions, lets first try to get a handle on multivaluedness in a simple case. We will consider the square root function,

$$w = z^{1/2} = r^{1/2}e^{i\left(\frac{\theta}{2}+k\pi\right)}, \quad k = 0, 1.$$

There are two branches, corresponding to each k value. If we follow a path not containing the origin, then we stay in the same branch, so the final argument (θ) will be equal to the initial argument. However, if we follow a path that encloses the origin, this will not be true. In particular, for an initial point on the unit circle, $z_0 = e^{i\theta_0}$, we have its image as $w_0 = e^{i\theta_0/2}$. However, if we go around a full revolution, $\theta = \theta_0 + 2\pi$, then

$$z_1 = e^{i\theta_0+2\pi i} = e^{i\theta_0},$$

but

$$w_1 = e^{\left(i\theta_0+2\pi i\right)/2} = e^{i\theta_0/2}e^{\pi i} \neq w_0.$$

Here we obtain a final argument (θ) that is not equal to the initial argument! Somewhere, we have crossed from one branch to another. Points, such as the origin in this example, are called branch points. Actually, there are two branch points, because we can view the closed path around the origin as a closed path around complex infinity in the compactified complex plane. However, we will not go into that at this time.

We can demonstrate this in the following figures. In Figure 8.5.29 we show how the points A-E are mapped from the z-plane into the w-plane under the square root function for the principal branch, $k = 0$. As we trace out the unit circle in the z-plane, we only trace out a semicircle in the w-plane. If we consider the branch $k = 1$, we then trace out a semicircle in the lower half plane, as shown in Figure 8.5.30 following the points from F to J.

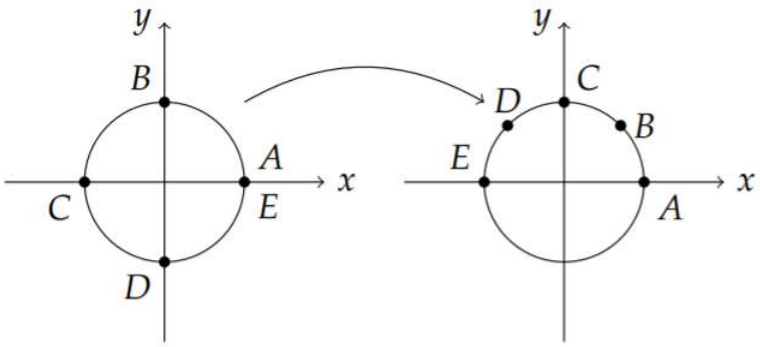


Figure 8.5.29: In this figure we show how points on the unit circle in the z-plane are mapped to points in the w-plane under the principal square root function.

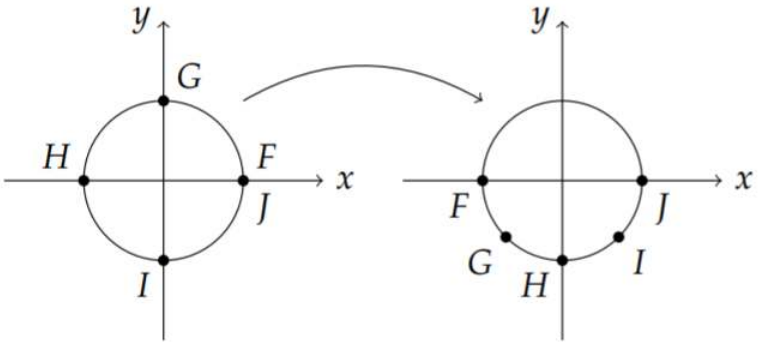


Figure 8.5.30: In this figure we show how points on the unit circle in the z-plane are mapped to points in the w-plane under the square root function for the second branch, $k = 1$.

We can combine these into one mapping depicting how the two complex planes corresponding to each branch provide a mapping to the w-plane. This is shown in Figure 8.5.31.

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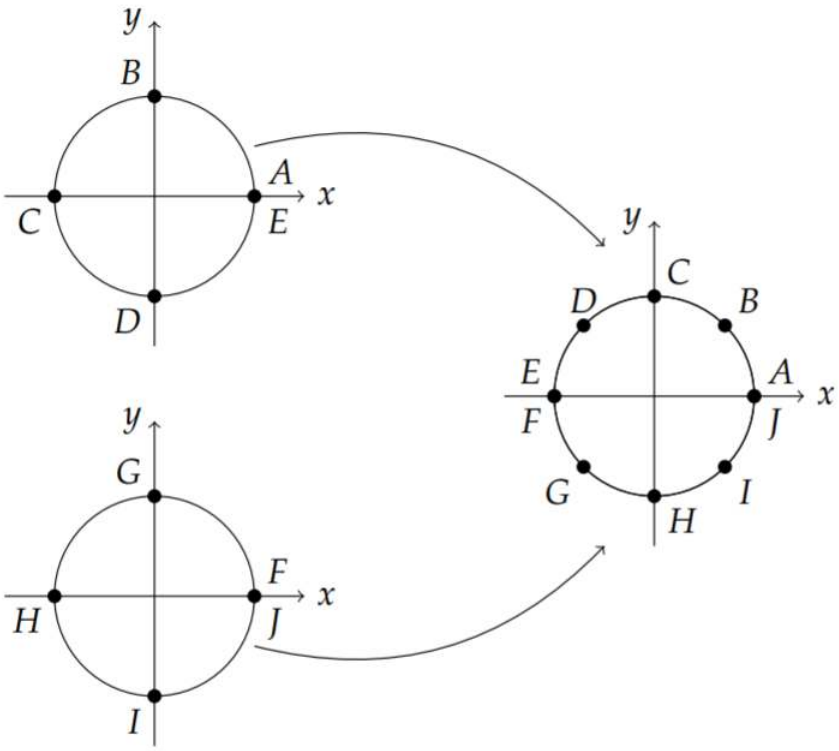


Figure 8.5.31: In this figure we show the combined mapping using two branches of the square root function.

A common way to draw this domain, which looks like two separate complex planes, would be to glue them together. Imagine cutting each plane along the positive x-axis, extending between the two branch points, $z = 0$ and $z = \infty$. As one approaches the cut on the principal branch, then one can move onto the glued second branch. Then one continues around the origin on this branch until one once again reaches the cut. This cut is glued to the principal branch in such a way that the path returns to its starting point. The resulting surface we obtain is the Riemann surface shown in Figure 8.5.32. Note that there is nothing that forces us to place the branch cut at a particular place. For example, the branch cut could be along the positive real axis, the negative real axis, or any path connecting the origin and complex infinity.

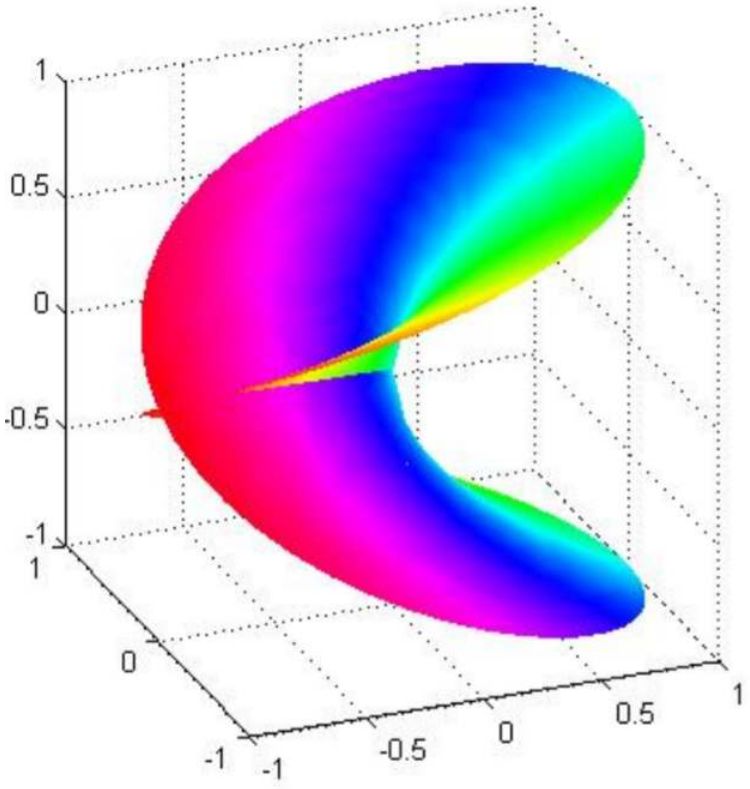


Figure 8.5.32: Riemann surface for $f(z) = z^{1/2}$.

We now look at examples involving integrals of multivalued functions.

Example 8.5.33

Evaluate $\int_0^\infty \frac{\sqrt{x}}{1+x^2}dx$.

Solution

We consider the contour integral $\oint_C \frac{\sqrt{z}}{1+z^2}dz$.

The first thing we can see in this problem is the square root function in the integrand. Being there is a multivalued function, we locate the branch point and determine where to draw the branch cut. In Figure 8.5.33 we show the contour that we will use in this problem. Note that we picked the branch cut along the positive x-axis.

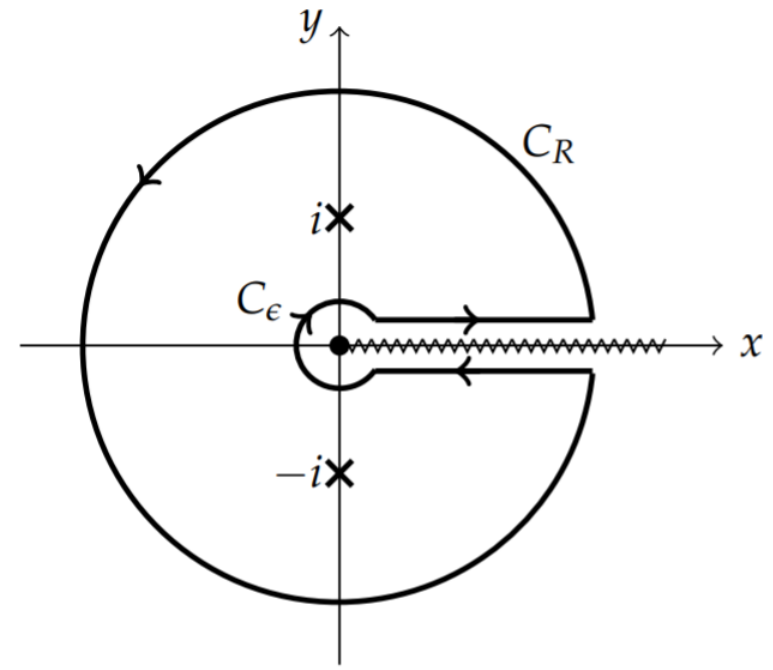


Figure 8.5.33: An example of a countour which accounts for a branch cut.

We take the contour C to be positively oriented, being careful to enclose the two poles and to hug the branch cut. It consists of two circles. The outer circle C_R is a circle of radius R and the inner circle C_e will have a radius of ϵ. The sought answer will be obtained by letting $R \to \infty$ and $\epsilon \to 0$. On the large circle we have that the integrand goes to zero fast enough as $R \to \infty$. The integral around the small circle vanishes as $\epsilon \to 0$. We can see this by parametrizing the circle as $z = \epsilon e^{i\theta}$ for $\theta \in [0, 2\pi]$:

$$\oint_{C_\epsilon} \frac{\sqrt{z}}{1+z^2} dz = \int_0^{2\pi} \frac{\sqrt{\epsilon e^{i\theta}}}{1+\left(\epsilon e^{i\theta}\right)^2} i\epsilon e^{i\theta} d\theta$$
$$= i\epsilon^{3/2} \int_0^{2\pi} \frac{e^{3i\theta/2}}{1+\left(\epsilon^2 e^{2i\theta}\right)} d\theta$$

It should now be easy to see that as $\epsilon \to 0$ this integral vanishes.

The integral above the branch cut is the one we are seeking, $\int_0^\infty \frac{\sqrt{x}}{1+x^2} dx$. The integral under the branch cut, where $z = re^{2\pi i}$, is

$$\int \frac{\sqrt{z}}{1+z^2} dz = \int_\infty^0 \frac{\sqrt{re^{2\pi i}}}{1+r^2 e^{4\pi i}} dr$$
$$= \int_0^\infty \frac{\sqrt{r}}{1+r^2} dr.$$

We note that this is the same as that above the cut.

Up to this point, we have that the contour integral, as $R \to \infty$ and $\epsilon \to 0$ is

$$\oint_C \frac{\sqrt{z}}{1+z^2} dz = 2\int_0^\infty \frac{\sqrt{x}}{1+x^2} dx.$$

In order to finish this problem, we need the residues at the two simple poles.

$$\text{Res}\left[\frac{\sqrt{z}}{1+z^2}; z=i\right] = \frac{\sqrt{i}}{2i} = \frac{\sqrt{2}}{4}(1+i),$$

$$\text{Res}\left[\frac{\sqrt{z}}{1+z^2}; z=-i\right] = \frac{\sqrt{-i}}{-2i} = \frac{\sqrt{2}}{4}(1-i).$$

So,

$$2\int_0^\infty \frac{\sqrt{x}}{1+x^2}dx = 2\pi i\left(\frac{\sqrt{2}}{4}(1+i) + \frac{\sqrt{2}}{4}(1-i)\right) = \pi\sqrt{2}.$$

Finally, we have the value of the integral that we were seeking,

$$\int_0^\infty \frac{\sqrt{x}}{1+x^2}dx = \frac{\pi\sqrt{2}}{2}.$$

Example 8.5.34

Compute $\int_a^\infty f(x)dx$ using contour integration involving logarithms.[2]

In this example we will apply contour integration to the integral

$$\oint_C f(z)\ln(a-z)dz$$

for the contour shown in Figure 8.5.34.

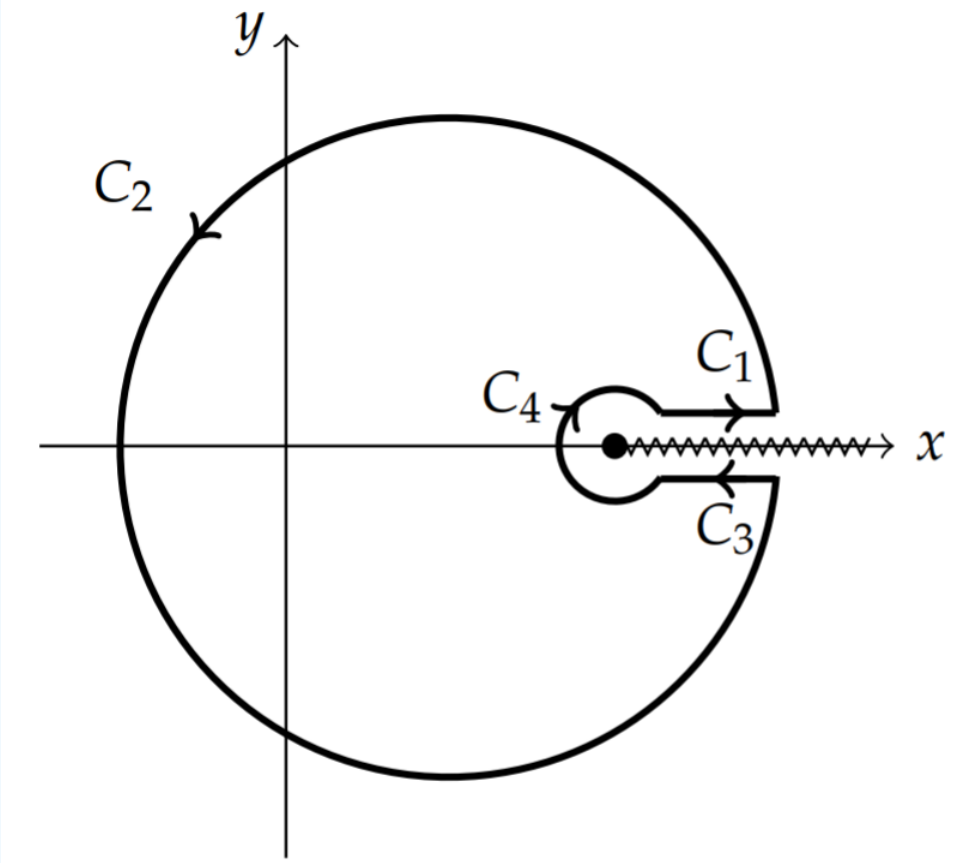


Figure 8.5.34: Contour needed to compute $\oint_C f(z)\ln(a-z)dz$.

We will assume that $f(z)$ is single valued and vanishes as $|z| \to \infty$. We will choose the branch cut to span from the origin along the positive real axis. Employing the Residue Theorem and breaking up the integrals over the pieces of the contour in Figure 8.5.34, we have schematically that

$$2\pi i \sum \text{Res}[f(z)\ln(a-z)] = \left(\int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}\right) f(z)\ln(a-z)dz$$

First of all, we assume that $f(z)$ is well behaved at $z = a$ and vanishes fast enough as $|z| = R \to \infty$. Then, the integrals over C_2 and C_4 will vanish. For example, for the path C_4, we let $z = a + \epsilon e^{i\theta}$, $0 < \theta < 2\pi$. Then,

$$\int_{C_4} f(z)\ln(a-z)dz. = \lim_{\epsilon \to 0} \int_{2\pi}^{0} f\left(a + \epsilon e^{i\theta}\right)\ln\left(\epsilon e^{i\theta}\right) i\epsilon e^{i\theta} d\theta.$$

If $f(a)$ is well behaved, then we only need to show that $\lim_{\epsilon \to 0} \epsilon \ln\epsilon = 0$. This is left to the reader.

Similarly, we consider the integral over C_2 as R gets large,

$$\int_{C_2} f(z)\ln(a-z)dz = \lim_{R \to \infty} \int_0^{2\pi} f\left(Re^{i\theta}\right)\ln\left(Re^{i\theta}\right) iRe^{i\theta} d\theta.$$

Thus, we need only require that

$$\lim_{R \to \infty} R\ln R\left|f\left(Re^{i\theta}\right)\right| = 0.$$

Next, we consider the two straight line pieces. For C_1, the integration along the real axis occurs for $z = x$, so

$$\int_{C_1} f(z)\ln(a-z)dz = \int_a^{\infty} f(x)\ln(a-x)dz.$$

However, integration over C_3 requires noting that we need the branch for the logarithm such that $\ln z = \ln(a-x) + 2\pi i$. Then,

$$\int_{C_3} f(z)\ln(a-z)dz = \int_{\infty}^{a} f(x)[\ln(a-x) + 2\pi i]dz.$$

Combining these results, we have

$$\begin{aligned} 2\pi i \sum \text{Res}[f(z)\ln(a-z)] &= \int_a^{\infty} f(x)\ln(a-x)dz \\ &\quad + \int_{\infty}^{a} f(x)[\ln(a-x) + 2\pi i]dz. \\ &= -2\pi i \int_a^{\infty} f(x)dz. \end{aligned}$$

Therefore,

$$\int_a^{\infty} f(x)dx = -\sum \text{Res}[f(z)\ln(a-z)]$$

Note

This approach was originally published in Neville, E. H., 1945, Indefinite integration by means of residues. *The Mathematical Student*, 13, 16-35, and discussed in Duffy, D. G., *Transform Methods for Solving Partial Differential Equations*, 1994.

Example 8.5.35

Compute $\int_1^{\infty} \frac{dx}{4x^2-1}$.

Solution

We can apply the last example to this case. We see from Figure 8.5.35 that the two poles at $z = \pm \frac{1}{2}$ are inside contour C. So, we compute the residues of $\frac{\ln(1-z)}{4z^2-1}$ at these poles and find that

$$\begin{aligned} \int_1^{\infty} \frac{dx}{4x^2-1} &= -\text{Res}\left[\frac{\ln(1-z)}{4z^2-1}; \frac{1}{2}\right] - \text{Res}\left[\frac{\ln(1-z)}{4z^2-1}; -\frac{1}{2}\right] \\ &= -\frac{\ln\frac{1}{2}}{4} + \frac{\ln\frac{3}{2}}{4} = \frac{\ln 3}{4} \end{aligned}$$

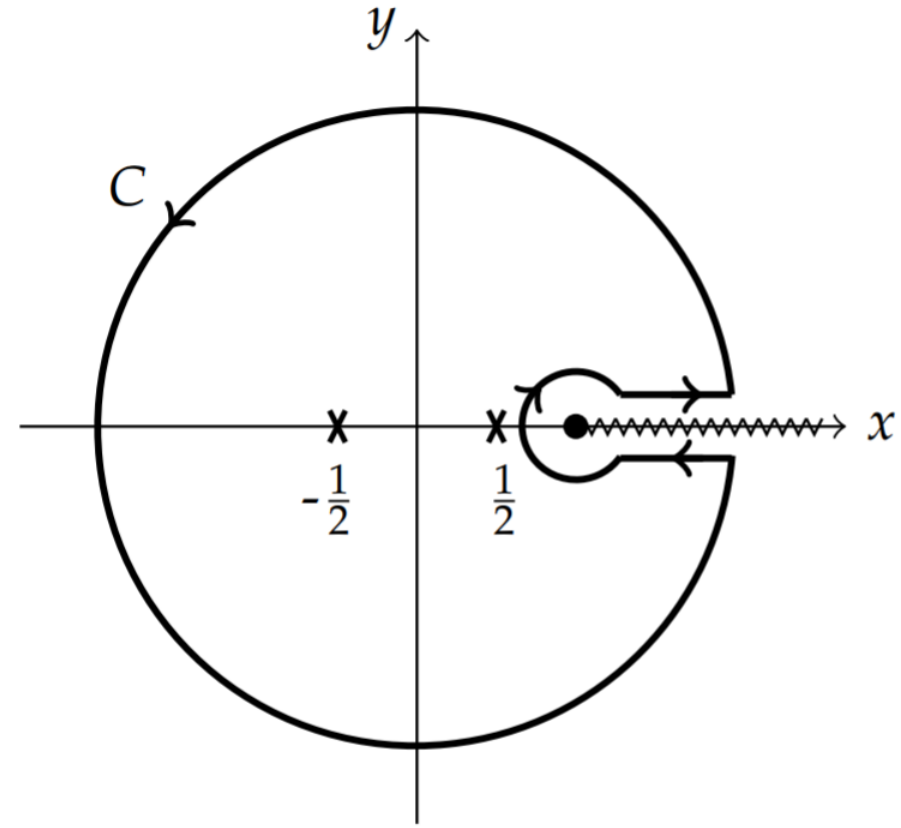


Figure 8.5.35: Contour needed to compute $\int_1^\infty \frac{dx}{4x^2-1}$.

Appendix: Jordan's Lemma

For completeness, we prove Jordan's Lemma

Theorem 8.5.8: Jordan's Lemma

If $f(z)$ converges uniformly to zero as $z \to \infty$, then

$$\lim_{R \to \infty} \int_{C_R} f(z) e^{ikz} dz = 0$$

where $k > 0$ and C_R is the upper half of the circle $|z| = R$.

Proof

We consider the integral

$$I_R = \int_{C_R} f(z) e^{ikz} dz,$$

where $k > 0$ and C_R is the upper half of the circle $|z| = R$ in the complex plane. Let $z = Re^{i\theta}$ be a parametrization of C_R. Then,

$$I_R = \int_0^\pi f\left(Re^{i\theta}\right) e^{ikR\cos\theta - aR\sin\theta} iRe^{i\theta} d\theta.$$

Since

$$\lim_{|z| \to \infty} f(z) = 0, \quad 0 \le \arg z \le \pi$$

then for large $|R|$, $|f(z)| < \epsilon$ for some $\epsilon > 0$. Then,

$$\begin{aligned} \left|I_R\right| &= \left|\int_0^\pi f\left(Re^{i\theta}\right)e^{ikR\cos\theta - aR\sin\theta} iRe^{i\theta}d\theta\right| \\ &\le \int_0^\pi \left|f\left(Re^{i\theta}\right)\right|\left|e^{ikR\cos\theta}\right|\left|e^{-aR\sin\theta}\right|\left|iRe^{i\theta}\right|d\theta \\ &\le \epsilon R\int_0^\pi e^{-aR\sin\theta}d\theta \\ &= 2\epsilon R\int_0^{\pi/2} e^{-aR\sin\theta}d\theta. \end{aligned}$$

The last integral still cannot be computed, but we can get a bound on it over the range $\theta \in [0, \pi/2]$. Note from Figure 8.5.36 that

$$\sin\theta \ge \frac{2}{\pi}\theta, \quad \theta \in [0, \pi/2].$$

Therefore, we have

$$\left|I_R\right| \le 2\epsilon R\int_0^{\pi/2} e^{-2aR\theta/\pi}d\theta = \frac{2\epsilon R}{2aR/\pi}\left(1 - e^{-aR}\right).$$

For large R we have

$$\lim_{R\to\infty}\left|I_R\right| \le \frac{\pi\epsilon}{a}.$$

So, as $\epsilon \to 0$, the integral vanishes.

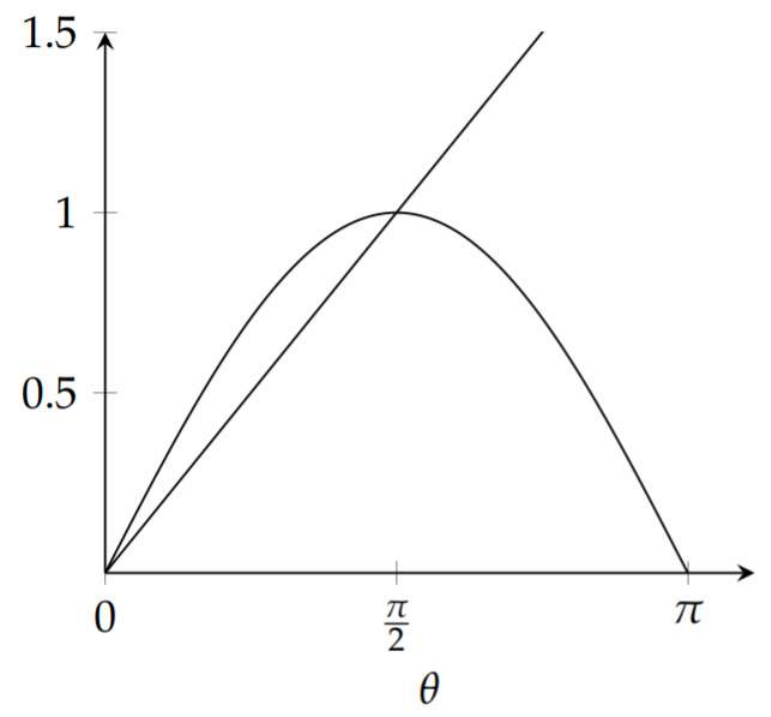


Figure 8.5.36: Plots of $y = \sin\theta$ and $y = \frac{2}{\pi}\theta$ to show where $\sin\theta \ge \frac{2}{\pi}\theta$.

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8.6: Laplace's Equation in 2D, Revisited

Harmonic functions are solutions of Laplace's equation. We have seen that the real and imaginary parts of a holomorphic function are harmonic. So, there must be a connection between complex functions and solutions of the two-dimensional Laplace equation. In this section we will describe how conformal mapping can be used to find solutions of Laplace's equation in two dimensional regions.

In Section 2.5 we had first seen applications in two-dimensional steadystate heat flow (or, diffusion), electrostatics, and fluid flow. For example, letting $\phi(\mathbf{r})$ be the electric potential, one has for a static charge distribution, $\rho(\mathbf{r})$, that the electric field, $\mathbf{E} = \nabla\phi$, satisfies

$$\nabla \cdot \mathbf{E} = \rho/\epsilon_0.$$

In regions devoid of charge, these equations yield the Laplace equation, $\nabla^2\phi = 0$.

Similarly, we can derive Laplace's equation for an incompressible, $\nabla \cdot \mathbf{v} = 0$, irrotational,,$\nabla \times \mathbf{v} = 0$, fluid flow. From well-known vector identities, we know that $\nabla \times \nabla\phi = 0$ for a scalar function, ϕ. Therefore, we can introduce a velocity potential, ϕ, such that $\mathbf{v} = \nabla\phi$. Thus, $\nabla \cdot \mathbf{v} = 0$ implies $\nabla^2\phi = 0$. So, the velocity potential satisfies Laplace's equation.

Fluid flow is probably the simplest and most interesting application of complex variable techniques for solving Laplace's equation. So, we will spend some time discussing how conformal mappings have been used to study two-dimensional ideal fluid flow, leading to the study of airfoil design.

Fluid Flow

The study of fluid flow and conformal mappings dates back to Euler, Riemann, and others.[1] The method was further elaborated upon by physicists like Lord Rayleigh (1877) and applications to airfoil theory we presented in papers by Kutta (1902) and Joukowski (1906) on later to be improved upon by others.

Note

"On the Use of Conformal Mapping in Shaping Wing Profiles," MAA lecture by R. S. Burington, 1939 , published (1940) in ... 362-373

The physics behind flight and the dynamics of wing theory relies on the ideas of drag and lift. Namely, as the the cross section of a wing, the airfoil, goes through the air, it will experience several forces. The air speed above and belong the wing will differ due to the distance the air has to travel across the top and bottom of the wing. According to Bernoulli's Principle, steady fluid flow satisfies the conservation of energy in the form

$$P + \frac{1}{2}\rho U^2 + \rho g h = \text{ constant}$$

at points on either side of the wing profile. Here P is the pressure, ρ is the air density, U is the fluid speed, h is a reference height, and g is the acceleration due to gravity. The gravitational potential energy, ρgh, is roughly constant on either side of the wing. So, this reduces to

$$P + \frac{1}{2}\rho U^2 = \text{ constant}.$$

Therefore, if the speed of the air below the wing is lower that than above the wing, the pressure below the wing will be higher, resulting in a net upward pressure. Since the pressure is the force per area, this will result in an upward force, a lift force, acting on the wing. This is the simplified version for the lift force. There is also a drag force acting in the direction of the flow. In general, we want to use complex variable methods to model the streamlines of the airflow as the air flows around an airfoil.

We begin by considering the fluid flow across a curve, C as shown in Figure 8.6.1. We assume that it is an ideal fluid with zero viscosity (i.e., does not flow like molasses) and is incompressible. It is a continuous, homogeneous flow with a constant thickness and represented by a velocity $\mathbf{U} = (u(x, y), v(x, y))$, where u and v are the horizontal components of the flow as shown in Figure 8.6.1.

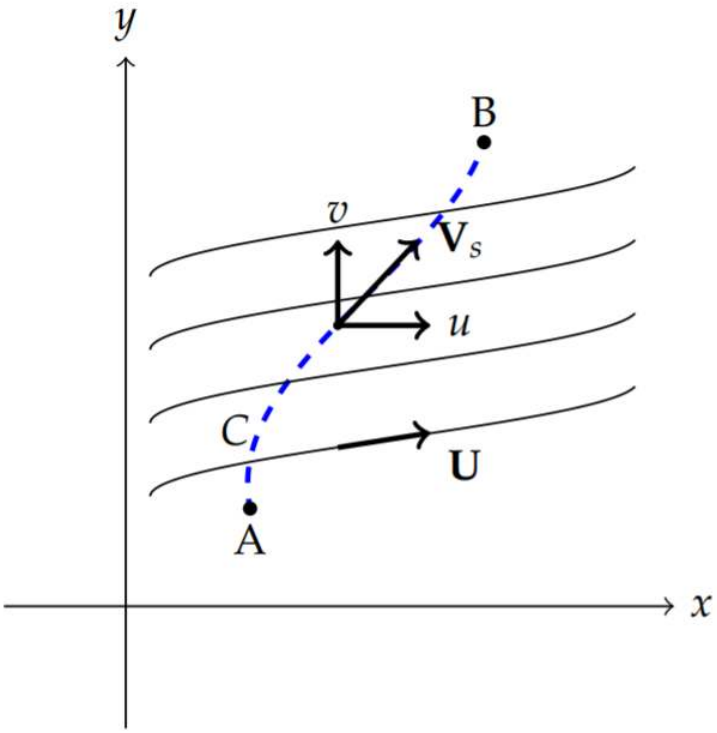


Figure 8.6.1: Fluid flow U across curve C between the points A and B.

We are interested in the flow of fluid across a given curve which crosses several streamlines. The mass that flows over C per unit thickness in time dt can be given by

$$dm = \rho \mathbf{U} \cdot \hat{\mathbf{n}} dA dt.$$

Here $\hat{\mathbf{n}} dA$ is the normal area to the flow and for unit thickness can be written as $\hat{\mathbf{n}} dA = \mathbf{i} dy - \mathbf{i} dx$. Therefore, for a unit thickness the mass flow rate is given by

$$\frac{dm}{dt} = \rho(u dy - v dx).$$

Since the total mass flowing across ds in time dt is given by $dm = \rho dV$, for constant density, this also gives the volume flow rate,

$$\frac{dV}{dt} = u dy - v dx$$

over a section of the curve. The total volume flow over C is therefore

$$\left.\frac{dV}{dt}\right|_{\text{total}} = \int_C u dy - v dx.$$

If this flow is independent of the curve, i.e., the path, then we have

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}.$$

[This is just a consequence of Green's Theorem in the Plane. See Equation (8.1.3).] Another way to say this is that there exists a function, $\psi(x, t)$, such that $d\psi = u dy - v dx$. Then,

$$\int_C u dy - v dx = \int_A^B d\psi = \psi_B - \psi_A.$$

However, from basic calculus of several variables, we know that

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = u dy - v dx.$$

Therefore,

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}.$$

It follows that if $\psi(x, y)$ has continuous second derivatives, then $u_x = -v_y$. This function is called the streamline function.

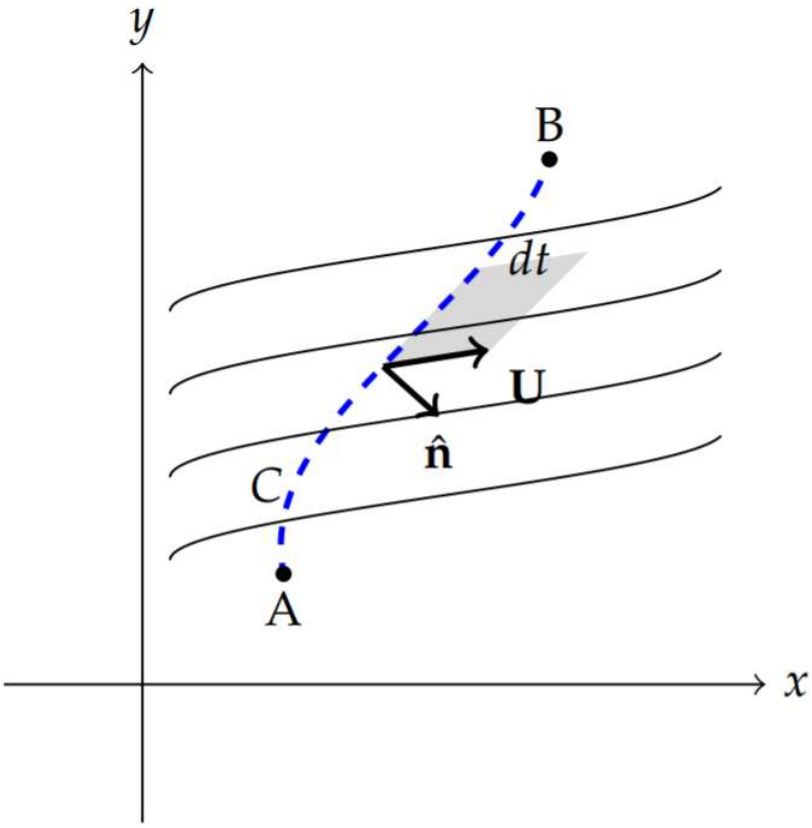


Figure 8.6.2: An amount of fluid crossing curve c in unit time.

Furthermore, for constant density, we have

$$\begin{aligned} \nabla \cdot (\rho \mathbf{U}) &= \rho \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \\ &= \rho \left(\frac{\partial^2 \psi}{\partial y \partial x} \frac{\partial^2 \psi}{\partial x \partial y} \right) = 0. \end{aligned} \tag{8.6.1}$$

This is the conservation of mass formula for constant density fluid flow.

We can also assume that the flow is irrotational. This means that the vorticity of the flow vanishes; i.e., $\nabla \times \mathbf{U} = 0$. Since the curl of the velocity field is zero, we can assume that the velocity is the gradient of a scalar function, $\mathbf{U} = \nabla \phi$. Then, a standard vector identity automatically gives

$$\nabla \times \mathbf{U} = \nabla \times \nabla \phi = 0.$$

For the two-dimensional flow with $\mathbf{U} = (u, v)$, we have

$$u = \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y}.$$

This is the velocity potential function for the flow.

Let's place the two-dimensional flow in the complex plane. Let an arbitrary point be $z = (x, y)$. Then, we have found two real-valued functions, $\psi(x, y)$ and $\psi(x, y)$, satisfying the relations

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$$
$$v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \tag{8.6.2}$$

These are the Cauchy-Riemann relations for the real and imaginary parts of a complex differentiable function,

$$F(z(x,y) = \phi(x,y) + i\psi(x,y).$$

Note

From its form, $\frac{dF}{dz}$ is called the complex velocity and $\sqrt{\left|\frac{dF}{dz}\right|} = \sqrt{u^2+v^2}$ is the flow speed.

Furthermore, we have

$$\frac{dF}{dz} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} = u - iv.$$

Integrating, we have

$$\begin{aligned} F &= \int_C (u - iv)dz \\ \phi(x,y) + i\psi(x,y) &= \int_{(x_0,y_0)}^{(x,y)} [u(x,y)dx + v(x,y)dy] \\ &\quad + i\int_{(x_0,y_0)}^{(x,y)} [-v(x,y)dx + u(x,y)dy]. \end{aligned} \tag{8.6.3}$$

Therefore, the streamline and potential functions are given by the integral forms

$$\begin{aligned} \phi(x,y) &= \int_{(x_0,y_0)}^{(x,y)} [u(x,y)dx + v(x,y)dy], \\ \psi(x,y) &= \int_{(x_0,y_0)}^{(x,y)} [-v(x,y)dx + u(x,y)dy]. \end{aligned} \tag{8.6.4}$$

These integrals give the circulation $\int_C V_s ds = \int_C udx + vdy$ and the fluid flow per time, $\int_C -vdx + udy$.

The streamlines for the flow are given by the level curves $\psi(x,y) = c_1$ and the potential lines are given by the level curves $\phi(x,y) = c_2$. These are two orthogonal families of curves; i.e., these families of curves intersect each other orthogonally at each point as we will see in the examples. Note that these families of curves also provide the field lines and equipotential curves for electrostatic problems.

Note

Streamliners and potential curves are orthogonal families of curves.

Example 8.6.1

Show that $\phi(x,y) = c_1$ and $\psi(x,y) = c_2$ are an orthogonal family of curves when $F(z) = \phi(x,y) + i\psi(x,y)$ is holomorphic.

Solution

In order to show that these curves are orthogonal, we need to find the slopes of the curves at an arbitrary point, (x,y). For $\phi(x,y) = c_1$, we recall from multivaribale calculus that

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy = 0.$$

So, the slope is found as

$$\frac{dy}{dx} = -\frac{\frac{\partial \phi}{\partial x}}{\frac{\partial \phi}{\partial y}}$$

Similarly, we have

$$\frac{dy}{dx} = -\frac{\frac{\partial \psi}{\partial x}}{\frac{\partial \psi}{\partial y}}.$$

Since $F(z)$ is differentiable, we can use the Cauchy-Riemann equations to find the product of the slopes satisfy

$$\frac{\partial \phi}{\frac{\partial \phi}{\partial \phi}\frac{\partial \psi}{\partial x}}\frac{\partial \psi}{\partial y} = -\frac{\frac{\partial \psi}{\partial y}}{\frac{\partial \psi}{\partial y}}\frac{\partial \psi}{\partial x} = -1.$$

Therefore, $\phi(x,y) = c_1$ and $\psi(x,y) = c_2$ form an orthogonal family of curves.

As an example, consider $F(z) = z^2 = x^2 - y^2 + 2ixy$. Then, $\phi(x,y) = x^2 - y^2$ and $\psi(x,y) = 2xy$. The slopes of the families of curves are given by

$$\begin{aligned} \frac{dy}{dx} &= -\frac{\partial \phi}{\partial x} \\ &= -\frac{2x}{\partial y} = \frac{x}{y}. \\ \frac{dy}{dx} &= -\frac{\frac{\partial \psi}{\partial x}}{\frac{\partial \psi}{\partial y}} \\ &= -\frac{2y}{2x} = -\frac{y}{x}. \end{aligned} \tag{8.6.5}$$

The products of these slopes is -1. The orthogonal families are depicted in Figure 8.6.3.

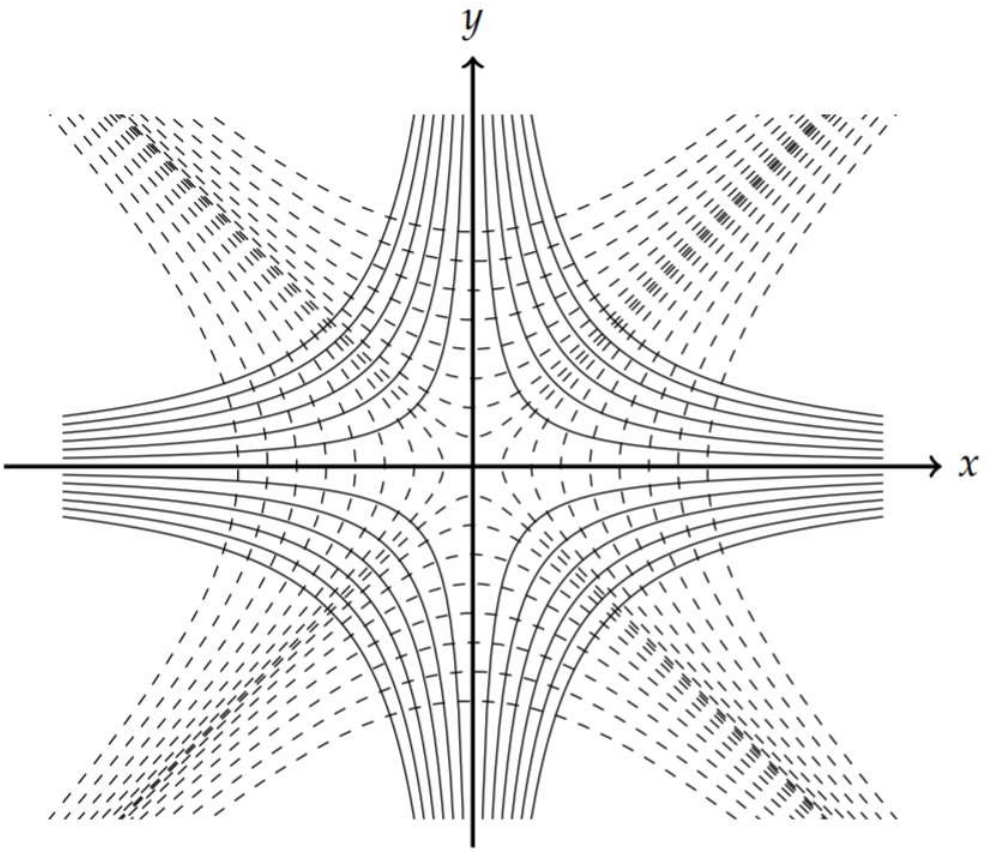


Figure 8.6.3: Plot of the orthogonal families $\phi = x^2 - y^2 = c_1$ (dashed) and $\phi(x, y) = 2xy = c_2$.

We will now turn to some typical examples by writing down some differentiable functions, $F(z)$, and determining the types of flows that result from these examples. We will then turn in the next section to using these basic forms to solve problems in slightly different domains through the use of conformal mappings.

Example 8.6.2

Describe the fluid flow associated with $F(z) = U_0 e^{-i\alpha} z$, where U_0 and α are real.

Solution

For this example, we have

$$\frac{dF}{dz} = U_0 e^{-i\alpha} = u - iv.$$

Thus, the velocity is constant,

$$\mathbf{U} = (U_0 \cos\alpha, U_0 \sin\alpha)$$

Thus, the velocity is a uniform flow at an angle of α.

Since

$$F(z) = U_0 e^{-i\alpha} z = U_0(x\cos\alpha + y\sin\alpha) + iU_0(y\cos\alpha - x\sin\alpha).$$

Thus, we have

$$\begin{aligned} \phi(x, y) &= U_0(x\cos\alpha + y\sin\alpha), \\ \psi(x, y) &= U_0(y\cos\alpha - x\sin\alpha). \end{aligned} \tag{8.6.6}$$

An example of this family of curves is shown in Figure 8.6.4.

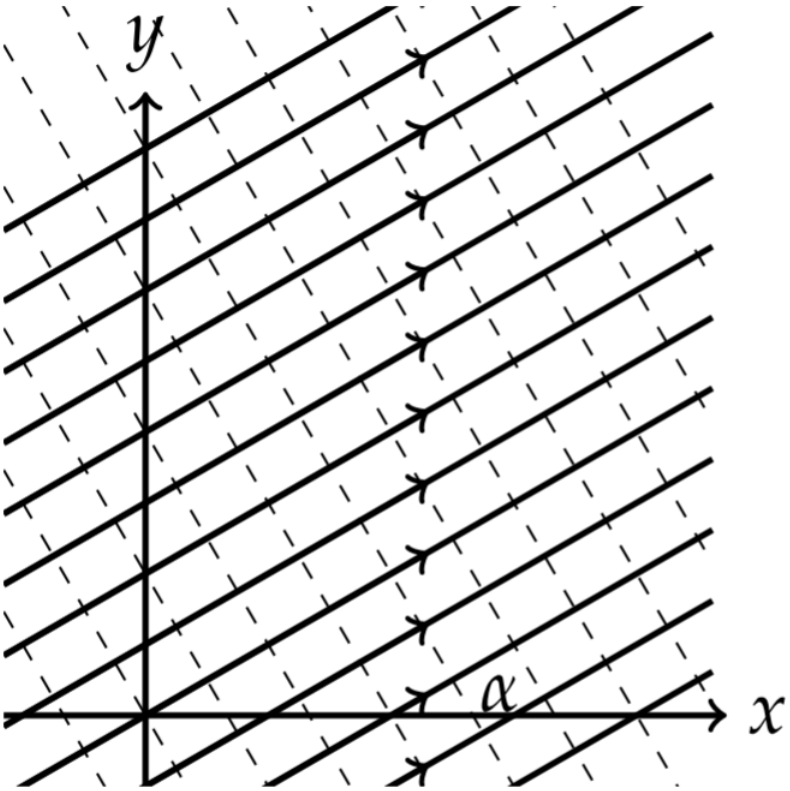


Figure 8.6.4: Stream lines (solid) and potential lines (dashed) for uniform flow at an angle of α, given by $F(z) = U_0 e^{-i\alpha} z$.

Example 8.6.3

Describe the flow given by $F(z) = \frac{u_0 e^{-i\alpha}}{z - z_0}$.

Solution

We write

$$\begin{aligned} F(z) &= \frac{U_0 e^{-i\alpha}}{z - z_0} \\ &= \frac{U_0(\cos\alpha + i\sin\alpha)}{(x-x_0)^2 + (y-y_0)^2}[(x-x_0) - i(y-y_0)] \\ &= \frac{U_0}{(x-x_0)^2 + (y-y_0)^2}[(x-x_0)\cos\alpha + (y-y_0)\sin\alpha] \\ &\quad + i\frac{U_0}{(x-x_0)^2 + (y-y_0)^2}[-(y-y_0)\cos\alpha + (x-x_0)\sin\alpha]. \end{aligned} \tag{8.6.7}$$

The level curves become

$$\begin{aligned} \phi(x,y) &= \frac{U_0}{(x-x_0)^2 + (y-y_0)^2}[(x-x_0)\cos\alpha + (y-y_0)\sin\alpha] = c_1, \\ \psi(x,y) &= \frac{U_0}{(x-x_0)^2 + (y-y_0)^2}[-(y-y_0)\cos\alpha + (x-x_0)\sin\alpha] = c_2. \end{aligned} \tag{8.6.8}$$

The level curves for the stream and potential functions satisfy equations of the form

$$\beta_i\left(\Delta x^2 + \Delta y^2\right) - \cos(\alpha + \delta_i)\Delta x - \sin(\alpha + \delta_i)\Delta y = 0,$$

where $\Delta x = x - x_0, \Delta y = y - y_0, \beta_i = \frac{c_i}{U_0}, \delta_1 = 0$, and $\delta_2 = \pi/2.$, These can be written in the more suggestive form

$$(\Delta x - \gamma_i \cos(\alpha - \delta_i))^2 + (\Delta y - \gamma_i \sin(\alpha - \delta_i))^2 = \gamma_i^2$$

for $\gamma_i = \frac{c_i}{2U_0}, i = 1, 2$. Thus, the stream and potential curves are circles with varying radii (γ_i) and centers $((x_0 + \gamma_i \cos(\alpha - \delta_i), y_0 + \gamma_i \sin(\alpha - \delta_i))$). Examples of this family of curves is shown for $\alpha = 0$ in in Figure 8.6.5 and for $\alpha = \pi/6$ in in Figure 8.6.6.

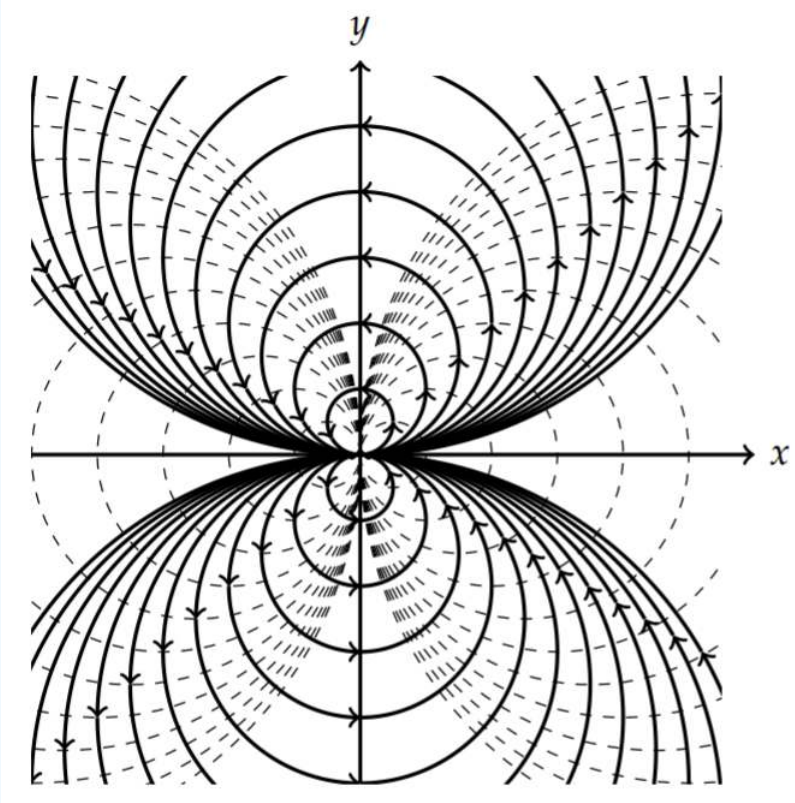


Figure 8.6.5: Stream lines (solid) and potential lines (dashed) for the flow given by $F(z) = \frac{U_0 e^{-i\alpha}}{z}$ for $\alpha = 0$.

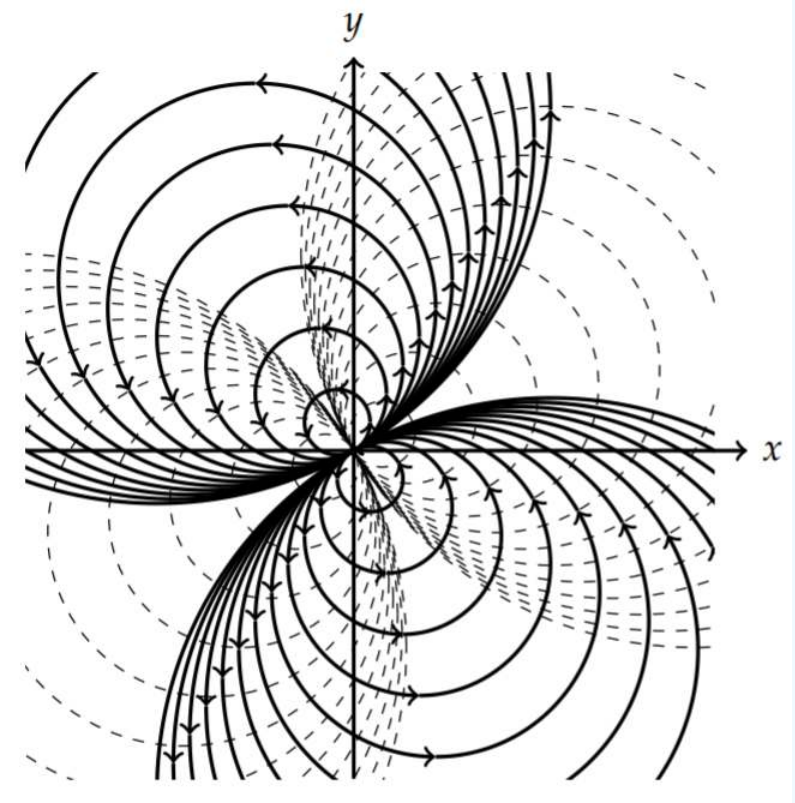


Figure 8.6.6: Stream lines (solid) and potential lines (dashed) for the flow given by $F(z)=\frac{u_0 e^{-iu}}{z}$ for $\alpha=\pi/6$.

The components of the velocity field for $\alpha=0$ are found from

$$\begin{aligned}
\frac{dF}{dz} &= \frac{d}{dz}\left(\frac{U_0}{z-z_0}\right) \\
&= -\frac{U_0}{(z-z_0)^2} \\
&= -\frac{U_0\left[(x-x_0)-i(y-y_0)\right]^2}{\left[(x-x_0)^2+(y-y_0)^2\right]^2} \\
&= -\frac{U_0\left[(x-x_0)^2+(y-y_0)^2-2i(x-x_0)(y-y_0)\right]}{\left[(x-x_0)^2+(y-y_0)^2\right]^2} \\
&= -\frac{U_0\left[(x-x_0)^2+(y-y_0)^2\right]}{\left[(x-x_0)^2+(y-y_0)^2\right]^2}+i\frac{U_0\left[2(x-x_0)(y-y_0)\right]}{\left[(x-x_0)^2+(y-y_0)^2\right]^2} \\
&= -\frac{U_0}{\left[(x-x_0)^2+(y-y_0)^2\right]}+i\frac{U_0\left[2(x-x_0)(y-y_0)\right]}{\left[(x-x_0)^2+(y-y_0)^2\right]^2}.
\end{aligned} \tag{8.6.9}$$

Thus, we have

$$\begin{aligned}
u &= -\frac{U_0}{\left[(x-x_0)^2+(y-y_0)^2\right]'} \\
v &= \frac{U_0\left[2(x-x_0)(y-y_0)\right]}{\left[(x-x_0)^2+(y-y_0)^2\right]^2}.
\end{aligned} \tag{8.6.10}$$

Example 8.6.4

Describe the flow given by $F(z) = \frac{m}{2\pi}\ln(z - z_0)$.

Solution

We write $F(z)$ in terms of its real and imaginary parts:

$$\begin{aligned} F(z) &= \frac{m}{2\pi}\ln(z - z_0) \\ &= \frac{m}{2\pi}\left[\ln\sqrt{(x-x_0)^2 + (y-y_0)^2} + i\tan^{-1}\frac{y-y_0}{x-x_0}\right]. \end{aligned} \tag{8.6.11}$$

The level curves become

$$\begin{aligned} \phi(x,y) &= \frac{m}{2\pi}\ln\sqrt{(x-x_0)^2 + (y-y_0)^2} = c_1, \\ \psi(x,y) &= \frac{m}{2\pi}\tan^{-1}\frac{y-y_0}{x-x_0} = c_2. \end{aligned} \tag{8.6.12}$$

Rewriting these equations, we have

$$\begin{aligned} (x-x_0)^2 + (y-y_0)^2 &= e^{4\pi c_1/m}, \\ y - y_0 &= (x-x_0)\tan\frac{2\pi c_2}{m}. \end{aligned} \tag{8.6.13}$$

In Figure 8.6.7 we see that the stream lines are those for a source or sink depending if $m > 0$ or $m < 0$, respectively.

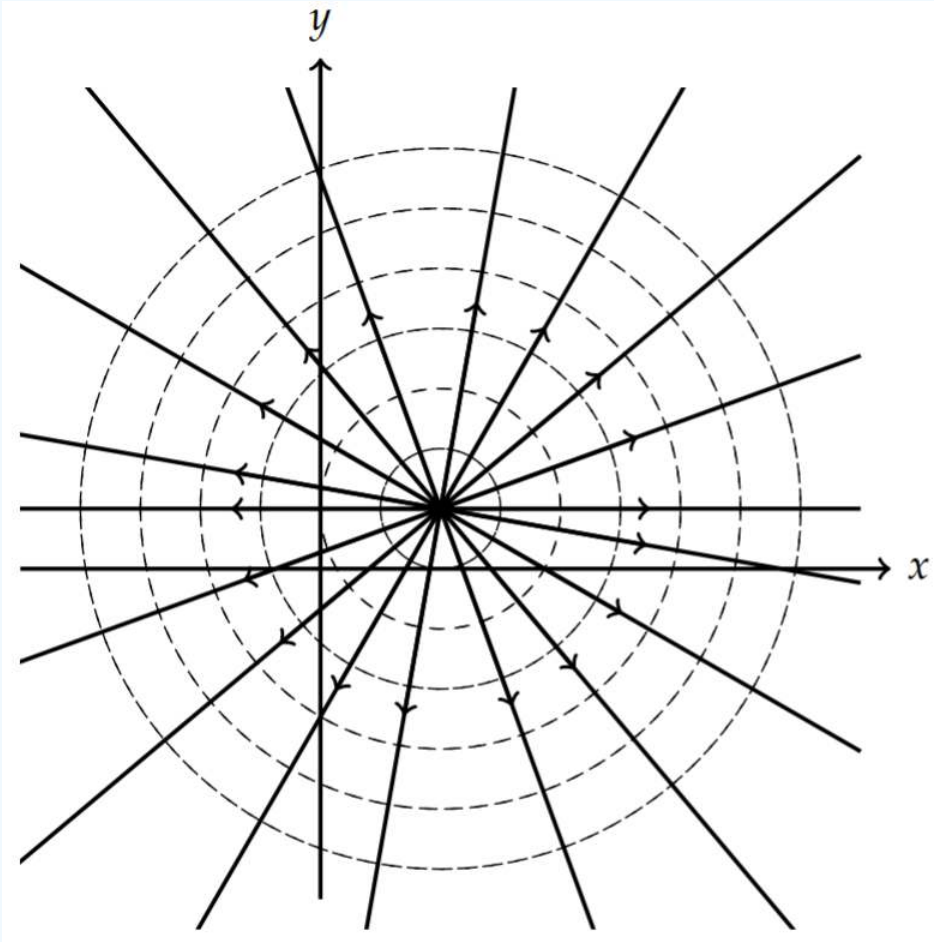


Figure 8.6.7: Stream lines (solid) and potential lines (dashed) for the flow given by $F(z) = \frac{m}{2\pi}\ln(z - z_0)$ for $(x_0, y_0) = (2, 1)$.

Example 8.6.5

Describe the flow given by $F(z) = -\frac{i\Gamma}{2\pi}\ln\frac{z-z_0}{a}$.

Solution

We write $F(z)$ in terms of its real and imaginary parts:

$$\begin{aligned} F(z) &= -\frac{i\Gamma}{2\pi}\ln\frac{z-z_0}{a} \\ &= -i\frac{\Gamma}{2\pi}\ln\sqrt{\left(\frac{x-x_0}{a}\right)^2+\left(\frac{y-y_0}{a}\right)^2} + \frac{\Gamma}{2\pi}\tan^{-1}\frac{y-y_0}{x-x_0} \end{aligned} \tag{8.6.14}$$

The level curves become

$$\begin{aligned} \phi(x,y) &= \frac{\Gamma}{2\pi}\tan^{-1}\frac{y-y_0}{x-x_0} = c_1, \\ \psi(x,y) &= -\frac{\Gamma}{2\pi}\ln\sqrt{\left(\frac{x-x_0}{a}\right)^2+\left(\frac{y-y_0}{a}\right)^2} = c_2. \end{aligned} \tag{8.6.15}$$

Rewriting these equations, we have

$$\begin{aligned} y-y_0 &= (x-x_0)\tan\frac{2\pi c_1}{\Gamma}, \\ \left(\frac{x-x_0}{a}\right)^2+\left(\frac{y-y_0}{a}\right)^2 &= e^{-2\pi c_2/\Gamma}. \end{aligned} \tag{8.6.16}$$

In Figure 8.6.8 we see that the stream lines circles, indicating rotational motion. Therefore, we have a vortex of counterclockwise, or clockwise flow, depending if $\Gamma > 0$ or $\Gamma < 0$, respectively.

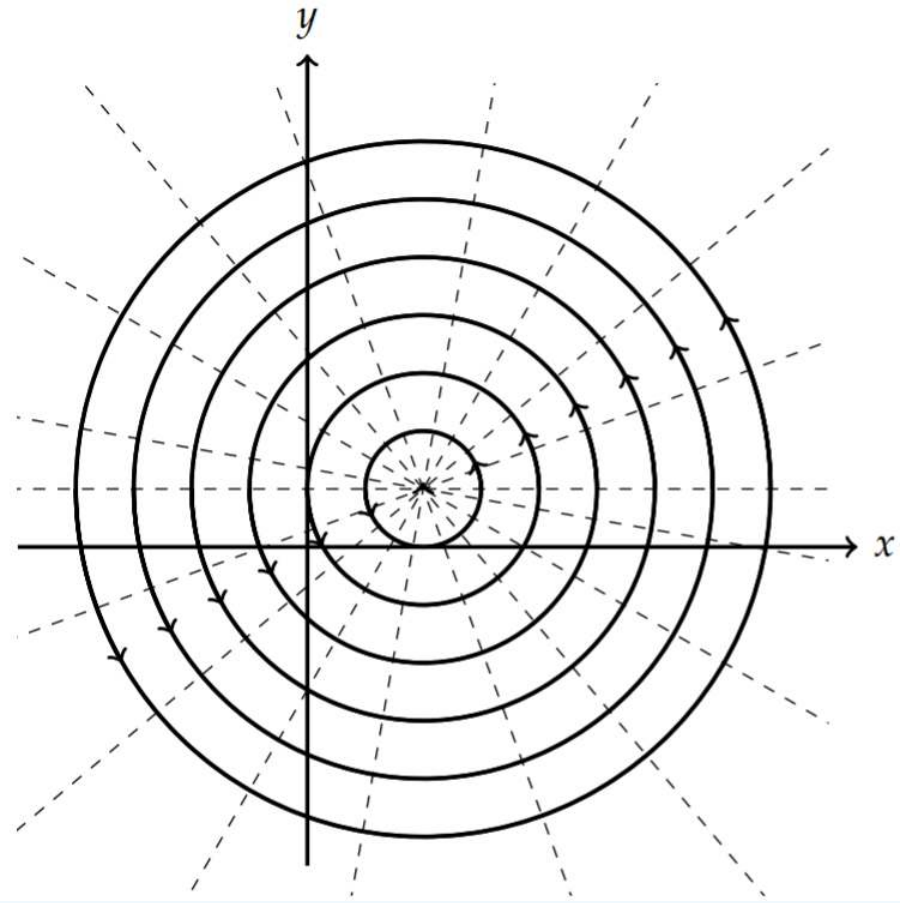


Figure 8.6.8: Stream lines (solid) and potential lines (dashed) for the flow given by $F(z) = \frac{m}{2\pi}\ln(z - z_0)$ for $(x_0, y_0) = (2, 1)$.

Example 8.6.6

Flow around a cylinder, $F(z) = U_0\left(z + \frac{a^2}{z}\right), a, U_0 \in R$.

Solution

For this example, we have

$$\begin{aligned} F(z) &= U_0\left(z + \frac{a^2}{z}\right) \\ &= U_0\left(x + iy + \frac{a^2}{x + iy}\right) \\ &= U_0\left(x + iy + \frac{a^2}{x^2 + y^2}(x - iy)\right) \\ &= U_0 x\left(1 + \frac{a^2}{x^2 + y^2}\right) + iU_0 y\left(1 - \frac{a^2}{x^2 + y^2}\right). \end{aligned} \tag{8.6.17}$$

The level curves become

$$\begin{aligned} \phi(x, y) &= U_0 x\left(1 + \frac{a^2}{x^2 + y^2}\right) = c_1, \\ \psi(x, y) &= U_0 y\left(1 - \frac{a^2}{x^2 + y^2}\right) = c_2. \end{aligned} \tag{8.6.18}$$

Note that for the streamlines when $|z|$ is large, then $\psi \sim Vy$, or horizontal lines. For $x^2+y^2=a^2$, we have $\psi=0$. This behavior is shown in Figure 8.6.9 where we have graphed the solution for $r \geq a$.

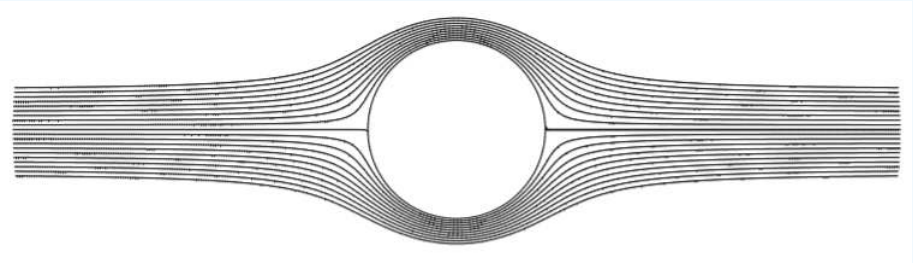


Figure 8.6.9: Stream lines for the flow given by $F(z)=U_0\left(z+\frac{a^2}{z}\right)$.

The level curves in Figure 8.6.9 can be obtained using the implicitplot feature of Maple. An example is shown below:

```
restart: with(plots):
k0:=20:
for k from 0 to k0 do
    P[k]:=implicitplot(sin(t)*(r-1/r)*1=(k0/2-k)/20, r=1..5,
    t=0..2*Pi, coords=polar,view=[-2..2, -1..1], axes=none,
    grid=[150,150],color=black):
    od:
display({seq(P[k],k=1..k0)},scaling=constrained);
```

A slight modification of the last example is if a circulation term is added:

$$F(z)=U_0\left(z+\frac{a^2}{z}\right)-\frac{i\Gamma}{2\pi}\ln\frac{r}{a}.$$

The combination of the two terms gives the streamlines,

$$\psi(x,y)=U_0 y\left(1-\frac{a^2}{x^2+y^2}\right)-\frac{\Gamma}{2\pi}\ln\frac{r}{a'},$$

which are seen in Figure 8.6.10. We can see interesting features in this flow including what is called a stagnation point. A stagnation point is a point where the flow speed, $\left|\frac{dF}{dz}\right|=0$.

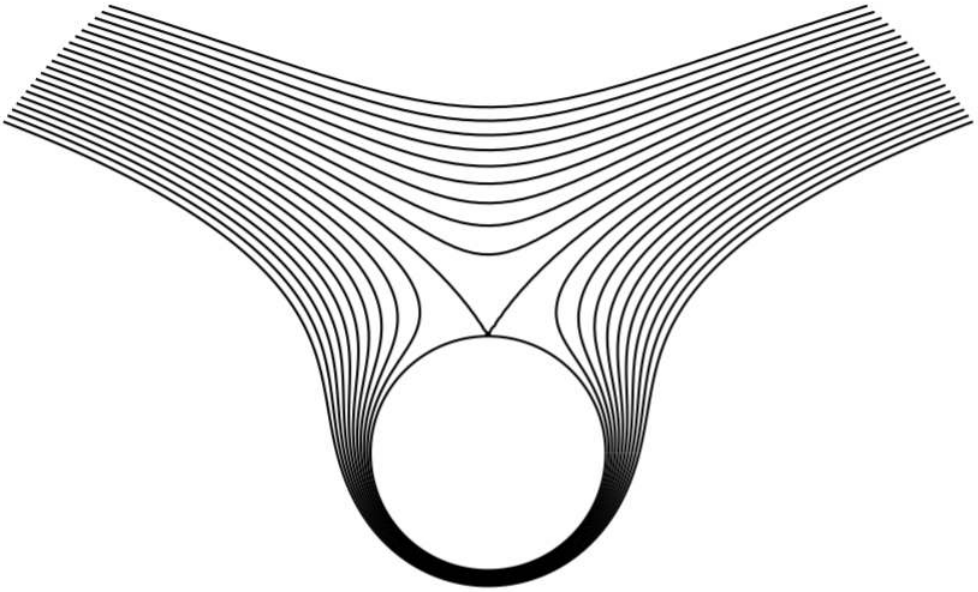


Figure 8.6.10: Stream lines for the flow given by $F(z) = U_0\left(z + \frac{a^2}{z}\right) - \frac{\Gamma}{2\pi}\ln\frac{z}{a}$.

Example 8.6.7

Find the stagnation point for the flow $F(z) = \left(z + \frac{1}{z}\right) - i\ln z$.

Solution

Since the flow speed vanishes at the stagnation points, we consider

$$\frac{dF}{dz} = 1 - \frac{1}{z^2} - \frac{i}{z} = 0.$$

This can be rewritten as

$$z^2 - iz - 1 = 0.$$

The solutions are $z = \frac{1}{2}(i \pm \sqrt{3})$. Thus, there are two stagnation points on the cylinder about which the flow is circulating. These are shown in Figure 8.6.11.

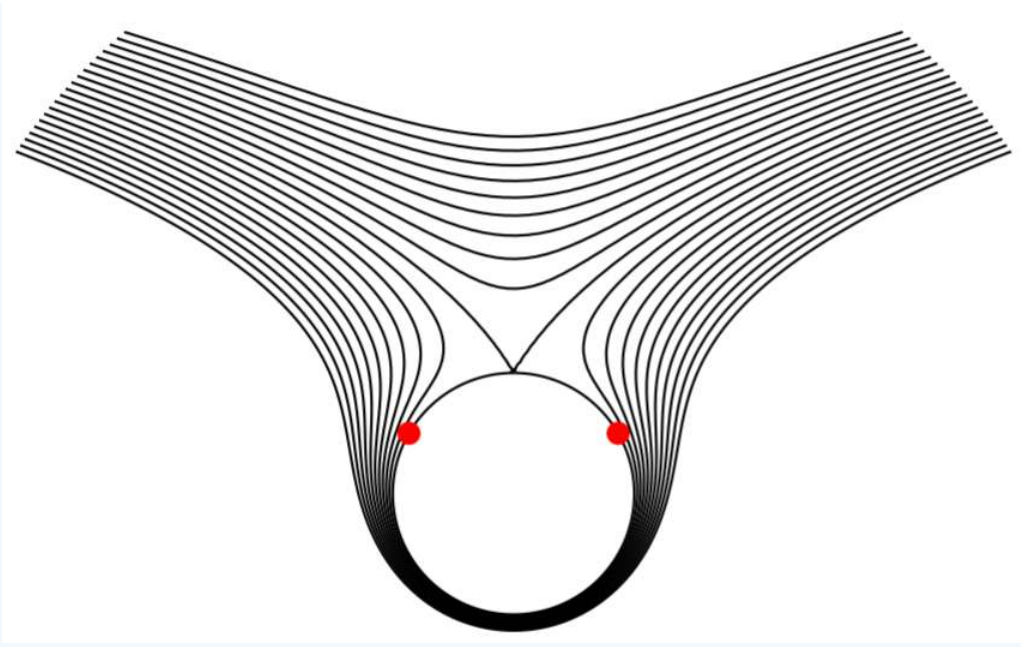


Figure 8.6.11: Stagnation points (red) on the cylinder are shown for the flow given by $F(z) = \left(z + \frac{1}{z}\right) - i \ln z$.

Example 8.6.8

Consider the complex potentials $F(z) = \frac{k}{2\pi} \ln \frac{z-a}{z-b}$, where $k = q$ and $k = -i$ for q real.

Solution

We first note that for $z = x + iy$,

$$\begin{aligned} \ln \frac{z-a}{z-b} =& \ln \sqrt{(x-a)^2 + y^2} - \ln \sqrt{(x-a)^2 + y^2}\,, \\ &+ i \tan^{-1} \frac{y}{x-a} - i \tan^{-1} \frac{y}{x-b}. \end{aligned} \tag{8.6.19}$$

For $k = q$, we have

$$\begin{aligned} \psi(x,y) &= \frac{q}{2\pi} \left[\ln \sqrt{(x-a)^2 + y^2} - \ln \sqrt{(x-a)^2 + y^2} \right] = c_1, \\ \phi(x,y) &= \frac{q}{2\pi} \left[\tan^{-1} \frac{y}{x-a} - \tan^{-1} \frac{y}{x-b} \right] = c_2. \end{aligned} \tag{8.6.20}$$

The potential lines are circles and the streamlines are circular arcs as shown in Figure 8.6.12 These correspond to a source at $z = a$ and a sink at $z = b$. One can also view these as the electric field lines and equipotentials for an electric dipole consisting of two point charges of opposite sign at the points $z = a$ and $z = b$.

