MATH 3A: COLLEGE ALGEBRA - FRESNO CITY COLLEGE



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Math 3A: College Algebra - Fresno City College

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Licensing

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CHAPTER OVERVIEW

1: Solving Equations

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1.2: Solving Absolute Value Equations
1.3: Solving Quadratic Equations
1.4: Solving Polynomial Equations
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1.1: Solving Linear Equations

Solve Equations Using the General Strategy

Until now we have dealt with solving one specific form of a linear equation. It is time now to lay out one overall strategy that can be used to solve any linear equation. Some equations we solve will not require all these steps to solve, but many will.

Beginning by simplifying each side of the equation makes the remaining steps easier.

Example 1.1.1: How to Solve Linear Equations Using the General Strategy

Solve: $-6(x+3)=24$.

Solution

Step 1. Simplify each side of the equation as much as possible.	Use the Distributive Property. Notice that each side of the equation is simplified as much as possible.	$-6(x+3)=24$ $-6x-18=24$
Step 2. Collect all variable terms on one side of the equation.	Nothing to do – all x's are on the left side.	
Step 3. Collect constant terms on the other side of the equation.	To get constants only on the right, add 18 to each side. Simplify.	$-6x-18+18=24+18$ $-6x=42$
Step 4. Make the coefficient of the variable term to equal to 1.	Divide each side by -6. Simplify.	$\frac{-6x}{-6}=\frac{42}{-6}$ $x=-7$
Step 5. Check the solution.	Let $x=-7$ Simplify. Multiply.	**Check:** $-6(x+3)=24$ $-6(-7+3)\stackrel{?}{=}24$ $-6(-4)\stackrel{?}{=}24$ $24=24$ ✓

Try It 1.1.2

Solve: $5(x+3)=35$

Answer

$x=4$

Try It 1.1.3

Solve: $6(y-4)=-18$

Answer

$y=1$

GENERAL STRATEGY FOR SOLVING LINEAR EQUATIONS.

1. **Simplify each side of the equation as much as possible.**
   Use the Distributive Property to remove any parentheses.
   Combine like terms.
2. **Collect all the variable terms on one side of the equation.**
   Use the Addition or Subtraction Property of Equality.
3. **Collect all the constant terms on the other side of the equation.**
   Use the Addition or Subtraction Property of Equality.
4. **Make the coefficient of the variable term to equal to 1.**
   Use the Multiplication or Division Property of Equality.
   State the solution to the equation.
5. **Check the solution.** Substitute the solution into the original equation to make sure the result is a true statement.

Example 1.1.4

Solve: $-(y+9)=8$

Solution

	$-(y+9)=8$
Simplify each side of the equation as much as possible by distributing.	$-y-9=8$
The only y term is on the left side, so all variable terms are on the left side of the equation.	
Add 9 to both sides to get all constant terms on the right side of the equation.	$-y-9+9=8+9$
Simplify.	$-y=17$
Rewrite $-y$ as $-1y$.	$-1y=17$
Make the coefficient of the variable term to equal to 1 by dividing both sides by -1.	$\frac{-1y}{-1}=\frac{17}{-1}$
Simplify.	$y=-17$
Check:	$-(y+9)=8$
Let $y=-17$.	$-(-17+9)\stackrel{?}{=}8$
	$-(-8)\stackrel{?}{=}8$
	$8=8\checkmark$

Try It 1.1.5

Solve: $-(y+8)=-2$

Answer

$y=-6$

Try It 1.1.6

Solve: $-(z+4)=-12$

Answer

$z=8$

Example 1.1.7

Solve: $5(a-3)+5=-10$

Solution

	$5(a-3)+5=-10$
Simplify each side of the equation as much as possible.	
Distribute.	$5a-15+5=-10$
Combine like terms.	$5a-10=-10$
The only a term is on the left side, so all variable terms are on one side of the equation.	
Add 10 to both sides to get all constant terms on the other side of the equation.	$5a-10+10=-10+10$
Simplify.	$5a=0$
Make the coefficient of the variable term to equal to 11 by dividing both sides by 55.	$\frac{5a}{5}=\frac{0}{5}$
Simplify.	$a=0$
Check:	$5(a-3)+5=-10$
Let a=0.	$5(0-3)+5\stackrel{?}{=}-10$
	$5(-3)+5\stackrel{?}{=}-10$
	$-15+5\stackrel{?}{=}-10$
	$-10=-10$ ✓

Try It 1.1.8

Solve: $2(m-4)+3=-1$

Answer

$m=2$

Try It 1.1.9

Solve:$7(n-3)-8=-15$

Answer

$n=2$

Example 1.1.10

Solve: $\frac{2}{3}(6m-3)=8-m$

Solution

	$\frac{2}{3}(6m-3)=8-m$
Distribute.	$4m-2=8-m$

Add m to get the variables only to the left.	$4m+m-2=8-m+m$
Simplify.	$5m-2=8$
Add 2 to get constants only on the right.	$5m-2+2=8+2$
Simplify.	$5m=10$
Divide by 5.	$\frac{5m}{5}=\frac{10}{5}$
Simplify.	$m=2$

Check:	$\frac{2}{3}(6m-3)=8-m$
Let m=2.	$\frac{2}{3}(6\cdot 2-3)\stackrel{?}{=}8-2$
	$\frac{2}{3}(12-3)\stackrel{?}{=}6$
	$\frac{2}{3}(9)\stackrel{?}{=}6$
	$6=6\checkmark$

Try It 1.1.11

Solve: $\frac{1}{3}(6u+3)=7-u$

Answer

$u=2$

Try It 1.1.12

Solve: $\frac{2}{3}(9x-12)=8+2x$

Answer

$x=4$

Example 1.1.13

Solve: $8-2(3y+5)=0$

Solution

	$8-2(3y+5)=0$
Simplify—use the Distributive Property.	$8-6y-10=0$
Combine like terms.	$-6y-2=0$
Add 2 to both sides to collect constants on the right.	$-6y-2+2=0+2$
Simplify.	$-6y=2$
Divide both sides by $-6-6$.	$\frac{-6y}{-6}=\frac{2}{-6}$
Simplify.	$y=-\frac{1}{3}$

Check: Let y=−13.

$$8-2(3y+5)=0$$

$$8-2\left[3\left(-\frac{1}{3}\right)+5\right]=0$$

$$8-2(-1+5)\stackrel{?}{=}0$$

$$8-2(4)\stackrel{?}{=}0$$

$$8-8\stackrel{?}{=}0$$

$$0=0\checkmark$$

Try It 1.1.14

Solve: $12-3(4j+3)=-17$

Answer

$j=\frac{5}{3}$

Try It 1.1.15

Solve: $-6-8(k-2)=-10$

Answer

$k=\frac{5}{2}$

Example 1.1.16

Solve: $4(x-1)-2=5(2x+3)+6$

Solution

	$4(x-1)-2=5(2x+3)+6$
Distribute.	$4x-4-2=10x+15+6$
Combine like terms.	$4x-6=10x+21$
Subtract 4x to get the variables only on the right side since $10>4$.	$4x-4x-6=10x-4x+21$
Simplify.	$-6=6x+21$
Subtract 21 to get the constants on left.	$-6-21=6x+21-21$
Simplify.	$-27=6x$
Divide by 6.	$\frac{-27}{6}=\frac{6x}{6}$
Simplify.	$-\frac{9}{2}=x$
Check:	$4(x-1)-2=5(2x+3)+6$
Let $x=-\frac{9}{2}$.	$4\left(-\frac{9}{2}-1\right)-2\stackrel{?}{=}5\left[2\left(-\frac{9}{2}\right)+3\right]+6$
	$4\left(-\frac{11}{2}\right)-2\stackrel{?}{=}5(-9+3)+6$

	$-22-2\stackrel{?}{=}5(-6)+6$	
	$-24\stackrel{?}{=}-30+6$	
	$-24=-24$ ✓	

? Try It 1.1.17

Solve: $6(p-3)-7=5(4p+3)-12$

Answer

$p=-2$

? Try It 1.1.18

Solve: $8(q+1)-5=3(2q-4)-1$

Answer

$q=-8$

✓ Example 1.1.19

Solve: $10[3-8(2s-5)]=15(40-5s)$

Solution

	$10[3-8(2s-5)]=15(40-5s)$
Simplify from the innermost parentheses first.	$10[3-16s+40]=15(40-5s)$
Combine like terms in the brackets.	$10[43-16s]=15(40-5s)$
Distribute.	$430-160s=600-75s$
Add 160s to get the s's to the right.	$430-160s+160s=600-75s+160s$
Simplify.	$430=600+85s$
Subtract 600 to get the constants to the left.	$430-600=600+85s-600$
Simplify.	$-170=85s$
Divide.	$\frac{-170}{85}=\frac{85s}{85}$
Simplify.	$-2=s$

Check:	$10[3-8(2s-5)]=15(40-5s)$
Substitute s=−2.	$10[3-8(2(-2)-5)]\stackrel{?}{=}15(40-5(-2))$
	$10[3-8(-4-5)]\stackrel{?}{=}15(40+10)$
	$10[3-8(-9)]\stackrel{?}{=}15(50)$
	$10[3+72]\stackrel{?}{=}750$
	$10[75]\stackrel{?}{=}750$
	$750=750$ ✓

Try It 1.1.20

Solve: $6[4-2(7y-1)]=8(13-8y)$.

Answer

$y=-\frac{17}{5}$

Try It 1.1.21

Solve: $12[1-5(4z-1)]=3(24+11z)$.

Answer

$z=0$

Example 1.1.22

Solve: $0.36(100n+5)=0.6(30n+15)$.

Solution

	$0.36(100n+5)=0.6(30n+15)$
Distribute.	$36n+1.8=18n+9$
Subtract 18n to get the variables to the left.	$36n-18n+1.8=18n-18n+9$
Simplify.	$18n+1.8=9$
Subtract 1.8 to get the constants to the right.	$18n+1.8-1.8=9-1.8$
Simplify.	$18n=7.2$
Divide.	$\frac{18n}{18}=\frac{7.2}{18}$
Simplify.	$n=0.4$

Check:	$0.36(100n+5)=0.6(30n+15)$
Let n=0.4.	$0.36(100(0.4)+5)\stackrel{?}{=}0.6(30(0.4)+15)$
	$0.36(40+5)\stackrel{?}{=}0.6(12+15)$
	$0.36(45)\stackrel{?}{=}0.6(27)$
	$16.2=16.2$ ✓

Try It 1.1.23

Solve: $0.55(100n+8)=0.6(85n+14)$.

Answer

$n=1$

Try It 1.1.24

Solve: $0.15(40m-120)=0.5(60m+12)$.

Answer

$m=-1$

Classify Equations

Consider the equation we solved at the start of the last section, 7x+8=−13. The solution we found was x=−3. This means the equation 7x+8=−13 is true when we replace the variable, *x*, with the value −3. We showed this when we checked the solution x=−3 and evaluated 7x+8=−13 for x=−3.

$$\begin{aligned} 7(-3)+8 &\stackrel{?}{=} -13 \\ -21+8 &\stackrel{?}{=} -13 \\ -13 &= -13 \checkmark \end{aligned}$$

If we evaluate 7x+8 for a different value of *x*, the left side will not be −13.

The equation 7x+8=−13 is true when we replace the variable, *x*, with the value −3, but not true when we replace *x* with any other value. Whether or not the equation 7x+8=−13 is true depends on the value of the variable. Equations like this are called conditional equations.

All the equations we have solved so far are conditional equations.

CONDITIONAL EQUATION

An equation that is true for one or more values of the variable and false for all other values of the variable is a **conditional equation**.

Now let's consider the equation 2y+6=2(y+3). Do you recognize that the left side and the right side are equivalent? Let's see what happens when we solve for *y*.

	$2y+6=2(y+3)$
Distribute.	$2y+6=2y+6$
Subtract 2y to get the y's to one side.	$2y-2y+6=2y-2y+6$
Simplify—the y's are gone!	$6=6$

But 6=6 is true.

This means that the equation 2y+6=2(y+3) is true for any value of *y*. We say the solution to the equation is all of the real numbers. An equation that is true for any value of the variable like this is called an **identity**.

IDENTITY

An equation that is true for any value of the variable is called an **identity**.

The solution of an identity is all real numbers.

What happens when we solve the equation 5z=5z−1?

	$5z=5z-1$
Subtract 5z to get the constant alone on the right.	$5z-5z=5z-5z-1$
Simplify—the z's are gone!	$0\neq -1$

But $0 \neq -1$.

Solving the equation 5z=5z−1 led to the false statement 0=−1. The equation 5z=5z−1 will not be true for any value of z. It has no solution. An equation that has no solution, or that is false for all values of the variable, is called a **contradiction**.

CONTRADICTION

An equation that is false for all values of the variable is called a **contradiction**.

A contradiction has no solution.

Example 1.1.25

Classify the equation as a conditional equation, an identity, or a contradiction. Then state the solution.

$6(2n-1)+3=2n-8+5(2n+1)$

Solution

$6(2n-1)+3=2n-8+5(2n+1)$	
Distribute.	$12n-6+3=2n-8+10n+5$
Combine like terms.	$12n-3=12n-3$
Subtract 12n to get the nn's to one side.	$12n-12n-3=12n-12n-3$
Simplify.	$-3=-3$
This is a true statement.	The equation is an identity. The solution is all real numbers.

Try It 1.1.26

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution:

$4+9(3x-7)=-42x-13+23(3x-2)$

Answer

identity; all real numbers

Try It 1.1.27

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution:

$8(1-3x)+15(2x+7)=2(x+50)+4(x+3)+1$

Answer

identity; all real numbers

Example 1.1.28

Classify as a conditional equation, an identity, or a contradiction. Then state the solution.

$10+4(p-5)=0$

Solution

	$10 + 4(p - 5) = 0$
Distribute.	$10 + 4p - 20 = 0$
Combine like terms.	$4p - 10 = 0$
Add 10 to both sides.	$4p - 10 + 10 = 0 + 10$
Simplify.	$4p = 10$
Divide.	$\frac{4p}{4} = \frac{10}{4}$
Simplify.	$p = \frac{5}{2}$
The equation is true when $p = frac52$.	This is a conditional equation. The solution is $p = frac52$.

Try It 1.1.29

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution: $11(q+3)-5=19$

Answer

conditional equation; \(q = \frac{9}{11}\

Try It 1.1.30

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution: $6+14(k-8)=95$

Answer

conditional equation; $k = \frac{193}{14}$

Example 1.1.31

Classify the equation as a conditional equation, an identity, or a contradiction. Then state the solution.

$5m+3(9+3m)=2(7m-11)$

Solution

	$5m + 3(9 + 3m) = 2(7m - 11)$
Distribute.	$5m + 27 + 9m = 14m - 22$
Combine like terms.	$14m + 27 = 14m - 22$
Subtract 14m from both sides.	$14m + 27 - 14m = 14m - 22 - 14m$
Simplify.	$27 \neq -22$
But $27 \neq -22$.	The equation is a contradiction. It has no solution.

Try It 1.1.32

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution:

$12c+5(5+3c)=3(9c-4)$

Answer

contradiction; no solution

? Try It 1.1.33

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution:

$4(7d+18)=13(3d-2)-11d$

Answer

contradiction; no solution

Type of equation	What happens when you solve it?	Solution
Conditional Equation	True for one or more values of the variables and false for all other values	One or more values
Identity	**True** for any value of the variable	All real numbers
Contradiction	**False** for all values of the variable	No solution

Key Concepts

- **General Strategy for Solving Linear Equations**
  1. Simplify each side of the equation as much as possible.
     Use the Distributive Property to remove any parentheses.
     Combine like terms.
  2. Collect all the variable terms on one side of the equation.
     Use the Addition or Subtraction Property of Equality.
  3. Collect all the constant terms on the other side of the equation.
     Use the Addition or Subtraction Property of Equality.
  4. Make the coefficient of the variable term to equal to 1.
     Use the Multiplication or Division Property of Equality.
     State the solution to the equation.
  5. Check the solution.
     Substitute the solution into the original equation.

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1.2: Solving Absolute Value Equations

Learning Objectives

By the end of this section, you will be able to:

- Solve absolute value equations
- Solve absolute value inequalities with “less than”
- Solve absolute value inequalities with “greater than”
- Solve applications with absolute value

Before you get started, take this readiness quiz.

1. Evaluate: $-|7|$.
   If you missed this problem, review [link].
2. Fill in $<, >, <, >$, or $=$ for each of the following pairs of numbers.
   ⓐ $|-8|\_\_\_-|-8|$ ⓑ $12\_\_\_-|-12|$ ⓒ $|-6|\_\_\_-6$ ⓓ $-(-15)\_\_\_-|-15|$
   If you missed this problem, review [link].
3. Simplify: $14-2|8-3(4-1)|$.
   If you missed this problem, review [link].

Solve Absolute Value Equations

As we prepare to solve absolute value equations, we review our definition of **absolute value**.

ABSOLUTE VALUE

The absolute value of a number is its distance from zero on the number line.

The absolute value of a number n is written as $|n|$ and $|n| \geq 0$ for all numbers.

Absolute values are always greater than or equal to zero.

We learned that both a number and its opposite are the same distance from zero on the number line. Since they have the same distance from zero, they have the same absolute value. For example:

- -5 is 5 units away from 0, so $|-5| = 5$.
- 5 is 5 units away from 0, so $|5| = 5$.

Figure $1.2.1$ illustrates this idea.

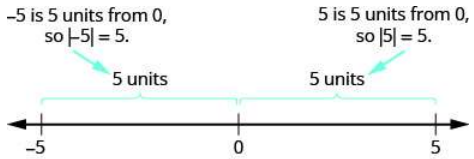


Figure 1.2.1: The numbers 5 and -5 are both five units away from zero.

For the equation |x|=5,|x|=5, we are looking for all numbers that make this a true statement. We are looking for the numbers whose distance from zero is 5. We just saw that both 5 and −5−5 are five units from zero on the number line. They are the solutions to the equation.

$$\begin{aligned} &\text{If} \quad |x| = 5 \\ &\text{then} \quad x = -5 \text{ or } x = 5 \end{aligned}$$

The solution can be simplified to a single statement by writing $x = \pm 5$. This is read, “x is equal to positive or negative 5”.

We can generalize this to the following property for absolute value equations.

ABSOLUTE VALUE EQUATIONS

For any algebraic expression, u, and any positive real number, a,

$$\begin{aligned} &\text{if} && |u| = a \\ &\text{then} && u = -a \text{ or } u = a \end{aligned}$$

Remember that an absolute value cannot be a negative number.

Example 1.2.1

Solve:

a. $|x| = 8$
b. $|y| = -6$
c. $|z| = 0$

Solution a

$$\begin{aligned} & && |x| = 8 \\ &\text{Write the equivalent equations.} && x = -8 \text{ or } x = 8 \\ & && x = \pm 8 \end{aligned}$$

Solution b

$$|y| = -6$$

No solution

Since an absolute value is always positive, there are no solutions to this equation.

Solution c

$$\begin{aligned} & && |z| = 0 \\ &\text{Write the equivalent equations.} && z = -0 \text{ or } z = 0 \\ &\text{Since } -0 = 0, && z = 0 \end{aligned}$$

Both equations tell us that z=0z=0 and so there is only one solution.

EXERCISE 1.2.2

Solve:

a. $|x| = 2$
b. $|y| = -4$
c. $|z| = 0$

Answer a

± 2

Answer b

no solution

Answer c

0

EXERCISE 1.2.3

Solve:

a. $|x| = 11$
b. $|y| = -5$

c. $|z| = 0$

Answer a

± 11

Answer b

no solution

Answer c

0

To solve an **absolute value equation**, we first isolate the absolute value expression using the same procedures we used to solve linear equations. Once we isolate the absolute value expression we rewrite it as the two equivalent equations.

How to Solve Absolute Value Equations

? Example 1.2.4

Solve $|5x - 4| - 3 = 8$.

Solution

Step 1. Isolate the absolute value expression.	Add 3 to both sides.	$\|5x-4\|-3=8$ $\|5x-4\|=11$
Step 2. Write the equivalent equations.		$5x-4=-11$ or $5x-4=11$
Step 3. Solve each equation.	Add 4 to each side. Divide each side by 5.	$5x=-7$ $\quad$ $5x=15$ $x=-\frac{7}{5}$ or $x=3$
Step 4. Check each solution.	Substitute 3 and $-\frac{7}{5}$ into the original equation.	$\|5x-4\|-3=8$ $x=3$ $\|5\cdot 3-4\|-3\stackrel{?}{=}8$ $\|15-4\|-3\stackrel{?}{=}8$ $\|11\|-3\stackrel{?}{=}8$ $11-3\stackrel{?}{=}8$ $8=8$ ✓ $\|5x-4\|-3=8$ $x=-\frac{7}{5}$ $\left\|5\left(-\frac{7}{5}\right)-4\right\|-3\stackrel{?}{=}8$ $\|-7-4\|-3\stackrel{?}{=}8$ $\|-11\|-3\stackrel{?}{=}8$ $11-3\stackrel{?}{=}8$ $8=8$ ✓

? EXERCISE 1.2.5

Solve: $|3x - 5| - 1 = 6$.

Answer

$x = 4,\ x = -\frac{2}{3}$

EXERCISE 1.2.6

Solve: $|4x - 3| - 5 = 2$.

Answer

$x = -1,\ x = \frac{5}{2}$

The steps for solving an absolute value equation are summarized here.

SOLVE ABSOLUTE VALUE EQUATIONS.

1. Isolate the absolute value expression.
2. Write the equivalent equations.
3. Solve each equation.
4. Check each solution.

Example 1.2.7

Solve $2|x - 7| + 5 = 9$.

Solution

	$2\|x - 7\| + 5 = 9$
Isolate the absolute value expression.	$2\|x - 7\| = 4$
	$\|x - 7\| = 2$
Write the equivalent equations.	$x - 7 = -2$ or $x - 7 = 2$
Solve each equation.	$x = 5$ or $x = 9$
Check: $2\|x-7\|+5=9$ $2\|5-7\|+5\stackrel{?}{=}9$ $2\|-2\|+5\stackrel{?}{=}9$ $2\cdot2+5\stackrel{?}{=}9$ $4+5\stackrel{?}{=}9$ $9=9$ ✓ $2\|x-7\|+5=9$ $2\|9-7\|+5\stackrel{?}{=}9$ $2\|2\|+5\stackrel{?}{=}9$ $2\cdot2+5\stackrel{?}{=}9$ $4+5\stackrel{?}{=}9$ $9=9$ ✓	

Exercise 1.2.8

Solve: $3|x - 4| - 4 = 8$.

Answer

$x = 8,\ x = 0$

Exercise 1.2.9

Solve: $2|x - 5| + 3 = 9$.

Answer

$x = 8,\ x = 2$

Remember, an absolute value is always positive!

Example 1.2.10

Solve: $|\frac{2}{3}x - 4| + 11 = 3$.

Solution

	$\|\frac{2}{3}x - 4\| = -8$
Isolate the absolute value term.	$\|\frac{2}{3}x - 4\| = -8$
An absolute value cannot be negative.	No solution

Exercise 1.2.11

Solve: $|\frac{3}{4}x - 5| + 9 = 4$.

Answer

No solution

Exercise 1.2.12

Solve: $|\frac{5}{6}x + 3| + 8 = 6$.

Answer

No solution

Some of our absolute value equations could be of the form $|u| = |v|$ where u and v are algebraic expressions. For example, $|x - 3| = |2x + 1|$.

How would we solve them? If two algebraic expressions are equal in absolute value, then they are either equal to each other or negatives of each other. The property for absolute value equations says that for any algebraic expression, u, and a positive real number, a, if $|u| = a$, then $u = -a$ or $u = a$.

This tell us that

if	$\|u\| = \|v\|$		
then	$\|u\| = v$	or	$\|u\| = -v$
and so	$u = v$ or $u = -v$	or	$u = -v$ or $u = -(-v)$

This leads us to the following property for equations with two absolute values.

EQUATIONS WITH TWO ABSOLUTE VALUES

For any algebraic expressions, u and v,

if	$\|u\| = \|v\|$
then	$u = -v$ or $u = v$

When we take the opposite of a quantity, we must be careful with the signs and to add parentheses where needed.

Example 1.2.13

Solve: $|5x-1|=|2x+3|$.

Solution

$$|5x-1|=|2x+3|$$

Write the equivalent equations.	$5x-1=-(2x+3)$	or	$5x-1=2x+3$
	$5x-1=-2x-3$	or	$3x-1=3$
Solve each equation.	$7x-1=-3$		$3x=4$
	$7x=-2$		$x=43$
	$x=-27$	or	$x=43$

Check.

We leave the check to you.

Exercise 1.2.14

Solve: $|7x-3|=|3x+7|$.

Answer

$x=-\frac{2}{5},\ x=\frac{5}{2}$

Exercise 1.2.15

Solve: $|6x-5|=|3x+4|$.

Answer

$x=3, x=19$

Solve Absolute Value Inequalities with "Less Than"

Let's look now at what happens when we have an **absolute value inequality**. Everything we've learned about solving inequalities still holds, but we must consider how the absolute value impacts our work. Again we will look at our definition of absolute value. The absolute value of a number is its distance from zero on the number line. For the equation $|x|=5$, we saw that both 5 and -5 are five units from zero on the number line. They are the solutions to the equation.

$$|x|=5$$
$$x=-5 \quad \text{or} \quad x=5$$

What about the inequality $|x|\leq 5$? Where are the numbers whose distance is less than or equal to 5? We know -5 and 5 are both five units from zero. All the numbers between -5 and 5 are less than five units from zero (Figure 1.2.2).

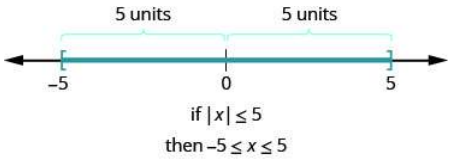


Figure 1.2.2.

In a more general way, we can see that if $|u|\leq a$, then $-a\leq u\leq a$ (Figure 1.2.3).

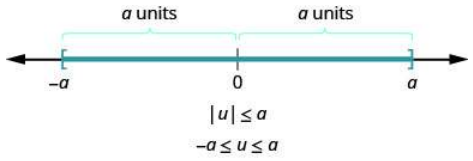


Figure 1.2.3.

This result is summarized here.

ABSOLUTE VALUE INEQUALITIES WITH $<$ OR $\leq$

For any algebraic expression, u, and any positive real number, a,

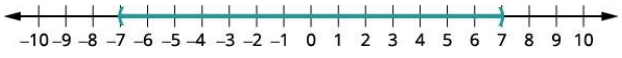
$$\text{if} \quad |u| < a, \quad \text{then} \quad -a < u < a$$
$$\text{if} \quad |u| \leq a, \quad \text{then} \quad -a \leq u \leq a$$

After solving an inequality, it is often helpful to check some points to see if the solution makes sense. The graph of the solution divides the number line into three sections. Choose a value in each section and substitute it in the original inequality to see if it makes the inequality true or not. While this is not a complete check, it often helps verify the solution.

Example 1.2.16

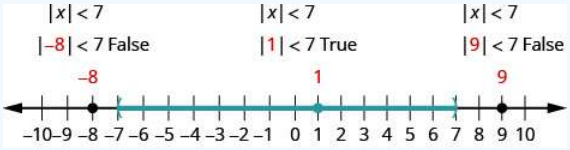
Solve $|x| < 7$. Graph the solution and write the solution in interval notation.

Solution

	$\lvert x\rvert < 7$
Write the equivalent inequality.	$-7 < x < 7$
Graph the solution.	
Write the solution using interval notation.	$(-7, 7)$

Check:

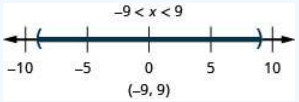
To verify, check a value in each section of the number line showing the solution. Choose numbers such as −8,−8, 1, and 9.



EXERCISE 1.2.17

Graph the solution and write the solution in interval notation: $|x| < 9$.

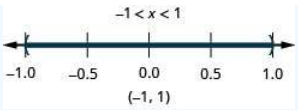
Answer



EXERCISE 1.2.18

Graph the solution and write the solution in interval notation: $|x| < 1$.

Answer



Example 1.2.19

Solve $|5x-6| \leq 4$. Graph the solution and write the solution in interval notation.

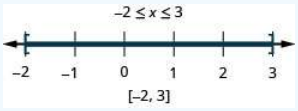
Solution

Step 1. Isolate the absolute value expression. It is isolated.	$\lvert 5x-6 \rvert \leq 4$
Step 2. Write the equivalent compound inequality.	$-4 \leq 5x-6 \leq 4$
Step 3. Solve the compound inequality.	$2 \leq 5x \leq 10$ $\frac{2}{5} \leq x \leq 2$
Step 4. Graph the solution.	
Step 5. Write the solution using interval notation.	$[\frac{2}{5}, 2]$
Check: The check is left to you.	

EXERCISE 1.2.20

Solve $|2x-1| \leq 5$. Graph the solution and write the solution in interval notation:

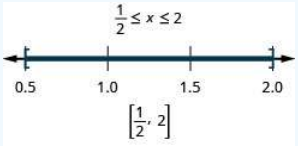
Answer



EXERCISE 1.2.21

Solve $|4x-5| \leq 3$. Graph the solution and write the solution in interval notation:

Answer



SOLVE ABSOLUTE VALUE INEQUALITIES WITH $<$ OR $\leq$

1. Isolate the absolute value expression.
2. Write the equivalent compound inequality.

$$|u| < a \quad \text{is equivalent to} \quad -a < u < a$$
$$|u| \leq a \quad \text{is equivalent to} \quad -a \leq u \leq a$$

3. Solve the compound inequality.
4. Graph the solution

5. Write the solution using interval notation.

Solve Absolute Value Inequalities with "Greater Than"

What happens for absolute value inequalities that have "greater than"? Again we will look at our definition of absolute value. The absolute value of a number is its distance from zero on the number line.

We started with the inequality $|x| \leq 5$. We saw that the numbers whose distance is less than or equal to five from zero on the number line were -5 and 5 and all the numbers between -5 and 5 (Figure 1.2.4).

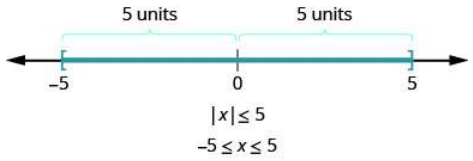


Figure 1.2.4.

Now we want to look at the inequality $|x| \geq 5$. Where are the numbers whose distance from zero is greater than or equal to five?

Again both -5 and 5 are five units from zero and so are included in the solution. Numbers whose distance from zero is greater than five units would be less than -5 and greater than 5 on the number line (Figure 1.2.5).

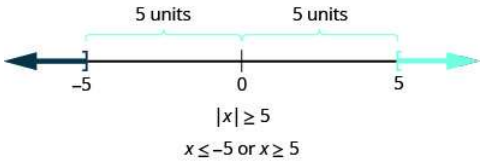


Figure 1.2.5.

In a more general way, we can see that if $|u| \geq a$, then $u \leq -a$ or $u \leq a$. See Figure.

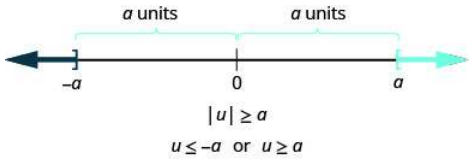


Figure 1.2.6.

This result is summarized here.

ABSOLUTE VALUE INEQUALITIES WITH $>$ OR $\geq$

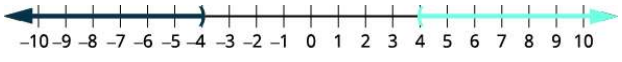
For any algebraic expression, *u*, and any positive real number, *a*,

$$\begin{aligned} &\text{if} \quad |u| > a, \quad &&\text{then } u < -a \text{ or } u > a \\ &\text{if} \quad |u| \geq a, \quad &&\text{then } u \leq -a \text{ or } u \geq a \end{aligned}$$

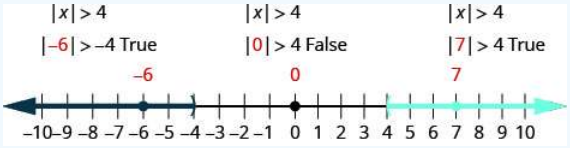
Example 1.2.22

Solve $|x| > 4$. Graph the solution and write the solution in interval notation.

Solution

	$\lvert x\rvert > 4$
Write the equivalent inequality.	$x < -4$ or $x > 4$
Graph the solution.	
Write the solution using interval notation.	$(-\inf, -4) \cup (4, \inf)$
Check:	

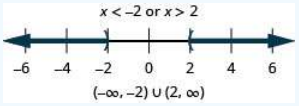
To verify, check a value in each section of the number line showing the solution. Choose numbers such as −6,−6, 0, and 7.



EXERCISE 1.2.23

Solve $|x| > 2$. Graph the solution and write the solution in interval notation.

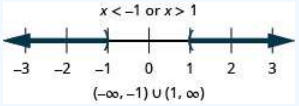
Answer



EXERCISE 1.2.24

Solve $|x| > 1$. Graph the solution and write the solution in interval notation.

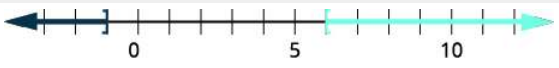
Answer



Example 1.2.25

Solve $|2x-3| \geq 5$. Graph the solution and write the solution in interval notation.

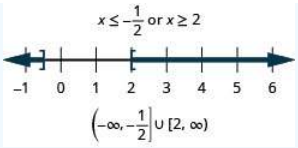
Solution

	$\lvert 2x-3\rvert \geq 5$
Step 1. Isolate the absolute value expression. It is isolated.	
Step 2. Write the equivalent compound inequality.	$2x-3 \leq -5$ or $2x-3 \geq 5$
Step 3. Solve the compound inequality.	$2x \leq -2$ or $2x \geq 8$ $x \leq -1$ or $x \geq 4$
Step 4. Graph the solution.	
Step 5. Write the solution using interval notation.	$(-\inf, -1] \cup [4, \inf)$
Check: The check is left to you.	

EXERCISE 1.2.26

Solve $|4x-3| \geq 5$. Graph the solution and write the solution in interval notation.

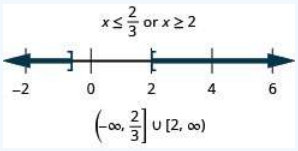
Answer



? EXERCISE 1.2.27

Solve $|3x - 4| \ge 2$. Graph the solution and write the solution in interval notation.

Answer



SOLVE ABSOLUTE VALUE INEQUALITIES WITH $>$ OR $\ge$.

1. Isolate the absolute value expression.
2. Write the equivalent compound inequality.

$$\begin{array}{llllll} |u| > a & \text{is equivalent to} & u < -a & \text{or} & u > a \\ |u| \ge a & \text{is equivalent to} & u \le -a & \text{or} & u \ge a \\ |u| > a & \text{is equivalent to} & u < -a & \text{or} & u > a \\ |u| \ge a & \text{is equivalent to} & u \le -a & \text{or} & u \ge a \end{array}$$

3. Solve the compound inequality.
4. Graph the solution
5. Write the solution using interval notation.

Solve Applications with Absolute Value

Absolute value inequalities are often used in the manufacturing process. An item must be made with near perfect specifications. Usually there is a certain *tolerance* of the difference from the specifications that is allowed. If the difference from the specifications exceeds the tolerance, the item is rejected.

$$|\text{actual-ideal}| \le \text{tolerance}$$

? Example 1.2.28

The ideal diameter of a rod needed for a machine is 60 mm. The actual diameter can vary from the ideal diameter by 0.075 mm. What range of diameters will be acceptable to the customer without causing the rod to be rejected?

Solution

	Let $x =$ the actual measurement
Use an absolute value inequality to express this situation.	$\|\text{actual-ideal}\| \le \text{tolerance}$
	$\|x - 60\| \le 0.075$
Rewrite as a compound inequality.	$-0.075 \le x - 60 \le 0.075$
Solve the inequality.	$59.925 \le x \le 60.075$
Answer the question.	The diameter of the rod can be between $59.925mm$ and $60.075mm$.

EXERCISE 1.2.29

The ideal diameter of a rod needed for a machine is 80 mm. The actual diameter can vary from the ideal diameter by 0.009 mm. What range of diameters will be acceptable to the customer without causing the rod to be rejected?

Answer

The diameter of the rod can be between 79.991 and 80.009 mm.

EXERCISE 1.2.30

The ideal diameter of a rod needed for a machine is 75 mm. The actual diameter can vary from the ideal diameter by 0.05 mm. What range of diameters will be acceptable to the customer without causing the rod to be rejected?

Answer

The diameter of the rod can be between 74.95 and 75.05 mm.

Access this online resource for additional instruction and practice with solving linear absolute value equations and inequalities.

- Solving Linear Absolute Value Equations and Inequalities

Key Concepts

- **Absolute Value**
  The absolute value of a number is its distance from 0 on the number line.
  The absolute value of a number *n* is written as $|n|$ and $|n| \geq 0$ for all numbers.
  Absolute values are always greater than or equal to zero.
- **Absolute Value Equations**
  For any algebraic expression, *u*, and any positive real number, *a*,
  if $|u| = a$
  then $u = -a$ or $u = a$
  Remember that an absolute value cannot be a negative number.
- **How to Solve Absolute Value Equations**
  1. Isolate the absolute value expression.
  2. Write the equivalent equations.
  3. Solve each equation.
  4. Check each solution.
- **Equations with Two Absolute Values**
  For any algebraic expressions, *u* and *v*,
  if $|u| = |v|$
  then $u = -v$ or $u = v$
- **Absolute Value Inequalities with** $<$ or $\leq$
  For any algebraic expression, *u*, and any positive real number, *a*,
  if $|u| = a$ then $-a < u < a$
  if $|u| \leq a$ then $-a \leq u \leq a$
- **How To Solve Absolute Value Inequalities with** $<$ or $\leq$
  1. Isolate the absolute value expression.
  2. Write the equivalent compound inequality.
     $|u| < a$ is equivalent to $-a < u < a$
     $|u| \leq a$ is equivalent to $-a \leq u \leq a$
  3. Solve the compound inequality.
  4. Graph the solution
  5. Write the solution using interval notation

- **Absolute Value Inequalities with $>$ or $\geq$**
  For any algebraic expression, u, and any positive real number, a,
  if $|u| > a$, then $u < -a$ or $u > a$
  if $|u| \geq a$, then $u \leq -a$ or $u \geq a$
- **How To Solve Absolute Value Inequalities with $>$ or $\geq$**
  1. Isolate the absolute value expression.
  2. Write the equivalent compound inequality.
     $|u| > a$ is equivalent to $u < -a$ or $u > a$
     $|u| \geq a$ is equivalent to $u \leq -a$ or $u \geq a$
  3. Solve the compound inequality.
  4. Graph the solution
  5. Write the solution using interval notation

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1.3: Solving Quadratic Equations

Solve Quadratic Equations Using the Zero Product Property

We will first solve some quadratic equations by using the Zero Product Property. The **Zero Product Property** says that if the product of two quantities is zero, it must be that at least one of the quantities is zero. The only way to get a product equal to zero is to multiply by zero itself.

ZERO PRODUCT PROPERTY

If $a \cdot b = 0,$ then either $a = 0$ or $b = 0$ or both.

We will now use the **Zero Product Property**, to solve a quadratic equation.

Exercise 1.3.1: How to Use the Zero Product Property to Solve a Quadratic Equation

Solve: $(x+1)(x-4)=0$

Answer

Step 1. Set each factor equal to zero.	The product equals zero, so at least one factor must equal zero.	$(x+1)(x-4)=0$ $x+1=0 \quad \text{or} \quad x-4=0$
Step 2. Solve the linear equations.	Solve each equation.	$x=-1 \quad \text{or} \quad x=4$
Step 3. Check.	Substitute each solution separately into the original equation.	$x=-1$ $(x+1)(x-4)=0$ $(-1+1)(-1-4) \stackrel{?}{=} 0$ $(0)(-5) \stackrel{?}{=} 0$ $0=0$ ✓ $x=4$ $(x+1)(x-4)=0$ $(4+1)(4-4) \stackrel{?}{=} 0$ $(5)(0) \stackrel{?}{=} 0$ $0=0$ ✓

Exercise 1.3.2

Solve: $(x-3)(x+5)=0$

Answer

$x=3, x=-5$

Exercise 1.3.3

Solve: $(y-6)(y+9)=0$

Answer

$y=6, y=-9$

We usually will do a little more work than we did in this last example to solve the linear equations that result from using the Zero Product Property.

Exercise 1.3.4

Solve: $(5n-2)(6n-1)=0$

Answer

	$(5n-2)(6n-1)=0$	
Use the Zero Product Property to set each factor to 0.	$5n-2=0$	$6n-1=0$
Solve the equations.	$n=\frac{2}{5}$	$n=\frac{1}{6}$
Check your answers.		

$n=\frac{2}{5}$	$n=\frac{1}{6}$
$(5n-2)(6n-1)=0$	$(5n-2)(6n-1)=0$
$\left(5\cdot\frac{2}{5}-2\right)\left(6\cdot\frac{2}{5}-1\right)\stackrel{?}{=}0$	$\left(5\cdot\frac{1}{6}-2\right)\left(6\cdot\frac{1}{6}-1\right)\stackrel{?}{=}0$
$(2-2)\left(\frac{12}{5}-\frac{5}{5}\right)\stackrel{?}{=}0$	$\left(\frac{5}{6}-\frac{12}{6}\right)(1-1)\stackrel{?}{=}0$
$(0)\left(\frac{7}{5}\right)\stackrel{?}{=}0$	$\left(-\frac{7}{6}\right)(0)\stackrel{?}{=}0$
$0=0$ ✓	$0=0$ ✓

Exercise 1.3.5

Solve: $(3m-2)(2m+1)=0$

Answer

$m=\frac{2}{3}, m=-\frac{1}{2}$

Exercise 1.3.6

Solve: $(4p+3)(4p-3)=0$

Answer

$p=-\frac{3}{4}, p=\frac{3}{4}$

Notice when we checked the solutions that each of them made just one factor equal to zero. But the product was zero for both solutions.

Exercise 1.3.7

Solve: $3p(10p+7)=0$

Answer

	$3p(10p+7)=0$	
Use the Zero Product Property to set each factor to 0.	3p=0	10p+7=0
Solve the equations.	p=0	10p=−7
		$p=-\frac{7}{10}$
Check your answers.		

$p=0$	$p=-\frac{7}{10}$
$3p(10p+7)=0$	$3p(10p+7)=0$
$3\cdot 0(10\cdot 0+7)\stackrel{?}{=}0$	$3\left(-\frac{7}{10}\right)10\left(-\frac{7}{10}\right)+7\stackrel{?}{=}0$
$0(0+7)\stackrel{?}{=}0$	$\left(-\frac{21}{10}\right)(-7+7)\stackrel{?}{=}0$
$0(7)\stackrel{?}{=}0$	$\left(-\frac{21}{10}\right)(0)\stackrel{?}{=}0$
$0=0$ ✓	$0=0$ ✓

Exercise 1.3.8

Solve: $2u(5u-1)=0$

Answer

$u=0, u=\frac{1}{5}$

Exercise 1.3.9

Solve: $w(2w+3)=0$

Answer

$w=0, w=-\frac{3}{2}$

It may appear that there is only one factor in the next example. Remember, however, that $(y-8)^2$ means $(y-8)(y-8)$.

Exercise 1.3.10

Solve: $(y-8)^2=0$

Answer

	$(y-8)^2=0$	
Rewrite the left side as a product.	(y−8)(y−8)=0	
Use the Zero Product Property and set each factor to 0.	y−8=0	y−8=0
Solve the equations.	y=8	y=8
When a solution repeats, we call it a double root.		
Check your answer.		
$y=8$ $(y-8)^2=0$ $(8-8)^2 \stackrel{?}{=} 0$ $(0)^2 \stackrel{?}{=} 0$ $0=0$ ✓		

Exercise 1.3.11

Solve: $(x+1)^2=0$

Answer

$x=1$

Exercise 1.3.12

Solve: $(v-2)^2=0$

Answer

$v=2$

Solve Quadratic Equations by Factoring

Each of the equations we have solved in this section so far had one side in factored form. In order to use the Zero Product Property, the quadratic equation must be factored, with zero on one side. So we be sure to start with the quadratic equation in standard form, $ax^2+bx+c=0$. Then we factor the expression on the left.

Exercise 1.3.13

Solve: $x^2+2x-8=0$

Answer

Step 1. Write the quadratic equation in standard form, $ax^2+bx+c=0$.	The equation is already in standard form.	$x^2+2x-8=0$
Step 2. Factor the quadratic expression.	Factor x^2+2x-8 $(x+4)(x-2)$	$(x+4)(x-2)=0$

Step 3. Use the Zero Product Property.	Set each factor equal to zero.	$x+4=0$ or $x-2=0$
Step 4. Solve the linear equations.	We have two linear equations.	$x=-4$ or $x=2$
Step 5. Check.	Substitute each solution separately into the original equation.	$x^2+2x-8=0$ $x=-4$ $(-4)^2-2(-4)-8\stackrel{?}{=}0$ $16+(-8)-8\stackrel{?}{=}0$ $0=0$ ✓ $x^2+2x-8=0$ $x=2$ $2^2-2(2)-8\stackrel{?}{=}0$ $4+4-8\stackrel{?}{=}0$ $0=0$ ✓

? Exercise 1.3.14

Solve: $x^2-x-12=0$

Answer

$x=4, x=-3$

? Exercise 1.3.15

Solve: $b^2+9b+14=0$

Answer

$b=-2, b=-7$

SOLVE A QUADRATIC EQUATION BY FACTORING.

1. Write the quadratic equation in standard form, $ax^2+bx+c=0$.
2. Factor the quadratic expression.
3. Use the Zero Product Property.
4. Solve the linear equations.
5. Check.

Before we factor, we must make sure the quadratic equation is in standard form.

? Exercise 1.3.16

Solve: $2y^2=13y+45$

Answer

	$2y^2=13y+45$	
Write the quadratic equation in standard form.	$2y^2-13y-45=0$	
Factor the quadratic expression.	$(2y+5)(y-9)=0$	
Use the Zero Product Property to set each factor to 0.	$2y+5=0$	$y-9=0$
Solve each equation.	$y=-\frac{5}{2}$	$y=9$
Check your answers.		

$y=-\frac{5}{2}$	$y=9$
$2y^2=13y+45$	$2y^2=13y+45$
$2\left(-\frac{5}{2}\right)^2 \stackrel{?}{=} 13\left(-\frac{5}{2}\right)+45$	$2(9)^2 \stackrel{?}{=} 13(9)+45$
$2\left(\frac{25}{4}\right) \stackrel{?}{=} \left(-\frac{65}{2}\right)+\frac{90}{2}$	$2(81) \stackrel{?}{=} 117+45$
$\frac{25}{2}=\frac{25}{2}$ ✓	$162=162$ ✓

Exercise 1.3.17

Solve: $3c^2 = 10c - 8$

Answer

$c = 0, c = \frac{4}{3}$

Exercise 1.3.18

Solve: $2d^2 - 5d = 3$

Answer

$d = 3, d = -\frac{1}{2}$

Exercise 1.3.19

Solve: $5x^2 - 13x = 7x$

Answer

	$5x^2 - 13x = 7x$	
Write the quadratic equation in standard form.	$5x^2 - 20x = 0$	
Factor the left side of the equation.	$5x(x-4) = 0$	
Use the Zero Product Property to set each factor to 0.	$5x = 0$	$x - 4 = 0$
Solve each equation.	$x = 0$	$x = 4$
Check your answers.		

$x=0$	$x=4$
$5x^2-13x=7x$	$5x^2-13x=7x$
$5(0)^2-13(0) \stackrel{?}{=} 7(0)$	$5(4)^2-13(4) \stackrel{?}{=} 7(4)$
$0-0 \stackrel{?}{=} 0$	$5(16)-52 \stackrel{?}{=} 28$
$0=0$ ✓	$28=28$ ✓

Exercise 1.3.20

Solve: $6a^2 + 9a = 3a$

Answer

$a = 0, a = -1$

Exercise 1.3.21

Solve: $45b^2 - 2b = -17b$

Answer

$b = 0, b = -\frac{1}{3}$

Solving quadratic equations by factoring will make use of all the factoring techniques you have learned in this chapter! Do you recognize the special product pattern in the next example?

Exercise 1.3.22

Solve: $144q^2 = 25$

Answer

	$144q^2$	$=$	25			
Write the quadratic equation in standard form.	$144q^2-25$	$=$	0			
Factor. It is a difference of squares.	$(12q-5)(12q+5)$	$=$	0			
Use the Zero Product Property to set each factor to 0.	$12q-5$	$=$	0	$12q+5$	$=$	0
Solve each equation.	$12q$	$=$	5	$12q$	$=$	-5
	q	$=$	$\frac{5}{12}$	q	$=$	$-\frac{5}{12}$
Check your answers.						

Exercise 1.3.23

Solve: $25p^2 = 49$

Answer

$p = \frac{7}{5}, p = -\frac{7}{5}$

Exercise 1.3.24

Solve: $36x^2 = 121$

Answer

$x = \frac{11}{6}, x = -\frac{11}{6}$

The left side in the next example is factored, but the right side is not zero. In order to use the Zero Product Property, one side of the equation must be zero. We'll multiply the factors and then write the equation in standard form.

Exercise 1.3.25

Solve: $(3x-8)(x-1) = 3x$

Answer

	$(3x-8)(x-1) = 3x$
Multiply the binomials.	$3x^2-11x+8 = 3x$
Write the quadratic equation in standard form.	$3x^2-14x+8 = 0$
Factor the trinomial.	$(3x-2)(x-4) = 0$
Use the Zero Product Property to set each factor to 0.	$3x-2 = 0 \quad x-4 = 0$
Solve each equation.	$3x = 2 \quad x = 4$
	$x = \frac{2}{3}$
Check your answers.	The check is left to you!

Exercise 1.3.26

Solve: $(2m+1)(m+3) = 12m$

Answer

$m = 1, m = \frac{3}{2}$

Exercise 1.3.27

Solve: $(k+1)(k-1) = 8$

Answer

$k = 3, k = -3$

The Zero Product Property also applies to the product of three or more factors. If the product is zero, at least one of the factors must be zero. We can solve some equations of degree more than two by using the Zero Product Property, just like we solved quadratic equations.

Exercise 1.3.28

Solve: $9m^3+100m=60m^2$

Answer

	$9m^3+100m$	$=$	$60m^2$					
Bring all the terms to one side so that the other side is zero.	$9m^3-60m^2+100m$	$=$	0					
Factor the greatest common factor first.	$m\left(9m^2-60m+100\right)$	$=$	0					
Factor the trinomial.	$m(3m-10)(3m-10)$	$=$	0					
Use the Zero Product Property to set each factor to 0.	m	$=$	0	$3m-10$	$=$	0	$3m-10$	$=$ 0
Solve each equation.	m	$=$	0	m	$=$	$\frac{10}{3}$	m	$=$ $\frac{10}{3}$
Check your answers.	The check is left to you.							

Exercise 1.3.29

Solve: $8x^3=24x^2-18x$

Answer

$x=0, x=\frac{3}{2}$

Exercise 1.3.30

Solve: $16y^2=32y^3+2y$

Answer

$y=0, y=\frac{1}{4}$

When we factor the quadratic equation in the next example we will get three factors. However the first factor is a constant. We know that factor cannot equal 0.

Exercise 1.3.31

Solve: $4x^2=16x+84$

Answer

	$4x^2$	$=$	$16x+84$					
Write the quadratic equation in standard form.	$4x^2-16x-84$	$=$	0					
Factor the greatest common factor first.	$4\left(x^2-4x-21\right)$	$=$	0					
Factor the trinomial.	$4(x-7)(x+3)$	$=$	0					
Use the Zero Product Property to set each factor to 0.	4	$\neq$	0	$x-7$	$=$	0	$x+3$	$=$ 0
Solve each equation.	4	$\neq$	0	x	$=$	7	x	$=$ -3
Check your answers.	The check is left to you.							

Exercise 1.3.32

Solve: $18a^2-30=-33a$

Answer

$a=-\frac{5}{2}, a=\frac{2}{3}$

Exercise 1.3.33

Solve: $123b=-6-60b^2$

Answer

$b=2, b=\frac{1}{20}$

Solve Quadratic Equations Using the Quadratic Formula

When we solved quadratic equations in the last section by completing the square, we took the same steps every time. By the end of the exercise set, you may have been wondering 'isn't there an easier way to do this?' The answer is 'yes'. Mathematicians look for patterns when they do things over and over in order to make their work easier. In this section we will derive and use a formula to find the solution of a quadratic equation.

We have already seen how to solve a formula for a specific variable 'in general', so that we would do the algebraic steps only once, and then use the new formula to find the value of the specific variable. Now we will go through the steps of completing the square using the general form of a quadratic equation to solve a quadratic equation for x.

We start with the standard form of a quadratic equation and solve it for x by completing the square.

	$ax^2+bx+c=0, \quad a\neq 0$
Isolate the variable terms on one side.	$ax^2+bx \quad =-c$
Make the coefficient of x^2 equal to 1, by dividing by a.	$\frac{ax^2}{a}+\frac{b}{a}x \quad =-\frac{c}{a}$
Simplify.	$x^2+\frac{b}{a}x \quad =-\frac{c}{a}$
To complete the square, find $\left(\frac{1}{2}\cdot\frac{b}{a}\right)^2$ and add it to both sides of the equation.	
$\left(\frac{1}{2}\frac{b}{a}\right)^2=\frac{b^2}{4a^2}$	$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=-\frac{c}{a}+\frac{b^2}{4a^2}$
The left side is a perfect square, factor it.	$\left(x+\frac{b}{2a}\right)^2=-\frac{c}{a}+\frac{b^2}{4a^2}$
Find the common denominator of the right side and write equivalent fractions with the common denominator.	$\left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c\cdot 4a}{a\cdot 4a}$
Simplify.	$\left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{4ac}{4a^2}$
Combine to one fraction.	$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$
Use the square root property.	$x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$
Simplify the radical.	$x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$
Add $-\frac{b}{2a}$ to both sides of the equation.	$x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$
Combine the terms on the right side.	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

The final equation is called the "Quadratic Formula."

Definition 1.3.1: Quadratic Formula

The solutions to a **quadratic equation** of the form $ax^2+bx+c=0$, where $a\neq 0$ are given by the formula:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \tag{1.3.1}$$

To use the **Quadratic Formula**, we substitute the values of a, b, and c from the standard form into the expression on the right side of the formula. Then we simplify the expression. The result is the pair of solutions to the quadratic equation.

Notice the Quadratic Formula (Equation 1.3.1) is an equation. Make sure you use both sides of the equation.

Example 1.3.1 How to Solve a Quadratic Equation Using the Quadratic Formula

Solve by using the Quadratic Formula: $2x^2+9x-5=0$.

Solution:

Step 1: Write the quadratic equation in standard form. Identify the a, b, c values.	This equation is in standard form.	$ax^2+bx+c=0$ $2x^2+9x-5=0$ $a=2, b=9, c=-5$

Step 2: Write the quadratic formula. Then substitute in the values of a, b, c.	Substitute in $a=2, b=9, c=-5$	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ $x=\frac{-9\pm\sqrt{9^2-4\cdot 2\cdot(-5)}}{2\cdot 2}$
Step 3: Simplify the fraction, and solve for x.		$x=\frac{-9\pm\sqrt{81-(-40)}}{4}$ $x=\frac{-9\pm\sqrt{121}}{4}$ $x=\frac{-9\pm 11}{4}$ $x=\frac{-9+11}{4} \quad x=\frac{-9-11}{4}$ $x=\frac{2}{4} \quad x=\frac{-20}{4}$ $x=\frac{1}{2} \quad x=-5$
Step 4: Check the solutions.	Put each answer in the original equation to check. Substitute $x=\frac{1}{2}$ and $x=-5$.	$2x^2+9x-5=0$ $2\left(\frac{1}{2}\right)^2+9\cdot\frac{1}{2}-5\stackrel{?}{=}0$ $2\cdot\frac{1}{4}+0\cdot\frac{1}{2}-5\stackrel{?}{=}0$ $2\cdot\frac{1}{4}+9\cdot\frac{1}{2}-5\stackrel{?}{=}0$ $\frac{1}{2}+\frac{9}{2}-5\stackrel{?}{=}0$ $\frac{10}{2}-5\stackrel{?}{=}0$ $5-5\stackrel{?}{=}0$ $0=0$ $2x^2+9x-5=0$ $2(-5)^2+9(-5)-5\stackrel{?}{=}0$ $2\cdot 25-45-5\stackrel{?}{=}0$ $50-45-5\stackrel{?}{=}0$ $0=0$

? Exercise 1.3.1

Solve by using the Quadratic Formula: $3y^2-5y+2=0$.

Answer

$y=1, y=\frac{2}{3}$

? Exercise 1.3.2

Solve by using the Quadratic Formula: $4z^2+2z-6=0$.

Answer

$z=1, z=-\frac{3}{2}$

HowTo: Solve a Quadratic Equation Using the Quadratic Formula

1. Write the quadratic equation in standard form, $ax^2+bx+c=0$. Identify the values of a, b, and c.
2. Write the Quadratic Formula. Then substitute in the values of a, b, and c.
3. Simplify.
4. Check the solutions.

If you say the formula as you write it in each problem, you'll have it memorized in no time! And remember, the Quadratic Formula is an EQUATION. Be sure you start with "$x=$".

Example 1.3.2

Solve by using the Quadratic Formula: $x^2 - 6x = -5$.

Solution:

	$x^2 - 6x = -5$
Write the equation in standard form by adding 5 to each side.	$x^2 - 6x + 5 = 0$
This equation is now in standard form.	$ax^2 + bx + c = 0$ $x^2 - 6x + 5 = 0$
Identify the values of a, b, c.	$a = 1, b = -6, c = 5$
Write the Quadratic Formula.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of a, b, c.	$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \cdot 1 \cdot (5)}}{2 \cdot 1}$
Simplify.	$x = \frac{6 \pm \sqrt{36 - 20}}{2}$ $x = \frac{6 \pm \sqrt{16}}{2}$ $x = \frac{6 \pm 4}{2}$
Rewrite to show two solutions.	$x = \frac{6+4}{2}, \quad x = \frac{6-4}{2}$
Simplify.	$x = \frac{10}{2}, \quad x = \frac{2}{2}$
	$x = 5, \quad x = 1$
Check: $x^2 - 6x + 5 = 0$ $5^2 - 6 \cdot 5 + 5 \stackrel{?}{=} 0$ $25 - 30 + 5 \stackrel{?}{=} 0$ $0 = 0$ ✓ $x^2 - 6x + 5 = 0$ $1^2 - 6 \cdot 1 + 5 \stackrel{?}{=} 0$ $1 - 6 + 5 \stackrel{?}{=} 0$ $0 = 0$ ✓	

Exercise 1.3.3

Solve by using the Quadratic Formula: $a^2 - 2a = 15$.

Answer

$a = -3, a = 5$

Exercise 1.3.4

Solve by using the Quadratic Formula: $b^2 + 24 = -10b$.

Answer

$b = -6, b = -4$

When we solved quadratic equations by using the Square Root Property, we sometimes got answers that had radicals. That can happen, too, when using the **Quadratic Formula**. If we get a **radical** as a solution, the final answer must have the radical in its simplified form.

Example 1.3.3

Solve by using the Quadratic Formula: $2x^2 + 10x + 11 = 0$.

Solution:

	$2x^2 + 10x + 11 = 0$
This equation is in standard form.	$ax^2 + bx + c = 0$ $2x^2 + 10x + 11 = 0$
Identify the values of a, b and c.	$a = 2, b = 10, c = 11$
Write the Quadratic Formula.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of a, b, and c.	$x = \frac{-(10) \pm \sqrt{(10)^2 - 4 \cdot 2 \cdot (11)}}{2 \cdot 2}$

Simplify.	$x=\frac{-10\pm\sqrt{100-88}}{4}$
	$x=\frac{-10\pm\sqrt{12}}{4}$
Simplify the radical.	$x=\frac{-10\pm2\sqrt{3}}{4}$
Factor out the common factor in the numerator.	$x=\frac{2(-5\pm\sqrt{3})}{4}$
Remove the common factors.	$x=\frac{-5\pm\sqrt{3}}{2}$
Rewrite to show two solutions.	$x=\frac{-5+\sqrt{3}}{2}, \quad x=\frac{-5-\sqrt{3}}{2}$
Check: We leave the check for you!	

? Exercise 1.3.5

Solve by using the Quadratic Formula: $3m^2+12m+7=0$.

Answer

$m=\frac{-6+\sqrt{15}}{3}, m=\frac{-6-\sqrt{15}}{3}$

? Exercise 1.3.6

Solve by using the Quadratic Formula: $5n^2+4n-4=0$.

Answer

$n=\frac{-2+2\sqrt{6}}{5}, n=\frac{-2-2\sqrt{6}}{5}$

When we substitute a, b, and c into the Quadratic Formula and the **radicand** is negative, the quadratic equation will have imaginary or complex solutions. We will see this in the next example.

✓ Example 1.3.4

Solve by using the Quadratic Formula: $3p^2+2p+9=0$.

Solution:

Table 9.3.5

	$3p^2+2p+9=0$
This equation is in standard form.	$\underset{3p^2+2p+9=0}{ax^2+bx+c=0}$
Identify the values of a, b, c.	$a=3, b=2, c=9$
Write the Quadratic Formula.	$p=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
Then substitute in the values of a, b, c.	$p=\frac{-(2)\pm\sqrt{(2)^2-4\cdot3\cdot(9)}}{2\cdot3}$
Simplify.	$p=\frac{-2\pm\sqrt{4-108}}{6}$
	$p=\frac{-2\pm\sqrt{-104}}{6}$
Simplify the radical using complex numbers.	$p=\frac{-2\pm\sqrt{104}\,i}{6}$
Simplify the radical.	$p=\frac{-2\pm2\sqrt{26}\,i}{6}$
Factor the common factor in the numerator.	$p=\frac{2(-1\pm\sqrt{26}\,i)}{6}$

Remove the common factors.	$p=\frac{-1\pm\sqrt{26}\,i}{3}$
Rewrite in standard $a+bi$ form.	$p=-\frac{1}{3}\pm\frac{\sqrt{26}\,i}{3}$
Write as two solutions.	$p=-\frac{1}{3}+\frac{\sqrt{26}\,i}{3},\quad p=-\frac{1}{3}-\frac{\sqrt{26}\,i}{3}$

Exercise 1.3.7

Solve by using the Quadratic Formula: $4a^2-2a+8=0$.

Answer

$$a=\frac{1}{4}+\frac{\sqrt{31}}{4}i,\quad a=\frac{1}{4}-\frac{\sqrt{31}}{4}i$$

Exercise 1.3.8

Solve by using the Quadratic Formula: $5b^2+2b+4=0$.

Answer

$$b=-\frac{1}{5}+\frac{\sqrt{19}}{5}i,\quad b=-\frac{1}{5}-\frac{\sqrt{19}}{5}i$$

Remember, to use the Quadratic Formula, the equation must be written in standard form, $ax^2+bx+c=0$. Sometimes, we will need to do some algebra to get the equation into standard form before we can use the Quadratic Formula.

Example 1.3.5

Solve by using the Quadratic Formula: $x(x+6)+4=0$.

Solution:

Our first step is to get the equation in standard form.

Table 9.3.6

	$x(x+6)+4=0$
Distribute to get the equation in standard form.	$x^2+6x+4=0$
This equation is now in standard form.	$ax^2+bx+c=0$ $x^2+6x+4=0$
Identify the values of a, b, c.	$a=1,\ b=6,\ c=4$
Write the Quadratic Formula.	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
Then substitute in the values of a, b, c.	$x=\frac{-(6)\pm\sqrt{(6)^2-4\cdot 1\cdot(4)}}{2\cdot 1}$
Simplify.	$x=\frac{-6\pm\sqrt{36-16}}{2}$
	$x=\frac{-6\pm\sqrt{20}}{2}$
Simplify the radical.	$x=\frac{-6\pm 2\sqrt{5}}{2}$
Factor the common factor in the numerator.	$x=\frac{2(-3\pm 2\sqrt{5})}{2}$
Remove the common factors.	$x=-3\pm 2\sqrt{5}$
Write as two solutions.	$x=-3+2\sqrt{5},\ \ x=-3-2\sqrt{5}$
Check: We leave the check for you!	

Exercise 1.3.9

Solve by using the Quadratic Formula: $x(x+2)-5=0$.

Answer

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$x = -1 + \sqrt{6}, x = -1 - \sqrt{6}$

Exercise 1.3.10

Solve by using the Quadratic Formula: $3y(y-2) - 3 = 0$.

Answer

$y = 1 + \sqrt{2}, y = 1 - \sqrt{2}$

When we solved linear equations, if an equation had too many fractions we cleared the fractions by multiplying both sides of the equation by the LCD. This gave us an equivalent equation—without fractions— to solve. We can use the same strategy with quadratic equations.

Example 1.3.6

Solve by using the Quadratic Formula: $\frac{1}{2}u^2 + \frac{2}{3}u = \frac{1}{3}$.

Solution:

Our first step is to clear the fractions.

Table 9.3.7

	$\frac{1}{2}u^2 + \frac{2}{3}u = \frac{1}{3}$
Multiply both sides by the LCD, 6, to clear the fractions.	$6\left(\frac{1}{2}u^2 + \frac{2}{3}u\right) = 6\left(\frac{1}{3}\right)$
Multiply.	$3u^2 + 4u = 2$
Subtract 2 to get the equation in standard form.	$3u^2 + 4u - 2 = 0$
Identify the values of a, b, and c.	$a = 3, b = 4, c = -2$
Write the Quadratic Formula.	$u = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of a, b, and c.	$u = \frac{-(4) \pm \sqrt{(4)^2 - 4 \cdot 3 \cdot (-2)}}{2 \cdot 3}$
Simplify.	$u = \frac{-4 \pm \sqrt{16 + 24}}{6}$
	$u = \frac{-4 \pm \sqrt{40}}{6}$
Simplify the radical.	$u = \frac{-4 \pm 2\sqrt{10}}{6}$
Factor the common factor in the numerator.	$u = \frac{2(-2 \pm \sqrt{10})}{6}$
Remove the common factors.	$u = \frac{-2 \pm \sqrt{10}}{3}$
Rewrite to show two solutions.	$u = \frac{-2 + \sqrt{10}}{3}, \quad u = \frac{-2 - \sqrt{10}}{3}$
Check: We leave the check for you!	

Exercise 1.3.11

Solve by using the Quadratic Formula: $\frac{1}{4}c^2 - \frac{1}{3}c = \frac{1}{12}$.

Answer

$c = \frac{2 + \sqrt{7}}{3}, \quad c = \frac{2 - \sqrt{7}}{3}$

Exercise 1.3.12

Solve by using the Quadratic Formula: $\frac{1}{9}d^2 - \frac{1}{2}d = -\frac{1}{3}$.

Answer

$$d = \frac{9+\sqrt{33}}{4}, d = \frac{9-\sqrt{33}}{4}$$

Think about the equation $(x-3)^2 = 0$. We know from the **Zero Product Property** that this equation has only one solution, $x = 3$.

We will see in the next example how using the **Quadratic Formula** to solve an equation whose standard form is a perfect square **trinomial** equal to 0 gives just one solution. Notice that once the radicand is simplified it becomes 0, which leads to only one solution.

Example 1.3.7

Solve by using the Quadratic Formula: $4x^2 - 20x = -25$.

Solution:

Table 9.3.8

	$4x^2 - 20x = -25$
Add 25 to get the equation in standard form.	$\underset{ax^2 + bx + c = 0}{4x^2 - 20x + 25 = 0}$
Identify the values of a, b, and c.	$a = 4,\ b = -20,\ c = 25$
Write the quadratic formula.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of a, b, and c.	$x = \frac{-(-20) \pm \sqrt{(-20)^2 - 4 \cdot 4 \cdot (25)}}{2 \cdot 4}$
Simplify.	$x = \frac{20 \pm \sqrt{400 - 400}}{8}$
	$x = \frac{20 \pm \sqrt{0}}{8}$
Simplify the radical.	$x = \frac{20}{8}$
Simplify the fraction.	$x = \frac{5}{2}$
Check: We leave the check for you!	

Did you recognize that $4x^2 - 20x + 25$ is a perfect square trinomial. It is equivalent to $(2x-5)^2$? If you solve $4x^2 - 20x + 25 = 0$ by factoring and then using the Square Root Property, do you get the same result?

Exercise 1.3.13

Solve by using the Quadratic Formula: $r^2 + 10r + 25 = 0$.

Answer

$r = -5$

Exercise 1.3.14

Solve by using the Quadratic Formula: $25t^2 - 40t = -16$.

Answer

$t = \frac{4}{5}$

Use the Discriminant to Predict the Number and Type of Solutions of a Quadratic Equation

When we solved the quadratic equations in the previous examples, sometimes we got two real solutions, one real solution, and sometimes two complex solutions. Is there a way to predict the number and type of solutions to a quadratic equation without actually solving the equation?

Yes, the expression under the radical of the Quadratic Formula makes it easy for us to determine the number and type of solutions. This expression is called the **discriminant**.

Definition 1.3.2

Discriminant

In the Quadratic Formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$,
the quantity $b^2 - 4ac$ is called the discriminant.

Figure 9.3.85

Let's look at the discriminant of the equations in some of the examples and the number and type of solutions to those quadratic equations.

Table 9.3.9

Quadratic Equation (in standard form)	Discriminat $b^2 - 4ac$	Value of the Discriminant	Number and Type of Solutions
$2x^2 + 9x - 5 = 0$	$9^2 - 4 \cdot 2(-5)$ 121	$+$	2 real
$4x^2 - 20x + 25 = 0$	$(-20)^2 - 4 \cdot 4 \cdot 25$ 0	0	1 real
$3p^2 + 2p + 9 = 0$	$2^2 - 4 \cdot 3 \cdot 9$ -104	$-$	2 complex

When the discriminant is positive, the quadratic equation has **2 real solutions**. $x = \frac{-b \pm \sqrt{+}}{2a}$

When the discriminant is zero, the quadratic equation has **1 real solution**. $x = \frac{-b \pm \sqrt{0}}{2a}$

When the discriminant is negative, the quadratic equation has **2 complex solutions**. $x = \frac{-b \pm \sqrt{-}}{2a}$

Figure 9.3.86

Using the Discriminant $b^2 - 4ac$, to Determine the Number and Type of Solutions of a Quadratic Equation

For a quadratic equation of the form $ax^2 + bx + c = 0$, $a \neq 0$,

- If $b^2 - 4ac > 0$, the equation has 2 real solutions.
- if $b^2 - 4ac = 0$, the equation has 1 real solution.
- if $b^2 - 4ac < 0$, the equation has 2 complex solutions.

Example 1.3.8

Determine the number of solutions to each quadratic equation.

a. $3x^2 + 7x - 9 = 0$
b. $5n^2 + n + 4 = 0$
c. $9y^2 - 6y + 1 = 0$

Solution:

To determine the number of solutions of each quadratic equation, we will look at its discriminant.

a.

$$3x^2 + 7x - 9 = 0$$

The equation is in standard form, identify a, b, and c.

$$a = 3, \quad b = 7, \quad c = -9$$

Write the discriminant.

$$b^2 - 4ac$$

Substitute in the values of a, b, and c.

$$(7)^2 - 4 \cdot 3 \cdot (-9)$$

Simplify.

$$49 + 108$$
$$157$$

Since the discriminant is positive, there are 2 real solutions to the equation.

b.

$$5n^2 + n + 4 = 0$$

The equation is in standard form, identify a, b, and c.

$$a = 5, \quad b = 1, \quad c = 4$$

Write the discriminant.

$$b^2-4ac$$

Substitute in the values of a, b, and c.

$$(1)^2-4\cdot 5\cdot 4$$

Simplify.

$$1-80$$
$$-79$$

Since the discriminant is negative, there are 2 complex solutions to the equation.

c.

$$9y^2-6y+1=0$$

The equation is in standard form, identify a, b, and c.

$$a=9,\quad b=-6,\quad c=1$$

Write the discriminant.

$$b^2-4ac$$

Substitute in the values of a, b, and c.

$$(-6)^2-4\cdot 9\cdot 1$$

Simplify.

$$36-36$$
$$0$$

Since the discriminant is 0, there is 1 real solution to the equation.

? Exercise 1.3.15

Determine the number and type of solutions to each quadratic equation.

a. $8m^2-3m+6=0$
b. $5z^2+6z-2=0$
c. $9w^2+24w+16=0$

Answer

a. 2 complex solutions
b. 2 real solutions
c. 1 real solution

? Exercise 1.3.16

Determine the number and type of solutions to each quadratic equation.

a. $b^2+7b-13=0$
b. $5a^2-6a+10=0$
c. $4r^2-20r+25=0$

Answer

a. 2 real solutions
b. 2 complex solutions
c. 1 real solution

Identify the Most Appropriate Method to Use to Solve a Quadratic Equation

We summarize the four methods that we have used to solve quadratic equations below.

Methods for Solving Quadratic Equations

1. Factoring
2. Square Root Property
3. Completing the Square
4. Quadratic Formula

Given that we have four methods to use to solve a quadratic equation, how do you decide which one to use? Factoring is often the quickest method and so we try it first. If the equation is $ax^2=k$ or $a(x-h)^2=k$ we use the Square Root Property. For any other equation, it is probably best to use the Quadratic

Formula. Remember, you can solve any quadratic equation by using the Quadratic Formula, but that is not always the easiest method.

What about the method of Completing the Square? Most people find that method cumbersome and prefer not to use it. We needed to include it in the list of methods because we completed the square in general to derive the Quadratic Formula. You will also use the process of Completing the Square in other areas of algebra.

Identify the Most Appropriate Method to Solve a Quadratic Equation

1. Try **Factoring** first. If the quadratic factors easily, this method is very quick.
2. Try the **Square Root Property** next. If the equation fits the form $ax^2 = k$ or $a(x-h)^2 = k$, it can easily be solved by using the Square Root Property.
3. Use the **Quadratic Formula**. Any other quadratic equation is best solved by using the Quadratic Formula.

The next example uses this strategy to decide how to solve each quadratic equation.

Example 1.3.9

Identify the most appropriate method to use to solve each quadratic equation.

a. $5z^2 = 17$
b. $4x^2 - 12x + 9 = 0$
c. $8u^2 + 6u = 11$

Solution:

a.

$$5z^2 = 17$$

Since the equation is in the $ax^2 = k$, the most appropriate method is to use the Square Root Property.

b.

$$4x^2 - 12x + 9 = 0$$

We recognize that the left side of the equation is a perfect square trinomial, and so factoring will be the most appropriate method.

c.

$$8u^2 + 6u = 11$$

Put the equation in standard form.

$$8u^2 + 6u - 11 = 0$$

While our first thought may be to try factoring, thinking about all the possibilities for trial and error method leads us to choose the Quadratic Formula as the most appropriate method.

Exercise 1.3.17

Identify the most appropriate method to use to solve each quadratic equation.

a. $x^2 + 6x + 8 = 0$
b. $(n-3)^2 = 16$
c. $5p^2 - 6p = 9$

Answer

a. Factoring
b. Square Root Property
c. Quadratic Formula

Exercise 1.3.18

Identify the most appropriate method to use to solve each quadratic equation.

a. $8a^2 + 3a - 9 = 0$
b. $4b^2 + 4b + 1 = 0$
c. $5c^2 = 125$

Answer

a. Quadratic Formula
b. Factoring or Square Root Property
c. Square Root Property

Access these online resources for additional instruction and practice with using the Quadratic Formula.

- Using the Quadratic Formula

- Solve a Quadratic Equation Using the Quadratic Formula with Complex Solutions
- Discriminant in Quadratic Formula

Key Concepts

- **Zero Product Property** If $a \cdot b = 0$, then either a=0 or b=0 or both. See Example.
- **Solve a quadratic equation by factoring** To solve a quadratic equation by factoring: See Example.
  1. Write the quadratic equation in standard form, $ax^2 + bx + c = 0$.
  2. Factor the quadratic expression.
  3. Use the Zero Product Property.
  4. Solve the linear equations.
  5. Check.
- **Use a problem solving strategy to solve word problems** See Example.
  1. **Read** the problem. Make sure all the words and ideas are understood.
  2. **Identify** what we are looking for.
  3. **Name** what we are looking for. Choose a variable to represent that quantity.
  4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebra equation.
  5. **Solve** the equation using good algebra techniques.
  6. **Check** the answer in the problem and make sure it makes sense.
  7. **Answer** the question with a complete sentence.

Key Concepts

- Quadratic Formula
  - The solutions to a quadratic equation of the form $ax^2 + bx + c = 0, a \neq 0$ are given by the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- How to solve a quadratic equation using the Quadratic Formula.
  1. Write the quadratic equation in standard form, $ax^2 + bx + c = 0$. Identify the values of a, b, c.
  2. Write the Quadratic Formula. Then substitute in the values of a, b, c.
  3. Simplify.
  4. Check the solutions.
- Using the Discriminant, $b^2 - 4ac$, to Determine the Number and Type of Solutions of a Quadratic Equation
  - For a quadratic equation of the form $ax^2 + bx + c = 0, a \neq 0$,
    - If $b^2 - 4ac > 0$, the equation has 2 real solutions.
    - If $b^2 - 4ac = 0$, the equation has 1 real solution.
    - If $b^2 - 4ac < 0$, the equation has 2 complex solutions.
- Methods to Solve Quadratic Equations:
  - Factoring
  - Square Root Property
  - Completing the Square
  - Quadratic Formula
- How to identify the most appropriate method to solve a quadratic equation.
  1. Try Factoring first. If the quadratic factors easily, this method is very quick.
  2. Try the **Square Root Property** next. If the equation fits the form $ax^2 = k$ or $a(x - h)^2 = k$, it can easily be solved by using the Square Root Property.
  3. Use the **Quadratic Formula.** Any other quadratic equation is best solved by using the Quadratic Formula.

Glossary

quadratic equations

are equations in which the variable is squared.

Zero Product Property

The Zero Product Property states that, if the product of two quantities is zero, at least one of the quantities is zero.

Glossary

discriminant

In the Quadratic Formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, the quantity $b^2 - 4ac$ is called the discriminant.

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1.4: Solving Polynomial Equations

Learning Objectives

By the end of this section, you will be able to:

- Use the Zero Product Property
- Solve quadratic equations by factoring
- Solve equations with polynomial functions
- Solve applications modeled by polynomial equations

Before you get started, take this readiness quiz.

1. Solve: $5y-3=0$.
   If you missed this problem, review [link].
2. Factor completely: n^3-9n^2-22n.
   If you missed this problem, review [link].
3. If $f(x)=8x-16$, find $f(3)$ and solve $f(x)=0$.
   If you missed this problem, review [link].

We have spent considerable time learning how to factor polynomials. We will now look at polynomial equations and solve them using factoring, if possible.

A **polynomial equation** is an equation that contains a polynomial expression. The **degree of the polynomial equation** is the degree of the polynomial.

POLYNOMIAL EQUATION

A **polynomial equation** is an equation that contains a polynomial expression.

The **degree of the polynomial equation** is the degree of the polynomial.

We have already solved polynomial equations of **degree one**. Polynomial equations of degree one are linear equations are of the form $ax+b=c$.

We are now going to solve polynomial equations of **degree two**. A polynomial equation of degree two is called a **quadratic equation**. Listed below are some examples of quadratic equations:

$$x^2+5x+6=0 \qquad 3y^2+4y=10 \qquad 64u^2-81=0 \qquad n(n+1)=42$$

The last equation doesn't appear to have the variable squared, but when we simplify the expression on the left we will get n^2+n.

The general form of a quadratic equation is $ax^2+bx+c=0$, with $a\neq 0$. (If $a=0$, then $0\cdot x^2=0$ and we are left with no quadratic term.)

QUADRATIC EQUATION

An equation of the form $ax^2+bx+c=0$ is called a quadratic equation.

$$a, b, \text{ and } c \text{ are real numbers and } a\neq 0$$

To solve quadratic equations we need methods different from the ones we used in solving linear equations. We will look at one method here and then several others in a later chapter.

Use the Zero Product Property

We will first solve some quadratic equations by using the **Zero Product Property**. The Zero Product Property says that if the product of two quantities is zero, then at least one of the quantities is zero. The only way to get a product equal to zero is to multiply by zero itself.

ZERO PRODUCT PROPERTY

If $a \cdot b = 0$, then either $a = 0$ or $b = 0$ or both.

We will now use the Zero Product Property, to solve a **quadratic equation**.

Example 1.4.1: How to Solve a Quadratic Equation Using the Zero Product Property

Solve: $(5n-2)(6n-1)=0$.

Answer

Step 1. Set each factor equal to zero.	The product equals zero, so at least one factor must equal zero.	$(5n-2)(6n-1)=0$ $5n-2=0$ or $6n-1=0$
Step 2. Solve the linear equations.	Solve each equation.	$n=\frac{2}{5}$ $n=\frac{1}{6}$
Step 3. Check.	Substitute each solution separately into the original equation.	$n=\frac{2}{5}$ $(5n-2)(6n-1)=0$ $\left(5\cdot\frac{2}{5}-2\right)\left(6\cdot\frac{2}{5}-1\right)\stackrel{?}{=}0$ $(2-2)\left(\frac{12}{5}-1\right)\stackrel{?}{=}0$ $0\cdot\frac{7}{5}\stackrel{?}{=}0$ $0=0$ ✓ $n=\frac{1}{6}$ $(5n-2)(6n-1)=0$ $\left(5\cdot\frac{1}{6}-2\right)\left(6\cdot\frac{1}{6}-1\right)\stackrel{?}{=}0$ $\left(\frac{5}{6}-\frac{12}{6}\right)(1-1)\stackrel{?}{=}0$ $\left(-\frac{7}{6}\right)(0)\stackrel{?}{=}0$ $0=0$ ✓

Example 1.4.2

Solve: $(3m-2)(2m+1)=0$.

Answer

$m=\frac{2}{3}$, $m=-\frac{1}{2}$

Example 1.4.3

Solve: $(4p+3)(4p-3)=0$.

Answer

$p=-\frac{3}{4}$, $p=\frac{3}{4}$

USE THE ZERO PRODUCT PROPERTY.

1. Set each factor equal to zero.
2. Solve the linear equations.
3. Check.

Solve Quadratic Equations by Factoring

The Zero Product Property works very nicely to solve quadratic equations. The quadratic equation must be factored, with zero isolated on one side. So we be sure to start with the quadratic equation in **standard form**, $ax^2+bx+c=0$. Then we factor the expression on the left.

Solve: $2y^2=13y+45$.

Answer

Step 1. Write the quadratic equation in standard form, $ax^2+bx+c=0$.	Write the equation in standard form.	$2y^2=13y+45$ $2y^2-13y-45=0$
Step 2. Factor the quadratic expression.	Factor $2y^2-13y+45$ $(2y+5)(y-9)$	$(2y+5)(y-9)=0$
Step 3. Use the Zero Product Property.	Set each factor equal to zero. We have two linear equations.	$2y+5=0$ $y-9=0$
Step 4. Solve the linear equations.		$y=-\frac{5}{2}$ $y=9$
Step 5. Check. Substitute each solution separately into the original equation.	Substitute each solution separately into the original equation.	$y=-\frac{5}{2}$ $2y^2=13y+45$ $2\left(-\frac{5}{2}\right)^2 \stackrel{?}{=} 13\left(-\frac{5}{2}\right)+45$ $2\left(\frac{25}{4}\right) \stackrel{?}{=} \left(-\frac{65}{2}\right)+\frac{90}{2}$ $\frac{25}{2}=\frac{25}{2}$ ✓ $y=9$ $2y^2=13y+45$ $2(9)^2 \stackrel{?}{=} 13(9)+45$ $2(81) \stackrel{?}{=} 117+45$ $162=162$ ✓

Example 1.4.5

Solve: $3c^2=10c-8$.

Answer

$c=2,\ c=\frac{4}{3}$

Example 1.4.6

Solve: $2d^2 - 5d = 3$.

Answer

$d = 3,\ d = -12$

SOLVE A QUADRATIC EQUATION BY FACTORING.

1. Write the quadratic equation in standard form, $ax^2 + bx + c = 0$.
2. Factor the quadratic expression.
3. Use the Zero Product Property.
4. Solve the linear equations.
5. Check. Substitute each solution separately into the original equation.

Before we factor, we must make sure the **quadratic equation** is in **standard form**.

Solving quadratic equations by factoring will make use of all the factoring techniques you have learned in this chapter! Do you recognize the special product pattern in the next example?

Example 1.4.7

Solve: $169q^2 = 49$.

Answer

	$169x^2 = 49$
Write the quadratic equation in standard form.	$169x^2 - 49 = 0$
Factor. It is a difference of squares.	$(13x - 7)(13x + 7) = 0$
Use the Zero Product Property to set each factor to 0.	
	$13x - 7 = 0 \quad 13x + 7 = 0$
Solve each equation.	$13x = 7 \quad 13x = -7$
	$x = \frac{7}{13} \quad x = -\frac{7}{13}$

Check:

We leave the check up to you.

Example 1.4.8

Solve: $25p^2 = 49$.

Answer

$p = \frac{7}{5}, p = -\frac{7}{5}$

Example 1.4.9

Solve: $36x^2 = 121$.

Answer

$x = \frac{11}{6}, x = -\frac{11}{6}$

In the next example, the left side of the equation is factored, but the right side is not zero. In order to use the **Zero Product Property**, one side of the equation must be zero. We'll multiply the factors and then write the equation in standard form.

? Example 1.4.10

Solve: $(3x-8)(x-1)=3x$.

Answer

	$(3x-8)(x-1)=3x$
Multiply the binomials.	$3x^2-11x+8=3x$
Write the quadratic equation in standard form.	$3x^2-14x+8=0$
Factor the trinomial.	$(3x-2)(x-4)=0$
Use the Zero Product Property to set each factor to 0.	$3x-2=0 \quad x-4=0$
Solve each equation.	$3x=2 \quad x=4$
	$x=\frac{2}{3}$
Check your answers.	The check is left to you.

? Example 1.4.11

Solve: $(2m+1)(m+3)=12m$.

Answer

$m=1,\ m=\frac{3}{2}$

? Example 1.4.12

Solve: $(k+1)(k-1)=8$.

Answer

$k=3,\ k=-3$

In the next example, when we factor the quadratic equation we will get three factors. However the first factor is a constant. We know that factor cannot equal 0.

? Example 1.4.13

Solve: $3x^2=12x+63$.

Answer

	$3x^2=12x+63$
Write the quadratic equation in standard form.	$3x^2-12x-63=0$
Factor the greatest common factor first.	$3(x^2-4x-21)=0$
Factor the trinomial.	$3(x-7)(x+3)=0$
Use the Zero Product Property to set each factor to 0.	$3\neq 0 \quad x-7=0 \quad x+3=0$
Solve each equation.	$3\neq 0 \quad x=7 \quad x=-3$
Check your answers.	The check is left to you.

? Example 1.4.14

Solve: $18a^2-30=-33a$.

Answer

$a=-\frac{5}{2}, a=\frac{2}{3}$

? Example 1.4.15

Solve: $123b = -6 - 60b^2$

Answer

$b = -2,\ b = -\frac{1}{20}$

The **Zero Product Property** also applies to the product of three or more factors. If the product is zero, at least one of the factors must be zero. We can solve some equations of degree greater than two by using the Zero Product Property, just like we solved quadratic equations.

? Example 1.4.16

Solve: $9m^3 + 100m = 60m^2$

Answer

	$9m^3 + 100m = 60m^2$
Bring all the terms to one side so that the other side is zero.	$9m^3 - 60m^2 + 100m = 0$
Factor the greatest common factor first.	$m(9m^2 - 60m + 100) = 0$
Factor the trinomial.	$m(3m - 10)^2 = 0$
Use the Zero Product Property to set each factor to 0.	
Solve each equation.	$m = 0 \quad 3m - 10 = 0$ $m = 0 \quad m = \frac{10}{3}$
Check your answers.	The check is left to you.

? Example 1.4.17

Solve: $8x^3 = 24x^2 - 18x$.

Answer

$x = 0,\ x = \frac{3}{2}$

? Example 1.4.18

Solve: $16y^2 = 32y^3 + 2y$.

Answer

$y = 0,\ y = 14$

Solve Equations with Polynomial Functions

As our study of polynomial functions continues, it will often be important to know when the function will have a certain value or what points lie on the graph of the function. Our work with the **Zero Product Property** will be help us find these answers.

? Example 1.4.19

For the function $f(x) = x^2 + 2x - 2$,

ⓐ find x when $f(x) = 6$

ⓑ find two points that lie on the graph of the function.

Answer

ⓐ

	$f(x)=x^2+2x-2$
Substitute 6 for $f(x)$.	$6=x^2+2x-2$
Put the quadratic in standard form.	$x^2+2x-8=0$
Factor the trinomial.	$(x+4)(x-2)=0$
Use the zero product property.	$x+4=0 \quad \text{or} \quad x-2=0$
Solve.	$x=-4 \quad \text{or} \quad x=2$
Check:	

$f(x)=x^2+2x-2$	$f(x)=x^2+2x-2$
$f(-4)=(-4)^2+2(-4)-2$	$f(2)=2^2+2\cdot 2-2$
$f(-4)=16-8-2$	$f(2)=4+4-2$
$f(-4)=6$✓	$f(2)=6$✓

ⓑ Since $f(-4)=6$ and $f(2)=6$, the points $(-4,6)$ and $(2,6)$ lie on the graph of the function.

? Example 1.4.20

For the function $f(x)=x^2-2x-8$,

ⓐ find x when $f(x)=7$
ⓑ Find two points that lie on the graph of the function.

Answer

ⓐ $x=-3$ or $x=5$
ⓑ $(-3,7)$ $(5,7)$

? Example 1.4.21

For the function $f(x)=x^2-8x+3$,

ⓐ find x when $f(x)=-4$
ⓑ Find two points that lie on the graph of the function.

Answer

ⓐ $x=1$ or $x=7$
ⓑ $(1,-4)$ $(7,-4)$

The **Zero Product Property** also helps us determine where the function is zero. A value of x where the function is 0, is called a **zero of the function**.

ZERO OF A FUNCTION

For any function f, if $f(x)=0$, then x is a **zero of the function**.

When $f(x)=0$, the point $(x,0)$ is a point on the graph. This point is an x-**intercept** of the graph. It is often important to know where the graph of a function crosses the axes. We will see some examples later.

Example 1.4.22

For the function $f(x)=3x^2+10x-8$, find

ⓐ the zeros of the function,
ⓑ any x-intercepts of the graph of the function
ⓒ any y-intercepts of the graph of the function

Answer

ⓐ To find the zeros of the function, we need to find when the function value is 0.

	$f(x)=3x^2+10x-8$
Substitute 0 for $f(x)$.	$0=3x^2+10x-8$
Factor the trinomial.	$(x+4)(3x-2)=0$
Use the zero product property.	$x+4=0 \quad \text{or} \quad 3x-2=0$
Solve.	$x=-4 \quad \text{or} \quad x=\frac{2}{3}$

ⓑ An x-intercept occurs when $y=0$. Since $f(-4)=0$ and $f(\frac{2}{3})=0$, the points $(-4,0)$ and $(\frac{2}{3},0)$ lie on the graph. These points are x-intercepts of the function.

ⓒ A y-intercept occurs when $x=0$. To find the y-intercepts we need to find $f(0)$.

	$f(x)=3x^2+10x-8$
Find $f(0)$ by substituting 0 for x.	$f(0)=3\cdot 0^2+10\cdot 0-8$
Simplify.	$f(0)=-8$

Since $f(0)=-8$, the point $(0,-8)$ lies on the graph. This point is the y-intercept of the function.

Example 1.4.23

For the function $f(x)=2x^2-7x+5$, find

ⓐ the zeros of the function
ⓑ any x-intercepts of the graph of the function
ⓒ any y-intercepts of the graph of the function.

Answer

ⓐ $x=1$ or $x=\frac{5}{2}$
ⓑ $(1,0),\ (\frac{5}{2},0)$ ⓒ $(0,5)$

Example 1.4.24

For the function $f(x)=6x^2+13x-15$, find

ⓐ the zeros of the function
ⓑ any x-intercepts of the graph of the function
ⓒ any y-intercepts of the graph of the function.

Answer

ⓐ $x=-3$ or $x=\frac{5}{6}$
ⓑ $(-3,0),\ (\frac{5}{6},0)$ ⓒ $(0,-15)$

Solve Applications Modeled by Polynomial Equations

The problem-solving strategy we used earlier for applications that translate to linear equations will work just as well for applications that translate to polynomial equations. We will copy the problem-solving strategy here so we can use it for reference.

USE A PROBLEM SOLVING STRATEGY TO SOLVE WORD PROBLEMS.

1. **Read** the problem. Make sure all the words and ideas are understood.
2. **Identify** what we are looking for.
3. **Name** what we are looking for. Choose a variable to represent that quantity.
4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebraic equation.
5. **Solve** the equation using appropriate algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

We will start with a number problem to get practice translating words into a polynomial equation.

Example 1.4.25

The product of two consecutive odd integers is 323. Find the integers.

Answer

Step 1. Read the problem.

Step 2. Identify what we are looking for. — We are looking for two consecutive integers.

Step 3. Name what we are looking for. — Let $n=$ the first integer.
$n+2=$ next consecutive odd integer

Step 4. Translate into an equation. Restate the problem in a sentence. — The product of the two consecutive odd integers is 323.
$n(n+2)=323$

Step 5. Solve the equation. $n^2+2n=323$

Bring all the terms to one side. $n^2+2n-323=0$

Factor the trinomial. $(n-17)(n+19)=0$

Use the Zero Product Property. $n-17=0$ $\quad$ $n+19=0$

Solve the equations. $n=17$ $\quad$ $n=-19$

There are two values for n that are solutions to this problem. So there are two sets of consecutive odd integers that will work.

If the first integer is $n=17$ then the next odd integer is

$$\begin{aligned} n+2 \\ 17+2 \\ 19 \end{aligned}$$

$17, 19$

If the first integer is $n=-19$ then the next odd integer is

$$\begin{aligned} n+2 \\ -19+2 \\ -17 \end{aligned}$$

$-17, -19$

Step 6. Check the answer.

The results are consecutive odd integers $17,\ 19$ and $-19,\ -17$.

$17\cdot 19=323$✓ $\quad$ $-19(-17)=323$✓

Both pairs of consecutive integers are solutions.

Step 7. Answer the question — The consecutive integers are $17, 19$ and $-19, -17$.

? Example 1.4.26

The product of two consecutive odd integers is 255. Find the integers.

Answer

$-15, -17$ and $15, 17$

? Example 1.4.27

The product of two consecutive odd integers is 483 Find the integers.

Answer

$-23, -21$ and $21, 23$

Were you surprised by the pair of negative integers that is one of the solutions to the previous example? The product of the two positive integers and the product of the two negative integers both give positive results.

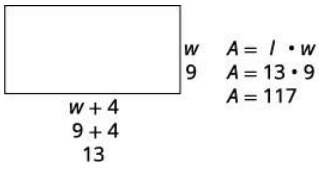
In some applications, negative solutions will result from the algebra, but will not be realistic for the situation.

? Example 1.4.28

A rectangular bedroom has an area 117 square feet. The length of the bedroom is four feet more than the width. Find the length and width of the bedroom.

Answer

Step 1. Read the problem. In problems involving geometric figures, a sketch can help you visualize the situation.	
Step 2. Identify what you are looking for.	We are looking for the length and width.
Step 3. Name what you are looking for.	Let $w =$ the width of the bedroom.
The length is four feet more than the width.	$w + 4 =$ the length of the garden
Step 4. Translate into an equation.	
Restate the important information in a sentence.	The area of the bedroom is 117 square feet.
Use the formula for the area of a rectangle.	$A = l \cdot w$
Substitute in the variables.	$117 = (w + 4)w$
Step 5. Solve the equation Distribute first.	$117 = w^2 + 4w$
Get zero on one side.	$117 = w^2 + 4w$
Factor the trinomial.	$0 = w^2 + 4w - 117$
Use the Zero Product Property.	$0 = (w^2 + 13)(w - 9)$
Solve each equation.	$0 = w + 13 \quad 0 = w - 9$
Since w is the width of the bedroom, it does not make sense for it to be negative. We eliminate that value for w.	$\cancel{w = -13} \quad w = 9$
	$w = 9$ The width is 9 feet.
Find the value of the length.	$w + 4$ $9 + 4$ 13 The length is 13 feet.

Step 6. Check the answer. Does the answer make sense?  $A = l \cdot w$ $A = 13 \cdot 9$ $A = 117$ Yes, this makes sense.	
Step 7. Answer the question.	The width of the bedroom is 9 feet and the length is 13 feet.

Example 1.4.29

A rectangular sign has area 30 square feet. The length of the sign is one foot more than the width. Find the length and width of the sign.

Answer

The width is 5 feet and length is 6 feet.

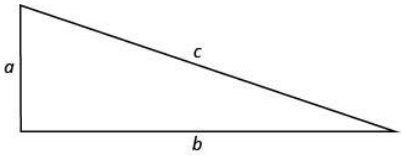
Example 1.4.30

A rectangular patio has area 180 square feet. The width of the patio is three feet less than the length. Find the length and width of the patio.

Answer

The length of the patio is 12 feet and the width 15 feet.

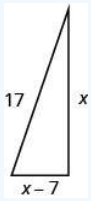
In the next example, we will use the Pythagorean Theorem $(a^2 + b^2 = c^2)$. This formula gives the relation between the legs and the hypotenuse of a right triangle.



We will use this formula to in the next example.

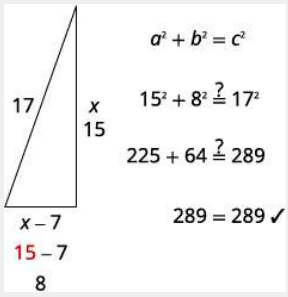
Example 1.4.31

A boat's sail is in the shape of a right triangle as shown. The hypotenuse will be 17 feet long. The length of one side will be 7 feet less than the length of the other side. Find the lengths of the sides of the sail.



Answer

Step 1. Read the problem

Step 2. Identify what you are looking for.	We are looking for the lengths of the sides of the sail.
Step 3. Name what you are looking for. One side is 7 less than the other.	Let $x =$ length of a side of the sail. $x - 7 =$ length of other side
Step 4. Translate into an equation. Since this is a right triangle we can use the Pythagorean Theorem.	$a^2 + b^2 = c^2$
Substitute in the variables.	$x^2 + (x-7)^2 = 17^2$
Step 5. Solve the equation Simplify.	$x^2 + x^2 - 14x + 49 = 289$
	$2x^2 - 14x + 49 = 289$
It is a quadratic equation, so get zero on one side.	$2x^2 - 14x - 240 = 0$
Factor the greatest common factor.	$2(x^2 - 7x - 120) = 0$
Factor the trinomial.	$2(x-15)(x+8) = 0$
Use the Zero Product Property.	$2 \neq 0 \quad x - 15 = 0 \quad x + 8 = 0$
Solve.	$2 \neq 0 \quad x = 15 \quad x = -8$
Since x is a side of the triangle, $x = -8$ does not make sense.	$2 \neq 0 \quad x = 15 \quad$ ~~$x = -8$~~
Find the length of the other side.	
If the length of one side is then the length of the other side is	$x = 15$ $x - 7$ $15 - 7$ 8 is the length of the other side.
Step 6. Check the answer in the problem Do these numbers make sense? 	
Step 7. Answer the question	The sides of the sail are 8, 15 and 17 feet.

? Example 1.4.32

Justine wants to put a deck in the corner of her backyard in the shape of a right triangle. The length of one side of the deck is 7 feet more than the other side. The hypotenuse is 13. Find the lengths of the two sides of the deck.

Answer

5 feet and 12 feet

? Example 1.4.33

A meditation garden is in the shape of a right triangle, with one leg 7 feet. The length of the hypotenuse is one more than the length of the other leg. Find the lengths of the hypotenuse and the other leg.

Answer

24 feet and 25 feet

The next example uses the function that gives the height of an object as a function of time when it is thrown from 80 feet above the ground.

? Example 1.4.34

Dennis is going to throw his rubber band ball upward from the top of a campus building. When he throws the rubber band ball from 80 feet above the ground, the function $h(t) = -16t^2 + 64t + 80$ models the height, h, of the ball above the ground as a function of time, t. Find:

ⓐ the zeros of this function which tell us when the ball hits the ground
ⓑ when the ball will be 80 feet above the ground
ⓒ the height of the ball at $t = 2$ seconds.

Answer

ⓐ The zeros of this function are found by solving $h(t) = 0$. This will tell us when the ball will hit the ground.

	$h(t) = 0$
Substitute in the polynomial for $h(t)$.	$-16t^2 + 64t + 80 = 0$
Factor the GCF, -16.	$-16(t^2 - 4t - 5) = 0$
Factor the trinomial.	$-16(t-5)(t+1) = 0$
Use the Zero Product Property.	$t-5=0 \quad t+1=0$
Solve.	$t=5 \quad t=-1$

The result $t = 5$ tells us the ball will hit the ground 5 seconds after it is thrown. Since time cannot be negative, the result $t = -1$ is discarded.

ⓑ The ball will be 80 feet above the ground when $h(t) = 80$.

	$h(t) = 80$
Substitute in the polynomial for $h(t)$.	$-16t^2 + 64t + 80 = 80$
Subtract 80 from both sides.	$-16t^2 + 64t = 0$
Factor the GCF, $-16t$.	$-16t(t-4) = 0$
Use the Zero Product Property.	$-16t = 0 \quad t-4=0$
Solve.	$t=0 \quad t=4$
	The ball will be at 80 feet the moment Dennis tosses the ball and then 4 seconds later, when the ball is falling.

ⓒ To find the height ball at $t = 2$ seconds we find $h(2)$.

	$h(t) = -16t^2 + 64t + 80$
To find $h(2)$ substitute 2 for t.	$h(2) = -16(2)^2 + 64 \cdot 2 + 80$
Simplify.	$h(2) = 144$
	After 2 seconds, the ball will be at 144 feet.

? Example 1.4.35

Genevieve is going to throw a rock from the top a trail overlooking the ocean. When she throws the rock upward from 160 feet above the ocean, the function $h(t) = -16t^2 + 48t + 160$ models the height, h, of the rock above the ocean as a function of time, t. Find:

ⓐ the zeros of this function which tell us when the rock will hit the ocean
ⓑ when the rock will be 160 feet above the ocean.
ⓒ the height of the rock at $t = 1.5$ seconds.

Answer

ⓐ 5 ⓑ 0;3 ⓒ 196

? Example 1.4.36

Calib is going to throw his lucky penny from his balcony on a cruise ship. When he throws the penny upward from 128 feet above the ground, the function $h(t) = -16t^2 + 32t + 128$ models the height, h, of the penny above the ocean as a function of time, t. Find:

ⓐ the zeros of this function which is when the penny will hit the ocean
ⓑ when the penny will be 128 feet above the ocean.
ⓒ the height the penny will be at $t = 1$ seconds which is when the penny will be at its highest point.

Answer

ⓐ 4 ⓑ 0;2 ⓒ 144

Access this online resource for additional instruction and practice with quadratic equations.

- Beginning Algebra & Solving Quadratics with the Zero Property

Key Concepts

- **Polynomial Equation:** A polynomial equation is an equation that contains a polynomial expression. The degree of the polynomial equation is the degree of the polynomial.
- **Quadratic Equation:** An equation of the form $ax^2 + bx + c = 0$ is called a quadratic equation.

$$a, b, c \text{ are real numbers and } a \neq 0$$

- **Zero Product Property:** If $a \cdot b = 0$, then either $a = 0$ or $b = 0$ or both.
- **How to use the Zero Product Property**
  1. Set each factor equal to zero.
  2. Solve the linear equations.
  3. Check.
- **How to solve a quadratic equation by factoring.**
  1. Write the quadratic equation in standard form, $ax^2 + bx + c = 0$.
  2. Factor the quadratic expression.
  3. Use the Zero Product Property.
  4. Solve the linear equations.
  5. Check. Substitute each solution separately into the original equation.
- **Zero of a Function:** For any function f, if $f(x) = 0$, then x is a zero of the function.
- **How to use a problem solving strategy to solve word problems.**
  1. **Read** the problem. Make sure all the words and ideas are understood.
  2. **Identify** what we are looking for.
  3. **Name** what we are looking for. Choose a variable to represent that quantity.

4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebraic equation.
5. **Solve** the equation using appropriate algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

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1.5: Solving Rational Equations

Learning Objectives

- Solve rational equations
- Use rational functions
- Solve a rational equation for a specific variable

Before you get started, take this readiness quiz.

1. Solve: $\frac{1}{6}x+\frac{1}{2}=\frac{1}{3}$
   If you missed this problem, review Example 2.2.9.
2. Solve: $n^2-5n-36=0$
   If you missed this problem, review Example 6.6.4.
3. Solve the formula $5x+2y=10$ for y.
   If you missed this problem, review Example 2.4.10.

After defining the terms ‘expression’ and ‘equation’ earlier, we have used them throughout this book. We have simplified many kinds of expressions and solved many kinds of equations. We have simplified many rational expressions so far in this chapter. Now we will solve a **rational equation**.

Rational Equation

A **rational equation** is an equation that contains a rational expression.

You must make sure to know the difference between rational expressions and rational equations. The equation contains an equal sign.

Rational Expression	Rational Equation
$\frac{1}{8}x+\frac{1}{2}$	$\frac{1}{8}x+\frac{1}{2}=\frac{1}{4}$
$\frac{y+6}{y^2-36}$	$\frac{y+6}{y^2-36}=y+1$
$\frac{1}{n-3}+\frac{1}{n+4}$	$\frac{1}{n-3}+\frac{1}{n+4}=\frac{15}{n^2+n-12}$

Solve Rational Equations

We have already solved linear equations that contained fractions. We found the LCD of all the fractions in the equation and then multiplied both sides of the equation by the LCD to “clear” the fractions.

We will use the same strategy to solve rational equations. We will multiply both sides of the equation by the LCD. Then, we will have an equation that does not contain rational expressions and thus is much easier for us to solve. But because the original equation may have a variable in a denominator, we must be careful that we don’t end up with a solution that would make a denominator equal to zero.

So before we begin solving a rational equation, we examine it first to find the values that would make any denominators zero. That way, when we solve a rational equation we will know if there are any algebraic solutions we must discard.

An algebraic solution to a rational equation that would cause any of the rational expressions to be undefined is called an **extraneous solution to a rational equation**.

Extraneous Solution to a Rational Equation

An **extraneous solution to a rational equation** is an algebraic solution that would cause any of the expressions in the original equation to be undefined.

We note any possible extraneous solutions, c, by writing $x \neq c$ next to the equation.

Example 1.5.1: How to Solve a Rational Equation

Solve:

$$\frac{1}{x}+\frac{1}{3}=\frac{5}{6}$$

Solution

Step 1. Note any value of the variable that would make any denominator zero.

If $x=0$, then $\frac{1}{x}$ is undefined. So we'll write $x \neq 0$ next to the equation.

$$\frac{1}{x}+\frac{1}{3}=\frac{5}{6}, x \neq 0$$

Step 2. Find the least common denominator of all denominators in the equation.

Find the LCD of $\frac{1}{x}$, $\frac{1}{3}$, and $\frac{5}{6}$

The LCD is $6x$.

Step 3. Clear the fractions by multiplying both sides of the equation by the LCD.

Multiply both sides of the equation by the LCD, $6x$.

$$6x \cdot \left(\frac{1}{x}+\frac{1}{3}\right)=6x \cdot \left(\frac{5}{6}\right)$$

Use the Distributive Property.

$$6x \cdot \frac{1}{x}+6x \cdot \frac{1}{3}=6x \cdot \left(\frac{5}{6}\right)$$

Simplify - and notice, no more fractions!

$$6+2x=5x$$

Step 4. Solve the resulting equation.

Simplify.

$$6=3x$$
$$2=x$$

Step 5. Check.

If any values found in Step 1 are algebraic solutions, discard them. Check any remaining solutions in the original equation.

We did not get 0 as an algebraic solution.

$$\frac{1}{x}+\frac{1}{3}=\frac{5}{6}$$

We substitute $x=2$ into the original equation.

$$\frac{1}{2}+\frac{1}{3} \stackrel{?}{=} \frac{5}{6}$$
$$\frac{3}{6}+\frac{2}{6} \stackrel{?}{=} \frac{5}{6}$$
$$\frac{5}{6}=\frac{5}{6}\checkmark$$

The solution is $x=2$

Exercise 1.5.1

Solve:

$$\frac{1}{y}+\frac{2}{3}=\frac{1}{5}$$

Answer

$$y=-\frac{15}{7}$$

Exercise 1.5.2

Solve:

$$\frac{2}{3}+\frac{1}{5}=\frac{1}{x}$$

Answer

$$x=\frac{15}{3}$$

The steps of this method are shown.

how to Solve equations with rational expressions.

- Step 1. Note any value of the variable that would make any denominator zero.
- Step 2. Find the least common denominator of all denominators in the equation.
- Step 3. Clear the fractions by multiplying both sides of the equation by the LCD.
- Step 4. Solve the resulting equation.
- Step 5. Check:
  - If any values found in Step 1 are algebraic solutions, discard them.
  - Check any remaining solutions in the original equation.

We always start by noting the values that would cause any denominators to be zero.

Example 1.5.2: How to Solve a Rational Equation using the Zero Product Property

Solve:

$$1-\frac{5}{y}=-\frac{6}{y^2}$$

Solution

Note any value of the variable that would make any denominator zero.

$$1-\frac{5}{y}=-\frac{6}{y^2}, y\neq 0$$

Find the least common denominator of all denominators in the equation. The LCD is y^2.

Clear the fractions by multiplying both sides of the equation by the LCD.

$$y^2\left(1-\frac{5}{y}\right)=y^2\left(-\frac{6}{y^2}\right)$$

Distribute.

$$y^2\cdot 1-y^2\left(\frac{5}{y}\right)=y^2\left(-\frac{6}{y^2}\right)$$

Multiply.

$$y^2-5y=-6$$

Solve the resulting equation. First write the quadratic equation in standard form.

$$y^2-5y+6=0$$

Factor.

$$(y-2)(y-3)=0$$

Use the Zero Product Property.

$$y-2=0 \text{ or } y-3=0$$

Solve.

$$y=2 \text{ or } y=3$$

Check. We did not get 0 as an algebraic solution.

Check $y=2$ and $y=3$ in the original equation.

$$1-\frac{5}{y}=-\frac{6}{y^2} \qquad 1-\frac{5}{y}=-\frac{6}{y^2}$$

$$1-\frac{5}{2}\stackrel{?}{=}-\frac{6}{2^2} \qquad 1-\frac{5}{3}\stackrel{?}{=}-\frac{6}{3^2}$$

$$1-\frac{5}{2}\stackrel{?}{=}-\frac{6}{4} \qquad 1-\frac{5}{3}\stackrel{?}{=}-\frac{6}{9}$$

$$\frac{2}{2}-\frac{5}{2}\stackrel{?}{=}-\frac{6}{4} \qquad \frac{3}{3}-\frac{5}{3}\stackrel{?}{=}-\frac{6}{9}$$

$$-\frac{3}{2}\stackrel{?}{=}-\frac{6}{4} \qquad -\frac{2}{3}\stackrel{?}{=}-\frac{6}{9}$$

$$-\frac{3}{2}=-\frac{3}{2}\surd \qquad -\frac{2}{3}=-\frac{2}{3}\surd$$

The solution is $y=2, y=3$

? Exercise 1.5.3

Solve:

$$1-\frac{2}{x}=\frac{15}{x^2}$$

Answer

$x=-3, x=5$

Exercise 1.5.4

Solve:

$$1-\frac{4}{y}=\frac{12}{y^2}$$

Answer

$y=-2, y=6$

In the next example, the last denominators is a difference of squares. Remember to factor it first to find the LCD.

Example 1.5.3

Solve:

$$\frac{2}{x+2}+\frac{4}{x-2}=\frac{x-1}{x^2-4}$$

Solution

Note any value of the variable that would make any denominator zero.

$$\frac{2}{x+2}+\frac{4}{x-2}=\frac{x-1}{(x+2)(x-2)}, x\neq -2, x\neq 2$$

Find the least common denominator of all denominators in the equation. The LCD is $(x+2)(x-2)$.

Clear the fractions by multiplying both sides of the equation by the LCD.

$$(x+2)(x-2)\left(\frac{2}{x+2}+\frac{4}{x-2}\right)=(x+2)(x-2)\left(\frac{x-1}{x^2-4}\right)$$

Distribute.

$$(x+2)(x-2)\frac{2}{x+2}+(x+2)(x-2)\frac{4}{x-2}=(x+2)(x-2)\left(\frac{x-1}{x^2-4}\right)$$

Remove common factors.

$$\cancel{(x+2)}(x-2)\frac{2}{\cancel{x+2}}+(x+2)\ \cancel{(x-2)}\frac{4}{\cancel{x-2}}=\ \cancel{(x+2)(x-2)}\left(\frac{x-1}{\cancel{x^2-4}}\right)$$

Simplify.

$$2(x-2)+4(x+2)=x-1$$

Distribute.

$$2x-4+4x+8=x-1$$

Solve.

$$\begin{aligned}6x+4&=x-1\\5x&=-5\\x&=-1\end{aligned}$$

Check: We did not get 2 or −2 as algebraic solutions.

Check $x=-1$ in the original equation.

$$\frac{2}{x+2}+\frac{4}{x-2}=\frac{x-1}{x^2-4}$$
$$\frac{2}{(-1)+2}+\frac{4}{(-1)-2}\stackrel{?}{=}\frac{(-1)-1}{(-1)^2-4}$$
$$\frac{2}{1}+\frac{4}{-3}\stackrel{?}{=}\frac{-2}{-3}$$
$$\frac{6}{3}-\frac{4}{3}\stackrel{?}{=}\frac{2}{3}$$
$$\frac{2}{3}=\frac{2}{3}\surd$$

The solution is $x=-1$.

Exercise 1.5.5

Solve:

$$\frac{2}{x+1}+\frac{1}{x-1}=\frac{1}{x^2-1}$$

Answer

$x=\frac{2}{3}$

Exercise 1.5.6

Solve:

$$\frac{5}{y+3}+\frac{2}{y-3}=\frac{5}{y^2-9}$$

Answer

$y=2$

In the next example, the first denominator is a trinomial. Remember to factor it first to find the LCD.

Example 1.5.4

Solve:

$$\frac{m+11}{m^2-5m+4}=\frac{5}{m-4}-\frac{3}{m-1}$$

Solution

Note any value of the variable that would make any denominator zero. Use the factored form of the quadratic denominator.

$$\frac{m+11}{(m-4)(m-1)}=\frac{5}{m-4}-\frac{3}{m-1}, m\neq 4, m\neq 1$$

Find the least common denominator of all denominators in the equation. The LCD is $(m-4)(m-1)$

Clear the fractions by multiplying both sides of the equation by the LCD.

$$(m-4)(m-1)\left(\frac{m+11}{(m-4)(m-1)}\right)=(m-4)(m-1)\left(\frac{5}{m-4}-\frac{3}{m-1}\right)$$

Distribute.

$$(m-4)(m-1)\left(\frac{m+11}{(m-4)(m-1)}\right)=(m-4)(m-1)\frac{5}{m-4}-(m-4)(m-1)\frac{3}{m-1}$$

Remove common factors.

$$\cancel{(m-4)(m-1)}\left(\frac{m+11}{\cancel{(m-4)(m-1)}}\right)=\cancel{(m-4)}(m-1)\frac{5}{\cancel{m-4}}-(m-4)\ \cancel{(m-1)}\frac{3}{\cancel{m-1}}$$

Simplify.

$$m+11=5(m-1)-3(m-4)$$

Solve the resulting equation.

$$\begin{aligned} m+11 &= 5m-5-3m+12 \\ 4 &= m \end{aligned}$$

Check. The only algebraic solution was 4, but we said that 4 would make a denominator equal to zero. The algebraic solution is an extraneous solution.

There is no solution to this equation.

? Exercise 1.5.7

Solve:

$$\frac{x+13}{x^2-7x+10}=\frac{6}{x-5}-\frac{4}{x-2}$$

Answer

There is no solution.

? Exercise 1.5.8

Solve:

$$\frac{y-6}{y^2+3y-4}=\frac{2}{y+4}+\frac{7}{y-1}$$

Answer

There is no solution.

The equation we solved in the previous example had only one algebraic solution, but it was an extraneous solution. That left us with no solution to the equation. In the next example we get two algebraic solutions. Here one or both could be extraneous solutions.

✓ Example 1.5.5

Solve:

$$\frac{y}{y+6}=\frac{72}{y^2-36}+4$$

Solution

Factor all the denominators, so we can note any value of the variable that would make any denominator zero.

$$\frac{y}{y+6}=\frac{72}{(y-6)(y+6)}+4, y\neq 6, y\neq -6$$

Find the least common denominator. The LCD is $(y-6)(y+6)$

Clear the fractions.

$$(y-6)(y+6)\left(\frac{y}{y+6}\right)=(y-6)(y+6)\left(\frac{72}{(y-6)(y+6)}+4\right)$$

Simplify.

$$(y-6)\cdot y=72+(y-6)(y+6)\cdot 4$$

Simplify.

$$y(y-6)=72+4\left(y^2-36\right)$$

Solve the resulting equation.

$$\begin{aligned} y^2-6y &= 72+4y^2-144 \\ 0 &= 3y^2+6y-72 \\ 0 &= 3\left(y^2+2y-24\right) \\ 0 &= 3(y+6)(y-4) \\ y &= -6, y=4 \end{aligned}$$

Check.

$y=-6$ is an extraneous solution. Check $y=4$ in the original equation.

$$\begin{aligned} \frac{y}{y+6} &= \frac{72}{y^2-36}+4 \\ \frac{4}{4+6} &\stackrel{?}{=} \frac{72}{4^2-36}+4 \\ \frac{4}{10} &\stackrel{?}{=} \frac{72}{-20}+4 \\ \frac{4}{10} &\stackrel{?}{=} -\frac{36}{10}+\frac{40}{10} \\ \frac{4}{10} &= \frac{4}{10}\surd \end{aligned}$$

The solution is $y=4$.

Exercise 1.5.9

Solve:

$$\frac{x}{x+4}=\frac{32}{x^2-16}+5$$

Answer

$x=3$

Exercise 1.5.10

Solve:

$$\frac{y}{y+8}=\frac{128}{y^2-64}+9$$

Answer

$y=7$

In some cases, all the algebraic solutions are extraneous.

Example 1.5.6

Solve:

$$\frac{x}{2x-2}-\frac{2}{3x+3}=\frac{5x^2-2x+9}{12x^2-12}$$

Solution

We will start by factoring all denominators, to make it easier to identify extraneous solutions and the LCD.

$$\frac{x}{2(x-1)}-\frac{2}{3(x+1)}=\frac{5x^2-2x+9}{12(x-1)(x+1)}$$

Note any value of the variable that would make any denominator zero.

$$\frac{x}{2(x-1)}-\frac{2}{3(x+1)}=\frac{5x^2-2x+9}{12(x-1)(x+1)}, x\neq 1, x\neq -1$$

Find the least common denominator. The LCD is $12(x-1)(x+1)$.

Clear the fractions.

$$12(x-1)(x+1)\left(\frac{x}{2(x-1)}-\frac{2}{3(x+1)}\right)=12(x-1)(x+1)\left(\frac{5x^2-2x+9}{12(x-1)(x+1)}\right)$$

Simplify.

$$6(x+1)\cdot x-4(x-1)\cdot 2=5x^2-2x+9$$

Simplify.

$$6x(x+1)-4\cdot 2(x-1)=5x^2-2x+9$$

Solve the resulting equation.

$$\begin{aligned} 6x^2+6x-8x+8 &= 5x^2-2x+9 \\ x^2-1 &= 0 \\ (x-1)(x+1) &= 0 \\ x &= 1 \text{ or } x=-1 \end{aligned}$$

Check.

$x=1$ and $x=-1$ are extraneous solutions.

The equation has no solution.

Exercise 1.5.11

Solve:

$$\frac{y}{5y-10}-\frac{5}{3y+6}=\frac{2y^2-19y+54}{15y^2-60}$$

Answer

There is no solution.

Exercise 1.5.12

Solve:

$$\frac{z}{2z+8}-\frac{3}{4z-8}=\frac{3z^2-16z-16}{8z^2+2z-64}$$

Answer

There is no solution.

Example 1.5.7

Solve:

$$\frac{4}{3x^2-10x+3}+\frac{3}{3x^2+2x-1}=\frac{2}{x^2-2x-3}$$

Solution

Factor all the denominators, so we can note any value of the variable that would make any denominator zero.

$$\frac{4}{(3x-1)(x-3)}+\frac{3}{(3x-1)(x+1)}=\frac{2}{(x-3)(x+1)}, x\neq -1, x\neq \frac{1}{3}, x\neq 3$$

Find the least common denominator. The LCD is $(3x-1)(x+1)(x-3)$.

Clear the fractions.

$$(3x-1)(x+1)(x-3)\left(\frac{4}{(3x-1)(x-3)}+\frac{3}{(3x-1)(x+1)}\right)=(3x-1)(x+1)(x-3)\left(\frac{2}{(x-3)(x+1)}\right)$$

Simplify.

$$4(x+1)+3(x-3)=2(3x-1)$$

Distribute.

$$4x+4+3x-9=6x-2$$

Simplify.

$$7x-5=6x-2$$

$$x=3$$

The only algebraic solution was $x=3$, but we said that $x=3$ would make a denominator equal to zero. The algebraic solution is an extraneous solution.

There is no solution to this equation.

Exercise 1.5.13

Solve:

$$\frac{15}{x^2+x-6}-\frac{3}{x-2}=\frac{2}{x+3}$$

Answer

There is no solution.

Exercise 1.5.14

Solve:

$$\frac{5}{x^2+2x-3}-\frac{3}{x^2+x-2}=\frac{1}{x^2+5x+6}$$

Answer

There is no solution.

Use Rational Functions

Working with functions that are defined by rational expressions often lead to rational equations. Again, we use the same techniques to solve them.

Example 1.5.8

For rational function, $f(x)=\dfrac{2x-6}{x^2-8x+15}$:

a. Find the domain of the function
b. Solve $f(x)=1$
c. Find the points on the graph at this function value.

Solution

a. The domain of a rational function is all real numbers except those that make the rational expression undefined. So to find them, we will set the denominator equal to zero and solve.

$$\begin{aligned} x^2-8x+15 &= 0 \\ (x-3)(x-5) &= 0 \quad \text{Factor the trinomial.} \\ x-3 &= 0 \quad \text{Use the Zero Product Property.} \\ x-5 &= 0 \quad \text{Use the Zero Product Property.} \\ x=3 \quad x &= 5 \quad \text{Solve.} \end{aligned}$$

The domain is all real numbers except $x\neq 3, x\neq 5$

b. $$f(x)=1$$

Substitute in the rational expression.

$$\frac{2x-6}{x^2-8x+15}=1$$

Factor the denominator.

$$\frac{2x-6}{(x-3)(x-5)}=1$$

Multiply both sides by the LCD, $(x-3)(x-5)$

$$(x-3)(x-5)\left(\frac{2x-6}{(x-3)(x-5)}\right)=(x-3)(x-5)(1)$$

Simplify.

$$2x-6=x^2-8x+15$$

Solve.

$$0=x^2-10x+21$$

Factor.

$$0 = (x-7)(x-3)$$

Use the Zero Product Property.

$$x-7=0 \quad x-3=0$$

Solve.

$$x=7 \quad x=3$$

c. The value of the function is 1 when $x=7, x=3$. So the points on the graph of this function when $f(x)=1$, will be $(7,1),(3,1)$. However, now we need to check for extraneous solutions. Since $x=3$ is not in the domain of the original function, we must remove $(3,1)$, since it would be considered extraneous.

? Exercise 1.5.15

For rational function, $f(x)=\dfrac{8-x}{x^2-7x+12}$

a. Find the domain of the function.
b. Solve $f(x)=3$.
c. Find the points on the graph at this function value.

Answer

a. The domain is all real numbers except $x \neq 3$ and $x \neq 4$
b. $x=2, x=\dfrac{14}{3}$
c. $(2,3), \left(\dfrac{14}{3},3\right)$

? Exercise 1.5.16

For rational function, $f(x)=\dfrac{x-1}{x^2-6x+5}$

a. Solve $f(x)=4$.
b. Find the points on the graph at this function value.

Answer

a. The domain is all real numbers except $x \neq 1$ and $x \neq 5$
b. $x=\dfrac{21}{4}$
c. $\left(\dfrac{21}{4},4\right)$

Solve a Rational Equation for a Specific Variable

When we solved linear equations, we learned how to solve a formula for a specific variable. Many formulas used in business, science, economics, and other fields use rational equations to model the relation between two or more variables. We will now see how to solve a rational equation for a specific variable.

When we developed the point-slope formula from our slope formula, we cleared the fractions by multiplying by the LCD.

$$
\begin{aligned}
m &= \frac{y-y_1}{x-x_1} \\
m\,(x-x_1) &= \left(\frac{y-y_1}{x-x_1}\right)(x-x_1) \quad \text{Multiply both sides of the equation by } x-x_1. \\
m\,(x-x_1) &= y-y_1 \quad \text{Simplify.} \\
y-y_1 &= m\,(x-x_1) \quad \text{Rewrite the equation with the y terms on the left.}
\end{aligned}
$$

In the next example, we will use the same technique with the formula for slope that we used to get the point-slope form of an equation of a line through the point $(2,3)$. We will add one more step to solve for y.

Example 1.5.9

Solve: $m = \dfrac{y-2}{x-3}$ for y.

Solution

$$m = \frac{y-2}{x-3}$$

Note any value of the variable that would make any denominator zero.

$$m = \frac{y-2}{x-3}, x \neq 3$$

Clear the fractions by multiplying both sides of the equation by the LCD, $x-3$.

$$(x-3)m = (x-3)\left(\frac{y-2}{x-3}\right)$$

Simplify.

$$xm-3m = y-2$$

Isolate the term with y.

$$xm-3m+2 = y$$

Exercise 1.5.17

Solve: $m = \dfrac{y-5}{x-4}$ for y.

Answer

$y = mx-4m+5$

Exercise 1.5.18

Solve: $m = \dfrac{y-1}{x+5}$ for y.

Answer

$y = mx+5m+1$

Remember to multiply both sides by the LCD in the next example.

Example 1.5.10

Solve: $\frac{1}{c}+\frac{1}{m}=1$ for c

Solution

$$\frac{1}{c}+\frac{1}{m}=1 \text{ for } c$$

Note any value of the variable that would make any denominator zero.

$$\frac{1}{c}+\frac{1}{m}=1, c\neq 0, m\neq 0$$

Clear the fractions by multiplying both sides of the equations by the LCD, cm.

$$cm\left(\frac{1}{c}+\frac{1}{m}\right)=cm(1)$$

Distribute.

$$cm\left(\frac{1}{c}\right)+cm\frac{1}{m}=cm(1)$$

Simplify.

$$m+c=cm$$

Collect the terms with c to the right.

$$m=cm-c$$

Factor the expression on the right.

$$m=c(m-1)$$

To isolate c, divide both sides by $m-1$.

$$\frac{m}{m-1}=\frac{c(m-1)}{m-1}$$

Simplify by removing common factors.

$$\frac{m}{m-1}=c$$

Notice that even though we excluded $c=0$ and $m=0$ from the original equation, we must also now state that $m\neq 1$.

Exercise 1.5.19

Solve: $\frac{1}{a}+\frac{1}{b}=c$ for a.

Answer

$$a=\frac{b}{cb-1}$$

Exercise 1.5.20

Solve: $\frac{2}{x}+\frac{1}{3}=\frac{1}{y}$ for y

Answer

$$y = \frac{3x}{x+6}$$

Media Access Additional Online Resources

Access this online resource for additional instruction and practice with equations with rational expressions.

- Equations with Rational Expressions

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1.6: Solving Radical Equations

Learning Objectives

By the end of this section, you will be able to:

- Solve radical equations
- Solve radical equations with two radicals
- Use radicals in applications

Before you get started, take this readiness quiz.

1. Simplify: $(y-3)^2$.
   If you missed this problem, review Example 5.31.
2. Solve: $2x-5=0$.
   If you missed this problem, review Example 2.2.
3. Solve $n^2-6n+8=0$.
   If you missed this problem, review Example 6.45.

Solve Radical Equations

In this section we will solve equations that have a variable in the radicand of a radical expression. An equation of this type is called a **radical equation**.

Definition: Radical Equation

An equation in which a variable is in the radicand of a radical expression is called a **radical equation**.

As usual, when solving these equations, what we do to one side of an equation we must do to the other side as well. Once we isolate the radical, our strategy will be to raise both sides of the equation to the power of the index. This will eliminate the radical.

Solving radical equations containing an even index by raising both sides to the power of the index may introduce an algebraic solution that would not be a solution to the original radical equation. Again, we call this an extraneous solution as we did when we solved rational equations.

In the next example, we will see how to solve a radical equation. Our strategy is based on raising a radical with index n to the nth power. This will eliminate the radical.

$$\text{For } a \geq 0, (\sqrt[n]{a})^n = a \text{ .}$$

Example 1.6.1 how to solve a radical equation

Solve: $\sqrt{5n-4}-9=0$.

Solution:

Table 8.6.1

Step 1: Isolate the radical on one side of the equation.	To isolate the radical, add 9 to both sides. Simplify.	$\sqrt{5n-4}-9=0$ $\sqrt{5n-4}-9+9=0+9$ $\sqrt{5n-4}=9$
Step 2: Raise both sides of the equation to the power of the index.	Since the index of a square root is 2, we square both sides.	$(\sqrt{5n-4})^2=(9)^2$
Step 3: Solve the new equation.	Remember, $(\sqrt{a})^2=a$.	$5n-4=81$ $5n=85$ $n=17$

Step 4: Check the answer in the original equation.		Check the answer. $\sqrt{5n-4}-9=0$ $\sqrt{5(17)-4}-9\stackrel{?}{=}0$ $\sqrt{85-4}-9\stackrel{?}{=}0$ $\sqrt{81}-9\stackrel{?}{=}0$ $9-9=0$ $0=0$ The solution is $n=17$.

Exercise 1.6.1

Solve: $\sqrt{3m+2}-5=0$.

Answer

$m=\frac{23}{3}$

Exercise 1.6.2

Solve: $\sqrt{10z+1}-2=0$.

Answer

$z=\frac{3}{10}$

Solve a Radical Equation With One Radical

1. Isolate the radical on one side of the equation.
2. Raise both sides of the equation to the power of the index.
3. Solve the new equation.
4. Check the answer in the original equation.

When we use a radical sign, it indicates the principal or positive root. If an equation has a radical with an even index equal to a negative number, that equation will have no solution.

Example 1.6.2

Solve: $\sqrt{9k-2}+1=0$.

Solution:

Table 8.6.2

	$\sqrt{9k-2}+1=0$
To isolate the radical, subtract 1 to both sides.	$\sqrt{9k-2}+1-1=0-1$
Simplify.	$\sqrt{9k-2}=-1$

Because the square root is equal to a negative number, the equation has no solution.

Exercise 1.6.3

Solve: $\sqrt{2r-3}+5=0$.

Answer

no solution

Exercise 1.6.4

Solve: $\sqrt{7s-3}+2=0$.

Answer

no solution

If one side of an equation with a square root is a binomial, we use the Product of Binomial Squares Pattern when we square it.

Definition: Binomial Squares

$(a+b)^2=a^2+2ab+b^2$
$(a-b)^2=a^2-2ab+b^2$

Don't forget the middle term!

Example 1.6.3

Solve: $\sqrt{p-1}+1=p$.

Solution:

Table 8.6.3

	$\sqrt{p-1}+1=p$
To isolate the radical, subtract 1 from both sides.	$\sqrt{p-1}+1-1=p-1$
Simplify.	$\sqrt{p-1}=p-1$
Square both sides of the equation.	$\left(\sqrt{p-1}\right)^2=(p-1)^2$
Simplify, using the Product of Binomial Squares Pattern on the right, Then solve the new equation.	$p-1=p^2-2p+1$
It is a quadratic equation, so get zero on one side.	$0=p^2-3p+2$
Factor the right side.	$0=(p-1)(p-2)$
Use the Zero Product Property.	$0=p-1 \quad 0=p-2$
Solve each equation.	$p=1 \quad p=2$
Check the answers.	
$p=1$ $\sqrt{p-1}+1=p$ $\sqrt{1-1}+1\stackrel{?}{=}1$ $\sqrt{0}+1\stackrel{?}{=}1$ $1=1$ ✓ $p=2$ $\sqrt{p-1}+1=p$ $\sqrt{2-1}+1\stackrel{?}{=}2$ $\sqrt{1}+1\stackrel{?}{=}2$ $2=2$ ✓	

The solutions are $p=1, p=2$.

Exercise 1.6.5

Solve: $\sqrt{x-2}+2=x$.

Answer

$x = 2, x = 3$

Exercise 1.6.6

Solve: $\sqrt{y-5}+5=y$.

Answer

$y = 5, y = 6$

When the index of the radical is 3, we cube both sides to remove the radical.

$$(\sqrt[3]{a})^3 = a$$

Example 1.6.4

Solve: $\sqrt[3]{5x+1}+8=4$.

Solution:

Table 8.6.4

	$\sqrt[3]{5x+1}+8=4$
To isolate the radical, subtract 8 from both sides.	$\sqrt[3]{5x+1}=-4$
Cube both sides of the equation.	$(\sqrt[3]{5x+1})^3=(-4)^3$
Simplify.	$5x+1=-64$
Solve the equation.	$5x=-65$
	$x=-13$
Check the answer.	
$x=-13$ $\quad \sqrt[3]{5x+1}+8=4$ $\sqrt[3]{5(-13)+1}+8\stackrel{?}{=}4$ $\sqrt[3]{-64}+8\stackrel{?}{=}4$ $-4+8\stackrel{?}{=}4$ $4=4\checkmark$	
	The solution is $x=-13$.

Exercise 1.6.7

Solve: $\sqrt[3]{4x-3}+8=5$

Answer

$x = -6$

Exercise 1.6.8

Solve: $\sqrt[3]{6x-10}+1=-3$

Answer

$x = -9$

Sometimes an equation will contain rational exponents instead of a radical. We use the same techniques to solve the equation as when we have a radical. We raise each side of the equation to the power of the denominator of the rational exponent. Since $(a^m)^n = a^{m \cdot n}$, we have for example,

$$\left(x^{\frac{1}{2}}\right)^2 = x, \left(x^{\frac{1}{3}}\right)^3 = x$$

Remember, $x^{\frac{1}{2}} = \sqrt{x}$ and $x^{\frac{1}{3}} = \sqrt[3]{x}$.

Example 1.6.5

Solve: $(3x-2)^{\frac{1}{4}} + 3 = 5$.

Solution:

Table 8.6.5

	$(3x-2)^{\frac{1}{4}} + 3 = 5$
To isolate the term with the rational exponent, subtract 3 from both sides.	$(3x-2)^{\frac{1}{4}} = 2$
Raise each side of the equation to the fourth power.	$\left((3x-2)^{\frac{1}{4}}\right)^4 = (2)^4$
Simplify.	$3x - 2 = 16$
Solve the equation.	$3x = 18$
	$x = 6$
Check the answer.	
$x = 6 \quad (3x-2)^{\frac{1}{4}} + 3 = 5$ $(3 \cdot 6 - 2)^{\frac{1}{4}} + 3 \stackrel{?}{=} 5$ $(16)^{\frac{1}{4}} + 3 \stackrel{?}{=} 5$ $2 + 3 \stackrel{?}{=} 5$ $5 = 5$ ✓	
	The solution is $x = 6$.

Exercise 1.6.9

Solve: $(9x+9)^{\frac{1}{4}} - 2 = 1$

Answer

$x = 8$

Exercise 1.6.10

Solve: $(4x-8)^{\frac{1}{4}} + 5 = 7$

Answer

$x = 6$

Sometimes the solution of a radical equation results in two algebraic solutions, but one of them may be an **extraneous solution**!

Example 1.6.6

Solve: $\sqrt{r+4}-r+2=0$.

Solution:

Table 8.6.6

	$\sqrt{r+4}-r+2=0$
Isolate the radical.	$\sqrt{r+4}=r-2$
Square both sides of the equation.	$(\sqrt{r+4})^2=(r-2)^2$
Simplify and then solve the equation.	$r+4=r^2-4r+4$
If it is a quadratic equation, so get zero on one side.	$0=r^2-5r$
Factor the right side.	$0=r(r-5)$
Use the Zero Product Property.	$0=r \quad 0=r-5$
Solve the equation.	$r=0 \quad r=5$
Check your answer.	
$r=0$, $\sqrt{r+4}-r+2=0$ $\sqrt{0+4}-0+2\stackrel{?}{=}0$ $\sqrt{4}+2\stackrel{?}{=}0$ $4\neq 0$ $r=5$, $\sqrt{r+4}-r+2=0$ $\sqrt{5+4}-5+2\stackrel{?}{=}0$ $\sqrt{9}-3\stackrel{?}{=}0$ $0=0\checkmark$	The solution is $r=5$.
	$r=0$ is an extreneous solution.

Exercise 1.6.11

Solve: $\sqrt{m+9}-m+3=0$

Answer

$m=7$

Exercise 1.6.12

Solve: $\sqrt{n+1}-n+1=0$.

Answer

$n=3$

When there is a coefficient in front of the radical, we must raise it to the power of the index, too.

Example 1.6.7

Solve: $3\sqrt{3x-5}-8=4$.

Solution:

Table 8.6.7

	$3\sqrt{3x-5}-8=4$
Isolate the radical term.	$3\sqrt{3x-5}=12$

Isolate the radical by dividing both sides by 3.	$\sqrt{3x-5}=4$
Square both sides of the equation.	$(\sqrt{3x-5})^2=(4)^2$
Simplify, then solve the new equation.	$3x-5=16$
	$3x=21$
Solve the equation.	$x=7$
Check the answer.	
$x=7$ $3\sqrt{3x-5}-8=4$ $3\sqrt{3(7)-5}-8\stackrel{?}{=}4$ $3\sqrt{21-5}-8\stackrel{?}{=}4$ $3\sqrt{16}-8\stackrel{?}{=}4$ $3(4)-8\stackrel{?}{=}4$ $4=4$ ✓	
	The solution is $x=7$.

Exercise 1.6.13

Solve: $2\sqrt{4a+4}-16=16$.

Answer

$a=63$

Exercise 1.6.14

Solve: $3\sqrt{2b+3}-25=50$

Answer

$b=311$

Solve Radical Equations with Two Radicals

If the radical equation has two radicals, we start out by isolating one of them. It often works out easiest to isolate the more complicated radical first.

In the next example, when one radical is isolated, the second radical is also isolated.

Example 1.6.8

Solve: $\sqrt[3]{4x-3}=\sqrt[3]{3x+2}$.

Solution:

The radical terms are isolated.

$$\sqrt[3]{4x-3}=\sqrt[3]{3x+2}$$

Since the index is 3, cube both sides of the equation.

$$(\sqrt[3]{4x-3})^3=(\sqrt[3]{3x+2})^3$$

Simplify, then solve the new equation.

$$\begin{aligned} 4x-3 &= 3x+2 \\ x-3 &= 2 \\ x &= 5 \end{aligned}$$

The solution is $x=5$.

Check the answer.

We leave it to you to show that 5 checks!

Exercise 1.6.15

Solve: $\sqrt[3]{5x-4}=\sqrt[3]{2x+5}$.

Answer

$x=3$

Exercise 1.6.16

Solve: $\sqrt[3]{7x+1}=\sqrt[3]{2x-5}$.

Answer

$x=-\frac{6}{5}$

Sometimes after raising both sides of an equation to a power, we still have a variable inside a radical. When that happens, we repeat Step 1 and Step 2 of our procedure. We isolate the radical and raise both sides of the equation to the power of the index again.

Example 1.6.9 how to solve a radical equation

Solve: $\sqrt{m}+1=\sqrt{m+9}$.

Solution:

Table 8.6.8

Step 1: Isolate one of the radical terms on one side of the equation.	The radical on the right is isolated.	$\sqrt{m}+1=\sqrt{m+9}$
Step 2: Raise both sides of the equation to the power of the index.	We square both sides. Simplify--be very careful as you multiply!	$(\sqrt{m}+1)^2=(\sqrt{m+9})^2$
Step 3: Are there any more radicals? If yes, repeat Step 1 and Step 2 again. If no, solve the new equation.	There is still a radical in the equation. So we must repeat the previous steps. Isolate the radical term. Here, we can easily isolate the radical by dividing both sides by 2. Square both sides.	$\begin{aligned} m+2\sqrt{m}+1 &= m+9 \\ 2\sqrt{m} &= 8 \\ \sqrt{m} &= 4 \\ (\sqrt{m})^2 &= (4)^2 \\ m &= 16 \end{aligned}$
Step 4: Check the answer in the original equation.		$\begin{aligned} \sqrt{m}+1 &= \sqrt{m+9} \\ \sqrt{16}+1 &\stackrel{?}{=} \sqrt{16+9} \\ 4+1 &\stackrel{?}{=} 5 \\ 5 &= 5 \end{aligned}$ The solution is $m=16$.

Exercise 1.6.17

Solve: $3-\sqrt{x}=\sqrt{x-3}$.

Answer

$x=4$

Exercise 1.6.18

Solve: $\sqrt{x}+2=\sqrt{x+16}$.

Answer

$x=9$

We summarize the steps here. We have adjusted our previous steps to include more than one radical in the equation This procedure will now work for any radical equations.

Solve a Radical Equation

1. Isolate one of the radical terms on one side of the equation.
2. Raise both sides of the equation to the power of the index.
3. Are there any more radicals?
   If yes, repeat Step 1 and Step 2 again.
   If no, solve the new equation.
4. Check the answer in the original equation.

Be careful as you square binomials in the next example. Remember the pattern in $(a+b)^2=a^2+2ab+b^2$ or $(a-b)^2=a^2-2ab+b^2$.

Example 1.6.10

Solve: $\sqrt{q-2}+3=\sqrt{4q+1}$.

Solution:

Table 8.6.9

	$\sqrt{q-2}+3=\sqrt{4q+1}$
The radical on the right is isolated. Square both sides.	$\left(\sqrt{q-2}+3\right)^2=\left(\sqrt{4q+1}\right)^2$
Simplify.	$q-2+6\sqrt{q-2}+9=4q+1$
There is still a radical in the equation so we must repeat the previous steps. Isolate the radical.	$6\sqrt{q-2}=3q-6$
Square both sides. It would not help to divide both sides by 6. Remember to square both the 6 and the $\sqrt{q-2}$.	$\left(6\sqrt{q-2}\right)^2=\left(3q-6\right)^2$ $a^2 - 2ab + b^2$ $6^2\left(\sqrt{q-2}\right)^2=(3q)^2-2\cdot 3q\cdot 6+6^2$
Simplify, then solve the new equation.	$36(q-2)=9q^2-36q+36$
Distribute.	$36q-72=9q^2-36q+36$
It is a quadratic equation, so get zero on one side.	$0=9q^2-72q+108$
Factor the right side.	$0=9(q^2-8q+12)$ $0=9(q-6)(q-2)$

Use the Zero Product Property.	$q-6=0 \quad q-2=0$ $q=6 \quad q=2$
The checks are left to you.	The solutions are $q=6$ and $q=2$.

Exercise 1.6.19

Solve: $\sqrt{x-1}+2=\sqrt{2x+6}$

Answer

$x=5$

Exercise 1.6.20

Solve: $\sqrt{x}+2=\sqrt{3x+4}$

Answer

$x=0x=4$

Use Radicals in Applications

As you progress through your college courses, you'll encounter formulas that include radicals in many disciplines. We will modify our Problem Solving Strategy for Geometry Applications slightly to give us a plan for solving applications with formulas from any discipline.

Use a Problem Solving Strategy for Applications With Formulas

1. **Read** the problem and make sure all the words and ideas are understood. When appropriate, draw a figure and label it with the given information.
2. **Identify** what we are looking for.
3. **Name** what we are looking for by choosing a variable to represent it.
4. **Translate** into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.
5. **Solve the equation** using good algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

One application of radicals has to do with the effect of **gravity** on falling objects. The formula allows us to determine how long it will take a fallen object to hit the gound.

Definition: Falling Objects

On Earth, if an object is dropped from a height of h feet, the time in seconds it will take to reach the ground is found by using the formula

$$t=\frac{\sqrt{h}}{4}$$

For example, if an object is dropped from a height of 64 feet, we can find the time it takes to reach the ground by substituting $h=64$ into the formula.

Table 8.6.10

	$t=\frac{\sqrt{h}}{4}$
	$t=\frac{\sqrt{64}}{4}$

Take the square root of 64.	$t=\frac{8}{4}$
Simplify the fraction.	$t=2$

It would take 2 seconds for an object dropped from a height of 64 feet to reach the ground.

Example 1.6.11

Marissa dropped her sunglasses from a bridge 400 feet above a river. Use the formula $t=\frac{\sqrt{h}}{4}$ to find how many seconds it took for the sunglasses to reach the river.

Solution:

Table 8.6.11

Step 1: **Read** the problem.	
Step 2: **Identify** what we are looking.	The time it takes for the sunglasses to reach the river.
Step 3: **Name** what we are looking.	Let (t=\) time.
Step 4: **Translate** into an equation by writing the appropriate formula. Substitute in the given information.	$t=\frac{\sqrt{h}}{4}$, and $h=400$ $t=\frac{\sqrt{400}}{4}$
Step 5: **Solve the equation**.	$t=\frac{20}{4}$
	$t=5$
Step 6: **Check** the answer in the problem and make sure it makes sense.	$5\stackrel{?}{=}\frac{\sqrt{400}}{4}$ $5\stackrel{?}{=}\frac{20}{4}$ $5=5\checkmark$
Does 5 seconds seem like a reasonable length of time?	Yes.
Step 7: **Answer** the equation.	It will take 5 seconds for the sunglasses to reach the river.

Exercise 1.6.21

A helicopter dropped a rescue package from a height of $1,296$ feet. Use the formula $t=\frac{\sqrt{h}}{4}$ to find how many seconds it took for the package to reach the ground.

Answer

9 seconds

Exercise 1.6.22

A window washer dropped a squeegee from a platform 196 feet above the sidewalk. Use the formula $t=\frac{\sqrt{h}}{4}$ to find how many seconds it took for the squeegee to reach the sidewalk.

Answer

3.5 seconds

Police officers investigating car accidents measure the length of the skid marks on the pavement. Then they use square roots to determine the **speed**, in miles per hour, a car was going before applying the brakes.

Definition: Skid Marks and Speed of a Car

If the length of the skid marks is d feet, then the speed, s, of the car before the brakes were applied can be found by using the formula

$$s = \sqrt{24d}$$

Example 1.6.12

After a car accident, the skid marks for one car measured 190 feet. Use the formula $s = \sqrt{24d}$ to find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.

Solution:

Table 8.6.12

Step 1: **Read** the problem.	
Step 2: **Identify** what we are looking for.	The speed of a car.
Step 3: **Name** what we are looking for.	Let $s =$ the speed.
Step 4: **Translate** into an equation by writing the appropriate formula. Substitute in the given information.	$s = \sqrt{24d}$, and $d = 190$ $s = \sqrt{24(190)}$
Step 5: **Solve the equation**.	$s = \sqrt{4{,}560}$
	$s = 67.52777...$
Round to 1 decimal place.	$s \approx 67.5$
	$67.5 \stackrel{?}{\approx} \sqrt{24(190)}$ $67.5 \stackrel{?}{\approx} \sqrt{4560}$ $67.5 \approx 67.5277...$ ✓
	The speed of the car before the brakes were applied was 67.5 miles per hour.

Exercise 1.6.23

An accident investigator measured the skid marks of the car. The length of the skid marks was 76 feet. Use the formula $s = \sqrt{24d}$ to find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.

Answer

42.7mph

Exercise 1.6.24

The skid marks of a vehicle involved in an accident were 122 feet long. Use the formula $s = \sqrt{24d}$ to find the speed of the vehicle before the brakes were applied. Round your answer to the nearest tenth.

Answer

54.1mph

Access these online resources for additional instruction and practice with solving radical equations.

- Solving an Equation Involving a Single Radical
- Solving Equations with Radicals and Rational Exponents
- Solving Radical Equations
- Solve Radical Equations
- Radical Equation Application

Key Concepts

- **Binomial Squares**
  $(a+b)^2 = a^2 + 2ab + b^2$
  $(a-b)^2 = a^2 - 2ab + b^2$
- **Solve a Radical Equation**
  1. Isolate one of the radical terms on one side of the equation.
  2. Raise both sides of the equation to the power of the index.
  3. Are there any more radicals?
     If yes, repeat Step 1 and Step 2 again.
     If no, solve the new equation.
  4. Check the answer in the original equation.
- **Problem Solving Strategy for Applications with Formulas**
  1. Read the problem and make sure all the words and ideas are understood. When appropriate, draw a figure and label it with the given information.
  2. Identify what we are looking for.
  3. Name what we are looking for by choosing a variable to represent it.
  4. Translate into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.
  5. Solve the equation using good algebra techniques.
  6. Check the answer in the problem and make sure it makes sense.
  7. Answer the question with a complete sentence.
- **Falling Objects**
  - On Earth, if an object is dropped from a height of h feet, the time in seconds it will take to reach the ground is found by using the formula $t = \frac{\sqrt{h}}{4}$.
- **Skid Marks and Speed of a Car**
  - If the length of the skid marks is d feet, then the speed, s, of the car before the brakes were applied can be found by using the formula $s = \sqrt{24d}$.

Glossary

radical equation

An equation in which a variable is in the radicand of a radical expression is called a radical equation.

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CHAPTER OVERVIEW

2: Functions

2.1: Functions and Function Notation
2.2: Domain and Range
2.3: Rates of Change and Behavior of Graphs
2.4: Composition of Functions
2.5: Inverse Functions
2.6: Transformation of Functions

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2.1: Functions and Function Notation

Determining Whether a Relation Represents a Function

A relation is a set of ordered pairs. The set of the first components of each ordered pair is called the domain and the set of the second components of each ordered pair is called the range. Consider the following set of ordered pairs. The first numbers in each pair are the first five natural numbers. The second number in each pair is twice that of the first.

$$\{(1,2),(2,4),(3,6),(4,8),(5,10)\} \tag{1.1.1}$$

The domain is $\{1,2,3,4,5\}$ The range is $\{2,4,6,8,10\}$

Note that each value in the domain is also known as an **input** value, or **independent variable**, and is often labeled with the lowercase letter x. Each value in the range is also known as an output value, or **dependent variable**, and is often labeled lowercase letter y.

A function f is a relation that assigns a single value in the range to each value in the domain. In other words, no x-values are repeated. For our example that relates the first five **natural numbers** to numbers double their values, this relation is a function because each element in the domain, {1, 2, 3, 4, 5}, is paired with exactly one element in the range, $\{2,4,6,8,10\}$

Now let's consider the set of ordered pairs that relates the terms "even" and "odd" to the first five natural numbers. It would appear as

$$\{(\text{odd},1),(\text{even},2),(\text{odd},3),(\text{even},4),(\text{odd},5)\} \tag{1.1.2}$$

Notice that each element in the domain, {even, odd} is not paired with exactly one element in the range, $\{1,2,3,4,5\}$ For example, the term "odd" corresponds to three values from the range, $\{1,3,5\}$,and the term "even" corresponds to two values from the range, $\{2,4\}$. This violates the definition of a function, so this relation is not a function.

Figure 2.1.1 compares relations that are functions and not functions.

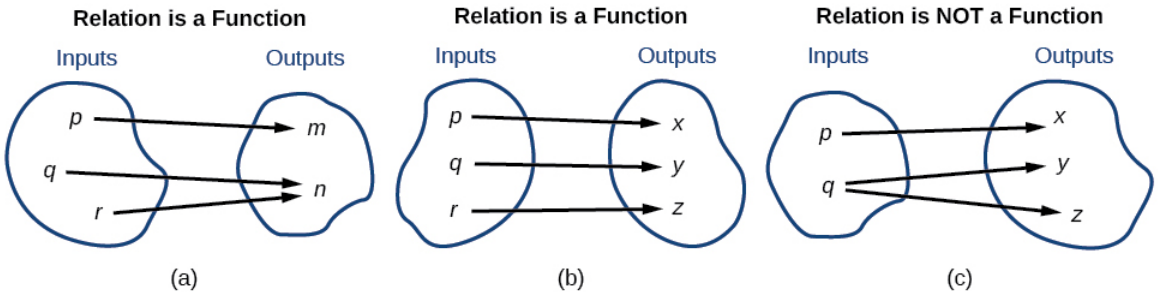


Figure 2.1.1: (a) This relationship is a function because each input is associated with a single output. Note that input q and r both give output n. (b) This relationship is also a function. In this case, each input is associated with a single output. (c) This relationship is not a function because input q is associated with two different outputs.

Function

A **function** is a relation in which each possible input value leads to exactly one output value. We say "the output is a function of the input."

The **input** values make up the **domain**, and the **output** values make up the **range**.

How To: Given a relationship between two quantities, determine whether the relationship is a function

1. Identify the input values.
2. Identify the output values.
3. If each input value leads to only one output value, classify the relationship as a function. If any input value leads to two or more outputs, do not classify the relationship as a function.

Example 2.1.1: Determining If Menu Price Lists Are Functions

The coffee shop menu, shown in Figure 2.1.2 consists of items and their prices.

a. Is price a function of the item?
b. Is the item a function of the price?

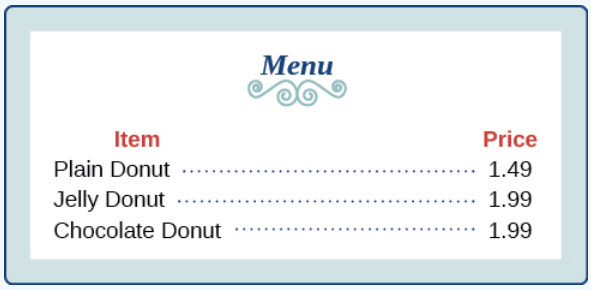


Figure 2.1.2: A menu of donut prices from a coffee shop where a plain donut is $1.49 and a jelly donut and chocolate donut are $1.99.

Solution

a. Let's begin by considering the input as the items on the menu. The output values are then the prices. See Figure 2.1.3.



Figure 2.1.3: A menu of donut prices from a coffee shop where a plain donut is $1.49 and a jelly donut and chocolate donut are $1.99.

Each item on the menu has only one price, so the price is a function of the item.

a. Two items on the menu have the same price. If we consider the prices to be the input values and the items to be the output, then the same input value could have more than one output associated with it. See Figure 2.1.4.

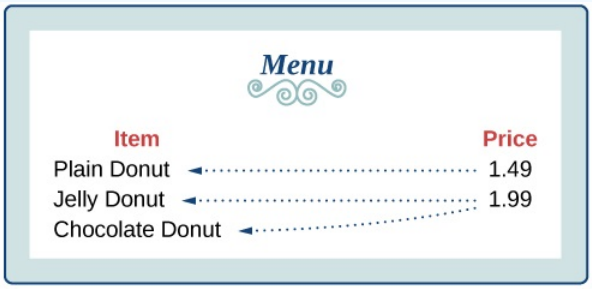


Figure 2.1.4: Association of the prices to the donuts.

Therefore, the item is a not a function of price.

Example 2.1.2: Determining If Class Grade Rules Are Functions

In a particular math class, the overall percent grade corresponds to a grade point average. Is grade point average a function of the percent grade? Is the percent grade a function of the grade point average? Table 2.1.1 shows a possible rule for assigning grade points.

Table 2.1.1: Class grade points.

Percent grade	0–56	57–61	62–66	67–71	72–77	78–86	87–91	92–100
Grade point average	0.0	1.0	1.5	2.0	2.5	3.0	3.5	4.0

Solution

For any percent grade earned, there is an associated grade point average, so the grade point average is a function of the percent grade. In other words, if we input the percent grade, the output is a specific grade point average.

In the grading system given, there is a range of percent grades that correspond to the same grade point average. For example, students who receive a grade point average of 3.0 could have a variety of percent grades ranging from 78 all the way to 86. Thus, percent grade is not a function of grade point average.

Exercise 2.1.2

Table 2.1.2 lists the five greatest baseball players of all time in order of rank.

Table 2.1.2: Five greatest baseball players.

Player	Rank
Babe Ruth	1
Willie Mays	2
Ty Cobb	3
Walter Johnson	4
Hank Aaron	5

a. Is the rank a function of the player name?
b. Is the player name a function of the rank?

Answer a

Yes

Answer b

yes. (Note: If two players had been tied for, say, 4th place, then the name would not have been a function of rank.)

Using Function Notation

Once we determine that a relationship is a function, we need to display and define the functional relationships so that we can understand and use them, and sometimes also so that we can program them into computers. There are various ways of representing functions. A standard function notation is one representation that facilitates working with functions.

To represent "height is a function of age," we start by identifying the descriptive variables h for height and a for age. The letters f, g,and h are often used to represent functions just as we use x, y,and z to represent numbers and A, B, and C to represent sets.

$$\begin{aligned} &h \text{ is } f \text{ of } a && \text{We name the function } f\text{; height is a function of age.} \\ &h = f(a) && \text{We use parentheses to indicate the function input.} \\ &f(a) && \text{We name the function } f \text{ ; the expression is read as "} f \text{ of } a\text{."} \end{aligned} \tag{2.1.1}$$

Remember, we can use any letter to name the function; the notation $h(a)$ shows us that h depends on a. The value a must be put into the function h to get a result. The parentheses indicate that age is input into the function; they do not indicate multiplication.

We can also give an algebraic expression as the input to a function. For example $f(a+b)$ means "first add a and b, and the result is the input for the function f." The operations must be performed in this order to obtain the correct result.

Function Notation

The notation $y=f(x)$ defines a function named f. This is read as "y is a function of x." The letter x represents the input value, or independent variable. The letter y, or $f(x)$, represents the output value, or dependent variable.

Example 2.1.3: Using Function Notation for Days in a Month

Use function notation to represent a function whose input is the name of a month and output is the number of days in that month.

Solution

Using Function Notation for Days in a Month

Use function notation to represent a function whose input is the name of a month and output is the number of days in that month.

The number of days in a month is a function of the name of the month, so if we name the function f, we write $\text{days}=f(\text{month})$ or $d=f(m)$. The name of the month is the input to a "rule" that associates a specific number (the output) with each input.

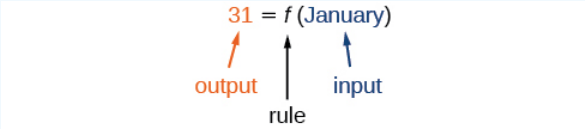


Figure 2.1.5: The function $31=f(January)$ where 31 is the output, f is the rule, and January is the input.

For example, $f(\text{March})=31$, because March has 31 days. The notation $d=f(m)$ reminds us that the number of days, d (the output), is dependent on the name of the month, m (the input).

Analysis

Note that the inputs to a function do not have to be numbers; function inputs can be names of people, labels of geometric objects, or any other element that determines some kind of output. However, most of the functions we will work with in this book will have numbers as inputs and outputs.

Example 2.1.3B: Interpreting Function Notation

A function $N=f(y)$ gives the number of police officers, N, in a town in year y. What does $f(2005)=300$ represent?

Solution

When we read $f(2005)=300$, we see that the input year is 2005. The value for the output, the number of police officers (N), is 300. Remember, $N=f(y)$. The statement $f(2005)=300$ tells us that in the year 2005 there were 300 police officers in the town.

Exercise 2.1.3

Use function notation to express the weight of a pig in pounds as a function of its age in days d.

Answer

$w=f(d)$

Q&A

Instead of a notation such as $y = f(x)$, could we use the same symbol for the output as for the function, such as $y = y(x)$, meaning "y is a function of x?"

Yes, this is often done, especially in applied subjects that use higher math, such as physics and engineering. However, in exploring math itself we like to maintain a distinction between a function such as f, which is a rule or procedure, and the output y we get by applying f to a particular input x. This is why we usually use notation such as $y = f(x)$, $P = W(d)$, and so on.

Finding Input and Output Values of a Function

When we know an input value and want to determine the corresponding output value for a function, we evaluate the function. Evaluating will always produce one result because each input value of a function corresponds to exactly one output value.

When we know an output value and want to determine the input values that would produce that output value, we set the output equal to the function's formula and solve for the input. Solving can produce more than one solution because different input values can produce the same output value.

Evaluation of Functions in Algebraic Forms

When we have a function in formula form, it is usually a simple matter to evaluate the function. For example, the function $f(x) = 5 - 3x^2$ can be evaluated by squaring the input value, multiplying by 3, and then subtracting the product from 5.

How To: Given the formula for a function, evaluate.

Given the formula for a function, evaluate.

1. Replace the input variable in the formula with the value provided.
2. Calculate the result.

Example 2.1.6A: Evaluating Functions at Specific Values

1. Evaluate $f(x) = x^2 + 3x - 4$ at
   a. 2
   b. a
   c. $a + h$
   d. Evaluate $\frac{f(a+h) - f(a)}{h}$

Solution

Replace the x in the function with each specified value.

a. Because the input value is a number, 2, we can use simple algebra to simplify.

$$\begin{aligned} f(2) &= 2^2 + 3(2) - 4 \\ &= 4 + 6 - 4 \\ &= 6 \end{aligned}$$

b. In this case, the input value is a letter so we cannot simplify the answer any further.

$$f(a) = a^2 + 3a - 4$$

c. With an input value of $a + h$, we must use the distributive property.

$$\begin{aligned} f(a+h) &= (a+h)^2 + 3(a+h) - 4 \\ &= a^2 + 2ah + h^2 + 3a + 3h - 4 \end{aligned}$$

d. In this case, we apply the input values to the function more than once, and then perform algebraic operations on the result. We already found that

$$f(a+h) = a^2 + 2ah + h^2 + 3a + 3h - 4$$

and we know that

$$f(a) = a^2 + 3a - 4$$

Now we combine the results and simplify.

$$\begin{aligned} \frac{f(a+h)-f(a)}{h} &= \frac{(a^2+2ah+h^2+3a+3h-4)-(a^2+3a-4)}{h} \\ &= \frac{(2ah+h^2+3h)}{h} \\ &= \frac{h(2a+h+3)}{h} && \text{Factor out h.} \\ &= 2a+h+3 && \text{Simplify.} \end{aligned}$$

Example 2.1.6B: Evaluating Functions

Given the function $h(p) = p^2 + 2p$, evaluate $h(4)$.

Solution

To evaluate $h(4)$, we substitute the value 4 for the input variable p in the given function.

$$\begin{aligned} h(p) &= p^2 + 2p \\ h(4) &= (4)^2 + 2(4) \\ &= 16 + 8 \\ &= 24 \end{aligned}$$

Therefore, for an input of 4, we have an output of 24.

Exercise 2.1.6

Given the function $g(m) = \sqrt{m-4}$, evaluate $g(5)$.

Answer

$g(5) = 1$

Example 2.1.7: Solving Functions

Given the function $h(p) = p^2 + 2p$, solve for $h(p) = 3$.

Solution

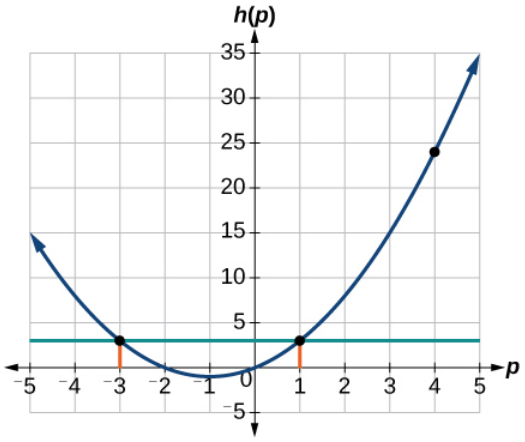
$$\begin{aligned} h(p) &= 3 \\ p^2 + 2p &= 3 && \text{Substitute the original function} \\ p^2 + 2p - 3 &= 0 && \text{Subtract 3 from each side.} \\ (p+3)(p-1) &= 0 && \text{Factor.} \end{aligned}$$

If $(p+3)(p-1) = 0$, either $(p+3) = 0$ or $(p-1) = 0$ (or both of them equal 0). We will set each factor equal to 0 and solve for p in each case.

$$(p+3) = 0,\ p = -3$$

$$(p-1) = 0,\ p = 1$$

This gives us two solutions. The output $h(p) = 3$ when the input is either $p = 1$ or $p = -3$. We can also verify by graphing as in Figure 2.1.6. The graph verifies that $h(1) = h(-3) = 3$ and $h(4) = 24$.



p	−3	−2	0	1	4
h(p)	3	0	0	3	24

Figure 2.1.6: Graph of $h(p) = p^2 + 2p$

Exercise 2.1.7

Given the function $g(m) = \sqrt{m-4}$, solve $g(m) = 2$.

Answer

$m = 8$

Evaluation of Functions in Algebraic Forms

When we have a function in formula form, it is usually a simple matter to evaluate the function. For example, the function $f(x) = 5 - 3x^2$ can be evaluated by squaring the input value, multiplying by 3, and then subtracting the product from 5.

How To: Given the formula for a function, evaluate.

Given the formula for a function, evaluate.

1. Replace the input variable in the formula with the value provided.
2. Calculate the result.

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1. Evaluate $f(x) = x^2 + 3x - 4$ at

a. 2
b. a
c. $a + h$
d. Evaluate $\frac{f(a+h) - f(a)}{h}$

Solution

Replace the x in the function with each specified value.

a. Because the input value is a number, 2, we can use simple algebra to simplify.

$$\begin{aligned} f(2) &= 2^2 + 3(2) - 4 \\ &= 4 + 6 - 4 \\ &= 6 \end{aligned}$$

b. In this case, the input value is a letter so we cannot simplify the answer any further.

$$f(a) = a^2 + 3a - 4$$

c. With an input value of $a+h$, we must use the distributive property.

$$\begin{aligned} f(a+h) &= (a+h)^2 + 3(a+h) - 4 \\ &= a^2 + 2ah + h^2 + 3a + 3h - 4 \end{aligned}$$

d. In this case, we apply the input values to the function more than once, and then perform algebraic operations on the result. We already found that

$$f(a+h) = a^2 + 2ah + h^2 + 3a + 3h - 4$$

and we know that

$$f(a) = a^2 + 3a - 4$$

Now we combine the results and simplify.

$$\begin{aligned} \frac{f(a+h) - f(a)}{h} &= \frac{(a^2 + 2ah + h^2 + 3a + 3h - 4) - (a^2 + 3a - 4)}{h} \\ &= \frac{(2ah + h^2 + 3h)}{h} \\ &= \frac{h(2a + h + 3)}{h} && \text{Factor out h.} \\ &= 2a + h + 3 && \text{Simplify.} \end{aligned}$$

Example 2.1.6B: Evaluating Functions

Given the function $h(p) = p^2 + 2p$, evaluate $h(4)$.

Solution

To evaluate $h(4)$, we substitute the value 4 for the input variable p in the given function.

$$\begin{aligned} h(p) &= p^2 + 2p \\ h(4) &= (4)^2 + 2(4) \\ &= 16 + 8 \\ &= 24 \end{aligned}$$

Therefore, for an input of 4, we have an output of 24.

Exercise 2.1.6

Given the function $g(m) = \sqrt{m-4}$, evaluate $g(5)$.

Answer

$g(5) = 1$

Example 2.1.7: Solving Functions

Given the function $h(p) = p^2 + 2p$, solve for $h(p) = 3$.

Solution

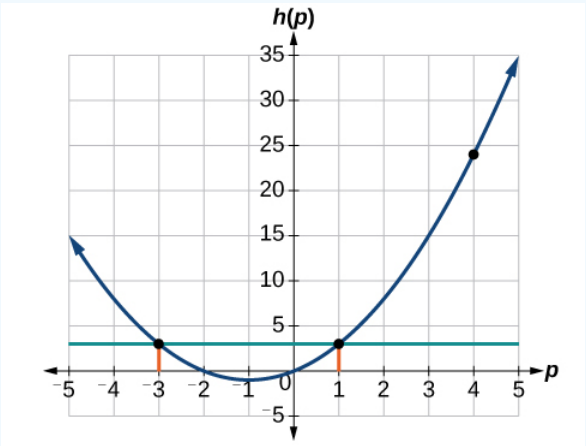
$$\begin{aligned} h(p) &= 3 \\ p^2 + 2p &= 3 && \text{Substitute the original function} \\ p^2 + 2p - 3 &= 0 && \text{Subtract 3 from each side.} \\ (p+3)(p-1) &= 0 && \text{Factor.} \end{aligned}$$

If $(p+3)(p-1)=0$, either $(p+3)=0$ or $(p-1)=0$ (or both of them equal 0). We will set each factor equal to 0 and solve for p in each case.

$$(p+3)=0,\ p=-3$$

$$(p-1)=0,\ p=1$$

This gives us two solutions. The output $h(p)=3$ when the input is either $p=1$ or $p=-3$. We can also verify by graphing as in Figure 2.1.6. The graph verifies that $h(1)=h(-3)=3$ and $h(4)=24$.



p	−3	−2	0	1	4
h(p)	3	0	0	3	24

Figure 2.1.6: Graph of $h(p)=p^2+2p$

Exercise 2.1.7

Given the function $g(m)=\sqrt{m-4}$, solve $g(m)=2$.

Answer

$m=8$

Evaluating Functions Expressed in Formulas

Some functions are defined by mathematical rules or procedures expressed in **equation** form. If it is possible to express the function output with a formula involving the input quantity, then we can define a function in algebraic form. For example, the equation $2n+6p=12$ expresses a functional relationship between n and p. We can rewrite it to decide if p is a function of n.

How to: Given a function in equation form, write its algebraic formula.

1. Solve the equation to isolate the output variable on one side of the equal sign, with the other side as an expression that involves only the input variable.
2. Use all the usual algebraic methods for solving equations, such as adding or subtracting the same quantity to or from both sides, or multiplying or dividing both sides of the equation by the same quantity.

Example 2.1.8A: Finding an Equation of a Function

Express the relationship $2n+6p=12$ as a function $p=f(n)$, if possible.

Solution

To express the relationship in this form, we need to be able to write the relationship where p is a function of n, which means writing it as $p=[\text{expression involving } n]$.

$$\begin{aligned} 2n+6p &= 12 \\ 6p &= 12-2n && \text{Subtract 2n from both sides.} \\ p &= \frac{12-2n}{6} && \text{Divide both sides by 6 and simplify.} \\ p &= \frac{12}{6}-\frac{2n}{6} \\ p &= 2-\frac{1}{3}n \end{aligned}$$

Therefore, p as a function of n is written as

$$p = f(n) = 2-\frac{1}{3}n$$

Analysis

It is important to note that not every relationship expressed by an equation can also be expressed as a function with a formula.

Example 2.1.8B: Expressing the Equation of a Circle as a Function

Does the equation $x^2+y^2=1$ represent a function with x as input and y as output? If so, express the relationship as a function $y=f(x)$.

Solution

First we subtract x^2 from both sides.

$$y^2 = 1-x^2$$

We now try to solve for y in this equation.

$$y = \pm\sqrt{1-x^2}$$

$$\text{so, } y = \sqrt{1-x^2} \text{ and } y = -\sqrt{1-x^2}$$

We get two outputs corresponding to the same input, so this relationship cannot be represented as a single function $y=f(x)$.

Exercise 2.1.8

If $x-8y^3=0$, express y as a function of x.

Answer

$$y = f(x) = \frac{\sqrt[3]{x}}{2}$$

Q & A

Are there relationships expressed by an equation that do represent a function but which still cannot be represented by an algebraic formula?

Yes, this can happen. For example, given the equation $x=y+2^y$, if we want to express y as a function of x, there is no simple algebraic formula involving only x that equals y. However, each x does determine a unique value for y, and there are mathematical procedures by which y can be found to any desired accuracy. In this case, we say that the equation gives an implicit (implied) rule for y as a function of x, even though the formula cannot be written explicitly.

Evaluating a Function Given in Tabular Form

As we saw above, we can represent functions in tables. Conversely, we can use information in tables to write functions, and we can evaluate functions using the tables. For example, how well do our pets recall the fond memories we share with them? There is an urban legend that a goldfish has a memory of 3 seconds, but this is just a myth. Goldfish can remember up to 3 months, while the

beta fish has a memory of up to 5 months. And while a puppy's memory span is no longer than 30 seconds, the adult dog can remember for 5 minutes. This is meager compared to a cat, whose memory span lasts for 16 hours.

The function that relates the type of pet to the duration of its memory span is more easily visualized with the use of a table (Table 2.1.10).

Table 2.1.10

Pet Memory	span in hours
Puppy	0.008
Adult Dog	0.083
Cat	3
Goldfish	2160
Beta Fish	3600

At times, evaluating a function in table form may be more useful than using equations. Here let us call the function P. The domain of the function is the type of pet and the range is a real number representing the number of hours the pet's memory span lasts. We can evaluate the function P at the input value of "goldfish." We would write $P(goldfish) = 2160$. Notice that, to evaluate the function in table form, we identify the input value and the corresponding output value from the pertinent row of the table. The tabular form for function P seems ideally suited to this function, more so than writing it in paragraph or function form.

How To: Given a function represented by a table, identify specific output and input values

1. Find the given input in the row (or column) of input values.
2. Identify the corresponding output value paired with that input value.
3. Find the given output values in the row (or column) of output values, noting every time that output value appears.
4. Identify the input value(s) corresponding to the given output value.

Example 2.1.9: Evaluating and Solving a Tabular Function

Using Table 2.1.11,

a. Evaluate $g(3)$.
b. Solve $g(n) = 6$.

Table 2.1.11

n	1	2	3	4	5
$g(n)$	8	6	7	6	8

Solution

a. Evaluating $g(3)$ means determining the output value of the function g for the input value of $n = 3$. The table output value corresponding to $n = 3$ is 7, so $g(3) = 7$.
b. Solving $g(n) = 6$ means identifying the input values, n,that produce an output value of 6. Table 2.1.12 shows two solutions: 2 and 4.

Table 2.1.12

n	1	2	3	4	5
$g(n)$	8	6	7	6	8

When we input 2 into the function g, our output is 6. When we input 4 into the function g, our output is also 6.

Exercise 2.1.1

Using Table 2.1.12, evaluate $g(1)$.

Answer

$g(1) = 8$

Finding Function Values from a Graph

Evaluating a function using a graph also requires finding the corresponding output value for a given input value, only in this case, we find the output value by looking at the graph. Solving a function equation using a graph requires finding all instances of the given output value on the graph and observing the corresponding input value(s).

Example 2.1.10: Reading Function Values from a Graph

Given the graph in Figure 2.1.7,

a. Evaluate $f(2)$.
b. Solve $f(x) = 4$.

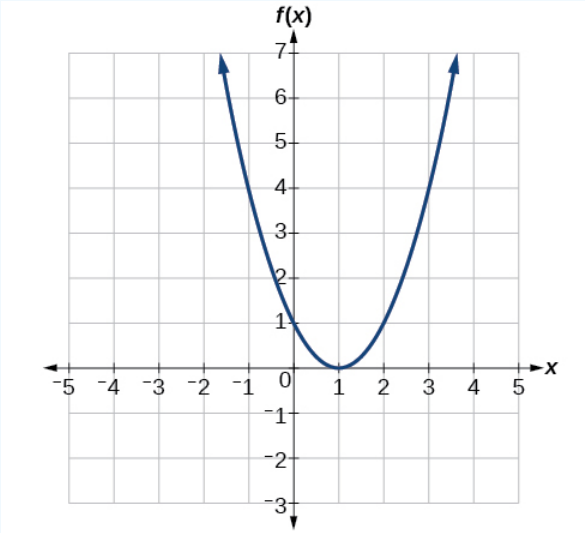
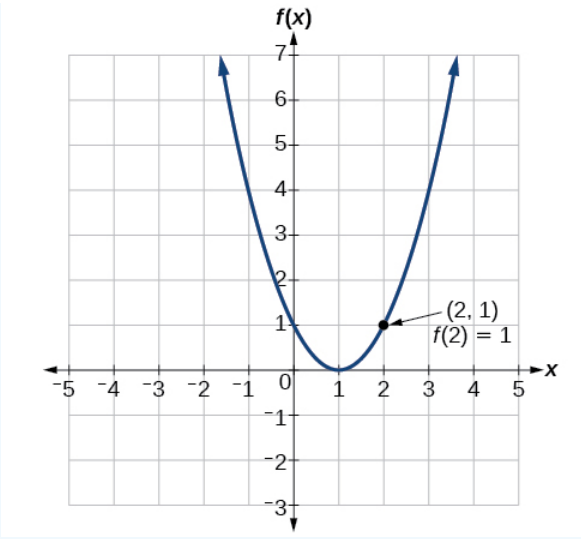


Figure 2.1.7: Graph of a positive parabola centered at $(1, 0)$.

Solution

To evaluate $f(2)$, locate the point on the curve where $x = 2$, then read the y-coordinate of that point. The point has coordinates $(2, 1)$, so $f(2) = 1$. See Figure 2.1.8.



2.1.8: Graph of a positive parabola centered at $(1,0)$ with the labeled point $(2,1)$ where $f(2)=1$.

To solve $f(x)=4$, we find the output value 4 on the vertical axis. Moving horizontally along the line $y=4$, we locate two points of the curve with output value 4: $(-1,4)$ and $(3,4)$. These points represent the two solutions to $f(x)=4$: -1 or 3. This means $f(-1)=4$ and $f(3)=4$, or when the input is -1 or 3, the output is 4. See Figure 2.1.9.

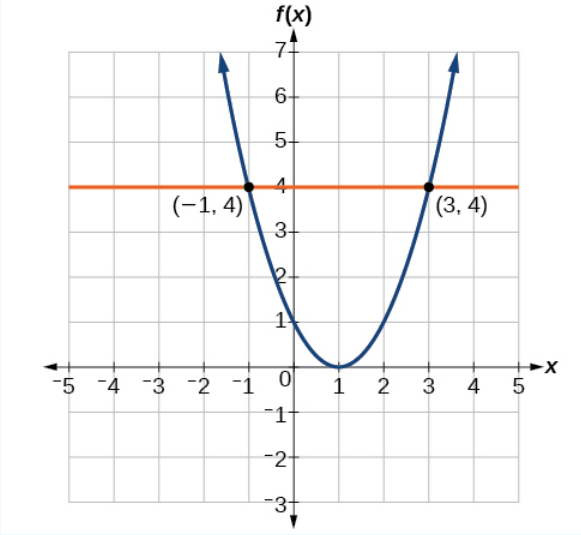


Figure 2.1.9: Graph of an upward-facing parabola with a vertex at $(0,1)$ and labeled points at $(-1,4)$ and $(3,4)$. A line at $y=4$ intersects the parabola at the labeled points.

Exercise 2.1.10

Given the graph in Figure 2.1.7, solve $f(x)=1$.

Answer

$x=0$ or $x=2$

Determining Whether a Function is One-to-One

Some functions have a given output value that corresponds to two or more input values. For example, in the stock chart shown in the Figure at the beginning of this chapter, the stock price was $1000 on five different dates, meaning that there were five different input values that all resulted in the same output value of $1000.

However, some functions have only one input value for each output value, as well as having only one output for each input. We call these functions one-to-one functions. As an example, consider a school that uses only letter grades and decimal equivalents, as listed in Table 2.1.13

Table 2.1.13: Letter grades and decimal equivalents.

Letter Grade	Grade Point Average
A	4.0
B	3.0
C	2.0
D	1.0

This grading system represents a one-to-one function, because each letter input yields one particular grade point average output and each grade point average corresponds to one input letter.

To visualize this concept, let's look again at the two simple functions sketched in Figures 2.1.1a and 2.1.1b The function in part (a) shows a relationship that is not a one-to-one function because inputs q and r both give output n. The function in part (b) shows a relationship that is a one-to-one function because each input is associated with a single output.

One-to-One Functions

A **one-to-one function** is a function in which each output value corresponds to exactly one input value.

Example 2.1.11: Determining Whether a Relationship Is a One-to-One Function

Is the area of a circle a function of its radius? If yes, is the function one-to-one?

Solution

A circle of radius r has a unique area measure given by $A = \pi r^2$, so for any input, r, there is only one output, A. The area is a function of radius$r$.

If the function is one-to-one, the output value, the area, must correspond to a unique input value, the radius. Any area measure A is given by the formula $A = \pi r^2$. Because areas and radii are positive numbers, there is exactly one solution: $\sqrt{\frac{A}{\pi}}$. So the area of a circle is a one-to-one function of the circle's radius.

Exercise 2.1.11A

a. Is a balance a function of the bank account number?
b. Is a bank account number a function of the balance?
c. Is a balance a one-to-one function of the bank account number?

Answer

a. yes, because each bank account has a single balance at any given time;

b. no, because several bank account numbers may have the same balance;

c. no, because the same output may correspond to more than one input.

Exercise 2.1.11B

Evaluate the following:

a. If each percent grade earned in a course translates to one letter grade, is the letter grade a function of the percent grade?
b. If so, is the function one-to-one?

Answer

a. Yes, letter grade is a function of percent grade;
b. No, it is not one-to-one. There are 100 different percent numbers we could get but only about five possible letter grades, so there cannot be only one percent number that corresponds to each letter grade.

Using the Vertical Line Test

As we have seen in some examples above, we can represent a function using a graph. Graphs display a great many input-output pairs in a small space. The visual information they provide often makes relationships easier to understand. By convention, graphs are typically constructed with the input values along the horizontal axis and the output values along the vertical axis.

The most common graphs name the input value x and the output y, and we say y is a function of x, or $y = f(x)$ when the function is named f. The graph of the function is the set of all points (x, y) in the plane that satisfies the equation $y = f(x)$. If the function is defined for only a few input values, then the graph of the function is only a few points, where the x-coordinate of each point is an input value and the y-coordinate of each point is the corresponding output value. For example, the black dots on the graph in Figure 2.1.10 tell us that $f(0) = 2$ and $f(6) = 1$. However, the set of all points (x, y) satisfying $y = f(x)$ is a curve. The curve shown includes $(0, 2)$ and $(6, 1)$ because the curve passes through those points

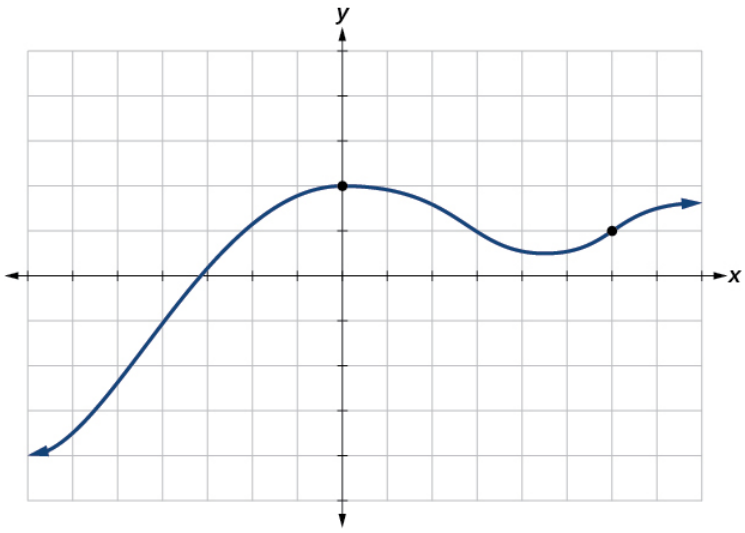


Figure 2.1.10: Graph of a polynomial.

The vertical line test can be used to determine whether a graph represents a function. If we can draw any vertical line that intersects a graph more than once, then the graph does not define a function because a function has only one output value for each input value. See Figure 2.1.11.

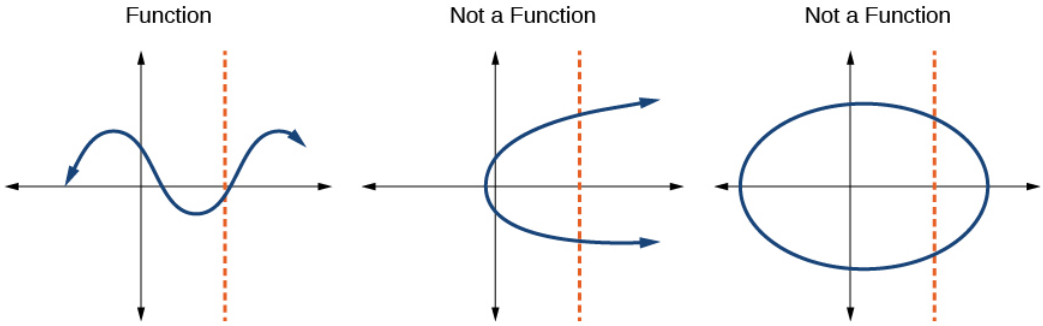


Figure 2.1.11: Three graphs visually showing what is and is not a function.

Howto: Given a graph, use the vertical line test to determine if the graph represents a function

1. Inspect the graph to see if any vertical line drawn would intersect the curve more than once.
2. If there is any such line, determine that the graph does not represent a function.

Example 2.1.12: Applying the Vertical Line Test

Which of the graphs in Figure 2.1.12 represent(s) a function $y = f(x)$?

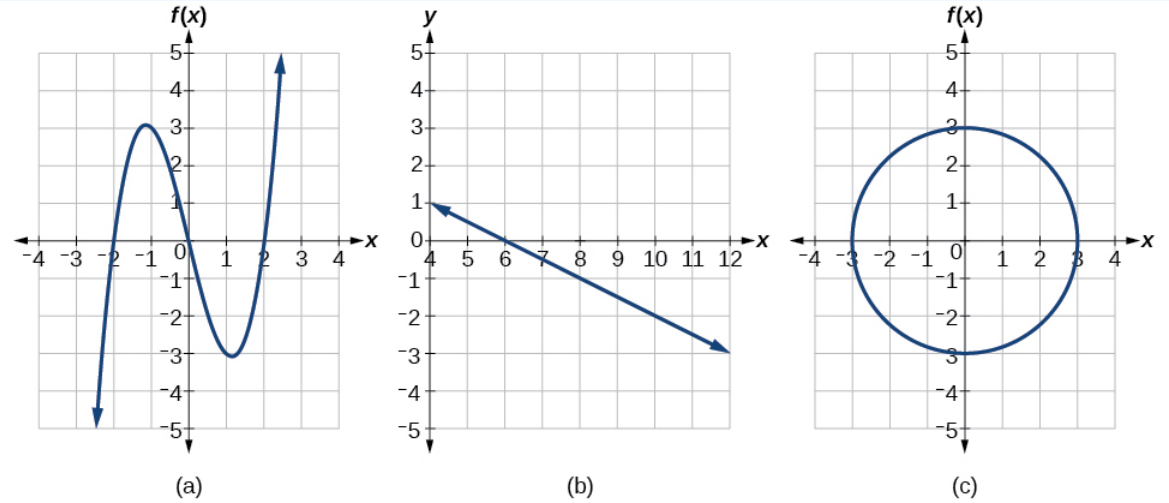


Figure 2.1.12: Graph of a polynomial (a), a downward-sloping line (b), and a circle (c).

Solution

If any vertical line intersects a graph more than once, the relation represented by the graph is not a function. Notice that any vertical line would pass through only one point of the two graphs shown in parts (a) and (b) of Figure 2.1.12 From this we can conclude that these two graphs represent functions. The third graph does not represent a function because, at most x-values, a vertical line would intersect the graph at more than one point, as shown in Figure 2.1.13

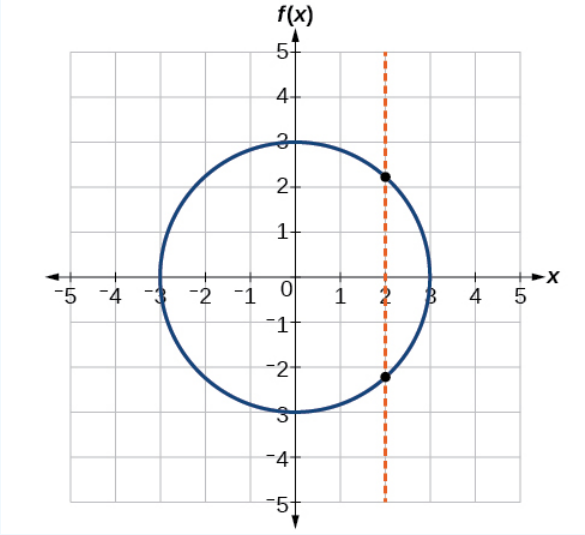


Figure 2.1.13: Graph of a circle.

Exercise 2.1.12

Does the graph in Figure 2.1.14 represent a function?

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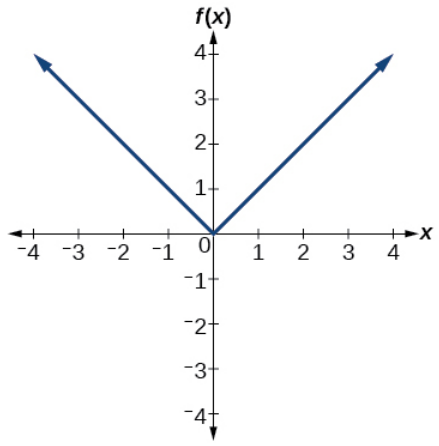


Figure 2.1.14: Graph of absolute value function.

Answer

yes

Using the Horizontal Line Test

Once we have determined that a graph defines a function, an easy way to determine if it is a one-to-one function is to use the **horizontal line test**. Draw horizontal lines through the graph. If any horizontal line intersects the graph more than once, then the graph does not represent a one-to-one function.

Howto: Given a graph of a function, use the horizontal line test to determine if the graph represents a one-to-one function

1. Inspect the graph to see if any horizontal line drawn would intersect the curve more than once.
2. If there is any such line, determine that the function is not one-to-one.

Example 2.1.13: Applying the Horizontal Line Test

Consider the functions shown in Figure $2.1.12a$ and Figure $2.1.12b$ Are either of the functions one-to-one?

Solution

The function in Figure $2.1.12a$ is not one-to-one. The horizontal line shown in Figure $2.1.15$ intersects the graph of the function at two points (and we can even find horizontal lines that intersect it at three points.)

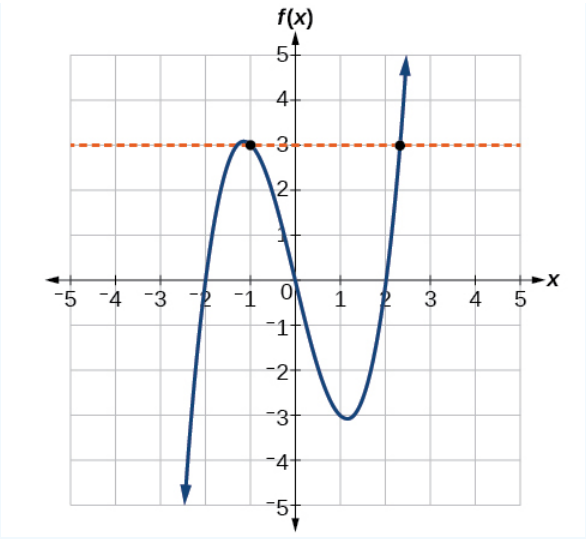


Figure 2.1.15: Graph of a polynomial with a horizontal line crossing through 2 points

The function in Figure $2.1.12b$is one-to-one. Any horizontal line will intersect a diagonal line at most once.

Exercise 2.1.13

Is the graph shown in Figure $2.1.13$one-to-one?

Answer

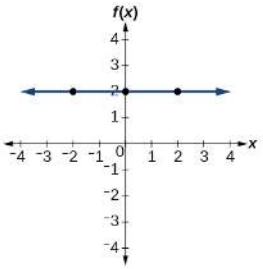
No, because it does not pass the horizontal line test.

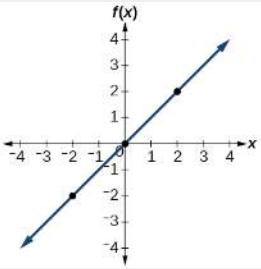
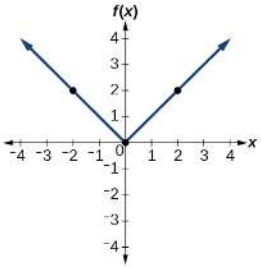
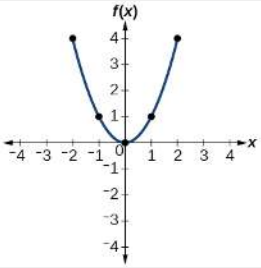
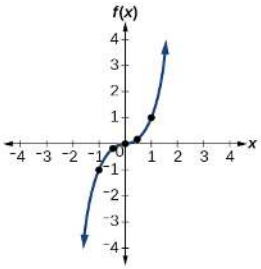
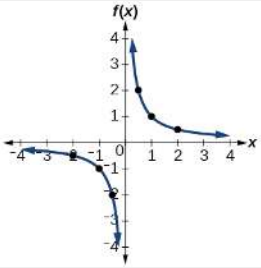
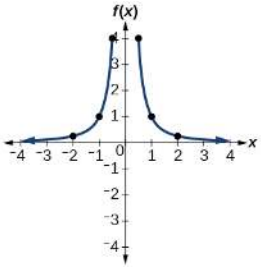
Identifying Basic Toolkit Functions

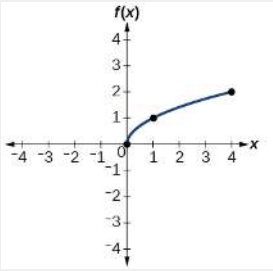
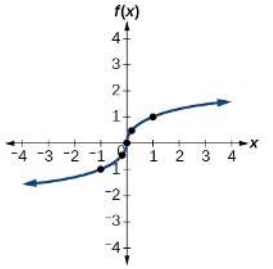
In this text, we will be exploring functions—the shapes of their graphs, their unique characteristics, their algebraic formulas, and how to solve problems with them. When learning to read, we start with the alphabet. When learning to do arithmetic, we start with numbers. When working with functions, it is similarly helpful to have a base set of building-block elements. We call these our "toolkit functions," which form a set of basic named functions for which we know the graph, formula, and special properties. Some of these functions are programmed to individual buttons on many calculators. For these definitions we will use x as the input variable and $y = f(x)$ as the output variable.

We will see these toolkit functions, combinations of toolkit functions, their graphs, and their transformations frequently throughout this book. It will be very helpful if we can recognize these toolkit functions and their features quickly by name, formula, graph, and basic table properties. The graphs and sample table values are included with each function shown in Table $2.1.14$

Table 2.1.14: Toolkit Functions

Name	Function	Graph
Constant	$f(x) = c$ where c is a constant	 <table><tr><th>x</th><th>f(x)</th></tr><tr><td>−2</td><td>2</td></tr><tr><td>0</td><td>2</td></tr><tr><td>2</td><td>2</td></tr></table>

Name	Function	Graph
Identity	$f(x) = x$	 <table><tr><th>x</th><th>f(x)</th></tr><tr><td>−2</td><td>−2</td></tr><tr><td>0</td><td>0</td></tr><tr><td>2</td><td>2</td></tr></table>
Absolute Value	$f(x) = \lvert x \rvert$	 <table><tr><th>x</th><th>f(x)</th></tr><tr><td>−2</td><td>2</td></tr><tr><td>0</td><td>0</td></tr><tr><td>2</td><td>2</td></tr></table>
Quadratic	$f(x) = x^2$	 <table><tr><th>x</th><th>f(x)</th></tr><tr><td>−2</td><td>4</td></tr><tr><td>−1</td><td>1</td></tr><tr><td>0</td><td>0</td></tr><tr><td>1</td><td>1</td></tr><tr><td>2</td><td>4</td></tr></table>
Cubic	$f(x) = x^3$	 <table><tr><th>x</th><th>f(x)</th></tr><tr><td>−1</td><td>−1</td></tr><tr><td>−0.5</td><td>−0.125</td></tr><tr><td>0</td><td>0</td></tr><tr><td>0.5</td><td>0.125</td></tr><tr><td>1</td><td>1</td></tr></table>
reciprocal	$f(x) = \frac{1}{x}$	 <table><tr><th>x</th><th>f(x)</th></tr><tr><td>−2</td><td>−0.5</td></tr><tr><td>−1</td><td>−1</td></tr><tr><td>−0.5</td><td>−2</td></tr><tr><td>0.5</td><td>2</td></tr><tr><td>1</td><td>1</td></tr><tr><td>2</td><td>0.5</td></tr></table>
Reciprocal squared	$f(x) = \frac{1}{x^2}$	 <table><tr><th>x</th><th>f(x)</th></tr><tr><td>−2</td><td>0.25</td></tr><tr><td>−1</td><td>1</td></tr><tr><td>−0.5</td><td>4</td></tr><tr><td>0.5</td><td>4</td></tr><tr><td>1</td><td>1</td></tr><tr><td>2</td><td>0.25</td></tr></table>

Name	Function	Graph
Square root	$f(x) = \sqrt{x}$	 <table><tr><th>x</th><th>f(x)</th></tr><tr><td>0</td><td>0</td></tr><tr><td>1</td><td>1</td></tr><tr><td>4</td><td>2</td></tr></table>
Cube root	$f(x) = \sqrt[3]{x}$	 <table><tr><th>x</th><th>f(x)</th></tr><tr><td>−1</td><td>−1</td></tr><tr><td>−0.125</td><td>−0.5</td></tr><tr><td>0</td><td>0</td></tr><tr><td>0.125</td><td>0.5</td></tr><tr><td>1</td><td>1</td></tr></table>

Key Equations

- Constant function $f(x) = c$, where c is a constant
- Identity function $f(x) = x$
- Absolute value function $f(x) = |x|$
- Quadratic function $f(x) = x^2$
- Cubic function $f(x) = x^3$
- Reciprocal function $f(x) = \dfrac{1}{x}$
- Reciprocal squared function $f(x) = \frac{1}{x^2}$
- Square root function $f(x) = \sqrt{x}$
- Cube root function $f(x) = 3\sqrt{x}$

Key Concepts

- A relation is a set of ordered pairs. A function is a specific type of relation in which each domain value, or input, leads to exactly one range value, or output.
- Function notation is a shorthand method for relating the input to the output in the form $y = f(x)$.
- In tabular form, a function can be represented by rows or columns that relate to input and output values.
- To evaluate a function, we determine an output value for a corresponding input value. Algebraic forms of a function can be evaluated by replacing the input variable with a given value.
- To solve for a specific function value, we determine the input values that yield the specific output value.
- An algebraic form of a function can be written from an equation.
- Input and output values of a function can be identified from a table.
- Relating input values to output values on a graph is another way to evaluate a function.
- A function is one-to-one if each output value corresponds to only one input value.
- A graph represents a function if any vertical line drawn on the graph intersects the graph at no more than one point.
- The graph of a one-to-one function passes the horizontal line test.

Footnotes

1 http://www.baseball-almanac.com/lege.../lisn100.shtml. Accessed 3/24/2014.
2 www.kgbanswers.com/how-long-i...y-span/4221590. Accessed 3/24/2014.

Glossary

dependent variable
an output variable

domain
the set of all possible input values for a relation

function
a relation in which each input value yields a unique output value

horizontal line test
a method of testing whether a function is one-to-one by determining whether any horizontal line intersects the graph more than once

independent variable
an input variable

input
each object or value in a domain that relates to another object or value by a relationship known as a function

one-to-one function
a function for which each value of the output is associated with a unique input value

output
each object or value in the range that is produced when an input value is entered into a function

range
the set of output values that result from the input values in a relation

relation
a set of ordered pairs

vertical line test
a method of testing whether a graph represents a function by determining whether a vertical line intersects the graph no more than once

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2.2: Domain and Range

Finding the Domain of a Function Defined by an Equation

In Functions and Function Notation, we were introduced to the concepts of **domain and range**. In this section, we will practice determining domains and ranges for specific functions. Keep in mind that, in determining domains and ranges, we need to consider what is physically possible or meaningful in real-world examples, such as tickets sales and year in the horror movie example above. We also need to consider what is mathematically permitted. For example, we cannot include any input value that leads us to take an even root of a negative number if the domain and range consist of real numbers. Or in a function expressed as a formula, we cannot include any input value in the domain that would lead us to divide by 0.

We can visualize the domain as a "holding area" that contains "raw materials" for a "function machine" and the range as another "holding area" for the machine's products (Figure 2.2.2).

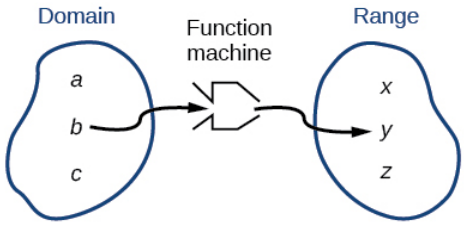


Figure 2.2.2: Diagram of how a function relates two relations

We can write the **domain and range** in **interval notation**, which uses values within brackets to describe a set of numbers. In interval notation, we use a square bracket [when the set includes the endpoint and a parenthesis (to indicate that the endpoint is either not included or the interval is unbounded. For example, if a person has $100 to spend, he or she would need to express the interval that is more than 0 and less than or equal to 100 and write $(0, 100]$. We will discuss interval notation in greater detail later.

Let's turn our attention to finding the domain of a function whose equation is provided. Oftentimes, finding the domain of such functions involves remembering three different forms. First, if the function has no denominator or an even root, consider whether the domain could be all real numbers. Second, if there is a denominator in the function's equation, exclude values in the domain that force the denominator to be zero. Third, if there is an even root, consider excluding values that would make the radicand negative.

Before we begin, let us review the conventions of interval notation:

- The smallest term from the interval is written first.
- The largest term in the interval is written second, following a comma.
- Parentheses, (or), are used to signify that an endpoint is not included, called exclusive.
- Brackets, [or], are used to indicate that an endpoint is included, called inclusive.

See Figure 2.2.3 for a summary of interval notation.

Inequality	Interval Notation	Graph on Number Line	Description
$x > a$	(a, ∞)		x is greater than a
$x < a$	$(-\infty, a)$		x is less than a
$x \geq a$	$[a, \infty)$		x is greater than or equal to a
$x \leq a$	$(-\infty, a]$		x is less than or equal to a
$a < x < b$	(a, b)		x is strictly between a and b
$a \leq x < b$	$[a, b)$		x is between a and b, to include a
$a < x \leq b$	$(a, b]$		x is between a and b, to include b
$a \leq x \leq b$	$[a, b]$		x is between a and b, to include a and b

Figure 2.2.3: Summary of interval notation.

Example 2.2.1: Finding the Domain of a Function as a Set of Ordered Pairs

Find the domain of the following function: $\{(2, 10), (3, 10), (4, 20), (5, 30), (6, 40)\}$

Solution

First identify the input values. The input value is the first coordinate in an ordered pair. There are no restrictions, as the ordered pairs are simply listed. The domain is the set of the first coordinates of the ordered pairs.

$$\{2, 3, 4, 5, 6\}$$

Exercse 2.2.1

Find the domain of the function:

$$\{(-5, 4), (0, 0), (5, -4), (10, -8), (15, -12)\}$$

Answer

$\{-5, 0, 5, 10, 15\}$

How To: Given a function written in equation form, find the domain.

1. Identify the input values.
2. Identify any restrictions on the input and exclude those values from the domain.
3. Write the domain in interval form, if possible.

Example 2.2.2: Finding the Domain of a Function

Find the domain of the function $f(x) = x^2 - 1$.

Solution

The input value, shown by the variable x in the equation, is squared and then the result is lowered by one. Any real number may be squared and then be lowered by one, so there are no restrictions on the domain of this function. The domain is the set of real numbers.

In interval form, the domain of f is $(-\infty, \infty)$.

Exercse 2.2.2

Find the domain of the function:

$$f(x) = 5 - x + x^3$$

Answer

$(-\infty, \infty)$

Howto: Given a function written in an equation form that includes a fraction, find the domain

1. Identify the input values.
2. Identify any restrictions on the input. If there is a denominator in the function's formula, set the denominator equal to zero and solve for x . If the function's formula contains an even root, set the radicand greater than or equal to 0, and then solve.
3. Write the domain in interval form, making sure to exclude any restricted values from the domain.

Example 2.2.3: Finding the Domain of a Function Involving a Denominator

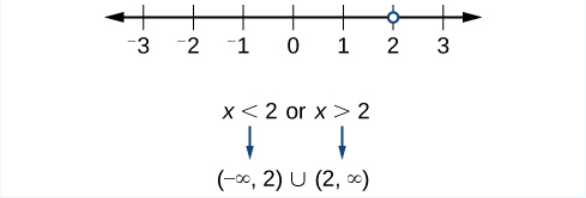
Find the domain of the function $f(x) = \dfrac{x+1}{2-x}$.

Solution

When there is a denominator, we want to include only values of the input that do not force the denominator to be zero. So, we will set the denominator equal to 0 and solve for x.

$$\begin{aligned} 2 - x &= 0 \\ -x &= -2 \\ x &= 2 \end{aligned}$$

Now, we will exclude 2 from the domain. The answers are all real numbers where $x < 2$ or $x > 2$. We can use a symbol known as the union, $\cup$,to combine the two sets. In interval notation, we write the solution:$(-\infty, 2) \cup (2, \infty)$.



In interval form, the domain of f is $(-\infty, 2) \cup (2, \infty)$.

Exercse 2.2.3

Find the domain of the function:

$$f(x) = \frac{1+4x}{2x-1}$$

Answer

$$(-\infty, \frac{1}{2}) \cup (\frac{1}{2}, \infty)$$

How To: Given a function written in equation form including an even root, find the domain.

1. Identify the input values.
2. Since there is an even root, exclude any real numbers that result in a negative number in the radicand. Set the radicand greater than or equal to zero and solve for x.
3. The solution(s) are the domain of the function. If possible, write the answer in interval form.

Example 2.2.4: Finding the Domain of a Function with an Even Root

Find the domain of the function:

$$f(x) = \sqrt{7-x}.$$

Solution

When there is an even root in the formula, we exclude any real numbers that result in a negative number in the radicand.

Set the radicand greater than or equal to zero and solve for x.

$$\begin{aligned} 7 - x &\geq 0 \\ -x &\geq -7 \\ x &\leq 7 \end{aligned}$$

Now, we will exclude any number greater than 7 from the domain. The answers are all real numbers less than or equal to 7, or $(-\infty, 7]$.

Exercse 2.2.4

Find the domain of the function

$$f(x) = \sqrt{5+2x}.$$

Answer

$$[-2.5, \infty)$$

Q&A: Can there be functions in which the domain and range do not intersect at all?

Yes. For example, the function $f(x) = -\frac{1}{\sqrt{x}}$ has the set of all positive real numbers as its domain but the set of all negative real numbers as its range. As a more extreme example, a function's inputs and outputs can be completely different categories (for example, names of weekdays as inputs and numbers as outputs, as on an attendance chart), in such cases the domain and range have no elements in common.

Using Notations to Specify Domain and Range

In the previous examples, we used inequalities and lists to describe the domain of functions. We can also use inequalities, or other statements that might define sets of values or data, to describe the behavior of the variable in set-builder notation. For example, $\{x|10 \leq x < 30\}$ describes the behavior of x in set-builder notation. The braces $\{\}$ are read as "the set of," and the vertical bar $|$ is read as "such that," so we would read $\{x|10 \leq x < 30\}$ as "the set of x-values such that 10 is less than or equal to x, and x is less than 30."

Figure $2.2.4$ compares inequality notation, set-builder notation, and interval notation.

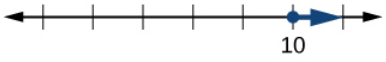
	Inequality Notation	Set-builder Notation	Interval Notation
	$5 < h \leq 10$	$\{h \mid 5 < h \leq 10\}$	$(5, 10]$
	$5 \leq h < 10$	$\{h \mid 5 \leq h < 10\}$	$[5, 10)$
	$5 < h < 10$	$\{h \mid 5 < h < 10\}$	$(5, 10)$
	$h < 10$	$\{h \mid h < 10\}$	$(-\infty, 10)$
	$h \geq 10$	$\{h \mid h \geq 10\}$	$[10, \infty)$
	All real numbers	$\mathbb{R}$	$(-\infty, \infty)$

Figure 2.2.4: Summary of notations for inequalities, set-builder, and interval.

To combine two intervals using inequality notation or set-builder notation, we use the word "or." As we saw in earlier examples, we use the union symbol, $\cup$,to combine two unconnected intervals. For example, the union of the sets $\{2, 3, 5\}$ and $\{4, 6\}$ is the set $\{2, 3, 4, 5, 6\}$ It is the set of all elements that belong to one or the other (or both) of the original two sets. For sets with a finite number of elements like these, the elements do not have to be listed in ascending order of numerical value. If the original two sets have some elements in common, those elements should be listed only once in the union set. For sets of real numbers on intervals, another example of a union is

$$\{x||x| \geq 3\} = (-\infty, -3] \cup [3, \infty) \tag{2.2.1}$$

Set-Builder Notation and Interval Notation

Set-builder notation is a method of specifying a set of elements that satisfy a certain condition. It takes the form $\{x|$ statement about x$\}$ which is read as, "the set of all x such that the statement about x is true." For example,

$$\{x|4 < x \leq 12\}$$

Interval notation is a way of describing sets that include all real numbers between a lower limit that may or may not be included and an upper limit that may or may not be included. The endpoint values are listed between brackets or parentheses. A square bracket indicates inclusion in the set, and a parenthesis indicates exclusion from the set. For example,

$$(4, 12]$$

How To: Given a line graph, describe the set of values using interval notation.

1. Identify the intervals to be included in the set by determining where the heavy line overlays the real line.
2. At the left end of each interval, use [with each end value to be included in the set (solid dot) or (for each excluded end value (open dot).
3. At the right end of each interval, use] with each end value to be included in the set (filled dot) or) for each excluded end value (open dot).
4. Use the union symbol $\cup$ to combine all intervals into one set.

Example 2.2.5: Describing Sets on the Real-Number Line

Describe the intervals of values shown in Figure 2.2.5 using inequality notation, set-builder notation, and interval notation.



Figure 2.2.5: Line graph of $1 \leq x \leq 3$ or $x > 5$.

Solution

To describe the values, x, included in the intervals shown, we would say, "x is a real number greater than or equal to 1 and less than or equal to 3, or a real number greater than 5."

Inequality

$$1 \leq x \leq 3 \text{ or } x > 5$$

Set-builder Notation

$$\{x | 1 \leq x \leq 3 \text{ or } x > 5\}$$

Interval notation

$$[1, 3] \cup (5, \infty)$$

Remember that, when writing or reading interval notation, using a square bracket means the boundary is included in the set. Using a parenthesis means the boundary is not included in the set.

Exercse 2.2.5

Given Figure 2.2.6, specify the graphed set in

a. words
b. set-builder notation
c. interval notation

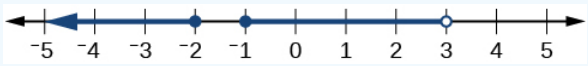


Figure 2.2.6: Line graph of $-2 \leq x$, $-1 \leq x < 3$.leq}x\), $-1 \leq x < 3$.

Answer a

Values that are less than or equal to –2, or values that are greater than or equal to –1 and less than 3;

Answer b

$\{x | x \leq -2 or -1 \leq x < 3\}$

Answer c

$(-\infty, -2] \cup [-1, 3)$

Finding Domain and Range from Graphs

Another way to identify the domain and range of functions is by using graphs. Because the domain refers to the set of possible input values, the domain of a graph consists of all the input values shown on the x-axis. The range is the set of possible output values, which are shown on the y-axis. Keep in mind that if the graph continues beyond the portion of the graph we can see, the domain and range may be greater than the visible values. See Figure 2.2.7.

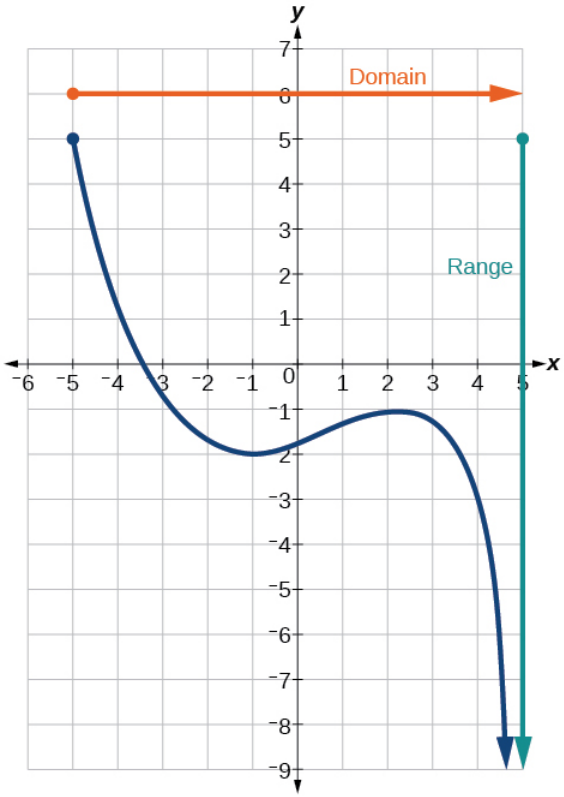


Figure 2.2.7: Graph of a polynomial that shows the x-axis is the domain and the y-axis is the range

We can observe that the graph extends horizontally from -5 to the right without bound, so the domain is $[-5, \infty)$. The vertical extent of the graph is all range values 5 and below, so the range is $(-\infty, 5]$. Note that the domain and range are always written from smaller to larger values, or from left to right for domain, and from the bottom of the graph to the top of the graph for range.

Example 2.2.6A: Finding Domain and Range from a Graph

Find the domain and range of the function f whose graph is shown in Figure 1.2.8.

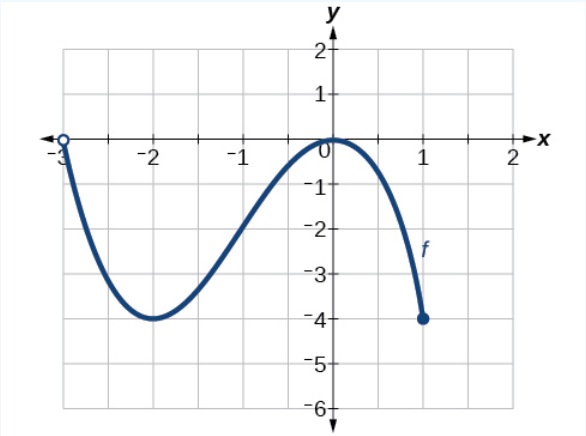


Figure 2.2.8: Graph of a function from (-3, 1].

Solution

We can observe that the horizontal extent of the graph is –3 to 1, so the domain of f is $(-3, 1]$.

The vertical extent of the graph is 0 to –4, so the range is $[-4, 0)$. See Figure 2.2.9.

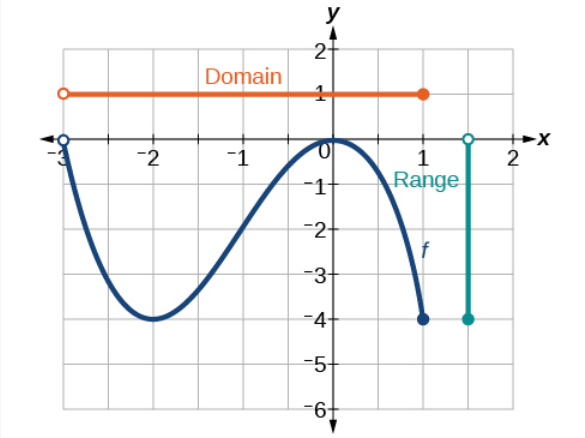


Figure 2.2.9: Graph of the previous function shows the domain and range

Example 2.2.6B: Finding Domain and Range from a Graph of Oil Production

Find the domain and range of the function f whose graph is shown in Figure 2.2.10.

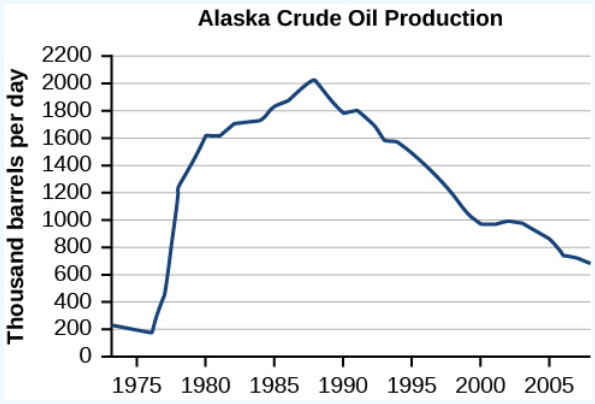


Figure 2.2.10: Graph of the Alaska Crude Oil Production where the y-axis is thousand barrels per day and the -axis is the years(credit: modification of work by the U.S. Energy Information Administration)

Solution

The input quantity along the horizontal axis is “years,” which we represent with the variable t for time. The output quantity is “thousands of barrels of oil per day,” which we represent with the variable b for barrels. The graph may continue to the left and right beyond what is viewed, but based on the portion of the graph that is visible, we can determine the domain as $1973 \le t \le 2008$ and the range as approximately $180 \le b \le 2010$.

In interval notation, the domain is $[1973, 2008]$ and the range is about $[180, 2010]$. For the domain and the range, we approximate the smallest and largest values since they do not fall exactly on the grid lines.

Exercse 2.2.6

Given Figure 2.2.11, identify the domain and range using interval notation.

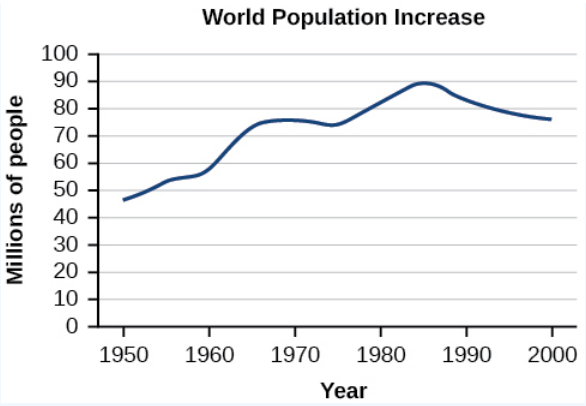


Figure 2.2.11: Graph of World Population Increase where the y-axis represents millions of people and the x-axis represents the year.

Answer

domain =$[1950, 2002]$

range = $[47,000,000, 89,000,000]$

Can a function's domain and range be the same?

Yes. For example, the domain and range of the cube root function are both the set of all real numbers.

Finding Domains and Ranges of the Toolkit Functions

We will now return to our set of toolkit functions to determine the domain and range of each.

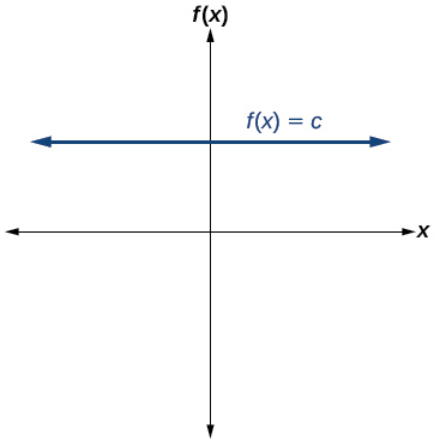
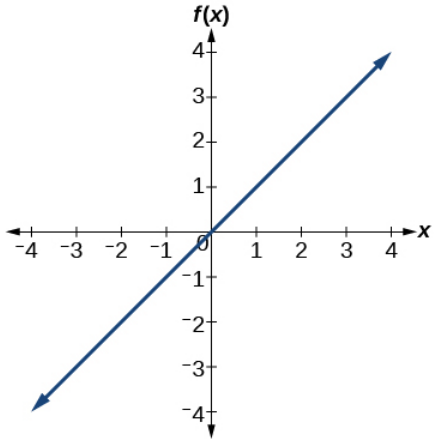


Figure 2.2.12: Constant function $f(x) = c$.

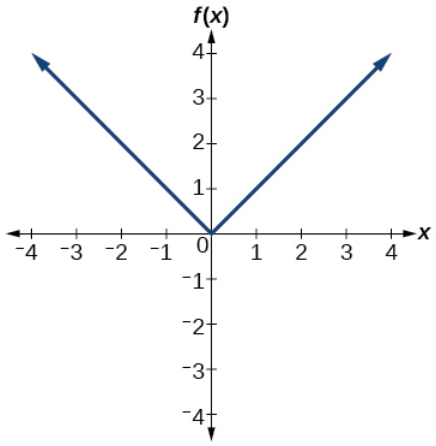
For the **constant function** $f(x) = c$, the domain consists of all real numbers; there are no restrictions on the input. The only output value is the constant c, so the range is the set $\{c\}$ that contains this single element. In interval notation, this is written as $[c, c]$, the interval that both begins and ends with c.



Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

Figure 2.2.13: *Identity function f(x)=x.*

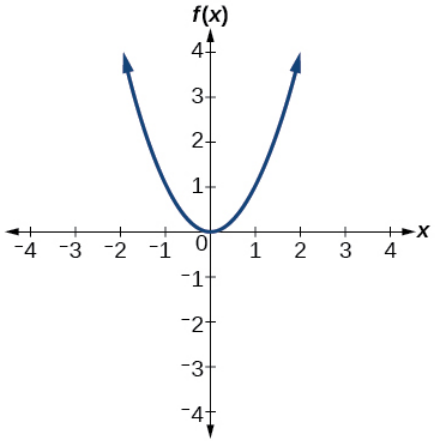
For the **identity function** $f(x) = x$, there is no restriction on x. Both the domain and range are the set of all real numbers.



Domain: $(-\infty, \infty)$
Range: $[0, \infty)$

Figure 2.2.14: Absolute function $f(x) = |x|$.

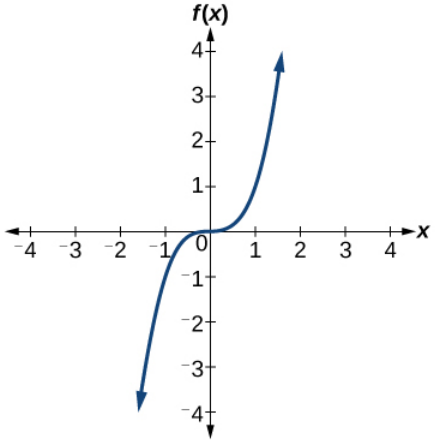
For the **absolute value function** $f(x) = |x|$, there is no restriction on x. However, because absolute value is defined as a distance from 0, the output can only be greater than or equal to 0.



Domain: $(-\infty, \infty)$
Range: $[0, \infty)$

Figure 2.2.15: Quadratic function $f(x) = x^2$.

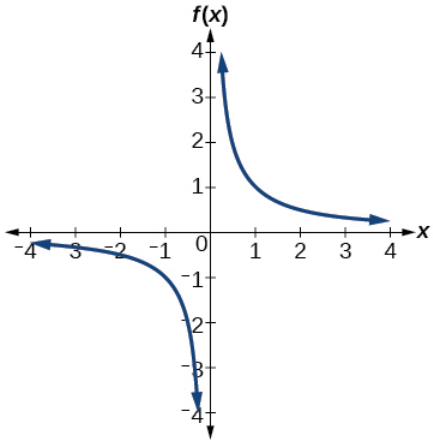
For the **quadratic function** $f(x) = x^2$, the domain is all real numbers since the horizontal extent of the graph is the whole real number line. Because the graph does not include any negative values for the range, the range is only nonnegative real numbers.



Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

Figure 2.2.16: Cubic function $f(x) - x^3$.

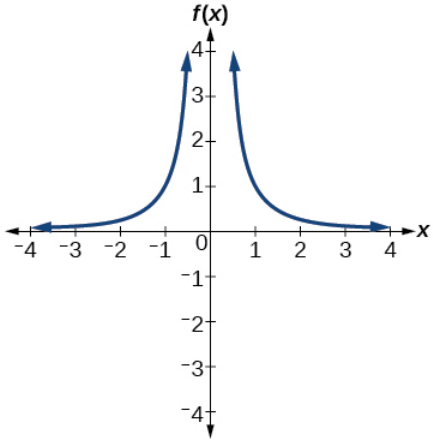
For the **cubic function** $f(x) = x^3$, the domain is all real numbers because the horizontal extent of the graph is the whole real number line. The same applies to the vertical extent of the graph, so the domain and range include all real numbers.



Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(-\infty, 0) \cup (0, \infty)$

Figure 2.2.17: Reciprocal function $f(x) = \frac{1}{x}$.

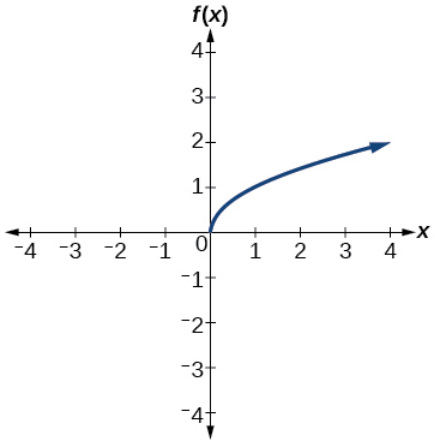
For the **reciprocal function** $f(x) = \frac{1}{x}$, we cannot divide by 0, so we must exclude 0 from the domain. Further, 1 divided by any value can never be 0, so the range also will not include 0. In set-builder notation, we could also write $\{x | x \neq 0\}$,the set of all real numbers that are not zero.



Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(0, \infty)$

Figure 2.2.18: Reciprocal squared function $f(x) = \frac{1}{x^2}$

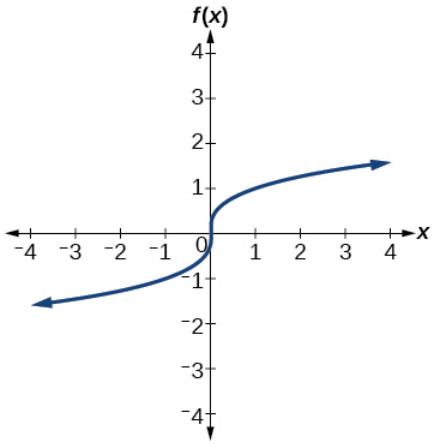
For the **reciprocal squared function** $f(x) = \frac{1}{x^2}$,we cannot divide by 0, so we must exclude 0 from the domain. There is also no x that can give an output of 0, so 0 is excluded from the range as well. Note that the output of this function is always positive due to the square in the denominator, so the range includes only positive numbers.



Domain: [0, ∞)
Range: [0, ∞)

Figure* 2.2.19*: *Square root function* $f(x) = \sqrt{(x)}$.

For the **square root function** $f(x) = \sqrt{x}$, we cannot take the square root of a negative real number, so the domain must be 0 or greater. The range also excludes negative numbers because the square root of a positive number x is defined to be positive, even though the square of the negative number $-\sqrt{x}$ also gives us x.



Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

Figure 2.2.20: Cube root function $f(x) = \sqrt[3]{x}$.

For the **cube root function** $f(x) = \sqrt[3]{x}$, the domain and range include all real numbers. Note that there is no problem taking a cube root, or any odd-integer root, of a negative number, and the resulting output is negative (it is an odd function).

Given the formula for a function, determine the domain and range.

1. Exclude from the domain any input values that result in division by zero.
2. Exclude from the domain any input values that have nonreal (or undefined) number outputs.
3. Use the valid input values to determine the range of the output values.
4. Look at the function graph and table values to confirm the actual function behavior.

Finding the Domain and Range Using Toolkit Functions

Find the domain and range of $f(x) = 2x^3 - x$.

Solution

There are no restrictions on the domain, as any real number may be cubed and then subtracted from the result.

The domain is $(-\infty, \infty)$ and the range is also $(-\infty, \infty)$.

Example 2.2.7B: Finding the Domain and Range

Find the domain and range of $f(x) = \frac{2}{x+1}$.

Solution

We cannot evaluate the function at -1 because division by zero is undefined. The domain is $(-\infty, -1) \cup (-1, \infty)$. Because the function is never zero, we exclude 0 from the range. The range is $(-\infty, 0) \cup (0, \infty)$.

Example 2.2.7C: Finding the Domain and Range

Find the domain and range of $f(x) = 2\sqrt{x+4}$.

Solution

We cannot take the square root of a negative number, so the value inside the radical must be nonnegative.

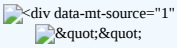
$x + 4 \geq 0$ when $x \geq -4$

The domain of $f(x)$ is $[-4, \infty)$.

We then find the range. We know that $f(-4) = 0$, and the function value increases as x increases without any upper limit. We conclude that the range of f is $[0, \infty)$.

Analysis

Figure 2.2.19represents the function f.



" src="/@api/deki/files/892/CNX_Precalc_Figure_01_02_020.jpg" fileid="892">

Figure 2.2.19: Graph of a square root function at $(-4, 0)$.

Exercise 2.2.7

Find the domain and range of

$f(x) = \sqrt{-2 - x}$.

Answer

domain: $(-\infty, -2]$

range: $[0, \infty)$

Key Concepts

- The domain of a function includes all real input values that would not cause us to attempt an undefined mathematical operation, such as dividing by zero or taking the square root of a negative number.
- The domain of a function can be determined by listing the input values of a set of ordered pairs.
- The domain of a function can also be determined by identifying the input values of a function written as an equation.
- Interval values represented on a number line can be described using inequality notation, set-builder notation, and interval notation.
- For many functions, the domain and range can be determined from a graph.
- An understanding of toolkit functions can be used to find the domain and range of related functions.
- A piecewise function is described by more than one formula.
- A piecewise function can be graphed using each algebraic formula on its assigned subdomain.

Footnotes

1 The Numbers: Where Data and the Movie Business Meet. “Box Office History for Horror Movies.” http://www.the-numbers.com/market/genre/Horror. Accessed 3/24/2014
2 www.eia.gov/dnav/pet/hist/Lea...s=MCRFPAK2&f=A.

Glossary

interval notation

a method of describing a set that includes all numbers between a lower limit and an upper limit; the lower and upper values are listed between brackets or parentheses, a square bracket indicating inclusion in the set, and a parenthesis indicating exclusion

piecewise function

a function in which more than one formula is used to define the output

set-builder notation

a method of describing a set by a rule that all of its members obey; it takes the form {x| statement about x}

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2.3: Rates of Change and Behavior of Graphs

Finding the Average Rate of Change of a Function

The price change per year is a **rate of change** because it describes how an output quantity changes relative to the change in the input quantity. We can see that the price of gasoline in Table 2.3.1 did not change by the same amount each year, so the rate of change was not constant. If we use only the beginning and ending data, we would be finding the **average rate of change** over the specified period of time. To find the average rate of change, we divide the change in the output value by the change in the input value.

$$\begin{aligned}\text{Average rate of change} &= \frac{\text{Change in output}}{\text{Change in input}} \\ &= \frac{\Delta y}{\Delta x} \\ &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{f(x_2) - f(x_1)}{x_2 - x_1}\end{aligned}$$

The Greek letter Δ (delta) signifies the change in a quantity; we read the ratio as "delta-y over delta-x" or "the change in y divided by the change in x." Occasionally we write Δf instead of Δy, which still represents the change in the function's output value resulting from a change to its input value. It does not mean we are changing the function into some other function.

In our example, the gasoline price increased by $1.37 from 2005 to 2012. Over 7 years, the average rate of change was

$$\frac{\Delta y}{\Delta x} = \frac{\$1.37}{7\text{years}} \approx 0.196 \text{ dollars per year.} \tag{2.3.1}$$

On average, the price of gas increased by about 19.6¢ each year. Other examples of rates of change include:

- A population of rats increasing by 40 rats per week
- A car traveling 68 miles per hour (distance traveled changes by 68 miles each hour as time passes)
- A car driving 27 miles per gallon (distance traveled changes by 27 miles for each gallon)
- The current through an electrical circuit increasing by 0.125 amperes for every volt of increased voltage
- The amount of money in a college account decreasing by $4,000 per quarter

Definition: Rate of Change

A **rate of change** describes how an output quantity changes relative to the change in the input quantity. The units on a rate of change are "output units per input units."

The average rate of change between two input values is the total change of the function values (output values) divided by the change in the input values.

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \tag{2.3.2}$$

How To...

Given the value of a function at different points, calculate the average rate of change of a function for the interval between two values x_1 and x_2.

1. Calculate the difference $y_2 - y_1 = \Delta y$.
2. Calculate the difference $x_2 - x_1 = \Delta x$.
3. Find the ratio $\frac{\Delta y}{\Delta x}$.

Example 2.3.1: Computing an Average Rate of Change

Using the data in Table 2.3.1, find the average rate of change of the price of gasoline between 2007 and 2009.

Solution

In 2007, the price of gasoline was \$2.84. In 2009, the cost was \$2.41. The average rate of change is

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{\$2.41 - \$2.84}{2009 - 2007} \\ &= \frac{-\$0.43}{2 \text{ years}} \\ &= -\$0.22 \text{ per year}\end{aligned}$$

Analysis

Note that a decrease is expressed by a negative change or "negative increase." A rate of change is negative when the output decreases as the input increases or when the output increases as the input decreases.

Exercise 2.3.1

Using the data in Table 2.3.1, find the average rate of change between 2005 and 2010.

Solution

$$\frac{\$2.84 - \$2.315}{5 \text{ years}} = \frac{\$0.535}{5 \text{ years}} = \$0.106 \text{per year.}$$

Example 2.3.2: Computing Average Rate of Change from a Graph

Given the function $g(t)$ shown in Figure 2.3.1, find the average rate of change on the interval $[-1, 2]$.

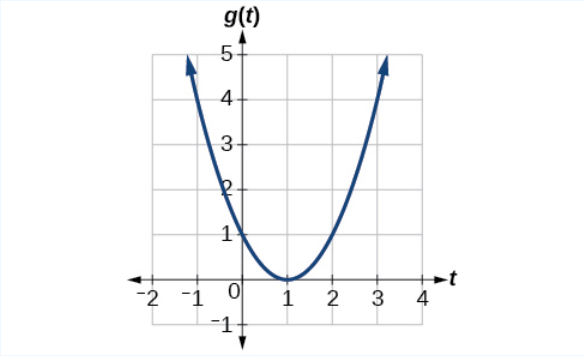


Figure 2.3.1: Graph of a parabola.

Solution

At $t = -1$, Figure 2.3.2 shows $g(-1) = 4$. At $t = 2$, the graph shows $g(2) = 1$.

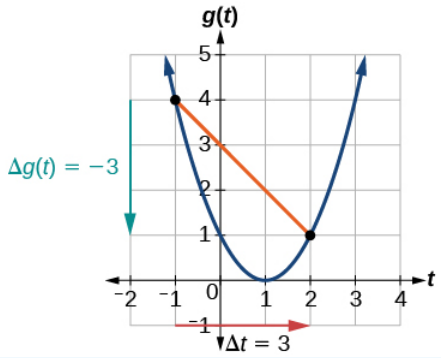


Figure 2.3.2: Graph of a parabola with a line from points (-1, 4) and (2, 1) to show the changes for g(t) and t.

The horizontal change $\Delta t = 3$ is shown by the red arrow, and the vertical change $\Delta g(t) = -3$ is shown by the turquoise arrow. The output changes by –3 while the input changes by 3, giving an average rate of change of

$$\frac{1-4}{2-(-1)} = \frac{-3}{3} = -1 \tag{2.3.3}$$

Analysis

Note that the order we choose is very important. If, for example, we use $\frac{y_2 - y_1}{x_1 - x_2}$, we will not get the correct answer. Decide which point will be 1 and which point will be 2, and keep the coordinates fixed as (x_1, y_1) and (x_2, y_2).

Example 2.3.3: Computing Average Rate of Change from a Table

After picking up a friend who lives 10 miles away, Anna records her distance from home over time. The values are shown in Table 2.3.2. Find her average speed over the first 6 hours.

Table 2.3.2

t (hours)	0	1	2	3	4	5	6	7
$D(t)$ (miles)	10	55	90	153	214	240	292	300

Solution

Here, the average speed is the average rate of change. She traveled 292 miles in 6 hours, for an average speed of

$$\frac{292-10}{6-0} = \frac{282}{6}$$
$$= 47$$

The average speed is about 47 miles per hour.

Analysis

Because the speed is not constant, the average speed depends on the interval chosen. For the interval $[2, 3]$, the average speed is 63 miles per hour.

Example 2.3.4: Computing Average Rate of Change for a Function Expressed as a Formula

Compute the average rate of change of $f(x) = x^2 - \frac{1}{x}$ on the interval $[2, 4]$.

Solution

We can start by computing the function values at each endpoint of the interval.

$$\begin{aligned} f(2) &= 2^2 - \frac{1}{2}f(4) & &= 4^2 - \frac{1}{4} \\ &= 4 - \frac{1}{2} & &= 16 - \frac{1}{4} \\ &= 72 & &= \frac{63}{4} \end{aligned}$$

Now we compute the average rate of change.

$$\begin{aligned} \text{Average rate of change} &= \frac{f(4) - f(2)}{4 - 2} \\ &= \frac{\frac{63}{4} - \frac{7}{2}}{4 - 2} \\ &= \frac{\frac{49}{4}}{2} \\ &= \frac{49}{8} \end{aligned}$$

Exercise 2.3.2

Find the average rate of change of $f(x) = x - 2\sqrt{x}$ on the interval $[1, 9]$.

Solution

$\frac{1}{2}$

Example 2.3.5: Finding the Average Rate of Change of a Force

The electrostatic force F, measured in newtons, between two charged particles can be related to the distance between the particles d,in centimeters, by the formula $F(d) = \frac{2}{d^2}$. Find the average rate of change of force if the distance between the particles is increased from 2 cm to 6 cm.

Solution

We are computing the average rate of change of $F(d) = \dfrac{2}{d^2}$ on the interval $[2, 6]$.

$$\begin{aligned} \text{Average rate of change} &= \frac{F(6) - F(2)}{6 - 2} \\ &= \frac{\frac{2}{6^2} - \frac{2}{2^2}}{6 - 2} && \text{Simplify} \\ &= \frac{\frac{2}{36} - \frac{2}{4}}{4} \\ &= \frac{-\frac{16}{36}}{4} && \text{Combine numerator terms.} \\ &= -\frac{1}{9} && \text{Simplify} \end{aligned}$$

The average rate of change is $-\frac{1}{9}$ newton per centimeter.

Example 2.3.6: Finding an Average Rate of Change as an Expression

Find the average rate of change of $g(t) = t^2 + 3t + 1$ on the interval $[0, a]$. The answer will be an expression involving a.

Solution

We use the average rate of change formula.

$$\begin{aligned}
\text{Average rate of change} &= \frac{g(a) - g(0)}{a - 0} && \text{Evaluate.} \\
&= \frac{(a^2 + 3a + 1) - (0^2 + 3(0) + 1)}{a - 0} && \text{Simplify.} \\
&= \frac{a^2 + 3a + 1 - 1}{a} && \text{Simplify and factor.} \\
&= \frac{a(a+3)}{a} && \text{Divide by the common factor a.} \\
&= a + 3
\end{aligned}$$

This result tells us the average rate of change in terms of a between $t = 0$ and any other point $t = a$. For example, on the interval $[0, 5]$, the average rate of change would be $5 + 3 = 8$.

Exercise 2.3.3

Find the average rate of change of $f(x) = x^2 + 2x - 8$ on the interval $[5, a]$.

Solution

$a + 7$

Using a Graph to Determine Where a Function is Increasing, Decreasing, or Constant

As part of exploring how functions change, we can identify intervals over which the function is changing in specific ways. We say that a function is increasing on an interval if the function values increase as the input values increase within that interval. Similarly, a function is decreasing on an interval if the function values decrease as the input values increase over that interval. The average rate of change of an increasing function is positive, and the average rate of change of a decreasing function is negative. Figure 2.3.3 shows examples of increasing and decreasing intervals on a function.