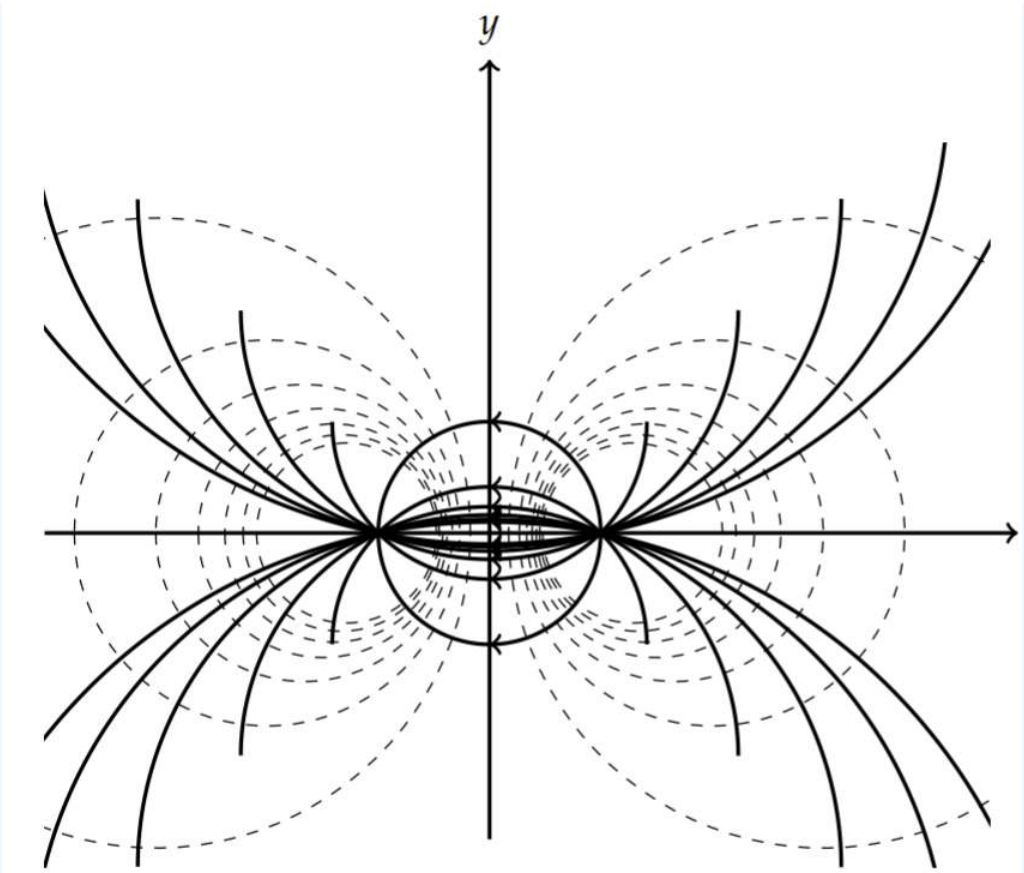


Figure 8.6.12: The electric field lines (solid) and equipotentials (dashed) for a dipole given by the complex potential $F(z)=\frac{q}{2\pi}\ln\frac{z-a}{z-b}$ for $b=-a$.

The equations for the curves are found from[2]

$$\begin{aligned}(x-a)^2+y^2 &= C_1\left[(x-b)^2+y^2\right],\\ (x-a)(x-b)+y^2 &= C_2 y(a-b),\end{aligned} \tag{8.6.21}$$

where these can be rewritten, respectively, in the more suggestive forms

$$\begin{aligned}\left(x-\frac{a-bC_1}{1-C_1}\right)^2+y^2 &= \frac{C_1(a-b)^2}{(1-C_1)^2}\\ \left(x-\frac{a+b}{2}\right)^2+\left(y-\frac{C_2(a-b)}{2}\right)^2 &= \left(1+C_2^2\right)\left(\frac{a-b}{2}\right)^2\end{aligned} \tag{8.6.22}$$

Note that the first family of curves are the potential curves and the second give the streamlines.

In the case that $k=-iq$ we have

$$\begin{aligned}F(z) &= \frac{-iq}{2\pi}\ln\frac{z-a}{z-b}\\ &= \frac{-iq}{2\pi}\left[\ln\sqrt{(x-a)^2+y^2}-\ln\sqrt{(x-a)^2+y^2}\right],\\ &\quad+\frac{q}{2\pi}\left[\tan^{-1}\frac{y}{x-a}-\tan^{-1}\frac{y}{x-b}\right].\end{aligned} \tag{8.6.23}$$

So, the roles of the streamlines and potential lines are reversed and the corresponding plots give a flow for a pair of vortices as shown in Figure 8.6.13

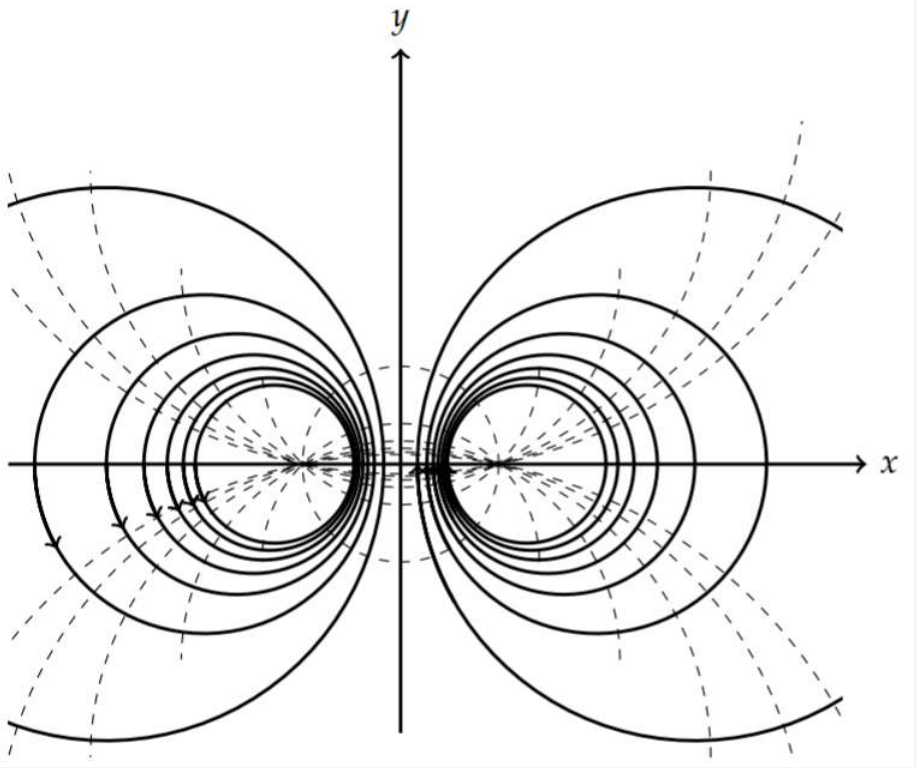


Figure 8.6.13: The streamlines (solid) and potentials (dashed) for a pair of vortices given by the complex potential $F(z) = \frac{q}{2\pi}\ln\frac{z-a}{-b}$ for $b = -a$.

Note

The streamlines are found using the identity

$$\tan^{-1}\alpha - \tan^{-1}\beta = \tan^{-1}\frac{\alpha-\beta}{1+\alpha\beta}$$

Conformal Mappings

It would be nice if the complex potentials in the last section could be mapped to a region of the complex plane such that the new stream functions and velocity potentials represent new flows. In order for this to be true, we would need the new families to once again be orthogonal families of curves. Thus, the mappings we seek must preserve angles. Such mappings are called conformal mappings.

We let $w = f(z)$ map points in the z-plane, (x, y), to points in the w plane, (u, v) by $f(x+iy) = u+iv$. We have shown this in Figure 8.3.1.

Example 8.6.9

Map lines in the z-plane to curves in the w-plane under $f(z) = z^2$.

Solution

We have seen how grid lines in the z-plane is mapped by $f(z) = z^2$ into the w plane in Figure 8.3.2, which is reproduced in Figure 8.6.14 The horizontal line $x = 1$ is mapped to $u(1, y) = 1 - y^2$ and $v(1, y) = 2y$. Eliminating the "parameter" y between these two equations, we have $u = 1 - v^2/4$. This is a parabolic curve. Similarly, the horizontal line $y = 1$ results in the curve $u = v^2/4 - 1$. These curves intersect at $w = 2i$.

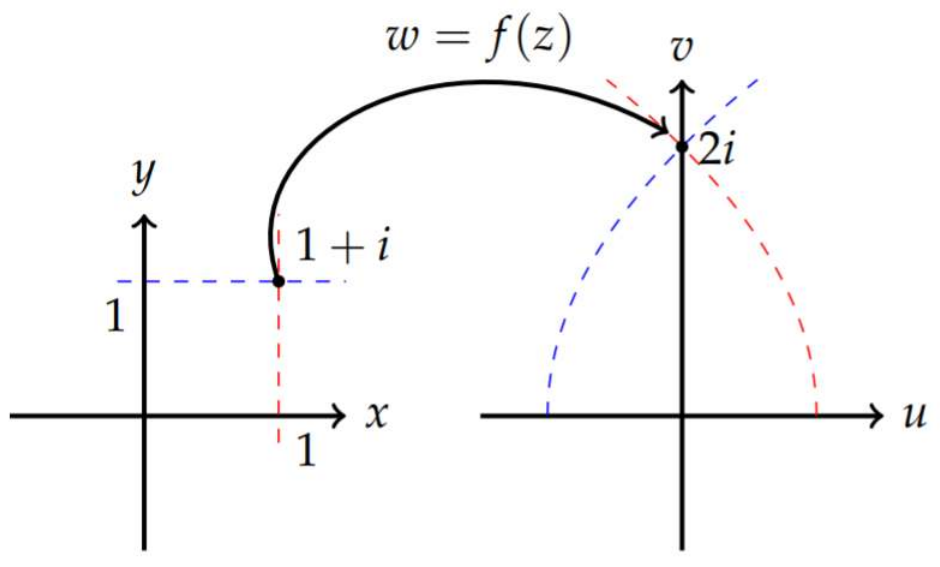


Figure 8.6.14: 2D plot showing how the function $f(z) = z^2$ maps the lines $x = 1$ and $y = 1$ in the z-plane into parabolae in the w-plane.

The lines in the z-plane intersect at $z = 1 + i$ at right angles. In the w-plane we see that the curves $u = 1 - v^2/4$ and $u = v^2/4 - 1$ intersect at $w = 2i$. The slopes of the tangent lines at $(0, 2)$ are -1 and 1 , respectively, as shown in Figure 8.6.15

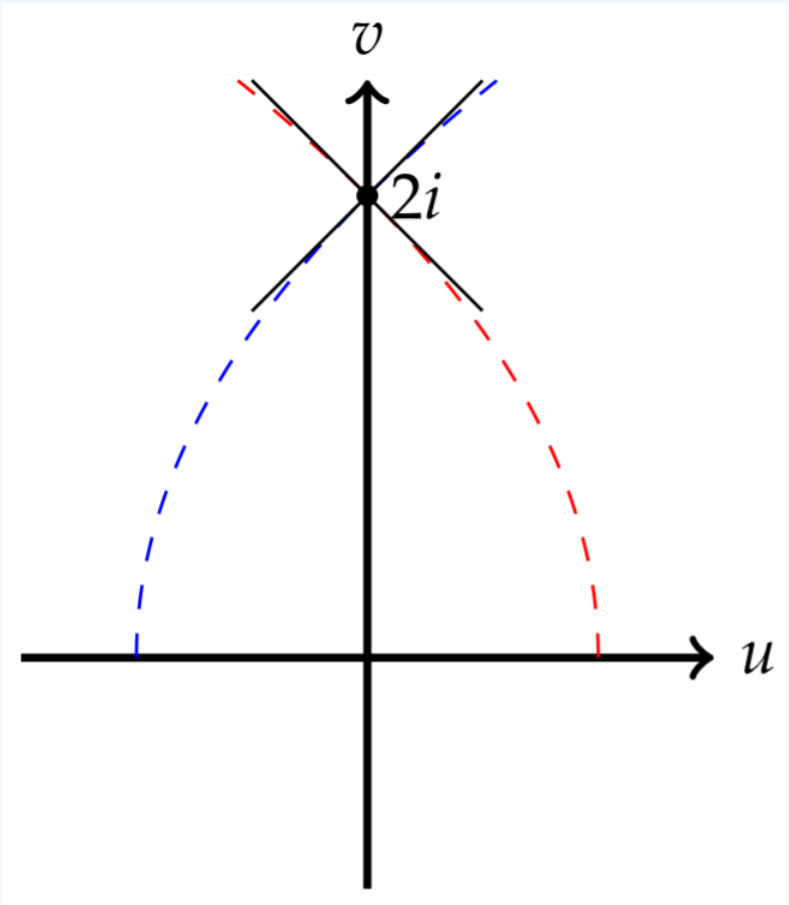


Figure 8.6.15: The tangents to the images of $x = 1$ and $y = 1$ under $f(z) = z^2$ are orthogonal.

In general, if two curves in the z-plane intersect orthogonally at $z = z_0$ and the corresponding curves in the w-plane under the mapping $w = f(z)$ are orthogonal at $w_0 = f(z_0)$, then the mapping is conformal. As we have seen, holomorphic functions are

conformal, but only at points where $f'(z) \neq 0$.

Note

Holomorphic functions are conformal at points where $f'(z) \neq 0$.

Example 8.6.10

Images of the real and imaginary axes under $f(z) = z^2$.

Solution

The line $z =$ iy maps to $w = z^2 = -y^2$ and the line $z = x$ maps to $w = z^2 = x^2$. The point of intersection $z_0 = 0$ maps to $w_0 = 0$. However, the image lines are the same line, the real axis in the w-plane. Obviously, the image lines are not orthogonal at the origin. Note that $f'(0) = 0$.

One special mapping is the inversion mapping, which is given by

$$f(z) = \frac{1}{z}.$$

This mapping maps the interior of the unit circle to the exterior of the unit circle in the w-plane as shown in Figure 8.6.16

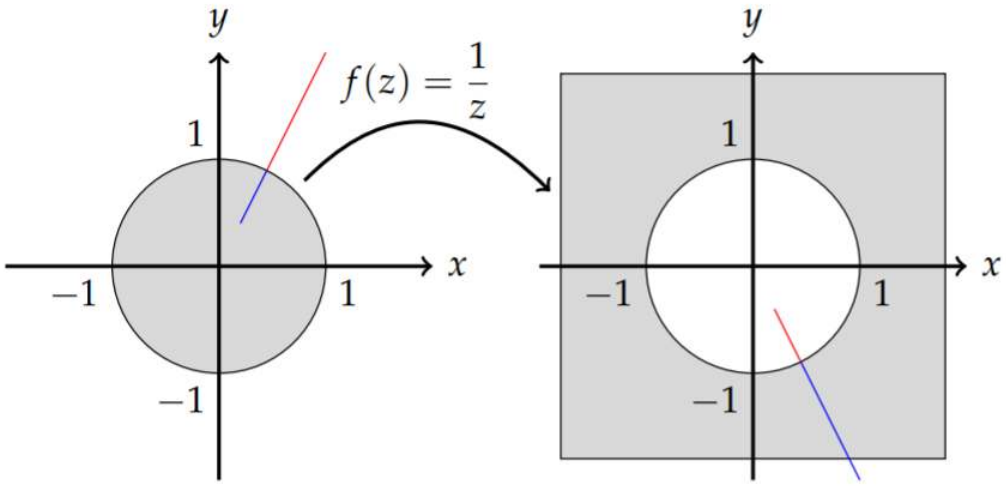


Figure 8.6.16: The inversion, $f(z) = \frac{1}{z}$, maps the interior of a unit circle to the external of a unit circle. Also, segments of aline through the origin, $y = 2x$, are mapped to the line $u = -2v$.

Let $z = x + iy$, where $x^2 + y^2 < 1$. Then,

$$w = \frac{1}{x+iy} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}.$$

Thus, $u = \frac{x}{x^2+y^2}$ and $v = -\frac{y}{x^2+y^2}$, and

$$\begin{aligned} u^2 + v^2 &= \left(\frac{x}{x^2+y^2}\right)^2 + \left(-\frac{y}{x^2+y^2}\right)^2 \\ &= \frac{x^2+y^2}{(x^2+y^2)^2} = \frac{1}{x^2+y^2} \end{aligned} \tag{8.6.24}$$

Thus, for $x^2 + y^2 < 1, u^2 + U^2 > 1$. Furthermore, for $x^2 + y^2 > 1, u^2 + U^2 < 1$, and for $x^2 + y^2 = 1, u^2 + U^2 = 1$.

In fact, an inversion maps circles into circles. Namely, for $z = z_0 + re^{i\theta}$, we have

$$\begin{aligned} w &= \frac{1}{z_0 + re^{i\theta}} \\ &= \frac{\bar{z}_0 + re^{-i\theta}}{\left|z_0 + re^{i\theta}\right|^2} \\ &= w_0 + Re^{-i\theta} \end{aligned} \tag{8.6.25}$$

Also, lines through the origin in the z-plane map into lines through the origin in the w-plane. Let $z = x + imx$. This corresponds to a line with slope m in the z-plane, $y = mx$. It maps to

$$\begin{aligned} f(z) &= \frac{1}{z} \\ &= \frac{1}{x + imx} \\ &= \frac{x - imx}{(1 + m^2)\, x}. \end{aligned} \tag{8.6.26}$$

So, $u = \frac{x}{(1+m^2)x}$ and $v = -\frac{mx}{(1+m^2)x} = -mu$. This is a line through the origin in the w-plane with slope $-m$. This is shown in Figure 8.6.16. Note how the potion of the line $y = 2x$ that is inside the unit disk maps to the outside of the disk in the w-plane.

Note

The bilinear transformation.

Another interesting class of transformation, of which the inversion is contained, is the bilinear transformation. The bilinear transformation is given by

$$w = f(z) = \frac{az+b}{cz+d}, \quad ad - bc \neq 0,$$

where $a, b, c,$ and d are complex constants. These transformations were studied by mappings was studied by August Ferdinand Möbius (1790-1868) and are also called Möbius transformations, or linear fractional transformations. We further note that if $ad - bc = 0$, then the transformation reduces to the constat function.

We can seek to invert the transformation. Namely, solving for z, we have

$$z = f^{-1}(w) = \frac{-dw+b}{cw-a}, \quad w \neq \frac{a}{c}.$$

Since $f^{-1}(w)$ is not defined for $w \neq \frac{a}{c}$, we can say that $w \neq \frac{a}{c}$ maps to the point at infinity, or $f^{-1}\left(\frac{a}{c}\right) = \infty$. Similarly, we can let $z \to \infty$ to obtain

$$f(\infty) = \lim_{n\to\infty} f(z) = -\frac{d}{c}.$$

Thus, we have that the bilinear transformation is a one-to-one mapping of the extended complex z-plane to the extended complex w-plane.

Note

The extended complex plane is the union of the complex plane plus the point at infinity. This is usually described in more detail using stereographic projection, which we will not review here.

If $c = 0$, $f(z)$ is easily seen to be a linear transformation. Linear transformations transform lines into lines and circles into circles.

When $c \neq 0$, we can write

$$
\begin{aligned}
f(z) &= \frac{az+b}{cz+d} \\
&= \frac{c(az+b)}{c(cz+d)} \\
&= \frac{acz+ad-ad+bc}{c(cz+d)} \\
&= \frac{a(cz+d)-ad+bc}{c(cz+d)} \\
&= \frac{a}{c} + \frac{bc-ad}{c}\frac{1}{cz+d}.
\end{aligned} \tag{8.6.27}
$$

We note that if $bc-ad=0$, then $f(z)=\frac{a}{c}$ is a constant, as noted above. The new form for $f(z)$ shows that it is the composition of a linear function $\zeta = cz+d$, an inversion, $g(\zeta)=\frac{1}{\zeta}$, and another linear transformation, $h(\zeta)= \frac{a}{c}+\frac{bc-ad}{c}\zeta$. Since linear transformations and inversions transform the set of circles and lines in the extended complex plane into circles and lines in the extended complex plane, then a bilinear does so as well.

What is important in out applications of complex analysis to the solution of Laplace's equation in the transformation of regions of the complex plane into other regions of the complex plane. Needed transformations can be found using the following property of bilinear transformations:

A given set of three points in the z-plane can be transformed into a given set of points in the w-plane using a bilinear transformation.

This statement is based on the following observation: There are three independent numbers that determine a bilinear transformation. If $a \neq 0$, then

$$
\begin{aligned}
f(z) &= \frac{az+b}{cz+d} \\
&= \frac{z+\frac{b}{a}}{\frac{c}{a}z+\frac{d}{a}} \\
&\equiv \frac{z+\alpha}{\beta z+\gamma}
\end{aligned} \tag{8.6.28}
$$

For $w=\frac{z+\alpha}{\beta z+\gamma}$, we have

$$
\begin{aligned}
w &= \frac{z+\alpha}{\beta z+\gamma} \\
w(\beta z+\gamma) &= z+\alpha \\
-\alpha + wz\beta + w\gamma &= z.
\end{aligned} \tag{8.6.29}
$$

Now, let $w_i = f(z_i), i=1,2,3$. This gives three equations for the three unknowns α, β, and γ. Namely,

$$
\begin{aligned}
-\alpha + w_1 z_1 \beta + w_1 \gamma &= z_1, \\
-\alpha + w_2 z_2 \beta + w_2 \gamma &= z_2, \\
-\alpha + w_3 z_3 \beta + w_3 \gamma &= z_3.
\end{aligned} \tag{8.6.30}
$$

This systems of linear equation can be put into matrix form as

$$
\begin{pmatrix} -1 & w_1 z_1 & w_1 \\ -1 & w_2 z_2 & w_2 \\ -1 & w_3 z_3 & w_3 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}.
$$

It is only a matter of solving this system for $(\alpha,\beta,\gamma)^T$ in order to find the bilinear transformation.

A quicker method is to use the implicit form of the transformation,

$$
\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} = \frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}.
$$

Note that this implicit relation works upon insertion of the values w_i, z_i, for $i = 1, 2, 3$.

Example 8.6.11

Find the bilinear transformation that maps the points $-1, i, 1$ to the points $-1, 0, 1$.

Solution

The implicit form of the transformation becomes

$$\begin{aligned} \frac{(z+1)(i-1)}{(z-1)(i+1)} &= \frac{(w+1)(0-1)}{(w-1)(0+1)} \\ \frac{z+1}{z-1}\,\frac{i-1}{i+1} &= -\frac{w+1}{w-1} \end{aligned} \tag{8.6.31}$$

Solving for w, we have

$$w = f(z) = \frac{(i-1)z+1+i}{(1+i)z-1+i}.$$

We can use the transformation in the last example to map the unit disk containing the points $-1, i$, and 1 to the half plane $w > 0$. We see that the unit circle gets mapped to the real axis with $z = -i$ mapped to the point at infinity. The point $z = 0$ gets mapped to

$$w = \frac{1+i}{-1+i} = \frac{1+i}{-1+i}\,\frac{-1-i}{-1-i} = \frac{2}{2} = 1.$$

Thus, interior points of the unit disk get mapped to the upper half plane. This is shown in Figure 8.6.17.

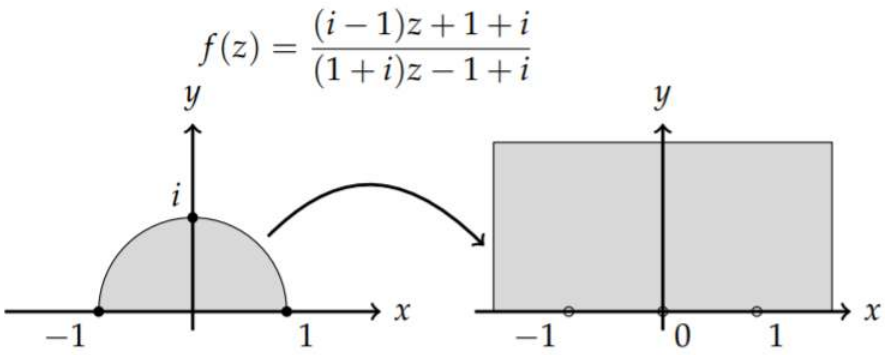


Figure 8.6.17: The binlinear transformation $f(z) = \frac{(i-1)z+1+i}{(1+i)z-1+i}$ maps the unit disk to the upper half plane.

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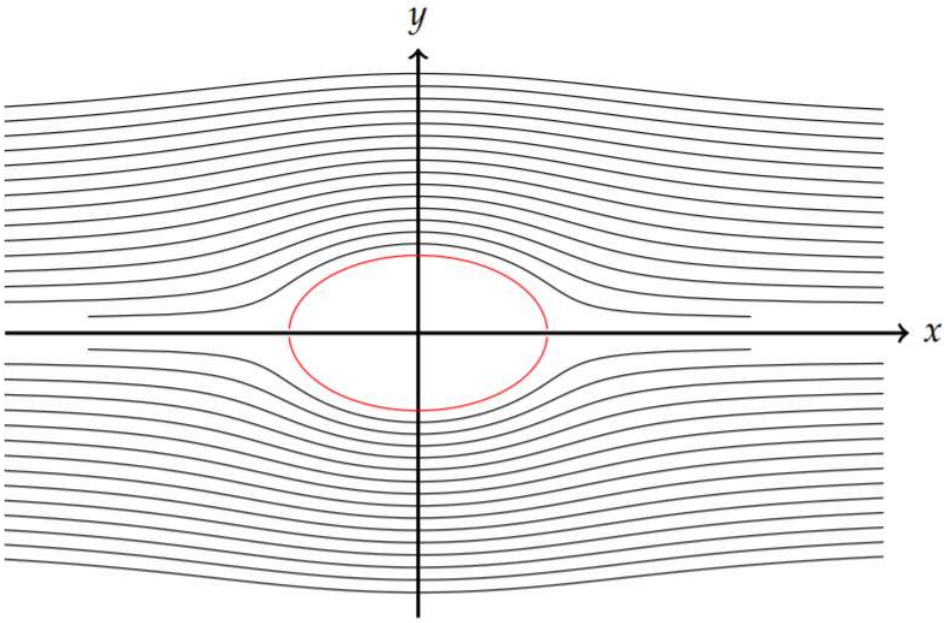


Figure 8.6.18: Flow ...

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8.7: Problems

Exercise 8.7.1

Write the following in standard form.

a. $(4+5i)(2-3i)$.
b. $(1+i)^3$.
c. $\frac{5+3i}{1-i}$.

Exercise 8.7.2

Write the following in polar form, $z=re^{i\theta}$.

a. $i-1$.
b. $-2i$.
c. $\sqrt{3}+3i$.

Exercise 8.7.3

Write the following in rectangular form, $z=a+ib$.

a. $4e^{i\pi/6}$.
b. $\sqrt{2}e^{5i\pi/4}$.
c. $(1-i)^{100}$.

Exercise 8.7.4

Find all z such that $z^4=16i$. Write the solutions in rectangular form, $z=a+ib$, with no decimal approximation or trig functions.

Exercise 8.7.5

Show that $\sin(x+iy)=\sin x\cosh y+i\cos x\sinh y$ using trigonometric identities and the exponential forms of these functions.

Exercise 8.7.6

Find all z such that $\cos z=2$, or explain why there are none. You will need to consider $\cos(x+iy)$ and equate real and imaginary parts of the resulting expression similar to problem 5.

Exercise 8.7.7

Find the principal value of i^i. Rewrite the base, i, as an exponential first.

Exercise 8.7.8

Consider the circle $|z-1|=1$.

a. Rewrite the equation in rectangular coordinates by setting $z=x+iy$.
b. Sketch the resulting circle using part a.
c. Consider the image of the circle under the mapping $f(z)=z^2$, given by $|z^2-1|=1$.
   i. By inserting $z=re^{i\theta}=r(\cos\theta+i\sin\theta)$, find the equation of the image curve in polar coordinates.
   ii. Sketch the image curve. You may need to refer to your Calculus II text for polar plots. [Maple might help.]

Exercise 8.7.9

Find the real and imaginary parts of the functions:

a. $f(z)=z^3$.
b. $f(z)=\sinh(z)$.
c. $f(z)=\cos\bar{z}$.

Exercise 8.7.10

Find the derivative of each function in Problem 9 when the derivative exists. Otherwise, show that the derivative does not exist.

Exercise 8.7.11

Let $f(z)=u+iv$ be differentiable. Consider the vector field given by $\mathbf{F}=v\mathbf{i}+u\mathbf{j}$. Show that the equations $\nabla\cdot\mathbf{F}=\mathbf{0}$ and $\nabla\times\mathbf{F}=\mathbf{0}$ are equivalent to the Cauchy-Riemann equations. [You will need to recall from multivariable calculus the del operator, $\nabla=\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}$.]

Exercise 8.7.12

What parametric curve is described by the function

$$\gamma(t)=(t-3)+i(2t+1),$$

$0\le t\le 2$? [Hint: What would you do if you were instead considering the parametric equations $x=t-3$ and $y=2t+1$?]

Exercise 8.7.13

Write the equation that describes the circle of radius 3 which is centered at $z=2-i$ in a) Cartesian form (in terms of x and y); b) polar form (in terms of θ and r); c) complex form (in terms of z, r, and $e^{i\theta}$).

Exercise 8.7.14

Consider the function $u(x,y)=x^3-3xy^2$.

a. Show that $u(x,y)$is harmonic; i.e., $\nabla^2u=0$.
b. Find its harmonic conjugate, $v(x,y)$.
c. Find a differentiable function, $f(z)$, for which $u(x,y)$is the real part.
d. Determine $f'(z)$ for the function in part c. [Use $f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$ and rewrite your answer as a function of z.]

Exercise 8.7.15

Evaluate the following integrals:

a. $\int_C\bar{z}dz$, where C is the parabola $y=x^2$ from $z=0$ to $z=1+i$.
b. $\int_C f(z)dz$, where $f(z)=2z-\bar{z}$ and C is the path from $z=0$ to $z=2+i$ consisting of two line segments from $z=0$ to $z=2$ and then $z=2$ to $z=2+i$.
c. $\int_C\frac{1}{z^2+4}dz$ for C the positively oriented circle, $|z|=2$. [Hint: Parametrize the circle as $z=2e^{i\theta}$, multiply numerator and denominator by $e^{-i\theta}$, and put in trigonometric form.]

Exercise 8.7.16

Let C be the positively oriented ellipse $3x^2+y^2=9$. Define

$$F(z_0)=\int_C \frac{z^2+2z}{z-z_0}dz.$$

Find $F(2i)$ and $F(2)$. [Hint: Sketch the ellipse in the complex plane. Use the Cauchy Integral Theorem with an appropriate $f(z)$, or Cauchy's Theorem if z_0 is outside the contour.]

Exercise 8.7.17

Show that

$$\int_C \frac{dz}{(z-1-i)^{n+1}}=\begin{cases}0, & n\neq 0\\ 2\pi i, & n=0\end{cases}$$

for C the boundary of the square $0\leq x\leq 2, 0\leq y\leq 2$ taken counterclockwise. [Hint: Use the fact that contours can be deformed into simpler shapes (like a circle) as long as the integrand is analytic in the region between them. After picking a simpler contour, integrate using parametrization.]

Exercise 8.7.18

Show that for g and h analytic functions at z_0, with $g(z_0)\neq 0, h(z_0)=0$, and $h'(z_0)\neq 0$,

$$\text{Res}\left[\frac{g(z)}{h(z)};z_0\right]=\frac{g(z_0)}{h'(z_0)}.$$

Exercise 8.7.19

For the following determine if the given point is a removable singularity, an essential singularity, or a pole (indicate its order).

a. $\frac{1-\cos z}{z^2}$, $z=0$.
b. $\frac{\sin z}{z^2}$, $z=0$.
c. $\frac{z^2-1}{(z-1)^2}$, $z=1$.
d. $ze^{1/z}$, $z=0$.
e. $\cos\frac{\pi}{z-\pi}$, $z=\pi$.

Exercise 8.7.20

Find the Laurent series expansion for $f(z)=\frac{\sinh z}{z^3}$ about $z=0$. [Hint: You need to first do a MacLaurin series expansion for the hyperbolic sine.]

Exercise 8.7.21

Find series representations for all indicated regions.

a. $f(z)=\frac{z}{z-1}, |z|<1, |z|>1$.
b. $f(z)=\frac{1}{(z-i)(z+2)}, |z|<1, 1<|z|<2, |z|>2$. [Hint: Use partial fractions to write this as a sum of two functions first.]

Exercise 8.7.22

Find the residues at the given points:

a. $\frac{2z^2+3z}{z-1}$ at $z=1$.

b. $\frac{\ln(1+2z)}{z}$ at $z=0$.
c. $\frac{\cos z}{(2z-\pi)^3}$ at $z=\frac{\pi}{2}$.

Exercise 8.7.23

Consider the integral $\int_0^{2\pi} \frac{d\theta}{5-4\cos\theta}$.

a. Evaluate this integral by making the substitution $2\cos\theta = z+\frac{1}{z}$, $z=e^{i\theta}$ and using complex integration methods.
b. In the 1800's Weierstrass introduced a method for computing integrals involving rational functions of sine and cosine. One makes the substitution $t=\tan\frac{\theta}{2}$ and converts the integrand into a rational function of t. Note that the integration around the unit circle corresponds to $t\in(-\infty,\infty)$.
   i. Show that
   $$\sin\theta=\frac{2t}{1+t^2},\quad \cos\theta=\frac{1-t^2}{1+t^2}.$$
   ii. Show that
   $$d\theta=\frac{2dt}{1+t^2}$$
   iii. Use the Weierstrass substitution to compute the above integral.

Exercise 8.7.24

Do the following integrals.

a. $$\oint_{|z-i|=3}\frac{e^z}{z^2+\pi^2}dz$$

b. $$\oint_{|z-i|=3}\frac{z^2-3z+4}{z^2-4z+3}dz.$$

c. $$\int_{-\infty}^{\infty}\frac{\sin x}{x^2+4}dx$$

[Hint: This is $\operatorname{Im}\int_{-\infty}^{\infty}\frac{e^{ix}}{x^2+4}dx$.]

Exercise 8.7.25

Evaluate the integral $\int_0^{\infty}\frac{(\ln x)^2}{1+x^2}dx$.

[Hint: Replace x with $z=e^t$ and use the rectangular contour in Figure 8.7.1 with $R\to\infty$.]

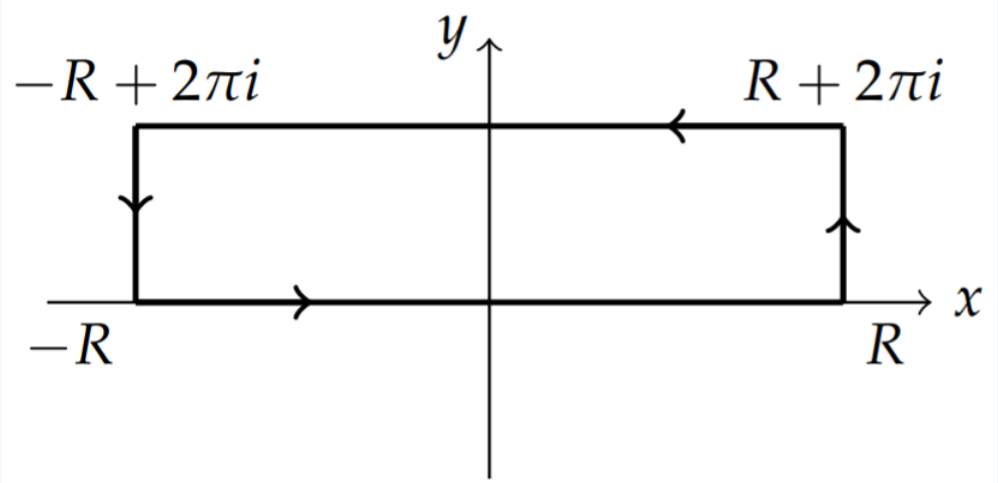


Figure 8.7.1: Rectangular controur for Problem 25.

Exercise 8.7.26

Do the following integrals for fun!

a. For C the boundary of the square $|x| \leq 2, |y| \leq 2$,

$$\oint_C \frac{dz}{z(z-1)(z-3)^2}.$$

b. $$\int_0^{\pi} \frac{\sin^2\theta}{13-12\cos\theta} d\theta.$$

c. $$\int_{-\infty}^{\infty} \frac{dx}{x^2+5x+6}.$$

d. $$\int_0^{\infty} \frac{\cos \pi x}{1-9x^2} dx$$

e. $$\int_{00}^{\infty} \frac{dx}{(x^2+9)(1-x)^2}$$

f. $$\int_0^{\infty} \frac{\sqrt{x}}{(1+x)^2} dx$$

g. $$\int_0^{\infty} \frac{\sqrt{x}}{(1+x)^2} dx$$

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CHAPTER OVERVIEW

9: Transform Techniques in Physics

> *"There is no branch of mathematics, however abstract, which may not some day be applied to phenomena of the real world.",*
>
> *~ Nikolai Lobatchevsky (1792-1856)*

9.1: Introduction
9.2: Complex Exponential Fourier Series
9.3: Exponential Fourier Transform
9.4: The Dirac Delta Function
9.5: Properties of the Fourier Transform
9.6: The Convolution Operation
9.7: The Laplace Transform
9.8: Applications of Laplace Transforms
9.9: The Convolution Theorem
9.10: The Inverse Laplace Transform
9.11: Transforms and Partial Differential Equations
9.12: Problems

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9.1: Introduction

Some of the most powerful tools for solving problems in physics are transform methods. The idea is that one can transform the problem at hand to a new problem in a different space, hoping that the problem in the new space is easier to solve. Such transforms appear in many forms.

As we had seen in Chapter 3 and will see later in the book, the solutions of linear partial differential equations can be found by using the method of separation of variables to reduce solving partial differential equations (PDEs) to solving ordinary differential equations (ODEs). We can also use transform methods to transform the given PDE into ODEs or algebraic equations. Solving these equations, we then construct solutions of the PDE (or, the ODE) using an inverse transform. A schematic of these processes is shown below and we will describe in this chapter how one can use Fourier and Laplace transforms to this effect.

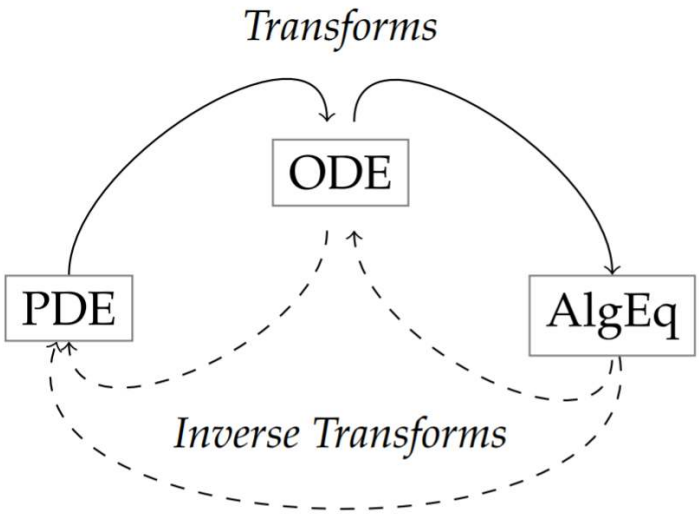


Figure 9.1.1: Schematic indicating that PDEs and ODEs can be transformed to simpler problems, solved in the new space and transformed back to the original space.

Note

In this chapter we will explore the use of integral transforms. Given a function $f(x)$, we define an integral transform to a new function $F(k)$ as

$$F(k) = \int_a^b f(x)K(x,k)dx.$$

Here $K(x,k)$ is called the kernel of the transform. We will concentrate specifically on Fourier transforms,

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x)e^{ikx}dx$$

and Laplace transforms

$$F(s) = \int_0^{\infty} f(t)e^{-st}dt.$$

Example 1 - The Linearized KdV Equation

As a relatively simple example, we consider the linearized Kortewegde Vries (KdV) equation:

$$u_t + cu_x + \beta u_{xxx} = 0, \quad -\infty < x < \infty. \tag{9.1.1}$$

This equation governs the propagation of some small amplitude water waves. Its nonlinear counterpart has been at the center of attention in the last 40 years as a generic nonlinear wave equation.

Note

The nonlinear counterpart to this equation is the Korteweg-de Vries (KdV) equation: $u_t + 6uu_x + u_{xxx} = 0$. This equation was derived by Diederik Johannes Korteweg (1848-1941) and his student Gustav de Vries (1866-1934). This equation governs the propagation of traveling waves called solutons. These were first observed by John Scott Russell (1808-1882) and were the source of a long debate on the existence of such waves. The history of this debate is interesting and the KdV turned up as a generic equation in many other fields in the latter part of the last century leading to many papers on nonlinear evolution equations.

We seek solutions that oscillate in space. So, we assume a solution of the form

$$u(x,t) = A(t)e^{ikx}. \tag{9.1.2}$$

Such behavior was seen in Chapters 3 and 6 for the wave equation for vibrating strings. In that case, we found plane wave solutions of the form $e^{ik(x\pm ct)}$, which we could write as $e^{i(kx\pm\omega t)}$ by defining $\omega = kc$. We further note that one often seeks complex solutions as a linear combination of such forms and then takes the real part in order to obtain physical solutions. In this case, we will find plane wave solutions for which the angular frequency $\omega = \omega(k)$ is a function of the wavenumber.

Inserting the guess (9.1.2) into the linearized KdV equation, we find that

$$\frac{dA}{dt} + i\left(ck - \beta k^3\right) A = 0. \tag{9.1.3}$$

Thus, we have converted the problem of seeking a solution of the partial differential equation into seeking a solution to an ordinary differential equation. This new problem is easier to solve. In fact, given an initial value, $A(0)$, we have

$$A(t) = A(0)e^{-i(ck-\beta k^3)t}. \tag{9.1.4}$$

Therefore, the solution of the partial differential equation is

$$u(x,t) = A(0)e^{ik(x-(c-\beta k^2)t)}. \tag{9.1.5}$$

We note that this solution takes the form $e^{i(kx-\omega t)}$, where

$$\omega = ck - \beta k^3.$$

Note

A dispersion relation is an expression giving the angular frequency as a function of the wave number, $\omega = \omega(k)$.

In general, the equation $\omega = \omega(k)$ gives the angular frequency as a function of the wave number, k, and is called a dispersion relation. For $\beta = 0$, we see that c is nothing but the wave speed. For $\beta \neq 0$, the wave speed is given as

$$v = \frac{\omega}{k} = c - \beta k^2.$$

This suggests that waves with different wave numbers will travel at different speeds. Recalling that wave numbers are related to wavelengths, $k = \frac{2\pi}{\lambda}$, this means that waves with different wavelengths will travel at different speeds. For example, an initial localized wave packet will not maintain its shape. It is said to disperse, as the component waves of differing wavelengths will tend to part company.

For a general initial condition, we write the solutions to the linearized KdV as a superposition of plane waves. We can do this since the partial differential equation is linear. This should remind you of what we had done when using separation of variables. We first sought product solutions and then took a linear combination of the product solutions to obtain the general solution.

For this problem, we will sum over all wave numbers. The wave numbers are not restricted to discrete values. We instead have a continuous range of values. Thus, "summing" over k means that we have to integrate over the wave numbers. Thus, we have the general solution[1]

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(k,0)e^{ik(x-(c-\beta k^2)t)}\,dk. \tag{9.1.6}$$

Note that we have indicated that A is a function of k. This is similar to introducing the A_n 's and B_n 's in the series solution for waves on a string.

> **Note**
>
> The extra 2π has been introduced to be consistent with the definition of the Fourier transform which is given later in the chapter.

How do we determine the $A(k,0)$'s? We introduce as an initial condition the initial wave profile $u(x,0)=f(x)$. Then, we have

$$f(x)=u(x,0)=\frac{1}{2\pi}\int_{-\infty}^{\infty}A(k,0)e^{ikx}dk. \tag{9.1.7}$$

Thus, given $f(x)$, we seek $A(k,0)$. In this chapter we will see that

$$A(k,0)=\int_{-\infty}^{\infty}f(x)e^{-ikx}dx$$

This is what is called the Fourier transform of $f(x)$. It is just one of the so-called integral transforms that we will consider in this chapter.

In Figure 9.1.2 we summarize the transform scheme. One can use methods like separation of variables to solve the partial differential equation directly, evolving the initial condition $u(x,0)$ into the solution $u(x,t)$ at a later time.

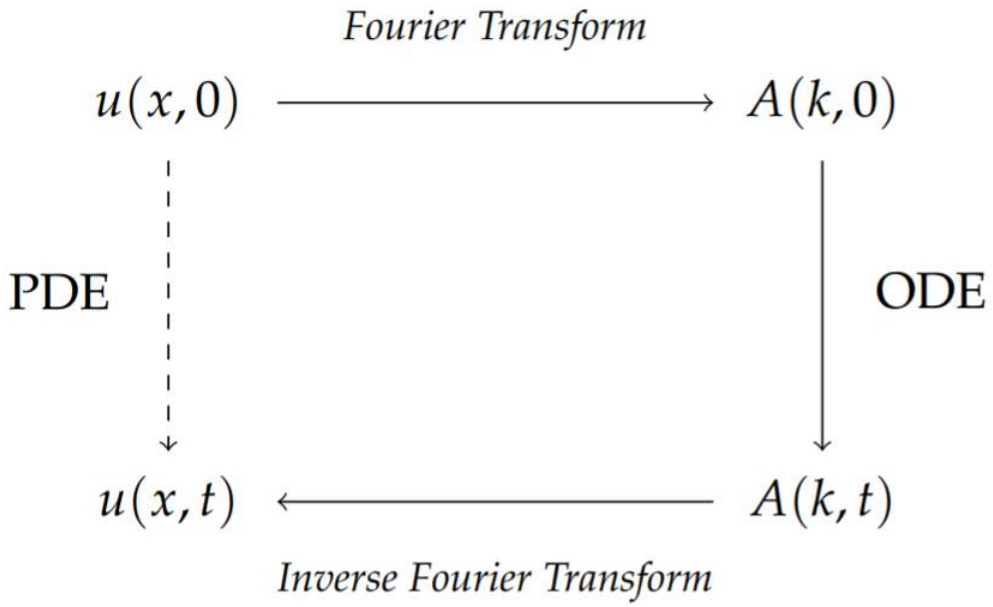


Figure 9.1.2: Schematic of using Fourier transforms to solve a linear evolution equation.

The transform method works as follows. Starting with the initial condition, one computes its Fourier Transform (FT) as[2]

$$A(k,0)=\int_{-\infty}^{\infty}f(x)e^{-ikx}dx$$

Applying the transform on the partial differential equation, one obtains an ordinary differential equation satisfied by $A(k,t)$ which is simpler to solve than the original partial differential equation. Once $A(k,t)$ has been found, then one applies the Inverse Fourier Transform (IFT) to $A(k,t)$ in order to get the desired solution:

$$\begin{aligned}u(x,t)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}A(k,t)e^{ikx}dk\\&=\frac{1}{2\pi}\int_{-\infty}^{\infty}A(k,0)e^{ik(x-(c-\beta k^2)t)}\,dk.\end{aligned} \tag{9.1.8}$$

Note

The Fourier transform as used in this section and the next section are defined slightly differently than how we will define them later. The sign of the exponentials has been reversed. The one dimensional time dependent Schrödinger equation.

Example 2 - The Free Particle Wave Function

A more familiar example in physics comes from quantum mechanics. The Schrödinger equation gives the wave function $\Psi(x,t)$ for a particle under the influence of forces, represented through the corresponding potential function $V(x)$. The one dimensional time dependent Schrödinger equation is given by

$$i\hbar\Psi_t = -\frac{\hbar^2}{2m}\Psi_{xx} + V\Psi. \tag{9.1.9}$$

We consider the case of a free particle in which there are no forces, $V = 0$. Then we have

$$i\hbar\Psi_t = -\frac{\hbar^2}{2m}\Psi_{xx}. \tag{9.1.10}$$

Taking a hint from the study of the linearized KdV equation, we will assume that solutions of Equation (9.1.10) take the form

$$\Psi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\phi(k,t)e^{ikx}dk.$$

[Here we have opted to use the more traditional notation, $\phi(k,t)$ instead of $A(k,t)$ as above.]

Inserting the expression for $\Psi(x,t)$ into (9.1.10), we have

$$i\hbar\int_{-\infty}^{\infty}\frac{d\phi(k,t)}{dt}e^{ikx}dk = -\frac{\hbar^2}{2m}\int_{-\infty}^{\infty}\phi(k,t)(ik)^2e^{ikx}dk.$$

Since this is true for all t, we can equate the integrands, giving

$$i\hbar\frac{d\phi(k,t)}{dt} = \frac{\hbar^2k^2}{2m}\phi(k,t).$$

As with the last example, we have obtained a simple ordinary differential equation. The solution of this equation is given by

$$\phi(k,t) = \phi(k,0)e^{-i\frac{\hbar k^2}{2m}t}.$$

Applying the inverse Fourier transform, the general solution to the time dependent problem for a free particle is found as

$$\Psi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\phi(k,0)e^{ik\left(x-\frac{\hbar k}{2m}t\right)}dk.$$

We note that this takes the familiar form

$$\Psi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\phi(k,0)e^{i(kx-\omega t)}dk,$$

where the dispersion relation is found as

$$\omega = \frac{\hbar k^2}{2m}.$$

The wave speed is given as

$$v = \frac{\omega}{k} = \frac{\hbar k}{2m}.$$

As a special note, we see that this is not the particle velocity! Recall that the momentum is given as $p = \hbar k$.[3] So, this wave speed is $v = \frac{p}{2m}$, which is only half the classical particle velocity! A simple manipulation of this result will clarify the "problem."

Note

Since $p = \hbar k$, we also see that the dispersion relation is given by

$$\omega = \frac{\hbar k^2}{2m} = \frac{p^2}{2m\hbar} = \frac{E}{\hbar}$$

We assume that particles can be represented by a localized wave function. This is the case if the major contributions to the integral are centered about a central wave number, k_0. Thus, we can expand $\omega(k)$ about k_0 :

$$\omega(k) = \omega_0 + \omega_0' (k - k_0) t + \ldots . \tag{9.1.11}$$

Here $\omega_0 = \omega(k_0)$ and $\omega_0' = \omega'(k_0)$. Inserting this expression into the integral representation for $\Psi(x,t)$, we have

$$\Psi(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(k,0) e^{i(kx - \omega_0 t - \omega_0'(k-k_0)t - \ldots)}\, dk,$$

We now make the change of variables, $s = k - k_0$, and rearrange the resulting factors to find

$$\begin{aligned} \Psi(x,t) &\approx \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(k_0 + s, 0)\, e^{i((k_0+s)x - (\omega_0 + \omega_0' s)t)}\, ds \\ &= \frac{1}{2\pi} e^{i(-\omega_0 t + k_0 \omega_0' t)} \int_{-\infty}^{\infty} \phi(k_0 + s, 0)\, e^{i(k_0+s)(x - \omega_0' t)}\, ds \\ &= e^{i(-\omega_0 t + k_0 \omega_0' t)}\, \Psi\left(x - \omega_0' t, 0\right). \end{aligned} \tag{9.1.12}$$

Note

Group and phase velocities, $v_g = \frac{d\omega}{dk}$, $v_p = \frac{\omega}{k}$.

Summarizing, for an initially localized wave packet, $\Psi(x,0)$ with wave numbers grouped around k_0 the wave function, $\Psi(x,t)$, is a translated version of the initial wave function up to a phase factor. In quantum mechanics we are more interested in the probability density for locating a particle, so from

$$|\Psi(x,t)|^2 = \left|\Psi\left(x - \omega_0' t, 0\right)\right|^2$$

we see that the "velocity of the wave packet" is found to be

$$\omega_0' = \left.\frac{d\omega}{dk}\right|_{k=k_0} = \frac{\hbar k}{m}.$$

This corresponds to the classical velocity of the particle $(v_{\text{part}} = p/m)$. Thus, one usually defines ω_0' to be the group velocity,

$$v_g = \frac{d\omega}{dk}$$

and the former velocity as the phase velocity,

$$v_p = \frac{\omega}{k}.$$

Transform Schemes

These examples have illustrated one of the features of transform theory. Given a partial differential equation, we can transform the equation from spatial variables to wave number space, or time variables to frequency space. In the new space the time evolution is simpler. In these cases, the evolution was governed by an ordinary differential equation. One solves the problem in the new space and then transforms back to the original space. This is depicted in Figure 9.1.3 for the Schrödinger equation and was shown in Figure 9.1.2 for the linearized KdV equation.

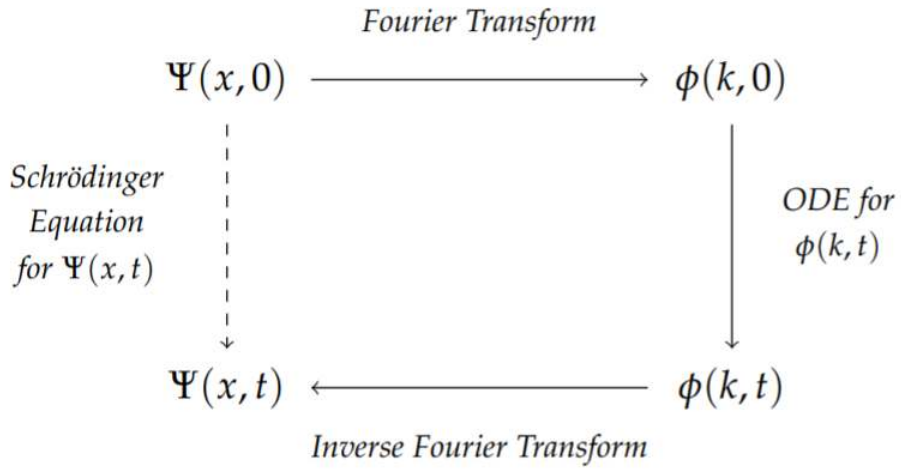


Figure 9.1.3: The scheme for solving the Schrödinger equation using Fourier transforms. The goal is to solve for $\Psi(x,t)$ given $\Psi(x,0)$. Instead of a direct solution in coordinate space (on the left side), one can first transform the initial condition obtaining $\phi(k,0)$ in wave number space. The governing equation in the new space is found by transforming the PDE to get an ODE. This simpler equation is solved to obtain $\phi(k,t)$. Then an inverse transform yields the solution of the original equation.

This is similar to the solution of the system of ordinary differential equations in Chapter 3, $\dot{\mathbf{x}} = A\mathbf{x}$ In that case we diagonalized the system using the transformation $\mathbf{x} = S\mathbf{y}$. This lead to a simpler system $\dot{\mathbf{y}} = \Lambda\mathbf{y}$, where $\Lambda = S^{-1}AS$. Solving for $\mathbf{y}$, we inverted the solution to obtain $\mathbf{x}$. Similarly, one can apply this diagonalization to the solution of linear algebraic systems of equations. The general scheme is shown in Figure 9.1.4.

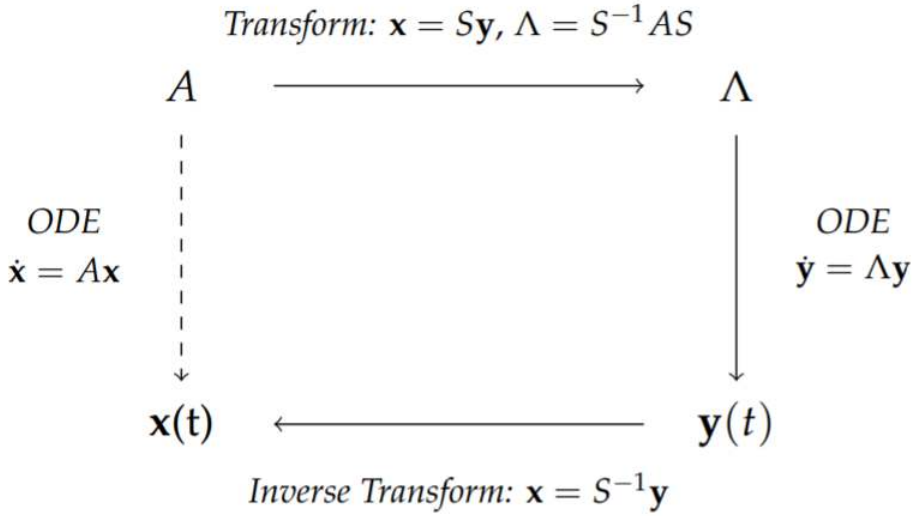


Figure 9.1.4: This shows the scheme for solving the linear system of ODEs $\dot{\mathbf{x}} = A\mathbf{x}$. One finds a transformation between $\mathbf{x}$ and $\mathbf{y}$ of the form $\mathbf{x} = S\mathbf{y}$ which diagonalizes the system. The resulting system is easier to solve for $\mathbf{y}$. Then, one uses the inverse transformation to obtain the solution to the original problem.

Similar transform constructions occur for many other type of problems. We will end this chapter with a study of Laplace transforms, which are useful in the study of initial value problems, particularly for linear ordinary differential equations with constant coefficients. A similar scheme for using Laplace transforms is depicted in Figure 9.8.1.

In this chapter we will begin with the study of Fourier transforms. These will provide an integral representation of functions defined on the real line. Such functions can also represent analog signals. Analog signals are continuous signals which can be represented as a sum over a continuous set of frequencies, as opposed to the sum over discrete frequencies, which Fourier series were used to represent in an earlier chapter. We will then investigate a related transform, the Laplace transform, which is useful in solving initial value problems such as those encountered in ordinary differential equations.

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9.2: Complex Exponential Fourier Series

Before deriving the Fourier transform, we will need to rewrite the trigonometric Fourier series representation as a complex exponential Fourier series. We first recall from Chapter ?? the trigonometric Fourier series representation of a function defined on $[-\pi, \pi]$ with period 2π. The Fourier series is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right), \tag{9.2.1}$$

where the Fourier coefficients were found as

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx, \quad n = 0, 1, \ldots, \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx, \quad n = 1, 2, \ldots \end{aligned} \tag{9.2.2}$$

In order to derive the exponential Fourier series, we replace the trigonometric functions with exponential functions and collect like exponential terms. This gives

$$\begin{aligned} f(x) \sim & \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \left(\frac{e^{inx} + e^{-inx}}{2} \right) + b_n \left(\frac{e^{inx} - e^{-inx}}{2i} \right) \right] \\ = & \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{a_n - ib_n}{2} \right) e^{inx} + \sum_{n=1}^{\infty} \left(\frac{a_n + ib_n}{2} \right) e^{-inx}. \end{aligned} \tag{9.2.3}$$

The coefficients of the complex exponentials can be rewritten by defining

$$c_n = \frac{1}{2}(a_n + ib_n), \quad n = 1, 2, \ldots. \tag{9.2.4}$$

This implies that

$$\bar{c}_n = \frac{1}{2}(a_n - ib_n), \quad n = 1, 2, \ldots \tag{9.2.5}$$

So far the representation is rewritten as

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \bar{c}_n e^{inx} + \sum_{n=1}^{\infty} c_n e^{-inx}.$$

Re-indexing the first sum, by introducing $k = -n$, we can write

$$f(x) \sim \frac{a_0}{2} + \sum_{k=-1}^{-\infty} \bar{c}_{-k} e^{-ikx} + \sum_{n=1}^{\infty} c_n e^{-inx}$$

Since k is a dummy index, we replace it with a new n as

$$f(x) \sim \frac{a_0}{2} + \sum_{n=-1}^{-\infty} \bar{c}_{-n} e^{-inx} + \sum_{n=1}^{\infty} c_n e^{-inx}.$$

We can now combine all of the terms into a simple sum. We first define c_n for negative n's by

$$c_n = \bar{c}_{-n}, \quad n = -1, -2, \ldots$$

Letting $c_0 = \frac{a_0}{2}$, we can write the complex exponential Fourier series representation as

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{-inx}, \tag{9.2.6}$$

where

$$
\begin{aligned}
c_n &= \frac{1}{2}(a_n + ib_n), \quad n = 1, 2, \ldots \\
c_n &= \frac{1}{2}(a_{-n} - ib_{-n}), \quad n = -1, -2, \ldots \\
c_0 &= \frac{a_0}{2}
\end{aligned} \tag{9.2.7}
$$

Given such a representation, we would like to write out the integral forms of the coefficients, c_n. So, we replace the a_n 's and b_n 's with their integral representations and replace the trigonometric functions with complex exponential functions. Doing this, we have for $n = 1, 2, \ldots$.

$$
\begin{aligned}
c_n &= \frac{1}{2}(a_n + ib_n) \\
&= \frac{1}{2}\left[\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx + \frac{i}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx\right] \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\left(\frac{e^{inx} + e^{-inx}}{2}\right)dx + \frac{i}{2\pi}\int_{-\pi}^{\pi} f(x)\left(\frac{e^{inx} - e^{-inx}}{2i}\right)dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}\,dx.
\end{aligned} \tag{9.2.8}
$$

It is a simple matter to determine the c_n 's for other values of n. For $n = 0$, we have that

$$
c_0 = \frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx.
$$

For $n = -1, -2, \ldots$, we find that

$$
c_n = \bar{c}_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{e^{-inx}}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}\,dx.
$$

Therefore, we have obtained the complex exponential Fourier series coefficients for all n. Now we can define the complex exponential Fourier series for the function $f(x)$ defined on $[-\pi, \pi]$ as shown below.

Complex Exponential Series for $f(x)$ defined on $[-\pi, \pi]$

$$
f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{-inx}, \tag{9.2.9}
$$

$$
c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}\,dx. \tag{9.2.10}
$$

We can easily extend the above analysis to other intervals. For example, for $x \in [-L, L]$ the Fourier trigonometric series is

$$
f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)
$$

with Fourier coefficients

$$
\begin{aligned}
a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx, \quad n = 0, 1, \ldots, \\
b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx, \quad n = 1, 2, \ldots.
\end{aligned}
$$

This can be rewritten as an exponential Fourier series of the form

Complex Exponential Series for $f(x)$ defined on $[-L, L]$.

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{-in\pi x/L}, \tag{9.2.11}$$

$$c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{in\pi x/L} dx. \tag{9.2.12}$$

We can now use this complex exponential Fourier series for function defined on $[-L, L]$ to derive the Fourier transform by letting L get large. This will lead to a sum over a continuous set of frequencies, as opposed to the sum over discrete frequencies, which Fourier series represent.

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9.3: Exponential Fourier Transform

Both the trigonometric and complex exponential Fourier series provide us with representations of a class of functions of finite period in terms of sums over a discrete set of frequencies. In particular, for functions defined on $x \in [-L, L]$, the period of the Fourier series representation is $2L$. We can write the arguments in the exponentials, $e^{-in\pi x/L}$, in terms of the angular frequency, $\omega_n = n\pi/L$, as $e^{-i\omega_n x}$. We note that the frequencies, v_n, are then defined through $\omega_n = 2\pi v_n = \frac{n\pi}{L}$. Therefore, the complex exponential series is seen to be a sum over a discrete, or countable, set of frequencies.

We would now like to extend the finite interval to an infinite interval, $x \in (-\infty, \infty)$, and to extend the discrete set of (angular) frequencies to a continuous range of frequencies, $\omega \in (-\infty, \infty)$. One can do this rigorously. It amounts to letting L and n get large and keeping $\frac{n}{L}$ fixed.

We first define $\Delta\omega = \frac{\pi}{L}$, so that $\omega_n = n\Delta\omega$. Inserting the Fourier coefficients (9.2.12) into Equation (9.2.11), we have

$$\begin{aligned} f(x) &\sim \sum_{n=-\infty}^{\infty} c_n e^{-in\pi x/L} \\ &= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2L} \int_{-L}^{L} f(\xi) e^{in\pi\tilde{\zeta}/L} d\xi \right) e^{-in\pi x/L} \\ &= \sum_{n=-\infty}^{\infty} \left(\frac{\Delta\omega}{2\pi} \int_{-L}^{L} f(\xi) e^{i\omega_n \tilde{}} d\xi \right) e^{-i\omega_n x}. \end{aligned} \tag{9.3.1}$$

Now, we let L get large, so that $\Delta\omega$ becomes small and ω_n approaches the angular frequency ω. Then,

$$\begin{aligned} f(x) &\sim \lim_{\Delta\omega \to 0, L\to\infty} \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left(\int_{-L}^{L} f(\xi) e^{i\omega_n \tilde{\zeta}} d\xi \right) e^{-i\omega_n x} \Delta\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(\xi) e^{i\omega\zeta} d\xi \right) e^{-i\omega x} d\omega. \end{aligned} \tag{9.3.2}$$

Note

Definitions of the Fourier transform and the inverse Fourier transform.

Looking at this last result, we formally arrive at the definition of the Fourier transform. It is embodied in the inner integral and can be written as

$$F[f] = \hat{f}(\omega) = \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx \tag{9.3.3}$$

This is a generalization of the Fourier coefficients (9.2.12).

Once we know the Fourier transform, $\hat{f}(\omega)$, then we can reconstruct the original function, $f(x)$, using the inverse Fourier transform, which is given by the outer integration,

$$F^{-1}[\hat{f}] = f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{-i\omega x} d\omega. \tag{9.3.4}$$

We note that it can be proven that the Fourier transform exists when $f(x)$ is absolutely integrable, i.e.,

$$\int_{-\infty}^{\infty} |f(x)| dx < \infty.$$

Such functions are said to be L_1.

We combine these results below, defining the Fourier and inverse Fourier transforms and indicating that they are inverse operations of each other. We will then prove the first of the equations, (9.3.7). [The second equation, (9.3.8), follows in a similar way.]

Note

The **Fourier transform** and **inverse Fourier transform** are inverse operations. Defining the Fourier transform as

$$F[f] = \hat{f}(\omega) = \int_{-\infty}^{\infty} f(x) e^{j\omega x} dx. \tag{9.3.5}$$

and the inverse Fourier transform as

$$F^{-1}[\hat{f}] = f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{-i\omega x} d\omega. \tag{9.3.6}$$

then

$$F^{-1}[F[f]] = f(x) \tag{9.3.7}$$

and

$$F[F^{-1}[\hat{f}]] = \hat{f}(\omega). \tag{9.3.8}$$

Proof 9.3.1

The proof is carried out by inserting the definition of the Fourier transform, (9.3.5), into the inverse transform definition, (9.3.6), and then interchanging the orders of integration. Thus, we have

$$\begin{aligned} F^{-1}[F[f]] &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F[f] e^{-i\omega x} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(\xi) e^{i\omega\xi} d\xi \right] e^{-i\omega x} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi) e^{i\omega(\xi - x)} d\xi d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} e^{i\omega(\xi - x)} d\omega \right] f(\xi) d\xi. \end{aligned} \tag{9.3.9}$$

In order to complete the proof, we need to evaluate the inside integral, which does not depend upon $f(x)$. This is an improper integral, so we first define

$$D_\Omega(x) = \int_{-\Omega}^{\Omega} e^{i\omega x} d\omega$$

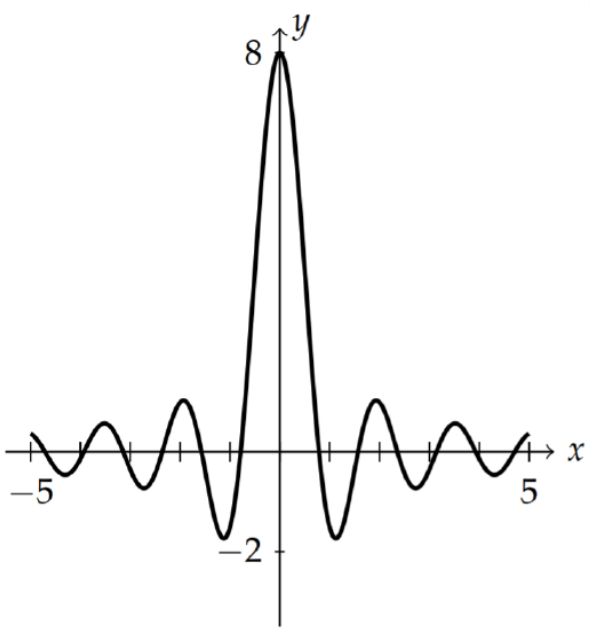
and compute the inner integral as

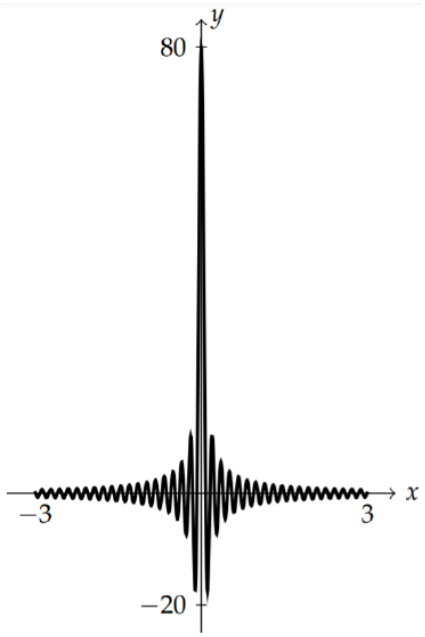
$$\int_{-\infty}^{\infty} e^{i\omega(\xi - x)} d\omega = \lim_{\Omega \to \infty} D_\Omega(\xi - x).$$

We can compute $D_\Omega(x)$. A simple evaluation yields

$$\begin{aligned} D_\Omega(x) &= \int_{-\Omega}^{\Omega} e^{-\omega x} d\omega \\ &= \left. \frac{e^{i\omega x}}{ix} \right|_{-\infty}^{\infty} \\ &= \frac{e^{ix\Omega} - e^{-ix\Omega}}{2ix} \\ &= \frac{2 \sin x\Omega}{x}. \end{aligned} \tag{9.3.10}$$

A plot of this function is in Figure 9.3.1 for $\Omega = 4$. For large Ω the peak grows and the values of $D_\Omega(x)$ for $x \neq 0$ tend to zero as shown in Figure 9.3.2. In fact, as x approaches 0, $D_\Omega(x)$ approaches 2Ω. For $x \neq 0$, the $D_\Omega(x)$ function tends to zero.





Figures 9.3.1 (left) and 9.3.2 (right): Figure 9.3.1 A plot of the function $D_\Omega(x)$ for $\Omega = 4$. Figure 9.3.2: A plot of the function $D_\Omega(x)$ for $\Omega = 40$.

We further note that

$$\lim_{\Omega\to\infty} D_\Omega(x) = 0, \quad x \neq 0,$$

and $\lim_{\Omega\to\infty} D_\Omega(x)$ is infinite at $x = 0$. However, the area is constant for each Ω. In fact,

$$\int_{-\infty}^{\infty} D_\Omega(x)dx = 2\pi.$$

We can show this by recalling the computation in Example 8.5.30,

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi$$

Then,

$$\begin{aligned} \int_{-\infty}^{\infty} D_\Omega(x)dx &= \int_{-\infty}^{\infty} \frac{2\sin x\Omega}{x} dx \\ &= \int_{-\infty}^{\infty} 2\frac{\sin y}{y} dy \\ &= 2\pi \end{aligned} \tag{9.3.11}$$

Another way to look at $D_\Omega(x)$ is to consider the sequence of functions $f_n(x) = \frac{\sin nx}{\pi x}, n = 1, 2, \ldots$ Then we have shown that this sequence of functions satisfies the two properties,

$$\lim_{n\to\infty} f_n(x) = 0, \quad x \neq 0,$$

$$\int_{-\infty}^{\infty} f_n(x)dx = 1.$$

This is a key representation of such generalized functions. The limiting value vanishes at all but one point, but the area is finite.

Such behavior can be seen for the limit of other sequences of functions. For example, consider the sequence of functions

$$f_n(x) = \begin{cases} 0, & |x| > \frac{1}{n}, \\ \frac{n}{2}, & |x| <\leq \text{ frac1n }. \end{cases}$$

This is a sequence of functions as shown in Figure 9.3.3. As $n \to \infty$, we find the limit is zero for $x \neq 0$ and is infinite for $x = 0$. However, the area under each member of the sequences is one. Thus, the limiting function is zero at most points but has area one.

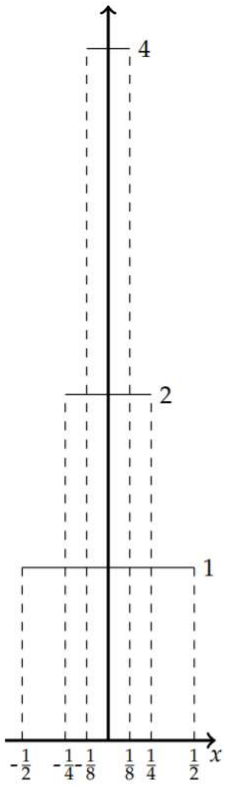


Figure 9.3.3: A plot of the functions $f_n(x)$ for $n = 2, 4, 8$.

The limit is not really a function. It is a generalized function. It is called the Dirac delta function, which is defined by

1. $\delta(x) = 0$ for $x \neq 0$.
2. $\int_{-\infty}^{\infty} \delta(x)dx = 1$.

Before returning to the proof that the inverse Fourier transform of the Fourier transform is the identity, we state one more property of the Dirac delta function, which we will prove in the next section. Namely, we will show that

$$\int_{-\infty}^{\infty} \delta(x-a)f(x)dx = f(a).$$

Returning to the proof, we now have that

$$\int_{-\infty}^{\infty} e^{i\omega(\xi-x)}d\omega = \lim_{\Omega\to\infty} D_\Omega(\xi - x) = 2\pi\delta(\xi - x).$$

Inserting this into (9.3.9), we have

$$\begin{aligned} F^{-1}[F[f]] &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} e^{i\omega(\xi-x)}\,d\omega\right] f(\xi)d\xi. \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\xi - x)f(\xi)d\xi. \\ &= f(x). \end{aligned} \tag{9.3.12}$$

Thus, we have proven that the inverse transform of the Fourier transform of f is f.

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9.4: The Dirac Delta Function

In the last section we introduced the Dirac delta function, $\delta(x)$. As noted above, this is one example of what is known as a generalized function, or a distribution. Dirac had introduced this function in the $1930'$s in his study of quantum mechanics as a useful tool. It was later studied in a general theory of distributions and found to be more than a simple tool used by physicists. The Dirac delta function, as any distribution, only makes sense under an integral.

Note

P. A. M. Dirac (1902-1984) introduced the δ function in his book, *The Principles of Quantum Mechanics*, 4th Ed., Oxford University Press, 1958, originally published in 1930, as part of his orthogonality statement for a basis of functions in a Hilbert space, $<\xi'|\xi''>=c\delta(\xi'-\xi'')$ in the same way we introduced discrete orthogonality using the **Kronecker delta**.

Two properties were used in the last section. First one has that the area under the delta function is one,

$$\int_{-\infty}^{\infty}\delta(x)dx=1.$$

Integration over more general intervals gives

$$\int_a^b \delta(x)dx=\begin{cases}1, & 0\in[a,b],\\ 0, & 0\notin[a,b].\end{cases} \tag{9.4.1}$$

The other property that was used was the sifting property:

$$\int_{-\infty}^{\infty}\delta(x-a)f(x)dx=f(a).$$

This can be seen by noting that the delta function is zero everywhere except at $x=a$. Therefore, the integrand is zero everywhere and the only contribution from $f(x)$ will be from $x=a$. So, we can replace $f(x)$ with $f(a)$ under the integral. Since $f(a)$ is a constant, we have that

$$\begin{aligned}\int_{-\infty}^{\infty}\delta(x-a)f(x)dx &= \int_{-\infty}^{\infty}\delta(x-a)f(a)dx\\ &= f(a)\int_{-\infty}^{\infty}\delta(x-a)dx=f(a).\end{aligned} \tag{9.4.2}$$

Note

Properties of the Dirac δ-function:

$$\int_{-\infty}^{\infty}\delta(x-a)f(x)dx=f(a)$$

$$\int_{-\infty}^{\infty}\delta(ax)dx=\frac{1}{|a|}\int_{-\infty}^{\infty}\delta(y)dy.$$

$$\int_{-\infty}^{\infty}\delta(f(x))dx=\int_{-\infty}^{\infty}\sum_{j=1}^{n}\frac{\delta\left(x-x_j\right)}{\left|f'\left(x_j\right)\right|}dx.$$

(For n simple roots.)