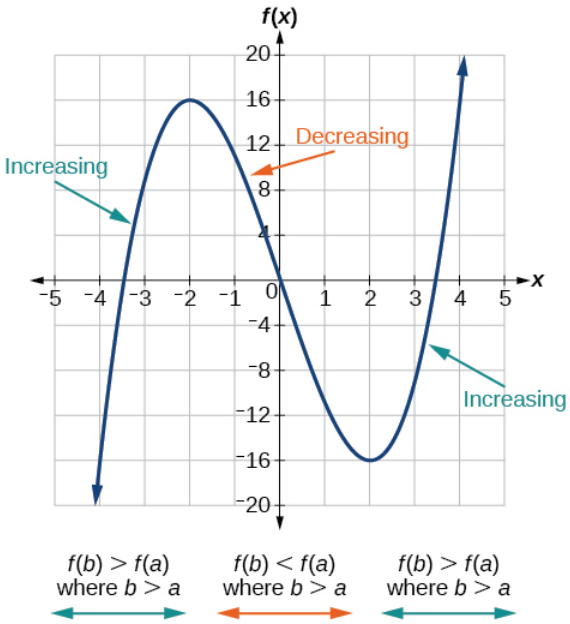


Figure 2.3.3: The function $f(x) = x^3 - 12x$ is increasing on $(-\infty, -2) \cup (2, \infty)$ and is decreasing on $(-2, 2)$.

While some functions are increasing (or decreasing) over their entire domain, many others are not. A value of the input where a function changes from increasing to decreasing (as we go from left to right, that is, as the input variable increases) is called a **local maximum**. If a function has more than one, we say it has local maxima. Similarly, a value of the input where a function changes from decreasing to increasing as the input variable increases is called a **local minimum**. The plural form is "local minima." Together, local maxima and minima are called **local extrema**, or local extreme values, of the function. (The singular form is "extremum.") Often, the term local is replaced by the term relative. In this text, we will use the term local.

Clearly, a function is neither increasing nor decreasing on an interval where it is constant. A function is also neither increasing nor decreasing at extrema. Note that we have to speak of local extrema, because any given local extremum as defined here is not necessarily the highest maximum or lowest minimum in the function's entire domain.

For the function whose graph is shown in Figure 2.3.4, the local maximum is 16, and it occurs at $x = -2$. The local minimum is -16 and it occurs at $x = 2$.

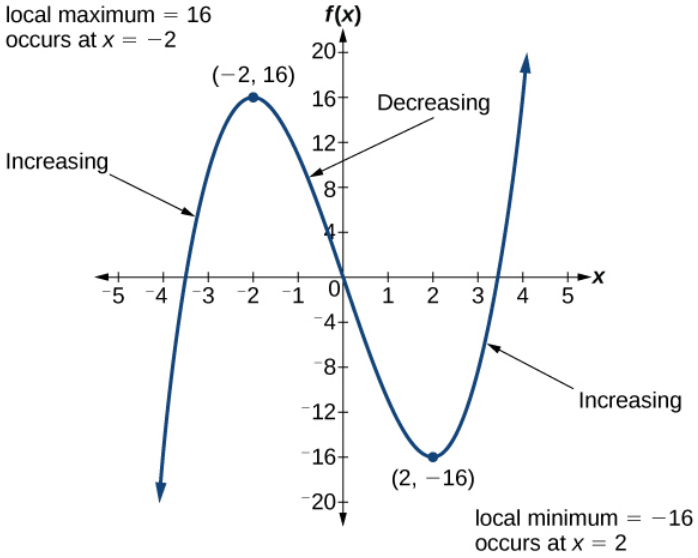


Figure 2.3.4: Graph of a polynomial that shows the increasing and decreasing intervals and local maximum.maximum

To locate the local maxima and minima from a graph, we need to observe the graph to determine where the graph attains its highest and lowest points, respectively, within an open interval. Like the summit of a roller coaster, the graph of a function is higher at a local maximum than at nearby points on both sides. The graph will also be lower at a local minimum than at neighboring points. Figure 2.3.5 illustrates these ideas for a local maximum.

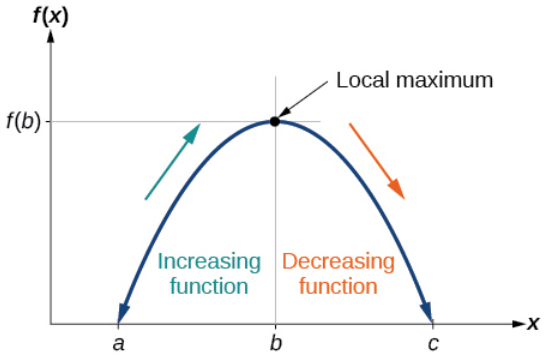


Figure 2.3.5: Definition of a local maximum

These observations lead us to a formal definition of local extrema.

Local Minima and Local Maxima

- A function f is an **increasing function** on an open interval if $f(b) > f(a)$ for every a, b interval where $b > a$.
- A function f is a **decreasing function** on an open interval if $f(b) < f(a)$ for every a, b interval where $b > a$.

A function f has a local maximum at a point b in an open interval (a, c) if $f(b)$ is greater than or equal to $f(x)$ for every point x (x does not equal b) in the interval. Likewise, f has a local minimum at a point b in (a, c) if $f(b)$ is less than or equal to $f(x)$ for every x (x does not equal b) in the interval.

Example 2.3.7 Finding Increasing and Decreasing Intervals on a Graph

Given the function $p(t)$ in Figure 2.3.6, identify the intervals on which the function appears to be increasing.

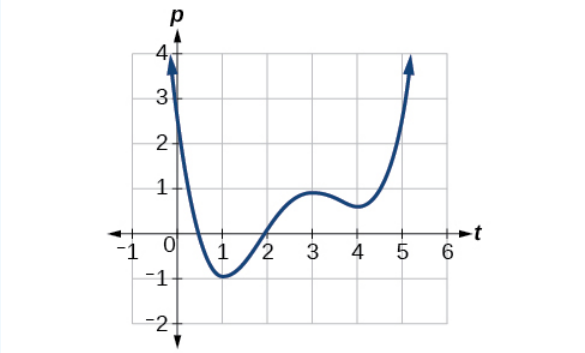


Figure 2.3.6: Graph of a polynomial.

Solution

We see that the function is not constant on any interval. The function is increasing where it slants upward as we move to the right and decreasing where it slants downward as we move to the right. The function appears to be increasing from $t = 1$ to $t = 3$ and from $t = 4$ on.

In interval notation, we would say the function appears to be increasing on the interval $(1, 3)$ and the interval $(4, \infty)$.

Analysis

Notice in this example that we used open intervals (intervals that do not include the endpoints), because the function is neither increasing nor decreasing at $t = 1$, $t = 3$, and $t = 4$. These points are the local extrema (two minima and a maximum).

Example 2.3.8: Finding Local Extrema from a Graph

Graph the function $f(x) = \frac{2}{x} + \frac{x}{3}$. Then use the graph to estimate the local extrema of the function and to determine the intervals on which the function is increasing.

Solution

Using technology, we find that the graph of the function looks like that in Figure 2.3.7. It appears there is a low point, or local minimum, between $x = 2$ and $x = 3$, and a mirror-image high point, or local maximum, somewhere between $x = -3$ and $x = -2$

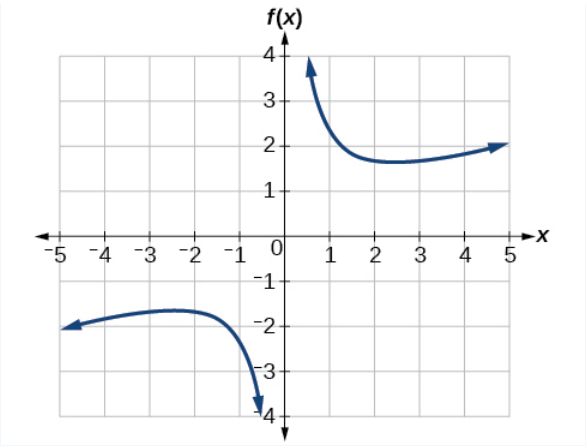


Figure 2.3.7: Graph of a reciprocal function.

Analysis

Most graphing calculators and graphing utilities can estimate the location of maxima and minima. Figure 2.3.8 provides screen images from two different technologies, showing the estimate for the local maximum and minimum.

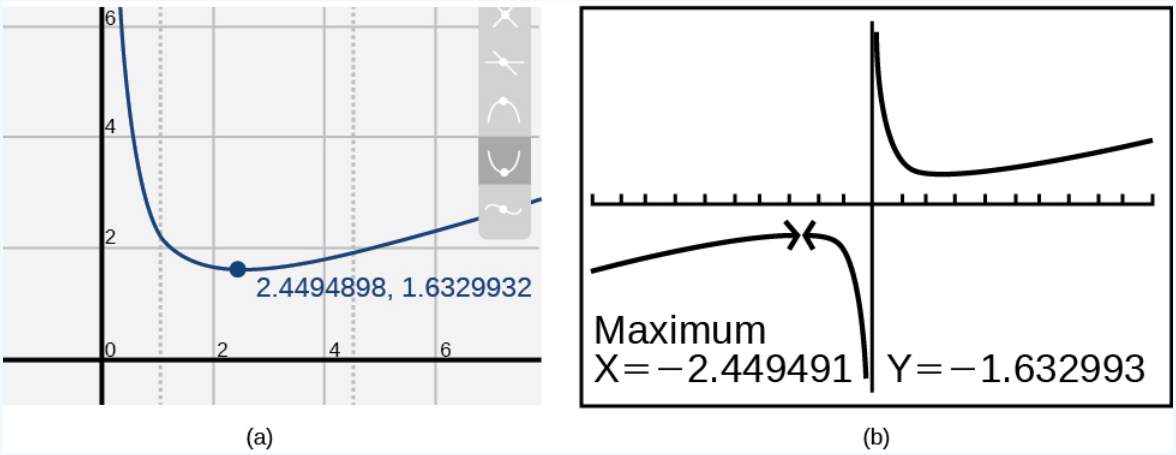


Figure 2.3.8: Graph of the reciprocal function on a graphing calculator.

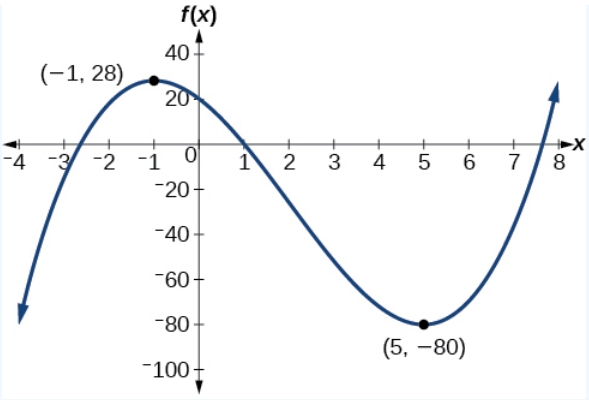
Based on these estimates, the function is increasing on the interval $(-\infty, -2.449)$ and $(2.449, \infty)$. Notice that, while we expect the extrema to be symmetric, the two different technologies agree only up to four decimals due to the differing approximation algorithms used by each. (The exact location of the extrema is at $\pm\sqrt{6}$, but determining this requires calculus.)

Exercise 2.3.8

Graph the function $f(x) = x^3 - 6x^2 - 15x + 20$ to estimate the local extrema of the function. Use these to determine the intervals on which the function is increasing and decreasing.

Solution

The local maximum appears to occur at $(-1, 28)$, and the local minimum occurs at $(5, -80)$. The function is increasing on $(-\infty, -1) \cup (5, \infty)$ and decreasing on $(-1, 5)$.



Graph of a polynomial with a local maximum at (-1, 28) and local minimum at (5, -80).

Example 2.3.9: Finding Local Maxima and Minima from a Graph

For the function f whose graph is shown in Figure 2.3.9, find all local maxima and minima.

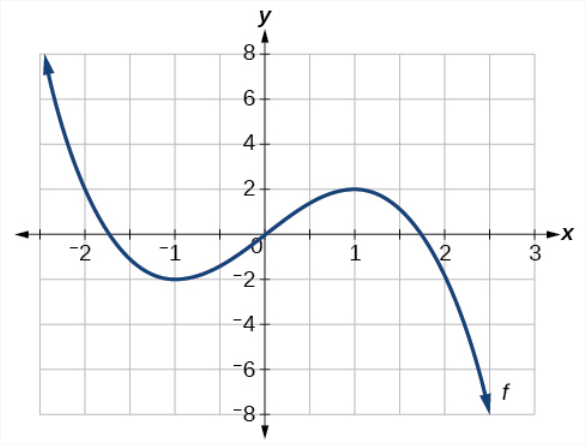


Figure 2.3.9: Graph of a polynomial.

Solution

Observe the graph of f. The graph attains a local maximum at $x = 1$ because it is the highest point in an open interval around $x = 1$.The local maximum is the y-coordinate at $x = 1$, which is 2.

The graph attains a local minimum at $x = -1$ because it is the lowest point in an open interval around $x = -1$. The local minimum is the y-coordinate at $x = -1$, which is −2.

Analyzing the Toolkit Functions for Increasing or Decreasing Intervals

We will now return to our toolkit functions and discuss their graphical behavior in Figure 2.3.10, Figure 2.3.11, and Figure 2.3.12

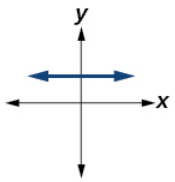
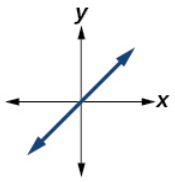
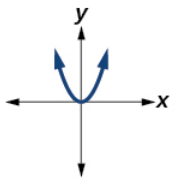
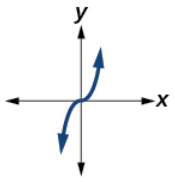
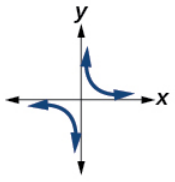
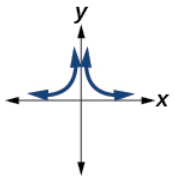
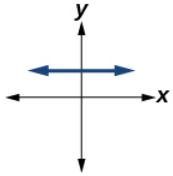
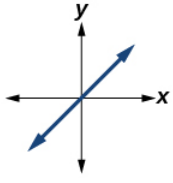
Function	Increasing/Decreasing	Example
Constant Function $f(x) = c$	Neither increasing nor decreasing	
Identity Function $f(x) = x$	Increasing	
Quadratic Function $f(x) = x^2$	Increasing on $(0, \infty)$ Decreasing on $(-\infty, 0)$ Minimum at $x = 0$	

Figure 2.3.10

Function	Increasing/Decreasing	Example
Cubic Function $f(x) = x^3$	Increasing	
Reciprocal $f(x) = \frac{1}{x}$	Decreasing $(-\infty, 0) \cup (0, \infty)$	
Reciprocal Squared $f(x) = \frac{1}{x^2}$	Increasing on $(-\infty, 0)$ Decreasing on $(0, \infty)$	

.

Figure 2.3.11

Function	Increasing/Decreasing	Example
Constant Function $f(x) = c$	Neither increasing nor decreasing	
Identity Function $f(x) = x$	Increasing	

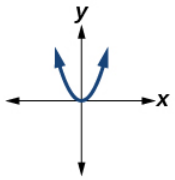
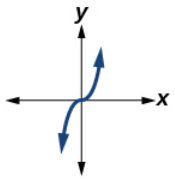
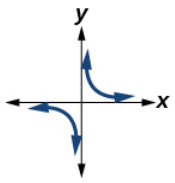
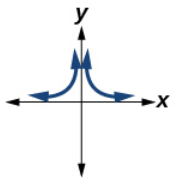
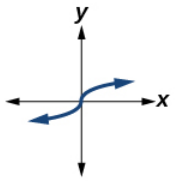
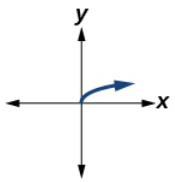
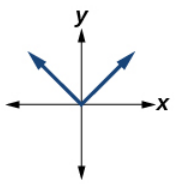
Quadratic Function $f(x) = x^2$	Increasing on $(0, \infty)$ Decreasing on $(-\infty, 0)$ Minimum at $x = 0$	
Cubic Function $f(x) = x^3$	Increasing	
Reciprocal $f(x) = \frac{1}{x}$	Decreasing $(-\infty, 0) \cup (0, \infty)$	
Reciprocal Squared $f(x) = \frac{1}{x^2}$	Increasing on $(-\infty, 0)$ Decreasing on $(0, \infty)$	
Cube Root $f(x) = \sqrt[3]{x}$	Increasing	
Square Root $f(x) = \sqrt{x}$	Increasing on $(0, \infty)$	
Absolute Value $f(x) = \lvert x \rvert$	Increasing on $(0, \infty)$ Decreasing on $(\infty, 0)$	

***Figure* 2.3.12**

Use A Graph to Locate the Absolute Maximum and Absolute Minimum

There is a difference between locating the highest and lowest points on a graph in a region around an open interval (locally) and locating the highest and lowest points on the graph for the entire domain. The y-coordinates (output) at the highest and lowest points are called the **absolute maximum** and **absolute minimum**, respectively. To locate absolute maxima and minima from a graph, we need to observe the graph to determine where the graph attains it highest and lowest points on the domain of the function (Figure 2.3.13).

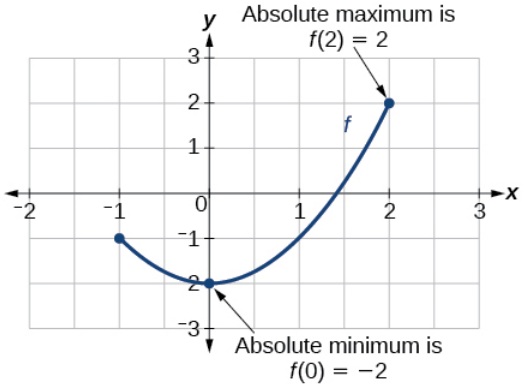


Figure 2.3.13: Graph of a segment of a parabola with an absolute minimum at (0, -2) and absolute maximum at (2, 2).

Not every function has an absolute maximum or minimum value. The toolkit function $f(x) = x^3$ is one such function.

Absolute Maxima and Minima

- The **absolute maximum** of f at $x = c$ is $f(c)$ where $f(c) \geq f(x)$ for all x in the domain of f.
- The **absolute minimum** of f at $x = d$ is $f(d)$ where $f(d) \leq f(x)$ for all x in the domain of f.

Example 2.3.10: Finding Absolute Maxima and Minima from a Graph

For the function f shown in Figure 2.3.14, find all absolute maxima and minima.

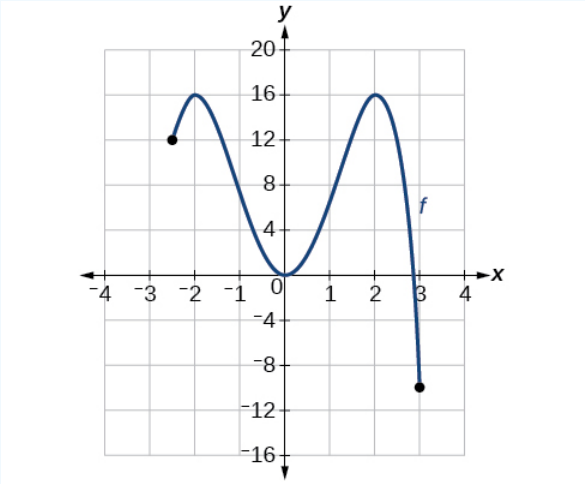


Figure 2.3.14: Graph of a polynomial.

Solution

Observe the graph of f. The graph attains an absolute maximum in two locations, $x = -2$ and $x = 2$, because at these locations, the graph attains its highest point on the domain of the function. The absolute maximum is the y-coordinate at $x = -2$ and $x = 2$, which is 16.

The graph attains an absolute minimum at x=3, because it is the lowest point on the domain of the function's graph. The absolute minimum is the y-coordinate at x=3,which is−10.

Key Equations

- Average rate of change: $\dfrac{\Delta y}{\Delta x} = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$

Key Concepts

- A rate of change relates a change in an output quantity to a change in an input quantity. The average rate of change is determined using only the beginning and ending data. See Example.

- Identifying points that mark the interval on a graph can be used to find the average rate of change. See Example.
- Comparing pairs of input and output values in a table can also be used to find the average rate of change. See Example.
- An average rate of change can also be computed by determining the function values at the endpoints of an interval described by a formula. See Example and Example.
- The average rate of change can sometimes be determined as an expression. See Example.
- A function is increasing where its rate of change is positive and decreasing where its rate of change is negative. See Example.
- A local maximum is where a function changes from increasing to decreasing and has an output value larger (more positive or less negative) than output values at neighboring input values.
- A local minimum is where the function changes from decreasing to increasing (as the input increases) and has an output value smaller (more negative or less positive) than output values at neighboring input values.
- Minima and maxima are also called extrema.
- We can find local extrema from a graph. See Example and Example.
- The highest and lowest points on a graph indicate the maxima and minima. See Example.

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2.4: Composition of Functions

Combining Functions Using Algebraic Operations

Function composition is only one way to combine existing functions. Another way is to carry out the usual algebraic operations on functions, such as addition, subtraction, multiplication and division. We do this by performing the operations with the function outputs, defining the result as the output of our new function.

Suppose we need to add two columns of numbers that represent a husband and wife's separate annual incomes over a period of years, with the result being their total household income. We want to do this for every year, adding only that year's incomes and then collecting all the data in a new column. If $w(y)$ is the wife's income and $h(y)$ is the husband's income in year y, and we want T to represent the total income, then we can define a new function.

$$T(y) = h(y) + w(y)$$

If this holds true for every year, then we can focus on the relation between the functions without reference to a year and write

$$T = h + w$$

Just as for this sum of two functions, we can define difference, product, and ratio functions for any pair of functions that have the same kinds of inputs (not necessarily numbers) and also the same kinds of outputs (which do have to be numbers so that the usual operations of algebra can apply to them, and which also must have the same units or no units when we add and subtract). In this way, we can think of adding, subtracting, multiplying, and dividing functions.

For two functions $f(x)$ and $g(x)$ with real number outputs, we define new functions $f+g$, $f-g$, fg, and $\frac{f}{g}$ by the relations.

$$\begin{aligned}(f+g)(x) &= f(x)+g(x)\\ (f-g)(x) &= f(x)-g(x)\\ (fg)(x) &= f(x)g(x)\\ \left(\frac{f}{g}\right)(x) &= \frac{f(x)}{g(x)}\end{aligned}$$

Example 2.4.1: Performing Algebraic Operations on Functions

Find and simplify the functions $(g-f)(x)$ and $\left(\frac{g}{f}\right)(x)$, given $f(x) = x-1$ and $g(x) = x^2-1$. Are they the same function?

Solution

Begin by writing the general form, and then substitute the given functions.

$$\begin{aligned}(g-f)(x) &= g(x)-f(x)\\ (g-f)(x) &= x^2-1-(x-1)\\ &= x^2-x\\ &= x(x-1)\\ \left(\frac{g}{f}\right)(x) &= g(x)f(x)\\ \left(\frac{g}{f}\right)(x) &= \frac{x^2-1}{x-1}\\ &= \frac{(x+1)(x-1)}{x-1}\\ &= x+1\end{aligned}$$

No, the functions are not the same.

Note: For $\left(\frac{g}{f}\right)(x)$, the condition $x \neq 1$ is necessary because when $x = 1$, the denominator is equal to 0, which makes the function undefined.

? Exercise 2.4.1

Find and simplify the functions $(fg)(x)$ and $(f-g)(x)$.

$$f(x) = x - 1$$

and

$$g(x) = x^2 - 1$$

Are they the same function?

Answer

$(fg)(x) = f(x)g(x) = (x-1)(x2-1) = x^3 - x^2 - x + 1$

$(f-g)(x) = f(x) - g(x) = (x-1) - (x^2-1) = x - x^2$

No, the functions are not the same.

Create a Function by Composition of Functions

Performing algebraic operations on functions combines them into a new function, but we can also create functions by composing functions. When we wanted to compute a heating cost from a day of the year, we created a new function that takes a day as input and yields a cost as output. The process of **combining functions** so that the output of one function becomes the input of another is known as a **composition of functions**. The resulting function is known as a **composite function**. We represent this combination by the following notation:

$$f \circ g(x) = f(g(x)) \tag{2.4.1}$$

We read the left-hand side as"f composed with g at x," and the right-hand side as"f of g of x."The two sides of the equation have the same mathematical meaning and are equal. The open circle symbol $\circ$ is called the composition operator. We use this operator mainly when we wish to emphasize the relationship between the functions themselves without referring to any particular input value. Composition is a binary operation that takes two functions and forms a new function, much as addition or multiplication takes two numbers and gives a new number. However, it is important not to confuse function composition with multiplication because, as we learned above, in most cases $f(g(x)) \neq f(x)g(x)$.

It is also important to understand the order of operations in evaluating a composite function. We follow the usual convention with parentheses by starting with the innermost parentheses first, and then working to the outside. In the equation above, the function g takes the input x first and yields an output $g(x)$. Then the function f takes $g(x)$ as an input and yields an output $f(g(x))$.

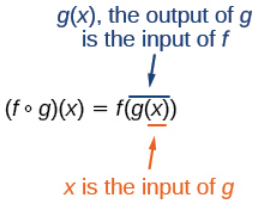


Figure 2.4.2: Explanation of the composite function.

In general, $f \circ g$ and $g \circ f$ are different functions. In other words, in many cases $f(g(x)) \neq g(f(x))$ for all x. We will also see that sometimes two functions can be composed only in one specific order.

For example, if $f(x) = x^2$ and $g(x) = x + 2$, then

$$\begin{aligned} f(g(x)) &= f(x+2) \\ &= (x+2)^2 \\ &= x^2+4x+4 \end{aligned}$$

but

$$\begin{aligned} g(f(x)) &= g(x^2) \\ &= x^2+2 \end{aligned}$$

These expressions are not equal for all values of x, so the two functions are not equal. It is irrelevant that the expressions happen to be equal for the single input value $x=-\frac{1}{2}$.

Note that the range of the inside function (the first function to be evaluated) needs to be within the domain of the outside function. Less formally, the composition has to make sense in terms of inputs and outputs.

Composition of Functions

When the output of one function is used as the input of another, we call the entire operation a composition of functions. For any input x and functions f and g, this action defines a **composite function**, which we write as $f \circ g$ such that

$$(f \circ g)(x) = f(g(x)) \tag{2.4.2}$$

The domain of the composite function $f \circ g$ is all x such that x is in the domain of g and $g(x)$ is in the domain of f.

It is important to realize that the product of functions fg is not the same as the function composition $f(g(x))$, because, in general, $f(x)g(x) \neq f(g(x))$.

Example 2.4.2: Determining whether Composition of Functions is Commutative

Using the functions provided, find $f(g(x))$ and $g(f(x))$. Determine whether the composition of the functions is **commutative**.

$$f(x) = 2x+1 \quad g(x) = 3-x$$

Solution

Let's begin by substituting $g(x)$ into $f(x)$.

$$\begin{aligned} f(g(x)) &= 2(3-x)+1 \\ &= 6-2x+1 \\ &= 7-2x \end{aligned}$$

Now we can substitute $f(x)$ into $g(x)$.

$$\begin{aligned} g(f(x)) &= 3-(2x+1) \\ &= 3-2x-1 \\ &= 2-2x \end{aligned}$$

We find that $g(f(x)) \neq f(g(x))$, so the operation of function composition is not commutative.

Example 2.4.3: Interpreting Composite Functions

The function $c(s)$ gives the number of calories burned completing s sit-ups, and $s(t)$ gives the number of sit-ups a person can complete in t minutes. Interpret $c(s(3))$.

Solution

The inside expression in the composition is $s(3)$. Because the input to the s-function is time, $t=3$ represents 3 minutes, and $s(3)$ is the number of sit-ups completed in 3 minutes.

Using $s(3)$ as the input to the function $c(s)$ gives us the number of calories burned during the number of sit-ups that can be completed in 3 minutes, or simply the number of calories burned in 3 minutes (by doing sit-ups).

Example 2.4.4: Investigating the Order of Function Composition

Suppose $f(x)$ gives miles that can be driven in x hours and $g(y)$ gives the gallons of gas used in driving y miles. Which of these expressions is meaningful: $f(g(y))$ or $g(f(x))$?

Solution

The function $y = f(x)$ is a function whose output is the number of miles driven corresponding to the number of hours driven.

$$\text{number of miles } = f(\text{number of hours})$$

The function $g(y)$ is a function whose output is the number of gallons used corresponding to the number of miles driven. This means:

$$\text{number of gallons } = g(\text{number of miles})$$

The expression $g(y)$ takes miles as the input and a number of gallons as the output. The function $f(x)$ requires a number of hours as the input. Trying to input a number of gallons does not make sense. The expression $f(g(y))$ is meaningless.

The expression $f(x)$ takes hours as input and a number of miles driven as the output. The function $g(y)$ requires a number of miles as the input. Using $f(x)$ (miles driven) as an input value for $g(y)$, where gallons of gas depends on miles driven, does make sense. The expression $g(f(x))$ makes sense, and will yield the number of gallons of gas used, g, driving a certain number of miles, $f(x)$, in x hours.

Question/Answer

Are there any situations where $f(g(y))$ and $g(f(x))$ would both be meaningful or useful expressions?

Yes. For many pure mathematical functions, both compositions make sense, even though they usually produce different new functions. In real-world problems, functions whose inputs and outputs have the same units also may give compositions that are meaningful in either order

Exercise 2.4.2

The gravitational force on a planet a distance r from the sun is given by the function $G(r)$. The acceleration of a planet subjected to any force F is given by the function $a(F)$. Form a meaningful composition of these two functions, and explain what it means.

Answer

A gravitational force is still a force, so $a(G(r))$ makes sense as the acceleration of a planet at a distance r from the Sun (due to gravity), but $G(a(F))$ does not make sense.

Evaluating Composite Functions

Once we compose a new function from two existing functions, we need to be able to evaluate it for any input in its domain. We will do this with specific numerical inputs for functions expressed as tables, graphs, and formulas and with variables as inputs to functions expressed as formulas. In each case, we evaluate the inner function using the starting input and then use the inner function's output as the input for the outer function.

Evaluating Composite Functions Using Tables

When working with functions given as tables, we read input and output values from the table entries and always work from the inside to the outside. We evaluate the inside function first and then use the output of the inside function as the input to the outside function.

Example 2.4.5: Using a Table to Evaluate a Composite Function

Using Table 2.4.1, evaluate $f(g(3))$ and $g(f(3))$.

Table 2.4.1

x	$f(x)$	$g(x)$
1	6	3
2	8	5
3	3	2
4	1	7

Solution

To evaluate $f(g(3))$, we start from the inside with the input value 3. We then evaluate the inside expression $g(3)$ using the table that defines the function g : $g(3) = 2$. We can then use that result as the input to the function f, so $g(3)$ is replaced by 2 and we get $f(2)$. Then, using the table that defines the function f, we find that $f(2) = 8$.

$$g(3) = 2$$

$$f(g(3)) = f(2) = 8$$

To evaluate $g(f(3))$, we first evaluate the inside expression $f(3)$ using the first table: $f(3) = 3$. Then, using the table for g, we can evaluate

$$g(f(3)) = g(3) = 2$$

Table 2.4.2 shows the composite functions $f \circ g$ and $g \circ f$ as tables.

Table 2.4.2

x	$g(x)$	$f(g(x))$	$f(x)$	$g(f(x))$
3	2	8	3	2

Exercise 2.4.3

Using Table 2.4.1, evaluate $f(g(1))$ and $g(f(4))$.

Answer

$f(g(1)) = f(3) = 3$ and $g(f(4)) = g(1) = 3$

Evaluating Composite Functions Using Graphs

When we are given individual functions as graphs, the procedure for evaluating composite functions is similar to the process we use for evaluating tables. We read the input and output values, but this time, from the x- and y-axes of the graphs.

How To ...

Given a composite function and graphs of its individual functions, evaluate it using the information provided by the graphs.

1. Locate the given input to the inner function on the x-axis of its graph.
2. Read off the output of the inner function from the y-axis of its graph.
3. Locate the inner function output on the x-axis of the graph of the outer function.
4. Read the output of the outer function from the y-axis of its graph. This is the output of the composite function.

Example 2.4.6: Using a Graph to Evaluate a Composite Function

Using Figure 2.4.3, evaluate $f(g(1))$.

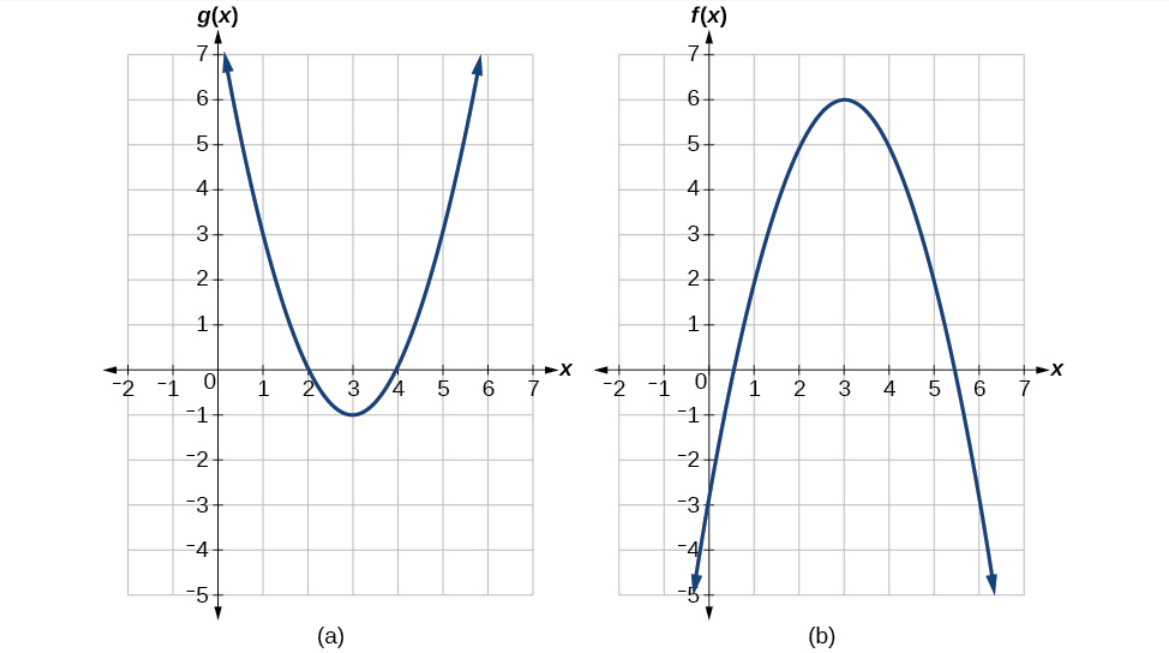


Figure 2.4.3: Two graphs of a positive and negative parabola.

Solution

To evaluate $f(g(1))$, we start with the inside evaluation. See Figure 2.4.4.

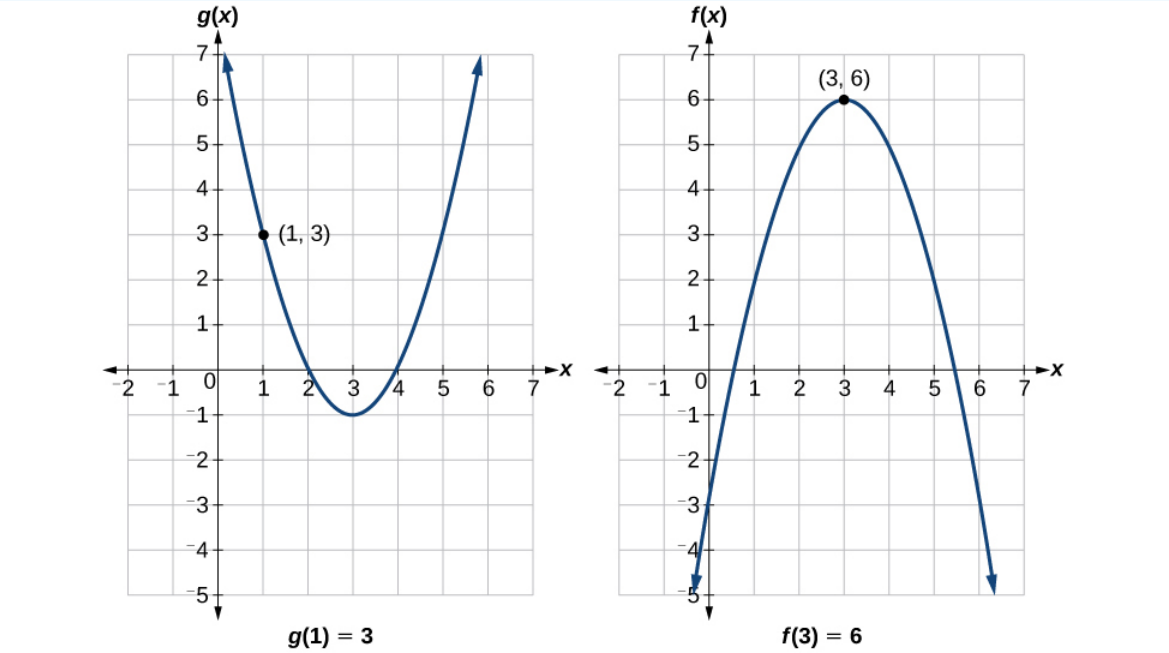


Figure 2.4.4: Two graphs of a positive parabola $g(x)$ and a negative parabola $f(x)$. The following points are plotted: $g(1) = 3$ and $f(3) = 6$.

We evaluate $g(1)$ using the graph of $g(x)$, finding the input of 1 on the x-axis and finding the output value of the graph at that input. Here, $g(1) = 3$. We use this value as the input to the function f.

$$f(g(1)) = f(3)$$

https://math.libretexts.org/@go/page/89775

We can then evaluate the composite function by looking to the graph of $f(x)$, finding the input of 3 on the x-axis and reading the output value of the graph at this input. Here, $f(3) = 6$, so $f(g(1)) = 6$.

Analysis

Figure 2.4.5 shows how we can mark the graphs with arrows to trace the path from the input value to the output value.

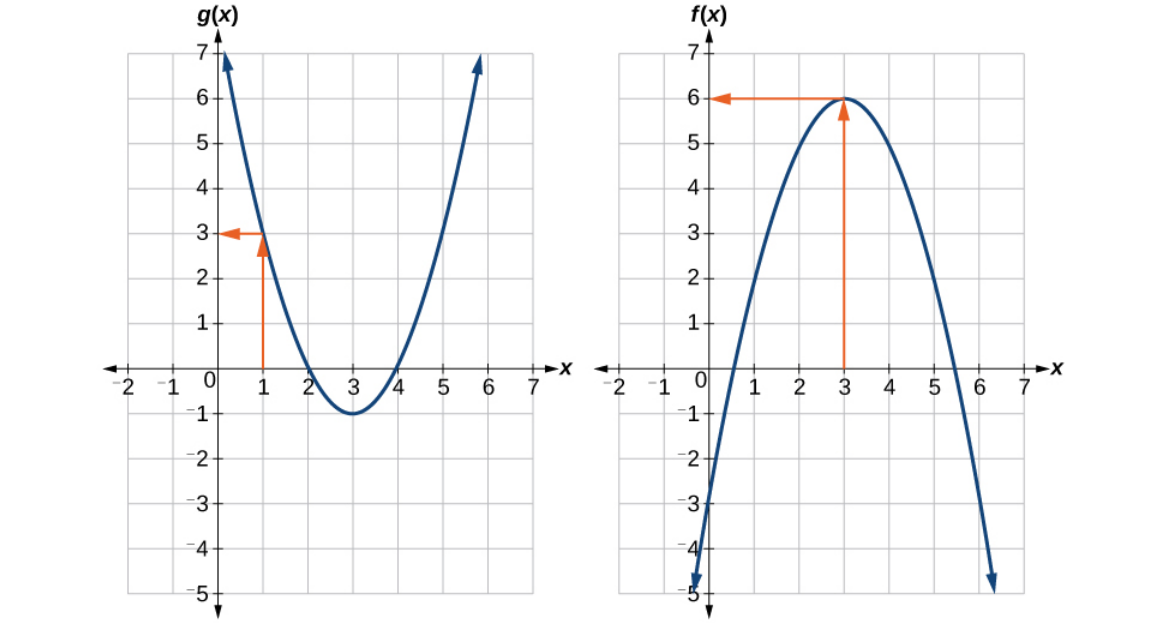


Figure 2.4.5: Two graphs of a positive and negative parabola.

? Exercise 2.4.4

Using Figure 2.4.3, evaluate $g(f(2))$.

Answer

$g(f(2)) = g(5) = 3$

Evaluating Composite Functions Using Formulas

When evaluating a composite function where we have either created or been given formulas, the rule of working from the inside out remains the same. The input value to the outer function will be the output of the inner function, which may be a numerical value, a variable name, or a more complicated expression.

While we can compose the functions for each individual input value, it is sometimes helpful to find a single formula that will calculate the result of a composition $f(g(x))$. To do this, we will extend our idea of function evaluation. Recall that, when we evaluate a function like $f(t) = t^2 - t$, we substitute the value inside the parentheses into the formula wherever we see the input variable.

How To...

Given a formula for a composite function, evaluate the function.

1. Evaluate the inside function using the input value or variable provided.
2. Use the resulting output as the input to the outside function.

Example 2.4.7: Evaluating a Composition of Functions Expressed as Formulas with a Numerical Input

Given $f(t)=t^2-t$ and $h(x)=3x+2$, evaluate $f(h(1))$.

Solution

Because the inside expression is $h(1)$, we start by evaluating $h(x)$ at 1.

$$h(1)=3(1)+2$$
$$h(1)=5$$

Then $f(h(1))=f(5)$, so we evaluate $f(t)$ at an input of 5.

$$f(h(1))=f(5)$$
$$f(h(1))=5^2-5$$
$$f(h(1))=20$$

Analysis

It makes no difference what the input variables t and x were called in this problem because we evaluated for specific numerical values.

Exercise 2.4.5

Given $f(t)=t^2-t$ and $h(x)=3x+2$, evaluate

a. $h(f(2))$
b. $h(f(-2))$

Answer a

8

Answer b

20

Key Equation

- Composite function $(f\circ g)(x)=f(g(x))$

Key Concepts

- We can perform algebraic operations on functions. See Example.
- When functions are combined, the output of the first (inner) function becomes the input of the second (outer) function.
- The function produced by combining two functions is a composite function. See Example and Example.
- The order of function composition must be considered when interpreting the meaning of composite functions. See Example.
- A composite function can be evaluated by evaluating the inner function using the given input value and then evaluating the outer function taking as its input the output of the inner function.
- A composite function can be evaluated from a table. See Example.
- A composite function can be evaluated from a graph. See Example.
- A composite function can be evaluated from a formula. See Example.
- The domain of a composite function consists of those inputs in the domain of the inner function that correspond to outputs of the inner function that are in the domain of the outer function. See Example and Example.
- Just as functions can be combined to form a composite function, composite functions can be decomposed into simpler functions.
- Functions can often be decomposed in more than one way. See Example.

Glossary

composite function

the new function formed by function composition, when the output of one function is used as the input of another

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2.5: Inverse Functions

Verifying That Two Functions Are Inverse Functions

Suppose a fashion designer traveling to Milan for a fashion show wants to know what the temperature will be. He is not familiar with the **Celsius** scale. To get an idea of how temperature measurements are related, he asks his assistant, Betty, to convert 75 degrees **Fahrenheit** to degrees Celsius. She finds the formula

$$C = \frac{5}{9}(F - 32) \tag{2.5.1}$$

and substitutes 75 for F to calculate

$$\frac{5}{9}(75 - 32) \approx 24^\circ \tag{2.5.2}$$

Knowing that a comfortable 75 degrees Fahrenheit is about 24 degrees Celsius, he sends his assistant the week's weather forecast from Figure 2.5.2 for Milan, and asks her to convert all of the temperatures to degrees Fahrenheit.

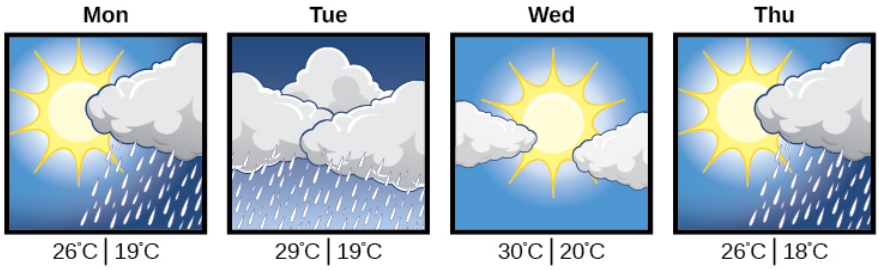


Figure 2.5.2: A forecast of Monday's through Thursday's weather.

At first, Betty considers using the formula she has already found to complete the conversions. After all, she knows her algebra, and can easily solve the equation for F after substituting a value for C. For example, to convert 26 degrees Celsius, she could write

$$26 = \frac{5}{9}(F - 32) \tag{2.5.3}$$

$$26 \cdot \frac{9}{5} = F - 32 \tag{2.5.4}$$

$$F = 26 \cdot \frac{9}{5} + 32 \approx 79 \tag{2.5.5}$$

After considering this option for a moment, however, she realizes that solving the equation for each of the temperatures will be awfully tedious. She realizes that since evaluation is easier than solving, it would be much more convenient to have a different formula, one that takes the Celsius temperature and outputs the Fahrenheit temperature.

The formula for which Betty is searching corresponds to the idea of an **inverse function**, which is a function for which the input of the original function becomes the output of the inverse function and the output of the original function becomes the input of the inverse function.

Given a function $f(x)$, we represent its inverse as $f^{-1}(x)$, read as "f inverse of x." The raised -1 is part of the notation. It is not an exponent; it does not imply a power of -1 . In other words, $f^{-1}(x)$ does not mean $\frac{1}{f(x)}$ because $\frac{1}{f(x)}$ is the reciprocal of f and not the inverse.

The "exponent-like" notation comes from an analogy between function composition and multiplication: just as $a^{-1}a = 1$ (1 is the identity element for multiplication) for any nonzero number a, so $f^{-1} \circ f$ equals the identity function, that is,

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(y) = x \tag{2.5.6}$$

This holds for all x in the domain of f. Informally, this means that inverse functions "undo" each other. However, just as zero does not have a **reciprocal**, some functions do not have inverses.

Given a function $f(x)$, we can verify whether some other function $g(x)$ is the inverse of $f(x)$ by checking whether either $g(f(x)) = x$ or $f(g(x)) = x$ is true. We can test whichever equation is more convenient to work with because they are logically

equivalent (that is, if one is true, then so is the other.)

For example, $y = 4x$ and $y = \frac{1}{4}x$ are inverse functions.

$$(f^{-1} \circ f)(x) = f^{-1}(4x) = \frac{1}{4}(4x) = x \tag{2.5.7}$$

and

$$(f \circ f^{-1})(x) = f\left(\frac{1}{4}x\right) = 4\left(\frac{1}{4}x\right) = x \tag{2.5.8}$$

A few coordinate pairs from the graph of the function $y = 4x$ are $(-2, -8)$, $(0, 0)$, and $(2, 8)$. A few coordinate pairs from the graph of the function $y = \frac{1}{4}x$ are $(-8, -2)$, $(0, 0)$, and $(8, 2)$. If we interchange the input and output of each coordinate pair of a function, the interchanged coordinate pairs would appear on the graph of the inverse function.

Definition: Inverse Function

For any **one-to-one function** $f(x) = y$, a function $f^{-1}(x)$ is an **inverse function** of f if $f^{-1}(y) = x$. This can also be written as $f^{-1}(f(x)) = x$ for all x in the domain of f. It also follows that $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1} if f^{-1} is the inverse of f.

The notation f^{-1} is read "f inverse." Like any other function, we can use any variable name as the input for f^{-1}, so we will often write $f^{-1}(x)$, which we read as "f inverse of x." Keep in mind that

$$f^{-1}(x) \neq \frac{1}{f(x)} \tag{2.5.9}$$

and not all functions have inverses.

Example 2.5.1: Identifying an Inverse Function for a Given Input-Output Pair

If for a particular one-to-one function $f(2) = 4$ and $f(5) = 12$, what are the corresponding input and output values for the inverse function?

Solution

The inverse function reverses the input and output quantities, so if

$$\begin{aligned} f(2) = 4, \text{ then } f^{-1}(4) = 2; \\ f(5) = 12, \text{ then } f^{-1}(12) = 5 \end{aligned} \tag{2.5.10}$$

.

Alternatively, if we want to name the inverse function g, then $g(4) = 2$ and $g(12) = 5$.

Analysis

Notice that if we show the coordinate pairs in a table form, the input and output are clearly reversed. See Table 2.5.1.

Table 2.5.1

$(x, f(x))$	$(x, g(x))$
$(2, 4)$	$(4, 2)$
$(5, 12)$	$(12, 5)$

Exercise 2.5.1

Given that $h^{-1}(6) = 2$, what are the corresponding input and output values of the original function h?

Answer

$h(2) = 6$

How To: Given two functions $f(x)$ and $g(x)$, test whether the functions are inverses of each other.

1. Determine whether $f(g(x))=x$ or $g(f(x))=x$.
2. If both statements are true, then $g=f^{-1}$ and $f=g^{-1}$. If either statement is false, then both are false, and $g\neq f^{-1}$ and $f\neq g^{-1}$.

Example 2.5.2: Testing Inverse Relationships Algebraically

If $f(x)=\frac{1}{x+2}$ and $g(x)=\frac{1}{x}-2$, is $g=f^{-1}$?

Solution

$$g(f(x))=\frac{1}{(\frac{1}{x+2})-2} \tag{2.5.11}$$
$$=x+2-2 \tag{2.5.12}$$
$$=x \tag{2.5.13}$$

so

$$g=f^{-1} \text{ and } f=g^{-1} \tag{2.5.14}$$

This is enough to answer yes to the question, but we can also verify the other formula.

$$f(g(x))=\frac{1}{\frac{1}{x}-2+2} \tag{2.5.15}$$
$$=\frac{1}{\frac{1}{x}} \tag{2.5.16}$$
$$=x \tag{2.5.17}$$

Analysis

Notice the inverse operations are in reverse order of the operations from the original function.

Exercise 2.5.2

If $f(x)=x^3-4$ and $g(x)=\sqrt[3]{x+4}$, is $g=f^{-1}$?

Answer

Yes

Example 2.5.3: Determining Inverse Relationships for Power Functions

If $f(x)=x^3$ (the cube function) and $g(x)=\frac{1}{3}x$, is $g=f^{-1}$?

Solution

$$f(g(x))=\frac{x^3}{27}\neq x \tag{2.5.18}$$

No, the functions are not inverses.

Analysis

The correct inverse to the cube is, of course, the cube root $\sqrt[3]{x}=x^{\frac{1}{3}}$, that is, the one-third is an exponent, not a multiplier.

Exercise 2.5.3

If $f(x)=(x-1)^3$ and $g(x)=\sqrt[3]{x}+1$, is $g=f^{-1}$?

Answer

Yes

Finding Domain and Range of Inverse Functions

The outputs of the function f are the inputs to f^{-1}, so the range of f is also the domain of f^{-1}. Likewise, because the inputs to f are the outputs of f^{-1}, the domain of f is the range of f^{-1}. We can visualize the situation as in Figure 2.5.3.

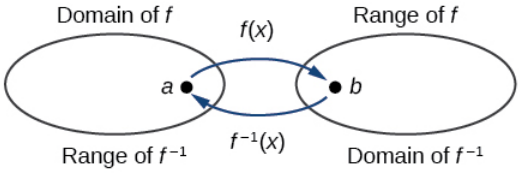


Figure 2.5.3: *Domain and range of a function and its inverse.*

When a function has no inverse function, it is possible to create a new function where that new function on a limited domain does have an inverse function. For example, the inverse of $f(x)=\sqrt{x}$ is $f^{-1}(x)=x^2$, because a square "undoes" a square root; but the square is only the inverse of the square root on the domain $[0,\infty)$, since that is the range of $f(x)=\sqrt{x}$.

We can look at this problem from the other side, starting with the square (toolkit quadratic) function $f(x)=x^2$. If we want to construct an inverse to this function, we run into a problem, because for every given output of the quadratic function, there are two corresponding inputs (except when the input is 0). For example, the output 9 from the quadratic function corresponds to the inputs 3 and –3. But an output from a function is an input to its inverse; if this inverse input corresponds to more than one inverse output (input of the original function), then the "inverse" is not a function at all! To put it differently, the quadratic function is not a one-to-one function; it fails the horizontal line test, so it does not have an inverse function. In order for a function to have an inverse, it must be a one-to-one function.

In many cases, if a function is not one-to-one, we can still restrict the function to a part of its domain on which it is one-to-one. For example, we can make a restricted version of the square function $f(x)=x^2$ with its range limited to $[0,\infty)$, which is a one-to-one function (it passes the horizontal line test) and which has an inverse (the square-root function).

If $f(x)=(x-1)^2$ on $[1,\infty)$, then the inverse function is $f^{-1}(x)=\sqrt{x}+1$.

- The domain of f = range of $f^{-1}=[1,\infty)$.
- The domain of f^{-1} = range of $f=[0,\infty)$.

Is it possible for a function to have more than one inverse?

No. If two supposedly different functions, say, g and h, both meet the definition of being inverses of another function f, then you can prove that $g=h$. We have just seen that some functions only have inverses if we restrict the domain of the original function. In these cases, there may be more than one way to restrict the domain, leading to different inverses. However, on any one domain, the original function still has only one unique inverse.

Note: Domain and Range of Inverse Functions

The range of a function $f(x)$ is the domain of the inverse function $f^{-1}(x)$.

The domain of $f(x)$ is the range of $f^{-1}(x)$.

How To: Given a function, find the domain and range of its inverse.

1. If the function is one-to-one, write the range of the original function as the domain of the inverse, and write the domain of the original function as the range of the inverse.
2. If the domain of the original function needs to be restricted to make it one-to-one, then this restricted domain becomes the range of the inverse function.

Example 2.5.4: Finding the Inverses of Toolkit Functions

Identify which of the toolkit functions besides the quadratic function are not one-to-one, and find a restricted domain on which each function is one-to-one, if any. The toolkit functions are reviewed in Table 2.5.2. We restrict the domain in such a fashion that the function assumes all y-values exactly once.

Table 2.5.2

Constant	Identity	Quadratic	Cubic	Reciprocal
$f(x) = c$	$f(x) = x$	$f(x) = x^2$	$f(x) = x^3$	$f(x) = \frac{1}{x}$
Reciprocal squared	Cube Root	Square Root	Absolute Value	
$f(x) = \frac{1}{x^2}$	$f(x) = \sqrt[3]{x}$	$f(x) = \sqrt{x}$	$f(x) = \lvert x \rvert$	

Solution

The constant function is not one-to-one, and there is no domain (except a single point) on which it could be one-to-one, so the constant function has no meaningful inverse.

The absolute value function can be restricted to the domain $[0, \infty)$,where it is equal to the identity function.

The reciprocal-squared function can be restricted to the domain $(0, \infty)$.

Analysis

We can see that these functions (if unrestricted) are not one-to-one by looking at their graphs, shown in Figure 2.5.4. They both would fail the horizontal line test. However, if a function is restricted to a certain domain so that it passes the horizontal line test, then in that restricted domain, it can have an inverse.

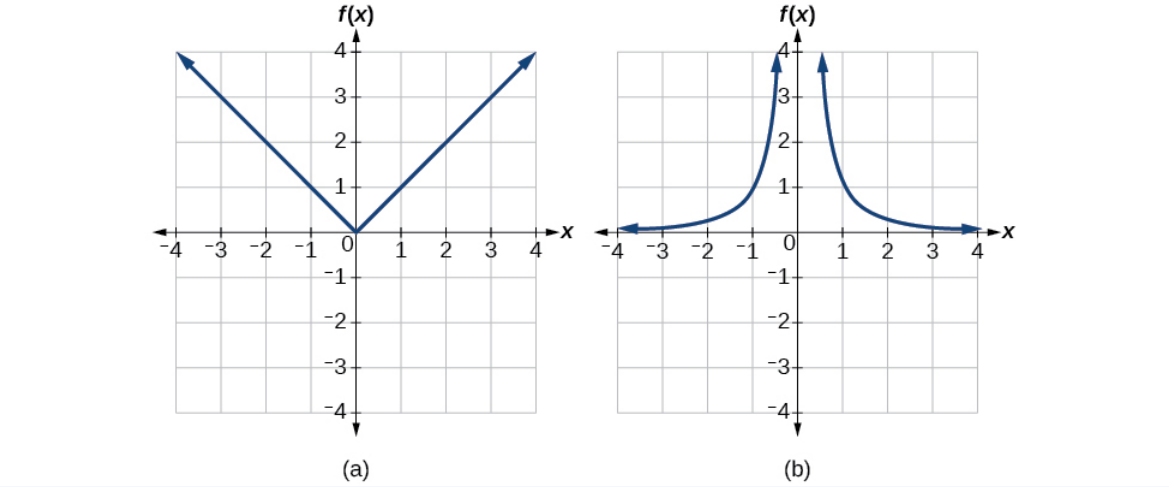


Figure 2.5.4: *(a) Absolute value (b) Reciprocal squared*

2.5.4: The domain of function f is $(1, \infty)$ and the range of function f is $(-\infty, -2)$. Find the domain and range of the inverse function.

Solution

The domain of function f^{-1} is $(-\infty, -2)$ and the range of function f^{-1} is $(1, \infty)$.

Finding and Evaluating Inverse Functions

Once we have a one-to-one function, we can evaluate its inverse at specific inverse function inputs or construct a complete representation of the inverse function in many cases.

Inverting Tabular Functions

Suppose we want to find the inverse of a function represented in table form. Remember that the domain of a function is the range of the inverse and the range of the function is the domain of the inverse. So we need to interchange the domain and range.

Each row (or column) of inputs becomes the row (or column) of outputs for the inverse function. Similarly, each row (or column) of outputs becomes the row (or column) of inputs for the inverse function.

Example 2.5.5: Interpreting the Inverse of a Tabular Function

A function $f(t)$ is given in Table 2.5.3, showing distance in miles that a car has traveled in t minutes. Find and interpret $f^{-1}(70)$

Table 2.5.3

t (minutes)	30	50	70	90
$f(t)$ (miles)	20	40	60	70

The inverse function takes an output of f and returns an input for f. So in the expression $f^{-1}(70)$, 70 is an output value of the original function, representing 70 miles. The inverse will return the corresponding input of the original function f, 90 minutes, so $f^{-1}(70) = 90$. The interpretation of this is that, to drive 70 miles, it took 90 minutes.

Alternatively, recall that the definition of the inverse was that if $f(a) = b$, then $f^{-1}(b) = a$. By this definition, if we are given $f^{-1}(70) = a$, then we are looking for a value a so that $f(a) = 70$. In this case, we are looking for a t so that $f(t) = 70$, which is when $t = 90$.

Exercise 2.5.5

Using Table 2.5.4, find and interpret (a) $f(60)$,and (b) $f^{-1}(60)$.

Table 2.5.4

t (minutes)	30	50	60	70	90
$f(t)$ (miles)	20	40	50	60	70

Answer

$f(60) = 50$. In 60 minutes, 50 miles are traveled.
$f^{-1}(60) = 70$. To travel 60 miles, it will take 70 minutes.

Evaluating the Inverse of a Function, Given a Graph of the Original Function

We saw in **Functions and Function Notation** that the domain of a function can be read by observing the horizontal extent of its graph. We find the domain of the inverse function by observing the *vertical* extent of the graph of the original function, because this corresponds to the horizontal extent of the inverse function. Similarly, we find the range of the inverse function by observing the *horizontal* extent of the graph of the original function, as this is the vertical extent of the inverse function. If we want to evaluate an inverse function, we find its input within its domain, which is all or part of the vertical axis of the original function's graph.

Given the graph of a function, evaluate its inverse at specific points.

1. Find the desired input on the y-axis of the given graph.
2. Read the inverse function's output from the x-axis of the given graph.

Example 2.5.6: Evaluating a Function and Its Inverse from a Graph at Specific Points

A function $g(x)$ is given in Figure 2.5.5. Find $g(3)$ and $g^{-1}(3)$.

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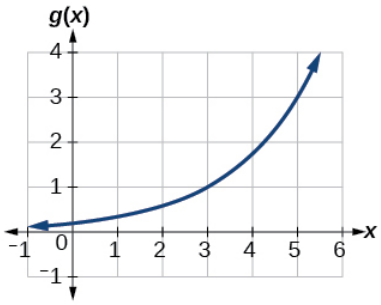


Figure 2.5.5: Graph of $g(x)$

Solution

To evaluate $g(3)$, we find 3 on the x-axis and find the corresponding output value on the y-axis. The point $(3, 1)$ tells us that $g(3) = 1$.

To evaluate $g^{-1}(3)$, recall that by definition $g^{-1}(3)$ means the value of x for which $g(x) = 3$. By looking for the output value 3 on the vertical axis, we find the point $(5, 3)$ on the graph, which means $g(5) = 3$, so by definition, $g^{-1}(3) = 5$. See Figure 2.5.6.

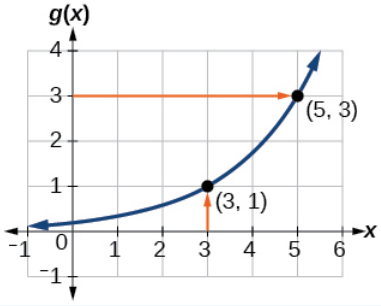


Figure 2.5.6: Graph of $g(x)$.

Exercise 2.5.6

Using the graph in Figure 2.5.6, (a) find $g^{-1}(1)$,and (b) estimate $g^{-1}(4)$.

Answer a

3

Answer b

5.6

Finding Inverses of Functions Represented by Formulas

Sometimes we will need to know an inverse function for all elements of its domain, not just a few. If the original function is given as a formula— for example, y as a function of x— we can often find the inverse function by solving to obtain x as a function of y.

How To: Given a function represented by a formula, find the inverse.

1. Make sure f is a one-to-one function.
2. Solve for x
3. Interchange x and y.

Example 2.5.7: Inverting the Fahrenheit-to-Celsius Function

Find a formula for the inverse function that gives Fahrenheit temperature as a function of Celsius temperature.

$$C = \frac{5}{9}(F - 32) \tag{2.5.19}$$

Solution

$$C = \frac{5}{9}(F-32) \tag{2.5.20}$$

$$C\cdot\frac{9}{5} = F-32 \tag{2.5.21}$$

$$F = \frac{9}{5}C+32 \tag{2.5.22}$$

By solving in general, we have uncovered the inverse function. If

$$C = h(F) = \frac{5}{9}(F-32) \tag{2.5.23}$$

,

then

$$F = h^{-1}(C) = \frac{9}{5}C+32. \tag{2.5.24}$$

In this case, we introduced a function h to represent the conversion because the input and output variables are descriptive, and writing C^{-1} could get confusing.

Exercise 2.5.7

Solve for x in terms of y given $y = \frac{1}{3}(x-5)$

Answer

$x = 3y+5$

Example 2.5.8: Solving to Find an Inverse Function

Find the inverse of the function $f(x) = \frac{2}{x-3}+4$.

Solution

$$y = \frac{2}{x-3+4} \qquad \text{Set up an equation.} \tag{2.5.25}$$

$$y-4 = \frac{2}{x-3} \qquad \text{Subtract 4 from both sides.} \tag{2.5.26}$$

$$x-3 = \frac{2}{y-4} \qquad \text{Multiply both sides by x−3 and divide by y−4.} \tag{2.5.27}$$

$$x = \frac{2}{y-4}+3 \qquad \text{Add 3 to both sides.} \tag{2.5.28}$$

So $f^{-1}(y) = \frac{2}{y-4}+3$ or $f^{-1}(x) = \frac{2}{x-4}+3$.

Analysis

The domain and range of f exclude the values 3 and 4, respectively. f and f^{-1} are equal at two points but are not the same function, as we can see by creating Table 2.5.5.

Table 2.5.5

x	1	2	5	$f^{-1}(y)$
$f(x)$	3	2	5	y

Example 2.5.9: Solving to Find an Inverse with Radicals

Find the inverse of the function $f(x) = 2 + \sqrt{x-4}$.

Solution

$$y = 2 + \sqrt{x-4} \quad (2.5.29)$$
$$(y-2)^2 = x - 4 \quad (2.5.30)$$
$$x = (y-2)^2 + 4 \quad (2.5.31)$$

So $f^{-1}(x) = (x-2)^2 + 4$.

The domain of f is $[4, \infty)$. Notice that the range of f is $[2, \infty)$, so this means that the domain of the inverse function f^{-1} is also $[2, \infty)$

Analysis

The formula we found for $f^{-1}(x)$ looks like it would be valid for all real x. However, f^{-1} itself must have an inverse (namely, f) so we have to restrict the domain of f^{-1} to $[2, \infty)$ in order to make f^{-1} a one-to-one function. This domain of f^{-1} is exactly the range of f.

Exercise 2.5.8

What is the inverse of the function $f(x) = 2 - \sqrt{x}$? State the domains of both the function and the inverse function.

Answer

$f^{-1}(x) = (2-x)^2$; domain of f: $[0, \infty)$; domain of f^{-1}: $(-\infty, 2]$

Inverting Tabular Functions

Suppose we want to find the inverse of a function represented in table form. Remember that the domain of a function is the range of the inverse and the range of the function is the domain of the inverse. So we need to interchange the domain and range.

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Using Table 2.5.4, find and interpret (a) $f(60)$,and (b) $f^{-1}(60)$.

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A function $g(x)$ is given in Figure 2.5.5. Find $g(3)$ and $g^{-1}(3)$.
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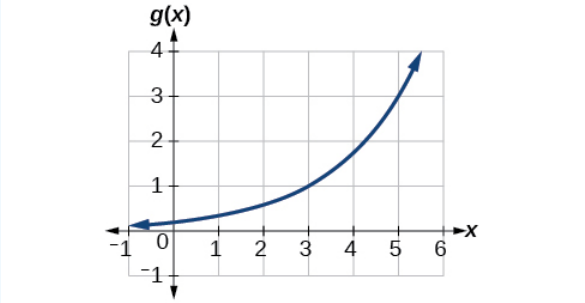


Figure 2.5.5: Graph of $g(x)$

Solution

To evaluate $g(3)$, we find 3 on the x-axis and find the corresponding output value on the y-axis. The point $(3, 1)$ tells us that $g(3) = 1$.

To evaluate $g^{-1}(3)$, recall that by definition $g^{-1}(3)$ means the value of x for which $g(x) = 3$. By looking for the output value 3 on the vertical axis, we find the point $(5, 3)$ on the graph, which means $g(5) = 3$, so by definition, $g^{-1}(3) = 5$. See Figure 2.5.6.

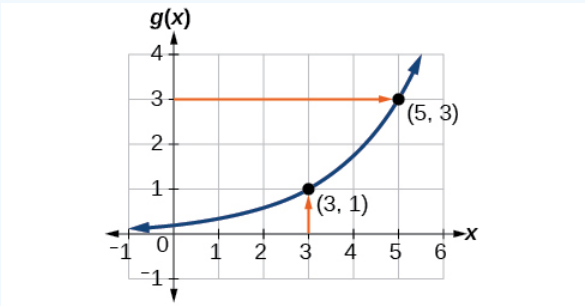


Figure 2.5.6: Graph of $g(x)$.

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Using the graph in Figure 2.5.6, (a) find $g^{-1}(1)$,and (b) estimate $g^{-1}(4)$.

Answer a

3

Answer b

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Sometimes we will need to know an inverse function for all elements of its domain, not just a few. If the original function is given as a formula— for example, y as a function of x— we can often find the inverse function by solving to obtain x as a function of y.

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Solution

$$C = \frac{5}{9}(F - 32) \tag{2.5.33}$$

$$C \cdot \frac{9}{5} = F - 32 \tag{2.5.34}$$

$$F = \frac{9}{5}C + 32 \tag{2.5.35}$$

By solving in general, we have uncovered the inverse function. If

$$C = h(F) = \frac{5}{9}(F - 32) \tag{2.5.36}$$

,

then

$$F = h^{-1}(C) = \frac{9}{5}C + 32. \tag{2.5.37}$$

In this case, we introduced a function h to represent the conversion because the input and output variables are descriptive, and writing C^{-1} could get confusing.

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Solve for x in terms of y given $y = \frac{1}{3}(x - 5)$

Answer

$x = 3y + 5$

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Find the inverse of the function $f(x)=\frac{2}{x-3}+4$.

Solution

$$y=\frac{2}{x-3}+4 \quad \text{Set up an equation.} \quad (2.5.38)$$

$$y-4=\frac{2}{x-3} \quad \text{Subtract 4 from both sides.} \quad (2.5.39)$$

$$x-3=\frac{2}{y-4} \quad \text{Multiply both sides by x−3 and divide by y−4.} \quad (2.5.40)$$

$$x=\frac{2}{y-4}+3 \quad \text{Add 3 to both sides.} \quad (2.5.41)$$

So $f^{-1}(y)=\frac{2}{y-4}+3$ or $f^{-1}(x)=\frac{2}{x-4}+3$.

Analysis

The domain and range of f exclude the values 3 and 4, respectively. f and f^{-1} are equal at two points but are not the same function, as we can see by creating Table 2.5.5.

Table 2.5.5

x	1	2	5	$f^{-1}(y)$
$f(x)$	3	2	5	y

Example 2.5.9: Solving to Find an Inverse with Radicals

Find the inverse of the function $f(x)=2+\sqrt{x-4}$.

Solution

$$y=2+\sqrt{x-4} \quad (2.5.42)$$

$$(y-2)^2=x-4 \quad (2.5.43)$$

$$x=(y-2)^2+4 \quad (2.5.44)$$

So $f^{-1}(x)=(x-2)^2+4$.

The domain of f is $[4,\infty)$. Notice that the range of f is $[2,\infty)$, so this means that the domain of the inverse function f^{-1} is also $[2,\infty)$

Analysis

The formula we found for $f^{-1}(x)$ looks like it would be valid for all real x. However, f^{-1} itself must have an inverse (namely, f) so we have to restrict the domain of f^{-1} to $[2,\infty)$ in order to make f^{-1} a one-to-one function. This domain of f^{-1} is exactly the range of f.

Exercise 2.5.8

What is the inverse of the function $f(x)=2-\sqrt{x}$? State the domains of both the function and the inverse function.

Answer

$f^{-1}(x)=(2-x)^2$; domain of f: $[0,\infty)$; domain of f^{-1}: $(-\infty,2]$

Finding Inverse Functions and Their Graphs

Now that we can find the inverse of a function, we will explore the graphs of functions and their inverses. Let us return to the quadratic function $f(x)=x^2$ restricted to the domain $[0,\infty)$, on which this function is one-to-one, and graph it as in Figure 2.5.7.

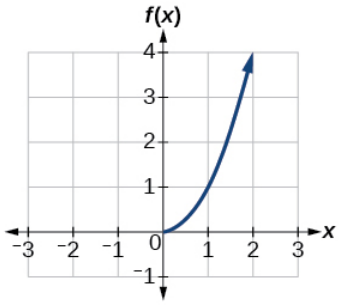


Figure* 2.5.7*: *Quadratic function with domain restricted to* $[0, \infty)$.

Restricting the domain to $[0, \infty)$ makes the function one-to-one (it will obviously pass the horizontal line test), so it has an inverse on this restricted domain.

We already know that the inverse of the toolkit quadratic function is the square root function, that is, $f^{-1}(x) = \sqrt{x}$. What happens if we graph both f and f^{-1} on the same set of axes, using the x-axis for the input to both f and f^{-1}?

We notice a distinct relationship: The graph of $f^{-1}(x)$ is the graph of $f(x)$ reflected about the diagonal line $y = x$, which we will call the identity line, shown in Figure 2.5.8.

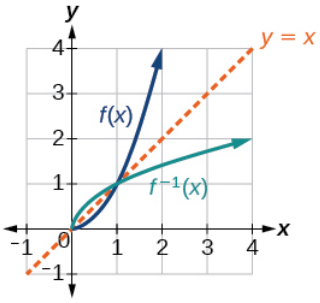


Figure* 2.5.8*: *Square and square-root functions on the non-negative domain*

This relationship will be observed for all one-to-one functions, because it is a result of the function and its inverse swapping inputs and outputs. This is equivalent to interchanging the roles of the vertical and horizontal axes.

Example 2.5.10: Finding the Inverse of a Function Using Reflection about the Identity Line

Given the graph of $f(x)$ in Figure 2.5.9, sketch a graph of $f^{-1}(x)$.

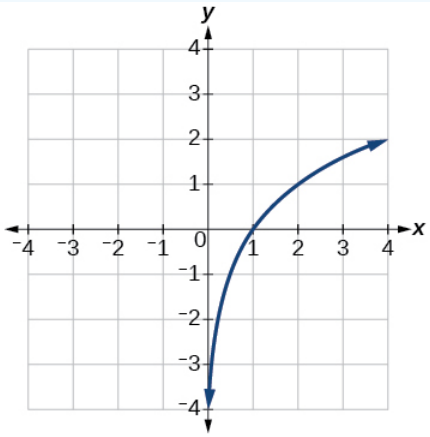


Figure 2.5.9: Graph of $f^(-1)(x)$.

This is a one-to-one function, so we will be able to sketch an inverse. Note that the graph shown has an apparent domain of $(0, \infty)$ and range of $(-\infty, \infty)$, so the inverse will have a domain of $(-\infty, \infty)$ and range of $(0, \infty)$.

If we reflect this graph over the line $y = x$, the point $(1, 0)$ reflects to $(0, 1)$ and the point $(4, 2)$ reflects to $(2, 4)$. Sketching the inverse on the same axes as the original graph gives Figure 2.5.10.

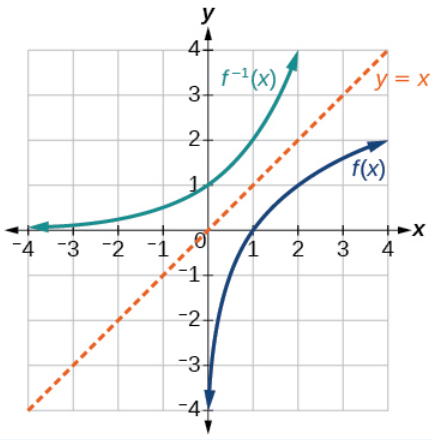


Figure 2.5.10: The function and its inverse, showing reflection about the identity line

Exercise 2.5.1

Draw graphs of the functions f and f^{-1} from Example 2.5.8.

Answer

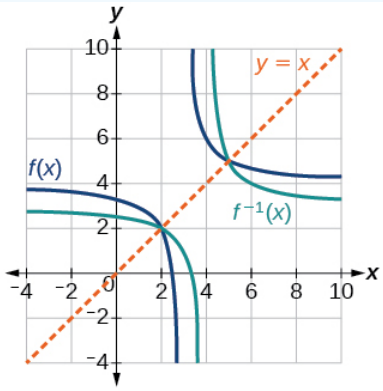


Figure 2.5.11: Graph of $f(x)$ and $f^(-1)(x)$.

Is there any function that is equal to its own inverse?

Yes. If $f = f^{-1}$, then $f(f(x)) = x$, and we can think of several functions that have this property. The identity function does, and so does the reciprocal function, because

$$\frac{1}{\frac{1}{x}} = x \tag{2.5.45}$$

Any function $f(x) = c - x$, where c is a constant, is also equal to its own inverse.

Key Concepts

- If $g(x)$ is the inverse of $f(x)$, then $g(f(x)) = f(g(x)) = x$.
- Each of the toolkit functions has an inverse.
- For a function to have an inverse, it must be one-to-one (pass the horizontal line test).
- A function that is not one-to-one over its entire domain may be one-to-one on part of its domain.
- For a tabular function, exchange the input and output rows to obtain the inverse.
- The inverse of a function can be determined at specific points on its graph.
- To find the inverse of a formula, solve the equation $y = f(x)$ for x as a function of y. Then exchange the labels x and y.
- The graph of an inverse function is the reflection of the graph of the original function across the line $y = x$.

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2.6: Transformation of Functions

Identifying Vertical Shifts

One simple kind of transformation involves shifting the entire graph of a function up, down, right, or left. The simplest shift is a vertical shift, moving the graph up or down, because this transformation involves adding a positive or negative constant to the function. In other words, we add the same constant to the output value of the function regardless of the input. For a function $g(x) = f(x) + k$, the function $f(x)$ is shifted vertically k units. See Figure $2.6.2$ for an example.

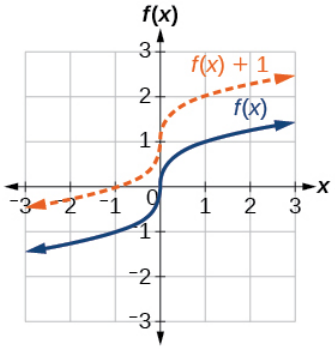


Figure 2.6.2: Vertical shift by $k = 1$ of the cube root function $f(x) = \sqrt[3]{x}$.

To help you visualize the concept of a vertical shift, consider that $y = f(x)$. Therefore, $f(x) + k$ is equivalent to $y + k$. Every unit of y is replaced by $y + k$, so the y-value increases or decreases depending on the value of k. The result is a shift upward or downward.

Definition: Vertical Shift

Given a function $f(x)$, a new function $g(x) = f(x) + k$, where k is a constant, is a **vertical shift** of the function $f(x)$. All the output values change by k units. If k is positive, the graph will shift up. If k is negative, the graph will shift down.

Example 2.6.1: Adding a Constant to a Function

To regulate temperature in a green building, airflow vents near the roof open and close throughout the day. Figure $2.6.3$ shows the area of open vents V (in square feet) throughout the day in hours after midnight, t. During the summer, the facilities manager decides to try to better regulate temperature by increasing the amount of open vents by 20 square feet throughout the day and night. Sketch a graph of this new function.

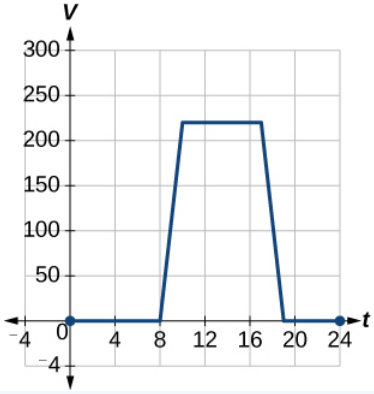


Figure 2.6.3

Solution

We can sketch a graph of this new function by adding 20 to each of the output values of the original function. This will have the effect of shifting the graph vertically up, as shown in Figure $2.6.4$.

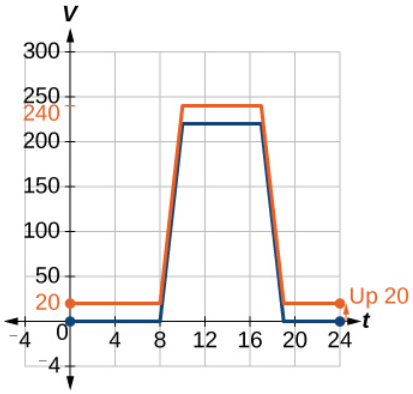


Figure 2.6.4

Notice that in Figure 2.6.4, for each input value, the output value has increased by 20, so if we call the new function $S(t)$,we could write

$$S(t) = V(t) + 20 \tag{2.6.1}$$

This notation tells us that, for any value of t,$S(t)$ can be found by evaluating the function V at the same input and then adding 20 to the result. This defines S as a transformation of the function V, in this case a vertical shift up 20 units. Notice that, with a vertical shift, the input values stay the same and only the output values change. See Table 2.6.1.

Table 2.6.1

t	0	8	10	17	19	24
$V(t)$	0	0	220	220	0	0
$S(t)$	20	20	240	240	20	20

How To...

Given a tabular function, create a new row to represent a vertical shift.

1. Identify the output row or column.
2. Determine the **magnitude** of the shift.
3. Add the shift to the value in each output cell. Add a positive value for up or a negative value for down.

Example 2.6.2: Shifting a Tabular Function Vertically

A function $f(x)$ is given in Table 2.6.2. Create a table for the function $g(x) = f(x) - 3$.

Table 2.6.2

x	2	4	6	8
$f(x)$	1	3	7	11

Solution

The formula $g(x) = f(x) - 3$ tells us that we can find the output values of g by subtracting 3 from the output values of f. For example:

$$\begin{aligned} f(x) &= 1 && \text{Given} \\ g(x) &= f(x) - 3 && \text{Given Transformation} \\ g(2) &= f(2) - 3 \\ &= 1 - 3 \\ &= -2 \end{aligned}$$

Subtracting 3 from each $f(x)$ value, we can complete a table of values for $g(x)$ as shown in Table 2.6.3.

Table 2.6.3

x	2	4	6	8
$f(x)$	1	3	7	11
$g(x)$	-2	0	4	8

Analysis

As with the earlier vertical shift, notice the input values stay the same and only the output values change.

Exercise 2.6.1

The function $h(t) = -4.9t^2 + 30t$ gives the height h of a ball (in meters) thrown upward from the ground after t seconds. Suppose the ball was instead thrown from the top of a 10-m building. Relate this new height function $b(t)$ to $h(t)$, and then find a formula for $b(t)$.

Answer

$b(t) = h(t) + 10 = -4.9t^2 + 30t + 10$

Identifying Horizontal Shifts

We just saw that the vertical shift is a change to the output, or outside, of the function. We will now look at how changes to input, on the inside of the function, change its graph and meaning. A shift to the input results in a movement of the graph of the function left or right in what is known as a **horizontal shift**, shown in Figure 2.6.4.

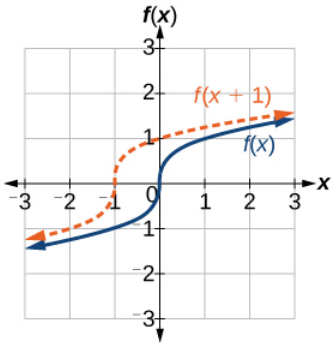


Figure 2.6.4: Horizontal shift of the function $f(x) = \sqrt[3]{x}$. Note that $h = -1$ shifts the graph to the left, that is, towards negative values of x.

For example, if $f(x) = x^2$, then $g(x) = (x-2)^2$ is a new function. Each input is reduced by 2 prior to squaring the function. The result is that the graph is shifted 2 units to the right, because we would need to increase the prior input by 2 units to yield the same output value as given in f.

Definition: Horizontal Shift

Given a function f, a new function $g(x) = f(x-h)$, where h is a constant, is a **horizontal shift** of the function f. If h is positive, the graph will shift right. If h is negative, the graph will shift left.

Example 2.6.4: Adding a Constant to an Input

Returning to our building airflow example from Figure 2.6.2, suppose that in autumn the facilities manager decides that the original venting plan starts too late, and wants to begin the entire venting program 2 hours earlier. Sketch a graph of the new function.

Solution

We can set $V(t)$ to be the original program and $F(t)$ to be the revised program.

$$V(t) = \text{ the original venting plan}$$

$$F(t) = \text{ starting 2 hrs sooner}$$

In the new graph, at each time, the airflow is the same as the original function V was 2 hours later. For example, in the original function V, the airflow starts to change at 8 a.m., whereas for the function F, the airflow starts to change at 6 a.m. The comparable function values are $V(8) = F(6)$. See Figure 2.6.5. Notice also that the vents first opened to 220ft^2 at 10 a.m. under the original plan, while under the new plan the vents reach 220ft^2 at 8 a.m., so $V(10) = F(8)$.

In both cases, we see that, because $F(t)$ starts 2 hours sooner, $h = -2$. That means that the same output values are reached when $F(t) = V(t-(-2)) = V(t+2)$.

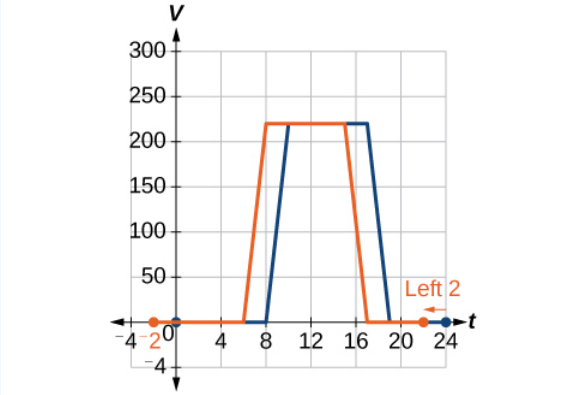


Figure 2.6.5

Analysis

Note that $V(t+2)$ has the effect of shifting the graph to the left.

Horizontal changes or "inside changes" affect the domain of a function (the input) instead of the range and often seem counterintuitive. The new function $F(t)$ uses the same outputs as $V(t)$, but matches those outputs to inputs 2 hours earlier than those of $V(t)$. Said another way, we must add 2 hours to the input of V to find the corresponding output for F : $F(t) = V(t+2)$.

How To...

Given a tabular function, create a new row to represent a horizontal shift.

1. Identify the input row or column.
2. Determine the magnitude of the shift.
3. Add the shift to the value in each input cell.

Example 2.6.5: Shifting a Tabular Function Horizontally

A function $f(x)$ is given in Table 2.6.4. Create a table for the function $g(x) = f(x-3)$.

Table 2.6.4

x	2	4	6	8
$f(x)$	1	3	7	11

Solution

The formula $g(x) = f(x-3)$ tells us that the output values of g are the same as the output value of f when the input value is 3 less than the original value. For example, we know that $f(2) = 1$. To get the same output from the function g, we will need an input value that is 3 larger. We input a value that is 3 larger for $g(x)$ because the function takes 3 away before evaluating the function f.

$$\begin{aligned} g(5) &= f(5-3) \\ &= f(2) \\ &= 1 \end{aligned}$$

We continue with the other values to create Table 2.6.5.

Table 2.6.5

x	5	7	9	11
$x-3$	2	4	6	8
$f(x)$	1	3	7	11
$g(x)$	1	3	7	11

The result is that the function $g(x)$ has been shifted to the right by 3. Notice the output values for $g(x)$ remain the same as the output values for $f(x)$, but the corresponding input values, x, have shifted to the right by 3. Specifically, 2 shifted to 5, 4 shifted to 7, 6 shifted to 9, and 8 shifted to 11.

Analysis

Figure 2.6.6 represents both of the functions. We can see the horizontal shift in each point.

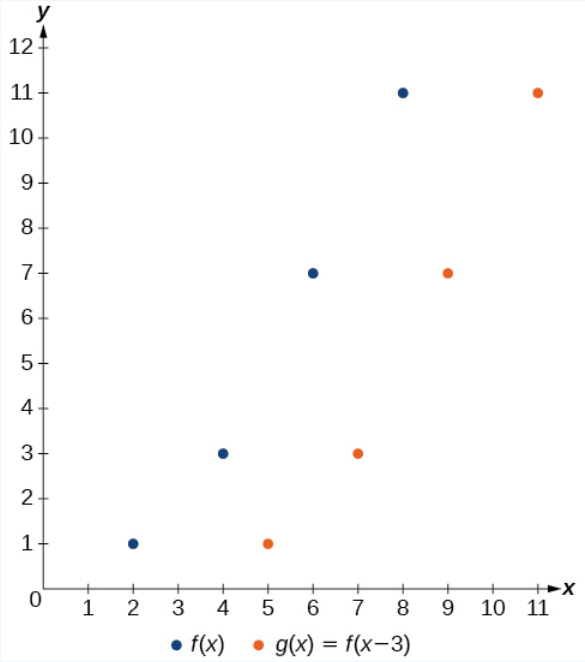


Figure 2.6.6: Graph of the points from the previous table for $f(x)$ and $g(x) = f(x-3)$

Example 2.6.6: Identifying a Horizontal Shift of a Toolkit Function

Figure 2.6.7 represents a transformation of the toolkit function $f(x) = x^2$. Relate this new function $g(x)$ to $f(x)$, and then find a formula for $g(x)$.

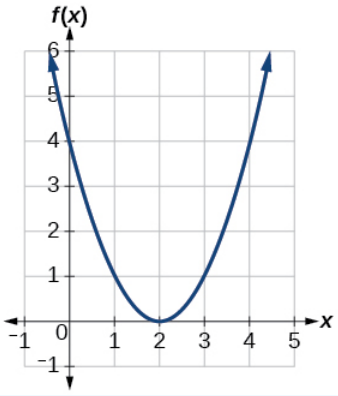


Figure 2.6.7: Graph of a parabola

Solution

Notice that the graph is identical in shape to the $f(x)=x^2$ function, but the x-values are shifted to the right 2 units. The vertex used to be at $(0,0)$, but now the vertex is at $(2,0)$. The graph is the basic quadratic function shifted 2 units to the right, so

$$g(x)=f(x-2)$$

Notice how we must input the value $x=2$ to get the output value $y=0$; the x-values must be 2 units larger because of the shift to the right by 2 units. We can then use the definition of the $f(x)$ function to write a formula for $g(x)$ by evaluating $f(x-2)$.

$$\begin{aligned} f(x) &= x^2 \\ g(x) &= f(x-2) \\ g(x) &= f(x-2)=(x-2)^2 \end{aligned}$$

Analysis

To determine whether the shift is $+2$ or -2, consider a single reference point on the graph. For a quadratic, looking at the vertex point is convenient. In the original function, $f(0)=0$. In our shifted function, $g(2)=0$. To obtain the output value of 0 from the function f, we need to decide whether a plus or a minus sign will work to satisfy $g(2)=f(x-2)=f(0)=0$. For this to work, we will need to subtract 2 units from our input values.

Example 2.6.7: Interpreting Horizontal versus Vertical Shifts

The function $G(m)$ gives the number of gallons of gas required to drive m miles. Interpret $G(m)+10$ and $G(m+10)$

Solution

$G(m)+10$ can be interpreted as adding 10 to the output, gallons. This is the gas required to drive m miles, plus another 10 gallons of gas. The graph would indicate a vertical shift.

$G(m+10)$ can be interpreted as adding 10 to the input, miles. So this is the number of gallons of gas required to drive 10 miles more than m miles. The graph would indicate a horizontal shift.

Exercise 2.6.7

Given the function $f(x)=\sqrt{x}$, graph the original function $f(x)$ and the transformation $g(x)=f(x+2)$ on the same axes. Is this a horizontal or a vertical shift? Which way is the graph shifted and by how many units?

Answer

The graphs of $f(x)$ and $g(x)$ are shown below. The transformation is a horizontal shift. The function is shifted to the left by 2 units.

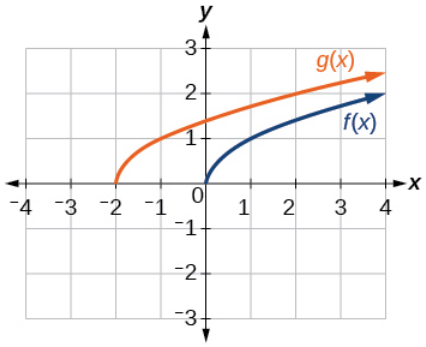


Figure 2.6.8

Combining Vertical and Horizontal Shifts

Now that we have two transformations, we can combine them together. Vertical shifts are outside changes that affect the output $(y-)$ axis values and shift the function up or down. Horizontal shifts are inside changes that affect the input $(x-)$ axis values and shift the function left or right. Combining the two types of shifts will cause the graph of a function to shift up or down and right or left.

How To...

Given a function and both a vertical and a horizontal shift, sketch the graph.

1. Identify the vertical and horizontal shifts from the formula.
2. The vertical shift results from a constant added to the output. Move the graph up for a positive constant and down for a negative constant.
3. The horizontal shift results from a constant added to the input. Move the graph left for a positive constant and right for a negative constant.
4. Apply the shifts to the graph in either order.

Example 2.6.8: Graphing Combined Vertical and Horizontal Shifts

Given $f(x)=|x|$, sketch a graph of $h(x)=f(x+1)-3$.

Solution

The function f is our toolkit absolute value function. We know that this graph has a V shape, with the point at the origin. The graph of h has transformed f in two ways: $f(x+1)$ is a change on the inside of the function, giving a horizontal shift left by 1, and the subtraction by 3 in $f(x+1)-3$ is a change to the outside of the function, giving a vertical shift down by 3. The transformation of the graph is illustrated in Figure 2.6.9.

Let us follow one point of the graph of $f(x)=|x|$.

- The point $(0,0)$ is transformed first by shifting left 1 unit:$(0,0)\rightarrow(-1,0)$
- The point $(-1,0)$ is transformed next by shifting down 3 units:$(-1,0)\rightarrow(-1,-3)$

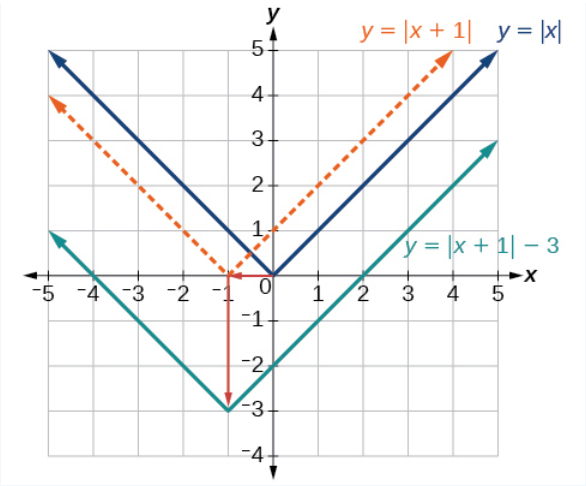


Figure 2.6.9: Graph of an absolute function, $y = |x|$, and how it was transformed to $y = |x + 1| - 3$

Figure 2.6.10 shows the graph of h.

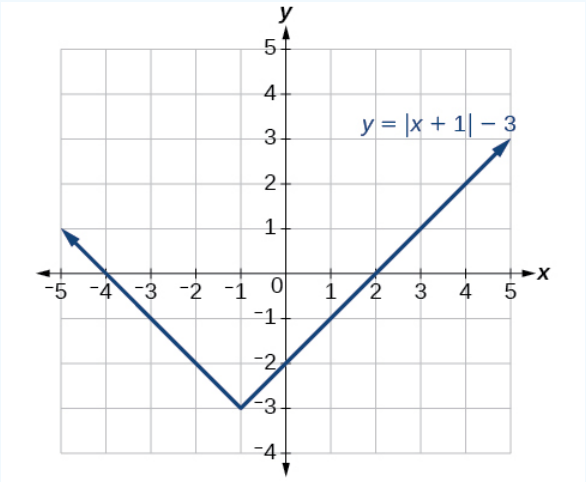


Figure 2.6.10: The final function $y = |x + 1| - 3$.

Exercise 2.6.8

Given $f(x) = |x|$, sketch a graph of $h(x) = f(x - 2) + 4$.

Answer

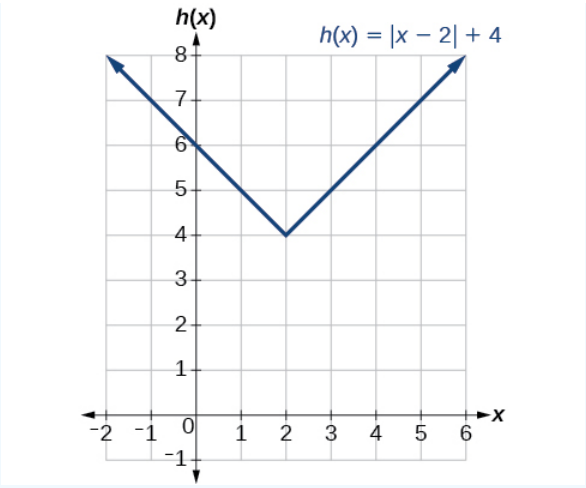


Figure 2.6.11

Example 2.6.9: Identifying Combined Vertical and Horizontal Shifts

Write a formula for the graph shown in Figure 2.6.12, which is a transformation of the toolkit square root function.

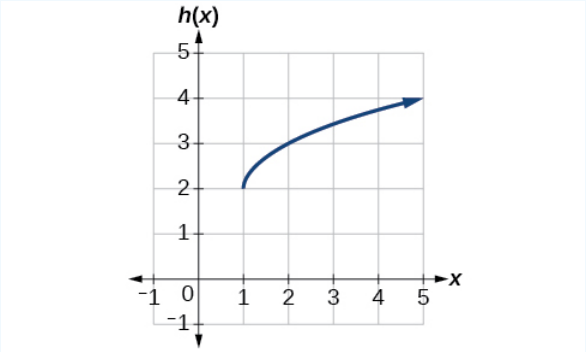


Figure 2.6.12: Graph of a square root function transposed right one unit and up 2.

Solution

The graph of the toolkit function starts at the origin, so this graph has been shifted 1 to the right and up 2. In function notation, we could write that as

$$h(x) = f(x-1)+2$$

Using the formula for the square root function, we can write

$$h(x) = \sqrt{x-1}+2$$

Analysis

Note that this transformation has changed the domain and range of the function. This new graph has domain $[1, \infty)$ and range $[2, \infty)$.

Exercise 2.6.9

Write a formula for a transformation of the toolkit reciprocal function $f(x) = \frac{1}{x}$ that shifts the function's graph one unit to the right and one unit up.

Answer

$$g(x) = \frac{1}{x-1}+1$$

Graphing Functions Using Reflections about the Axes

Another transformation that can be applied to a function is a reflection over the x- or y-axis. A **vertical reflection** reflects a graph vertically across the x-axis, while a **horizontal reflection** reflects a graph horizontally across the y-axis. The reflections are shown in Figure 2.6.13.

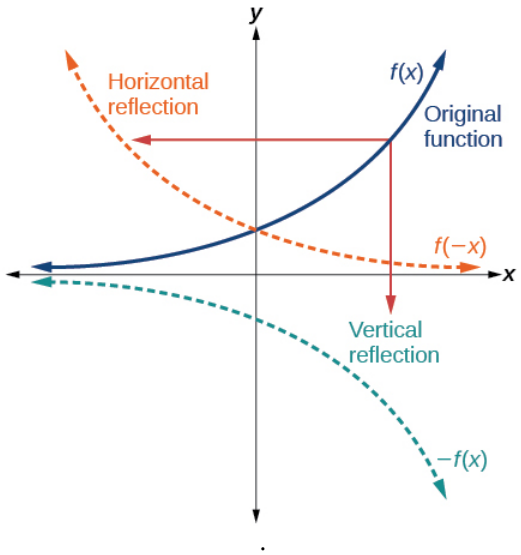


Figure 2.6.13: Graph of the vertical and horizontal reflection of a function.

Notice that the vertical reflection produces a new graph that is a mirror image of the base or original graph about the x-axis. The horizontal reflection produces a new graph that is a mirror image of the base or original graph about the y-axis.

Definitions: Reflections

Given a function $f(x)$, a new function $g(x) = -f(x)$ is a **vertical reflection** of the function $f(x)$, sometimes called a reflection about (or over, or through) the x-axis.

Given a function $f(x)$, a new function $g(x) = f(-x)$ is a **horizontal reflection** of the function $f(x)$, sometimes called a reflection about the y-axis.

How To...

Given a function, reflect the graph both vertically and horizontally.

1. Multiply all outputs by –1 for a vertical reflection. The new graph is a reflection of the original graph about the x-axis.
2. Multiply all inputs by –1 for a horizontal reflection. The new graph is a reflection of the original graph about the y-axis.

Example 2.6.10: Reflecting a Graph Horizontally and Vertically

Reflect the graph of $s(t) = \sqrt{t}$ (a) vertically and (b) horizontally.

Solution

a. Reflecting the graph vertically means that each output value will be reflected over the horizontal t-axis as shown in Figure 2.6.14

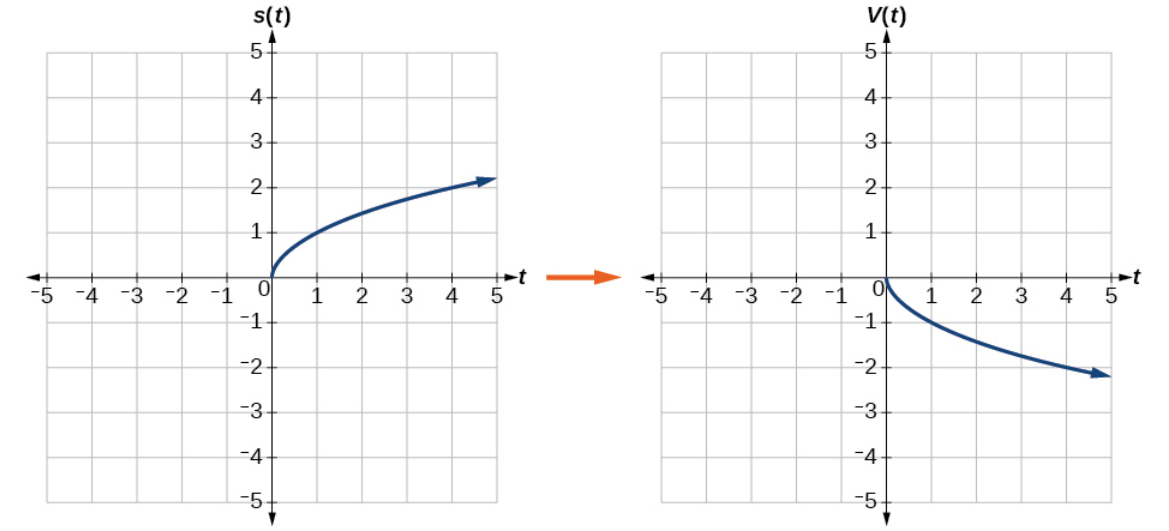


Figure 2.6.14: Graph of the vertical reflection of the square root function.

Because each output value is the opposite of the original output value, we can write

$$V(t) = -s(t) \text{ or } V(t) = -\sqrt{t}$$

Notice that this is an outside change, or vertical shift, that affects the output $s(t)$ values, so the negative sign belongs outside of the function.

b. Reflecting horizontally means that each input value will be reflected over the vertical axis as shown in Figure 2.6.15.

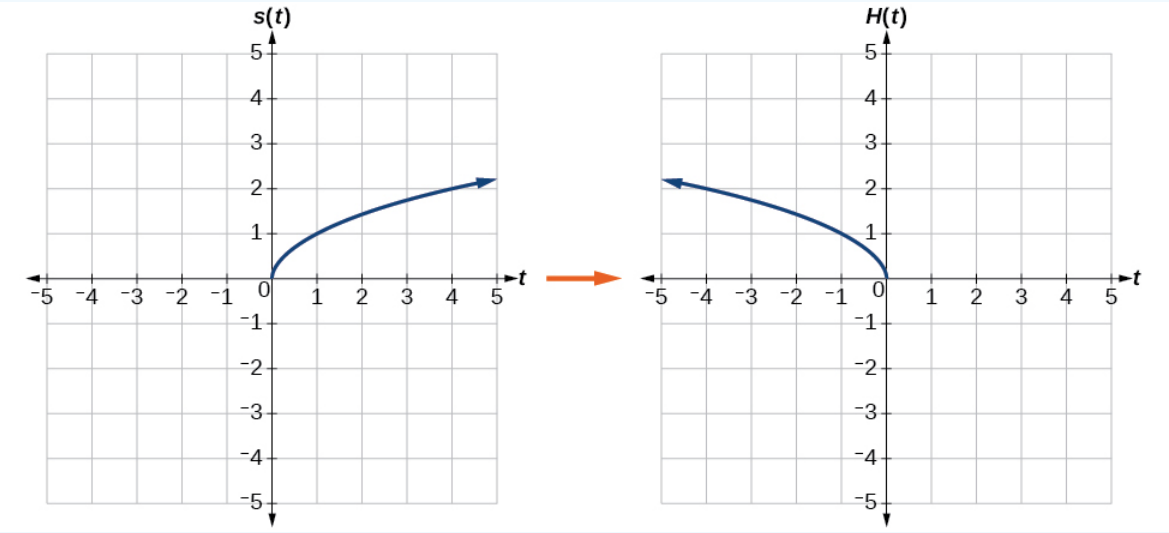


Figure 2.6.15: Horizontal reflection of the square root function

Because each input value is the opposite of the original input value, we can write

$$H(t) = s(-t) \text{ or } H(t) = \sqrt{-t}$$

Notice that this is an inside change or horizontal change that affects the input values, so the negative sign is on the inside of the function.

Note that these transformations can affect the domain and range of the functions. While the original square root function has domain $[0, \infty)$ and range $[0, \infty)$, the vertical reflection gives the $V(t)$ function the range $(-\infty, 0]$ and the horizontal reflection gives the $H(t)$ function the domain $(-\infty, 0]$.

Exercise 2.6.5

Reflect the graph of $f(x) = |x - 1|$ (a) vertically and (b) horizontally.

Answer

a.

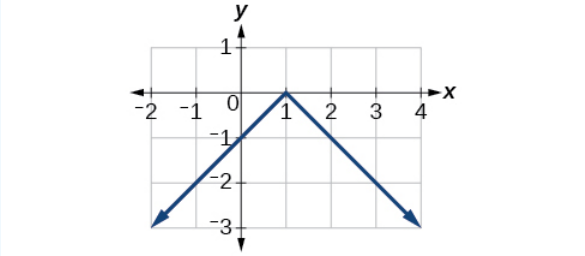


Figure 2.6.16: Graph of a vertically reflected absolute function.

b.

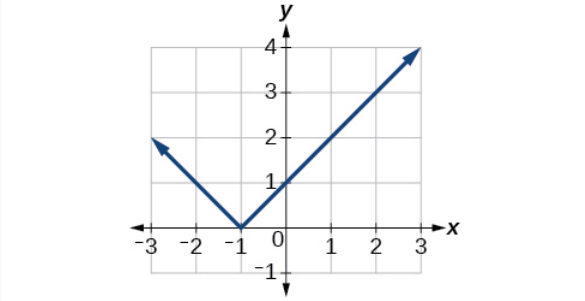


Figure 2.6.17: Graph of an absolute function translated one unit left.

Example 2.6.11: Reflecting a Tabular Function Horizontally and Vertically

A function $f(x)$ is given as Table 2.6.6. Create a table for the functions below.

a. $g(x) = -f(x)$
b. $h(x) = f(-x)$

Table 2.6.6

x	2	4	6	8
$f(x)$	1	3	7	11

a. For $g(x)$, the negative sign outside the function indicates a vertical reflection, so the x-values stay the same and each output value will be the opposite of the original output value. See Table 2.6.7.

Table 2.6.7

x	2	4	6	8
$g(x)$	-1	-3	-7	-11

b. For $h(x)$, the negative sign inside the function indicates a horizontal reflection, so each input value will be the opposite of the original input value and the $h(x)$ values stay the same as the $f(x)$ values. See Table 2.6.8.

Table 2.6.8

x	-2	-4	-6	-8
$h(x)$	1	3	7	11

Exercise 2.6.6

A function $f(x)$ is given as Table 2.6.9. Create a table for the functions below.

a. $g(x) = -f(x)$
b. $h(x) = f(-x)$

Table 2.6.9

x	-2	0	2	4
$f(x)$	5	10	15	20

Answer

a. $g(x) = -f(x)$

Table 2.6.10

x	-2	0	2	4
$g(x)$	-5	-10	-15	-20

b. $h(x) = f(-x)$

Table 2.6.11

x	-2	0	2	-4
$h(x)$	15	10	5	20

Example 2.6.12: Applying a Learning Model Equation

A common model for learning has an equation similar to $k(t) = -2^{-t} + 1$, where k is the percentage of mastery that can be achieved after t practice sessions. This is a transformation of the function $f(t) = 2^t$ shown in Figure 2.6.18. Sketch a graph of $k(t)$.

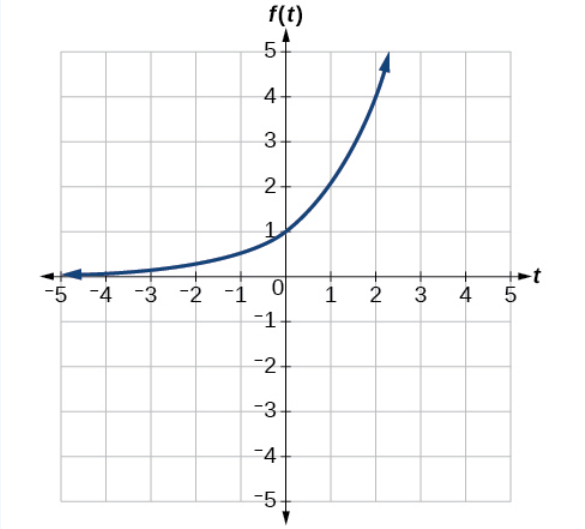


Figure 2.6.18: Graph of $k(t)$

Solution

This equation combines three transformations into one equation.

- A horizontal reflection: $f(-t) = 2^{-t}$

- A vertical reflection: $-f(-t) = -2^{-t}$
- A vertical shift: $-f(-t)+1 = -2^{-t}+1$

We can sketch a graph by applying these transformations one at a time to the original function. Let us follow two points through each of the three transformations. We will choose the points $(0, 1)$ and $(1, 2)$.

- First, we apply a horizontal reflection: $(0, 1)$ $(-1, 2)$.
- Then, we apply a vertical reflection: $(0, -1)$ $(-1, -2)$.
- Finally, we apply a vertical shift: $(0, 0)$ $(-1, -1)$.

This means that the original points, $(0, 1)$ and $(1, 2)$ become $(0, 0)$ and $(-1, -1)$ after we apply the transformations.

In Figure 2.6.19, the first graph results from a horizontal reflection. The second results from a vertical reflection. The third results from a vertical shift up 1 unit.

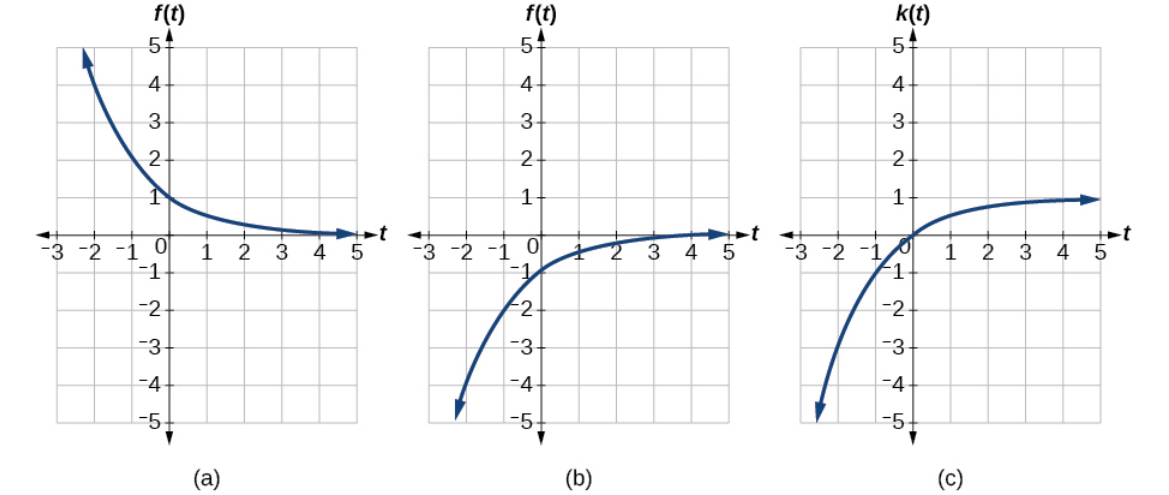


Figure 2.6.19: Graphs of all the transformations.

Analysis

As a model for learning, this function would be limited to a domain of $t \geq 0$, with corresponding range $[0, 1)$.

Exercise 2.6.7

Given the toolkit function $f(x) = x^2$, graph $g(x) = -f(x)$ and $h(x) = f(-x)$. Take note of any surprising behavior for these functions.

Answer

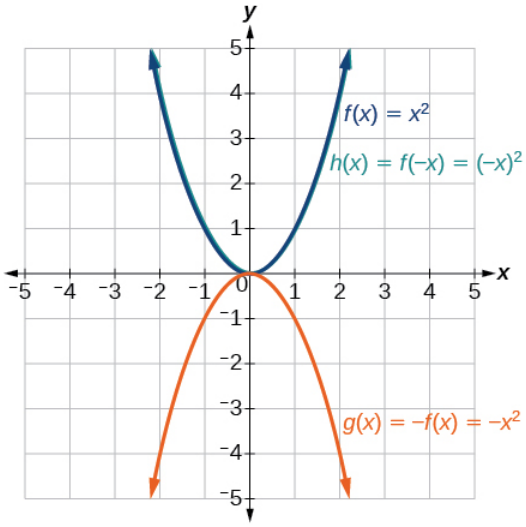


Figure 2.6.20: Graph of x^2 and its reflections.

Notice: $g(x) = f(-x)$ looks the same as $f(x)$.

Determining Even and Odd Functions

Some functions exhibit symmetry so that reflections result in the original graph. For example, horizontally reflecting the toolkit functions $f(x) = x^2$ or $f(x) = |x|$ will result in the original graph. We say that these types of graphs are symmetric about the y-axis. Functions whose graphs are symmetric about the y-axis are called **even functions**.

If the graphs of $f(x) = x^3$ or $f(x) = \frac{1}{x}$ were reflected over both axes, the result would be the original graph, as shown in Figure 2.6.21.

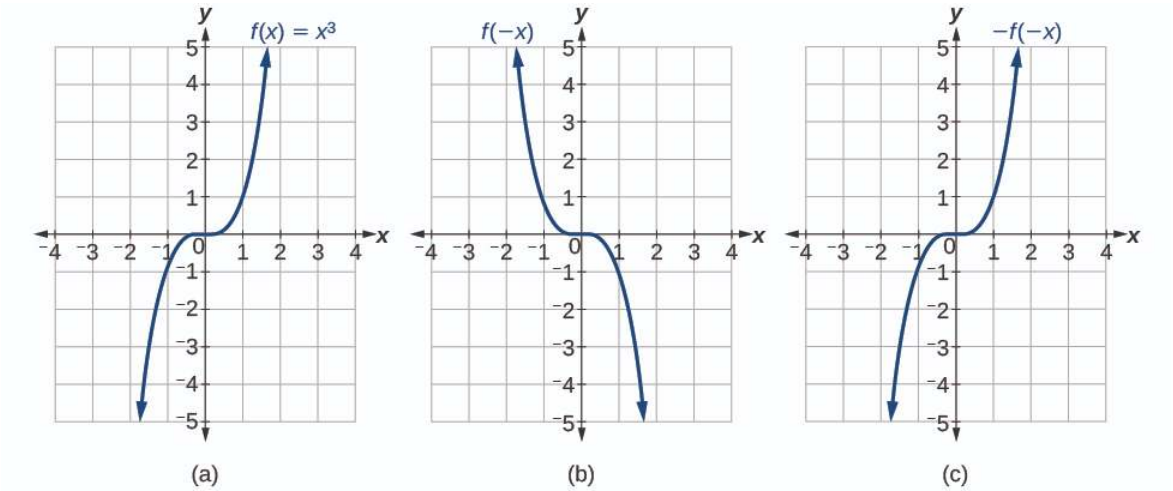


Figure 2.6.21: (a) The cubic toolkit function (b) Horizontal reflection of the cubic toolkit function (c) Horizontal and vertical reflections reproduce the original cubic function.

We say that these graphs are symmetric about the origin. A function with a graph that is symmetric about the origin is called an **odd function**.

Note: A function can be neither even nor odd if it does not exhibit either symmetry. For example, $f(x) = 2^x$ is neither even nor odd. Also, the only function that is both even and odd is the constant function $f(x) = 0$.

Definitions: Even and Odd Functions

A function is called an **even function** if for every input x

$f(x) = f(-x)$

The graph of an even function is symmetric about the y-axis.

A function is called an **odd function** if for every input x

$f(x) = -f(-x)$

The graph of an odd function is symmetric about the origin.

How To...

Given the formula for a function, determine if the function is even, odd, or neither.

1. Determine whether the function satisfies $f(x) = f(-x)$. If it does, it is even.
2. Determine whether the function satisfies $f(x) = -f(-x)$. If it does, it is odd.
3. If the function does not satisfy either rule, it is neither even nor odd.

Example 2.6.13: Determining whether a Function Is Even, Odd, or Neither

Is the function $f(x) = x^3 + 2x$ even, odd, or neither?

Solution

Without looking at a graph, we can determine whether the function is even or odd by finding formulas for the reflections and determining if they return us to the original function. Let's begin with the rule for even functions.

$$f(-x) = (-x)^3 + 2(-x) = -x^3 - 2x$$

This does not return us to the original function, so this function is not even. We can now test the rule for odd functions.

$$-f(-x) = -(-x^3 - 2x) = x^3 + 2x$$

Because $-f(-x) = f(x)$, this is an odd function.

Analysis

Consider the graph of f in Figure 2.6.22 Notice that the graph is symmetric about the origin. For every point (x, y) on the graph, the corresponding point $(-x, -y)$ is also on the graph. For example, $(1, 3)$ is on the graph of f, and the corresponding point $(-1, -3)$ is also on the graph.

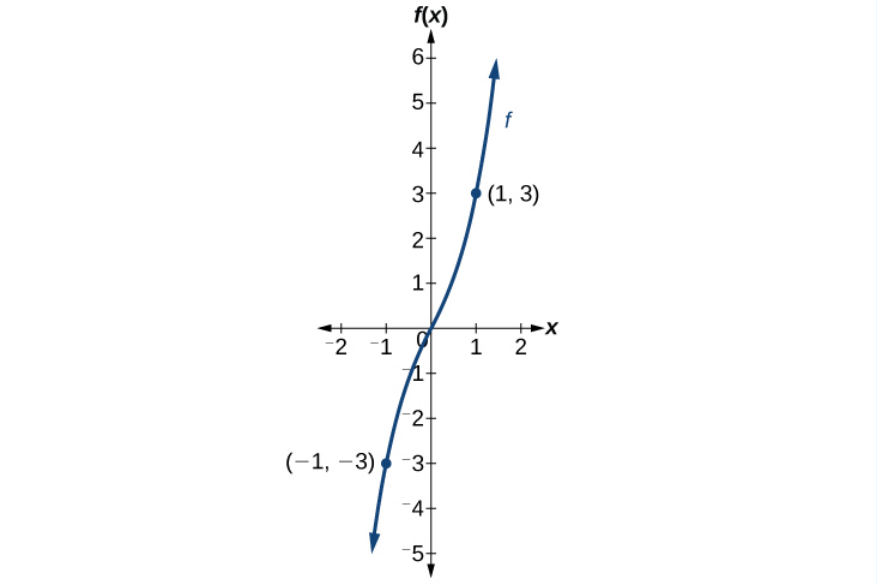


Figure 2.6.22: Graph of $f(x)$ with labeled points at $(1, 3)$ and $(-1, -3)$.

Exercise 2.6.8

Is the function $f(s) = s^4 + 3s^2 + 7$ even, odd, or neither?

Answer

even

Graphing Functions Using Stretches and Compressions

Adding a constant to the inputs or outputs of a function changed the position of a graph with respect to the axes, but it did not affect the shape of a graph. We now explore the effects of multiplying the inputs or outputs by some quantity.

We can transform the inside (input values) of a function or we can transform the outside (output values) of a function. Each change has a specific effect that can be seen graphically.

Vertical Stretches and Compressions

When we multiply a function by a positive constant, we get a function whose graph is stretched or compressed vertically in relation to the graph of the original function. If the constant is greater than 1, we get a **vertical stretch**; if the constant is between 0 and 1, we get a **vertical compression**. Figure 2.6.23 shows a function multiplied by constant factors 2 and 0.5 and the resulting vertical stretch and compression.

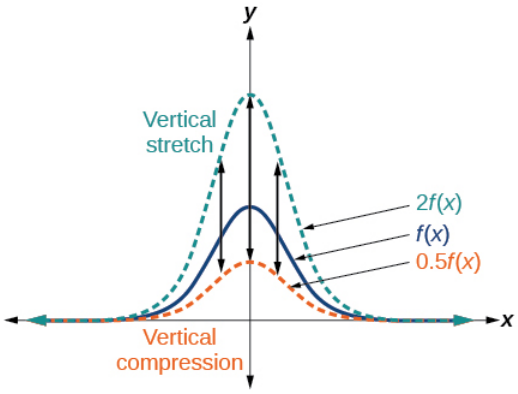


Figure 2.6.23: Vertical stretch and compression

Definitions: Vertical Stretches and Compressions

Given a function $f(x)$, a new function $g(x) = af(x)$, where a is a constant, is a **vertical stretch** or **vertical compression** of the function $f(x)$.