These and other properties are often written outside the integral:

$$\delta(ax) = \frac{1}{|a|}\delta(x)$$

$$\delta(-x) = \delta(x)$$

$$\delta((x-a)(x-b)) = \frac{[\delta(x-a)+\delta(x-a)]}{|a-b|}$$

$$(x-b)) = \frac{[\delta(x-a)+\delta}{|a-b}$$

$$\delta(f(x)) = \sum_j \frac{\delta(x-x_j)}{|f'(x_j)|}$$

for $f(x_j) = 0, f'(x_j) \neq 0$.

Another property results from using a scaled argument, ax. In this case we show that

$$\delta(ax) = |a|^{-1}\delta(x). \tag{9.4.3}$$

As usual, this only has meaning under an integral sign. So, we place $\delta(ax)$ inside an integral and make a substitution $y = ax$:

$$\begin{aligned}\int_{-\infty}^{\infty}\delta(ax)dx &= \lim_{L\to\infty}\int_{-L}^{L}\delta(ax)dx \\ &= \lim_{L\to\infty}\frac{1}{a}\int_{-aL}^{aL}\delta(y)dy.\end{aligned} \tag{9.4.4}$$

If $a > 0$ then

$$\int_{-\infty}^{\infty}\delta(ax)dx = \frac{1}{a}\int_{-\infty}^{\infty}\delta(y)dy.$$

However, if $a < 0$ then

$$\int_{-\infty}^{\infty}\delta(ax)dx = \frac{1}{a}\int_{\infty}^{-\infty}\delta(y)dy = -\frac{1}{a}\int_{-\infty}^{\infty}\delta(y)dy.$$

The overall difference in a multiplicative minus sign can be absorbed into one expression by changing the factor $1/a$ to $1/|a|$. Thus,

$$\int_{-\infty}^{\infty}\delta(ax)dx = \frac{1}{|a|}\int_{-\infty}^{\infty}\delta(y)dy. \tag{9.4.5}$$

Example 9.4.1

Evaluate $\int_{-\infty}^{\infty}(5x+1)\delta(4(x-2))dx$.

Solution

This is a straight forward integration:

$$\int_{-\infty}^{\infty}(5x+1)\delta(4(x-2))dx = \frac{1}{4}\int_{-\infty}^{\infty}(5x+1)\delta(x-2)dx = \frac{11}{4}.$$

The first step is to write $\delta(4(x-2)) = \frac{1}{4}\delta(x-2)$. Then, the final evaluation is given by

$$\frac{1}{4}\int_{-\infty}^{\infty}(5x+1)\delta(x-2)dx = \frac{1}{4}(5(2)+1) = \frac{11}{4}.$$

Even more general than $\delta(ax)$ is the delta function $\delta(f(x))$. The integral of $\delta(f(x))$ can be evaluated depending upon the number of zeros of $f(x)$. If there is only one zero, $f(x_1) = 0$, then one has that

$$\int_{-\infty}^{\infty}\delta(f(x))dx=\int_{-\infty}^{\infty}\frac{1}{|f'(x_1)|}\delta(x-x_1)\,dx.$$

This can be proven using the substitution $y=f(x)$ and is left as an exercise for the reader. This result is often written as

$$\delta(f(x))=\frac{1}{|f'(x_1)|}\delta(x-x_1),$$

again keeping in mind that this only has meaning when placed under an integral.

Example 9.4.2

Evaluate $\int_{-\infty}^{\infty}\delta(3x-2)x^2dx$.

Solution

This is not a simple $\delta(x-a)$. So, we need to find the zeros of $f(x)=3x-2$. There is only one, $x=\frac{2}{3}$. Also, $|f'(x)|=3$. Therefore, we have

$$\int_{-\infty}^{\infty}\delta(3x-2)x^2dx=\int_{-\infty}^{\infty}\frac{1}{3}\delta\left(x-\frac{2}{3}\right)x^2dx=\frac{1}{3}\left(\frac{2}{3}\right)^2=\frac{4}{27}.$$

Note that this integral can be evaluated the long way by using the substitution $y=3x-2$. Then, $dy=3dx$ and $x=(y+2)/3$. This gives

$$\int_{-\infty}^{\infty}\delta(3x-2)x^2dx=\frac{1}{3}\int_{-\infty}^{\infty}\delta(y)\left(\frac{y+2}{3}\right)^2dy=\frac{1}{3}\left(\frac{4}{9}\right)=\frac{4}{27}.$$

More generally, one can show that when $f(x_j)=0$ and $f'(x_j)\neq 0$ for $j=1,2,\ldots,n$, (i.e.; when one has n simple zeros), then

$$\delta(f(x))=\sum_{j=1}^{n}\frac{1}{|f'(x_j)|}\delta(x-x_j)$$

Example 9.4.3

Evaluate $\int_0^{2\pi}\cos x\delta\left(x^2-\pi^2\right)dx$.

Solution

In this case the argument of the delta function has two simple roots. Namely, $f(x)=x^2-\pi^2=0$ when $x=\pm\pi$. Furthermore, $f'(x)=2x$. Therefore, $|f'(\pm\pi)|=2\pi$. This gives

$$\delta\left(x^2-\pi^2\right)=\frac{1}{2\pi}[\delta(x-\pi)+\delta(x+\pi)].$$

Inserting this expression into the integral and noting that $x=-\pi$ is not in the integration interval, we have

$$\begin{aligned}\int_0^{2\pi}\cos x\delta\left(x^2-\pi^2\right)dx &=\frac{1}{2\pi}\int_0^{2\pi}\cos x[\delta(x-\pi)+\delta(x+\pi)]dx\\ &=\frac{1}{2\pi}\cos\pi=-\frac{1}{2\pi}\end{aligned}\tag{9.4.6}$$

Example 9.4.4

Show $H'(x)=\delta(x)$, where the Heaviside function (or, step function) is defined as

$$H(x)=\begin{cases}0, & x<0\\ 1, & x>0\end{cases}$$

and is shown in Figure 9.4.1.

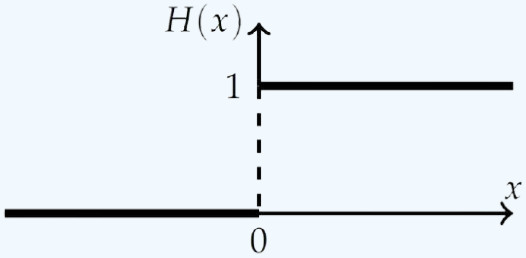


Figure 9.4.1: The Heaviside step function, $H(x)$.

Solution

Looking at the plot, it is easy to see that $H'(x) = 0$ for $x \neq 0$. In order to check that this gives the delta function, we need to compute the area integral. Therefore, we have

$$\int_{-\infty}^{\infty} H'(x)dx = H(x)|_{-\infty}^{\infty} = 1 - 0 = 1.$$

Thus, $H'(x)$ satisfies the two properties of the Dirac delta function.

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9.5: Properties of the Fourier Transform

We now return to the Fourier transform. Before actually computing the Fourier transform of some functions, we prove a few of the properties of the Fourier transform.

First we note that there are several forms that one may encounter for the Fourier transform. In applications functions can either be functions of time, $f(t)$, or space, $f(x)$. The corresponding Fourier transforms are then written as

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt \tag{9.5.1}$$

or

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{ikx} dx \tag{9.5.2}$$

ω is called the angular frequency and is related to the frequency v by $\omega = 2\pi v$. The units of frequency are typically given in Hertz (Hz). Sometimes the frequency is denoted by f when there is no confusion. k is called the wavenumber. It has units of inverse length and is related to the wavelength, λ, by $k = \frac{2\pi}{\lambda}$.

We explore a few basic properties of the Fourier transform and use them in examples in the next section.

1. and

$$F[af] = aF[f].$$

2. **Transform of a Derivative**: $F\left[\frac{df}{dx}\right] = -ik\hat{f}(k)$

   Here we compute the Fourier transform (9.3.5) of the derivative by inserting the derivative in the Fourier integral and using integration by parts.

$$\begin{aligned} F\left[\frac{df}{dx}\right] &= \int_{-\infty}^{\infty} \frac{df}{dx} e^{ikx} dx \\ &= \lim_{L\to\infty} \left[f(x)e^{ikx}\right]_{-L}^{L} - ik\int_{-\infty}^{\infty} f(x)e^{ikx} dx \end{aligned} \tag{9.5.3}$$

   The limit will vanish if we assume that $\lim_{x\to\pm\infty} f(x) = 0$. The last integral is recognized as the Fourier transform of f, proving the given property.

3. **Higher Order Derivatives**: $F\left[\frac{d^n f}{dx^n}\right] = (-ik)^n \hat{f}(k)$

   The proof of this property follows from the last result, or doing several integration by parts. We will consider the case when $n = 2$. Noting that the second derivative is the derivative of $f'(x)$ and applying the last result, we have

$$\begin{aligned} F\left[\frac{d^2 f}{dx^2}\right] &= F\left[\frac{d}{dx} f'\right] \\ &= -ikF\left[\frac{df}{dx}\right] = (-ik)^2 \hat{f}(k). \end{aligned} \tag{9.5.4}$$

   This result will be true if

$$\lim_{x\to\pm\infty} f(x) = 0 \text{ and } \lim_{x\to\pm\infty} f'(x) = 0.$$

   The generalization to the transform of the nth derivative easily follows.

4. **Multiplication by** x : $F[xf(x)] = -i\frac{d}{dk}\hat{f}(k)$

   This property can be shown by using the fact that $\frac{d}{dk}e^{ikx} = ixe^{ikx}$ and the ability to differentiate an integral with respect to a parameter.

$$\begin{aligned} F[xf(x)] &= \int_{-\infty}^{\infty} xf(x)e^{ikx}\,dx \\ &= \int_{-\infty}^{\infty} f(x)\frac{d}{dk}\left(\frac{1}{i}e^{ikx}\right)dx \\ &= -i\frac{d}{dk}\int_{-\infty}^{\infty} f(x)e^{ikx}\,dx \\ &= -i\frac{d}{dk}\hat{f}\,(k). \end{aligned} \tag{9.5.5}$$

This result can be generalized to $F\left[x^n f(x)\right]$ as an exercise.

5. Here we have denoted the Fourier transform pairs using a double arrow as $f(x) \leftrightarrow \hat{f}\,(k)$. These are easily proven by inserting the desired forms into the definition of the Fourier transform (9.3.5), or inverse Fourier transform (9.3.6). The first shift property (???) is shown by the following argument. We evaluate the Fourier transform.

$$F[f(x-a)] = \int_{-\infty}^{\infty} f(x-a)e^{ikx}\,dx.$$

$$\begin{aligned} F[f(x-a)] &= \int_{-\infty}^{\infty} f(y)e^{ik(y+a)}\,dy \\ &= e^{ika}\int_{-\infty}^{\infty} f(y)e^{iky}\,dy \\ &= e^{ika}\hat{f}\,(k). \end{aligned} \tag{9.5.6}$$

The second shift property (???) follows in a similar way.

6. Then, the Fourier transform of the convolution is the product of the Fourier transforms of the individual functions:

$$F[f*g] = \hat{f}\,(k)\hat{g}(k). \tag{9.5.7}$$

Fourier Transform Examples

In this section we will compute the Fourier transforms of several functions.

Example 9.5.1

Find the Fourier transform of a Gaussian, $f(x) = e^{-ax^2/2}$.

Solution

This function, shown in Figure 9.5.1 is called the Gaussian function. It has many applications in areas such as quantum mechanics, molecular theory, probability and heat diffusion. We will compute the Fourier transform of this function and show that the Fourier transform of a Gaussian is a Gaussian. In the derivation we will introduce classic techniques for computing such integrals.

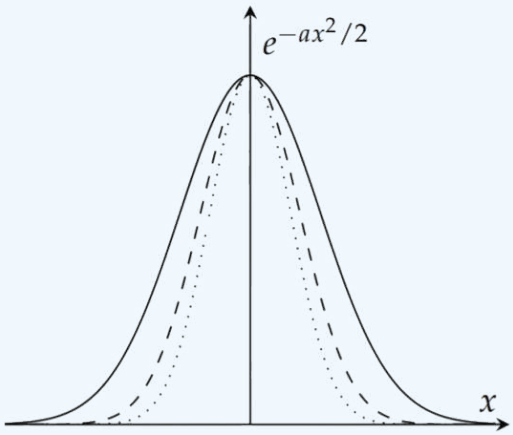


Figure 9.5.1: Plots of the Gaussian function $f(x) = e^{-ax^2/2}$ for $a = 1, 2, 3$.

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We begin by applying the definition of the Fourier transform,

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{ikx}\, dx = \int_{-\infty}^{\infty} e^{-ax^2/2 + ikx}\, dx. \tag{9.5.8}$$

The first step in computing this integral is to complete the square in the argument of the exponential. Our goal is to rewrite this integral so that a simple substitution will lead to a classic integral of the form $\int_{-\infty}^{\infty} e^{\beta y^2}\, dy$, which we can integrate. The completion of the square follows as usual:

$$\begin{aligned} -\frac{a}{2}x^2 + ikx &= -\frac{a}{2}\left[x^2 - \frac{2ik}{a}x\right] \\ &= -\frac{a}{2}\left[x^2 - \frac{2ik}{a}x + \left(-\frac{ik}{a}\right)^2 - \left(-\frac{ik}{a}\right)^2\right] \\ &= -\frac{a}{2}\left(x - \frac{ik}{a}\right)^2 - \frac{k^2}{2a}. \end{aligned} \tag{9.5.9}$$

We now put this expression into the integral and make the substitutions $y = x - \frac{ik}{a}$ and $\beta = \frac{a}{2}$.

$$\begin{aligned} \hat{f}(k) &= \int_{-\infty}^{\infty} e^{-ax^2/2 + ikx}\, dx \\ &= e^{-\frac{k^2}{2a}} \int_{-\infty}^{\infty} e^{-\frac{a}{2}\left(x - \frac{ik}{a}\right)^2}\, dx \\ &= e^{-\frac{k^2}{2a}} \int_{-\infty - \frac{ik}{a}}^{\infty - \frac{ik}{a}} e^{-\beta y^2}\, dy. \end{aligned} \tag{9.5.10}$$

One would be tempted to absorb the $-\frac{ik}{a}$ terms in the limits of integration. However, we know from our previous study that the integration takes place over a contour in the complex plane as shown in Figure 9.5.2.

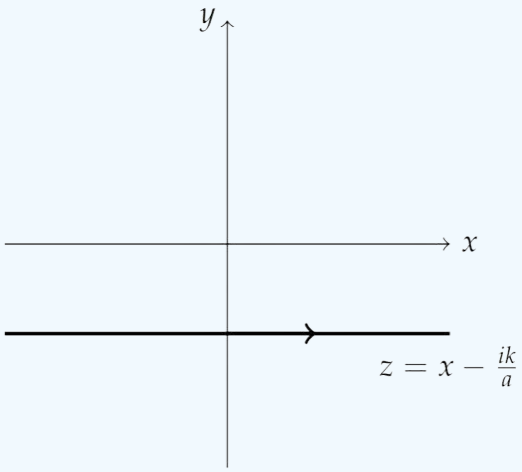


Figure 9.5.2: Simple horizontal contour.

In this case we can deform this horizontal contour to a contour along the real axis since we will not cross any singularities of the integrand. So, we now safely write

$$\hat{f}(k) = e^{-\frac{k^2}{2a}} \int_{-\infty}^{\infty} e^{-\beta y^2}\, dy.$$

The resulting integral is a classic integral and can be performed using a standard trick. Define I by[1]

$$I = \int_{-\infty}^{\infty} e^{-\beta y^2}\, dy.$$

Then,

$$I^2 = \int_{-\infty}^{\infty} e^{-\beta y^2}\,dy \int_{-\infty}^{\infty} e^{-\beta x^2}\,dx.$$

Note that we needed to change the integration variable so that we can write this product as a double integral:

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\beta(x^2+y^2)}\,dx\,dy.$$

This is an integral over the entire xy-plane. We now transform to polar coordinates to obtain

$$\begin{aligned} I^2 &= \int_0^{2\pi}\int_0^{\infty} e^{-\beta r^2}\,r\,dr\,d\theta \\ &= 2\pi\int_0^{\infty} e^{-\beta r^2}\,r\,dr \\ &= -\frac{\pi}{\beta}\left[e^{-\beta r^2}\right]_0^{\infty} = \frac{\pi}{\beta}. \end{aligned} \tag{9.5.11}$$

The final result is gotten by taking the square root, yielding

$$I = \sqrt{\frac{\pi}{\beta}}.$$

We can now insert this result to give the Fourier transform of the Gaussian function:

$$\hat{f}(k) = \sqrt{\frac{2\pi}{a}}e^{-k^2/2a}. \tag{9.5.12}$$

Therefore, we have shown that the Fourier transform of a Gaussian is a Gaussian.

Note

Here we show

$$\int_{-\infty}^{\infty} e^{-\beta y^2}\,dy = \sqrt{\frac{\pi}{\beta}}.$$

Note that we solved the $\beta = 1$ case in Example 5.4.1, so a simple variable transformation $z = \sqrt{\beta}y$ is all that is needed to get the answer. However, it cannot hurt to see this classic derivation again.

Example 9.5.2

Find the Fourier transform of the Box, or Gate, Function,

$$f(x) = \begin{cases} b, & |x| \le a \\ 0, & |x| > a \end{cases}.$$

Solution

This function is called the box function, or gate function. It is shown in Figure 9.5.3.

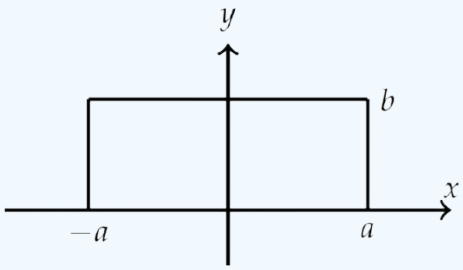


Figure 9.5.3: A plot of the box function in Example 9.5.2.

The Fourier transform of the box function is relatively easy to compute. It is given by

$$\begin{aligned}\hat{f}(k) &= \int_{-\infty}^{\infty} f(x)e^{ikx}dx \\ &= \int_{-a}^{a} be^{ikx}dx \\ &= \frac{b}{ik}e^{ikx}\Big|_{-a}^{a} \\ &= \frac{2b}{k}\sin ka. \end{aligned} \tag{9.5.13}$$

We can rewrite this as

$$\hat{f}(k) = 2ab\frac{\sin ka}{ka} \equiv 2ab\text{sinc } ka.$$

Here we introduced the sinc function

$$\text{sinc } x = \frac{sinx}{x}.$$

A plot of this function is shown in Figure 9.5.4. This function appears often in signal analysis and it plays a role in the study of diffraction.

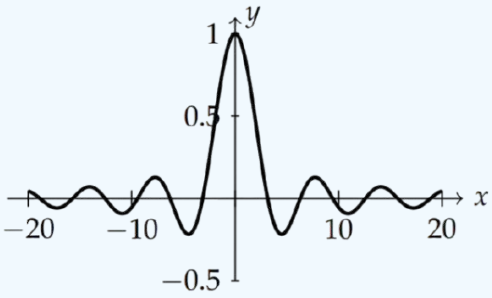


Figure 9.5.4: A plot of the Fourier transform of the box function in Example 9.5.2. This is the general shape of the sinc function.

We will now consider special limiting values for the box function and its transform. This will lead us to the Uncertainty Principle for signals, connecting the relationship between the localization properties of a signal and its transform.

1. $a \to \infty$ and b fixed
   In this case, as a gets large the box function approaches the constant function $f(x) = b$. At the same time, we see that the Fourier transform approaches a Dirac delta function. We had seen this function earlier when we first defined the Dirac delta function. Compare Figure 9.5.4 with Figure 9.3.1. In fact, $\hat{f}(k) = bD_a(k)$. [Recall the definition of $D_\Omega(x)$ in Equation (9.3.10).] So, in the limit we obtain $\hat{f}(k) = 2\pi b\delta(k)$. This limit implies fact that the Fourier transform of $f(x) = 1$ is $\hat{f}(k) = 2\pi\delta(k)$. As the width of the box becomes wider, the Fourier transform becomes more localized. In fact, we have arrived at the important result that

$$\int_{-\infty}^{\infty} e^{ikx} = 2\pi\delta(k). \tag{9.5.14}$$

2. $b \to \infty, a \to 0$, and $2ab = 1$.
   In this case the box narrows and becomes steeper while maintaining a constant area of one. This is the way we had found a representation of the Dirac delta function previously. The Fourier transform approaches a constant in this limit. As a approaches zero, the sinc function approaches one, leaving $\hat{f}(k) \to 2ab = 1$. Thus, the Fourier transform of the Dirac delta function is one. Namely, we have

$$\int_{-\infty}^{\infty} \delta(x)e^{ikx} = 1. \tag{9.5.15}$$

   In this case we have that the more localized the function $f(x)$ is, the more spread out the Fourier transform, $\hat{f}(k)$, is. We will summarize these notions in the next item by relating the widths of the function and its Fourier transform.

3. The Uncertainty Principle, $\Delta x \Delta k = 4\pi$.

The widths of the box function and its Fourier transform are related as we have seen in the last two limiting cases. It is natural to define the width, Δx of the box function as

$$\Delta x = 2a.$$

The width of the Fourier transform is a little trickier. This function actually extends along the entire k-axis. However, as $\hat{f}(k)$ became more localized, the central peak in Figure 9.5.4 became narrower. So, we define the width of this function, Δk as the distance between the first zeros on either side of the main lobe as shown in Figure 9.5.5. This gives

$$\Delta k = \frac{2\pi}{a}.$$

Combining these two relations, we find that

$$\Delta x \Delta k = 4\pi.$$

Thus, the more localized a signal, the less localized its transform and vice versa. This notion is referred to as the Uncertainty Principle. For general signals, one needs to define the effective widths more carefully, but the main idea holds:

$$\Delta x \Delta k \geq c > 0.$$

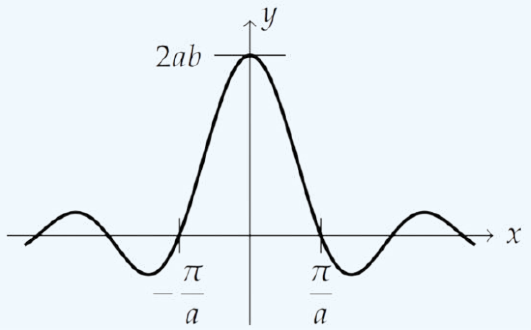


Figure 9.5.5: The width of the function $2ab\frac{\sin ka}{ka}$ is defined as the distance between the smallest magnitude zeros.

We now turn to other examples of Fourier transforms.

Example 9.5.3

Find the Fourier transform of $f(x) = \begin{cases} e^{-ax}, & x \geq 0 \\ 0, & x < 0 \end{cases}, a > 0$

Solution

The Fourier transform of this function is

$$\begin{aligned} \hat{f}(k) &= \int_{-\infty}^{\infty} f(x) e^{ikx} dx \\ &= \int_{0}^{\infty} e^{ikx - ax} dx \\ &= \frac{1}{a - ik}. \end{aligned} \tag{9.5.16}$$

Next, we will compute the inverse Fourier transform of this result and recover the original function.

Note

More formally, the uncertainty principle for signals is about the relation between duration and bandwidth, which are defined by $\Delta t = \frac{\|tf\|_2}{\|f\|_2}$ and $\Delta\omega = \frac{\|\omega\hat{f}\|_2}{\|f\|_2}$, respectively, where $\|f\|_2 = \int_{-\infty}^{\infty} |f(t)|^2 dt$ and $\|\hat{f}\|_2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 d\omega$. Under appropriate conditions, one can prove that $\Delta t \Delta\omega \geq \frac{1}{2}$. Equality holds for Gaussian signals. Werner Heisenberg (1901-1976) introduced

the uncertainty principle into quantum physics in 1926 , relating uncertainties in the position (Δx) and momentum (Δp_x) of particles. In this case, $\Delta x \Delta p_x \geq \frac{1}{2}\hbar$. Here, the uncertainties are defined as the positive square roots of the quantum mechanical variances of the position and momentum.

Example 9.5.4

Find the inverse Fourier transform of $\hat{f}(k) = \frac{1}{a-ik}$.

Solution

The inverse Fourier transform of this function is

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k) e^{-ikx}\, dk = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-ikx}}{a-ik} dk.$$

This integral can be evaluated using contour integral methods. We evaluate the integral

$$I = \int_{-\infty}^{\infty} \frac{e^{-ixz}}{a-iz} dz,$$

using Jordan's Lemma from Section 8.5.8. According to Jordan's Lemma, we need to enclose the contour with a semicircle in the upper half plane for $x < 0$ and in the lower half plane for $x > 0$ as shown in Figure 9.5.6.

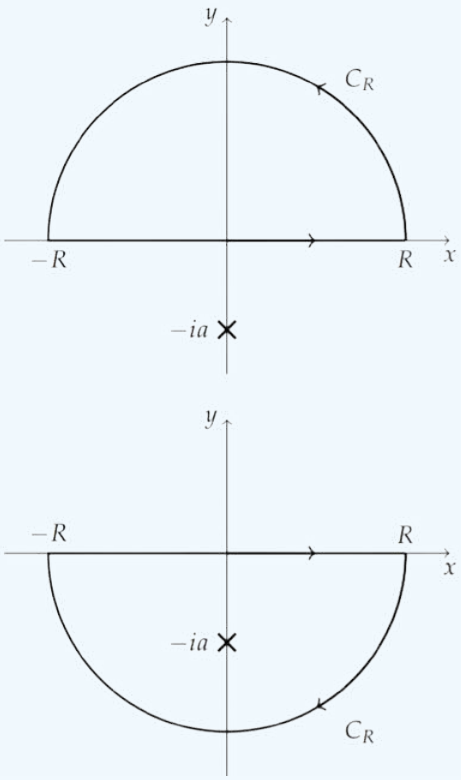


Figure 9.5.6: Contours for inverting $\hat{f}(k) = \frac{1}{a-ik}$.

The integrations along the semicircles will vanish and we will have

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-ikx}}{a-ik}dk \\ &= \pm\frac{1}{2\pi}\oint_C \frac{e^{-ixz}}{a-iz}dz \\ &= \begin{cases} 0, & x<0 \\ -\frac{1}{2\pi}2\pi i\,\text{Res}[z=-ia], & x<0 \end{cases} \\ &= \begin{cases} 0, & x<0 \\ e^{-ax}, & x>0 \end{cases} \end{aligned} \tag{9.5.17}$$

Note that without paying careful attention to Jordan's Lemma one might not retrieve the function from the last example.

Example 9.5.5

Find the inverse Fourier transform of $\hat{f}(\omega) = \pi\delta(\omega+\omega_0) + \pi\delta(\omega-\omega_0)$.

Solution

We would like to find the inverse Fourier transform of this function. Instead of carrying out any integration, we will make use of the properties of Fourier transforms. Since the transforms of sums are the sums of transforms, we can look at each term individually. Consider $\delta(\omega-\omega_0)$. This is a shifted function. From the shift theorems in Equations (???)-(???) we have the Fourier transform pair

$$e^{i\omega_0 t}f(t) \leftrightarrow \hat{f}(\omega-\omega_0).$$

Recalling from Example 9.5.2 that

$$\int_{-\infty}^{\infty} e^{i\omega t}dt = 2\pi\delta(\omega),$$

we have from the shift property that

$$F^{-1}[\delta(\omega-\omega_0)] = \frac{1}{2\pi}e^{-i\omega_0 t}.$$

The second term can be transformed similarly. Therefore, we have

$$F^{-1}\left[\pi\delta(\omega+\omega_0)+\pi\delta(\omega-\omega_0\right] = \frac{1}{2}e^{i\omega_0 t}+\frac{1}{2}e^{-i\omega_0 t} = \cos\omega_0 t.$$

Example 9.5.6

Find the Fourier transform of the finite wave train.

$$f(t) = \begin{cases} \cos\omega_0 t, & |t|\le a \\ 0, & |t|>a \end{cases}.$$

Solution

For the last example, we consider the finite wave train, which will reappear in the last chapter on signal analysis. In Figure 9.5.7 we show a plot of this function.

A straight forward computation gives

$$\begin{aligned}\hat{f}(\omega) &= \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt \\ &= \int_{-a}^{a} \left[\cos\omega_0 t + i \sin\omega_0 t\right] e^{i\omega t} dt \\ &= \int_{-a}^{a} \cos\omega_0 t \cos\omega t dt + i \int_{-a}^{a} \sin\omega_0 t \sin\omega t dt \\ &= \frac{1}{2} \int_{-a}^{a} \left[\cos((\omega+\omega_0)\, t) + \cos((\omega-\omega_0)\, t)\right] dt \\ &= \frac{\sin((\omega+\omega_0)\, a)}{\omega+\omega_0} + \frac{\sin((\omega-\omega_0)\, a)}{\omega-\omega_0}. \end{aligned} \tag{9.5.18}$$

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9.6: The Convolution Operation

In the list of properties of the Fourier transform, we defined the convolution of two functions, $f(x)$ and $g(x)$ to be the integral

$$(f*g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)dt. \tag{9.6.1}$$

In some sense one is looking at a sum of the overlaps of one of the functions and all of the shifted versions of the other function. The German word for convolution is faltung, which means "folding" and in old texts this is referred to as the **Faltung Theorem**. In this section we will look into the convolution operation and its Fourier transform.

Before we get too involved with the convolution operation, it should be noted that there are really two things you need to take away from this discussion. The rest is detail. First, the convolution of two functions is a new functions as defined by (9.6.1) when dealing wit the Fourier transform. The second and most relevant is that the Fourier transform of the convolution of two functions is the product of the transforms of each function. The rest is all about the use and consequences of these two statements. In this section we will show how the convolution works and how it is useful.

Note

The convolution is commutative.

First, we note that the convolution is commutative: $f*g = g*f$. This is easily shown by replacing $x-t$ with a new variable, $y = x-t$ and $dy = -dt$.

$$\begin{aligned}(g*f)(x) &= \int_{-\infty}^{\infty} g(t)f(x-t)dt \\ &= -\int_{\infty}^{-\infty} g(x-y)f(y)dy \\ &= \int_{-\infty}^{\infty} f(y)g(x-y)dy \\ &= (f*g)(x).\end{aligned} \tag{9.6.2}$$

The best way to understand the folding of the functions in the convolution is to take two functions and convolve them. The next example gives a graphical rendition followed by a direct computation of the convolution. The reader is encouraged to carry out these analyses for other functions.

Example 9.6.1

Graphical Convolution of the box function and a triangle function.

Solution

In order to understand the convolution operation, we need to apply it to specific functions. We will first do this graphically for the box function

$$f(x) = \begin{cases} 1, & |x| \le 1, \\ 0, & |x| > 1 \end{cases}$$

and the triangular function

$$g(x) = \begin{cases} x, & 0 \le x \le 1, \\ 0, & \text{otherwise} \end{cases}$$

as shown in Figure 9.6.1.

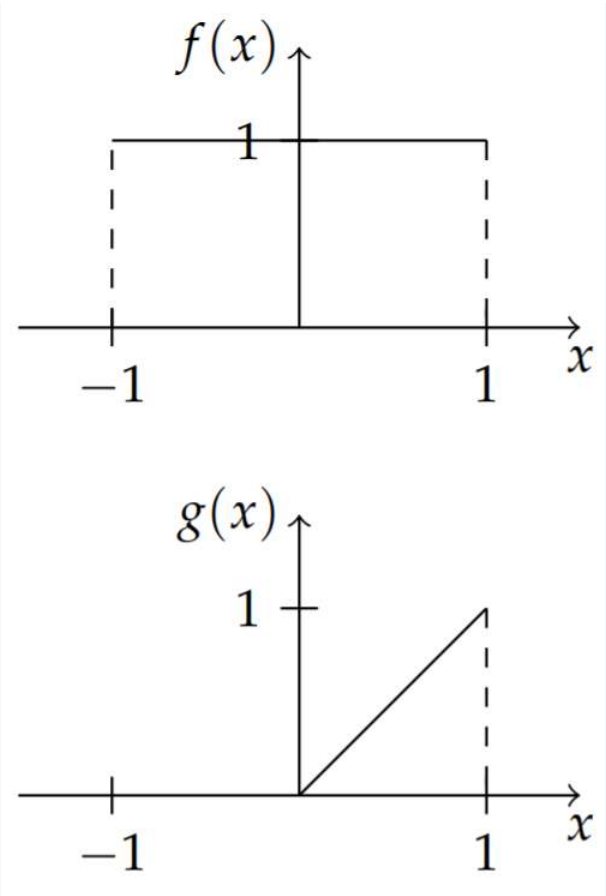


Figure 9.6.1: A plot of the box function $f(x)$ and the triangle function $g(x)$.

Next, we determine the contributions to the integrand. We consider the shifted and reflected function $g(t-x)$ in Equation (9.6.1) for various values of t. For $t=0$, we have $g(x-0)=g(-x)$. This function is a reflection of the triangle function, $g(x)$, as shown in Figure 9.6.2.

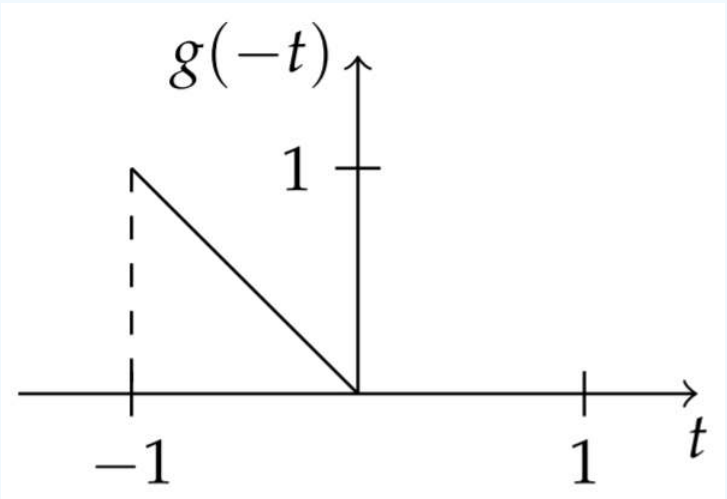


Figure 9.6.2: A plot of the reflected triangle function, $g(-t)$.

We then translate the triangle function performing horizontal shifts by t. In Figure 9.6.3 we show such a shifted and reflected $g(x)$ for $t=2$, or $g(2-x)$.

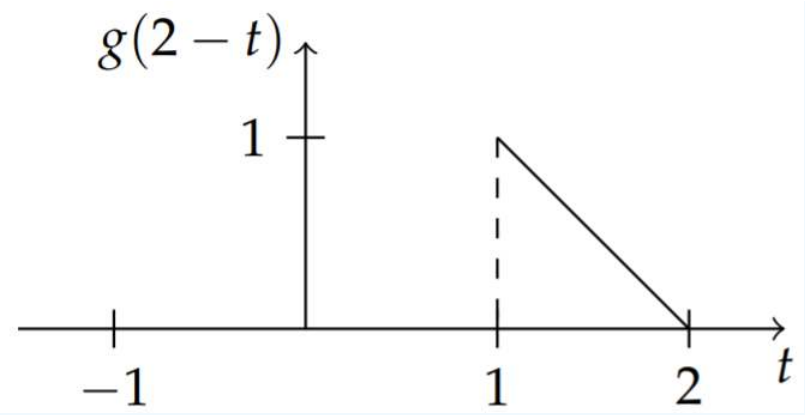


Figure 9.6.3: A plot of the reflected triangle function shifted by 2 units, $g(2-t)$.

In Figure 9.6.3 we show several plots of other shifts, $g(x-t)$, superimposed on $f(x)$.

The integrand is the product of $f(t)$ and $g(x-t)$ and the integral of the product $f(t)g(x-t)$ is given by the sum of the shaded areas for each value of x.

In the first plot of Figure 9.6.4 the area is zero, as there is no overlap of the functions. Intermediate shift values are displayed in the other plots in Figure 9.6.4. The value of the convolution at x is shown by the area under the product of the two functions for each value of x.

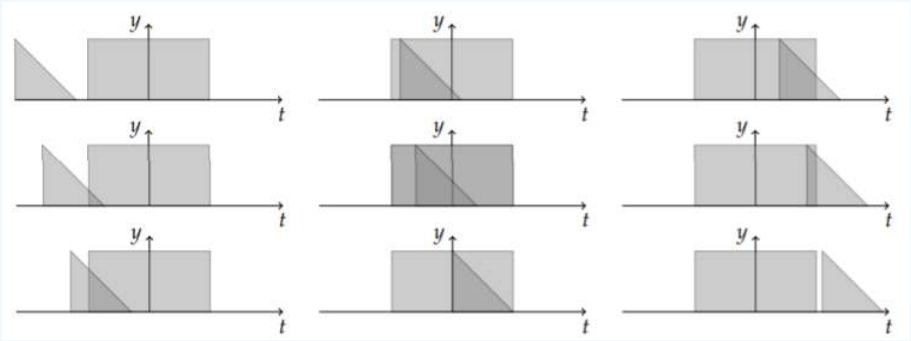


Figure 9.6.4: A plot of the box and triangle functions with the overlap indicated by the shaded area.

Plots of the areas of the convolution of the box and triangle functions for several values of x are given in Figure 9.6.3. We see that the value of the convolution integral builds up and then quickly drops to zero as a function of x. In Figure 9.6.5 the values of these areas is shown as a function of x.

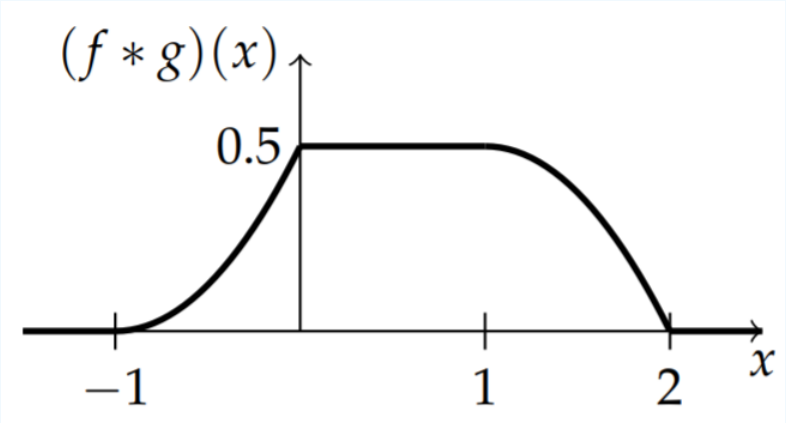


Figure 9.6.5: Copy and Paste Caption here. (Copyright; author via source)

The plot of the convolution in Figure 9.6.5 is not easily determined using the graphical method. However, we can directly compute the convolution as shown in the next example.

Example 9.6.2

Analytically find the convolution of the box function and the triangle function.

Solution

The nonvanishing contributions to the convolution integral are when both $f(t)$ and $g(x-t)$ do not vanish. $f(t)$ is nonzero for $|t| \leq 1$, or $-1 \leq t \leq 1$. $g(x-t)$ is nonzero for $0 \leq x-t \leq 1$, or $x-1 \leq t \leq x$. These two regions are shown in Figure 9.6.6. On this region, $f(t)g(x-t) = x-t$.

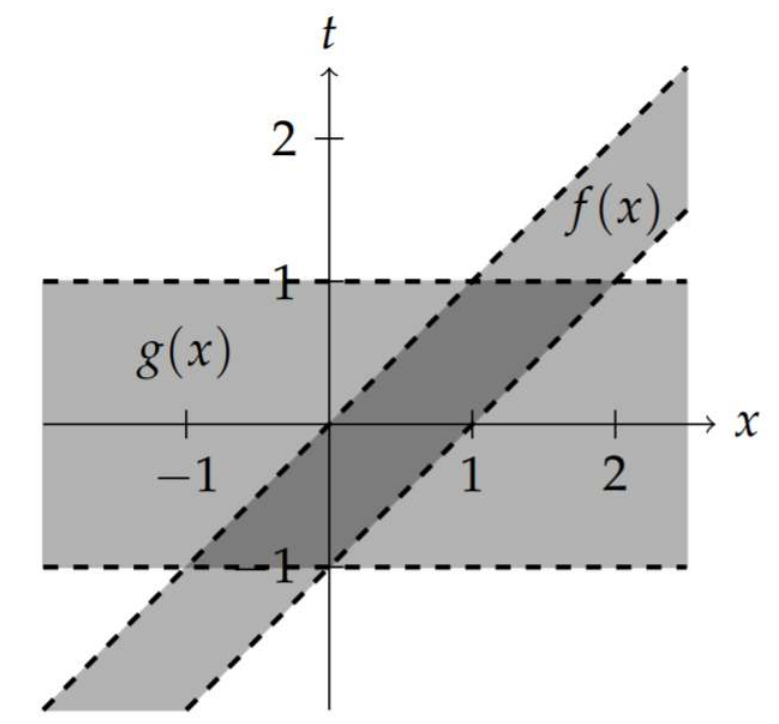


Figure 9.6.6: Intersection of the support of $g(x)$ and $f(x)$.

Isolating the intersection in Figure 9.6.7, we see in Figure 9.6.7 that there are three regions as shown by different shadings. These regions lead to a piecewise defined function with three different branches of nonzero values for $-1 < x < 0$, $0 < x < 1$, and $1 < x < 2$.

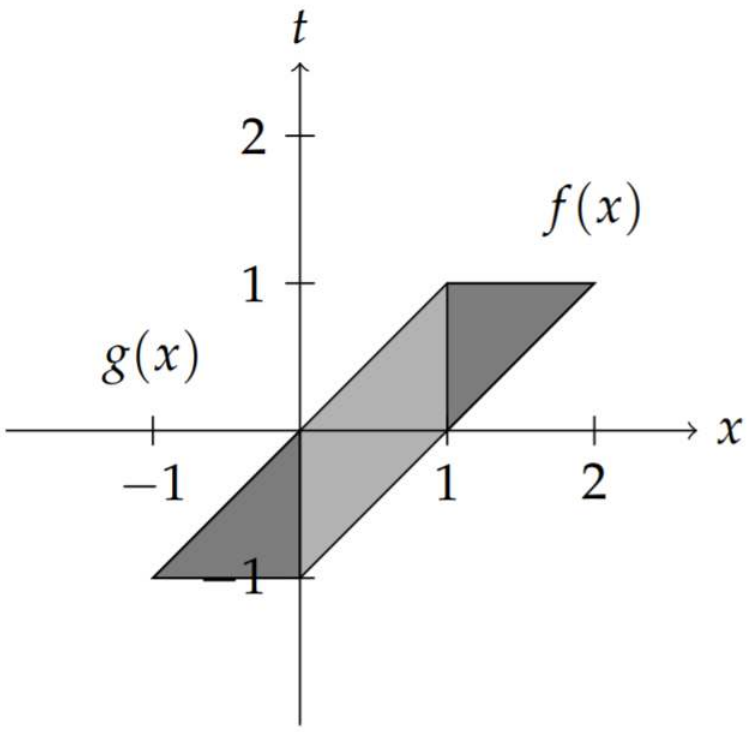


Figure 9.6.7: Intersection of the support of $g(x)$ and $f(x)$ showing the integration regions.

The values of the convolution can be determined through careful integration. The resulting integrals are given as

$$\begin{aligned}(f*g)(x) &= \int_{-\infty}^{\infty} f(t)g(x-t)dt \\ &= \begin{cases} \int_{-1}^{x}(x-t)dt, & -1<x<0 \\ \int_{x-1}^{x}(x-t)dt, & 0<x<1 \\ \int_{x-1}^{1}(x-t)dt, & 1<x<2 \end{cases} \\ &= \begin{cases} \frac{1}{2}(x+1)^2, & -1<x<0 \\ \frac{1}{2}, & 0<x<1 \\ \frac{1}{2}[1-(x-1)^2] & 1<x<2 \end{cases} \end{aligned} \tag{9.6.3}$$

A plot of this function is shown in Figure 9.6.5.

Convolution Theorem for Fourier Transforms

In this section we compute the Fourier transform of the convolution integral and show that the Fourier transform of the convolution is the product of the transforms of each function,

$$F[f*g] = \hat{f}(k)\hat{g}(k). \tag{9.6.4}$$

First, we use the definitions of the Fourier transform and the convolution to write the transform as

$$\begin{aligned} F[f*g] &= \int_{-\infty}^{\infty}(f*g)(x)e^{ikx}dx \\ &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}f(t)g(x-t)dt\right)e^{ikx}dx \\ &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}g(x-t)e^{ikx}dx\right)f(t)dt. \end{aligned} \tag{9.6.5}$$

We now substitute $y = x - t$ on the inside integral and separate the integrals:

$$\begin{aligned} F[f*g] &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} g(x-t)e^{ikx}\,dx\right) f(t)dt \\ &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} g(y)e^{ik(y+t)}\,dy\right) f(t)dt \\ &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} g(y)e^{iky}\,dy\right) f(t)e^{ikt}dt \\ &= \left(\int_{-\infty}^{\infty} f(t)e^{ikt}\,dt\right)\left(\int_{-\infty}^{\infty} g(y)e^{iky}\,dy\right) \end{aligned} \tag{9.6.6}$$

We see that the two integrals are just the Fourier transforms of f and g. Therefore, the Fourier transform of a convolution is the product of the Fourier transforms of the functions involved:

$$F[f*g] = \hat{f}\,(k)\hat{g}(k).$$

Example 9.6.3

Compute the convolution of the box function of height one and width two with itself.

Solution

Let $\hat{f}\,(k)$ be the Fourier transform of $f(x)$. Then, the Convolution Theorem says that $F[f*f](k) = \hat{f}^{\,2}(k)$, or

$$(f*f)(x) = F^{-1}\left[\hat{f}^{\,2}(k)\right].$$

For the box function, we have already found that

$$\hat{f}\,(k) = \frac{2}{k}\sin k.$$

So, we need to compute

$$\begin{aligned} (f*f)(x) &= F^{-1}\left[\frac{4}{k^2}\sin^2 k\right] \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{4}{k^2}\sin^2 k\right)e^{-ikx}\,dk. \end{aligned} \tag{9.6.7}$$

One way to compute this integral is to extend the computation into the complex k-plane. We first need to rewrite the integrand. Thus,

$$\begin{aligned} (f*f)(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{4}{k^2}\sin^2 k e^{-ikx}\,dk \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{k^2}[1-\cos 2k]e^{-ikx}\,dk \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{k^2}\left[1-\frac{1}{2}\left(e^{ik}+e^{-ik}\right)\right]e^{-ikx}\,dk \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{k^2}\left[e^{-ikx}-\frac{1}{2}\left(e^{-i(1-k)}+e^{-i(1+k)}\right)\right]dk. \end{aligned} \tag{9.6.8}$$

We can compute the above integrals if we know how to compute the integral

$$I(y) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{e^{-iky}}{k^2}\,dk.$$

Then, the result can be found in terms of $I(y)$ as

$$(f*f)(x) = I(x) - \frac{1}{2}[I(1-k)+I(1+k)].$$

We consider the integral

$$\oint_C \frac{e^{-iyz}}{\pi z^2} dz$$

over the contour in Figure 9.6.8.

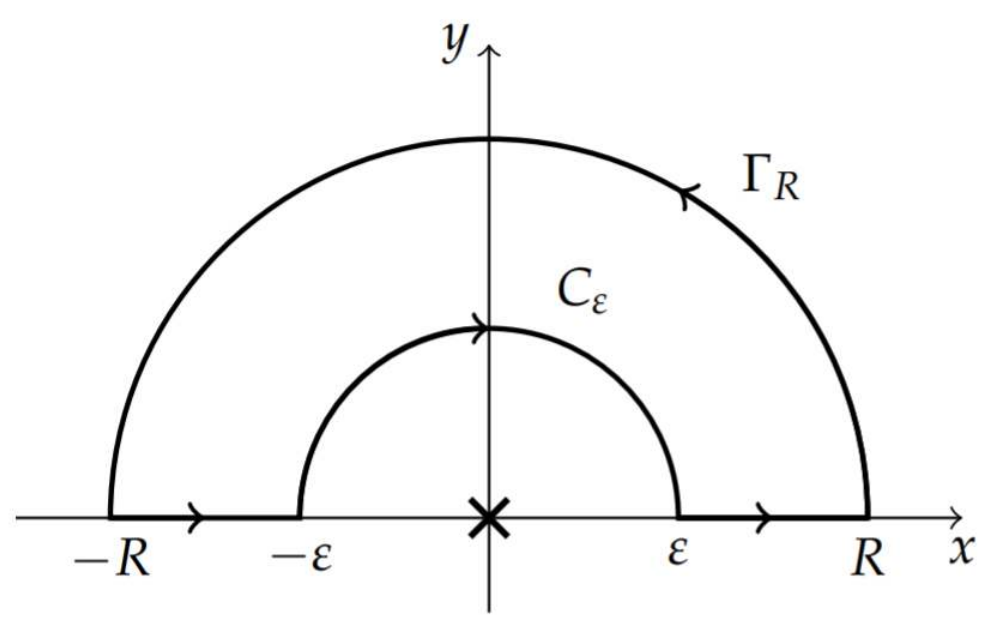


Figure 9.6.8: Contour for computing $P\int_{-\infty}^{\infty} \frac{e^{-iyz}}{\pi z^2} dz$.

We can see that there is a double pole at $z = 0$. The pole is on the real axis. So, we will need to cut out the pole as we seek the value of the principal value integral.

Recall from Chapter 8 that

$$\oint_{C_R} \frac{e^{-iyz}}{\pi z^2} dz = \int_{\Gamma_R} \frac{e^{-iyz}}{\pi z^2} dz + \int_{-R}^{-\epsilon} \frac{e^{-iyz}}{\pi z^2} dz + \int_{C_\epsilon} \frac{e^{-iyz}}{\pi z^2} dz + \int_{\epsilon}^{R} \frac{e^{-iyz}}{\pi z^2} dz.$$

The integral $\oint_{C_R} \frac{e^{-iyz}}{\pi z^2} dz$ vanishes since there are no poles enclosed in the contour! The sum of the second and fourth integrals gives the integral we seek as $\epsilon \to 0$ and $R \to \infty$. The integral over Γ_R will vanish as R gets large according to Jordan's Lemma provided $y < 0$. That leaves the integral over the small semicircle.

As before, we can show that

$$\lim_{\epsilon \to 0} \int_{\mathcal{C}_\epsilon} f(z) dz = -\pi i \operatorname{Res}[f(z); z = 0].$$

Therefore, we find

$$I(y) = P\int_{-\infty}^{\infty} \frac{e^{-iyz}}{\pi z^2} dz = \pi i \operatorname{Res}\left[\frac{e^{-iyz}}{\pi z^2}; z = 0\right].$$

A simple computation of the reside gives $I(y) = -y$, for $y < 0$.

When $y > 0$, we need to close the contour in the lower half plane in order to apply Jordan's Lemma. Carrying out the computation, one finds $I(y) = y$, for $y > 0$. Thus,

$$I(y) = \begin{cases} -y, & y > 0, \\ y, & y < 0, \end{cases} \tag{9.6.9}$$

We are now ready to finish the computation of the convolution. We have to combine the integrals $I(y), I(y+1),$ and $I(y-1)$, since $(f * f)(x) = I(x) - \frac{1}{2}[I(1-k) + I(1+k)]$. This gives different results in four intervals:

$$
\begin{aligned}
(f*f)(x) &= x - \frac{1}{2}[(x-2)+(x+2)] = 0, \quad x < -2, \\
&= x - \frac{1}{2}[(x-2)-(x+2)] = 2 + x \quad -2 < x < 0, \\
&= -x - \frac{1}{2}[(x-2)-(x+2)] = 2 - x, \quad 0 < x < 2, \\
&= -x - \frac{1}{2}[-(x-2)-(x+2)] = 0, \quad x > 2.
\end{aligned} \tag{9.6.10}
$$

A plot of this solution is the triangle function,

$$
(f*f)(x) = \begin{cases} 0, & x < -2 \\ 2+x, & -2 < x < 0 \\ 2-x, & 0 < x < 2 \\ 0, & x > 2 \end{cases} \tag{9.6.11}
$$

which was shown in the last example.

Example 9.6.4

Find the convolution of the box function of height one and width two with itself using a direct computation of the convolution integral.

Solution

The nonvanishing contributions to the convolution integral are when both $f(t)$ and $f(x-t)$ do not vanish. $f(t)$ is nonzero for $|t| \le 1$, or $-1 \le t \le 1$. $f(x-t)$ is nonzero for $|x-t| \le 1$, or $x-1 \le t \le x+1$. These two regions are shown in Figure 9.6.10. On this region, $f(t)g(x-t) = 1$.

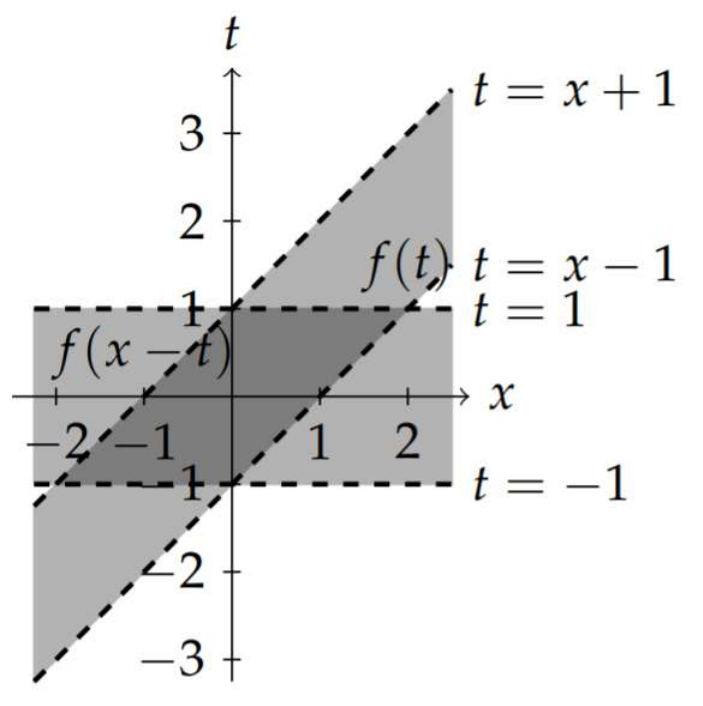


Figure 9.6.9: Plot of the regions of support for $f(t)$ and $f(x-t)$.

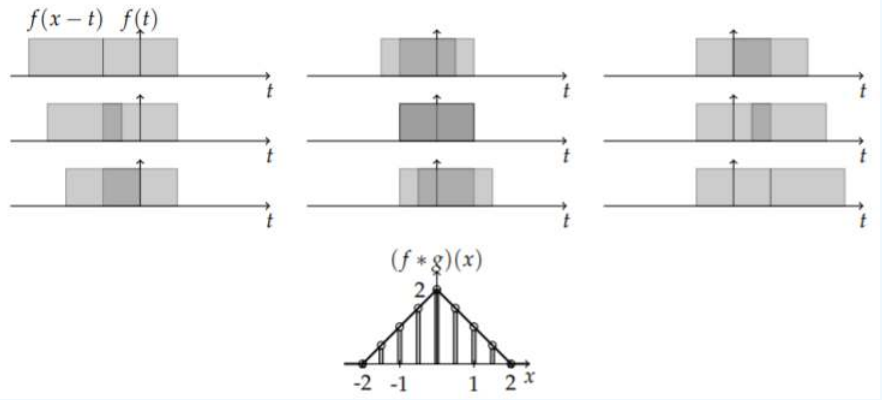


Figure 9.6.10: A plot of the convolution of a box function with itself. The areas of the overlaps of as $f(x-t)$ is translated across $f(t)$ are shown as well. The result is the triangular function.

Thus, the nonzero contributions to the convolution are

$$(f*f)(x) = \begin{cases} \int_{-1}^{x+1} dt, & 0 \le x \le 2, \\ \int_{x-1}^{1} dt, & -2 \le x \le 0, \end{cases} = \begin{cases} 2+x, & 0 \le x \le 2, \\ 2-x, & -2 \le x \le 0. \end{cases}$$

Once again, we arrive at the triangle function.

In the last section we showed the graphical convolution. For completeness, we do the same for this example. In Figure 9.6.10 we show the results. We see that the convolution of two box functions is a triangle function.

Example 9.6.5

Show the graphical convolution of the box function of height one and width two with itself.

Let's consider a slightly more complicated example, the convolution of two Gaussian functions.

Example 9.6.6

Convolution of two Gaussian functions $f(x) = e^{-ax^2}$.

Solution

In this example we will compute the convolution of two Gaussian functions with different widths. Let $f(x) = e^{-ax^2}$ and $g(x) = e^{-bx^2}$. A direct evaluation of the integral would be to compute

$$(f*g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)dt = \int_{-\infty}^{\infty} e^{-at^2-b(x-t)^2}\,dt.$$

This integral can be rewritten as

$$(f*g)(x) = e^{-bx^2}\int_{-\infty}^{\infty} e^{-(a+b)t^2+2bxt}dt.$$

One could proceed to complete the square and finish carrying out the integration. However, we will use the Convolution Theorem to evaluate the convolution and leave the evaluation of this integral to Problem 12.

Recalling the Fourier transform of a Gaussian from Example 9.5.1, we have

$$\hat{f}\,(k) = F[e^{-ax^2}] = \sqrt{\frac{\pi}{a}}e^{-k^2/4a} \tag{9.6.12}$$

and

$$\hat{g}(k) = F[e^{-bx^2}] = \sqrt{\frac{\pi}{b}}e^{-k^2/4b}.$$

Denoting the convolution function by $h(x) = (f * g)(x)$, the Convolution Theorem gives

$$\hat{h}(k) = \hat{f}(k)\hat{g}(k) = \frac{\pi}{\sqrt{ab}} e^{-k^2/4a} e^{-k^2/4b}.$$

This is another Gaussian function, as seen by rewriting the Fourier transform of $h(x)$ as

$$\hat{h}(k) = \frac{\pi}{\sqrt{ab}} e^{-\frac{1}{4}\left(\frac{1}{a}+\frac{1}{b}\right)k^2} = \frac{\pi}{\sqrt{ab}} e^{-\frac{a+b}{4ab}k^2}. \tag{9.6.13}$$

In order to complete the evaluation of the convolution of these two Gaussian functions, we need to find the inverse transform of the Gaussian in Equation (9.6.13). We can do this by looking at Equation (9.6.12). We have first that

$$F^{-1}\left[\sqrt{\frac{\pi}{a}} e^{-k^2/4a}\right] = e^{-ax^2}.$$

Moving the constants, we then obtain

$$F^{-1}[e^{-k^2/4a}] = \sqrt{\frac{a}{\pi}} e^{-ax^2}.$$

We now make the substitution $\alpha = \frac{1}{4a}$,

$$F^{-1}[e^{-\alpha k^2}] = \sqrt{\frac{1}{4\pi\alpha}} e^{-x^2/4\alpha}.$$

This is in the form needed to invert (9.6.13). Thus, for $\alpha = \frac{a+b}{4ab}$ we find

$$(f * g)(x) = h(x) = \sqrt{\frac{\pi}{a+b}} e^{-\frac{ab}{a+b}x^2}.$$

Application to Signal Analysis

There are many applications of the convolution operation. One of these areas is the study of analog signals. An analog signal is a continuous signal and may contain either a finite, or continuous, set of frequencies. Fourier transforms can be used to represent such signals as a sum over the frequency content of these signals. In this section we will describe how convolutions can be used in studying signal analysis.

The first application is filtering. For a given signal there might be some noise in the signal, or some undesirable high frequencies. For example, a device used for recording an analog signal might naturally not be able to record high frequencies. Let $f(t)$ denote the amplitude of a given analog signal and $\hat{f}(\omega)$ be the Fourier transform of this signal such the example provided in Figure 9.6.11. Recall that the Fourier transform gives the frequency content of the signal.

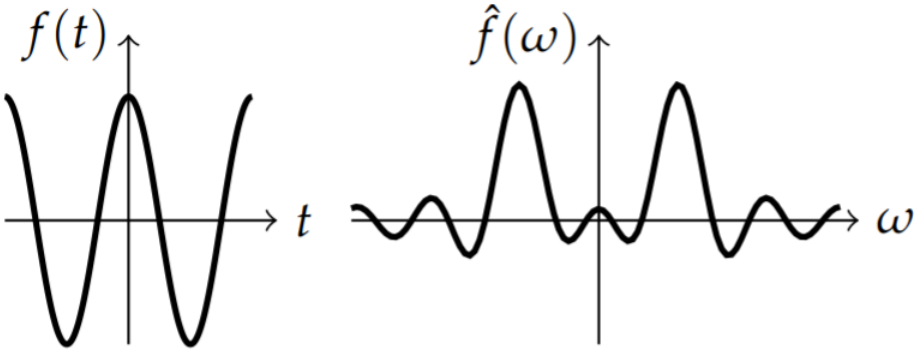


Figure 9.6.11: Schematic plot of a signal $f(t)$ and its Fourier transform $\hat{f}(\omega)$.

There are many ways to filter out unwanted frequencies. The simplest would be to just drop all of the high (angular) frequencies. For example, for some cutoff frequency ω_0 frequencies $|\omega| > \omega_0$ will be removed. The Fourier transform of the filtered signal would then be zero for $|\omega| > \omega_0$. This could be accomplished by multiplying the Fourier transform of the signal by a function that vanishes for $|\omega| > \omega_0$. For example, we could use the gate function

$$p_{\omega_0}(\omega) = \begin{cases} 1, & |\omega| \leq \omega_0 \\ 0, & |\omega| > \omega_0 \end{cases}, \tag{9.6.14}$$

as shown in Figure 9.6.12

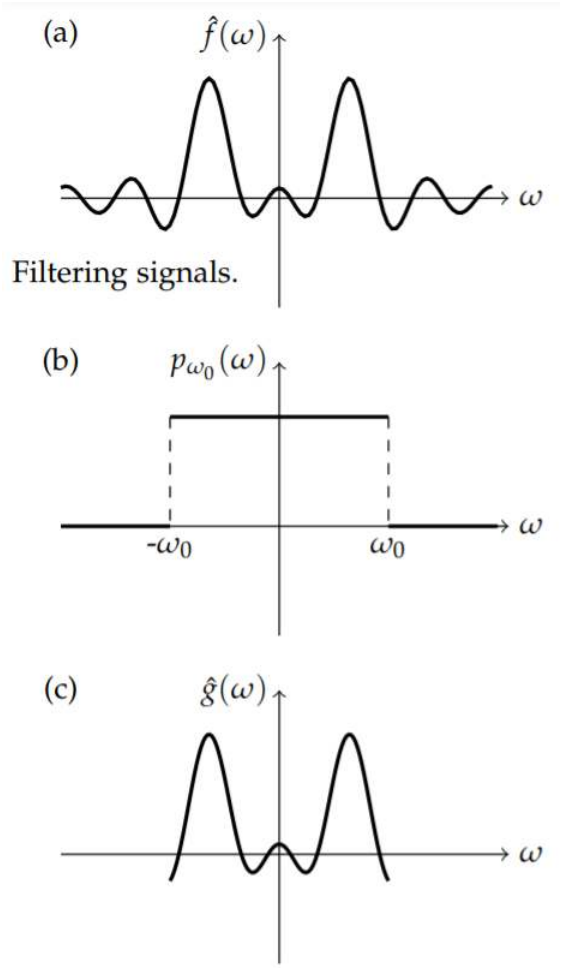


Figure 9.6.12: (a) Plot of the Fourier transform $f(\omega)$ of a signal. (b) The gate function $p_{\omega_0}(\omega)$ used to filter out high frequencies. (c) The product of the functions, $\hat{g}(\omega) = \hat{f}(\omega)p_{\omega_0}(\omega)$, in (a) and (b) shows how the filters cuts out high frequencies, $|\omega| > \omega_0$.

In general, we multiply the Fourier transform of the signal by some filtering function $\hat{h}(t)$ to get the Fourier transform of the filtered signal,

$$\hat{g}(\omega) = \hat{f}(\omega)\hat{h}(\omega).$$

The new signal, $g(t)$ is then the inverse Fourier transform of this product, giving the new signal as a convolution:

$$g(t) = F^{-1}[\hat{f}(\omega)\hat{h}(\omega)] = \int_{-\infty}^{\infty} h(t-\tau)f(\tau)d\tau. \tag{9.6.15}$$

Such processes occur often in systems theory as well. One thinks of $f(t)$ as the input signal into some filtering device which in turn produces the output, $g(t)$. The function $h(t)$ is called the impulse response. This is because it is a response to the impulse function, $\delta(t)$. In this case, one has

$$\int_{-\infty}^{\infty} h(t-\tau)\delta(\tau)d\tau = h(t).$$

Another application of the convolution is in windowing. This represents what happens when one measures a real signal. Real signals cannot be recorded for all values of time. Instead data is collected over a finite time interval. If the length of time the data is collected is T, then the resulting signal is zero outside this time interval. This can be modeled in the same way as with filtering,

except the new signal will be the product of the old signal with the windowing function. The resulting Fourier transform of the new signal will be a convolution of the Fourier transforms of the original signal and the windowing function.

Example 9.6.7: Finite Wave Train, Revisited

We return to the finite wave train in Example 9.5.6 given by

$$h(t) = \begin{cases} \cos\omega_0 t, & |t| \le a \\ 0, & |t| > a \end{cases}.$$

Solution

We can view this as a windowed version of $f(t) = \cos\omega_0$ t obtained by multiplying $f(t)$ by the gate function

$$g_a(t) = \begin{cases} 1, & |x| \le a \\ 0, & |x| > a \end{cases} \tag{9.6.16}$$

This is shown in Figure 9.6.13. Then, the Fourier transform is given as a convolution,

$$\begin{aligned} \hat{h}(\omega) &= \left(\hat{f} * \hat{g}_a\right)(\omega) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}\,(\omega - v)\hat{g}_a(v)dv. \end{aligned} \tag{9.6.17}$$

Note that the convolution in frequency space requires the extra factor of $1/(2\pi)$.

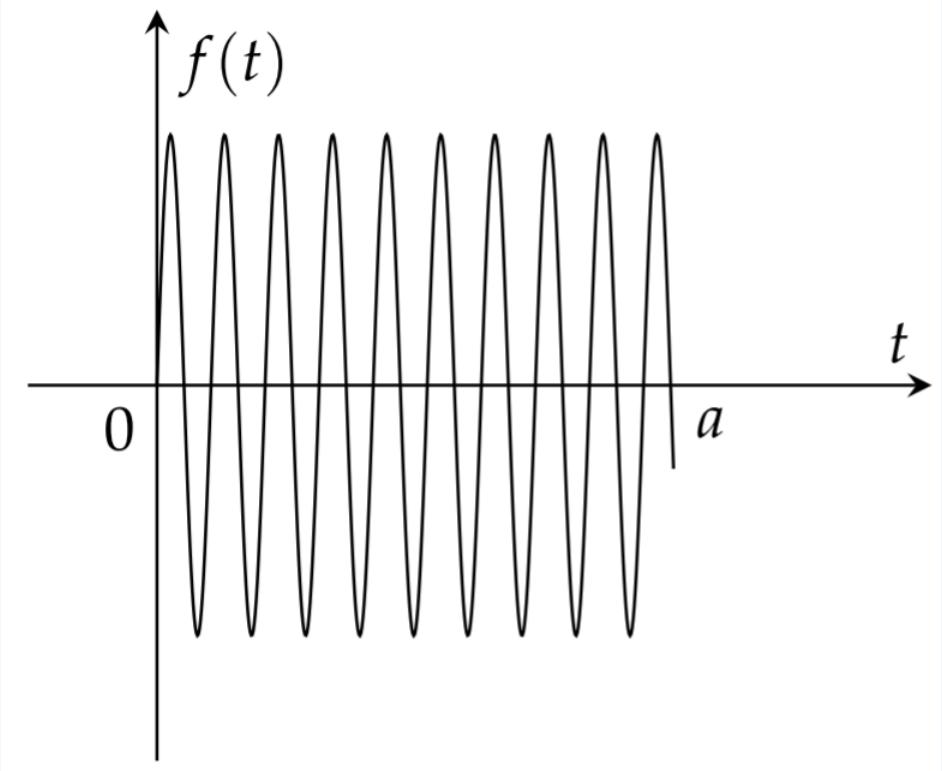


Figure 9.6.13: A plot of the finite wave train.

Note

The convolution in spectral space is defined with an extra factor of $1/2\pi$ so as to preserve the idea that the inverse Fourier transform of a convolution is the product of the corresponding signals.

We need the Fourier transforms of f and g_a in order to finish the computation. The Fourier transform of the box function was found in Example 9.5.2 as

$$\hat{g}_a(\omega) = \frac{2}{\omega}\sin\omega a.$$

The Fourier transform of the cosine function, $f(t) = \cos\omega_0 t$, is

$$\begin{aligned}\hat{f}(\omega) &= \int_{-\infty}^{\infty} \cos(\omega_0 t) e^{i\omega t} dt \\ &= \int_{-\infty}^{\infty} \frac{1}{2}\left(e^{i\omega_0 t} + e^{-i\omega_0 t}\right) e^{i\omega t} dt \\ &= \frac{1}{2}\int_{-\infty}^{\infty} \left(e^{i(\omega+\omega_0)t} + e^{i(\omega-\omega_0)t}\right) dt \\ &= \pi\left[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)\right] \end{aligned} \tag{9.6.18}$$

Note that we had earlier computed the inverse Fourier transform of this function in Example 9.5.5.

Inserting these results in the convolution integral, we have

$$\begin{aligned}\hat{h}(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega - v)\hat{g}_a(v) dv \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \pi\left[\delta(\omega - v + \omega_0) + \delta(\omega - v - \omega_0)\right] \frac{2}{v}\sin v a dv \\ &= \frac{\sin(\omega+\omega_0)a}{\omega+\omega_0} + \frac{\sin(\omega-\omega_0)a}{\omega-\omega_0}. \end{aligned} \tag{9.6.19}$$

This is the same result we had obtained in Example 9.5.6.

Parseval's Equality

Note

The integral/sum of the (modulus) square of a function is the integral/sum of the (modulus) square of the transform.

As another example of the convolution theorem, we derive Parseval's Equality (named after Marc-Antoine Parseval (1755-1836)):

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 d\omega. \tag{9.6.20}$$

This equality has a physical meaning for signals. The integral on the left side is a measure of the energy content of the signal in the time domain. The right side provides a measure of the energy content of the transform of the signal. Parseval's equality, is simply a statement that the energy is invariant under the Fourier transform. Parseval's equality is a special case of Plancherel's formula (named after Michel Plancherel, 1885-1967).

Let's rewrite the Convolution Theorem in its inverse form

$$F^{-1}[\hat{f}(k)\hat{g}(k)] = (f * g)(t). \tag{9.6.21}$$

Then, by the definition of the inverse Fourier transform, we have

$$\int_{-\infty}^{\infty} f(t-u)g(u)du = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\hat{g}(\omega)e^{-i\omega t} d\omega.$$

Setting $t = 0$,

$$\int_{-\infty}^{\infty} f(-u)g(u)du = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\hat{g}(\omega) d\omega. \tag{9.6.22}$$

Now, let $g(t) = \overline{f(-t)}$, or $f(-t) = \overline{g(t)}$. We note that the Fourier transform of $g(t)$ is related to the Fourier transform of $f(t)$:

$$\begin{aligned}\hat{g}(\omega) &= \int_{-\infty}^{\infty} \overline{f(-t)} e^{i\omega t} dt \\ &= -\int_{\infty}^{-\infty} \overline{f(\tau) e^{-i\omega\tau}} d\tau \\ &= \int_{-\infty}^{\infty} f(\tau) e^{i\omega\tau} d\tau = \bar{f}(\omega). \end{aligned} \tag{9.6.23}$$

So, inserting this result into Equation (9.6.22), we find that

$$\int_{-\infty}^{\infty} f(-u)\overline{f(-u)} du = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 d\omega$$

which yields Parseval's Equality in the form (9.6.20) after substituting $t = -u$ on the left.

As noted above, the forms in Equations (9.6.20) and (9.6.22) are often referred to as the Plancherel formula or Parseval formula. A more commonly defined Parseval equation is that given for Fourier series. For example, for a function $f(x)$ defined on $[-\pi, \pi]$, which has a Fourier series representation, we have

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right) = \frac{1}{\pi}\int_{-\pi}^{\pi} [f(x)]^2 dx.$$

In general, there is a Parseval identity for functions that can be expanded in a complete sets of orthonormal functions, $\{\phi_n(x)\}, n = 1, 2, \ldots$, which is given by

$$\sum_{n=1}^{\infty} < f, \phi_n >^2 = \|f\|^2.$$

Here $\|f\|^2 = \langle f, f \rangle$. The Fourier series example is just a special case of this formula.

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9.7: The Laplace Transform

Up to this point we have only explored Fourier exponential transforms as one type of integral transform. The Fourier transform is useful on infinite domains. However, students are often introduced to another integral transform, called the **Laplace transform**, in their introductory differential equations class. These transforms are defined over semi-infinite domains and are useful for solving initial value problems for ordinary differential equations.

Note

The Laplace transform is named after Pierre-Simon de Laplace ($1749-1827$). Laplace made major contributions, especially to celestial mechanics, tidal analysis, and probability.

Note

Integral transform on $[a,b]$ with respect to the integral kernel, $K(x,k)$.

The Fourier and Laplace transforms are examples of a broader class of transforms known as **integral transforms**. For a function $f(x)$ defined on an interval (a,b), we define the integral transform

$$F(k)=\int_a^b K(x,k)f(x)dx,$$

where $K(x,k)$ is a specified kernel of the transform. Looking at the Fourier transform, we see that the interval is stretched over the entire real axis and the kernel is of the form, $K(x,k)=e^{ikx}$. In Table $9.7.1$ we show several types of integral transforms.