- If $a > 1$, then the graph will be stretched.
- If $0 < a < 1$, then the graph will be compressed.
- If $a < 0$, then there will be combination of a vertical stretch or compression with a vertical reflection.

How To...

Given a function, graph its vertical stretch.

a. Identify the value of a.
b. Multiply all range values by a
c. If $a > 1$, the graph is stretched by a factor of a.
d. If $0 < a < 1$, the graph is compressed by a factor of a.
e. If $a < 0$, the graph is either stretched or compressed and also reflected about the x-axis.

Example 1.5.14: Graphing a Vertical Stretch

A function $P(t)$ models the population of fruit flies. The graph is shown in Figure $2.6.24$

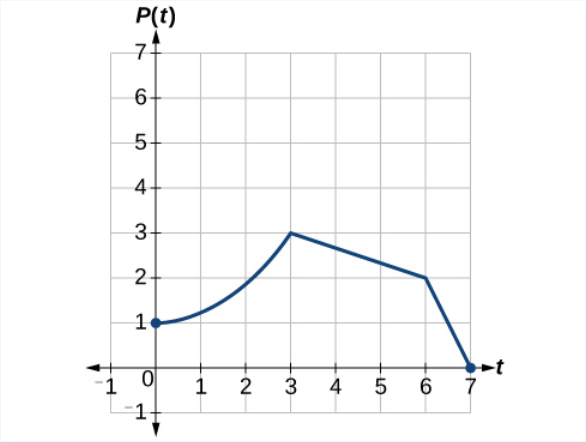


Figure 2.6.24: Graph to represent the growth of the population of fruit flies.

A scientist is comparing this population to another population, Q, whose growth follows the same pattern, but is twice as large. Sketch a graph of this population.

Solution

Because the population is always twice as large, the new population's output values are always twice the original function's output values. Graphically, this is shown in Figure $2.6.25$

If we choose four reference points, $(0, 1)$, $(3, 3)$, $(6, 2)$ and $(7, 0)$ we will multiply all of the outputs by 2.

The following shows where the new points for the new graph will be located.

$$(0, 1) \to (0, 2) \tag{2.6.2}$$
$$(3, 3) \to (3, 6) \tag{2.6.3}$$
$$(6, 2) \to (6, 4) \tag{2.6.4}$$
$$(7, 0) \to (7, 0) \tag{2.6.5}$$

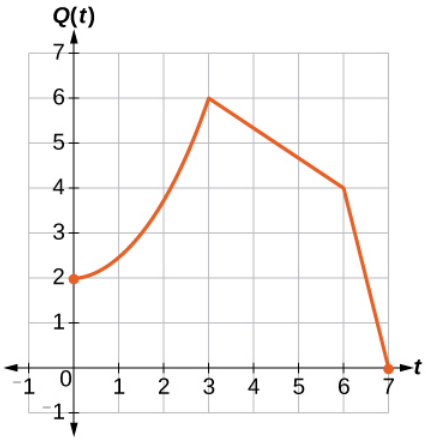


Figure 2.6.25: Graph of the population function doubled.

Symbolically, the relationship is written as

$$Q(t) = 2P(t)$$

This means that for any input t, the value of the function Q is twice the value of the function P. Notice that the effect on the graph is a vertical stretching of the graph, where every point doubles its distance from the horizontal axis. The input values, t, stay the same while the output values are twice as large as before.

How To...

Given a tabular function and assuming that the transformation is a vertical stretch or compression, create a table for a vertical compression.

1. Determine the value of a.
2. Multiply all of the output values by a.

Example 2.6.15: Finding a Vertical Compression of a Tabular Function

A function f is given as Table 2.6.12 Create a table for the function $g(x) = \frac{1}{2}f(x)$.

Table 2.6.12

x	2	4	6	8
$f(x)$	1	3	7	11

Solution

The formula $g(x) = \frac{1}{2}f(x)$ tells us that the output values of g are half of the output values of f with the same inputs. For example, we know that $f(4) = 3$. Then

$$g(4) = \frac{1}{2}f(4) = \frac{1}{2}(3) = \frac{3}{2}$$

We do the same for the other values to produce Table 2.6.13

Table 2.6.13

x	2	4	6	8
$g(x)$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{7}{2}$	$\frac{11}{2}$

Analysis

The result is that the function $g(x)$ has been compressed vertically by $\frac{1}{2}$. Each output value is divided in half, so the graph is half the original height.

Exercise 2.6.9

A function f is given as Table 2.6.14 Create a table for the function $g(x) = \frac{3}{4}f(x)$.

Table 2.6.14

x	2	4	6	8
$f(x)$	12	16	20	0

Answer

Table 2.6.15

x	2	4	6	8
$g(x)$	9	12	15	0

Example 2.6.16: Recognizing a Vertical Stretch

The graph in Figure 2.6.26 is a transformation of the toolkit function $f(x) = x^3$. Relate this new function $g(x)$ to $f(x)$, and then find a formula for $g(x)$.

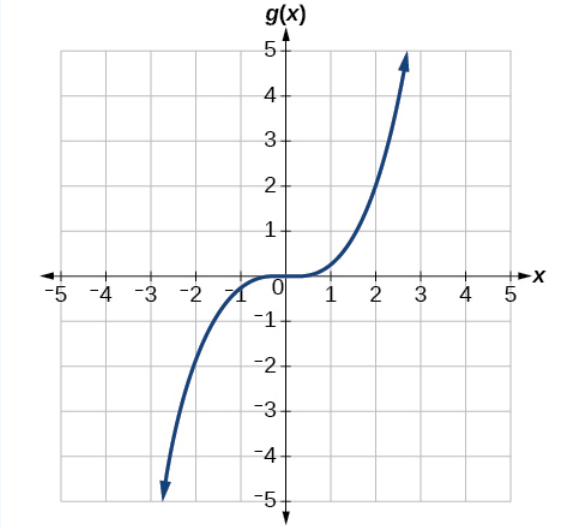


Figure 2.6.26: Graph of a transformation of $f(x) = x^3$

When trying to determine a vertical stretch or shift, it is helpful to look for a point on the graph that is relatively clear. In this graph, it appears that $g(2) = 2$. With the basic cubic function at the same input, $f(2) = 2^3 = 8$. Based on that, it appears that the outputs of g are $\frac{1}{4}$ the outputs of the function f because $g(2) = \frac{1}{4}f(2)$. From this we can fairly safely conclude that $g(x) = \frac{1}{4}f(x)$.

We can write a formula for g by using the definition of the function f.

$$g(x) = \frac{1}{4}f(x) = \frac{1}{4}x^3. \tag{2.6.6}$$

Exercise 2.6.1

Write the formula for the function that we get when we stretch the identity toolkit function by a factor of 3, and then shift it down by 2 units.

Answer

$g(x) = 3x - 2$

Horizontal Stretches and Compressions

Now we consider changes to the inside of a function. When we multiply a function's input by a positive constant, we get a function whose graph is stretched or compressed horizontally in relation to the graph of the original function. If the constant is between 0 and 1, we get a **horizontal stretch**; if the constant is greater than 1, we get a **horizontal compression** of the function.

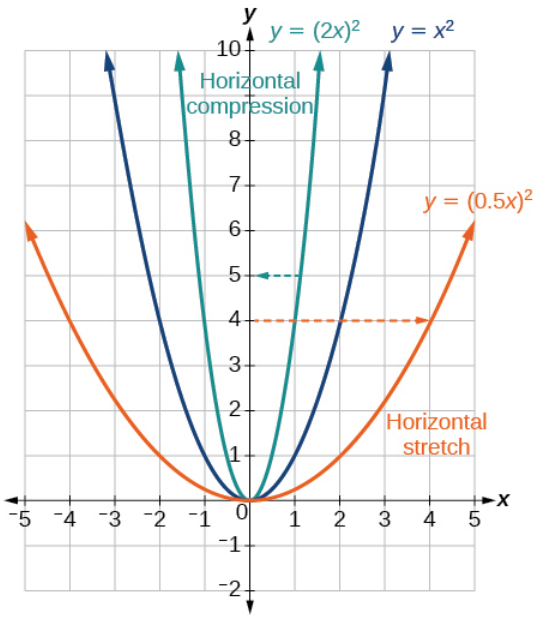


Figure 2.6.27: Graph of the vertical stretch and compression of x^2.

Given a function $y = f(x)$, the form $y = f(bx)$ results in a horizontal stretch or compression. Consider the function $y = x^2$. Observe Figure 2.6.27. The graph of $y = (0.5x)^2$ is a horizontal stretch of the graph of the function $y = x^2$ by a factor of 2. The graph of $y = (2x)^2$ is a horizontal compression of the graph of the function $y = x^2$ by a factor of 2.

Definitions: Horizontal Stretches and Compressions

Given a function $f(x)$, a new function $g(x) = f(bx)$, where b is a constant, is a **horizontal stretch** or **horizontal compression** of the function $f(x)$.

- If $b > 1$, then the graph will be compressed by $\frac{1}{b}$.
- If $0 < b < 1$, then the graph will be stretched by $\frac{1}{b}$.
- If $b < 0$, then there will be combination of a horizontal stretch or compression with a horizontal reflection.

How To...

Given a description of a function, sketch a horizontal compression or stretch.

1. Write a formula to represent the function.
2. Set $g(x) = f(bx)$ where $b > 1$ for a compression or $0 < b < 1$ for a stretch.

Example 2.6.17: Graphing a Horizontal Compression

Suppose a scientist is comparing a population of fruit flies to a population that progresses through its lifespan twice as fast as the original population. In other words, this new population, R, will progress in 1 hour the same amount as the original population does in 2 hours, and in 2 hours, it will progress as much as the original population does in 4 hours. Sketch a graph of this population.

Solution

Symbolically, we could write

$$R(1) = P(2), \tag{2.6.7}$$
$$R(2) = P(4), \quad \text{and in general,} \tag{2.6.8}$$
$$R(t) = P(2t). \tag{2.6.9}$$

See Figure 2.6.28 for a graphical comparison of the original population and the compressed population.

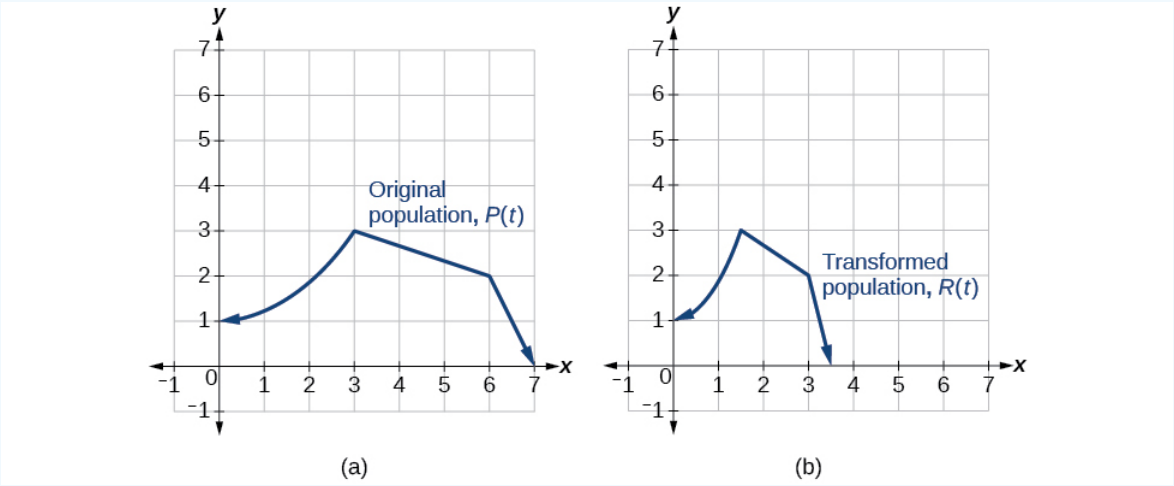


Figure 2.6.28: (a) Original population graph (b) Compressed population graph

Analysis

Note that the effect on the graph is a horizontal compression where all input values are half of their original distance from the vertical axis.

Example 2.6.18: Finding a Horizontal Stretch for a Tabular Function

A function $f(x)$ is given as Table 2.6.16. Create a table for the function $g(x) = f(\frac{1}{2}x)$.

Table 2.6.16

x	2	4	6	8
$f(x)$	1	3	7	11

The formula $g(x) = f(\frac{1}{2}x)$ tells us that the output values for g are the same as the output values for the function f at an input half the size. Notice that we do not have enough information to determine $g(2)$ because $g(2) = f(\frac{1}{2} \cdot 2) = f(1)$, and we do not have a value for $f(1)$ in our table. Our input values to g will need to be twice as large to get inputs for f that we can evaluate. For example, we can determine $g(4)$.

$$g(4) = f(\frac{1}{2} \cdot 4) = f(2) = 1 \tag{2.6.10}$$

We do the same for the other values to produce Table 2.6.17.

Table 2.6.17

x	4	8	12	16

$g(x)$	1	3	7	11

Figure 2.6.29 shows the graphs of both of these sets of points.

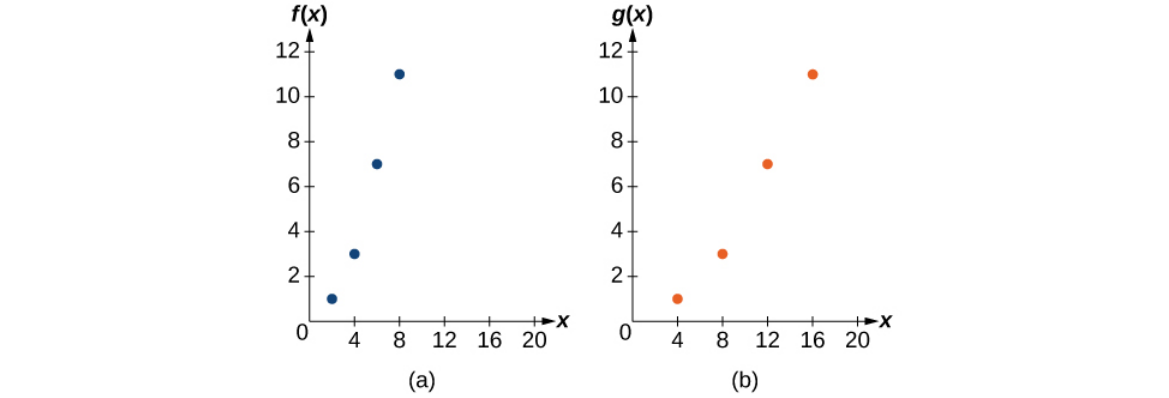


Figure 2.6.29: Graph of the previous table.

Analysis

Because each input value has been doubled, the result is that the function $g(x)$ has been stretched horizontally by a factor of 2.

Example 2.6.19: Recognizing a Horizontal Compression on a Graph

Relate the function $g(x)$ to $f(x)$ in Figure 2.6.30.

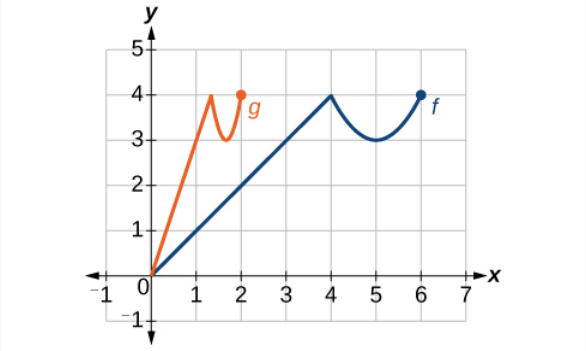


Figure 2.6.30: Graph of $f(x)$ being vertically compressed to $g(x)$.

Solution

The graph of $g(x)$ looks like the graph of $f(x)$ horizontally compressed. Because $f(x)$ ends at (6,4) and $g(x)$ ends at (2,4), we can see that the x-values have been compressed by $\frac{1}{3}$, because $6(\frac{1}{3}) = 2$. We might also notice that $g(2) = f(6)$ and $g(1) = f(3)$. Either way, we can describe this relationship as $g(x) = f(3x)$. This is a horizontal compression by $\frac{1}{3}$.

Analysis

Notice that the coefficient needed for a horizontal stretch or compression is the reciprocal of the stretch or compression. So to stretch the graph horizontally by a scale factor of 4, we need a coefficient of $\frac{1}{4}$ in our function: $f(\frac{1}{4}x)$. This means that the input values must be four times larger to produce the same result, requiring the input to be larger, causing the horizontal stretching.

Exercise 2.6.11

Write a formula for the toolkit square root function horizontally stretched by a factor of 3.

Answer

$g(x) = f(\frac{1}{3}x)$, so using the square root function we get $g(x) = \sqrt{\frac{1}{3}x}$

Vertical Stretches and Compressions

When we multiply a function by a positive constant, we get a function whose graph is stretched or compressed vertically in relation to the graph of the original function. If the constant is greater than 1, we get a **vertical stretch**; if the constant is between 0 and 1, we get a **vertical compression**. Figure $2.6.23$ shows a function multiplied by constant factors 2 and 0.5 and the resulting vertical stretch and compression.

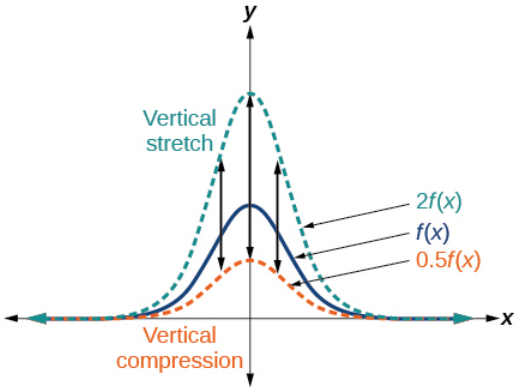


Figure 2.6.23: Vertical stretch and compression

Definitions: Vertical Stretches and Compressions

Given a function $f(x)$, a new function $g(x) = af(x)$, where a is a constant, is a **vertical stretch** or **vertical compression** of the function $f(x)$.

- If $a > 1$, then the graph will be stretched.
- If $0 < a < 1$, then the graph will be compressed.
- If $a < 0$, then there will be combination of a vertical stretch or compression with a vertical reflection.

How To...

Given a function, graph its vertical stretch.

a. Identify the value of a.
b. Multiply all range values by a
c. If $a > 1$, the graph is stretched by a factor of a.
d. If $0 < a < 1$, the graph is compressed by a factor of a.
e. If $a < 0$, the graph is either stretched or compressed and also reflected about the x-axis.

Example 1.5.14: Graphing a Vertical Stretch

A function $P(t)$ models the population of fruit flies. The graph is shown in Figure $2.6.24$

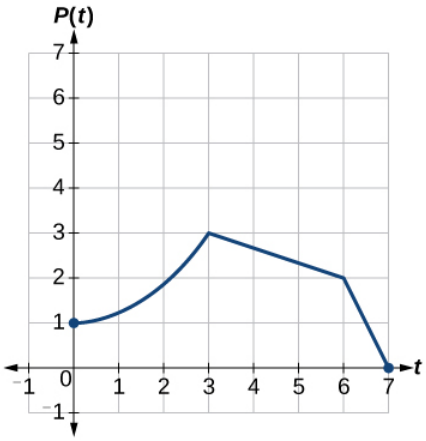


Figure 2.6.24: Graph to represent the growth of the population of fruit flies.

A scientist is comparing this population to another population, Q, whose growth follows the same pattern, but is twice as large. Sketch a graph of this population.

Solution

Because the population is always twice as large, the new population's output values are always twice the original function's output values. Graphically, this is shown in Figure 2.6.25.

If we choose four reference points, $(0, 1)$, $(3, 3)$, $(6, 2)$ and $(7, 0)$ we will multiply all of the outputs by 2.

The following shows where the new points for the new graph will be located.

$$(0, 1) \to (0, 2) \tag{2.6.11}$$

$$(3, 3) \to (3, 6) \tag{2.6.12}$$

$$(6, 2) \to (6, 4) \tag{2.6.13}$$

$$(7, 0) \to (7, 0) \tag{2.6.14}$$

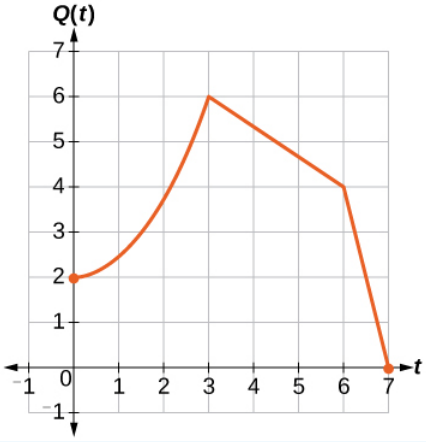


Figure 2.6.25: Graph of the population function doubled.

Symbolically, the relationship is written as

$$Q(t) = 2P(t)$$

This means that for any input t, the value of the function Q is twice the value of the function P. Notice that the effect on the graph is a vertical stretching of the graph, where every point doubles its distance from the horizontal axis. The input values, t, stay the same while the output values are twice as large as before.

How To...

Given a tabular function and assuming that the transformation is a vertical stretch or compression, create a table for a vertical compression.

1. Determine the value of a.
2. Multiply all of the output values by a.

Example 2.6.15: Finding a Vertical Compression of a Tabular Function

A function f is given as Table 2.6.12 Create a table for the function $g(x) = \frac{1}{2} f(x)$.

Table 2.6.12

x	2	4	6	8
$f(x)$	1	3	7	11

Solution

The formula $g(x) = \frac{1}{2} f(x)$ tells us that the output values of g are half of the output values of f with the same inputs. For example, we know that $f(4) = 3$. Then

$$g(4) = \frac{1}{2} f(4) = \frac{1}{2}(3) = \frac{3}{2}$$

We do the same for the other values to produce Table 2.6.13

Table 2.6.13

x	2	4	6	8
$g(x)$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{7}{2}$	$\frac{11}{2}$

Analysis

The result is that the function $g(x)$ has been compressed vertically by $\frac{1}{2}$. Each output value is divided in half, so the graph is half the original height.

Exercise 2.6.9

A function f is given as Table 2.6.14 Create a table for the function $g(x) = \frac{3}{4} f(x)$.

Table 2.6.14

x	2	4	6	8
$f(x)$	12	16	20	0

Answer

Table 2.6.15

x	2	4	6	8
$g(x)$	9	12	15	0

Example 2.6.16: Recognizing a Vertical Stretch

The graph in Figure 2.6.26 is a transformation of the toolkit function $f(x) = x^3$. Relate this new function $g(x)$ to $f(x)$, and then find a formula for $g(x)$.

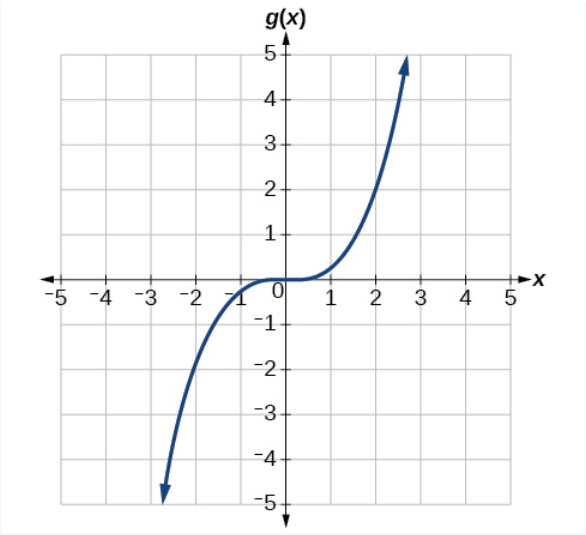


Figure 2.6.26: Graph of a transformation of $f(x) = x^3$

When trying to determine a vertical stretch or shift, it is helpful to look for a point on the graph that is relatively clear. In this graph, it appears that $g(2) = 2$. With the basic cubic function at the same input, $f(2) = 2^3 = 8$. Based on that, it appears that the outputs of g are $\frac{1}{4}$ the outputs of the function f because $g(2) = \frac{1}{4}f(2)$. From this we can fairly safely conclude that $g(x) = \frac{1}{4}f(x)$.

We can write a formula for g by using the definition of the function f.

$$g(x) = \frac{1}{4}f(x) = \frac{1}{4}x^3. \tag{2.6.15}$$

Exercise 2.6.1

Write the formula for the function that we get when we stretch the identity toolkit function by a factor of 3, and then shift it down by 2 units.

Answer

$g(x) = 3x - 2$

Horizontal Stretches and Compressions

Now we consider changes to the inside of a function. When we multiply a function's input by a positive constant, we get a function whose graph is stretched or compressed horizontally in relation to the graph of the original function. If the constant is between 0 and 1, we get a **horizontal stretch**; if the constant is greater than 1, we get a **horizontal compression** of the function.

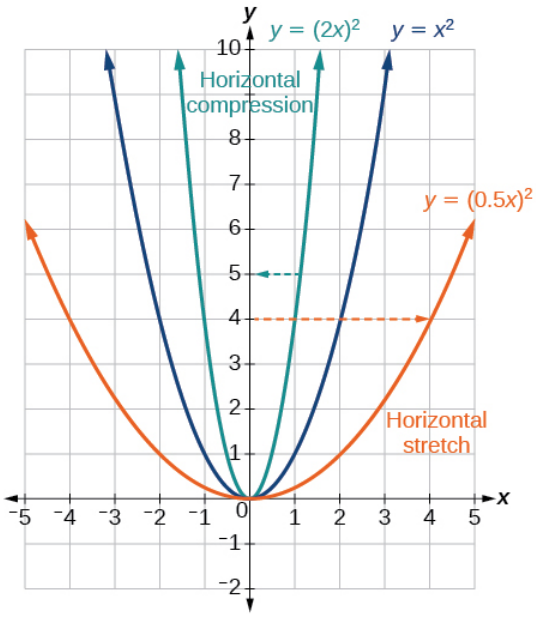


Figure 2.6.27: Graph of the vertical stretch and compression of x^2.

Given a function $y = f(x)$, the form $y = f(bx)$ results in a horizontal stretch or compression. Consider the function $y = x^2$. Observe Figure 2.6.27. The graph of $y = (0.5x)^2$ is a horizontal stretch of the graph of the function $y = x^2$ by a factor of 2. The graph of $y = (2x)^2$ is a horizontal compression of the graph of the function $y = x^2$ by a factor of 2.

Definitions: Horizontal Stretches and Compressions

Given a function $f(x)$, a new function $g(x) = f(bx)$, where b is a constant, is a **horizontal stretch** or **horizontal compression** of the function $f(x)$.

- If $b > 1$, then the graph will be compressed by $\frac{1}{b}$.
- If $0 < b < 1$, then the graph will be stretched by $\frac{1}{b}$.
- If $b < 0$, then there will be combination of a horizontal stretch or compression with a horizontal reflection.

How To...

Given a description of a function, sketch a horizontal compression or stretch.

1. Write a formula to represent the function.
2. Set $g(x) = f(bx)$ where $b > 1$ for a compression or $0 < b < 1$ for a stretch.

Example 2.6.17: Graphing a Horizontal Compression

Suppose a scientist is comparing a population of fruit flies to a population that progresses through its lifespan twice as fast as the original population. In other words, this new population, R, will progress in 1 hour the same amount as the original population does in 2 hours, and in 2 hours, it will progress as much as the original population does in 4 hours. Sketch a graph of this population.

Solution

Symbolically, we could write

$$R(1) = P(2), \tag{2.6.16}$$
$$R(2) = P(4), \quad \text{and in general,} \tag{2.6.17}$$
$$R(t) = P(2t). \tag{2.6.18}$$

See Figure 2.6.28 for a graphical comparison of the original population and the compressed population.

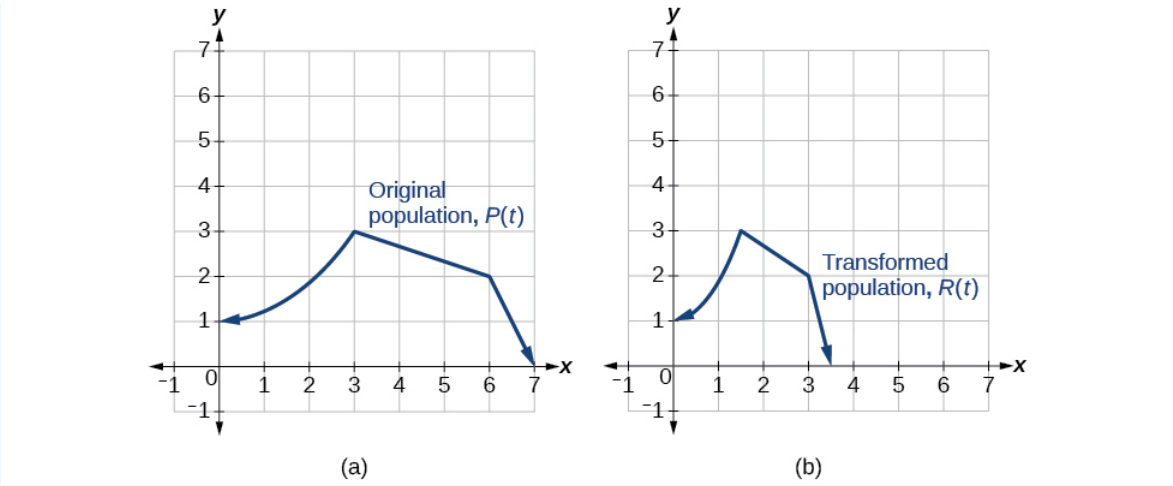


Figure 2.6.28: (a) Original population graph (b) Compressed population graph

Analysis

Note that the effect on the graph is a horizontal compression where all input values are half of their original distance from the vertical axis.

Example 2.6.18: Finding a Horizontal Stretch for a Tabular Function

A function $f(x)$ is given as Table 2.6.16. Create a table for the function $g(x) = f(\frac{1}{2}x)$.

Table 2.6.16

x	2	4	6	8
$f(x)$	1	3	7	11

The formula $g(x) = f(\frac{1}{2}x)$ tells us that the output values for g are the same as the output values for the function f at an input half the size. Notice that we do not have enough information to determine $g(2)$ because $g(2) = f(\frac{1}{2} \cdot 2) = f(1)$, and we do not have a value for $f(1)$ in our table. Our input values to g will need to be twice as large to get inputs for f that we can evaluate. For example, we can determine $g(4)$.

$$g(4) = f(\frac{1}{2} \cdot 4) = f(2) = 1 \tag{2.6.19}$$

We do the same for the other values to produce Table 2.6.17.

Table 2.6.17

x	4	8	12	16
$g(x)$	1	3	7	11

Figure 2.6.29 shows the graphs of both of these sets of points.

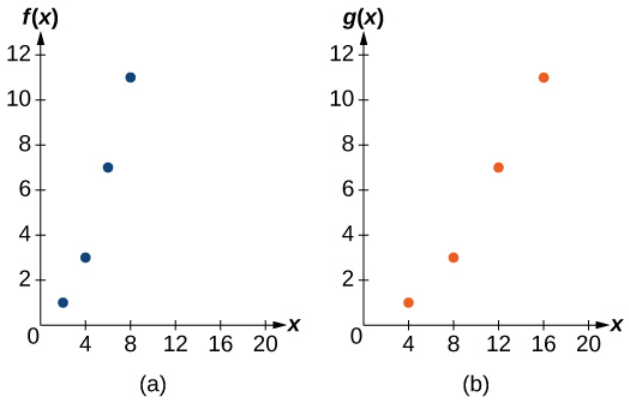


Figure 2.6.29: Graph of the previous table.

Analysis

Because each input value has been doubled, the result is that the function $g(x)$ has been stretched horizontally by a factor of 2.

Example 2.6.19: Recognizing a Horizontal Compression on a Graph

Relate the function $g(x)$ to $f(x)$ in Figure 2.6.30

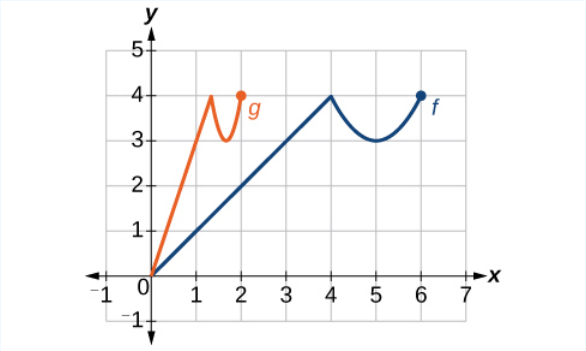


Figure 2.6.30: Graph of $f(x)$ being vertically compressed to $g(x)$.

Solution

The graph of $g(x)$ looks like the graph of $f(x)$ horizontally compressed. Because $f(x)$ ends at (6,4) and $g(x)$ ends at (2,4), we can see that the x-values have been compressed by $\frac{1}{3}$, because $6(\frac{1}{3}) = 2$. We might also notice that $g(2) = f(6)$ and $g(1) = f(3)$. Either way, we can describe this relationship as $g(x) = f(3x)$. This is a horizontal compression by $\frac{1}{3}$.

Analysis

Notice that the coefficient needed for a horizontal stretch or compression is the reciprocal of the stretch or compression. So to stretch the graph horizontally by a scale factor of 4, we need a coefficient of $\frac{1}{4}$ in our function: $f(\frac{1}{4}x)$. This means that the input values must be four times larger to produce the same result, requiring the input to be larger, causing the horizontal stretching.

Exercise 2.6.11

Write a formula for the toolkit square root function horizontally stretched by a factor of 3.

Answer

$g(x) = f(\frac{1}{3}x)$, so using the square root function we get $g(x) = \sqrt{\frac{1}{3}x}$

Performing a Sequence of Transformations

When combining transformations, it is very important to consider the order of the transformations. For example, vertically shifting by 3 and then vertically stretching by 2 does not create the same graph as vertically stretching by 2 and then vertically shifting by 3, because when we shift first, both the original function and the shift get stretched, while only the original function gets stretched when we stretch first.

When we see an expression such as $2f(x)+3$, which transformation should we start with? The answer here follows nicely from the order of operations. Given the output value of $f(x)$, we first multiply by 2, causing the vertical stretch, and then add 3, causing the vertical shift. In other words, multiplication before addition.

Horizontal transformations are a little trickier to think about. When we write $g(x)=f(2x+3)$, for example, we have to think about how the inputs to the function g relate to the inputs to the function f. Suppose we know $f(7)=12$. What input to g would produce that output? In other words, what value of x will allow $g(x)=f(2x+3)=12$? We would need $2x+3=7$. To solve for x, we would first subtract 3, resulting in a horizontal shift, and then divide by 2, causing a horizontal compression.

This format ends up being very difficult to work with, because it is usually much easier to horizontally stretch a graph before shifting. We can work around this by factoring inside the function.

$$f(bx+p)=f(b(x+\frac{p}{b}))$$

Let's work through an example.

$$f(x)=(2x+4)^2$$

We can factor out a 2.

$$f(x)=(2(x+2))^2$$

Now we can more clearly observe a horizontal shift to the left 2 units and a horizontal compression. Factoring in this way allows us to horizontally stretch first and then shift horizontally.

Combining Transformations

- When combining vertical transformations written in the form $af(x)+k$, first vertically stretch by a and then vertically shift by k.
- When combining horizontal transformations written in the form $f(bx+h)$, first horizontally shift by h and then horizontally stretch by $\frac{1}{b}$.
- When combining horizontal transformations written in the form $f(b(x+h))$, first horizontally stretch by $\frac{1}{b}$ and then horizontally shift by h.
- Horizontal and vertical transformations are independent. It does not matter whether horizontal or vertical transformations are performed first.

Example 2.6.20: Finding a Triple Transformation of a Tabular Function

Given Table 2.6.18 for the function $f(x)$, create a table of values for the function $g(x)=2f(3x)+1$.

Table 2.6.18

x	6	12	18	24
$f(x)$	10	14	15	17

Solution

There are three steps to this transformation, and we will work from the inside out. Starting with the horizontal transformations, $f(3x)$ is a horizontal compression by $\frac{1}{3}$, which means we multiply each x-value by $\frac{1}{3}$. See Table 2.6.19

Table 2.6.19

x	2	4	6	8

$f(3x)$	10	14	15	17

Looking now to the vertical transformations, we start with the vertical stretch, which will multiply the output values by 2. We apply this to the previous transformation. See Table 2.6.20

Table 2.6.20

x	2	4	6	8
$2f(3x)$	20	28	30	34

Finally, we can apply the vertical shift, which will add 1 to all the output values. See Table 2.6.21

Table 2.6.21

x	2	4	6	8
$g(x)=2f(3x)+1+1$	21	29	31	35

Example 2.6.21: Finding a Triple Transformation of a Graph

Use the graph of $f(x)$ in Figure 2.6.31 to sketch a graph of $k(x)=f\left(\frac{1}{2}x+1\right)-3$.

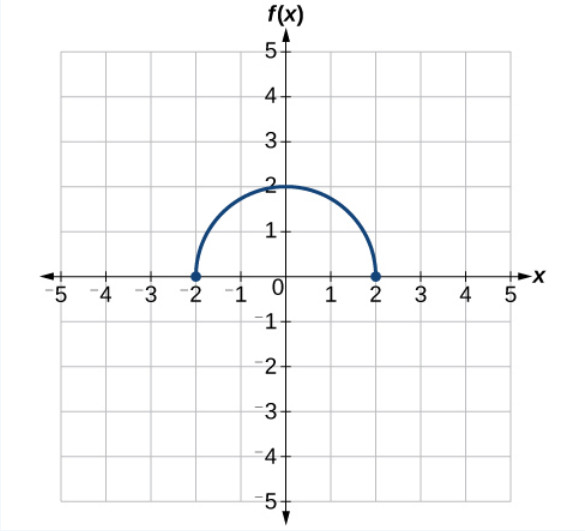


Figure 2.6.31: Graph of a half-circle.

To simplify, let's start by factoring out the inside of the function.

$$f\left(\frac{1}{2}x+1\right)-3=f\left(\frac{1}{2}(x+2)\right)-3 \tag{2.6.20}$$

By factoring the inside, we can first horizontally stretch by 2, as indicated by the $\frac{1}{2}$ on the inside of the function. Remember that twice the size of 0 is still 0, so the point $(0,2)$ remains at $(0,2)$ while the point $(2,0)$ will stretch to $(4,0)$. See Figure 2.6.32

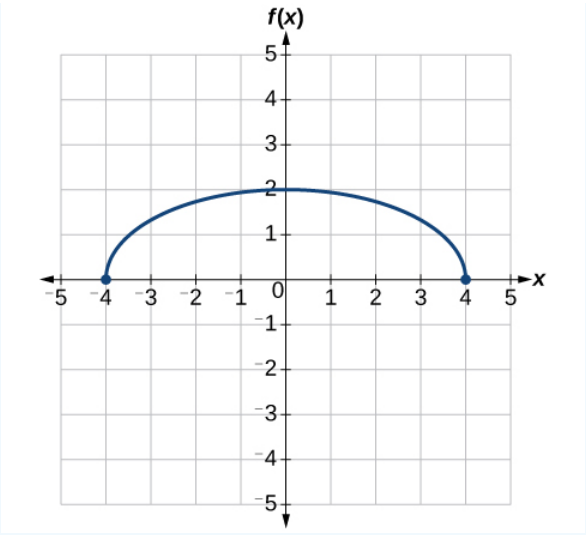


Figure 2.6.32: Graph of a vertically stretch half-circle.

Next, we horizontally shift left by 2 units, as indicated by $x+2$. See Figure 2.6.33

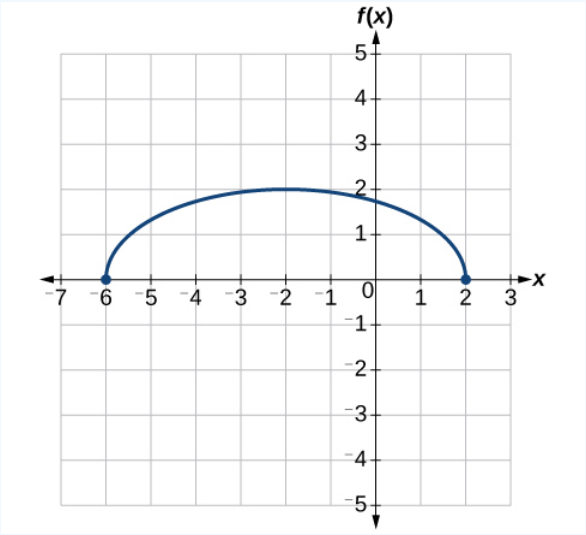


Figure 2.6.33: Graph of a vertically stretch and translated half-circle.

Last, we vertically shift down by 3 to complete our sketch, as indicated by the -3 on the outside of the function. See Figure 2.6.34

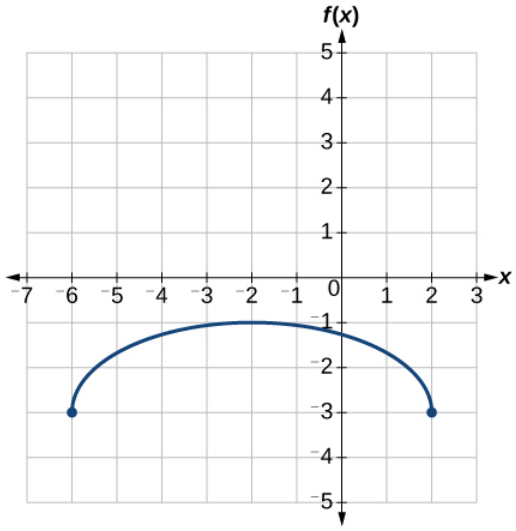


Figure 2.6.34: Graph of a vertically stretch and translated half-circle.

Key Equations

- Vertical shift $g(x) = f(x) + k$ (up for $k > 0$)
- Horizontal shift $g(x) = f(x - h)$ (right) for $h > 0$
- Vertical reflection $g(x) = -f(x)$
- Horizontal reflection $g(x) = f(-x)$
- Vertical stretch $g(x) = af(x)$ (a>0)
- Vertical compression $g(x) = af(x)$ (0<a<1)
- Horizontal stretch $g(x) = f(bx)(0 < b < 1)$
- Horizontal compression $g(x) = f(bx)$ (b>1)

Key Concepts

- A function can be shifted vertically by adding a constant to the output.
- A function can be shifted horizontally by adding a constant to the input.
- Relating the shift to the context of a problem makes it possible to compare and interpret vertical and horizontal shifts.
- Vertical and horizontal shifts are often combined.
- A vertical reflection reflects a graph about the x-axis. A graph can be reflected vertically by multiplying the output by –1.
- A horizontal reflection reflects a graph about the y-axis. A graph can be reflected horizontally by multiplying the input by –1.
- A graph can be reflected both vertically and horizontally. The order in which the reflections are applied does not affect the final graph.
- A function presented in tabular form can also be reflected by multiplying the values in the input and output rows or columns accordingly.
- A function presented as an equation can be reflected by applying transformations one at a time.
- Even functions are symmetric about the y-axis, whereas odd functions are symmetric about the origin.
- Even functions satisfy the condition $f(x) = f(-x)$.
- Odd functions satisfy the condition $f(x) = -f(-x)$.
- A function can be odd, even, or neither.
- A function can be compressed or stretched vertically by multiplying the output by a constant.
- A function can be compressed or stretched horizontally by multiplying the input by a constant.
- The order in which different transformations are applied does affect the final function. Both vertical and horizontal transformations must be applied in the order given. However, a vertical transformation may be combined with a horizontal transformation in any order.

Glossary

even function

a function whose graph is unchanged by horizontal reflection, $f(x) = f(-x)$, and is symmetric about the y-axis

horizontal compression
a transformation that compresses a function's graph horizontally, by multiplying the input by a constant b>1

horizontal reflection
a transformation that reflects a function's graph across the y-axis by multiplying the input by −1

horizontal shift
a transformation that shifts a function's graph left or right by adding a positive or negative constant to the input

horizontal stretch
a transformation that stretches a function's graph horizontally by multiplying the input by a constant 0<b<1

odd function
a function whose graph is unchanged by combined horizontal and vertical reflection, $f(x) = -f(-x)$, and is symmetric about the origin

vertical compression
a function transformation that compresses the function's graph vertically by multiplying the output by a constant 0<a<1

vertical reflection
a transformation that reflects a function's graph across the x-axis by multiplying the output by −1

vertical shift
a transformation that shifts a function's graph up or down by adding a positive or negative constant to the output

vertical stretch
a transformation that stretches a function's graph vertically by multiplying the output by a constant a>1

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CHAPTER OVERVIEW

3: Linear and Quadratic Functions

Recall that a function is a relation that assigns to every element in the domain exactly one element in the range. Linear functions are a specific type of function that can be used to model many real-world applications, such as plant growth over time. In this chapter, we will explore linear functions, their graphs, and how to relate them to data.

3.1: Linear Functions
3.2: Graphs of Linear Functions
3.3: Modeling with Linear Functions
3.4: Quadratic Functions
3.5: Applications with Quadratic Functions

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3.1: Linear Functions

Just as with the growth of a bamboo plant, there are many situations that involve constant change over time. Consider, for example, the first commercial maglev train in the world, the Shanghai MagLev Train (maglev train in the world, the Shanghai MagLev Train (Figure 3.1.1). It carries passengers comfortably for a 30-kilometer trip from the airport to the subway station in only eight minutes.



Figure 3.1.1: Shanghai MagLev Train (credit: “kanegen”/Flickr)

Suppose a maglev train were to travel a long distance, and that the train maintains a constant speed of 83 meters per second for a period of time once it is 250 meters from the station. How can we analyze the train’s distance from the station as a function of time? In this section, we will investigate a kind of function that is useful for this purpose, and use it to investigate real-world situations such as the train’s distance from the station at a given point in time.maglev train were to travel a long distance, and that the train maintains a constant speed of 83 meters per second for a period of time once it is 250 meters from the station. How can we analyze the train’s distance from the station as a function of time? In this section, we will investigate a kind of function that is useful for this purpose, and use it to investigate real-world situations such as the train’s distance from the station at a given point in time.

Representing Linear Functions

The function describing the train’s motion is a **linear function**, which is defined as a function with a constant rate of change, that is, a polynomial of degree 1. There are several ways to represent a linear function, including word form, function notation, tabular form, and graphical form. We will describe the train’s motion as a function using each method.

Representing a Linear Function in Word Form

Let’s begin by describing the linear function in words. For the train problem we just considered, the following word sentence may be used to describe the function relationship.

- *The train’s distance from the station is a function of the time during which the train moves at a constant speed plus its original distance from the station when it began moving at constant speed.*

The speed is the rate of change. Recall that a rate of change is a measure of how quickly the dependent variable changes with respect to the independent variable. The rate of change for this example is constant, which means that it is the same for each input value. As the time (input) increases by 1 second, the corresponding distance (output) increases by 83 meters. The train began moving at this constant speed at a distance of 250 meters from the station.

Representing a Linear Function in Function Notation

Another approach to representing linear functions is by using function notation. One example of function notation is an equation written in the form known as the slope-intercept form of a line, where xis the input value, m is the rate of change, and b is the initial value of the dependent variable.

Equation form	$y = mx + b$
Equation notation	$f(x) = mx + b$

In the example of the train, we might use the notation $D(t)$ in which the total distance D is a function of the time t. The rate, m, is 83 meters per second. The initial value of the dependent variable b is the original distance from the station, 250 meters. We can write a generalized equation to represent the motion of the train.

$$D(t) = 83t + 250 \tag{3.1.1}$$

Representing a Linear Function in Tabular Form

A third method of representing a linear function is through the use of a table. The relationship between the distance from the station and the time is represented in Figure 3.1.2. From the table, we can see that the distance changes by 83 meters for every 1 second increase in time.

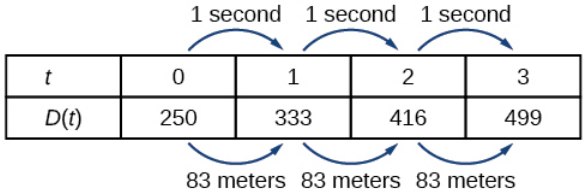


Figure 3.1.2: Tabular representation of the function D showing selected input and output values

Q/A? Can the input in the previous example be any real number?

No. The input represents time, so while nonnegative rational and irrational numbers are possible, negative real numbers are not possible for this example. The input consists of non-negative real numbers.

Representing a Linear Function in Graphical Form

Another way to represent linear functions is visually, using a graph. We can use the function relationship from above, $D(t) = 83t + 250$, to draw a graph, represented in Figure 3.1.3. Notice the graph is a line. When we plot a linear function, the graph is always a line.

The rate of change, which is constant, determines the slant, or **slope** of the line. The point at which the input value is zero is the vertical intercept, or **y-intercept**, of the line. We can see from the graph in Figure 3.1.3 that the y-intercept in the train example we just saw is $(0, 250)$ and represents the distance of the train from the station when it began moving at a constant speed.

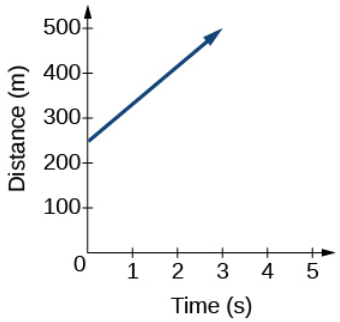


Figure 3.1.3: The graph of $D(t) = 83t + 250$. Graphs of linear functions are lines because the rate of change is constant.

Notice that the graph of the train example is restricted, but this is not always the case. Consider the graph of the line $f(x) = 2x + 1$. Ask yourself what numbers can be input to the function, that is, what is the domain of the function? The domain is comprised of all real numbers because any number may be doubled, and then have one added to the product.

Definition: Linear Function

A **linear function** is a function whose graph is a line. Linear functions can be written in the slope-intercept form of a line

$$f(x) = mx + b \tag{3.1.2}$$

where b is the initial or starting value of the function (when input, $x = 0$), and m is the constant rate of change, or **slope** of the function. The **y-intercept** is at $(0, b)$.

Example : Using a Linear Function to Find the Pressure on a Diver

The pressure, P, in pounds per square inch (PSI) on the diver in Figure 3.1.4 depends upon her depth below the water surface, d, in feet. This relationship may be modeled by the equation, $P(d) = 0.434d + 14.696$. Restate this function in words.



Figure 3.1.4: (credit: Ilse Reijs and Jan-Noud Hutten)Hutten)

To restate the function in words, we need to describe each part of the equation. The pressure as a function of depth equals four hundred thirty-four thousandths times depth plus fourteen and six hundred ninety-six thousandths.

Analysis

The initial value, 14.696, is the pressure in PSI on the diver at a depth of 0 feet, which is the surface of the water. The rate of change, or slope, is 0.434 PSI per foot. This tells us that the pressure on the diver increases 0.434 PSI for each foot her depth increases.

Determining whether a Linear Function Is Increasing, Decreasing, or Constant

The linear functions we used in the two previous examples increased over time, but not every linear function does. A linear function may be increasing, decreasing, or constant. For an increasing function, as with the train example, the output values increase as the input values increase. The graph of an **increasing function** has a positive slope. A line with a positive slope slants upward from left to right as in Figure 3.1.5(a). For a **decreasing function**, the slope is negative. The output values decrease as the input values increase. A line with a negative slope slants downward from left to right as in Figure 3.1.5(b). If the function is constant, the output values are the same for all input values so the slope is zero. A line with a slope of zero is horizontal as in Figure 3.1.5(c).

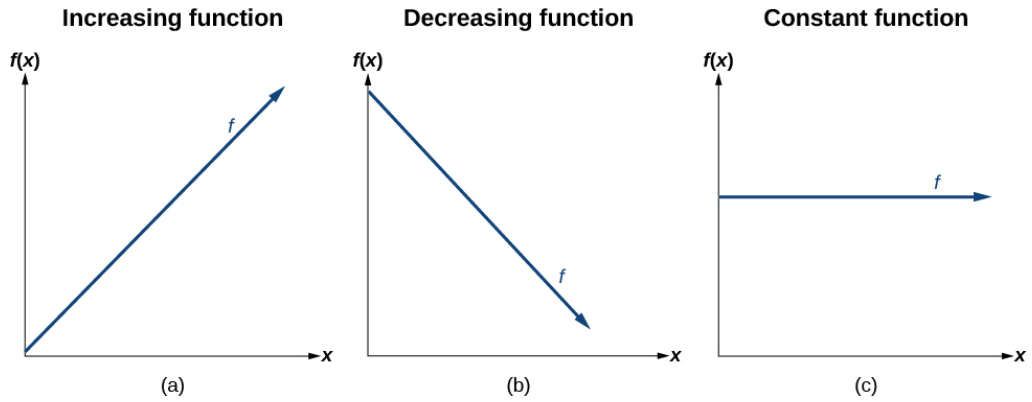


Figure 3.1.5: Three graphs depicting an increasing function, a decreasing function, and a constant function.]
Increasing and Decreasing Functions

Increasing and Decreasing Functions

The slope determines if the function is an **increasing linear function**, a **decreasing linear function**, or a constant function.

- $f(x) = mx + b$ is an increasing function if $m > 0$.
- $f(x) = mx + b$ is an decreasing function if $m < 0$.
- $f(x) = mx + b$ is a constant function if $m = 0$.

Example : Deciding whether a Function Is Increasing, Decreasing, or Constant

Some recent studies suggest that a teenager sends an average of 60 texts per day. For each of the following scenarios, find the linear function that describes the relationship between the input value and the output value. Then, determine whether the graph of the function is increasing, decreasing, or constant.

a. The total number of texts a teen sends is considered a function of time in days. The input is the number of days, and output is the total number of texts sent.
b. A teen has a limit of 500 texts per month in his or her data plan. The input is the number of days, and output is the total number of texts remaining for the month.
c. A teen has an unlimited number of texts in his or her data plan for a cost of $50 per month. The input is the number of days, and output is the total cost of texting each month.

Solution

Analyze each function.

a. The function can be represented as $f(x) = 60x$ where x is the number of days. The slope, 60, is positive so the function is increasing. This makes sense because the total number of texts increases with each day.
b. The function can be represented as $f(x) = 500 - 60x$ where x is the number of days. In this case, the slope is negative so the function is decreasing. This makes sense because the number of texts remaining decreases each day and this function represents the number of texts remaining in the data plan after x days.
c. The cost function can be represented as $f(x) = 50$ because the number of days does not affect the total cost. The slope is 0 so the function is constant.

Calculating and Interpreting Slope

In the examples we have seen so far, we have had the slope provided for us. However, we often need to calculate the slope given input and output values. Given two values for the input, x_1 and x_2, and two corresponding values for the output, y_1 and y_2—which can be represented by a set of points, (x_1, y_1) and (x_2, y_2)—we can calculate the slope m, as follows

$$m = \frac{\text{change in output (rise)}}{\text{change in input (run)}}$$
$$= \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

where Δy is the vertical displacement and Δx is the horizontal displacement. Note in function notation two corresponding values for the output y_1 and y_2 for the function f, $y_1 = f(x_1)$ and $y_2 = f(x_2)$, so we could equivalently write

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Figure 3.1.6 indicates how the slope of the line between the points, (x_1, y_1) and (x_2, y_2), is calculated. Recall that the slope measures steepness. The greater the absolute value of the slope, the steeper the line is.

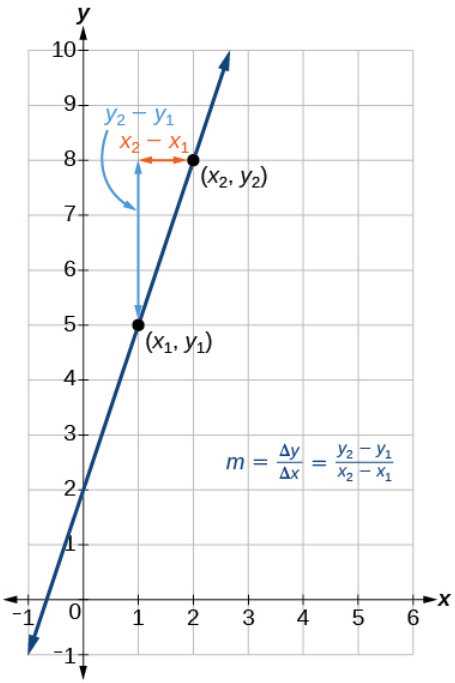


Figure 3.1.6: The slope of a function is calculated by the change in y divided by the change in x. It does not matter which coordinate is used as the (x_2, y_2) and which is the (x_1, y_1),as long as each calculation is started with the elements from the same coordinate pair.

Q & A

Are the units for slope always $\frac{\text{units for the output}}{\text{units for the input}}$?

Yes. Think of the units as the change of output value for each unit of change in input value. An example of slope could be miles per hour or dollars per day. Notice the units appear as a ratio of units for the output per units for the input.

Calculate Slope

The slope, or rate of change, of a function m can be calculated according to the following:

$$m = \frac{\text{change in output (rise)}}{\text{change in input (run)}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \tag{3.1.3}$$

where x_1 and x_2 are input values, y_1 and y_2 are output values.

How To: Given two points from a linear function, calculate and interpret the slope.

1. Determine the units for output and input values.
2. Calculate the change of output values and change of input values.
3. Interpret the slope as the change in output values per unit of the input value.

Example : Finding the Slope of a Linear Function

If $f(x)$ is a linear function, and $(3, -2)$ and $(8, 1)$ are points on the line, find the slope. Is this function increasing or decreasing?

Solution

The coordinate pairs are $(3, -2)$ and $(8, 1)$. To find the rate of change, we divide the change in output by the change in input.

$$m = \frac{\text{change in output (rise)}}{\text{change in input (run)}} = \frac{1 - (-2)}{8 - 3} = \frac{3}{5} \tag{3.1.4}$$

We could also write the slope as $m = 0.6$. The function is increasing because $m > 0$.

Analysis

As noted earlier, the order in which we write the points does not matter when we compute the slope of the line as long as the first output value, or *y*-coordinate, used corresponds with the first input value, or *x*-coordinate, used.

? Try it! 3.1.1

If $f(x)$ is a linear function, and $(2, 3)$ and $(0, 4)$ are points on the line, find the slope. Is this function increasing or decreasing?

Answer

$m = \frac{4-3}{0-2} = \frac{1}{-2} = -\frac{1}{2}$; decreasing because $m < 0$.

✓ Example : Finding the Population Change from a Linear Function

The population of a city increased from 23,400 to 27,800 between 2008 and 2012. Find the change of population per year if we assume the change was constant from 2008 to 2012.

The rate of change relates the change in population to the change in time. The population increased by $27,800 - 23,400 = 4400$ people over the four-year time interval. To find the rate of change, divide the change in the number of people by the number of years.

$$\frac{4,400 \text{ people}}{4 \text{ years}} = 1,100 \frac{\text{people}}{\text{year}} \tag{3.1.5}$$

So the population increased by 1,100 people per year.

Analysis

Because we are told that the population increased, we would expect the slope to be positive. This positive slope we calculated is therefore reasonable.

? Try it! 3.1.2

The population of a small town increased from 1,442 to 1,868 between 2009 and 2012. Find the change of population per year if we assume the change was constant from 2009 to 2012.

Answer

$m = \frac{1,868-1,442}{2,012-2,009} = \frac{426}{3} = 142$ people per year

Writing the Point-Slope Form of a Linear Equation

Up until now, we have been using the slope-intercept form of a linear equation to describe linear functions. Here, we will learn another way to write a linear function, the **point-slope form**.

$$y - y_1 = m(x - x_1) \tag{3.1.6}$$

The point-slope form is derived from the slope formula.

$$\begin{aligned} m &= \frac{y - y_1}{x - x_1} && \text{assuming } x \neq x_1 \\ m(x - x_1) &= \frac{y - y_1}{x - x_1}(x - x_1) && \text{Multiply both sides by } (x - x_1). \\ m(x - x_1) &= y - y_1 && \text{Simplify} \\ y - y_1 &= m(x - x_1) && \text{Rearrange} \end{aligned}$$

Keep in mind that the slope-intercept form and the point-slope form can be used to describe the same function. We can move from one form to another using basic algebra. For example, suppose we are given an equation in point-slope form, $y-4=-\frac{1}{2}(x-6)$. We can convert it to the slope-intercept form as shown.

$$\begin{aligned} y-4 &= -\frac{1}{2}(x-6) \\ y-4 &= -\frac{1}{2}x+3 \qquad \text{Distribute the } -\frac{1}{2}. \\ y &= -\frac{1}{2}x+7 \qquad \text{Add 4 to each side.} \end{aligned}$$

Therefore, the same line can be described in slope-intercept form as $y=\frac{1}{2}x+7$.

Point-Slope Form of a Linear Equation

The **point-slope** form of a linear equation takes the form

$$y-y_1=m(x-x_1) \tag{3.1.7}$$

where m is the slope, x_1 and y_1 are the x and y coordinates of a specific point through which the line passes.

Writing the Equation of a Line Using a Point and the Slope

The point-slope form is particularly useful if we know one point and the slope of a line. Suppose, for example, we are told that a line has a slope of 2 and passes through the point $(4,1)$. We know that $m=2$ and that $x_1=4$ and $y_1=1$. We can substitute these values into the general point-slope equation.

$$\begin{aligned} y-y_1 &= m(x-x_1) \\ y-1 &= 2(x-4) \end{aligned}$$

If we wanted to then rewrite the equation in slope-intercept form, we apply algebraic techniques.

$$\begin{aligned} y-1 &= 2(x-4) \\ y-1 &= 2x-8 \qquad \text{Distribute the 2.} \\ y &= 2x-7 \qquad \text{Add 1 to each side.} \end{aligned}$$

Both equations, $y-1=2(x-4)$ and $y=2x-7$, describe the same line. See Figure 3.1.7.

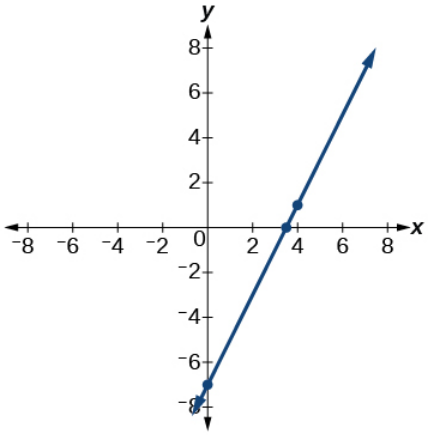


Figure 3.1.7

Example : Writing Linear Equations Using a Point and the Slope

Write the point-slope form of an equation of a line with a slope of 3 that passes through the point $(6,-1)$. Then rewrite it in the slope-intercept form.

Solution

Let's figure out what we know from the given information. The slope is 3, so $m = 3$. We also know one point, so we know $x_1 = 6$ and $y_1 = -1$. Now we can substitute these values into the general point-slope equation.

$$\begin{aligned} y - y_1 &= m(x - x_1) & \\ y - (-1) &= 3(x - 6) & \text{Substitute known values.} \\ y + 1 &= 3(x - 6) & \text{Distribute } -1 \text{ to find point-slope form.} \end{aligned}$$

Then we use algebra to find the slope-intercept form.

$$\begin{aligned} y + 1 &= 3(x - 6) & \\ y + 1 &= 3x - 18 & \text{Distribute 3.} \\ y &= 3x - 19 & \text{Simplify to slope-intercept form.} \end{aligned}$$

? Try it! 3.1.3

Write the point-slope form of an equation of a line with a slope of –2 that passes through the point $(-2, 2)$. Then rewrite it in the slope-intercept form.

Answer

$y - 2 = -2(x + 2)$; $y = -2x - 2$

Writing the Equation of a Line Using Two Points

The point-slope form of an equation is also useful if we know any two points through which a line passes. Suppose, for example, we know that a line passes through the points $(0, 1)$ and $(3, 2)$. We can use the coordinates of the two points to find the slope.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{2 - 1}{3 - 0} \\ &= \frac{1}{3} \end{aligned}$$

Now we can use the slope we found and the coordinates of one of the points to find the equation for the line. Let use $(0, 1)$ for our point.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 1 &= \frac{1}{3}(x - 0) \end{aligned}$$

As before, we can use algebra to rewrite the equation in the slope-intercept form.

$$\begin{aligned} y - 1 &= \frac{1}{3}(x - 0) & \\ y - 1 &= \frac{1}{3}x & \text{Distribute the } \frac{1}{3}. \\ y &= \frac{1}{3}x + 1 & \text{Add 1 to each side.} \end{aligned}$$

Both equations describe the line shown in Figure 3.1.8.

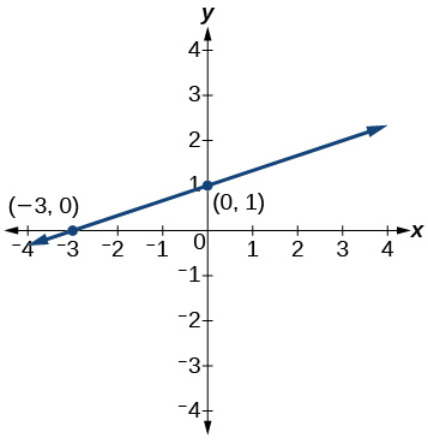


Figure 3.1.8

Example : Writing Linear Equations Using Two Points

Write the point-slope form of an equation of a line that passes through the points $(5, 1)$ and $(8, 7)$. Then rewrite it in the slope-intercept form.

Let's begin by finding the slope.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{7-1}{8-5} \\ &= \frac{6}{3} \\ &= 2 \end{aligned}$$

So $m = 2$. Next, we substitute the slope and the coordinates for one of the points into the general point-slope equation. We can choose either point, but we will use $(5, 1)$.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 1 &= 2(x - 5) \end{aligned}$$

The point-slope equation of the line is $y_2 - 1 = 2(x_2 - 5)$. To rewrite the equation in slope-intercept form, we use algebra.

$$\begin{aligned} y - 1 &= 2(x - 5) \\ y - 1 &= 2x - 10 \\ y &= 2x - 9 \end{aligned}$$

The slope-intercept equation of the line is $y = 2x - 9$.

3.1.4: Write the point-slope form of an equation of a line that passes through the points $(-1, 3)$ and $(0, 0)$. Then rewrite it in the slope-intercept form.

Solution

$y - 0 = -3(x - 0)$; $y = -3x$

Writing and Interpreting an Equation for a Linear Function

Now that we have written equations for linear functions in both the slope-intercept form and the point-slope form, we can choose which method to use based on the information we are given. That information may be provided in the form of a graph, a point and a slope, two points, and so on. Look at the graph of the function f in Figure 3.1.9.

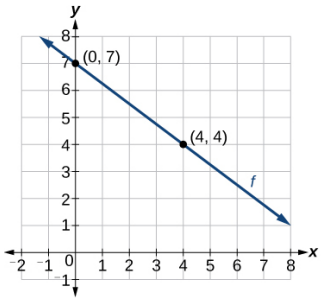


Figure 3.1.9: Graph depicting how to calculate the slope of a line.

We are not given the slope of the line, but we can choose any two points on the line to find the slope. Let's choose $(0, 7)$ and $(4, 4)$. We can use these points to calculate the slope.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{4-7}{4-0} \\ &= -\frac{3}{4} \end{aligned}$$

Now we can substitute the slope and the coordinates of one of the points into the point-slope form.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 4 &= -\frac{3}{4}(x-4) \end{aligned}$$

If we want to rewrite the equation in the slope-intercept form, we would find

$$\begin{aligned} y - 4 &= -\frac{3}{4}(x-4) \\ y - 4 &= -\frac{3}{4}x + 3 \\ y &= -\frac{3}{4}x + 7 \end{aligned}$$

If we wanted to find the slope-intercept form without first writing the point-slope form, we could have recognized that the line crosses the y-axis when the output value is 7. Therefore, $b = 7$. We now have the initial value b and the slope m so we can substitute m and b into the slope-intercept form of a line.

$$f(x) = mx + b$$

$-\frac{3}{4}$ $\quad$ 7

$$f(x) = -\tfrac{3}{4}x + 7$$

Figure 3.1.10

So the function is $f(x) = -\frac{3}{4}x + 7$, and the linear equation would be $y = -\frac{3}{4}x + 7$.

How To: Given the graph of a linear function, write an equation to represent the function.

1. Identify two points on the line.
2. Use the two points to calculate the slope.
3. Determine where the line crosses the y-axis to identify the y-intercept by visual inspection.
4. Substitute the slope and y-intercept into the slope-intercept form of a line equation.

Example : Writing an Equation for a Linear Function

Write an equation for a linear function given a graph of f shown in Figure 3.1.11.

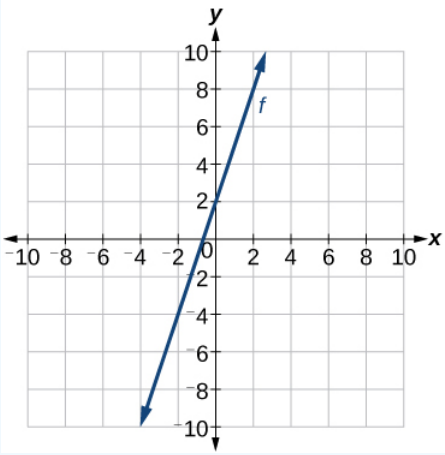


Figure 3.1.11

Solution

Identify two points on the line, such as $(0, 2)$ and $(-2, -4)$. Use the points to calculate the slope.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{4-2}{-2-0} \\ &= \frac{-6}{-2} \\ &= 3 \end{aligned}$$

Substitute the slope and the coordinates of one of the points into the point-slope form.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - (-4) &= 3(x - (-2)) \\ y + 4 &= 3(x + 2) \end{aligned}$$

We can use algebra to rewrite the equation in the slope-intercept form.

$$\begin{aligned} y + 4 &= 3(x + 2) \\ y + 4 &= 3x + 6 \\ y &= 3x + 2 \end{aligned}$$

Analysis

This makes sense because we can see from Figure 3.1.12 that the line crosses the y-axis at the point $(0, 2)$, which is the y-intercept, so $b = 2$.

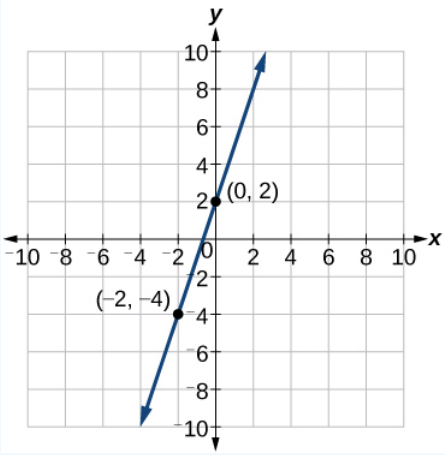


Figure 3.1.12: Graph of an increasing line with points at $(0, 2)$ and $(-2, -4)$.

Example : Writing an Equation for a Linear Cost Function

Suppose Ben starts a company in which he incurs a fixed cost of $1,250 per month for the overhead, which includes his office rent. His production costs are $37.50 per item. Write a linear function C where $C(x)$ is the cost for x items produced in a given month.

Solution

The fixed cost is present every month, $1,250. The costs that can vary include the cost to produce each item, which is $37.50 for Ben. The variable cost, called the marginal cost, is represented by 37.5. The cost Ben incurs is the sum of these two costs, represented by $C(x) = 1250 + 37.5x$.

Analysis

If Ben produces 100 items in a month, his monthly cost is represented by

$$\begin{aligned} C(100) &= 1250 + 37.5(100) \\ &= 5000 \end{aligned}$$

So his monthly cost would be $5,000.

Example : Writing an Equation for a Linear Function Given Two Points

If f is a linear function, with $f(3) = -2$, and $f(8) = 1$, find an equation for the function in slope-intercept form.

Solution

We can write the given points using coordinates.

$$\begin{aligned} f(3) = -2 &\rightarrow (3, 2) \\ f(8) = 1 &\rightarrow (8, 1) \end{aligned}$$

We can then use the points to calculate the slope.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{1 - (-2)}{8 - 3} \\ &= \frac{3}{5} \end{aligned}$$

Substitute the slope and the coordinates of one of the points into the point-slope form.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - (-2) &= \frac{3}{5}(x - 3) \end{aligned}$$

We can use algebra to rewrite the equation in the slope-intercept form.

$$\begin{aligned} y + 2 &= \frac{3}{5}(x - 3) \\ y + 2 &= \frac{3}{5}x - \frac{9}{5} \\ y &= \frac{3}{5}x - \frac{19}{5} \end{aligned}$$

? Tri It! 3.1.1

If $f(x)$ is a linear function, with $f(2) = -11$, and $f(4) = -25$, find an equation for the function in slope-intercept form.

Answer

$y = -7x + 3$

Modeling Real-World Problems with Linear Functions

In the real world, problems are not always explicitly stated in terms of a function or represented with a graph. Fortunately, we can analyze the problem by first representing it as a linear function and then interpreting the components of the function. As long as we know, or can figure out, the initial value and the rate of change of a linear function, we can solve many different kinds of real-world problems.

How To: Given a linear function f and the initial value and rate of change, evaluate $f(c)$.

1. Determine the initial value and the rate of change (slope).
2. Substitute the values into $f(x) = mx + b$.
3. Evaluate the function at $x = c$.

✓ Example : Using a Linear Function to Determine the number of Songs in a Music Collection

Marcus currently has 200 songs in his music collection. Every month, he adds 15 new songs. Write a formula for the number of songs, N, in his collection as a function of time, t, the number of months. How many songs will he own in a year?

Solution

The initial value for this function is 200 because he currently owns 200 songs, so $N(0) = 200$, which means that $b = 200$.

The number of songs increases by 15 songs per month, so the rate of change is 15 songs per month. Therefore we know that $m = 15$. We can substitute the initial value and the rate of change into the slope-intercept form of a line.

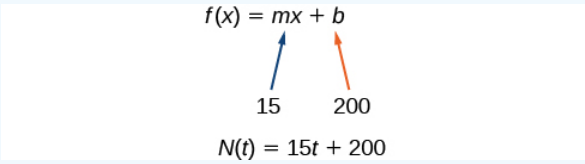


Figure 3.1.13

We can write the formula $N(t) = 15t + 200$.

With this formula, we can then predict how many songs Marcus will have in 1 year (12 months). In other words, we can evaluate the function at $t = 12$.

$$\begin{aligned} N(12) &= 15(12) + 200 \\ &= 180 + 200 \\ &= 380 \end{aligned}$$

Marcus will have 380 songs in 12 months.

Analysis

Notice that N is an increasing linear function. As the input (the number of months) increases, the output (number of songs) increases as well.

Example : Using a Linear Function to Calculate Salary Plus Commission

Working as an insurance salesperson, Ilya earns a base salary plus a commission on each new policy. Therefore, Ilya's weekly income, I,depends on the number of new policies, n, he sells during the week. Last week he sold 3 new policies, and earned \$760 for the week. The week before, he sold 5 new policies and earned \$920. Find an equation for $I(n)$, and interpret the meaning of the components of the equation.

Solution

The given information gives us two input-output pairs: $(3, 760)$ and $(5, 920)$. We start by finding the rate of change.

$$\begin{aligned} m &= \frac{920-760}{5-3} \\ &= \frac{\$160}{2 \text{ policies}} \\ &= \$80 \text{ per policy} \end{aligned}$$

Keeping track of units can help us interpret this quantity. Income increased by \$160 when the number of policies increased by 2, so the rate of change is \$80 per policy. Therefore, Ilya earns a commission of \$80 for each policy sold during the week.

We can then solve for the initial value.

$$\begin{aligned} I(n) &= 80n+b \\ 760 &= 80(3)+b \text{ when } n=3, I(3)=760 \\ 760-80(3) &= b \\ 520 &= b \end{aligned}$$

The value of b is the starting value for the function and represents Ilya's income when $n=0$, or when no new policies are sold. We can interpret this as Ilya's base salary for the week, which does not depend upon the number of policies sold.

We can now write the final equation.

$$I(n) = 80n+520$$

Our final interpretation is that Ilya's base salary is \$520 per week and he earns an additional \$80 commission for each policy sold.

Example : Using Tabular Form to Write an Equation for a Linear Function

Table $3.1.1$ relates the number of rats in a population to time, in weeks. Use the table to write a linear equation.

Table 3.1.1

w, number of weeks	0	2	4	6
P(w), number of rats	1000	1080	1160	1240

Solution

We can see from the table that the initial value for the number of rats is 1000, so $b = 1000$.

Rather than solving for m, we can tell from looking at the table that the population increases by 80 for every 2 weeks that pass. This means that the rate of change is 80 rats per 2 weeks, which can be simplified to 40 rats per week.

$$P(w) = 40w+1000$$

If we did not notice the rate of change from the table we could still solve for the slope using any two points from the table. For example, using $(2, 1080)$ and $(6, 1240)$

$$\begin{aligned} m &= \frac{1240 - 1080}{6 - 2} \\ &= \frac{160}{4} \\ &= 40 \end{aligned}$$

Is the initial value always provided in a table of values like Table 3.1.1?

No. Sometimes the initial value is provided in a table of values, but sometimes it is not. If you see an input of 0, then the initial value would be the corresponding output. If the initial value is not provided because there is no value of input on the table equal to 0, find the slope, substitute one coordinate pair and the slope into $f(x) = mx + b$, and solve for b.

3.1.5: A new plant food was introduced to a young tree to test its effect on the height of the tree. Table 3.1.2 shows the height of the tree, in feet, x months since the measurements began. Write a linear function, $H(x)$, where x is the number of months since the start of the experiment.

Table 3.1.2

x	0	2	4	8	12
$H(x)$	12.5	13.5	14.5	16.5	18.5

Solution

$H(x) = 0.5x + 12.5$

Key Equations

- slope-intercept form of a line: $f(x) = mx + b$
- slope: $m = \dfrac{\text{change in output (rise)}}{\text{change in input (run)}} = \dfrac{\Delta y}{\Delta x} = \dfrac{y_2 - y_1}{x_2 - x_1}$
- point-slope form of a line: $y - y_1 = m(x - x_1)$

Key Concepts

- The ordered pairs given by a linear function represent points on a line.
- Linear functions can be represented in words, function notation, tabular form, and graphical form.
- The rate of change of a linear function is also known as the slope.
- An equation in the slope-intercept form of a line includes the slope and the initial value of the function.
- The initial value, or y-intercept, is the output value when the input of a linear function is zero. It is the y-value of the point at which the line crosses the y-axis.
- An increasing linear function results in a graph that slants upward from left to right and has a positive slope.
- A decreasing linear function results in a graph that slants downward from left to right and has a negative slope.
- A constant linear function results in a graph that is a horizontal line.
- Analyzing the slope within the context of a problem indicates whether a linear function is increasing, decreasing, or constant.
- The slope of a linear function can be calculated by dividing the difference between y-values by the difference in corresponding x-values of any two points on the line.
- The slope and initial value can be determined given a graph or any two points on the line.
- One type of function notation is the slope-intercept form of an equation.
- The point-slope form is useful for finding a linear equation when given the slope of a line and one point.
- The point-slope form is also convenient for finding a linear equation when given two points through which a line passes.
- The equation for a linear function can be written if the slope m and initial value b are known.
- A linear function can be used to solve real-world problems.
- A linear function can be written from tabular form.

Footnotes

1 www.chinahighlights.com/shang...glev-train.htm
2 www.cbsnews.com/8301-501465_1...ay-study-says/

Glossary

decreasing linear function

a function with a negative slope: If $f(x) = mx + b$, then $m < 0$.

increasing linear function

a function with a positive slope: If $f(x) = mx + b$, then $m > 0$.

linear function

a function with a constant rate of change that is a polynomial of degree 1, and whose graph is a straight line

point-slope form

the equation for a line that represents a linear function of the form \(y−y_1=m(x−x_1)

slope

the ratio of the change in output values to the change in input values; a measure of the steepness of a line

slope-intercept form

the equation for a line that represents a linear function in the form $f(x) = mx + b$

y-intercept

the value of a function when the input value is zero; also known as initial value

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3.2: Graphs of Linear Functions

Two competing telephone companies offer different payment plans. The two plans charge the same rate per long distance minute, but charge a different monthly flat fee. A consumer wants to determine whether the two plans will ever cost the same amount for a given number of long distance minutes used. The total cost of each payment plan can be represented by a linear function. To solve the problem, we will need to compare the functions. In this section, we will consider methods of comparing functions using graphs.

Graphing Linear Functions

Previously, we saw that that the graph of a linear function is a straight line. We were also able to see the points of the function as well as the initial value from a graph. By graphing two functions, then, we can more easily compare their characteristics. There are three basic methods of graphing linear functions:

1. Plot the points and then drawing a line through the points.
2. Use the y-intercept and slope.
3. Use transformations of the identity function $f(x) = x$.

Graphing a Function by Plotting Points

To find points of a function, we can choose input values, evaluate the function at these input values, and calculate output values. The input values and corresponding output values form coordinate pairs. We then plot the coordinate pairs on a grid. In general, we should evaluate the function at a minimum of two inputs in order to find at least two points on the graph. For example, given the function, $f(x) = 2x$, we might use the input values 1 and 2. Evaluating the function for an input value of 1 yields an output value of 2, which is represented by the point $(1, 2)$. Evaluating the function for an input value of 2 yields an output value of 4, which is represented by the point $(2, 4)$. Choosing three points is often advisable because if all three points do not fall on the same line, we know we made an error.

How To: Given a linear function, graph by plotting points.

1. Choose a minimum of two input values.
2. Evaluate the function at each input value.
3. Use the resulting output values to identify coordinate pairs.
4. Plot the coordinate pairs on a grid.
5. Draw a line through the points.

Example 3.2.1: Graphing by Plotting Points

Graph $f(x) = -\frac{2}{3}x + 5$ by plotting points.

Solution

Begin by choosing input values. This function includes a fraction with a denominator of 3, so let's choose multiples of 3 as input values. We will choose 0, 3, and 6.

Evaluate the function at each input value, and use the output value to identify coordinate pairs.

$$x = 0 \qquad f(0) = -\frac{2}{3}(0) + 5 = 5 \rightarrow (0, 5)$$
$$x = 3 \qquad f(3) = -\frac{2}{3}(3) + 5 = 3 \rightarrow (3, 3)$$
$$x = 6 \qquad f(6) = -\frac{2}{3}(6) + 5 = 1 \rightarrow (6, 1)$$

Plot the coordinate pairs and draw a line through the points. Figure 3.2.1 represents the graph of the function $f(x) = -\frac{2}{3}x + 5$.

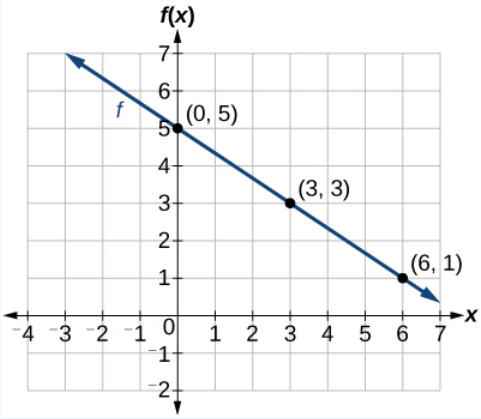


Figure 3.2.1: The graph of the linear function $f(x) = -\frac{2}{3}x + 5$.

Analysis

The graph of the function is a line as expected for a linear function. In addition, the graph has a downward slant, which indicates a negative slope. This is also expected from the negative constant rate of change in the equation for the function.

? Exercise 3.2.1

Graph $f(x) = -\frac{3}{4}x + 6$ by plotting points.

Answer

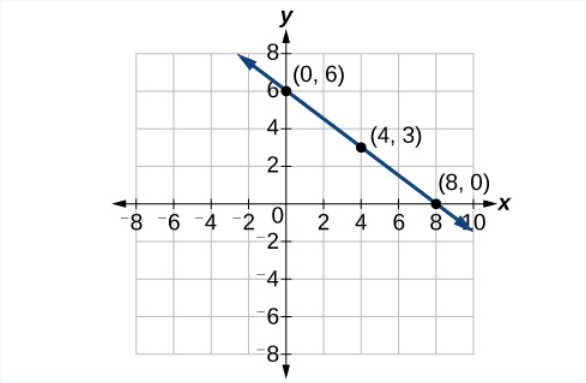


Figure 3.2.2

Graphing a Function Using y-intercept and Slope

Another way to graph linear functions is by using specific characteristics of the function rather than plotting points. The first characteristic is its **y-intercept**, which is the point at which the input value is zero. To find the y-intercept, we can set $x = 0$ in the equation.

The other characteristic of the linear function is its slope m, which is a measure of its steepness. Recall that the slope is the rate of change of the function. The slope of a function is equal to the ratio of the change in outputs to the change in inputs. Another way to think about the slope is by dividing the vertical difference, or rise, by the horizontal difference, or run. We encountered both the y-intercept and the slope in Linear Functions.

Let's consider the following function.

$$f(x) = \frac{1}{2}x + 1 \tag{3.2.1}$$

The slope is $\frac{1}{2}$. Because the slope is positive, we know the graph will slant upward from left to right. The y-intercept is the point on the graph when $x = 0$. The graph crosses the y-axis at $(0, 1)$. Now we know the slope and the y-intercept. We can begin graphing by plotting the point $(0, 1)$ We know that the slope is rise over run, $m = \frac{\text{rise}}{\text{run}}$. From our example, we have $m = \frac{1}{2}$, which means

that the rise is 1 and the run is 2. So starting from our y-intercept $(0, 1)$, we can rise 1 and then run 2, or run 2 and then rise 1. We repeat until we have a few points, and then we draw a line through the points as shown in Figure 3.2.3.

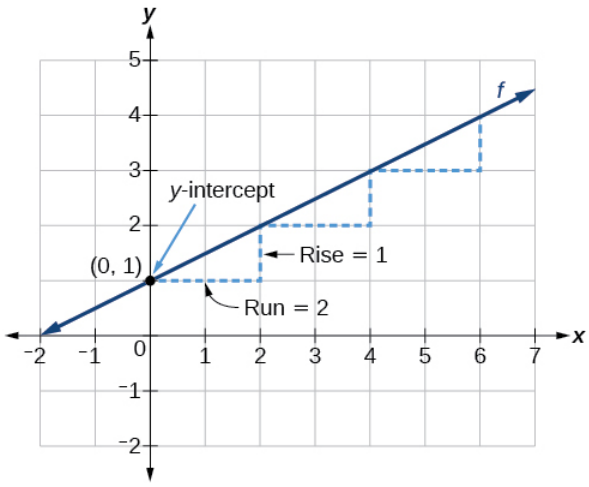


Figure 3.2.3: Graph of $f(x)$ showing slope and y-intercept

Graphical Interpretation of a Linear Function

In the equation $f(x) = mx + b$

- b is the y-intercept of the graph and indicates the point $(0, b)$ at which the graph crosses the y-axis.
- m is the slope of the line and indicates the vertical displacement (rise) and horizontal displacement (run) between each successive pair of points. Recall the formula for the slope:

$$m = \frac{\text{change in output (rise)}}{\text{change in input (run)}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \tag{3.2.2}$$

Exercise 3.2.1

Do all linear functions have y-intercepts?

Answer

Yes. All linear functions cross the y-axis and therefore have y-intercepts. (Note: A vertical line parallel to the y-axis does not have a y-intercept, but it is not a function.)

How To: Given the equation for a linear function, graph the function using the y-intercept and slope.

1. Evaluate the function at an input value of zero to find the y-intercept.
2. Identify the slope as the rate of change of the input value.
3. Plot the point represented by the y-intercept.
4. Use $\frac{\text{rise}}{\text{run}}$ to determine at least two more points on the line.
5. Sketch the line that passes through the points.

Example 3.2.2: Graphing by Using the y-intercept and Slope

Graph $f(x) = -\frac{2}{3}x + 5$ using the y-intercept and slope.

Solution

Evaluate the function at $x = 0$ to find the y-intercept. The output value when $x = 0$ is 5, so the graph will cross the y-axis at $(0, 5)$.

According to the equation for the function, the slope of the line is $-\frac{2}{3}$. This tells us that for each vertical decrease in the "rise" of –2 units, the "run" increases by 3 units in the horizontal direction. We can now graph the function by first plotting the y-intercept on the graph in Figure 3.2.4. From the initial value $(0, 5)$ we move down 2 units and to the right 3 units. We can extend the line to the left and right by repeating, and then draw a line through the points.

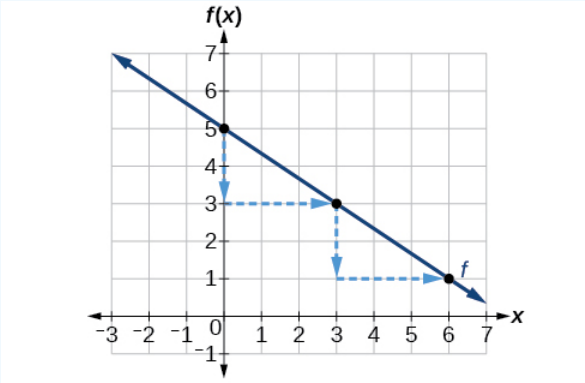


Figure 3.2.4: Graph of $f(x)$

Analysis

The graph slants downward from left to right, which means it has a negative slope as expected.

? Exercise 3.2.2

Find a point on the graph we drew in Example 3.2.2 that has a negative x-value.

Answer

Possible answers include $(-3, 7)$, $(-6, 9)$, or $(-9, 11)$.

Graphing a Function Using Transformations

Another option for graphing is to use **transformations** of the identity function $f(x) = x$. A function may be transformed by a shift up, down, left, or right. A function may also be transformed using a reflection, stretch, or compression.

Vertical Stretch or Compression

In the equation $f(x) = mx$, the m is acting as the **vertical stretch** or **compression** of the identity function. When m is negative, there is also a vertical reflection of the graph. Notice in Figure 3.2.5 that multiplying the equation of $f(x) = x$ by m stretches the graph of f by a factor of m units if $m > 1$ and compresses the graph of f by a factor of m units if $0 < m < 1$. This means the larger the absolute value of m, the steeper the slope.

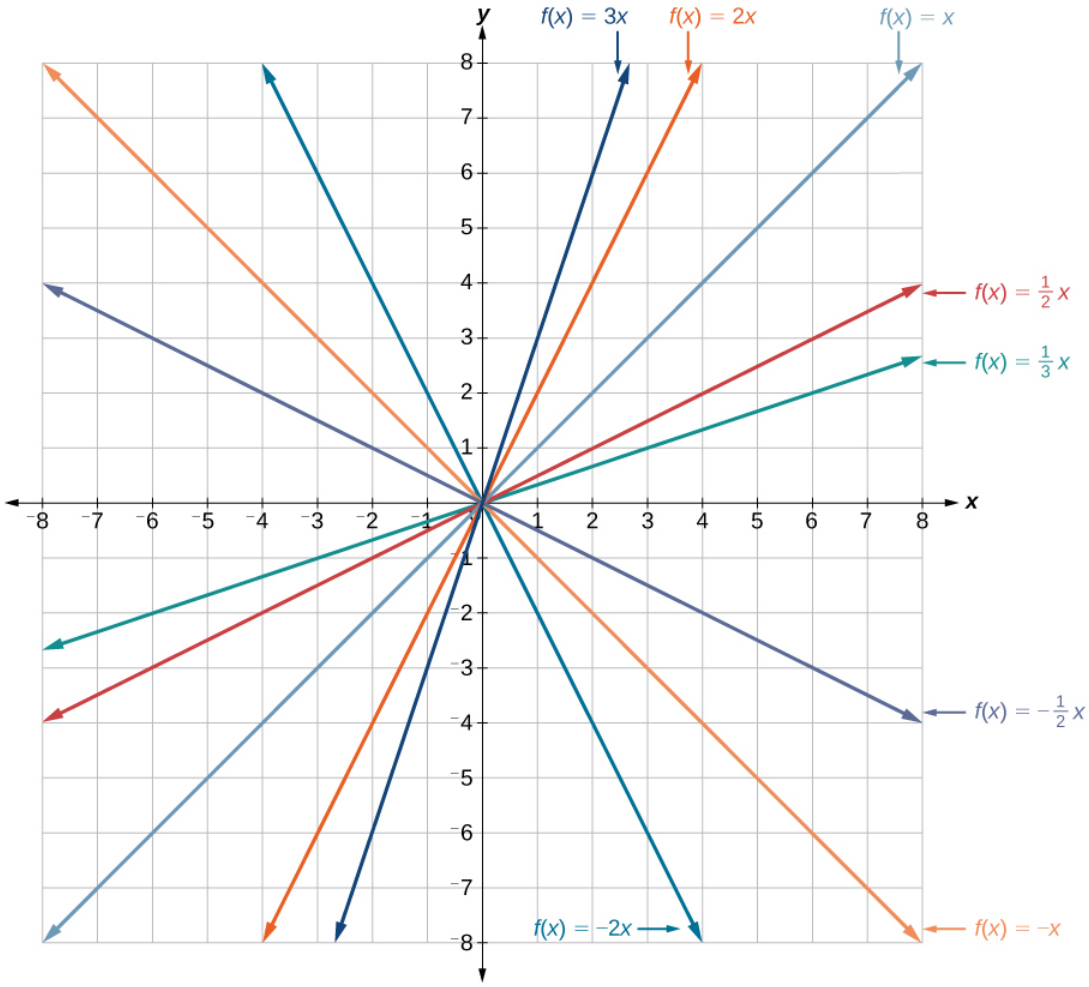


Figure 3.2.5: Vertical stretches and compressions and reflections on the function $f(x) = x$.

Vertical Shift

In $f(x) = mx + b$, the b acts as the **vertical shift**, moving the graph up and down without affecting the slope of the line. Notice in Figure 3.2.6 that adding a value of b to the equation of $f(x) = x$ shifts the graph of f a total of b units up if b is positive and $|b|$ units down if b is negative.

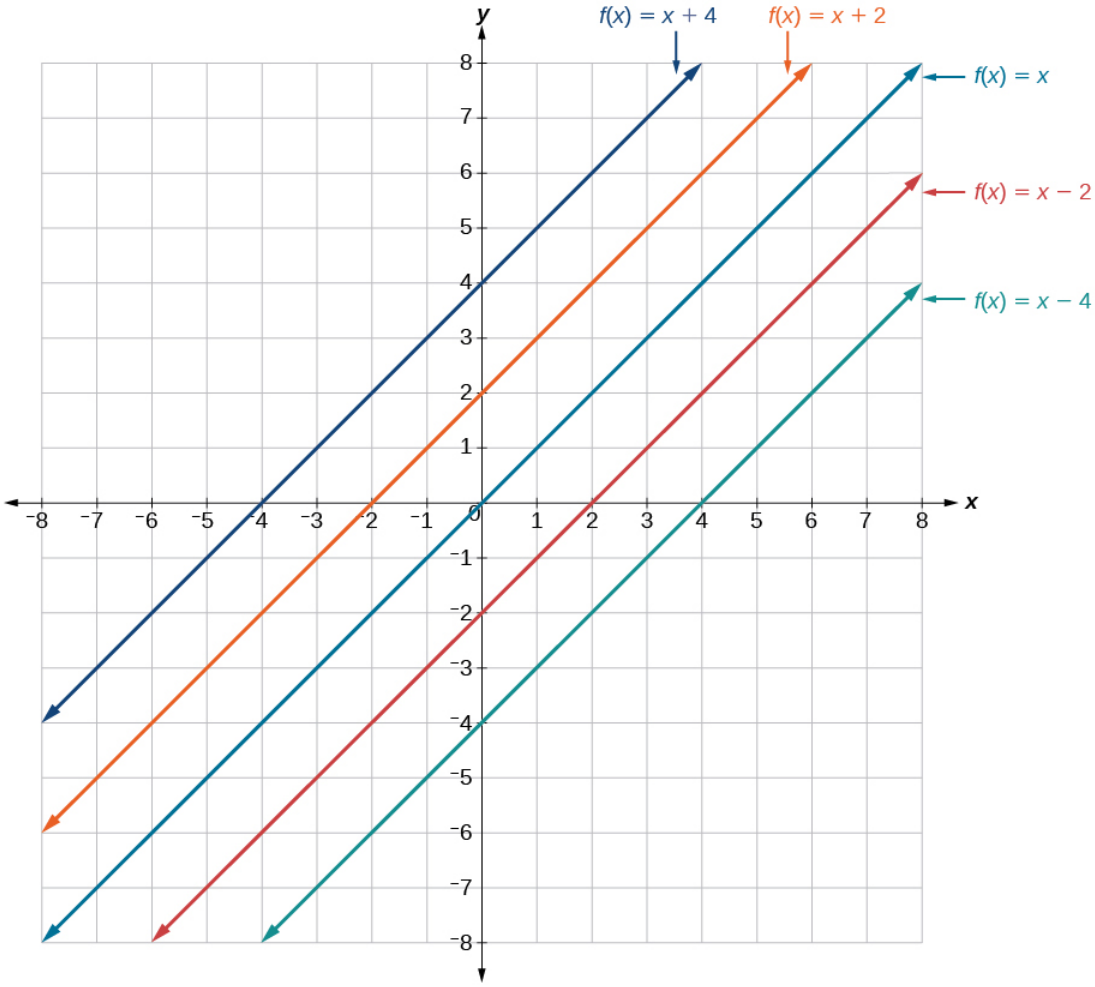


Figure 3.2.6: This graph illustrates vertical shifts of the function $f(x) = x$.

Using vertical stretches or compressions along with vertical shifts is another way to look at identifying different types of linear functions. Although this may not be the easiest way to graph this type of function, it is still important to practice each method.

How To: Given the equation of a linear function, use transformations to graph the linear function in the form $f(x) = mx + b$.

1. Graph $f(x) = x$.
2. Vertically stretch or compress the graph by a factor m.
3. Shift the graph up or down b units.

Example 3.2.3: Graphing by Using Transformations

Graph $f(x) = \frac{1}{2}x - 3$ using transformations.

Solution

The equation for the function shows that $m = \frac{1}{2}$ so the identity function is vertically compressed by $\frac{1}{2}$. The equation for the function also shows that $b = -3$ so the identity function is vertically shifted down 3 units. First, graph the identity function, and show the vertical compression as in Figure 3.2.7.

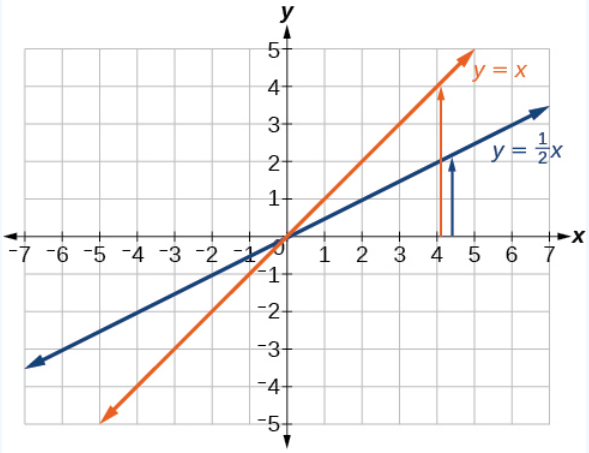


Figure 3.2.7: The function, $y = x$, compressed by a factor of $\frac{1}{2}$.

Then show the vertical shift as in Figure 3.2.8.

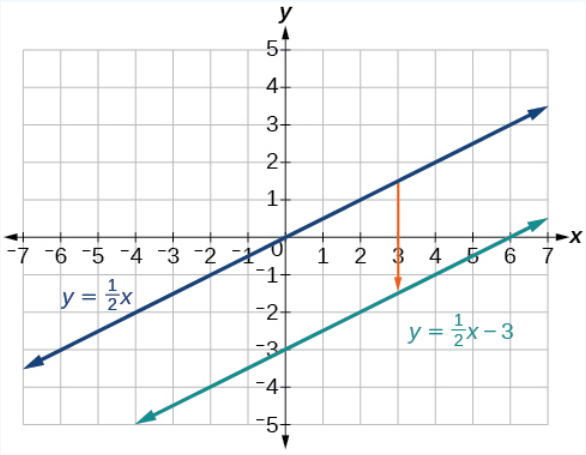


Figure 3.2.8: The function $y = \frac{1}{2}x$, shifted down 3 units.

? Exercise 3.2.3

Graph $f(x) = 4 + 2x$, using transformations.

Answer

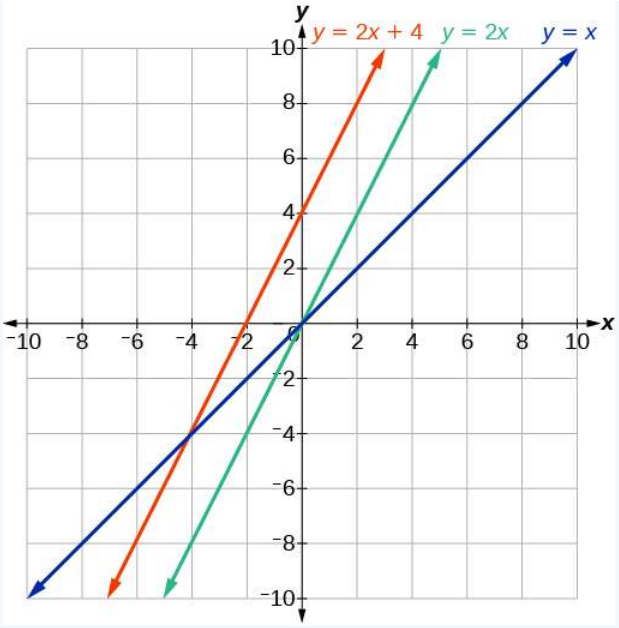


Figure 3.2.9

In Example 2.2.3, could we have sketched the graph by reversing the order of the transformations?

No. The order of the transformations follows the order of operations. When the function is evaluated at a given input, the corresponding output is calculated by following the order of operations. This is why we performed the compression first. For example, following the order: Let the input be 2.

$$f(2) = \frac{1}{2}(2) - 3 \tag{3.2.3}$$
$$= 1 - 3 \tag{3.2.4}$$
$$= -2 \tag{3.2.5}$$

Writing the Equation for a Function from the Graph of a Line

Recall that in Linear Functions, we wrote the equation for a linear function from a graph. Now we can extend what we know about graphing linear functions to analyze graphs a little more closely. Begin by taking a look at Figure 3.2.10. We can see right away that the graph crosses the y-axis at the point $(0, 4)$ so this is the y-intercept.

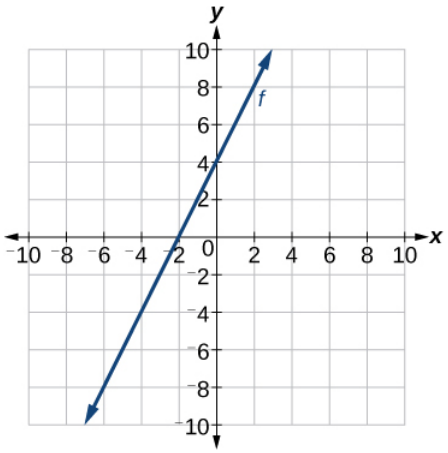


Figure 3.2.10: Graph of a linear function

Then we can calculate the slope by finding the rise and run. We can choose any two points, but let's look at the point (−2,0). To get from this point to the y-intercept, we must move up 4 units (rise) and to the right 2 units (run). So the slope must be

$$m = \frac{\text{rise}}{\text{run}} = \frac{4}{2} = 2 \tag{3.2.6}$$

Substituting the slope and y-intercept into the slope-intercept form of a line gives

$$y = 2x + 4 \tag{3.2.7}$$

How To: Given a graph of linear function, find the equation to describe the function.

1. Identify the y-intercept of an equation.
2. Choose two points to determine the slope.
3. Substitute the y-intercept and slope into the slope-intercept form of a line.

Example 3.2.4: Matching Linear Functions to Their Graphs

Match each equation of the linear functions with one of the lines in Figure 3.2.11.

a. $f(x) = 2x + 3$
b. $g(x) = 2x - 3$
c. $h(x) = -2x + 3$
d. $j(x) = \frac{1}{2}x + 3$

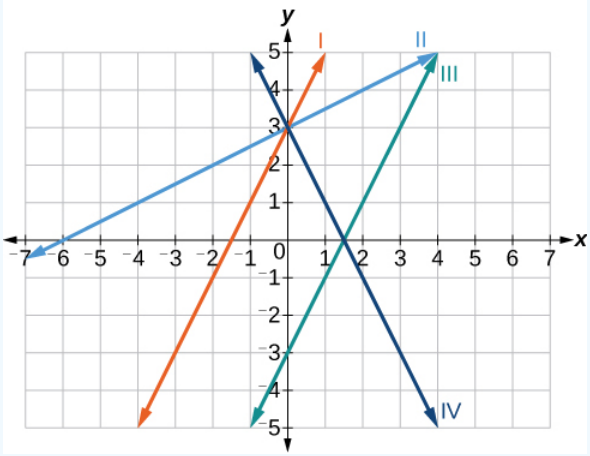


Figure 3.2.11: Graph of four linear functions

Solution

Analyze the information for each function.

a. This function has a slope of 2 and a y-intercept of 3. It must pass through the point $(0, 3)$ and slant upward from left to right. We can use two points to find the slope, or we can compare it with the other functions listed. Function g has the same slope, but a different y-intercept. Lines I and III have the same slant because they have the same slope. Line III does not pass through $(0, 3)$ so f must be represented by line I.
b. This function also has a slope of 2, but a y-intercept of −3. It must pass through the point $(0, -3)$ and slant upward from left to right. It must be represented by line III.
c. This function has a slope of –2 and a y-intercept of 3. This is the only function listed with a negative slope, so it must be represented by line IV because it slants downward from left to right.
d. This function has a slope of $\frac{1}{2}$ and a y-intercept of 3. It must pass through the point (0, 3) and slant upward from left to right. Lines I and II pass through $(0, 3)$, but the slope of j is less than the slope of f so the line for j must be flatter. This function is represented by Line II.

Now we can re-label the lines as in Figure 3.2.12

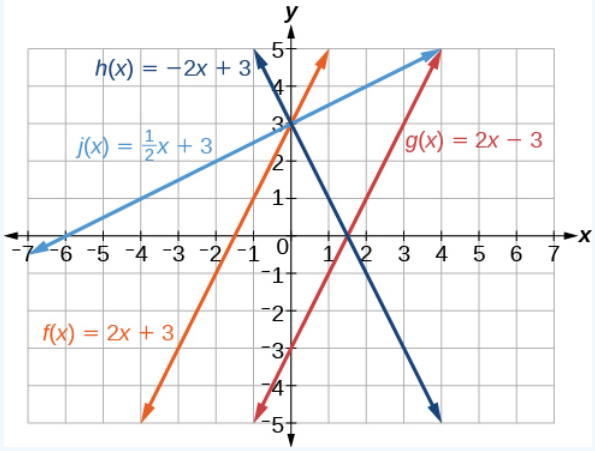


Figure 3.2.12

Finding the x-intercept of a Line

So far, we have been finding the y-intercepts of a function: the point at which the graph of the function crosses the y-axis. A function may also have an x-intercept, which is the x-coordinate of the point where the graph of the function crosses the x-axis. In other words, it is the input value when the output value is zero.

To find the x-intercept, set a function $f(x)$ equal to zero and solve for the value of x. For example, consider the function shown.

$$f(x) = 3x - 6 \tag{3.2.8}$$

Set the function equal to 0 and solve for x.

$$0 = 3x - 6 \tag{3.2.9}$$
$$6 = 3x \tag{3.2.10}$$
$$2 = x \tag{3.2.11}$$
$$x = 2 \tag{3.2.12}$$

The graph of the function crosses the x-axis at the point $(2, 0)$.

Do all linear functions have x-intercepts?

Figure 3.2.13.

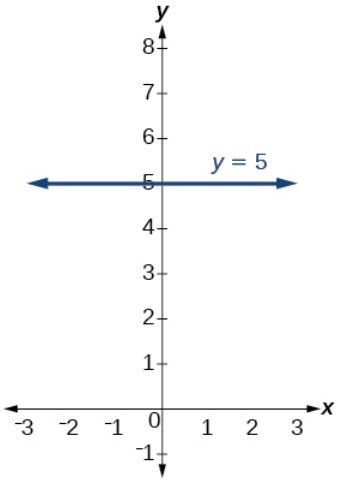


Figure 3.2.13: Graph of $y = 5$.

Note: x-intercept

The **x-intercept** of the function is value of x when $f(x) = 0$. It can be solved by the equation $0 = mx + b$.

Example 3.2.5: Finding an x-intercept

Find the x-intercept of $f(x) = \frac{1}{2} - 3$.

Solution

Set the function equal to zero to solve for x.

$$\begin{aligned} 0 &= \frac{1}{2}x - 3 \\ 3 &= \frac{1}{2}x \\ 6 &= x \\ x &= 6 \end{aligned}$$

The graph crosses the x-axis at the point $(6, 0)$.

Analysis

A graph of the function is shown in Figure 3.2.14 We can see that the x-intercept is $(6, 0)$ as we expected.

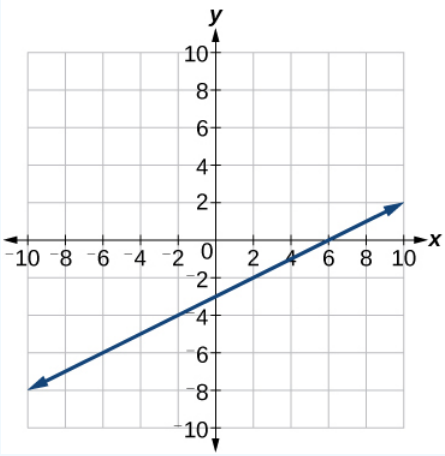


Figure 3.2.14: The graph of the linear function $f(x) = \frac{1}{2}x - 3$.

Exercise 3.2.5

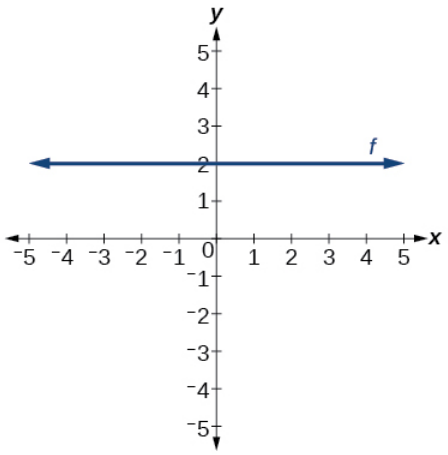
Find the x-intercept of $f(x) = \frac{1}{4}x - 4$.

Answer

$(16, 0)$

Describing Horizontal and Vertical Lines

There are two special cases of lines on a graph—horizontal and vertical lines. A **horizontal line** indicates a constant output, or y-value. In Figure 3.2.15, we see that the output has a value of 2 for every input value. The change in outputs between any two points, therefore, is 0. In the slope formula, the numerator is 0, so the slope is 0. If we use $m = 0$ in the equation $f(x) = mx + b$, the equation simplifies to $f(x) = b$. In other words, the value of the function is a constant. This graph represents the function $f(x) = 2$.



x	-4	-2	0	2	4
y	2	2	2	2	2

Figure 3.2.15: A horizontal line representing the function $f(x) = 2$.

A **vertical line** indicates a constant input, or x-value. We can see that the input value for every point on the line is 2, but the output value varies. Because this input value is mapped to more than one output value, a vertical line does not represent a function. Notice that between any two points, the change in the input values is zero. In the slope formula, the denominator will be zero, so the slope of a vertical line is undefined.

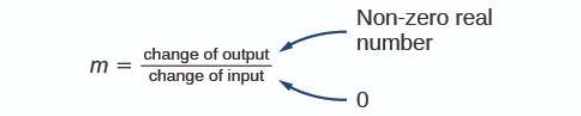
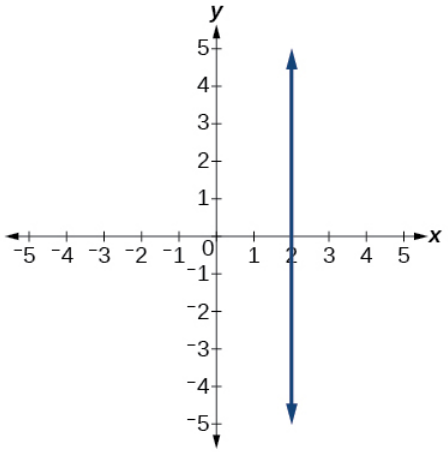


Figure 3.2.16

Notice that a vertical line, such as the one in Figure 3.2.17, has an x-intercept, but no y-intercept unless it's the line $x = 0$. This graph represents the line $x = 2$.



x	2	2	2	2	2
y	-4	-2	0	2	4

Figure 3.2.17: The vertical line, $x = 2$, which does not represent a function.

Definitions: Horizontal and Vertical Lines

Lines can be horizontal or vertical.

- A **horizontal line** is a line defined by an equation in the form $f(x) = b$.

- A **vertical line** is a line defined by an equation in the form $x = a$.

Example 3.2.6: Writing the Equation of a Horizontal Line

Write the equation of the line graphed in Figure 3.2.18.

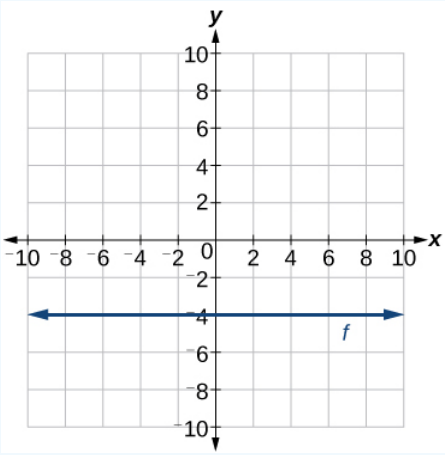


Figure 3.2.18: Graph of $x = 7$.

Solution

For any x-value, the y-value is -4, so the equation is $y = -4$.

Example 3.2.7: Writing the Equation of a Vertical Line

Write the equation of the line graphed in Figure 3.2.19.

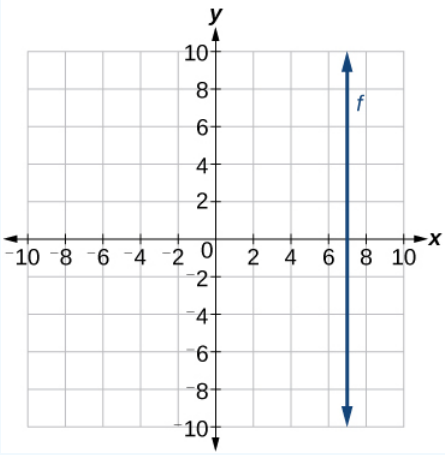


Figure 3.2.19: Graph of a vertical line

Solution

The constant x-value is 7, so the equation is $x = 7$.

Key Concepts

- Linear functions may be graphed by plotting points or by using the y-intercept and slope.
- Graphs of linear functions may be transformed by using shifts up, down, left, or right, as well as through stretches, compressions, and reflections.
- The y-intercept and slope of a line may be used to write the equation of a line.
- The x-intercept is the point at which the graph of a linear function crosses the x-axis.
- Horizontal lines are written in the form, $f(x) = b$.

- Vertical lines are written in the form, $x = b$.

Glossary

horizontal line
a line defined by $f(x) = b$, where b is a real number. The slope of a horizontal line is 0.

parallel lines
two or more lines with the same slope

perpendicular lines
two lines that intersect at right angles and have slopes that are negative reciprocals of each other

vertical line
a line defined by $x = a$, where a is a real number. The slope of a vertical line is undefined.

x-intercept
the point on the graph of a linear function when the output value is 0; the point at which the graph crosses the horizontal axis

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3.3: Modeling with Linear Functions

Identifying Steps to Model and Solve Problems

When **modeling** scenarios with linear functions and solving problems involving quantities with a **constant rate of change**, we typically follow the same problem strategies that we would use for any type of function. Let's briefly review them:

Identify changing quantities, and then define descriptive variables to represent those quantities. When appropriate, sketch a picture or define a coordinate system.
Carefully read the problem to identify important information. Look for information that provides values for the variables or values for parts of the functional model, such as slope and initial value.
Carefully read the problem to determine what we are trying to find, identify, solve, or interpret.
Identify a solution pathway from the provided information to what we are trying to find. Often this will involve checking and tracking units, building a table, or even finding a formula for the function being used to model the problem.
When needed, write a formula for the function.
Solve or evaluate the function using the formula.
Reflect on whether your answer is reasonable for the given situation and whether it makes sense mathematically.
Clearly convey your result using appropriate units, and answer in full sentences when necessary.

Building Linear Models

Now let's take a look at the student in Seattle. In her situation, there are two changing quantities: time and money. The amount of money she has remaining while on vacation depends on how long she stays. We can use this information to define our variables, including units.

- Output: M, money remaining, in dollars
- Input: t, time, in weeks

So, the amount of money remaining depends on the number of weeks: $M(t)$

We can also identify the initial value and the rate of change.

- Initial Value: She saved $3,500, so $3,500 is the initial value for M.
- Rate of Change: She anticipates spending $400 each week, so –$400 per week is the rate of change, or slope.

Notice that the unit of dollars per week matches the unit of our output variable divided by our input variable. Also, because the slope is negative, the linear function is decreasing. This should make sense because she is spending money each week.

The **rate of change is constant**, so we can start with the linear model $M(t) = mt + b$. Then we can substitute the intercept and slope provided.

$$M(t) = mt + b$$

$$-400 \qquad 3500$$

$$M(t) = -400t + 3500$$

Figure 3.3.2

To find the x-intercept, we set the output to zero, and solve for the input.

$$\begin{aligned} 0 &= -400t + 3500 \\ t &= \frac{3500}{400} \\ &= 8.75 \end{aligned}$$

The x-intercept is 8.75 weeks. Because this represents the input value when the output will be zero, we could say that Emily will have no money left after 8.75 weeks.

When modeling any real-life scenario with functions, there is typically a limited domain over which that model will be valid—almost no trend continues indefinitely. Here the domain refers to the number of weeks. In this case, it doesn't make sense to talk about input values less than zero. A negative input value could refer to a number of weeks before she saved $3,500, but the

scenario discussed poses the question once she saved $3,500 because this is when her trip and subsequent spending starts. It is also likely that this model is not valid after the x-intercept, unless Emily will use a credit card and goes into debt. The domain represents the set of input values, so the reasonable domain for this function is $0 \le t \le 8.75$.

In the above example, we were given a written description of the situation. We followed the steps of modeling a problem to analyze the information. However, the information provided may not always be the same. Sometimes we might be provided with an intercept. Other times we might be provided with an output value. We must be careful to analyze the information we are given, and use it appropriately to build a linear model.

Using a Given Intercept to Build a Model

Some real-world problems provide the y-intercept, which is the constant or initial value. Once the y-intercept is known, the x-intercept can be calculated. Suppose, for example, that Hannah plans to pay off a no-interest loan from her parents. Her loan balance is $1,000. She plans to pay $250 per month until her balance is $0. The y-intercept is the initial amount of her debt, or $1,000. The rate of change, or slope, is -$250 per month. We can then use the slope-intercept form and the given information to develop a linear model.

$$\begin{aligned} f(x) &= mx + b \\ &= -250x + 1000 \end{aligned}$$

Now we can set the function equal to 0, and solve for x to find the x-intercept.

$$\begin{aligned} 0 &= -250 + 1000 \\ 1000 &= 250x \\ 4 &= x \\ x &= 4 \end{aligned}$$

The x-intercept is the number of months it takes her to reach a balance of $0. The x-intercept is 4 months, so it will take Hannah four months to pay off her loan.

Using a Given Input and Output to Build a Model

Many real-world applications are not as direct as the ones we just considered. Instead they require us to identify some aspect of a linear function. We might sometimes instead be asked to evaluate the linear model at a given input or set the equation of the linear model equal to a specified output.

How To: Given a word problem that includes two pairs of input and output values, use the linear function to solve a problem.

1. Identify the input and output values.
2. Convert the data to two coordinate pairs.
3. Find the slope.
4. Write the linear model.
5. Use the model to make a prediction by evaluating the function at a given x-value.
6. Use the model to identify an x-value that results in a given y-value.
7. Answer the question posed.

Example 3.3.1: Using a Linear Model to Investigate a Town's Population

A town's population has been growing linearly. In 2004 the population was 6,200. By 2009 the population had grown to 8,100. Assume this trend continues.

a. Predict the population in 2013.
b. Identify the year in which the population will reach 15,000.

Solution

The two changing quantities are the population size and time. While we could use the actual year value as the input quantity, doing so tends to lead to very cumbersome equations because the y-intercept would correspond to the year 0, more than 2000 years ago!

To make computation a little nicer, we will define our input as the number of years since 2004:

- Input: t, years since 2004
- Output: $P(t)$, the town's population

To predict the population in 2013 ($t = 9$), we would first need an equation for the population. Likewise, to find when the population would reach 15,000, we would need to solve for the input that would provide an output of 15,000. To write an equation, we need the initial value and the rate of change, or slope.

To determine the rate of change, we will use the change in output per change in input.

$$m = \frac{\text{change in output}}{\text{change in input}} \tag{3.3.1}$$

The problem gives us two input-output pairs. Converting them to match our defined variables, the year 2004 would correspond to $t = 0$, giving the point $(0, 6200)$. Notice that through our clever choice of variable definition, we have "given" ourselves the y-intercept of the function. The year 2009 would correspond to $t = 5$, giving the point $(5, 8100)$.

The two coordinate pairs are $(0, 6200)$ and $(5, 8100)$. Recall that we encountered examples in which we were provided two points earlier in the chapter. We can use these values to calculate the slope.

$$\begin{aligned} m &= \frac{8100 - 6200}{5 - 0} \\ &= \frac{1900}{5} \\ &= 380 \text{ people per year} \end{aligned}$$

We already know the y-intercept of the line, so we can immediately write the equation:

$$P(t) = 380t + 6200 \tag{3.3.2}$$

To predict the population in 2013, we evaluate our function at $t = 9$.

$$\begin{aligned} P(9) &= 380(9) + 6,200 \\ &= 9,620 \end{aligned}$$

If the trend continues, our model predicts a population of 9,620 in 2013.

To find when the population will reach 15,000, we can set $P(t) = 15000$ and solve for t.

$$\begin{aligned} 15000 &= 380t + 6200 \\ 8800 &= 380t \\ t &\approx 23.158 \end{aligned}$$

Our model predicts the population will reach 15,000 in a little more than 23 years after 2004, or somewhere around the year 2027.

Exercise 3.3.1A

A company sells doughnuts. They incur a fixed cost of \$25,000 for rent, insurance, and other expenses. It costs \$0.25 to produce each doughnut.

a. Write a linear model to represent the cost C of the company as a function of x, the number of doughnuts produced.
b. Find and interpret the y-intercept.

Solution

a. $C(x) = 0.25x + 25,000$ b. The y-intercept is $(0, 25,000)$. If the company does not produce a single doughnut, they still incur a cost of \$25,000.

Exercise 3.3.1B

A city's population has been growing linearly. In 2008, the population was 28,200. By 2012, the population was 36,800. Assume this trend continues.

a. Predict the population in 2014.
b. Identify the year in which the population will reach 54,000.

Solution

a. 41,100 b. 2020

Using a Diagram to Model a Problem

It is useful for many real-world applications to draw a picture to gain a sense of how the variables representing the input and output may be used to answer a question. To draw the picture, first consider what the problem is asking for. Then, determine the input and the output. The diagram should relate the variables. Often, geometrical shapes or figures are drawn. Distances are often traced out. If a right triangle is sketched, the Pythagorean Theorem relates the sides. If a rectangle is sketched, labeling width and height is helpful.

Example 3.3.2: Using a Diagram to Model Distance Walked

Anna and Emanuel start at the same intersection. Anna walks east at 4 miles per hour while Emanuel walks south at 3 miles per hour. They are communicating with a two-way radio that has a range of 2 miles. How long after they start walking will they fall out of radio contact?

Solution

In essence, we can partially answer this question by saying they will fall out of radio contact when they are 2 miles apart, which leads us to ask a new question:

"How long will it take them to be 2 miles apart?"

In this problem, our changing quantities are time and position, but ultimately we need to know how long will it take for them to be 2 miles apart. We can see that time will be our input variable, so we'll define our input and output variables.

- Input: t, time in hours.
- Output: $A(t)$, distance in miles, and $E(t)$, distance in miles

Because it is not obvious how to define our output variable, we'll start by drawing a picture such as Figure 3.3.3.

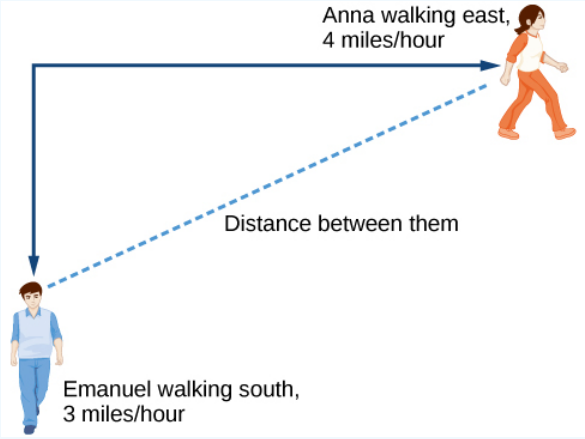


Figure 3.3.3

- Initial Value: They both start at the same intersection so when $t = 0$, the distance traveled by each person should also be 0. Thus the initial value for each is 0.
- Rate of Change: Anna is walking 4 miles per hour and Emanuel is walking 3 miles per hour, which are both rates of change. The slope for A is 4 and the slope for E is 3.

Using those values, we can write formulas for the distance each person has walked.

$$A(t) = 4t \tag{3.3.3}$$

$$E(t) = 3t \tag{3.3.4}$$

For this problem, the distances from the starting point are important. To notate these, we can define a coordinate system, identifying the "starting point" at the intersection where they both started. Then we can use the variable, A, which we introduced above, to represent Anna's position, and define it to be a measurement from the starting point in the eastward direction. Likewise, can use the variable, E, to represent Emanuel's position, measured from the starting point in the southward direction. Note that in defining the coordinate system, we specified both the starting point of the measurement and the direction of measure.

We can then define a third variable, D, to be the measurement of the distance between Anna and Emanuel. Showing the variables on the diagram is often helpful, as we can see from Figure 3.3.4.

Recall that we need to know how long it takes for D, the distance between them, to equal 2 miles. Notice that for any given input t, the outputs $A(t)$, $E(t)$, and $D(t)$ represent distances.

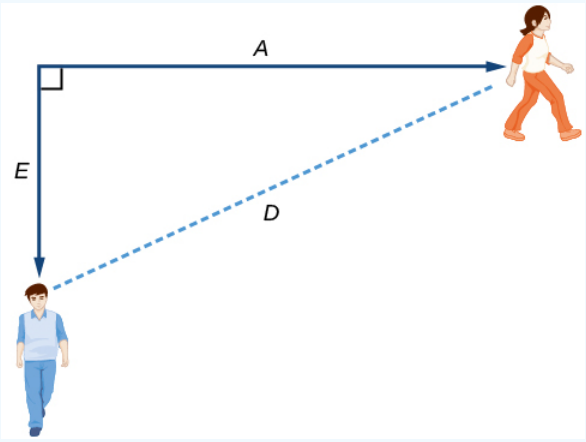


Figure 3.3.4: We can use the Pythagorean Theorem because we have drawn a right angle.

Using the Pythagorean Theorem, we get:

$$\begin{aligned} d(t)^2 &= A(t)^2 + E(t)^2 \\ &= (4t)^2 + (3t)^2 \\ &= 16t^2 + 9t^2 \\ &= 25t^2 \\ D(t) &= \pm\sqrt{25t^2} \qquad \text{Solve for } D(t) \text{ using the square root} \\ &= \pm 5|t| \end{aligned}$$

In this scenario we are considering only positive values of t, so our distance $D(t)$ will always be positive. We can simplify this answer to $D(t) = 5t$. This means that the distance between Anna and Emanuel is also a linear function. Because D is a linear function, we can now answer the question of when the distance between them will reach 2 miles. We will set the output $D(t) = 2$ and solve for t.

$$\begin{aligned} D(t) &= 2 \\ 5t &= 2 \\ t &= \frac{2}{5} = 0.4 \end{aligned}$$

They will fall out of radio contact in 0.4 hours, or 24 minutes.

Should I draw diagrams when given information based on a geometric shape?

Yes. Sketch the figure and label the quantities and unknowns on the sketch.

Example 3.3.3: Using a Diagram to Model Distance between Cities

There is a straight road leading from the town of Westborough to Agritown 30 miles east and 10 miles north. Partway down this road, it junctions with a second road, perpendicular to the first, leading to the town of Eastborough. If the town of Eastborough is located 20 miles directly east of the town of Westborough, how far is the road junction from Westborough?

Solution

It might help here to draw a picture of the situation. See Figure 3.3.5. It would then be helpful to introduce a coordinate system. While we could place the origin anywhere, placing it at Westborough seems convenient. This puts Agritown at coordinates $(30, 10)$, and Eastborough at $(20, 0)$.

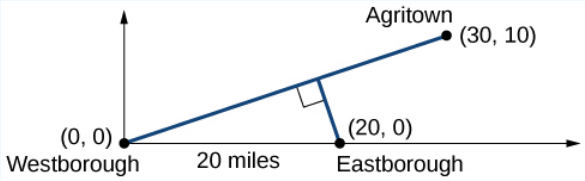


Figure 3.3.5

Using this point along with the origin, we can find the slope of the line from Westborough to Agritown:

$$m = \frac{10-0}{30-0} = \frac{1}{3} \tag{3.3.5}$$

The equation of the road from Westborough to Agritown would be

$$W(x) = \frac{1}{3}x \tag{3.3.6}$$

From this, we can determine the perpendicular road to Eastborough will have slope $m = -3$. Because the town of Eastborough is at the point $(20, 0)$, we can find the equation:

$$\begin{aligned} E(x) &= -3x + b \\ 0 &= -3(20) + b \qquad \text{Substitute in } (20, 0) \\ b &= 60 \\ E(x) &= -3x + 60 \end{aligned}$$

We can now find the coordinates of the junction of the roads by finding the intersection of these lines. Setting them equal,

$$\begin{aligned} \frac{1}{3}x &= -3x + 60 \\ \frac{10}{3}x &= 60 \\ 10x &= 180 \\ x &= 18 \qquad \text{Substituting this back into } W(x) \\ y &= W(18) \\ &= \frac{1}{3}(18) \\ &= 6 \end{aligned}$$

The roads intersect at the point $(18, 6)$. Using the distance formula, we can now find the distance from Westborough to the junction.

$$\begin{aligned} \text{distance} &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(18-0)^2 + (6-0)^2} \\ &\approx 18.743 \text{miles} \end{aligned}$$

Analysis

One nice use of linear models is to take advantage of the fact that the graphs of these functions are lines. This means real-world applications discussing maps need linear functions to model the distances between reference points.

Exercise 3.3.2

There is a straight road leading from the town of Timpson to Ashburn 60 miles east and 12 miles north. Partway down the road, it junctions with a second road, perpendicular to the first, leading to the town of Garrison. If the town of Garrison is located 22 miles directly east of the town of Timpson, how far is the road junction from Timpson?

Solution

21.15 miles

Key Concepts

- We can use the same problem strategies that we would use for any type of function.
- When modeling and solving a problem, identify the variables and look for key values, including the slope and y-intercept.
- Draw a diagram, where appropriate.
- Check for reasonableness of the answer.
- Linear models may be built by identifying or calculating the slope and using the y-intercept.
- The x-intercept may be found by setting $y = 0$, which is setting the expression $mx + b$ equal to 0.
- The point of intersection of a system of linear equations is the point where the x- and y-values are the same.
- A graph of the system may be used to identify the points where one line falls below (or above) the other line.

Footnotes

1 Rates retrieved Aug 2, 2010 from www.budgettruck.com and http://www.uhaul.com/

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3.4: Quadratic Functions

Recognizing Characteristics of Parabolas

The graph of a quadratic function is a U-shaped curve called a parabola. One important feature of the graph is that it has an extreme point, called the **vertex**. If the parabola opens up, the vertex represents the lowest point on the graph, or the **minimum value** of the quadratic function. If the parabola opens down, the vertex represents the highest point on the graph, or the **maximum value**. In either case, the vertex is a turning point on the graph. The graph is also symmetric with a vertical line drawn through the vertex, called the **axis of symmetry**. These features are illustrated in Figure 3.4.2.

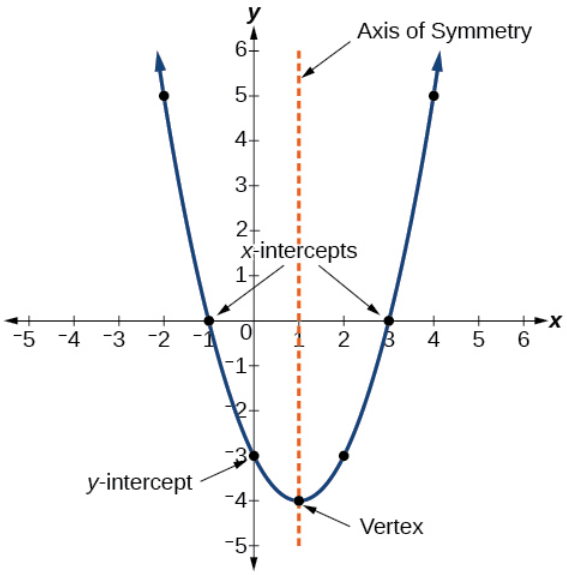


Figure 3.4.2: Graph of a parabola showing where the x and y intercepts, vertex, and axis of symmetry are.

The y-intercept is the point at which the parabola crosses the y-axis. The x-intercepts are the points at which the parabola crosses the x-axis. If they exist, the x-intercepts represent the **zeros**, or **roots**, of the quadratic function, the values of x at which $y = 0$.

Example 3.4.1: Identifying the Characteristics of a Parabola

Determine the vertex, axis of symmetry, zeros, and y-intercept of the parabola shown in Figure 3.4.3.

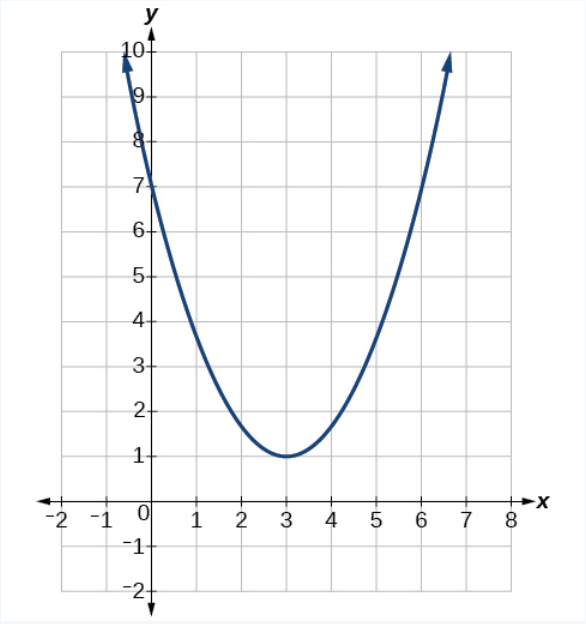


Figure 3.4.3.

Solution

The vertex is the turning point of the graph. We can see that the vertex is at $(3, 1)$. Because this parabola opens upward, the axis of symmetry is the vertical line that intersects the parabola at the vertex. So the axis of symmetry is $x = 3$. This parabola does not cross the x-axis, so it has no zeros. It crosses the y-axis at $(0, 7)$ so this is the y-intercept.

Understanding How the Graphs of Parabolas are Related to Their Quadratic Functions

The **general form of a quadratic function** presents the function in the form

$$f(x) = ax^2 + bx + c \tag{3.4.1}$$

where a, b, and c are real numbers and $a \neq 0$. If $a > 0$, the parabola opens upward. If $a < 0$, the parabola opens downward. We can use the general form of a parabola to find the equation for the axis of symmetry.

The axis of symmetry is defined by $x = -\frac{b}{2a}$. If we use the quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, to solve $ax^2 + bx + c = 0$ for the x-intercepts, or zeros, we find the value of x halfway between them is always $x = -\frac{b}{2a}$, the equation for the axis of symmetry.

Figure 3.4.4 represents the graph of the quadratic function written in general form as $y = x^2 + 4x + 3$. In this form, $a = 1$, $b = 4$, and $c = 3$. Because $a > 0$, the parabola opens upward. The axis of symmetry is $x = -\frac{4}{2(1)} = -2$. This also makes sense because we can see from the graph that the vertical line $x = -2$ divides the graph in half. The vertex always occurs along the axis of symmetry. For a parabola that opens upward, the vertex occurs at the lowest point on the graph, in this instance, $(-2, -1)$. The x-intercepts, those points where the parabola crosses the x-axis, occur at $(-3, 0)$ and $(-1, 0)$.

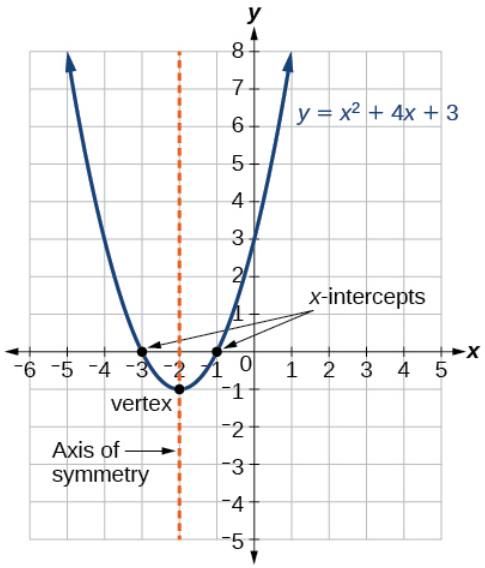


Figure 3.4.4: Graph of a parabola showing where the x and y intercepts, vertex, and axis of symmetry are for the function $y = x^2 + 4x + 3$.

The **standard form of a quadratic function** presents the function in the form

$$f(x) = a(x - h)^2 + k \tag{3.4.2}$$

where (h, k) is the vertex. Because the vertex appears in the standard form of the quadratic function, this form is also known as the **vertex form of a quadratic function**.

As with the general form, if $a > 0$, the parabola opens upward and the vertex is a minimum. If $a < 0$, the parabola opens downward, and the vertex is a maximum. Figure 3.4.5 represents the graph of the quadratic function written in standard form as $y = -3(x + 2)^2 + 4$. Since $x - h = x + 2$ in this example, $h = -2$. In this form, $a = -3$, $h = -2$, and $k = 4$. Because $a < 0$, the parabola opens downward. The vertex is at $(-2, 4)$.

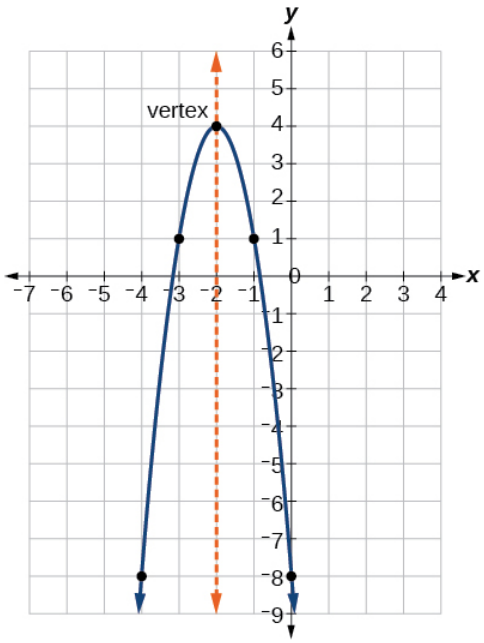


Figure 3.4.5: Graph of a parabola showing where the x and y intercepts, vertex, and axis of symmetry are for the function $y = -3(x+2)^2 + 4$.

The standard form is useful for determining how the graph is transformed from the graph of $y = x^2$. Figure 3.4.6 is the graph of this basic function.

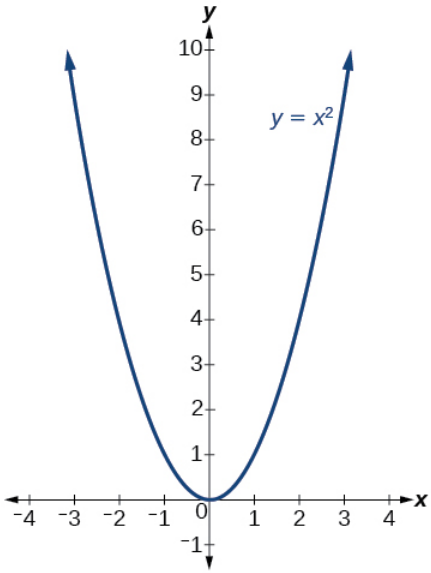


Figure 3.4.6: Graph of $y = x^2$.

If $k > 0$, the graph shifts upward, whereas if $k < 0$, the graph shifts downward. In Figure 3.4.5, $k > 0$, so the graph is shifted 4 units upward. If $h > 0$, the graph shifts toward the right and if $h < 0$, the graph shifts to the left. In Figure 3.4.5, $h < 0$, so the graph is shifted 2 units to the left. The magnitude of a indicates the stretch of the graph. If $|a| > 1$, the point associated with a particular x-value shifts farther from the x-axis, so the graph appears to become narrower, and there is a vertical stretch. But if $|a| < 1$, the point associated with a particular x-value shifts closer to the x-axis, so the graph appears to become wider, but in fact there is a vertical compression. In Figure 3.4.5, $|a| > 1$, so the graph becomes narrower.

The standard form and the general form are equivalent methods of describing the same function. We can see this by expanding out the general form and setting it equal to the standard form.

$$\begin{aligned} a(x-h)^2+k &= ax^2+bx+c \\ ax^2-2ahx+(ah^2+k) &= ax^2+bx+c \end{aligned}$$

For the linear terms to be equal, the coefficients must be equal.

$$-2ah=b, \text{ so } h=-\frac{b}{2a}.$$

This is the **axis of symmetry** we defined earlier. Setting the constant terms equal:

$$\begin{aligned} ah^2+k &= c \\ k &= c-ah^2 \\ &= c-a\cdot\left(-\frac{b}{2a}\right)^2 \\ &= c-\frac{b^2}{4a} \end{aligned}$$

In practice, though, it is usually easier to remember that k is the output value of the function when the input is h, so $f(h)=k$.

Definitions: Forms of Quadratic Functions

A quadratic function is a function of degree two. The graph of a *quadratic function* is a parabola.

- The *general form* of a quadratic function is $f(x)=ax^2+bx+c$ where a, b, and c are real numbers and $a\neq 0$.
- The standard form of a quadratic function is $f(x)=a(x-h)^2+k$.
- The vertex (h,k) is located at

$$h=-\frac{b}{2a},\ k=f(h)=f(\frac{-b}{2a}). \tag{3.4.3}$$

HOWTO: Write a quadratic function in a general form

Given a graph of a quadratic function, write the equation of the function in general form.

1. Identify the horizontal shift of the parabola; this value is h. Identify the vertical shift of the parabola; this value is k.
2. Substitute the values of the horizontal and vertical shift for h and k. in the function $f(x)=a(x-h)^2+k$.
3. Substitute the values of any point, other than the vertex, on the graph of the parabola for x and $f(x)$.
4. Solve for the stretch factor, $|a|$.
5. If the parabola opens up, $a>0$. If the parabola opens down, $a<0$ since this means the graph was reflected about the x-axis.
6. Expand and simplify to write in general form.

Example 3.4.2: Writing the Equation of a Quadratic Function from the Graph

Write an equation for the quadratic function g in Figure 3.4.7 as a transformation of $f(x)=x^2$, and then expand the formula, and simplify terms to write the equation in general form.

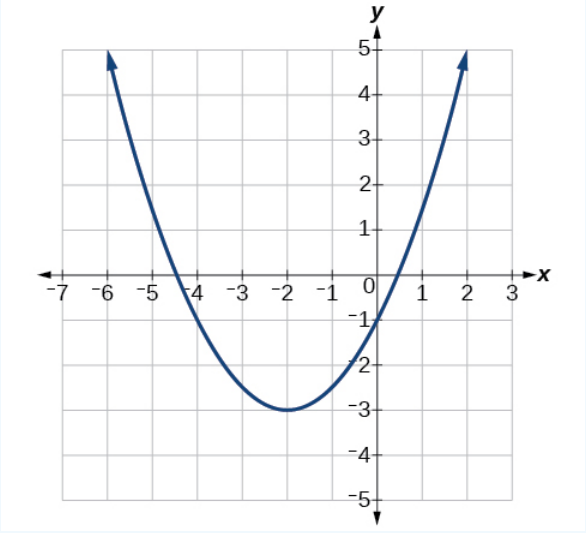


Figure 3.4.7: Graph of a parabola with its vertex at $(-2, -3)$.

Solution

We can see the graph of g is the graph of $f(x) = x^2$ shifted to the left 2 and down 3, giving a formula in the form $g(x) = a(x+2)^2 - 3$.

Substituting the coordinates of a point on the curve, such as $(0, -1)$, we can solve for the stretch factor.

$$-1 = a(0+2)^2 - 3 \tag{3.4.4}$$
$$2 = 4a \tag{3.4.5}$$
$$a = \frac{1}{2} \tag{3.4.6}$$

In standard form, the algebraic model for this graph is $g(x) = \frac{1}{2}(x+2)^2 - 3$.

To write this in general polynomial form, we can expand the formula and simplify terms.

$$g(x) = \frac{1}{2}(x+2)^2 - 3 \tag{3.4.7}$$
$$= \frac{1}{2}(x+2)(x+2) - 3 \tag{3.4.8}$$
$$= \frac{1}{2}(x^2 + 4x + 4) - 3 \tag{3.4.9}$$
$$= \frac{1}{2}x^2 + 2x + 2 - 3 \tag{3.4.10}$$
$$= \frac{1}{2}x^2 + 2x - 1 \tag{3.4.11}$$

Notice that the horizontal and vertical shifts of the basic graph of the quadratic function determine the location of the vertex of the parabola; the vertex is unaffected by stretches and compressions.

Analysis

We can check our work using the table feature on a graphing utility. First enter $\mathrm{Y1} = \frac{1}{2}(\mathrm{x}+2)^2 - 3$. Next, select TBLSET, then use $\mathrm{TblStart} = -6$ and $\Delta\mathrm{Tbl} = 2$, and select TABLE. See Table 3.4.1

Table 3.4.1

x	-6	-4	-2	0	2
y	-5	-1	-3	-1	5

The ordered pairs in the table correspond to points on the graph.

Exercise 3.4.2

A coordinate grid has been superimposed over the quadratic path of a basketball in Figure 3.4.8. Find an equation for the path of the ball. Does the shooter make the basket?

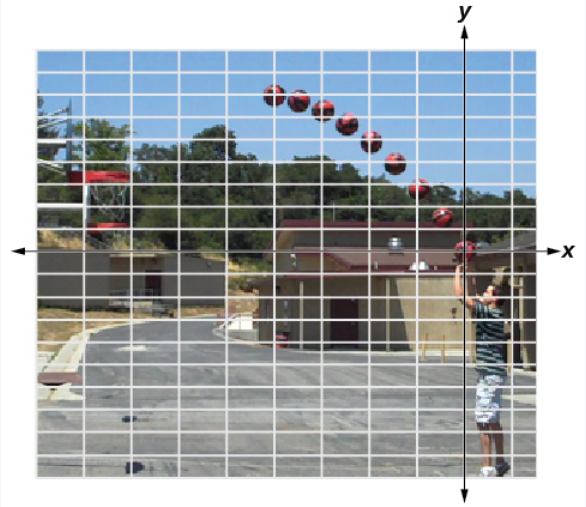


Figure 3.4.8: *Stop motioned picture of a boy throwing a basketball into a hoop to show the parabolic curve it makes. (credit: modification of work by Dan Meyer)*

Answer

The path passes through the origin and has vertex at $(-4, 7)$, so $h(x) = -\frac{7}{16}(x+4)^2 + 7$. To make the shot, $h(-7.5)$ would need to be about 4 but $h(-7.5) \approx 1.64$; he doesn't make it.

How To: Given a quadratic function in general form, find the vertex of the parabola.

1. Identify a, b, and c.
2. Find h, the x-coordinate of the vertex, by substituting a and b into $h = -\frac{b}{2a}$.
3. Find k, the y-coordinate of the vertex, by evaluating $k = f(h) = f\left(-\frac{b}{2a}\right)$.

Example 3.4.3: Finding the Vertex of a Quadratic Function

Find the vertex of the quadratic function $f(x) = 2x^2 - 6x + 7$. Rewrite the quadratic in standard form (vertex form).

Solution

The horizontal coordinate of the vertex will be at

$$h = -\frac{b}{2a} \tag{3.4.12}$$
$$= -\frac{-6}{2(2)} \tag{3.4.13}$$
$$= \frac{6}{4} \tag{3.4.14}$$
$$= \frac{3}{2} \tag{3.4.15}$$

The vertical coordinate of the vertex will be at

$$k = f(h) \tag{3.4.16}$$
$$= f\left(\frac{3}{2}\right) \tag{3.4.17}$$
$$= 2\left(\frac{3}{2}\right)^2 - 6\left(\frac{3}{2}\right) + 7 \tag{3.4.18}$$
$$= \frac{5}{2} \tag{3.4.19}$$

Rewriting into standard form, the stretch factor will be the same as the a in the original quadratic.

$$f(x) = ax^2 + bx + c \tag{3.4.20}$$
$$f(x) = 2x^2 - 6x + 7$$

Using the vertex to determine the shifts,

$$f(x) = 2\left(x - \frac{3}{2}\right)^2 + \frac{5}{2} \tag{3.4.21}$$

Analysis

One reason we may want to identify the vertex of the parabola is that this point will inform us what the maximum or minimum value of the function is, (k),and where it occurs, (h).

Exercise 3.4.3

Given the equation $g(x) = 13 + x^2 - 6x$, write the equation in general form and then in standard form.

Answer

$g(x) = x^2 - 6x + 13$ in general form; $g(x) = (x-3)^2 + 4$ in standard form.

Finding the Domain and Range of a Quadratic Function

Any number can be the input value of a quadratic function. Therefore, the domain of any quadratic function is all real numbers. Because parabolas have a maximum or a minimum point, the range is restricted. Since the vertex of a parabola will be either a maximum or a minimum, the range will consist of all y-values greater than or equal to the y-coordinate at the turning point or less than or equal to the y-coordinate at the turning point, depending on whether the parabola opens up or down.

Definition: Domain and Range of a Quadratic Function

The domain of any **quadratic function** is all real numbers.

The range of a quadratic function written in general form $f(x) = ax^2 + bx + c$ with a positive a value is $f(x) \geq f\left(-\frac{b}{2a}\right)$, or $[f(-\frac{b}{2a}), \infty)$; the range of a quadratic function written in general form with a negative a value is $f(x) \leq f(-\frac{b}{2a})$, or $(-\infty, f(-\frac{b}{2a})]$.

The range of a quadratic function written in standard form $f(x) = a(x-h)^2 + k$ with a positive a value is $f(x) \geq k$; the range of a quadratic function written in standard form with a negative a value is $f(x) \leq k$.

How To: Given a quadratic function, find the domain and range.

1. Identify the domain of any quadratic function as all real numbers.
2. Determine whether a is positive or negative. If a is positive, the parabola has a minimum. If a is negative, the parabola has a maximum.
3. Determine the maximum or minimum value of the parabola, k.
4. If the parabola has a minimum, the range is given by $f(x) \geq k$, or $[k, \infty)$. If the parabola has a maximum, the range is given by $f(x) \leq k$, or $(-\infty, k]$.

Example 3.4.4: Finding the Domain and Range of a Quadratic Function

Find the domain and range of $f(x) = -5x^2 + 9x - 1$.

Solution

As with any quadratic function, the domain is all real numbers.

Because a is negative, the parabola opens downward and has a maximum value. We need to determine the maximum value. We can begin by finding the x-value of the vertex.

$$h = -\frac{b}{2a} \tag{3.4.22}$$
$$= -\frac{9}{2(-5)} \tag{3.4.23}$$
$$= \frac{9}{10} \tag{3.4.24}$$

The maximum value is given by $f(h)$.

$$f(\frac{9}{10}) = 5(\frac{9}{10})^2 + 9(\frac{9}{10}) - 1 \tag{3.4.25}$$
$$= \frac{61}{20} \tag{3.4.26}$$

The range is $f(x) \leq \frac{61}{20}$, or $\left(-\infty, \frac{61}{20}\right]$.

Exercise 3.4.4

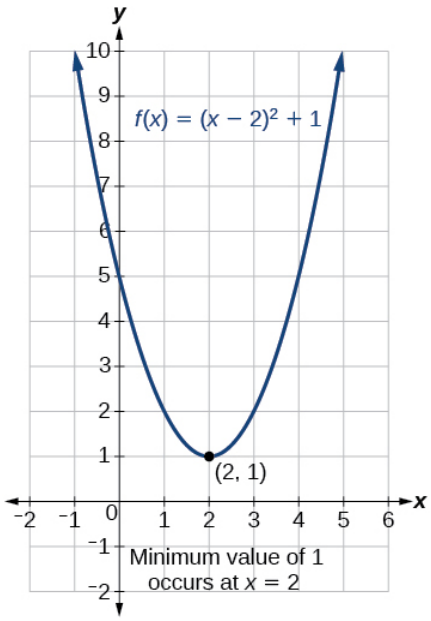
Find the domain and range of $f(x) = 2\left(x - \frac{4}{7}\right)^2 + \frac{8}{11}$.

Answer

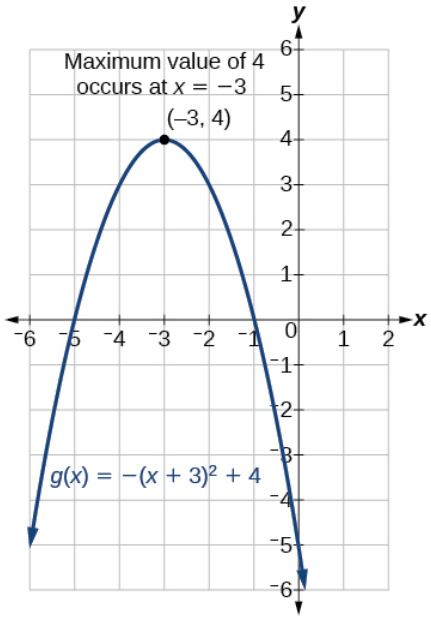
The domain is all real numbers. The range is $f(x) \geq \frac{8}{11}$, or $\left[\frac{8}{11}, \infty\right)$.

Determining the Maximum and Minimum Values of Quadratic Functions

The output of the quadratic function at the vertex is the maximum or minimum value of the function, depending on the orientation of the **parabola**. We can see the maximum and minimum values in Figure 3.4.9.



(a)



(b)

Figure 3.4.9: Minimum and maximum of two quadratic functions.

There are many real-world scenarios that involve finding the maximum or minimum value of a quadratic function, such as applications involving area and revenue.

Example 3.4.5: Finding the Maximum Value of a Quadratic Function

A backyard farmer wants to enclose a rectangular space for a new garden within her fenced backyard. She has purchased 80 feet of wire fencing to enclose three sides, and she will use a section of the backyard fence as the fourth side.

a. Find a formula for the area enclosed by the fence if the sides of fencing perpendicular to the existing fence have length L.
b. What dimensions should she make her garden to maximize the enclosed area?

Solution

Let's use a diagram such as Figure 3.4.10 to record the given information. It is also helpful to introduce a temporary variable, W, to represent the width of the garden and the length of the fence section parallel to the backyard fence.

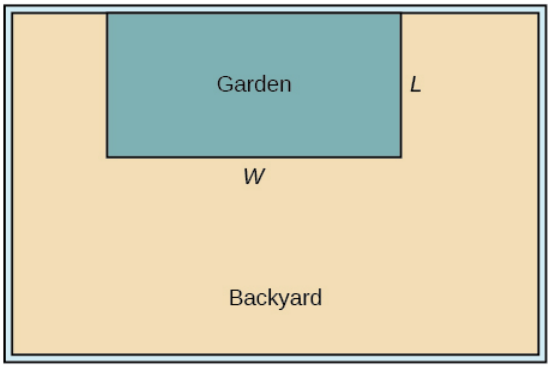


Figure 3.4.10: Diagram of the garden and the backyard.

a. We know we have only 80 feet of fence available, and $L+W+L=80$, or more simply, $2L+W=80$. This allows us to represent the width, W, in terms of L.

$$W=80-2L \tag{3.4.27}$$

Now we are ready to write an equation for the area the fence encloses. We know the area of a rectangle is length multiplied by width, so

$$A = LW = L(80-2L) \tag{3.4.28}$$
$$A(L) = 80L-2L^2 \tag{3.4.29}$$

This formula represents the area of the fence in terms of the variable length L. The function, written in general form, is

$$A(L)=-2L^2+80L \tag{3.4.30}$$

.

The quadratic has a negative leading coefficient, so the graph will open downward, and the vertex will be the maximum value for the area. In finding the vertex, we must be careful because the equation is not written in standard polynomial form with decreasing powers. This is why we rewrote the function in general form above. Since a is the coefficient of the squared term, $a=-2$, $b=80$, and $c=0$.

To find the vertex:

$$h=-\frac{80}{2(-2)} \qquad k=A(20) \tag{3.4.31}$$
$$=20 \qquad \text{and} \qquad =80(20)-2(20)^2 \tag{3.4.32}$$
$$=800 \tag{3.4.33}$$

The maximum value of the function is an area of 800 square feet, which occurs when $L=20$ feet. When the shorter sides are 20 feet, there is 40 feet of fencing left for the longer side. To maximize the area, she should enclose the garden so the two shorter sides have length 20 feet and the longer side parallel to the existing fence has length 40 feet.

Analysis

This problem also could be solved by graphing the quadratic function. We can see where the maximum area occurs on a graph of the quadratic function in Figure 3.4.11.

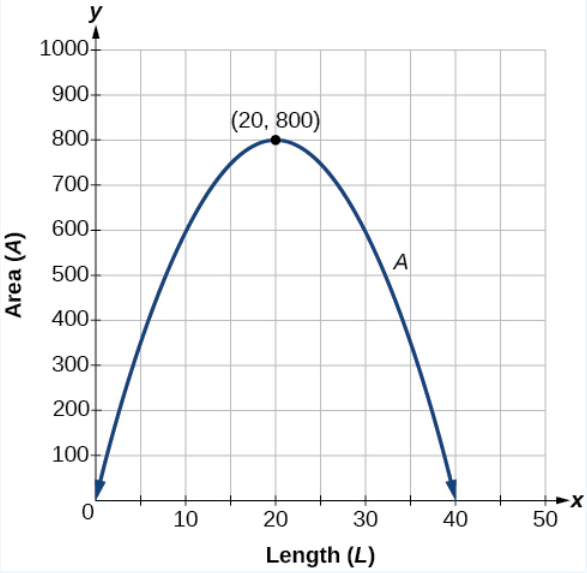


Figure 3.4.11: Graph of the parabolic function $A(L) = -2L^2 + 80L$

How To: Given an application involving revenue, use a quadratic equation to find the maximum.

1. Write a quadratic equation for revenue.
2. Find the vertex of the quadratic equation.
3. Determine the y-value of the vertex.

Example 3.4.6: Finding Maximum Revenue

The unit price of an item affects its supply and demand. That is, if the unit price goes up, the demand for the item will usually decrease. For example, a local newspaper currently has 84,000 subscribers at a quarterly charge of $30. Market research has suggested that if the owners raise the price to $32, they would lose 5,000 subscribers. Assuming that subscriptions are linearly related to the price, what price should the newspaper charge for a quarterly subscription to maximize their revenue?

Solution

Revenue is the amount of money a company brings in. In this case, the revenue can be found by multiplying the price per subscription times the number of subscribers, or quantity. We can introduce variables, p for price per subscription and Q for quantity, giving us the equation $\text{Revenue} = pQ$.

Because the number of subscribers changes with the price, we need to find a relationship between the variables. We know that currently $p = 30$ and $Q = 84,000$. We also know that if the price rises to $32, the newspaper would lose 5,000 subscribers, giving a second pair of values, $p = 32$ and $Q = 79,000$. From this we can find a linear equation relating the two quantities. The slope will be

$$m = \frac{79,000 - 84,000}{32 - 30} \tag{3.4.34}$$

$$= -\frac{5,000}{2} \tag{3.4.35}$$

$$= -2,500 \tag{3.4.36}$$

This tells us the paper will lose 2,500 subscribers for each dollar they raise the price. We can then solve for the y-intercept.

$$Q = -2500p + b \quad \text{Substitute in the point } Q = 84,000 \text{ and } p = 30 \tag{3.4.37}$$

$$84,000 = -2500(30) + b \quad \text{Solve for } b \tag{3.4.38}$$

$$b = 159,000 \tag{3.4.39}$$

This gives us the linear equation $Q = -2,500p + 159,000$ relating cost and subscribers. We now return to our revenue equation.

$$\text{Revenue} = pQ \tag{3.4.40}$$
$$\text{Revenue} = p(-2,500p + 159,000) \tag{3.4.41}$$
$$\text{Revenue} = -2,500p^2 + 159,000p \tag{3.4.42}$$

We now have a quadratic function for revenue as a function of the subscription charge. To find the price that will maximize revenue for the newspaper, we can find the vertex.

$$h = -\frac{159,000}{2(-2,500)} \tag{3.4.43}$$
$$= 31.8 \tag{3.4.44}$$

The model tells us that the maximum revenue will occur if the newspaper charges $31.80 for a subscription. To find what the maximum revenue is, we evaluate the revenue function.

$$\text{maximum revenue} = -2,500(31.8)^2 + 159,000(31.8) \tag{3.4.45}$$
$$= 2,528,100 \tag{3.4.46}$$

Analysis

This could also be solved by graphing the quadratic as in Figure 3.4.12 We can see the maximum revenue on a graph of the quadratic function.

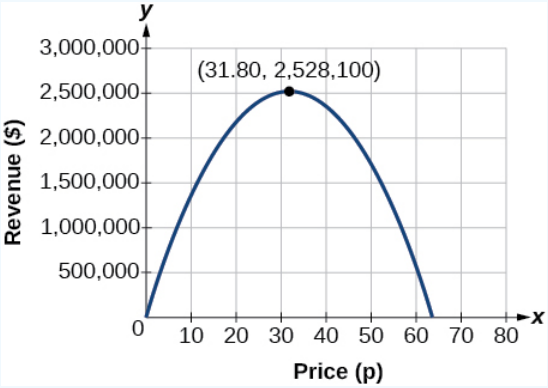


Figure 3.4.12: Graph of the parabolic function

Finding the x- and y-Intercepts of a Quadratic Function

Much as we did in the application problems above, we also need to find intercepts of quadratic equations for graphing parabolas. Recall that we find the y-intercept of a quadratic by evaluating the function at an input of zero, and we find the x-intercepts at locations where the output is zero. Notice in Figure 3.4.13 that the number of x-intercepts can vary depending upon the location of the graph.

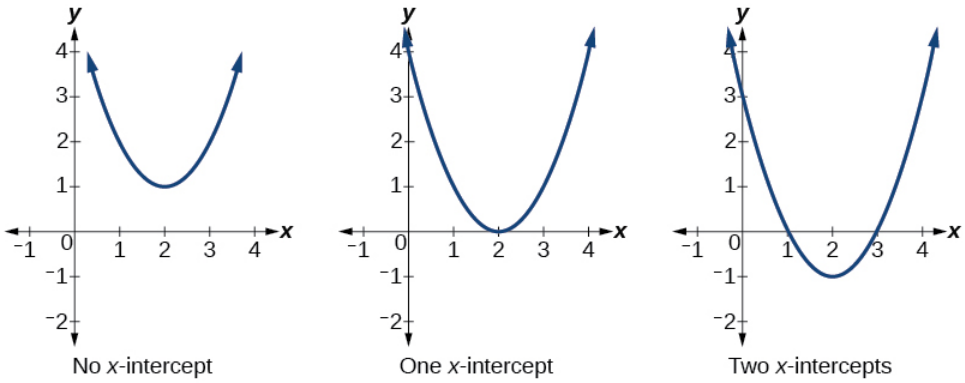


Figure 3.4.13: Number of x-intercepts of a parabola.

Given a quadratic function $f(x)$, find the y- and x-intercepts.

1. Evaluate $f(0)$ to find the y-intercept.
2. Solve the quadratic equation $f(x)=0$ to find the x-intercepts.

Example 3.4.7: Finding the y- and x-Intercepts of a Parabola

Find the y- and x-intercepts of the quadratic $f(x)=3x^2+5x-2$.

Solution

We find the y-intercept by evaluating $f(0)$.

$$f(0) = 3(0)^2+5(0)-2 \tag{3.4.47}$$
$$= -2 \tag{3.4.48}$$

So the y-intercept is at $(0,-2)$.

For the x-intercepts, we find all solutions of $f(x)=0$.

$$0=3x^2+5x-2 \tag{3.4.49}$$

In this case, the quadratic can be factored easily, providing the simplest method for solution.

$$0=(3x-1)(x+2) \tag{3.4.50}$$
$$0=3x-1 \qquad\qquad 0=x+2 \tag{3.4.51}$$
$$x=\frac{1}{3} \qquad \text{or} \qquad x=-2 \tag{3.4.52}$$

So the x-intercepts are at $(\frac{1}{3},0)$ and $(-2,0)$.

Analysis

By graphing the function, we can confirm that the graph crosses the y-axis at $(0,-2)$. We can also confirm that the graph crosses the x-axis at $\left(\frac{1}{3},0\right)$ and $(-2,0)$. See Figure 3.4.14

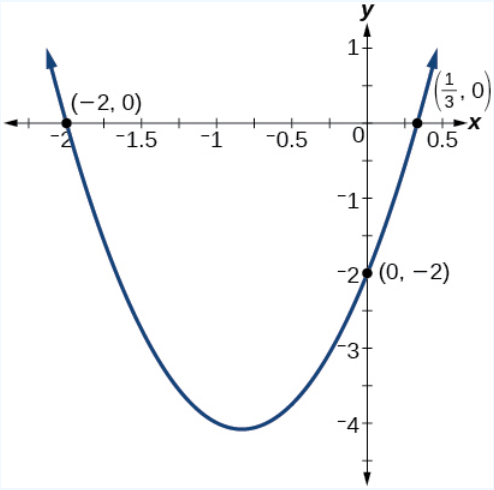


Figure 3.4.14: Graph of a parabola.

Rewriting Quadratics in Standard Form

In Example 3.4.7, the quadratic was easily solved by factoring. However, there are many quadratics that cannot be factored. We can solve these quadratics by first rewriting them in standard form.

How To: Given a quadratic function, find the x-intercepts by rewriting in standard form.

1. Substitute a and b into $h=-\frac{b}{2a}$.
2. Substitute $x=h$ into the general form of the quadratic function to find k.
3. Rewrite the quadratic in standard form using h and k.

4. Solve for when the output of the function will be zero to find the x-intercepts.

Example 3.4.8: Finding the x-Intercepts of a Parabola

Find the x-intercepts of the quadratic function $f(x) = 2x^2 + 4x - 4$.

Solution

We begin by solving for when the output will be zero.

$$0 = 2x^2 + 4x - 4$$

Because the quadratic is not easily factorable in this case, we solve for the intercepts by first rewriting the quadratic in standard form.

$$f(x) = a(x-h)^2 + k$$

We know that $a = 2$. Then we solve for h and k.

$$\begin{aligned} h &= -\frac{b}{2a} & k &= f(-1) \\ &= -\frac{4}{2(2)} & &= 2(-1)^2 + 4(-1) - 4 \\ &= -1 & &= -6 \end{aligned}$$

So now we can rewrite in standard form.

$$f(x) = 2(x+1)^2 - 6$$

We can now solve for when the output will be zero.

$$\begin{aligned} 0 &= 2(x+1)^2 - 6 \\ 6 &= 2(x+1)^2 \\ 3 &= (x+1)^2 \\ x+1 &= \pm\sqrt{3} \\ x &= -1 \pm \sqrt{3} \end{aligned}$$

The graph has x-intercepts at $(-1-\sqrt{3}, 0)$ and $(-1+\sqrt{3}, 0)$.

Analysis

We can check our work by graphing the given function on a graphing utility and observing the x-intercepts. See Figure 3.4.15.

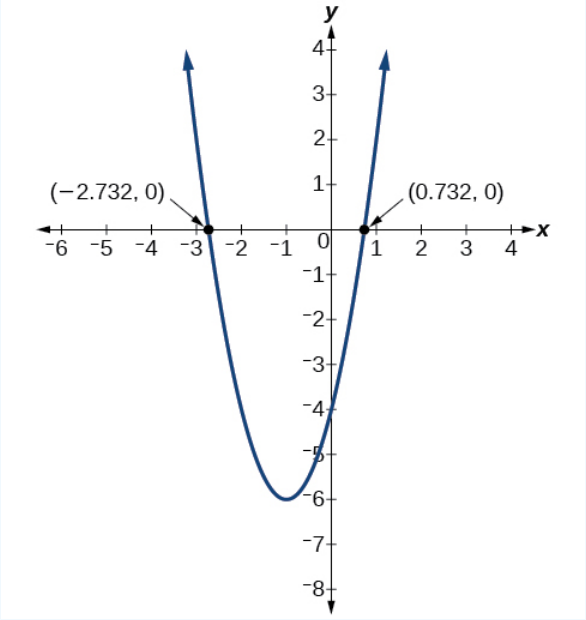


Figure 3.4.15: Graph of a parabola which has the following x-intercepts: $(-2.732, 0)$ and $(0.732, 0)$.

Exercise 3.4.1

In Try It 3.4.1, we found the standard and general form for the function $g(x) = 13 + x^2 - 6x$. Now find the y- and x-intercepts (if any).

Answer

y-intercept at $(0, 13)$, No x-intercepts

Example 3.4.9: Solving a Quadratic Equation with the Quadratic Formula

Solve $x^2 + x + 2 = 0$.

Solution

Let's begin by writing the quadratic formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

When applying the quadratic formula, we identify the coefficients a, b and c. For the equation $x^2 + x + 2 = 0$, we have $a = 1$, $b = 1$, and $c = 2$. Substituting these values into the formula we have:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot (2)}}{2 \cdot 1} \\ &= \frac{-1 \pm \sqrt{1 - 8}}{2} \\ &= \frac{-1 \pm \sqrt{-7}}{2} \\ &= \frac{-1 \pm i\sqrt{7}}{2} \end{aligned}$$

The solutions to the equation are $x = \frac{-1+i\sqrt{7}}{2}$ and $x = \frac{-1-i\sqrt{7}}{2}$ or $x = -\frac{1}{2} + \frac{i\sqrt{7}}{2}$ and $x = \frac{-1}{2} - \frac{i\sqrt{7}}{2}$.

Example 3.4.10: Applying the Vertex and x-Intercepts of a Parabola

A ball is thrown upward from the top of a 40 foot high building at a speed of 80 feet per second. The ball's height above ground can be modeled by the equation $H(t) = -16t^2 + 80t + 40$.

When does the ball reach the maximum height?
What is the maximum height of the ball?
When does the ball hit the ground?

The ball reaches the maximum height at the vertex of the parabola.

$$h = -\frac{80}{2(-16)} \tag{3.4.53}$$
$$= \frac{80}{32} \tag{3.4.54}$$
$$= \frac{5}{2} \tag{3.4.55}$$
$$= 2.5 \tag{3.4.56}$$

The ball reaches a maximum height after 2.5 seconds.

To find the maximum height, find the y-coordinate of the vertex of the parabola.

$$k = H(-\frac{b}{2a}) \tag{3.4.57}$$
$$= H(2.5) \tag{3.4.58}$$
$$= -16(2.5)^2 + 80(2.5) + 40 \tag{3.4.59}$$
$$= 140 \tag{3.4.60}$$

The ball reaches a maximum height of 140 feet.

To find when the ball hits the ground, we need to determine when the height is zero, $H(t) = 0$.

We use the quadratic formula.

$$t = \frac{-80 \pm \sqrt{80^2 - 4(-16)(40)}}{2(-16)} \tag{3.4.61}$$
$$= \frac{-80 \pm \sqrt{8960}}{-32} \tag{3.4.62}$$

Because the square root does not simplify nicely, we can use a calculator to approximate the values of the solutions.

$$t = \frac{-80 - \sqrt{8960}}{-32} \approx 5.458 \text{ or } t = \frac{-80 + \sqrt{8960}}{-32} \approx -0.458 \tag{3.4.63}$$

The second answer is outside the reasonable domain of our model, so we conclude the ball will hit the ground after about 5.458 seconds. See Figure 3.4.16.

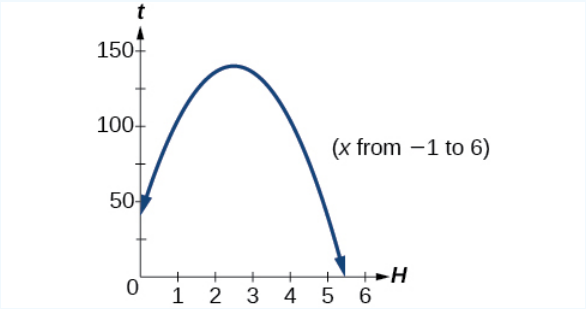


Figure 3.4.16

https://math.libretexts.org/@go/page/89789

3.4.5: A rock is thrown upward from the top of a 112-foot high cliff overlooking the ocean at a speed of 96 feet per second. The rock's height above ocean can be modeled by the equation $H(t) = -16t^2 + 96t + 112$.

a. When does the rock reach the maximum height?
b. What is the maximum height of the rock?
c. When does the rock hit the ocean?

Solution

a. 3 seconds b. 256 feet c. 7 seconds

Key Equations

- general form of a quadratic function: $f(x) = ax^2 + bx + c$
- the quadratic formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- standard form of a quadratic function: $f(x) = a(x - h)^2 + k$

Key Concepts

- A polynomial function of degree two is called a quadratic function.
- The graph of a quadratic function is a parabola. A parabola is a U-shaped curve that can open either up or down.
- The axis of symmetry is the vertical line passing through the vertex. The zeros, or x-intercepts, are the points at which the parabola crosses the x-axis. The y-intercept is the point at which the parabola crosses the y-axis.
- Quadratic functions are often written in general form. Standard or vertex form is useful to easily identify the vertex of a parabola. Either form can be written from a graph.
- The vertex can be found from an equation representing a quadratic function. .
- The domain of a quadratic function is all real numbers. The range varies with the function.
- A quadratic function's minimum or maximum value is given by the y-value of the vertex.
- The minimum or maximum value of a quadratic function can be used to determine the range of the function and to solve many kinds of real-world problems, including problems involving area and revenue.
- Some quadratic equations must be solved by using the quadratic formula.
- The vertex and the intercepts can be identified and interpreted to solve real-world problems.

Glossary

axis of symmetry
a vertical line drawn through the vertex of a parabola around which the parabola is symmetric; it is defined by $x = -\frac{b}{2a}$.

general form of a quadratic function
the function that describes a parabola, written in the form $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers and a≠0.

standard form of a quadratic function
the function that describes a parabola, written in the form $f(x) = a(x - h)^2 + k$, where (h, k) is the vertex.

vertex
the point at which a parabola changes direction, corresponding to the minimum or maximum value of the quadratic function

vertex form of a quadratic function
another name for the standard form of a quadratic function

zeros
in a given function, the values of x at which $y = 0$, also called roots

---

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3.5: Applications with Quadratic Functions

Find the Extreme Value

Definition: Extreme Value

The ***extreme value*** of a quadratic function $f(x)$ is either the *minimum* or *maximum* value of the quadratic function $f(x)$. The extreme value is given as

$$f\left(-\frac{b}{2a}\right)$$

The *minimum* value is located at the lowest point of an upwards parabola and the *maximum* value is located at the highest point of a downwards parabola.

Note

Notice in the definition that the extreme value is the y-coordinate of the vertex. Recall, the vertex is

$$\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$$

and we can see the y-coordinate is just the extreme value. Hence, when we use the words vertex, minimum or maximum value, or extreme value, they are all associated with the vertex of a parabola.

Example 11.6.1

Find the vertex and the extreme value of the function $f(x) = x^2 - 4x + 5$.

Solution

Notice $a = 1$, which means $a > 0$. Hence, $f(x)$ is an upwards parabola and, from the definition, we expect $f(x)$ to have a minimum value. Let's use the formula to find the vertex, where $a = 1$ and $b = -4$.

$$\begin{aligned} x &= -\frac{b}{2a} && \text{Plug-n-chug} \\ x &= -\frac{-4}{2(1)} && \text{Simplify} \\ x &= 2 && \text{The } x\text{-coordinate of the vertex} \end{aligned}$$

Next, we find the y-coordinate of the vertex by obtaining $f(2)$.

$$\begin{aligned} f(x) &= x^2 - 4x + 5 && \text{Plug-n-chug} \\ f(2) &= (2)^2 - 4(2) + 5 && \text{Evaluate} \\ f(2) &= 1 && \text{The } y\text{-coordinate of the vertex} \end{aligned}$$

Hence, the vertex is $(2, 1)$. Next, we find the extreme value of $f(x)$. From the vertex calculation, we see $f\left(-\frac{b}{2a}\right) = f(2) = 1$. Thus, the extreme value is 1, the y-coordinate of the vertex.

Example 11.6.2

Find the vertex and the extreme value of the function $q(n) = -3n^2 - 5n + 3$.

Solution

Notice $a = -3$, which means $a < 0$. Hence, $q(n)$ is an downwards parabola and, from the definition, we expect $q(n)$ to have a maximum value. Let's use the formula to find the vertex, where $a = -3$ and $b = -5$.

$$n = -\frac{b}{2a} \quad \text{Plug-n-chug}$$
$$n = -\frac{-5}{2(-3)} \quad \text{Simplify}$$
$$n = -\frac{5}{6} \quad \text{The } x\text{-coordinate of the vertex}$$

Next, we find the y-coordinate of the vertex by obtaining $q\left(-\frac{5}{6}\right)$.

$$q(n) = -3n^2 - 5n + 3 \quad \text{Plug-n-chug}$$
$$q\left(-\frac{5}{6}\right) = -3\left(-\frac{5}{6}\right)^2 - 5\left(-\frac{5}{6}\right) + 3 \quad \text{Evaluate}$$
$$q\left(-\frac{5}{6}\right) = \frac{61}{12} \quad \text{The } y\text{-coordinate of the vertex}$$

Hence, the vertex is $\left(-\frac{5}{6}, \frac{61}{12}\right)$. Next, we find the extreme value of $q(n)$. From the vertex calculation, we see $q\left(-\frac{b}{2a}\right) = q\left(-\frac{5}{6}\right) = \frac{61}{12}$. Thus, the extreme value is $\frac{61}{12}$, the y-coordinate of the vertex.

It may seem a little redundant to find the vertex and the extreme value, but, remember, the goal is to apply this concept to real-world applications with quadratic functions. Let's take a look at a few applications where we find the extreme value in context of a real-world model.

Projectile Motion

Example 11.6.3

A rocket is launched at $t = 0$ seconds. Its height, in meters above sea-level, as a function of time is given by $h(t) = -4.9t^2 + 46t + 157$. At what time does the rocket reach its maximum height? At what height does the rocket reach its maximum height above the water? *Round the answers to 2 decimal places.*

Solution

When we read the word *maximum*, we should think about the vertex of $h(t)$. Since we need to find the time in which the maximum height occurs, then we can find the x-coordinate of the vertex.

$$t = -\frac{b}{2a}$$
$$t = -\frac{46}{2(-4.9)}$$
$$t \approx 4.69$$

Thus, the maximum height occurs after 4.69 seconds. Next, we find the height of the rocket when it reaches its maximum height above the water. Since we need to find the maximum height, then we need to find the y-coordinate of the vertex, or $h(4.69)$.

$$h(4.69) = -4.9(4.69)^2 + 46(4.69) + 157 \approx 264.96$$

Thus, the maximum height of the rocket is 264.96 meters after 4.69 seconds the rocket is launched.

Revenue and Cost Functions

Example 11.6.4

The revenue, $R(x)$, of producing and selling x *Awesome Hearing Aids* is modeled by the function $R(x) = -6x^2 + 108x$. What is the maximum revenue?

Solution

To find the maximum revenue, we need to find $R\left(-\frac{b}{2a}\right)$, where $a=-6$ and $b=108$. Let's plug-n-chug this into $R(x)$ to find the maximum revenue.

$$\begin{aligned} R(x) &= -6x^2+108x \\ R\left(-\frac{b}{2a}\right) &= -6\left(-\frac{b}{2a}\right)^2+108\left(-\frac{b}{2a}\right) \\ R\left(-\frac{108}{2(-6)}\right) &= -6\left(-\frac{108}{2(-6)}\right)^2+108\left(-\frac{108}{2(-6)}\right) \\ R\left(-\frac{108}{2(-6)}\right) &= 486 \end{aligned}$$

Thus, the maximum revenus is $486.

Example 11.6.5

The cost, $C(x)$, of producing x *Totally Cool Coolers* is modeled by the function $C(x)=0.005x^2-0.3x+17$. What is the minimum cost?

Solution

To find the minimum cost, we need to find $C\left(-\frac{b}{2a}\right)$, where $a=0.005$ and $b=-0.3$. Let's plug-n-chug this into $C(x)$ to find the minimum cost.

$$\begin{aligned} C(x) &= 0.005x^2-0.3x+17 \\ C\left(-\frac{b}{2a}\right) &= 0.005\left(-\frac{b}{2a}\right)^2-0.3\left(-\frac{b}{2a}\right)+17 \\ C\left(-\frac{-0.3}{2(0.005)}\right) &= 0.005\left(-\frac{-0.3}{2(0.005)}\right)^2-0.3\left(-\frac{-0.3}{2(0.005)}\right)+17 \\ C\left(-\frac{-0.3}{2(0.005)}\right) &= 12.5 \end{aligned}$$

Note

From all the examples, we see the variety of methods in obtaining the extreme value of a quadratic function. We can either graph the function, find each coordinate of the vertex, or directly calculate the extreme value. It is at the discretion of the student to use any method. However, directly calculating the extreme value is recommended when only the extreme value is needed.

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CHAPTER OVERVIEW

4: Polynomial and Rational Functions

In this chapter, we will learn about these concepts and discover how mathematics can be used in such applications.

4.1: Power Functions and Polynomial Functions
4.2: Graphs of Polynomial Functions
4.3: Dividing Polynomials
4.4: Zeros of Polynomial Functions
4.5: Simplifying Rational Expressions
4.6: Rational Functions

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4.1: Power Functions and Polynomial Functions

Identifying Power Functions

In order to better understand the bird problem, we need to understand a specific type of function. A power function is a function with a single term that is the product of a real number, a coefficient, and a variable raised to a fixed real number. (A number that multiplies a variable raised to an exponent is known as a coefficient.)

As an example, consider functions for area or volume. The function for the area of a circle with radius r is

$$A(r) = \pi r^2$$

and the function for the volume of a sphere with radius r is

$$V(r) = \frac{4}{3}\pi r^3$$

Both of these are examples of power functions because they consist of a coefficient, π or $\frac{4}{3}\pi$, multiplied by a variable r raised to a power.

Definition: Power Function

A *power function* is a function that can be represented in the form

$$f(x) = kx^p \tag{4.1.1}$$

where k and p are real numbers, and k is known as the *coefficient*.

Q&A: Is $f(x) = 2^x$ a power function?

No. A power function contains a variable base raised to a fixed power (Equation 4.1.1). This function has a constant base raised to a variable power. This is called an exponential function, not a power function. This function will be discussed later.

Example 4.1.1: Identifying Power Functions

Which of the following functions are power functions?

$f(x) = 1$	Constant function
$f(x) = x$	Identify function
$f(x) = x^2$	Quadratic function
$f(x) = x^3$	Cubic function
$f(x) = \frac{1}{x}$	Reciprocal function
$f(x) = \frac{1}{x^2}$	Reciprocal squared function
$f(x) = \sqrt{x}$	Square root function
$f(x) = \sqrt[3]{x}$	Cube root function

Solution

All of the listed functions are power functions.

The constant and identity functions are power functions because they can be written as $f(x) = x^0$ and $f(x) = x^1$ respectively.

The quadratic and cubic functions are power functions with whole number powers $f(x) = x^2$ and $f(x) = x^3$.

The **reciprocal** and reciprocal squared functions are power functions with negative whole number powers because they can be written as $f(x) = x^{-1}$ and $f(x) = x^{-2}$.

The square and **cube root** functions are power functions with fractional powers because they can be written as $f(x) = x^{1/2}$ or $f(x) = x^{1/3}$.

Exercise 4.1.1

Which functions are power functions?

- $f(x) = 2x^2 \cdot 4x^3$
- $g(x) = -x^5 + 5x^3 - 4x$
- $h(x) = \frac{2x^5 - 1}{3x^2 + 4}$

Answer

$f(x)$ is a power function because it can be written as $f(x) = 8x^5$. The other functions are not power functions.

Identifying End Behavior of Power Functions

Figure 4.1.2 shows the graphs of $f(x) = x^2$, $g(x) = x^4$ and and $h(x) = x^6$, which are all power functions with even, whole-number powers. Notice that these graphs have similar shapes, very much like that of the quadratic function in the toolkit. However, as the power increases, the graphs flatten somewhat near the origin and become steeper away from the origin.

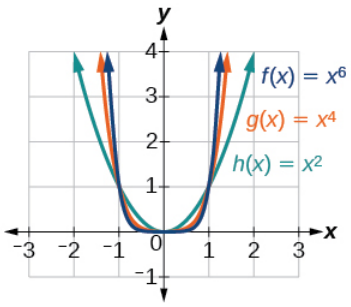


Figure 4.1.2: Even-power functions

To describe the behavior as numbers become larger and larger, we use the idea of infinity. We use the symbol ∞ for positive infinity and $-\infty$ for negative infinity. When we say that "x approaches infinity," which can be symbolically written as $x \to \infty$, we are describing a behavior; we are saying that x is increasing without bound.

With the even-power function, as the input increases or decreases without bound, the output values become very large, positive numbers. Equivalently, we could describe this behavior by saying that as x approaches positive or negative infinity, the $f(x)$ values increase without bound. In symbolic form, we could write

$$\text{as } x \to \pm\infty,\ f(x) \to \infty$$

Figure 4.1.3 shows the graphs of $f(x) = x^3$, $g(x) = x^5$, and $h(x) = x^7$, which are all power functions with odd, whole-number powers. Notice that these graphs look similar to the cubic function in the toolkit. Again, as the power increases, the graphs flatten near the origin and become steeper away from the origin.

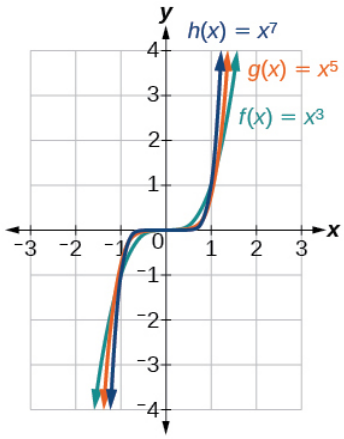


Figure 4.1.3: Odd-power function

These examples illustrate that functions of the form $f(x) = x^n$ reveal symmetry of one kind or another. First, in Figure 4.1.2 we see that even functions of the form $f(x) = x^n$, n even, are symmetric about the y-axis. In Figure 4.1.3 we see that odd functions of the form $f(x) = x^n$, n odd, are symmetric about the origin.

For these odd power functions, as x approaches negative infinity, $f(x)$ decreases without bound. As x approaches positive infinity, $f(x)$ increases without bound. In symbolic form we write

$$\text{as } x \to -\infty,\ f(x) \to -\infty$$
$$\text{as } x \to \infty,\ f(x) \to \infty$$

The behavior of the graph of a function as the input values get very small $(x \to -\infty)$ and get very large $x \to \infty$ is referred to as the *end behavior* of the function. We can use words or symbols to describe end behavior.

Figure 4.1.4 shows the end behavior of power functions in the form $f(x) = kx^n$ where n is a non-negative integer depending on the power and the constant.

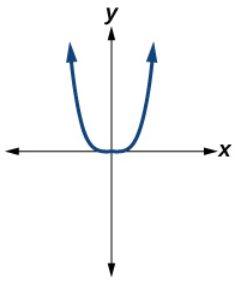
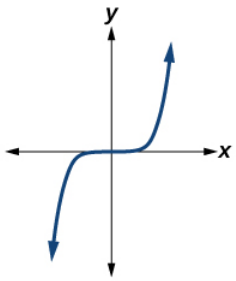
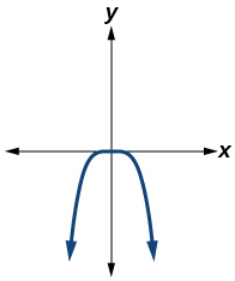
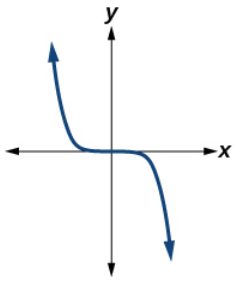
	Even power	Odd power
Positive constant $k > 0$	 $x \to -\infty$, $f(x) \to \infty$ and $x \to \infty$, $f(x) \to \infty$	 $x \to -\infty$, $f(x) \to -\infty$ and $x \to \infty$, $f(x) \to \infty$
Negative constant $k < 0$	 $x \to -\infty$, $f(x) \to -\infty$ and $x \to \infty$, $f(x) \to -\infty$	 $x \to -\infty$, $f(x) \to \infty$ and $x \to \infty$, $f(x) \to -\infty$

Figure 4.1.4

How To: Given a power function $f(x) = kx^n$ where n is a non-negative integer, identify the end behavior.

1. Determine whether the power is even or odd.
2. Determine whether the constant is positive or negative.
3. Use Figure 4.1.4 to identify the end behavior.

Example 4.1.2: Identifying the End Behavior of a Power Function

Describe the end behavior of the graph of $f(x) = x^8$.

Solution

The coefficient is 1 (positive) and the exponent of the power function is 8 (an even number). As x approaches infinity, the output (value of $f(x)$) increases without bound. We write as $x \to \infty$, $f(x) \to \infty$. As x approaches negative infinity, the output increases without bound. In symbolic form, as $x \to -\infty$, $f(x) \to \infty$. We can graphically represent the function as shown in Figure 4.1.5.

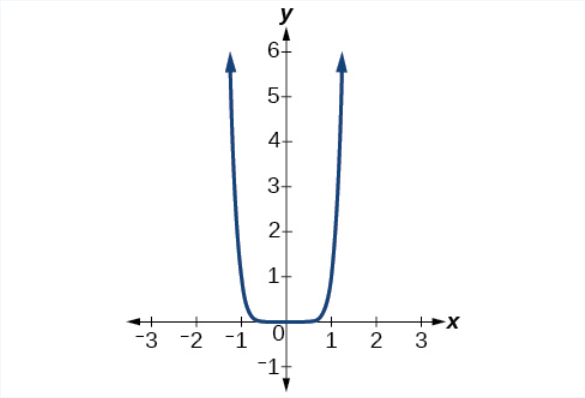


Figure 4.1.5: Graph of $f(x) = x^8$.

Example 4.1.3: Identifying the End Behavior of a Power Function.

Describe the end behavior of the graph of $f(x) = -x^9$.

Solution

The exponent of the power function is 9 (an odd number). Because the coefficient is –1 (negative), the graph is the reflection about the x-axis of the graph of $f(x) = x^9$. Figure 4.1.6 shows that as x approaches infinity, the output decreases without bound. As x approaches negative infinity, the output increases without bound. In symbolic form, we would write

$$\text{as } x \to -\infty,\ f(x) \to \infty$$
$$\text{as } x \to \infty,\ f(x) \to -\infty$$

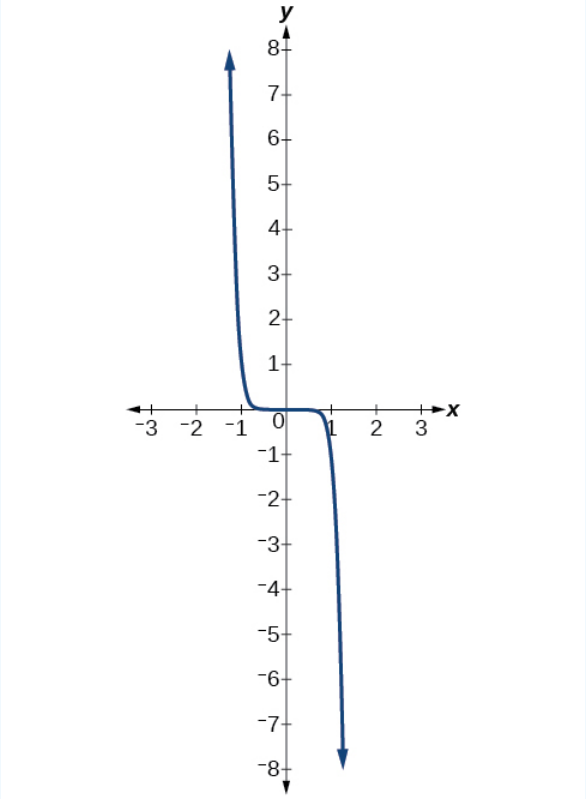


Figure 4.1.6: Graph of $f(x) = -x^9$.

Analysis

We can check our work by using the table feature on a graphing utility.

Table 4.1.2

x	$f(x)$
-10	1,000,000,000
-5	1,953,125
0	0
5	-1,953,125
10	-1,000,000,000

We can see from Table 4.1.2 that, when we substitute very small values for x, the output is very large, and when we substitute very large values for x, the output is very small (meaning that it is a very large negative value).

Exercise 4.1.2

Describe in words and symbols the end behavior of $f(x) = -5x^4$.

Answer

As x approaches positive or negative infinity, $f(x)$ decreases without bound: as $x \to \pm\infty$, $f(x) \to -\infty$ because of the negative coefficient.

Identifying Polynomial Functions

An oil pipeline bursts in the Gulf of Mexico, causing an oil slick in a roughly circular shape. The slick is currently 24 miles in radius, but that radius is increasing by 8 miles each week. We want to write a formula for the area covered by the oil slick by combining two functions. The radius r of the spill depends on the number of weeks w that have passed. This relationship is linear.

$$r(w) = 24 + 8w$$

We can combine this with the formula for the area A of a circle.

$$A(r) = \pi r^2$$

Composing these functions gives a formula for the area in terms of weeks.

$$\begin{aligned} A(w) &= A(r(w)) \\ &= A(24 + 8w) \\ &= \pi(24 + 8w)^2 \end{aligned}$$

Multiplying gives the formula.

$$A(w) = 576\pi + 384\pi w + 64\pi w^2$$

This formula is an example of a polynomial function. A polynomial function consists of either zero or the sum of a finite number of non-zero terms, each of which is a product of a number, called the coefficient of the term, and a variable raised to a non-negative integer power.

Definition: Polynomial Functions

Let n be a non-negative integer. A *polynomial function* is a function that can be written in the form

$$f(x) = a_n x^n + \ldots + a_2 x^2 + a_1 x + a_0 \tag{4.1.2}$$

This is called the general form of a polynomial function. Each a_i is a coefficient and can be any real number. Each product $a_i x^i$ is a *term of a polynomial function.*

Example 4.1.4: Identifying Polynomial Functions

Which of the following are polynomial functions?

- $f(x) = 2x^3 \cdot 3x + 4$
- $g(x) = -x(x^2 - 4)$
- $h(x) = 5\sqrt{x} + 2$

Solution

The first two functions are examples of polynomial functions because they can be written in the form of Equation 4.1.2, where the powers are non-negative integers and the coefficients are real numbers.

- $f(x)$ can be written as $f(x) = 6x^4 + 4$.
- $g(x)$ can be written as $g(x) = -x^3 + 4x$.
- $h(x)$ cannot be written in this form and is therefore not a polynomial function.

Identifying the Degree and Leading Coefficient of a Polynomial Function

Because of the form of a polynomial function, we can see an infinite variety in the number of terms and the power of the variable. Although the order of the terms in the polynomial function is not important for performing operations, we typically arrange the terms in descending order of power, or in general form. The degree of the polynomial is the highest power of the variable that occurs in the polynomial; it is the power of the first variable if the function is in general form. The leading term is the term containing the highest power of the variable, or the term with the highest degree. The leading coefficient is the coefficient of the leading term.

Terminology of Polynomial Functions

We often rearrange polynomials so that the powers are descending.

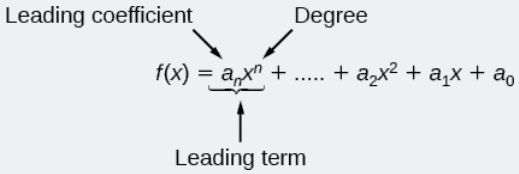


Figure 4.1.7

When a polynomial is written in this way, we say that it is in *general form*.

How To: Given a polynomial function, identify the degree and leading coefficient

1. Find the highest power of x to determine the degree function.
2. Identify the term containing the highest power of x to find the leading term.
3. Identify the coefficient of the leading term.

Example 4.1.5: Identifying the Degree and Leading Coefficient of a Polynomial Function

Identify the degree, leading term, and leading coefficient of the following polynomial functions.

$$f(x) = 3 + 2x^2 - 4x^3$$
$$g(t) = 5t^5 - 2t^3 + 7t$$
$$h(p) = 6p - p^3 - 2$$

Solution

For the function $f(x)$, the highest power of x is 3, so the degree is 3. The leading term is the term containing that degree, $-4x^3$. The leading coefficient is the coefficient of that term, -4.

For the function $g(t)$, the highest power of t is 5, so the degree is 5. The leading term is the term containing that degree, $5t^5$. The leading coefficient is the coefficient of that term, 5.

For the function $h(p)$, the highest power of p is 3, so the degree is 3. The leading term is the term containing that degree, $-p^3$; the leading coefficient is the coefficient of that term, -1.

Exercise 4.1.3

Identify the degree, leading term, and leading coefficient of the polynomial $f(x) = 4x^2 - x^6 + 2x - 6$.

Answer

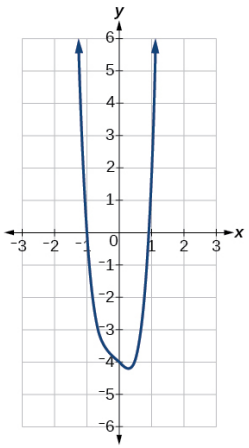
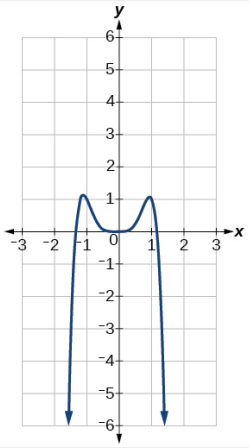
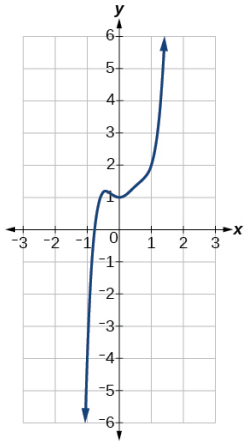
The degree is 6. The leading term is $-x^6$. The leading coefficient is -1.

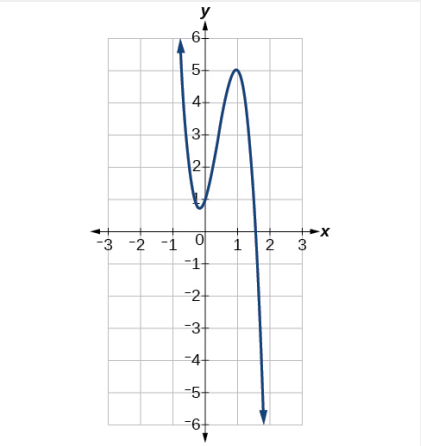
Identifying End Behavior of Polynomial Functions

Knowing the degree of a polynomial function is useful in helping us predict its end behavior. To determine its end behavior, look at the leading term of the polynomial function. Because the power of the leading term is the highest, that term will grow significantly faster than the other terms as x gets very large or very small, so its behavior will dominate the graph. For any polynomial, the end behavior of the polynomial will match the end behavior of the term of highest degree (Table $4.1.3$).

Table 4.1.3

Polynomial Function	Leading Term	Graph of Polynomial Function

Polynomial Function	Leading Term	Graph of Polynomial Function
$f(x) = 5x^4 + 2x^3 - x - 4$	$5x^4$	
$f(x) = -2x^6 - x^5 + 3x^4 + x^3$	$-2x^6$	
$f(x) = 3x^5 - 4x^4 + 2x^2 + 1$	$3x^5$	

Polynomial Function	Leading Term	Graph of Polynomial Function
$f(x) = -6x^3 + 7x^2 + 3x + 1$	$-6x^3$	

Example 4.1.6: Identifying End Behavior and Degree of a Polynomial Function

Describe the end behavior and determine a possible degree of the polynomial function in Figure 4.1.8.

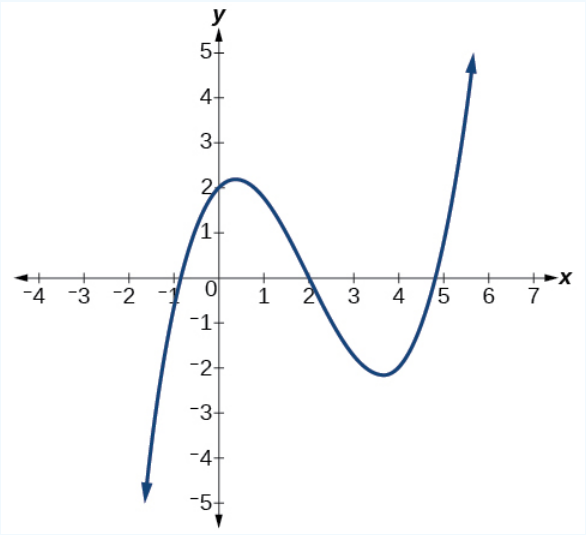


Figure 4.1.8.

Solution

As the input values x get very large, the output values $f(x)$ increase without bound. As the input values x get very small, the output values $f(x)$ decrease without bound. We can describe the end behavior symbolically by writing

$$\text{as } x \to \infty,\ f(x) \to \infty$$

$$\text{as } x \to -\infty,\ f(x) \to -\infty$$

In words, we could say that as x values approach infinity, the function values approach infinity, and as x values approach negative infinity, the function values approach negative infinity.

We can tell this graph has the shape of an odd degree power function that has not been reflected, so the degree of the polynomial creating this graph must be odd and the leading coefficient must be positive.

Exercise 4.1.1

Describe the end behavior, and determine a possible degree of the polynomial function in Figure 4.1.9.

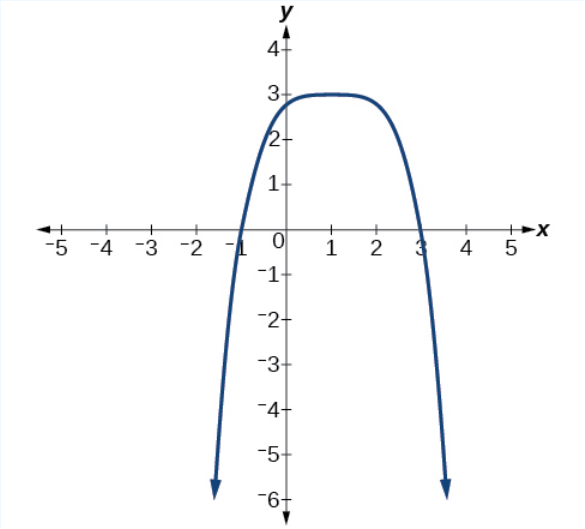


Figure 4.1.9

Answer

As $x \to \infty$, $f(x) \to -\infty$; as $x \to -\infty$, $f(x) \to -\infty$. It has the shape of an even degree power function with a negative coefficient.

Example 4.1.7: Identifying End Behavior and Degree of a Polynomial Function

Given the function $f(x) = -3x^2(x-1)(x+4)$, express the function as a polynomial in general form, and determine the leading term, degree, and end behavior of the function.

Solution

Obtain the general form by expanding the given expression for $f(x)$.

$$\begin{aligned} f(x) &= -3x^2(x-1)(x+4) \\ &= -3x^2(x^2+3x-4) \\ &= -3x^4-9x^3+12x^2 \end{aligned}$$

The general form is $f(x) = -3x^4 - 9x^3 + 12x^2$. The leading term is $-3x^4$; therefore, the degree of the polynomial is 4. The degree is even (4) and the leading coefficient is negative (–3), so the end behavior is

$$\text{as } x \to -\infty,\ f(x) \to -\infty$$

$$\text{as } x \to \infty,\ f(x) \to -\infty$$

Exercise 4.1.7

Given the function $f(x) = 0.2(x-2)(x+1)(x-5)$, express the function as a polynomial in general form and determine the leading term, degree, and end behavior of the function.

Answer

The leading term is $0.2x^3$, so it is a degree 3 polynomial. As x approaches positive infinity, $f(x)$ increases without bound; as x approaches negative infinity, $f(x)$ decreases without bound.

Identifying Local Behavior of Polynomial Functions

In addition to the end behavior of polynomial functions, we are also interested in what happens in the "middle" of the function. In particular, we are interested in locations where graph behavior changes. A **turning point** is a point at which the function values change from increasing to decreasing or decreasing to increasing.

We are also interested in the intercepts. As with all functions, the y-intercept is the point at which the graph intersects the vertical axis. The point corresponds to the coordinate pair in which the input value is zero. Because a polynomial is a function, only one output value corresponds to each input value so there can be only one y-intercept $(0, a_0)$. The x-intercepts occur at the input values that correspond to an output value of zero. It is possible to have more than one x-intercept. See Figure $4.1.10$.

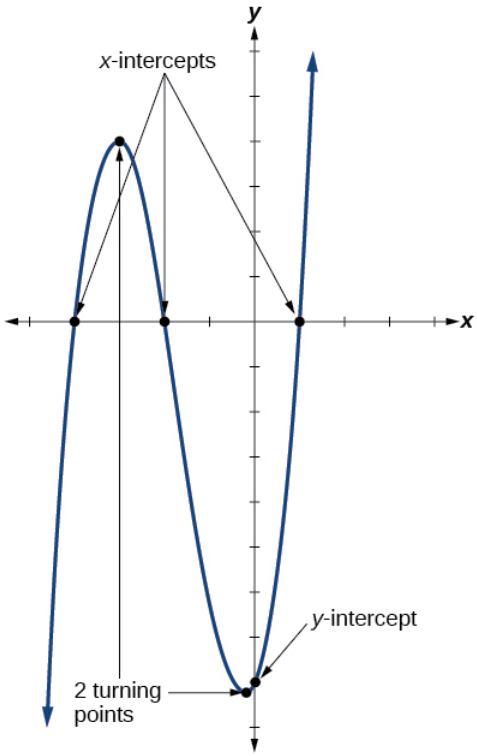


Figure 4.1.10

Defintion: Intercepts and Turning Points of Polynomial Functions

A **turning point** of a graph is a point at which the graph changes direction from increasing to decreasing or decreasing to increasing. The y-intercept is the point at which the function has an input value of zero. The x-intercepts are the points at which the output value is zero.

How To: Given a polynomial function, determine the intercepts.

1. Determine the y-intercept by setting $x = 0$ and finding the corresponding output value.
2. Determine the x-intercepts by solving for the input values that yield an output value of zero.

Example 4.1.8: Determining the Intercepts of a Polynomial Function

Given the polynomial function $f(x) = (x-2)(x+1)(x-4)$, written in factored form for your convenience, determine the y- and x-intercepts.

Solution

The y-intercept occurs when the input is zero, so substitute 0 for x.

$$\begin{aligned} f(0) &= (0-2)(0+1)(0-4) \\ &= (-2)(1)(-4) \\ &= 8 \end{aligned}$$

The y-intercept is $(0, 8)$.

The x-intercepts occur when the output is zero.

$$0 = (x-2)(x+1)(x-4)$$

$$\begin{aligned} x-2 &= 0 \quad \text{or} \quad x+1 = 0 \quad \text{or} \quad x-4 = 0 \\ x &= 2 \quad \text{or} \quad x = -1 \quad \text{or} \quad x = 4 \end{aligned}$$

The x-intercepts are $(2, 0)$,$(-1, 0)$, and $(4, 0)$.

We can see these intercepts on the graph of the function shown in Figure 4.1.11.

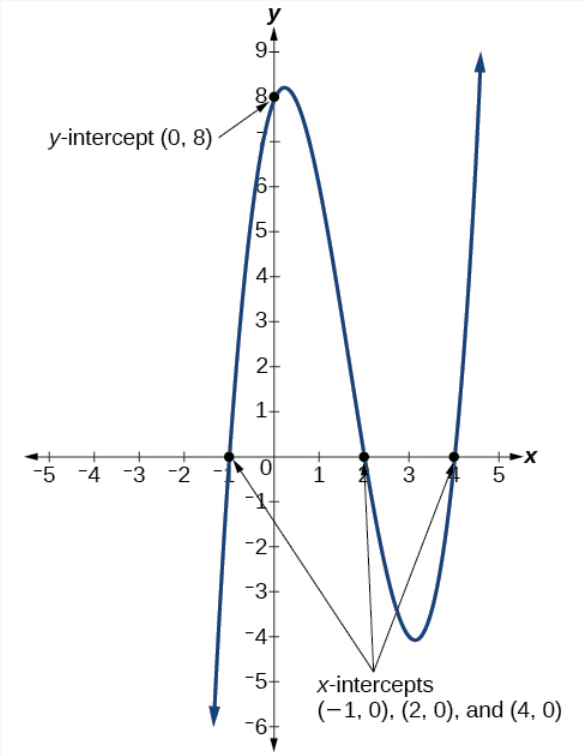


Figure 4.1.11: Graph of $f(x) = (x-2)(x+1)(x-4)$.

Example 4.1.9: Determining the Intercepts of a Polynomial Function with Factoring

Given the polynomial function $f(x) = x^4 - 4x^2 - 45$, determine the y- and x-intercepts.

Solution

The y-intercept occurs when the input is zero.

$$\begin{aligned} f(0) &= (0)^4 - 4(0)^2 - 45 \\ &= -45 \end{aligned}$$

The y-intercept is $(0, -45)$.

The x-intercepts occur when the output is zero. To determine when the output is zero, we will need to factor the polynomial.

$$\begin{aligned} f(x) &= x^4 - 4x^2 - 45 \\ &= (x^2 - 9)(x^2 + 5) \\ &= (x-3)(x+3)(x^2+5) \end{aligned}$$

$$0 = (x-3)(x+3)(x^2+5)$$

$$\begin{array}{lllll} x-3=0 & \text{or} & x+3=0 & \text{or} & x^2+5=0 \\ x=3 & \text{or} & x=-3 & \text{or} & \text{(no real solution)} \end{array}$$

The x-intercepts are $(3,0)$ and $(-3,0)$.

We can see these intercepts on the graph of the function shown in Figure 4.1.12 We can see that the function is even because $f(x)=f(-x)$.

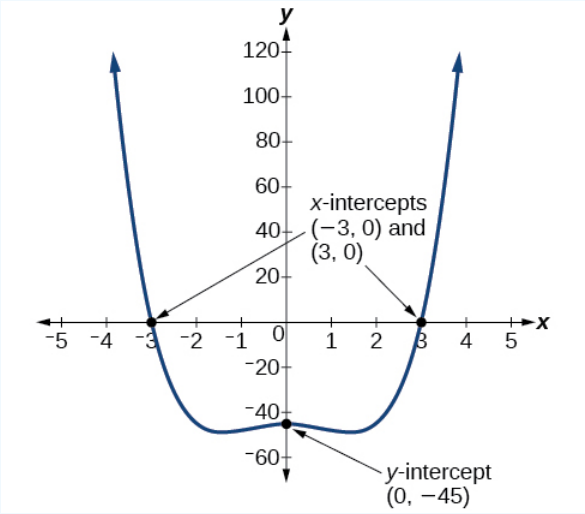


Figure 4.1.12: Graph of $f(x)=x^4-4x^2-45$.

Exercise 4.1.5

4.1.5: Given the polynomial function $f(x)=2x^3-6x^2-20x$, determine the y- and x-intercepts.

Solution

y-intercept $(0,0)$; x-intercepts $(0,0)$,$(-2,0)$, and $(5,0)$

Comparing Smooth and Continuous Graphs

The degree of a polynomial function helps us to determine the number of x-intercepts and the number of turning points. A polynomial function of nth degree is the product of n factors, so it will have at most n roots or zeros, or x-intercepts. The graph of the polynomial function of degree n must have at most $n-1$ turning points. This means the graph has at most one fewer turning point than the degree of the polynomial or one fewer than the number of factors.

A **continuous function** has no breaks in its graph: the graph can be drawn without lifting the pen from the paper. A smooth curve is a graph that has no sharp corners. The turning points of a smooth graph must always occur at rounded curves. The graphs of polynomial functions are both continuous and smooth.

Intercepts and Turning Points of Polynomials

A polynomial of degree n will have, at most, n x-intercepts and $n-1$ turning points.

Example 4.1.10: Determining the Number of Intercepts and Turning Points of a Polynomial

Without graphing the function, determine the local behavior of the function by finding the maximum number of x-intercepts and turning points for $f(x)=-3x^{10}+4x^7-x^4+2x^3$.

Solution

The polynomial has a degree of **10**, so there are at most n x-intercepts and at most $n-1$ turning points.

Exercise 4.1.6

Without graphing the function, determine the maximum number of x-intercepts and turning points for $f(x) = 108 - 13x^9 - 8x^4 + 14x^{12} + 2x^3$

Answer

There are at most 12 x-intercepts and at most 11 turning points.

Example 4.1.11: Drawing Conclusions about a Polynomial Function from the Graph

What can we conclude about the polynomial represented by the graph shown in Figure 4.1.12 based on its intercepts and turning points?

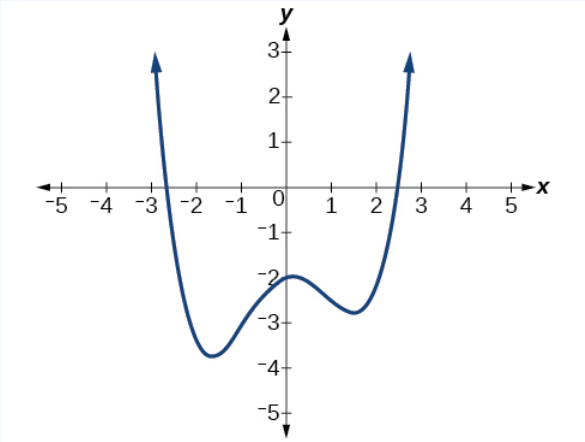


Figure 4.1.13.

Solution

The end behavior of the graph tells us this is the graph of an even-degree polynomial. See Figure 4.1.14

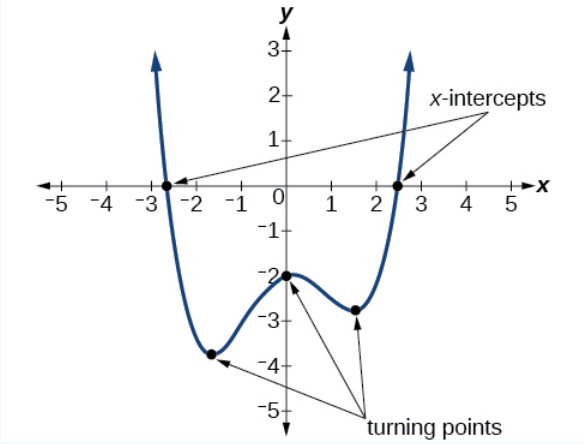


Figure 4.1.14: Graph of an even-degree polynomial.

The graph has 2 x-intercepts, suggesting a degree of 2 or greater, and 3 turning points, suggesting a degree of 4 or greater. Based on this, it would be reasonable to conclude that the degree is even and at least 4.

Exercise 4.1.7

What can we conclude about the polynomial represented by the graph shown in Figure 4.1.15 based on its intercepts and turning points?

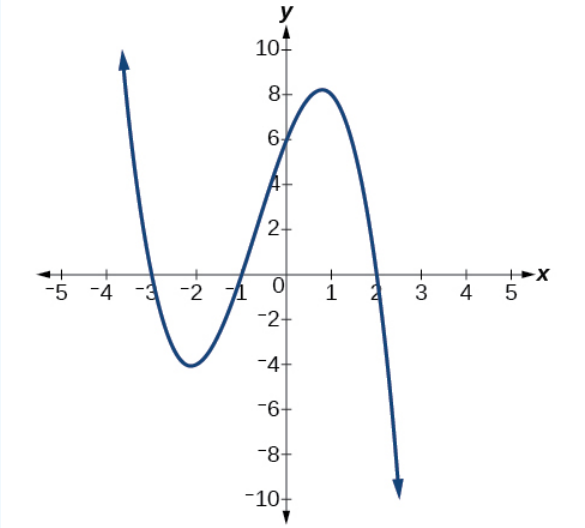


Figure 4.1.15.

Answer

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Solution

The end behavior indicates an odd-degree polynomial function; there are 3 x-intercepts and 2 turning points, so the degree is odd and at least 3. Because of the end behavior, we know that the lead coefficient must be negative.

Example 4.1.12: Drawing Conclusions about a Polynomial Function from the Factors

Given the function $f(x) = -4x(x+3)(x-4)$, determine the local behavior.

Solution

The y-intercept is found by evaluating $f(0)$.

$$\begin{aligned} f(0) &= -4(0)(0+3)(0-4) \\ &= 0 \end{aligned}$$

The y-intercept is $(0, 0)$.

The x-intercepts are found by determining the zeros of the function.

$$\begin{aligned} 0 &= -4x(x+3)(x-4) \\ x &= 0 \quad \text{or} \quad x+3 = 0 \quad \text{or} \quad x-4 = 0 \\ x &= 0 \quad \text{or} \quad x = -3 \quad \text{or} \quad x = 4 \end{aligned}$$

The x-intercepts are $(0, 0)$,$(-3, 0)$, and $(4, 0)$.

The degree is 3 so the graph has at most 2 turning points.

Exercise 4.1.8

Given the function $f(x) = 0.2(x-2)(x+1)(x-5)$, determine the local behavior.

Answer

The x-intercepts are $(2,0)$, $(-1,0)$, and $(5,0)$, the y-intercept is $(0,2)$, and the graph has at most 2 turning points.

Key Equations

- general form of a polynomial function: $f(x) = a_n x^n + a_{n-1} x^{n-1} \ldots + a_2 x^2 + a_1 x + a_0$

Key Concepts

- A power function is a variable base raised to a number power.
- The behavior of a graph as the input decreases without bound and increases without bound is called the end behavior.
- The end behavior depends on whether the power is even or odd.
- A polynomial function is the sum of terms, each of which consists of a transformed power function with positive whole number power.
- The degree of a polynomial function is the highest power of the variable that occurs in a polynomial. The term containing the highest power of the variable is called the leading term. The coefficient of the leading term is called the leading coefficient.
- The end behavior of a polynomial function is the same as the end behavior of the power function represented by the leading term of the function.
- A polynomial of degree n will have at most n x-intercepts and at most $n-1$ turning points.

Glossary

coefficient

a nonzero real number that is multiplied by a variable raised to an exponent (only the number factor is the coefficient)

continuous function

a function whose graph can be drawn without lifting the pen from the paper because there are no breaks in the graph

degree

the highest power of the variable that occurs in a polynomial

end behavior

the behavior of the graph of a function as the input decreases without bound and increases without bound

leading coefficient

the coefficient of the leading term

leading term

the term containing the highest power of the variable

polynomial function

a function that consists of either zero or the sum of a finite number of non-zero terms, each of which is a product of a number, called the coefficient of the term, and a variable raised to a non-negative integer power.

power function

a function that can be represented in the form $f(x) = kx^p$ where k is a constant, the base is a variable, and the exponent, p, is a constant

smooth curve

a graph with no sharp corners

term of a polynomial function

any $a_i x^i$ of a polynomial function in the form $f(x) = a_n x^n + a_{n-1} x^{n-1} \ldots + a_2 x^2 + a_1 x + a_0$

turning point

the location at which the graph of a function changes direction

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4.2: Graphs of Polynomial Functions

Recognizing Characteristics of Graphs of Polynomial Functions

Polynomial functions of degree 2 or more have graphs that do not have sharp corners; recall that these types of graphs are called smooth curves. Polynomial functions also display graphs that have no breaks. Curves with no breaks are called continuous. Figure 4.2.1 shows a graph that represents a **polynomial function** and a graph that represents a function that is not a polynomial.

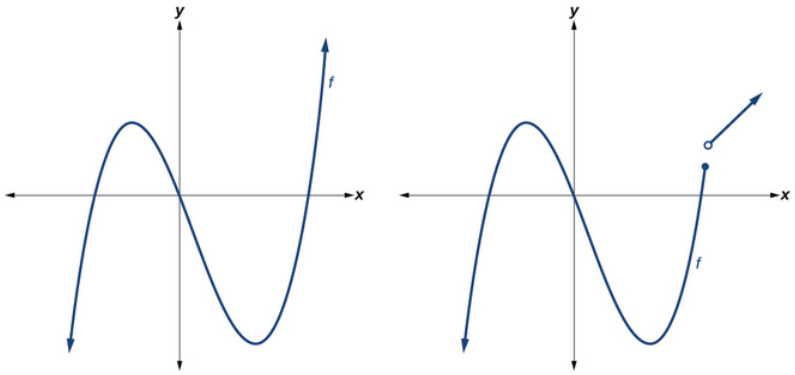


Figure 4.2.1: Graph of $f(x) = x^3 - 0.01x$.

Example 4.2.1: Recognizing Polynomial Functions

Which of the graphs in Figure 4.2.2 represents a polynomial function?

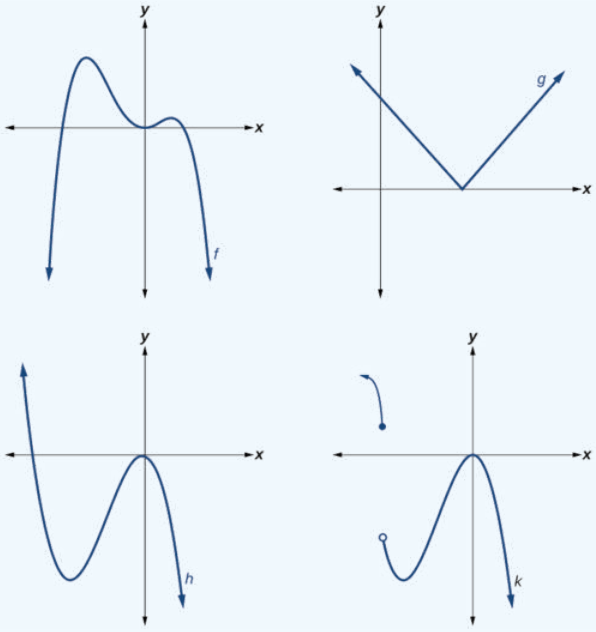


***Figure* 4.2.2**

Solution

- The graphs of f and h are graphs of polynomial functions. They are smooth and **continuous**.
- The graphs of g and k are graphs of functions that are not polynomials. The graph of function g has a sharp corner. The graph of function k is not continuous.

Q&A

Do all polynomial functions have as their domain all real numbers?

- Yes. Any real number is a valid input for a polynomial function.

Using Factoring to Find Zeros of Polynomial Functions

Recall that if f is a polynomial function, the values of x for which $f(x)=0$ are called **zeros** of f. If the equation of the polynomial function can be factored, we can set each factor equal to zero and solve for the zeros.

We can use this method to find x-intercepts because at the x-intercepts we find the input values when the output value is zero. For general polynomials, this can be a challenging prospect. While quadratics can be solved using the relatively simple quadratic formula, the corresponding formulas for cubic and fourth-degree polynomials are not simple enough to remember, and formulas do not exist for general higher-degree polynomials. Consequently, we will limit ourselves to three cases in this section:

The polynomial can be factored using known methods: greatest common factor and trinomial factoring.
The polynomial is given in factored form.
Technology is used to determine the intercepts.

HOwTO: Given a polynomial function f, find the x-intercepts by factoring

1. Set $f(x)=0$.
2. If the polynomial function is not given in factored form:
   a. Factor out any common monomial factors.
   b. Factor any factorable binomials or trinomials.
3. Set each factor equal to zero and solve to find the x-intercepts.

Example 4.2.2: Finding the x-Intercepts of a Polynomial Function by Factoring

Find the x-intercepts of $f(x)=x^6-3x^4+2x^2$.

Solution

We can attempt to factor this polynomial to find solutions for $f(x)=0$.

$$\begin{aligned} x^6-3x^4+2x^2 &= 0 && \text{Factor out the greatest common factor.}\\ x^2(x^4-3x^2+2) &= 0 && \text{Factor the trinomial.}\\ x^2(x^2-1)(x^2-2) &= 0 && \text{Set each factor equal to zero.}\end{aligned}$$

$$\begin{aligned} x^2 &= 0 \\ x^2 &= 0 \\ x &= 0 \end{aligned} \quad \text{or} \quad \begin{aligned} (x^2-1) &= 0 \\ x^2 &= 1 \\ x &= \pm 1 \end{aligned} \quad \text{or} \quad \begin{aligned} (x^2-2) &= 0 \\ x^2 &= 2 \\ x &= \pm\sqrt{2} \end{aligned}$$

.

This gives us five x-intercepts: $(0,0)$, $(1,0)$, $(-1,0)$, $(\sqrt{2},0)$,and $(-\sqrt{2},0)$ (Figure 4.2.3). We can see that this is an even function.

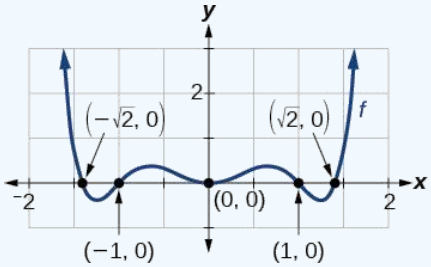


Figure 4.2.3.

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Example 4.2.3: Finding the x-Intercepts of a Polynomial Function by Factoring

Find the x-intercepts of $f(x) = x^3 - 5x^2 - x + 5$.

Solution

Find solutions for $f(x) = 0$ by factoring.

$$\begin{aligned} x^3 - 5x^2 - x + 5 &= 0 && \text{Factor by grouping.} \\ x^2(x-5) - (x-5) &= 0 && \text{Factor out the common factor.} \\ (x^2-1)(x-5) &= 0 && \text{Factor the difference of squares.} \\ (x+1)(x-1)(x-5) &= 0 && \text{Set each factor equal to zero.} \end{aligned}$$

$$\begin{aligned} x+1 &= 0 \quad \text{or} \quad x-1 = 0 \quad \text{or} \quad x-5 = 0 \\ x &= -1 \qquad\quad x = 1 \qquad\quad x = 5 \end{aligned}$$

There are three x-intercepts: $(-1, 0)$, $(1, 0)$, and $(5, 0)$ (Figure 4.2.4).

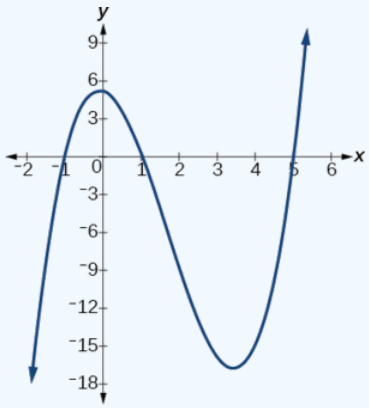


Figure 4.2.4: Graph of $f(x)$.

Example 4.2.4: Finding the y- and x-Intercepts of a Polynomial in Factored Form

Find the y- and x-intercepts of $g(x) = (x-2)^2(2x+3)$.

Solution

The y-intercept can be found by evaluating $g(0)$.

$$\begin{aligned} g(0) &= (0-2)^2(2(0)+3) \\ &= 12 \end{aligned}$$

So the y-intercept is $(0, 12)$.

The x-intercepts can be found by solving $g(x) = 0$.

$$(x-2)^2(2x+3) = 0 \tag{4.2.1}$$

$$\begin{aligned} (x-2)^2 &= 0 \qquad\quad (2x+3) = 0 \\ x-2 &= 0 \quad \text{or} \quad x = -\frac{3}{2} \\ x &= 2 \end{aligned}$$

So the x-intercepts are $(2, 0)$ and $\left(-\frac{3}{2}, 0\right)$.

Analysis

We can always check that our answers are reasonable by using a graphing calculator to graph the polynomial as shown in Figure 4.2.5.

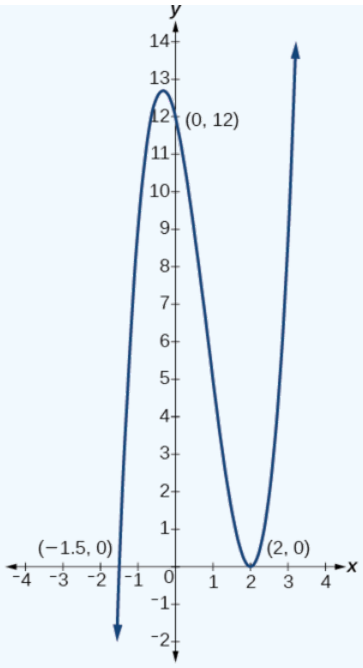


Figure 4.2.5: Graph of $g(x)$.

Example 4.2.5: Finding the x-Intercepts of a Polynomial Function Using a Graph

Find the x-intercepts of $h(x) = x^3 + 4x^2 + x - 6$.

Solution

This polynomial is not in factored form, has no common factors, and does not appear to be factorable using techniques previously discussed. Fortunately, we can use technology to find the intercepts. Keep in mind that some values make graphing difficult by hand. In these cases, we can take advantage of graphing utilities.

Looking at the graph of this function, as shown in Figure 4.2.6, it appears that there are x-intercepts at $x = -3, -2, \text{ and } 1$.

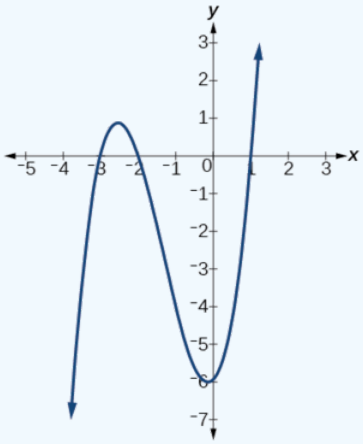


Figure 4.2.6: Graph of $h(x)$.

We can check whether these are correct by substituting these values for x and verifying that

$$h(-3) = h(-2) = h(1) = 0.$$

Since $h(x) = x^3 + 4x^2 + x - 6$, we have:

$$\begin{aligned} h(-3) &= (-3)^3+4(-3)^2+(-3)-6=-27+36-3-6=0 \\ h(-2) &= (-2)^3+4(-2)^2+(-2)-6=-8+16-2-6=0 \\ h(1) &= (1)^3+4(1)^2+(1)-6=1+4+1-6=0 \end{aligned}$$

Each x-intercept corresponds to a zero of the polynomial function and each zero yields a factor, so we can now write the polynomial in factored form.

$$\begin{aligned} h(x) &= x^3+4x^2+x-6 \\ &= (x+3)(x+2)(x-1) \end{aligned}$$

Exercise 4.2.1

Find the y-and x-intercepts of the function $f(x)=x^4-19x^2+30x$.

Answer

- y-intercept $(0,0)$;
- x-intercepts $(0,0)$, $(-5,0)$, $(2,0)$, and $(3,0)$

Identifying Zeros and Their Multiplicities

Graphs behave differently at various x-intercepts. Sometimes, the graph will cross over the horizontal axis at an intercept. Other times, the graph will touch the horizontal axis and bounce off. Suppose, for example, we graph the function

$$f(x)=(x+3)(x-2)^2(x+1)^3. \tag{4.2.2}$$

Notice in Figure $4.2.7$ that the behavior of the function at each of the x-intercepts is different.

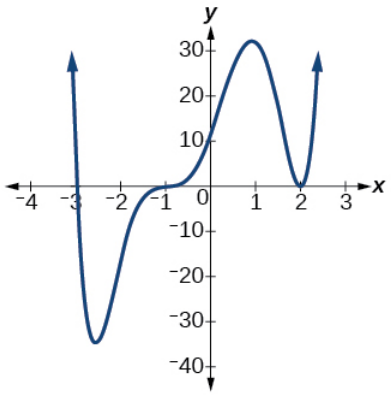


Figure 4.2.7: Identifying the behavior of the graph at an x-intercept by examining the multiplicity of the zero.

The x-intercept −3 is the solution of equation $(x+3)=0$. The graph passes directly through thex-intercept at $x=-3$. The factor is linear (has a degree of 1), so the behavior near the intercept is like that of a line—it passes directly through the intercept. We call this a single zero because the zero corresponds to a single factor of the function.

The x-intercept 2 is the repeated solution of equation $(x-2)^2=0$. The graph touches the axis at the intercept and changes direction. The factor is quadratic (degree 2), so the behavior near the intercept is like that of a quadratic—it bounces off of the horizontal axis at the intercept.

$$(x-2)^2=(x-2)(x-2) \tag{4.2.3}$$

The factor is repeated, that is, the factor $(x-2)$ appears twice. The number of times a given factor appears in the factored form of the equation of a polynomial is called the **multiplicity**. The zero associated with this factor, $x=2$, has multiplicity 2 because the factor $(x-2)$ occurs twice.

The x-intercept −1 is the repeated solution of factor $(x+1)^3=0$ .The graph passes through the axis at the intercept, but flattens out a bit first. This factor is cubic (degree 3), so the behavior near the intercept is like that of a cubic—with the same S-shape near

the intercept as the toolkit function $f(x) = x^3$. We call this a triple zero, or a zero with multiplicity 3.

For **zeros** with even multiplicities, the graphs touch or are tangent to the x-axis. For zeros with odd multiplicities, the graphs cross or intersect the x-axis. See Figure $4.2.8$ for examples of graphs of polynomial functions with multiplicity 1, 2, and 3.

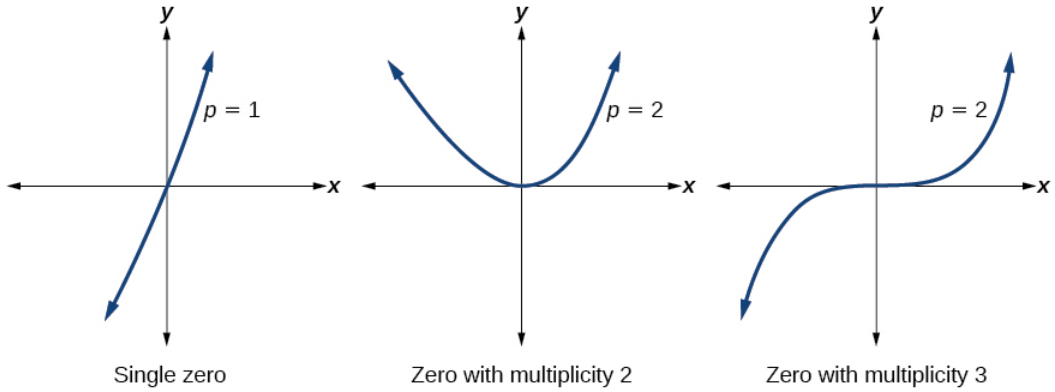


Figure $4.2.8$: Three graphs showing three different polynomial functions with multiplicity 1, 2, and 3.

For higher even powers, such as 4, 6, and 8, the graph will still touch and bounce off of the horizontal axis but, for each increasing even power, the graph will appear flatter as it approaches and leaves the x-axis.

For higher odd powers, such as 5, 7, and 9, the graph will still cross through the horizontal axis, but for each increasing odd power, the graph will appear flatter as it approaches and leaves the x-axis.

Graphical Behavior of Polynomials at x-Intercepts

If a polynomial contains a factor of the form $(x-h)^p$, the behavior near the x-intercepth is determined by the power p. We say that $x = h$ is a zero of **multiplicity** p.

The graph of a polynomial function will touch the x-axis at zeros with even multiplicities. The graph will cross the x-axis at zeros with odd multiplicities.

The sum of the multiplicities is the degree of the polynomial function.

HOWTO: Given a graph of a polynomial function of degree n, identify the zeros and their multiplicities

1. If the graph crosses the x-axis and appears almost linear at the intercept, it is a single zero.
2. If the graph touches the x-axis and bounces off of the axis, it is a zero with even multiplicity.
3. If the graph crosses the x-axis at a zero, it is a zero with odd multiplicity.
4. The sum of the multiplicities is n.

Example $4.2.6$: Identifying Zeros and Their Multiplicities

Use the graph of the function of degree 6 in Figure $4.2.9$ to identify the zeros of the function and their possible multiplicities.

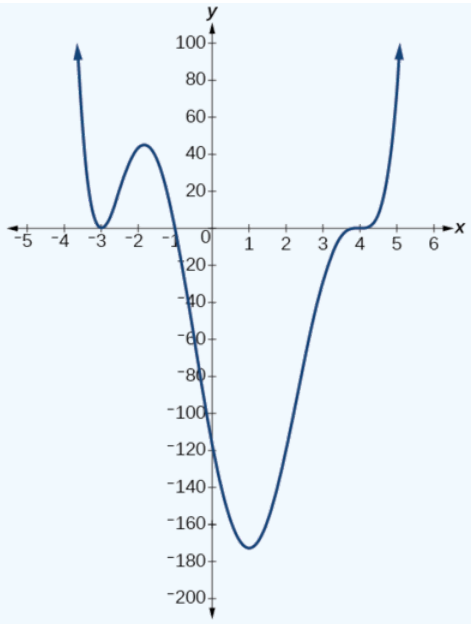


Figure 4.2.9: Graph of a polynomial function with degree 5.

Solution

The polynomial function is of degree n. The sum of the multiplicities must be n.

Starting from the left, the first zero occurs at $x = -3$. The graph touches the x-axis, so the multiplicity of the zero must be even. The zero of −3 has multiplicity 2.

The next zero occurs at $x = -1$. The graph looks almost linear at this point. This is a single zero of multiplicity 1.

The last zero occurs at $x = 4$.The graph crosses the x-axis, so the multiplicity of the zero must be odd. We know that the multiplicity is likely 3 and that the sum of the multiplicities is likely 6.

Exercise 4.2.2

Use the graph of the function of degree 5 in Figure 4.2.10to identify the zeros of the function and their multiplicities.

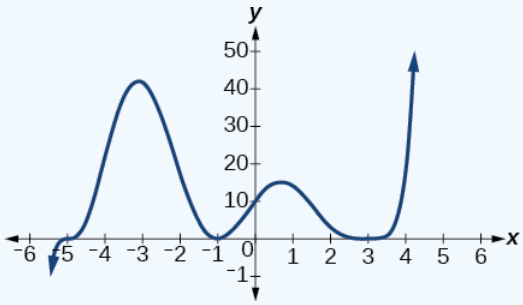


Figure* 4.2.10*: *Graph of a polynomial function with degree 5.*

Answer

The graph has a zero of –5 with multiplicity 1, a zero of –1 with multiplicity 2, and a zero of 3 with even multiplicity.

Determining End Behavior

As we have already learned, the behavior of a graph of a polynomial function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \tag{4.2.4}$$

will either ultimately rise or fall as x increases without bound and will either rise or fall as x decreases without bound. This is because for very large inputs, say 100 or 1,000, the leading term dominates the size of the output. The same is true for very small inputs, say –100 or –1,000.

Recall that we call this behavior the end behavior of a function. As we pointed out when discussing quadratic equations, when the leading term of a polynomial function, $a_n x^n$, is an even power function, as x increases or decreases without bound, $f(x)$ increases without bound. When the leading term is an odd power function, as x decreases without bound, $f(x)$ also decreases without bound; as x increases without bound, $f(x)$ also increases without bound. If the leading term is negative, it will change the direction of the end behavior. Figure $4.2.11$ summarizes all four cases.