Table $9.7.1$: A table of common integral transforms.

Laplace Transform	$F(s)=\int_0^\infty e^{-sx}f(x)dx$
Fourier Transform	$F(k)=\int_{-\infty}^\infty e^{ikx}f(x)dx$
Fourier Cosine Transform	$F(k)=\int_0^\infty \cos(kx)f(x)dx$
Fourier Sine Transform	$F(k)=\int_0^\infty \sin(kx)f(x)dx$
Mellin Transform	$F(k)=\int_0^\infty x^{k-1}f(x)dx$
Hankel Transform	$F(k)=\int_0^\infty xJ_n(kx)f(x)dx$

It should be noted that these integral transforms inherit the linearity of integration. Namely. let $h(x)=\alpha f(x)+\beta g(x)$, where α and β are constants. Then,

$$\begin{aligned} H(k) &= \int_a^b K(x,k)h(x)dx, \\ &= \int_a^b K(x,k)(\alpha f(x)+\beta g(x))dx, \\ &= \alpha\int_a^b K(x,k)f(x)dx+\beta\int_a^b K(x,k)g(x)dx, \\ &= \alpha F(x)+\beta G(x). \end{aligned} \tag{9.7.1}$$

Therefore, we have shown linearity of the integral transforms. We have seen the linearity property used for Fourier transforms and we will use linearity in the study of Laplace transforms.

Note

The Laplace transform of f, $F=\mathcal{L}[f]$.

We now turn to Laplace transforms. The Laplace transform of a function $f(t)$ is defined as

$$F(s) = \mathcal{L}[f](s) = \int_0^\infty f(t)e^{-st}\,dt, \quad s > 0. \tag{9.7.2}$$

This is an improper integral and one needs

$$\lim_{t\to\infty} f(t)e^{-st} = 0$$

to guarantee convergence.

Laplace transforms also have proven useful in engineering for solving circuit problems and doing systems analysis. In Figure 9.7.1 it is shown that a signal $x(t)$ is provided as input to a linear system, indicated by $h(t)$. One is interested in the system output, $y(t)$, which is given by a convolution of the input and system functions. By considering the transforms of $x(t)$ and $h(t)$, the transform of the output is given as a product of the Laplace transforms in the s-domain. In order to obtain the output, one needs to compute a convolution product for Laplace transforms similar to the convolution operation we had seen for Fourier transforms earlier in the chapter. Of course, for us to do this in practice, we have to know how to compute Laplace transforms.

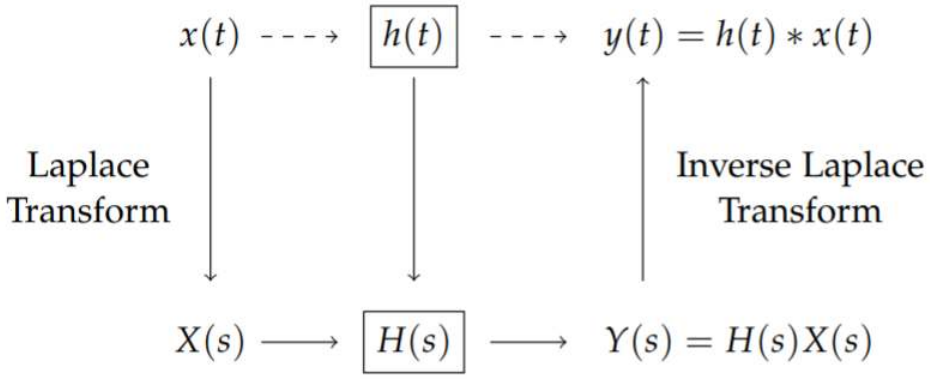


Figure 9.7.1: A schematic depicting the use of Laplace transforms in systems theory.

Properties and Examples of Laplace Transforms

It is typical that one makes use of Laplace transforms by referring to a Table of transform pairs. A sample of such pairs is given in Table 9.7.2. Combining some of these simple Laplace transforms with the properties of the Laplace transform, as shown in Table 9.7.3, we can deal with many applications of the Laplace transform. We will first prove a few of the given Laplace transforms and show how they can be used to obtain new transform pairs. In the next section we will show how these transforms can be used to sum infinite series and to solve initial value problems for ordinary differential equations.

Table 9.7.2: Table of selected Laplace transform pairs.

$f(t)$	$F(s)$	$f(t)$	$F(s)$
c	$\frac{c}{s}$	e^{at}	$\frac{1}{s-a'} s > a$
t^n	$\frac{n!}{s^{n+1}}, s > 0$	$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$	$e^{at} \sin \omega t$	$\frac{\omega}{(s-a)^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$	$e^{at} \cos \omega t$	$\frac{s-a}{(s-a)^2+\omega^2}$
$t \sin \omega t$	$\frac{2\omega s}{(s^2+\omega^2)^2}$	$t \cos \omega t$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$	$\cosh at$	$\frac{s}{s^2-a^2}$
$H(t-a)$	$\frac{e^{-as}}{s}, s > 0$	$\delta(t-a)$	$e^{-as}, a \geq 0, s > 0$

We begin with some simple transforms. These are found by simply using the definition of the Laplace transform.

Example 9.7.1

Show that $\mathcal{L}[1] = \frac{1}{\varsigma}$.

Solution

For this example, we insert $f(t) = 1$ into the definition of the Laplace transform:

$$\mathcal{L}[1] = \int_0^\infty e^{-st}\,dt$$

This is an improper integral and the computation is understood by introducing an upper limit of a and then letting $a \to \infty$. We will not always write this limit, but it will be understood that this is how one computes such improper integrals. Proceeding with the computation, we have

$$\begin{aligned}\mathcal{L}[1] &= \int_0^\infty e^{-st}\,dt \\ &= \lim_{a\to\infty} \int_0^a e^{-st}\,dt \\ &= \lim_{a\to\infty} \left(-\frac{1}{s}e^{-st}\right)_0^a \\ &= \lim_{a\to\infty} \left(-\frac{1}{s}e^{-sa} + \frac{1}{s}\right) = \frac{1}{s}.\end{aligned} \tag{9.7.3}$$

Thus, we have found that the Laplace transform of 1 is $\frac{1}{\frac{1}{s}}$. This result can be extended to any constant c, using the linearity of the transform, $\mathcal{L}[c] = c\mathcal{L}[1]$. Therefore,

$$\mathcal{L}[c] = \frac{c}{s}.$$

Example 9.7.2

Show that $\mathcal{L}\left[e^{at}\right] = \frac{1}{s-a}$, for $s > a$.

Solution

For this example, we can easily compute the transform. Again, we only need to compute the integral of an exponential function.

$$\begin{aligned}\mathcal{L}\left[e^{at}\right] &= \int_0^\infty e^{at}e^{-st}\,dt \\ &= \int_0^\infty e^{(a-s)t}\,dt \\ &= \left(\frac{1}{a-s}e^{(a-s)t}\right)_0^\infty \\ &= \lim_{t\to\infty} \frac{1}{a-s}e^{(a-s)t} - \frac{1}{a-s} = \frac{1}{s-a}.\end{aligned} \tag{9.7.4}$$

Note that the last limit was computed as $\lim_{t\to\infty} e^{(a-s)t} = 0$. This is only true if a $-s < 0$, or $s >$ a. [Actually, a could be complex. In this case we would only need s to be greater than the real part of a, $s > \mathrm{Re}(a).1$

Example 9.7.3

Show that $\mathcal{L}[\cos at] = \frac{s}{s^2+a^2}$ and $\mathcal{L}[\sin at] = \frac{a}{s^2+a^2}$.

Solution

For these examples, we could again insert the trigonometric functions directly into the transform and integrate. For example,

$$\mathcal{L}[\cos at] = \int_0^\infty e^{-st}\cos at dt.$$

Recall how one evaluates integrals involving the product of a trigonometric function and the exponential function. One integrates by parts two times and then obtains an integral of the original unknown integral. Rearranging the resulting integral expressions, one arrives at the desired result. However, there is a much simpler way to compute these transforms.

Recall that $e^{\text{iat}} = \cos at + i\sin$ at. Making use of the linearity of the Laplace transform, we have

$$\mathcal{L}\left[e^{iat}\right] = \mathcal{L}[\cos at] + i\mathcal{L}[\sin at].$$

Thus, transforming this complex exponential will simultaneously provide the Laplace transforms for the sine and cosine functions!

The transform is simply computed as

$$\mathcal{L}\left[e^{iat}\right] = \int_0^\infty e^{iat}e^{-st}dt = \int_0^\infty e^{-(s-ia)t}dt = \frac{1}{s-ia}.$$

Note that we could easily have used the result for the transform of an exponential, which was already proven. In this case $s > \text{Re(ia)} = 0$.

We now extract the real and imaginary parts of the result using the complex conjugate of the denominator:

$$\frac{1}{s-ia} = \frac{1}{s-ia}\frac{s+ia}{s+ia} = \frac{s+ia}{s^2+a^2}.$$

Reading off the real and imaginary parts, we find the sought transforms,

$$\begin{aligned}\mathcal{L}[\cos at] &= \frac{s}{s^2+a^2}\\ \mathcal{L}[\sin at] &= \frac{a}{s^2+a^2}.\end{aligned} \tag{9.7.5}$$

Example 9.7.4

Show that $\mathcal{L}[t] = \frac{1}{s^2}$.

Solution

For this example we evaluate

$$\mathcal{L}[t] = \int_0^\infty te^{-st}dt$$

This integral can be evaluated using the method of integration by parts:

$$\begin{aligned}\int_0^\infty te^{-st}dt &= -t\frac{1}{s}e^{-st}\Big|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}dt\\ &= \frac{1}{s^2}.\end{aligned} \tag{9.7.6}$$

Example 9.7.5

Show that $\mathcal{L}\left[t^n\right] = \frac{n!}{s^n+1}$ for nonnegative integer n.

Solution

We have seen the $n = 0$ and $n = 1$ cases: $\mathcal{L}[1] = \frac{1}{s}$ and $\mathcal{L}[t] = \frac{1}{s^2}$. We now generalize these results to nonnegative integer powers, $n > 1$, of t. We consider the integral

$$\mathcal{L}\left[t^n\right] = \int_0^\infty t^n e^{-st}dt.$$

Following the previous example, we again integrate by parts:[1]

$$\begin{aligned}\int_0^\infty t^n e^{-st}\,dt &= -t^n \frac{1}{s} e^{-st}\Big|_0^\infty + \frac{n}{s}\int_0^\infty t^{-n} e^{-st}\,dt \\ &= \frac{n}{s}\int_0^\infty t^{-n} e^{-st}\,dt. \end{aligned} \tag{9.7.7}$$

We could continue to integrate by parts until the final integral is computed. However, look at the integral that resulted after one integration by parts. It is just the Laplace transform of t^{n-1}. So, we can write the result as

$$\mathcal{L}\left[t^n\right] = \frac{n}{s}\mathcal{L}\left[t^{n-1}\right].$$

Note

We compute $\int_0^\infty t^n e^{-st}\,dt$ by turning it into an initial value problem for a first order difference equation and finding the solution using an iterative method.

This is an example of a recursive definition of a sequence. In this case we have a sequence of integrals. Denoting

$$I_n = \mathcal{L}\left[t^n\right] = \int_0^\infty t^n e^{-st}\,dt$$

and noting that $I_0 = \mathcal{L}[1] = \frac{1}{s}$, we have the following:

$$I_n = \frac{n}{s} I_{n-1}, \quad I_0 = \frac{1}{s}. \tag{9.7.8}$$

This is also what is called a difference equation. It is a first order difference equation with an "initial condition," I_0. The next step is to solve this difference equation.

Finding the solution of this first order difference equation is easy to do using simple iteration. Note that replacing n with $n-1$, we have

$$I_{n-1} = \frac{n-1}{s} I_{n-2}.$$

Repeating the process, we find

$$\begin{aligned} I_n &= \frac{n}{s} I_{n-1} \\ &= \frac{n}{s}\left(\frac{n-1}{s} I_{n-2}\right) \\ &= \frac{n(n-1)}{s^2} I_{n-2} \\ &= \frac{n(n-1)(n-2)}{s^3} I_{n-3} \end{aligned} \tag{9.7.9}$$

We can repeat this process until we get to I_0, which we know. We have to carefully count the number of iterations. We do this by iterating k times and then figure out how many steps will get us to the known initial value. A list of iterates is easily written out:

$$\begin{aligned} I_n &= \frac{n}{s} I_{n-1} \\ &= \frac{n(n-1)}{s^2} I_{n-2} \\ &= \frac{n(n-1)(n-2)}{s^3} I_{n-3} \\ &= \dots \\ &= \frac{n(n-1)(n-2)\dots(n-k+1)n(n-k)}{s^k} I_{n-k}. \end{aligned} \tag{9.7.10}$$

Since we know $I_0 = \frac{1}{s}$, we choose to stop at $k = n$ obtaining

$$I_n = \frac{n(n-1)(n-2)\dots(2)(1)}{s^n} I_0 = \frac{n!}{s^{n+1}}.$$

Therefore, we have shown that $\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$.

Such iterative techniques are useful in obtaining a variety of integrals, such as $I_n = \int_{-\infty}^{\infty} x^{2n} e^{-x^2} dx$.

Note

This integral can just as easily be done using differentiation. We note that

$$\left(-\frac{d}{ds}\right)^n \int_0^\infty e^{-st} dt = \int_0^\infty t^n e^{-st} dt.$$

Since

$$\int_0^\infty e^{-st} dt = \frac{1}{s},$$
$$\int_0^\infty t^n e^{-st} dt = \left(-\frac{d}{ds}\right)^n \frac{1}{s} = \frac{n!}{s^{n+1}}.$$

As a final note, one can extend this result to cases when n is not an integer. To do this, we use the Gamma function, which was discussed in Section 5.4. Recall that the Gamma function is the generalization of the factorial function and is defined as

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt. \tag{9.7.11}$$

Note the similarity to the Laplace transform of t^{x-1} :

$$\mathcal{L}\left[t^{x-1}\right] = \int_0^\infty t^{x-1} e^{-st} dt.$$

For $x-1$ an integer and $s=1$, we have that

$$\Gamma(x) = (x-1)!.$$

Thus, the Gamma function can be viewed as a generalization of the factorial and we have shown that

$$\mathcal{L}[t^p] = \frac{\Gamma(p+1)}{s^{p+1}}$$

for $p > -1$.

Now we are ready to introduce additional properties of the Laplace transform in Table 9.7.3. We have already discussed the first property, which is a consequence of linearity of the integral transforms. We will prove the other properties in this and the following sections.

Table 9.7.3: Table of selected Laplace transform properties.

Laplace Transform Properties
$\mathcal{L}[af(t)+bg(t)] = aF(s)+bG(s)$
$\mathcal{L}[tf(t)] = -\frac{d}{ds}F(s)$
$\mathcal{L}\left[\frac{df}{dt}\right] = sF(s)-f(0)$
$\mathcal{L}\left[\frac{d^2f}{dt^2}\right] = s^2F(s)-sf(0)-f'(0)$
$\mathcal{L}\left[e^{at}f(t)\right] = F(s-a)$
$\mathcal{L}[H(t-a)f(t-a)] = e^{-as}F(s)$

Laplace Transform Properties
$\mathcal{L}[(f*g)(t)] = \mathcal{L}\left[\int_0^t f(t-u)g(u)du\right] = F(s)G(s)$

Example 9.7.6

Show that $\mathcal{L}\left[\frac{df}{dt}\right] = sF(s) - f(0)$.

Solution

We have to compute

$$\mathcal{L}\left[\frac{df}{dt}\right] = \int_0^\infty \frac{df}{dt} e^{-st}\, dt.$$

We can move the derivative off f by integrating by parts. This is similar to what we had done when finding the Fourier transform of the derivative of a function. Letting $u = e^{-st}$ and $v = f(t)$, we have

$$\begin{aligned} \mathcal{L}\left[\frac{df}{dt}\right] &= \int_0^\infty \frac{df}{dt} e^{-st}\, dt \\ &= f(t)e^{-st}\Big|_0^\infty + s\int_0^\infty f(t)e^{-st}\, dt \\ &= -f(0) + sF(s). \end{aligned} \tag{9.7.12}$$

Here we have assumed that $f(t)e^{-\mathrm{st}}$ vanishes for large t.

The final result is that

$$\mathcal{L}\left[\frac{df}{dt}\right] = sF(s) - f(0).$$

Example 9.7.7

Show that $\mathcal{L}\left[\frac{d^2f}{dt^2}\right] = s^2F(s) - sf(0) - f'(0)$.

Solution

We can compute this Laplace transform using two integrations by parts, or we could make use of the last result. Letting $g(t) = \frac{df(t)}{dt}$, we have

$$\mathcal{L}\left[\frac{d^2f}{dt^2}\right] = \mathcal{L}\left[\frac{dg}{dt}\right] = sG(s) - g(0) = sG(s) - f'(0).$$

But,

$$G(s) = \mathcal{L}\left[\frac{df}{dt}\right] = sF(s) - f(0)$$

So,

$$\begin{aligned} \mathcal{L}\left[\frac{d^2f}{dt^2}\right] &= sG(s) - f'(0) \\ &= s[sF(s) - f(0)] - f'(0) \\ &= s^2F(s) - sf(0) - f'(0). \end{aligned} \tag{9.7.13}$$

We will return to the other properties in Table 9.7.3 after looking at a few applications.

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9.8: Applications of Laplace Transforms

Although the Laplace transform is a very useful transform, it is often encountered only as a method for solving initial value problems in introductory differential equations. In this section we will show how to solve simple differential equations. Along the way we will introduce step and impulse functions and show how the Convolution Theorem for Laplace transforms plays a role in finding solutions. However, we will first explore an unrelated application of Laplace transforms. We will see that the Laplace transform is useful in finding sums of infinite series.

Series Summation Using Laplace Transforms

We saw in Chapter ?? that Fourier series can be used to sum series. For example, in Problem ??.13, one proves that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

In this section we will show how Laplace transforms can be used to sum series.[1] There is an interesting history of using integral transforms to sum series. For example, Richard Feynman[2] $(1918-1988)$ described how one can use the convolution theorem for Laplace transforms to sum series with denominators that involved products. We will describe this and simpler sums in this section.

Note

Albert D. Wheelon, Tables of Summable Series and Integrals Involving Bessel Functions, Holden-Day, 1968.

Note

R. P. Feynman, 1949 , Phys. Rev. 76, p. 769

We begin by considering the Laplace transform of a known function,

$$F(s) = \int_0^{\infty} f(t)e^{-st}\,dt.$$

Inserting this expression into the sum $\sum_n F(n)$ and interchanging the sum and integral, we find

$$\begin{aligned} \sum_{n=0}^{\infty} F(n) &= \sum_{n=0}^{\infty} \int_0^{\infty} f(t)e^{-nt}\,dt \\ &= \int_0^{\infty} f(t) \sum_{n=0}^{\infty} \left(e^{-t}\right)^n dt \\ &= \int_0^{\infty} f(t) \frac{1}{1-e^{-t}}\,dt. \end{aligned} \tag{9.8.1}$$

The last step was obtained using the sum of a geometric series. The key is being able to carry out the final integral as we show in the next example.

Example 9.8.1

Evaluate the sum $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$.

Solution

Since, $\mathcal{L}[1] = 1/s$, we have

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = \sum_{n=1}^{\infty}\int_0^{\infty}(-1)^{n+1}e^{-nt}dt$$
$$= \int_0^{\infty}\frac{e^{-t}}{1+e^{-t}}dt$$
$$= \int_1^2 \frac{du}{u} = \ln 2 \tag{9.8.2}$$

Example 9.8.2

Evaluate the sum $\sum_{n=1}^{\infty}\frac{1}{n^2}$.

Solution

This is a special case of the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}. \tag{9.8.3}$$

The Riemann zeta function[3] is important in the study of prime numbers and more recently has seen applications in the study of dynamical systems. The series in this example is $\zeta(2)$. We have already seen in ??.13 that

$$\zeta(2) = \frac{\pi^2}{6}.$$

Using Laplace transforms, we can provide an integral representation of $\zeta(2)$.

The first step is to find the correct Laplace transform pair. The sum involves the function $F(n) = 1/n^2$. So, we look for a function $f(t)$ whose Laplace transform is $F(s) = 1/s^2$. We know by now that the inverse Laplace transform of $F(s) = 1/s^2$ is $f(t) = t$. As before, we replace each term in the series by a Laplace transform, exchange the summation and integration, and sum the resulting geometric series:

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \sum_{n=1}^{\infty}\int_0^{\infty}te^{-nt}dt$$
$$= \int_0^{\infty}\frac{t}{e^t-1}dt. \tag{9.8.4}$$

So, we have that

$$\int_0^{\infty}\frac{t}{e^t-1}dt = \sum_{n=1}^{\infty}\frac{1}{n^2} = \zeta(2)$$

Integrals of this type occur often in statistical mechanics in the form of Bose-Einstein integrals. These are of the form

$$G_n(z) = \int_0^{\infty}\frac{x^{n-1}}{z^{-1}e^x-1}dx.$$

Note that $G_n(1) = \Gamma(n)\zeta(n)$.

Note

A translation of Riemann, Bernhard (1859), "Über die Anzahl der Primzahlen unter einer gegebenen Grösse" is in H. M. Edwards (1974). *Riemann's Zeta Function*. Academic Press. Riemann had shown that the Riemann zeta function can be obtained through contour integral representation, $2\sin(\pi s)\Gamma\zeta(s) = i\oint_C \frac{(-x)^{s-1}}{e^x-1}dx$, for a specific contour C.

In general the Riemann zeta function has to be tabulated through other means. In some special cases, one can closed form expressions. For example,

$$\zeta(2n) = \frac{2^{2n-1}\pi^{2n}}{(2n)!}B_n,$$

where the B_n 's are the Bernoulli numbers. Bernoulli numbers are defined through the Maclaurin series expansion

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!}x^n.$$

The first few Riemann zeta functions are

$$\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \zeta(6) = \frac{\pi^6}{945}.$$

We can extend this method of using Laplace transforms to summing series whose terms take special general forms. For example, from Feynman's 1949 paper we note that

$$\frac{1}{(a+bn)^2} = -\frac{\partial}{\partial a}\int_0^{\infty} e^{-s(a+bn)}\,ds.$$

This identity can be shown easily by first noting

$$\int_0^{\infty} e^{-s(a+bn)}\,ds = \left[\frac{-e^{-s(a+bn)}}{a+bn}\right]_0^{\infty} = \frac{1}{a+bn}.$$

Now, differentiate the result with respect to a and the result follows.

The latter identity can be generalized further as

$$\frac{1}{(a+bn)^{k+1}} = \frac{(-1)^k}{k!}\frac{\partial^k}{\partial a^k}\int_0^{\infty} e^{-s(a+bn)}\,ds.$$

In Feynman's 1949 paper, he develops methods for handling several other general sums using the convolution theorem. Wheelon gives more examples of these. We will just provide one such result and an example. First, we note that

$$\frac{1}{ab} = \int_0^1 \frac{du}{[a(1-u)+bu]^2}.$$

However,

$$\frac{1}{[a(1-u)+bu]^2} = \int_0^{\infty} te^{-t[a(1-u)+bu]}\,dt.$$

So, we have

$$\frac{1}{ab} = \int_0^1 du \int_0^{\infty} te^{-t[a(1-u)+bu]}\,dt.$$

We see in the next example how this representation can be useful.

Example 9.8.3

Evaluate $\sum_{n=0}^{\infty} \frac{1}{(2n+1)(2n+2)}$.

Solution

We sum this series by first letting $a = 2n+1$ and $b = 2n+2$ in the formula for $1/ab$. Collecting the n-dependent terms, we can sum the series leaving a double integral computation in ut-space. The details are as follows:

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{1}{(2n+1)(2n+2)} &= \sum_{n=0}^{\infty}\int_0^1 \frac{du}{[(2n+1)(1-u)+(2n+2)u]^2} \\ &= \sum_{n=0}^{\infty}\int_0^1 du\int_0^{\infty} te^{-t(2n+1+u)}\,dt \\ &= \int_0^1 du\int_0^{\infty} te^{-t(1+u)}\sum_{n=0}^{\infty}e^{-2nt}dt \\ &= \int_0^{\infty}\frac{te^{-t}}{1-e^{-2t}}\int_0^1 e^{-tu}\,dudt \\ &= \int_0^{\infty}\frac{te^{-t}}{1-e^{-2t}}\frac{1-e^{-t}}{t}dt \\ &= \int_0^{\infty}\frac{e^{-t}}{1+e^{-t}}dt \\ &= -\ln\left(1+e^{-t}\right)\Big|_0^{\infty} = \ln 2 \end{aligned} \tag{9.8.5}$$

Solution of ODEs Using Laplace Transforms

One of the typical applications of Laplace transforms is the solution of nonhomogeneous linear constant coefficient differential equations. In the following examples we will show how this works.

The general idea is that one transforms the equation for an unknown function $y(t)$ into an algebraic equation for its transform, $Y(t)$. Typically, the algebraic equation is easy to solve for $Y(s)$ as a function of s. Then, one transforms back into t-space using Laplace transform tables and the properties of Laplace transforms. The scheme is shown in Figure 9.8.1.

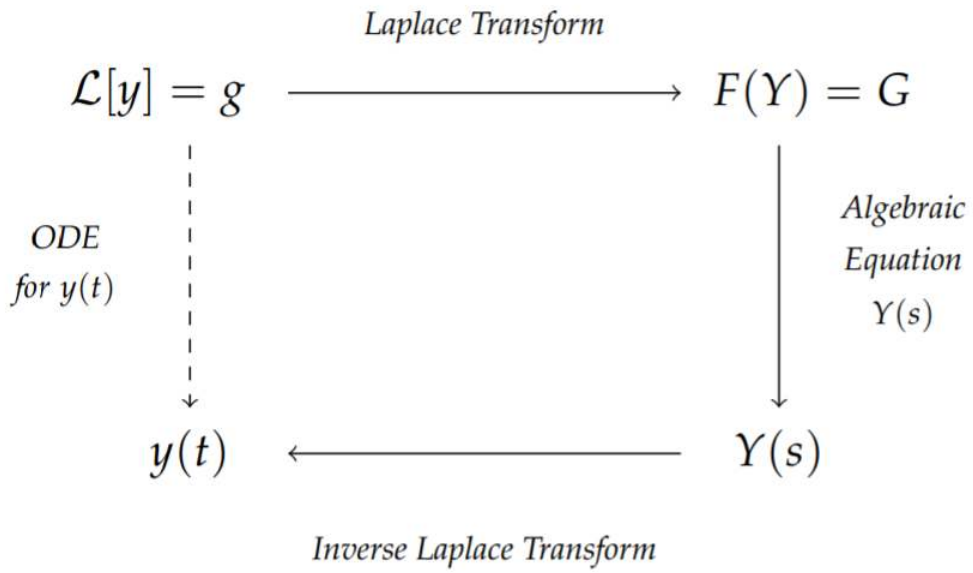


Figure 9.8.1: The scheme for solving an ordinary differential equation using Laplace transforms. One transforms the initial value problem for $y(t)$ and obtains an algebraic equation for $Y(s)$. Solve for $Y(s)$ and the inverse transform give the solution to the initial value problem.

Example 9.8.4

Solve the initial value problem $y' + 3y = e^{2t}, y(0) = 1$.

Solution

The first step is to perform a Laplace transform of the initial value problem. The transform of the left side of the equation is

$$\mathcal{L}\left[y' + 3y\right] = sY - y(0) + 3Y = (s+3)Y - 1.$$

Transforming the right hand side, we have

$$\mathcal{L}\left[e^{2t}\right] = \frac{1}{s-2}$$

Combining these two results, we obtain

$$(s+3)Y - 1 = \frac{1}{s-2}.$$

The next step is to solve for $Y(s)$:

$$Y(s) = \frac{1}{s+3} + \frac{1}{(s-2)(s+3)}.$$

Now, we need to find the inverse Laplace transform. Namely, we need to figure out what function has a Laplace transform of the above form. We will use the tables of Laplace transform pairs. Later we will show that there are other methods for carrying out the Laplace transform inversion.

The inverse transform of the first term is e^{-3t}. However, we have not seen anything that looks like the second form in the table of transforms that we have compiled; but, we can rewrite the second term by using a partial fraction decomposition. Let's recall how to do this.

The goal is to find constants, A and B, such that

$$\frac{1}{(s-2)(s+3)} = \frac{A}{s-2} + \frac{B}{s+3}. \tag{9.8.6}$$

We picked this form because we know that recombining the two will have the same denominator. We just need to make sure the afterwards. So, adding the two terms, we have

$$\frac{1}{(s-2)(s+3)} = \frac{A(s+3)+B(s-2)}{(s-2)(s+3)}.$$

Equating numerators,

$$1 = A(s+3)+B(s-2).$$

There are several ways to proceed at this point.

Note

This is an example of carrying out a partial fraction decomposition.

a. Method 1.
We can rewrite the equation by gathering terms with common powers of s, we have

$$(A+B)s+3A-2B=1.$$

The only way that this can be true for all s is that the coefficients of the different powers of s agree on both sides. This leads to two equations for A and B :

$$\begin{aligned} A+B &= 0 \\ 3A-2B &= 1. \end{aligned} \tag{9.8.7}$$

The first equation gives $A=-B$, so the second equation becomes $-5B=1$. The solution is then $A=-B=\frac{1}{5}$.

b. Method 2.
Since the equation $\frac{1}{(s-2)(s+3)} = \frac{A}{s-2} + \frac{B}{s+3}$ is true for all s, we can pick specific values. For $s=2$, we find $1=5A$, or $A=\frac{1}{5}$. For $s=-3$, we find $1=-5B$, or $B=-\frac{1}{5}$. Thus, we obtain the same result as Method 1, but much quicker.

c. Method 3.
We could just inspect the original partial fraction problem. Since the numerator has no s terms, we might guess the form

$$\frac{1}{(s-2)(s+3)} = \frac{1}{s-2} - \frac{1}{s+3}.$$

But, recombining the terms on the right hand side, we see that

$$\frac{1}{s-2}-\frac{1}{s+3}=\frac{5}{(s-2)(s+3)}.$$

Since we were off by 5 , we divide the partial fractions by 5 to obtain

$$\frac{1}{(s-2)(s+3)}=\frac{1}{5}\left[\frac{1}{s-2}-\frac{1}{s+3}\right],$$

which once again gives the desired form.

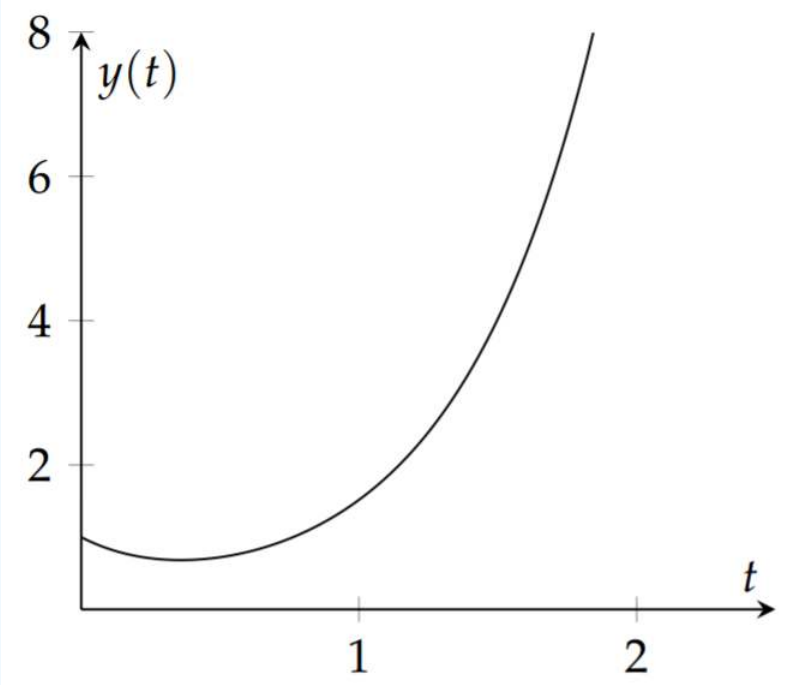


Figure 9.8.2: A plot of the solution to Example 9.8.4.

Returning to the problem, we have found that

$$Y(s)=\frac{1}{s+3}+\frac{1}{5}\left(\frac{1}{s-2}-\frac{1}{s+3}\right).$$

We can now see that the function with this Laplace transform is given by

$$y(t)=\mathcal{L}^{-1}\left[\frac{1}{s+3}+\frac{1}{5}\left(\frac{1}{s-2}-\frac{1}{s+3}\right)\right]=e^{-3t}+\frac{1}{5}\left(e^{2t}-e^{-3t}\right)$$

works. Simplifying, we have the solution of the initial value problem

$$y(t)=\frac{1}{5}e^{2t}+\frac{4}{5}e^{-3t}.$$

We can verify that we have solved the initial value problem.

$$y'+3y=\frac{2}{5}e^{2t}-\frac{12}{5}e^{-3t}+3\left(\frac{1}{5}e^{2t}+\frac{4}{5}e^{-3t}\right)=e^{2t}$$

and $y(0)=\frac{1}{5}+\frac{4}{5}=1$.

Example 9.8.5

Solve the initial value problem $y'' + 4y = 0, y(0) = 1, y'(0) = 3$.

Solution

We can probably solve this without Laplace transforms, but it is a simple exercise. Transforming the equation, we have

$$\begin{aligned} 0 &= s^2Y - sy(0) - y'(0) + 4Y \\ &= (s^2+4)Y - s - 3. \end{aligned} \tag{9.8.8}$$

Solving for Y, we have

$$Y(s) = \frac{s+3}{s^2+4}.$$

We now ask if we recognize the transform pair needed. The denominator looks like the type needed for the transform of a sine or cosine. We just need to play with the numerator. Splitting the expression into two terms, we have

$$Y(s) = \frac{s}{s^2+4} + \frac{3}{s^2+4}.$$

The first term is now recognizable as the transform of $\cos 2t$. The second term is not the transform of $\sin 2t$. It would be if the numerator were a 2 . This can be corrected by multiplying and dividing by 2:

$$\frac{3}{s^2+4} = \frac{3}{2}\left(\frac{2}{s^2+4}\right).$$

The solution is then found as

$$y(t) = \mathcal{L}^{-1}\left[\frac{s}{s^2+4} + \frac{3}{2}\left(\frac{2}{s^2+4}\right)\right] = \cos 2t + \frac{3}{2}\sin 2t.$$

The reader can verify that this is the solution of the initial value problem.

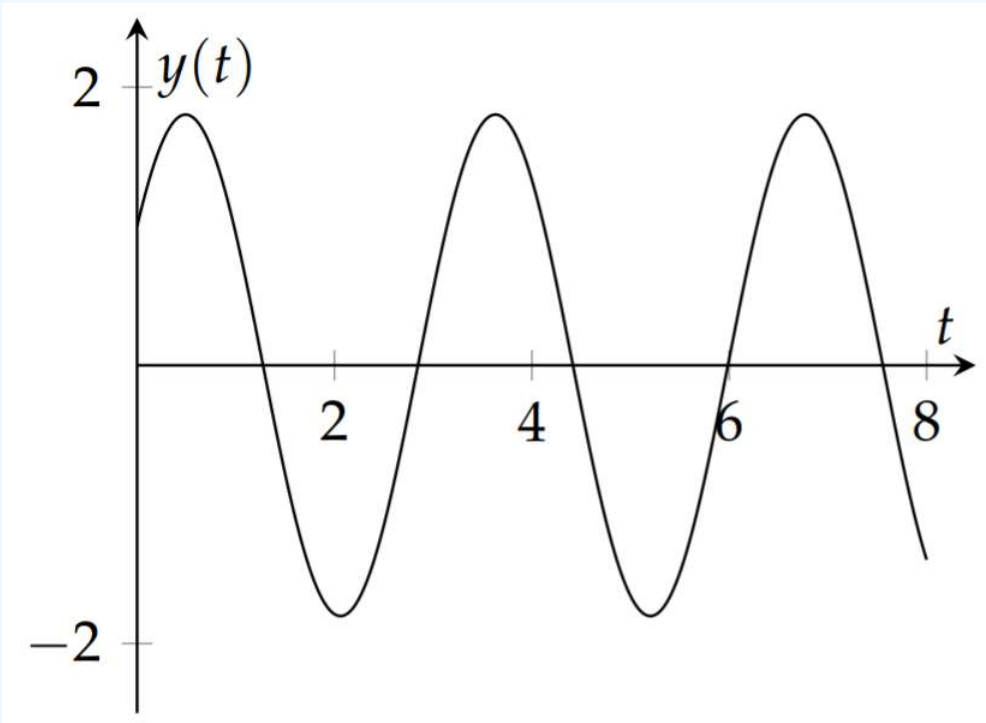


Figure 9.8.3: A plot of the solution to Example 9.8.5.

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Step and Impulse Functions

Often the initial value problems that one faces in differential equations courses can be solved using either the Method of Undetermined Coefficients or the Method of Variation of Parameters. However, using the latter can be messy and involves some skill with integration. Many circuit designs can be modeled with systems of differential equations using Kirchoff's Rules. Such systems can get fairly complicated. However, Laplace transforms can be used to solve such systems and electrical engineers have long used such methods in circuit analysis.

In this section we add a couple of more transform pairs and transform properties that are useful in accounting for things like turning on a driving force, using periodic functions like a square wave, or introducing impulse forces.

We first recall the Heaviside step function, given by

$$H(t) = \begin{cases} 0, & t < 0, \\ 1, & t > 0. \end{cases} \tag{9.8.9}$$

A more general version of the step function is the horizontally shifted step function, $H(t-a)$. This function is shown in Figure 9.8.4. The Laplace transform of this function is found for $a > 0$ as

$$\begin{aligned} \mathcal{L}[H(t-a)] &= \int_0^\infty H(t-a)e^{-st}\,dt \\ &= \int_a^\infty e^{-st}\,dt \\ &= \frac{e^{-st}}{s}\Big|_a^\infty = \frac{e^{-as}}{s}. \end{aligned} \tag{9.8.10}$$

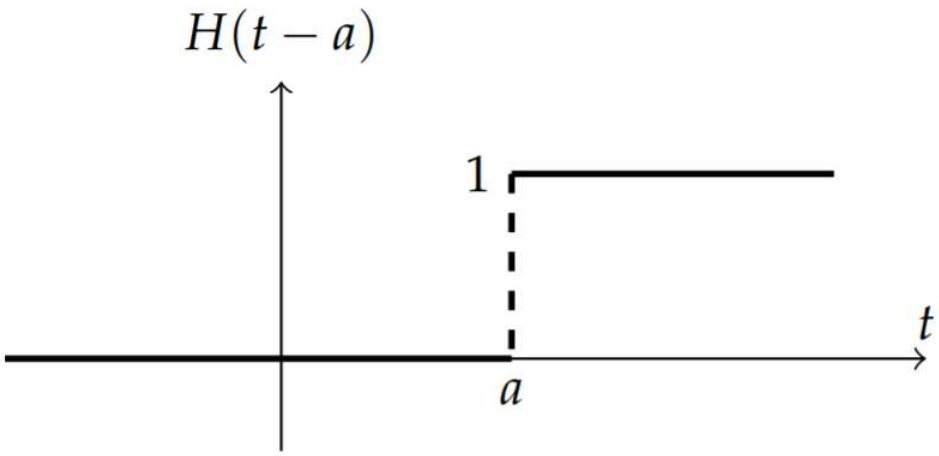


Figure 9.8.4: A shifted Heaviside function, $H(t-a)$.

Just like the Fourier transform, the Laplace transform has two shift theorems involving the multiplication of the function, $f(t)$, or its transform, $F(s)$, by exponentials. The first and second shifting properties/theorems are given by

$$\mathcal{L}\left[e^{at}f(t)\right] = F(s-a) \tag{9.8.11}$$

$$\mathcal{L}[f(t-a)H(t-a)] = e^{-as}F(s) \tag{9.8.12}$$

We prove the First Shift Theorem and leave the other proof as an exercise for the reader. Namely,

$$\begin{aligned} \mathcal{L}\left[e^{at}f(t)\right] &= \int_0^\infty e^{at}f(t)e^{-st}\,dt \\ &= \int_0^\infty f(t)e^{-(s-a)t}\,dt = F(s-a). \end{aligned} \tag{9.8.13}$$

Example 9.8.6

Compute the Laplace transform of $e^{-at}\sin\omega t$.

Solution

This function arises as the solution of the underdamped harmonic oscillator. We first note that the exponential multiplies a sine function. The shift theorem tells us that we first need the transform of the sine function. So, for $f(t)=\sin\omega t$, we have

$$F(s)=\frac{\omega}{s^2+\omega^2}.$$

Using this transform, we can obtain the solution to this problem as

$$\mathcal{L}\left[e^{-at}\sin\omega t\right]=F(s+a)=\frac{\omega}{(s+a)^2+\omega^2}.$$

More interesting examples can be found using piecewise defined functions. First we consider the function $H(t)-H(t-a)$. For $t<0$ both terms are zero. In the interval $[0,a]$ the function $H(t)=1$ and $H(t-a)=0$. Therefore, $H(t)-H(t-a)=1$ for $t\in[0,a]$. Finally, for $t>a$, both functions are one and therefore the difference is zero. The graph of $H(t)-H(t-a)$ is shown in Figure 9.8.5.

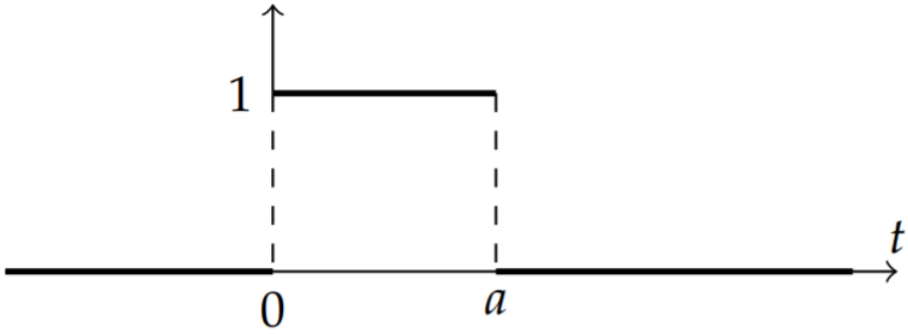


Figure 9.8.5: The box function, $H(t)-H(t-a)$.

We now consider the piecewise defined function

$$g(t)=\begin{cases} f(t), & 0\le t\le a,\\ 0, & t<0, t>a.\end{cases}$$

This function can be rewritten in terms of step functions. We only need to multiply $f(t)$ by the above box function,

$$g(t)=f(t)[H(t)-H(t-a)]$$

We depict this in Figure 9.8.6.

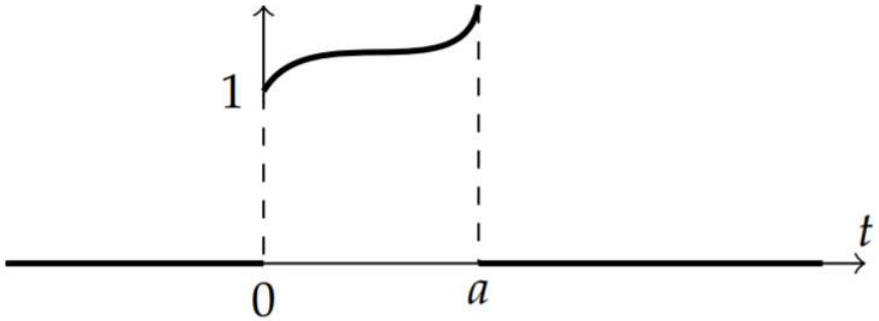


Figure 9.8.6: Formation of a piecewise function, $f(t)[H(t)-H(t-a)]$.

Even more complicated functions can be written in terms of step functions. We only need to look at sums of functions of the form $f(t)[H(t-a)-H(t-b)]$ for $b>a$. This is similar to a box function. It is nonzero between a and b has height $f(t)$.

We show as an example the square wave function in Figure 9.8.7. It can be represented as a sum of an infinite number of boxes,

$$f(t) = \sum_{n=-\infty}^{\infty} [H(t-2na) - H(t-(2n+1)a)],$$

for $a > 0$.

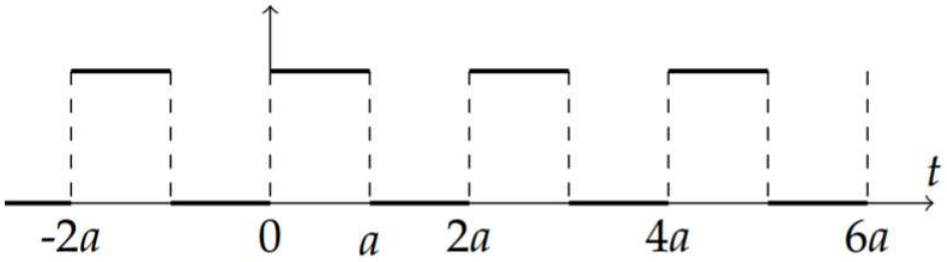


Figure 9.8.7: A square wave, $f(t) = \sum_{n=-\infty}^{\infty}[H(t-2na) - H(t-(2n+1)a)]$.

Example 9.8.7

Find the Laplace Transform of a square wave "turned on" at $t = 0$.

Solution

We let

$$f(t) = \sum_{n=0}^{\infty}[H(t-2na) - H(t-(2n+1)a)], \quad a > 0.$$

Using the properties of the Heaviside function, we have

$$\begin{aligned}
\mathcal{L}[f(t)] &= \sum_{n=0}^{\infty}[\mathcal{L}[H(t-2na)] - \mathcal{L}[H(t-(2n+1)a)]] \\
&= \sum_{n=0}^{\infty}\left[\frac{e^{-2nas}}{s} - \frac{e^{-(2n+1)as}}{s}\right] \\
&= \frac{1-e^{-as}}{s}\sum_{n=0}^{\infty}\left(e^{-2as}\right)^n \\
&= \frac{1-e^{-as}}{s}\left(\frac{1}{1-e^{-2as}}\right) \\
&= \frac{1-e^{-as}}{s\left(1-e^{-2as}\right)}
\end{aligned} \tag{9.8.14}$$

Note that the third line in the derivation is a geometric series. We summed this series to get the answer in a compact form since $e^{-2as} < 1$.

Other interesting examples are provided by the delta function. The Dirac delta function can be used to represent a unit impulse. Summing over a number of impulses, or point sources, we can describe a general function as shown in Figure 9.8.8. The sum of impulses located at points $a_i, i = 1, \ldots, n$ with strengths $f(a_i)$ would be given by

$$f(x) = \sum_{i=1}^{n} f(a_i)\,\delta(x - a_i).$$

A continuous sum could be written as

$$f(x) = \int_{-\infty}^{\infty} f(\xi)\delta(x-\xi)d\xi.$$

This is simply an application of the sifting property of the delta function. We will investigate a case when one would use a single impulse. While a mass on a spring is undergoing simple harmonic motion, we hit it for an instant at time $t = a$. In such a case, we

could represent the force as a multiple of $\delta(t-a)$.

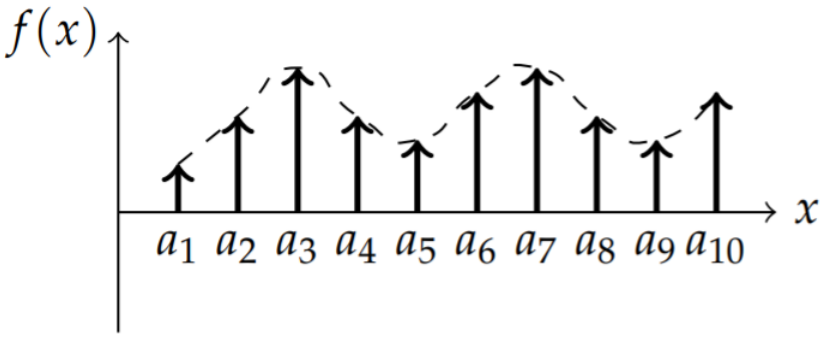


Figure 9.8.8: Plot representing impulse forces of height $f(a_i)$. The sum $\sum_{i=1}^{n} f(a_i)\,\delta(x-a_i)$ describes a general impulse function.

Note

$\mathcal{L}[\delta(t-a)] = e^{-as}$.

One would then need the Laplace transform of the delta function to solve the associated initial value problem. Inserting the delta function into the Laplace transform, we find that for $a > 0$

$$\begin{aligned}\mathcal{L}[\delta(t-a)] &= \int_0^\infty \delta(t-a)e^{-st}\,dt \\ &= \int_{-\infty}^\infty \delta(t-a)e^{-st}\,dt \\ &= e^{-as}.\end{aligned} \tag{9.8.15}$$

Example 9.8.8

Solve the initial value problem $y'' + 4\pi^2 y = \delta(t-2), y(0) = \ y'(0) = 0$.

Solution

This initial value problem models a spring oscillation with an impulse force. Without the forcing term, given by the delta function, this spring is initially at rest and not stretched. The delta function models a unit impulse at $t = 2$. Of course, we anticipate that at this time the spring will begin to oscillate. We will solve this problem using Laplace transforms.

First, we transform the differential equation:

$$s^2Y - sy(0) - y'(0) + 4\pi^2 Y = e^{-2s}.$$

Inserting the initial conditions, we have

$$\left(s^2 + 4\pi^2\right) Y = e^{-2s}.$$

Solving for $Y(s)$, we obtain

$$Y(s) = \frac{e^{-2s}}{s^2 + 4\pi^2}.$$

We now seek the function for which this is the Laplace transform. The form of this function is an exponential times some Laplace transform, $F(s)$. Thus, we need the Second Shift Theorem since the solution is of the form $Y(s) = e^{-2s}F(s)$ for

$$F(s) = \frac{1}{s^2 + 4\pi^2}.$$

We need to find the corresponding $f(t)$ of the Laplace transform pair. The denominator in $F(s)$ suggests a sine or cosine. Since the numerator is constant, we pick sine. From the tables of transforms, we have

$$\mathcal{L}[\sin 2\pi t] = \frac{2\pi}{s^2+4\pi^2}.$$

So, we write

$$F(s) = \frac{1}{2\pi}\frac{2\pi}{s^2+4\pi^2}.$$

This gives $f(t) = (2\pi)^{-1}\sin 2\pi t$.

We now apply the Second Shift Theorem, $\mathcal{L}[f(t-a)H(t-a)] = e^{-as}F(s)$, or

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left[e^{-2s}F(s)\right] \\ &= H(t-2)f(t-2) \\ &= \frac{1}{2\pi}H(t-2)\sin 2\pi(t-2). \end{aligned} \tag{9.8.16}$$

This solution tells us that the mass is at rest until $t=2$ and then begins to oscillate at its natural frequency. A plot of this solution is shown in Figure 9.8.9.

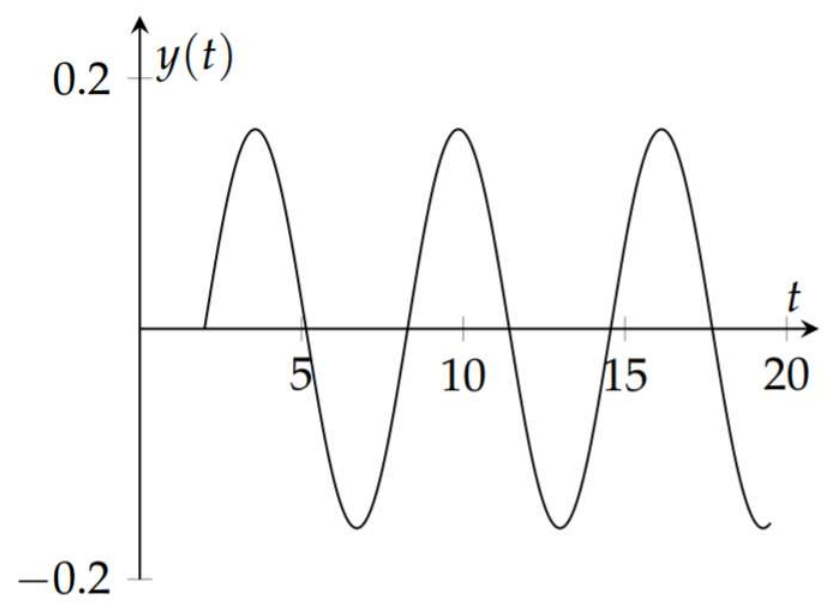


Figure 9.8.9: A plot of the solution to Example 9.8.8 in which a spring at rest experiences an impulse force at $t=2$.

Example 9.8.9

Solve the initial value problem

$$y'' + y = f(t), \quad y(0) = 0, y'(0) = 0,$$

where

$$f(t) = \begin{cases} \cos \pi t, & 0 \le t \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

Solution

We need the Laplace transform of $f(t)$. This function can be written in terms of a Heaviside function, $f(t) = \cos \pi t H(t-2)$. In order to apply the Second Shift Theorem, we need a shifted version of the cosine function. We find the shifted version by noting that $\cos \pi(t-2) = \cos \pi t$. Thus, we have

$$\begin{aligned} f(t) &= \cos \pi t[H(t) - H(t-2)] \\ &= \cos \pi t - \cos \pi(t-2)H(t-2), \quad t \ge 0. \end{aligned} \tag{9.8.17}$$

The Laplace transform of this driving term is

$$F(s) = \left(1 - e^{-2s}\right) \mathcal{L}[\cos \pi t] = \left(1 - e^{-2s}\right) \frac{s}{s^2 + \pi^2}.$$

Now we can proceed to solve the initial value problem. The Laplace transform of the initial value problem yields

$$\left(s^2 + 1\right) Y(s) = \left(1 - e^{-2s}\right) \frac{s}{s^2 + \pi^2}.$$

Therefore,

$$Y(s) = \left(1 - e^{-2s}\right) \frac{s}{\left(s^2 + \pi^2\right)\left(s^2 + 1\right)}.$$

We can retrieve the solution to the initial value problem using the Second Shift Theorem. The solution is of the form $Y(s) = \left(1 - e^{-2s}\right) G(s)$ for

$$G(s) = \frac{s}{\left(s^2 + \pi^2\right)\left(s^2 + 1\right)}.$$

Then, the final solution takes the form

$$y(t) = g(t) - g(t-2)H(t-2).$$

We only need to find $g(t)$ in order to finish the problem. This is easily done by using the partial fraction decomposition

$$G(s) = \frac{s}{\left(s^2 + \pi^2\right)\left(s^2 + 1\right)} = \frac{1}{\pi^2 - 1}\left[\frac{s}{s^2 + 1} - \frac{s}{s^2 + \pi^2}\right].$$

Then,

$$g(t) = \mathcal{L}^{-1}\left[\frac{s}{\left(s^2 + \pi^2\right)\left(s^2 + 1\right)}\right] = \frac{1}{\pi^2 - 1}(\cos t - \cos \pi t).$$

The final solution is then given by

$$y(t) = \frac{1}{\pi^2 - 1}[\cos t - \cos \pi t - H(t-2)(\cos(t-2) - \cos \pi t)].$$

A plot of this solution is shown in Figure 9.8.10

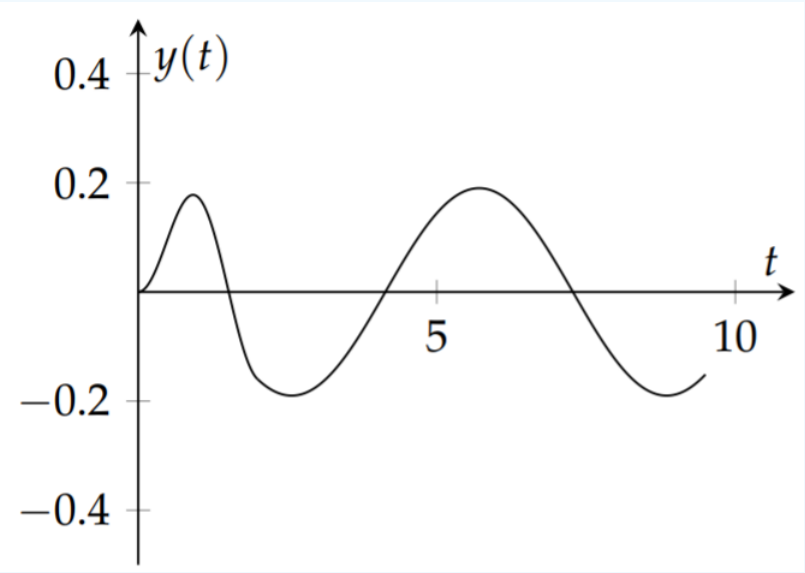


Figure 9.8.10: A plot of the solution to Example 9.8.9 in which a spring at rest experiences an piecewise defined force.

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9.9: The Convolution Theorem

Finally, we consider the convolution of two functions. Often we are faced with having the product of two Laplace transforms that we know and we seek the inverse transform of the product. For example, let's say we have obtained $Y(s) = \frac{1}{(s-1)(s-2)}$ while trying to solve an initial value problem. In this case we could find a partial fraction decomposition. But, are other ways to find the inverse transform, especially if we cannot perform a partial fraction decomposition. We could use the Convolution Theorem for Laplace transforms or we could compute the inverse transform directly. We will look into these methods in the next two sections. We begin with defining the convolution.

We define the convolution of two functions defined on $[0, \infty)$ much the same way as we had done for the Fourier transform. The convolution $f * g$ is defined as

$$(f * g)(t) = \int_0^t f(u)g(t-u)du. \tag{9.9.1}$$

Note that the convolution integral has finite limits as opposed to the Fourier transform case.

The convolution operation has two important properties:

1. The convolution is commutative: $f * g = g * f$
   Proof. The key is to make a substitution $y = t - u$ in the integral. This makes f a simple function of the integration variable.

$$\begin{aligned}(g * f)(t) &= \int_0^t g(u)f(t-u)du \\ &= -\int_t^0 g(t-y)f(y)dy \\ &= \int_0^t f(y)g(t-y)dy \\ &= (f * g)(t).\end{aligned} \tag{9.9.2}$$

2. The Convolution Theorem: The Laplace transform of a convolution is the product of the Laplace transforms of the individual functions:

$$\mathcal{L}[f * g] = F(s)G(s)$$

   Proof. Proving this theorem takes a bit more work. We will make some assumptions that will work in many cases. First, we assume that the functions are causal, $f(t) = 0$ and $g(t) = 0$ for $t < 0$. Secondly, we will assume that we can interchange integrals, which needs more rigorous attention than will be provided here. The first assumption will allow us to write the finite integral as an infinite integral. Then a change of variables will allow us to split the integral into the product of two integrals that are recognized as a product of two Laplace transforms.
   Carrying out the computation, we have

$$\begin{aligned}\mathcal{L}[f * g] &= \int_0^\infty \left(\int_0^t f(u)g(t-u)du \right) e^{-st}\,dt \\ &= \int_0^\infty \left(\int_0^\infty f(u)g(t-u)du \right) e^{-st}\,dt \\ &= \int_0^\infty f(u) \left(\int_0^\infty g(t-u)e^{-st}\,dt \right) du\end{aligned} \tag{9.9.3}$$

   Now, make the substitution $\tau = t - u$. We note that $\int_0^\infty f(u) \left(\int_0^\infty g(t-u)e^{-st}\,dt \right) du = \int_0^\infty f(u) \left(\int_{-u}^\infty g(\tau)e^{-s(\tau+u)}\,d\tau \right) du$ However, since $g(\tau)$ is a causal function, we have that it vanishes for $\tau < 0$ and we can change the integration interval to $[0, \infty)$. So, after a little rearranging, we can proceed to the result.

$$\begin{aligned}\mathcal{L}[f*g] &= \int_0^\infty f(u)\left(\int_0^\infty g(\tau)e^{-s(\tau+u)}\,d\tau\right)du \\ &= \int_0^\infty f(u)e^{-su}\left(\int_0^\infty g(\tau)e^{-s\tau}\,d\tau\right)du \\ &= \left(\int_0^\infty f(u)e^{-su}du\right)\left(\int_0^\infty g(\tau)e^{-s\tau}\,d\tau\right) \\ &= F(s)G(s). \end{aligned} \tag{9.9.4}$$

We make use of the Convolution Theorem to do the following examples.

Example 9.9.1

Find $y(t)=\mathcal{L}^{-1}\left[\frac{1}{(s-1)(s-2)}\right]$.

Solution

We note that this is a product of two functions

$$Y(s)=\frac{1}{(s-1)(s-2)}=\frac{1}{s-1}\,\frac{1}{s-2}=F(s)G(s).$$

We know the inverse transforms of the factors: $f(t)=e^t$ and $g(t)=e^{2t}$.

Using the Convolution Theorem, we find $y(t)=(f*g)(t)$. We compute the convolution:

$$\begin{aligned} y(t) &= \int_0^t f(u)g(t-u)du \\ &= \int_0^t e^u e^{2(t-u)}\,du \\ &= e^{2t}\int_0^t e^{-u}du \\ &= e^{2t}\left[-e^t+1\right]=e^{2t}-e^t. \end{aligned} \tag{9.9.5}$$

One can also confirm this by carrying out a partial fraction decomposition.

Example 9.9.2

Consider the initial value problem, $y''+9y=2\sin 3t, y(0)=1$, $y'(0)=0$.

Solution

The Laplace transform of this problem is given by

$$\left(s^2+9\right)Y-s=\frac{6}{s^2+9}.$$

Solving for $Y(s)$, we obtain

$$Y(s)=\frac{6}{\left(s^2+9\right)^2}+\frac{s}{s^2+9}.$$

The inverse Laplace transform of the second term is easily found as $\cos(3t)$; however, the first term is more complicated.

We can use the Convolution Theorem to find the Laplace transform of the first term. We note that

$$\frac{6}{\left(s^2+9\right)^2}=\frac{2}{3}\,\frac{3}{(s^2+9)}\,\frac{3}{(s^2+9)}$$

is a product of two Laplace transforms (up to the constant factor). Thus,

$$\mathcal{L}^{-1}\left[\frac{6}{(s^2+9)^2}\right] = \frac{2}{3}(f*g)(t),$$

where $f(t) = g(t) = \sin 3t$. Evaluating this convolution product, we have

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{6}{(s^2+9)^2}\right] &= \frac{2}{3}(f*g)(t)\\ &= \frac{2}{3}\int_0^t \sin 3u \sin 3(t-u)du\\ &= \frac{1}{3}\int_0^t [\cos 3(2u-t) - \cos 3t]du\\ &= \frac{1}{3}\left[\frac{1}{6}\sin(6u-3t) - u\cos 3t\right]_0^t\\ &= \frac{1}{9}\sin 3t - \frac{1}{3}t\cos 3t. \end{aligned} \tag{9.9.6}$$

Combining this with the inverse transform of the second term of $Y(s)$, the solution to the initial value problem is

$$y(t) = -\frac{1}{3}t\cos 3t + \frac{1}{9}\sin 3t + \cos 3t.$$

Note that the amplitude of the solution will grow in time from the first term. You can see this in Figure 9.9.1. This is known as a resonance.

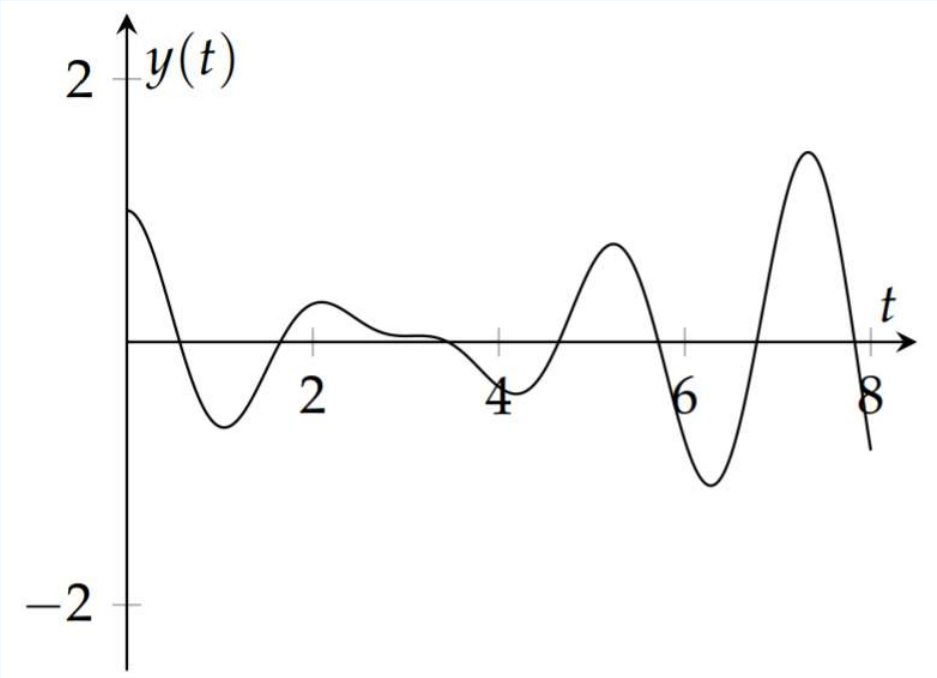


Figure 9.9.1: Plot of the solution to Example 9.9.2 showing a resonance.

Example 9.9.3

Find $\mathcal{L}^{-1}\left[\frac{6}{(s^2+9)^2}\right]$ using partial fraction decomposition.

Solution

If we look at Table 9.7.2, we see that the Laplace transform pairs with the denominator $\left(s^2+\omega^2\right)^2$ are

$$\mathcal{L}[t\sin\omega t] = \frac{2\omega s}{\left(s^2+\omega^2\right)^2}$$

and

$$\mathcal{L}[t\cos\omega t] = \frac{s^2-\omega^2}{\left(s^2+\omega^2\right)^2}.$$

So, we might consider rewriting a partial fraction decomposition as

$$\frac{6}{\left(s^2+9\right)^2} = \frac{A6s}{\left(s^2+9\right)^2} + \frac{B\left(s^2-9\right)}{\left(s^2+9\right)^2} + \frac{Cs+D}{s^2+9}.$$

Combining the terms on the right over a common denominator, we find

$$6 = 6As + B\left(s^2-9\right) + (Cs+D)\left(s^2+9\right).$$

Collecting like powers of s, we have

$$Cs^3 + (D+B)s^2 + 6As + (D-B) = 6.$$

Therefore, $C=0, A=0, D+B=0$, and $D-B=\frac{2}{3}$. Solving the last two equations, we find $D=-B=\frac{1}{3}$.

Using these results, we find

$$\frac{6}{\left(s^2+9\right)^2} = -\frac{1}{3}\frac{\left(s^2-9\right)}{\left(s^2+9\right)^2} + \frac{1}{3}\frac{1}{s^2+9}.$$

This is the result we had obtained in the last example using the Convolution Theorem.

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9.10: The Inverse Laplace Transform

Until this point we have seen that the inverse Laplace transform can be found by making use of Laplace transform tables and properties of Laplace transforms. This is typically the way Laplace transforms are taught and used in a differential equations course. One can do the same for Fourier transforms. However, in the case of Fourier transforms we introduced an inverse transform in the form of an integral. Does such an inverse integral transform exist for the Laplace transform? Yes, it does! In this section we will derive the inverse Laplace transform integral and show how it is used.

We begin by considering a causal function $f(t)$ which vanishes for $t<0$ and define the function $g(t)=f(t)e^{-ct}$ with $c>0$. For $g(t)$ absolutely integrable,

$$\int_{-\infty}^{\infty}|g(t)|dt=\int_{0}^{\infty}|f(t)|e^{-ct}dt<\infty,$$

we can write the Fourier transform,

$$\hat{g}(\omega)=\int_{-\infty}^{\infty}g(t)e^{i\omega t}dt=\int_{0}^{\infty}f(t)e^{i\omega t-ct}\,dt$$

and the inverse Fourier transform,

$$g(t)=f(t)e^{-ct}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{g}(\omega)e^{-i\omega t}d\omega.$$

Multiplying by e^{ct} and inserting $\hat{g}(\omega)$ into the integral for $g(t)$, we find

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{0}^{\infty}f(\tau)e^{(i\omega-c)\tau}d\tau e^{-(i\omega-c)t}d\omega.$$

Letting $s=c-i\omega($ so $d\omega=ids)$, we have

$$f(t)=\frac{i}{2\pi}\int_{c+i\infty}^{c-i\infty}\int_{0}^{\infty}f(\tau)e^{-s\tau}d\tau e^{st}ds$$

Note that the inside integral is simply $F(s)$. So, we have

$$f(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}F(s)e^{st}ds. \tag{9.10.1}$$

The integral in the last equation is the inverse Laplace transform, called the Bromwich integral and is named after Thomas John I'Anson Bromwich (1875-1929). This inverse transform is not usually covered in differential equations courses because the integration takes place in the complex plane. This integral is evaluated along a path in the complex plane called the Bromwich contour. The typical way to compute this integral is to first chose c so that all poles are to the left of the contour. This guarantees that $f(t)$ is of exponential type. The contour is closed a semicircle enclosing all of the poles. One then relies on a generalization of Jordan's lemma to the second and third quadrants.[1]

Note

Closing the contour to the left of the contour can be reasoned in a manner similar to what we saw in Jordan's Lemma. Write the exponential as $e^{st}=e^{(s_R+is_I)t}=e^{s_Rt}e^{is_It}$. The second factor is an oscillating factor and the growth in the exponential can only come from the first factor. In order for the exponential to decay as the radius of the semicircle grows, $s_Rt<0$. Since $t>0$, we need $s<0$ which is done by closing the contour to the left. If $t<0$, then the contour to the right would enclose no singularities and preserve the causality of $f(t)$.

Example 9.10.1

Find the inverse Laplace transform of $F(s)=\frac{1}{s(s+1)}$.

Solution

The integral we have to compute is

$$f(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{st}}{s(s+1)}ds$$

This integral has poles at $s=0$ and $s=-1$. The contour we will use is shown in Figure 9.10.1. We enclose the contour with a semicircle to the left of the path in the complex s-plane. One has to verify that the integral over the semicircle vanishes as the radius goes to infinity. Assuming that we have done this, then the result is simply obtained as $2\pi i$ times the sum of the residues. The residues in this case are:

$$\text{Res}\left[\frac{e^{zt}}{z(z+1)};z=0\right]=\lim_{z\to 0}\frac{e^{zt}}{(z+1)}=1$$

and

$$\text{Res}\left[\frac{e^{zt}}{z(z+1)};z=-1\right]=\lim_{z\to -1}\frac{e^{zt}}{z}=-e^{-t}.$$

Therefore, we have

$$f(t)=2\pi i\left[\frac{1}{2\pi i}(1)+\frac{1}{2\pi i}\left(-e^{-t}\right)\right]=1-e^{-t}.$$

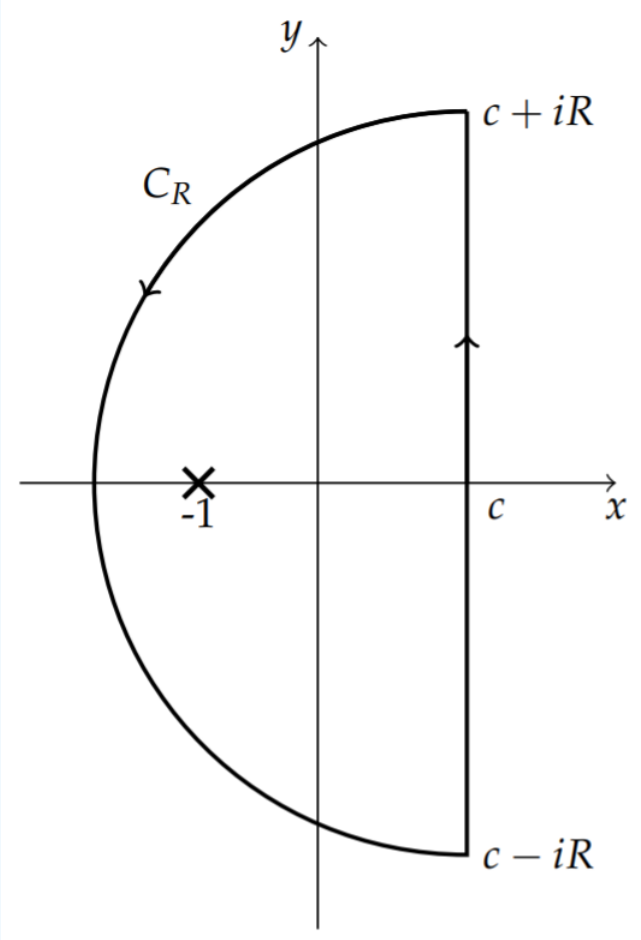


Figure 9.10.1: The contour used for applying the Bromwich integral to the Laplace transform $F(s)=\frac{1}{s(s+1)}$.

We can verify this result using the Convolution Theorem or using a partial fraction decomposition. The latter method is simplest. We note that

$$\frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}.$$

The first term leads to an inverse transform of 1 and the second term gives e^{-t}. So,

$$\mathcal{L}^{-1}\left[\frac{1}{s} - \frac{1}{s+1}\right] = 1 - e^{-t}.$$

Thus, we have verified the result from doing contour integration.

Example 9.10.2

Find the inverse Laplace transform of $F(s) = \frac{1}{s(1+e^s)}$.

Solution

In this case, we need to compute

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{st}}{s\left(1+e^s\right)} ds.$$

This integral has poles at complex values of s such that $1 + e^s = 0$, or $e^s = -1$. Letting $s = x + iy$, we see that

$$e^s = e^{x+iy} = e^x(\cos y + i \sin y) = -1.$$

We see $x = 0$ and y satisfies $\cos y = -1$ and $\sin y = 0$. Therefore, $y = n\pi$ for n an odd integer. Therefore, the integrand has an infinite number of simple poles at $s = n\pi i, n = \pm 1, \pm 3, \ldots$. It also has a simple pole at $s = 0$.