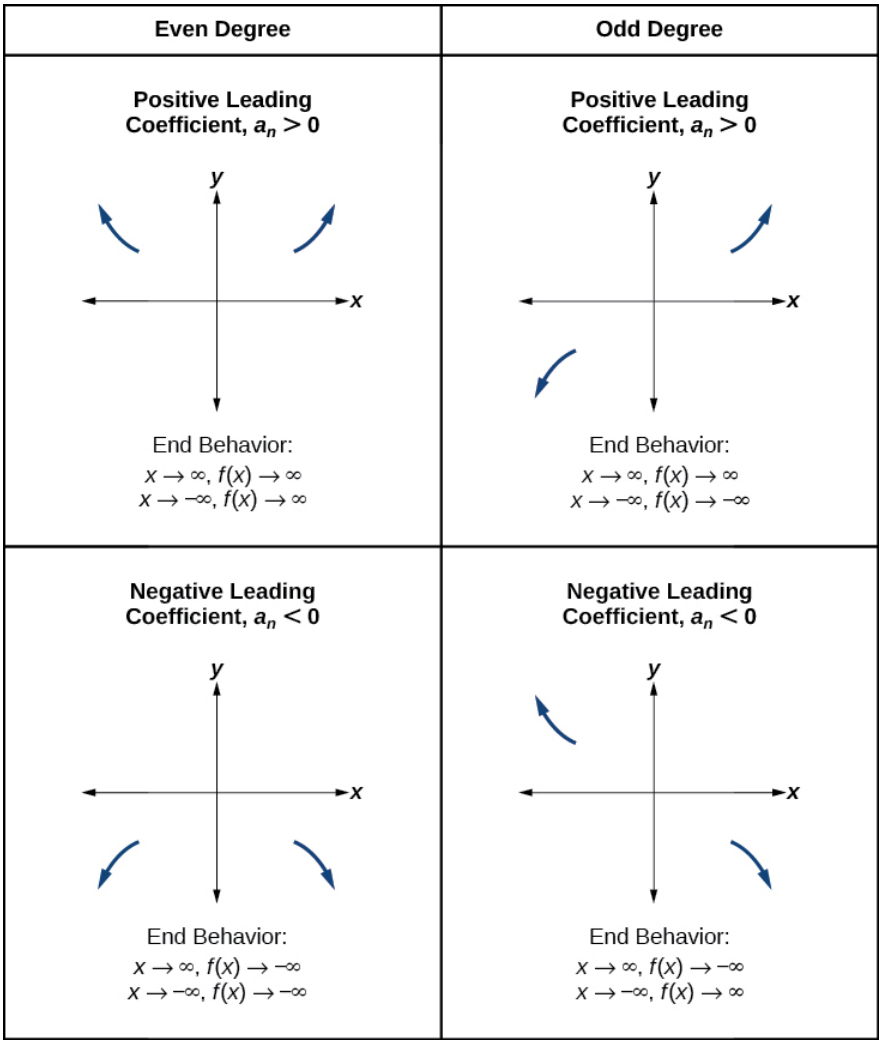


Figure 4.2.11.

Understanding the Relationship between Degree and Turning Points

In addition to the end behavior, recall that we can analyze a polynomial function's local behavior. It may have a turning point where the graph changes from increasing to decreasing (rising to falling) or decreasing to increasing (falling to rising). Look at the graph of the polynomial function $f(x) = x^4 - x^3 - 4x^2 + 4x$ in Figure $4.2.12$ The graph has three turning points.

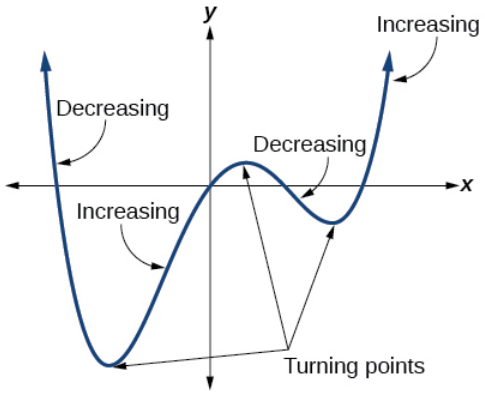


Figure 4.2.12: Graph of $f(x) = x^4 - x^3 - 4x^2 + 4x$

This function f is a 4th degree polynomial function and has 3 turning points. The maximum number of turning points of a polynomial function is always one less than the degree of the function.

Definition: Interpreting Turning Points

A turning point is a point of the graph where the graph changes from increasing to decreasing (rising to falling) or decreasing to increasing (falling to rising). A polynomial of degree n will have at most $n-1$ turning points.

Example 4.2.7: Finding the Maximum Number of Turning Points Using the Degree of a Polynomial Function

Find the maximum number of turning points of each polynomial function.

a. $f(x) = -x^3 + 4x^5 - 3x^2 + 1$
b. $f(x) = -(x-1)^2(1+2x^2)$

Solution

a. $f(x) = -x^3 + 4x^5 - 3x^2 + 1$

First, rewrite the polynomial function in descending order: $f(x) = 4x^5 - x^3 - 3x^2 + 1$

Identify the degree of the polynomial function. This polynomial function is of degree 5.

The maximum number of turning points is $5 - 1 = 4$.

b. $f(x) = -(x-1)^2(1+2x^2)$

First, identify the leading term of the polynomial function if the function were expanded.

$$f(x) = -(x-1)^2\ (1+2x^2)$$
$$a_n = -(x^2)\ (2x^2) - 2x^4$$

Then, identify the degree of the polynomial function. This polynomial function is of degree 4.

The maximum number of turning points is $4 - 1 = 3$.

Graphing Polynomial Functions

We can use what we have learned about multiplicities, end behavior, and turning points to sketch graphs of polynomial functions. Let us put this all together and look at the steps required to graph polynomial functions.

Howto: Given a polynomial function, sketch the graph

1. Find the intercepts.
2. Check for symmetry. If the function is an even function, its graph is symmetrical about the y-axis, that is, $f(-x) = f(x)$. If a function is an odd function, its graph is symmetrical about the origin, that is, $f(-x) = -f(x)$.
3. Use the multiplicities of the zeros to determine the behavior of the polynomial at the x-intercepts.

4. Determine the end behavior by examining the leading term.
5. Use the end behavior and the behavior at the intercepts to sketch a graph.
6. Ensure that the number of turning points does not exceed one less than the degree of the polynomial.
7. Optionally, use technology to check the graph.

Example 4.2.8: Sketching the Graph of a Polynomial Function

Sketch a graph of $f(x) = -2(x+3)^2(x-5)$.

Solution

This graph has two x-intercepts. At $x = -3$, the factor is squared, indicating a multiplicity of 2. The graph will bounce at this x-intercept. At $x = 5$,the function has a multiplicity of one, indicating the graph will cross through the axis at this intercept.

The y-intercept is found by evaluating $f(0)$.

$$\begin{aligned} f(0) &= -2(0+3)^2(0-5) \\ &= -2 \cdot 9 \cdot (-5) \\ &= 90 \end{aligned}$$

The y-intercept is $(0, 90)$.

Additionally, we can see the leading term, if this polynomial were multiplied out, would be $-2x3$, so the end behavior is that of a vertically reflected cubic, with the outputs decreasing as the inputs approach infinity, and the outputs increasing as the inputs approach negative infinity. See Figure 4.2.13

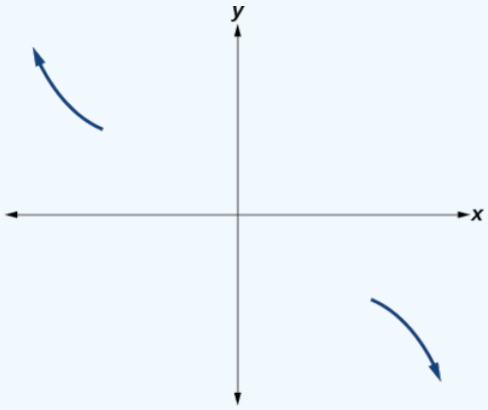


Figure 4.2.13: Showing the distribution for the leading term.

To sketch this, we consider that:

- As $x \to -\infty$ the function $f(x) \to \infty$,so we know the graph starts in the second quadrant and is decreasing toward the x-axis.
- Since $f(-x) = -2(-x+3)^2(-x-5)$ is not equal to $f(x)$, the graph does not display symmetry.
- At $(-3, 0)$, the graph bounces off of thex-axis, so the function must start increasing.
- At $(0, 90)$, the graph crosses the y-axis at the y-intercept. See Figure 4.2.14

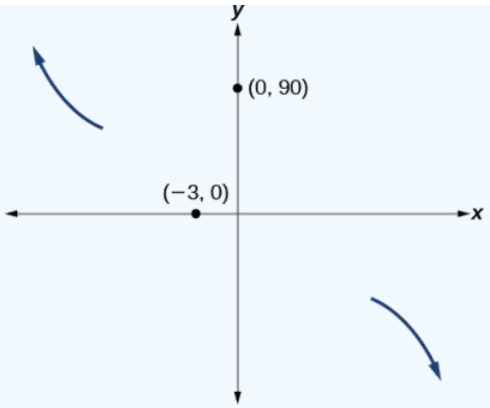


Figure 4.2.14: Graph of the end behavior and intercepts, $(-3,0)$ and $(0,90)$, for the function $f(x)=-2(x+3)^2(x-5)$.

Somewhere after this point, the graph must turn back down or start decreasing toward the horizontal axis because the graph passes through the next intercept at $(5,0)$. See Figure 4.2.15.

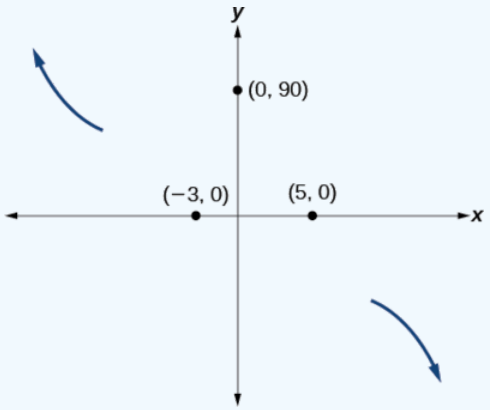


Figure 4.2.15: Graph of the end behavior and intercepts, $(-3,0)$, $(0,90)$ and $(5,0)$, for the function $f(x)=-2(x+3)^2(x-5)$.

As $x\to\infty$ the function $f(x)\to-\infty$,

so we know the graph continues to decrease, and we can stop drawing the graph in the fourth quadrant.

Using technology, we can create the graph for the polynomial function, shown in Figure 4.2.16, and verify that the resulting graph looks like our sketch in Figure 4.2.15.

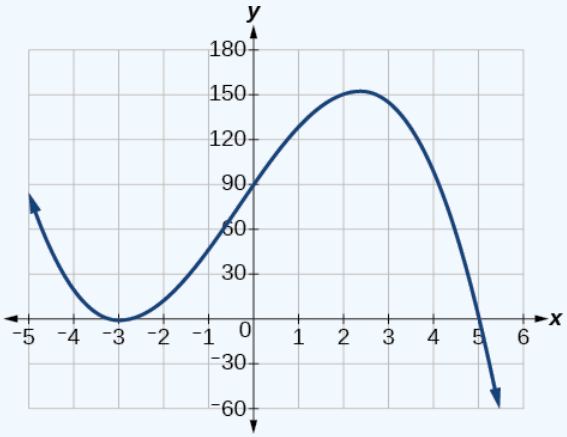


***Figure* 4.2.16:** *The complete graph of the polynomial function* $f(x)=-2(x+3)^2(x-5)$.

https://math.libretexts.org/@go/page/89791

Exercise 4.2.8

Sketch a graph of $f(x) = \frac{1}{4}x(x-1)^4(x+3)^3$.

Answer

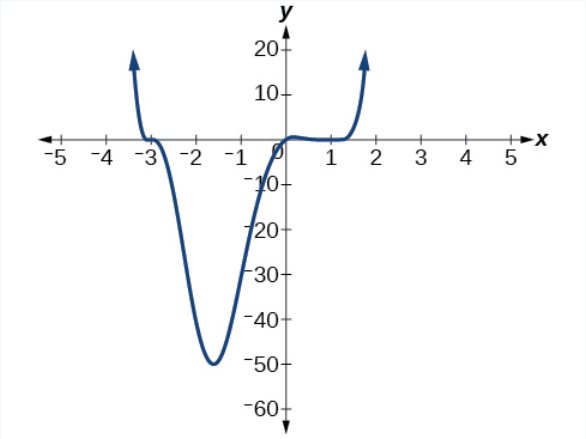


Figure 4.2.17: Graph of $f(x) = \frac{1}{4}x(x-1)^4(x+3)^3$

Writing Formulas for Polynomial Functions

Now that we know how to find zeros of polynomial functions, we can use them to write formulas based on graphs. Because a **polynomial function** written in factored form will have an x-intercept where each factor is equal to zero, we can form a function that will pass through a set of x-intercepts by introducing a corresponding set of factors.

Note: Factored Form of Polynomials

If a polynomial of lowest degree p has horizontal intercepts at $x = x_1, x_2, \ldots, x_n$, then the polynomial can be written in the factored form: $f(x) = a(x-x_1)^{p_1}(x-x_2)^{p_2}\cdots(x-x_n)^{p_n}$ where the powers p_i on each factor can be determined by the behavior of the graph at the corresponding intercept, and the stretch factor a can be determined given a value of the function other than the x-intercept.

How to: Given a graph of a polynomial function, write a formula for the function.

1. Identify the x-intercepts of the graph to find the factors of the polynomial.
2. Examine the behavior of the graph at the x-intercepts to determine the multiplicity of each factor.
3. Find the polynomial of least degree containing all the factors found in the previous step.
4. Use any other point on the graph (the y-intercept may be easiest) to determine the stretch factor.

Example 4.2.10: Writing a Formula for a Polynomial Function from the Graph

Write a formula for the polynomial function shown in Figure 4.2.20

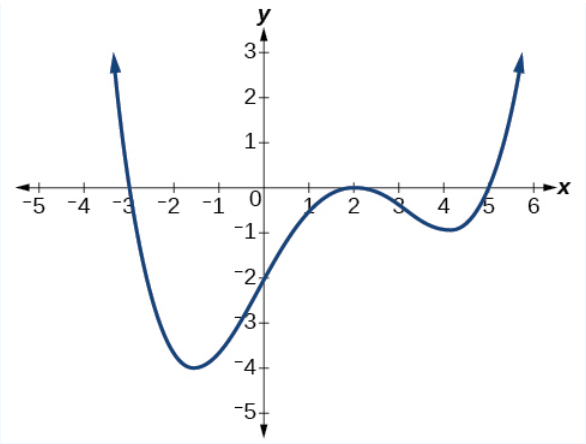


Figure 4.2.20.

Solution

This graph has three x-intercepts: $x=-3,\ 2,$ and 5. The y-intercept is located at $(0,2)$.At $x=-3$ and $x=5$, the graph passes through the axis linearly, suggesting the corresponding factors of the polynomial will be linear. At $x=2$, the graph bounces at the intercept, suggesting the corresponding factor of the polynomial will be second degree (quadratic). Together, this gives us

$$f(x)=a(x+3)(x-2)^2(x-5) \tag{4.2.5}$$

To determine the stretch factor, we utilize another point on the graph. We will use the y-intercept $(0,-2)$, to solve for a.

$$\begin{aligned} f(0) &= a(0+3)(0-2)^2(0-5) \\ -2 &= a(0+3)(0-2)^2(0-5) \\ -2 &= -60a \\ a &= \frac{1}{30} \end{aligned}$$

The graphed polynomial appears to represent the function $f(x)=\frac{1}{30}(x+3)(x-2)^2(x-5)$.

Exercise 4.2.5

Given the graph shown in Figure $4.2.21$, write a formula for the function shown.

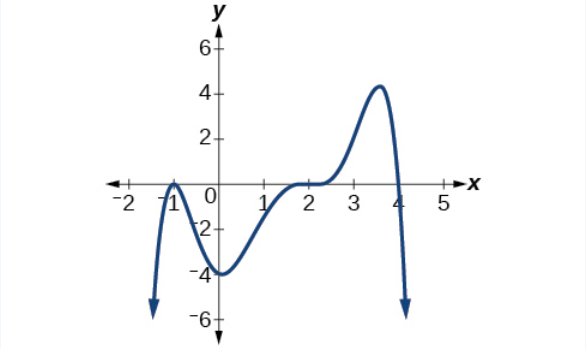


Figure 4.2.21.

Answer

$f(x)=-\frac{1}{8}(x-2)^3(x+1)^2(x-4)$

Key Concepts

- Polynomial functions of degree 2 or more are smooth, continuous functions.
- To find the zeros of a polynomial function, if it can be factored, factor the function and set each factor equal to zero.
- Another way to find the x-intercepts of a polynomial function is to graph the function and identify the points at which the graph crosses the x-axis.
- The multiplicity of a zero determines how the graph behaves at the x-intercepts.
- The graph of a polynomial will cross the horizontal axis at a zero with odd multiplicity.
- The graph of a polynomial will touch the horizontal axis at a zero with even multiplicity.
- The end behavior of a polynomial function depends on the leading term.
- The graph of a polynomial function changes direction at its turning points.
- A polynomial function of degree n has at most $n-1$ turning points.
- To graph polynomial functions, find the zeros and their multiplicities, determine the end behavior, and ensure that the final graph has at most $n-1$ turning points.
- Graphing a polynomial function helps to estimate local and global extremas.
- The Intermediate Value Theorem tells us that if $f(a)$ and $f(b)$ have opposite signs, then there exists at least one value c between a and b for which $f(c)=0$.

Glossary

global maximum
highest turning point on a graph; $f(a)$ where $f(a) \geq f(x)$ for all x.

global minimum
lowest turning point on a graph; $f(a)$ where $f(a) \leq f(x)$ for all x.

Intermediate Value Theorem
for two numbers a and b in the domain of f, if $a<b$ and $f(a) \neq f(b)$, then the functionf takes on every value between $f(a)$ and $f(b)$; specifically, when a polynomial function changes from a negative value to a positive value, the function must cross the x-axis

multiplicity
the number of times a given factor appears in the factored form of the equation of a polynomial; if a polynomial contains a factor of the form $(x-h)^p$, $x=h$ is a zero of multiplicity p.

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4.3: Dividing Polynomials

Using Long Division to Divide Polynomials

We are familiar with the **long division** algorithm for ordinary arithmetic. We begin by dividing into the digits of the dividend that have the greatest place value. We divide, multiply, subtract, include the digit in the next place value position, and repeat. For example, let's divide 178 by 3 using long division.

Long Division

$$\begin{array}{r} 59 \\ 3\overline{)178} \\ \underline{-15\downarrow} \\ 28 \\ \underline{-27} \\ 1 \end{array}$$

Step 1: $5 \times 3 = 15$ and $17 - 15 = 2$
Step 2: Bring down the 8
Step 3: $9 \times 3 = 27$ and $28 - 27 = 1$

Answer: $59\,R\,1$ or $59\frac{1}{3}$

Another way to look at the solution is as a sum of parts. This should look familiar, since it is the same method used to check division in elementary arithmetic.

$$\begin{aligned} \text{dividend} &= (\text{divisor}\cdot\text{quotient}) + \text{remainder} \\ 178 &= (3\cdot 59) + 1 \\ &= 177 + 1 \\ &= 178 \end{aligned}$$

We call this the **Division Algorithm** and will discuss it more formally after looking at an example.

Division of polynomials that contain more than one term has similarities to long division of whole numbers. We can write a polynomial dividend as the product of the divisor and the quotient added to the remainder. The terms of the polynomial division correspond to the digits (and place values) of the whole number division. This method allows us to divide two polynomials. For example, if we were to divide $2x^3 - 3x^2 + 4x + 5$ by $x + 2$ using the long division algorithm, it would look like this:

$$x+2\overline{)2x^3-3x^2+4x+5}$$ Set up the division problem.

$$\begin{array}{r} 2x^2 \\ x+2\overline{)2x^3-3x^2+4x+5} \end{array}$$ $2x^3$ divided by x is $2x^2$.

$$\begin{array}{r} 2x^2 \\ x+2\overline{)2x^3-3x^2+4x+5} \\ \underline{-\left(2x^3+4x^2\right)} \\ -7x^2+4x \end{array}$$ Multiply $x+2$ by $2x^2$ and subtract.
Bring down the next term.

$$\begin{array}{r} 2x^2-7x \\ x+2\overline{)2x^3-3x^2+4x+5} \\ \underline{-\left(2x^3+4x^2\right)} \\ -7x^2+4x \end{array}$$ $-7x^2$ divided by x is $-7x$.

$$\begin{array}{r} 2x^2-7x \\ x+2\overline{)2x^3-3x^2+4x+5} \\ \underline{-\left(2x^3+4x^2\right)} \\ -7x^2+4x \\ \underline{-\left(-7x^2-14x\right)} \\ 18x+5 \end{array}$$ Multiply $x+2$ by $-7x$.
Subtract and bring down the next term.

$$\begin{array}{r} 2x^2-7x+18 \\ x+2\overline{)2x^3-3x^2+4x+5} \\ \underline{-\left(2x^3+4x^2\right)} \\ -7x^2+4x \\ \underline{-\left(-7x^2-14x\right)} \\ 18x+5 \\ \underline{-(18x+36)} \\ -31 \end{array}$$ $18x$ divided by x is 18.
Multiply $x+2$ by 18.
Subtract.

We have found

$$\frac{2x^3-3x^2+4x+5}{x+2}=2x^2-7x+18-\frac{31}{x+2}$$

or

$$2x^3-3x^2+4x+5=(x+2)(2x^2-7x+18)-31$$

We can identify the dividend, the **divisor**, the **quotient**, and the **remainder**.

Writing the result in this manner illustrates the Division Algorithm.

The Division Algorithm

The Division Algorithm states that, given a polynomial dividend $f(x)$ and a non-zero polynomial divisor $d(x)$ where the degree of $d(x)$ is less than or equal to the degree of $f(x)$, there exist unique polynomials $q(x)$ and $r(x)$ such that

$$f(x)=d(x)q(x)+r(x) \tag{4.3.1}$$

$q(x)$ is the quotient and $r(x)$ is the remainder. The remainder is either equal to zero or has degree strictly less than $d(x)$.

If $r(x)=0$, then $d(x)$ divides evenly into $f(x)$. This means that, in this case, both $d(x)$ and $q(x)$ are factors of $f(x)$.

Given a polynomial and a binomial, use long division to divide the polynomial by the binomial

1. Set up the division problem.
2. Determine the first term of the quotient by dividing the leading term of the dividend by the leading term of the divisor.
3. Multiply the answer by the divisor and write it below the like terms of the dividend.
4. Subtract the bottom **binomial** from the top binomial.

5. Bring down the next term of the dividend.
6. Repeat steps 2–5 until reaching the last term of the dividend.
7. If the remainder is non-zero, express as a fraction using the divisor as the denominator.

Example 4.3.1: Using Long Division to Divide a Second-Degree Polynomial

Divide $5x^2+3x-2$ by $x+1$.

Solution

$$
\begin{array}{rl}
x+1\,\overline{)\,5x^2+3x-2} & \text{Set up division problem.}\\
\quad 5x & \\
x+1\,\overline{)\,5x^2+3x-2} & 5x^2 \text{ divided by } x \text{ is } 5x.\\
\quad 5x & \\
x+1\,\overline{)\,5x^2+3x-2} & \text{Multiply } x+1 \text{ by } 5x.\\
\quad -(5x^2+5x) & \\
\hline
\quad -2x-2 & \text{Subtract.}\\
\quad 5x-2 & \text{Bring down the next term.}\\
x+1\,\overline{)\,5x^2+3x-2} & -2x \text{ divided by } x \text{ is } -2.\\
\quad -(5x^2+5x) & \\
\hline
\quad -2x-2 & \\
\quad -(-2x-2) & \text{Multiply } x+1 \text{ by } -2.\\
\hline
\quad 0 & \text{Subtract.}
\end{array}
$$

The quotient is $5x-2$. The remainder is 0. We write the result as

$$\frac{5x^2+3x-2}{x+1}=5x-2$$

or

$$5x^2+3x-2=(x+1)(5x-2)$$

Analysis

This division problem had a remainder of 0. This tells us that the dividend is divided evenly by the divisor, and that the divisor is a factor of the dividend.

Example 4.3.2: Using Long Division to Divide a Third-Degree Polynomial

Divide $6x^3+11x^2-31x+15$ by $3x-2$.

Solution

$$
\begin{array}{rl}
\quad 2x^2+\ 5x-\ 7 & \\
3x-2\,\overline{)\,6x^3+11x^2-31x+15} & 6x^3 \text{ divided by } 3x \text{ is } 2x^2.\\
\quad -\left(6x^3-4x^2\right) & \text{Multiply } 3x-2 \text{ by } 2x^2.\\
\hline
\quad 15x^2-31x & \text{Subtract. Bring down next term. } 15x^2 \text{ divided by } 3x \text{ is } 5x.\\
\quad -\left(15x^2-10x\right) & \text{Multiply } 3x-2 \text{ by } 5x.\\
\hline
\quad -21x+15 & \text{Subtract. Bring down next term. } -21x \text{ divided by } 3x \text{ is } -7.\\
\quad -(-21x+14) & \text{Multiply } 3x-2 \text{ by } -7.\\
\hline
\quad 1 & \text{Subtract. The remainder is 1.}
\end{array}
$$

There is a remainder of 1. We can express the result as:

$$\frac{6x^3+11x^2-31x+15}{3x-2}=2x^2+5x-7+\frac{1}{3x-2}$$

Analysis

We can check our work by using the Division Algorithm to rewrite the solution. Then multiply.

$$(3x-2)(2x^2+5x-7)+1=6x^3+11x^2-31x+15$$

Notice, as we write our result,

- the dividend is $6x^3+11x^2-31x+15$
- the divisor is $3x-2$
- the quotient is $2x^2+5x-7$
- the remainder is 1

Try It! 4.3.2

Divide $16x^3-12x^2+20x-3$ by $4x+5$.

Solution

$$4x^2-8x+15-\frac{78}{4x+5}$$

Using Synthetic Division to Divide Polynomials

As we've seen, long division of polynomials can involve many steps and be quite cumbersome. **Synthetic division** is a shorthand method of dividing polynomials for the special case of dividing by a linear factor whose leading coefficient is 1.

To illustrate the process, recall the example at the beginning of the section.

Divide $2x^3-3x^2+4x+5$ by $x+2$ using the long division algorithm.

The final form of the process looked like this:

$$\begin{array}{r} 2x^2-7x+18 \\ x+2\overline{)\,2x^3-3x^2+4x+5} \\ \underline{-\left(2x^3+4x^2\right)} \\ -7x^2+4x \\ \underline{-\left(-7x^2-14x\right)} \\ 18x+5 \\ \underline{-(18x+36)} \\ -31 \end{array}$$

There is a lot of repetition in the table. If we don't write the variables but, instead, line up their coefficients in columns under the division sign and also eliminate the partial products, we already have a simpler version of the entire problem.

$$\begin{array}{r|rrrr} 2 & 2 & -3 & 4 & 5 \\ & -2 & -4 & & \\ \hline & & -7 & 14 & \\ & & & 18 & -36 \\ \hline & & & & -31 \end{array}$$

Synthetic division carries this simplification even a few more steps. Collapse the table by moving each of the rows up to fill any vacant spots. Also, instead of dividing by 2, as we would in division of whole numbers, then multiplying and subtracting the middle product, we change the sign of the "divisor" to –2, multiply and add. The process starts by bringing down the leading coefficient.

$$\begin{array}{r|rrrr} -2 & 2 & -3 & 4 & 5 \\ & & -4 & 14 & -36 \\ \hline & 2 & -7 & 18 & -31 \end{array}$$

We then multiply it by the "divisor" and add, repeating this process column by column, until there are no entries left. The bottom row represents the coefficients of the quotient; the last entry of the bottom row is the remainder. In this case, the quotient is $2x^2-7x+18$ and the remainder is -31. The process will be made more clear in Example 4.3.3.

Synthetic Division

Synthetic division is a shortcut that can be used when the divisor is a binomial in the form $x-k$. In *synthetic division*, only the coefficients are used in the division process.

Given two polynomials, use synthetic division to divide

1. Write k for the divisor.
2. Write the coefficients of the dividend.
3. Bring the lead coefficient down.
4. Multiply the lead coefficient by k. Write the product in the next column.
5. Add the terms of the second column.
6. Multiply the result by k. Write the product in the next column.
7. Repeat steps 5 and 6 for the remaining columns.
8. Use the bottom numbers to write the quotient. The number in the last column is the remainder and has degree 0, the next number from the right has degree 1, the next number from the right has degree 2, and so on.

Example 4.3.3: Using Synthetic Division to Divide a Second-Degree Polynomial

Use synthetic division to divide $5x^2-3x-36$ by $x-3$.

Solution

Begin by setting up the synthetic division. Write k and the coefficients.

$$\begin{array}{r|rrr} 3 & 5 & -3 & -36 \\ \hline & & & \end{array}$$

Bring down the lead coefficient. Multiply the lead coefficient by k.

$$\begin{array}{r|rrr} 3 & 5 & -3 & -36 \\ & & 15 & \\ \hline & 5 & & \end{array}$$

Continue by adding the numbers in the second column. Multiply the resulting number by k.Write the result in the next column. Then add the numbers in the third column.

$$\begin{array}{r|rrr} 3 & 5 & -3 & -36 \\ & & 15 & 36 \\ \hline & 5 & 12 & 0 \end{array}$$

The result is $5x+12$. The remainder is 0. So $x-3$ is a factor of the original polynomial.

Analysis

Just as with long division, we can check our work by multiplying the quotient by the divisor and adding the remainder.

$$(x-3)(5x+12)+0=5x^2-3x-36$$

Example 4.3.4: Using Synthetic Division to Divide a Third-Degree Polynomial

Use synthetic division to divide $4x^3+10x^2-6x-20$ by $x+2$.

Solution

The binomial divisor is $x+2$ so $k=-2$. Add each column, multiply the result by –2, and repeat until the last column is reached.

$$\begin{array}{r|rrrr} -2 & 1 & -1 & -11 & 18 \\ & & 2 & 2 & -18 \\ \hline & 1 & 1 & -9 & 0 \end{array}$$

The result is $4x^2+2x-10$.

The remainder is 0. Thus, $x+2$ is a factor of $4x^3+10x^2-6x-20$.

Analysis

The graph of the polynomial function $f(x)=4x^3+10x^2-6x-20$ in Figure 4.3.2 shows a zero at $x=k=-2$. This confirms that $x+2$ is a factor of $4x^3+10x^2-6x-20$.

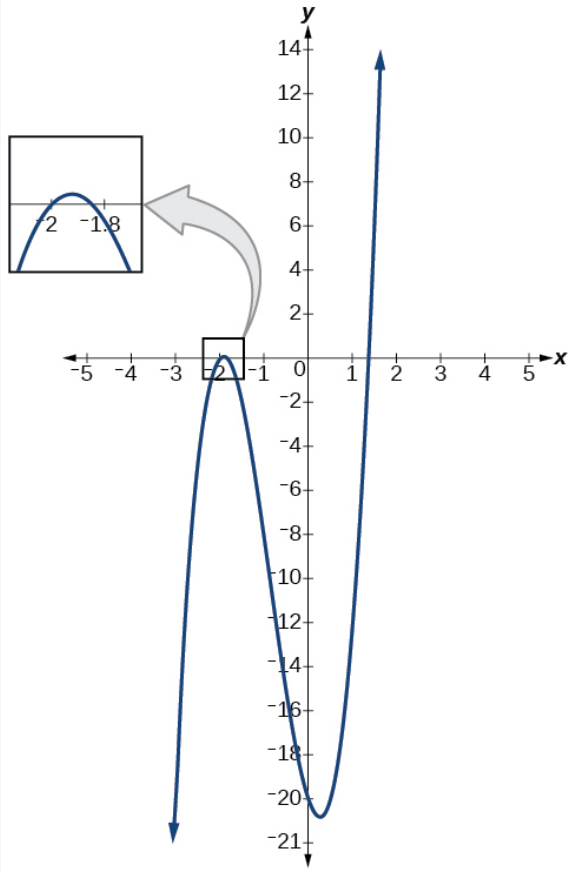


Figure 4.3.2

Example 4.3.5: Using Synthetic Division to Divide a Fourth-Degree Polynomial

Use synthetic division to divide $-9x^4+10x^3+7x^2-6$ by $x-1$.

Solution

Notice there is no x-term. We will use a zero as the coefficient for that term.

$$\begin{array}{r|rrrrr} 1 & -9 & 10 & 7 & 0 & -6 \\ & & -9 & 1 & 8 & 8 \\ \hline & -9 & 1 & 8 & 8 & 2 \end{array}$$

The result is $-9x^3+x^2+8x+8+\dfrac{2}{x-1}$.

Try It! 4.3.3

Use synthetic division to divide $3x^4+18x^3-3x+40$ by $x+7$.

Solution

$$3x^3-3x^2+21x-150+\frac{1090}{x+7}$$

Key Equations

Division Algorithm $f(x)=d(x)q(x)+r(x)$ where $q(x)\neq 0$

Key Concepts

- Polynomial long division can be used to divide a polynomial by any polynomial with equal or lower degree.
- The Division Algorithm tells us that a polynomial dividend can be written as the product of the divisor and the quotient added to the remainder.
- Synthetic division is a shortcut that can be used to divide a polynomial by a binomial in the form $x-k$.
- Polynomial division can be used to solve application problems, including area and volume.

Footnotes

[1] National Park Service. "Lincoln Memorial Building Statistics." www.nps.gov/linc/historycultu...statistics.htm. Accessed 4/3/2014

Glossary

Division Algorithm

given a polynomial dividend $f(x)$ and a non-zero polynomial divisor $d(x)$ where the degree of $d(x)$ is less than or equal to the degree of $f(x)$, there exist unique polynomials $q(x)$ and $r(x)$ such that $f(x)=d(x)q(x)+r(x)$ where $q(x)$ is the quotient and $r(x)$ is the remainder. The remainder is either equal to zero or has degree strictly less than $d(x)$.

synthetic division

a shortcut method that can be used to divide a polynomial by a binomial of the form $x-k$

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4.4: Zeros of Polynomial Functions

Evaluating a Polynomial Using the Remainder Theorem

In the last section, we learned how to divide polynomials. We can now use polynomial division to evaluate polynomials using the **Remainder Theorem**. If the polynomial is divided by $x-k$, the remainder may be found quickly by evaluating the polynomial function at k, that is, $f(k)$. Let's walk through the proof of the theorem.

Recall that the Division Algorithm states that, given a polynomial dividend $f(x)$ and a non-zero polynomial divisor $d(x)$ where the degree of $d(x)$ is less than or equal to the degree of $f(x)$,there exist unique polynomials $q(x)$ and $r(x)$ such that

$$f(x)=d(x)q(x)+r(x)$$

If the divisor, $d(x)$, is $x-k$, this takes the form

$$f(x)=(x-k)q(x)+r$$

Since the divisor $x-k$

is linear, the remainder will be a constant, r. And, if we evaluate this for $x=k$, we have

$$\begin{aligned} f(k) &= (k-k)q(k)+r \\ &= 0\cdot q(k)+r \\ &= r \end{aligned}$$

In other words, $f(k)$ is the remainder obtained by dividing $f(x)$by $x-k$.

The Remainder Theorem

If a polynomial $f(x)$ is divided by $x-k$,then the remainder is the value $f(k)$.

How To: Given a polynomial function f, evaluate $f(x)$ at $x=k$ using the Remainder Theorem.

1. Use synthetic division to divide the polynomial by $x-k$.
2. The remainder is the value $f(k)$.

Example 4.4.1: Using the Remainder Theorem to Evaluate a Polynomial

Use the Remainder Theorem to evaluate $f(x)=6x^4-x^3-15x^2+2x-7$ at $x=2$.

Solution

To find the remainder using the Remainder Theorem, use synthetic division to divide the polynomial by $x-2$.

$$\begin{array}{r|ccccc} 2 & 6 & -1 & -15 & 2 & -7 \\ & & 12 & 22 & 14 & 32 \\ \hline & 6 & 11 & 7 & 16 & 25 \end{array} \tag{4.4.1}$$

The remainder is 25. Therefore, $f(2)=25$.

Analysis

We can check our answer by evaluating $f(2)$.

$$\begin{aligned} f(x) &= 6x^4-x^3-15x^2+2x-7 \\ f(2) &= 6(2)^4-(2)^3-15(2)^2+2(2)-7 \\ &= 25 \end{aligned}$$

Exercise 4.4.1

se the Remainder Theorem to evaluate $f(x)=2x^5-3x^4-9x^3+8x^2+2$ at $x=-3$.

Answer

$f(-3)=-412$

Using the Factor Theorem to Solve a Polynomial Equation

The **Factor Theorem** is another theorem that helps us analyze polynomial equations. It tells us how the zeros of a polynomial are related to the factors. Recall that the Division Algorithm.

$$f(x)=(x-k)q(x)+r \tag{4.4.2}$$

If k is a zero, then the remainder r is $f(k)=0$ and $f(x)=(x-k)q(x)+0$ or $f(x)=(x-k)q(x)$.

Notice, written in this form, $x-k$ is a factor of $f(x)$. We can conclude if k is a zero of $f(x)$, then $x-k$ is a factor of $f(x)$.

Similarly, if $x-k$ is a factor of $f(x)$, then the remainder of the Division Algorithm $f(x)=(x-k)q(x)+r$ is 0. This tells us that k is a zero.

This pair of implications is the Factor Theorem. As we will soon see, a polynomial of degree n in the complex number system will have n zeros. We can use the Factor Theorem to completely factor a polynomial into the product of n factors. Once the polynomial has been completely factored, we can easily determine the zeros of the polynomial.

THE FACTOR THEOREM

According to the Factor Theorem, k is a zero of $f(x)$ if and only if $(x-k)$ is a factor of $f(x)$.

How to: Given a factor and a third-degree polynomial, use the Factor Theorem to factor the polynomial

1. Use synthetic division to divide the polynomial by $(x-k)$.
2. Confirm that the remainder is 0.
3. Write the polynomial as the product of $(x-k)$ and the quadratic quotient.
4. If possible, factor the quadratic.
5. Write the polynomial as the product of factors.

Example 4.4.2: Using the Factor Theorem to Solve a Polynomial Equation

Show that $(x+2)$ is a factor of x^3-6x^2-x+30. Find the remaining factors. Use the factors to determine the zeros of the **polynomial**.

Solution

We can use synthetic division to show that $(x+2)$ is a factor of the polynomial.

$$\begin{array}{r|cccc} -2 & 1 & -6 & -1 & 30 \\ & & -2 & 16 & -30 \\ \hline & 1 & -8 & 15 & 0 \end{array} \tag{4.4.3}$$

The remainder is zero, so $(x+2)$ is a factor of the polynomial. We can use the Division Algorithm to write the polynomial as the product of the divisor and the quotient:

$$(x+2)(x^2-8x+15) \tag{4.4.4}$$

We can factor the quadratic factor to write the polynomial as

$$(x+2)(x-3)(x-5) \tag{4.4.5}$$

By the Factor Theorem, the zeros of x^3-6x^2-x+30 are –2, 3, and 5.

Exercise 4.4.2

Use the Factor Theorem to find the zeros of $f(x) = x^3 + 4x^2 - 4x - 16$ given that $(x - 2)$ is a factor of the polynomial.

Answer

The zeros are 2, –2, and –4.

Using the Rational Zero Theorem to Find Rational Zeros

Another use for the Remainder Theorem is to test whether a rational number is a zero for a given polynomial. But first we need a pool of rational numbers to test. The **Rational Zero Theorem** helps us to narrow down the number of possible rational zeros using the ratio of the factors of the constant term and factors of the leading **coefficient** of the polynomial

Consider a quadratic function with two zeros, $x = \frac{2}{5}$ and $x = \frac{3}{4}$. By the Factor Theorem, these zeros have factors associated with them. Let us set each factor equal to 0, and then construct the original quadratic function absent its stretching factor.

$x - \frac{2}{5} = 0$ or $x - \frac{3}{4} = 0$	Set each factor equal to 0.
$5x - 2 = 0$ or $4x - 3 = 0$	Multiply both sides of the equation to eliminate fractions.
$f(x) = (5x - 2)(4x - 3)$	Create the quadratic function, multiplying the factors.
$f(x) = 20x^2 - 23x + 6$	Expand the polynomial.
$f(x) = (5 \cdot 4)x^2 - 23x + (2 \cdot 3)$	

Notice that two of the factors of the constant term, 6, are the two numerators from the original rational roots: 2 and 3. Similarly, two of the factors from the leading coefficient, 20, are the two denominators from the original rational roots: 5 and 4.

We can infer that the numerators of the rational roots will always be factors of the constant term and the denominators will be factors of the leading coefficient. This is the essence of the Rational Zero Theorem; it is a means to give us a pool of possible rational zeros.

THE RATIONAL ZERO THEOREM

The Rational Zero Theorem states that, if the polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ has integer coefficients, then every rational zero of $f(x)$ has the form $\frac{p}{q}$ where p is a factor of the constant term a_0 and q is a factor of the leading coefficient a_n.

When the leading coefficient is 1, the possible rational zeros are the factors of the constant term.

How to: Given a polynomial function $f(x)$, use the Rational Zero Theorem to find rational zeros.

1. Determine all factors of the constant term and all factors of the leading coefficient.
2. Determine all possible values of $\frac{p}{q}$, where p is a factor of the constant term and q is a factor of the leading coefficient. Be sure to include both positive and negative candidates.
3. Determine which possible zeros are actual zeros by evaluating each case of $f(\frac{p}{q})$.

Example 4.4.3: Listing All Possible Rational Zeros

List all possible rational zeros of $f(x) = 2x^4 - 5x^3 + x^2 - 4$.

Solution

The only possible rational zeros of $f(x)$ are the quotients of the factors of the last term, –4, and the factors of the leading coefficient, 2.

The constant term is –4; the factors of –4 are $p = \pm 1, \pm 2, \pm 4$.

The leading coefficient is 2; the factors of 2 are $q = \pm 1, \pm 2$.

If any of the four real zeros are rational zeros, then they will be of one of the following factors of –4 divided by one of the factors of 2.

$$\frac{p}{q}=\pm\frac{1}{1},\pm\frac{1}{2} \qquad \frac{p}{q}=\pm\frac{2}{1},\pm\frac{2}{2} \qquad \frac{p}{q}=\pm\frac{4}{1},\pm\frac{4}{2}$$

Note that $\frac{2}{2}=1$ and $\frac{4}{2}=2$, which have already been listed. So we can shorten our list.

$$\frac{p}{q}=\frac{\text{Factors of the last}}{\text{Factors of the first}}=\pm1,\pm2,\pm4,\pm\frac{1}{2}$$

Example 4.4.4: Using the Rational Zero Theorem to Find Rational Zeros

Use the Rational Zero Theorem to find the rational zeros of $f(x)=2x^3+x^2-4x+1$.

Solution

The Rational Zero Theorem tells us that if $\frac{p}{q}$ is a zero of $f(x)$, then p is a factor of 1 and q is a factor of 2.

$$\frac{p}{q}=\frac{factor\ of\ constant\ term}{factor\ of\ leading\ coefficient}$$

$$=\frac{factor\ of\ 1}{factor\ of\ 2}$$

The factors of 1 are ±1 and the factors of 2 are ±1 and ±2. The possible values for $\frac{p}{q}$ are ±1 and $\pm\frac{1}{2}$. These are the possible rational zeros for the function. We can determine which of the possible zeros are actual zeros by substituting these values for x in $f(x)$.

$$f(-1)=2(-1)^3+(-1)^2-4(-1)+1=4 \tag{4.4.6}$$

$$f(1)=1(1)^3+(1)^2-4(1)+1=0 \tag{4.4.7}$$

$$f(-\frac{1}{2})=2(-\frac{1}{2})^3+(-\frac{1}{2})^2-4(-\frac{1}{2})+1=3 \tag{4.4.8}$$

$$f(\frac{1}{2})=2(\frac{1}{2})^3+(\frac{1}{2})^2-4(\frac{1}{2})+1=-\frac{1}{2} \tag{4.4.9}$$

Of those, $-1,-\frac{1}{2}$, and $\frac{1}{2}$ are not zeros of $f(x)$. 1 is the only rational zero of $f(x)$.

Exercise 4.4.3

Use the Rational Zero Theorem to find the rational zeros of $f(x)=x^3-5x^2+2x+1$.

Answer

There are no rational zeros.

Finding the Zeros of Polynomial Functions

The Rational Zero Theorem helps us to narrow down the list of possible rational zeros for a polynomial function. Once we have done this, we can use **synthetic division** repeatedly to determine all of the **zeros** of a polynomial function.

How to: Given a polynomial function f, use synthetic division to find its zeros.

1. Use the Rational Zero Theorem to list all possible rational zeros of the function.
2. Use synthetic division to evaluate a given possible zero by synthetically dividing the candidate into the polynomial. If the remainder is 0, the candidate is a zero. If the remainder is not zero, discard the candidate.
3. Repeat step two using the quotient found with synthetic division. If possible, continue until the quotient is a quadratic.

4. Find the zeros of the quadratic function. Two possible methods for solving quadratics are factoring and using the quadratic formula.

Example 4.4.5: Finding the Zeros of a Polynomial Function with Repeated Real Zeros

Find the zeros of $f(x) = 4x^3 - 3x - 1$.

Solution

The Rational Zero Theorem tells us that if $\frac{p}{q}$ is a zero of $f(x)$, then p is a factor of –1 and q is a factor of 4.

$$\frac{p}{q} = \frac{factor\ of\ constant\ term}{factor\ of\ leading\ coefficient}$$
$$= \frac{factor\ of\ -1}{factor\ of\ 4}$$

The factors of –1 are ±1 and the factors of 4 are ±1,±2, and ±4. The possible values for $\frac{p}{q}$ are $\pm 1, \pm\frac{1}{2}$, and $\pm\frac{1}{4}$. These are the possible rational zeros for the function. We will use synthetic division to evaluate each possible zero until we find one that gives a remainder of 0. Let's begin with 1.

Dividing by $(x-1)$ gives a remainder of 0, so 1 is a zero of the function. The polynomial can be written as

$$(x-1)(4x^2+4x+1)$$

The quadratic is a perfect square. $f(x)$ can be written as

$$(x-1)(2x+1)^2$$

We already know that 1 is a zero. The other zero will have a multiplicity of 2 because the factor is squared. To find the other zero, we can set the factor equal to 0.

$$2x+1=0$$
$$x = -\frac{1}{2}$$

The zeros of the function are 1 and $-\frac{1}{2}$ with multiplicity 2.

Analysis

Look at the graph of the function f in Figure 4.4.1. Notice, at $x=-0.5$, the graph bounces off the x-axis, indicating the even multiplicity (2,4,6…) for the zero −0.5. At $x=1$, the graph crosses the x-axis, indicating the odd multiplicity (1,3,5…) for the zero $x=1$.

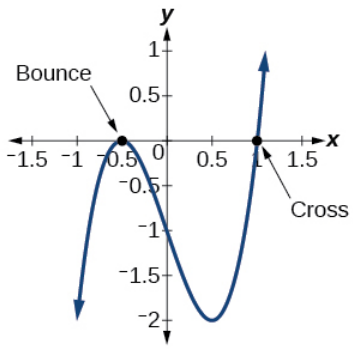


Figure 4.4.1.

Using the Fundamental Theorem of Algebra

Now that we can find rational zeros for a polynomial function, we will look at a theorem that discusses the number of complex zeros of a polynomial function. The **Fundamental Theorem of Algebra** tells us that every polynomial function has at least one

complex zero. This theorem forms the foundation for solving polynomial equations.

Suppose f is a polynomial function of degree four, and $f(x)=0$. The Fundamental Theorem of Algebra states that there is at least one complex solution, call it c_1. By the Factor Theorem, we can write $f(x)$ as a product of $x-c_1$ and a polynomial quotient. Since $x-c_1$ is linear, the polynomial quotient will be of degree three. Now we apply the Fundamental Theorem of Algebra to the third-degree polynomial quotient. It will have at least one complex zero, call it c_2. So we can write the polynomial quotient as a product of $x-c_2$ and a new polynomial quotient of degree two. Continue to apply the Fundamental Theorem of Algebra until all of the zeros are found. There will be four of them and each one will yield a factor of $f(x)$.

THE FUNDAMENTAL THEOREM OF ALGEBRA

The *Fundamental Theorem of Algebra* states that, if $f(x)$ is a polynomial of degree $n>0$, then $f(x)$ has at least one complex zero.

We can use this theorem to argue that, if $f(x)$ is a polynomial of degree $n>0$, and a is a non-zero real number, then $f(x)$ has exactly n linear factors

$$f(x)=a(x-c_1)(x-c_2)\ldots(x-c_n) \tag{4.4.10}$$

where $c_1, c_2,...,c_n$ are complex numbers. Therefore, $f(x)$ has n roots if we allow for multiplicities.

Q&A: Does every polynomial have at least one imaginary zero?

No. Real numbers are a subset of complex numbers, but not the other way around. A complex number is not necessarily imaginary. Real numbers are also complex numbers.

Example 4.4.6: Finding the Zeros of a Polynomial Function with Complex Zeros

Find the zeros of $f(x)=3x^3+9x^2+x+3$.

Solution

The Rational Zero Theorem tells us that if $\frac{p}{q}$ is a zero of $f(x)$, then p is a factor of 3 and q is a factor of 3.

$$\frac{p}{q}=\frac{factor\ of\ constant\ term}{factor\ of\ leading\ coefficient}$$
$$=\frac{factor\ of\ 3}{factor\ of\ 3}$$

The factors of 3 are ±1 and ±3. The possible values for $\frac{p}{q}$, and therefore the possible rational zeros for the function, are ±3,±1, and $\pm\frac{1}{3}$. We will use synthetic division to evaluate each possible zero until we find one that gives a remainder of 0. Let's begin with –3.

Dividing by $(x+3)$ gives a remainder of 0, so –3 is a zero of the function. The polynomial can be written as

$$(x+3)(3x^2+1)$$

We can then set the quadratic equal to 0 and solve to find the other zeros of the function.

$$3x^2+1=0$$
$$x^2=-\frac{1}{3}$$
$$x=\pm-\sqrt{\frac{1}{3}}$$
$$=\pm\frac{i\sqrt{3}}{3}$$

The zeros of $f(x)$ are -3 and $\pm\frac{i\sqrt{3}}{3}$.

Analysis

Look at the graph of the function f in Figure 4.4.2. Notice that, at $x=-3$, the graph crosses the x-axis, indicating an odd multiplicity (1) for the zero $x=-3$. Also note the presence of the two turning points. This means that, since there is a 3rd degree polynomial, we are looking at the maximum number of turning points. So, the end behavior of increasing without bound to the right and decreasing without bound to the left will continue. Thus, all the x-intercepts for the function are shown. So either the multiplicity of $x=-3$ is 1 and there are two complex solutions, which is what we found, or the multiplicity at $x=-3$ is three. Either way, our result is correct.

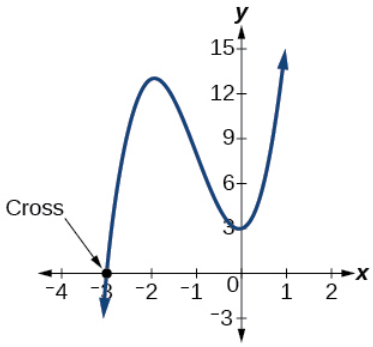


Figure 4.4.2.

4.4.4

Find the zeros of $f(x)=2x^3+5x^2-11x+4$.

Solution

The zeros are -4, $\frac{1}{2}$, and 1.

Using the Linear Factorization Theorem to Find Polynomials with Given Zeros

A vital implication of the **Fundamental Theorem of Algebra**, as we stated above, is that a polynomial function of degree n will have n zeros in the set of complex numbers, if we allow for multiplicities. This means that we can factor the polynomial function into n factors. The **Linear Factorization Theorem** tells us that a polynomial function will have the same number of factors as its degree, and that each factor will be in the form $(x-c)$, where c is a complex number.

Let f be a polynomial function with real coefficients, and suppose $a+bi$, $b\neq 0$, is a zero of $f(x)$. Then, by the Factor Theorem, $x-(a+bi)$ is a factor of $f(x)$. For f to have real coefficients, $x-(a-bi)$ must also be a factor of $f(x)$. This is true because any factor other than $x-(a-bi)$, when multiplied by $x-(a+bi)$, will leave imaginary components in the product. Only multiplication with conjugate pairs will eliminate the imaginary parts and result in real coefficients. In other words, if a polynomial function f with real coefficients has a complex zero $a+bi$, then the complex conjugate $a-bi$ must also be a zero of $f(x)$. This is called the **Complex Conjugate Theorem**.

COMPLEX CONJUGATE THEOREM

According to the Linear Factorization Theorem, a polynomial function will have the same number of factors as its degree, and each factor will be in the form $(x-c)$, where c is a complex number.

If the polynomial function f has real coefficients and a complex zero in the form $a+bi$, then the complex conjugate of the zero, $a-bi$, is also a zero.

How to

Given the zeros of a polynomial function f and a point $(c,f(c))$ on the graph of f, use the Linear Factorization Theorem to find the polynomial function.

1. Use the zeros to construct the linear factors of the polynomial.

2. Multiply the linear factors to expand the polynomial.
3. Substitute $(c, f(c))$ into the function to determine the leading coefficient.
4. Simplify.

Example 4.4.7: Using the Linear Factorization Theorem to Find a Polynomial with Given Zeros

Find a fourth degree polynomial with real coefficients that has zeros of -3, 2, i, such that $f(-2) = 100$.

Solution

Because $x = i$ is a zero, by the Complex Conjugate Theorem $x = -i$ is also a zero. The polynomial must have factors of $(x+3)$, $(x-2)$, $(x-i)$, and $(x+i)$. Since we are looking for a degree 4 polynomial, and now have four zeros, we have all four factors. Let's begin by multiplying these factors.

$$f(x) = a(x+3)(x-2)(x-i)(x+i) \tag{4.4.11}$$
$$f(x) = a(x^2+x-6)(x^2+1) \tag{4.4.12}$$
$$f(x) = a(x^4+x^3-5x^2+x-6) \tag{4.4.13}$$

We need to find a to ensure $f(-2) = 100$. Substitute $x = -2$ and $f(-2) = 100$ into $f(x)$.

$$100 = a((-2)^4+(-2)^3-5(-2)^2+(-2)-6) \tag{4.4.14}$$
$$100 = a(-20) \tag{4.4.15}$$
$$-5 = a \tag{4.4.16}$$

So the polynomial function is

$$f(x) = -5(x^4+x^3-5x^2+x-6) \tag{4.4.17}$$

or

$$f(x) = -5x^4-5x^3+25x^2-5x+30 \tag{4.4.18}$$

Analysis

We found that both i and $-i$ were zeros, but only one of these zeros needed to be given. If i is a zero of a polynomial with real coefficients, then $-i$ must also be a zero of the polynomial because $-i$ is the complex conjugate of i.

Q&A

If $2+3i$ were given as a zero of a polynomial with real coefficients, would $2-3i$ also need to be a zero?

Yes. When any complex number with an imaginary component is given as a zero of a polynomial with real coefficients, the conjugate must also be a zero of the polynomial.

4.4.5

Find a third degree polynomial with real coefficients that has zeros of 5 and $-2i$ such that $f(1) = 10$.

Solution

$f(x) = -\frac{1}{2}x^3+\frac{5}{2}x^2-2x+10$

Using Descartes' Rule of Signs

There is a straightforward way to determine the possible numbers of positive and negative real zeros for any polynomial function. If the polynomial is written in descending order, **Descartes' Rule of Signs** tells us of a relationship between the number of sign changes in $f(x)$ and the number of positive real zeros. For example, the polynomial function below has one sign change.

$$f(x) = x^4 + x^3 + x^2 + x - 1$$

This tells us that the function must have 1 positive real zero.

There is a similar relationship between the number of sign changes in $f(-x)$ and the number of negative real zeros.

$$f(-x) = (-x)^4 + (-x)^3 + (-x)^2 + (-x) - 1$$
$$f(-x) = +x^4 - x^3 + x^2 - x - 1$$

In this case, $f(-x)$ has 3 sign changes. This tells us that $f(x)$ could have 3 or 1 negative real zeros.

DESCARTES' RULE OF SIGNS

According to *Descartes' Rule of Signs*, if we let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ be a polynomial function with real coefficients:

- The number of positive real zeros is either equal to the number of sign changes of $f(x)$ or is less than the number of sign changes by an even integer.
- The number of negative real zeros is either equal to the number of sign changes of $f(-x)$ or is less than the number of sign changes by an even integer.

Example 4.4.8: Using Descartes' Rule of Signs

Use Descartes' Rule of Signs to determine the possible numbers of positive and negative real zeros for $f(x) = -x^4 - 3x^3 + 6x^2 - 4x - 12$.

Solution

Begin by determining the number of sign changes.

There are two sign changes, so there are either 2 or 0 positive real roots. Next, we examine $f(-x)$ to determine the number of negative real roots.

$$f(-x) = -(-x)^4 - 3(-x)^3 + 6(-x)^2 - 4(-x) - 12 \quad (4.4.19)$$
$$f(-x) = -x^4 + 3x^3 + 6x^2 + 4x - 12 \quad (4.4.20)$$

$$f(-x) = -x^4 + 3x^3 + 6x^2 + 4x - 12$$

Again, there are two sign changes, so there are either 2 or 0 negative real roots.

There are four possibilities, as we can see in Table 4.4.1.

Table 4.4.1

Positive Real Zeros	Negative Real Zeros	Complex Zeros	Total Zeros
2	2	0	4
2	0	2	4
0	2	2	4
0	0	4	4

Analysis

We can confirm the numbers of positive and negative real roots by examining a graph of the function. See Figure 4.4.3. We can see from the graph that the function has 0 positive real roots and 2 negative real roots.

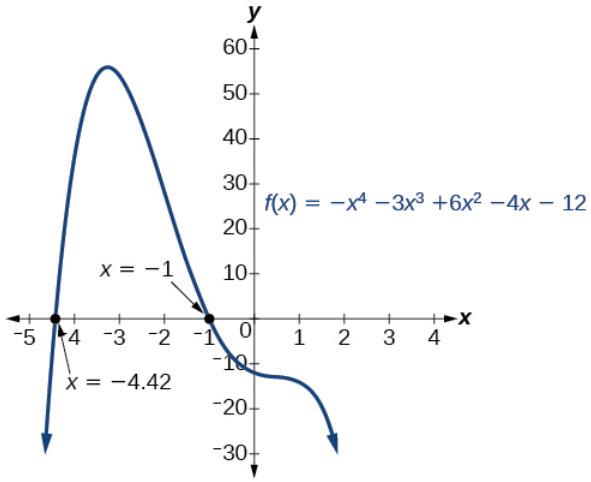


Figure 4.4.3.

4.4.6

Use Descartes' Rule of Signs to determine the maximum possible numbers of positive and negative real zeros for $f(x) = 2x^4 - 10x^3 + 11x^2 - 15x + 12$. Use a graph to verify the numbers of positive and negative real zeros for the function.

Solution

There must be 4, 2, or 0 positive real roots and 0 negative real roots. The graph shows that there are 2 positive real zeros and 0 negative real zeros.

Key Concepts

- To find $f(k)$, determine the remainder of the polynomial $f(x)$ when it is divided by $x - k$. This is known as the Remainder Theorem. See Example 4.4.1.
- According to the Factor Theorem, k is a zero of $f(x)$ if and only if $(x - k)$ is a factor of $f(x)$. See Example4.4.2.
- According to the Rational Zero Theorem, each rational zero of a polynomial function with integer coefficients will be equal to a factor of the constant term divided by a factor of the leading coefficient. See Example 4.4.3 and Example 4.4.4.
- When the leading coefficient is 1, the possible rational zeros are the factors of the constant term.
- Synthetic division can be used to find the zeros of a polynomial function. See Example 4.4.5.
- According to the Fundamental Theorem, every polynomial function with degree greater than 0 has at least one complex zero. See Example 4.4.6.
- Allowing for multiplicities, a polynomial function will have the same number of factors as its degree. Each factor will be in the form $(x - c)$, where c is a complex number. See Example 4.4.7.
- The number of positive real zeros of a polynomial function is either the number of sign changes of the function or less than the number of sign changes by an even integer.
- The number of negative real zeros of a polynomial function is either the number of sign changes of $f(-x)$ or less than the number of sign changes by an even integer. See Example 4.4.8.
- Polynomial equations model many real-world scenarios. Solving the equations is easiest done by synthetic division. See Example 4.4.9.

Glossary

Descartes' Rule of Signs

a rule that determines the maximum possible numbers of positive and negative real zeros based on the number of sign changes of $f(x)$ and $f(-x)$

Factor Theorem

k is a zero of polynomial function $f(x)$ if and only if $(x - k)$ is a factor of $f(x)$

Fundamental Theorem of Algebra

a polynomial function with degree greater than 0 has at least one complex zero

Linear Factorization Theorem

allowing for multiplicities, a polynomial function will have the same number of factors as its degree, and each factor will be in the form $(x-c)$, where c is a complex number

Rational Zero Theorem

the possible rational zeros of a polynomial function have the form $\frac{p}{q}$ where p is a factor of the constant term and q is a factor of the leading coefficient.

Remainder Theorem

if a polynomial $f(x)$ is divided by $x-k$,then the remainder is equal to the value $f(k)$

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4.5: Simplifying Rational Expressions

Reducing Rational Expressions in x

Now that we've discussed some fundamental ideas and techniques, let's apply what we've learned to rational expressions that are functions of an independent variable (usually x). Let's start with a simple example.

Example 4.5.4

Reduce the rational expression

$$\frac{2x-6}{x^2-7x+12}$$

to lowest terms. For what values of x is your result valid?

Solution

In the numerator, factor out a 2, as in 2x − 6 = 2(x − 3).

The denominator is a quadratic trinomial with ac = (1)(12) = 12. The integer pair −3 and −4 has product 12 and sum −7, so the denominator factors as shown.

$$\frac{2x-6}{x^2-7x+12}=\frac{2(x-3)}{(x-3)(x-4)}$$

Now that both numerator and denominator are factored, we can cancel common factors.

\[\frac{2 x-6}{x^{2}-7 x+12}=\frac{2(x-3)}{(x-3)(x-4)}=\frac{2}{x-4} \nonumber \]

Thus, we have shown that

$$\frac{2x-6}{x^2-7x+12}=\frac{2}{x-4}$$

In equation (15), we are stating that the expression on the left (the original expression) is identical to the expression on the right for all values of x.

Actually, there are two notable exceptions, the first of which is x = 3. If we substitute x = 3 into the left-hand side of equation (15), we get

$$\frac{2x-6}{x^2-7x+12}=\frac{2(3)-6}{(3)^2-7(3)+12}=\frac{0}{0}$$

We cannot divide by zero, so the left-hand side of equation (15) is undefined if x = 3. Therefore, the result in equation (15) is not valid if x = 3.

Similarly, if we insert x = 4 in the left-hand side of equation (15),

$$\frac{2x-6}{x^2-7x+12}=\frac{2(4)-6}{(4)^2-7(4)+12}=\frac{2}{0}$$

Again, division by zero is undefined. The left-hand side of equation (15) is undefined if x = 4, so the result in equation (15) is not valid if x = 4. Note that the right-hand side of equation (15) is also undefined at x = 4.

However, the algebraic work we did above guarantees that the left-hand side of equation (15) will be identical to the right-hand side of equation (15) for all other values of x. For example, if we substitute x = 5 into the left-hand side of equation (15),

$$\frac{2x-6}{x^2-7x+12}=\frac{2(5)-6}{(5)^2-7(5)+12}=\frac{4}{2}=2$$

On the other hand, if we substitute x = 5 into the right-hand side of equation (15),

$$\frac{2}{x-4}=\frac{2}{5-4}=2$$

Hence, both sides of equation (15) are identical when x = 5. In a similar manner, we could check the validity of the identity in equation (15) for all other values of x

You can use the graphing calculator to verify the identity in equation (15). Load the left- and right-hand sides of equation (15) in Y= menu, as shown in Figure 4.5.1(a). Press 2nd TBLSET and adjust settings as shown in Figure 4.5.1(b). Be sure that you highlight AUTO for both independent and dependent variables and press ENTER on each to make the selection permanent. In Figure 4.5.1(b), note that we've set TblStart = 0 and ΔTbl = 1. Press 2nd TABLE to produce the tabular results shown in Figure 4.5.1(c).

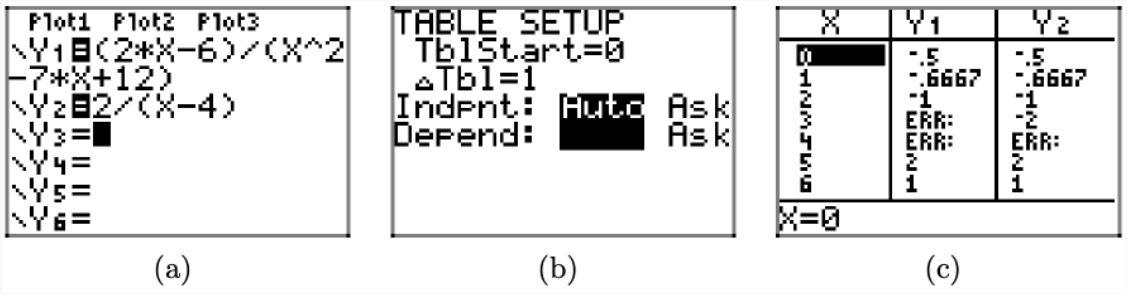


Figure 4.5.1. Using the graphing calculator to check that the left- and right-hand sides of equation (15) are identical.

Remember that we placed the left- and right-hand sides of equation (15) in Y1 and Y2, respectively.

- In the tabular results of Figure 4.5.1(c), note the ERR (error) message in Y1 when x = 3 and x = 4. This agrees with our findings above, where the left-hand side of equation (15) was undefined because of the presence of zero in the denominator when x = 3 or x = 4.
- In the tabular results of Figure 4.5.1(c), note that the value of Y1 and Y2 agree for all other values of x.

We are led to the following key result.

Restrictions

In general, when you reduce a rational expression to lowest terms, the expression obtained should be **identical** to the original expression for all values of the variables in each expression, save those values of the variables that make any denominator equal to zero. This applies to the denominator in the original expression, all intermediate expressions in your work, and the final result. We will refer to any values of the variable that make any denominator equal to zero as **restrictions**.

Let's look at another example.

Example 4.5.5

Reduce the expression

$$\frac{2x^2+5x-12}{4x^3+16x^2-9x-36}$$

to lowest terms. State all restrictions.

Solution

The numerator is a quadratic trinomial with ac = (2)(−12) = −24. The integer pair −3 and 8 have product −24 and sum 5. Break the middle term of the polynomial in the numerator into a sum using this integer pair, then factor by grouping.

$$\begin{aligned} 2x^2+5x-12 &= 2x^2-3x+8x-12 \\ &= x(2x-3)+4(2x-3) \\ &= (x+4)(2x-3) \end{aligned}$$

Factor the denominator by grouping.

$$\begin{aligned} 4x^3+16x^2-9x-36 &= 4x^2(x+4)-9(x+4) \\ &= \left(4x^2-9\right)(x+4) \\ &= (2x+3)(2x-3)(x+4) \end{aligned}$$

Note how the difference of two squares pattern was used to factor $4x^2-9=(2x+3)(2x-3)$ in the last step.

Now that we've factored both numerator and denominator, we cancel common factors.

\[\begin{aligned} \frac{2 x^{2}+5 x-12}{4 x^{3}+16 x^{2}-9 x-36} &=\frac{(x+4)(2 x-3)}{(2 x+3)(2 x-3)(x+4)} \\ &=\frac{(x+4)(2 x-3)}{(2 x+3)(2 x-3)(x+4)} \\ &=\frac{1}{2 x+3} \end{aligned} \nonumber \]

We must now determine the restrictions. This means that we must find those values of x that make any denominator equal to zero.

- In the body of our work, we have the denominator (2x + 3)(2x − 3)(x + 4). If we set this equal to zero, the zero product property implies that

$$2x+3=0 \quad \text{or} \quad 2x-3=0 \quad \text{or} \quad x+4=0$$

Each of these linear factors can be solved independently

$$x=-3/2 \quad \text{or} \quad x=3/2 \quad \text{or} \quad x=-4$$

Each of these x-values is a restriction.

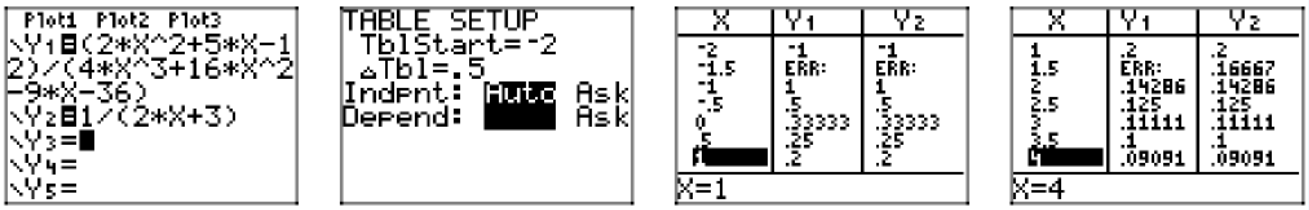
- In the final rational expression, the denominator is 2x + 3. This expression equals zero when x = −3/2 and provides no new restrictions.
- Because the denominator of the original expression, namely $4x^3+16x^2-9x-36$, is identical to its factored form in the body our work, this denominator will produce no new restrictions.

Thus, for all values of x

$$\frac{2x^2+5x-12}{4x^3+16x^2-9x-36}=\frac{1}{2x+3}$$

provided $x \neq -3/2, 3/2, or -4$. These are the restrictions. The two expressions are identical for all other values of x.

Finally, let's check this result with our graphing calculator. Load each side of equation (18) into the Y= menu, as shown in Figure 4.5.2(a). We know that we have a restriction at x = −3/2, so let's set TblStart = −2 and $\Delta \text{Tbl} = 0.5$, as shown in Figure 4.5.2(b). Be sure that you have AUTO set for both independent and dependent variables. Push the TABLE button to produce the tabular display shown in Figure 4.5.2(c).



(a) (b) (c) (c)

Figure 4.5.2. Using the graphing calculator to check that the left- and right-hand sides of equation (18) are identical.

Remember that we placed the left- and right-hand sides of equation (18) in Y1 and Y2, respectively.

- In Figure 4.5.2(c), note that the expressions Y1 and Y2 agree at all values of x except x = −1.5. This is the restriction −3/2 we found above.
- Use the down arrow key to scroll down in the table shown in Figure 4.5.2(c) to produce the tabular view shown in Figure 4.5.2(d). Note that Y1 and Y2 agree for all values of x except x = 1.5. This is the restriction 3/2 we found above.
- We leave it to our readers to uncover the restriction at x = −4 by using the uparrow to scroll up in the table until you reach an x-value of −4. You should uncover another ERR (error) message at this x-value because it is a restriction. You get the ERR message due to the fact that the denominator of the left-hand side of equation (18) is zero at x = −4.

Sign Changes

It is not uncommon that you will have to manipulate the signs in a fraction in order to obtain common factors that can be then cancelled. Consider, for example, the rational expression

$$\frac{3-x}{x-3}$$

One possible approach is to factor −1 out of the numerator to obtain

$$\frac{3-x}{x-3}=\frac{-(x-3)}{x-3}$$

You can now cancel common factors.

\[\frac{3-x}{x-3}=\frac{-(x-3)}{x-3}=\frac{-(x-3)}{x-3}=-1 \nonumber \]

This result is valid for all values of x, provided $x \neq 3$.

Let's look at another example.

Example 4.5.6

Reduce the rational expression

$$\frac{2x-2x^3}{3x^3+4x^2-3x-4}$$

to lowest terms. State all restrictions.

Solution

In the numerator, factor out 2x, then complete the factorization using the difference of two squares pattern.

$$2x-2x^3=2x\left(1-x^2\right)=2x(1+x)(1-x)$$

The denominator can be factored by grouping.

$$\begin{aligned} 3x^3+4x^2-3x-4 &= x^2(3x+4)-1(3x+4) \\ &= \left(x^2-1\right)(3x+4) \\ &= (x+1)(x-1)(3x+4) \end{aligned}$$

Note how the difference of two squares pattern was applied in the last step.

At this point,

$$\frac{2x-2x^3}{3x^3+4x^2-3x-4}=\frac{2x(1+x)(1-x)}{(x+1)(x-1)(3x+4)}$$

Because we have 1 − x in the numerator and x − 1 in the denominator, we will factor out a −1 from 1 − x, and because the order of factors does not affect their product, we will move the −1 out to the front of the numerator.

$$\frac{2x-2x^3}{3x^3+4x^2-3x-4}=\frac{2x(1+x)(-1)(x-1)}{(x+1)(x-1)(3x+4)}=\frac{-2x(1+x)(x-1)}{(x+1)(x-1)(3x+4)}$$

We can now cancel common factors.

\[\begin{aligned} \frac{2 x-2 x^{3}}{3 x^{3}+4 x^{2}-3 x-4} &=\frac{-2 x(1+x)(x-1)}{(x+1)(x-1)(3 x+4)} \\ &=\frac{-2 x(1+x)(x-1)}{(x+1)(x-1)(3 x+4)} \\ &=\frac{-2 x}{3 x+4} \end{aligned} \nonumber \]

Note that x + 1 is identical to 1 + x and cancels. Thus,

$$\frac{2x-2x^3}{3x^3+4x^2-3x-4}=\frac{-2x}{3x+4}$$

for all values of x, provided $x \neq -1, 1$, or $-4/3$. These are the restrictions, values of x that make denominators equal to zero.

The Sign Change Rule for Fractions

Let's look at an alternative approach to the last example. First, let's share the precept that every fraction has three signs, one on the numerator, one on the denominator, and a third on the fraction bar. Thus,

$$\frac{-2}{3} \quad \text{has understood signs} \quad +\frac{-2}{+3}$$

Let's state the sign change rule for fractions.

The Sign Change Rule for Fractions

Every fraction has three signs, one on the numerator, one on the denominator, and one on the fraction bar. If you don't see an explicit sign, then a plus sign is understood. If you negate any two of these parts,

- numerator and denominator, or
- numerator and fraction bar, or
- fraction bar and denominator,

then the fraction remains unchanged.

For example, let's start with −2/3, then do two negations: numerator and fraction bar. Then,

$$+\frac{-2}{+3} = -\frac{+2}{+3}, \quad \text{or with understood plus signs,} \quad \frac{-2}{3} = -\frac{2}{3}$$

This is a familiar result, as negative two divided by a positive three equals a negative two-thirds.

On another note, we might decide to negate numerator and denominator. Then −2/3 becomes

$$+\frac{-2}{+3} = \frac{+2}{-3}, \quad \text{or with understood plus signs,} \quad \frac{-2}{3} = \frac{2}{-3}$$

Again, a familiar result. Certainly, negative two divided by positive three is the same as positive two divided by negative three. They both equal minus two-thirds.

So there you have it. Negate any two parts of a fraction and it remains unchanged. On the surface, this seems a trivial remark, but it can be put to good use when reducing rational expressions. Suppose, for example, that we take the original rational expression from Example 4.5.6 and negate the numerator and fraction bar.

$$\frac{2x - 2x^3}{3x^3 + 4x^2 - 3x - 4} = -\frac{2x^3 - 2x}{3x^3 + 4x^2 - 3x - 4}$$

Note how we've made two sign changes. We've negated the fraction bar, we've negated the numerator $\left(-\left(2x - 2x^3\right) = 2x^3 - 2x\right)$, and left the denominator alone. Therefore, the fraction is unchanged according to our sign change rule.

Now, factor and cancel common factors (we leave the steps for our readers — they're similar to those we took in Example 4.5.6).

\[\begin{aligned} \frac{2 x-2 x^{3}}{3 x^{3}+4 x^{2}-3 x-4} &=-\frac{2 x^{3}-2 x}{3 x^{3}+4 x^{2}-3 x-4} \\ &=-\frac{2 x(x+1)(x-1)}{(x+1)(x-1)(3 x+4)} \\ &=-\frac{2 x(x+1)(x-1)}{(x+1)(x-1)(3 x+4)} \\ &=-\frac{2 x}{3 x+4} \end{aligned} \nonumber \]

But does this answer match the answer in equation (22)? It does, as can be seen by making two negations, fraction bar and numerator.

$$-\frac{2x}{3x+4} = \frac{-2x}{3x+4}$$

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4.6: Rational Functions

Using Arrow Notation

We have seen the graphs of the basic **reciprocal function** and the squared reciprocal function from our study of toolkit functions. Examine these graphs, as shown in Figure 4.6.1, and notice some of their features.

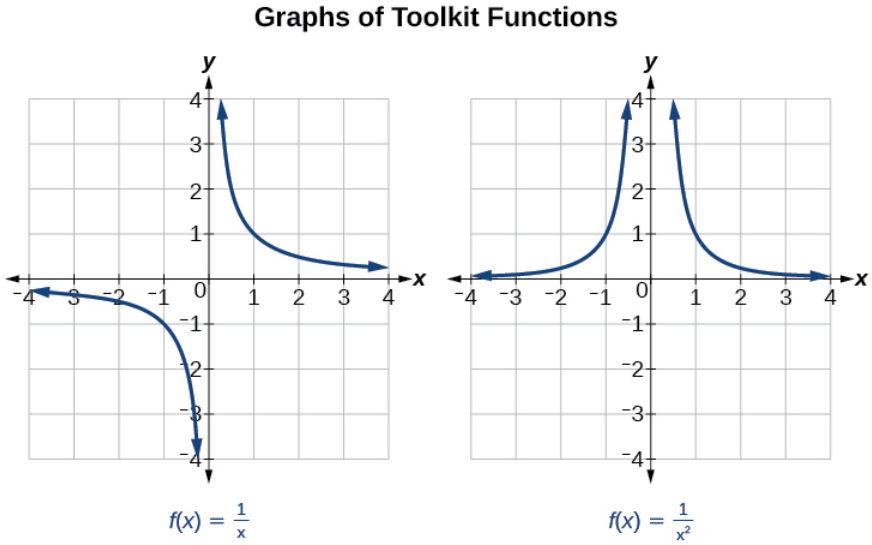


Figure 4.6.1

Several things are apparent if we examine the graph of $f(x) = \frac{1}{x}$.

- On the left branch of the graph, the curve approaches the x-axis $(y = 0)$ as $x \to -\infty$.
- As the graph approaches $x = 0$ from the left, the curve drops, but as we approach zero from the right, the curve rises.
- Finally, on the right branch of the graph, the curves approaches the x-axis $(y = 0)$ as $x \to \infty$.

To summarize, we use **arrow notation** to show that x or $f(x)$ is approaching a particular value (Table 4.6.1).

Table 4.6.1

Symbol	Meaning
$x \to a^-$	x approaches a from the left ($x < a$ but close to a)
$x \to a^+$	x approaches a from the right ($x > a$ but close to a)
$x \to \infty$	x approaches infinity (x increases without bound)
$x \to -\infty$	x approaches negative infinity (x decreases without bound)
$f(x) \to \infty$	the output approaches infinity (the output increases without bound)
$f(x) \to -\infty$	the output approaches negative infinity (the output decreases without bound)
$f(x) \to a$	the output approaches a

Finding the Domains of Rational Functions

A *vertical asymptote* represents a value at which a rational function is undefined, so that value is not in the domain of the function. A reciprocal function cannot have values in its domain that cause the denominator to equal zero. In general, to find the domain of a rational function, we need to determine which inputs would cause division by zero.

Definition: DOMAIN OF A RATIONAL FUNCTION

The domain of a rational function includes all real numbers except those that cause the denominator to equal zero.

How To: Given a rational function, find the domain.

1. Set the denominator equal to zero.
2. Solve to find the x-values that cause the denominator to equal zero.
3. The domain is all real numbers except those found in Step 2.

Example 4.6.4: Finding the Domain of a Rational Function

Find the domain of $f(x)=\dfrac{x+3}{x^2-9}$.

Solution

Begin by setting the denominator equal to zero and solving.

$$x^2-9=0$$

$$x^2=9$$

$$x=\pm 3$$

The denominator is equal to zero when $x=\pm 3$. The domain of the function is all real numbers except $x=\pm 3$.

Analysis

A graph of this function, as shown in Figure 4.6.9, confirms that the function is not defined when $x=\pm 3$.

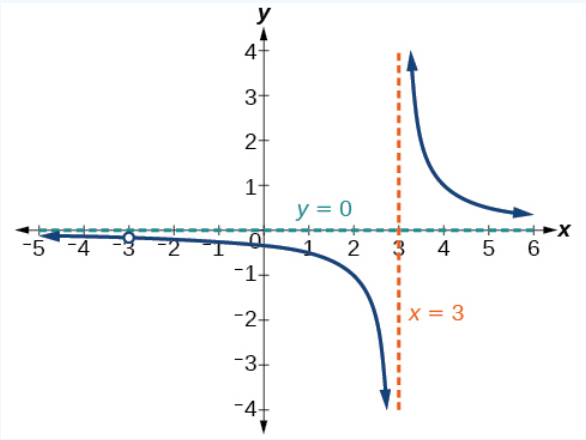


Figure 4.6.9.

There is a vertical asymptote at $x=3$ and a hole in the graph at $x=-3$. We will discuss these types of holes in greater detail later in this section.

Exercise 4.6.4

Find the domain of $f(x)=\dfrac{4x}{5(x-1)(x-5)}$.

Answer

The domain is all real numbers except $x=1$ and $x=5$.

Identifying Vertical Asymptotes of Rational Functions

By looking at the graph of a rational function, we can investigate its local behavior and easily see whether there are asymptotes. We may even be able to approximate their location. Even without the graph, however, we can still determine whether a given rational function has any asymptotes, and calculate their location.

Vertical Asymptotes

The vertical asymptotes of a rational function may be found by examining the factors of the denominator that are not common to the factors in the numerator. Vertical asymptotes occur at the zeros of such factors.

How To: Given a rational function, identify any vertical asymptotes of its graph

1. Factor the numerator and denominator.
2. Note any restrictions in the domain of the function.
3. Reduce the expression by canceling common factors in the numerator and the denominator.
4. Note any values that cause the denominator to be zero in this simplified version. These are where the vertical asymptotes occur.
5. Note any restrictions in the domain where asymptotes do not occur. These are removable discontinuities, or "holes."

Example 4.6.5: Identifying Vertical Asymptotes

Find the vertical asymptotes of the graph of $k(x) = \dfrac{5+2x^2}{2-x-x^2}$.

Solution

First, factor the numerator and denominator.

$$k(x) = \frac{5+2x^2}{2-x-x^2}$$
$$= \frac{5+2x^2}{(2+x)(1-x)}$$

To find the vertical asymptotes, we determine where this function will be undefined by setting the denominator equal to zero:

$$(2+x)(1-x) = 0$$
$$x = -2,\ x = 1$$

Neither $x = -2$ nor $x = 1$ are zeros of the numerator, so the two values indicate two vertical asymptotes. The graph in Figure 4.6.10 confirms the location of the two vertical asymptotes.

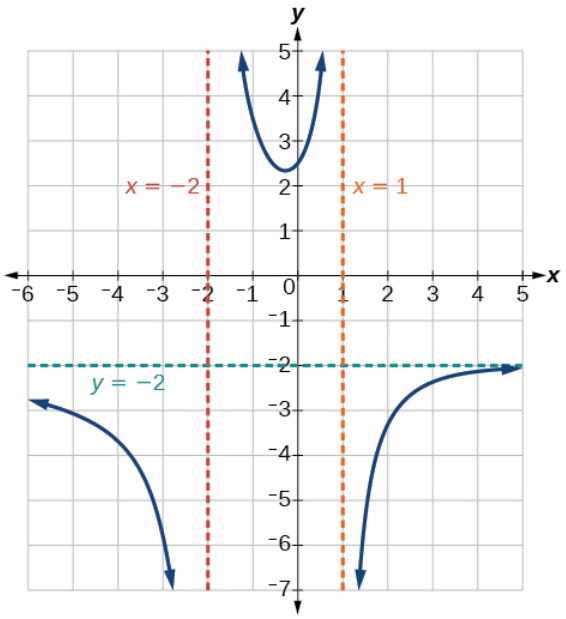


***Figure* 4.6.10.**

Removable Discontinuities

Occasionally, a graph will contain a hole: a single point where the graph is not defined, indicated by an open circle. We call such a hole a *removable discontinuity*. For example, the function $f(x) = \dfrac{x^2 - 1}{x^2 - 2x - 3}$ may be re-written by factoring the numerator and the denominator.

$$f(x) = \frac{(x+1)(x-1)}{(x+1)(x-3)}$$

Notice that $x+1$ is a common factor to the numerator and the denominator. The zero of this factor, $x = -1$, is the location of the removable discontinuity. Notice also that $(x-3)$ is not a factor in both the numerator and denominator. The zero of this factor, $x = 3$, is the vertical asymptote. See Figure 4.6.11. [Note that removable discontinuities may not be visible when we use a graphing calculator, depending upon the window selected.]

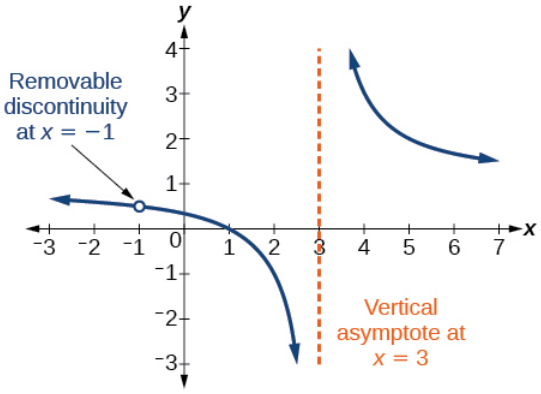


Figure 4.6.11.

REMOVABLE DISCONTINUITIES OF RATIONAL FUNCTIONS

A **removable discontinuity** occurs in the graph of a rational function at $x = a$ if a is a zero for a factor in the denominator that is common with a factor in the numerator. We factor the numerator and denominator and check for common factors. If we find any, we set the common factor equal to 0 and solve. This is the location of the removable discontinuity. This is true if the

multiplicity of this factor is greater than or equal to that in the denominator. If the multiplicity of this factor is greater in the denominator, then there is still an asymptote at that value.

Example 4.6.6: Identifying Vertical Asymptotes and Removable Discontinuities for a Graph

Find the vertical asymptotes and removable discontinuities of the graph of $k(x) = \dfrac{x-2}{x^2-4}$.

Solution

Factor the numerator and the denominator.

$$k(x) = \frac{x-2}{(x-2)(x+2)}$$

Notice that there is a common factor in the numerator and the denominator, $x-2$. The zero for this factor is $x=2$. This is the location of the removable discontinuity.

Notice that there is a factor in the denominator that is not in the numerator, $x+2$. The zero for this factor is $x=-2$. The vertical asymptote is $x=-2$. See Figure 4.6.12

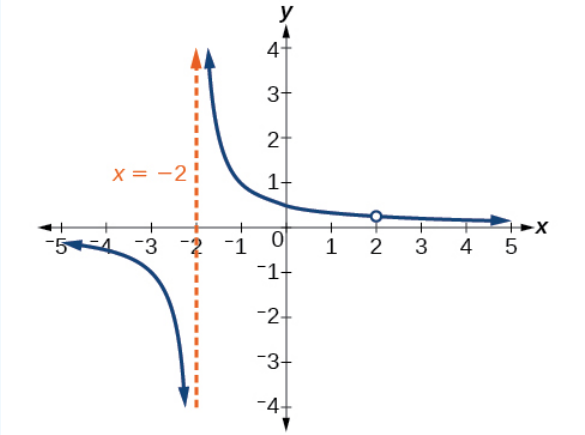


Figure 4.6.12.

The graph of this function will have the vertical asymptote at $x=-2$, but at $x=2$ the graph will have a hole.

Exercise 4.6.5

Find the vertical asymptotes and removable discontinuities of the graph of $f(x) = \dfrac{x^2-25}{x^3-6x^2+5x}$.

Answer

Removable discontinuity at $x=5$.

Vertical asymptotes: $x=0$, $x=1$.

Vertical Asymptotes

The vertical asymptotes of a rational function may be found by examining the factors of the denominator that are not common to the factors in the numerator. Vertical asymptotes occur at the zeros of such factors.

How To: Given a rational function, identify any vertical asymptotes of its graph

1. Factor the numerator and denominator.
2. Note any restrictions in the domain of the function.
3. Reduce the expression by canceling common factors in the numerator and the denominator.

4. Note any values that cause the denominator to be zero in this simplified version. These are where the vertical asymptotes occur.
5. Note any restrictions in the domain where asymptotes do not occur. These are removable discontinuities, or "holes."

Example 4.6.5: Identifying Vertical Asymptotes

Find the vertical asymptotes of the graph of $k(x) = \dfrac{5+2x^2}{2-x-x^2}$.

Solution

First, factor the numerator and denominator.

$$k(x) = \frac{5+2x^2}{2-x-x^2}$$
$$= \frac{5+2x^2}{(2+x)(1-x)}$$

To find the vertical asymptotes, we determine where this function will be undefined by setting the denominator equal to zero:

$$(2+x)(1-x) = 0$$
$$x = -2,\ x = 1$$

Neither $x = -2$ nor $x = 1$ are zeros of the numerator, so the two values indicate two vertical asymptotes. The graph in Figure 4.6.10 confirms the location of the two vertical asymptotes.

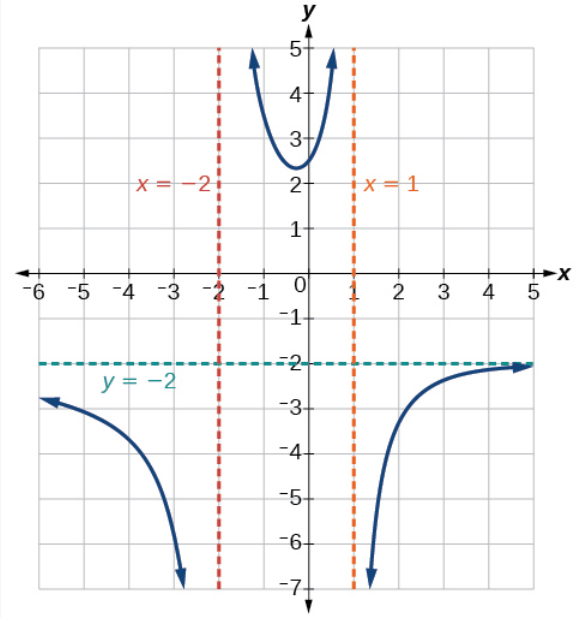


***Figure* 4.6.10.**

Removable Discontinuities

Occasionally, a graph will contain a hole: a single point where the graph is not defined, indicated by an open circle. We call such a hole a *removable discontinuity*. For example, the function $f(x) = \dfrac{x^2-1}{x^2-2x-3}$ may be re-written by factoring the numerator and the denominator.

$$f(x) = \frac{(x+1)(x-1)}{(x+1)(x-3)}$$

Notice that $x+1$ is a common factor to the numerator and the denominator. The zero of this factor, $x=-1$, is the location of the removable discontinuity. Notice also that $(x-3)$ is not a factor in both the numerator and denominator. The zero of this factor, $x=3$, is the vertical asymptote. See Figure 4.6.11. [Note that removable discontinuities may not be visible when we use a graphing calculator, depending upon the window selected.]

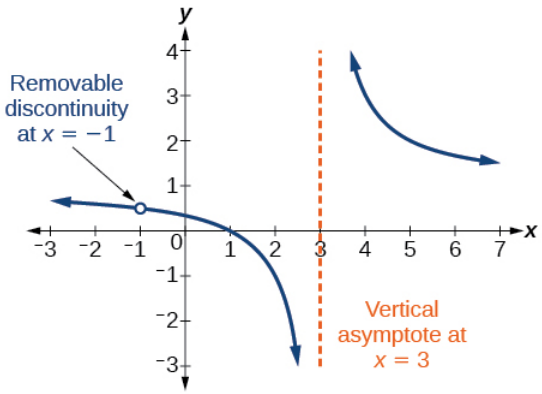


Figure 4.6.11.

REMOVABLE DISCONTINUITIES OF RATIONAL FUNCTIONS

A **removable discontinuity** occurs in the graph of a rational function at $x=a$ if a is a zero for a factor in the denominator that is common with a factor in the numerator. We factor the numerator and denominator and check for common factors. If we find any, we set the common factor equal to 0 and solve. This is the location of the removable discontinuity. This is true if the multiplicity of this factor is greater than or equal to that in the denominator. If the multiplicity of this factor is greater in the denominator, then there is still an asymptote at that value.

Example 4.6.6: Identifying Vertical Asymptotes and Removable Discontinuities for a Graph

Find the vertical asymptotes and removable discontinuities of the graph of $k(x)=\dfrac{x-2}{x^2-4}$.

Solution

Factor the numerator and the denominator.

$$k(x)=\frac{x-2}{(x-2)(x+2)}$$

Notice that there is a common factor in the numerator and the denominator, $x-2$. The zero for this factor is $x=2$. This is the location of the removable discontinuity.

Notice that there is a factor in the denominator that is not in the numerator, $x+2$. The zero for this factor is $x=-2$. The vertical asymptote is $x=-2$. See Figure 4.6.12

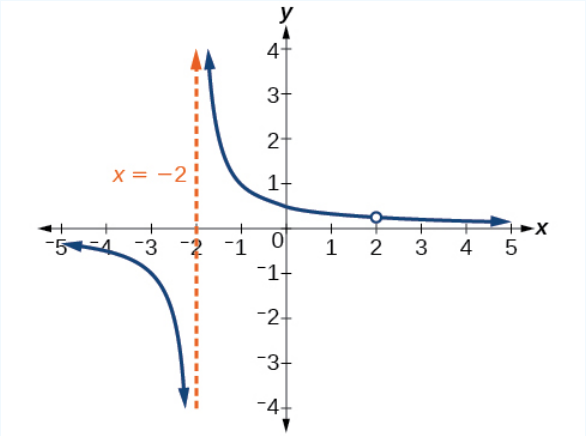


Figure 4.6.12.

The graph of this function will have the vertical asymptote at $x = -2$, but at $x = 2$ the graph will have a hole.

Exercise 4.6.5

Find the vertical asymptotes and removable discontinuities of the graph of $f(x) = \dfrac{x^2 - 25}{x^3 - 6x^2 + 5x}$.

Answer

Removable discontinuity at $x = 5$.

Vertical asymptotes: $x = 0$, $x = 1$.

Identifying Horizontal Asymptotes of Rational Functions

While vertical asymptotes describe the behavior of a graph as the *output* gets very large or very small, horizontal asymptotes help describe the behavior of a graph as the *input* gets very large or very small. Recall that a polynomial's end behavior will mirror that of the leading term. Likewise, a rational function's end behavior will mirror that of the ratio of the function that is the ratio of the leading terms.

There are three distinct outcomes when checking for horizontal asymptotes:

Case 1: If the degree of the denominator > degree of the numerator, there is a **horizontal asymptote** at $y = 0$.

Example: $f(x) = \dfrac{4x + 2}{x^2 + 4x - 5}$

In this case, the end behavior is $f(x) \approx \dfrac{4x}{x^2} = \dfrac{4}{x}$. This tells us that, as the inputs increase or decrease without bound, this function will behave similarly to the function $g(x) = \dfrac{4}{x}$, and the outputs will approach zero, resulting in a horizontal asymptote at $y = 0$. See **Figure** 4.6.13. Note that this graph crosses the horizontal asymptote.

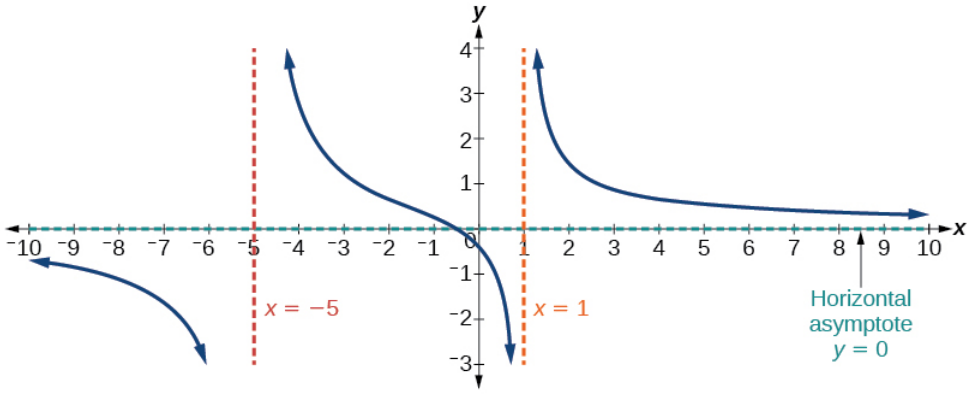


Figure 4.6.13: Horizontal asymptote $y = 0$ when $f(x) = \dfrac{p(x)}{q(x)}$, $q(x) \neq 0$ where degree of p < degree of q.

Case 2: If the degree of the denominator < degree of the numerator by one, we get a slant asymptote.

Example: $f(x) = \dfrac{3x^2 - 2x + 1}{x - 1}$

In this case, the end behavior is $f(x) \approx \dfrac{3x^2}{x} = 3x$. This tells us that as the inputs increase or decrease without bound, this function will behave similarly to the function $g(x) = 3x$. As the inputs grow large, the outputs will grow and not level off, so this graph has no horizontal asymptote. However, the graph of $g(x) = 3x$ looks like a diagonal line, and since f will behave similarly to g, it will approach a line close to $y = 3x$. This line is a slant asymptote.

To find the equation of the slant asymptote, divide $\frac{3x^2-2x+1}{x-1}$. The quotient is $3x+1$, and the remainder is 2. The slant asymptote is the graph of the line $g(x)=3x+1$. See Figure 4.6.14

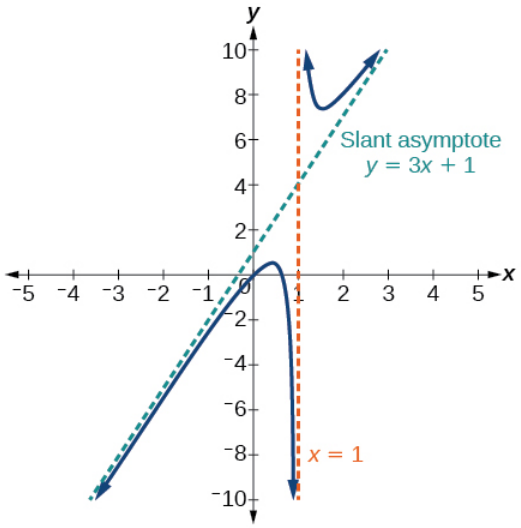


Figure 4.6.14: Slant asymptote when $f(x)=\frac{p(x)}{q(x)}$, $q(x)\neq 0$ where degree of p >degree of q by 1.

Case 3: If the degree of the denominator = degree of the numerator, there is a horizontal asymptote at $y=\frac{a_n}{b_n}$, where a_n and b_n are respectively the leading coefficients of $p(x)$ and $q(x)$ for $f(x)=\frac{p(x)}{q(x)}$, $q(x)\neq 0$.

Example: $f(x)=\frac{3x^2+2}{x^2+4x-5}$

In this case, the end behavior is $f(x)\approx\frac{3x^2}{x^2}=3$. This tells us that as the inputs grow large, this function will behave like the function $g(x)=3$, which is a horizontal line. As $x\to\pm\infty$, $f(x)\to 3$, resulting in a horizontal asymptote at $y=3$. See Figure 4.6.15. Note that this graph crosses the horizontal asymptote.

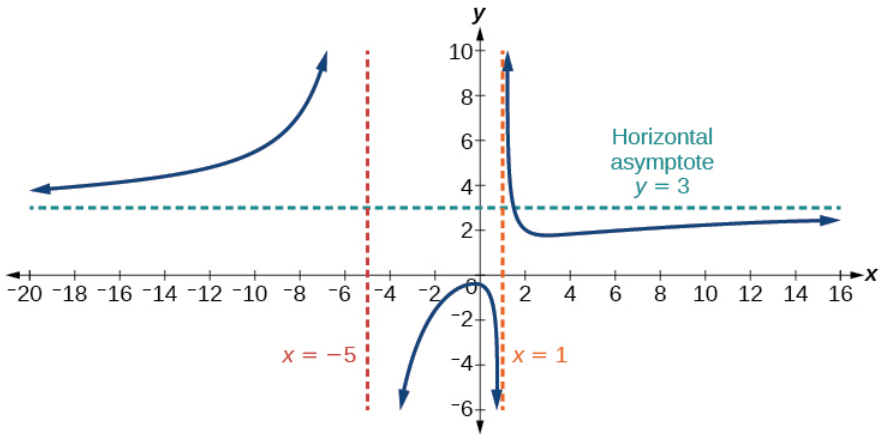


Figure 4.6.15: Horizontal asymptote when $f(x)=\frac{p(x)}{q(x)}$, $q(x)\neq 0$ where degree of p = degree of q.

Notice that, while the graph of a rational function will never cross a **vertical asymptote**, the graph may or may not cross a horizontal or slant asymptote. Also, although the graph of a rational function may have many vertical asymptotes, the graph will have at most one horizontal (or slant) asymptote.

It should be noted that, if the degree of the numerator is larger than the degree of the denominator by more than one, the **end behavior** of the graph will mimic the behavior of the reduced end behavior fraction. For instance, if we had the function

$$f(x) = \frac{3x^5 - x^2}{x+3}$$

with end behavior

$$f(x) \approx \frac{3x^5}{x} = 3x^4$$

the end behavior of the graph would look similar to that of an even polynomial with a positive leading coefficient.

$$x \to \pm\infty, f(x) \to \infty$$

HORIZONTAL ASYMPTOTES OF RATIONAL FUNCTIONS

The **horizontal asymptote** of a rational function can be determined by looking at the degrees of the numerator and denominator.

- Degree of numerator is less than degree of denominator: horizontal asymptote at $y = 0$.
- Degree of numerator is greater than degree of denominator by one: no horizontal asymptote; slant asymptote.
- Degree of numerator is equal to degree of denominator: horizontal asymptote at ratio of leading coefficients.

Example 4.6.7: Identifying Horizontal and Slant Asymptotes

For the functions listed, identify the horizontal or slant asymptote.

1. $g(x) = \dfrac{6x^3 - 10x}{2x^3 + 5x^2}$
2. $h(x) = \dfrac{x^2 - 4x + 1}{x+2}$
3. $k(x) = \dfrac{x^2 + 4x}{x^3 - 8}$

Solution

For these solutions, we will use $f(x) = \dfrac{p(x)}{q(x)},\ q(x) \neq 0$.

1. $g(x) = \dfrac{6x^3 - 10x}{2x^3 + 5x^2}$: The degree of $p =$ degree of $q = 3$, so we can find the horizontal asymptote by taking the ratio of the leading terms. There is a horizontal asymptote at $y = \frac{6}{2}$ or $y = 3$.
2. $h(x) = \dfrac{x^2 - 4x + 1}{x+2}$: The degree of $p = 2$ and degree of $q = 1$. Since $p > q$ by 1, there is a slant asymptote found at $\dfrac{x^2 - 4x + 1}{x+2}$.

$$\begin{array}{r|rrr} -2 & 1 & -4 & 1 \\ & & -2 & 12 \\ \hline & 1 & -6 & 13 \end{array}$$

3. $k(x) = \dfrac{x^2 + 4x}{x^3 - 8}$: The degree of $p = 2 <$ degree of $q = 3$, so there is a horizontal asymptote $y = 0$.

Example 4.6.8 Identifying Horizontal Asymptotes

In the sugar concentration problem earlier, we created the equation $C(t) = \dfrac{5+t}{100+10t}$.

Find the horizontal asymptote and interpret it in context of the problem.

Solution

Both the numerator and denominator are linear (degree 1). Because the degrees are equal, there will be a horizontal asymptote at the ratio of the leading coefficients. In the numerator, the leading term is t, with coefficient 1. In the denominator, the leading term is 10t, with coefficient 10. The horizontal asymptote will be at the ratio of these values:

$$t \to \infty,\ C(t) \to \tfrac{1}{10}$$

This function will have a horizontal asymptote at $y = \frac{1}{10}$.

This tells us that as the values of t increase, the values of C will approach $\frac{1}{10}$. In context, this means that, as more time goes by, the concentration of sugar in the tank will approach one-tenth of a pound of sugar per gallon of water or $\frac{1}{10}$ pounds per gallon.

Example 4.6.9: Identifying Horizontal and Vertical Asymptotes

Find the horizontal and vertical asymptotes of the function $f(x) = \dfrac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}$

Solution

First, note that this function has no common factors, so there are no potential removable discontinuities.

The function will have vertical asymptotes when the denominator is zero, causing the function to be undefined. The denominator will be zero at $x = 1, -2,$ and 5, indicating vertical asymptotes at these values.

The numerator has degree 2, while the denominator has degree 3. Since the degree of the denominator is greater than the degree of the numerator, the denominator will grow faster than the numerator, causing the outputs to tend towards zero as the inputs get large, and so as $x \to \pm\infty$, $f(x) \to 0$. This function will have a horizontal asymptote at $y = 0$. See Figure 4.6.16.

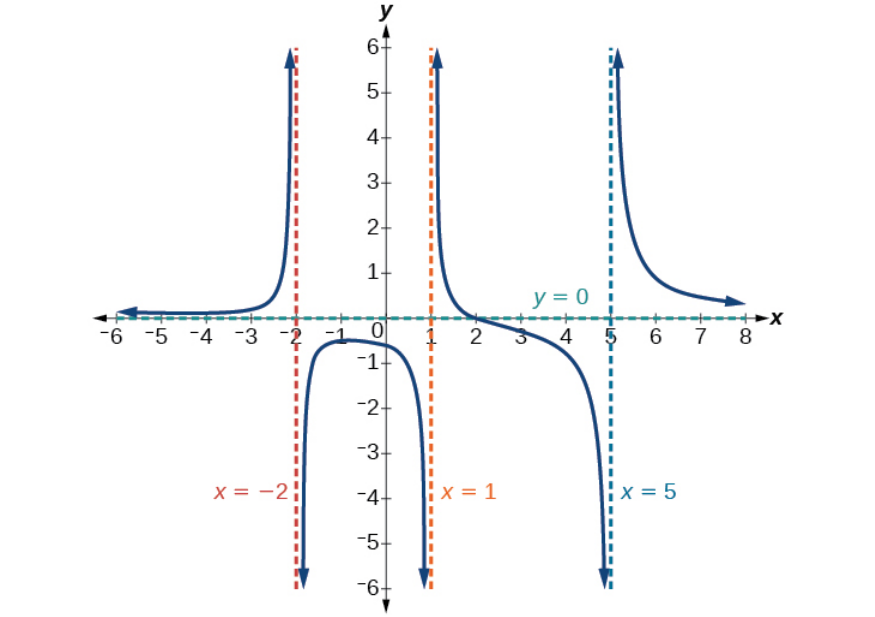


Figure 4.6.16.

Exercise 4.6.6

Find the vertical and horizontal asymptotes of the function:

$$f(x) = \frac{(2x-1)(2x+1)}{(x-2)(x+3)}$$

Answer

Vertical asymptotes at $x = 2$ and $x = -3$

horizontal asymptote at $y = 4$.

INTERCEPTS OF RATIONAL FUNCTIONS

A *rational function* will have a y-intercept at $f(0)$, if the function is defined at zero. A rational function will not have a y-intercept if the function is not defined at zero.

Likewise, a rational function will have x-intercepts at the inputs that cause the output to be zero. Since a fraction is only equal to zero when the numerator is zero, x-intercepts can only occur when the numerator of the rational function is equal to zero.

Example 4.6.10: Finding the Intercepts of a Rational Function

Find the intercepts of $f(x) = \dfrac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}$.

Solution

We can find the y-intercept by evaluating the function at zero

$$\begin{aligned} f(0) &= \frac{(0-2)(0+3)}{(0-1)(0+2)(0-5)} \\ &= -\frac{6}{10} \\ &= -\frac{3}{5} \\ &= -0.6 \end{aligned}$$

The x-intercepts will occur when the function is equal to zero:

$$0 = \frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)} \text{ This is zero when the numerator is zero.}$$

$$0 = (x-2)(x+3)$$

$$x = 2, x = -3$$

The y-intercept is $(0, -0.6)$, the x-intercepts are $(2, 0)$ and $(-3, 0)$. See Figure 4.6.17.

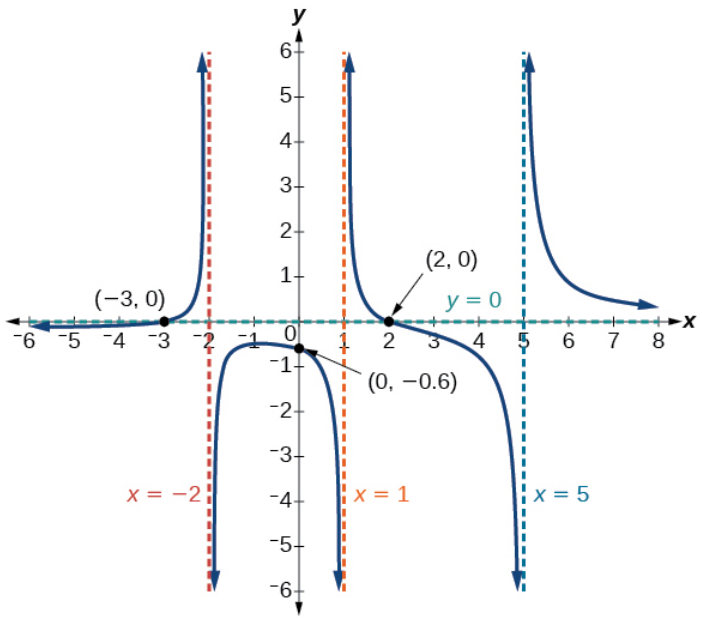


Figure 4.6.17.

Exercise 4.6.7

Given the reciprocal squared function that is shifted right 3 units and down 4 units, write this as a rational function. Then, find the x- and y-intercepts and the horizontal and vertical asymptotes.

Answer

For the transformed reciprocal squared function, we find the rational form.

$$f(x)=\frac{1}{(x-3)^2}-4=\frac{1-4(x-3)^2}{(x-3)^2}=\frac{1-4(x^2-6x+9)}{(x-3)(x-3)}=\frac{-4x^2+24x-35}{x^2-6x+9}$$

Because the numerator is the same degree as the denominator we know that as $x\to\pm\infty$, $f(x)\to -4$; so $y=-4$ is the horizontal asymptote. Next, we set the denominator equal to zero, and find that the vertical asymptote is $x=3$, because as $x\to 3$, $f(x)\to\infty$. We then set the numerator equal to 0 and find the x-intercepts are at $(2.5,0)$ and $(3.5,0)$. Finally, we evaluate the function at 0 and find the y-intercept to be at $(0,-\frac{35}{9})$.

Graphing Rational Functions

In Example 4.6.10, we see that the numerator of a rational function reveals the *x*-intercepts of the graph, whereas the denominator reveals the vertical asymptotes of the graph. As with polynomials, factors of the numerator may have integer powers greater than one. Fortunately, the effect on the shape of the graph at those intercepts is the same as we saw with polynomials.

The vertical asymptotes associated with the factors of the denominator will mirror one of the two toolkit reciprocal functions. When the degree of the factor in the denominator is odd, the distinguishing characteristic is that on one side of the vertical asymptote the graph heads towards positive infinity, and on the other side the graph heads towards negative infinity. See Figure 4.6.18

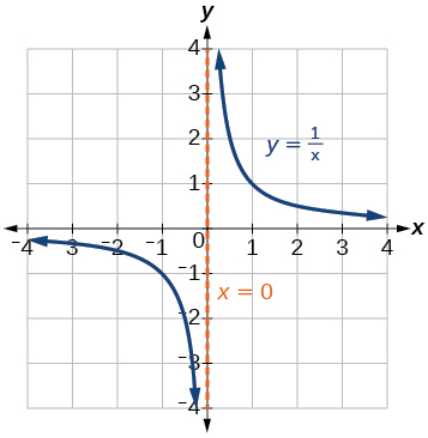


Figure 4.6.18.

When the degree of the factor in the denominator is even, the distinguishing characteristic is that the graph either heads toward positive infinity on both sides of the vertical asymptote or heads toward negative infinity on both sides. See Figure 4.6.19

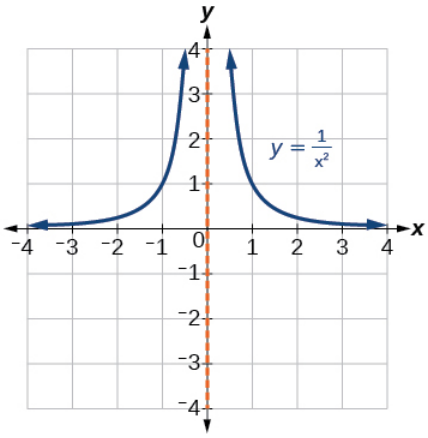


Figure 4.6.19.

For example, the graph of $f(x) = \dfrac{(x+1)^2(x-3)}{(x+3)^2(x-2)}$ is shown in Figure 4.6.20

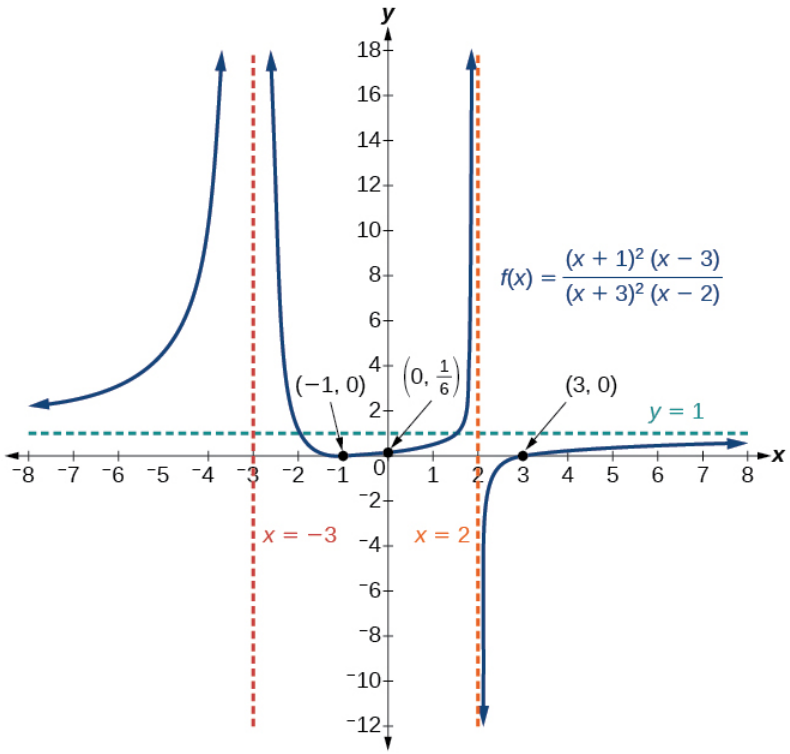


Figure 4.6.20.

- At the x-intercept $x = -1$ corresponding to the $(x+1)^2$ factor of the numerator, the graph "bounces", consistent with the quadratic nature of the factor.
- At the x-intercept $x = 3$ corresponding to the $(x-3)$ factor of the numerator, the graph passes through the axis as we would expect from a linear factor.
- At the vertical asymptote $x = -3$ corresponding to the $(x+3)^2$ factor of the denominator, the graph heads towards positive infinity on both sides of the asymptote, consistent with the behavior of the function $f(x) = \frac{1}{x^2}$.
- At the vertical asymptote $x = 2$, corresponding to the $(x-2)$ factor of the denominator, the graph heads towards positive infinity on the left side of the asymptote and towards negative infinity on the right side, consistent with the behavior of the function $f(x) = \frac{1}{x}$.

Howto: Given a rational function, sketch a graph.

1. Evaluate the function at 0 to find the y-intercept.
2. Factor the numerator and denominator.
3. For factors in the numerator not common to the denominator, determine where each factor of the numerator is zero to find the x-intercepts.
4. Find the multiplicities of the x-intercepts to determine the behavior of the graph at those points.
5. For factors in the denominator, note the multiplicities of the zeros to determine the local behavior. For those factors not common to the numerator, find the vertical asymptotes by setting those factors equal to zero and then solve.
6. For factors in the denominator common to factors in the numerator, find the removable discontinuities by setting those factors equal to 0 and then solve.
7. Compare the degrees of the numerator and the denominator to determine the horizontal or slant asymptotes.
8. Sketch the graph.

Example 4.6.11: Graphing a Rational Function

Sketch a graph of $f(x) = \frac{(x+2)(x-3)}{(x+1)^2(x-2)}$.

Solution

We can start by noting that the function is already factored, saving us a step.

Next, we will find the intercepts. Evaluating the function at zero gives the y-intercept:

$$f(0)=\frac{(0+2)(0-3)}{(0+1)^2(0-2)}=3$$

To find the x-intercepts, we determine when the numerator of the function is zero. Setting each factor equal to zero, we find x-intercepts at $x=-2$ and $x=3$. At each, the behavior will be linear (multiplicity 1), with the graph passing through the intercept.

We have a y-intercept at $(0,3)$ and x-intercepts at $(-2,0)$ and $(3,0)$.

To find the vertical asymptotes, we determine when the denominator is equal to zero. This occurs when $x+1=0$ and when $x-2=0$, giving us vertical asymptotes at $x=-1$ and $x=2$.

There are no common factors in the numerator and denominator. This means there are no removable discontinuities.

Finally, the degree of denominator is larger than the degree of the numerator, telling us this graph has a horizontal asymptote at $y=0$.

To sketch the graph, we might start by plotting the three intercepts. Since the graph has no x-intercepts between the vertical asymptotes, and the y-intercept is positive, we know the function must remain positive between the asymptotes, letting us fill in the middle portion of the graph as shown in Figure 4.6.21.

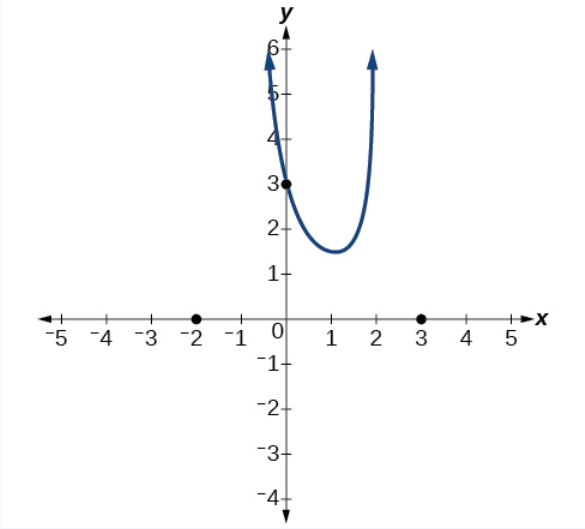


Figure 4.6.21.

The factor associated with the vertical asymptote at $x=-1$ was squared, so we know the behavior will be the same on both sides of the asymptote. The graph heads toward positive infinity as the inputs approach the asymptote on the right, so the graph will head toward positive infinity on the left as well.

For the vertical asymptote at $x=2$, the factor was not squared, so the graph will have opposite behavior on either side of the asymptote. See Figure 4.6.22. After passing through the x-intercepts, the graph will then level off toward an output of zero, as indicated by the horizontal asymptote.

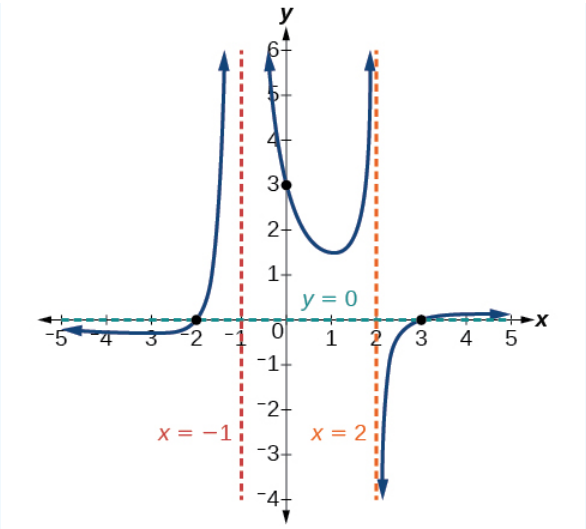


Figure 4.6.22.

Exercise 4.6.8

Given the function $f(x) = \frac{(x+2)^2(x-2)}{2(x-1)^2(x-3)}$, use the characteristics of polynomials and rational functions to describe its behavior and sketch the function.