In Figure 9.10.2we indicate the poles. We need to compute the resides at each pole. At $s = n\pi i$ we have

$$\begin{aligned} \text{Res}\left[\frac{e^{st}}{s(1+e^s)};\ s = n\pi i\right] &= \lim_{s\to n\pi i}(s - n\pi i)\frac{e^{st}}{s(1+e^s)} \\ &= \lim_{s\to n\pi i}\frac{e^{st}}{se^s} \\ &= -\frac{e^{n\pi it}}{n\pi i}, \quad n \text{ odd.} \end{aligned} \tag{9.10.2}$$

At $s = 0$, the residue is

$$\text{Res}\left[\frac{e^{st}}{s\left(1+e^s\right)}; s = 0\right] = \lim_{s\to 0}\frac{e^{st}}{1+e^s} = \frac{1}{2}.$$

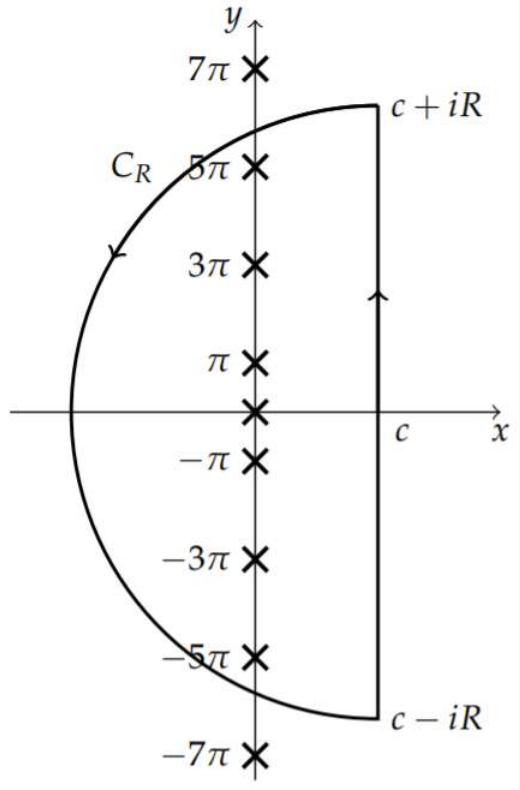


Figure 9.10.2: The contour used for applying the Bromwich integral to the Laplace transform $F(s) = \frac{1}{1+e^s}$.

Summing the residues and noting the exponentials for $\pm n$ can be combined to form sine functions, we arrive at the inverse transform.

$$\begin{aligned} f(t) &= \frac{1}{2} - \sum_{n \text{ odd}} \frac{e^{n\pi it}}{n\pi i} \\ &= \frac{1}{2} - 2\sum_{k=1}^{\infty} \frac{\sin(2k-1)\pi t}{(2k-1)\pi}. \end{aligned} \tag{9.10.3}$$

The series in this example might look familiar. It is a Fourier sine series with odd harmonics whose amplitudes decay like $1/n$. It is a vertically shifted square wave. In fact, we had computed the Laplace transform of a general square wave in Example 9.8.7.

In that example we found

$$\begin{aligned} \mathcal{L}\left[\sum_{n=0}^{\infty}[H(t-2na) - H(t-(2n+1)a)]\right] &= \frac{1-e^{-as}}{s\left(1-e^{-2as}\right)} \\ &= \frac{1}{s\left(1+e^{-as}\right)}. \end{aligned} \tag{9.10.4}$$

In this example, one can show that

$$f(t) = \sum_{n=0}^{\infty}[H(t-2n+1) - H(t-2n)].$$

The reader should verify that this result is indeed the square wave shown in Figure 9.10.3

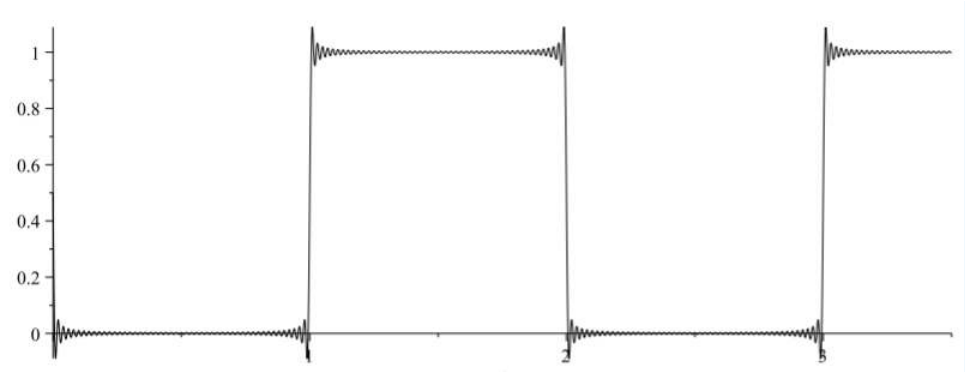


Figure 9.10.3: Plot of the square wave result as the inverse Laplace transform of $F(s) = \frac{1}{s(1+e^s)}$ with 50 terms.

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9.11: Transforms and Partial Differential Equations

As another application of the transforms, we will see that we can use transforms to solve some linear partial differential equations. We will first solve the one dimensional heat equation and the two dimensional Laplace equations using Fourier transforms. The transforms of the partial differential equations lead to ordinary differential equations which are easier to solve. The final solutions are then obtained using inverse transforms.

We could go further by applying a Fourier transform in space and a Laplace transform in time to convert the heat equation into an algebraic equation. We will also show that we can use a finite sine transform to solve nonhomogeneous problems on finite intervals. Along the way we will identify several Green's functions.

Fourier Transform and the Heat Equation

We will first consider the solution of the heat equation on an infinite interval using Fourier transforms. The basic scheme has been discussed earlier and is outlined in Figure 9.11.1.

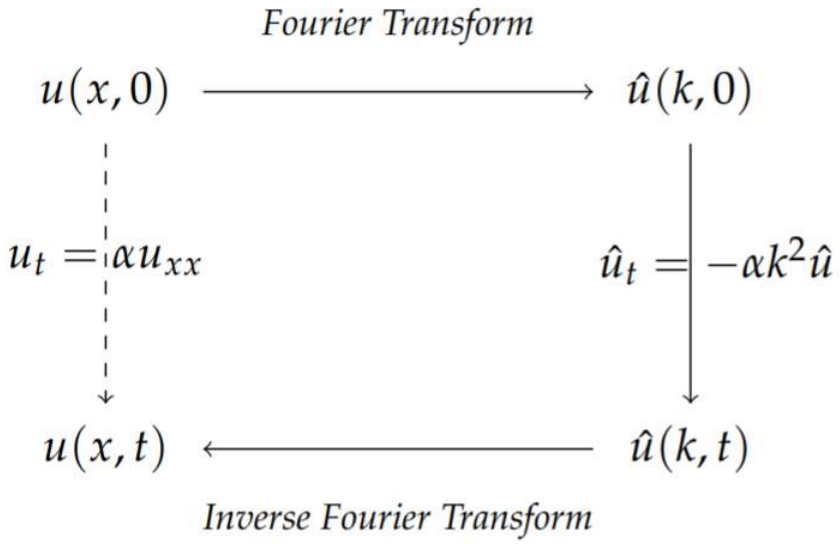


Figure 9.11.1: Using Fourier transforms to solve a linear partial differential equation.

Consider the heat equation on the infinite line,

$$\begin{aligned} &u_t = \alpha u_{xx}, && -\infty < x < \infty, t > 0. \\ &u(x,0) = f(x), && -\infty < x < \infty. \end{aligned} \tag{9.11.1}$$

We can Fourier transform the heat equation using the Fourier transform of $u(x,t)$,

$$\mathcal{F}[u(x,t)] = \hat{u}(k,t) = \int_{-\infty}^{\infty} u(x,t) e^{ikx} dx$$

We need to transform the derivatives in the equation. First we note that

$$\begin{aligned} \mathcal{F}[u_t] &= \int_{-\infty}^{\infty} \frac{\partial u(x,t)}{\partial t} e^{ikx} dx \\ &= \frac{\partial}{\partial t} \int_{-\infty}^{\infty} u(x,t) e^{ikx} dx \\ &= \frac{\partial \hat{u}(k,t)}{\partial t} \end{aligned} \tag{9.11.2}$$

Assuming that $\lim_{|x|\to\infty} u(x,t) = 0$ and $\lim_{|x|\to\infty} u_x(x,t) = 0$, then we also have that

$$\begin{aligned} \mathcal{F}[u_{xx}] &= \int_{-\infty}^{\infty} \frac{\partial^2 u(x,t)}{\partial x^2} e^{ikx} dx \\ &= -k^2 \hat{u}(k,t) \end{aligned} \tag{9.11.3}$$

Therefore, the heat equation becomes

$$\frac{\partial \hat{u}(k,t)}{\partial t} = -\alpha k^2 \hat{u}(k,t)$$

This is a first order differential equation which is readily solved as

$$\hat{u}(k,t) = A(k)e^{-\alpha k^2 t}$$

where $A(k)$ is an arbitrary function of k. The inverse Fourier transform is

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}(k,t)e^{-ikx}\,dk \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{A}(k)e^{-\alpha k^2 t}e^{-ikx}\,dk \end{aligned} \tag{9.11.4}$$

We can determine $A(k)$ using the initial condition. Note that

$$\mathcal{F}[u(x,0)] = \hat{u}(k,0) = \int_{-\infty}^{\infty} f(x)e^{ikx}\,dx.$$

But we also have from the solution,

$$u(x,0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{A}(k)e^{-ikx}\,dk.$$

Comparing these two expressions for $\hat{u}(k,0)$, we see that

$$A(k) = \mathcal{F}[f(x)].$$

We note that $\hat{u}(k,t)$ is given by the product of two Fourier transforms, $\hat{u}(k,t) = A(k)e^{-\alpha k^2 t}$. So, by the Convolution Theorem, we expect that $u(x,t)$ is the convolution of the inverse transforms,

$$u(x,t) = (f*g)(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi,t)g(x-\xi,t)d\xi,$$

where

$$g(x,t) = \mathcal{F}^{-1}\left[e^{-\alpha k^2 t}\right]$$

In order to determine $g(x,t)$, we need only recall Example 9.5.1. In that example we saw that the Fourier transform of a Gaussian is a Gaussian. Namely, we found that

$$\mathcal{F}\left[e^{-ax^2/2}\right] = \sqrt{\frac{2\pi}{a}}e^{-k^2/2a},$$

or,

$$\mathcal{F}^{-1}\left[\sqrt{\frac{2\pi}{a}}e^{-k^2/2a}\right] = e^{-ax^2/2}.$$

Applying this to the current problem, we have

$$g(x) = \mathcal{F}^{-1}\left[e^{-\alpha k^2 t}\right] = \sqrt{\frac{\pi}{\alpha t}}e^{-x^2/4t}.$$

Finally, we can write down the solution to the problem:

$$u(x,t) = (f*g)(x,t) = \int_{-\infty}^{\infty} f(\xi,t)\frac{e^{-(x-\xi)^2/4t}}{\sqrt{4\pi\alpha t}}d\xi,$$

The function in the integrand,

$$K(x,t) = \frac{e^{-x^2/4t}}{\sqrt{4\pi\alpha t}}$$

is called the heat kernel.

Laplace's Equation on the Half Plane

We consider a steady state solution in two dimensions. In particular, we look for the steady state solution, $u(x,y)$, satisfying the two-dimensional Laplace equation on a semi-infinite slab with given boundary conditions as shown in Figure 9.11.2 The boundary value problem is given as

$$\begin{aligned} u_{xx} + u_{yy} &= 0, \quad -\infty < x < \infty, \quad y > 0, \\ u(x,0) &= f(x), \quad -\infty < x < \infty \\ \lim_{y\to\infty} u(x,y) &= 0, \quad \lim_{|x|\to\infty} u(x,y) = 0. \end{aligned} \tag{9.11.5}$$

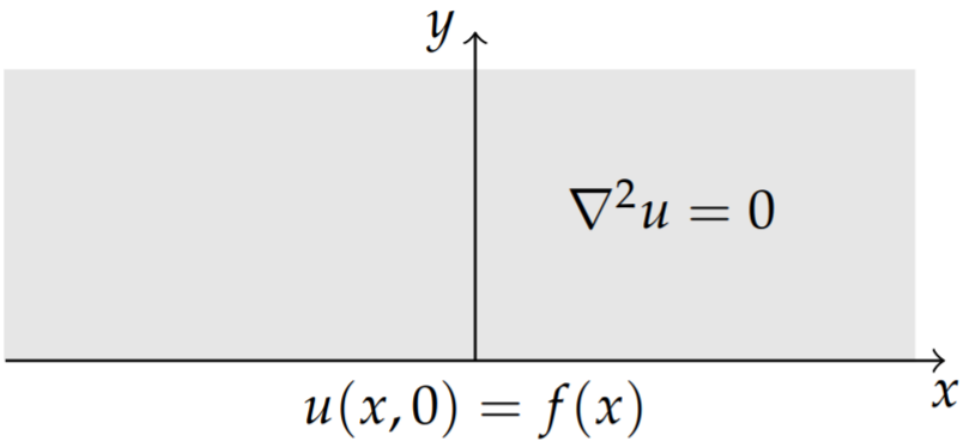


Figure 9.11.2: This is the domain for a semi-infinite slab with boundary value $u(x,0) = f(x)$ and governed by Laplace's equation.

This problem can be solved using a Fourier transform of $u(x,y)$ with respect to x. The transform scheme for doing this is shown in Figure 9.11.3 We begin by defining the Fourier transform

$$\hat{u}(k,y) = \mathcal{F}[u] = \int_{-\infty}^{\infty} u(x,y) e^{ikx}\, dx.$$

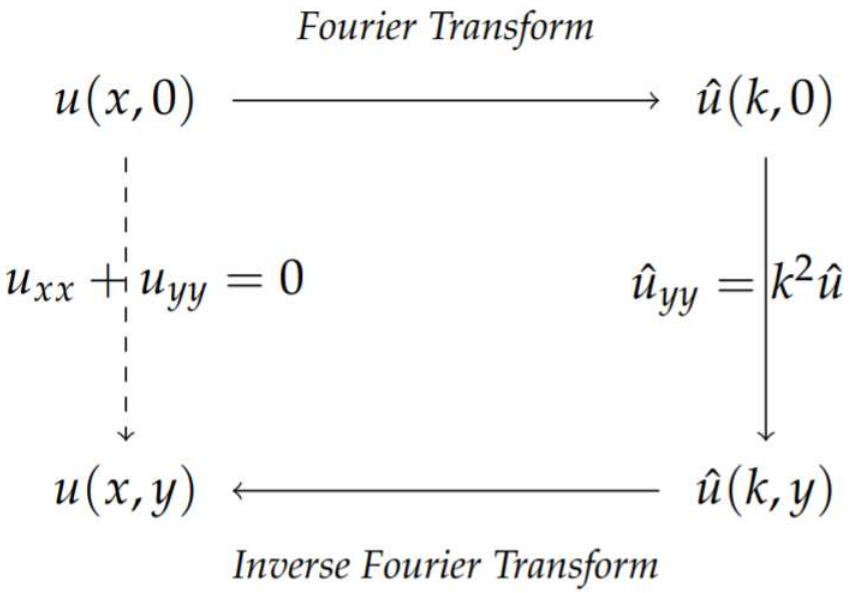


Figure 9.11.3: The transform scheme used to convert Laplace's equation to an ordinary differential equation which is easier to solve.

We can transform Laplace's equation. We first note from the properties of Fourier transforms that

$$\mathcal{F}\left[\frac{\partial^2 u}{\partial x^2}\right] = -k^2\hat{u}(k,y),$$

if $\lim_{|x|\to\infty} u(x,y) = 0$ and $\lim_{|x|\to\infty} u_x(x,y) = 0$. Also,

$$\mathcal{F}\left[\frac{\partial^2 u}{\partial y^2}\right] = \frac{\partial^2 \hat{u}(k,y)}{\partial y^2}.$$

Thus, the transform of Laplace's equation gives $\hat{u}_{yy} = k^2\hat{u}$.

This is a simple ordinary differential equation. We can solve this equation using the boundary conditions. The general solution is

$$\hat{u}(k,y) = a(k)e^{ky} + b(k)e^{-ky}.$$

Since $\lim_{y\to\infty} u(x,y) = 0$ and k can be positive or negative, we have that $\hat{u}(k,y) = a(k)e^{-|k|y}$. The coefficient $a(k)$ can be determined using the remaining boundary condition, $u(x,0) = f(x)$. We find that $a(k) = \hat{f}\,(k)$ since

$$a(k) = \hat{u}(k,0) = \int_{-\infty}^{\infty} u(x,0)e^{ikx}dx = \int_{-\infty}^{\infty} f(x)e^{ikx}dx = \hat{f}\,(k).$$

We have found that $\hat{u}(k,y) = \hat{f}\,(k)e^{-|k|y}$. We can obtain the solution using the inverse Fourier transform,

$$u(x,t) = \mathcal{F}^{-1}\left[\hat{f}\,(k)e^{-|k|y}\right].$$

We note that this is a product of Fourier transforms and use the Convolution Theorem for Fourier transforms. Namely, we have that $a(k) = \mathcal{F}[f]$ and $e^{-|k|y} = \mathcal{F}[g]$ for $g(x,y) = \frac{2y}{x^2+y^2}$. This last result is essentially proven in Problem 6.

Then, the Convolution Theorem gives the solution

$$\begin{aligned} u(x,y) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)g(x-\xi)d\xi \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)\frac{2y}{(x-\xi)^2+y^2}d\xi \end{aligned} \tag{9.11.6}$$

We note for future use, that this solution is in the form

$$u(x,y) = \int_{-\infty}^{\infty} f(\xi)G(x,\xi;y,0)d\xi,$$

where

$$G(x,\xi;y,0) = \frac{2y}{\pi\left((x-\xi)^2+y^2\right)}$$

is the Green's function for this problem.

Heat Equation on Infinite Interval, Revisited

We will reconsider the initial value problem for the heat equation on an infinite interval,

$$\begin{aligned} &u_t = u_{xx}, && -\infty < x < \infty, \quad t > 0 \\ &u(x,0) = f(x), && -\infty < x < \infty. \end{aligned} \tag{9.11.7}$$

We can apply both a Fourier and a Laplace transform to convert this to an algebraic problem. The general solution will then be written in terms of an initial value Green's function as

$$u(x,t) = \int_{-\infty}^{\infty} G(x,t;\xi,0)f(\xi)d\xi.$$

For the time dependence we can use the Laplace transform and for the spatial dependence we use the Fourier transform. These combined transforms lead us to define

$$\hat{u}(k,s) = \mathcal{F}[\mathcal{L}[u]] = \int_{-\infty}^{\infty}\int_{0}^{\infty} u(x,t)e^{-st}e^{ikx}\,dtdx.$$

Applying this to the terms in the heat equation, we have

$$\begin{aligned}\mathcal{F}\left[\mathcal{L}\left[u_t\right]\right] &= s\hat{u}(k,s) - \mathcal{F}[u(x,0)] \\ &= s\hat{u}(k,s) - \hat{f}\,(k) \\ \mathcal{F}\left[\mathcal{L}\left[u_{xx}\right]\right] &= -k^2\hat{u}(k,s). \end{aligned} \tag{9.11.8}$$

Here we have assumed that

$$\lim_{t\to\infty} u(x,t)e^{-st} = 0, \quad \lim_{|x|\to\infty} u(x,t) = 0, \quad \lim_{|x|\to\infty} u_x(x,t) = 0.$$

Therefore, the heat equation can be turned into an algebraic equation for the transformed solution,

$$\left(s+k^2\right)\hat{u}(k,s) = \hat{f}\,(k),$$

or

$$\hat{u}(k,s) = \frac{\hat{f}\,(k)}{s+k^2}.$$

The solution to the heat equation is obtained using the inverse transforms for both the Fourier and Laplace transform. Thus, we have

$$\begin{aligned} u(x,t) &= \mathcal{F}^{-1}\left[\mathcal{L}^{-1}[\hat{u}]\right] \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{1}{2\pi i}\int_{c-i\infty}^{c+\infty}\frac{\hat{f}\,(k)}{s+k^2}e^{st}\,ds\right)e^{-ikx}\,dk. \end{aligned} \tag{9.11.9}$$

Since the inside integral has a simple pole at $s=-k^2$, we can compute the Bromwich integral by choosing $c > -k^2$. Thus,

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+\infty}\frac{\hat{f}\,(k)}{s+k^2}e^{st}\,ds = \operatorname{Res}\left[\frac{\hat{f}\,(k)}{s+k^2}e^{st}; s=-k^2\right] = e^{-k^2t}\hat{f}\,(k).$$

Inserting this result into the solution, we have

$$\begin{aligned} u(x,t) &= \mathcal{F}^{-1}\left[\mathcal{L}^{-1}[\hat{u}]\right] \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}\,(k)e^{-k^2t}e^{-ikx}\,dk. \end{aligned} \tag{9.11.10}$$

This solution is of the form

$$u(x,t) = \mathcal{F}^{-1}[\hat{f}\,\hat{g}]$$

for $\hat{g}(k) = e^{-k^2t}$. So, by the Convolution Theorem for Fourier transforms, the solution is a convolution,

$$u(x,t) = \int_{-\infty}^{\infty} f(\xi)g(x-\xi)d\xi$$

All we need is the inverse transform of $\hat{g}(k)$.

We note that $\hat{g}(k) = e^{-k^2t}$ is a Gaussian. Since the Fourier transform of a Gaussian is a Gaussian, we need only recall Example 9.5.1,

$$\mathcal{F}\left[e^{-ax^2/2}\right] = \sqrt{\frac{2\pi}{a}}e^{-k^2/2a}.$$

Setting $a = 1/2t$, this becomes

$$\mathcal{F}\left[e^{-x^2/4t}\right] = \sqrt{4\pi t}e^{-k^2t}.$$

So,

$$g(x) = \mathcal{F}^{-1}\left[e^{-k^2 t}\right] = \frac{e^{-x^2/4t}}{\sqrt{4\pi t}}$$

Inserting $g(x)$ into the solution, we have

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} f(\xi) e^{-(x-\xi)^2/4t} d\xi \\ &= \int_{-\infty}^{\infty} G(x,t;\zeta,0) f(\xi) d\xi \end{aligned} \tag{9.11.11}$$

Here we have identified the initial value Green's function

$$G(x,t;\xi,0) = \frac{1}{\sqrt{4\pi t}} e^{-(x-\xi)^2/4t}.$$

Nonhomogeneous Heat Equation

We now consider the nonhomogeneous heat equation with homogeneous boundary conditions defined on a finite interval.

$$\begin{aligned} u_t - k u_{xx} &= h(x,t), \quad 0 \le x \le L, \quad t > 0 \\ u(0,t) &= 0, \quad u(L,t) = 0, \quad t > 0 \\ u(x,0) &= f(x), \quad 0 \le x \le \end{aligned} \tag{9.11.12}$$

We know that when $h(x,t) \equiv 0$ the solution takes the form

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

So, when $h(x,t) \neq 0$, we might assume that the solution takes the form

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin \frac{n\pi x}{L}$$

where the b_n' s are the finite Fourier sine transform of the desired solution,

$$b_n(t) = \mathcal{F}_s[u] = \frac{2}{L} \int_0^L u(x,t) \sin \frac{n\pi x}{L} dx$$

Note that the finite Fourier sine transform is essentially the Fourier sine transform which we encountered in Section 3.4.

The idea behind using the finite Fourier Sine Transform is to solve the given heat equation by transforming the heat equation to a simpler equation for the transform, $b_n(t)$, solve for $bn(t)$, and then do an inverse transform, i.e., insert the $b_n(t)$'s back into the series representation. This is depicted in Figure 9.11.4. Note that we had explored similar diagram earlier when discussing the use of transforms to solve differential equations.

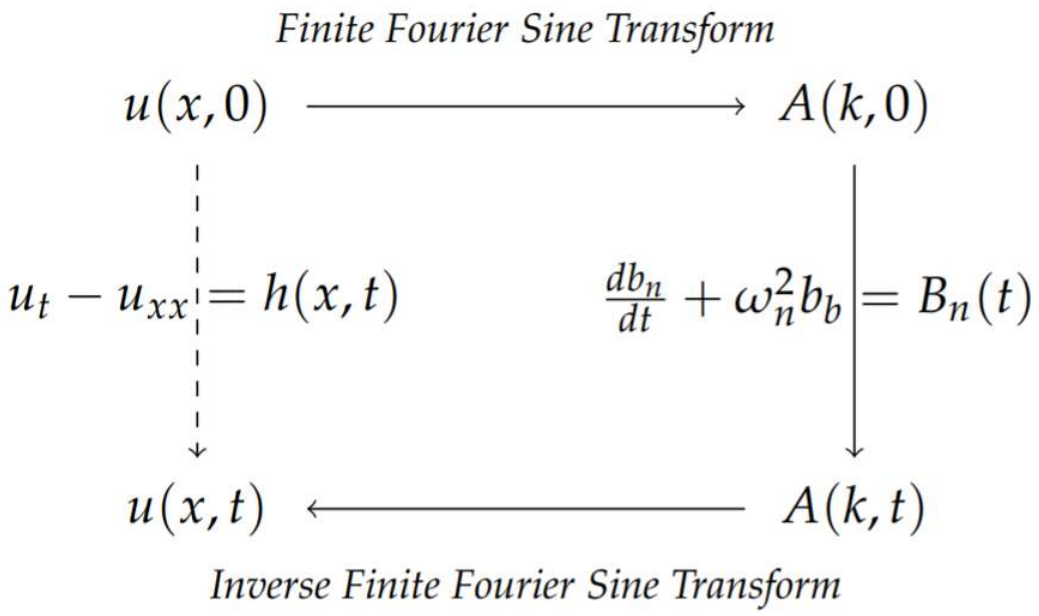


Figure 9.11.4: Using finite Fourier transforms to solve the heat equation by solving an ODE instead of a PDE.

First, we need to transform the partial differential equation. The finite transforms of the derivative terms are given by

$$\begin{aligned}\mathcal{F}_s\left[u_t\right] &= \frac{2}{L}\int_0^L \frac{\partial u}{\partial t}(x,t)\sin\frac{n\pi x}{L}dx \\ &= \frac{d}{dt}\left(\frac{2}{L}\int_0^L u(x,t)\sin\frac{n\pi x}{L}dx\right) \\ &= \frac{db_n}{dt}.\end{aligned} \tag{9.11.13}$$

$$\begin{aligned}\mathcal{F}_s\left[u_{xx}\right] &= \frac{2}{L}\int_0^L \frac{\partial^2 u}{\partial x^2}(x,t)\sin\frac{n\pi x}{L}dx \\ &= \left[u_x\sin\frac{n\pi x}{L}\right]_0^L - \left(\frac{n\pi}{L}\right)\frac{2}{L}\int_0^L \frac{\partial u}{\partial x}(x,t)\cos\frac{n\pi x}{L}dx \\ &= -\left[\frac{n\pi}{L}u\cos\frac{n\pi x}{L}\right]_0^L - \left(\frac{n\pi}{L}\right)^2\frac{2}{L}\int_0^L u(x,t)\sin\frac{n\pi x}{L}dx \\ &= \frac{n\pi}{L}[u(0,t)-u(L,0)\cos n\pi] - \left(\frac{n\pi}{L}\right)^2 b_n^2 \\ &= -\omega_n^2 b_{n,}^2\end{aligned} \tag{9.11.14}$$

where $\omega_n = \frac{n\pi}{L}$.

Furthermore, we define

$$H_n(t) = \mathcal{F}_s[h] = \frac{2}{L}\int_0^L h(x,t)\sin\frac{n\pi x}{L}dx.$$

Then, the heat equation is transformed to

$$\frac{db_n}{dt} + \omega_n^2 b_n = H_n(t), \quad n = 1,2,3,\ldots.$$

This is a simple linear first order differential equation. We can supplement this equation with the initial condition

$$b_n(0) = \frac{2}{L}\int_0^L u(x,0)\sin\frac{n\pi x}{L}dx.$$

The differential equation for b_n is easily solved using the integrating factor, $\mu(t) = e^{\omega_n^2 t}$. Thus,

$$\frac{d}{dt}\left(e^{\omega_n^2 t}b_n(t)\right) = H_n(t)e^{\omega_n^2 t}$$

and the solution is

$$b_n(t) = b_n(0)e^{-\omega_n^2 t} + \int_0^t H_n(\tau)e^{-\omega_n^2(t-\tau)}\,d\tau$$

The final step is to insert these coefficients (finite Fourier sine transform) into the series expansion (inverse finite Fourier sine transform) for $u(x,t)$. The result is

$$u(x,t) = \sum_{n=1}^{\infty} b_n(0)e^{-\omega_n^2 t}\sin\frac{n\pi x}{L} + \sum_{n=1}^{\infty}\left[\int_0^t H_n(\tau)e^{-\omega_n^2(t-\tau)}\,d\tau\right]\sin\frac{n\pi x}{L}.$$

This solution can be written in a more compact form in order to identify the Green's function. We insert the expressions for $b_n(0)$ and $H_n(t)$ in terms of the initial profile and source term and interchange sums and integrals. This leads to

$$\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty}\left(\frac{2}{L}\int_0^L u(\xi,0)\sin\frac{n\pi\xi}{L}d\xi\right)e^{-\omega_n^2 t}\sin\frac{n\pi x}{L} \\
&\quad + \sum_{n=1}^{\infty}\left[\int_0^t\left(\frac{2}{L}\int_0^L h(\xi,\tau)\sin\frac{n\pi\xi}{L}d\xi\right)e^{-\omega_n^2(t-\tau)}\,d\tau\right]\sin\frac{n\pi x}{L} \\
&= \int_0^L u(\xi,0)\left[\frac{2}{L}\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}\sin\frac{n\pi\xi}{L}e^{-\omega_n^2 t}\right]d\xi \\
&\quad + \int_0^t\int_0^L h(\xi,\tau)\left[\frac{2}{L}\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}\sin\frac{n\pi\xi}{L}e^{-\omega_n^2(t-\tau)}\right] \\
&= \int_0^L u(\xi,0)G(x,\xi;t,0)d\xi + \int_0^t\int_0^L h(\xi,\tau)G(x,\xi;t,\tau)d\xi d\tau
\end{aligned} \tag{9.11.15}$$

Here we have defined the Green's function

$$G(x,\xi;t,\tau) = \frac{2}{L}\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}\sin\frac{n\pi\xi}{L}e^{-\omega_n^2(t-\tau)}.$$

We note that $G(x,\xi;t,0)$ gives the initial value Green's function.

Note that at $t=\tau$,

$$G(x,\xi;t,t) = \frac{2}{L}\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}\sin\frac{n\pi\xi}{L}.$$

This is actually the series representation of the Dirac delta function. The Fourier sine transform of the delta function is

$$\mathcal{F}_s[\delta(x-\xi)] = \frac{2}{L}\int_0^L \delta(x-\xi)\sin\frac{n\pi x}{L}dx = \frac{2}{L}\sin\frac{n\pi\xi}{L}.$$

Then, the representation becomes

$$\delta(x-\xi) = \frac{2}{L}\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}\sin\frac{n\pi\xi}{L}.$$

Also, we note that

$$\begin{aligned}
\frac{\partial G}{\partial t} &= -\omega_n^2 G \\
\frac{\partial^2 G}{\partial x^2} &= -\left(\frac{n\pi}{L}\right)^2 G.
\end{aligned}$$

Therefore, $G_t = G_{xx}$ at least for $\tau \neq t$ and $\xi \neq x$.

We can modify this problem by adding nonhomogeneous boundary conditions.

$$\begin{aligned} u_t - ku_{xx} &= h(x,t), \quad 0 \le x \le L, \quad t > 0, \\ u(0,t) &= A, \quad u(L,t) = B, \quad t > 0, \\ u(x,0) &= f(x), \quad 0 \le x \le L. \end{aligned} \tag{9.11.16}$$

One way to treat these conditions is to assume $u(x,t) = w(x) + v(x,t)$ where $v_t - kv_{xx} = h(x,t)$ and $w_{xx} = 0$. Then, $u(x,t) = w(x) + v(x,t)$ satisfies the original nonhomogeneous heat equation.

If $v(x,t)$ satisfies $v(0,t) = v(L,t) = 0$ and $w(x)$ satisfies $w(0) = A$ and $w(L) = B$, then $u(0,t) = w(0) + v(0,t) = Au(L,t) = w(L) + v(L,t) = B$

Finally, we note that

$$v(x,0) = u(x,0) - w(x) = f(x) - w(x).$$

Therefore, $u(x,t) = w(x) + v(x,t)$ satisfies the original problem if

$$\begin{aligned} v_t - kv_{xx} &= h(x,t), \quad 0 \le x \le L, \quad t > 0, \\ v(0,t) &= 0, \quad v(L,t) = 0, \quad t > 0, \\ v(x,0) &= f(x) - w(x), \quad 0 \le x \le L. \end{aligned} \tag{9.11.17}$$

and

$$\begin{aligned} w_{xx} &= 0, 0 \le x \le L, \\ w(0) &= A, w(L) = B. \end{aligned} \tag{9.11.18}$$

We can solve the last problem to obtain $w(x) = A + \frac{B-A}{L}x$. The solution to the problem for $v(x,t)$ is simply the problem we had solved already in terms of Green's functions with the new initial condition, $f(x) - A - \frac{B-A}{L}x$.

Solution of the ${}_3D$ Poisson Equation

We recall from electrostatics that the gradient of the electric potential gives the electric field, $\mathrm{E} = -\nabla\phi$. However, we also have from Gauss' Law for electric fields $\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$, where $\rho(\mathbf{r})$ is the charge distribution at position $\mathbf{r}$. Combining these equations, we arrive at Poisson's equation for the electric potential,

$$\nabla^2\phi = -\frac{\rho}{\epsilon_0}.$$

We note that Poisson's equation also arises in Newton's theory of gravitation for the gravitational potential in the form $\nabla^2\phi = -4\pi G\rho$ where ρ is the matter density.

We consider Poisson's equation in the form

$$\nabla^2\phi(\mathbf{r}) = -4\pi f(\mathbf{r})$$

for $\mathbf{r}$ defined throughout all space. We will seek a solution for the potential function using a three dimensional Fourier transform. In the electrostatic problem $f = \rho(\mathbf{r})/4\pi\epsilon_0$ and the gravitational problem has $f = G\rho(\mathbf{r})$

The Fourier transform can be generalized to three dimensions as

$$\hat{\phi}(\mathbf{k}) = \int_V \phi(\mathbf{r}) e^{i\mathbf{k}\cdot\mathbf{r}} d^3r$$

where the integration is over all space, V, $d^3r = dxdydz$, and $\mathbf{k}$ is a three dimensional wavenumber, $\mathbf{k} = k_x\mathbf{i} + k_y\mathbf{j} + k_z\mathbf{k}$. The inverse Fourier transform can then be written as

$$\phi(\mathbf{r}) = \frac{1}{(2\pi)^3}\int_{V_k} \hat{\phi}(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{r}} d^3k,$$

where $d^3k = dk_x dk_y dk_z$ and V_k is all of k-space.

The Fourier transform of the Laplacian follows from computing Fourier transforms of any derivatives that are present. Assuming that ϕ and its gradient vanish for large distances, then

$$\mathcal{F}\left[\nabla^2\phi\right] = -\left(k_x^2+k_y^2+k_z^2\right)\hat{\phi}(\mathbf{k}).$$

Defining $k^2 = k_x^2+k_y^2+k_z^2$, then Poisson's equation becomes the algebraic equation

$$k^2\hat{\phi}(\mathbf{k}) = 4\pi\hat{f}\,(\mathbf{k}).$$

Solving for $\hat{\phi}(\mathbf{k})$, we have

$$\phi(\mathbf{k}) = \frac{4\pi}{k^2}\hat{f}\,(\mathbf{k}).$$

The solution to Poisson's equation is then determined from the inverse Fourier transform,

$$\phi(\mathbf{r}) = \frac{4\pi}{(2\pi)^3}\int_{V_k}\hat{f}\,(\mathbf{k})\frac{e^{-i\mathbf{k}\cdot\mathbf{r}}}{k^2}d^3k. \tag{9.11.19}$$

First we will consider an example of a point charge (or mass in the gravitational case) at the origin. We will set $f(\mathbf{r}) = f_0\delta^3(\mathbf{r})$ in order to represent a point source. For a unit point charge, $f_0 = 1/4\pi\epsilon_0$.

Note

The three dimensional Dirac delta function, $\delta^3\left(\mathbf{r}-\mathbf{r}_0\right)$.

Here we have introduced the three dimensional Dirac delta function which, like the one dimensional case, vanishes outside the origin and satisfies a unit volume condition,

$$\int_V \delta^3(\mathbf{r})d^3r = 1.$$

Also, there is a sifting property, which takes the form

$$\int_V \delta^3\left(\mathbf{r}-\mathbf{r}_0\right)f(\mathbf{r})d^3r = f\left(\mathbf{r}_0\right).$$

In Cartesian coordinates,

$$\delta^3(\mathbf{r}) = \delta(x)\delta(y)\delta(z),$$
$$\int_V \delta^3(\mathbf{r})d^3r = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(x)\delta(y)\delta(z)dxdydz = 1,$$

and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta\left(x-x_0\right)\delta\left(y-y_0\right)\delta\left(z-z_0\right)f(x,y,z)dxdydz = f\left(x_0,y_0,z_0\right)$$

One can define similar delta functions operating in two dimensions and n dimensions.

We can also transform the Cartesian form into curvilinear coordinates. From Section 6.9 we have that the volume element in curvilinear coordinates is

$$d^3r = dxdydz = h_1h_2h_3du_1du_2du_3,$$

where .

This gives

$$\int_V \delta^3(\mathbf{r})d^3r = \int_V \delta^3(\mathbf{r})h_1h_2h_3du_1du_2du_3 = 1.$$

Therefore,

$$\delta^3(\mathbf{r}) = \frac{\delta(u_1)}{\left|\frac{\partial \mathbf{r}}{\partial u_1}\right|}\frac{\delta(u_2)}{\left|\frac{\partial \mathbf{r}}{\partial u_2}\right|}\frac{\delta(u_3)}{\left|\frac{\partial \mathbf{r}}{\partial u_2}\right|}$$
$$= \frac{1}{h_1 h_2 h_3}\delta(u_1)\,\delta(u_2)\,\delta(u_3) \tag{9.11.20}$$

So, for cylindrical coordinates,

$$\delta^3(\mathbf{r}) = \frac{1}{r}\delta(r)\delta(\theta)\delta(z).$$

Example 9.11.1

Find the solution of Poisson's equation for a point source of the form $f(\mathbf{r}) = f_0\delta^3(\mathbf{r})$.

Solution

The solution is found by inserting the Fourier transform of this source into Equation (9.11.19) and carrying out the integration. The transform of $f(\mathbf{r})$ is found as

$$\hat{f}(\mathbf{k}) = \int_V f_0\delta^3(\mathbf{r})e^{i\mathbf{k}\cdot\mathbf{r}}d^3r = f_0.$$

Inserting $\hat{f}(\mathbf{k})$ into the inverse transform in Equation (9.11.19) and carrying out the integration using spherical coordinates in k-space, we find

$$\begin{aligned}\phi(\mathbf{r}) &= \frac{4\pi}{(2\pi)^3}\int_{V_k} f_0\frac{e^{-i\mathbf{k}\cdot\mathbf{r}}}{k^2}d^3k \\ &= \frac{f_0}{2\pi^2}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\frac{e^{-ikx\cos\theta}}{k^2}k^2\sin\theta dk d\theta d\phi \\ &= \frac{f_0}{\pi}\int_0^{\pi}\int_0^{\infty}e^{-ikx\cos\theta}\sin\theta dk d\theta \\ &= \frac{f_0}{\pi}\int_0^{\infty}\int_{-1}^{1}e^{-ikxy}dk dy, \quad y = \cos\theta \\ &= \frac{2f_0}{\pi r}\int_0^{\infty}\frac{\sin z}{z}dz = \frac{f_0}{r}\end{aligned} \tag{9.11.21}$$

If the last example is applied to a unit point charge, then $f_0 = 1/4\pi\epsilon_0$. So, the electric potential outside a unit point charge located at the origin becomes

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0 r}.$$

This is the form familiar from introductory physics.

Also, by setting $f_0 = 1$, we have also shown in the last example that

$$\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta^3(\mathbf{r}).$$

Since $\nabla\left(\frac{1}{r}\right) = -\frac{\mathbf{r}}{r^3}$, then we have also shown that

$$\nabla\cdot\left(\frac{\mathbf{r}}{r^3}\right) = 4\pi\delta^3(\mathbf{r}).$$

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9.12: Problems

Exercise 9.12.1

In this problem you will show that the sequence of functions

$$f_n(x) = \frac{n}{\pi}\left(\frac{1}{1+n^2x^2}\right)$$

approaches $\delta(x)$ as $n \to \infty$. Use the following to support your argument:

a. Show that $\lim_{n\to\infty} f_n(x) = 0$ for $x \neq 0$.
b. Show that the area under each function is one.

Exercise 9.12.2

Verify that the sequence of functions $\{f_n(x)\}_{n=1}^{\infty}$, defined by $f_n(x) = \frac{n}{2}e^{-n|x|}$, approaches a delta function.

Exercise 9.12.3

Evaluate the following integrals:

a. $\int_0^{\pi} \sin x \delta\left(x - \frac{\pi}{2}\right) dx$.
b. $\int_{-\infty}^{\infty} \delta\left(\frac{x-5}{3}e^{2x}\right)\left(3x^2 - 7x + 2\right) dx$.
c. $\int_0^{\pi} x^2 \delta\left(x + \frac{\pi}{2}\right) dx$.
d. $\int_0^{\infty} e^{-2x}\delta\left(x^2 - 5x + 6\right) dx$. [See Problem 4.]
e. $\int_{-\infty}^{\infty} \left(x^2 - 2x + 3\right)\delta\left(x^2 - 9\right) dx$. [See Problem 4.]

Exercise 9.12.4

For the case that a function has multiple roots, $f(x_i) = 0, i = 1, 2, \ldots$, it can be shown that

$$\delta(f(x)) = \sum_{i=1}^{n} \frac{\delta(x - x_i)}{|f'(x_i)|}.$$

Use this result to evaluate $\int_{-\infty}^{\infty} \delta\left(x^2 - 5x - 6\right)\left(3x^2 - 7x + 2\right) dx$.

Exercise 9.12.5

Find a Fourier series representation of the Dirac delta function, $\delta(x)$, on $[-L, L]$.

Exercise 9.12.6

For $a > 0$, find the Fourier transform, $\hat{f}(k)$, of $f(x) = e^{-a|x|}$.

Exercise 9.12.7

Use the result from the last problem plus properties of the Fourier transform to find the Fourier transform, of $f(x) = x^2 e^{-a|x|}$ for $a > 0$.

Exercise 9.12.8

Find the Fourier transform, $\hat{f}(k)$, of $f(x) = e^{-2x^2+x}$.

Exercise 9.12.9

Prove the second shift property in the form

$$F\left[e^{i\beta x} f(x)\right] = \hat{f}(k+\beta).$$

Exercise 9.12.10

A damped harmonic oscillator is given by

$$f(t) = \begin{cases} Ae^{-\alpha t} e^{i\omega_0 t}, & t \geq 0, \\ 0, & t < 0. \end{cases}$$

a. Find $\hat{f}(\omega)$ and
b. the frequency distribution $|\hat{f}(\omega)|^2$.
c. Sketch the frequency distribution.

Exercise 9.12.11

Show that the convolution operation is associative: $(f * (g * h))(t) = ((f * g) * h)(t)$.

Exercise 9.12.12

In this problem you will directly compute the convolution of two Gaussian functions in two steps.

a. Use completing the square to evaluate

$$\int_{-\infty}^{\infty} e^{-\alpha t^2+\beta t}\, dt.$$

b.

Exercise 9.12.13

You will compute the (Fourier) convolution of two box functions of the same width. Recall the box function is given by

$$f_a(x) = \begin{cases} 1, & |x| \leq a \\ 0, & |x| > a. \end{cases}$$

Consider $(f_a * f_a)(x)$ for different intervals of x. A few preliminary sketches would help. In Figure 9.12.1 the factors in the convolution integrand are show for one value of x. The integrand is the product of the first two functions. The convolution at x is the area of the overlap in the third figure. Think about how these pictures change as you vary x. Plot the resulting areas as a function of x. This is the graph of the desired convolution.

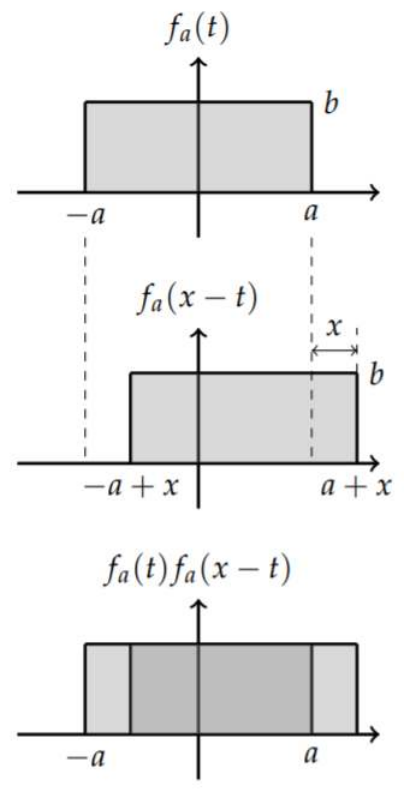


Figure 9.12.1: Sketch used to compute the convolution of the box function with itself. In the top figure is the box function. The second figure shows the box shifted by x. The last figure indicates the overlap of the functions.

? Exercise 9.12.14

Define the integrals $I_n = \int_{-\infty}^{\infty} x^{2n} e^{-x^2}\, dx$. Noting that $I_0 = \sqrt{\pi}$,

a. Find a recursive relation between I_n and I_{n-1}.
b. Use this relation to determine I_1, I_2 and I_3.
c. Find an expression in terms of n for I_n.

? Exercise 9.12.15

Find the Laplace transform of the following functions.

a. $f(t) = 9t^2 - 7$.
b. $f(t) = e^{5t-3}$.
c. $f(t) = \cos 7t$.
d. $f(t) = e^{4t} \sin 2t$.
e. $f(t) = e^{2t}(t + \cosh t)$.
f. $f(t) = t^2 H(t-1)$.
g. $f(t) = \begin{cases} \sin t, & t < 4\pi \\ \sin t + \cos t, & t > 4\pi \end{cases}$.
h. $f(t) = \int_0^t (t-u)^2 \sin u\, du$.
i. $f(t) = (t+5)^2 + te^{2t}\cos 3t$ and write the answer in the simplest form.

Exercise 9.12.16

Find the inverse Laplace transform of the following functions using the properties of Laplace transforms and the table of Laplace transform pairs.

a. $F(s) = \frac{18}{s^3} + \frac{7}{s}$.
b. $F(s) = \frac{1}{s-5} - \frac{2}{s^2+4}$.
c. $F(s) = \frac{s+1}{s^2+1}$.
d. $F(s) = \frac{3}{s^2+2s+2}$.
e. $F(s) = \frac{1}{(s-1)^2}$.
f. $F(s) = \frac{e^{-3s}}{s^2-1}$.
g. $F(s) = \frac{1}{s^2+4s-5}$.
h. $F(s) = \frac{s+3}{s^2+8s+17}$.

Exercise 9.12.17

Compute the convolution $(f * g)(t)$ (in the Laplace transform sense) and its corresponding Laplace transform $\mathcal{L}[f * g]$ for the following functions:

a. $f(t) = t^2, g(t) = t^3$.
b. $f(t) = t^2, g(t) = \cos 2t$.
c. $f(t) = 3t^2 - 2t + 1, g(t) = e^{-3t}$.
d. $f(t) = \delta(t - \frac{\pi}{4}), g(t) = \sin 5t$.

Exercise 9.12.18

For the following problems draw the given function and find the Laplace transform in closed form.

a. $f(t) = 1 + \sum_{n=1}^{\infty}(-1)^n H(t-n)$.
b. $f(t) = \sum_{n=0}^{\infty}[H(t-2n+1) - H(t-2n)]$.
c. $f(t) = \sum_{n=0}^{\infty}(t-2n)[H(t-2n) - H(t-2n-1)] + (2n+2-t)[H(t-\quad 2n-1) - H(t-2n-2)]$.

Exercise 9.12.19

Use the convolution theorem to compute the inverse transform of the following:

a. $F(s) = \frac{2}{s^2(s^2+1)}$.
b. $F(s) = \frac{e^{-3s}}{s^2}$.
c. $F(s) = \frac{1}{s(s^2+2s+5)}$.

Exercise 9.12.20

Find the inverse Laplace transform two different ways: i) Use Tables. ii) Use the Bromwich Integral.

a. $F(s) = \frac{1}{s^3(s+4)^2}$.
b. $F(s) = \frac{1}{s^2-4s-5}$.
c. $F(s) = \frac{s+3}{s^2+8s+17}$.
d. $F(s) = \frac{s+1}{(s-2)^2(s+4)}$.
e. $F(s) = \frac{s^2+8s-3}{(s^2+2s+1)(s^2+1)}$.

Exercise 9.12.21

Use Laplace transforms to solve the following initial value problems. Where possible, describe the solution behavior in terms of oscillation and decay.

a. $y''-5y'+6y=0,\ y(0)=2,\ y'(0)=0.$
b. $y''-y=te^{2t},\ y(0)=0,\ y'(0)=1.$
c. $y''+4y=\delta(t-1),\ y(0)=3,\ y'(0)=0.$
d. $y''+6y'+18y=2H(\pi-t),\ y(0)=0,\ y'(0)=0.$

Exercise 9.12.22

Use Laplace transforms to convert the following system of differential equations into an algebraic system and find the solution of the differential equations.

$$\begin{aligned} x'' &= 3x-6y, \quad x(0) = 1, \quad x'(0) = 0 \\ y'' &= x+y, \qquad\quad y(0) = 0, \quad y'(0) = 0. \end{aligned}$$

Exercise 9.12.23

Use Laplace transforms to convert the following nonhomogeneous systems of differential equations into an algebraic system and find the solutions of the differential equations.

a.
$$\begin{aligned} x' &= 2x+3y+2\sin 2t, \quad x(0)=1, \\ y' &= -3x+2y, \quad\ \ y(0)=0. \end{aligned}$$

b.
$$\begin{aligned} x' &= -4x-y+e^{-t}, \quad x(0)=2, \\ y' &= x-2y+2e^{-3t}, \quad y(0)=-1. \end{aligned}$$

c.
$$\begin{aligned} x' &= x-y+2\cos t, \quad x(0)=3, \\ y' &= x+y-3\sin t, \quad y(0)=2. \end{aligned}$$

Exercise 9.12.24

Consider the series circuit in Problem 2.8.20 and in Figure ?? with $L=1.00\text{H}, R=1.00\times 10^2\Omega, C=1.00\times 10^{-4}\ \text{F}$, and $V_0=1.00\times 10^3\ \text{V}$.

a. Write the second order differential equation for this circuit.
b. Suppose that no charge is present and no current is flowing at time $t=0$ when V_0 is applied. Use Laplace transforms to find the current and the charge on the capacitor as functions of time.
c. Replace the battery with the alternating source $V(t)=V_0\sin 2\pi ft$ with $V_0=1.00\times 10^3\ \text{V}$ and $f=150\ \text{Hz}$. Again, suppose that no charge is present and no current is flowing at time $t=0$ when the AC source is applied. Use Laplace transforms to find the current and the charge on the capacitor as functions of time.
d. Plot your solutions and describe how the system behaves over time.

Exercise 9.12.25

Use Laplace transforms to sum the following series.

a. $\sum_{n=0}^{\infty}\frac{(-1)^n}{1+2n}$.
b. $\sum_{n=1}^{\infty}\frac{1}{n(n+3)}$.
c. $\sum_{n=1}^{\infty}\frac{(-1)^n}{n(n+3)}$.
d. $\sum_{n=0}^{\infty}\frac{(-1)^n}{n^2-a^2}$.
e. $\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2-a^2}$.

f. $\sum_{n=1}^{\infty} \frac{1}{n} e^{-an}$.

Exercise 9.12.26

Use Laplace transforms to prove

$$\sum_{n=1}^{\infty} \frac{1}{(n+a)(n+b)} = \frac{1}{b-a} \int_0^1 \frac{u^a - u^b}{1-u} du.$$

Use this result to evaluate the sums

a. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.
b. $\sum_{n=1}^{\infty} \frac{1}{(n+2)(n+3)}$.

Exercise 9.12.27

Do the following.

a. Find the first four nonvanishing terms of the Maclaurin series expansion of $f(x) = \frac{x}{e^x - 1}$.
b. Use the result in part a. to determine the first four nonvanishing Bernoulli numbers, B_n.
c. Use these results to compute $\zeta(2n)$ for $n = 1, 2, 3, 4$.

Exercise 9.12.28

Given the following Laplace transforms, $F(s)$, find the function $f(t)$. Note that in each case there are an infinite number of poles, resulting in an infinite series representation.

a. $F(s) = \frac{1}{s^2(1+e^{-s})}$.
b. $F(s) = \frac{1}{s \sinh s}$.
c. $F(s) = \frac{\sinh s}{s^2 \cosh s}$.
d. $F(s) = \frac{\sinh(\beta\sqrt{s}x)}{s \sinh(\beta\sqrt{s}L)}$.

Exercise 9.12.29

Consider the initial boundary value problem for the heat equation:

$$\begin{aligned} u_t &= 2u_{xx}, & 0 < t, \quad 0 \le x \le 1, \\ u(x,0) &= x(1-x), & 0 < x < 1, \\ u(0,t) &= 0, & t > 0, \\ u(1,t) &= 0, & t > 0. \end{aligned}$$

Use the finite transform method to solve this problem. Namely, assume that the solution takes the form $u(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin n\pi x$ and obtain an ordinary differential equation for b_n and solve for the b_n 's for each n.

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CHAPTER OVERVIEW

10: Numerical Solutions of PDEs

> *There's no sense in being precise when you don't even know what you're talking about. - John von Neumann (1903-1957)*

Most of the book has dealt with finding exact solutions to some generic problems. However, most problems of interest cannot be solved exactly. The heat, wave, and Laplace equations are linear partial differential equations and can be solved using separation of variables in geometries in which the Laplacian is separable. However, once we introduce nonlinearities, or complicated non-constant coefficients intro the equations, some of these methods do not work. Even when separation of variables or the method of eigenfunction expansions gave us exact results, the computation of the resulting series had to be done on a computer and inevitably one could only use a finite number of terms of the expansion. So, therefore, it is sometimes useful to be able to solve differential equations numerically.

In this chapter we will introduce the idea of numerical solutions of partial differential equations. However, we will first begin with a discussion of the solution of ordinary differential equations in order to get a feel for some common problems in the solution of differential equations and the notion of convergence rates of numerical schemes. Then, we turn to the finite difference method and the ideas of stability. Other common approaches may be added later.

10.1: Ordinary Differential Equations

10.2: The Heat Equation

10.3: Truncation Error

10.4: Stability

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10.1: Ordinary Differential Equations

Euler's Method

In this section we will look at the simplest method for solving first order equations, Euler's Method. While it is not the most efficient method, it does provide us with a picture of how one proceeds and can be improved by introducing better techniques, which are typically covered in a numerical analysis text.

Let's consider the class of first order initial value problems of the form

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0. \tag{10.1.1}$$

We are interested in finding the solution $y(x)$ of this equation which passes through the initial point (x_0, y_0) in the xy-plane for values of x in the interval $[a, b]$, where $a = x_0$. We will seek approximations of the solution at N points, labeled x_n for $n = 1, \ldots, N$. For equally spaced points we have $\Delta x = x_1 - x_0 = x_2 - x_1$, etc. We can write these as

$$x_n = x_0 + n\Delta x.$$

In Figure 10.1.1 we show three such points on the x-axis.

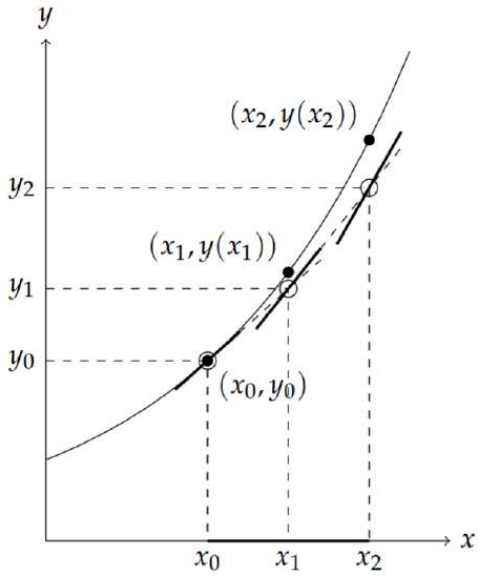


Figure 10.1.1: The basics of Euler's Method are shown. An interval of the x axis is broken into N subintervals. The approximations to the solutions are found using the slope of the tangent to the solution, given by $f(x, y)$. Knowing previous approximations at (x_{n-1}, y_{n-1}), one can determine the next approximation, y_n.

The first step of Euler's Method is to use the initial condition. We represent this as a point on the solution curve, $(x_0, y(x_0)) = (x_0, y_0)$, as shown in Figure 10.1.1. The next step is to develop a method for obtaining approximations to the solution for the other x_n 's.

We first note that the differential equation gives the slope of the tangent line at $(x, y(x))$ of the solution curve since the slope is the derivative, $y'(x)'$ From the differential equation the slope is $f(x, y(x))$. Referring to Figure 10.1.1, we see the tangent line drawn at (x_0, y_0). We look now at $x = x_1$. The vertical line $x = x_1$ intersects both the solution curve and the tangent line passing through (x_0, y_0). This is shown by a heavy dashed line.

While we do not know the solution at $x = x_1$, we can determine the tangent line and find the intersection point that it makes with the vertical. As seen in the figure, this intersection point is in theory close to the point on the solution curve. So, we will designate y_1 as the approximation of the solution $y(x_1)$. We just need to determine y_1.

The idea is simple. We approximate the derivative in the differential equation by its difference quotient:

$$\frac{dy}{dx} \approx \frac{y_1 - y_0}{x_1 - x_0} = \frac{y_1 - y_0}{\Delta x}. \tag{10.1.2}$$

Since the slope of the tangent to the curve at (x_0, y_0) is $y'(x_0) = f(x_0, y_0)$, we can write

$$\frac{y_1 - y_0}{\Delta x} \approx f(x_0, y_0). \tag{10.1.3}$$

Solving this equation for y_1, we obtain

$$y_1 = y_0 + \Delta x f(x_0, y_0). \tag{10.1.4}$$

This gives y_1 in terms of quantities that we know.

We now proceed to approximate $y(x_2)$. Referring to Figure 10.1.1, we see that this can be done by using the slope of the solution curve at (x_1, y_1). The corresponding tangent line is shown passing though (x_1, y_1) and we can then get the value of y_2 from the intersection of the tangent line with a vertical line, $x = x_2$. Following the previous arguments, we find that

$$y_2 = y_1 + \Delta x f(x_1, y_1). \tag{10.1.5}$$

Continuing this procedure for all $x_n, n = 1, \ldots N$, we arrive at the following scheme for determining a numerical solution to the initial value problem:

$$\begin{aligned} y_0 &= y(x_0), \\ y_n &= y_{n-1} + \Delta x f(x_{n-1}, y_{n-1}), \quad n = 1, \ldots, N. \end{aligned} \tag{10.1.6}$$

This is referred to as Euler's Method.

Example 10.1.1

Use Euler's Method to solve the initial value problem $\frac{dy}{dx} = x + y, \quad y(0) = 1$ and obtain an approximation for $y(1)$.

Solution

First, we will do this by hand. We break up the interval $[0, 1]$, since we want the solution at $x = 1$ and the initial value is at $x = 0$. Let $\Delta x = 0.50$. Then, $x_0 = 0$, $x_1 = 0.5$ and $x_2 = 1.0$. Note that there are $N = \frac{b-a}{\Delta x} = 2$ subintervals and thus three points.

We next carry out Euler's Method systematically by setting up a table for the needed values. Such a table is shown in Table 10.1.1. Note how the table is set up. There is a column for each x_n and y_n. The first row is the initial condition. We also made use of the function $f(x, y)$ in computing the y_n 's from (10.1.6). This sometimes makes the computation easier. As a result, we find that the desired approximation is given as $y_2 = 2.5$.

Table 10.1.1: Application of Euler's Method for $y' = x + y, y(0) = 1$ and $\Delta x = 0.5$.

n	x_n	$y_n = y_{n-1} + \Delta x f(x_{n-1}, y_{n-1} = 0.5x_{n-1} + 1$
o	o	1
1	o.5	$0.5(0) + 1.5(1.0) = 1.5$
2	1. o	$0.5(0.5) + 1.5(1.5) = 2.5$

Is this a good result? Well, we could make the spatial increments smaller. Let's repeat the procedure for $\Delta x = 0.2$, or $N = 5$. The results are in Table 10.1.2.

Now we see that the approximation is $y_1 = 2.97664$. So, it looks like the value is near 3, but we cannot say much more. Decreasing Δx more shows that we are beginning to converge to a solution. We see this in Table 10.1.3.

Table 10.1.2: Application of Euler's Method for $y' = x + y, y(0) = 1$ and $\Delta x = 0.2$.

n	x_n	$y_n = 0.2x_{n-1} + 1.2y_{n-1}$
o	○	1
1	0.2	$0.2(0) + 1.2(1.0) = 1.2$
2	0.4	$0.2(0.2) + 1.2(1.2) = 1.48$
3	0.6	$0.2(0.4) + 1.2(1.48) = 1.856$
4	0.8	$0.2(0.6) + 1.2(1.856) = 2.3472$
5	1.0	$0.2(0.8) + 1.2(2.3472) = 2.97664$

Table 10.1.3: Results of Euler's Method for $y' = x + y, y(0) = 1$ and varying Δx

Δx	$y_N \approx y(1)$
0.5	2.5
0.2	2.97664
0.1	3.187484920

Δx	$y_N \approx y(1)$
0.01	3.409627659
0.001	3.433847864
0.0001	3.436291854

Of course, these values were not done by hand. The last computation would have taken 1000 lines in the table, or at least 40 pages! One could use a computer to do this. A simple code in Maple would look like the following:

```
> restart:
> f:=(x,y)->y+x;
> a:=0: b:=1: N:=100: h:=(b-a)/N;
> x[0]:=0: y[0]:=1:
    for i from 1 to N do
    y[i]:=y[i-1]+h*f(x[i-1],y[i-1]):
    x[i]:=x[0]+h*(i):
    od:
    evalf(y[N]);
```

In this case we could simply use the exact solution. The exact solution is easily found as

$$y(x) = 2e^x - x - 1.$$

(The reader can verify this.) So, the value we are seeking is

$$y(1) = 2e - 2 = 3.4365636\ldots.$$

Thus, even the last numerical solution was off by about 0.00027.

Adding a few extra lines for plotting, we can visually see how well the approximations compare to the exact solution. The Maple code for doing such a plot is given below.

```
> with(plots):
> Data:=[seq([x[i],y[i]],i=0..N)]:
> P1:=pointplot(Data,symbol=DIAMOND):
> Sol:=t->-t-1+2*exp(t);
> P2:=plot(Sol(t),t=a..b,Sol=0..Sol(b)):
> display({P1,P2});
```

We show in Figures 10.1.2-10.1.3 the results for $N = 10$ and $N = 100$. In Figure 10.1.2 we can see how quickly the numerical solution diverges from the exact solution. In Figure 10.1.3 we can see that visually the solutions agree, but we note that from Table 10.1.3 that for $\Delta x = 0.01$, the solution is still off in the second decimal place with a relative error of about 0.8%.

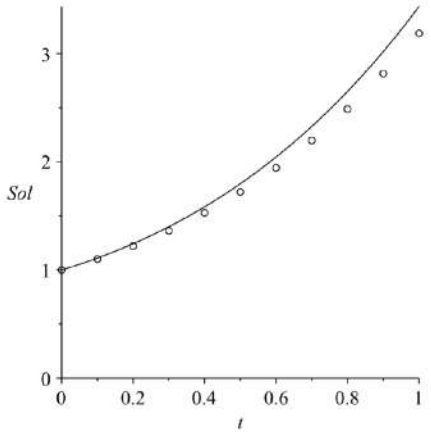


Figure 10.1.2: A comparison of the results Euler's Method to the exact solution for $y' = x + y, y(0) = 1$ and $N = 10$.

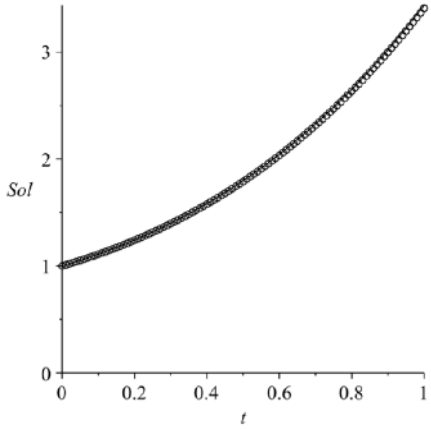


Figure 10.1.3: A comparison of the results Euler's Method to the exact solution for $y' = x + y, y(0) = 1$ and $N = 100$.

Why would we use a numerical method when we have the exact solution? Exact solutions can serve as test cases for our methods. We can make sure our code works before applying them to problems whose solution is not known.

There are many other methods for solving first order equations. One commonly used method is the fourth order Runge-Kutta method. This method has smaller errors at each step as compared to Euler's Method. It is well suited for programming and comes built-in in many packages like Maple and MATLAB. Typically, it is set up to handle systems of first order equations.

In fact, it is well known that nth order equations can be written as a system of n first order equations. Consider the simple second order equation

$$y'' = f(x, y).$$

This is a larger class of equations than the second order constant coefficient equation. We can turn this into a system of two first order differential equations by letting $u = y$ and $v = y' = u'$. Then, $v' = y'' = f(x, u)$. So, we have the first order system

$$\begin{aligned} u' &= v \\ v' &= f(x, u). \end{aligned} \tag{10.1.7}$$

We will not go further into the Runge-Kutta Method here. You can find more about it in a numerical analysis text. However, we will see that systems of differential equations do arise naturally in physics. Such systems are often coupled equations and lead to interesting behaviors.

Higher Order Taylor Methods

Euler's Method for solving differential equations is easy to understand but is not efficient in the sense that it is what is called a first order method. The error at each step, the local truncation error, is of order Δx, for x the independent variable. The accumulation of the local truncation errors results in what is called the global error. In order to generalize Euler's Method, we need to rederive it. Also, since these methods are typically used for initial value problems, we will cast the problem to be solved as

$$\frac{dy}{dt} = f(t, y), \quad y(a) = y_0, \quad t \in [a, b]. \tag{10.1.8}$$

The first step towards obtaining a numerical approximation to the solution of this problem is to divide the t-interval, $[a, b]$, into N subintervals,

$$t_i = a + ih, \quad i = 0, 1, \ldots, N, \quad t_0 = a, \quad t_N = b,$$

where

$$h = \frac{b-a}{N}.$$

We then seek the numerical solutions

$$\tilde{y}_i \approx y(t_i), \quad i = 1, 2, \ldots, N,$$

with $\tilde{y}_0 = y(t_0) = y_0$. Figure 10.1.4graphically shows how these quantities are related.

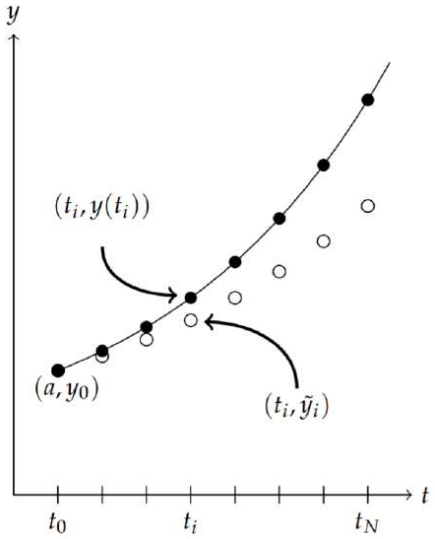


Figure 10.1.4: The interval $[a, b]$ is divided into N equally spaced subintervals. The exact solution $y(t_i)$ is shown with the numerical solution, $\tilde{y}_i$ with $t_i = a + ih$, $i = 0, 1, \ldots, N$.

Euler's Method can be derived using the Taylor series expansion of of the solution $y(t_i + h)$ about $t = t_i$ for $i = 1, 2, \ldots, N$. This is given by

$$\begin{aligned} y(t_{i+1}) &= y(t_i + h) \\ &= y(t_i) + y'(t_i)\, h + \frac{h^2}{2} y''(\xi_i), \quad \xi_i \in (t_i, t_{i+1}). \end{aligned} \tag{10.1.9}$$

Here the term $\frac{h^2}{2} y''(\xi_i)$ captures all of the higher order terms and represents the error made using a linear approximation to $y(t_i + h)$.

Dropping the remainder term, noting that $y'(t) = f(t, y)$, and defining the resulting numerical approximations by $\tilde{y}_i \approx y(t_i)$, we have

$$\begin{aligned} \tilde{y}_{i+1} &= \tilde{y}_i + h f(t_i, \tilde{y}_i), \quad i = 0, 1, \ldots, N-1, \\ \tilde{y}_0 &= y(a) = y_0. \end{aligned} \tag{10.1.10}$$

This is Euler's Method.

Euler's Method is not used in practice since the error is of order h. However, it is simple enough for understanding the idea of solving differential equations numerically. Also, it is easy to study the numerical error, which we will show next.

The error that results for a single step of the method is called the local truncation error, which is defined by

$$\tau_{i+1}(h) = \frac{y(t_{i+1}) - \tilde{y}_i}{h} - f(t_i, y_i).$$

A simple computation gives

$$\tau_{i+1}(h) = \frac{h}{2} y''(\xi_i), \quad \xi_i \in (t_i, t_{i+1}).$$

Since the local truncation error is of order h, this scheme is said to be of order one. More generally, for a numerical scheme of the form

$$\begin{aligned} \tilde{y}_{i+1} &= \tilde{y}_i + hF(t_i, \tilde{y}_i), \quad i = 0, 1, \ldots, N-1, \\ \tilde{y}_0 &= y(a) = y_0, \end{aligned} \tag{10.1.11}$$

the local truncation error is defined by

$$\tau_{i+1}(h) = \frac{y(t_{i+1}) - \tilde{y}_i}{h} - F(t_i, y_i).$$

The accumulation of these errors leads to the global error. In fact, one can show that if f is continuous, satisfies the Lipschitz condition,

$$|f(t, y_2) - f(t, y_1)| \leq L|y_2 - y_1|$$

for a particular domain $D \subset R^2$, and

$$|y''(t)| \leq M, \quad t \in [a, b],$$

then

$$|y(t_i) - \tilde{y}_| \leq \frac{hM}{2L}\left(e^{L(t_i - a)} - 1\right), \quad i = 0, 1, \ldots, N.$$

Furthermore, if one introduces round-off errors, bounded by δ, in both the initial condition and at each step, the global error is modified as

$$\left|y(t_i) - \tilde{y}_| \leq \frac{1}{L}\left(\frac{hM}{2} + \frac{\delta}{h}\right)\left(e^{L(t_i - a)} - 1\right) + \right|\delta_0 \mid e^{L(t_i - a)}, \quad i = 0, 1, \ldots, N.$$

Then for small enough steps h, there is a point when the round-off error will dominate the error. [See Burden and Faires, *Numerical Analysis* for the details.]

Can we improve upon Euler's Method? The natural next step towards finding a better scheme would be to keep more terms in the Taylor series expansion. This leads to Taylor series methods of order n.

Taylor series methods of order n take the form

$$\begin{aligned} \tilde{y}_{i+1} &= \tilde{y}_i + hT^{(n)}(t_i, \tilde{y}_i), \quad i = 0, 1, \ldots, N-1, \\ \tilde{y}_0 &= y_0, \end{aligned} \tag{10.1.12}$$

where we have defined

$$T^{(n)}(t, y) = y'(t) + \frac{h}{2}y''(t) + \cdots + \frac{h^{(n-1)}}{n!}y^{(n)}(t).$$

However, since $y'(t) = f(t, y)$, we can write

$$T^{(n)}(t, y) = f(t, y) + \frac{h}{2}f'(t, y) + \cdots + \frac{h^{(n-1)}}{n!}f^{(n-1)}(t, y).$$

We note that for $n = 1$, we retrieve Euler's Method as a special case. We demonstrate a third order Taylor's Method in the next example.

Example 10.1.2

Apply the third order Taylor's Method to

$$\frac{dy}{dt} = t + y, \quad y(0) = 1$$

and obtain an approximation for $y(1)$ for $h = 0.1$.

Solution

The third order Taylor's Method takes the form

$$\begin{aligned} \tilde{y}_{i+1} &= \tilde{y}_i + hT^{(3)}(t_i, \tilde{y}_i), \quad i = 0, 1, \ldots, N-1, \\ \tilde{y}_0 &= y_0, \end{aligned} \tag{10.1.13}$$

where

$$T^{(3)}(t, y) = f(t, y) + \frac{h}{2}f'(t, y) + \frac{h^2}{3!}f''(t, y)$$

and $f(t, y) = t + y(t)$.

In order to set up the scheme, we need the first and second derivative of $f(t, y)$:

$$
\begin{aligned}
f'(t,y) &= \frac{d}{dt}(t+y) \\
&= 1+y' \\
&= 1+t+y \qquad (10.1.14)\\
f''(t,y) &= \frac{d}{dt}(1+t+y) \qquad (10.1.15)\\
&= 1+y' \\
&= 1+t+y \qquad (10.1.16)
\end{aligned}
$$

Inserting these expressions into the scheme, we have

$$
\begin{aligned}
\tilde{y}_{i+1} &= \tilde{y}_i + h\left[(t_i+y_i)+\frac{h}{2}(1+t_i+y_i)+\frac{h^2}{3!}(1+t_i+y_i)\right], \\
&= \tilde{y}_i + h\,(t_i+y_i)+h^2\left(\frac{1}{2}+\frac{h}{6}\right)(1+t_i+y_i), \\
\tilde{y}_0 &= y_0, \qquad (10.1.17)
\end{aligned}
$$

for $i = 0, 1, \ldots, N-1$.

In Figure 10.1.2 we show the results comparing Euler's Method, the 3rd Order Taylor's Method, and the exact solution for $N = 10$. In Table 10.1.4 we provide are the numerical values. The relative error in Euler's method is about 7% and that of the 3rd Order Taylor's Method is about 0.006%. Thus, the 3rd Order Taylor's Method is significantly better than Euler's Method.

Table 10.1.4: Numerical values for Euler's Method, 3 rd Order Taylor's Method, and exact solution for solving Example 10.2 with $N = 10$..

Euler	Taylor	Exact
1.0000	1.0000	1.0000
1.1000	1.1103	1.1103
1.2200	1.2428	1.2428
1.3620	1.3997	1.3997
1.5282	1.5836	1.5836
1.7210	1.7974	1.7974
1.9431	2.0442	2.0442
2.1974	2.3274	2.3275
2.4872	2.6509	2.6511
2.8159	3.0190	3.0192
3.1875	3.4364	3.4366

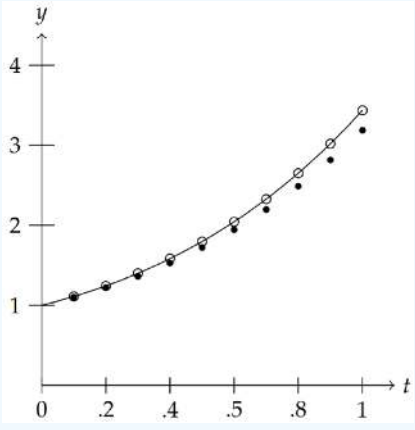


Figure 10.1.5: Numerical results for Euler's Method (filled circle) and 3rd Order Taylor's Method (open circle) for solving Example 10.1.2 as compared to exact solution (solid line).

In the last section we provided some Maple code for performing Euler's method. A similar code in MATLAB looks like the following:

```
a=0;
b=1;
N=10;
h=(b-a)/N;

% Slope function
    f = inline('t+y','t','y');
    sol = inline('2*exp(t)-t-1','t');

% Initial Condition
    t(1)=0;
    y(1)=1;

% Euler's Method
    for i=2:N+1
        y(i)=y(i-1)+h*f(t(i-1),y(i-1));
        t(i)=t(i-1)+h;
    end
```

A simple modification can be made for the 3rd Order Taylor's Method by replacing the Euler's method part of the preceding code by

```
% Taylor's Method, Order 3
    y(1)=1;
    h3 = h^2*(1/2+h/6);
    for i=2:N+1
        y(i)=y(i-1)+h*f(t(i-1),y(i-1))+h3*(1+t(i-1)+y(i-1));
        t(i)=t(i-1)+h;
    end
```

While the accuracy in the last example seemed sufficient, we have to remember that we only stopped at one unit of time. How can we be confident that the scheme would work as well if we carried out the computation for much longer times. For example, if the time unit were only a second, then one would need 86,400 times longer to predict a day forward. Of course, the scale matters. But, often we need to carry out numerical schemes for long times and we hope that the scheme not only converges to a solution, but that it converges to the solution to the given problem. Also, the previous example was relatively easy to program because we could provide a relatively simple form for $T^{(3)}(t,y)$ with a quick computation of the derivatives of $f(t,y)$. This is not always the case and higher order Taylor methods in this form are not typically used. Instead, one can approximate $T^{(n)}(t,y)$ by evaluating the known function $f(t,y)$ at selected values of t and y, leading to Runge-Kutta methods.

Runge-Kutta Methods

As we had seen in the last section, we can use higher order Taylor methods to derive numerical schemes for solving

$$\frac{dy}{dt} = f(t,y), \quad y(a) = y_0, \quad t \in [a,b], \tag{10.1.18}$$

using a scheme of the form

$$\begin{aligned} \tilde{y}_{i+1} &= \tilde{y}_i + hT^{(n)}(t_i, \tilde{y}_i), \quad i = 0,1,\ldots,N-1, \\ \tilde{y}_0 &= y_0, \end{aligned} \tag{10.1.19}$$

where we have defined

$$T^{(n)}(t,y) = y'(t) + \frac{h}{2}y''(t) + \cdots + \frac{h^{(n-1)}}{n!}y^{(n)}(t).$$

In this section we will find approximations of $T^{(n)}(t,y)$ which avoid the need for computing the derivatives.

For example, we could approximate

$$T^{(2)}(t,y) = f(t,y) + \frac{h}{2}\text{fracdfdt}(t,y)$$

by

$$T^{(2)}(t,y) \approx af(t+\alpha, y+\beta)$$

for selected values of a, α, and β. This requires use of a generalization of Taylor's series to functions of two variables. In particular, for small α and β we have

$$\begin{aligned} af(t+\alpha, y+\beta) = a\Bigg[& f(t,y) + \frac{\partial f}{\partial t}(t,y)\alpha + \frac{\partial f}{\partial y}(t,y)\beta \\ & + \frac{1}{2}\left(\frac{\partial^2 f}{\partial t^2}(t,y)\alpha^2 + 2\frac{\partial^2 f}{\partial t \partial y}(t,y)\alpha\beta + \frac{\partial^2 f}{\partial y^2}(t,y)\beta^2 \right) \Bigg] \\ & + \text{ higher order terms.} \end{aligned} \tag{10.1.20}$$

Furthermore, we need $\frac{df}{dt}(t,y)$. Since $y = y(t)$, this can be found using a generalization of the Chain Rule from Calculus III:

$$\frac{df}{dt}(t,y) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Thus,

$$T^{(2)}(t,y) = f(t,y) + \frac{h}{2}\left[\frac{\partial f}{\partial t} + \frac{\partial f}{\partial y}\frac{dy}{dt} \right].$$

Comparing this expression to the linear (Taylor series) approximation of af $(t+\alpha, y+\beta)$, we have

$$\begin{aligned} T^{(2)} &\approx af(t+\alpha, y+\beta) \\ f + \frac{h}{2}\frac{\partial f}{\partial t} + \frac{h}{2}f\frac{\partial f}{\partial y} &\approx af + a\alpha\frac{\partial f}{\partial t} + \beta\frac{\partial f}{\partial y}. \end{aligned} \tag{10.1.21}$$

We see that we can choose

$$a = 1, \quad \alpha = \frac{h}{2}, \quad \beta = \frac{h}{2}f.$$

This leads to the numerical scheme

$$\begin{aligned} \tilde{y}_{i+1} &= \tilde{y}_i + hf\left(t_i + \frac{h}{2}, \tilde{y}_i + \frac{h}{2}f\left(t_i, \tilde{y}_i\right) \right), \quad i = 0, 1, \ldots, N-1, \\ \tilde{y}_0 &= y_0, \end{aligned} \tag{10.1.22}$$

This Runge-Kutta scheme is called the Midpoint Method, or Second Order Runge-Kutta Method, and it has order 2 if all second order derivatives of $f(t,y)$ are bounded.

Often, in implementing Runge-Kutta schemes, one computes the arguments separately as shown in the following MATLAB code snippet. (This code snippet could replace the Euler's Method section in the code in the last section.)

```
% Midpoint Method
    y(1)=1;
    for i=2:N+1
        k1=h/2*f(t(i-1),y(i-1));
        k2=h*f(t(i-1)+h/2,y(i-1)+k1);
        y(i)=y(i-1)+k2;
        t(i)=t(i-1)+h;
    end
```

Example 10.1.3

Compare the Midpoint Method with the 2nd Order Taylor's Method for the problem

$$y' = t^2 + y, \quad y(0) = 1, \quad t \in [0,1]. \tag{10.1.23}$$

Solution

The solution to this problem is $y(t) = 3e^t - 2 - 2t - t^2$. In order to implement the 2nd Order Taylor's Method, we need

$$\begin{aligned} T^{(2)} &= f(t,y)+\frac{h}{2}f'(t,y) \\ &= t^2+y+\frac{h}{2}\left(2t+t^2+y\right). \end{aligned}$$

The results of the implementation are shown in Table 10.1.3

There are other way to approximate higher order Taylor polynomials. For example, we can approximate $T^{(3)}(t,y)$ using four parameters by

$$T^{(3)}(t,y) \approx af(t,y)+bf(t+\alpha, y+\beta f(t,y).$$

Expanding this approximation and using

$$T^{(3)}(t,y) \approx f(t,y)+\frac{h}{2}\frac{df}{dt}(t,y)+\frac{h^2}{6}\frac{df}{dt}(t,y),$$

we find that we cannot get rid of $O\left(h^2\right)$ terms. Thus, the best we can do is derive second order schemes. In fact, following a procedure similar to the derivation of the Midpoint Method, we find that

$$a+b=1, \quad ,\alpha b=\frac{h}{2}, \beta=\alpha.$$

There are three equations and four unknowns. Therefore there are many second order methods. Two classic methods are given by the modified Euler method $\left(a=b=\frac{1}{2}, \alpha=\beta=h\right)$ and Huen's method $\left(a=\frac{1}{4}, b=\frac{3}{4}, \alpha=\beta=\frac{2}{3}h\right.$.

Table 10.1.5: Numerical values for 2nd Order Taylor's Method, Midpoint Method, exact solution, and errors for solving Example 10.1.3 with $N=10$..

Exact	Taylor	Error	Midpoint	Error
1.0000	1.0000	0.0000	1.0000	0.0000
1.1055	1.1050	0.0005	1.1053	0.0003
1.2242	1.2231	0.0011	1.2236	0.0006
1.3596	1.3577	0.0019	1.3585	0.0010
1.5155	1.5127	0.0028	1.5139	0.0016
1.6962	1.6923	0.0038	1.6939	0.0023
1.9064	1.9013	0.0051	1.9032	0.0031
2.1513	2.1447	0.0065	2.1471	0.0041
2.4366	2.4284	0.0083	2.4313	0.0053
2.7688	2.7585	0.0103	2.7620	0.0068
3.1548	3.1422	0.0126	3.1463	0.0085

The Fourth Order Runge-Kutta Method, which is most often used, is given by the scheme

$$\begin{aligned} \tilde{y}_0 &= y_0, \\ k_1 &= hf\left(t_i, \tilde{y}_i\right), \\ k_2 &= hf\left(t_i+\frac{h}{2}, \tilde{y}_i+\frac{1}{2}k_1\right), \\ k_3 &= hf\left(t_i+\frac{h}{2}, \tilde{y}_i+\frac{1}{2}k_2\right), \\ k_4 &= hf\left(t_i+h, \tilde{y}_i+k_3\right), \\ \tilde{y}_{i+1} &= \tilde{y}_i+\frac{1}{6}\left(k_1+2k_2+2k_3+k_4\right), \quad i=0,1,\ldots,N-1. \end{aligned}$$

Again, we can test this on Example 10.1.3with $N=10$. The MATLAB implementation is given by

```
% Runge-Kutta 4th Order to solve dy/dt = f(t,y), y(a)=y0, on [a,b]
    clear

    a=0;
```

```
    b=1;
    N=10;
    h=(b-a)/N;

% Slope function
    f = inline('t^2+y','t','y');
    sol = inline('-2-2*t-t^2+3*exp(t)','t');

% Initial Condition
    t(1)=0;
    y(1)=1;

% RK4 Method
    y1(1)=1;
    for i=2:N+1
        k1=h*f(t(i-1),y1(i-1));
        k2=h*f(t(i-1)+h/2,y1(i-1)+k1/2);
        k3=h*f(t(i-1)+h/2,y1(i-1)+k2/2);
        k4=h*f(t(i-1)+h,y1(i-1)+k3);
        y1(i)=y1(i-1)+(k1+2*k2+2*k3+k4)/6;
        t(i)=t(i-1)+h;
    end
```

MATLAB has built-in ODE solvers, such as **ode45** for a fourth order Runge-Kutta method. Its implementation is given by

```
[t,y]=ode45(f,[0 1],1);
```

Note

MATLAB has built-in ODE solvers, as do other software packages, like Maple and Mathematica. You should also note that there are currently open source packages, such as Python based NumPy and Matplotlib, or Octave, of which some packages are contained within the Sage Project.

In this case f is given by an inline function like in the above RK 4 code. The time interval is entered as $[0, 1]$ and the 1 is the initial condition, $y(0) = 1$.

However, **ode 45** is not a straight forward RK_4 implementation. It is a hybrid method in which a combination of 4 th and 5 th order methods are combined allowing for adaptive methods to handled subintervals of the integration region which need more care. In this case, it implements a fourth order Runge-Kutta-Fehlberg method. Running this code for the above example actually results in values for $N = 41$ and not $N = 10$. If we wanted to have the routine output numerical solutions at specific times, then one could use the following form

```
tspan=0:h:1;
[t,y]=ode45(f,tspan,1);
```

In Table 10.1.6 we show the solutions which results for Example 10.1.3 comparing the RK_4 snippet above with **ode45**. As you can see RK $_4$ is much better than the previous implementation of the second order RK (Midpoint) Method. However, the MATLAB routine is two orders of magnitude better that RK_4.

Table 10.1.6: Numerical values for Fourth Order Range-Kutta Method, rk45, exact solution, and errors for solving Example 10.1.3 with $N = 10$.

Exact	Taylor	Error	Midpoint	Error
1.0000	1.0000	0.0000	1.0000	0.0000
1.1055	1.1055	$4.5894\mathrm{e}-08$	1.1055	$-2.5083\mathrm{e}-10$
1.2242	1.2242	$1.2335\mathrm{e}-07$	1.2242	$-6.0935\mathrm{e}-10$
1.3596	1.3596	$2.3850\mathrm{e}-07$	1.3596	$-1.0954\mathrm{e}-09$

Exact	Taylor	Error	Midpoint	Error
1.5155	1.5155	3.9843e − 07	1.5155	−1.7319e − 09
1.6962	1.6962	6.1126e − 07	1.6962	−2.5451e − 09
1.9064	1.9064	8.8636e − 07	1.9064	−3.5651e − 09
2.1513	2.1513	1.2345e − 06	2.1513	−4.8265e − 09
2.4366	2.4366	1.6679e − 06	2.4366	−6.3686e − 09
2.7688	2.7688	2.2008e − 06	2.7688	−8.2366e − 09
3.1548	3.1548	2.8492e − 06	3.1548	−1.0482e − 08

There are many ODE solvers in MATLAB. These are typically useful if RK_4 is having difficulty solving particular problems. For the most part, one is fine using RK_4, especially as a starting point. For example, there is **ode 23**, which is similar to **ode 45** but combining a second and third order scheme. Applying the results to Example 10.1.3 we obtain the results in Table 10.1.6. We compare these to the second order Runge-Kutta method. The code snippets are shown below.

```
% Second Order RK Method
    y1(1)=1;
    for i=2:N+1
        k1=h*f(t(i-1),y1(i-1));
        k2=h*f(t(i-1)+h/2,y1(i-1)+k1/2);
        y1(i)=y1(i-1)+k2;
        t(i)=t(i-1)+h;
    end
```

```
tspan=0:h:1;
[t,y]=ode23(f,tspan,1);
```

Table 10.1.7: Numerical values for Second Order Runge-Kutta Method, rk23, exact solution, and errors for solving Example 10.1.3 with $N = 10$.

Exact	Taylor	Error	Midpoint	Error
1.0000	1.0000	o.0000	1.0000	o.0000
1.1055	1.1053	0.0003	1.1055	2.7409e − 06
1.2242	1.2236	0.0006	1.2242	8.7114e − 06
1.3596	1.3585	0.0010	1.3596	1.6792e − 05
1.5155	1.5139	0.0016	1.5154	2.7361e − 05
1.6962	1.6939	0.0023	1.6961	4.0853e − 05
1.9064	1.9032	0.0031	1.9063	5.7764e − 05
2.1513	2.1471	0.0041	2.1512	7.8665e − 05
2.4366	2.4313	0.0053	2.4365	0.0001
2.7688	2.7620	0.0068	2.7687	0.0001
3.1548	3.1463	0.0085	3.1547	0.0002

We have seen several numerical schemes for solving initial value problems. There are other methods, or combinations of methods, which aim to refine the numerical approximations efficiently as if the step size in the current methods were taken to be much smaller. Some methods extrapolate solutions to obtain information outside of the solution interval. Others use one scheme to get a guess to the solution while refining, or correcting, this to obtain better solutions as the iteration through time proceeds. Such methods are described in courses in numerical analysis and in the literature. At this point we will apply these methods to several physics problems before continuing with analytical solutions.

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10.2: The Heat Equation

Finite Difference Method

The heat equation can be solved using separation of variables. However, many partial differential equations cannot be solved exactly and one needs to turn to numerical solutions. The heat equation is a simple test case for using numerical methods. Here we will use the simplest method, finite differences.

Let us consider the heat equation in one dimension,

$$u_t = ku_{xx}.$$

Boundary conditions and an initial condition will be applied later. The starting point is figuring out how to approximate the derivatives in this equation.

Recall that the partial derivative, u_t, is defined by

$$\frac{\partial u}{\partial t} = \lim_{\Delta t \to \infty} \frac{u(x,t+\Delta t) - u(x,t)}{\Delta t}.$$

Therefore, we can use the approximation

$$\frac{\partial u}{\partial t} \approx \frac{u(x,t+\Delta t) - u(x,t)}{\Delta t}. \tag{10.2.1}$$

This is called a **forward difference approximation**.

In order to find an approximation to the second derivative, u_{xx}, we start with the forward difference

$$\frac{\partial u}{\partial x} \approx \frac{u(x+\Delta x,t) - u(x,t)}{\Delta x}.$$

Then,

$$\frac{\partial u_x}{\partial x} \approx \frac{u_x(x+\Delta x,t) - u_x(x,t)}{\Delta x}.$$

We need to approximate the terms in the numerator. It is customary to use a backward difference approximation. This is given by letting $\Delta x \to -\Delta x$ in the forward difference form,

$$\frac{\partial u}{\partial x} \approx \frac{u(x,t) - u(x-\Delta x,t)}{\Delta t}. \tag{10.2.2}$$

Applying this to u_x evaluated at $x = x$ and $x = x + \Delta x$, we have

$$u_x(x,t) \approx \frac{u(x,t) - u(x-\Delta x,t)}{\Delta x},$$

and

$$u_x(x+\Delta x,t) \approx \frac{u(x+\Delta x,t) - u(x,t)}{\Delta x}$$

Inserting these expressions into the approximation for u_{xx}, we have

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \frac{\partial u_x}{\partial x} \\ &\approx \frac{u_x(x+\Delta x,t) - u_x(x,t)}{\Delta x} \\ &\approx \frac{\frac{u(x+\Delta x,t)-u(x,t)}{\Delta x}}{\Delta x} - \frac{\frac{u(x,t)-u(x-\Delta x,t)}{\Delta x}}{\Delta x} \\ &= \frac{u(x+\Delta x,t) - 2u(x,t) + u(x-\Delta x,t)}{(\Delta x)^2}. \end{aligned} \tag{10.2.3}$$

This approximation for u_{xx} is called the **central difference approximation** of u_{xx}.

Combining Equation (10.2.1) with (10.2.3) in the heat equation, we have

$$\frac{u(x,t+\Delta t)-u(x,t)}{\Delta t} \approx k\frac{u(x+\Delta x,t)-2u(x,t)+u(x-\Delta x,t)}{(\Delta x)^2}.$$

Solving for $u(x,t+\Delta t)$, we find

$$u(x,t+\Delta t) \approx u(x,t)+\alpha[u(x+\Delta x,t)-2u(x,t)+u(x-\Delta x,t)], \tag{10.2.4}$$

where $\alpha = k\frac{\Delta t}{(\Delta x)^2}$.

In this equation we have a way to determine the solution at position x and time $t+\Delta t$ given that we know the solution at three positions, $x, x+\Delta x$, and $x+2\Delta x$ at time t.

$$u(x,t+\Delta t) \approx u(x,t)+\alpha[u(x+\Delta x,t)-2u(x,t)+u(x-\Delta x,t)]. \tag{10.2.5}$$

A shorthand notation is usually used to write out finite difference schemes. The domain of the solution is $x \in [a,b]$ and $t \geq 0$. We seek approximate values of $u(x,t)$ at specific positions and times. We first divide the interval $[a,b]$ into N subintervals of width $\Delta x = (b-a)/N$. Then, the endpoints of the subintervals are given by

$$x_i = a+i\Delta x, \quad i=0,1,\ldots,N.$$

Similarly, we take time steps of Δt, at times

$$t_j = j\Delta t, \quad j=0,1,2,\ldots$$

This gives a grid of points (x_i,t_j) in the domain.

At each grid point in the domain we seek an approximate solution to the heat equation, $u_{i,j} \approx u(x_i,t_j)$. Equation (10.2.5) becomes

$$u_{i,j+1} \approx u_{i,j}+\alpha\left[u_{i+1,j}-2u_{i,j}+u_{i-1,j}\right]. \tag{10.2.6}$$

Equation (10.2.7) is the finite difference scheme for solving the heat equation. This equation is represented by the stencil shown in Figure 10.2.1. The black circles represent the four terms in the equation, $u_{i,j}u_{i-1,j}u_{i+1,j}$ and $u_{i,j+1}$.

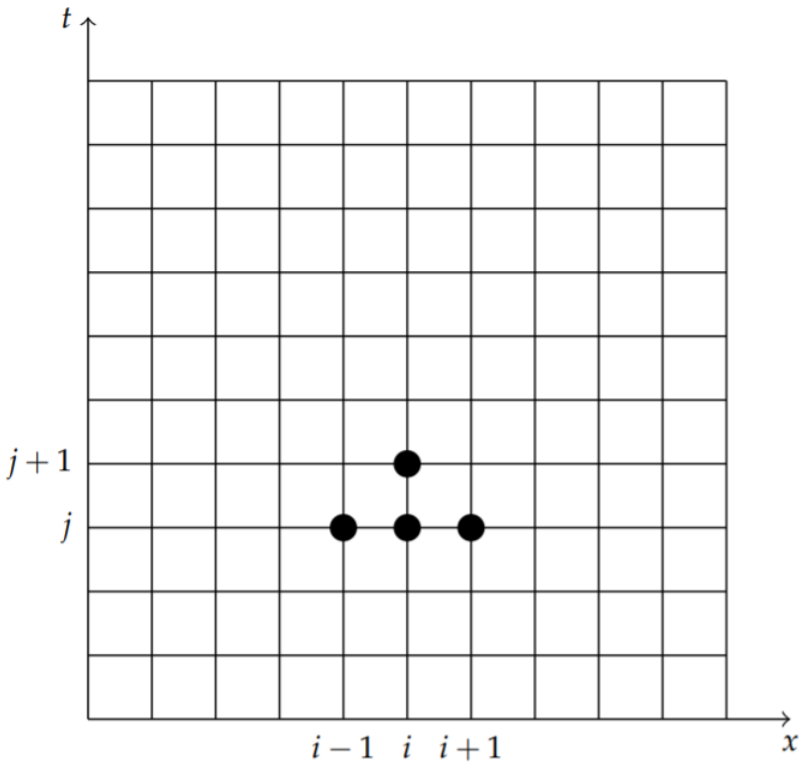


Figure 10.2.1: This stencil indicates the four types of terms in the finite difference scheme in Equation (10.2.7). The black circles represent the four terms in the equation, $u_{i,j}u_{i-1,j}u_{i+1,j}$ and $u_{i,j+1}$.

Let's assume that the initial condition is given by

$$u(x,0) = f(x).$$

Then, we have $u_{i,0} = f(x_i)$. Knowing these values, denoted by the open circles in Figure 10.2.2, we apply the stencil to generate the solution on the $j = 1$ row. This is shown in Figure 10.2.2

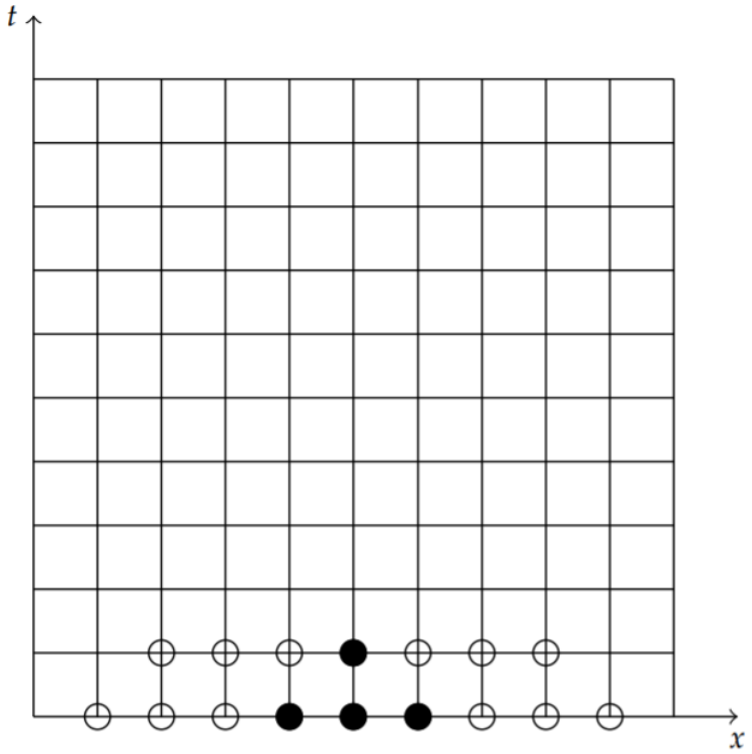


Figure 10.2.2: Applying the stencil to the row of initial values gives the solution at the next time step.

Further rows are generated by successively applying the stencil on each row, using the known approximations of $u_{i,j}$ at each level. This gives the values of the solution at the open circles shown in Figure 10.2.3. We notice that the solution can only be obtained at a finite number of points on the grid.

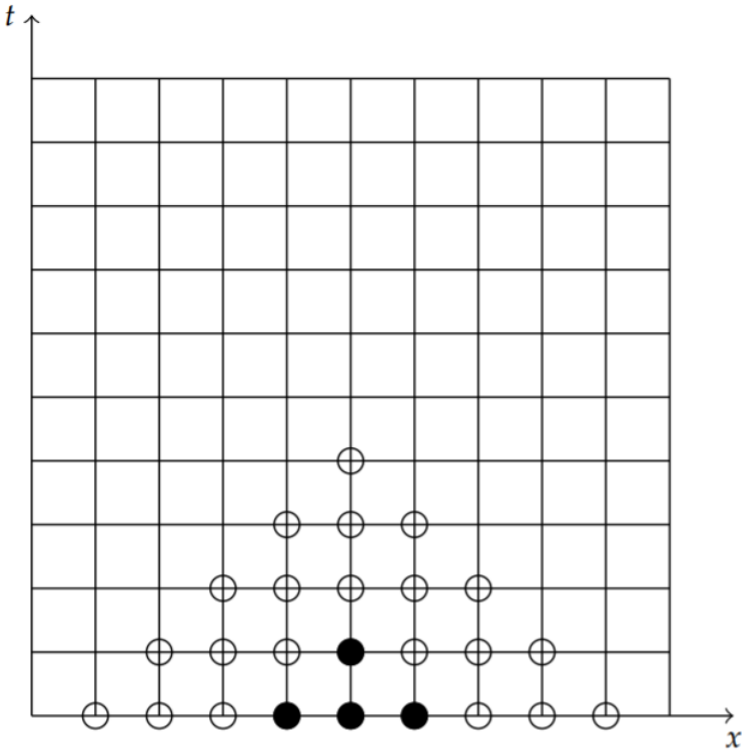


Figure 10.2.3: Continuation of the process provides solutions at the indicated points.

In order to obtain the missing values, we need to impose boundary conditions. For example, if we have Dirichlet conditions at $x = a$,

$$u(a, t) = 0,$$

or $u_{0,j} = 0$ for $j = 0, 1, \ldots$, then we can fill in some of the missing data points as seen in Figure 10.2.4

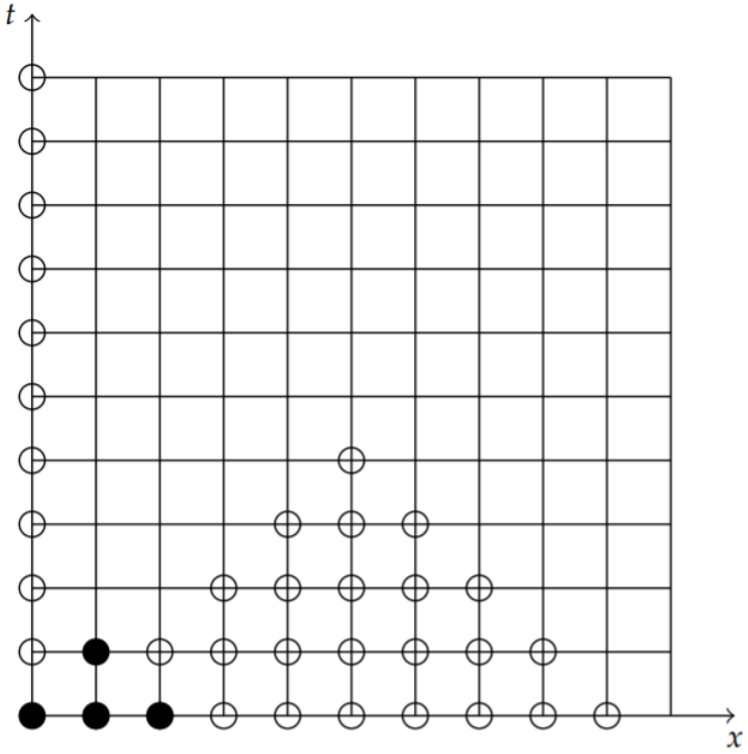


Figure 10.2.4: Knowing the values of the solution at $x = a$, we can fill in more of the grid.

The process continues until we again go as far as we can. This is shown in Figure 10.2.5

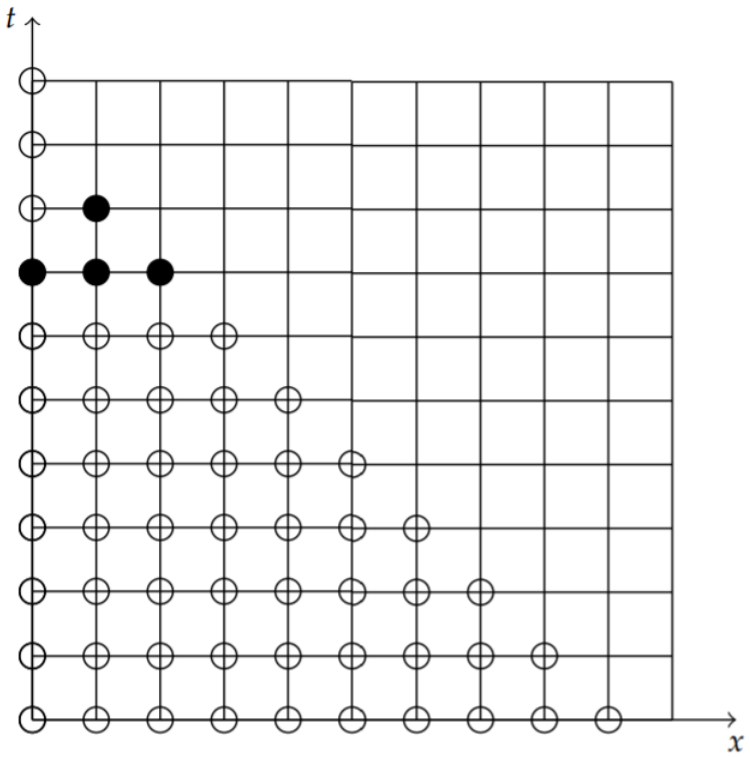


Figure 10.2.5: Knowing the values of the solution at $x = a$, we can fill in more of the grid until we stop.

We can fill in the rest of the grid using a boundary condition at $x = b$. For Dirichlet conditions at $x = b$,

$$u(b, t) = 0,$$

or $u_{N,j} = 0$ for $j = 0, 1, \ldots$, then we can fill in the rest of the missing data points as seen in Figure 10.2.6

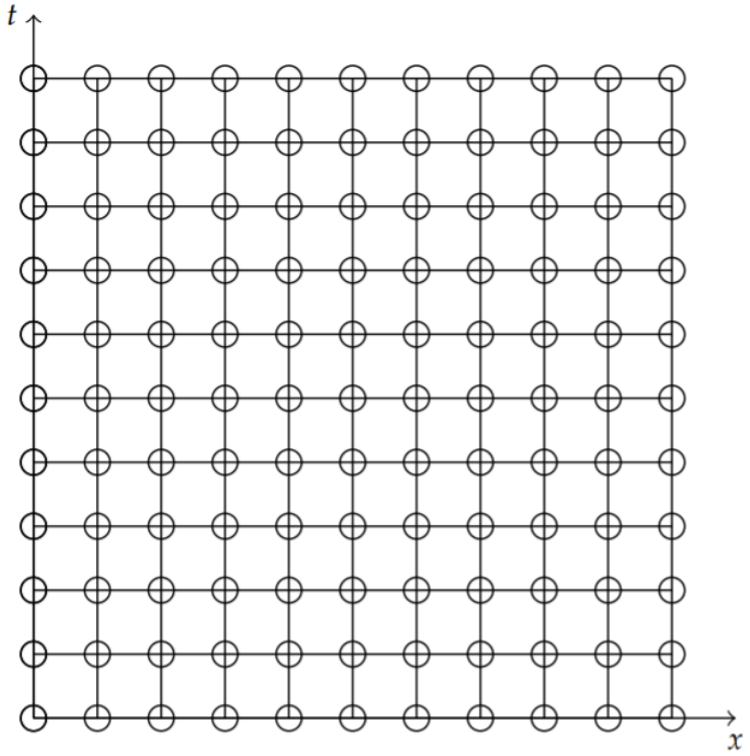


Figure 10.2.6: Using boundary conditions and the initial condition, the grid can be fill in through any time level.

We could also use Neumann conditions. For example, let

$$u_x(a,t)=0.$$

The approximation to the derivative gives

$$\left.\frac{\partial u}{\partial x}\right|_{x=a} \approx \frac{u(a+\Delta x,t)-u(a,t)}{\Delta x}=0.$$

Then,

$$u(a+\Delta x,t)-u(a,t)$$

or $u_{0,j}=u_{1,j}$, for $j=0,1,\ldots$. Thus, we know the values at the boundary and can generate the solutions at the grid points as before.

We now have to code this using software. We can use MATLAB to do this. An example of the code is given below. In this example we specify the length of the rod, $L=1$, and the heat constant, $k=1$. The code is run for $t\in[0,0.1]$.

The grid is created using $N=10$ subintervals in space and $M=50$ time steps. This gives dx $=\Delta x$ and dt $=\Delta t$. Using these values, we find the numerical scheme constant $\alpha=k\Delta t/(\Delta x)^2$.

Nest, we define $x_i=i*dx, i=0,1,\ldots,N$. However, in MATLAB, we cannot have an index of 0 . We need to start with $i=1$. Thus, $x_i=(i-1)*dx$, $i=1,2,\ldots,N+1$.

Next, we establish the initial condition. We take a simple condition of

$$u(x,0)=\sin\pi x.$$

We have enough information to begin the numerical scheme as developed earlier. Namely, we cycle through the time steps using the scheme. There is one loop for each time step. We will generate the new time step from the last time step in the form

$$u_i^{\text{new}}=u_i^{\text{old}}+\alpha\left[u_{i+1}^{\text{old}}-2u_i^{\text{old}}+u_{i-1}^{\text{old}}\right]. \tag{10.2.7}$$

This is done using $u0(i)=u_i^{\text{new}}$ and $u1(i)=u_i^{\text{old}}$.

At the end of each time loop we update the boundary points so that the grid can be filled in as discussed. When done, we can plot the final solution. If we want to show solutions at intermediate steps, we can plot the solution earlier.

```
% Solution of the Heat Equation Using a Forward Difference Scheme

% Initialize Data
%     Length of Rod, Time Interval
%     Number of Points in Space, Number of Time Steps

L=1;
T=0.1;
k=1;
N=10;
M=50;
dx=L/N;
dt=T/M;
alpha=k*dt/dx^2;

% Position

for i=1:N+1
    x(i)=(i-1)*dx;
```

```
end

% Initial Condition

for i=1:N+1
    u0(i)=sin(pi*x(i));
end

% Partial Difference Equation (Numerical Scheme)

for j=1:M
    for i=2:N
        u1(i)=u0(i)+alpha*(u0(i+1)-2*u0(i)+u0(i-1));
    end
    u1(1)=0;
    u1(N+1)=0;
    u0=u1;
end

% Plot solution
plot(x, u1);
```

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10.3: Truncation Error

In the previous section we found a finite difference scheme for numerically solving the one dimensional heat equation. We have from Equations (10.2.5) and (10.2.7),

$$u(x,t+\Delta t) \approx u(x,t)+\alpha[u(x+\Delta x,t)-2u(x,t)+u(x-\Delta x,t)] \tag{10.3.1}$$

$$u_{i,j+1} \approx u_{i,j}+\alpha\left[u_{i+1,j}-2u_{i,j}+u_{i-1,j}\right], \tag{10.3.2}$$

where $\alpha = k\Delta t/(\Delta x)^2$. For points $x \in [a,b]$ and $t \geq 0$, we use the scheme to find approximate values of $u(x_i,t_i) = u_{i,j}$ at positions $x_i = a+i\Delta x, \quad i = 0,1,\ldots,N$, and times $t_j = j\Delta t, \quad j = 0,1,2,\ldots$.

In implementing the scheme we have found that there are errors introduced just like when using Euler's Method for ordinary differential equations. These truncations errors can be found by applying Taylor approximations just like we had for ordinary differential equations. In the schemes (10.3.1) and (10.3.2), we have not use equality. In order to replace the approximation by an equality, we need t estimate the order of the terms neglected in a Taylor series approximation of the time and space derivatives we have approximated.

We begin with the time derivative approximation. We used the **forward difference approximation** (10.2.1),

$$\frac{\partial u}{\partial t} \approx \frac{u(x,t+\Delta t)-u(x,t)}{\Delta t}. \tag{10.3.3}$$

This can be derived from the Taylor series expansion of $u(x,t+\Delta t)$ about $\Delta t = 0$,

$$u(x,t+\Delta t) = u(x,t)+\frac{\partial u}{\partial t}(x,t)\Delta t+\frac{1}{2!}\frac{\partial^2 u}{\partial t^2}(x,t)(\Delta t)^2+O\left((\Delta t)^3\right).$$

Solving for $\dfrac{\partial u}{\partial t}(x,t)$, we obtain

$$\frac{\partial u}{\partial t}(x,t) = \frac{u(x,t+\Delta t)-u(x,t)}{\Delta t}-\frac{1}{2!}\frac{\partial^2 u}{\partial t^2}(x,t)\Delta t+O\left((\Delta t)^2\right).$$

We see that we have obtained the forward difference approximation (10.2.1) with the added benefit of knowing something about the error terms introduced in the approximation. Namely, when we approximate u_t with the forward difference approximation (10.2.1), we are making an error of

$$E(x,t,\Delta t) = -\frac{1}{2!}\frac{\partial^2 u}{\partial t^2}(x,t)\Delta t+O\left((\Delta t)^2\right).$$

We have truncated the Taylor series to obtain this approximation and we say that

$$\frac{\partial u}{\partial t} = \frac{u(x,t+\Delta t)-u(x,t)}{\Delta t}+O(\Delta t) \tag{10.3.4}$$

is a first order approximation in Δt.

In a similar manor, we can obtain the truncation error for the $u_x x$-term. However, instead of starting with the approximation we used in Equation ??uxx), we will derive a term using the Taylor series expansion of $u(x+\Delta x,t)$ about $\Delta x = 0$. Namely, we begin with the expansion

$$\begin{aligned} u(x+\Delta x,t) = {} & u(x,t)+u_x(x,t)\Delta x+\frac{1}{2!}u_{xx}(x,t)(\Delta x)^2+\frac{1}{3!}u_{xxx}(x,t)(\Delta x)^3 \\ & +\frac{1}{4!}u_{xxxx}(x,t)(\Delta x)^4+\ldots \end{aligned} \tag{10.3.5}$$

We want to solve this equation for u_{xx}. However, there are some obstructions, like needing to know the u_x term. So, we seek a way to eliminate lower order terms. On way is to note that replacing Δx by $-\Delta x$ gives

$$\begin{aligned} u(x-\Delta x,t) = {} & u(x,t)-u_x(x,t)\Delta x+\frac{1}{2!}u_{xx}(x,t)(\Delta x)^2-\frac{1}{3!}u_{xxx}(x,t)(\Delta x)^3 \\ & +\frac{1}{4!}u_{xxxx}(x,t)(\Delta x)^4+\ldots \end{aligned} \tag{10.3.6}$$

Adding these Taylor series, we have

$$u(x+\Delta x,t)+u(x+\Delta x,t)=2u(x,t)+u_{xx}(x,t)(\Delta x)^2 + \frac{2}{4!}u_{xxxx}(x,t)(\Delta x)^4+O\left((\Delta x)^6\right). \tag{10.3.7}$$

We can now solve for u_{xx} to find

$$u_{xx}(x,t)=\frac{u(x+\Delta x,t)-2u(x,t)+u(x+\Delta x,t)}{(\Delta x)^2} + \frac{2}{4!}u_{xxxx}(x,t)(\Delta x)^2+O\left((\Delta x)^4\right) \tag{10.3.8}$$

Thus, we have that

$$u_{xx}(x,t)=\frac{u(x+\Delta x,t)-2u(x,t)+u(x+\Delta x,t)}{(\Delta x)^2}+O\left((\Delta x)^2\right)$$

is the second order in Δx approximation of u_{xx}.

Combining these results, we find that the heat equation is approximated by

$$\frac{u(x,t+\Delta t)-u(x,t)}{\Delta t}=\frac{u(x+\Delta x,t)-2u(x,t)+u(x+\Delta x,t)}{(\Delta x)^2}+O\left((\Delta x)^2,\Delta t\right).$$

This has local truncation error that is first order in time and second order in space.

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10.4: Stability

Another consideration for numerical schemes for the heat equation is the stability of the scheme. In implementing the finite difference scheme,

$$u_{m,j+1} = u_{m,j} + \alpha \left[u_{m+1,j} - 2u_{m,j} + u_{m-1,j} \right] \tag{10.4.1}$$

with $\alpha = k\Delta t/(\Delta x)^2$, one finds that the solution goes crazy when α is too big. In other words, if you try to push the individual time steps too far into the future, then something goes haywire. We can determine the onset of instability by looking at the solution of this equation for $u_{m,j}$. [Note: We changed index i to m to avoid confusion later in this section.]

The scheme is actually what is called a partial difference equation for $u_{m,j}$. We could write it in terms of difference, such as $u_{m+1,j} - u_{m,j}$ and $u_{m,j+1} - u_{m,j}$. The furthest apart the time steps are are one unit and the spatial points are two units apart. We can see this in the stencils in Figure 10.6. So, this is a second order partial difference equation similar to the idea that the heat equation is a second order partial differential equation. The heat equation can be solved using the method of separation of variables. The difference scheme can also be solved in a similar fashion. We will show how this can lead to product solutions.

We begin by assuming that $u_{mj} = X_m T_j$, a product of functions of the indices m and j. Inserting this guess into the finite difference equation, we have

$$\begin{aligned} u_{m,j+1} &= u_{m,j} + \alpha \left[u_{m+1,j} - 2u_{m,j} + u_{m-1,j} \right] \\ X_m T_{j+1} &= X_m T_j + \alpha \left[X_{m+1} - 2X_m + X_{m-1} \right] T_j, \\ \frac{T_{j+1}}{T_j} &= \frac{\alpha X_{m+1} + (1-2\alpha) X_m + \alpha X_{m-1}}{X_m}. \end{aligned} \tag{10.4.2}$$

Noting that we have a function of j equal to a function of m, then we can set each of these to a constant, λ. Then, we obtain two ordinary difference equations:

$$T_{j+1} = \lambda T_j, \tag{10.4.3}$$

$$\alpha X_{m+1} + (1-2\alpha) X_m + \alpha X_{m-1} = \lambda X_m. \tag{10.4.4}$$

The first equation is a simple first order difference equation and can be solved by iteration:

$$\begin{aligned} T_{j+1} &= \lambda T_j, \\ &= \lambda \left(\lambda T_{j-1} \right) = \lambda^2 T_{j-1}, \\ &= \lambda^3 T_{j-2}, \\ &= \lambda^{j+1} T_0, \end{aligned} \tag{10.4.5}$$

The second difference equation can be solved by making a guess in the same spirit as solving a second order constant coefficient differential equation.Namely, let $X_m = \xi^m$ for some number ξ. This gives

$$\begin{aligned} \alpha X_{m+1} + (1-2\alpha) X_m + \alpha X_{m-1} &= \lambda X_m, \\ \xi^{m-1} \left[\alpha \xi^2 + (1-2\alpha)\xi + \alpha \right] &= \lambda \xi^m \\ \alpha \xi^2 + (1-2\alpha-\lambda)\xi + \alpha &= 0. \end{aligned} \tag{10.4.6}$$

This is an equation foe ξ in terms of α and λ. Due to the boundary conditions, we expect to have oscillatory solutions. So, we can guess that $\xi = |\xi| e^{i\theta}$, where i here is the imaginary unit. We assume that $|\xi| = 1$, and thus $\xi = e^{i\theta}$ and $X_m = \xi^m = e^{im\theta}$. Since $x_m = m\Delta x$, we have $X_m = e^{ix_m\theta/\Delta x}$. We define $\beta =$ theta / Δ, to give $X_m = e^{i\beta x_m}$ and $\xi = e^{i\beta\Delta x}$.

Inserting this value for ξ into the quadratic equation for ξ, we have

$$\begin{aligned} 0 &= \alpha\xi^2 + (1-2\alpha-\lambda)\xi + \alpha \\ &= \alpha e^{2i\beta\Delta x} + (1-2\alpha-\lambda) e^{i\beta\Delta x} + \alpha \\ &= e^{i\beta\Delta x} \left[\alpha \left(e^{i\beta\Delta x} + e^{-i\beta\Delta x} \right) + (1-2\alpha-\lambda) \right] \\ &= e^{i\beta\Delta x} \left[2\alpha \cos(\beta\Delta x) + (1-2\alpha-\lambda) \right] \\ \lambda &= 2\alpha\cos(\beta\Delta x) + 1 - 2\alpha. \end{aligned} \tag{10.4.7}$$

So, we have found that

$$u_{mj} = X_m T_j = \lambda^m \left(a \cos \alpha x_m + b \sin \alpha x_m \right), \quad a^2 + b^2 = h_0^2,$$

and

$$\lambda = 2\alpha \cos(\beta \Delta x) + 1 - 2\alpha$$

For the solution to remain bounded, or stable, we need $|\lambda| \leq 1$.

Therefore, we have the inequality

$$-1 \leq 2\alpha \cos(\beta \Delta x) + 1 - 2\alpha \leq 1.$$

Since $\cos(\beta \Delta x) \leq 1$, the upper bound is obviously satisfied. Since $-1 \leq \cos(\beta \Delta x)$, the lower bound is satisfied for $-1 \leq -2s + 1 - 2s$, or $s \leq \frac{1}{2}$. Therefore, the stability criterion is satisfied when

$$\alpha = k \frac{\Delta t}{\Delta x^2} \leq \frac{1}{2}.$$

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CHAPTER OVERVIEW

11: A - Calculus Review - What Do I Need to Know From Calculus?

> *"Ordinary language is totally unsuited for expressing what physics really asserts, since the words of everyday life are not sufficiently abstract. Only mathematics and mathematical logic can say as little as the physicist means to say."*
>
> *-Bertrand Russell (1872-1970)*

Before you begin our study of differential equations perhaps you should review some things from calculus. You definitely need to know something before taking this class. It is assumed that you have taken Calculus and are comfortable with differentiation and integration. Of course, you are not expected to know every detail from these courses. However, there are some topics and methods that will come up and it would be useful to have a handy reference to what it is you should know.

Most importantly, you should still have your calculus text to which you can refer throughout the course. Looking back on that old material, you will find that it appears easier than when you first encountered the material. That is the nature of learning mathematics and other subjects. Your understanding is continually evolving as you explore topics more in depth. It does not always sink in the first time you see it. In this chapter we will give a quick review of these topics. We will also mention a few new methods that might be interesting.

11.1: Introduction
11.2: Trigonometric Functions
11.3: Hyperbolic Functions
11.4: Derivatives
11.5: Integrals
11.6: Geometric Series
11.7: Power Series
11.8: The Binomial Expansion
11.9: Problems

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11.1: Introduction

There are two main topics in calculus: derivatives and integrals. You learned that derivatives are useful in providing rates of change in either time or space. Integrals provide areas under curves, but also are useful in providing other types of sums over continuous bodies, such as lengths, areas, volumes, moments of inertia, or flux integrals. In physics, one can look at graphs of position versus time and the slope (derivative) of such a function gives the velocity. (See Figure 11.1.1) By plotting velocity versus time you can either look at the derivative to obtain acceleration, or you could look at the area under the curve and get the displacement:

$$x = \int_{t_0}^{t} v\,dt \tag{11.1.1}$$

This is shown in Figure 11.1.2

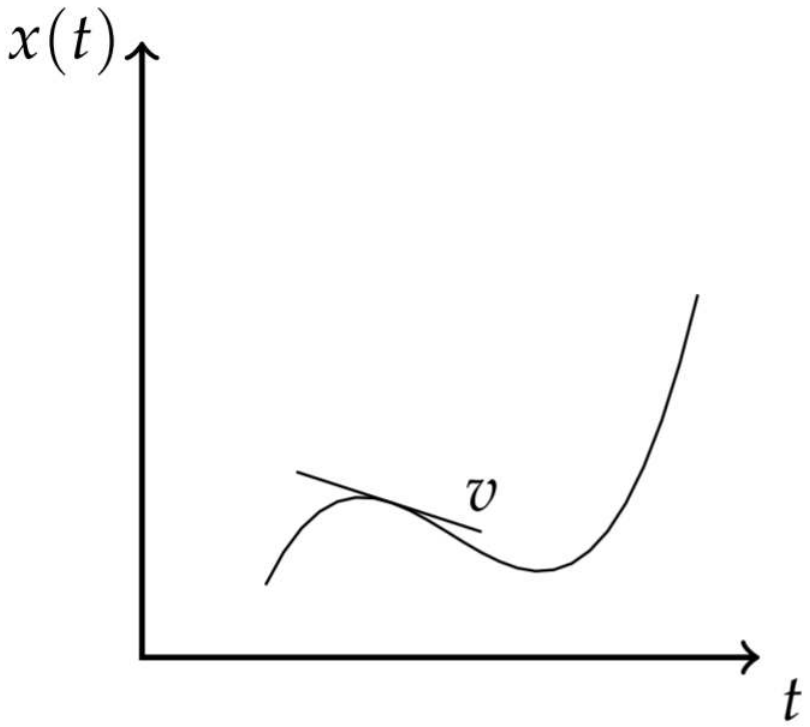


Figure 11.1.1: Plot of position vs time.

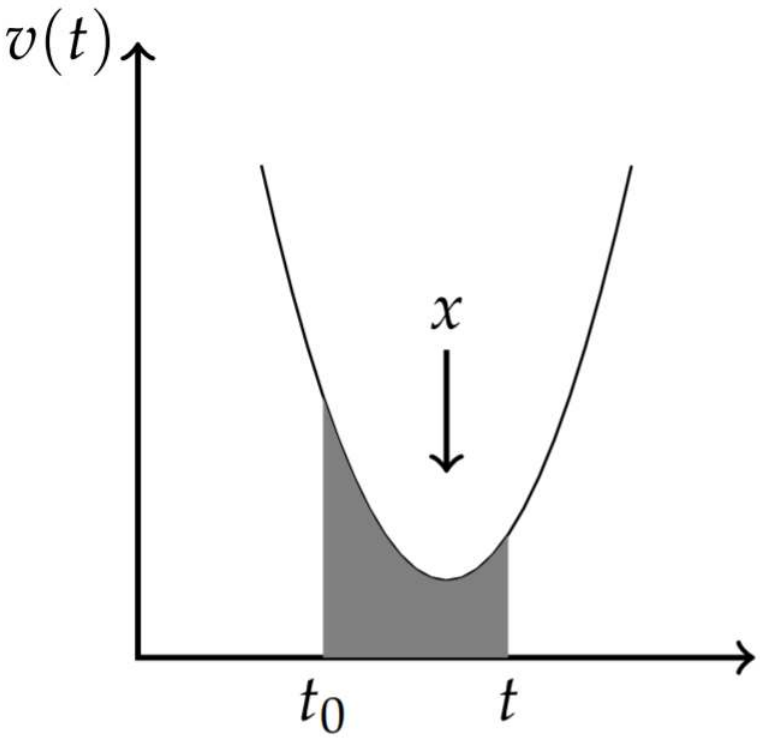


Figure 11.1.2: Plot of velocity vs time.

Of course, you need to know how to differentiate and integrate given functions. Even before getting into differentiation and integration, you need to have a bag of functions useful in physics. Common functions are the polynomial and rational functions. You should be fairly familiar with these. Polynomial functions take the general form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \tag{11.1.2}$$

where $a_n \neq 0$. This is the form of a polynomial of degree n. Rational functions, $f(x) = \frac{g(x)}{h(x)}$, consist of ratios of polynomials. Their graphs can exhibit vertical and horizontal asymptotes.

Next are the exponential and logarithmic functions. The most common are the natural exponential and the natural logarithm. The natural exponential is given by $f(x) = e^x$, where $e \approx 2.718281828\ldots$ The natural logarithm is the inverse to the exponential, denoted by $\ln x$. (One needs to be careful, because some mathematics and physics books use $\log$ to mean natural exponential, whereas many of us were first trained to use this notation to mean the common logarithm, which is the 'log base 10'. Here we will use $\ln x$ for the natural logarithm.)

The properties of the exponential function follow from the basic properties for exponents. Namely, we have:

$$e^0 = 1 \tag{11.1.3}$$

$$e^{-a} = \frac{1}{e^a} \tag{11.1.4}$$

$$e^a e^b = e^{a+b} \tag{11.1.5}$$

$$(e^a)^b = e^{ab} \tag{11.1.6}$$

The relation between the natural logarithm and natural exponential is given by

$$y = e^x \Leftrightarrow x = \ln y \tag{11.1.7}$$

Some common logarithmic properties are

$$\ln 1 = 0 \tag{11.1.8}$$
$$\ln \frac{1}{a} = -\ln a \tag{11.1.9}$$
$$\ln(ab) = \ln a + \ln b \tag{11.1.10}$$
$$\ln \frac{a}{b} = \ln a - \ln b \tag{11.1.11}$$
$$\ln \frac{1}{b} = -\ln b \tag{11.1.12}$$

We will see applications of these relations as we progress through the course.

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11.2: Trigonometric Functions

Another set of useful functions are the trigonometric functions. These functions have probably plagued you since high school. They have their origins as far back as the building of the pyramids. Typical applications in your introductory math classes probably have included finding the heights of trees, flag poles, or buildings. It was recognized a long time ago that similar right triangles have fixed ratios of any pair of sides of the two similar triangles. These ratios only change when the non-right angles change.

Thus, the ratio of two sides of a right triangle only depends upon the angle. Since there are six possible ratios (think about it!), then there are six possible functions. These are designated as sine, cosine, tangent and their reciprocals (cosecant, secant and cotangent). In your introductory physics class, you really only needed the first three. You also learned that they are represented as the ratios of the opposite to hypotenuse, adjacent to hypotenuse, etc. Hopefully, you have this down by now.

You should also know the exact values of these basic trigonometric functions for the special angles $\theta = 0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{2}$, and their corresponding angles in the second, third and fourth quadrants. This becomes internalized after much use, but we provide these values in Table 11.2.1just in case you need a reminder.

Table 11.2.1: Table of Trigonometric Values

θ	$\cos\theta$	$\sin\theta$	$\tan\theta$
0	1	0	0
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\sqrt{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$\frac{\pi}{2}$	0	1	undefined

The problems students often have using trigonometric functions in later courses stem from using, or recalling, identities. We will have many an occasion to do so in this class as well. What is an identity? It is a relation that holds true all of the time. For example, the most common identity for trigonometric functions is the Pythagorean identity

$$\sin^2\theta + \cos^2\theta = 1 \tag{11.2.1}$$

This holds true for every angle θ! An even simpler identity is

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \tag{11.2.2}$$

Other simple identities can be derived from the Pythagorean identity. Dividing the identity by $\cos^2\theta$, or $sin^2\theta$, yields

$$\tan^2\theta + 1 = \sec^2\theta \tag{11.2.3}$$
$$1 + \cot^2\theta = \csc^2\theta \tag{11.2.4}$$

Several other useful identities stem from the use of the sine and cosine of the sum and difference of two angles. Namely, we have that

$$\sin(A \pm B) = \sin A\cos B \pm \sin B\cos A \tag{11.2.5}$$
$$\cos(A \pm B) = \cos A\cos B \pm \sin A\sin B \tag{11.2.6}$$

Note that the upper (lower) signs are taken together.

Example 11.2.1

Evaluate $\sin\frac{\pi}{12}$.

Solution

$$\begin{aligned}\sin\frac{\pi}{12} &= \sin\left(\frac{\pi}{3}-\frac{\pi}{4}\right)\\ &= \sin\frac{\pi}{3}\cos\frac{\pi}{4}-\sin\frac{\pi}{4}\cos\frac{\pi}{3}\\ &= \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\frac{1}{2}\\ &= \frac{\sqrt{2}}{4}(\sqrt{3}-1) \qquad (11.2.7)\end{aligned}$$

The double angle formulae are found by setting $A = B$:

$$\sin(2A) = 2\sin A\cos B \qquad (11.2.8)$$
$$\cos(2A) = \cos^2 A - \sin^2 A \qquad (11.2.9)$$

Using Equation (11.2.13), we can rewrite (11.2.21)as

$$\cos(2A) = 2\cos^2 A - 1 \qquad (11.2.10)$$
$$= 1 - 2\sin^2 A \qquad (11.2.11)$$

These, in turn, lead to the half angle formulae. Solving for $\cos^2 A$ and $sin^2 A$, we find that

$$\sin^2 A = \frac{1-\cos 2A}{2} \qquad (11.2.12)$$
$$\cos^2 A = \frac{1+\cos 2A}{2} \qquad (11.2.13)$$

Example 11.2.2

Evaluate $\cos\frac{\pi}{12}$. In the last example, we used the sum/difference identities to evaluate a similar expression. We could have also used a half angle identity.

Solution

In this example, we have

$$\begin{aligned}\cos^2\frac{\pi}{12} &= \frac{1}{2}\left(1+\cos\frac{\pi}{6}\right)\\ &= \frac{1}{2}\left(1+\frac{\sqrt{3}}{2}\right)\\ &= \frac{1}{4}(2+\sqrt{3}) \qquad (11.2.14)\end{aligned}$$

So, $\cos\frac{\pi}{12} = \frac{1}{2}\sqrt{2+\sqrt{3}}$. This is not the simplest form and is called a nested radical. In fact, if we proceeded using the difference identity for cosines, then we would obtain

$$\cos\pi 12 = \frac{\sqrt{2}}{4}(1+\sqrt{3})$$

So, how does one show that these answers are the same?

Note

It is useful at times to know when one can reduce square roots of such radicals, called denesting. More generally, one seeks to write $\sqrt{a+b\sqrt{q}} = c+d\sqrt{q}$. Following the procedure in this example, one has $d = \frac{b}{2c}$ and

$$c^2 = \frac{1}{2}\left(a \pm \sqrt{a^2-qb^2}\right)$$

As long as $a^2 - qb^2$ is a perfect square, there is a chance to reduce the expression to a simpler form.

Let's focus on the factor $\sqrt{2+\sqrt{3}}$. We seek to write this in the form $c+d\sqrt{3}$. Equating the two expressions and squaring, we have

$$\begin{aligned} 2+\sqrt{3} &= (c+d\sqrt{3})^2 \\ &= c^2+3d^2+2cd\sqrt{3} \end{aligned} \tag{11.2.15}$$

In order to solve for c and d, it would seem natural to equate the coefficients of $\sqrt{3}$ and the remaining terms. We obtain a system of two nonlinear algebraic equations,

$$c^2 3d^2 = 2 \tag{11.2.16}$$

$$2cd = 1 \tag{11.2.17}$$

Solving the second equation for $d = 1/2c$, and substituting the result into the first equation, we find

$$4c^4 - 8c^2 + 3 = 0$$

This fourth order equation has four solutions

$$c + \pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{6}}{2}$$

and

$$b = \pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{6}}{6}$$

Thus,

$$\begin{aligned} \cos\pi 12 &= \frac{1}{2}\sqrt{2+\sqrt{3}} \\ &= \pm\frac{1}{2}\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\sqrt{3}\right) \\ &= \pm\frac{\sqrt{2}}{4}(1+\sqrt{3}) \end{aligned} \tag{11.2.18}$$

and

$$\begin{aligned} \cos\frac{\pi}{12} &= \frac{1}{2}\sqrt{2+\sqrt{3}} \\ &= \pm\frac{1}{2}\left(\frac{\sqrt{6}}{2}+\frac{\sqrt{6}}{6}\sqrt{3}\right) \\ &= \pm\frac{\sqrt{6}}{12}(3+\sqrt{3}) \end{aligned} \tag{11.2.19}$$

Of the four solutions, two are negative and we know the value of the cosine for this angle has to be positive. The remaining two solutions are actually equal! A quick computation will verify this

$$\begin{aligned} \frac{\sqrt{6}}{12}(3+\sqrt{3}) &= \frac{\sqrt{3}\sqrt{2}}{12}(3+\sqrt{3}) \\ &= \frac{\sqrt{2}}{12}(3\sqrt{3}+3) \\ &= \frac{\sqrt{2}}{4}(\sqrt{3}+1) \end{aligned} \tag{11.2.20}$$

We could have bypassed this situation be requiring that the solutions for b and c were not simply proportional to $\sqrt{3}$ like they are in the second case.

Finally, another useful set of identities are the product identities. For example, if we add the identities for $\sin(A+B)$ and $\sin(A-B)$, the second terms cancel and we have

$$\sin(A+B)+\sin(A-B) = 2\sin A\cos B$$

Thus, we have that

$$\sin A \cos B = \frac{1}{2}(\sin(A+B)+\sin(A-B)) \tag{11.2.21}$$

Similarly, we have

$$\cos A \cos B = \frac{1}{2}(\cos(A+B)+\cos(A-B)) \tag{11.2.22}$$

and

$$\sin A \sin B = \frac{1}{2}(\cos(A-B)-\cos(A+B)) \tag{11.2.23}$$

These boxed equations are the most common trigonometric identities. They appear often and should just roll off of your tongue.

We will also need to understand the behaviors of trigonometric functions. In particular, we know that the sine and cosine functions are periodic. They are not the only periodic functions, as we shall see. [Just visualize the teeth on a carpenter's saw.] However, they are the most common periodic functions.

A periodic function $f(x)$ satisfies the relation

$$f(x+p) = f(x), \quad \text{for all } x$$

for some constant p. If p is the smallest such number, then p is called the period. Both the sine and cosine functions have period 2π. This means that the graph repeats its form every 2π units. Similarly, $\sin bx$ and $\cos bx$ have the common period $p = \frac{2\pi}{b}$. We will make use of this fact in later chapters.

Related to these are the inverse trigonometric functions. For example, $f(x) = \sin^{-1} x$, or $f(x) = \arcsin x$. Inverse functions give back angles, so you should think

$$\theta = \sin^{-1} x \quad \Leftrightarrow \quad x = \sin\theta \tag{11.2.24}$$

Also, you should recall that $y = \sin^{-1} x = \arcsin x$ is only a function if $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. Similar relations exist for $y = \cos^{-1} x = \arccos x$ and $\tan^{-1} x = \arctan x$.

Note

In Feynman's *Surely You're Joking Mr. Feynman*!, Richard Feynman (1918-1988) talks about his invention of his own notation for both trigonometric and inverse trigonometric functions as the standard notation did not make sense to him.

Once you think about these functions as providing angles, then you can make sense out of more complicated looking expressions, like $\tan\left(\sin^{-1} x\right)$. Such expressions often pop up in evaluations of integrals. We can untangle this in order to produce a simpler form by referring to expression (11.2.24). $\theta = \sin^{-1} x$ is simple an angle whose sine is x. Knowing the sine is the opposite side of a right triangle divided by its hypotenuse, then one just draws a triangle in this proportion as shown in Figure 11.2.1. Namely, the side opposite the angle has length x and the hypotenuse has length 1. Using the Pythagorean Theorem, the missing side (adjacent to the angle) is simply $\sqrt{1-x^2}$. Having obtained the lengths for all three sides, we can now produce the tangent of the angle as

$$\tan\left(\sin^{-1} x\right) = \frac{x}{\sqrt{1-x^2}}.$$

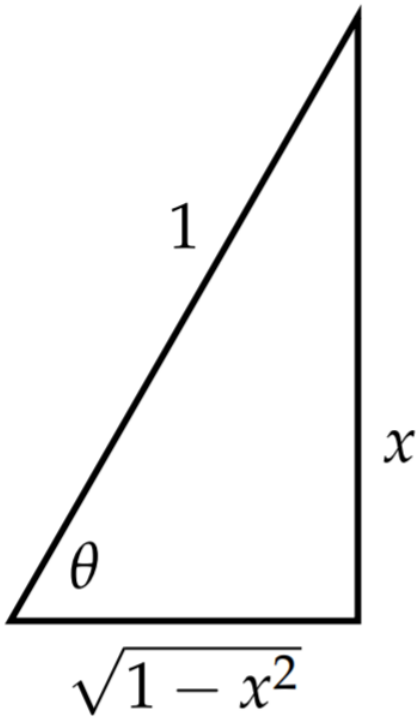


Figure 11.2.1: $\theta = \sin^{-1} x \Rightarrow \tan\theta = \frac{x}{\sqrt{1-x^2}}$.

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11.3: Hyperbolic Functions

So, are there any other functions that are useful in physics? Actually, there are many more. However, you have probably not see many of them to date. We will see by the end of the semester that there are many important functions that arise as solutions of some fairly generic, but important, physics problems. In your calculus classes you have also seen that some relations are represented in parametric form. However, there is at least one other set of elementary functions, which you should already know about. These are the **hyperbolic functions**. Such functions are useful in representing hanging cables, unbounded orbits, and special traveling waves called **solitons**. They also play a role in special and general relativity.

Solitons

Solitons are special solutions to some generic nonlinear wave equations. They typically experience elastic collisions and play special roles in a variety of fields in physics, such as hydrodynamics and optics. A simple soliton solution is of the form

$$u(x,t) = 2\eta^2 \operatorname{sech}^2 \eta \left(x - 4\eta^2 t\right).$$

Note

We will later see the connection between the hyperbolic and trigonometric functions in Chapter 8.

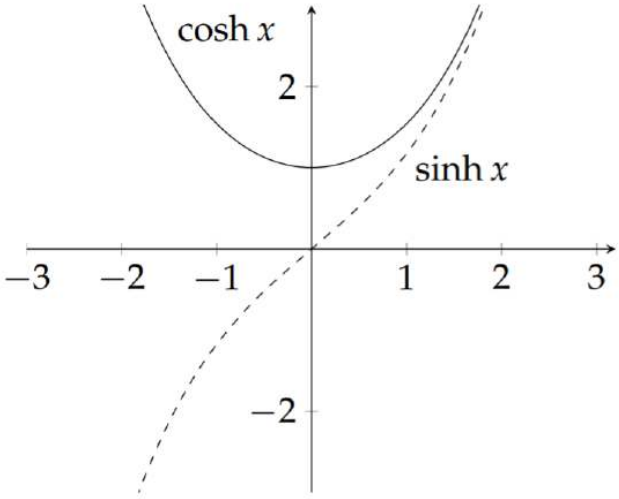


Figure 11.3.1: Plots of $\cosh x$ and $\sinh x$. Note that $\sinh 0 = 0$, $\cosh 0 = 1$, and $\cosh x \geq 1$.

Hyperbolic functions are actually related to the trigonometric functions, as we shall see after a little bit of complex function theory. For now, we just want to recall a few definitions and identities. Just as all of the trigonometric functions can be built from the sine and the cosine, the hyperbolic functions can be defined in terms of the hyperbolic sine and hyperbolic cosine (shown in Figure 11.3.1):

$$\sinh x = \frac{e^x - e^{-x}}{2}, \tag{11.3.1}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}. \tag{11.3.2}$$

There are four other hyperbolic functions. These are defined in terms of the above functions similar to the relations between the trigonometric functions. We have

$$\tanh x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}, \tag{11.3.3}$$

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}, \tag{11.3.4}$$

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}, \tag{11.3.5}$$

$$\coth x = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}. \tag{11.3.6}$$

There are also a whole set of identities, similar to those for the trigonometric functions. For example, the Pythagorean identity for trigonometric functions, $\sin^2\theta + \cos^2\theta = 1$, is replaced by the identity

$$\cosh^2 x - \sinh^2 x = 1.$$

This is easily shown by simply using the definitions of these functions. This identity is also useful for providing a parametric set of equations describing hyperbolae. Letting $x = a\cosh t$ and $y = b\sinh t$, one has

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \cosh^2 t - \sinh^2 t = 1.$$

A list of commonly needed hyperbolic function identities are given by the following:

$$\cosh^2 x - \sinh^2 x = 1, \tag{11.3.7}$$

$$\tanh^2 x + \operatorname{sech}^2 x = 1, \tag{11.3.8}$$

$$\cosh(A \pm B) = \cosh A \cosh B \pm \sinh A \sinh B, \tag{11.3.9}$$

$$\sinh(A \pm B) = \sinh A \cosh B \pm \sinh B \cosh A, \tag{11.3.10}$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x, \tag{11.3.11}$$

$$\sinh 2x = 2\sinh x \cosh x, \tag{11.3.12}$$

$$\cosh^2 x = \frac{1}{2}(1 + \cosh 2x), \tag{11.3.13}$$

$$\sinh^2 x = \frac{1}{2}(\cosh 2x - 1). \tag{11.3.14}$$

Note the similarity with the trigonometric identities. Other identities can be derived from these.

There also exist inverse hyperbolic functions and these can be written in terms of logarithms. As with the inverse trigonometric functions, we begin with the definition

$$y = \sinh^{-1} x \quad \Leftrightarrow \quad x = \sinh y. \tag{11.3.15}$$

The aim is to write y in terms of x without using the inverse function. First, we note that

$$x = \frac{1}{2}\left(e^y - e^{-y}\right). \tag{11.3.16}$$

Next we solve for e^y. This is done by noting that $e^{-y} = \frac{1}{e^y}$ and rewriting the previous equation as

$$0 = \left(e^y\right)^2 - 2xe^y - 1. \tag{11.3.17}$$

This equation is in quadratic form which we can solve using the quadratic formula as

$$e^y = x + \sqrt{1 + x^2}.$$

(There is only one root as we expect the exponential to be positive.)

The final step is to solve for y,

$$y = \ln\left(x + \sqrt{1 + x^2}\right). \tag{11.3.18}$$

Inverse Hyperbolic Functions

The **inverse hyperbolic functions** are given by

$$\sinh^{-1} x = \ln\left(x + \sqrt{1+x^2}\right)$$

$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right)$$

$$\tanh^{-1} x = \frac{1}{2}\ln\frac{1+x}{1-x}$$

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11.4: Derivatives

Now that we know some elementary functions, we seek their derivatives. We will not spend time exploring the appropriate limits in any rigorous way. We are only interested in the results. We provide these in Table 11.4.1. We expect that you know the meaning of the derivative and all of the usual rules, such as the product and quotient rules.

Table 11.4.1: Table of Common Derivatives (a is a constant).

Function	Derivative
a	0
x^n	nx^{n-1}
e^{ax}	ae^{ax}
$\ln ax$	$\frac{1}{x}$
$\sin ax$	$a\cos ax$
$\cos ax$	$-a\sin ax$
$\tan ax$	$a\sec^2 ax$
$\csc ax$	$-a\csc ax\cot ax$
$\sec ax$	$a\sec ax\tan ax$
$\cot ax$	$-a\csc^2 ax$
$\sinh ax$	$a\cosh ax$
$\cosh ax$	$a\sinh ax$
$\tanh ax$	$a\operatorname{sech}^2 ax$
$\operatorname{csch} ax$	$-a\operatorname{csch} ax\coth ax$
$\operatorname{sech} ax$	$-a\operatorname{sech} ax\tanh ax$
$\coth ax$	$-a\operatorname{csch}^2 ax$

Also, you should be familiar with the Chain Rule. Recall that this rule tells us that if we have a composition of functions, such as the elementary functions above, then we can compute the derivative of the composite function. Namely, if $h(x)=f(g(x))$, then

$$\frac{dh}{dx}=\frac{d}{dx}(f(g(x)))=\left.\frac{df}{dg}\right|_{g(x)}\frac{dg}{dx}=f'(g(x))g'(x). \tag{11.4.1}$$

Example 11.4.1

Differentiate $H(x)=5\cos\left(\pi\tanh 2x^2\right)$.

Solution

This is a composition of three functions, $H(x)=f(g(h(x)))$, where $f(x)=5\cos x$, $g(x)=\pi\tanh x$, and $h(x)=2x^2$. Then the derivative becomes

$$\begin{aligned} H'(x) &= 5\left(-\sin\left(\pi\tanh 2x^2\right)\right)\frac{d}{dx}\left(\left(\pi\tanh 2x^2\right)\right) \\ &= -5\pi\sin\left(\pi\tanh 2x^2\right)\operatorname{sech}^2 2x^2\frac{d}{dx}\left(2x^2\right) \\ &= -20\pi x\sin\left(\pi\tanh 2x^2\right)\operatorname{sech}^2 2x^2. \end{aligned} \tag{11.4.2}$$

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11.5: Integrals

Integration is typically a bit harder. Imagine being given the last result in (11.4.2) and having to figure out what was differentiated in order to get the given function. As you may recall from the Fundamental Theorem of Calculus, the integral is the inverse operation to differentiation:

$$\int \frac{df}{dx}dx = f(x) + C. \tag{11.5.1}$$

It is not always easy to evaluate a given integral. In fact some integrals are not even doable! However, you learned in calculus that there are some methods that could yield an answer. While you might be happier using a computer algebra system, such as Maple or WolframAlpha.com, or a fancy calculator, you should know a few basic integrals and know how to use tables for some of the more complicated ones. In fact, it can be exhilarating when you can do a given integral without reference to a computer or a Table of Integrals. However, you should be prepared to do some integrals using what you have been taught in calculus. We will review a few of these methods and some of the standard integrals in this section.

First of all, there are some integrals you are expected to know without doing any work. These integrals appear often and are just an application of the Fundamental Theorem of Calculus to the previous Table 11.4.1. The basic integrals that students should know off the top of their heads are given in Table 11.5.1.

These are not the only integrals you should be able to do. We can expand the list by recalling a few of the techniques that you learned in calculus, the Method of Substitution, Integration by Parts, integration using partial fraction decomposition, and trigonometric integrals, and trigonometric substitution. There are also a few other techniques that you had not seen before. We will look at several examples.

Example 11.5.1

Evaluate $\int \frac{x}{\sqrt{x^2+1}}dx$.

Solution

When confronted with an integral, you should first ask if a simple substitution would reduce the integral to one you know how to do.