Answer

Horizontal asymptote at $y = \frac{1}{2}$. Vertical asymptotes at $x = 1$ and $x = 3$. y-intercept at $(0, \frac{4}{3})$.

x-intercepts at $(2, 0)$ and $(-2, 0)$. $(-2, 0)$ is a zero with multiplicity 2, and the graph bounces off the x-axis at this point. $(2, 0)$ is a single zero and the graph crosses the axis at this point.

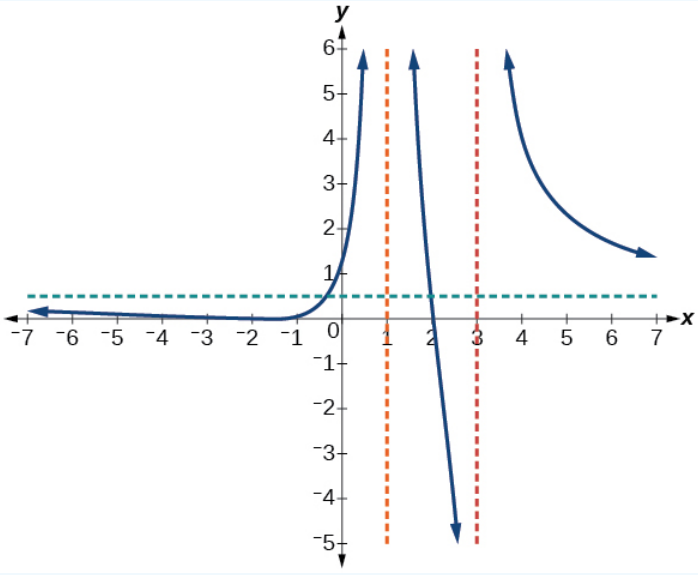


Figure 4.6.23.

Key Equations

Rational Function $$f(x) = \frac{P(x)}{Q(x)} = \frac{a_p x^p + a_{p-1} x^{p-1} + \ldots + a_1 x + a_0}{b_q x^q + b_{q-1} x^{q-1} + \ldots + b_1 x + b_0},\ Q(x) \neq 0$$

Key Concepts

- We can use arrow notation to describe local behavior and end behavior of the toolkit functions $f(x) = \frac{1}{x}$ and $f(x) = \frac{1}{x^2}$. See Example 4.6.1.
- A function that levels off at a horizontal value has a horizontal asymptote. A function can have more than one vertical asymptote. See Example.
- Application problems involving rates and concentrations often involve rational functions. See Example.
- The domain of a rational function includes all real numbers except those that cause the denominator to equal zero. See Example.
- The vertical asymptotes of a rational function will occur where the denominator of the function is equal to zero and the numerator is not zero. See Example.
- A removable discontinuity might occur in the graph of a rational function if an input causes both numerator and denominator to be zero. See Example.
- A rational function's end behavior will mirror that of the ratio of the leading terms of the numerator and denominator functions. See Example, Example, Example, and Example.
- Graph rational functions by finding the intercepts, behavior at the intercepts and asymptotes, and end behavior. See Example.
- If a rational function has x-intercepts at $x = x_1, x_2, \ldots, x_n$, vertical asymptotes at $x = v_1, v_2, \ldots, v_m$, and no $x_i =$ any v_j, then the function can be written in the form

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CHAPTER OVERVIEW

5: Other Functions

5.1: Absolute Value Functions
5.2: Radical Functions

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5.1: Absolute Value Functions

Understanding Absolute Value

Recall that in its basic form $f(x) = |x|$, the **absolute value** function, is one of our toolkit functions. The absolute value function is commonly thought of as providing the distance the number is from zero on a number line. Algebraically, for whatever the input value is, the output is the value without regard to sign.

Absolute Value Function

The absolute value function can be defined as a piecewise function

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \tag{5.1.1}$$

Example 5.1.1: Determine a Number within a Prescribed Distance

Describe all values x within or including a distance of 4 from the number 5.

Solution

We want the distance between x and 5 to be less than or equal to 4. We can draw a number line, such as the one in , to represent the condition to be satisfied.

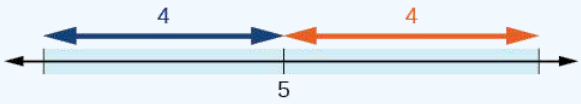


Figure 5.1.2: Number line describing the difference of the distance of 4 away from 5.

The distance from x to 5 can be represented using the absolute value as $|x-5|$. We want the values of x that satisfy the condition $|x-5| \leq 4$.

Analysis

Note that

$$\begin{aligned} -4 \leq x-5 \qquad & x-5 \leq 4 \\ 1 \leq x \qquad & x \leq 9 \end{aligned}$$

So $|x-5| \leq 4$ is equivalent to $1 \leq x \leq 9$.

However, mathematicians generally prefer absolute value notation.

Exercise 5.1.1

Describe all values x within a distance of 3 from the number 2.

Answer

$|x-2| \leq 3$

Example 5.1.2: Resistance of a Resistor

Electrical parts, such as resistors and capacitors, come with specified values of their operating parameters: resistance, capacitance, etc. However, due to imprecision in manufacturing, the actual values of these parameters vary somewhat from piece to piece, even when they are supposed to be the same. The best that manufacturers can do is to try to guarantee that the variations will stay within a specified range, often ±1%, ±5%, or ±10%.

Suppose we have a resistor rated at 680 ohms, ±5%. Use the absolute value function to express the range of possible values of the actual resistance.

Solution

5% of 680 ohms is 34 ohms. The absolute value of the difference between the actual and nominal resistance should not exceed the stated variability, so, with the resistance R in ohms,

$$|R-680| \leq 34$$

Exercise 5.1.2

Students who score within 20 points of 80 will pass a test. Write this as a distance from 80 using absolute value notation.

Answer

Using the variable p for passing, $|p-80| \leq 20$

Graphing an Absolute Value Function

The most significant feature of the absolute value graph is the corner point at which the graph changes direction. This point is shown at the origin in Figure 5.1.3.

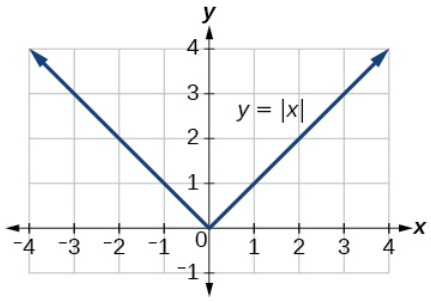


Figure 5.1.3: Graph of an absolute function.

Figure 5.1.3 shows the graph of $y = 2|x-3|+4$. The graph of $y = |x|$ has been shifted right 3 units, vertically stretched by a factor of 2, and shifted up 4 units. This means that the corner point is located at $(3, 4)$ for this transformed function.

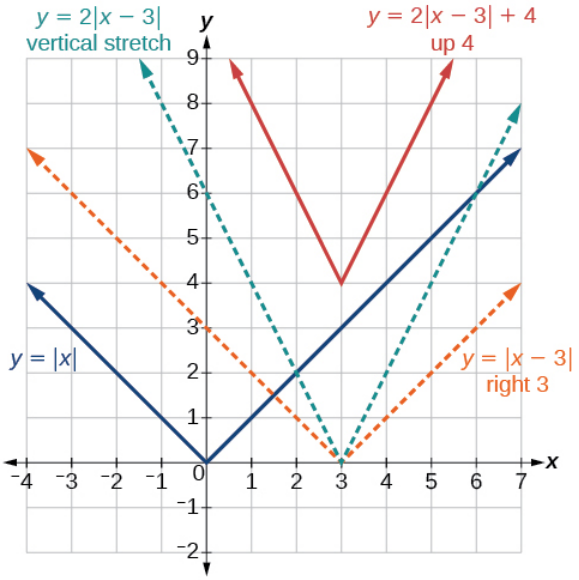


Figure 5.1.4: Graph of the different types of transformations for an absolute function.

Example 5.1.3: Writing an Equation for an Absolute Value Function

Write an equation for the function graphed in Figure 5.1.5.

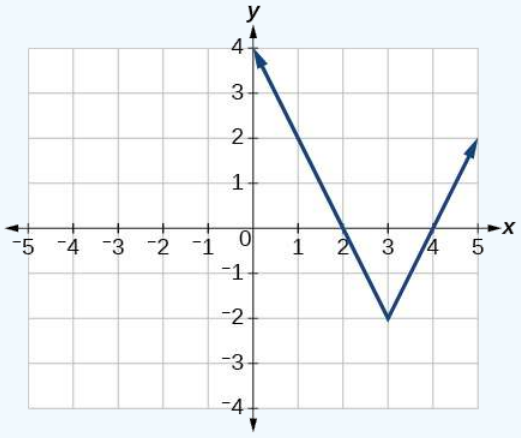


Figure 5.1.5: Graph of an absolute function.

Solution

The basic absolute value function changes direction at the origin, so this graph has been shifted to the right 3 units and down 2 units from the basic toolkit function. See Figure 5.1.6.

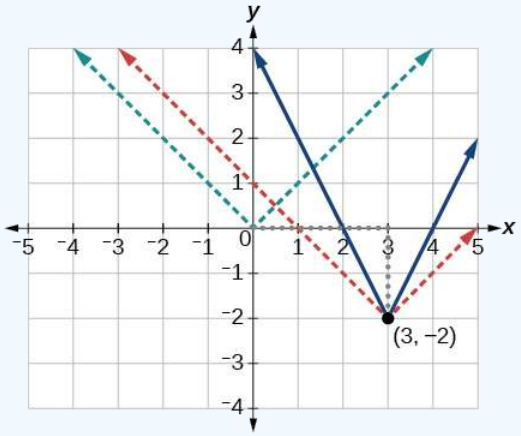


Figure 5.1.6: Graph of two transformations for an absolute function at $(3, -2)$.

We also notice that the graph appears vertically stretched, because the width of the final graph on a horizontal line is not equal to 2 times the vertical distance from the corner to this line, as it would be for an unstretched absolute value function. Instead, the width is equal to 1 times the vertical distance as shown in Figure 5.1.7.

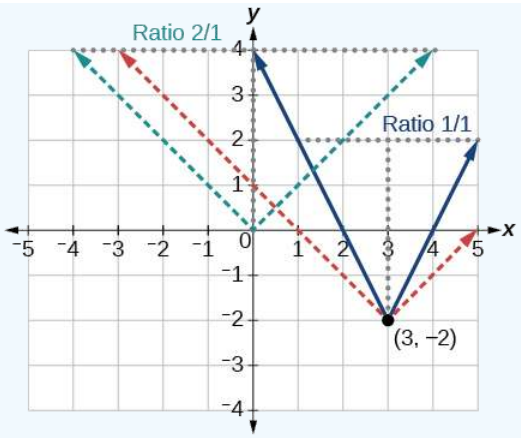


Figure 5.1.7: Graph of two transformations for an absolute function at $(3,-2)$ and the ratios between the two different transformations.

From this information we can write the equation

$$\begin{aligned} f(x) &= 2|x-3|-2, && \text{treating the stretch as a vertial stretch, or} \\ f(x) &= |2(x-3)|-2, && \text{treating the stretch as a horizontal compression.} \end{aligned}$$

Analysis

Note that these equations are algebraically equivalent—the stretch for an absolute value function can be written interchangeably as a vertical or horizontal stretch or compression.

Q & A

If we couldn't observe the stretch of the function from the graphs, could we algebraically determine it?

Answer

Yes. If we are unable to determine the stretch based on the width of the graph, we can solve for the stretch factor by putting in a known pair of values for x and $f(x)$.

$$f(x) = a|x-3|-2$$

Now substituting in the point $(1,2)$

$$\begin{aligned} 2 &= a|1-3|-2 \\ 4 &= 2a \\ a &= 2 \end{aligned}$$

Exercise 5.1.3

Write the equation for the absolute value function that is horizontally shifted left 2 units, is vertically flipped, and vertically shifted up 3 units.

Answer

$f(x) = -|x+2|+3$

Q & A

Do the graphs of absolute value functions always intersect the vertical axis? The horizontal axis?

Answer

Yes, they always intersect the vertical axis. The graph of an absolute value function will intersect the vertical axis when the input is zero.

No, they do not always intersect the horizontal axis. The graph may or may not intersect the horizontal axis, depending on how the graph has been shifted and reflected. It is possible for the absolute value function to intersect the horizontal axis at zero, one, or two points (Figure 5.1.8).

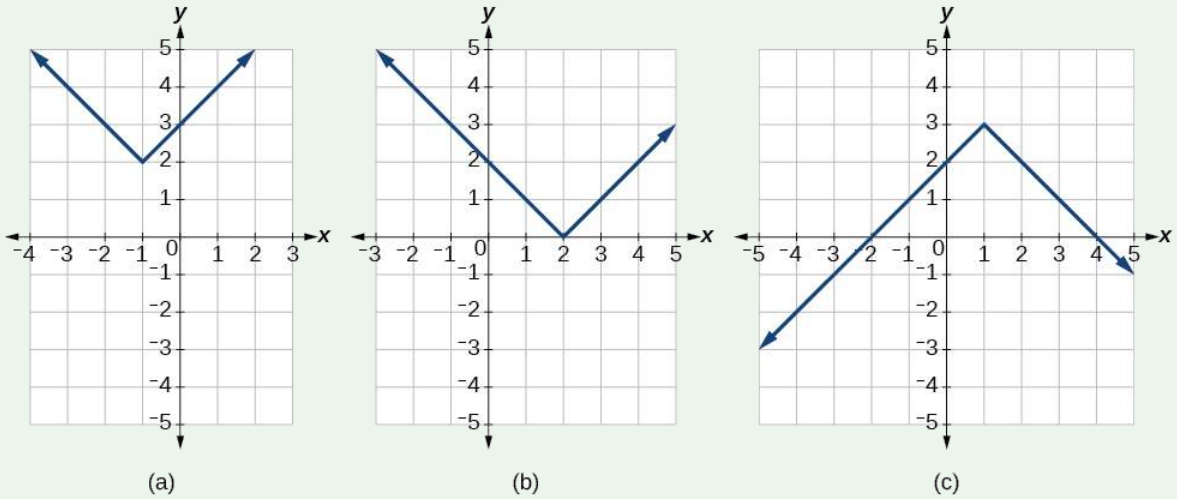


Figure 5.1.8: (a) The absolute value function does not intersect the horizontal axis. (b) The absolute value function intersects the horizontal axis at one point. (c) The absolute value function intersects the horizontal axis at two points.

Solving an Absolute Value Equation

Now that we can graph an absolute value function, we will learn how to solve an absolute value equation. To solve an equation such as $8 = |2x - 6|$, we notice that the absolute value will be equal to 8 if the quantity inside the absolute value is 8 or -8. This leads to two different equations we can solve independently.

$$2x - 6 = 8 \quad \text{or} \quad 2x - 6 = -8$$

$$2x = 14 \qquad 2x = -2$$
$$x = 7 \qquad x = -1$$

Knowing how to solve problems involving absolute value functions is useful. For example, we may need to identify numbers or points on a line that are at a specified distance from a given reference point.

An **absolute value equation** is an equation in which the unknown variable appears in absolute value bars. For example,

$$|x| = 4,$$
$$|2x - 1| = 3,$$
$$|5x + 2| - 4 = 9.$$

Solutions to Absolute Value Equations

For real numbers A and B, an equation of the form $|A| = B$, with $B \geq 0$, will have solutions when $A = B$ or $A = -B$. If $B < 0$, the equation $|A| = B$ has no solution.

How To ...

Given the formula for an absolute value function, find the horizontal intercepts of its graph.

1. Isolate the absolute value term.
2. Use $|A| = B$ to write $A = B$ or $-A = B$, assuming $B > 0$.
3. Solve for x.

Example 5.1.4: Finding the Zeros of an Absolute Value Function

For the function $f(x) = |4x+1| - 7$, find the values of x such that $f(x) = 0$.

Solution

$$
\begin{aligned}
0 &= |4x+1| - 7 && && \text{Substitute 0 for f(x).}\\
7 &= |4x+1| && && \text{Isolate the absolute value on one side of the equation.}\\
7 &= 4x+1 && \text{or } -7 = 4x+1 && \text{Break into two separate equations and solve.}\\
6 &= 4x && -8 = 4x &&\\
x &= \frac{6}{4} = 1.5 && x = \frac{-8}{4} = -2 &&
\end{aligned}
$$

The function outputs 0 when $x = 1.5$ or $x = -2$ (Figure 5.1.9).

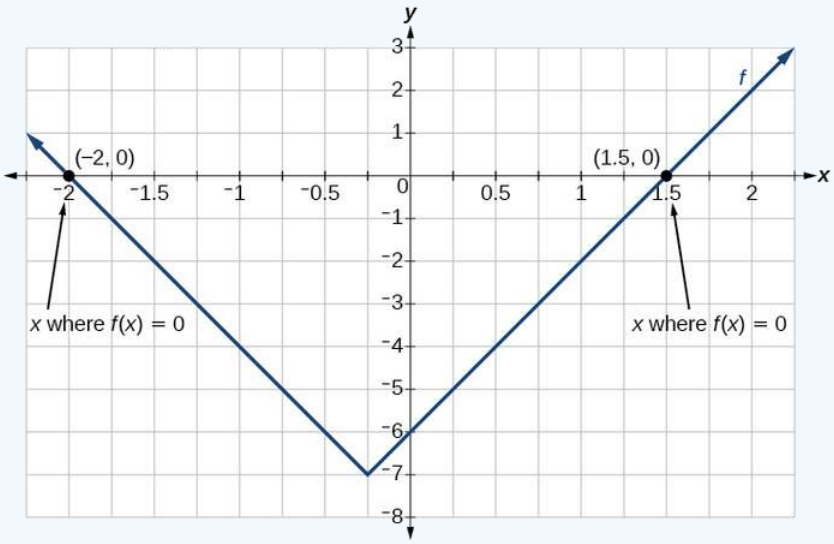


Figure 5.1.9: Graph of an absolute function with x-intercepts at -2 and 1.5.

Exercise 5.1.4

For the function $f(x) = |2x-1| - 3$, find the values of x such that $f(x) = 0$.

Solution

$x = -1$ or $x = 2$

Q & A

Should we always expect two answers when solving $|A| = B$?

Answer

No. We may find one, two, or even no answers. For example, there is no solution to $2 + |3x-5| = 1$.

How To ...

Given an absolute value equation, solve it.

1. Isolate the absolute value term.
2. Use $|A| = B$ to write $A = B$ or $A = -B$.
3. Solve for x.

Example 5.1.5: Solving an Absolute Value Equation

Solve $1 = 4|x-2|+2$.

Solution

Isolating the absolute value on one side of the equation gives the following.

$$\begin{aligned} 1 &= 4|x-2|+2 \\ -1 &= 4|x-2| \\ -\frac{1}{4} &= |x-2| \end{aligned}$$

The absolute value always returns a positive value, so it is impossible for the absolute value to equal a negative value. At this point, we notice that this equation has no solutions.

Q & A

In Example 5.1.5, if $f(x) = 1$ and $g(x) = 4|x-2|+2$ were graphed on the same set of axes, would the graphs intersect?

Answer

No. The graphs of f and g would not intersect, as shown in Figure 5.1.10. This confirms, graphically, that the equation $1 = 4|x-2|+2$ has no solution.

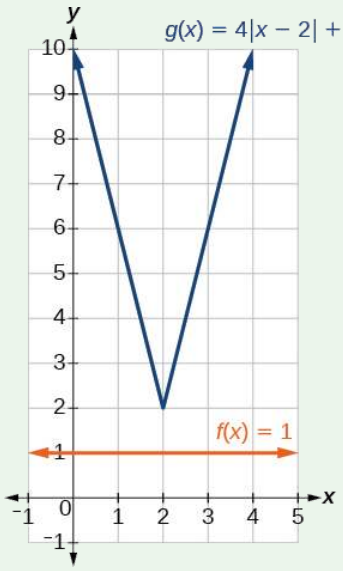


Figure 5.1.10: Graph of $g(x) = 4|x-2|+2$ and $f(x) = 1$.

Find where the graph of the function $f(x) = -|x+2|+3$ intersects the horizontal and vertical axes.

$f(0) = 1$, so the graph intersects the vertical axis at $(0, 1)$. $f(x) = 0$ when $x = -5$ and $x = 1$ so the graph intersects the horizontal axis at $(-5, 0)$ and $(1, 0)$.

Solving an Absolute Value Inequality

Absolute value equations may not always involve equalities. Instead, we may need to solve an equation within a range of values. We would use an **absolute value inequality** to solve such an equation. An absolute value inequality is an equation of the form

$$|A| < B,\ |A| \leq B, |A| > B,$$

or

$$|A| \geq B,$$

where an expression A (and possibly but not usually B) depends on a variable x. Solving the inequality means finding the set of all x that satisfy the inequality. Usually this set will be an interval or the union of two intervals.

There are two basic approaches to solving absolute value inequalities: graphical and algebraic. The advantage of the graphical approach is we can read the solution by interpreting the graphs of two functions. The advantage of the algebraic approach is it yields solutions that may be difficult to read from the graph.

For example, we know that all numbers within 200 units of 0 may be expressed as

$$|x| < 200$$

or

$$-200 < x < 200$$

Suppose we want to know all possible returns on an investment if we could earn some amount of money within $200 of $600. We can solve algebraically for the set of values x such that the distance between x and 600 is less than 200. We represent the distance between x and 600 as $|x-600|$.

$$|x-600| < 200 \tag{5.1.2}$$

or

$$-200 < x-600 < 200 \tag{5.1.3}$$

$$-200+600 < x-600+600 < 200+600$$

$$400 < x < 800$$

This means our returns would be between $400 and $800.

Sometimes an absolute value inequality problem will be presented to us in terms of a shifted and/or stretched or compressed absolute value function, where we must determine for which values of the input the function's output will be negative or positive.

How To ...

Given an absolute value inequality of the form $|x-A| \leq B$ for real numbers a and b where b is positive, solve the absolute value inequality algebraically.

1. Find boundary points by solving $|x-A| = B$.
2. Test intervals created by the boundary points to determine where $|x-A| \leq B$.
3. Write the interval or union of intervals satisfying the inequality in interval, inequality, or set-builder notation.

Example 5.1.6: Solving an Absolute Value Inequality

Solve $|x-5| \leq 4$.

Solution

With both approaches, we will need to know first where the corresponding equality is true. In this case we first will find where $|x-5| = 4$. We do this because the absolute value is a function with no breaks, so the only way the function values can switch from being less than 4 to being greater than 4 is by passing through where the values equal 4. Solve $|x-5| = 4$.

$$\begin{aligned} x-5 &= 4 \quad \text{or} \quad x = 9 \\ x-5 &= -4 \quad\quad\;\; x = 1 \end{aligned}$$

After determining that the absolute value is equal to 4 at $x = 1$ and $x = 9$, we know the graph can change only from being less than 4 to greater than 4 at these values. This divides the number line up into three intervals:

$$x < 1,\ 1 < x < 9,\ \text{and}\ x > 9.$$

To determine when the function is less than 4, we could choose a value in each interval and see if the output is less than or greater than 4, as shown in Table 5.1.1.

Table 5.1.1

Interval test x	$f(x)$	<4 or >4	
$x<1$	0	$\lvert 0-5\rvert=5$	Greater than
$1<x<9$	6	$\lvert 6-5\rvert=1$	Less than
$x>9$	11	$\lvert 11-5\rvert=6$	Greater than

Because $1\leq x\leq 9$ is the only interval in which the output at the test value is less than 4, we can conclude that the solution to $|x-5|\leq 4$ is $1\leq x\leq 9$, or $[1,9]$.

To use a graph, we can sketch the function $f(x)=|x-5|$. To help us see where the outputs are 4, the line $g(x)=4$ could also be sketched as in Figure 5.1.11.

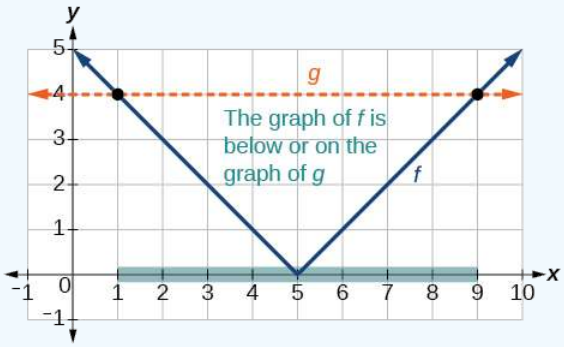


Figure 5.1.11: Graph to find the points satisfying an absolute value inequality.

We can see the following:

- The output values of the absolute value are equal to 4 at $x=1$ and $x=9$.
- The graph of f is below the graph of g on $1<x<9$. This means the output values of $f(x)$ are less than the output values of $g(x)$.
- The absolute value is less than or equal to 4 between these two points, when $1\leq x\leq 9$. In interval notation, this would be the interval $[1,9]$.

Analysis

For absolute value inequalities,

$$\begin{array}{ll} |x-A|<C, & |x-A|>C, \\ -C<x-A<C, & x-A<-C \text{ or } x-A>C. \end{array}$$

The $<$ or $>$ symbol may be replaced by $\leq$ or $\geq$.

So, for this example, we could use this alternative approach.

$$\begin{array}{ll} |x-5|\leq 4 & \\ -4\leq x-5\leq 4 & \text{Rewrite by removing the absolute value bars.} \\ -4+5\leq x-5+5\leq 4+5 & \text{Isolate the x.} \\ 1\leq x\leq 9 & \end{array}$$

Exercise 5.1.5

Solve $|x+2|\leq 6$.

Answer

$-8\leq x\leq 4$

How To ...

Given an absolute value function, solve for the set of inputs where the output is positive (or negative).

1. Set the function equal to zero, and solve for the boundary points of the solution set.
2. Use test points or a graph to determine where the function's output is positive or negative.

Example 5.1.7: Using a Graphical Approach to Solve Absolute Value Inequalities

Given the function $f(x) = -\frac{1}{2}|4x-5|+3$, determine the x-values for which the function values are negative.

Solution

We are trying to determine where $f(x) < 0$, which is when $-\frac{1}{2}|4x-5|+3 < 0$. We begin by isolating the absolute value.

$$\begin{aligned} -\frac{1}{2}|4x-5| &< -3 \quad \text{Multiply both sides by } -2, \text{ and reverse the inequality.} \\ |4x-5| &> 6 \end{aligned}$$

Next we solve for the equality $|4x-5| = 6$.

$$\begin{aligned} 4x-5 &= 6 \qquad 4x-5 = -6 \\ 4x-6 &= 6 \end{aligned}$$

or

$$\begin{aligned} 4x &= -1 \\ x &= \frac{11}{4} \qquad x = -\frac{1}{4} \end{aligned}$$

Now, we can examine the graph of f to observe where the output is negative. We will observe where the branches are below the x-axis. Notice that it is not even important exactly what the graph looks like, as long as we know that it crosses the horizontal axis at $x = -\frac{1}{4}$ and $x = \frac{11}{4}$ and that the graph has been reflected vertically. See Figure 5.1.12

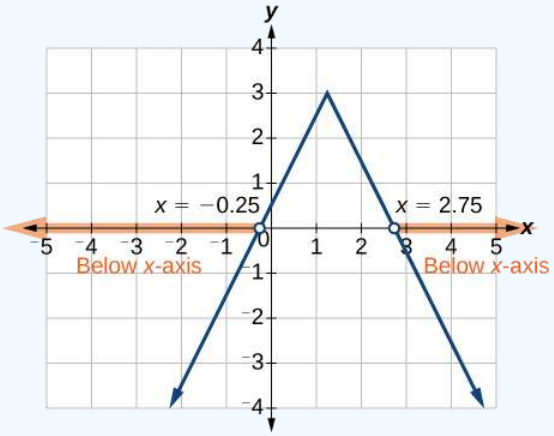


Figure 5.1.12: Graph of an absolute function with x-intercepts at -0.25 and 2.75.]

We observe that the graph of the function is below the x-axis left of $x = -\frac{1}{4}$ and right of $x = \frac{11}{4}$. This means the function values are negative to the left of the first horizontal intercept at $x = -\frac{1}{4}$, and negative to the right of the second intercept at $x = \frac{11}{4}$. This gives us the solution to the inequality.

$$x < -\frac{1}{4} \text{ or } x > 1\frac{1}{4}$$

In interval notation, this would be $(-\infty, -0.25) \cup (2.75, \infty)$.

Exercise 5.1.6

Solve $-2|k-4| \leq -6$.

Answer

$k \leq 1$ or $k \geq 7$; in interval notation, this would be $(-\infty, 1] \cup [7, \infty)$

Key Concepts

- The absolute value function is commonly used to measure distances between points.
- Applied problems, such as ranges of possible values, can also be solved using the absolute value function.
- The graph of the absolute value function resembles a letter V. It has a corner point at which the graph changes direction.
- In an absolute value equation, an unknown variable is the input of an absolute value function.
- If the absolute value of an expression is set equal to a positive number, expect two solutions for the unknown variable.
- An absolute value equation may have one solution, two solutions, or no solutions.
- An absolute value inequality is similar to an absolute value equation but takes the form | A |<B, | A |≤B, | A |>B, or | A |≥B.It can be solved by determining the boundaries of the solution set and then testing which segments are in the set.
- Absolute value inequalities can also be solved graphically.

Glossary

absolute value equation
an equation of the form $|A| = B$, with $B \geq 0$; it will have solutions when $A = B$ or $A = -B$

absolute value inequality
a relationship in the form $|A| < B$, $|A| \leq B$, $|A| > B$, or $|A| \geq B$

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5.3: Radical Functions

Evaluate a Radical Function

In this section we will extend our previous work with functions to include radicals. If a function is defined by a radical expression, we call it a **radical function**.

- The square root function is $f(x)=\sqrt{x}$.
- The cube root function is $f(x)=\sqrt[3]{x}$.

Definition 5.3.1: radical function

A **radical function** is a function that is defined by a radical expression.

To evaluate a radical function, we find the value of $f(x)$ for a given value of x just as we did in our previous work with functions.

Example 5.3.1

For the function $f(x)=\sqrt{2x-1}$, find

a. $f(5)$
b. $f(-2)$

Solution:

a.

$$f(x)=\sqrt{2x-1}$$

To evaluate $f(5)$, substitute 5 for x.

$$f(5)=\sqrt{2\cdot 5-1}$$

Simplify.

$$f(5)=\sqrt{9}$$

Take the square root.

$$f(5)=3$$

b.

$$f(x)=\sqrt{2x-1}$$

To evaluate $f(-2)$, substitute -2 for x.

$$f(-2)=\sqrt{2(-2)-1}$$

Simplify.

$$f(-2)=\sqrt{-5}$$

Since the square root of a negative number is not a real number, the function does not have a value at $x=-2$.

Exercise 5.3.1

For the function $f(x)=\sqrt{3x-2}$, find

a. $f(6)$
b. $f(0)$

Answer

a. $f(6)=4$
b. no value at $x=0$

Exercise 5.3.2

For the function $g(x)=\sqrt{5x+5}$, find

a. $g(4)$
b. $g(-3)$

Answer

a. $g(4)=5$
b. no value at $f(-3)$

We follow the same procedure to evaluate cube roots.

Example 5.3.2

For the function $g(x)=\sqrt[3]{x-6}$, find

a. $g(14)$
b. $g(-2)$

Solution:

a.

$$g(x)=\sqrt[3]{x-6}$$

To evaluate $g(14)$, substitute 14 for x.

$$g(14)=\sqrt[3]{14-6}$$

Simplify.

$$g(14)=\sqrt[3]{8}$$

Take the cube root.

$$g(14)=2$$

b.

$$g(x)=\sqrt[3]{x-6}$$

To evaluate $g(-2)$, substitute -2 for x.

$$g(-2)=\sqrt[3]{-2-6}$$

Simplify.

$$g(-2)=\sqrt[3]{-8}$$

Take the cube root.

$$g(-2)=-2$$

Exercise 5.3.3

For the function $g(x)=\sqrt[3]{3x-4}$, find

a. $g(4)$
b. $g(1)$

Answer

a. $g(4)=2$
b. $g(1)=-1$

Exercise 5.3.4

For the function $h(x) = \sqrt[3]{5x-2}$, find

a. $h(2)$
b. $h(-5)$

Answer

a. $h(2) = 2$
b. $h(-5) = -3$

The next example has fourth roots.

Example 5.3.3

For the function $f(x) = \sqrt[4]{5x-4}$, find

a. $f(4)$
b. $f(-12)$

Solution:

a.

$$f(x) = \sqrt[4]{5x-4}$$

To evaluate $f(4)$, substitute 4 for x.

$$f(4) = \sqrt[4]{5 \cdot 4 - 4}$$

Simplify.

$$f(4) = \sqrt[4]{16}$$

Take the fourth root.

$$f(4) = 2$$

b.

$$f(x) = \sqrt[4]{5x-4}$$

To evaluate $f(-12)$, substitute -12 for x.

$$f(-12) = \sqrt[4]{5(-12)-4}$$

Simplify.

$$f(-12) = \sqrt[4]{-64}$$

Since the fourth root of a negative number is not a real number, the function does not have a value at $x = -12$.

Exercise 5.3.5

For the function $f(x) = \sqrt[4]{3x+4}$, find

a. $f(4)$
b. $f(-1)$

Answer

a. $f(4) = 2$
b. $f(-1) = 1$

Exercise 5.3.6

For the function $g(x) = \sqrt[4]{5x+1}$, find

a. $g(16)$
b. $g(3)$

Answer

a. $g(16) = 3$
b. $g(3) = 2$

Find the Domain of a Radical Function

To find the **domain** and **range** of radical functions, we use our properties of radicals. For a radical with an even index, we said the radicand had to be greater than or equal to zero as even roots of negative numbers are not real numbers. For an odd index, the radicand can be any real number. We restate the properties here for reference.

Properties of $\sqrt[n]{a}$

When n is an **even** number and:

- $a \geq 0$, then $\sqrt[n]{a}$ is a real number.
- $a < 0$, then $\sqrt[n]{a}$ is not a real number.

When n is an **odd** number, $\sqrt[n]{a}$ is a real number for all values of a.

So, to find the domain of a radical function with even index, we set the radicand to be greater than or equal to zero. For an odd index radical, the radicand can be any real number.

Domain of a Radical Function

When the **index** of the radical is **even**, the radicand must be greater than or equal to zero.

When the **index** of the radical is **odd**, the radicand can be any real number.

Example 5.3.4

Find the domain of the function, $f(x) = \sqrt{3x-4}$. Write the domain in interval notation.

Solution:

Since the function, $f(x) = \sqrt{3x-4}$ has a radical with an index of 2, which is even, we know the radicand must be greater than or equal to 0. We set the radicand to be greater than or equal to 0 and then solve to find the domain.

Solve.

$$\begin{aligned} 3x-4 &\geq 0 \\ 3x &\geq 4 \\ x &\geq \frac{4}{3} \end{aligned}$$

The domain of $f(x) = \sqrt{3x-4}$ is all values $x \geq \frac{4}{3}$ and we write it in interval notation as $\left[\frac{4}{3}, \infty\right)$.

Exercise 5.3.7

Find the domain of the function, $f(x) = \sqrt{6x-5}$. Write the domain in interval notation.

Answer

$\left[\frac{5}{6}, \infty\right)$

Exercise 5.3.8

Find the domain of the function, $f(x)=\sqrt{4-5x}$. Write the domain in interval notation.

Answer

$\left(-\infty, \frac{4}{5}\right]$

Example 5.3.5

Find the domain of the function, $g(x)=\sqrt{\frac{6}{x-1}}$. Write the domain in interval notation.

Solution:

Solve the function, $g(x)=\sqrt{\frac{6}{x-1}}$ has a radical with an index of 2, which is even, we know the radicand must be greater than or equal to 0.

The radicand cannot be zero since the numerator is not zero.

For $\frac{6}{x-1}$ to be greater than zero, the denominator must be positive since the numerator is positive. We know a positive divided by a positive is positive.

We set $x-1>0$ and solve.

$$x-1>0$$

Solve.

$$x>1$$

Also, since the radicand is a fraction, we must realize that the denominator cannot be zero.

We solve $x-1=0$ to find the value that must be eliminated from the domain.

$$x-1=0$$

Solve.

$$x=1 \text{ so } x/neq1 \text{ in the domain.}$$

Putting this together we get the domain is $x>1$ and we write it as $(1, \infty)$.

Exercise 5.3.9

Find the domain of the function, $f(x)=\sqrt{\frac{4}{x+3}}$. Write the domain in interval notation.

Answer

$(-3, \infty)$

Exercise 5.3.10

Find the domain of the function, $h(x)=\sqrt{\frac{9}{x-5}}$. Write the domain in interval notation.

Answer

$(5, \infty)$

The next example involves a cube root and so will require different thinking.

Example 5.3.6

Find the domain of the function, $f(x)=\sqrt[3]{2x^2+3}$. Write the domain in interval notation.

Solution:

Since the function, $f(x)=\sqrt[3]{2x^2+3}$ has a radical with an index of 3, which is odd, we know the radicand can be any real number. This tells us the domain is any real number. In interval notation, we write $(-\infty,\infty)$.

The domain of $f(x)=\sqrt[3]{2x^2+3}$ is all real numbers and we write it in interval notation as $(-\infty,\infty)$.

Exercise 5.3.11

Find the domain of the function, $f(x)=\sqrt[3]{3x^2-1}$. Write the domain in interval notation.

Answer

$(-\infty,\infty)$

Exercise 5.3.12

Find the domain of the function, $g(x)=\sqrt[3]{5x-4}$. Write the domain in interval notation.

Answer

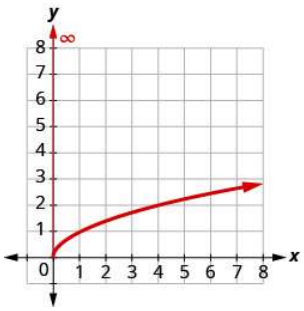
$(-\infty,\infty)$

Graph Radical Functions

Before we graph any radical function, we first find the **domain** of the function. For the function, $f(x)=\sqrt{x}$, the index is even, and so the radicand must be greater than or equal to 0.

This tells us the domain is $x\geq 0$ and we write this in interval notation as $[0,\infty)$.

Previously we used point plotting to graph the function, $f(x)=\sqrt{x}$. We chose x-values, substituted them in and then created a chart. Notice we chose points that are perfect squares in order to make taking the square root easier.



x	$f(x)=\sqrt{x}$	$(x, f(x))$
0	0	(0, 0)
1	1	(1, 1)
4	2	(4, 2)
9	3	(9, 3)

Figure 8.7.1

Once we see the graph, we can find the range of the function. The y-values of the function are greater than or equal to zero. The range then is $[0,\infty)$.

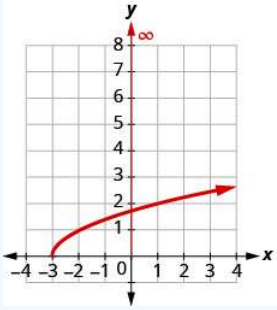
Example 5.3.7

For the function $f(x)=\sqrt{x+3}$,

a. find the domain
b. graph the function
c. use the graph to determine the range

Solution:

a. Since the radical has index 2, we know the radicand must be greater than or equal to zero. If $x+3 \geq 0$, then $x \geq -3$. This tells us the domain is all values $x \geq -3$ and written in interval notation as $[-3, \infty)$.
b. To graph the function, we choose points in the interval $[-3, \infty)$ that will also give us a radicand which will be easy to take the square root.



x	$f(x)=\sqrt{x+3}$	$(x, f(x))$
–3	0	(–3, 0)
–2	1	(–2, 1)
1	2	(1, 2)
6	3	(6, 3)

Figure 8.7.2

c. Looking at the graph, we see the y-values of the function are greater than or equal to zero. The range then is $[0, \infty)$.

Exercise 5.3.13

For the function $f(x)=\sqrt{x+2}$,

a. find the domain
b. graph the function
c. use the graph to determine the range

Answer

a. domain: $[-2, \infty)$
b.

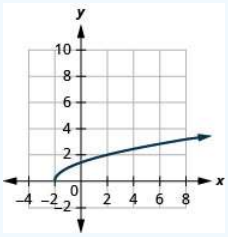


Figure 8.7.3

c. range: $[0, \infty)$

Exercise 5.3.14

For the function $f(x)=\sqrt{x-2}$,

a. find the domain
b. graph the function
c. use the graph to determine the range

Answer

a. domain: $[2, \infty)$

b.

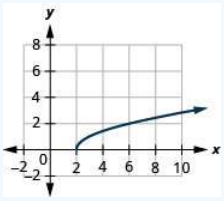


Figure 8.7.4

c. range: $[0, \infty)$

In our previous work graphing functions, we graphed $f(x) = x^3$ but we did not graph the function $f(x) = \sqrt[3]{x}$. We will do this now in the next example.

Example 5.3.8

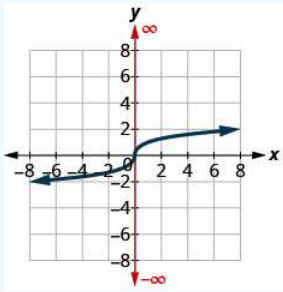
For the function, $f(x) = \sqrt[3]{x}$,

a. find the domain
b. graph the function
c. use the graph to determine the range

Solution:

a. Since the radical has index 3, we know the radicand can be any real number. This tells us the domain is all real numbers and written in interval notation as $(-\infty, \infty)$

b. To graph the function, we choose points in the interval $(-\infty, \infty)$ that will also give us a radicand which will be easy to take the cube root.



x	$f(x) = \sqrt[3]{x}$	$(x, f(x))$
–8	–2	(–8, –2)
–1	–1	(–1, –1)
0	0	(0, 0)
1	1	(1, 1)
8	2	(8, 2)

Figure 8.7.5

c. Looking at the graph, we see the y-values of the function are all real numbers. The range then is $(-\infty, \infty)$.

Exercise 5.3.15

For the function $f(x) = -\sqrt[3]{x}$,

a. find the domain
b. graph the function
c. use the graph to determine the range

Answer

a. domain: $(-\infty, \infty)$

b.

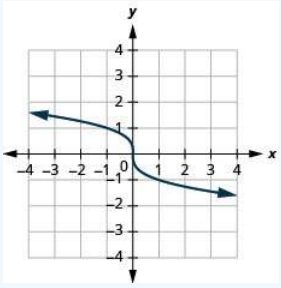


Figure 8.7.6

c. range: $(-\infty, \infty)$

? Exercise 5.3.16

For the function $f(x) = \sqrt[3]{x-2}$,

a. find the domain
b. graph the function
c. use the graph to determine the range

Answer

a. domain: $(-\infty, \infty)$
b.

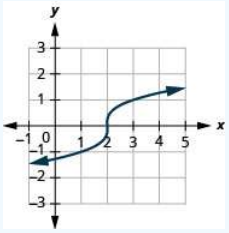


Figure 8.7.7

c. range: $(-\infty, \infty)$

Access these online resources for additional instruction and practice with radical functions.

- Domain of a Radical Function
- Domain of a Radical Function 2
- Finding Domain of a Radical Function

Key Concepts

- **Properties of $\sqrt[n]{a}$**
  - When n is an **even** number and:
    $a \geq 0$, then $\sqrt[n]{a}$ is a real number.
    $a < 0$, then $\sqrt[n]{a}$ is not a real number.
  - When n is an **odd** number, $\sqrt[n]{a}$ is a real number for all values of a.
- **Domain of a Radical Function**
  - When the **index** of the radical is **even**, the radicand must be greater than or equal to zero.
  - When the **index** of the radical is **odd**, the radicand can be any real number.

Glossary

radical function

A radical function is a function that is defined by a radical expression.

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CHAPTER OVERVIEW

6: Exponential and Logarithmic Functions

In this chapter, we will explore exponential functions, which can be used for, among other things, modeling growth patterns such as those found in bacteria. We will also investigate logarithmic functions, which are closely related to exponential functions. Both types of functions have numerous real-world applications when it comes to modeling and interpreting data.

6.1: Evaluate and Graph Exponential Functions
6.2: Evaluate and Graph Logarithmic Functions
6.3: Properties of the Logarithm
6.4: Solving Exponential and Logarithmic Equations
6.5: Applications of Exponential and Logarithmic Functions

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6.1: Evaluate and Graph Exponential Functions

Graph Exponential Functions

The functions we have studied so far do not give us a model for many naturally occurring phenomena. From the growth of populations and the spread of viruses to radioactive decay and compounding interest, the models are very different from what we have studied so far. These models involve exponential functions.

An **exponential function** is a function of the form $f(x) = a^x$ where $a > 0$ and $a \neq 1$.

Definition 6.1.1

An exponential function, where $a > 0$ and $a \neq 1$, is a function of the form

$$f(x) = a^x$$

Notice that in this function, the variable is the exponent. In our functions so far, the variables were the base.

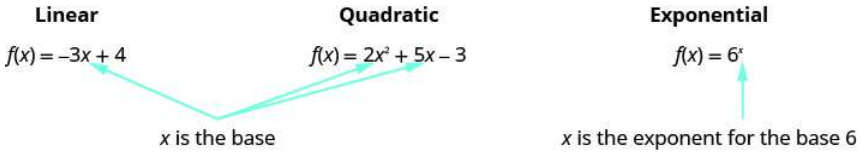


Figure 10.2.1

Our definition says $a \neq 1$. If we let $a = 1$, then $f(x) = a^x$ becomes $f(x) = 1^x$. Since $1^x = 1$ for all real numbers, $f(x) = 1$. This is the constant function.

Our definition also says $a > 0$. If we let a base be negative, say -4, then $f(x) = (-4)^x$ is not a real number when $x = \frac{1}{2}$.

$$f(x) = (-4)^x$$
$$f\left(\frac{1}{2}\right) = (-4)^{\frac{1}{2}}$$
$$f\left(\frac{1}{2}\right) = \sqrt{-4} \text{ not a real number}$$

In fact, $f(x) = (-4)^x$ would not be a real number any time x is a fraction with an even denominator. So our definition requires $a > 0$.

By graphing a few exponential functions, we will be able to see their unique properties.

Example 6.1.1

On the same coordinate system graph $f(x) = 2^x$ and $g(x) = 3^x$.

Solution:

We will use point plotting to graph the functions.

x	$f(x) = 2^x$	$(x, f(x))$	$g(x) = 3^x$	$(x, g(x))$
–2	$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$	$\left(-2, \frac{1}{4}\right)$	$3^{-2} = \frac{1}{3^2} = \frac{1}{9}$	$\left(-2, \frac{1}{9}\right)$
–1	$2^{-1} = \frac{1}{2^1} = \frac{1}{2}$	$\left(-1, \frac{1}{2}\right)$	$3^{-1} = \frac{1}{3^1} = \frac{1}{3}$	$\left(-1, \frac{1}{3}\right)$
0	$2^0 = 1$	(0, 1)	$3^0 = 1$	(0, 1)
1	$2^1 = 2$	(1, 2)	$3^1 = 3$	(1, 3)
2	$2^2 = 4$	(2, 4)	$3^2 = 9$	(2, 9)
3	$2^3 = 8$	(3, 8)	$3^3 = 27$	(3, 27)

Figure 10.2.2

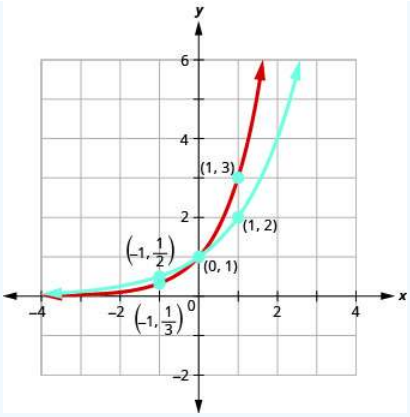


Figure 10.2.3

? Exercise 6.1.1

Graph: $f(x) = 4^x$.

Answer

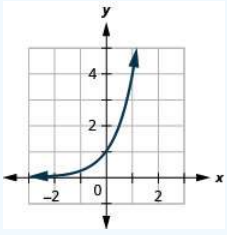


Figure 10.2.4

? Exercise 6.1.2

Graph: $g(x) = 5^x$

Answer

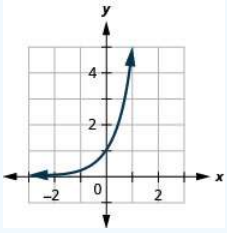


Figure 10.2.5

If we look at the graphs from the previous Example 10.2.1 and Exercises 10.2.1 and 10.2.2, we can identify some of the properties of exponential functions.

The graphs of $f(x) = 2^x$ and $g(x) = 3^x$, as well as the graphs of $f(x) = 4^x$ and $g(x) = 5^x$, all have the same basic shape. This is the shape we expect from an exponential function where $a > 1$.

We notice, that for each function, the graph contains the point $(0, 1)$. This make sense because $a^0 = 1$ for any a.

The graph of each function, $f(x) = a^x$ also contains the point $(1, a)$. The graph of $f(x) = 2^x$ contained $(1, 2)$ and the graph of $g(x) = 3^x$ contained $(1, 3)$. This makes sense as $a^1 = a$.

Notice too, the graph of each function $f(x) = a^x$ also contains the point $(-1, \frac{1}{a})$. The graph of $f(x) = 2^x$ contained $(-1, \frac{1}{2})$ and the graph of $g(x) = 3^x$ contained $(-1, \frac{1}{3})$.This makes sense as $a^{-1} = \frac{1}{a}$.

What is the domain for each function? From the graphs we can see that the domain is the set of all real numbers. There is no restriction on the domain. We write the domain in interval notation as $(-\infty, \infty)$.

Look at each graph. What is the range of the function? The graph never hits the x-axis. The range is all positive numbers. We write the range in interval notation as $(0, \infty)$.

Whenever a graph of a function approaches a line but never touches it, we call that line an **asymptote**. For the exponential functions we are looking at, the graph approaches the x-axis very closely but will never cross it, we call the line $y = 0$, the x-axis, a horizontal asymptote.

Properties of the Graph of $f(x) = a^x$ when $a > 1$

Table 10.2.1

Domain	$(-\infty, \infty)$
Range	$(0, \infty)$
x-intercept	None
y-intercept	$(0, 1)$
Contains	$(1, a), \left(-1, \frac{1}{a}\right)$
Asymptote	x-axis, the line $y = 0$

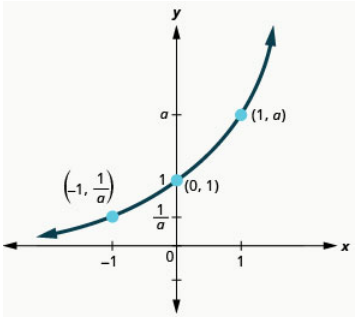


Figure 10.2.6

Our definition of an exponential function $f(x) = a^x$ says $a > 0$, but the examples and discussion so far has been about functions where $a > 1$. What happens when $0 < a < 1$ The next example will explore this possibility.

Example 6.1.2

On the same coordinate system, graph $f(x) = \left(\frac{1}{2}\right)^x$ and $g(x) = \left(\frac{1}{3}\right)^x$.

Solution:

We will use point plotting to graph the functions.

x	$f(x)=\left(\frac{1}{2}\right)^x$	$(x, f(x))$	$g(x)=\left(\frac{1}{3}\right)^x$	$(x, g(x))$
-2	$\left(\frac{1}{2}\right)^{-2}=2^2=4$	$(-2, 4)$	$\left(\frac{1}{3}\right)^{-2}=3^2=9$	$(-2, 9)$
-1	$\left(\frac{1}{2}\right)^{-1}=2^1=2$	$(-1, 2)$	$\left(\frac{1}{3}\right)^{-1}=3^1=3$	$(-1, 3)$
0	$\left(\frac{1}{2}\right)^0=1$	$(0, 1)$	$\left(\frac{1}{3}\right)^0=1$	$(0, 1)$
1	$\left(\frac{1}{2}\right)^1=\frac{1}{2}$	$\left(1, \frac{1}{2}\right)$	$\left(\frac{1}{3}\right)^1=\frac{1}{3}$	$\left(1, \frac{1}{3}\right)$
2	$\left(\frac{1}{2}\right)^2=\frac{1}{4}$	$\left(2, \frac{1}{4}\right)$	$\left(\frac{1}{3}\right)^2=\frac{1}{9}$	$\left(2, \frac{1}{9}\right)$
3	$\left(\frac{1}{2}\right)^3=\frac{1}{8}$	$\left(3, \frac{1}{8}\right)$	$\left(\frac{1}{3}\right)^3=\frac{1}{27}$	$\left(3, \frac{1}{27}\right)$

Figure 10.2.7

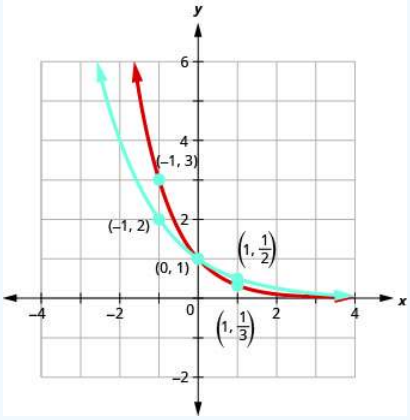


Figure 10.2.8

Exercise 6.1.3

Graph: $f(x)=\left(\frac{1}{4}\right)^x$.

Answer

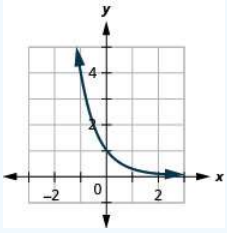


Figure 10.2.9

Exercise 6.1.4

Graph: $g(x)=\left(\frac{1}{5}\right)^x$.

Answer

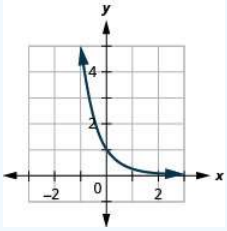


Figure 10.2.10

Now let's look at the graphs from the previous Example 10.2.2 and Exercises 10.2.3 and 10.2.4 so we can now identify some of the properties of exponential functions where $0 < a < 1$.

The graphs of $f(x) = \left(\frac{1}{2}\right)^x$ and $g(x) = \left(\frac{1}{3}\right)^x$ as well as the graphs of $f(x) = \left(\frac{1}{4}\right)^x$ and $g(x) = \left(\frac{1}{5}\right)^x$ all have the same basic shape. While this is the shape we expect from an exponential function where $0 < a < 1$, the graphs go down from left to right while the previous graphs, when $a > 1$, went from up from left to right.

We notice that for each function, the graph still contains the point $(0, 1)$. This make sense because $a^0 = 1$ for any a.

As before, the graph of each function, $f(x) = a^x$, also contains the point $(1, a)$. The graph of $f(x) = \left(\frac{1}{2}\right)^x$ contained $\left(1, \frac{1}{2}\right)$ and the graph of $g(x) = \left(\frac{1}{3}\right)^x$ contained $\left(1, \frac{1}{3}\right)$. This makes sense as $a^1 = a$.

Notice too that the graph of each function, $f(x) = a^x$, also contains the point $\left(-1, \frac{1}{a}\right)$. The graph of $f(x) = \left(\frac{1}{2}\right)^x$ contained $(-1, 2)$ and the graph of $g(x) = \left(\frac{1}{3}\right)^x$ contained $(-1, 3)$. This makes sense as $a^{-1} = \frac{1}{a}$.

What is the domain and range for each function? From the graphs we can see that the domain is the set of all real numbers and we write the domain in interval notation as $(-\infty, \infty)$. Again, the graph never hits the x-axis. The range is all positive numbers. We write the range in interval notation as $(0, \infty)$.

We will summarize these properties in the chart below. Which also include when $a > 1$.

Properties of the Graph of $f(x) = a^x$

Table 10.2.2

When $a > 1$		When $0 < a < 1$	
1\)">Domain	$(-\infty, \infty)$	Domain	$(-\infty, \infty)$
1\)">Range	$(0, \infty)$	Range	$(0, \infty)$
1\)">x-intercept	none	x-intercept	none
1\)">y-intercept	$(0, 1)$	y-intercept	$(0, 1)$
1\)">Contains	$(1, a)$, $\left(-1, \frac{1}{a}\right)$	Contains	$(1, a)$, $\left(-1, \frac{1}{a}\right)$
1\)">Asymptote	x-axis, the line $y = 0$	Asymptote	x-axis, the line $y = 0$
1\)">Basic Shape	increasing	Basic Shape	decreasing

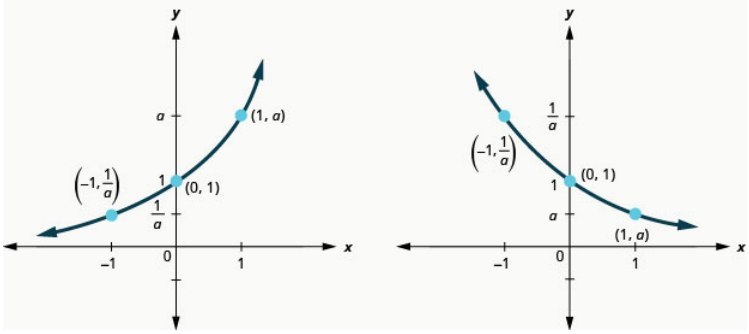


Figure 10.2.11

It is important for us to notice that both of these graphs are one-to-one, as they both pass the horizontal line test. This means the exponential function will have an inverse. We will look at this later.

When we graphed quadratic functions, we were able to graph using translation rather than just plotting points. Will that work in graphing exponential functions?

Example 6.1.3

On the same coordinate system graph $f(x) = 2^x$ and $g(x) = 2^{x+1}$.

Solution:

We will use point plotting to graph the functions.

x	$f(x) = 2^x$	$(x, f(x))$	$g(x) = 2^{x+1}$	$(x, g(x))$
-2	$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$	$\left(-2, \frac{1}{4}\right)$	$2^{-2+1} = \frac{1}{2^1} = \frac{1}{2}$	$\left(-2, \frac{1}{2}\right)$
-1	$2^{-1} = \frac{1}{2^1} = \frac{1}{2}$	$\left(-1, \frac{1}{2}\right)$	$2^{-1+1} = 2^0 = 1$	$(-1, 1)$
0	$2^0 = 1$	$(0, 1)$	$2^{0+1} = 2^1 = 2$	$(0, 2)$
1	$2^1 = 2$	$(1, 2)$	$2^{1+1} = 2^2 = 4$	$(1, 4)$
2	$2^2 = 4$	$(2, 4)$	$2^{2+1} = 2^3 = 8$	$(2, 8)$
3	$2^3 = 8$	$(3, 8)$	$2^{3+1} = 2^4 = 16$	$(3, 16)$

Figure 10.2.12

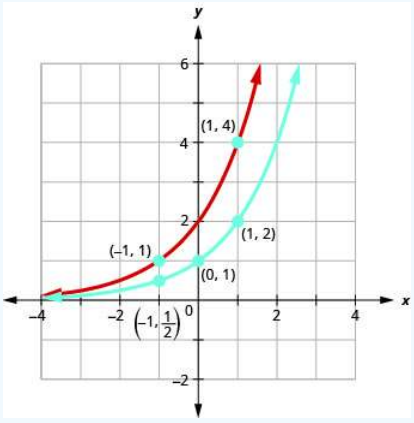


Figure 10.2.13

Exercise 6.1.5

On the same coordinate system, graph: $f(x) = 2^x$ and $g(x) = 2^{x-1}$.

Answer

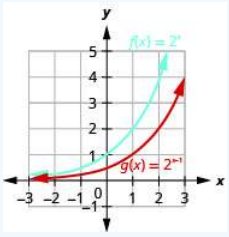


Figure 10.2.14

Exercise 6.1.6

On the same coordinate system, graph $f(x) = 3^x$ and $g(x) = 3^{x+1}$.

Answer

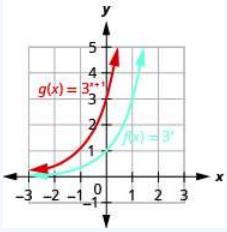


Figure 10.2.15

Looking at the graphs of the functions $f(x) = 2^x$ and $g(x) = 2^{x+1}$ in the last example, we see that adding one in the exponent caused a horizontal shift of one unit to the left. Recognizing this pattern allows us to graph other functions with the same pattern by translation.

Let's now consider another situation that might be graphed more easily by translation, once we recognize the pattern.

Example 6.1.4

On the same coordinate system graph $f(x) = 3^x$ and $g(x) = 3^x - 2$.

Solution:

We will use point plotting to graph the functions.

x	$f(x) = 3^x$	$(x, g(x))$	$g(x) = 3^x - 2$	$(x, g(x))$
–2	$3^{-2} = \frac{1}{9}$	$\left(-2, \frac{1}{9}\right)$	$3^{-2} - 2 = \frac{1}{9} - 2 = -\frac{17}{9}$	$\left(-2, -\frac{17}{9}\right)$
–1	$3^{-1} = \frac{1}{3}$	$\left(-1, \frac{1}{3}\right)$	$3^{-1} - 2 = \frac{1}{3} - 2 = -\frac{5}{3}$	$\left(-1, -\frac{5}{3}\right)$
0	$3^0 = 1$	(0, 1)	$3^0 - 2 = 1 - 2 = -1$	(0, –1)
1	$3^1 = 3$	(1, 3)	$3^1 - 2 = 3 - 2 = 1$	(1, 1)
2	$3^2 = 9$	(2, 9)	$3^2 - 2 = 9 - 2 = 7$	(2, 8)

Figure 10.2.16

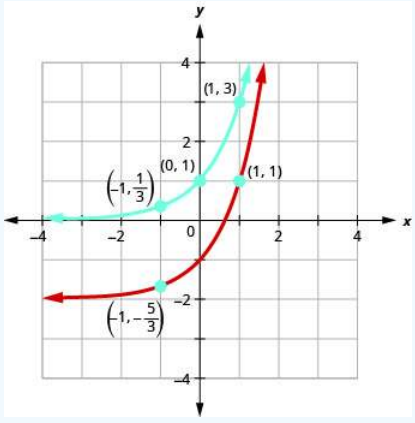


Figure 10.2.17

https://math.libretexts.org/@go/page/92117

Exercise 6.1.7

On the same coordinate system, graph $f(x) = 3^x$ and $g(x) = 3^x + 2$.

Answer

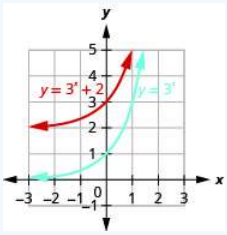


Figure 10.2.18

Exercise 6.1.8

On the same coordinate system, graph $f(x) = 4^x$ and $g(x) = 4^x - 2$.

Answer

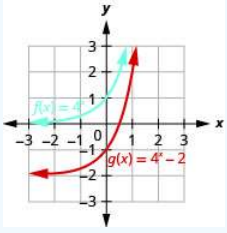


Figure 10.2.19

Looking at the graphs of the functions $f(x) = 3^x$ and $g(x) = 3^x - 2$ in the last example, we see that subtracting 2 caused a vertical shift of down two units. Notice that the horizontal asymptote also shifted down 2 units. Recognizing this pattern allows us to graph other functions with the same pattern by translation.

All of our exponential functions have had either an integer or a rational number as the base. We will now look at an exponential function with an irrational number as the base.

Before we can look at this exponential function, we need to define the irrational number, e. This number is used as a base in many applications in the sciences and business that are modeled by exponential functions. The number is defined as the value of $\left(1 + \frac{1}{n}\right)^n$ as n gets larger and larger. We say, as n approaches infinity, or increases without bound. The table shows the value of $\left(1 + \frac{1}{n}\right)^n$ for several values of n.

Table 10.2.3

n	$\left(1 + \frac{1}{n}\right)^n$
1	2
2	2.25
5	2.48832
10	2.59374246
100	2.704813829 . . .
1, 000	2.716923932 . . .
10, 000	2.718145927 . . .
100, 000	2.718268237 . . .

n	$\left(1+\frac{1}{n}\right)^n$
$1,000,000$	$2.718280469\ldots$
$1,000,000,000$	$2.718281827\ldots$

$$e \approx 2.718281827$$

The number e is like the number π in that we use a symbol to represent it because its decimal representation never stops or repeats. The irrational number e is called the **natural base**.

Definition 6.1.2

Natural Base e

The number e is defined as the value of $\left(1+\frac{1}{n}\right)^n$, as n increases without bound. We say, as n approaches infinity,

$$e \approx 2.718281827$$

The exponential function whose base is e, $f(x)=e^x$ is called the **natural exponential function**.

Definition 6.1.3

Natural Exponential Function

The natural exponential function is an exponential function whose base is e

$$f(x)=e^x$$

The domain is $(-\infty,\infty)$ and the range is $(0,\infty)$.

Let's graph the function $f(x)=e^x$ on the same coordinate system as $g(x)=2^x$ and $h(x)=3^x$.

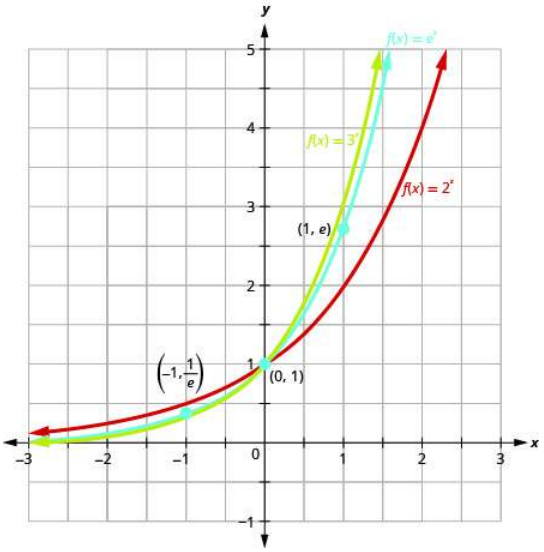


Figure 10.2.20

Notice that the graph of $f(x)=e^x$ is "between" the graphs of $g(x)=2^x$ and $h(x)=3^x$.Does this make sense as $2<e<3$?

Use Exponential Models in Applications

Exponential functions model many situations. If you own a bank account, you have experienced the use of an exponential function. There are two formulas that are used to determine the balance in the account when interest is earned. If a principal, P, is invested at an interest rate, r, for t years, the new balance, A, will depend on how often the interest is compounded. If the interest is

compounded n times a year we use the formula $A = P\left(1 + \frac{r}{n}\right)^{nt}$. If the interest is compounded continuously, we use the formula $A = Pe^{rt}$. These are the formulas for **compound interest**.

Definition 6.1.5

Compound Interest

For a principal, P, invested at an interest rate, r, for t years, the new balance, A, is:

$$\begin{aligned} A &= P\left(1 + \tfrac{r}{n}\right)^{nt} && \text{when compounded } n \text{ times a year.} \\ A &= Pe^{rt} && \text{when compounded continuously.} \end{aligned}$$

As you work with the Interest formulas, it is often helpful to identify the values of the variables first and then substitute them into the formula.

Example 6.1.7

A total of $\$10,000$ was invested in a college fund for a new grandchild. If the interest rate is 5%, how much will be in the account in 18 years by each method of compounding?

a. compound quarterly
b. compound monthly
c. compound continuously

Solution:

Identify the values of each variable in the formulas. Remember to express the percent as a decimal.

$$\begin{aligned} A &= ? \\ P &= \$10,000 \\ r &= 0.05 \\ t &= 18 \text{ years} \end{aligned}$$

a. For quarterly compounding, $n = 4$. There are 4 quarters in a year.

$$A = P\left(1 + \tfrac{r}{n}\right)^{nt}$$

Substitute the values in the formula.

$$A = 10,000\left(1 + \tfrac{0.05}{4}\right)^{4 \cdot 18}$$

Compute the amount. Be careful to consider the order of operations as you enter the expression into your calculator.

$$A = \$24,459.20$$

b. For monthly compounding, $n = 12$.There are 12 months in a year.

$$A = P\left(1 + \tfrac{r}{n}\right)^{nt}$$

Substitute the values in the formula.

$$A = 10,000\left(1 + \tfrac{0.05}{12}\right)^{12 \cdot 18}$$

Compute the amount.

$$A = \$24,550.08$$

c. For compounding continuously,

$$A = Pe^{rt}$$

Substitute the values in the formula.

$$A = 10,000e^{0.05 \cdot 18}$$

Compute the amount.

$$A = \$24,596.03$$

Exercise 6.1.13

Angela invested $\$15,000$ in a savings account. If the interest rate is 4%, how much will be in the account in 10 years by each method of compounding?

a. compound quarterly
b. compound monthly
c. compound continuously

Answer

a. $\$22,332.96$
b. $\$22,362.49$
c. $\$22,377.37$

Exercise 6.1.14

Allan invested $\$10,000$ in a mutual fund. If the interest rate is 5%, how much will be in the account in 15 years by each method of compounding?

a. compound quarterly
b. compound monthly
c. compound continuously

Answer

a. $\$21,071.81$
b. $\$21,137.04$
c. $\$21,170.00$

Other topics that are modeled by exponential functions involve growth and decay. Both also use the formula $A = Pe^{rt}$ we used for the growth of money. For growth and decay, generally we use A_0, as the original amount instead of calling it P, the principal. We see that **exponential growth** has a positive rate of growth and **exponential decay** has a negative rate of growth.

Definition 6.1.6

Exponential Growth and Decay

For an original amount, A_0, that grows or decays at a rate, r, for a certain time, t, the final amount, A, is:

$$A = A_0 e^{rt}$$

Exponential growth is typically seen in the growth of populations of humans or animals or bacteria. Our next example looks at the growth of a virus.

Example 6.1.8

Chris is a researcher at the Center for Disease Control and Prevention and he is trying to understand the behavior of a new and dangerous virus. He starts his experiment with 100 of the virus that grows at a rate of 25% per hour. He will check on the virus in 24 hours. How many viruses will he find?

Solution:

Identify the values of each variable in the formulas. Be sure to put the percent in decimal form. Be sure the units match--the rate is per hour and the time is in hours.

$$A = ?$$
$$A_0 = 100$$
$$r = 0.25/\text{ hour}$$
$$t = 24 \text{ hours}$$

Substitute the values in the formula: $A = A_0 e^{rt}$.

$$A = 100e^{0.25 \cdot 24}$$

Compute the amount.

$$A = 40,342.88$$

Round to the nearest whole virus.

$$A = 40,343$$

The researcher will find $40,343$ viruses.

? Exercise 6.1.15

Another researcher at the Center for Disease Control and Prevention, Lisa, is studying the growth of a bacteria. She starts his experiment with 50 of the bacteria that grows at a rate of 15% per hour. He will check on the bacteria every 8 hours. How many bacteria will he find in 8 hours?

Answer

She will find 166 bacteria.

? Exercise 6.1.16

Maria, a biologist is observing the growth pattern of a virus. She starts with 100 of the virus that grows at a rate of 10% per hour. She will check on the virus in 24 hours. How many viruses will she find?

Answer

She will find $1,102$ viruses.

Access these online resources for additional instruction and practice with evaluating and graphing exponential functions.

- Graphing Exponential Functions
- Solving Exponential Equations
- Applications of Exponential Functions
- Continuously Compound Interest
- Radioactive Decay and Exponential Growth

Key Concepts

- **Properties of the Graph of $f(x) = a^x$:**

Table 10.2.2

When $a > 1$		When $0 < a < 1$	
1\)">Domain	$(-\infty, \infty)$	Domain	$(-\infty, \infty)$
1\)">Range	$(0, \infty)$	Range	$(0, \infty)$
1\)">x-intercept	none	x-intercept	none
1\)">y-intercept	$(0, 1)$	y-intercept	$(0, 1)$
1\)">Contains	$(1, a), \left(-1, \frac{1}{a}\right)$	Contains	$(1, a), \left(-1, \frac{1}{a}\right)$

When $a > 1$		When $0 < a < 1$	
1\)">Asymptote	x-axis, the line $y = 0$	Asymptote	x-axis, the line $y = 0$
1\)">Basic Shape	increasing	Basic Shape	decreasing

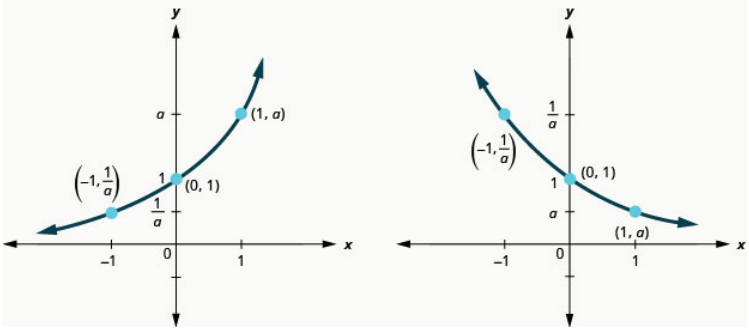


Figure 10.2.11

- **One-to-One Property of Exponential Equations:**
  For $a > 0$ and $a \neq 1$,
  $$A = A_0 e^{rt}$$
- **How to Solve an Exponential Equation**
  1. Write both sides of the equation with the same base, if possible.
  2. Write a new equation by setting the exponents equal.
  3. Solve the equation.
  4. Check the solution.
- **Compound Interest:** For a principal, P, invested at an interest rate, r, for t years, the new balance, A, is
  $A = P\left(1 + \frac{r}{n}\right)^{nt}$ when compounded n times a year.
  $A = Pe^{rt}$ when compounded continuously.
- **Exponential Growth and Decay:** For an original amount, A_0 that grows or decays at a rate, r, for a certain time t, the final amount, A, is $A = A_0 e^{rt}$.

Glossary

asymptote

A line which a graph of a function approaches closely but never touches.

exponential function

An exponential function, where $a > 0$ and $a \neq 1$, is a function of the form $f(x) = a^x$.

natural base

The number e is defined as the value of $(1 + \frac{1}{n})^n$, as n gets larger and larger. We say, as n increases without bound, $e \approx 2.718281827\ldots$

natural exponential function

The natural exponential function is an exponential function whose base is e: $f(x) = e^x$. The domain is $(-\infty, \infty)$ and the range is $(0, \infty)$.

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6.2: Evaluate and Graph Logarithmic Functions

Definition 6.2.1: Logarithmic Function

The function $f(x) = \log_a x$ is the **logarithmic function** with base a, where $a > 0, x > 0$, and $a \neq 1$.

$$y = \log_a x \text{ is equivalent to } x = a^y$$

Convert Between Exponential and Logarithmic Form

Since the equations $y = \log_a x$ and $x = a^y$ are equivalent, we can go back and forth between them. This will often be the method to solve some exponential and logarithmic equations. To help with converting back and forth let's take a close look at the equations. See *Figure 10.3.1*. Notice the positions of the exponent and base.

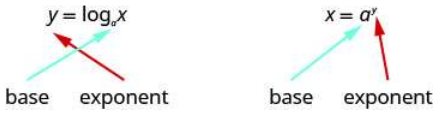


Figure 10.3.1

If we realize the logarithm is the exponent it makes the conversion easier. You may want to repeat, "base to the exponent give us the number."

Example 6.2.1

Convert to logarithmic form:

a. $2^3 = 8$
b. $5^{\frac{1}{2}} = \sqrt{5}$
c. $\left(\frac{1}{2}\right)^x = \frac{1}{16}$

Solution:

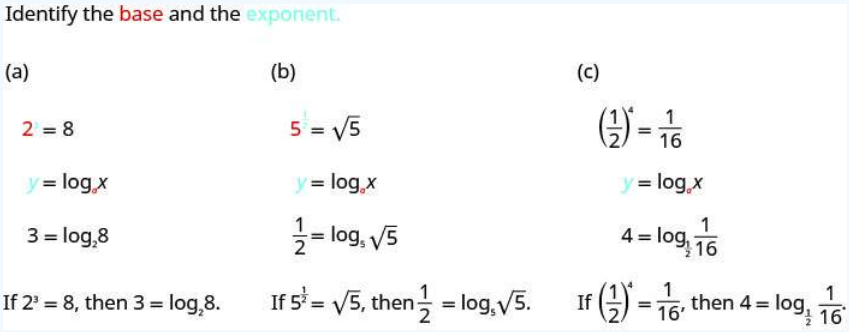


Figure 10.3.2

Exercise 6.2.1

Convert to logarithmic form:

a. $3^2 = 9$
b. $7^{\frac{1}{2}} = \sqrt{7}$
c. $\left(\frac{1}{3}\right)^x = \frac{1}{27}$

Answer

a. $\log_3 9 = 2$
b. $\log_7 \sqrt{7} = \frac{1}{2}$
c. $\log_{\frac{1}{3}} \frac{1}{27} = x$

Exercise 6.2.2

Convert to logarithmic form:

a. $4^3 = 64$
b. $4^{\frac{1}{3}} = \sqrt[3]{4}$
c. $\left(\frac{1}{2}\right)^x = \frac{1}{32}$

Answer

a. $\log_4 64 = 3$
b. $\log_4 \sqrt[3]{4} = \frac{1}{3}$
c. $\log_{\frac{1}{2}} \frac{1}{32} = x$

In the next example we do the reverse—convert logarithmic form to exponential form.

Example 6.2.2

Convert to exponential form:

a. $2 = \log_8 64$
b. $0 = \log_4 1$
c. $-3 = \log_{10} \frac{1}{1000}$

Solution:

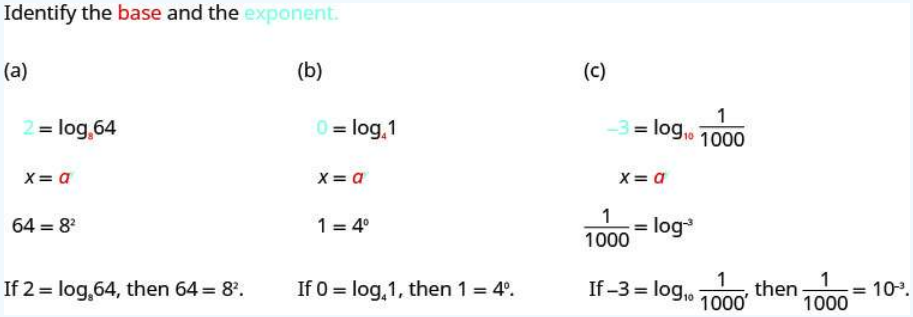


Figure 10.3.3

Exercise 6.2.3

Convert to exponential form:

a. $3 = \log_4 64$
b. $0 = \log_x 1$
c. $-2 = \log_{10} \frac{1}{100}$

Answer

a. $64 = 4^3$
b. $1 = x^0$
c. $\frac{1}{100} = 10^{-2}$

Exercise 6.2.4

Convert to exponential form:

a. $3 = \log_3 27$
b. $0 = \log_x 1$

c. $-1 = \log_{10} \frac{1}{10}$

Answer

a. $27 = 3^3$
b. $1 = x^0$
c. $\frac{1}{10} = 10^{-1}$

Evaluate Logarithmic Functions

We can solve and evaluate logarithmic equations by using the technique of converting the equation to its equivalent exponential equation.

Example 6.2.3

Find the value of x:

a. $\log_x 36 = 2$
b. $\log_4 x = 3$
c. $\log_{\frac{1}{2}} \frac{1}{8} = x$

Solution:

a.

$$\log_x 36 = 2$$

Convert to exponential form.

$$x^2 = 36$$

Solve the quadratic.

$$x = 6, \quad x = \cancel{-6}$$

The base of a logarithmic function must be positive, so we eliminate $x = -6$.

$$x = 6 \quad \text{Therefore, } \log_6 36 = 2$$

b.

$$\log_4 x = 3$$

Convert to exponential form.

$$4^3 = x$$

Simplify.

$$x = 64 \quad \text{Therefore , } \log_4 64 = 3$$

c.

$$\log_{\frac{1}{2}} \frac{1}{8} = x$$

Convert to exponential form.

$$\left(\frac{1}{2}\right)^x = \frac{1}{8}$$

Rewrite $\frac{1}{8}$ as $\left(\frac{1}{2}\right)^3$.

$$\left(\frac{1}{2}\right)^x = \left(\frac{1}{2}\right)^3$$

With the same base, the exponents must be equal.

$$x = 3 \quad \text{Therefore } \log_{\frac{1}{2}} \frac{1}{8} = 3$$

Exercise 6.2.5

Find the value of x:

a. $\log_x 64 = 2$
b. $\log_5 x = 3$
c. $\log_{\frac{1}{2}} \frac{1}{4} = x$

Answer

a. $x = 8$
b. $x = 125$
c. $x = 2$

Exercise 6.2.6

Find the value of x:

a. $\log_x 81 = 2$
b. $\log_3 x = 5$
c. $\log_{\frac{1}{3}} \frac{1}{27} = x$

Answer

a. $x = 9$
b. $x = 243$
c. $x = 3$

When see an expression such as $log_3 27$, we can find its exact value two ways. By inspection we realize it means "3 to what power will be 27"? Since $3^3 = 27$, we know $log_3 27 = 3$. An alternate way is to set the expression equal to x and then convert it into an exponential equation.

Example 6.2.4

Find the exact value of each logarithm without using a calculator:

a. $\log_5 25$
b. $\log_9 3$
c. $\log_2 \frac{1}{16}$

Solution:

a.

$$\log_5 25$$

5 to what power will be 25?

$$\log_5 25 = 2$$

Or

Set the expression equal to x.

$$\log_5 25 = x$$

Change to exponential form.

$$5^x = 25$$

Rewrite 25 as 5^2.

$$5^x = 5^2$$

With the same base the exponents must be equal.

$$x = 2 \quad \text{Therefore , } \log_5 25 = 2.$$

b.

$$\log_9 3$$

Set the expression equal to x.

$$\log_9 3 = x$$

Change to exponential form.

$$9^x = 3$$

Rewrite 9 as 3^2.

$$\left(3^2\right)^x = 3^1$$

Simplify the exponents.

$$3^{2x} = 3^1$$

With the same base the exponents must be equal.

$$2x = 1$$

Solve the equation.

$$x = \tfrac{1}{2} \quad \text{Therefore , } \log_9 3 = \tfrac{1}{2}.$$

c.

$$\log_2 \tfrac{1}{16}$$

Set the expression equal to x.

$$\log_2 \tfrac{1}{16} = x$$

Change to exponential form.

$$2^x = \tfrac{1}{16}$$

Rewrite 16 as 2^4.

$$2^x = \tfrac{1}{2^4}$$

$$2^x = 2^{-4}$$

With the same base the exponents must be equal.

$$x = -4 \quad \text{Therefore , } \log_2 \tfrac{1}{16} = -4.$$

Exercise 6.2.7

Find the exact value of each logarithm without using a calculator:

a. $\log_{12} 144$
b. $\log_4 2$
c. $\log_2 \frac{1}{32}$

Answer

a. 2
b. $\frac{1}{2}$
c. -5

Exercise 6.2.8

Find the exact value of each logarithm without using a calculator:

a. $\log_9 81$
b. $\log_8 2$
c. $\log_3 \frac{1}{9}$

Answer

a. 2
b. $\frac{1}{3}$
c. -2

Graph Logarithmic Functions

To graph a logarithmic function $y = log_a x$, it is easiest to convert the equation to its exponential form, $x = a^y$. Generally, when we look for ordered pairs for the graph of a function, we usually choose an x-value and then determine its corresponding y-value. In this case you may find it easier to choose y-values and then determine its corresponding x-value.

Example 6.2.5

Graph $y = \log_2 x$.

Solution:

To graph the function, we will first rewrite the logarithmic equation, $y = \log_2 x$, in exponential form, $2^y = x$.

We will use point plotting to graph the function. It will be easier to start with values of y and then get x.

Table 10.3.1

y	$2^y = x$	(x, y)
-2	$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$	$(\frac{1}{4}, 2)$
-1	$2^{-1} = \frac{1}{2^1} = \frac{1}{2}$	$(\frac{1}{2}, -1)$
0	$2^0 = 1$	$(1, 0)$
1	$2^1 = 2$	$(2, 1)$
2	$2^2 = 4$	$(4, 2)$
3	$2^3 = 8$	$(8, 3)$

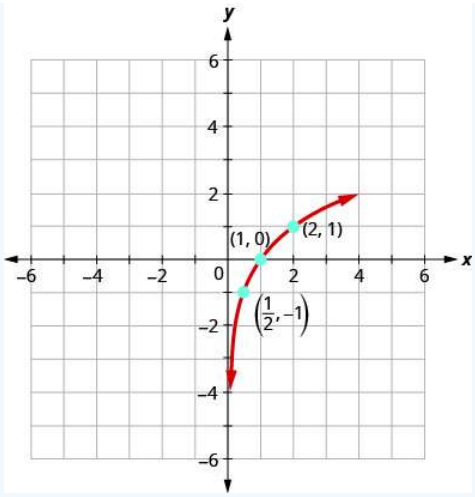


Figure 10.3.4

? Exercise 6.2.9

Graph: $y = \log_3 x$.

Answer

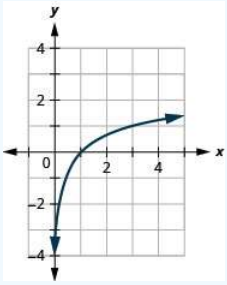


Figure 10.3.5

? Exercise 6.2.10

Graph: $y = \log_5 x$.

Answer

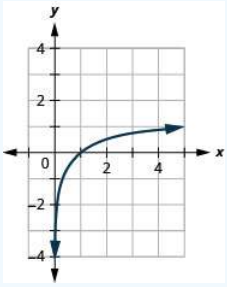


Figure 10.3.6

The graphs of $y = \log_2 x, y = \log_3 x$, and $y = \log_5 x$ are the shape we expect from a logarithmic function where $a > 1$.

We notice that for each function the graph contains the point $(1, 0)$. This make sense because $0 = log_a 1$ means $a^0 = 1$ which is true for any a.

The graph of each function, also contains the point $(a, 1)$. This makes sense as $1 = \log_a a$ means $a^1 = a$. which is true for any a.

Notice too, the graph of each function $y = \log_a x$ also contains the point $\left(\frac{1}{a}, -1\right)$. This makes sense as $-1 = \log_a \frac{1}{a}$ means $a^{-1} = \frac{1}{a}$, which is true for any a.

Look at each graph again. Now we will see that many characteristics of the logarithm function are simply 'mirror images' of the characteristics of the corresponding exponential function.

What is the domain of the function? The graph never hits the y-axis. The domain is all positive numbers. We write the domain in interval notation as $(0, \infty)$.

What is the range for each function? From the graphs we can see that the range is the set of all real numbers. There is no restriction on the range. We write the range in interval notation as $(-\infty, \infty)$.

When the graph approaches the y-axis so very closely but will never cross it, we call the line $x = 0$, the y-axis, a vertical asymptote.

Table 10.3.2: Properties of the Graph of $y = \log_a x$ When $a > 1$

Domain	$(0, \infty)$
Range	$(-\infty, \infty)$
x-intercept	$(1, 0)$
y-intercept	None
Contains	$(a, 1), \left(\frac{1}{a}, -1\right)$
Asymptote	y-axis

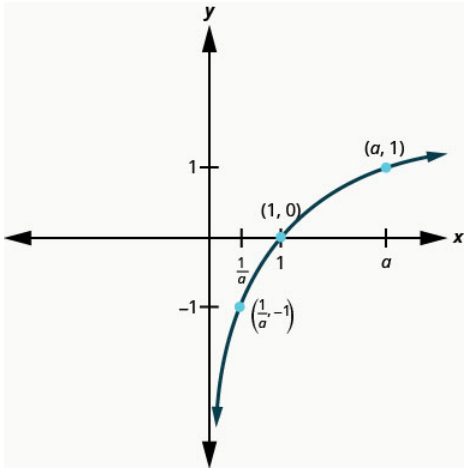


Figure 10.3.7

Our next example looks at the graph of $y = log_a x$ when $0 < a < 1$.

Example 6.2.6

Graph $y = \log_{\frac{1}{3}} x$.

Solution:

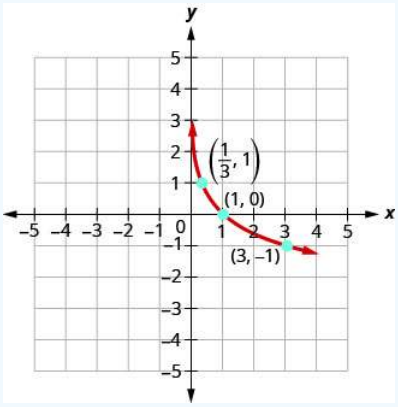
To graph the function, we will first rewrite the logarithmic equation, $y = \log_{\frac{1}{3}} x$, in exponential form, $\left(\frac{1}{3}\right)^y = x$.

We will use point plotting to graph the function. It will be easier to start with values of y and then get x.

Table 10.3.3

y	$\left(\frac{1}{3}\right)^y = x$	(x, y)
-2	$\left(\frac{1}{3}\right)^{-2} = 3^2 = 9$	$(9, -2)$

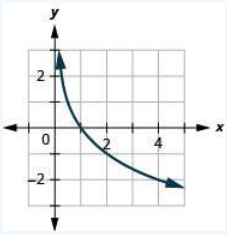
y	$\left(\frac{1}{3}\right)^y = x$	(x, y)
-1	$\left(\frac{1}{3}\right)^{-1} = 3^1 = 3$	$(3, -1)$
0	$\left(\frac{1}{3}\right)^0 = 1$	$(1, 0)$
1	$\left(\frac{1}{3}\right)^1 = \frac{1}{3}$	$\left(\frac{1}{3}, 1\right)$
2	$\left(\frac{1}{3}\right)^2 = \frac{1}{9}$	$\left(\frac{1}{9}, 2\right)$
3	$\left(\frac{1}{3}\right)^3 = \frac{1}{27}$	$\left(\frac{1}{27}, 3\right)$



Exercise 6.2.11

Graph: $y = \log_{\frac{1}{2}} x$.

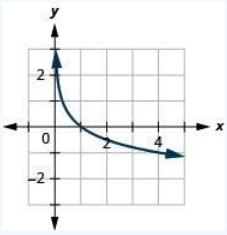
Answer



Exercise 6.2.12

Graph: $y = \log_{\frac{1}{4}} x$.

Answer



Now, let's look at the graphs $y = \log_{\frac{1}{2}} x, y = \log_{\frac{1}{3}} x$ and $y = \log_{\frac{1}{4}} x$, so we can identify some of the properties of logarithmic functions where $0 < a < 1$.

The graphs of all have the same basic shape. While this is the shape we expect from a logarithmic function where $0 < a < 1$.

We notice, that for each function again, the graph contains the points,$(1, 0), (a, 1), \left(\frac{1}{a}, -1\right)$. This make sense for the same reasons we argued above.

We notice the domain and range are also the same—the domain is $(0, \infty)$ and the range is $(-\infty, \infty)$. The y-axis is again the vertical asymptote.

We will summarize these properties in the chart below. Which also include when $a > 1$.

Table 10.3.4: Properties of the Graph of $y = \log_a x$

When $a > 1$		**When $0 < a < 1$**	
Domain	$(0, \infty)$	Domain	$(0, \infty)$
Range	$(-\infty, \infty)$	Range	$(-\infty, \infty)$
x-intercept	$(1, 0)$	x-intercept	$(1, 0)$
y-intercept	None	y-intercept	None
Contains	$(a, 1), \left(\frac{1}{a}, -1\right)$	Contains	$(a, 1), \left(\frac{1}{a}, -1\right)$
Asymptote	y -axis	Asymptote	y -axis
Basic shape	Increasing	Basic shape	Decreasing

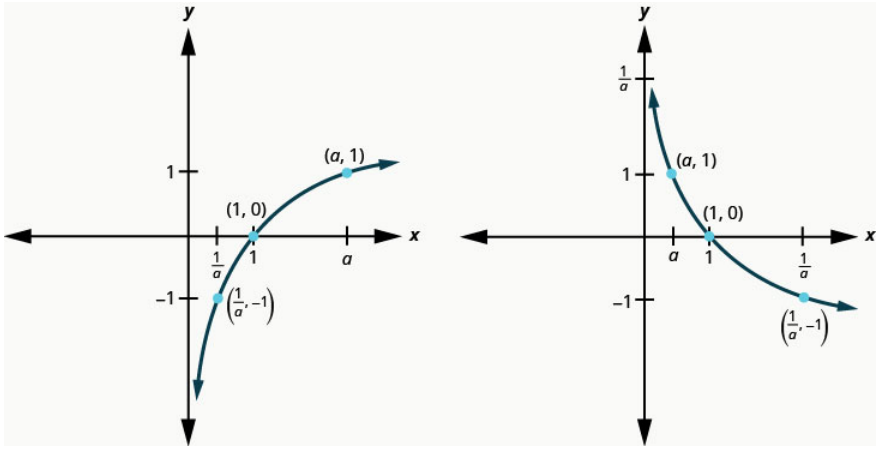


Figure 10.3.11

We talked earlier about how the logarithmic function $f^{-1}(x) = \log_a x$ is the inverse of the exponential function $f(x) = a^x$. The graphs in *Figure 10.3.12* show both the exponential (blue) and logarithmic (red) functions on the same graph for both $a > 1$ and $0 < a < 1$.

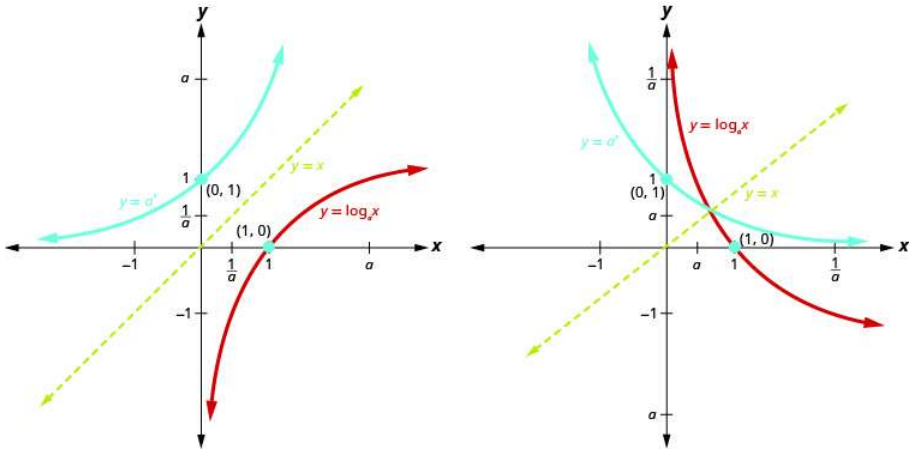


Figure 10.3.12

Notice how the graphs are reflections of each other through the line $y = x$. We know this is true of inverse functions. Keeping a visual in your mind of these graphs will help you remember the domain and range of each function. Notice the x-axis is the horizontal asymptote for the exponential functions and the y-axis is the vertical asymptote for the logarithmic functions.

Use Logarithmic Models in Applications

There are many applications that are modeled by logarithmic equations. We will first look at the logarithmic equation that gives the decibel (dB) level of sound. Decibels range from 0, which is barely audible to 160, which can rupture an eardrum. The 10^{-12} in the formula represents the intensity of sound that is barely audible.

Definition 6.2.4

Decibel Level of Sound

The loudness level, D, measured in decibels, of a sound of intensity, I, measured in watts per square inch is

$$D = 10 \log\left(\frac{I}{10^{-12}}\right)$$

Example 6.2.9

Extended exposure to noise that measures 85 dB can cause permanent damage to the inner ear which will result in hearing loss. What is the decibel level of music coming through ear phones with intensity 10^{-2} watts per square inch?

Solution:

	$D = 10 \log\left(\frac{I}{10^{-12}}\right)$
Substitute in the intensity level, I.	$D = 10 \log\left(\frac{10^{-2}}{10^{-12}}\right)$
Simplify.	$D = 10 \log(10^{10})$
Since $\log 10^{10} = 10$.	$D = 10 \cdot 10$
Multiply.	$D = 100$
	The decibel level of music coming through earphones is 100 dB.

Exercise 6.2.17

What is the decibel level of one of the new quiet dishwashers with intensity 10^{-7} watts per square inch?

Answer

The quiet dishwashers have a decibel level of 50 dB.

Exercise 6.2.18

What is the decibel level heavy city traffic with intensity 10^{-3} watts per square inch?

Answer

The decibel level of heavy traffic is 90 dB.

The magnitude R of an earthquake is measured by a logarithmic scale called the Richter scale. The model is $R = \log I$, where I is the intensity of the shock wave. This model provides a way to measure **earthquake intensity**.

Definition 6.2.5: Earthquake Intensity

The magnitude R of an earthquake is measured by$R = \log I$, where I is the intensity of its shock wave.

Example 6.2.10

In 1906, San Francisco experienced an intense earthquake with a magnitude of 7.8 on the Richter scale. Over 80% of the city was destroyed by the resulting fires. In 2014, Los Angeles experienced a moderate earthquake that measured 5.1 on the Richter scale and caused $108 million dollars of damage. Compare the intensities of the two earthquakes.

Solution:

To compare the intensities, we first need to convert the magnitudes to intensities using the log formula. Then we will set up a ratio to compare the intensities.

Convert the magnitudes to intensities.

$$R = \log I$$

1906 earthquake

$$7.8 = \log I$$

Convert to exponential form.

$$I = 10^{7.8}$$

2014 earthquake

$$5.1 = \log I$$

Convert to exponential form.

$$I = 10^{5.1}$$

Form a ratio of the intensities.

$$\frac{\text{Intensity for 1906}}{\text{Intensity for 2014}}$$

Substitute in the values.

$$\frac{10^{7.8}}{10^{5.1}}$$

Divide by subtracting the exponents.

$$10^{2.7}$$

Evaluate.

$$501$$

The intensity of the 1906 earthquake was about 501 times the intensity of the 2014 earthquake.

Exercise 6.2.19

In 1906, San Francisco experienced an intense earthquake with a magnitude of 7.8 on the Richter scale. In 1989, the Loma Prieta earthquake also affected the San Francisco area, and measured 6.9 on the Richter scale. Compare the intensities of the two earthquakes.

Answer

The intensity of the 1906 earthquake was about 8 times the intensity of the 1989 earthquake.

Exercise 6.2.20

In 2014, Chile experienced an intense earthquake with a magnitude of 8.2 on the Richter scale. In 2014, Los Angeles also experienced an earthquake which measured 5.1 on the Richter scale. Compare the intensities of the two earthquakes.

Answer

The intensity of the earthquake in Chile was about $1,259$ times the intensity of the earthquake in Los Angeles.

Access these online resources for additional instruction and practice with evaluating and graphing logarithmic functions.

- Re-writing logarithmic equations in exponential form
- Simplifying Logarithmic Expressions
- Graphing logarithmic functions
- Using logarithms to calculate decibel levels

Key Concepts

- **Properties of the Graph of $y = \log_a x$:**

When $a > 1$		**When $0 < a < 1$**	
Domain	$(0, \infty)$	Domain	$(0, \infty)$
Range	$(-\infty, \infty)$	Range	$(-\infty, \infty)$
x-intercept	$(1, 0)$	x-intercept	$(1, 0)$
y-intercept	None	y-intercept	None
Contains	$(a, 1), \left(\frac{1}{a}, -1\right)$	Contains	$(a, 1), \left(\frac{1}{a}, -1\right)$
Asymptote	y -axis	Asymptote	y -axis
Basic shape	Increasing	Basic shape	Decreasing

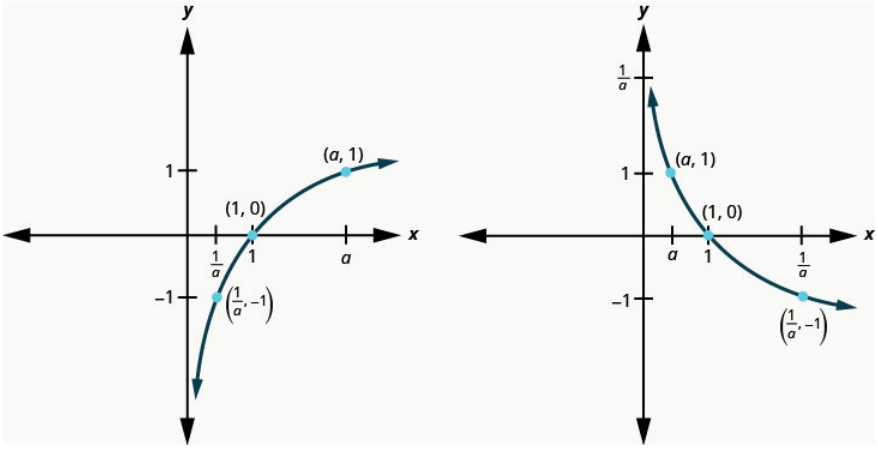


Figure 10.3.11

- **Decibel Level of Sound:** The loudness level, D, measured in decibels, of a sound of intensity, I, measured in watts per square inch is $D = 10 \log\left(\frac{I}{10^{-12}}\right)$.
- **Earthquake Intensity:** The magnitude R of an earthquake is measured by $R = \log I$, where I is the intensity of its shock wave.

Glossary

common logarithmic function

The function $f(x) = \log x$ is the common logarithmic function with base 10, where $x > 0$.

$$y = \log x \text{ is equivalent to } x = 10^y$$

logarithmic function

The function $f(x) = \log_a x$ is the logarithmic function with base a, where $a > 0, x > 0$, and $a \neq 1$.

$$y = \log_a x \text{ is equivalent to } x = a^y$$

natural logarithmic function

The function $f(x) = \ln x$ is the natural logarithmic function with base e, where $x > 0$.

$$y = \ln x \text{ is equivalent to } x = e^y$$

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6.3: Properties of the Logarithm

Logarithms and Their Inverse Properties

Recall the definition of the base-b logarithm: given $b > 0$ where $b \neq 1$,

$$y = \log_b x \text{ if and only if } x = b^y$$

Use this definition to convert logarithms to exponential form. Doing this, we can derive a few properties:

$$\log_b 1 = 0 \text{ because } b^0 = 1$$
$$\log_b b = 1 \text{ because } b^1 = b$$
$$\log_b\left(\frac{1}{b}\right) = -1 \text{ because } b^{-1} = \frac{1}{b}$$

Example 6.3.1

Evaluate:

a. $\log 1$
b. $\ln e$
c. $\log_5\left(\frac{1}{5}\right)$

Solution

a. When the base is not written, it is assumed to be 10. This is the common logarithm,

$$\log 1 = \log_{10} 1 = 0$$

b. The natural logarithm, by definition, has base e,

$$\ln e = \log_e e = 1$$

c. Because $5^{-1} = \frac{1}{5}$ we have,

$$\log_5\left(\frac{1}{5}\right) = -1$$

Furthermore, consider fractional bases of the form $1/b$ where $b > 1$.

$$\log_{1/b} b = -1 \quad \text{because } \left(\tfrac{1}{b}\right)^{-1} = \frac{1^{-1}}{b^{-1}} = \frac{b}{1} = b$$

Example 6.3.2

Evaluate:

a. $\log_{1/4} 4$
b. $\log_{2/3}\left(\frac{3}{2}\right)$

Solution

a. $\log_{1/4} 4 = -1$ because $\left(\frac{1}{4}\right)^{-1} = 4$
b. $\log_{2/3}\left(\frac{3}{2}\right) = -1$ because $\left(\frac{2}{3}\right)^{-1} = \frac{3}{2}$

Given an exponential function defined by $f(x) = b^x$, where $b > 0$ and $b \neq 1$, its inverse is the base-b logarithm, $f^{-1}(x) = log_b x$. And because $f\left(f^{-1}(x)\right) = x$ and $f^{-1}(f(x)) = x$, we have the following **inverse properties of the logarithm**[11]:

$$f^{-1}(f(x)) = \log_b b^x = x \text{ and}$$
$$f\left(f^{-1}(x)\right) = b^{\log_b x} = x, x > 0$$

Since $f^{-1}(x) = \log_b x$ has a domain consisting of positive values $(0, \infty)$, the property $b^{\log_b x} = x$ is restricted to values where $x > 0$.

Example 6.3.3

Evaluate

a. $\log_5 625$
b. $5^{\log_5 3}$
c. $e^{\ln 5}$

Solution

Apply the inverse properties of the logarithm.

a. $\log_5 625 = \log_5 5^4 = 4$
b. $5^{\log_5 3} = 3$
c. $e^{\ln 5} = 5$

In summary, when $b > 0$ and $b \neq 1$, we have the following properties:

Table 6.3.1

$\log_b 1 = 0$	$\log_b b = 1$
$\log_{1/b} b = -1$	$\log_b \left(\frac{1}{b}\right) = -1$
$\log_b b^x = x$	$b^{\log_b x} = x, x > 0$

Exercise 6.3.1

Evaluate: $\log 0.00001$

Answer

-5

www.youtube.com/v/YFsZxKiMCTg

Product, Quotient, and Power Properties of Logarithms

In this section, three very important properties of the logarithm are developed. These properties will allow us to expand our ability to solve many more equations. We begin by assigning u and v to the following logarithms and then write them in exponential form:

$$\log_b x = u \Longrightarrow b^u = x$$
$$\log_b y = v \Longrightarrow b^v = y$$

Substitute $x = b^u$ and $y = b^v$ into the logarithm of a product $log_b(xy)$ and the logarithm of a quotient $\log_b \left(\frac{x}{y}\right)$.Then simplify using the rules of exponents and the inverse properties of the logarithm.

Table 6.3.2

Logarithm of a Product	Logarithm of a Quotient
$\begin{aligned}\log_b(xy) &= \log_b(b^u b^v) \\ &= \log_b b^{u+v} \\ &= u+v \\ &= \log_b x + \log_b y\end{aligned}$	$\begin{aligned}\log_b\left(\frac{x}{y}\right) &= \log_b\left(\frac{b^u}{b^v}\right) \\ &= \log_b b^{u-v} \\ &= u-v \\ &= \log_b x - \log_b y\end{aligned}$

This gives us two essential properties: the **product property of logarithms**[12],

$$\log_b(xy) = \log_b x + \log_b y$$

and the **quotient property of logarithms**[13],

$$\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$$

In words, the logarithm of a product is equal to the sum of the logarithm of the factors. Similarly, the logarithm of a quotient is equal to the difference of the logarithm of the numerator and the logarithm of the denominator.

Example 6.3.4

Write as a sum $\log_2(8x)$.

Solution

Apply the product property of logarithms and then simplify.

$$\begin{aligned}\log_2(8x) &= \log_2 8 + \log_2 x \\ &= \log_2 2^3 + \log_2 x \\ &= 3 + \log_2 x\end{aligned}$$

Answer

$$3 + \log_2 x$$

Example 6.3.5

Write as a difference $\log\left(\frac{x}{10}\right)$.

Solution

Apply the quotient property of logarithms and then simplify.

$$\begin{aligned}\log\left(\frac{x}{10}\right) &= \log x - \log 10 \\ &= \log x - 1\end{aligned}$$

Answer

$$\log x - 1$$

Next we begin with $log_b x = u$ and rewrite it in exponential form. After raising both sides to the nth power, convert back to logarithmic form, and then back substitute.

$$\begin{aligned}\log_b x = u &\Longrightarrow b^u = x \\ (b^u)^n &= (x)^n \\ \log_b x^n = nu &\Longleftarrow b^{nu} = x^n \\ \log_b x^n &= n\log_b x\end{aligned}$$

This leads us to the **power property of logarithms**[14],

$$\log_b x^n = n\log_b x$$

In words, the logarithm of a quantity raised to a power is equal to that power times the logarithm of the quantity.

Example 6.3.6

Write as a product:

a. $\log_2 x^4$
b. $\log_5(\sqrt{x})$

Solution

a. Apply the power property of logarithms.

$$\log_2 x^4 = 4\log_2 x \tag{6.3.1}$$

b. Recall that a square root can be expressed using rational exponents, $\sqrt{x} = x^{1/2}$. Make this replacement and then apply the power property of logarithms.

$$\begin{aligned}\log_5(\sqrt{x}) &= \log_5 x^{1/2} \\ &= \frac{1}{2}\log_5 x\end{aligned}$$

In summary,

Table 6.3.3

Product Property of Logarithms	$\log_b(xy) = \log_b x + \log_b y$
Quotient Property of Logarithms	$\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$
Power Property of Logarithms	$\log_b x^n = n\log_b x$

We can use these properties to expand logarithms involving products, quotients, and powers using sums, differences and coefficients. A logarithmic expression is completely expanded when the properties of the logarithm can no further be applied.

Caution

It is important to point out the following:

$$\log(xy) \neq \log x \cdot \log y \tag{6.3.2}$$

and

$$\log\left(\frac{x}{y}\right) \neq \frac{\log x}{\log y} \tag{6.3.3}$$

Example 6.3.7

Expand completely: $\ln\left(2x^3\right)$.

Solution

Recall that the natural logarithm is a logarithm base e, $\ln x = \log_e x$. Therefore, all of the properties of the logarithm apply.

$$\begin{aligned}\ln\left(2x^3\right) &= \ln 2 + \ln x^3 \quad \textit{Product rule for logarithms} \\ &= \ln 2 + 3\ln x \quad \textit{Power rule for logarithms}\end{aligned}$$

Answer

$$\ln 2 + 3\ln x$$

Example 6.3.8:

Expand completely: $\log\sqrt[3]{10xy^2}$.

Solution

Begin by rewriting the cube root using the rational exponent $\frac{1}{3}$ and then apply the properties of the logarithm.

$$\begin{aligned}\log\sqrt[3]{10xy^2} &= \log\left(10xy^2\right)^{1/3} \\ &= \frac{1}{3}\log\left(10xy^2\right) \\ &= \frac{1}{3}\left(\log 10 + \log x + \log y^2\right) \\ &= \frac{1}{3}\left(1 + \log x + 2\log y\right) \\ &= \frac{1}{3} + \frac{1}{3}\log x + \frac{2}{3}\log y\end{aligned}$$

Answer

$$\frac{1}{3}+\frac{1}{3}\log x+\frac{2}{3}\log y$$

Example 6.3.9:

Expand completely: $\log_2\left(\frac{(x+1)^2}{5y}\right)$.

Solution

When applying the product property to the denominator, take care to distribute the negative obtained from applying the quotient property.

$$\begin{aligned}\log_2\left(\frac{(x+1)^2}{5y}\right) &= \log_2(x+1)^2-\log_2(5y)\\ &= \log_2(x+1)^2-(\log_2 5+\log_2 y) \quad \textit{Distribute.}\\ &= \log_2(x+1)^2-\log_2 5-\log_2 y\\ &= 2\log_2(x+1)-\log_2 5-\log_2 y\end{aligned}$$

Answer

$$2\log_2(x+1)-\log_2 5-\log_2 y$$

Caution

There is no rule that allows us to expand the logarithm of a sum or difference. In other words,

$$\log(x\pm y)\neq \log x\pm\log y$$

Exercise 6.3.2

Expand completely: $\ln\left(\frac{5y^4}{\sqrt{x}}\right)$

Answer

$\ln 5+4\ln y-\frac{1}{2}\ln x$

www.youtube.com/v/h_Zu-SkATl0

Example 6.3.10:

Given that $\log_2 x=a$, $\log_2 y=b$, and that $\log_2 z=c$, write the following in terms of a, b, and c:

a. $\log_2\left(8x^2y\right)$
b. $\log_2\left(\frac{2x^4}{\sqrt{z}}\right)$

Solution

a. Begin by expanding using sums and coefficients and then replace a and b with the appropriate logarithm.

$$\begin{aligned}\log_2\left(8x^2y\right) &= \log_2 8+\log_2 x^2+\log_2 y\\ &= \log_2 8+2\log_2 x+\log_2 y\\ &= 3+2a+b\end{aligned}$$

b. Expand and then replace a, b, and c where appropriate.

$$\begin{aligned}\log_2\left(\frac{2x^4}{\sqrt{z}}\right) &= \log_2\left(2x^4\right) - \log_2 z^{1/2} \\ &= \log_2 2 + \log_2 x^4 - \log_2 z^{1/2} \\ &= \log_2 2 + 4\log_2 x - \frac{1}{2}\log_2 z \\ &= 1 + 4a - \frac{1}{2}b\end{aligned}$$

Next we will condense logarithmic expressions. As we will see, it is important to be able to combine an expression involving logarithms into a single logarithm with coefficient 1. This will be one of the first steps when solving logarithmic equations.

Example 6.3.11:

Write as a single logarithm with coefficient 1: $3\log_3 x - \log_3 y + 2\log_3 5$.

Solution

Begin by rewriting all of the logarithmic terms with coefficient 1. Use the power rule to do this. Then use the product and quotient rules to simplify further.

$$\begin{aligned}3\log_3 x - \log_3 y + 2\log_3 5 &= \left\{\log_3 x^3 - \log_3 y\right\} + \log_3 5^2 \quad \textit{quotient property} \\ &= \left\{\log_3\left(\frac{x^3}{y}\right) + \log_3 25\right\} \quad \textit{product property} \\ &= \log_3\left(\frac{x^3}{y}\cdot 25\right) \\ &= \log_3\left(\frac{25x^3}{y}\right)\end{aligned}$$

Answer

$$\log_3\left(\frac{25x^3}{y}\right)$$

Example 6.3.12:

Write as a single logarithm with coefficient 1: $\frac{1}{2}\ln x - 3\ln y - \ln z$.

Solution

Begin by writing the coefficients of the logarithms as powers of their argument, after which we will apply the quotient rule twice working from left to right.

$$\begin{aligned}\frac{1}{2}\ln x - 3\ln y - \ln z &= \ln x^{1/2} - \ln y^3 - \ln z \\ &= \ln\left(\frac{x^{1/2}}{y^3}\right) - \ln z \\ &= \ln\left(\frac{x^{1/2}}{y^3} \div z\right) \\ &= \ln\left(\frac{x^{1/2}}{y^3}\cdot\frac{1}{z}\right) \\ &= \ln\left(\frac{x^{1/2}}{y^3 z}\right) \quad \text{or} \quad = \ln\left(\frac{\sqrt{x}}{y^3 z}\right)\end{aligned}$$

Answer

$$\ln\left(\frac{\sqrt{x}}{y^3 z}\right)$$

? Exercise 6.3.3

Write as a single logarithm with coefficient 1: $3\log(x+y)-6\log z+2\log 5$

Answer

$\log\left(\frac{25(x+y)^3}{z^6}\right)$

www.youtube.com/v/o1AVbmS9AU8

Key Takeaways

- Given any base $b>0$ and $b\neq 1$, we can say that $log_b 1=0$, $log_b b=1$, $log_{1/b} b=-1$ and that $log_b(\frac{1}{b})=-1$.
- The inverse properties of the logarithm are $log_b b^x=x$ and $b^{log_b x}=x$ where $x>0$.
- The product property of the logarithm allows us to write a product as a sum: $\log_b(xy)=\log_b x+\log_b y$.
- The quotient property of the logarithm allows us to write a quotient as a difference: $\log_b\left(\frac{x}{y}\right)=\log_b x-\log_b y$.
- The power property of the logarithm allows us to write exponents as coefficients: $\log_b x^n=n\log_b x$.
- Since the natural logarithm is a base-e logarithm, $\ln x=\log_e x$, all of the properties of the logarithm apply to it.
- We can use the properties of the logarithm to expand logarithmic expressions using sums, differences, and coefficients. A logarithmic expression is completely expanded when the properties of the logarithm can no further be applied.
- We can use the properties of the logarithm to combine expressions involving logarithms into a single logarithm with coefficient 1. This is an essential skill to be learned in this chapter.

Footnotes

[11]Given $b>0$ we have $\log_b b^x=x$ and $b^{\log_b x}=x$ when $x>0$.

[12]$\log_b(xy)=\log_b x+\log_b y$; the logarithm of a product is equal to the sum of the logarithm of the factors.

[13]$\log_b\left(\frac{x}{y}\right)=\log_b x-\log_b y$; the logarithm of a quotient is equal to the difference of the logarithm of the numerator and the logarithm of the denominator.

[14]$\log_b x^n=n\log_b x$; the logarithm of a quantity raised to a power is equal to that power times the logarithm of the quantity.]

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6.4: Solving Exponential and Logarithmic Equations

Solving Exponential Equations

An **exponential equation**[15] is an equation that includes a variable as one of its exponents. In this section we describe two methods for solving exponential equations. First, recall that exponential functions defined by $f(x) = b^x$ where $b > 0$ and $b \neq 1$, are one-to-one; each value in the range corresponds to exactly one element in the domain. Therefore, $f(x) = f(y)$ implies $x = y$. The converse is true because f is a function. This leads to the very important **one-to-one property of exponential functions**[16]:

$$b^x = b^y \quad \text{if and only if} \quad x = y$$

Use this property to solve special exponential equations where each side can be written in terms of the same base.

Example 6.4.1:

Solve $3^{2x-1} = 27$.

Solution

Begin by writing 27 as a power of 3.

$$\begin{aligned} 3^{2x-1} &= 27 \\ 3^{2x-1} &= 3^3 \end{aligned}$$

Next apply the one-to-one property of exponential functions. In other words, set the exponents equal to each other and then simplify.

$$\begin{aligned} 2x - 1 &= 3 \\ 2x &= 4 \\ x &= 2 \end{aligned}$$

Answer:

2

Example 6.4.2:

Solve: $16^{1-3x} = 2$.

Solution

Begin by writing 16 as a power of 2 and then apply the power rule for exponents.

$$\begin{aligned} 16^{1-3x} &= 2 \\ \left(2^4\right)^{1-3x} &= 2 \\ 2^{4(1-3x)} &= 2^1 \end{aligned}$$

Now that the bases are the same we can set the exponents equal to each other and simplify.

$$\begin{aligned} 4(1-3x) &= 1 \\ 4 - 12x &= 1 \\ -12x &= -3 \\ x &= \frac{-3}{-12} = \frac{1}{4} \end{aligned}$$

Answer:

$\frac{1}{4}$

Exercise 6.4.1

Solve: $25^{2x+3} = 125$.

Answer

$-\frac{3}{4}$

www.youtube.com/v/yU_tJg6Unnc

In many cases we will not be able to equate the bases. For this reason we develop a second method for solving exponential equations. Consider the following equations:

$$3^2 = 9$$
$$3^? = 12$$
$$3^3 = 27$$

We can see that the solution to $3^x = 12$ should be somewhere between 2 and 3. A graphical interpretation follows.

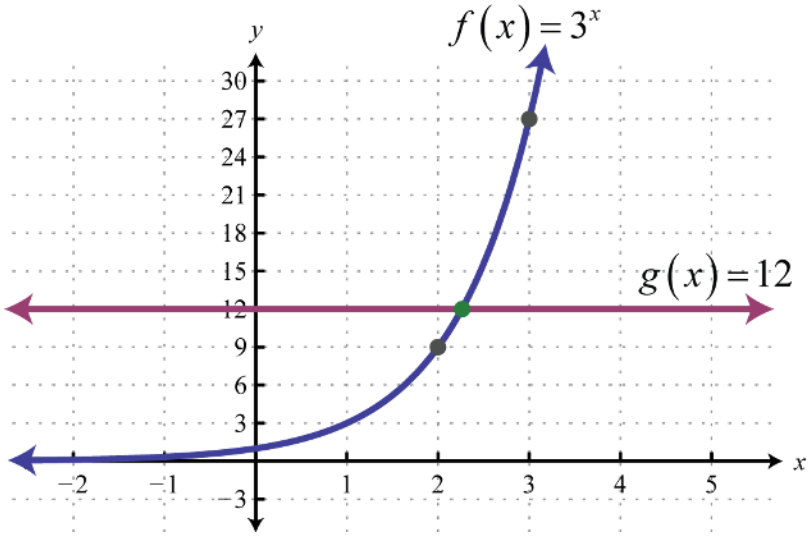


Figure 6.4.1

To solve this we make use of fact that logarithms are one-to-one functions. Given $x, y > 0$ the **one-to-one property of logarithms**[17] follows:

$$\log_b x = \log_b y \quad \text{if and only if} \quad x = y$$

This property, as well as the properties of the logarithm, allows us to solve exponential equations. For example, to solve $3^x = 12$ apply the common logarithm to both sides and then use the properties of the logarithm to isolate the variable.

$$\begin{aligned} 3^x &= 12 \\ \log 3^x &= \log 12 \quad \textit{One-to-one property of logarithms} \\ x \log 3 &= \log 12 \quad \textit{Power rule for logarithms} \\ x &= \frac{\log 12}{\log 3} \end{aligned}$$

Approximating to four decimal places on a calculator.

$$x = \log(12)/\log(3) \approx 2.2619$$

An answer between 2 and 3 is what we expected. Certainly we can check by raising 3 to this power to verify that we obtain a good approximation of 12.

$$3 \wedge 2.2618 \approx 12 \ \checkmark$$

Note that we are *not* multiplying both sides by "log"; we are applying the one-to-one property of logarithmic functions — which is often expressed as "*taking the log of both sides*." The general steps for solving exponential equations are outlined in the following example.