The ugly part of this integral is the x^2+1 under the square root. So, we let $u = x^2+1$.

Noting that when $u = f(x)$, we have $du = f'(x)dx$. For our example, $du = 2xdx$.

Looking at the integral, part of the integrand can be written as $xdx = \frac{1}{2}udu$. Then, the integral becomes

$$\int \frac{x}{\sqrt{x^2+1}}dx = \frac{1}{2}\int \frac{du}{\sqrt{u}}.$$

The substitution has converted our integral into an integral over u. Also, this integral is doable! It is one of the integrals we should know. Namely, we can write it as

$$\frac{1}{2}\int \frac{du}{\sqrt{u}} = \frac{1}{2}\int u^{-1/2}du.$$

This is now easily finished after integrating and using the substitution variable to give

$$\int \frac{x}{\sqrt{x^2+1}}dx = \frac{1}{2}\frac{u^{1/2}}{\frac{1}{2}} + C = \sqrt{x^2+1} + C.$$

Note that we have added the required integration constant and that the derivative of the result easily gives the original integrand (after employing the Chain Rule).

Table 11.5.1: Table of Common Integrals.

Function	Indefinite Integral
a	ax

Function	Indefinite Integral
x^n	$\frac{x^{n+1}}{n+1}$
e^{ax}	$\frac{1}{a}e^{ax}$
$\frac{1}{x}$	$\ln x$
$\sin ax$	$-\frac{1}{a}\cos ax$
$\cos ax$	$\frac{1}{a}\sin ax$
$\sec^2 ax$	$\frac{1}{a}\tan ax$
$\sinh ax$	$\frac{1}{a}\cosh ax$
$\cosh ax$	$\frac{1}{a}\sinh ax$
$\operatorname{sech}^2 ax$	$\frac{1}{a}\tanh ax$
$\sec x$	$\ln\lvert\sec x+\tan x\rvert$
$\frac{1}{a+bx}$	$\frac{1}{b}\ln(a+bx)$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}\frac{x}{a}$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\frac{x}{a}$
$\frac{1}{x\sqrt{x^2-a^2}}$	$\frac{1}{a}\sec^{-1}\frac{x}{a}$
$\frac{1}{\sqrt{x^2-a^2}}$	$\cosh^{-1}\frac{x}{a}=\ln\lvert\sqrt{x^2-a^2}+x\rvert$

Often we are faced with definite integrals, in which we integrate between two limits. There are several ways to use these limits. However, students often forget that a change of variables generally means that the limits have to change.

Example 11.5.2

Evaluate $\int_0^2 \frac{x}{\sqrt{x^2+1}}dx$.

Solution

This is the last example but with integration limits added. We proceed as before. We let $u=x^2+1$. As x goes from 0 to 2, u takes values from 1 to 5. So, this substitution gives

$$\int_0^2 \frac{x}{\sqrt{x^2+1}}dx=\frac{1}{2}\int_1^5 \frac{du}{\sqrt{u}}=\sqrt{u}\Big|_1^5=\sqrt{5}-1.$$

When you becomes proficient at integration, you can bypass some of these steps. In the next example we try to demonstrate the thought process involved in using substitution without explicitly using the substitution variable. Example 11.5.3

Example 11.5.3

Evaluate $\int_0^2 \frac{x}{\sqrt{9+4x^2}}dx$

Solution

As with the previous example, one sees that the derivative of $9+4x^2$ is proportional to x, which is in the numerator of the integrand. Thus a substitution would give an integrand of the form $u^{-1/2}$. So, we expect the answer to be proportional to $\sqrt{u}=\sqrt{9+4x^2}$. The starting point is therefore,

$$\int \frac{x}{\sqrt{9+4x^2}}dx = A\sqrt{9+4x^2},$$

where A is a constant to be determined.

We can determine A through differentiation since the derivative of the answer should be the integrand. Thus,

$$\begin{aligned}\frac{d}{dx}A\left(9+4x^2\right)^{\frac{1}{2}} &= A\left(9+4x^2\right)^{-\frac{1}{2}}\left(\frac{1}{2}\right)(8x)\\ &= 4xA\left(9+4x^2\right)^{-\frac{1}{2}}\end{aligned} \tag{11.5.2}$$

Comparing this result with the integrand, we see that the integrand is obtained when $A = \frac{1}{4}$. Therefore,

$$\int \frac{x}{\sqrt{9+4x^2}}dx = \frac{1}{4}\sqrt{9+4x^2}.$$

We now complete the integral,

$$\int_0^2 \frac{x}{\sqrt{9+4x^2}}dx = \frac{1}{4}[5-3] = \frac{1}{2}.$$

Note

The function

$$\text{gd}(x) = \int_0^x \frac{dx}{\cosh x} = 2\tan^{-1}e^x - \frac{\pi}{2}$$

is called the Gudermannian and connects trigonometric and hyperbolic functions. This function was named after Christoph Gudermann ($1798-1852$), but introduced by Iohann Heinrich Lambert ($1728-1777$), who was one of the first to introduce hyperbolic functions.

Example 11.5.4

Evaluate $\int \frac{dx}{\cosh x}$.

Solution

This integral can be performed by first using the definition of $\cosh x$ followed by a simple substitution.

$$\begin{aligned}\int \frac{dx}{\cosh x} &= \int \frac{2}{e^x+e^{-x}}dx\\ &= \int \frac{2e^x}{e^{2x}+1}dx.\end{aligned} \tag{11.5.3}$$

Now, we let $u = e^x$ and $du = e^x dx$. Then,

$$\begin{aligned}\int \frac{dx}{\cosh x} &= \int \frac{2}{1+u^2}du\\ &= 2\tan^{-1}u + C\\ &= 2\tan^{-1}e^x + C.\end{aligned} \tag{11.5.4}$$

11.5.1 Integration by Parts

When the Method of Substitution fails, there are other methods you can try. One of the most used is the Method of Integration by Parts. Recall the Integration by Parts Formula:

$$\int udv = uv - \int vdu \tag{11.5.5}$$

The idea is that you are given the integral on the left and you can relate it to an integral on the right. Hopefully, the new integral is one you can do, or at least it is an easier integral than the one you are trying to evaluate.

However, you are not usually given the functions u and v. You have to determine them. The integral form that you really have is a function of another variable, say x. Another form of the Integration by Parts Formula can be written as

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx. \tag{11.5.6}$$

This form is a bit more complicated in appearance, though it is clearer than the $u-v$ form as to what is happening. The derivative has been moved from one function to the other. Recall that this formula was derived by integrating the product rule for differentiation. (See your calculus text.)

Note

Often in physics one needs to move a derivative between functions inside an integrand. The key - use integration by parts to move the derivative from one function to the other under an integral.

These two formulae can be related by using the differential relations

$$\begin{aligned} u = f(x) \quad &\rightarrow \quad du = f'(x)dx \\ v = g(x) \quad &\rightarrow \quad dv = g'(x)dx. \end{aligned} \tag{11.5.7}$$

This also gives a method for applying the Integration by Parts Formula.

Example 11.5.5

Consider the integral $\int x \sin 2x dx$.

Solution

We choose $u = x$ and $dv = \sin 2x dx$. This gives the correct left side of the Integration by Parts Formula. We next determine v and du :

$$\begin{aligned} du &= \frac{du}{dx}dx = dx, \\ v &= \int dv = \int \sin 2x dx = -\frac{1}{2}\cos 2x. \end{aligned}$$

We note that one usually does not need the integration constant. Inserting these expressions into the Integration by Parts Formula, we have

$$\int x \sin 2x dx = -\frac{1}{2}x\cos 2x + \frac{1}{2}\int \cos 2x dx.$$

We see that the new integral is easier to do than the original integral. Had we picked $u = \sin 2x$ and $dv = xdx$, then the formula still works, but the resulting integral is not easier.

For completeness, we finish the integration. The result is

$$\int x \sin 2x dx = -\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + C.$$

As always, you can check your answer by differentiating the result, a step students often forget to do. Namely,

$$\begin{aligned} \frac{d}{dx}\left(-\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + C\right) &= -\frac{1}{2}\cos 2x + x\sin 2x + \frac{1}{4}(2\cos 2x) \\ &= x\sin 2x. \end{aligned} \tag{11.5.8}$$

So, we do get back the integrand in the original integral.

We can also perform integration by parts on definite integrals. The general formula is written as

$$\int_a^b f(x)g'(x)dx = f(x)g(x)|_a^b - \int_a^b g(x)f'(x)dx. \tag{11.5.9}$$

Example 11.5.6

Consider the integral

$$\int_0^\pi x^2 \cos x dx.$$

Solution

This will require two integrations by parts. First, we let $u = x^2$ and $dv = \cos x$.

$$du = 2xdx. \quad v = \sin x$$

Inserting into the Integration by Parts Formula, we have

$$\begin{aligned} \int_0^\pi x^2 \cos x dx &= x^2 \sin x|_0^\pi - 2\int_0^\pi x \sin x dx \\ &= -2\int_0^\pi x \sin x dx. \end{aligned} \tag{11.5.10}$$

We note that the resulting integral is easier that the given integral, but we still cannot do the integral off the top of our head (unless we look at Example 3!). So, we need to integrate by parts again. (Note: In your calculus class you may recall that there is a tabular method for carrying out multiple applications of the formula. We will show this method in the next example.)

We apply integration by parts by letting $U = x$ and $dV = \sin x dx$. This gives $dU = dx$ and $V = -\cos x$. Therefore, we have

$$\begin{aligned} \int_0^\pi x \sin x dx &= -x \cos x|_0^\pi + \int_0^\pi \cos x dx \\ &= \pi + \sin x|_0^\pi \\ &= \pi. \end{aligned} \tag{11.5.11}$$

The final result is

$$\int_0^\pi x^2 \cos x dx = -2\pi.$$

There are other ways to compute integrals of this type. First of all, there is the Tabular Method to perform integration by parts. A second method is to use differentiation of parameters under the integral. We will demonstrate this using examples.

Example 11.5.7

Compute the integral $\int_0^\pi x^2 \cos x dx$ using the Tabular Method.

Solution

First we identify the two functions under the integral, x^2 and $\cos x$. We then write the two functions and list the derivatives and integrals of each, respectively. This is shown in Table 11.5.2 Note that we stopped when we reached zero in the left column.

Next, one draws diagonal arrows, as indicated, with alternating signs attached, starting with $+$. The indefinite integral is then obtained by summing the products of the functions at the ends of the arrows along with the signs on each arrow:

$$\int x^2 \cos x dx = x^2 \sin x + 2x \cos x - 2 \sin x + C.$$

To find the definite integral, one evaluates the antiderivative at the given limits.

$$\begin{aligned}\int_0^{\pi} x^2 \cos x dx &= \left[x^2 \sin x + 2x \cos x - 2 \sin x\right]_0^{\pi} \\ &= \left(\pi^2 \sin \pi + 2\pi \cos \pi - 2 \sin \pi\right) - 0 \\ &= -2\pi. \end{aligned} \tag{11.5.12}$$

Actually, the Tabular Method works even if a zero does not appear in the left column. One can go as far as possible, and if a zero does not appear, then one needs only integrate, if possible, the product of the functions in the last row, adding the next sign in the alternating sign progression. The next example shows how this works.

Table 11.5.2 Tabular Method

D		I
x^2	$+$ →	$\cos x$
$2x$	$-$ →	$\sin x$
2	$+$ →	$-\cos x$
0		$-\sin x$

Example 11.5.8

Use the Tabular Method to compute $\int e^{2x} \sin 3x dx$.

Solution

As before, we first set up the table as shown in Table 11.5.3

Table 11.5.3 Tabular Method, showing a nonterminating example.

D		I
$\sin 3x$	$+$ →	e^{2x}
$3\cos 3x$	$-$ →	$\frac{1}{2}e^{2x}$
$-9\sin 3x$		$\frac{1}{4}e^{2x}$

Putting together the pieces, noting that the derivatives in the left column will never vanish, we have

$$\int e^{2x} \sin 3x dx = \left(\frac{1}{2}\sin 3x - \frac{3}{4}\cos 3x\right) e^{2x} + \int (-9 \sin 3x) \left(\frac{1}{4} e^{2x}\right) dx.$$

The integral on the right is a multiple of the one on the left, so we can combine them,

$$\frac{13}{4} \int e^{2x} \sin 3x dx = \left(\frac{1}{2}\sin 3x - \frac{3}{4}\cos 3x\right) e^{2x},$$

or

$$\int e^{2x} \sin 3x dx = \left(\frac{2}{13}\sin 3x - \frac{3}{13}\cos 3x\right) e^{2x}.$$

11.5.2 Differentiation Under the Integral

Another method that one can use to evaluate this integral is to differentiate under the integral sign. This is mentioned in the Richard Feynman's memoir *Surely You're Joking, Mr. Feynman!*. In the book Feynman recounts using this "trick" to be able to do integrals that his MIT classmates could not do. This is based on a theorem found in Advanced Calculus texts. Reader's unfamiliar with partial derivatives should be able to grasp their use in the following example.

Theorem 11.5.1

Let the functions $f(x,t)$ and $\frac{\partial f(x,t)}{\partial x}$ be continuous in both t, and x, in the region of the (t,x) plane which includes $a(x) \leq t \leq b(x), x_0 \leq x \leq x_1$, where the functions $a(x)$ and $b(x)$ are continuous and have continuous derivatives for $x_0 \leq x \leq x_1$. Defining

$$F(x) \equiv \int_{a(x)}^{b(x)} f(x,t)dt,$$

then

$$\begin{aligned} \frac{dF(x)}{dx} &= \left(\frac{\partial F}{\partial b}\right)\frac{db}{dx} + \left(\frac{\partial F}{\partial a}\right)\frac{da}{dx} + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)dt \\ &= f(x,b(x))b'(x) - f(x,a(x))a'(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)dt. \end{aligned} \tag{11.5.13}$$

for $x_0 \leq x \leq x_1$. This is a generalized version of the Fundamental Theorem of Calculus.

In the next examples we show how we can use this theorem to bypass integration by parts.

Example 11.5.9

Use differentiation under the integral sign to evaluate $\int xe^x dx$.

Solution

First, consider the integral

$$I(x,a) = \int e^{ax} dx = \frac{e^{ax}}{a}.$$

Then,

$$\frac{\partial I(x,a)}{\partial a} = \int xe^{ax} dx$$

So,

$$\begin{aligned} \int xe^{ax} dx &= \frac{\partial I(x,a)}{\partial a} \\ &= \frac{\partial}{\partial a}\left(\int e^{ax} dx\right) \\ &= \frac{\partial}{\partial a}\left(\frac{e^{ax}}{a}\right) \\ &= \left(\frac{x}{a} - \frac{1}{a^2}\right)e^{ax} \end{aligned} \tag{11.5.14}$$

Evaluating this result at $a = 1$, we have

$$\int xe^x dx = (x-1)e^x.$$

The reader can verify this result by employing the previous methods or by just differentiating the result.

Example 11.5.10

We will do the integral $\int_0^\pi x^2 \cos x dx$ once more.

Solution

First, consider the integral

$$\begin{aligned} I(a) &\equiv \int_0^\pi \cos ax dx \\ &= \frac{\sin ax}{a}\Big|_0^\pi \\ &= \frac{\sin a\pi}{a}. \end{aligned} \tag{11.5.15}$$

Differentiating the integral $I(a)$ with respect to a twice gives

$$\frac{d^2 I(a)}{da^2} = -\int_0^\pi x^2 \cos ax dx. \tag{11.5.16}$$

Evaluation of this result at $a = 1$ leads to the desired result. Namely,

$$\begin{aligned} \int_0^\pi x^2 \cos x dx &= -\frac{d^2 I(a)}{da^2}\Big|_{a=1} \\ &= -\frac{d^2}{da^2}\left(\frac{\sin a\pi}{a}\right)\Big|_{a=1} \\ &= -\frac{d}{da}\left(\frac{a\pi\cos a\pi - \sin a\pi}{a^2}\right)\Big|_{a=1} \\ &= -\left(\frac{a^2\pi^2 \sin a\pi + 2a\pi\cos a\pi - 2\sin a\pi}{a^3}\right)\Big|_{a=1} \\ &= -2\pi. \end{aligned} \tag{11.5.17}$$

11.5.3 Trigonometric Integrals

Other types of integrals that you will see often are trigonometric integrals. In particular, integrals involving powers of sines and cosines. For odd powers, a simple substitution will turn the integrals into simple powers.

Example 11.5.11

For example, consider

$$\int \cos^3 x dx.$$

Solution

This can be rewritten as

$$\int \cos^3 x dx = \int \cos^2 x \cos x dx.$$

Let $u = \sin x$. Then, $du = \cos x dx$. Since $\cos^2 x = 1 - \sin^2 x$, we have

$$\begin{aligned} \int \cos^3 x dx &= \int \cos^2 x \cos x dx \\ &= \int \left(1 - u^2\right) du \\ &= u - \frac{1}{3}u^3 + C \\ &= \sin x - \frac{1}{3}\sin^3 x + C. \end{aligned} \tag{11.5.18}$$

A quick check confirms the answer:

$$\begin{aligned}\frac{d}{dx}\left(\sin x-\frac{1}{3}\sin^3 x+C\right) &= \cos x-\sin^2 x\cos x\\ &= \cos x\left(1-\sin^2 x\right)\\ &= \cos^3 x. \end{aligned} \tag{11.5.19}$$

Even powers of sines and cosines are a little more complicated, but doable. In these cases we need the half angle formulae (11.2.12)-(11.2.13).

Example 11.5.12

As an example, we will compute

$$\int_0^{2\pi}\cos^2 x dx.$$

Solution

Substituting the half angle formula for $\cos^2 x$, we have

$$\begin{aligned}\int_0^{2\pi}\cos^2 x dx &= \frac{1}{2}\int_0^{2\pi}(1+\cos 2x)dx\\ &= \frac{1}{2}\left(x-\frac{1}{2}\sin 2x\right)_0^{2\pi}\\ &= \pi \end{aligned} \tag{11.5.20}$$

Note

Recall that RMS averages refer to the root mean square average. This is computed by first computing the average, or mean, of the square of some quantity. Then one takes the square root. Typical examples are RMS voltage, RMS current, and the average energy in an electromagnetic wave. AC currents oscillate so fast that the measured value is the RMS voltage.

We note that this result appears often in physics. When looking at root mean square averages of sinusoidal waves, one needs the average of the square of sines and cosines. Recall that the average of a function on interval $[a,b]$ is given as

$$f \text{ ave } = \frac{1}{b-a}\int_a^b f(x)dx. \tag{11.5.21}$$

So, the average of $\cos^2 x$ over one period is

$$\frac{1}{2\pi}\int_0^{2\pi}\cos^2 x dx = \frac{1}{2}. \tag{11.5.22}$$

The root mean square is then found by taking the square root, $\frac{1}{\sqrt{2}}$.

11.5.4 Trigonometric Function Substitution

Another class of integrals typically studied in calculus are those involving the forms $\sqrt{1-x^2}$, $\sqrt{1+x^2}$, or $\sqrt{x^2-1}$. These can be simplified through the use of trigonometric substitutions. The idea is to combine the two terms under the radical into one term using trigonometric identities. We will consider some typical examples.

Example 11.5.13

Evaluate $\int\sqrt{1-x^2}\,dx$.

Solution

Since $1-\sin^2\theta=\cos^2\theta$, we perform the sine substitution

$$x=\sin\theta, \quad dx=\cos\theta d\theta.$$

Then,

$$\begin{aligned}\int \sqrt{1-x^2}\,dx &= \int \sqrt{1-\sin^2\theta}\cos\theta d\theta \\ &= \int \cos^2\theta d\theta. \end{aligned} \tag{11.5.23}$$

Using the last example, we have

$$\int \sqrt{1-x^2}\,dx = \frac{1}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right) + C.$$

However, we need to write the answer in terms of x. We do this by first using the double angle formula for $\sin 2\theta$ and $\cos\theta = \sqrt{1-x^2}$ to obtain

$$\int \sqrt{1-x^2}\,dx = \frac{1}{2}\left(\sin^{-1} x - x\sqrt{1-x^2}\right) + C.$$

Note

In any of these computations careful attention has to be paid to simplifying the radical. This is because

$$\sqrt{x^2} = |x|.$$

For example, $\sqrt{(-5)^2} = \sqrt{25} = 5$. For $x = \sin\theta$, one typically specifies the domain $-\pi/2 \le \theta \le \pi/2$. In this domain we have $|\cos\theta| = \cos\theta$.

Similar trigonometric substitutions result for integrands involving $\sqrt{1+x^2}$ and $\sqrt{x^2-1}$. The substitutions are summarized in Table 11.5.4 The simplification of the given form is then obtained using trigonometric identities. This can also be accomplished by referring to the right triangles shown in Figure 11.5.1.

Table 11.5.4: Standard trigonometric substitutions.

Form	Substitution	Differential
$\sqrt{a^2-x^2}$	$x = a\sin\theta$	$dx = a\cos\theta d\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta$	$dx = a\sec^2\theta d\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$	$dx = a\sec\theta\tan\theta d\theta$

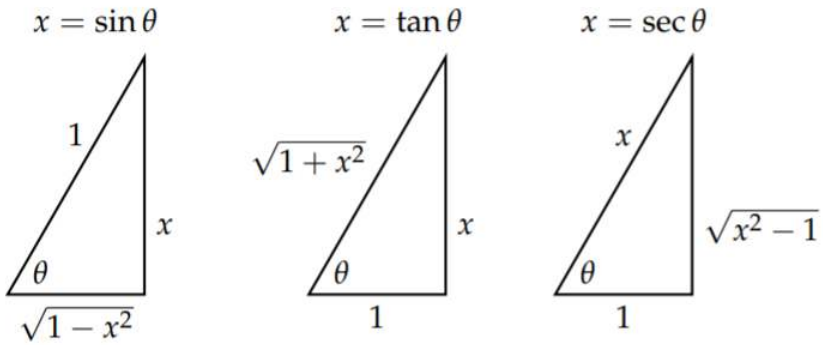


Figure 11.5.1: Geometric relations used in trigonometric substitution.

Example 11.5.14

Evaluate $\int_0^2 \sqrt{x^2+4}\,dx$.

Solution

Let $x = 2\tan\theta$. Then, $dx = 2\sec^2\theta d\theta$ and

$$\sqrt{x^2+4} = \sqrt{4\tan^2\theta+4} = 2\sec\theta.$$

So, the integral becomes

$$\int_0^2 \sqrt{x^2+4}dx = 4\int_0^{\pi/4} \sec^3\theta d\theta$$

One has to recall, or look up,

$$\int \sec^3\theta d\theta = \frac{1}{2}(\tan\theta\sec\theta + \ln|\sec\theta+\tan\theta|) + C.$$

This gives

$$\begin{aligned}\int_0^2 \sqrt{x^2+4}dx &= 2[\tan\theta\sec\theta+\ln|\sec\theta+\tan\theta|]_0^{\pi/4} \\ &= 2(\sqrt{2}+\ln|\sqrt{2}+1| - (0+\ln(1))) \\ &= 2(\sqrt{2}+\ln(\sqrt{2}+1)). \end{aligned} \quad (11.5.24)$$

Example 11.5.15

Evaluate $\int \frac{dx}{\sqrt{x^2-1}}, x \geq 1$.

Solution

In this case one needs the secant substitution. This yields

$$\begin{aligned}\int \frac{dx}{\sqrt{x^2-1}} &= \int \frac{\sec\theta\tan\theta d\theta}{\sqrt{\sec^2\theta-1}} \\ &= \int \frac{\sec\theta\tan\theta d\theta}{\tan\theta} \\ &= \int \sec\theta d\theta \\ &= \ln(\sec\theta+\tan\theta)+C \\ &= \ln\left(x+\sqrt{x^2-1}\right)+C. \end{aligned} \quad (11.5.25)$$

Example 11.5.16

Evaluate $\int \frac{dx}{x\sqrt{x^2-1}}, x \geq 1$

Solution

Again we can use a secant substitution. This yields

$$\begin{aligned}\int \frac{dx}{x\sqrt{x^2-1}} &= \int \frac{\sec\theta\tan\theta d\theta}{\sec\theta\sqrt{\sec^2\theta-1}} \\ &= \int \frac{\sec\theta\tan\theta}{\sec\theta\tan\theta}d\theta \\ &= \int d\theta = \theta + C = \sec^{-1}x + C. \end{aligned} \quad (11.5.26)$$

11.5.5 Hyperbolic Function Substitution

Even though trigonometric substitution plays a role in the calculus program, students often see hyperbolic function substitution used in physics courses. The reason might be because hyperbolic function substitution is sometimes simpler. The idea is the same as for trigonometric substitution. We use an identity to simplify the radical.

Example 11.5.17

Evaluate $\int_0^2 \sqrt{x^2+4}\,dx$ using the substitution $x = 2\sinh u$.

Solution

Since $x = 2\sinh u$, we have $dx = 2\cosh u du$. Also, we can use the identity $\cosh^2 u - \sinh^2 u = 1$ to rewrite

$$\sqrt{x^2+4} = \sqrt{4\sinh^2 u + 4} = 2\cosh u.$$

The integral can be now be evaluated using these substitutions and some hyperbolic function identities,

$$\begin{aligned}\int_0^2 \sqrt{x^2+4}dx &= 4\int_0^{\sinh^{-1}1} \cosh^2 u du \\ &= 2\int_0^{\sinh^{-1}1} (1+\cosh 2u)du \\ &= 2\left[u + \frac{1}{2}\sinh 2u\right]_0^{\sinh^{-1}1} \\ &= 2[u+\sinh u\cosh u]_0^{\sinh^{-1}1} \\ &= 2\left(\sinh^{-1}1 + \sqrt{2}\right). \end{aligned} \tag{11.5.27}$$

In Example 11.5.14we used a trigonometric substitution and found

$$\int_0^2 \sqrt{x^2+4} = 2(\sqrt{2}+\ln(\sqrt{2}+1)).$$

This is the same result since $\sinh^{-1}1 = \ln(1+\sqrt{2})$.

Example 11.5.18

Evaluate $\int \frac{dx}{\sqrt{x^2-1}}$ for $x \geq 1$ using hyperbolic function substitution.

Solution

This integral was evaluated in Example 11.5.16using the trigonometric substitution $x = \sec\theta$ and the resulting integral of $\sec\theta$ had to be recalled. Here we will use the substitution

$$x = \cosh u, \quad dx = \sinh u du, \quad \sqrt{x^2-1} = \sqrt{\cosh^2 u - 1} = \sinh u.$$

Then,

$$\begin{aligned}\int \frac{dx}{\sqrt{x^2-1}} &= \int \frac{\sinh u du}{\sinh u} \\ &= \int du = u + C \\ &= \cosh^{-1} x + C \\ &= \frac{1}{2}\ln\left(x+\sqrt{x^2-1}\right) + C, \quad x \geq 1 \end{aligned} \tag{11.5.28}$$

This is the same result as we had obtained previously, but this derivation was a little cleaner.

Also, we can extend this result to values $x \leq -1$ by letting $x = -\cosh u$. This gives

$$\int \frac{dx}{\sqrt{x^2-1}} = \frac{1}{2}\ln\left(x+\sqrt{x^2-1}\right)+C, \quad x \le -1.$$

Combining these results, we have shown

$$\int \frac{dx}{\sqrt{x^2-1}} = \frac{1}{2}\ln\left(|x|+\sqrt{x^2-1}\right)+C, \quad x^2 \ge 1.$$

We have seen in the last example that the use of hyperbolic function substitution allows us to bypass integrating the secant function in Example 11.5.16 when using trigonometric substitutions. In fact, we can use hyperbolic substitutions to evaluate integrals of powers of secants. Comparing Examples 11.5.16 and 11.5.18, we consider the transformation $\sec\theta = \cosh u$. The relation between differentials is found by differentiation, giving

$$\sec\theta\tan\theta d\theta = \sinh u du$$

Since

$$\tanh^2\theta = \sec^2\theta - 1,$$

we have $\tan\theta = \sinh u$, therefore

$$d\theta = \frac{du}{\cosh u}.$$

In the next example we show how useful this transformation is.

Example 11.5.19

Evaluate $\int \sec\theta d\theta$ using hyperbolic function substitution.

Solution

From the discussion in the last paragraph, we have

$$\begin{aligned}\int \sec\theta d\theta &= \int du \\ &= u + C \\ &= \cosh^{-1}(\sec\theta) + C \end{aligned} \tag{11.5.29}$$

We can express this result in the usual form by using the logarithmic form of the inverse hyperbolic cosine,

$$\cosh^{-1} x = \ln\left(x+\sqrt{x^2-1}\right).$$

The result is

$$\int \sec\theta d\theta = \ln(\sec\theta + \tan\theta).$$

This example was fairly simple using the transformation $\sec\theta = \cosh u$. Another common integral that arises often is integrations of $\sec^3\theta$. In a typical calculus class this integral is evaluated using integration by parts. However. that leads to a tricky manipulation that is a bit scary the first time it is encountered (and probably upon several more encounters.) In the next example, we will show how hyperbolic function substitution is simpler.

Example 11.5.20

Evaluate $\int \sec^3\theta d\theta$ using hyperbolic function substitution.

Solution

First, we consider the transformation $\sec\theta = \cosh u$ with $d\theta = \frac{du}{\cosh u}$. Then,

$$\int \sec^3\theta d\theta = \int \frac{du}{\cosh u}.$$

This integral was done in Example 11.5.4, leading to

$$\int \sec^3\theta d\theta = 2\tan^{-1} e^u + C.$$

While correct, this is not the form usually encountered. Instead, we make the slightly different transformation $\tan\theta = \sinh u$. Since $\sec^2\theta = 1 + \tan^2\theta$, we find $\sec\theta = \cosh u$. As before, we find

$$d\theta = \frac{du}{\cosh u}.$$

Using this transformation and several identities, the integral becomes

$$\begin{aligned}
\int \sec^3\theta d\theta &= \int \cosh^2 u du \\
&= \frac{1}{2}\int (1 + \cosh 2u) du \\
&= \frac{1}{2}\left(u + \frac{1}{2}\sinh 2u\right) \\
&= \frac{1}{2}(u + \sinh u \cosh u) \\
&= \frac{1}{2}\left(\cosh^{-1}(\sec\theta) + \tan\theta\sec\theta\right) \\
&= \frac{1}{2}(\sec\theta\tan\theta + \ln(\sec\theta + \tan\theta)). \qquad (11.5.30)
\end{aligned}$$

There are many other integration methods, some of which we will visit in other parts of the book, such as partial fraction decomposition and numerical integration. Another topic which we will revisit is power series.

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11.6: Geometric Series

Infinite series occur often in mathematics and physics. Two series which occur often are the geometric series and the binomial series. we will discuss these next.

Note

Geometric series are fairly common and will be used throughout the book. You should learn to recognize them and work with them.

A geometric series is of the form

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \ldots + ar^n + \ldots. \tag{11.6.1}$$

Here a is the first term and r is called the ratio. It is called the ratio because the ratio of two consecutive terms in the sum is r.

Example 11.6.1

For example,

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$$

is an example of a geometric series.

Solution

We can write this using summation notation,

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \sum_{n=0}^{\infty} 1\left(\frac{1}{2}\right)^n.$$

Thus, $a = 1$ is the first term and $r = \frac{1}{2}$ is the common ratio of successive terms. Next, we seek the sum of this infinite series, if it exists.

The sum of a geometric series, when it exists, can easily be determined. We consider the nth partial sum:

$$s_n = a + ar + \ldots + ar^{n-2} + ar^{n-1}. \tag{11.6.2}$$

Now, multiply this equation by r.

$$rs_n = ar + ar^2 + \ldots + ar^{n-1} + ar^n. \tag{11.6.3}$$

Subtracting these two equations, while noting the many cancelations, we have

$$\begin{aligned}(1-r)s_n = &\left(a + ar + \ldots + ar^{n-2} + ar^{n-1}\right) \\ &- \left(ar + ar^2 + \ldots + ar^{n-1} + ar^n\right) \\ =&a - ar^n \\ =&a\left(1 - r^n\right)\end{aligned} \tag{11.6.4}$$

Thus, the nth partial sums can be written in the compact form

$$s_n = \frac{a\left(1 - r^n\right)}{1 - r}. \tag{11.6.5}$$

The sum, if it exists, is given by $S = \lim_{n\to\infty} s_n$. Letting n get large in the partial sum (11.6.5), we need only evaluate $\lim_{n\to\infty} r^n$. From the special limits in the Appendix we know that this limit is zero for $|r| < 1$. Thus, we have

Geometric Series

The sum of the geometric series exists for $|r| < 1$ and is given by

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r'} \quad |r| < 1. \tag{11.6.6}$$

The reader should verify that the geometric series diverges for all other values of r. Namely, consider what happens for the separate cases $|r| > 1$, $r = 1$ and $r = -1$.

Next, we present a few typical examples of geometric series.

Example 11.6.2

$\sum_{n=0}^{\infty} \frac{1}{2^n}$

Solution

In this case we have that $a = 1$ and $r = \frac{1}{2}$. Therefore, this infinite series converges and the sum is

$$S = \frac{1}{1-\frac{1}{2}} = 2.$$

Example 11.6.3

$\sum_{k=2}^{\infty} \frac{4}{3^k}$

Solution

In this example we first note that the first term occurs for $k = 2$. It sometimes helps to write out the terms of the series,

$$\sum_{k=2}^{\infty} \frac{4}{3^k} = \frac{4}{3^2} + \frac{4}{3^3} + \frac{4}{3^4} + \frac{4}{3^5} + \ldots$$

Looking at the series, we see that $a = \frac{4}{9}$ and $r = \frac{1}{3}$. Since $|r| < 1$, the geometric series converges. So, the sum of the series is given by

$$S = \frac{\frac{4}{9}}{1-\frac{1}{3}} = \frac{2}{3}.$$

Example 11.6.4

$\sum_{n=1}^{\infty} \left(\frac{3}{2^n} - \frac{2}{5^n}\right)$

Solution

Finally, in this case we do not have a geometric series, but we do have the difference of two geometric series. Of course, we need to be careful whenever rearranging lutely convergent. (See the Appendix.) infinite series. In this case it is allowed[1]. Thus, we have

$$\sum_{n=1}^{\infty} \left(\frac{3}{2^n} - \frac{2}{5^n}\right) = \sum_{n=1}^{\infty} \frac{3}{2^n} - \sum_{n=1}^{\infty} \frac{2}{5^n}.$$

Now we can add both geometric series to obtain

$$\sum_{n=1}^{\infty} \left(\frac{3}{2^n} - \frac{2}{5^n}\right) = \frac{\frac{3}{2}}{1-\frac{1}{2}} - \frac{\frac{2}{5}}{1-\frac{1}{5}} = 3 - \frac{1}{2} = \frac{5}{2}.$$

Note

A rearrangement of terms in an infinite series is allowed when the series is absolutely convergent. (See the Appendix.)

Geometric series are important because they are easily recognized and summed. Other series which can be summed include special cases of Taylor series and telescoping series. Next, we show an example of a telescoping series.

Example 11.6.5

$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

Solution

The first few terms of this series are

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots$$

It does not appear that we can sum this infinite series. However, if we used the partial fraction expansion

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1},$$

then we find the kth partial sum can be written as

$$\begin{aligned} s_k &= \sum_{n=1}^{k} \frac{1}{n(n+1)} \\ &= \sum_{n=1}^{k} \left(\frac{1}{n} - \frac{1}{n+1} \right) \\ &= \left(\frac{1}{1} - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \cdots + \left(\frac{1}{k} - \frac{1}{k+1} \right). \end{aligned} \tag{11.6.7}$$

We see that there are many cancelations of neighboring terms, leading to the series collapsing (like a retractable telescope) to something manageable:

$$s_k = 1 - \frac{1}{k+1}.$$

Taking the limit as $k \to \infty$, we find $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.

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11.7: Power Series

Another example of an infinite series that the student has encountered in previous courses is the power series. Examples of such series are provided by Taylor and Maclaurin series.

> **Note**
>
> Actually, what are now known as Taylor and Maclaurin series were known long before they were named. James Gregory (1638-1675) has been recognized for discovering Taylor series, which were later named after Brook Taylor (1685-1731). Similarly, Colin Maclaurin (1698-1746) did not actually discover Maclaurin series, but the name was adopted because of his particular use of series.

A power series expansion about $x = a$ with coefficient sequence c_n is given by $\sum_{n=0}^{\infty} c_n (x-a)^n$. For now we will consider all constants to be real numbers with x in some subset of the set of real numbers.

Consider the following expansion about $x = 0$:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots \tag{11.7.1}$$

We would like to make sense of such expansions. For what values of x will this infinite series converge? Until now we did not pay much attention to which infinite series might converge. However, this particular series is already familiar to us. It is a geometric series. Note that each term is gotten from the previous one through multiplication by $r = x$. The first term is $a = 1$. So, from Equation (11.6.6), we have that the sum of the series is given by

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad |x| < 1.$$

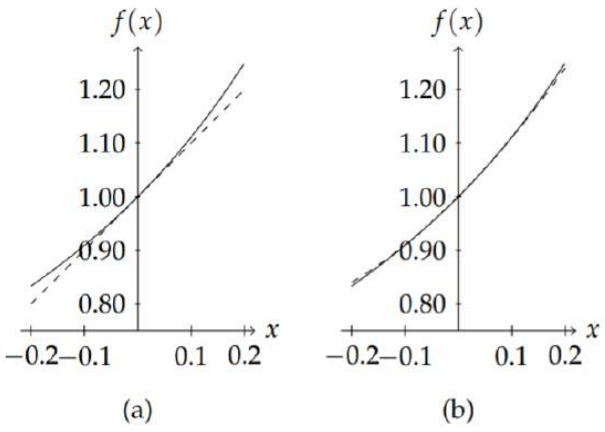


Figure 11.7.1: (a) Comparison of $\frac{1}{1-x}$ (solid) to $1+x$ (dashed) for $x \in [-0.2, 0.2]$. (b) Comparison of $\frac{1}{1-x}$ (solid) to $1+x+x^2$ (dashed) for $x \in [-0.2, 0.2]$.

In this case we see that the sum, when it exists, is a simple function. In fact, when x is small, we can use this infinite series to provide approximations to the function $(1-x)^{-1}$. If x is small enough, we can write

$$(1-x)^{-1} \approx 1 + x.$$

In Figure 11.7.1(a) we see that for small values of x these functions do agree.

Of course, if we want better agreement, we select more terms. In Figure 11.7.1(b) we see what happens when we do so. The agreement is much better. But extending the interval, we see in Figure 11.7.2 that keeping only quadratic terms may not be good enough. Keeping the cubic terms gives better agreement over the interval.

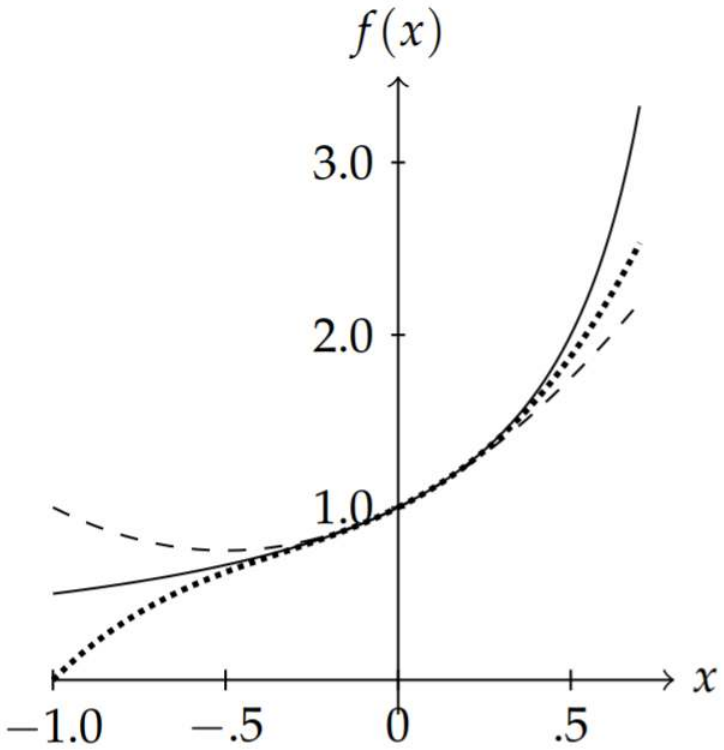


Figure 11.7.2: Comparison of $\frac{1}{1-x}$ (solid) to $1+x+x^2$ (dashed) and $1+x+x^2+x^3$ (dotted) for $x \in [-1.0, 0.7]$.

Finally, in Figure 11.7.3 we show the sum of the first 21 terms over the entire interval $[-1, 1]$. Note that there are problems with approximations near the endpoints of the interval, $x = \pm 1$.

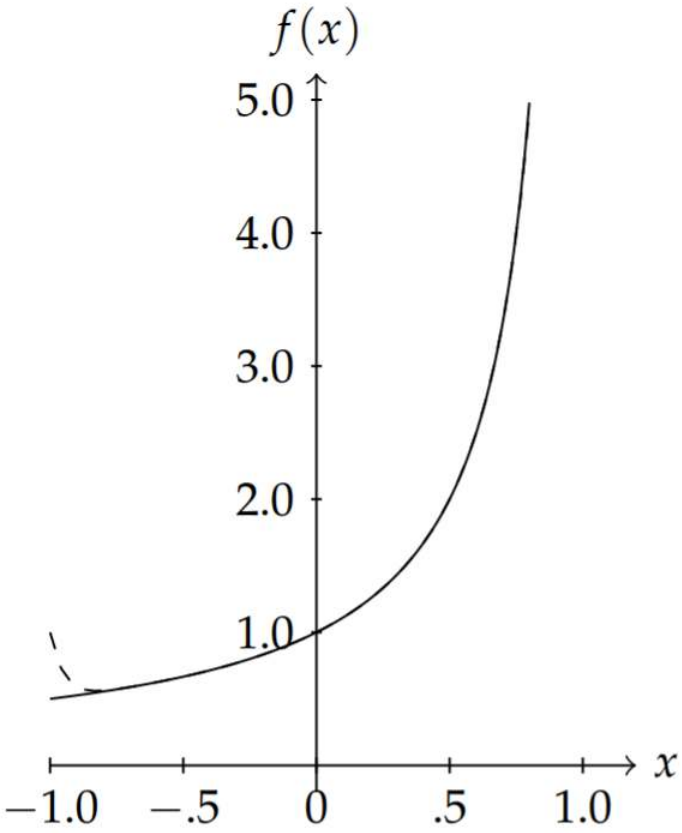


Figure 11.7.3: Comparison of $\frac{1}{1-x}$ (solid) to $\sum_{n=0}^{20} x^n$ for $x \in [-1, 1]$.

Such polynomial approximations are called Taylor polynomials. Thus, $T_3(x) = 1 + x + x^2 + x^3$ is the third order Taylor polynomial approximation of $f(x) = \frac{1}{1-x}$.

With this example we have seen how useful a series representation might be for a given function. However, the series representation was a simple geometric series, which we already knew how to sum. Is there a way to begin with a function and then find its series representation? Once we have such a representation, will the series converge to the function with which we started? For what values of x will it converge? These questions can be answered by recalling the definitions of Taylor and Maclaurin series.

A Taylor series expansion of $f(x)$ about $x = a$ is the series

$$f(x) \sim \sum_{n=0}^{\infty} c_n (x-a)^n, \tag{11.7.2}$$

where

$$c_n = \frac{f^{(n)}(a)}{n!}. \tag{11.7.3}$$

Note that we use $\sim$ to indicate that we have yet to determine when the series may converge to the given function. A special class of series are those Taylor series for which the expansion is about $x = 0$. These are called Maclaurin series.

A Maclaurin series expansion of $f(x)$ is a Taylor series expansion of $f(x)$ about $x = 0$, or

$$f(x) \sim \sum_{n=0}^{\infty} c_n x^n, \tag{11.7.4}$$

where

$$c_n = \frac{f^{(n)}(0)}{n!}. \tag{11.7.5}$$

Example 11.7.1

Expand $f(x) = e^x$ about $x = 0$.

Solution

We begin by creating a table. In order to compute the expansion coefficients, c_n, we will need to perform repeated differentiations of $f(x)$. So, we provide a table for these derivatives. Then, we only need to evaluate the second column at $x = 0$ and divide by $n!$.

Table 11.7.1

n	$f^{(n)}(x)$	$f^{(n)}(0)$	c_n
0	e^x	$e^0 = 1$	$\frac{1}{0!} = 1$
1	e^x	$e^0 = 1$	$\frac{1}{1!} = 1$
2	e^x	$e^0 = 1$	$\frac{1}{2!}$
3	e^x	$e^0 = 1$	$\frac{1}{3!}$

Next, we look at the last column and try to determine a pattern so that we can write down the general term of the series. If there is only a need to get a polynomial approximation, then the first few terms may be sufficient. In this case, the pattern is obvious: $c_n = \frac{1}{n!}$. So,

$$e^x \sim \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Example 11.7.2

Expand $f(x) = e^x$ about $x = 1$.

Solution

Here we seek an expansion of the form $e^x \sim \sum_{n=0}^{\infty} c_n (x-1)^n$. We could create a table like the last example. In fact, the last column would have values of the form $\frac{e}{v1}$. (You should confirm this.) However, we will make use of the Maclaurin series expansion for e^x and get the result quicker. Note that $e^x = e^{x-1+1} = ee^{x-1}$. Now, apply the known expansion for e^x :

$$e^x \sim e\left(1 + (x-1) + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3!} + \ldots\right) = \sum_{n=0}^{\infty} \frac{e(x-1)^n}{n!}$$

Example 11.7.3

Expand $f(x) = \frac{1}{1-x}$ about $x = 0$.

Solution

This is the example with which we started our discussion. We can set up a table in order to find the Maclaurin series coefficients. We see from the last column of the table that we get back the geometric series (11.7.1).

Table 11.7.2

n	$f^{(n)}(x)$	$f^{(n)}(0)$	c_n
0	$\frac{1}{1-x}$	1	$\frac{1}{0!} = 1$
1	$\frac{1}{(1-x)^2}$	1	$\frac{1}{1!} = 1$
2	$\frac{2(1)}{(1-x)^3}$	$2(1)$	$\frac{2!}{2!} = 1$
3	$\frac{3(2)(1)}{(1-x)^4}$	$3(2)(1)$	$\frac{3!}{3!} = 1$

So, we have found

$$\frac{1}{1-x} \sim \sum_{n=0}^{\infty} x^n.$$

We can replace $\sim$ by equality if we can determine the range of x-values for which the resulting infinite series converges. We will investigate such convergence shortly.

Series expansions for many elementary functions arise in a variety of applications. Some common expansions are provided in Table 11.7.3

We still need to determine the values of x for which a given power series converges. The first five of the above expansions converge for all reals, but the others only converge for $|x| < 1$.

We consider the convergence of $\sum_{n=0}^{\infty} c_n (x-a)^n$. For $x = a$ the series obviously converges. Will it converge for other points? One can prove

Theorem 11.7.1

If $\sum_{n=0}^{\infty} c_n (b-a)^n$ converges for $b \neq a$, then $\sum_{n=0}^{\infty} c_n (x-a)^n$ converges absolutely for all x satisfying $|x-a| < |b-a|$.

This leads to three possibilities

1. $\sum_{n=0}^{\infty} c_n (x-a)^n$ may only converge at $x = a$.
2. $\sum_{n=0}^{\infty} c_n (x-a)^n$ may converge for all real numbers.

3. $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges for $|x-a|<R$ and diverges for $|x-a|>R$.

The number R is called the radius of convergence of the power series and $(a-R, a+R)$ is called the interval of convergence. Convergence at the endpoints of this interval has to be tested for each power series.

Table 11.7.3: Common Mclaurin Series Expansions

Series Expansions You Should Know		
e^x	$=1+x+\frac{x^2}{2}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots$	$=\sum_{n=0}^{\infty}\frac{x^n}{n!}$
$\cos x$	$=1-\frac{x^2}{2}+\frac{x^4}{4!}-\ldots$	$=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$
$\sin x$	$=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots$	$=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$
$\cosh x$	$=1+\frac{x^2}{2}+\frac{x^4}{4!}+\ldots$	$=\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$
$\sinh x$	$=x+\frac{x^3}{3!}+\frac{x^5}{5!}+\ldots$	$=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}$
$\frac{1}{1-x}$	$=1+x+x^2+x^3+\ldots$	$=\sum_{n=0}^{\infty}x^n$
$\frac{1}{1+x}$	$=1-x+x^2-x^3+\ldots$	$=\sum_{n=0}^{\infty}(-x)^n$
$\tan^{-1}x$	$=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots$	$=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$
$\ln(1+x)$	$=x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots$	$=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$

In order to determine the interval of convergence, one needs only note that when a power series converges, it does so absolutely. So, we need only test the convergence of $\sum_{n=0}^{\infty}|c_n(x-a)^n|=\sum_{n=0}^{\infty}|c_n|\,|x-a|^n$. This is easily done using either the ratio test or the nth root test. We first identify the nonnegative terms $a_n=|c_n|\,|x-a|^n$, using the notation from Section ??. Then, we apply one of the convergence tests.

For example, the nth Root Test gives the convergence condition for $a_n=|c_n|\,|x-a|^n$,

$$\rho=\lim_{n\to\infty}\sqrt[n]{a_n}=\lim_{n\to\infty}\sqrt[n]{|c_n|\,|x-a|}<1.$$

Since $|x-a|$ is independent of n, we can factor it out of the limit and divide the value of the limit to obtain

$$|x-a|<\left(\lim_{n\to\infty}\sqrt[n]{|c_n|}\right)^{-1}\equiv R.$$

Thus, we have found the radius of convergence, R.

Similarly, we can apply the Ratio Test.

$$\rho=\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{|c_{n+1}|}{|c_n|}|x-a|<1.$$

Again, we rewrite this result to determine the radius of convergence:

$$|x-a|<\left(\lim_{n\to\infty}\frac{|c_{n+1}|}{|c_n|}\right)^{-1}\equiv R.$$

Example 11.7.4

Find the radius of convergence of the series $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$.

Solution

Since there is a factorial, we will use the Ratio Test.

$$\rho=\lim_{n\to\infty}\frac{|n!|}{|(n+1)!|}|x|=\lim_{n\to\infty}\frac{1}{n+1}|x|=0.$$

Since $\rho = 0$, it is independent of $|x|$ and thus the series converges for all x. We also can say that the radius of convergence is infinite.

Example 11.7.5

Find the radius of convergence of the series $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$.

Solution

In this example we will use the nth Root Test.

$$\rho = \lim_{n\to\infty} \sqrt[n]{1}|x| = |x| < 1.$$

Thus, we find that we have absolute convergence for $|x| < 1$. Setting $x = 1$ or $x = -1$, we find that the resulting series do not converge. S_0, the endpoints are not included in the complete interval of convergence.

In this example we could have also used the Ratio Test. Thus,

$$\rho = \lim_{n\to\infty} \frac{1}{1}|x| = |x| < 1.$$

We have obtained the same result as when we used the nth Root Test.

Example 11.7.6

Find the radius of convergence of the series $\sum_{n=1}^{\infty} \frac{3^n(x-2)^n}{n}$.

Solution

In this example, we have an expansion about $x = 2$. Using the nth Root Test we find that

$$\rho = \lim_{n\to\infty} \sqrt[n]{\frac{3^n}{n}}|x-2| = 3|x-2| < 1.$$

Solving for $|x-2|$ in this inequality, we find $|x-2| < \frac{1}{3}$. Thus, the radius of convergence is $R = \frac{1}{3}$ and the interval of convergence is $\left(2-\frac{1}{3}, 2+\frac{1}{3}\right) = \left(\frac{5}{3}, \frac{7}{3}\right)$.

As for the endpoints, we first test the point $x = \frac{7}{3}$. The resulting series is $\sum_{n=1}^{\infty} \frac{3^n\left(\frac{1}{3}\right)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$. This is the harmonic series, and thus it does not converge. Inserting $x = \frac{5}{3}$, we get the alternating harmonic series. This series does converge. So, we have convergence on $\left[\frac{5}{3}, \frac{7}{3}\right)$. However, it is only conditionally convergent at the left endpoint, $x = \frac{5}{3}$.

Example 11.7.7

Find an expansion of $f(x) = \frac{1}{x+2}$ about $x = 1$.

Solution

Instead of explicitly computing the Taylor series expansion for this function, we can make use of an already known function. We first write $f(x)$ as a function of $x-1$, since we are expanding about $x = 1$; i.e., we are seeking a series whose terms are powers of $x-1$.

This expansion is easily done by noting that $\frac{1}{x+2} = \frac{1}{(x-1)+3}$. Factoring out a 3, we can rewrite this expression as a sum of a geometric series. Namely, we use the expansion for

$$\begin{aligned} g(z) &= \frac{1}{1+z} \\ &= 1 - z + z^2 - z^3 + \ldots \end{aligned} \tag{11.7.6}$$

and then we rewrite $f(x)$ as

$$\begin{aligned} f(x) &= \frac{1}{x+2} \\ &= \frac{1}{(x-1)+3} \\ &= \frac{1}{3\left[1+\frac{1}{3}(x-1)\right]} \\ &= \frac{1}{3}\frac{1}{1+\frac{1}{3}(x-1)}. \end{aligned} \tag{11.7.7}$$

Note that $f(x) = \frac{1}{3}g\left(\frac{1}{3}(x-1)\right)$ for $g(z) = \frac{1}{1+z}$. So, the expansion becomes

$$f(x) = \frac{1}{3}\left[1 - \frac{1}{3}(x-1) + \left(\frac{1}{3}(x-1)\right)^2 - \left(\frac{1}{3}(x-1)\right)^3 + \ldots\right].$$

This can further be simplified as

$$f(x) = \frac{1}{3} - \frac{1}{9}(x-1) + \frac{1}{27}(x-1)^2 - \ldots$$

Convergence is easily established. The expansion for $g(z)$ converges for $|z| < 1$. So, the expansion for $f(x)$ converges for $\left|-\frac{1}{3}(x-1)\right| < 1$. This implies that $|x-1| < 3$. Putting this inequality in interval notation, we have that the power series converges absolutely for $x \in (-2, 4)$. Inserting the endpoints, one can show that the series diverges for both $x = -2$ and $x = 4$. You should verify this!

Example 11.7.8

Prove Euler's Formula: $e^{i\theta} = \cos\theta + i\sin\theta$.

Solution

As a final application, we can derive Euler's Formula,

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

where $i = \sqrt{-1}$. We naively use the expansion for e^x with $x = i\theta$. This leads us to

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \ldots.$$

Next we note that each term has a power of i. The sequence of powers of i is given as $\{1, i, -1, -i, 1, i, -1, -i, 1, i, -1, -i, \ldots\}$See the pattern? We conclude that

$$i^n = i^r, \text{ where } r = \text{ remainder after dividing } n \text{ by } 4.$$

This gives

$$e^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots\right).$$

We recognize the expansions in the parentheses as those for the cosine and sine functions. Thus, we end with Euler's Formula.

Note

Euler's Formula, $e^{i\theta} = \cos\theta + i\sin\theta$, is an important formula and is used throughout the text.

We further derive relations from this result, which will be important for our next studies. From Euler's formula we have that for integer n:

$$e^{in\theta} = \cos(n\theta) + i\sin(n\theta).$$

We also have

$$e^{in\theta} = \left(e^{i\theta}\right)^n = (\cos\theta + i\sin\theta)^n.$$

Equating these two expressions, we are led to de Moivre's Formula, named after Abraham de Moivre ($1667 - 1754$),

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta). \tag{11.7.8}$$

This formula is useful for deriving identities relating powers of sines or cosines to simple functions. For example, if we take $n = 2$ in Equation (11.7.8), we find

$$\cos 2\theta + i\sin 2\theta = (\cos\theta + i\sin\theta)^2 = \cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta.$$

Looking at the real and imaginary parts of this result leads to the well known double angle identities

$$\cos 2\theta = \cos^2\theta - \sin^2\theta, \quad \sin 2\theta = 2\sin\theta\cos\theta.$$

Note

Here we see elegant proofs of well known trigonometric identities.

$$\begin{aligned} \cos 2\theta &= \cos^2\theta - \sin^2\theta \\ \sin 2\theta &= 2\sin\theta\cos\theta \\ \cos^2\theta &= \frac{1}{2}(1 + \cos 2\theta) \\ \sin^2\theta &= \frac{1}{2}(1 - \cos 2\theta) \end{aligned} \tag{11.7.9}$$

Replacing $\cos^2\theta = 1 - \sin^2\theta$ or $\sin^2\theta = 1 - \cos^2\theta$ leads to the half angle formulae:

$$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta), \quad \sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$$

Note

Trigonometric functions can be written in terms of complex exponentials:

$$\cos\theta = \frac{e^{-\theta} + e^{-i\theta}}{2},$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

We can also use Euler's Formula to write sines and cosines in terms of complex exponentials. We first note that due to the fact that the cosine is an even function and the sine is an odd function, we have

$$e^{-i\theta} = \cos\theta - i\sin\theta.$$

Combining this with Euler's Formula, wehave that

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

Note

Hyperbolic functions and trigonometric functions are intimately related.

$$\begin{aligned} \cos(ix) &= \cosh x \\ \sin(ix) &= -i\sinh x \end{aligned}$$

We finally note that there is a simple relationship between hyperbolic functions and trigonometric functions. Recall that

$$\cosh x = \frac{e^x + e^{-x}}{2}.$$

If we let $x = i\theta$, then we have that $\cosh(i\theta) = \cos\theta$ and $\cos(ix) = \cosh x$. Similarly, we can show that $\sinh(i\theta) = i\sin\theta$ and $\sin(ix) = -i\sinh x$.

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11.8: The Binomial Expansion

Another series expansion which occurs often in examples and applications is the **binomial expansion**. This is simply the expansion of the expression $(a+b)^p$ in powers of a and b. We will investigate this expansion first for nonnegative integer powers p and then derive the expansion for other values of p. While the binomial expansion can be obtained using Taylor series, we will provide a more intuitive derivation to show that

$$(a+b)^n = \sum_{r=0}^{n} C_r^n a^{n-r} b^r, \tag{11.8.1}$$

where the C_r^n are called the **binomial coefficients**.

Pascal's Triangle

The binomial expansion is a special series expansion used to approximate expressions of the form $(a+b)^p$ for $b \ll a$, or $(1+x)^p$ for $|x| \ll 1$.

Lets list some of the common expansions for nonnegative integer powers.

$$\begin{aligned}
(a+b)^0 &= 1 \\
(a+b)^1 &= a+b \\
(a+b)^2 &= a^2+2ab+b^2 \\
(a+b)^3 &= a^3+3a^2b+3ab^2+b^3 \\
(a+b)^4 &= a^4+4a^3b+6a^2b^2+4ab^3+b^4 \\
&\dots
\end{aligned} \tag{11.8.2}$$

We now look at the patterns of the terms in the expansions. First, we note that each term consists of a product of a power of a and a power of b. The powers of a are decreasing from n to 0 in the expansion of $(a+b)^n$. Similarly, the powers of b increase from 0 to n. The sums of the exponents in each term is n. So, we can write the $(k+1)$ st term in the expansion as $a^{n-k}b^k$. For example, in the expansion of $(a+b)^{51}$ the 6th term is $a^{51-5}b^5 = a^{46}b^5$. However, we do not yet know the numerical coefficients in the expansion.

Let's list the coefficients for the above expansions.

$$\begin{aligned}
n=0: &\qquad\qquad 1 \\
n=1: &\qquad\quad 1 \quad\; 1 \\
n=2: &\qquad\; 1 \quad\; 2 \quad\; 1 \\
n=3: &\qquad 1 \quad\; 3 \quad\; 3 \quad\; 1 \\
n=4: &\quad\; 1 \quad\; 4 \quad\; 6 \quad\; 4 \quad\; 1
\end{aligned} \tag{11.8.3}$$

This pattern is the famous **Pascal's triangle**.[1] There are many interesting features of this triangle. But we will first ask how each row can be generated.

Blaise Pascal

Pascal's triangle is named after Blaise Pascal (1623-1662). While such configurations of numbers were known earlier in history, Pascal published them and applied them to probability theory.

Pascal's triangle has many unusual properties and a variety of uses:

- Horizontal rows add to powers of 2.
- The horizontal rows are powers of 11 $(1, 11, 121, 1331$ etc.).
- Adding any two successive numbers in the diagonal $1-3-6-10-15-21-28\dots$ results in a perfect square
- When the first number to the right of the I in any row is a prime number, all numbers in that row are divisible by that prime number. The reader can readily check this for the $n=5$ and $n=7$ rows.
- Sums along certain diagonals leads to the Fibonacci sequence. These diagonals are parallel to the line connecting the first 1 for $n=3$ row and the 2 in the $n=2$ row.

We see that each row begins and ends with a one. The second term and next to last term have a coefficient of n. Next we note that consecutive pairs in each row can be added to obtain entries in the next row. For example, we have for rows $n=2$ and $n=3$ that $1+2=3$ and $2+1=3$:

$$\begin{array}{lccccccc} n=2: & & 1 & & 2 & & 1 & \\ & & \searrow & \swarrow & & \searrow & \swarrow & \\ n=3: & 1 & & 3 & & 3 & & 1 \end{array} \tag{11.8.4}$$

With this in mind, we can generate the next several rows of our triangle.

$$\begin{array}{lccccccccccc} n & =3: & & & 1 & & 3 & & 3 & & 1 & \\ n & =4: & & 1 & & 4 & & 6 & & 4 & & 1 \\ n & =5: & 1 & & 5 & & 10 & & 10 & & 5 & & 1 \\ n & =6: & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \end{array} \tag{11.8.5}$$

So, we use the numbers in row $n=4$ to generate entries in row $n=5$: $1+4=5, 4+6=10$. We then use row $n=5$ to get row $n=6$, etc.

Of course, it would take a while to compute each row up to the desired n. Fortunately, there is a simple expression for computing a specific coefficient. Consider the k th term in the expansion of $(a+b)^n$. Let $r=k-1$, for $k=1,\ldots,n+1$. Then this term is of the form $C_r^n a^{n-r}b^r$. We have seen that the coefficients satisfy

$$C_r^n = C_r^{n-1} + C_{r-1}^{n-1}.$$

Actually, the binomial coefficients, C_r^n, have been found to take a simple form,

$$C_r^n = \frac{n!}{(n-r)!r!} \equiv \binom{n}{r}.$$

This is nothing other than the combinatoric symbol for determining how to choose n objects r at a time. In the binomial expansions this makes sense. We have to count the number of ways that we can arrange r products of b with $n-r$ products of a. There are n slots to place the b' s. For example, the $r=2$ case for $n=4$ involves the six products: $aabb, abab, abba, baab, baba$, and $bbaa$. Thus, it is natural to use this notation.

Andreas Freiherr von Ettingshausen

Andreas Freiherr von Ettingshausen (1796-1878) was a German mathematician and physicist who in 1826 introduced the notation $\binom{n}{r}$. However, the binomial coefficients were known by the Hindus centuries beforehand.

So, we have found that

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r}b^r. \tag{11.8.6}$$

Now consider the geometric series $1+x+x^2+\ldots$ We have seen that such this geometric series converges for $|x|<1$, giving

$$1+x+x^2+\ldots = \frac{1}{1-x}.$$

But, $\frac{1}{1-x} = (1-x)^{-1}$. This is a binomial to a power, but the power is not an integer.

It turns out that the coefficients of such a binomial expansion can be written similar to the form in Equation(A.108). This example suggests that our sum may no longer be finite. So, for p a real number, $a=1$ and $b=x$, we generalize Equation (11.8.6) as

$$(1+x)^p = \sum_{r=0}^{\infty} \binom{p}{r} x^r \tag{11.8.7}$$

and see if the resulting series makes sense. However, we quickly run into problems with the coefficients in the series.

Consider the coefficient for $r=1$ in an expansion of $(1+x)^{-1}$. This is given by

$$\binom{-1}{1} = \frac{(-1)!}{(-1-1)!1!} = \frac{(-1)!}{(-2)!1!}.$$

But what is (-1) !? By definition, it is

$$(-1)! = (-1)(-2)(-3)\cdots.$$

This product does not seem to exist! But with a little care, we note that

$$\frac{(-1)!}{(-2)!} = \frac{(-1)(-2)!}{(-2)!} = -1.$$

So, we need to be careful not to interpret the combinatorial coefficient literally. There are better ways to write the general binomial expansion. We can write the general coefficient as

$$\begin{aligned}\binom{p}{r} &= \frac{p!}{(p-r)!r!} \\ &= \frac{p(p-1)\cdots(p-r+1)(p-r)!}{(p-r)!r!} \\ &= \frac{p(p-1)\cdots(p-r+1)}{r!}.\end{aligned} \tag{11.8.8}$$

With this in mind we now state the theorem:

Theorem 11.8.1: General Binomial Expansion

The general binomial expansion for $(1+x)^p$ is a simple generalization of Equation (A.108). For p real, we have the following binomial series:

$$(1+x)^p = \sum_{r=0}^{\infty} \frac{p(p-1)\cdots(p-r+1)}{r!} x^r, \quad |x| < 1. \tag{11.8.9}$$

Often in physics we only need the first few terms for the case that $x \ll 1$:

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2}x^2 + O\left(x^3\right). \tag{11.8.10}$$

Example 11.8.1

Approximate $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ for $v \ll c$.

Solution

For $v \ll c$ the first approximation is found inserting $v/c = 0$. Thus, one obtains $\gamma = 1$. This is the Newtonian approximation and does not provide enough of an approximation for terrestrial speeds. Thus, we need to expand γ in powers of v/c.

First, we rewrite γ as

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \left[1 - \left(\frac{v}{c}\right)^2\right]^{-1/2}.$$

Using the binomial expansion for $p = -1/2$, we have

$$\gamma \approx 1 + \left(-\frac{1}{2}\right)\left(-\frac{v^2}{c^2}\right) = 1 + \frac{v^2}{2c^2}.$$

Note

The factor $\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$ is important in special relativity. Namely, this is the factor relating differences in time and length measurements by observers moving relative inertial frames. For terrestrial speeds, this gives an appropriate approximation.

Example 11.8.2: Time Dilation Example

The average speed of a large commercial jet airliner is about 500mph. If you flew for an hour (measured from the ground), then how much younger would you be than if you had not taken the flight, assuming these reference frames obeyed the postulates of special relativity?

Solution

This is the problem of time dilation. Let Δt be the elapsed time in a stationary reference frame on the ground and $\Delta\tau$ be that in the frame of the moving plane. Then from the Theory of Special Relativity these are related by

$$\Delta t = \gamma \Delta \tau.$$

The time differences would then be

$$\begin{aligned} \Delta t - \Delta \tau &= \left(1 - \gamma^{-1}\right) \Delta t \\ &= \left(1 - \sqrt{1 - \frac{v^2}{c^2}}\right) \Delta t. \end{aligned} \tag{11.8.11}$$

The plane speed, 500mph, is roughly 225 m/s and $c = 3.00 \times 10^8$ m/s. Since $V \ll c$, we would need to use the binomial approximation to get a nonzero result.

$$\begin{aligned} \Delta t - \Delta \tau &= \left(1 - \sqrt{1 - \frac{v^2}{c^2}}\right) \Delta t \\ &= \left(1 - \left(1 - \frac{v^2}{2c^2} + \ldots\right)\right) \Delta t \\ &\approx \frac{v^2}{2c^2} \Delta t \\ &= \frac{(225)^2}{2\left(3.00 \times 10^8\right)^2} (1h) = 1.01ns. \end{aligned} \tag{11.8.12}$$

Thus, you have aged one nanosecond less than if you did not take the flight.

Example 11.8.3: Small differences in large numbers

Compute $f(R,h) = \sqrt{R^2 + h^2} - R$ for $R = 6378.164$ km and $h = 1.0$ m.

Solution

Inserting these values into a scientific calculator, one finds that

$$f(6378164, 1) = \sqrt{6378164^2 + 1} - 6378164 = 1 \times 10^{-7} \text{ m}.$$

In some calculators one might obtain 0, in other calculators, or computer algebra systems like Maple, one might obtain other answers. What answer do you get and how accurate is your answer?

The problem with this computation is that $R \gg$ h. Therefore, the computation of $f(R,h)$ depends on how many digits the computing device can handle. The best way to get an answer is to use the binomial approximation. Writing $h = Rx$, or $x = \frac{h}{R}$, we have

$$\begin{aligned} f(R,h) &= \sqrt{R^2+h^2} - R \\ &= R\sqrt{1+x^2} - R \\ &\simeq R\left[1+\frac{1}{2}x^2\right] - R \\ &= \frac{1}{2}Rx^2 \\ &= \frac{1}{2}\frac{h}{R^2} = 7.83926\times 10^{-8}\text{ m}. \end{aligned} \tag{11.8.13}$$

Of course, you should verify how many digits should be kept in reporting the result.

In the next examples, we generalize this example. Such general computations appear in proofs involving general expansions without specific numerical values given.

Example 11.8.4

Obtain an approximation to $(a+b)^p$ when a is much larger than b, denoted by $a \gg b$.

Solution

If we neglect b then $(a+b)^p \simeq a^p$. How good of an approximation is this? This is where it would be nice to know the order of the next term in the expansion. Namely, what is the power of b/a of the first neglected term in this expansion?

In order to do this we first divide out a as

$$(a+b)^p = a^p\left(1+\frac{b}{a}\right)^p.$$

Now we have a small parameter, $\frac{b}{a}$. According to what we have seen earlier, we can use the binomial expansion to write

$$\left(1+\frac{b}{a}\right)^n = \sum_{r=0}^{\infty}\binom{p}{r}\left(\frac{b}{a}\right)^r. \tag{11.8.14}$$

Thus, we have a sum of terms involving powers of $\frac{b}{a}$. Since $a \gg b$, most of these terms can be neglected. So, we can write

$$\left(1+\frac{b}{a}\right)^p = 1+p\frac{b}{a}+O\left(\left(\frac{b}{a}\right)^2\right).$$

Here we used $O()$, big-Oh notation, to indicate the size of the first neglected term.

Summarizing, we have

$$\begin{aligned} (a+b)^p &= a^p\left(1+\frac{b}{a}\right)^p \\ &= a^p\left(1+p\frac{b}{a}+O\left(\left(\frac{b}{a}\right)^2\right)\right) \\ &= a^p + pa^p\frac{b}{a} + a^pO\left(\left(\frac{b}{a}\right)^2\right). \end{aligned} \tag{11.8.15}$$

Therefore, we can approximate $(a+b)^p \simeq a^p + pba^{p-1}$, with an error on the order of b^2a^{p-2}. Note that the order of the error does not include the constant factor from the expansion. We could also use the approximation that $(a+b)^p \simeq a^p$, but it is not typically good enough in applications because the error in this case is of the order ba^{p-1}.

Example 11.8.5

Approximate $f(x) = (a+x)^p - a^p$ for $x \ll a$.

Solution

In an earlier example we computed $f(R,h)=\sqrt{R^2+h^2}-R$ for $R=6378.164$ km and $h=1.0$ m. We can make use of the binomial expansion to determine the behavior of similar functions in the form $f(x)=(a+x)^p-a^p$. Inserting the binomial expression into $f(x)$, we have as $\frac{x}{a}\to 0$ that

$$\begin{aligned} f(x) &= (a+x)^p - a^p \\ &= a^p\left[\left(1+\frac{x}{a}\right)^p - 1\right] \\ &= a^p\left[\frac{px}{a}+O\left(\left(\frac{x}{a}\right)^2\right)\right] \\ &= O\left(\frac{x}{a}\right) \quad \text{as } \frac{x}{a}\to 0. \end{aligned} \tag{11.8.16}$$

This result might not be the approximation that we desire. So, we could back up one step in the derivation to write a better approximation as

$$(a+x)^p - a^p = a^{p-1}px + O\left(\left(\frac{x}{a}\right)^2\right) \quad \text{as } \frac{x}{a}\to 0.$$

We now use this approximation to compute $f(R,h)=\sqrt{R^2+h^2}-R$ for $R=$ 6378.164 km and $h=1.0$ m in the earlier example. We let $a=R^2, x=1$ and $p=\frac{1}{2}$. Then, the leading order approximation would be of order

$$O\left(\left(\frac{x}{a}\right)^2\right) = O\left(\left(\frac{1}{6378164^2}\right)^2\right) \sim 2.4\times 10^{-14}.$$

Thus, we have

$$\sqrt{6378164^2+1}-6378164 \approx a^{p-1}px$$

where

$$a^{p-1}px = \left(6378164^2\right)^{-1/2}(0.5)1 = 7.83926\times 10^{-8}.$$

This is the same result we had obtained before. However, we have also an estimate of the size of the error and this might be useful in indicating how many digits we should trust in the answer.

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11.9: Problems

Exercise 11.9.1

Prove the following identities using only the definitions of the trigonometric functions, the Pythagorean identity, or the identities for sines and cosines of sums of angles.

a. $\cos 2x = 2\cos^2 x - 1$.
b. $\sin 3x = A\sin^3 x + B\sin x$, for what values of A and B?
c. $\sec\theta + \tan\theta = \tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right)$.

Exercise 11.9.2

Determine the exact values of

a. $\sin\frac{\pi}{8}$.
b. $\tan 15^\circ$
c. $\cos 105^\circ$.

Exercise 11.9.3

Denest the following if possible.

a. $\sqrt{3-2\sqrt{2}}$.
b. $\sqrt{1+\sqrt{2}}$.
c. $\sqrt{5+2\sqrt{6}}$.
d. $\sqrt[3]{\sqrt{5}+2} - \sqrt[3]{\sqrt{5}-2}$
e. Find the roots of $x^2 + 6x - 4\sqrt{5} = 0$ in simplified form.

Exercise 11.9.4

Determine the exact values of

a. $\sin\left(\cos^{-1}\frac{3}{5}\right)$.
b. $\tan\left(\sin^{-1}\frac{x}{7}\right)$
c. $\sin^{-1}\left(\sin\frac{3\pi}{2}\right)$.

Exercise 11.9.5

Do the following.

a. Write $(\cosh x - \sinh x)^6$ in terms of exponentials.
b. Prove $\cosh(x-y) = \cosh x\cosh y - \sinh x\sinh y$ using the exponential forms of the hyperbolic functions.
c. Prove $\cosh 2x = \cosh^2 x + \sinh^2 x$.
d. If $\cosh x = \frac{13}{12}$ and $x < 0$, find $\sinh x$ and $\tanh x$.
e. Find the exact value of $\sinh(\operatorname{arccosh} 3)$.

Exercise 11.9.6

Prove that the inverse hyperbolic functions are the following logarithms:

a. $\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right)$.
b. $\tanh^{-1} x = \frac{1}{2}\ln\frac{1+x}{1-x}$.

Exercise 11.9.7

Write the following in terms of logarithms:

a. $\cosh^{-1}\frac{4}{3}$.
b. $\tanh^{-1}\frac{1}{2}$.
c. $\sinh^{-1}2$.

Exercise 11.9.8

Solve the following equations for x.

a. $\cosh(x+\ln 3)=3$.
b. $2\tanh^{-1}\frac{x-2}{x-1}=\ln 2$.
c. $\sinh^2 x-7\cosh x+13=0$.

Exercise 11.9.9

Compute the following integrals.

a. $\int xe^{2x^2}dx$.
b. $\int_0^3 \frac{5x}{\sqrt{x^2+16}}dx$.
c. $\int x^3\sin 3x\,dx$. (Do this using integration by parts, the Tabular Method, and differentiation under the integral sign.)
d. $\int \cos^4 3x\,dx$.
e. $\int_0^{\pi/4}\sec^3 x\,dx$.
f. $\int e^x\sinh x\,dx$
g. $\int\sqrt{9-x^2}\,dx$
h. $\int\frac{dx}{(4-x^2)^2}$, using the substitution $x=2\tanh u$.
i. $\int_0^4\frac{dx}{\sqrt{9+x^2}}$, using a hyperbolic function substitution.
j. $\int\frac{dx}{1-x^2}$, using the substitution $x=\tanh u$.
k. $\int\frac{dx}{(x^2+4)^{3/2}}$, using the substitutions $x=2\tan\theta$ and $x=2\sinh u$.
l. $\int\frac{dx}{\sqrt{3x^2-6x+4}}$.

Exercise 11.9.10

Find the sum for each of the series:

a. $5+\frac{25}{7}+\frac{125}{49}+\frac{625}{343}+\cdots$.
b. $\sum_{n=0}^{\infty}\frac{(-1)^n 3}{4^n}$.
c. $\sum_{n=2}^{\infty}\frac{2}{5^n}$.
d. $\sum_{n=-1}^{\infty}(-1)^{n+1}\left(\frac{e}{\pi}\right)^n$.
e. $\sum_{n=0}^{\infty}\left(\frac{5}{2^n}+\frac{1}{3^n}\right)$.
f. $\sum_{n=1}^{\infty}\frac{3}{n(n+3)}$.
g. What is $0.56\overline{9}$?

Exercise 11.9.11

A superball is dropped from a 2.00 m height. After it rebounds, it reaches a new height of 1.65 m. Assuming a constant coefficient of restitution, find the (ideal) total distance the ball will travel as it keeps bouncing.

Exercise 11.9.12

Here are some telescoping series problems.

a. Verify that

$$\sum_{n=1}^{\infty} \frac{1}{(n+2)(n+1)} = \sum_{n=1}^{\infty} \left(\frac{n+1}{n+2} - \frac{n}{n+1} \right).$$

b. Find the nth partial sum of the series $\sum_{n=1}^{\infty} \left(\frac{n+1}{n+2} - \frac{n}{n+1} \right)$ and use it to determine the sum of the resulting telescoping series.

c. Sum the series $\sum_{n=1}^{\infty} \left[\tan^{-1} n - \tan^{-1}(n+1) \right]$ by first writing the N th partial sum and then computing $\lim_{N \to \infty} s_N$.

Exercise 11.9.13

Determine the radius and interval of convergence of the following infinite series:

a. $\sum_{n=1}^{\infty} (-1)^n \frac{(x-1)^n}{n}$.
b. $\sum_{n=1}^{\infty} \frac{x^n}{2^n n!}$.
c. $\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{x}{5} \right)^n$.
d. $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{\sqrt{n}}$.

Exercise 11.9.14

Find the Taylor series centered at $x = a$ and its corresponding radius of convergence for the given function. In most cases, you need not employ the direct method of computation of the Taylor coefficients.

a. $f(x) = \sinh x, a = 0$.
b. $f(x) = \sqrt{1+x}, a = 0$.
c. $f(x) = \ln \frac{1+x}{1-x}, a = 0$.
d. $f(x) = xe^x, a = 1$.
e. $f(x) = \frac{1}{\sqrt{x}}, a = 1$.
f. $f(x) = x^4 + x - 2, a = 2$.
g. $f(x) = \frac{x-1}{2+x}, a = 1$.

Exercise 11.9.15

Consider Gregory's expansion

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}.$$

a. Derive Gregory's expansion by using the definition

$$\tan^{-1} x = \int_0^x \frac{dt}{1+t^2},$$

expanding the integrand in a Maclaurin series, and integrating the resulting series term by term.

b. From this result, derive Gregory's series for π by inserting an appropriate value for x in the series expansion for $\tan^{-1} x$.

Exercise 11.9.16

In the event that a series converges uniformly, one can consider the derivative of the series to arrive at the summation of other infinite series.

a. Differentiate the series representation for $f(x)=\frac{1}{1-x}$ to sum the series $\sum_{n=1}^{\infty} nx^n, |x|<1$.
b. Use the result from part a to sum the series $\sum_{n=1}^{\infty} \frac{n}{5^n}$.
c. Sum the series $\sum_{n=2}^{\infty} n(n-1)x^n, |x|<1$.
d. Use the result from part c to sum the series $\sum_{n=2}^{\infty} \frac{n^2-n}{5^n}$.
e. Use the results from this problem to sum the series $\sum_{n=4}^{\infty} \frac{n^2}{5^n}$.

Exercise 11.9.17

Evaluate the integral $\int_0^{\pi/6} \sin^2 x dx$ by doing the following:

a. Compute the integral exactly.
b. Integrate the first three terms of the Maclaurin series expansion of the integrand and compare with the exact result.

Exercise 11.9.18

Determine the next term in the time dilation example, 11.8.2. That is, find the $\frac{v^4}{c^2}$ term and determine a better approximation to the time difference of 1 ns.

Exercise 11.9.19

Evaluate the following expressions at the given point. Use your calculator or your computer (such as Maple). Then use series expansions to find an approximation to the value of the expression to as many places as you trust.

a. $\frac{1}{\sqrt{1+x^3}}-\cos x^2$ at $x=0.015$.
b. $\ln\sqrt{\frac{1+x}{1-x}}-\tan x$ at $x=0.0015$.
c. $f(x)=\frac{1}{\sqrt{1+2x^2}}-1+x^2$ at $x=5.00\times10^{-3}$.
d. $f(R,h)=R-\sqrt{R^2+h^2}$ for $R=1.374\times10^3$ km and $h=1.00$ m.
e. $f(x)=1-\frac{1}{\sqrt{1-x}}$ for $x=2.5\times10^{-13}$.

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CHAPTER OVERVIEW

12: B - Ordinary Differential Equations Review

> *"The profound study of nature is the most fertile source of mathematical discoveries."*
>
> *- Joseph Fourier (1768-1830)*

12.1: First Order Differential Equations
12.2: Second Order Linear Differential Equations
12.3: Forced Systems
12.4: Cauchy-Euler Equations
12.5: Problems

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12.1: First Order Differential Equations

Before moving on, we first define an n-th order ordinary differential equation. It is an equation for an unknown function $y(x)$ that expresses a relationship between the unknown function and its first n derivatives. One could write this generally as

$$F\left(y^{(n)}(x), y^{(n-1)}(x), \ldots, y'(x), y(x), x\right) = 0. \tag{12.1.1}$$

Here $y^{(n)}(x)$ represents the nth derivative of $y(x)$.

An initial value problem consists of the differential equation plus the values of the first $n-1$ derivatives at a particular value of the independent variable, say x_0 :

$$y^{(n-1)}(x_0) = y_{n-1}, \quad y^{(n-2)}(x_0) = y_{n-2}, \quad \ldots, \quad y(x_0) = y_0. \tag{12.1.2}$$

A linear nth order differential equation takes the form

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \ldots + a_1(x)y'(x) + a_0(x)y(x)\Big) = f(x). \tag{12.1.3}$$

If $f(x) \equiv 0$, then the equation is said to be homogeneous, otherwise it is called nonhomogeneous.

Typically, the first differential equations encountered are first order equations. A first order differential equation takes the form

$$F(y', y, x) = 0. \tag{12.1.4}$$

There are two common first order differential equations for which one can formally obtain a solution. The first is the separable case and the second is a first order equation. We indicate that we can formally obtain solutions, as one can display the needed integration that leads to a solution. However, the resulting integrals are not always reducible to elementary functions nor does one obtain explicit solutions when the integrals are doable.

Separable Equations

A first order equation is separable if it can be written the form

$$\frac{dy}{dx} = f(x)g(y). \tag{12.1.5}$$

Special cases result when either $f(x) = 1$ or $g(y) = 1$. In the first case the equation is said to be autonomous.

The general solution to equation (12.1.5) is obtained in terms of two integrals:

$$\int \frac{dy}{g(y)} = \int f(x)dx + C, \tag{12.1.6}$$

where C is an integration constant. This yields a 1-parameter family of solutions to the differential equation corresponding to different values of C. If one can solve (12.1.6) for $y(x)$, then one obtains an explicit solution. Otherwise, one has a family of implicit solutions. If an initial condition is given as well, then one might be able to find a member of the family that satisfies this condition, which is often called a particular solution.

Example 12.1.1

$y' = 2xy, y(0) = 2.$

Solution

Applying (12.1.6), one has

$$\int \frac{dy}{y} = \int 2x dx + C.$$

Integrating yields

$$\ln|y| = x^2 + C.$$

Exponentiating, one obtains the general solution,

$$y(x) = \pm e^{x^2+C} = Ae^{x^2}.$$

Here we have defined $A = \pm e^C$. Since C is an arbitrary constant, A is an arbitrary constant. Several solutions in this 1-parameter family are shown in Figure 12.1.1.

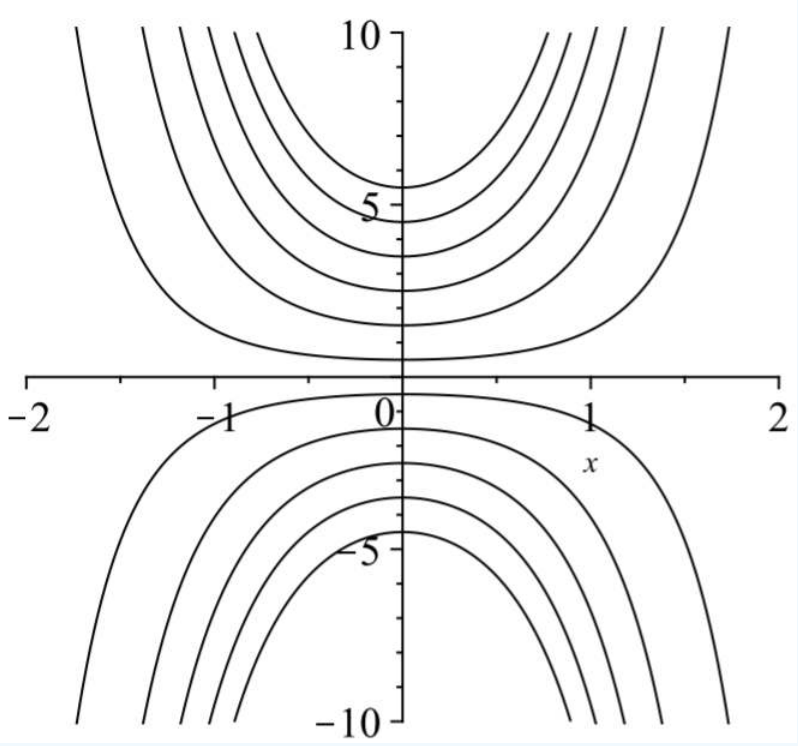


Figure 12.1.1: Plots of solutions from the 1−parameter family of solutions of Example 12.1.1 for several initial conditions.

Next, one seeks a particular solution satisfying the initial condition. For $y(0) = 2$, one finds that $A = 2$. So, the particular solution satisfying the initial condition is $y(x) = 2e^{x^2}$.

Example 12.1.2

$yy' = -x$.

Solution

Following the same procedure as in the last example, one obtains:

$$\int ydy = -\int xdx + C \Rightarrow y^2 = -x^2 + A, \text{ where } A = 2C.$$

Thus, we obtain an implicit solution. Writing the solution as $x^2 + y^2 = A$, we see that this is a family of circles for $A > 0$ and the origin for $A = 0$. Plots of some solutions in this family are shown in Figure 12.1.2

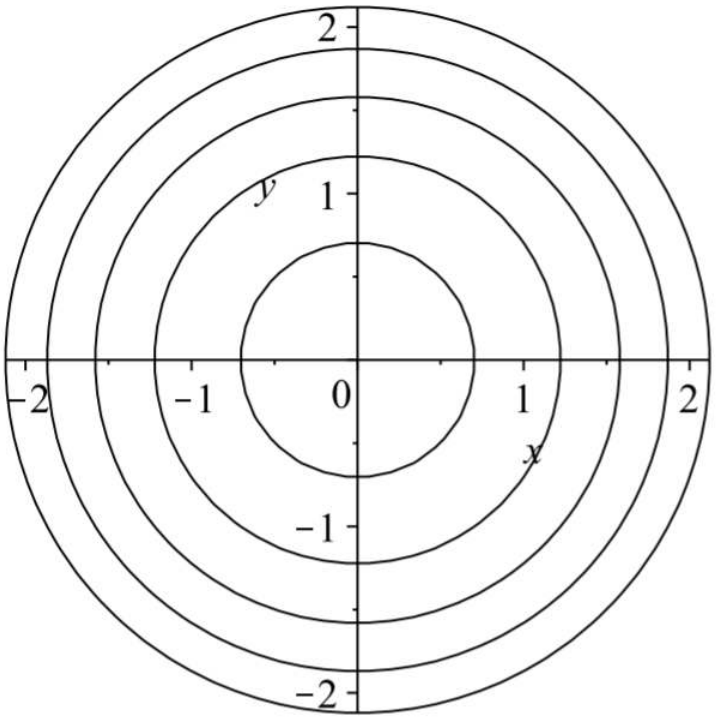


Figure 12.1.2: Plots of solutions of Example 12.1.2 for several initial conditions.

Linear First Order Equations

The second type of first order equation encountered is the linear first order differential equation in the standard form

$$y'(x)+p(x)y(x)=q(x). \tag{12.1.7}$$

In this case one seeks an integrating factor, $\mu(x)$, which is a function that one can multiply through the equation making the left side a perfect derivative. Thus, obtaining,

$$\frac{d}{dx}[\mu(x)y(x)]=\mu(x)q(x). \tag{12.1.8}$$

The integrating factor that works is $\mu(x)=\exp\left(\int^x p(\xi)d\xi\right)$. One can derive $\mu(x)$ by expanding the derivative in Equation (12.1.8),

$$\mu(x)y'(x)+\mu'(x)y(x)=\mu(x)q(x), \tag{12.1.9}$$

and comparing this equation to the one obtained from multiplying (12.1.7) by $\mu(x)$:

$$\mu(x)y'(x)+\mu(x)p(x)y(x)=\mu(x)q(x). \tag{12.1.10}$$

Note that these last two equations would be the same if the second terms were the same. Thus, we will require that

$$\frac{d\mu(x)}{dx}=\mu(x)p(x).$$

This is a separable first order equation for $\mu(x)$ whose solution is the integrating factor:

$$\mu(x)=\exp\left(\int^x p(\xi)d\xi\right). \tag{12.1.11}$$

Equation (12.1.8) is now easily integrated to obtain the general solution to the linear first order differential equation:

$$y(x)=\frac{1}{\mu(x)}\left[\int^x \mu(\xi)q(\xi)d\xi+C\right]. \tag{12.1.12}$$

Example 12.1.3

$xy' + y = x, \quad x > 0, y(1) = 0$.

Solution

One first notes that this is a linear first order differential equation. Solving for y', one can see that the equation is not separable. Furthermore, it is not in the standard form (12.1.7). So, we first rewrite the equation as

$$\frac{dy}{dx} + \frac{1}{x}y = 1. \tag{12.1.13}$$

Noting that $p(x) = \frac{1}{x}$, we determine the integrating factor

$$\mu(x) = \exp\left[\int^x \frac{d\xi}{\xi}\right] = e^{\ln x} = x.$$

Multiplying equation (12.1.13) by $\mu(x) = x$, we actually get back the original equation! In this case we have found that $xy' + y$ must have been the derivative of something to start. In fact, $(xy)' = xy' + x$. Therefore, the differential equation becomes

$$(xy)' = x.$$

Integrating, one obtains

$$xy = \frac{1}{2}x^2 + C$$

or

$$y(x) = \frac{1}{2}x + \frac{C}{x}.$$

Inserting the initial condition into this solution, we have $0 = \frac{1}{2} + C$. Therefore, $C = -\frac{1}{2}$. Thus, the solution of the initial value problem is

$$y(x) = \frac{1}{2}\left(x - \frac{1}{x}\right).$$

We can verify that this is the solution. Since $y' = \frac{1}{2} + \frac{1}{2x^2}$, we have

$$xy' + y = \frac{1}{2}x + \frac{1}{2x} + \frac{1}{2}\left(x - \frac{1}{x}\right) = x.$$

Also, $y(1) = \frac{1}{2}(1-1) = 0$.

Example 12.1.4

$(\sin x)y' + (\cos x)y = x^2$.

Solution

Actually, this problem is easy if you realize that the left hand side is a perfect derivative. Namely,

$$\frac{d}{dx}((\sin x)y) = (\sin x)y' + (\cos x)y.$$

But, we will go through the process of finding the integrating factor for practice.

First, we rewrite the original differential equation in standard form. We divide the equation by $\sin x$ to obtain

$$y' + (\cot x)y = x^2 \csc x$$

Then, we compute the integrating factor as

$$\mu(x) = \exp\left(\int^x \cot\xi d\xi\right) = e^{\ln(\sin x)} = \sin x.$$

Using the integrating factor, the standard form equation becomes

$$\frac{d}{dx}((\sin x)y) = x^2.$$

Integrating, we have

$$y\sin x = \frac{1}{3}x^3 + C$$

So, the solution is

$$y(x) = \left(\frac{1}{3}x^3 + C\right)\csc x.$$

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12.2: Second Order Linear Differential Equations

Second order differential equations are typically harder than first order. In most cases students are only exposed to second order linear differential equations. A general form for a *second order linear differential equation* is given by

$$a(x)y''(x)+b(x)y'(x)+c(x)y(x)=f(x). \tag{12.2.1}$$

One can rewrite this equation using operator terminology. Namely, one first defines the differential operator $L=a(x)D^2+b(x)D+c(x)$, where $D=\frac{d}{dx}$. Then equation (12.2.1) becomes

$$Ly=f. \tag{12.2.2}$$

The solutions of linear differential equations are found by making use of the linearity of L. Namely, we consider the *vector space*[1] consisting of real-valued functions over some domain. Let f and g be vectors in this function space. L is a *linear operator* if for two vectors f and g and scalar a, we have

a. $L(f+g)=Lf+Lg$
b. $L(af)=aLf$.

Note

We assume that the reader has been introduced to concepts in linear algebra. Later in the text we will recall the definition of a vector space and see that linear algebra is in the background of the study of many concepts in the solution of differential equations.

One typically solves (12.2.1) by finding the general solution of the homogeneous problem,

$$Ly_h=0$$

and a particular solution of the nonhomogeneous problem,

$$Ly_p=f.$$

Then, the general solution of (12.2.1) is simply given as $y=y_h+y_p$. This is true because of the linearity of L. Namely,

$$\begin{aligned} Ly &= L\left(y_h+y_p\right) \\ &= Ly_h+Ly_p \\ &= 0+f=f. \end{aligned} \tag{12.2.3}$$

There are methods for finding a particular solution of a nonhomogeneous differential equation. These methods range from pure guessing, the Method of Undetermined Coefficients, the Method of Variation of Parameters, or Green's functions. We will review these methods later in the chapter.

Determining solutions to the homogeneous problem, $Ly_h=0$, is not always easy. However, many now famous mathematicians and physicists have studied a variety of second order linear equations and they have saved us the trouble of finding solutions to the differential equations that often appear in applications. We will encounter many of these in the following chapters. We will first begin with some simple homogeneous linear differential equations.

Linearity is also useful in producing the general solution of a homogeneous linear differential equation. If y_1 and y_2 are solutions of the homogeneous equation, then the *linear combination* $y=c_1y_1+c_2y_2$ is also a solution of the homogeneous equation. In fact, if y_1 and y_2 are *linearly independent*,[2] then $y=c_1y_1+c_2y_2$ is the general solution of the homogeneous problem.

Note

A set of functions $\{y_i(x)\}_{i=1}^n$ is a linearly independent set if and only if

$$c_1y_1(x)+\ldots+c_ny_n(x)=0$$

implies $c_i=0$, for $i=1,\ldots,n$.

For $n=2, c_1y_1(x)+c_2y_2(x)=0$. If y_1 and y_2 are linearly dependent, then the coefficients are not zero and $y_2(x)=-\frac{c_1}{c_2}y_1(x)$ and is a multiple of $y_1(x)$.

Linear independence can also be established by looking at the Wronskian of the solutions. For a second order differential equation the Wronskian is defined as

$$W(y_1,y_2)=y_1(x)y_2'(x)-y_1'(x)y_2(x). \tag{12.2.4}$$

The solutions are linearly independent if the Wronskian is not zero.

Constant Coefficient Equations

The simplest second order differential equations are those with constant coefficients. The general form for a homogeneous constant coefficient second order linear differential equation is given as

$$ay''(x)+by'(x)+cy(x)=0, \tag{12.2.5}$$

where a, b, and c are constants.

Solutions to (12.2.5) are obtained by making a guess of $y(x)=e^{rx}$. Inserting this guess into (12.2.5) leads to the characteristic equation

$$ar^2+br+c=0. \tag{12.2.6}$$

Namely, we compute the derivatives of $y(x)=e^{rx}$, to get $y(x)=re^{rx}$, and $y(x)=r^2e^{rx}$. Inserting into (12.2.5), we have

$$0=ay''(x)+by'(x)+cy(x)=\left(ar^2+br+c\right)e^{rx}.$$

Since the exponential is never zero, we find that $ar^2+br+c=0$.

Note

The characteristic equation for $ay''+by'+cy=0$ is $ar^2+br+c=0$. Solutions of this quadratic equation lead to solutions of the differential equation.

Two real, distinct roots, r_1 and r_2, give solutions of the form

$$y(x)=c_1e^{r_1x}+c_2e^{r_2x}.$$

The roots of this equation, r_1, r_2, in turn lead to three types of solutions depending upon the nature of the roots. In general, we have two linearly independent solutions, $y_1(x)=e^{r_1x}$ and $y_2(x)=e^{r_2x}$, and the general solution is given by a linear combination of these solutions,

$$y(x)=c_1e^{r_1x}+c_2e^{r_2x}.$$

For two real distinct roots, we are done. However, when the roots are real, but equal, or complex conjugate roots, we need to do a little more work to obtain usable solutions.

Example 12.2.1

$y''-y'-6y=0 y(0)=2, y'(0)=0$.

Solution

The characteristic equation for this problem is $r^2-r-6=0$. The roots of this equation are found as $r=-2,3$. Therefore, the general solution can be quickly written down:

$$y(x)=c_1e^{-2x}+c_2e^{3x}.$$

Note that there are two arbitrary constants in the general solution. Therefore, one needs two pieces of information to find a particular solution. Of course, we have the needed information in the form of the initial conditions.

One also needs to evaluate the first derivative

$$y'(x) = -2c_1 e^{-2x} + 3c_2 e^{3x}$$

in order to attempt to satisfy the initial conditions. Evaluating y and y' at $x = 0$ yields

$$\begin{aligned} 2 &= c_1 + c_2 \\ 0 &= -2c_1 + 3c_2 \end{aligned} \tag{12.2.7}$$

These two equations in two unknowns can readily be solved to give $c_1 = 6/5$ and $c_2 = 4/5$. Therefore, the solution of the initial value problem is obtained as $y(x) = \frac{6}{5}e^{-2x} + \frac{4}{5}e^{3x}$.

Note

Repeated roots, $r_1 = r_2 = r$, give solutions of the form

$$y(x) = (c_1 + c_2 x)\, e^{rx}.$$

In the case when there is a repeated real root, one has only one solution, $y_1(x) = e^{rx}$. The question is how does one obtain the second linearly independent solution? Since the solutions should be independent, we must have that the ratio $y_2(x)/y_1(x)$ is not a constant. So, we guess the form $y_2(x) = v(x)y_1(x) = v(x)e^{rx}$. (This process is called the Method of Reduction of Order.)

For constant coefficient second order equations, we can write the equation as

$$(D - r)^2 y = 0,$$

where $D = \frac{d}{dx}$. We now insert $y_2(x) = v(x)e^{rx}$ into this equation. First we compute

$$(D - r)ve^{rx} = v'e^{rx}.$$

Then,

$$0 = (D - r)^2 ve^{rx} = (D - r)v'e^{rx} = v''e^{rx}.$$

So, if $y_2(x)$ is to be a solution to the differential equation, then $v''(x)e^{rx} = 0$ for all x. So, $v''(x) = 0$, which implies that

$$v(x) = ax + b.$$

So,

$$y_2(x) = (ax + b)e^{rx}.$$

Without loss of generality, we can take $b = 0$ and $a = 1$ to obtain the second linearly independent solution, $y_2(x) = xe^{rx}$. The general solution is then

$$y(x) = c_1 e^{rx} + c_2 xe^{rx}.$$

Example 12.2.2

$y'' + 6y' + 9y = 0$.

Solution

In this example we have $r^2 + 6r + 9 = 0$. There is only one root, $r = -3$. From the above discussion, we easily find the solution $y(x) = (c_1 + c_2 x)\, e^{-3x}$.

When one has complex roots in the solution of constant coefficient equations, one needs to look at the solutions

$$y_{1,2}(x) = e^{(\alpha \pm i\beta)x}.$$

We make use of Euler's formula (See Chapter 6 for more on complex variables)

$$e^{i\beta x} = \cos \beta x + i \sin \beta x \tag{12.2.8}$$

Then, the linear combination of $y_1(x)$ and $y_2(x)$ becomes

$$\begin{aligned} Ae^{(\alpha+i\beta)x} + Be^{(\alpha-i\beta)x} &= e^{\alpha x}\left[Ae^{i\beta x} + Be^{-i\beta x}\right] \\ &= e^{\alpha x}[(A+B)\cos\beta x + i(A-B)\sin\beta x] \\ &\equiv e^{\alpha x}\left(c_1\cos\beta x + c_2\sin\beta x\right) \end{aligned} \tag{12.2.9}$$

Thus, we see that we have a linear combination of two real, linearly independent solutions, $e^{\alpha x}\cos\beta x$ and $e^{\alpha x}\sin\beta x$.

Note

Complex roots, $r = \alpha \pm i\beta$, give solutions of the form

$$y(x) = e^{\alpha x}\left(c_1\cos\beta x + c_2\sin\beta x\right).$$

Example 12.2.3

$y'' + 4y = 0$.

Solution

The characteristic equation in this case is $r^2 + 4 = 0$. The roots are pure imaginary roots, $r = \pm 2i$, and the general solution consists purely of sinusoidal functions, $y(x) = c_1\cos(2x) + c_2\sin(2x)$, since $\alpha = 0$ and $\beta = 2$.

Example 12.2.4

$y'' + 2y' + 4y = 0$.

Solution

The characteristic equation in this case is $r^2 + 2r + 4 = 0$. The roots are complex, $r = -1 \pm \sqrt{3}i$ and the general solution can be written as

$$y(x) = \left[c_1\cos(\sqrt{3}x) + c_2\sin(\sqrt{3}x)\right]e^{-x}.$$

Example 12.2.5

$y'' + 4y = \sin x$.

Solution

This is an example of a nonhomogeneous problem. The homogeneous problem was actually solved in Example 12.2.3 According to the theory, we need only seek a particular solution to the nonhomogeneous problem and add it to the solution of the last example to get the general solution.

The particular solution can be obtained by purely guessing, making an educated guess, or using the Method of Variation of Parameters. We will not review all of these techniques at this time. Due to the simple form of the driving term, we will make an intelligent guess of $y_p(x) = A\sin x$ and determine what A needs to be. Inserting this guess into the differential equation gives $(-A + 4A)\sin x = \sin x$. So, we see that $A = 1/3$ works. The general solution of the nonhomogeneous problem is therefore $y(x) = c_1\cos(2x) + c_2\sin(2x) + \frac{1}{3}\sin x$.

The three cases for constant coefficient linear second order differential equations are summarized below.

Classification of Roots of the Characteristic Equation for Second Order Constant Coefficient ODEs

1. **Real, distinct roots** r_1, r_2. In this case the solutions corresponding to each root are linearly independent. Therefore, the general solution is simply $y(x) = c_1e^{r_1x} + c_2e^{r_2x}$.
2. **Real, equal roots** $r_1 = r_2 = r$. In this case the solutions corresponding to each root are linearly dependent. To find a second linearly independent solution, one uses the Method of Reduction of Order. This gives the second solution as xe^{rx}. Therefore, the general solution is found as $y(x) = (c_1 + c_2x)e^{rx}$.

3. **Complex conjugate roots** $r_1, r_2 = \alpha \pm i\beta$. In this case the solutions corresponding to each root are linearly independent. Making use of Euler's identity, $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, these complex exponentials can be rewritten in terms of trigonometric functions. Namely, one has that $e^{\alpha x}\cos(\beta x)$ and $e^{\alpha x}\sin(\beta x)$ are two linearly independent solutions. Therefore, the general solution becomes $y(x) = e^{\alpha x}(c_1\cos(\beta x) + c_2\sin(\beta x))$.

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12.3: Forced Systems

Many problems can be modeled by nonhomogeneous second order equations. Thus, we want to find solutions of equations of the form

$$\text{Ly}\,(x) = a(x)y''(x) + b(x)y'(x) + c(x)y(x) = f(x). \tag{12.3.1}$$

As noted in Section 12.2, one solves this equation by finding the general solution of the homogeneous problem,

$$Ly_h = 0$$

and a particular solution of the nonhomogeneous problem,

$$Ly_p = f.$$

Then, the general solution of (12.2.1) is simply given as $y = y_h + y_p$

So far, we only know how to solve constant coefficient, homogeneous equations. So, by adding a nonhomogeneous term to such equations we will need to find the particular solution to the nonhomogeneous equation.

We could guess a solution, but that is not usually possible without a little bit of experience. So, we need some other methods. There are two main methods. In the first case, the Method of Undetermined Coefficients, one makes an intelligent guess based on the form of $f(x)$. In the second method, one can systematically developed the particular solution. We will come back to the Method of Variation of Parameters and we will also introduce the powerful machinery of Green's functions later in this section.

Method of Undetermined Coefficients

Let's solve a simple differential equation highlighting how we can handle nonhomogeneous equations.

Example 12.3.1

Consider the equation

$$y'' + 2y' - 3y = 4. \tag{12.3.2}$$

Solution

The first step is to determine the solution of the homogeneous equation. Thus, we solve

$$y_h'' + 2y_h' - 3y_h = 0. \tag{12.3.3}$$

The characteristic equation is $r^2 + 2r - 3 = 0$. The roots are $r = 1, -3$. So, we can immediately write the solution

$$y_h(x) = c_1 e^x + c_2 e^{-3x}.$$

The second step is to find a particular solution of (12.3.2). What possible function can we insert into this equation such that only a 4 remains? If we try something proportional to x, then we are left with a linear function after inserting x and its derivatives. Perhaps a constant function you might think. $y = 4$ does not work. But, we could try an arbitrary constant, $y = A$.

Let's see. Inserting $y = A$ into (12.3.2), we obtain

$$-3A = 4.$$

Ah ha! We see that we can choose $A = -\frac{4}{3}$ and this works. So, we have a particular solution, $y_p(x) = -\frac{4}{3}$. This step is done.

Combining the two solutions, we have the general solution to the original nonhomogeneous equation (12.3.2). Namely,

$$y(x) = y_h(x) + y_p(x) = c_1 e^x + c_2 e^{-3x} - \frac{4}{3}.$$

Insert this solution into the equation and verify that it is indeed a solution. If we had been given initial conditions, we could now use them to determine the arbitrary constants.

Example 12.3.2

What if we had a different source term? Consider the equation

$$y'' + 2y' - 3y = 4x. \tag{12.3.4}$$

The only thing that would change is the particular solution. So, we need a guess.

Solution

We know a constant function does not work by the last example. So, let's try $y_p = Ax$. Inserting this function into Equation (12.3.4), we obtain

$$2A - 3Ax = 4x.$$

Picking $A = -4/3$ would get rid of the x terms, but will not cancel everything. We still have a constant left. So, we need something more general.

Let's try a linear function, $y_p(x) = Ax + B$. Then we get after substitution into (12.3.4)

$$2A - 3(Ax + B) = 4x.$$

Equating the coefficients of the different powers of x on both sides, we find a system of equations for the undetermined coefficients:

$$\begin{aligned} 2A - 3B &= 0 \\ -3A &= 4. \end{aligned} \tag{12.3.5}$$

These are easily solved to obtain

$$\begin{aligned} A &= -\frac{4}{3} \\ B &= \frac{2}{3}A = -\frac{8}{9}. \end{aligned} \tag{12.3.6}$$

So, the particular solution is

$$y_p(x) = -\frac{4}{3}x - \frac{8}{9}$$

This gives the general solution to the nonhomogeneous problem as

$$y(x) = y_h(x) + y_p(x) = c_1 e^x + c_2 e^{-3x} - \frac{4}{3}x - \frac{8}{9}.$$

There are general forms that you can guess based upon the form of the driving term, $f(x)$. Some examples are given in Table 12.3.1. More general applications are covered in a standard text on differential equations. However, the procedure is simple. Given $f(x)$ in a particular form, you make an appropriate guess up to some unknown parameters, or coefficients. Inserting the guess leads to a system of equations for the unknown coefficients. Solve the system and you have the solution. This solution is then added to the general solution of the homogeneous differential equation.

Table 12.3.1: Forms used in the Method of Undetermined Coefficients.

$f(x)$	Guess
$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$	$A_n x^n + A_{n-1} x^{n-1} + \cdots + A_1 x + A_0$
ae^{bx}	Ae^{bx}
$a\cos\omega x + b\sin\omega x$	$A\cos\omega x + B\sin\omega x$

Example 12.3.3

Solve

$$y'' + 2y' - 3y = 2e^{-3x}. \tag{12.3.7}$$

Solution

According to the above, we would guess a solution of the form $y_p = Ae^{-3x}$. Inserting our guess, we find

$$0 = 2e^{-3x}.$$

Oops! The coefficient, A, disappeared! We cannot solve for it. What went wrong?

The answer lies in the general solution of the homogeneous problem. Note that e^{x} and e^{-3x} are solutions to the homogeneous problem. So, a multiple of e^{-3x} will not get us anywhere. It turns out that there is one further modification of the method. If the driving term contains terms that are solutions of the homogeneous problem, then we need to make a guess consisting of the smallest possible power of x times the function which is no longer a solution of the homogeneous problem. Namely, we guess $y_p(x) = Axe^{-3x}$ and differentiate this guess to obtain the derivatives $y_p' = A(1-3x)e^{-3x}$ and $y_p'' = A(9x-6)e^{-3x}$.

Inserting these derivatives into the differential equation, we obtain

$$[(9x-6)+2(1-3x)-3x]Ae^{-3x} = 2e^{-3x}.$$

Comparing coefficients, we have

$$-4A = 2.$$

So, $A = -1/2$ and $y_p(x) = -\frac{1}{2}xe^{-3x}$. Thus, the solution to the problem is

$$y(x) = \left(2 - \frac{1}{2}x\right)e^{-3x}.$$

Modified Method of Undetermined Coefficients

In general, if any term in the guess $y_p(x)$ is a solution of the homogeneous equation, then multiply the guess by x^k, where k is the smallest positive integer such that no term in $x^k y_p(x)$ is a solution of the homogeneous problem.

Periodically Forced Oscillations

A special type of forcing is periodic forcing. Realistic oscillations will dampen and eventually stop if left unattended. For example, mechanical clocks are driven by compound or torsional pendula and electric oscillators are often designed with the need to continue for long periods of time. However, they are not perpetual motion machines and will need a periodic injection of energy. This can be done systematically by adding periodic forcing. Another simple example is the motion of a child on a swing in the park. This simple damped pendulum system will naturally slow down to equilibrium (stopped) if left alone. However, if the child pumps energy into the swing at the right time, or if an adult pushes the child at the right time, then the amplitude of the swing can be increased.

There are other systems, such as airplane wings and long bridge spans, in which external driving forces might cause damage to the system. A well know example is the wind induced collapse of the Tacoma Narrows Bridge due to strong winds. Of course, if one is not careful, the child in the last example might get too much energy pumped into the system causing a similar failure of the desired motion.

Note

The Tacoma Narrows Bridge opened in Washington State (U.S.) in mid 1940. However, in November of the same year the winds excited a transverse mode of vibration, which eventually (in a few hours) lead to large amplitude oscillations and then collapse.

While there are many types of forced systems, and some fairly complicated, we can easily get to the basic characteristics of forced oscillations by modifying the mass-spring system by adding an external, time-dependent, driving force. Such as system satisfies the equation

$$m\ddot{x}+b(x)+kx=F(t), \tag{12.3.8}$$

where m is the mass, b is the damping constant, k is the spring constant, and $F(t)$ is the driving force. If $F(t)$ is of simple form, then we can employ the Method of Undetermined Coefficients. Since the systems we have considered so far are similar, one could easily apply the following to pendula or circuits.

As the damping term only complicates the solution, we will consider the simpler case of undamped motion and assume that $b=0$. Furthermore, we will introduce a sinusoidal driving force, $F(t)=F_0\cos\omega t$ in order to study periodic forcing. This leads to the simple periodically driven mass on a spring system

$$m\ddot{x}+kx=F_0\cos\omega t. \tag{12.3.9}$$

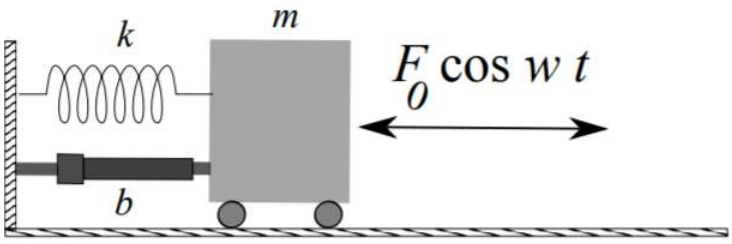


Figure 12.3.1: An external driving force is added to the spring-mass-damper system.

In order to find the general solution, we first obtain the solution to the homogeneous problem,

$$x_h=c_1\cos\omega_0 t+c_2\sin\omega_0 t$$

where $\omega_0=\sqrt{\frac{k}{m}}$. Next, we seek a particular solution to the nonhomogeneous problem. We will apply the Method of Undetermined Coefficients.

A natural guess for the particular solution would be to use $x_p=A\cos\omega t+ B\sin\omega t$. However, recall that the guess should not be a solution of the homogeneous problem. Comparing x_p with x_h, this would hold if $\omega\neq\omega_0$. Otherwise, one would need to use the Modified Method of Undetermined Coefficients as described in the last section. So, we have two cases to consider.

Note

Dividing through by the mass, we solve the simple driven system,

$$\ddot{x}+\omega_0^2x=\frac{F_0}{m}\cos\omega t.$$

Example 12.3.4

Solve $\ddot{x}+\omega_0^2x=\frac{F_0}{m}\cos\omega t$, for $\omega\neq\omega_0$.

Solution

In this case we continue with the guess $x_p=A\cos\omega t+B$ sinct. Since there is no damping term, one quickly finds that $B=0$. Inserting $x_p=A\cos\omega t$ into the differential equation, we find that

$$\left(-\omega^2+\omega_0^2\right)A\cos\omega t=\frac{F_0}{m}\cos\omega t.$$

Solving for A, we obtain

$$A=\frac{F_0}{m\left(\omega_0^2-\omega^2\right)}.$$

The general solution for this case is thus,

$$x(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{F_0}{m\left(\omega_0^2 - \omega^2\right)} \cos \omega t. \tag{12.3.10}$$

Example 12.3.5

Solve $\ddot{x} + \omega_0^2 x = \frac{F_0}{m} \cos \omega_0 t$.

Solution

In this case, we need to employ the Modified Method of Undetermined Coefficients. So, we make the guess $x_p = t(A \cos \omega_0 t + B \sin \omega_0 t)$. Since there is no damping term, one finds that $A = 0$. Inserting the guess in to the differential equation, we find that

$$B = \frac{F_0}{2m\omega_0},$$

or the general solution is

$$x(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{F_0}{2m\omega} t \sin \omega t. \tag{12.3.11}$$

The general solution to the problem is thus

$$x(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \begin{cases} \frac{F_0}{m\left(\omega_0^2 - \omega^2\right)} \cos \omega t \\ \frac{F_0}{2m\omega_0} t \sin \omega_0 t \end{cases} \omega = \omega_0, \tag{12.3.12}$$

Special cases of these solutions provide interesting physics, which can be explored by the reader in the homework. In the case that $\omega = \omega_0$, we see that the solution tends to grow as t gets large. This is what is called a resonance. Essentially, one is driving the system at its natural frequency. As the system is moving to the left, one pushes it to the left. If it is moving to the right, one is adding energy in that direction. This forces the amplitude of oscillation to continue to grow until the system breaks. An example of such an oscillation is shown in Figure 12.3.2

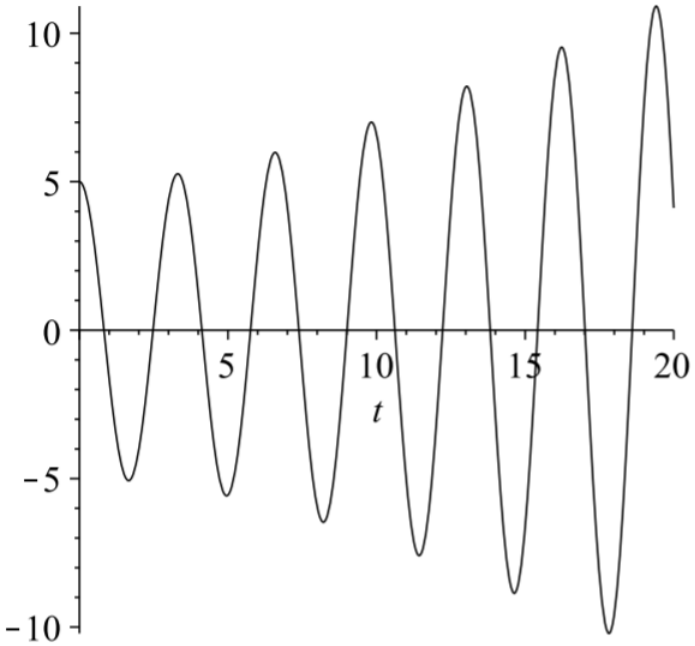


Figure 12.3.2: Plot of $x(t) = 5 \cos 2t + \frac{1}{2} t \sin 2t$,

In the case that $\omega \neq \omega_0$, one can rewrite the solution in a simple form. Let's choose the initial conditions that $c_1 = -F_0 / \left(m\left(\omega_0^2 - \omega^2\right)\right), c_2 = 0$. Then one has (see Problem ??)

$$x(t) = \frac{2F_0}{m\left(\omega_0^2 - \omega^2\right)} \sin\frac{(\omega_0 - \omega)\,t}{2} \sin\frac{(\omega_0 + \omega)\,t}{2}. \tag{12.3.13}$$

For values of ω near ω_0, one finds the solution consists of a rapid oscillation, due to the $\sin\frac{(\omega_0+\omega)t}{2}$ factor, with a slowly varying amplitude, $\frac{2F_0}{m(\omega_0^2-\omega^2)}\sin\frac{(\omega_0-\omega)t}{2}$. The reader can investigate this solution.

This slow variation is called a beat and the beat frequency is given by $f = \frac{|\omega_0-\omega|}{4\pi}$. In Figure 12.3.3 we see the high frequency oscillations are contained by the lower beat frequency, $f = \frac{0.15}{4\pi}$s. This corresponds to a period of $T = 1/f \approx 83.7$ Hz, which looks about right from the figure.

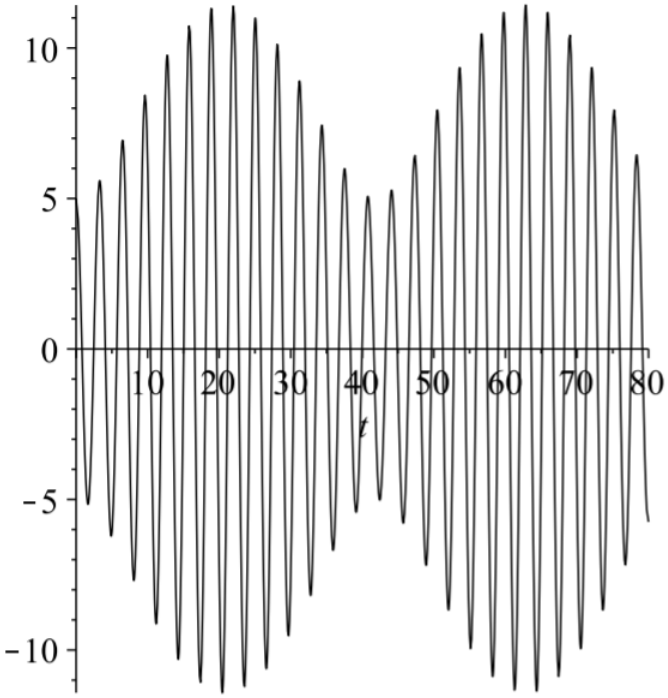


Figure 12.3.3: Plot of $x(t) = \frac{1}{249}\left(2045\cos 2t - 800\cos\frac{43}{20}t\right)$, a solution of $\ddot{x} + 4x = 2\cos 2.15t$.

Example 12.3.6

Solve $\ddot{x} + x = 2\cos\omega t, x(0) = 0, \dot{x}(0) = 0$, for $\omega = 1, 1.15$.

Solution

For each case, we need the solution of the homogeneous problem,

$$x_h(t) = c_1\cos t + c_2\sin t.$$

The particular solution depends on the value of ω.

For $\omega = 1$, the driving term, $2\cos\omega t$, is a solution of the homogeneous problem. Thus, we assume

$$x_p(t) = At\cos t + Bt\sin t.$$

Inserting this into the differential equation, we find $A = 0$ and $B = 1$. So, the general solution is

$$x(t) = c_1\cos t + c_2\sin t + t\sin t.$$

Imposing the initial conditions, we find

$$x(t) = t\sin t.$$

This solution is shown in Figure 12.3.4

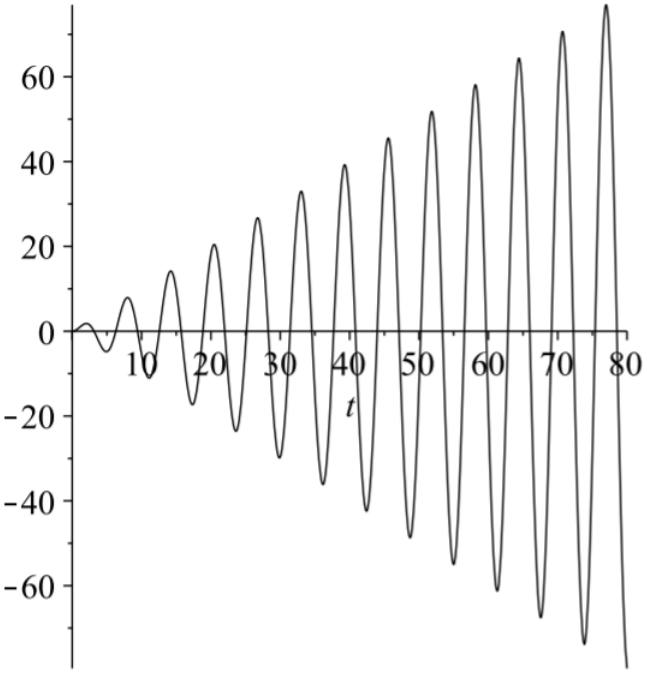


Figure 12.3.4: Plot of $x(t) = t \sin 2t$, a solution of $\ddot{x} + x = 2 \cos t$.

For $\omega = 1.15$, the driving term, $2 \cos \omega 1.15t$, is not a solution of the homogeneous problem. Thus, we assume

$$x_p(t) = A \cos 1.15t + B \sin 1.15t.$$

Inserting this into the differential equation, we find $A = -\frac{800}{129}$ and $B = 0$. So, the general solution is

$$x(t) = c_1 \cos t + c_2 \sin t - \frac{800}{129} \cos t.$$

Imposing the initial conditions, we find

$$x(t) = \frac{800}{129}(\cos t - \cos 1.15t).$$

This solution is shown in Figure 12.3.5. The beat frequency in this case is the same as with Figure 12.3.3.

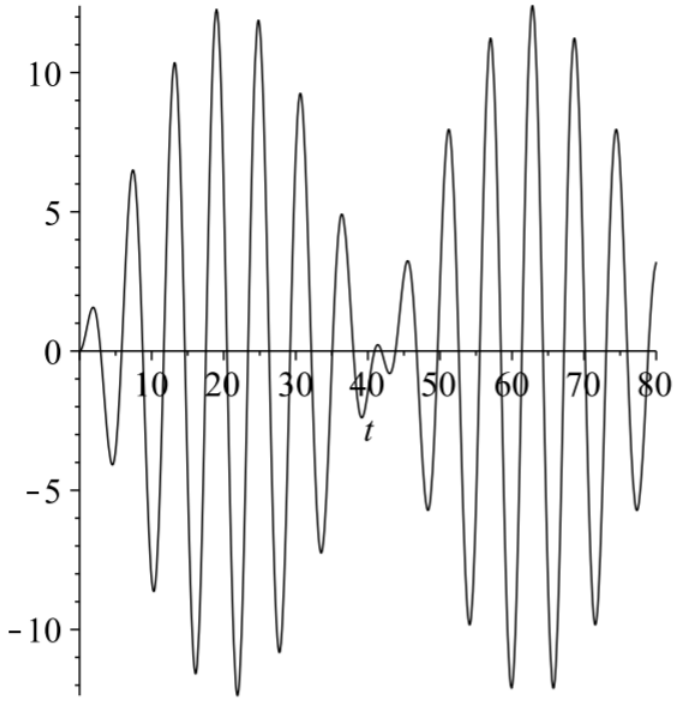


Figure 12.3.5: Plot of $x(t) = \frac{800}{129}\left(\cos t - \cos \frac{23}{20}t\right)$, a solution of $\ddot{x} + x = 2\cos 1.15t$.

Method of Variation of Parameters

A more systematic way to find particular solutions is through the use of the Method of Variation of Parameters. The derivation is a little detailed and the solution is sometimes messy, but the application of the method is straight forward if you can do the required integrals. We will first derive the needed equations and then do some examples.

We begin with the nonhomogeneous equation. Let's assume it is of the standard form

$$a(x)y''(x) + b(x)y'(x) + c(x)y(x) = f(x). \tag{12.3.14}$$

We know that the solution of the homogeneous equation can be written in terms of two linearly independent solutions, which we will call $y_1(x)$ and $y_2(x)$:

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x).$$

Replacing the constants with functions, then we no longer have a solution to the homogeneous equation. Is it possible that we could stumble across the right functions with which to replace the constants and somehow end up with $f(x)$ when inserted into the left side of the differential equation? It turns out that we can.

So, let's assume that the constants are replaced with two unknown functions, which we will call $c_1(x)$ and $c_2(x)$. This change of the parameters is where the name of the method derives. Thus, we are assuming that a particular solution takes the form

$$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x). \tag{12.3.15}$$

If this is to be a solution, then insertion into the differential equation should make the equation hold. To do this we will first need to compute some derivatives.

Note

We assume the nonhomogeneous equation has a particular solution of the form

$$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x).$$

The first derivative is given by

$$y_p'(x) = c_1(x)y_1'(x) + c_2(x)y_2'(x) + c_1'(x)y_1(x) + c_2'(x)y_2(x). \tag{12.3.16}$$

Next we will need the second derivative. But, this will yield eight terms. So, we will first make a simplifying assumption. Let's assume that the last two terms add to zero:

$$c_1'(x)y_1(x) + c_2'(x)y_2(x) = 0. \tag{12.3.17}$$

It turns out that we will get the same results in the end if we did not assume this. The important thing is that it works!

Under the assumption the first derivative simplifies to

$$y_p'(x) = c_1(x)y_1'(x) + c_2(x)y_2'(x). \tag{12.3.18}$$

The second derivative now only has four terms:

$$y_p'(x) = c_1(x)y_1''(x) + c_2(x)y_2''(x) + c_1'(x)y_1'(x) + c_2'(x)y_2'(x). \tag{12.3.19}$$

Now that we have the derivatives, we can insert the guess into the differential equation. Thus, we have

$$\begin{aligned} f(x) = a(x)&\left[c_1(x)y_1''(x) + c_2(x)y_2''(x) + c_1'(x)y_1'(x) + c_2'(x)y_2'(x)\right] \\ &+ b(x)\left[c_1(x)y_1'(x) + c_2(x)y_2'(x)\right] \\ &+ c(x)\left[c_1(x)y_1(x) + c_2(x)y_2(x)\right]. \end{aligned} \tag{12.3.20}$$

Regrouping the terms, we obtain

$$\begin{aligned} f(x) = c_1(x)&\left[a(x)y_1''(x) + b(x)y_1'(x) + c(x)y_1(x)\right] \\ &+ c_2(x)\left[a(x)y_2''(x) + b(x)y_2'(x) + c(x)y_2(x)\right] \\ &+ a(x)\left[c_1'(x)y_1'(x) + c_2'(x)y_2'(x)\right]. \end{aligned} \tag{12.3.21}$$

Note that the first two rows vanish since y_1 and y_2 are solutions of the homogeneous problem. This leaves the equation

$$f(x) = a(x)\left[c_1'(x)y_1'(x) + c_2'(x)y_2'(x)\right],$$

which can be rearranged as

$$c_1'(x)y_1'(x) + c_2'(x)y_2'(x) = \frac{f(x)}{a(x)}. \tag{12.3.22}$$

Note

In order to solve the differential equation $Ly = f$, we assume

$$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x),$$

for $Ly_{1,2} = 0$. Then, one need only solve a simple system of equations (???).

In summary, we have assumed a particular solution of the form

$$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x).$$

This is only possible if the unknown functions $c_1(x)$ and $c_2(x)$ satisfy the system of equations

\[\begin{align} &c_{1}^{\prime}(x) y_{1}(x)+c_{2}^{\prime}(x) y_{2}(x)=0\nonumber \\ &c_{1}^{\prime}(x) y_{1}^{\prime}(x)+c_{2}^{\prime}(x) y_{2}^{\prime}(x)=\frac{f(x)}{a(x)}\label{eq:23} \end{align} \]

It is standard to solve this system for the derivatives of the unknown functions and then present the integrated forms. However, one could just as easily start from this system and solve the system for each problem encountered.

Example 12.3.7

Find the general solution of the nonhomogeneous problem: $y'' - y = e^{2x}$.

Solution

The general solution to the homogeneous problem $y_h'' - y_h = 0$ is

$$y_h(x) = c_1 e^x + c_2 e^{-x}.$$

In order to use the Method of Variation of Parameters, we seek a solution of the form

$$y_p(x) = c_1(x)e^x + c_2(x)e^{-x}.$$

We find the unknown functions by solving the system in (???), which in this case becomes

$$\begin{aligned} c_1'(x)e^x + c_2'(x)e^{-x} &= 0 \\ c_1'(x)e^x - c_2'(x)e^{-x} &= e^{2x}. \end{aligned} \tag{12.3.23}$$

Adding these equations we find that

$$2c_1' e^x = e^{2x} \rightarrow c_1' = \frac{1}{2}e^x$$

Solving for $c_1(x)$ we find

$$c_1(x) = \frac{1}{2}\int e^x dx = \frac{1}{2}e^x.$$

Subtracting the equations in the system yields

$$2c_2' e^{-x} = -e^{2x} \rightarrow c_2' = -\frac{1}{2}e^{3x}$$

Thus,

$$c_2(x) = -\frac{1}{2}\int e^{3x} dx = -\frac{1}{6}e^{3x}.$$

The particular solution is found by inserting these results into y_p :

$$\begin{aligned} y_p(x) &= c_1(x)y_1(x) + c_2(x)y_2(x) \\ &= \left(\frac{1}{2}e^x\right)e^x + \left(-\frac{1}{6}e^{3x}\right)e^{-x} \\ &= \frac{1}{3}e^{2x} \end{aligned} \tag{12.3.24}$$

Thus, we have the general solution of the nonhomogeneous problem as

$$y(x) = c_1 e^x + c_2 e^{-x} + \frac{1}{3}e^{2x}.$$

Example 12.3.8

Now consider the problem: $y'' + 4y = \sin x$.

Solution

The solution to the homogeneous problem is

$$y_h(x) = c_1 \cos 2x + c_2 \sin 2x. \tag{12.3.25}$$

We now seek a particular solution of the form

$$y_h(x) = c_1(x)\cos 2x + c_2(x)\sin 2x.$$

We let $y_1(x) = \cos 2x$ and $y_2(x) = \sin 2x, a(x) = 1, f(x) = \sin x$ in system (???):

$$\begin{aligned} c_1'(x)\cos 2x + c_2'(x)\sin 2x &= 0 \\ -2c_1'(x)\sin 2x + 2c_2'(x)\cos 2x &= \sin x. \end{aligned} \tag{12.3.26}$$

Now, use your favorite method for solving a system of two equations and two unknowns. In this case, we can multiply the first equation by $2\sin 2x$ and the second equation by $\cos 2x$. Adding the resulting equations will eliminate the c_1' terms. Thus, we

have

$$c_2'(x) = \frac{1}{2}\sin x \cos 2x = \frac{1}{2}\left(2\cos^2 x - 1\right)\sin x.$$

Inserting this into the first equation of the system, we have

$$c_1'(x) = -c_2'(x)\frac{\sin 2x}{\cos 2x} = -\frac{1}{2}\sin x \sin 2x = -\sin^2 x \cos x.$$

These can easily be solved:

$$c_2(x) = \frac{1}{2}\int \left(2\cos^2 x - 1\right)\sin x dx = \frac{1}{2}\left(\cos x - \frac{2}{3}\cos^3 x\right).$$

$$c_1(x) = -\int \sin^x \cos x dx = -\frac{1}{3}\sin^3 x.$$

The final step in getting the particular solution is to insert these functions into $y_p(x)$. This gives

$$\begin{aligned} y_p(x) &= c_1(x)y_1(x) + c_2(x)y_2(x) \\ &= \left(-\frac{1}{3}\sin^3 x\right)\cos 2x + \left(\frac{1}{2}\cos x - \frac{1}{3}\cos^3 x\right)\sin x \\ &= \frac{1}{3}\sin x. \end{aligned} \tag{12.3.27}$$

So, the general solution is

$$y(x) = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{3}\sin x. \tag{12.3.28}$$

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12.4: Cauchy-Euler Equations

Another class of solvable linear differential equations that is of interest are the Cauchy-Euler type of equations, also referred to in some books as Euler's equation. These are given by

$$ax^2y''(x)+bxy'(x)+cy(x)=0. \tag{12.4.1}$$

Note that in such equations the power of x in each of the coefficients matches the order of the derivative in that term. These equations are solved in a manner similar to the constant coefficient equations.

One begins by making the guess $y(x)=x^r$. Inserting this function and its derivatives,

$$y'(x)=rx^{r-1}, \quad y''(x)=r(r-1)x^{r-2},$$

into Equation (12.4.1), we have

$$[ar(r-1)+br+c]x^r=0.$$

Since this has to be true for all x in the problem domain, we obtain the characteristic equation

$$ar(r-1)+br+c=0. \tag{12.4.2}$$

Note

The solutions of Cauchy-Euler equations can be found using the characteristic equation $\mathrm{ar}(r-1)+br+c=0$.

Just like the constant coefficient differential equation, we have a quadratic equation and the nature of the roots again leads to three classes of solutions. If there are two real, distinct roots, then the general solution takes the form $y(x)=c_1x^{r_1}+c_2x^{r_2}$.

Note

For two real, distinct roots, the general solution takes the form

$$y(x)=c_1x^{r_1}+c_2x^{r_2}.$$

Example 12.4.1

Find the general solution: $x^2y''+5xy'+12y=0$.

Solution

As with the constant coefficient equations, we begin by writing down the characteristic equation. Doing a simple computation,

$$\begin{aligned} 0 &= r(r-1)+5r+12 \\ &= r^2+4r+12 \\ &= (r+2)^2+8, \\ -8 &= (r+2)^2, \end{aligned} \tag{12.4.3}$$

one determines the roots are $r=-2\pm 2\sqrt{2}$. Therefore, the general solution is $y(x)=\left[c_1\cos(2\sqrt{2}\ln|x|)+c_2\sin(2\sqrt{2}\ln|x|)\right]x^{-2}$

Deriving the solution for Case 2 for the Cauchy-Euler equations works in the same way as the second for constant coefficient equations, but it is a bit messier. First note that for the real root, $r=r_1$, the characteristic equation has to factor as $(r-r_1)^2=0$. Expanding, we have

$$r^2-2r_1r+r_1^2=0.$$

The general characteristic equation is

$$\mathrm{ar}(r-1)+br+c=0.$$

Dividing this equation by a and rewriting, we have

$$r^2+\left(\frac{b}{a}-1\right)r+\frac{c}{a}=0.$$

Comparing equations, we find

$$\frac{b}{a}=1-2r_1, \quad \frac{c}{a}=r_1^2.$$

So, the Cauchy-Euler equation for this case can be written in the form

$$x^2y''+(1-2r_1)\,xy'+r_1^2y=0.$$

Now we seek the second linearly independent solution in the form $y_2(x)=v(x)x^{r_1}$. We first list this function and its derivatives,

$$\begin{aligned} y_2(x) &= vx^{r_1}, \\ y_2'(x) &= (xv'+r_1v)\,x^{r_1-1}, \\ y_2''(x) &= \left(x^2v''+2r_1xv'+r_1\left(r_1-1\right)v\right)x^{r_1-2}. \end{aligned} \tag{12.4.4}$$

Inserting these forms into the differential equation, we have

$$\begin{aligned} 0 &= x^2y''+(1-2r_1)\,xy'+r_1^2y \\ &= \left(xv''+v'\right)x^{r_1+1}. \end{aligned} \tag{12.4.5}$$

Thus, we need to solve the equation

$$xv''+v'=0,$$

or

$$\frac{v''}{v'}=-\frac{1}{x}$$

Integrating, we have

$$\ln|v'|=-\ln|x|+C,$$

where $A=\pm e^C$ absorbs C and the signs from the absolute values. Exponentiating, we obtain one last differential equation to solve,

$$v'=\frac{A}{x}.$$

Thus,

$$v(x)=A\ln|x|+k.$$

So, we have found that the second linearly independent equation can be written as

$$y_2(x)=x^{r_1}\ln|x|.$$

Therefore, the general solution is found as $y(x)=(c_1+c_2\ln|x|)\,x^r$.

Note

For one root, $r_1=r_2=r$, the general solution is of the form

$$y(x)=(c_1+c_2\ln|x|)\,x^r.$$

Example 12.4.2

Solve the initial value problem: $t^2y'' + 3ty' + y = 0$, with the initial conditions $y(1) = 0, y'(1) = 1$.

Solution

For this example the characteristic equation takes the form

$$r(r-1) + 3r + 1 = 0,$$

or

$$r^2 + 2r + 1 = 0.$$

There is only one real root, $r = -1$. Therefore, the general solution is

$$y(t) = (c_1 + c_2 \ln|t|)\, t^{-1}.$$

However, this problem is an initial value problem. At $t = 1$ we know the values of y and y'. Using the general solution, we first have that

$$0 = y(1) = c_1.$$

Thus, we have so far that $y(t) = c_2 \ln|t| t^{-1}$. Now, using the second condition and

$$y'(t) = c_2(1 - \ln|t|)t^{-2},$$

we have

$$1 = y(1) = c_2.$$

Therefore, the solution of the initial value problem is $y(t) = \ln|t| t^{-1}$.

We now turn to the case of complex conjugate roots, $r = \alpha \pm i\beta$. When dealing with the Cauchy-Euler equations, we have solutions of the form $y(x) = x^{\alpha+i\beta}$. The key to obtaining real solutions is to first rewrite x^y :

$$x^y = e^{\ln x^y} = e^{y \ln x}.$$

Thus, a power can be written as an exponential and the solution can be written as

$$y(x) = x^{\alpha+i\beta} = x^\alpha e^{i\beta \ln x}, \quad x > 0.$$

Recalling that

$$e^{i\beta \ln x} = \cos(\beta \ln|x|) + i \sin(\beta \ln|x|),$$

we can now find two real, linearly independent solutions, $x^\alpha \cos(\beta \ln|x|)$ and $x^\alpha \sin(\beta \ln|x|)$ following the same steps as earlier for the constant coefficient case. This gives the general solution as

$$y(x) = x^\alpha \left(c_1 \cos(\beta \ln|x|) + c_2 \sin(\beta \ln|x|)\right).$$

Note

For complex conjugate roots, $r = \alpha \pm i\beta$, the general solution takes the form $y(x) = x^\alpha \left(c_1 \cos(\beta \ln|x|) + c_2 \sin(\beta \ln|x|)\right)$.

Example 12.4.3

Solve: $x^2y'' - xy' + 5y = 0$.

Solution

The characteristic equation takes the form

$$r(r-1) - r + 5 = 0,$$

or

$$r^2-2r+5=0$$

The roots of this equation are complex, $r_{1,2}=1\pm 2$. Therefore, the general solution is $y(x)=x\left(c_1\cos(2\ln|x|)+c_2\sin(2\ln|x|)\right)$.

The three cases are summarized in the table below.

Table 12.4.1

Classification of Roots of the Characteristic Equation for Cauchy-Euler Differential Equations
1. Real, distinct roots r_1, r_2. In this case the solutions corresponding to each root are linearly independent. Therefore, the general solution is simply $y(x)=c_1x^{r_1}+c_2x^{r_2}$.
2. Real, equal roots $r_1=r_2=r$. In this case the solutions corresponding to each root are linearly dependent. To find a second linearly independent solution, one uses the Method of Reduction of Order. This gives the second solution as $x^r\ln|x|$. Therefore, the general solution is found as $y(x)=(c_1+c_2\ln|x|)\,x^r$.
3. Complex conjugate roots $r_1, r_2=\alpha\pm i\beta$. In this case the solutions corresponding to each root are linearly independent. These complex exponentials can be rewritten in terms of trigonometric functions. Namely, one has that $x^\alpha\cos(\beta\ln|x|)$ and $x^\alpha\sin(\beta\ln|x|)$ are two lin early independent solutions. Therefore, the general solution becomes $y(x)=x^\alpha\left(c_1\cos(\beta\ln|x|)+c_2\sin(\beta\ln|x|)\right)$.

Nonhomogeneous Cauchy-Euler Equations

We can also solve some nonhomogeneous Cauchy-Euler equations using the Method of Undetermined Coefficients or the Method of Variation of Parameters. We will demonstrate this with a couple of examples.

Example 12.4.4

Find the solution of $x^2y''-xy'-3y=2x^2$.

Solution

First we find the solution of the homogeneous equation. The characteristic equation is $r^2-2r-3=0$. So, the roots are $r=-1,3$ and the solution is $y_h(x)=c_1x^{-1}+c_2x^3$.

We next need a particular solution. Let's guess $y_p(x)=Ax^2$. Inserting the guess into the nonhomogeneous differential equation, we have

$$\begin{aligned}2x^2 &= x^2y''-xy'-3y=2x^2\\ &=2Ax^2-2Ax^2-3Ax^2\\ &=-3Ax^2.\end{aligned}\tag{12.4.6}$$

So, $A=-2/3$. Therefore, the general solution of the problem is

$$y(x)=c_1x^{-1}+c_2x^3-\frac{2}{3}x^2.$$

Example 12.4.5

Find the solution of $x^2y''-xy'-3y=2x^3$.

Solution

In this case the nonhomogeneous term is a solution of the homogeneous problem, which we solved in the last example. So, we will need a modification of the method. We have a problem of the form

$$ax^2y''+bxy'+cy=dx^r,$$

where r is a solution of ar $(r-1)+br+c=0$. Let's guess a solution of the form $y=Ax^r\ln x$. Then one finds that the differential equation reduces to $Ax^r(2ar-a+b)=dx^r$. [You should verify this for yourself.]

With this in mind, we can now solve the problem at hand. Let $y_p = Ax^3 \ln x$. Inserting into the equation, we obtain $4Ax^3 = 2x^3$, or $A = 1/2$. The general solution of the problem can now be written as

$$y(x) = c_1 x^{-1} + c_2 x^3 + \frac{1}{2} x^3 \ln x.$$

Example 12.4.6

Find the solution of $x^2 y'' - xy' - 3y = 2x^3$ using Variation of Parameters.

Solution

As noted in the previous examples, the solution of the homogeneous problem has two linearly independent solutions, $y_1(x) = x^{-1}$ and $y_2(x) = x^3$. Assuming a particular solution of the form $y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x)$, we need to solve the system (12.3.23):

$$\begin{aligned} c_1'(x)x^{-1} + c_2'(x)x^3 &= 0 \\ -c_1'(x)x^{-2} + 3c_2'(x)x^2 &= \frac{2x^3}{x^2} = 2x. \end{aligned} \tag{12.4.7}$$

From the first equation of the system we have $c_1'(x) = -x^4 c_2'(x)$. Substituting this into the second equation gives $c_2'(x) = \frac{1}{2x}$. So, $c_2(x) = \frac{1}{2}\ln|x|$ and, therefore, $c_1(x) = \frac{1}{8}x^4$. The particular solution is

$$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x) = \frac{1}{8}x^3 + \frac{1}{2}x^3 \ln|x|.$$

Adding this to the homogeneous solution, we obtain the same solution as in the last example using the Method of Undetermined Coefficients. However, since $\frac{1}{8}x^3$ is a solution of the homogeneous problem, it can be absorbed into the first terms, leaving

$$y(x) = c_1 x^{-1} + c_2 x^3 + \frac{1}{2} x^3 \ln x.$$

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12.5: Problems

Exercise 12.5.1

Find all of the solutions of the first order differential equations. When an initial condition is given, find the particular solution satisfying that condition.

a. $\frac{dy}{dx} = \frac{e^x}{2y}$.
b. $\frac{dy}{dt} = y^2\left(1+t^2\right), y(0) = 1$.
c. $\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{x}$.
d. $xy' = y(1-2y), \quad y(1) = 2$.
e. $y' - (\sin x)y = \sin x$.
f. $xy' - 2y = x^2, y(1) = 1$.
g. $\frac{ds}{dt} + 2s = st^2, \quad , s(0) = 1$.
h. $x' - 2x = te^{2t}$.
i. $\frac{dy}{dx} + y = \sin x, y(0) = 0$.
j. $\frac{dy}{dx} - \frac{3}{x}y = x^3, y(1) = 4$.

Exercise 12.5.2

Consider the differential equation

$$\frac{dy}{dx} = \frac{x}{y} - \frac{x}{1+y}.$$

a. Find the 1-parameter family of solutions (general solution) of this equation.
b. Find the solution of this equation satisfying the initial condition $y(0) = 1$. Is this a member of the 1-parameter family?

Exercise 12.5.3

Identify the type of differential equation. Find the general solution and plot several particular solutions. Also, find the singular solution if one exists.

a. $y = xy' + \frac{1}{y'}$.
b. $y = 2xy' + \ln y'$.
c. $y' + 2xy = 2xy^2$.
d. $y' + 2xy = y^2e^{x^2}$.

Exercise 12.5.4

Find all of the solutions of the second order differential equations. When an initial condition is given, find the particular solution satisfying that condition.

a. $y'' - 9y' + 20y = 0$.
b. $y'' - 3y' + 4y = 0, \quad y(0) = 0, \quad y'(0) = 1$.
c. $8y'' + 4y' + y = 0, \quad y(0) = 1, \quad y'(0) = 0$.
d. $x'' - x' - 6x = 0$ for $x = x(t)$.

Exercise 12.5.5

Verify that the given function is a solution and use Reduction of Order to find a second linearly independent solution.

a. $x^2y'' - 2xy' - 4y = 0, \quad y_1(x) = x^4$.
b. $xy'' - y' + 4x^3y = 0, \quad y_1(x) = \sin\left(x^2\right)$.

Exercise 12.5.6

Prove that $y_1(x) = \sinh x$ and $y_2(x) = 3\sinh x - 2\cosh x$ are linearly independent solutions of $y'' - y = 0$. Write $y_3(x) = \cosh x$ as a linear combination of y_1 and y_2.

Exercise 12.5.7

Consider the nonhomogeneous differential equation $x'' - 3x' + 2x = 6e^{3t}$.

a. Find the general solution of the homogenous equation.
b. Find a particular solution using the Method of Undetermined Coefficients by guessing $x_p(t) = Ae^{3t}$.
c. Use your answers in the previous parts to write down the general solution for this problem.

Exercise 12.5.8

Find the general solution of the given equation by the method given.

a. $y'' - 3y' + 2y = 10$. Method of Undetermined Coefficients.
b. $y'' + y' = 3x^2$. Variation of Parameters.

Exercise 12.5.9

Use the Method of Variation of Parameters to determine the general solution for the following problems.

a. $y'' + y = \tan x$.
b. $y'' - 4y' + 4y = 6xe^{2x}$.

Exercise 12.5.10

Instead of assuming that $c_1'y_1 + c_2'y_2 = 0$ in the derivation of the solution using Variation of Parameters, assume that $c_1'y_1 + c_2'y_2 = h(x)$ for an arbitrary function $h(x)$ and show that one gets the same particular solution.

Exercise 12.5.11

Find all of the solutions of the second order differential equations for $x > 0$.. When an initial condition is given, find the particular solution satisfying that condition.

a. $x^2y'' + 3xy' + 2y = 0$.
b. $x^2y'' - 3xy' + 3y = 0$.
c. $x^2y'' + 5xy' + 4y = 0$.
d. $x^2y'' - 2xy' + 3y = 0$.
e. $x^2y'' + 3xy' - 3y = x^2$.

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