Example 6.4.3:

Solve: $5^{2x-1}+2=9$.

Solution

Step 1: Isolate the exponential expression.

$$\begin{aligned} 5^{2x-1}+2 &= 9 \\ 5^{2x-1} &= 7 \end{aligned}$$

Step 2: Take the logarithm of both sides. In this case, we will take the common logarithm of both sides so that we can approximate our result on a calculator.

$$\log 5^{2x-1} = \log 7$$

Step 3: Apply the power rule for logarithms and then solve.

$$\begin{aligned} \log 5^{2x-1} &= \log 7 \\ (2x-1)\log 5 &= \log 7 \qquad \textit{Distribute.} \\ 2x\log 5 - \log 5 &= \log 7 \\ 2x\log 5 &= \log 5 + \log 7 \\ x &= \frac{\log 5+\log 7}{2\log 5} \end{aligned}$$

This is an irrational number which can be approximated using a calculator. Take care to group the numerator and the product in the denominator when entering this into your calculator. To do this, make use of the parenthesis buttons (and) :

$$x = (\log 5 + \log(7))/(2 * \log(5)) \approx 1.1045$$

Answer:

$\frac{\log 5+\log 7}{2\log 5} \approx 1.1045$

Example 6.4.4:

Solve: $e^{5x+3}=1$.

Solution

The exponential function is already isolated and the base is e. Therefore, we choose to apply the natural logarithm to both sides.

$$\begin{aligned} e^{5x+3} &= 1 \\ \ln e^{5x+3} &= \ln 1 \end{aligned}$$

Apply the power rule for logarithms and then simplify.

$$\begin{aligned} \ln e^{5x+3} &= \ln 1 \\ (5x+3)\ln e &= \ln 1 \quad \textit{Recall } \ln e = 1 \textit{ and } \ln 1 = 0. \\ (5x+3)\cdot 1 &= 0 \\ 5x+3 &= 0 \\ x &= -\frac{3}{5} \end{aligned}$$

Answer:

$-\frac{3}{5}$

On most calculators there are only two logarithm buttons, the common logarithm LOG and the natural logarithm LN. If we want to approximate $\log_3 10$ we have to somehow change this base to 10 or e. The idea begins by rewriting the logarithmic function $y = \log_a x$, in exponential form.

$$\log_a x = y \Longrightarrow x = a^y$$

Here $x > 0$ and so we can apply the one-to-one property of logarithms. Apply the logarithm base b to both sides of the function in exponential form.

$$x = a^y$$
$$\log_b x = \log_b a^y$$

And then solve for y.

$$\log_b x = y \log_b a$$
$$\frac{\log_b x}{\log_b a} = y$$

Replace y into the original function and we have the very important **change of base formula**[18]:

$$\log_a x = \frac{\log_b x}{\log_b a}$$

We can use this to approximate $\log_3 10$ as follows.

$$\log_3 10 = \frac{\log 10}{\log 3} \approx 2.0959 \text{ or } \log_3 10 = \frac{\ln 10}{\ln 3} \approx 2.0959$$

Notice that the result is independent of the choice of base. In words, we can approximate the logarithm of any given base on a calculator by dividing the logarithm of the argument by the logarithm of that given base.

Example 6.4.5:

Approximate $\log_7 120$ the nearest hundredth.

Solution

Apply the change of base formula and use a calculator.

$$\log_7 120 = \frac{\log 120}{\log 7}$$

On a calculator,

$$\log(120) / \log(7) \approx 2.46$$

Answer:

2.46

Exercise 6.4.2

Solve: $2^{3x+1} - 4 = 1$. Give the exact and approximate answer rounded to four decimal places.

Answer

$\frac{\log 5 - \log 2}{3 \log 2} \approx 0.4406$

www.youtube.com/v/kQf1-cEQUg8

Solving Logarithmic Equations

A **logarithmic equation**[19] is an equation that involves a logarithm with a variable argument. Some logarithmic equations can be solved using the one-to-one property of logarithms. This is true when a single logarithm with the same base can be obtained on both sides of the equal sign.

Example 6.4.6:

Solve: $\log_2(2x-5)-\log_2(x-2)=0$.

Solution

We can obtain two equal logarithms base 2 by adding $\log_2(x-2)$ to both sides of the equation.

$$\begin{aligned}\log_2(2x-5)-\log_2(x-2) &= 0\\ \log_2(2x-5) &= \log_2(x-2)\end{aligned}$$

Here the bases are the same and so we can apply the one-to-one property and set the arguments equal to each other.

$$\begin{aligned}\log_2(2x-5) &= \log_2(x-2)\\ 2x-5 &= x-2\\ x &= 3\end{aligned}$$

Checking $x=3$ in the original equation:

$$\begin{aligned}\log_2(2(3)-5) &= \log_2((3)-2)\\ \log_2 1 &= \log_2 1\\ 0 &= 0\ \checkmark\end{aligned}$$

Answer:

3

When solving logarithmic equations the check is very important because extraneous solutions can be obtained. The properties of the logarithm only apply for values in the domain of the given logarithm. And when working with variable arguments, such as $\log(x-2)$, the value of x is not known until the end of this process. The logarithmic expression $\log(x-2)$ is only defined for values $x>2$.

Example 6.4.7:

Solve: $\log(3x-4)=\log(x-2)$.

Solution

Apply the one-to-one property of logarithms (set the arguments equal to each other) and then solve for x.

$$\begin{aligned}\log(3x-4) &= \log(x-2)\\ 3x-4 &= x-2\\ 2x &= 2\\ x &= 1\end{aligned}$$

When performing the check we encounter a logarithm of a negative number:

$$\begin{aligned}\log(x-2) &= \log(1-2)\\ &= \log(-1) \quad \textit{Undefined}\end{aligned}$$

Try this on a calculator, what does it say? Here $x=1$ is not in the domain of $\log(x-2)$. Therefore our only possible solution is extraneous and we conclude that there are no solutions to this equation.

Answer

No solution, $\emptyset$.

Caution: Solving logarithmic equations sometimes leads to extraneous solutions — we must check our answers.

Exercise 6.4.3

Solve: $\ln\left(x^2-15\right)-\ln(2x)=0$.

Answer

5

www.youtube.com/v/-kBkk-q2pNs

In many cases we will not be able to obtain two equal logarithms. To solve such equations we make use of the definition of the logarithm. If $b>0$, where $b\neq 1$, then $\log_b x=y$ implies that $b^y=x$. Consider the following common logarithmic equations (base 10),

$$\log x=0 \Longrightarrow =1 \quad \textit{Because } 10^{10}=1.$$
$$\log x=0.5 \Longrightarrow =\,?$$
$$\log x=1 \Longrightarrow =10 \quad \textit{Because } 10^{1}=10.$$

We can see that the solution to $\log x=0.5$ will be somewhere between 1 and 10. A graphical interpretation follows.

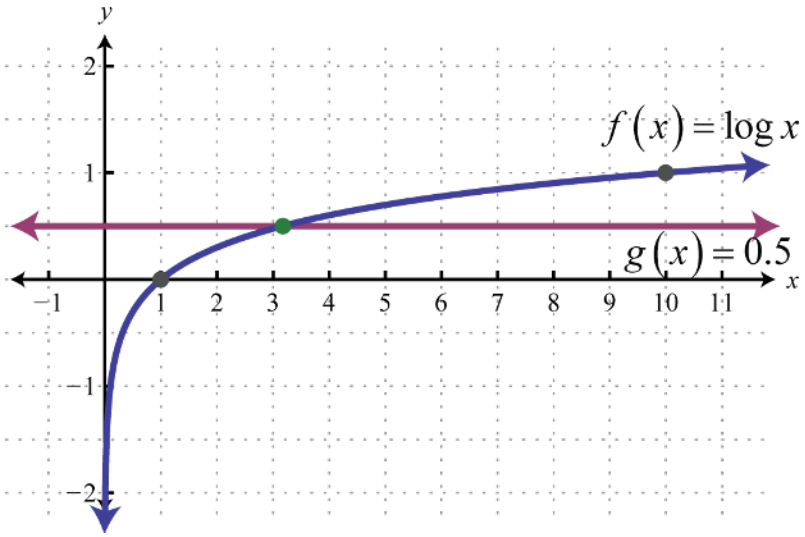


Figure 6.4.2

To find x we can apply the definition as follows.

$$\log_{10} x=0.5 \Longrightarrow 10^{0.5}=x$$

This can be approximated using a calculator,

$$x=10^{0.5}=10\text{^}0.5\approx 3.1623$$

An answer between 1 and 10 is what we expected. Check this on a calculator.

$$\log 3.1623\approx 5 \ \checkmark$$

Example 6.4.8:

Solve: $\log_3(2x-5)=2$.

Solution

Apply the definition of the logarithm.

$$\log_3(2x-5)=2 \Longrightarrow 2x-5=3^2$$

Solve the resulting equation.

$$\begin{aligned} 2x-5 &= 9 \\ 2x &= 14 \\ x &= 7 \end{aligned}$$

Check.

$$\log_3(2(7)-5) \stackrel{?}{=} 2$$
$$\log_3(9) = 2 \ \checkmark$$

Answer:

7

In order to apply the definition, we will need to rewrite logarithmic expressions as a single logarithm with coefficient 1. The general steps for solving logarithmic equations are outlined in the following example.

Example 6.4.9:

Solve: $\log_2(x-2)+\log_2(x-3)=1$.

Solution

Step 1: Write all logarithmic expressions as a single logarithm with coefficient 1. In this case, apply the product rule for logarithms.

$$\begin{aligned}\log_2(x-2)+\log_2(x-3) &= 1\\ \log_2[(x-2)(x-3)] &= 1\end{aligned}$$

Step 2: Use the definition and rewrite the logarithm in exponential form,

$$\log_2[(x-2)(x-3)]=1 \Longrightarrow (x-2)(x-3)=2^1$$

Step 3: Solve the resulting equation. Here we can solve by factoring.

$$\begin{aligned}(x-2)(x-3) &= 2\\ x^2-5x+6 &= 2\\ x^2-5x+4 &= 0\\ (x-4)(x-1) &= 0\\ x-4=0 \quad &\text{or} \quad x-1=0\\ x=4 \quad &\quad\ \ x=1\end{aligned}$$

Step 4: Check. This step is required.

Table 6.4.1

Check $x=4$	Check $x=1$
$\log_2(x-2)+\log_2(x-3) = 1$ $\log_2(4-2)+\log_2(4-3) = 1$ $\log_2(2)+\log_2(1) = 1$ $1+0=1$ ✓	\(\begin{aligned}\log _{2}(x-2)+\log _{2}(x-3)&=1 \\ \log _{2}(\color{Cerulean}{1}\color{black}{-}2)+\log _{2}(\color{Cerulean}{1}\color{black}{-}3)&=1 \\ \log _{2}(-1)+\log _{2}(-2)&=1\\N/A&\neq1\:\:\color{red}{✗}\end{aligned}\)

In this example, $x=1$ is not in the domain of the given logarithmic expression and is extraneous. The only solution is $x=4$.

Answer:

4

Example 6.4.10:

Solve: $\log(x+15)-1=\log(x+6)$

Solution

Begin by writing all logarithmic expressions on one side and constants on the other.

$$\begin{aligned}\log(x+15)-1 &= \log(x+6)\\ \log(x+15)-\log(x+6) &= 1\end{aligned}$$

Apply the quotient rule for logarithms as a means to obtain a single logarithm with coefficient 1.

$$\log(x+15)-\log(x+6)=1$$
$$\log\left(\tfrac{x+15}{x+6}\right)=1$$

This is a common logarithm; therefore use 10 as the base when applying the definition.

$$\begin{aligned}\frac{x+15}{x+6} &= 10^1\\ x+15 &= 10(x+6)\\ x+15 &= 10x+60\\ -9x &= 45\\ x &= -5\end{aligned}$$

Check.

$$\begin{aligned}\log(x+15)-1 &= \log(x+6)\\ \log(-5+15)-1 &= \log(-5+6)\\ \log 10-1 &= \log 1\\ 1-1 &= 0\\ 0 &= 0\ \checkmark\end{aligned}$$

Answer:

-5

? Exercise 6.4.4

Solve: $\log_2(x)+\log_2(x-1)=1$.

Answer

2

www.youtube.com/v/Lzwh1jkj-EY

✔ Example 6.4.11:

Find the inverse: $f(x)=\log_2(3x-4)$.

Solution

Begin by replacing the function notation $f(x)$ with y.

$$\begin{aligned}f(x) &= \log_2(3x-4)\\ y &= \log_2(3x-4)\end{aligned}$$

Interchange x and y and then solve for y.

$$\begin{aligned}x=\log_2(3y-4)\Longrightarrow 3y-4 &= 2^x\\ 3y &= 2^x+4\\ y &= \frac{2^x+4}{3}\end{aligned}$$

The resulting function is the inverse of f. Present the answer using function notation.

Answer:

$f^{-1}(x)=\frac{2^x+4}{3}$

Key Takeaways

- If each side of an exponential equation can be expressed using the same base, then equate the exponents and solve.
- To solve a general exponential equation, first isolate the exponential expression and then apply the appropriate logarithm to both sides. This allows us to use the properties of logarithms to solve for the variable.
- The change of base formula allows us to use a calculator to calculate logarithms. The logarithm of a number is equal to the common logarithm of the number divided by the common logarithm of the given base.
- If a single logarithm with the same base can be isolated on each side of an equation, then equate the arguments and solve.
- To solve a general logarithmic equation, first isolate the logarithm with coefficient 1 and then apply the definition. Solve the resulting equation.
- The steps for solving logarithmic equations sometimes produce extraneous solutions. Therefore, the check is required.

Footnotes

[15] An equation which includes a variable as an exponent.

[16] Given $b > 0$ and $b \neq 1$ we have $b^x = b^y$ if and only if $x = y$.

[17] Given $b > 0$ and $b \neq 1$ where $x, y > 0$ we have $log_b x = log_b y$ if and only if $x = y$.

[18] $\log_a x = \frac{\log_b x}{\log_b a}$; we can write any base-$a$ logarithm in terms of base-b logarithms using this formula.

[19] An equation that involves a logarithm with a variable argument.

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6.5: Applications of Exponential and Logarithmic Functions

Compound and Continuous Interest Formulas

Recall that compound interest occurs when interest accumulated for one period is added to the principal investment before calculating interest for the next period. The amount A accrued in this manner over time t is modeled by the compound interest formula:

$$A(t) = P\left(1+\tfrac{r}{n}\right)^{nt}$$

Here the initial principal P is accumulating compound interest at an annual rate r where the value n represents the number of times the interest is compounded in a year.

Example 6.5.1:

Susan invested $\$500$ in an account earning $4\frac{1}{2}\%$ annual interest that is compounded monthly.

a. How much will be in the account after 3 years?

b. How long will it take for the amount to grow to $\$750$?

Solution

In this example, the principal $P = \$500$, the interest rate $r = 4\frac{1}{2}\% = 0.045$, and because the interest is compounded monthly, $n = 12$. The investment can be modeled by the following function:

$$A(t) = 500\left(1+\tfrac{0.045}{12}\right)^{12t}$$
$$A(t) = 500(1.00375)^{12t}$$

a. Use this model to calculate the amount in the account after $t = 3$ years.

$$\begin{aligned} A(3) &= 500(1.00375)^{12(3)} \\ &= 500(1.00375)^{36} \\ &\approx 572.12 \end{aligned}$$

Rounded off to the nearest cent, after 3 years, the amount accumulated will be $\$572.12$

b. To calculate the time it takes to accumulate $\$750$, set $A(t) = 750$ and solve for t.

$$A(t) = 500(1.00375)^{12t}$$
$$750 = 500(1.00375)^{12t}$$

This results in an exponential equation that can be solved by first isolating the exponential expression.

$$750 = 500(1.00375)^{12t}$$
$$\frac{750}{500} = (1.00375)^{12t}$$
$$1.5 = (1.00375)^{12t}$$

At this point take the common logarithm of both sides, apply the power rule for logarithms, and then solve for t.

$$\log(1.5) = \log(1.00375)^{12t}$$
$$\log(1.5) = 12t\log(1.00375)$$
$$\frac{\log(1.5)}{12\log(1.00375)} = \frac{12t\ \log(1.00375)}{12\ \log(1.00375)}$$
$$\frac{\log(1.5)}{12\ \log(1.00375)} = t$$

Using a calculator we can approximate the time it takes.

$$t = \log(1.5)/(12 * \log(1.00375)) \approx 9\text{years}$$

Answer:

a. $572.12

b. Approximately 9 years

The period of time it takes a quantity to double is called the **doubling time**[20]. We next outline a technique for calculating the time it takes to double an initial investment earning compound interest.

Example 6.5.2:

Mario invested $1000 in an account earning 6.3% annual interest, that is compounded semi-annually. How long will it take the investment to double?

Solution

Here the principal $P = \$1,000$, the interest rate $r = 6.3\% = 0.063$, and because the interest is compounded semi-annually $n = 2$. This investment can be modeled as follows:

$$A(t) = 1,000\left(1 + \tfrac{0.063}{2}\right)^{2t}$$
$$A(t) = 1,000(1.0315)^{2t}$$

Since we are looking for the time it takes to double $1,000, substitute $2,000 for the resulting amount $A(t)$ and then solve for t.

$$\begin{aligned} 2,000 &= 1,000(1.0315)^{2t} \\ \frac{2,000}{1,000} &= (1.0315)^{2t} \\ 2 &= (1.0315)^{2t} \end{aligned}$$

At this point we take the common logarithm of both sides.

$$\begin{aligned} 2 &= (1.0315)^{2t} \\ \log 2 &= \log(1.0315)^{2t} \\ \log 2 &= 2t\log(1.0315) \\ \frac{\log 2}{2\log(1.0315)} &= t \end{aligned}$$

Using a calculator we can approximate the time it takes:

$$t = \log(2)/(2 * \log(1.0315)) \approx 11.17 \text{years}$$

Answer:

Approximately 11.17 years to double at 6.3%.

If the investment in the previous example was one million dollars, how long would it take to double? To answer this we would use $P = \$1,000,000$ and $A(t) = \$2,000,000$

$$A(t) = 1,000(1.0315)^{2t}$$
$$2,000,000 = 1,000,000(1.0315)^{2t}$$

Dividing both sides by $1,000,000$ we obtain the same exponential function as before.

$$2 = (1.0315)^{2t}$$

Hence, the result will be the same, about 11.17 years. In fact, doubling time is independent of the initial investment P.

Interest is typically compounded semi-annually $(n = 2)$, quarterly $(n = 4)$, monthly $(n = 12)$, or daily $(n = 365)$. However if interest is compounded every instant we obtain a formula for continuously compounding interest:

$$A(t) = Pe^{rt}$$

Here P represents the initial principal amount invested, r represents the annual interest rate, and t represents the time in years the investment is allowed to accrue continuously compounded interest.

Example 6.5.3:

Mary invested \$200 in an account earning $5\frac{3}{4}\%$ annual interest that is compounded continuously. How long will it take the investment to grow to \$350?

Solution

Here the principal $P = \$200$ and the interest rate $r = 5\frac{3}{4}\% = 5.75\% = 0.0575$. Since the interest is compounded continuously, use the formula $A(t) = Pe^{rt}$. Hence, the investment can be modeled by the following,

$$A(t) = 200e^{0.0575t}$$

To calculate the time it takes to accumulate to \$350, set $A(t) = 350$ and solve for t.

$$\begin{aligned} A(t) &= 200e^{0.0575t} \\ 350 &= 200e^{0.0575t} \end{aligned}$$

Begin by isolating the exponential expression.

$$\begin{aligned} \frac{350}{200} &= e^{0.0575t} \\ \frac{7}{4} &= e^{0.0575t} \\ 1.75 &= e^{0.0575t} \end{aligned}$$

Because this exponential has base e, we choose to take the natural logarithm of both sides and then solve for t.

$$\begin{aligned} \ln(1.75) &= \ln e^{0.0575t} && \textit{Apply the power rule for logarithms.} \\ \ln(1.75) &= 0.0575t \ln e && \textit{Recall that } \ln e = 1. \\ \ln(1.75) &= 0.0575t \cdot 1 \\ \tfrac{\ln(1.75)}{0.0575} &= t \end{aligned}$$

Using a calculator we can approximate the time it takes:

$$t = \ln(1.75)/0.0575 \approx 9.73 \quad years$$

Answer:

It will aprroximately 9.73 years.

When solving applications involving compound interest, look for the keyword “continuous,” or the keywords that indicate the number of annual compoundings. It is these keywords that determine which formula to choose.

Exercise 6.5.1

Mario invested $\$1,000$ in an account earning 6.3% annual interest that is compounded continuously. How long will it take the investment to double?

Answer

Approximately 11 years.

www.youtube.com/v/Z_ilJHIQvWQ

Modeling Exponential Growth and Decay

In the sciences, when a quantity is said to grow or decay exponentially, it is specifically meant to be modeled using the **exponential growth/decay formula**[21]:

$$P(t) = P_0 e^{kt}$$

Here P_0, read “P naught,” or “P zero,” represents the initial amount, k represents the growth rate, and t represents the time the initial amount grows or decays exponentially. If k is negative, then the function models exponential decay. Notice that the function

looks very similar to that of continuously compounding interest formula. We can use this formula to model population growth when conditions are optimal.

Example 6.5.4:

It is estimated that the population of a certain small town is $93,000$ people with an annual growth rate of 2.6%. If the population continues to increase exponentially at this rate:

a. Estimate the population in 7 years' time.
b. Estimate the time it will take for the population to reach 120,000 people.

Solution

We begin by constructing a mathematical model based on the given information. Here the initial population $P_0 = 93,000$ people and the growth rate $r = 2.6\% = 0.026$. The following model gives population in terms of time measured in years:

$$P(t) = 93,000e^{0.026t}$$

a. Use this function to estimate the population in $t = 7$ years.

$$\begin{aligned} P(t) &= 93,000e^{0006(7)} \\ &= 93,000e^{0.182} \\ &\approx 111,564 \quad people \end{aligned}$$

b. Use the model to determine the time it takes to reach $P(t) = 120,000$ people.

$$\begin{aligned} P(t) &= 93,000e^{0.026t} \\ 120,000 &= 93,000e^{0.026t} \\ \frac{120,000}{93,000} &= e^{0.026t} \\ \frac{40}{31} &= e^{0.026t} \end{aligned}$$

Take the natural logarithm of both sides and then solve for t.

$$\begin{aligned} \ln\left(\tfrac{40}{31}\right) &= \ln e^{0.026t} \\ \ln\left(\tfrac{40}{31}\right) &= 0.026t \ln e \\ \ln\left(\tfrac{40}{31}\right) &= 0.026t \cdot 1 \\ \frac{\ln\left(\frac{40}{31}\right)}{0.026} &= t \end{aligned}$$

Using a calculator,

$$t = \ln(40/31)/0.026 \approx 9.8 \quad years$$

Answer:

a. $111,564$ people
b. 9.8 years

Often the growth rate k is not given. In this case, we look for some other information so that we can determine it and then construct a mathematical model. The general steps are outlined in the following example.

Example 6.5.5:

Under optimal conditions *Escherichia coli (E. coli)* bacteria will grow exponentially with a doubling time of 20 minutes. If $1,000$ *E. coli* cells are placed in a Petri dish and maintained under optimal conditions, how many *E. coli* cells will be present in 2 hours?

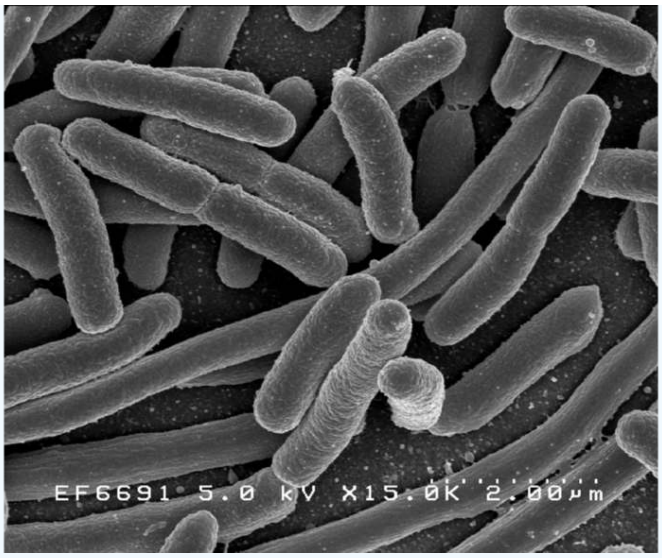


***Figure* 6.5.1**: *Escherichia coli (E. coli)*

Solution

The goal is to use the given information to construct a mathematical model based on the formula $P(t) = P_0 e^{kt}$.

Step 1: Find the growth rate k. Use the fact that the initial amount, $P_0 = 1,000$ cells, doubles in 20 minutes. That is, $P(t) = 2,000$ cells when $t = 20$ minutes.

$$\begin{aligned} P(t) &= P_0 e^{kt} \\ 2,000 &= 1,000 e^{k20} \end{aligned}$$

Solve for the only variable k.

$$\begin{aligned} 2,000 &= 1,000 e^{k20} \\ \frac{2,000}{1,000} &= e^{k20} \\ 2 &= e^{k20} \\ \ln(2) &= \ln e^{k20} \\ \ln(2) &= k20 \ln e \\ \ln(2) &= k20 \cdot 1 \\ \frac{\ln(2)}{20} &= k \end{aligned}$$

Step 2: Write a mathematical model based on the given information. Here $k \approx 0.0347$, which is about 3.5% growth rate per minute. However, we will use the exact value for k in our model. This will allow us to avoid round-off error in the final result. Use $P_0 = 1,000$ and $k = \ln(2)/20$:

$$P(t) = 1,000 e^{(\ln(2)/20)t}$$

This equation models the number of *E. coli* cells in terms of time in minutes.

Step 3: Use the function to answer the questions. In this case, we are asked to find the number of cells present in 2 hours. Because time is measured in minutes, use $t = 120$ minutes to calculate the number of *E. coli* cells.

$$\begin{aligned} P(120) &= 1,000 e^{(\ln(2)/20)(120)} \\ &= 1,000 e^{\ln(2) \cdot 6} \\ &= 1,000 e^{\ln 2^6} \\ &= 1,000 \cdot 2^6 \\ &= 64,000 \text{ cells} \end{aligned}$$

Answer:

In two hours $64,000$ cells will be present.

When the growth rate is negative the function models exponential decay. We can describe decreasing quantities using a **half-life**[22], or the time it takes to decay to one-half of a given quantity.

Example 6.5.6:

Due to radioactive decay, caesium-137 has a half-life of 30 years. How long will it take a 50-milligram sample to decay to 10 milligrams?

Solution

Use the half-life information to determine the rate of decay k. In $t = 30$ years the initial amount $P_0 = 50$ milligrams will decay to half $P(30) = 25$ milligrams.

$$\begin{aligned} P(t) &= P_0 e^{kt} \\ 25 &= 50e^{k30} \end{aligned}$$

Solve for the only variable, k.

$$\begin{aligned} 25 &= 50e^{130} \\ \frac{25}{50} &= e^{30k} \\ \ln\left(\frac{1}{2}\right) &= \ln e^{30k} \\ \ln\left(\frac{1}{2}\right) &= 30k \ln e \\ \frac{\ln 1 - \ln 2}{30} &= k \qquad \textit{Recall that } \ln 1 = 0. \\ -\frac{\ln 2}{30} &= k \end{aligned}$$

Note that $k = -\ln \frac{2}{30} \approx -0.0231$ is negative. However, we will use the exact value to construct a model that gives the amount of cesium-137 with respect to time in years.

$$P(t) = 50e^{(-\ln 2/30)t}$$

Use this model to find t when $P(t) = 10$ milligrams.

$$\begin{aligned} 10 &= 50e^{(-\ln 2/30)t} \\ \frac{10}{50} &= e^{(-\ln 2/30)t} \\ \ln\left(\frac{1}{5}\right) &= \ln e^{(-\ln 2/30)t} \\ \ln 1 - \ln 5 &= \left(-\frac{\ln 2}{30}\right) t \ln e \quad \textit{Recall that } \ln e = 1. \\ -30\frac{(\ln 1 - \ln 5)}{\ln 2} &= t \\ -\frac{30(0 - \ln 5)}{\ln 2} &= t \\ \frac{30 \ln 5}{\ln 2} &= t \end{aligned}$$

Answer:

Using a calculator, it will take $t \approx 69.66$ years to decay to 10 milligrams.

Radiocarbon dating is a method used to estimate the age of artifacts based on the relative amount of carbon-14 present in it. When an organism dies, it stops absorbing this naturally occurring radioactive isotope, and the carbon-14 begins to decay at a known rate. Therefore, the amount of carbon-14 present in an artifact can be used to estimate the age of the artifact.

Example 6.5.7:

An ancient bone tool is found to contain 25% of the carbon-14 normally found in bone. Given that carbon-14 has a half-life of $5,730$ years, estimate the age of the tool.

Solution

Begin by using the half-life information to find k. Here the initial amount P_0 of carbon-14 is not given, however, we know that in $t = 5,730$ years, this amount decays to half, $\frac{1}{2}P_0$.

$$P(t) = P_0 e^{kt}$$
$$\tfrac{1}{2}P_0 = P_0 e^{k5,730}$$

Dividing both sides by P_0 leaves us with an exponential equation in terms of k. This shows that half-life is independent of the initial amount.

$$\tfrac{1}{2} = e^{k5,730}$$

Solve for k.

$$\begin{aligned} \ln\left(\frac{1}{2}\right) &= \ln e^{k5,730} \\ \ln 1 - \ln 2 &= 5,730k \ln e \\ \frac{0 - \ln 2}{5,730} &= k \\ -\frac{\ln 2}{5,730} &= k \end{aligned}$$

Therefore we have the model,

$$P(t) = P_0 e^{(-\ln 2/5,730)t}$$

Next we wish to the find time it takes the carbon-14 to decay to 25% of the initial amount, or $P(t) = 0.25P_0$

$$.0.25P_0 = P_0 e^{(-\ln 2/5,730)t}$$

Divide both sides by P_0 and solve for t.

$$\begin{aligned} 0.25 &= e^{(-\ln 2/5,730)t} \\ \ln(0.25) &= \ln e^{(-\ln 2/5,730)t} \\ \ln(0.25) &= \left(-\frac{\ln 2}{5,730}\right)_{t \ln e} \\ -\frac{5,730 \ln(0.25)}{\ln 2} &= t \\ 11,460 &\approx t \end{aligned}$$

Answer:

The tool is approximately $11,460$ years old.

Exercise 6.5.2

The half-life of strontium-90 is about 28 years. How long will it take a 36 milligram sample of strontium-90 to decay to 30 milligrams?

Answer

7.4 years

www.youtube.com/v/Oto0ihiyvBc

Key Takeaways

- When interest is compounded a given number of times per year use the formula $A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$.
- When interest is to be compounded continuously use the formula $A(t) = Pe^{rt}$.
- Doubling time is the period of time it takes a given amount to double. Doubling time is independent of the principal.
- When amounts are said to be increasing or decaying exponentially, use the formula $P(t) = P_0 e^{kt}$.
- Half-life is the period of time it takes a given amount to decrease to one-half. Half-life is independent of the initial amount.
- To model data using the exponential growth/decay formula, use the given information to determine the growth/decay rate k. Once k is determined, a formula can be written to model the problem. Use the formula to answer the questions.

Footnotes

[20] The period of time it takes a quantity to double.

[21] A formula that models exponential growth or decay: $P(t) = P_0 e^{kt}$.

[22] The period of time it takes a quantity to decay to one-half of the initial amount.

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CHAPTER OVERVIEW

7: Systems of Equations

In this chapter, we will investigate matrices and their inverses, and various ways to use matrices to solve systems of equations. First, however, we will study systems of equations on their own: linear and nonlinear, and then partial fractions.

7.1: Systems of Linear Equations - Two Variables

7.2: Systems of Linear Equations - Three Variables

7.3: Partial Fractions

7.4: Solve Systems of Equations Using Matrices

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7.1: Systems of Linear Equations - Two Variables

Learning Objectives

- Solve systems of equations by graphing.
- Solve systems of equations by substitution.
- Solve systems of equations by addition.
- Identify inconsistent systems of equations containing two variables.
- Express the solution of a system of dependent equations containing two variables.

A skateboard manufacturer introduces a new line of boards. The manufacturer tracks its costs, which is the amount it spends to produce the boards, and its revenue, which is the amount it earns through sales of its boards. How can the company determine if it is making a profit with its new line? How many skateboards must be produced and sold before a profit is possible? In this section, we will consider linear equations with two variables to answer these and similar questions.



Figure 7.1.1

Introduction to Systems of Equations

In order to investigate situations such as that of the skateboard manufacturer, we need to recognize that we are dealing with more than one variable and likely more than one equation. **A system of linear equations** consists of two or more linear equations made up of two or more variables such that all equations in the system are considered simultaneously. To find the unique solution to a system of linear equations, we must find a numerical value for each variable in the system that will satisfy all equations in the system at the same time. Some linear systems may not have a solution and others may have an infinite number of solutions. In order for a linear system to have a unique solution, there must be at least as many equations as there are variables. Even so, this does not guarantee a unique solution.

In this section, we will look at systems of linear equations in two variables, which consist of two equations that contain two different variables. For example, consider the following system of linear equations in two variables.

$$\begin{aligned} 2x+y &= 15 \\ 3x-y &= 5 \end{aligned}$$

The *solution* to a system of linear equations in two variables is any ordered pair that satisfies each equation independently. In this example, the ordered pair $(4,7)$ is the solution to the system of linear equations. We can verify the solution by substituting the values into each equation to see if the ordered pair satisfies both equations. Shortly we will investigate methods of finding such a solution if it exists.

$$\begin{aligned} 2(4)+(7) &= 15 \text{ True} \\ 3(4)-(7) &= 5 \text{ True} \end{aligned}$$

In addition to considering the number of equations and variables, we can categorize systems of linear equations by the number of solutions. A consistent system of equations has at least one solution. A consistent system is considered to be an **independent system** if it has a single solution, such as the example we just explored. The two lines have different slopes and intersect at one point in the plane. A consistent system is considered to be a **dependent system** if the equations have the same slope and the same y-intercepts. In other words, the lines coincide so the equations represent the same line. Every point on the line represents a coordinate pair that satisfies the system. Thus, there are an infinite number of solutions.

Another type of system of linear equations is an **inconsistent system**, which is one in which the equations represent two parallel lines. The lines have the same slope and different y-intercepts. There are no points common to both lines; hence, there is no solution to the system.

TYPES OF LINEAR SYSTEMS

There are three types of systems of linear equations in two variables, and three types of solutions.

- An independent system has exactly one solution pair (x, y). The point where the two lines intersect is the only solution.
- An inconsistent system has no solution. Notice that the two lines are parallel and will never intersect.
- A dependent system has infinitely many solutions. The lines are coincident. They are the same line, so every coordinate pair on the line is a solution to both equations.

Figure $7.1.2$ compares graphical representations of each type of system.

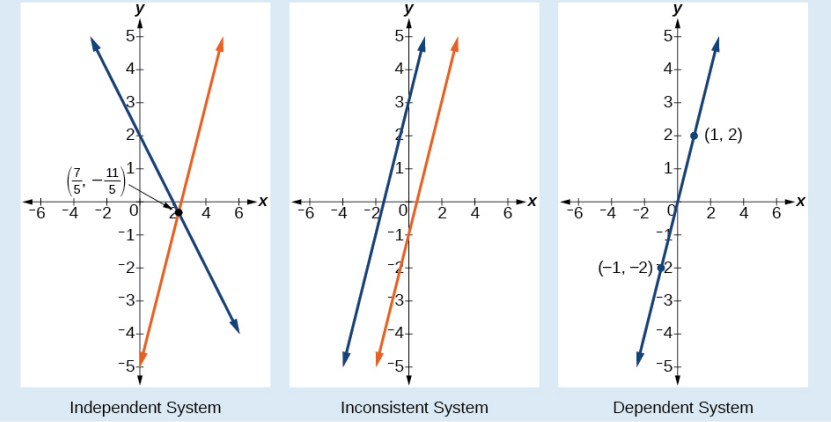


Figure 7.1.2

Given a system of linear equations and an ordered pair, determine whether the ordered pair is a solution

1. Substitute the ordered pair into each equation in the system.
2. Determine whether true statements result from the substitution in both equations; if so, the ordered pair is a solution.

Example 7.1.1: Determining Whether an Ordered Pair Is a Solution to a System of Equations

Determine whether the ordered pair $(5, 1)$ is a solution to the given system of equations.

$$\begin{aligned} x + 3y &= 8 \\ 2x - 9 &= y \end{aligned}$$

Solution

Substitute the ordered pair $(5, 1)$ into both equations.

$$\begin{aligned} (5) + 3(1) &= 8 \\ 8 &= 8 \text{ True} \\ 2(5) - 9 &= (1) \\ 1 &= 1 \text{ True} \end{aligned}$$

The ordered pair $(5, 1)$ satisfies both equations, so it is the solution to the system.

Analysis

We can see the solution clearly by plotting the graph of each equation. Since the solution is an ordered pair that satisfies both equations, it is a point on both of the lines and thus the point of intersection of the two lines. See Figure $7.1.3$.

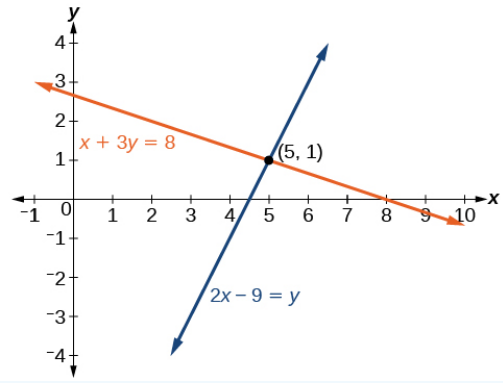


Figure 7.1.3

? Exercise 7.1.1

Determine whether the ordered pair $(8, 5)$ is a solution to the following system.

$$\begin{aligned} 5x - 4y &= 20 \\ 2x + 1 &= 3y \end{aligned}$$

Answer

Not a solution.

Solving Systems of Equations by Graphing

There are multiple methods of solving systems of linear equations. For a system of linear equations in two variables, we can determine both the type of system and the solution by graphing the system of equations on the same set of axes.

✓ Example 7.1.2: Solving a System of Equations in Two Variables by Graphing

Solve the following system of equations by graphing. Identify the type of system.

$$\begin{aligned} 2x + y &= -8 \\ x - y &= -1 \end{aligned}$$

Solution

Solve the first equation for y.

$$\begin{aligned} 2x + y &= -8 \\ y &= -2x - 8 \end{aligned}$$

Solve the second equation for y.

$$\begin{aligned} x - y &= -1 \\ y &= x + 1 \end{aligned}$$

Graph both equations on the same set of axes as in Figure 7.1.4.

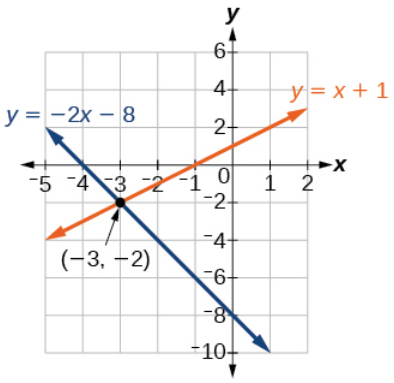


Figure 7.1.4

The lines appear to intersect at the point $(-3,-2)$. We can check to make sure that this is the solution to the system by substituting the ordered pair into both equations.

$$\begin{aligned} 2(-3)+(-2) &= -8 \\ -8 &= -8 \text{ True} \\ (-3)-(-2) &= -1 \\ -1 &= -1 \text{ True} \end{aligned}$$

The solution to the system is the ordered pair $(-3,-2)$,so the system is independent.

Exercise 7.1.2

Solve the following system of equations by graphing.

$$\begin{aligned} 2x-5y &= -25 \\ -4x+5y &= 35 \end{aligned}$$

Answer

The solution to the system is the ordered pair $(-5,3)$.

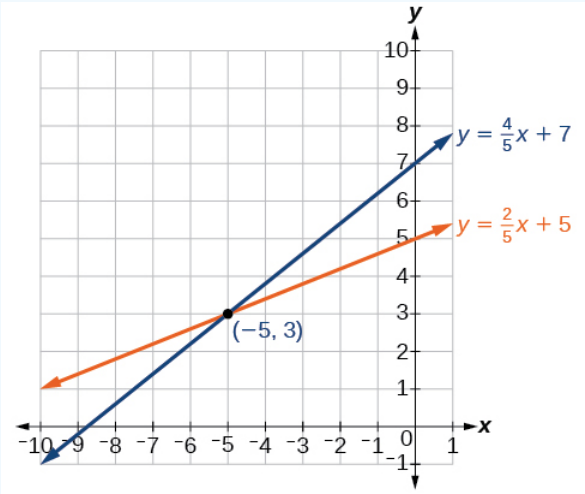


Figure 7.1.5

Q&A

Can graphing be used if the system is inconsistent or dependent?

Yes, in both cases we can still graph the system to determine the type of system and solution. If the two lines are parallel, the system has no solution and is inconsistent. If the two lines are identical, the system has infinite solutions and is a dependent

system.

Solving Systems of Equations by Substitution

Solving a linear system in two variables by graphing works well when the solution consists of integer values, but if our solution contains decimals or fractions, it is not the most precise method. We will consider two more methods of solving a **system of linear equations** that are more precise than graphing. One such method is solving a system of equations by the **substitution method**, in which we solve one of the equations for one variable and then substitute the result into the second equation to solve for the second variable. Recall that we can solve for only one variable at a time, which is the reason the substitution method is both valuable and practical.

How to: Given a system of two equations in two variables, solve using the substitution method.

1. Solve one of the two equations for one of the variables in terms of the other.
2. Substitute the expression for this variable into the second equation, then solve for the remaining variable.
3. Substitute that solution into either of the original equations to find the value of the first variable. If possible, write the solution as an ordered pair.
4. Check the solution in both equations.

Example 7.1.3: Solving a System of Equations in Two Variables by Substitution

Solve the following system of equations by substitution.

$$\begin{aligned} -x+y &= -5 \\ 2x-5y &= 1 \end{aligned}$$

Solution

First, we will solve the first equation for y.

$$\begin{aligned} -x+y &= -5 \\ y &= x-5 \end{aligned}$$

Now we can substitute the expression $x-5$ for y in the second equation.

$$\begin{aligned} 2x-5y &= 1 \\ 2x-5(x-5) &= 1 \\ 2x-5x+25 &= 1 \\ -3x &= -24 \\ x &= 8 \end{aligned}$$

Now, we substitute $x=8$ into the first equation and solve for y.

$$\begin{aligned} -(8)+y &= -5 \\ y &= 3 \end{aligned}$$

Our solution is $(8,3)$.

Check the solution by substituting $(8,3)$ into both equations.

$$\begin{aligned} -x+y &= -5 \\ -(8)+(3) &= -5 \text{ True} \\ 2x-5y &= 1 \\ 2(8)-5(3) &= 1 \text{ True} \end{aligned}$$

Exercise 7.1.3

Solve the following system of equations by substitution.

$$x = y + 3$$
$$4 = 3x - 2y$$

Answer

$(-2, -5)$

Q&A

Can the substitution method be used to solve any linear system in two variables?

Yes, but the method works best if one of the equations contains a coefficient of 1 *or* -1 *so that we do not have to deal with fractions.*

Solving Systems of Equations in Two Variables by the Addition Method

A third method of solving systems of linear equations is the addition method. In this method, we add two terms with the same variable, but opposite coefficients, so that the sum is zero. Of course, not all systems are set up with the two terms of one variable having opposite coefficients. Often we must adjust one or both of the equations by multiplication so that one variable will be eliminated by addition.

How to: Given a system of equations, solve using the addition method.

1. Write both equations with *x*- and *y*-variables on the left side of the equal sign and constants on the right.
2. Write one equation above the other, lining up corresponding variables. If one of the variables in the top equation has the opposite coefficient of the same variable in the bottom equation, add the equations together, eliminating one variable. If not, use multiplication by a nonzero number so that one of the variables in the top equation has the opposite coefficient of the same variable in the bottom equation, then add the equations to eliminate the variable.
3. Solve the resulting equation for the remaining variable.
4. Substitute that value into one of the original equations and solve for the second variable.
5. Check the solution by substituting the values into the other equation.

Example 7.1.4: Solving a System by the Addition Method

Solve the given system of equations by addition.

$$x + 2y = -1$$
$$-x + y = 3$$

Solution

Both equations are already set equal to a constant. Notice that the coefficient of x in the second equation, -1, is the opposite of the coefficient of x in the first equation, 1. We can add the two equations to eliminate x without needing to multiply by a constant.

$$\begin{aligned} x + 2y &= -1 \\ \underline{-x + y} &= \underline{3} \\ 3y &= 2 \end{aligned}$$

Now that we have eliminated x, we can solve the resulting equation for y.

$$3y = 2$$
$$y = \frac{2}{3}$$

Then, we substitute this value for y into one of the original equations and solve for x.

$$\begin{aligned} -x+y &= 3 \\ -x+\frac{2}{3} &= 3 \\ -x &= 3-\frac{2}{3} \\ -x &= \frac{7}{3} \\ x &= -\frac{7}{3} \end{aligned}$$

The solution to this system is $\left(-\frac{7}{3}, \frac{2}{3}\right)$.

Check the solution in the first equation.

$$\begin{aligned} x+2y &= -1 \\ \left(-\frac{7}{3}\right)+2\left(\frac{2}{3}\right) &= \\ -\frac{7}{3}+\frac{4}{3} &= -\frac{3}{3} \\ -1 &= -1 \qquad \text{True} \end{aligned}$$

Analysis

We gain an important perspective on systems of equations by looking at the graphical representation. See Figure 7.1.6 to find that the equations intersect at the solution. We do not need to ask whether there may be a second solution because observing the graph confirms that the system has exactly one solution.

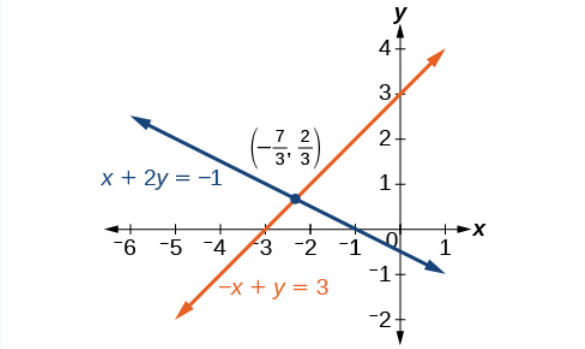


Figure 7.1.6

Example 7.1.5: Using the Addition Method When Multiplication of One Equation Is Required

Solve the given system of equations by the addition method.

$$\begin{aligned} 3x+5y &= -11 \\ x-2y &= 11 \end{aligned}$$

Solution

Adding these equations as presented will not eliminate a variable. However, we see that the first equation has $3x$ in it and the second equation has x. So if we multiply the second equation by -3,the x-terms will add to zero.

$$\begin{aligned} x-2y &= 11 \\ -3(x-2y) &= -3(11) \qquad \text{Multiply both sides by } -3. \\ -3x+6y &= -33 \qquad \text{Use the distributive property.} \end{aligned}$$

Now, let's add them.

$$\begin{aligned} 3x+5y &= -11 \\ \underline{-3x+6y} &= \underline{-33} \\ 11y &= -44 \\ y &= -4 \end{aligned}$$

For the last step, we substitute $y=-4$ into one of the original equations and solve for x.

$$\begin{aligned} 3x+5y &= -11 \\ 3x+5(-4) &= -11 \\ 3x-20 &= -11 \\ 3x &= 9 \\ x &= 3 \end{aligned}$$

Our solution is the ordered pair $(3,-4)$. See Figure 7.1.7. Check the solution in the original second equation.

$$\begin{aligned} x-2y &= 11 \\ (3)-2(-4) &= 3+8 \\ &= 11 \qquad \text{True} \end{aligned}$$

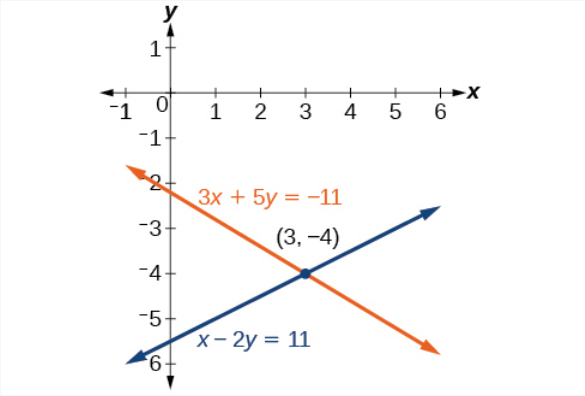


Figure 7.1.7

Exercise 7.1.4

Solve the system of equations by addition.

$$\begin{aligned} 2x-7y &= 2 \\ 3x+y &= -20 \end{aligned}$$

Answer

$(-6,-2)$

Example 7.1.6: Using the Addition Method When Multiplication of Both Equations Is Required

Solve the given system of equations in two variables by addition.

$$\begin{aligned} 2x+3y &= -16 \\ 5x-10y &= 30 \end{aligned}$$

Solution

One equation has $2x$ and the other has $5x$. The least common multiple is $10x$ so we will have to multiply both equations by a constant in order to eliminate one variable. Let's eliminate x by multiplying the first equation by -5 and the second equation by 2.

$$\begin{aligned} -5(2x+3y) &= -5(-16) \\ -10x-15y &= 80 \\ 2(5x-10y) &= 2(30) \\ 10x-20y &= 60 \end{aligned}$$

Then, we add the two equations together.

$$\begin{aligned} -10x-15y &= 80 \\ \underline{10x-20y} &= \underline{60} \\ -35y &= 140 \\ y &= -4 \end{aligned}$$

Substitute $y=-4$ into the original first equation.

$$\begin{aligned} 2x+3(-4) &= -16 \\ 2x-12 &= -16 \\ 2x &= -4 \\ x &= -2 \end{aligned}$$

The solution is $(-2,-4)$. Check it in the other equation.

$$\begin{aligned} 5x-10y &= 30 \\ 5(-2)-10(-4) &= 30 \\ -10+40 &= 30 \\ 30 &= 30 \end{aligned}$$

See Figure 7.1.8.

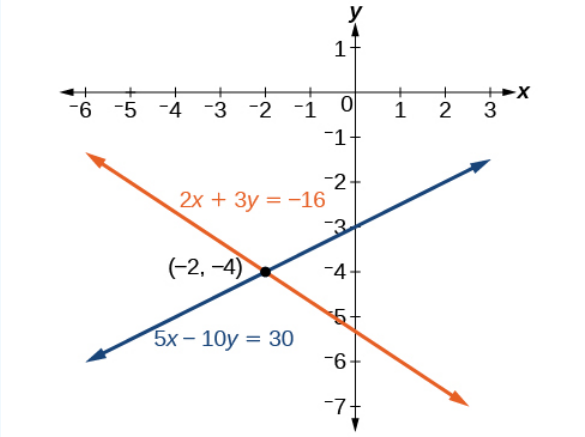


Figure 7.1.8

Example 7.1.7: Using the Addition Method in Systems of Equations Containing Fractions

Solve the given system of equations in two variables by addition.

$$\begin{aligned} \frac{x}{3}+\frac{y}{6} &= 3 \\ \frac{x}{2}-\frac{y}{4} &= 1 \end{aligned}$$

Solution

First clear each equation of fractions by multiplying both sides of the equation by the least common denominator.

$$6\left(\frac{x}{3}+\frac{y}{6}\right)=6(3)$$
$$2x+y=18$$
$$4\left(\frac{x}{2}-\frac{y}{4}\right)=4(1)$$
$$2x-y=4$$

Now multiply the second equation by -1 so that we can eliminate the x-variable.

$$-1(2x-y)=-1(4)$$
$$-2x+y=-4$$

Add the two equations to eliminate the x-variable and solve the resulting equation.

$$2x+y=18$$
$$-2x+y=-4$$
$$2y=14$$
$$y=7$$

Substitute $y=7$ into the first equation.

$$2x+(7)=18$$
$$2x=11$$
$$x=\frac{11}{2}$$
$$=7.5$$

The solution is $\left(\frac{11}{2},7\right)$. Check it in the other equation.

$$\frac{x}{2}-\frac{y}{4}=1$$
$$\frac{\frac{11}{2}}{2}-\frac{7}{4}=1$$
$$\frac{11}{4}-\frac{7}{4}=1$$
$$\frac{4}{4}=1$$

? Exercise 7.1.5

Solve the system of equations by addition.

$$2x+3y=8$$
$$3x+5y=10$$

Answer

$(10,-4)$

Identifying Inconsistent Systems of Equations Containing Two Variables

Now that we have several methods for solving systems of equations, we can use the methods to identify inconsistent systems. Recall that an inconsistent system consists of parallel lines that have the same slope but different y-intercepts. They will never intersect. When searching for a solution to an inconsistent system, we will come up with a false statement, such as $12=0$.

Example 7.1.8: Solving an Inconsistent System of Equations

Solve the following system of equations.

$$\begin{aligned} x &= 9-2y \\ x+2y &= 13 \end{aligned}$$

Solution

We can approach this problem in two ways. Because one equation is already solved for x, the most obvious step is to use substitution.

$$\begin{aligned} x+2y &= 13 \\ (9-2y)+2y &= 13 \\ 9+0y &= 13 \\ 9 &= 13 \end{aligned}$$

Clearly, this statement is a contradiction because $9 \neq 13$. Therefore, the system has no solution.

The second approach would be to first manipulate the equations so that they are both in slope-intercept form. We manipulate the first equation as follows.

$$\begin{aligned} x &= 9-2y \\ 2y &= -x+9 \\ y &= -\frac{1}{2}x+\frac{9}{2} \end{aligned}$$

We then convert the second equation expressed to slope-intercept form.

$$\begin{aligned} x+2y &= 13 \\ 2y &= -x+13 \\ y &= -\frac{1}{2}x+\frac{13}{2} \end{aligned}$$

Comparing the equations, we see that they have the same slope but different y-intercepts. Therefore, the lines are parallel and do not intersect.

$$\begin{aligned} y &= -\frac{1}{2}x+\frac{9}{2} \\ y &= -\frac{1}{2}x+\frac{13}{2} \end{aligned}$$

Analysis

Writing the equations in slope-intercept form confirms that the system is inconsistent because all lines will intersect eventually unless they are parallel. Parallel lines will never intersect; thus, the two lines have no points in common. The graphs of the equations in this example are shown in Figure 7.1.9.

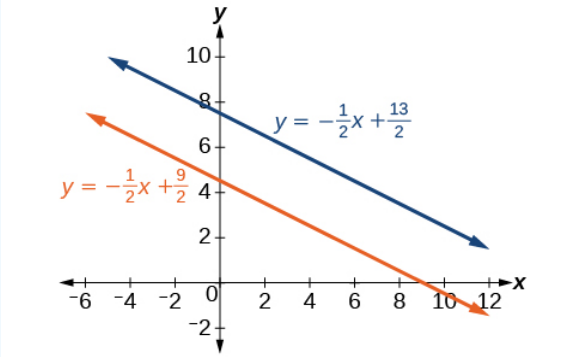


Figure 7.1.9

? Exercise 7.1.6

Solve the following system of equations in two variables.

$$\begin{aligned} 2y-2x &= 2 \\ 2y-2x &= 6 \end{aligned}$$

Answer

No solution. It is an inconsistent system.

Expressing the Solution of a System of Dependent Equations Containing Two Variables

Recall that a dependent system of equations in two variables is a system in which the two equations represent the same line. Dependent systems have an infinite number of solutions because all of the points on one line are also on the other line. After using substitution or addition, the resulting equation will be an identity, such as $0=0$.

✓ Example 7.1.9: Finding a Solution to a Dependent System of Linear Equations

Find a solution to the system of equations using the **addition method**.

$$\begin{aligned} x+3y &= 2 \\ 3x+9y &= 6 \end{aligned}$$

Solution

With the addition method, we want to eliminate one of the variables by adding the equations. In this case, let's focus on eliminating x. If we multiply both sides of the first equation by -3, then we will be able to eliminate the x-variable.

$$\begin{aligned} x+3y &= 2 \\ (-3)(x+3y) &= (-3)(2) \\ -3x-9y &= -6 \end{aligned}$$

Now add the equations.

$$\begin{aligned} -3x-9y &= -6 \\ \underline{+\,3x+9y} &= \underline{6} \\ 0 &= 0 \end{aligned}$$

We can see that there will be an infinite number of solutions that satisfy both equations.

Analysis

If we rewrote both equations in the slope-intercept form, we might know what the solution would look like before adding. Let's look at what happens when we convert the system to slope-intercept form.

$$\begin{aligned} x+3y &= 2 \\ 3y &= -x+2 \\ y &= -\frac{1}{3}x+\frac{2}{3} \\ 3x+9y &= 6 \\ 9y &= -3x+6 \\ y &= -\frac{3}{9}x+\frac{6}{9} \\ y &= -\frac{1}{3}x+\frac{2}{3} \end{aligned}$$

See Figure 7.1.10. Notice the results are the same. The general solution to the system is $\left(x, -\frac{1}{3}x+\frac{2}{3}\right)$.

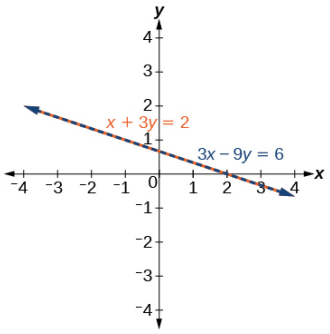


Figure 7.1.10

? Exercise 7.1.7

Solve the following system of equations in two variables.

$$\begin{aligned} y-2x &= 5 \\ -3y+6x &= -15 \end{aligned}$$

Answer

The system is dependent so there are infinite solutions of the form $(x, 2x+5)$.

Using Systems of Equations to Investigate Profits

Using what we have learned about systems of equations, we can return to the skateboard manufacturing problem at the beginning of the section. The skateboard manufacturer's revenue function is the function used to calculate the amount of money that comes into the business. It can be represented by the equation $R = xp$, where x=quantity and p=price. The revenue function is shown in orange in Figure 7.1.11.

The cost function is the function used to calculate the costs of doing business. It includes fixed costs, such as rent and salaries, and variable costs, such as utilities. The cost function is shown in blue in Figure 7.1.11. The x-axis represents quantity in hundreds of units. The y-axis represents either cost or revenue in hundreds of dollars.

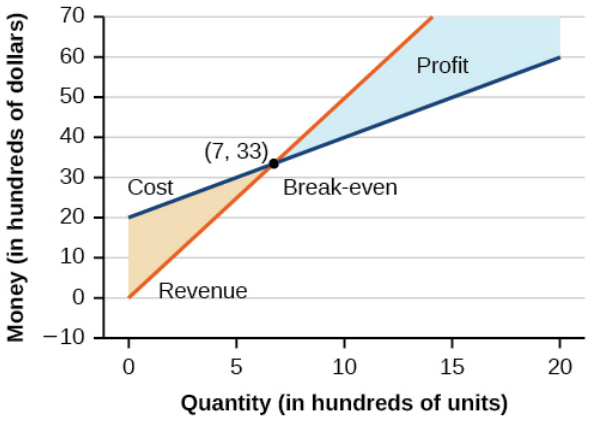


Figure 7.1.11

The point at which the two lines intersect is called the break-even point. We can see from the graph that if 700 units are produced, the cost is $\$3,300$ and the revenue is also $\$3,300$. In other words, the company breaks even if they produce and sell 700 units. They neither make money nor lose money.

The shaded region to the right of the break-even point represents quantities for which the company makes a profit. The shaded region to the left represents quantities for which the company suffers a loss. The profit function is the revenue function minus the cost function, written as $P(x) = R(x) - C(x)$. Clearly, knowing the quantity for which the cost equals the revenue is of great importance to businesses.

Example 7.1.10: Finding the Break-Even Point and the Profit Function Using Substitution

Given the cost function $C(x) = 0.85x + 35,000$ and the revenue function $R(x) = 1.55x$, find the break-even point and the profit function.

Solution

Write the system of equations using y to replace function notation.

$$\begin{aligned} y &= 0.85x + 35,000 \\ y &= 1.55x \end{aligned}$$

Substitute the expression $0.85x + 35,000$ from the first equation into the second equation and solve for x.

$$\begin{aligned} 0.85x + 35,000 &= 1.55x \\ 35,000 &= 0.7x \\ 50,000 &= x \end{aligned}$$

Then, we substitute $x = 50,000$ into either the cost function or the revenue function.

$$1.55(50,000) = 77,500$$

The break-even point is $(50,000, 77,500)$.

The profit function is found using the formula $P(x) = R(x) - C(x)$.

$$\begin{aligned} P(x) &= 1.55x - (0.85x + 35,000) \\ &= 0.7x - 35,000 \end{aligned}$$

The profit function is $P(x) = 0.7x - 35,000$.

Analysis

The cost to produce $50,000$ units is $\$77,500$, and the revenue from the sales of $50,000$ units is also $\$77,500$. To make a profit, the business must produce and sell more than $50,000$ units. See Figure 7.1.12.

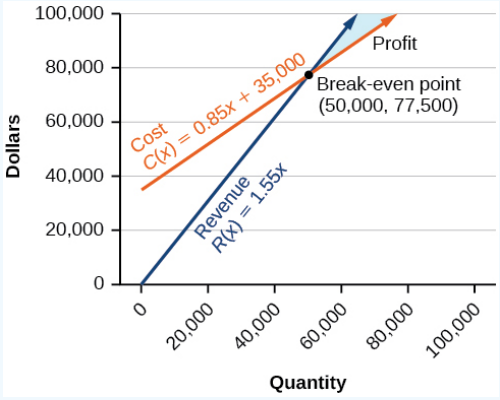


Figure 7.1.12

We see from the graph in Figure 7.1.13 that the profit function has a negative value until $x = 50,000$, when the graph crosses the x-axis. Then, the graph emerges into positive y-values and continues on this path as the profit function is a straight line. This illustrates that the break-even point for businesses occurs when the profit function is 0. The area to the left of the break-even point represents operating at a loss.

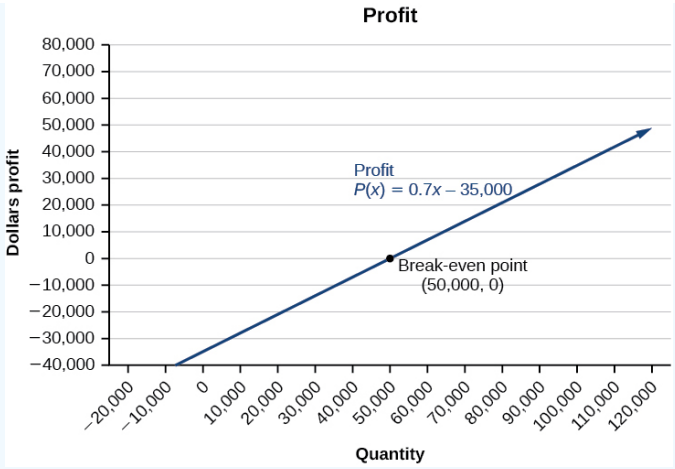


Figure 7.1.13

Example 7.1.11: Writing and Solving a System of Equations in Two Variables

The cost of a ticket to the circus is $\$25.00$ for children and $\$50.00$ for adults. On a certain day, attendance at the circus is $2,000$ and the total gate revenue is $\$70,000$. How many children and how many adults bought tickets?

Solution

Let c = the number of children and a = the number of adults in attendance.

The total number of people is $2,000$. We can use this to write an equation for the number of people at the circus that day.

$$c+a=2,000$$

The revenue from all children can be found by multiplying $\$25.00$ by the number of children, $25c$. The revenue from all adults can be found by multiplying $\$50.00$ by the number of adults, $50a$. The total revenue is $\$70,000$. We can use this to write an equation for the revenue.

$$25c+50a=70,000$$

We now have a system of linear equations in two variables.

$$c+a=2,000$$
$$25c+50a=70,000$$

In the first equation, the coefficient of both variables is 1. We can quickly solve the first equation for either c or a. We will solve for a.

$$\begin{aligned} c+a &= 2,000 \\ a &= 2,000-c \end{aligned}$$

Substitute the expression $2,000-c$ in the second equation fora a and solve for c.

$$\begin{aligned} 25c+50(2,000-c) &= 70,000 \\ 25c+100,000-50c &= 70,000 \\ -25c &= -30,000 \\ c &= 1,200 \end{aligned}$$

Substitute $c=1,200$ into the first equation to solve for a.

$$\begin{aligned} 1,200+a &= 2,000 \\ a &= 800 \end{aligned}$$

We find that $1,200$ children and 800 adults bought tickets to the circus that day.

Exercise 7.1.8

Meal tickets at the circus cost $\$4.00$ for children and $\$12.00$ for adults. If $1,650$ meal tickets were bought for a total of, $\$14,200$, how many children and how many adults bought meal tickets?

Answer

700 children, 950 adults

Media

Access these online resources for additional instruction and practice with systems of linear equations.

- Solving Systems of Equations Using Substitution
- Solving Systems of Equations Using Elimination
- Applications of Systems of Equations

Key Concepts

- A system of linear equations consists of two or more equations made up of two or more variables such that all equations in the system are considered simultaneously.
- The solution to a system of linear equations in two variables is any ordered pair that satisfies each equation independently. See Example 7.1.1.
- Systems of equations are classified as independent with one solution, dependent with an infinite number of solutions, or inconsistent with no solution.
- One method of solving a system of linear equations in two variables is by graphing. In this method, we graph the equations on the same set of axes. See Example 7.1.2.
- Another method of solving a system of linear equations is by substitution. In this method, we solve for one variable in one equation and substitute the result into the second equation. See Example 7.1.3.
- A third method of solving a system of linear equations is by addition, in which we can eliminate a variable by adding opposite coefficients of corresponding variables. See Example 7.1.4.
- It is often necessary to multiply one or both equations by a constant to facilitate elimination of a variable when adding the two equations together. See Example 7.1.5, Example 7.1.6, and Example 7.1.7.
- Either method of solving a system of equations results in a false statement for inconsistent systems because they are made up of parallel lines that never intersect. See Example 7.1.8.
- The solution to a system of dependent equations will always be true because both equations describe the same line. See Example 7.1.9.
- Systems of equations can be used to solve real-world problems that involve more than one variable, such as those relating to revenue, cost, and profit. See Example 7.1.10 and Example 7.1.11.

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7.2: Systems of Linear Equations - Three Variables

Learning Objectives

- Solve systems of three equations in three variables.
- Identify inconsistent systems of equations containing three variables.
- Express the solution of a system of dependent equations containing three variables.

John received an inheritance of $\$12,000$ that he divided into three parts and invested in three ways: in a money-market fund paying 3% annual interest; in municipal bonds paying 4% annual interest; and in mutual funds paying 7% annual interest. John invested $\$4,000$ more in municipal funds than in municipal bonds. He earned $\$670$ in interest the first year. How much did John invest in each type of fund?



***Figure* 7.2.1:** *(credit: "Elembis," Wikimedia Commons)*

Understanding the correct approach to setting up problems such as this one makes finding a solution a matter of following a pattern. We will solve this and similar problems involving three equations and three variables in this section. Doing so uses similar techniques as those used to solve systems of two equations in two variables. However, finding solutions to systems of three equations requires a bit more organization and a touch of visual gymnastics.

Solving Systems of Three Equations in Three Variables

In order to solve systems of equations in three variables, known as three-by-three systems, the primary tool we will be using is called Gaussian elimination, named after the prolific German mathematician Karl Friedrich Gauss. While there is no definitive order in which operations are to be performed, there are specific guidelines as to what type of moves can be made. We may number the equations to keep track of the steps we apply. The goal is to eliminate one variable at a time to achieve upper triangular form, the ideal form for a three-by-three system because it allows for straightforward back-substitution to find a solution (x, y, z), which we call an ordered triple. A system in upper triangular form looks like the following:

$$\begin{aligned} Ax + By + Cz &= D \\ Ey + Fz &= G \\ Hz &= K \end{aligned}$$

The third equation can be solved for z,and then we back-substitute to find y and x. To write the system in upper triangular form, we can perform the following operations:

1. Interchange the order of any two equations.
2. Multiply both sides of an equation by a nonzero constant.
3. Add a nonzero multiple of one equation to another equation.

The **solution set** to a three-by-three system is an ordered triple (x, y, z). Graphically, the ordered triple defines the point that is the intersection of three planes in space. You can visualize such an intersection by imagining any corner in a rectangular room. A corner is defined by three planes: two adjoining walls and the floor (or ceiling). Any point where two walls and the floor meet represents the intersection of three planes.

NUMBER OF POSSIBLE SolutionS

Figure 7.2.2 and Figure 7.2.3 illustrate possible solution scenarios for three-by-three systems.

- Systems that have a single solution are those which, after elimination, result in a solution set consisting of an ordered triple (x, y, z). Graphically, the ordered triple defines a point that is the intersection of three planes in space.
- Systems that have an infinite number of solutions are those which, after elimination, result in an expression that is always true, such as $0 = 0$. Graphically, an infinite number of solutions represents a line or coincident plane that serves as the intersection of three planes in space.
- Systems that have no solution are those that, after elimination, result in a statement that is a contradiction, such as $3 = 0$. Graphically, a system with no solution is represented by three planes with no point in common.

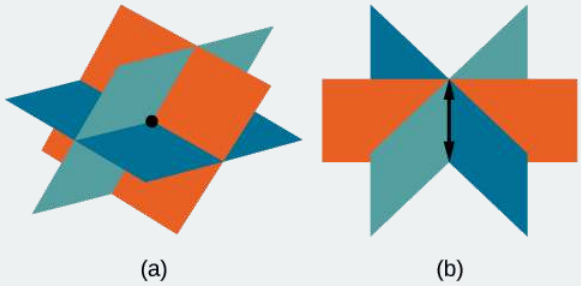


Figure 7.2.2: *(a)Three planes intersect at a single point, representing a three-by-three system with a single solution. (b) Three planes intersect in a line, representing a three-by-three system with infinite solutions.*

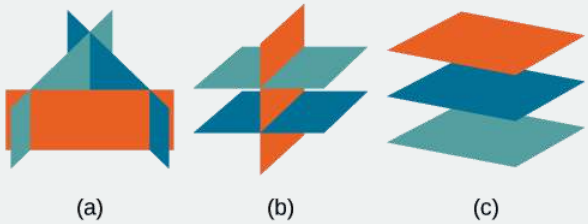


Figure 7.2.3: *All three figures represent three-by-three systems with no solution. (a) The three planes intersect with each other, but not at a common point. (b) Two of the planes are parallel and intersect with the third plane, but not with each other. (c) All three planes are parallel, so there is no point of intersection.*

Example 7.2.1: Determining Whether an Ordered Triple Is a Solution to a System

Determine whether the ordered triple $(3, -2, 1)$ is a solution to the system.

$$\begin{aligned} x + y + z &= 2 \\ 6x - 4y + 5z &= 31 \\ 5x + 2y + 2z &= 13 \end{aligned}$$

Solution

We will check each equation by substituting in the values of the ordered triple for x, y, and z.

$x + y + z = 2$	$6x - 4y + 5z = 31$	$5x + 2y + 2z = 13$
$(3) + (-2) + (1) = 2$	$6(3) - 4(-2) + 5(1) = 31$	$5(3) + 2(-2) + 2(1) = 13$
True	$18 + 8 + 5 = 31$	$15 - 4 + 2 = 13$
	True	True

The ordered triple $(3, -2, 1)$ is indeed a solution to the system.

How to: Given a linear system of three equations, solve for three unknowns

1. Pick any pair of equations and solve for one variable.
2. Pick another pair of equations and solve for the same variable.
3. You have created a system of two equations in two unknowns. Solve the resulting two-by-two system.
4. Back-substitute known variables into any one of the original equations and solve for the missing variable.

Example 7.2.2: Solving a System of Three Equations in Three Variables by Elimination

Find a solution to the following system:

$$\begin{aligned} x-2y+3z &= 9 \quad (1)\\ -x+3y-z &= -6 \quad (2)\\ 2x-5y+5z &= 17 \quad (3) \end{aligned}$$

Solution

There will always be several choices as to where to begin, but the most obvious first step here is to eliminate x by adding equations (1) and (2).

$$\begin{aligned} x-2y+3z &= 9 \quad (1)\\ -x+3y-z &= -6 \quad (2)\\ \hline y+2z &= 3 \quad (4) \end{aligned}$$

The second step is multiplying equation (1) by -2 and adding the result to equation (3). These two steps will eliminate the variable x.

$$\begin{aligned} -2x+4y-6z &= -18 \quad (1) \quad \text{multiplied by } -2\\ 2x-5y+5z &= 17 \quad (3)\\ \hline -y-z &= -1 \quad (5) \end{aligned}$$

In equations (4) and (5), we have created a new two-by-two system. We can solve for z by adding the two equations.

$$\begin{aligned} y+2z &= 3 \quad (4)\\ -y-z &= -1 \quad (5)\\ \hline z &= 2 \quad (6) \end{aligned}$$

Choosing one equation from each new system, we obtain the upper triangular form:

$$\begin{aligned} x-2y+3z &= 9 \quad (1)\\ y+2z &= 3 \quad (4)\\ z &= 2 \quad (6) \end{aligned}$$

Next, we back-substitute $z=2$ into equation (4) and solve for y.

$$\begin{aligned} y+2(2) &= 3\\ y+4 &= 3\\ y &= -1 \end{aligned}$$

Finally, we can back-substitute $z=2$ and $y=-1$ into equation (1). This will yield the solution for x.

$$\begin{aligned} x-2(-1)+3(2) &= 9\\ x+2+6 &= 9\\ x &= 1 \end{aligned}$$

The solution is the ordered triple $(1, -1, 2)$. See Figure 7.2.4.

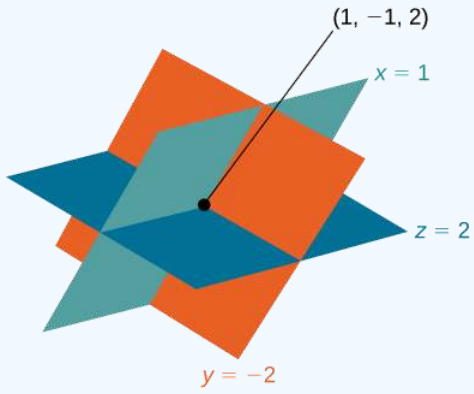


***Figure* 7.2.4**

Example 7.2.3: Solving a Real-World Problem Using a System of Three Equations in Three Variables

In the problem posed at the beginning of the section, John invested his inheritance of $\$12,000$ in three different funds: part in a money-market fund paying 3% interest annually; part in municipal bonds paying 4% annually; and the rest in mutual funds paying 7% annually. John invested $\$4,000$ more in mutual funds than he invested in municipal bonds. The total interest earned in one year was $\$670$. How much did he invest in each type of fund?

Solution

To solve this problem, we use all of the information given and set up three equations. First, we assign a variable to each of the three investment amounts:

$$x = \text{amount invested in money-market fund}$$
$$y = \text{amount invested in municipal bonds}$$
$$z = \text{amount invested in mutual funds}$$

The first equation indicates that the sum of the three principal amounts is $\$12,000$.

$$x + y + z = 12,000$$

We form the second equation according to the information that John invested $\$4,000$ more in mutual funds than he invested in municipal bonds.

$$z = y + 4,000$$

The third equation shows that the total amount of interest earned from each fund equals $\$670$.

$$0.03x + 0.04y + 0.07z = 670$$

Then, we write the three equations as a system.

$$\begin{aligned} x + y + z &= 12,000 \\ -y + z &= 4,000 \\ 0.03x + 0.04y + 0.07z &= 670 \end{aligned}$$

To make the calculations simpler, we can multiply the third equation by 100. Thus,

$$\begin{aligned} x + y + z &= 12,000 \quad (1) \\ -y + z &= 4,000 \quad (2) \\ 3x + 4y + 7z &= 67,000 \quad (3) \end{aligned}$$

Step 1. Interchange equation (2) and equation (3) so that the two equations with three variables will line up.

$$\begin{aligned} x+y+z &= 12,000 \\ 3x+4y+7z &= 67,000 \\ -y+z &= 4,000 \end{aligned}$$

Step 2. Multiply equation (1) by -3 and add to equation (2). Write the result as row 2.

$$\begin{aligned} x+y+z &= 12,000 \\ y+4z &= 31,000 \\ -y+z &= 4,000 \end{aligned}$$

Step 3. Add equation (2) to equation (3) and write the result as equation (3).

$$\begin{aligned} x+y+z &= 12,000 \\ y+4z &= 31,000 \\ 5z &= 35,000 \end{aligned}$$

Step 4. Solve for z in equation (3). Back-substitute that value in equation (2) and solve for y. Then, back-substitute the values for z and y into equation (1) and solve for x.

$$\begin{aligned} 5z &= 35,000 \\ z &= 7,000 \end{aligned}$$

$$\begin{aligned} y+4(7,000) &= 31,000 \\ y &= 3,000 \end{aligned}$$

$$\begin{aligned} x+3,000+7,000 &= 12,000 \\ x &= 2,000 \end{aligned}$$

John invested $\$2,000$ in a money-market fund, $\$3,000$ in municipal bonds, and $\$7,000$ in mutual funds.

? Exercise 7.2.1

Solve the system of equations in three variables.

$$\begin{aligned} 2x+y-2z &= -1 \\ 3x-3y-z &= 5 \\ x-2y+3z &= 6 \end{aligned}$$

Answer

$(1,-1,1)$

Identifying Inconsistent Systems of Equations Containing Three Variables

Just as with systems of equations in two variables, we may come across an inconsistent system of equations in three variables, which means that it does not have a solution that satisfies all three equations. The equations could represent three parallel planes, two parallel planes and one intersecting plane, or three planes that intersect the other two but not at the same location. The process of elimination will result in a false statement, such as $3=7$ or some other contradiction.

✓ Example 7.2.4: Solving an Inconsistent System of Three Equations in Three Variables

Solve the following system.

$$x-3y+z=4 \quad (7.2.1)$$
$$-x+2y-5z=3 \quad (7.2.2)$$
$$5x-13y+13z=8 \quad (7.2.3)$$

Solution

Looking at the coefficients of x, we can see that we can eliminate x by adding Equation 7.2.1 to Equation 7.2.2.

$$\begin{aligned} x-3y+z&=4\,(1)\\ -x+2y-5z&=3\,(2)\\ \hline -y-4z&=7\,(4)\end{aligned}$$

Next, we multiply equation (1) by -5 and add it to equation (3).

$$\begin{aligned} -5x+15y-5z&=-20\,(1) \quad \text{multiplied by } -5\\ 5x-13y+13z&=8\,(3)\\ \hline 2y+8z&=-12\,(5)\end{aligned}$$

Then, we multiply equation (4) by 2 and add it to equation (5).

$$\begin{aligned} -2y-8z&=14\,(4) \quad \text{multiplied by } 2\\ 2y+8z&=-12\,(5)\\ \hline 0&=2\end{aligned}$$

The final equation $0=2$ is a contradiction, so we conclude that the system of equations in inconsistent and, therefore, has no solution.

Analysis

In this system, each plane intersects the other two, but not at the same location. Therefore, the system is inconsistent.

? Exercise 7.2.2

Solve the system of three equations in three variables.

$$x+y+z=2$$
$$y-3z=1$$
$$2x+y+5z=0$$

Answer

No solution.

Expressing the Solution of a System of Dependent Equations Containing Three Variables

We know from working with systems of equations in two variables that a dependent system of equations has an infinite number of solutions. The same is true for dependent systems of equations in three variables. An infinite number of solutions can result from several situations. The three planes could be the same, so that a solution to one equation will be the solution to the other two equations. All three equations could be different but they intersect on a line, which has infinite solutions. Or two of the equations could be the same and intersect the third on a line.

✓ Example 7.2.5: Finding the Solution to a Dependent System of Equations

Find the solution to the given system of three equations in three variables.

$$\begin{aligned} 2x+y-3z &= 0 \quad (1)\\ 4x+2y-6z &= 0 \quad (2)\\ x-y+z &= 0 \quad (3) \end{aligned}$$

Solution

First, we can multiply equation (1) by -2 and add it to equation (2).

$$\begin{aligned} -4x-2y+6z &= 0 \ (1) \quad \text{multiplied by } -2\\ \underline{4x+2y-6z} &= \underline{0 \ (2)}\\ 0 &= 0 \end{aligned}$$

We do not need to proceed any further. The result we get is an identity, $0=0$, which tells us that this system has an infinite number of solutions. There are other ways to begin to solve this system, such as multiplying equation (3) by -2, and adding it to equation (1). We then perform the same steps as above and find the same result, $0=0$.

When a system is dependent, we can find general expressions for the solutions. Adding equations (1) and (3), we have

$$\begin{aligned} 2x+y-3z &= 0\\ x-y+z &= 0\\ 3x-2z &= 0 \end{aligned}$$

We then solve the resulting equation for z.

$$\begin{aligned} 3x-2z &= 0\\ z &= \frac{3}{2}x \end{aligned}$$

We back-substitute the expression for z into one of the equations and solve for y.

$$\begin{aligned} 2x+y-3\left(\frac{3}{2}x\right) &= 0\\ 2x+y-\frac{9}{2}x &= 0\\ y &= \frac{9}{2}x-2x\\ y &= \frac{5}{2}x \end{aligned}$$

So the general solution is $\left(x, \frac{5}{2}x, \frac{3}{2}x\right)$. In this solution, x can be any real number. The values of y and z are dependent on the value selected for x.

Analysis

As shown in Figure 7.2.5, two of the planes are the same and they intersect the third plane on a line. The solution set is infinite, as all points along the intersection line will satisfy all three equations.

***Figure* 7.2.5**

Q&A: Does the generic solution to a dependent system always have to be written in terms of x?

No, you can write the generic solution in terms of any of the variables, but it is common to write it in terms of x and if needed x and y.

Exercise 7.2.3:

Solve the following system.

$$\begin{aligned} x+y+z &= 7 \\ 3x-2y-z &= 4 \\ x+6y+5z &= 24 \end{aligned}$$

Answer

Infinite number of solutions of the form $(x, 4x-11, -5x+18)$.

Media

Access these online resources for additional instruction and practice with systems of equations in three variables.

- Ex 1: System of Three Equations with Three Unknowns Using Elimination
- Ex. 2: System of Three Equations with Three Unknowns Using Elimination

Key Concepts

- A solution set is an ordered triple {(x,y,z)} that represents the intersection of three planes in space. See Example 7.2.1.
- A system of three equations in three variables can be solved by using a series of steps that forces a variable to be eliminated. The steps include interchanging the order of equations, multiplying both sides of an equation by a nonzero constant, and adding a nonzero multiple of one equation to another equation. See Example 7.2.2.
- Systems of three equations in three variables are useful for solving many different types of real-world problems. See Example 7.2.3.
- A system of equations in three variables is inconsistent if no solution exists. After performing elimination operations, the result is a contradiction. See Example 7.2.4.
- Systems of equations in three variables that are inconsistent could result from three parallel planes, two parallel planes and one intersecting plane, or three planes that intersect the other two but not at the same location.
- A system of equations in three variables is dependent if it has an infinite number of solutions. After performing elimination operations, the result is an identity. See Example 7.2.5.
- Systems of equations in three variables that are dependent could result from three identical planes, three planes intersecting at a line, or two identical planes that intersect the third on a line.

Contributors

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7.3: Partial Fractions

Learning Objectives

Decompose $\frac{P(x)}{Q(x)}$, where

- $Q(x)$ has only nonrepeated linear factors.
- $Q(x)$ has repeated linear factors.
- $Q(x)$ has a nonrepeated irreducible quadratic factor.
- $Q(x)$ has a repeated irreducible quadratic factor.

Earlier in this chapter, we studied systems of two equations in two variables, systems of three equations in three variables, and nonlinear systems. Here we introduce another way that systems of equations can be utilized—the decomposition of rational expressions. Fractions can be complicated; adding a variable in the denominator makes them even more so. The methods studied in this section will help simplify the concept of a rational expression.

Decomposing $\frac{P(x)}{Q(x)}$ where $Q(x)$ Has Only Nonrepeated Linear Factors

Recall the algebra regarding adding and subtracting rational expressions. These operations depend on finding a common denominator so that we can write the sum or difference as a single, simplified rational expression. In this section, we will look at **partial fraction decomposition**, which is the undoing of the procedure to add or subtract rational expressions. In other words, it is a return from the single simplified **rational expression** to the original expressions, called the **partial fractions**.

For example, suppose we add the following fractions:

$$\frac{2}{x-3}+\frac{-1}{x+2}$$

We would first need to find a common denominator: $(x+2)(x-3)$.

Next, we would write each expression with this common denominator and find the sum of the terms.

$$\begin{aligned}\frac{2}{x-3}\left(\frac{x+2}{x+2}\right)+\frac{-1}{x+2}\left(\frac{x-3}{x-3}\right) &= \frac{2x+4-x+3}{(x+2)(x-3)}\\ &= \frac{x+7}{x^2-x-6}\end{aligned}$$

Partial fraction **decomposition** is the reverse of this procedure. We would start with the solution and rewrite (decompose) it as the sum of two fractions.

$$\underbrace{\frac{x+7}{x^2-x-6}}_{\text{Simplified sum}}=\underbrace{\frac{2}{x-3}+\frac{-1}{x+2}}_{\text{Partial fraction decomposition}}$$

We will investigate rational expressions with linear factors and quadratic factors in the denominator where the degree of the numerator is less than the degree of the denominator. Regardless of the type of expression we are decomposing, the first and most important thing to do is factor the denominator.

When the denominator of the simplified expression contains distinct linear factors, it is likely that each of the original rational expressions, which were added or subtracted, had one of the linear factors as the denominator. In other words, using the example above, the factors of x^2-x-6 are $(x-3)(x+2)$, the denominators of the decomposed rational expression. So we will rewrite the simplified form as the sum of individual fractions and use a variable for each numerator. Then, we will solve for each numerator using one of several methods available for partial fraction decomposition.

PARTIAL FRACTION DECOMPOSITION OF $\frac{P(x)}{Q(x)}$: $Q(x)$ HAS NONREPEATED LINEAR FACTORS

The *partial fraction decomposition* of $\dfrac{P(x)}{Q(x)}$ when $Q(x)$ has nonrepeated linear factors and the degree of $P(x)$ is less than the degree of $Q(x)$ is

$$\frac{P(x)}{Q(x)}=\frac{A_1}{(a_1x+b_1)}+\frac{A_2}{(a_2x+b_2)}+\frac{A_3}{(a_3x+b_3)}+\cdots+\frac{A_n}{(a_nx+b_n)} \tag{7.3.1}$$

How to: Given a rational expression with distinct linear factors in the denominator, decompose it

1. Use a variable for the original numerators, usually A, B, or C, depending on the number of factors, placing each variable over a single factor. For the purpose of this definition, we use A_n for each numerator
$$\frac{P(x)}{Q(x)}=\frac{A_1}{(a_1x+b_1)}+\frac{A_2}{(a_2x+b_2)}+\frac{A_3}{(a_3x+b_3)}+\cdots+\frac{A_n}{(a_nx+b_n)}$$
2. Multiply both sides of the equation by the common denominator to eliminate fractions.
3. Expand the right side of the equation and collect like terms.
4. Set coefficients of like terms from the left side of the equation equal to those on the right side to create a system of equations to solve for the numerators.

Example 7.3.1: Decomposing a Rational Function with Distinct Linear Factors

Decompose the given rational expression with distinct linear factors.

$$\frac{3x}{(x+2)(x-1)}$$

Solution

We will separate the denominator factors and give each numerator a symbolic label, like A, B ,or C.

$$\frac{3x}{(x+2)(x-1)}=\frac{A}{(x+2)}+\frac{B}{(x-1)}$$

Multiply both sides of the equation by the common denominator to eliminate the fractions:

$$(x+2)(x-1)\left[\frac{3x}{(x+2)(x-1)}\right]=(x+2)(x-1)\left[\frac{A}{(x+2)}\right]+(x+2)(x-1)\left[\frac{B}{(x-1)}\right]$$

The resulting equation is

$$3x=A(x-1)+B(x+2)$$

Expand the right side of the equation and collect like terms.

$$\begin{aligned}3x&=Ax-A+Bx+2B\\3x&=(A+B)x-A+2B\end{aligned}$$

Set up a system of equations associating corresponding coefficients.

$$\begin{aligned}3&=A+B\\0&=-A+2B\end{aligned}$$

Add the two equations and solve for B.

$$\begin{aligned}3&=A+B\\\underline{0}&=\underline{-A+2B}\\3&=0+3B\\1&=B\end{aligned}$$

Substitute $B=1$ into one of the original equations in the system.

$$\begin{aligned} 3 &= A+1 \\ 2 &= A \end{aligned}$$

Thus, the partial fraction decomposition is

$$\frac{3x}{(x+2)(x-1)} = \frac{2}{(x+2)} + \frac{1}{(x-1)}$$

Another method to use to solve for A or B is by considering the equation that resulted from eliminating the fractions and substituting a value for x that will make either the $A-$ or $B-$term equal 0. If we let $x=1$, the

$A-$ term becomes 0 and we can simply solve for B.

$$\begin{aligned} 3x &= A(x-1)+B(x+2) \\ 3(1) &= A[(1)-1]+B[(1)+2] \\ 3 &= 0+3B \\ 1 &= B \end{aligned}$$

Next, either substitute $B=1$ into the equation and solve for A, or make the $B-$term 0 by substituting $x=-2$ into the equation.

$$\begin{aligned} 3x &= A(x-1)+B(x+2) \\ 3(-2) &= A[(-2)-1]+B[(-2)+2] \\ -6 &= -3A+0 \\ \frac{-6}{-3} &= A \\ 2 &= A \end{aligned}$$

We obtain the same values for A and B using either method, so the decompositions are the same using either method.

$$\frac{3x}{(x+2)(x-1)} = \frac{2}{(x+2)} + \frac{1}{(x-1)}$$

Although this method is not seen very often in textbooks, we present it here as an alternative that may make some partial fraction decompositions easier. It is known as the **Heaviside method**, named after Charles Heaviside, a pioneer in the study of electronics.

? Exercise 7.3.1

Find the partial fraction decomposition of the following expression.

$$\frac{x}{(x-3)(x-2)}$$

Answer

$$\frac{3}{x-3} - \frac{2}{x-2}$$

Decomposing $\frac{P(x)}{Q(x)}$ where $Q(x)$ Has Repeated Linear Factors

Some fractions we may come across are special cases that we can decompose into partial fractions with repeated linear factors. We must remember that we account for repeated factors by writing each factor in increasing powers.

PARTIAL FRACTION DECOMPOSITION OF $\frac{P(x)}{Q(x)}$: $Q(x)$ HAS REPEATED LINEAR FACTORS

The partial fraction decomposition of $\frac{P(x)}{Q(x)}$, when $Q(x)$ has a repeated linear factor occurringn n times and the degree of $P(x)$ is less than the degree of $Q(x)$, is

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(a_1x+b_1)} + \frac{A_2}{(a_2x+b_2)} + \frac{A_3}{(a_3x+b_3)} + \cdots + \frac{A_n}{(a_nx+b_n)} \tag{7.3.2}$$

Write the denominator powers in increasing order.

How to: decompose a rational expression with repeated linear factors

1. Use a variable like A, B, or C for the numerators and account for increasing powers of the denominators.

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(a_1x+b_1)} + \frac{A_2}{(a_2x+b_2)} + \frac{A_3}{(a_3x+b_3)} + \cdots + \frac{A_n}{(a_nx+b_n)} \tag{7.3.3}$$

2. Multiply both sides of the equation by the common denominator to eliminate fractions.
3. Expand the right side of the equation and collect like terms.
4. Set coefficients of like terms from the left side of the equation equal to those on the right side to create a system of equations to solve for the numerators.

Example 7.3.2: Decomposing with Repeated Linear Factors

Decompose the given rational expression with repeated linear factors.

$$\frac{-x^2+2x+4}{x^3-4x^2+4x}$$

Solution

The denominator factors are $x(x-2)^2$. To allow for the repeated factor of $(x-2)$, the decomposition will include three denominators: x, $(x-2)$, and $(x-2)^2$. Thus,

$$\frac{-x^2+2x+4}{x^3-4x^2+4x} = \frac{A}{x} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2}$$

Next, we multiply both sides by the common denominator.

$$x(x-2)^2\left[\frac{-x^2+2x+4}{x(x-2)^2}\right] = \left[\frac{A}{x} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2}\right] x(x-2)^2$$

$$-x^2+2x+4 = A(x-2)^2 + Bx(x-2) + Cx$$

On the right side of the equation, we expand and collect like terms.

$$\begin{aligned} -x^2+2x+4 &= A(x^2-4x+4) + B(x^2-2x) + Cx \\ &= Ax^2 - 4Ax + 4A + Bx^2 - 2Bx + Cx \\ &= (A+B)x^2 + (-4A-2B+C)x + 4A \end{aligned}$$

Next, we compare the coefficients of both sides. This will give the system of equations in three variables:

$$-x^2+2x+4 = (A+B)x^2 + (-4A-2B+C)x + 4A \tag{7.3.4}$$

$$A+B = -1 \tag{7.3.5}$$

$$-4A-2B+C = 2 \tag{7.3.6}$$

$$4A = 4 \tag{7.3.7}$$

Solving for A in Equation 7.3.7, we have

$$\begin{aligned} 4A &= 4 \\ A &= 1 \end{aligned}$$

Substitute $A = 1$ into Equation 7.3.5.

$$\begin{aligned} A + B &= -1 \\ (1) + B &= -1 \\ B &= -2 \end{aligned}$$

Then, to solve for C, substitute the values for A and B into Equation 7.3.6.

$$\begin{aligned} -4A - 2B + C &= 2 \\ -4(1) - 2(-2) + C &= 2 \\ -4 + 4 + C &= 2 \\ C &= 2 \end{aligned}$$

Thus,

$$\frac{-x^2 + 2x + 4}{x^3 - 4x^2 + 4x} = \frac{1}{x} - \frac{2}{(x-2)} + \frac{2}{(x-2)^2}$$

? Exercise 7.3.2

Find the partial fraction decomposition of the expression with repeated linear factors.

$$\frac{6x - 11}{(x-1)^2}$$

Answer

$$\frac{6}{x-1} - \frac{5}{(x-1)^2}$$

Decomposing $\frac{P(x)}{Q(x)}$, where $Q(x)$ Has a Nonrepeated Irreducible Quadratic Factor

So far, we have performed partial fraction decomposition with expressions that have had linear factors in the denominator, and we applied numerators A, B, or C representing constants. Now we will look at an example where one of the factors in the denominator is a quadratic expression that does not factor. This is referred to as an irreducible quadratic factor. In cases like this, we use a linear numerator such as $Ax + B$, $Bx + C$, etc.

DECOMPOSITION OF $\frac{P(x)}{Q(x)}$: $Q(x)$ HAS A NONREPEATED IRREDUCIBLE QUADRATIC FACTOR

The partial fraction decomposition of $\dfrac{P(x)}{Q(x)}$ such that $Q(x)$ has a nonrepeated irreducible quadratic factor and the degree of $P(x)$ is less than the degree of $Q(x)$ is written as

$$\frac{P(x)}{Q(x)} = \frac{A_1 x + B_1}{(a_1 x^2 + b1_x + c_1)} + \frac{A_2 x + B_2}{(a_2 x^2 + b_2 x + c_2)} + \cdots + \frac{A_n x + B_n}{(a_n x^2 + b_n x + c_n)} \tag{7.3.8}$$

The decomposition may contain more rational expressions if there are linear factors. Each linear factor will have a different constant numerator: A, B, C, and so on.

Howto: decompose a rational expression where the factors of the denominator are distinct, irreducible quadratic factors

1. Use variables such as A, B, or C for the constant numerators over linear factors, and linear expressions such as A_1x+B_1, A_2x+B_2, etc., for the numerators of each quadratic factor in the denominator.

$$\frac{P(x)}{Q(x)}=\frac{A}{ax+b}+\frac{A_1x+B_1}{(a_1x^2+b1_x+c_1)}+\frac{A_2x+B_2}{(a_2x^2+b_2x+c_2)}+\cdots+\frac{A_nx+B_n}{(a_nx^2+b_nx+c_n)}$$

2. Multiply both sides of the equation by the common denominator to eliminate fractions.
3. Expand the right side of the equation and collect like terms.
4. Set coefficients of like terms from the left side of the equation equal to those on the right side to create a system of equations to solve for the numerators.

Example 7.3.3: Decomposing $\frac{P(x)}{Q(x)}$ When $Q(x)$ Contains a Nonrepeated Irreducible Quadratic Factor

Find a partial fraction decomposition of the given expression.

$$\frac{8x^2+12x-20}{(x+3)(x^2+x+2)}$$

Solution

We have one linear factor and one irreducible quadratic factor in the denominator, so one numerator will be a constant and the other numerator will be a linear expression. Thus,

$$\frac{8x^2+12x-20}{(x+3)(x^2+x+2)}=\frac{A}{(x+3)}+\frac{Bx+C}{(x^2+x+2)}$$

We follow the same steps as in previous problems. First, clear the fractions by multiplying both sides of the equation by the common denominator.

$$\begin{aligned}(x+3)(x^2+x+2)\left[\frac{8x^2+12x-20}{(x+3)(x^2+x+2)}\right] &= \left[\frac{A}{(x+3)}+\frac{Bx+C}{(x^2+x+2)}\right](x+3)(x^2+x+2)\\ 8x^2+12x-20 &= A(x^2+x+2)+(Bx+C)(x+3)\end{aligned}$$

Notice we could easily solve for A by choosing a value for x that will make the $Bx+C$ term equal 0. Let $x=-3$ and substitute it into the equation.

$$\begin{aligned}8x^2+12x-20 &= A(x^2+x+2)+(Bx+C)(x+3)\\ 8(-3)^2+12(-3)-20 &= A((-3)^2+(-3)+2)+(B(-3)+C)((-3)+3)\\ 16 &= 8A\\ A &= 2\end{aligned}$$

Now that we know the value of A, substitute it back into the equation. Then expand the right side and collect like terms.

$$\begin{aligned}8x^2+12x-20 &= 2(x^2+x+2)+(Bx+C)(x+3)\\ 8x^2+12x-20 &= 2x^2+2x+4+Bx^2+3B+Cx+3C\\ 8x^2+12x-20 &= (2+B)x^2+(2+3B+C)x+(4+3C)\end{aligned}$$

Setting the coefficients of terms on the right side equal to the coefficients of terms on the left side gives the system of equations.

$$2+B=8 \tag{7.3.9}$$

$$2+3B+C=12 \tag{7.3.10}$$

$$4+3C=-20 \tag{7.3.11}$$

Solve for B using Equation 7.3.9

\[

$$2+B=8$$

$$B=6$$

and solve for C using Equation 7.3.11.

$$4+3C=-20$$

$$3C=-24$$

$$C=-8$$

Thus, the partial fraction decomposition of the expression is

$$\frac{8x^2+12x-20}{(x+3)(x^2+x+2)}=\frac{2}{(x+3)}+\frac{6x-8}{(x^2+x+2)}$$

Q&A: Could we have just set up a system of equations to solve the example above?

Yes, we could have solved it by setting up a system of equations without solving for A first. The expansion on the right would be:

$$8x^2+12x-20 = Ax^2+Ax+2A+Bx^2+3B+Cx+3C$$

$$8x^2+12x-20 = (A+B)x^2+(A+3B+C)x+(2A+3C)$$

So the system of equations would be:

$$A+B=8$$

$$A+3B+C=12$$

$$2A+3C=-20$$

Exercise 7.3.3

Find the partial fraction decomposition of the expression with a nonrepeating irreducible quadratic factor.

$$\frac{5x^2-6x+7}{(x-1)(x^2+1)}$$

Answer

$$\frac{3}{x-1}+\frac{2x-4}{x^2+1}$$

Decomposing $\frac{P(x)}{Q(x)}$ When $Q(x)$ Has a Repeated Irreducible Quadratic Factor

Now that we can decompose a simplified rational expression with an irreducible quadratic factor, we will learn how to do partial fraction decomposition when the simplified rational expression has repeated irreducible quadratic factors. The decomposition will consist of partial fractions with linear numerators over each irreducible quadratic factor represented in increasing powers.

DECOMPOSITION OF $\frac{P(x)}{Q(x)}$ WHEN $Q(X)$ HAS A REPEATED IRREDUCIBLE QUADRATIC FACTOR

The partial fraction decomposition of $\frac{P(x)}{Q(x)}$, when $Q(x)$ has a repeated irreducible quadratic factor and the degree of $P(x)$ is less than the degree of $Q(x)$, is

$$\frac{P(x)}{(ax^2+bx+c)^n}=\frac{A_1x+B_1}{(ax^2+bx+c)}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\frac{A_3x+B_3}{(ax^2+bx+c)^3}+\cdots+\frac{A_nx+B_n}{(ax^2+bx+c)^n} \quad (7.3.12)$$

Write the denominators in increasing powers.

How to: decompose a rational expression that has a repeated irreducible factor

1. Use variables like A, B, or C for the constant numerators over linear factors, and linear expressions such as A_1x+B_1, A_2x+B_2, etc., for the numerators of each quadratic factor in the denominator written in increasing powers, such as
$$\frac{P(x)}{Q(x)}=\frac{A}{ax+b}+\frac{A_1x+B_1}{(ax^2+bx+c)}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_nx+B_n}{(ax^2+bx+c)^n}$$
2. Multiply both sides of the equation by the common denominator to eliminate fractions.
3. Expand the right side of the equation and collect like terms.
4. Set coefficients of like terms from the left side of the equation equal to those on the right side to create a system of equations to solve for the numerators.

Example 7.3.4: Decomposing a Rational Function with a Repeated Irreducible Quadratic Factor in the Denominator

Decompose the given expression that has a repeated irreducible factor in the denominator.
$$\frac{x^4+x^3+x^2-x+1}{x(x^2+1)^2}$$

Solution

The factors of the denominator are x, (x^2+1), and $(x^2+1)^2$. Recall that, when a factor in the denominator is a quadratic that includes at least two terms, the numerator must be of the linear form $Ax+B$. So, let's begin the decomposition.
$$\frac{x^4+x^3+x^2-x+1}{x(x^2+1)^2}=\frac{A}{x}+\frac{Bx+C}{(x^2+1)}+\frac{Dx+E}{(x^2+1)^2}$$

We eliminate the denominators by multiplying each term by $x(x^2+1)^2$. Thus,
$$\begin{aligned} x^4+x^3+x^2-x+1 &= A(x^2+1)^2+(Bx+C)(x)(x^2+1)+(Dx+E)(x) \\ x^4+x^3+x^2-x+1 &= A(x^4+2x^2+1)+Bx^4+Bx^2+Cx^3+Cx+Dx^2+Ex \qquad \text{Expand the right side.} \\ &= Ax^4+2Ax^2+A+Bx^4+Bx^2+Cx^3+Cx+Dx^2+Ex \end{aligned}$$

Now we will collect like terms.
$$x^4+x^3+x^2-x+1=(A+B)x^4+(C)x^3+(2A+B+D)x^2+(C+E)x+A$$

Set up the system of equations matching corresponding coefficients on each side of the equal sign.
$$\begin{aligned} A+B &= 1 \\ C &= 1 \\ 2A+B+D &= 1 \\ C+E &= -1 \\ A &= 1 \end{aligned}$$

We can use substitution from this point. Substitute $A=1$ into the first equation.
$$\begin{aligned} 1+B &= 1 \\ B &= 0 \end{aligned}$$

Substitute $A=1$ and $B=0$ into the third equation.
$$\begin{aligned} 2(1)+0+D &= 1 \\ D &= -1 \end{aligned}$$

Substitute $C = 1$ into the fourth equation.

$$1 + E = -1$$
$$E = -2$$

Now we have solved for all of the unknowns on the right side of the equal sign. We have $A = 1$, $B = 0$, $C = 1$, $D = -1$, and $E = -2$. We can write the decomposition as follows:

$$\frac{x^4 + x^3 + x^2 - x + 1}{x(x^2+1)^2} = \frac{1}{x} + \frac{1}{(x^2+1)} - \frac{x+2}{(x^2+1)^2}$$

Exercise 7.3.4

Find the partial fraction decomposition of the expression with a repeated irreducible quadratic factor.

$$\frac{x^3 - 4x^2 + 9x - 5}{(x^2 - 2x + 3)^2}$$

Answer

$$\frac{x-2}{x^2-2x+3} + \frac{2x+1}{(x^2-2x+3)^2}$$

Media

Access these online resources for additional instruction and practice with partial fractions.

- Partial Fraction Decomposition
- Partial Fraction Decomposition With Repeated Linear Factors
- Partial Fraction Decomposition With Linear and Quadratic Factors

Key Concepts

- Decompose $\frac{P(x)}{Q(x)}$ by writing the partial fractions as

  $$\frac{A}{a_1 x + b_1} + \frac{B}{a_2 x + b_2}.$$

  Solve by clearing the fractions, expanding the right side, collecting like terms, and setting corresponding coefficients equal to each other, then setting up and solving a system of equations (see Example 7.3.1).
- The decomposition of $\frac{P(x)}{Q(x)}$ with repeated linear factors must account for the factors of the denominator in increasing powers (see Example 7.3.2).
- The decomposition of $\frac{P(x)}{Q(x)}$ with a nonrepeated irreducible quadratic factor needs a linear numerator over the quadratic factor, as in $\frac{A}{x} + \frac{Bx + C}{(ax^2 + bx + c)}$ (see Example 7.3.3).
- In the decomposition of $\frac{P(x)}{Q(x)}$, where $Q(x)$ has a repeated irreducible quadratic factor, when the irreducible quadratic factors are repeated, powers of the denominator factors must be represented in increasing powers as

  $$\frac{A_1 x + B_1}{ax^2 + bx + c} + \frac{A_2 x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_n x + B_n}{(ax^2 + bx + c)^n}$$

  See Example 7.3.4.

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7.4: Solve Systems of Equations Using Matrices

Summary

By the end of this section, you will be able to:

- Write the augmented matrix for a system of equations
- Use row operations on a matrix
- Solve systems of equations using matrices

Before you get started, take this readiness quiz.

1. Solve: $3(x+2)+4=4(2x-1)+9$.
   If you missed this problem, review [link].
2. Solve: $0.25p+0.25(x+4)=5.20$.
   If you missed this problem, review [link].
3. Evaluate when $x=-2$ and $y=3$: $2x^2-xy+3y^2$.
   If you missed this problem, review [link].

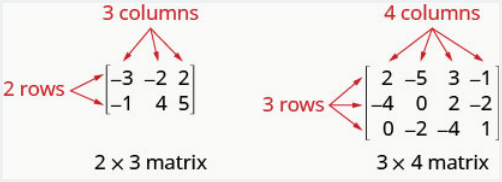
Write the Augmented Matrix for a System of Equations

Solving a system of equations can be a tedious operation where a simple mistake can wreak havoc on finding the solution. An alternative method which uses the basic procedures of elimination but with notation that is simpler is available. The method involves using a **matrix**. A matrix is a rectangular array of numbers arranged in rows and columns.

MATRIX

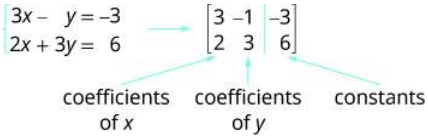
A **matrix** is a rectangular array of numbers arranged in rows and columns.

A matrix with *m* rows and *n* columns has order $m \times n$. The matrix on the left below has 2 rows and 3 columns and so it has order 2×3. We say it is a 2 by 3 matrix.



Each number in the matrix is called an element or entry in the matrix.

We will use a matrix to represent a system of linear equations. We write each equation in standard form and the coefficients of the variables and the constant of each equation becomes a row in the matrix. Each column then would be the coefficients of one of the variables in the system or the constants. A vertical line replaces the equal signs. We call the resulting matrix the augmented matrix for the system of equations.



Notice the first column is made up of all the coefficients of *x*, the second column is the all the coefficients of *y*, and the third column is all the constants.

? Example 7.4.1

ⓐ $\begin{cases} 5x-3y=-1 \\ y=2x-2 \end{cases}$ ⓑ $\begin{cases} 6x-5y+2z=3 \\ 2x+y-4z=5 \\ 3x-3y+z=-1 \end{cases}$

Answer

ⓐ The second equation is not in standard form. We rewrite the second equation in standard form.

$$y=2x-2$$
$$-2x+y=-2$$

We replace the second equation with its standard form. In the augmented matrix, the first equation gives us the first row and the second equation gives us the second row. The vertical line replaces the equal signs.

$$\begin{array}{l} \\ 5x-3y=-1 \\ 2x-\ y=\ 2 \end{array} \qquad \begin{array}{c} x\ \ y \\ \left[\begin{array}{cc|c} 5 & -3 & -1 \\ 2 & -1 & 2 \end{array}\right] \end{array}$$

ⓑ All three equations are in standard form. In the augmented matrix the first equation gives us the first row, the second equation gives us the second row, and the third equation gives us the third row. The vertical line replaces the equal signs.

$$\begin{array}{l} \\ 6x-5y+2z=\ 3 \\ 2x+\ y-4z=\ 5 \\ 3x-3y+\ z=-1 \end{array} \qquad \begin{array}{c} x\ \ y\ \ z \\ \left[\begin{array}{ccc|c} 6 & -5 & 2 & 3 \\ 2 & 1 & -4 & 5 \\ 3 & -3 & 1 & -1 \end{array}\right] \end{array}$$

? Example 7.4.2

Write each system of linear equations as an augmented matrix:

ⓐ $\begin{cases} 3x+8y=-3 \\ 2x=-5y-3 \end{cases}$ ⓑ $\begin{cases} 2x-5y+3z=8 \\ 3x-y+4z=7 \\ x+3y+2z=-3 \end{cases}$

Answer

ⓐ $\left[\begin{array}{ccc} 3 & 8 & -3 \\ 2 & 5 & -3 \end{array}\right]$

ⓑ $\left[\begin{array}{cccc} 2 & 3 & 1 & -5 \\ -1 & 3 & 3 & 4 \\ 2 & 8 & 7 & -3 \end{array}\right]$

? Example 7.4.3

Write each system of linear equations as an augmented matrix:

ⓐ $\begin{cases} 11x=-9y-5 \\ 7x+5y=-1 \end{cases}$ ⓑ $\begin{cases} 5x-3y+2z=-5 \\ 2x-y-z=4 \\ 3x-2y+2z=-7 \end{cases}$

Answer

ⓐ $\begin{bmatrix} 11 & 9 & -5 \\ 7 & 5 & -1 \end{bmatrix}$

ⓑ $\begin{bmatrix} 5 & -3 & 2 & -5 \\ 2 & -1 & -1 & 4 \\ 3 & -2 & 2 & -7 \end{bmatrix}$

It is important as we solve systems of equations using matrices to be able to go back and forth between the system and the matrix. The next example asks us to take the information in the matrix and write the system of equations.

Example 7.4.4

Write the system of equations that corresponds to the augmented matrix:

$$\left[\begin{array}{ccc|c} 4 & -3 & 3 & -1 \\ 1 & 2 & -1 & 2 \\ -2 & -1 & 3 & -4 \end{array}\right].$$

Answer

We remember that each row corresponds to an equation and that each entry is a coefficient of a variable or the constant. The vertical line replaces the equal sign. Since this matrix is a 4×3, we know it will translate into a system of three equations with three variables.

$$\begin{array}{cccc} x & y & z & \\ \left[\begin{array}{ccc|c} 4 & -3 & 3 & -1 \\ 1 & 2 & -1 & 2 \\ -2 & -1 & 3 & -4 \end{array}\right] & \longrightarrow & \begin{array}{r} 4x-3y+3z=-1 \\ x+2y-z=2 \\ -2x-y+3z=-4 \end{array} \end{array}$$

Example 7.4.5

Write the system of equations that corresponds to the augmented matrix: $\begin{bmatrix} 1 & -1 & 2 & 3 \\ 2 & 1 & -2 & 1 \\ 4 & -1 & 2 & 0 \end{bmatrix}$.

Answer

$$\begin{cases} x-y+2z=3 \\ 2x+y-2z=1 \\ 4x-y+2z=0 \end{cases}$$

Example 7.4.6

Write the system of equations that corresponds to the augmented matrix: $\begin{bmatrix} 1 & 1 & 1 & 4 \\ 2 & 3 & -1 & 8 \\ 1 & 1 & -1 & 3 \end{bmatrix}$.

Answer

$$\begin{cases} x+y+z=4 \\ 2x+3y-z=8 \\ x+y-z=3 \end{cases}$$

Use Row Operations on a Matrix

Once a system of equations is in its augmented matrix form, we will perform operations on the rows that will lead us to the solution.

To solve by elimination, it doesn't matter which order we place the equations in the system. Similarly, in the matrix we can interchange the rows.

When we solve by elimination, we often multiply one of the equations by a constant. Since each row represents an equation, and we can multiply each side of an equation by a constant, similarly we can multiply each entry in a row by any real number except 0.

In elimination, we often add a multiple of one row to another row. In the matrix we can replace a row with its sum with a multiple of another row.

These actions are called row operations and will help us use the matrix to solve a system of equations.

ROW OPERATIONS

In a matrix, the following operations can be performed on any row and the resulting matrix will be equivalent to the original matrix.

1. Interchange any two rows.
2. Multiply a row by any real number except 0.
3. Add a nonzero multiple of one row to another row.

Performing these operations is easy to do but all the arithmetic can result in a mistake. If we use a system to record the row operation in each step, it is much easier to go back and check our work.

We use capital letters with subscripts to represent each row. We then show the operation to the left of the new matrix. To show interchanging a row:

$$\left[\begin{array}{cc|c} 5 & -3 & -1 \\ 2 & -1 & 2 \end{array}\right] \xrightarrow{\substack{R_2 \\ R_1}} \left[\begin{array}{cc|c} 2 & -1 & 2 \\ 5 & -3 & -1 \end{array}\right]$$

To multiply row 2 by -3:

$$\left[\begin{array}{cc|c} 5 & -3 & -1 \\ 2 & -1 & 2 \end{array}\right] \xrightarrow{-3R_2} \left[\begin{array}{cc|c} 5 & -3 & -1 \\ -6 & 3 & -6 \end{array}\right]$$

To multiply row 2 by -3 and add it to row 1:

$$\left[\begin{array}{cc|c} 5 & -3 & -1 \\ 2 & -1 & 2 \end{array}\right] \xrightarrow{-3R_2 + R_1} \left[\begin{array}{cc|c} -1 & 0 & -7 \\ 2 & -1 & 2 \end{array}\right]$$

Example 7.4.7

Perform the indicated operations on the augmented matrix:

ⓐ Interchange rows 2 and 3.

ⓑ Multiply row 2 by 5.

ⓒ Multiply row 3 by $-2-2$ and add to row 1.

$$\left[\begin{array}{ccc|c} 6 & -5 & 2 & 3 \\ 2 & 1 & -4 & 5 \\ 3 & -3 & 1 & -1 \end{array}\right]$$

Answer

ⓐ We interchange rows 2 and 3.

$$\left[\begin{array}{ccc|c} 6 & -5 & 2 & 3 \\ 2 & 1 & -4 & 5 \\ 3 & -3 & 1 & -1 \end{array}\right] \xrightarrow{\substack{R_3 \\ R_2}} \left[\begin{array}{ccc|c} 6 & -5 & 2 & 3 \\ 3 & -3 & 1 & -1 \\ 2 & 1 & -4 & 5 \end{array}\right]$$

ⓑ We multiply row 2 by 5.

$$\left[\begin{array}{rrr|r} 6 & -5 & 2 & 3 \\ 2 & 1 & -4 & 5 \\ 3 & -3 & 1 & -1 \end{array}\right] \xrightarrow{5R_2} \left[\begin{array}{rrr|r} 6 & -5 & 2 & 3 \\ 10 & 5 & -20 & 25 \\ 3 & -3 & 1 & -1 \end{array}\right]$$

ⓒ We multiply row 3 by -2 and add to row 1.

$$\left[\begin{array}{rrr|r} 6 & -5 & 2 & 3 \\ 2 & 1 & -4 & 5 \\ 3 & -3 & 1 & -1 \end{array}\right] \xrightarrow{-2R_3+R_1} \left[\begin{array}{rrrr} (-2\cdot 3)+6 & (-2\cdot(-3))+(-5) & (-2\cdot 1)+2 & (-2\cdot(-1))+3 \\ 2 & 1 & -4 & 5 \\ 3 & -3 & 1 & -1 \end{array}\right]$$

$$\xrightarrow{-2R_3+R_1} \left[\begin{array}{rrr|r} 0 & 1 & 0 & 5 \\ 2 & 1 & -4 & 5 \\ 3 & -3 & 1 & -1 \end{array}\right]$$

Example 7.4.8

Perform the indicated operations on the augmented matrix:

ⓐ Interchange rows 1 and 3.

ⓑ Multiply row 3 by 3.

ⓒ Multiply row 3 by 2 and add to row 2.

$$\left[\begin{array}{rrr|r} 5 & -2 & -2 & -2 \\ 4 & -1 & -4 & 4 \\ -2 & 3 & 0 & -1 \end{array}\right]$$

Answer

ⓐ $\left[\begin{array}{rrrr} -2 & 3 & 0 & -2 \\ 4 & -1 & -4 & 4 \\ 5 & -2 & -2 & -2 \end{array}\right]$

ⓑ $\left[\begin{array}{rrrr} -2 & 3 & 0 & -2 \\ 4 & -1 & -4 & 4 \\ 15 & -6 & -6 & -6 \end{array}\right]$

ⓒ $\left[\begin{array}{rrrrr} -2 & 3 & 0 & 2 & \\ 3 & 4 & -13 & -16 & -8 \\ 15 & -6 & -6 & -6 & \end{array}\right]$

Example 7.4.9

Perform the indicated operations on the augmented matrix:

ⓐ Interchange rows 1 and 2,

ⓑ Multiply row 1 by 2,

ⓒ Multiply row 2 by 3 and add to row 1.

$$\left[\begin{array}{rrr|r} 2 & -3 & -2 & -4 \\ 4 & 1 & -3 & 2 \\ 5 & 0 & 4 & -1 \end{array}\right]$$

Answer

ⓐ $\begin{bmatrix} 4 & 1 & -3 & 2 \\ 2 & -3 & -2 & -4 \\ 5 & 0 & 4 & -1 \end{bmatrix}$

ⓑ $\begin{bmatrix} 8 & 2 & -6 & 4 \\ 2 & -3 & -2 & -4 \\ 5 & 0 & 4 & -1 \end{bmatrix}$

ⓒ $\begin{bmatrix} 14 & -7 & -12 & -8 \\ 2 & -3 & -2 & -4 \\ 5 & 0 & 4 & -1 \end{bmatrix}$

Now that we have practiced the row operations, we will look at an augmented matrix and figure out what operation we will use to reach a goal. This is exactly what we did when we did elimination. We decided what number to multiply a row by in order that a variable would be eliminated when we added the rows together.

Given this system, what would you do to eliminate x?

$$\begin{matrix} x-y=2 \\ 4x-8y=0 \end{matrix} \xrightarrow{\text{multiply the first equation by } -4} \begin{matrix} -4x+4y=-8 \\ 4x-8y=0 \end{matrix} \xrightarrow{\text{then add}} \begin{matrix} -4x+4y=-8 \\ 4x-8y=0 \\ \hline -4y=-8 \end{matrix}$$

This next example essentially does the same thing, but to the matrix.

Example 7.4.10

Perform the needed row operation that will get the first entry in row 2 to be zero in the augmented matrix: $\left[\begin{array}{cc|c} 1 & -1 & 2 \\ 4 & -8 & 0 \end{array}\right]$

Answer

To make the 4 a 0, we could multiply row 1 by -4 and then add it to row 2.

$$\begin{bmatrix} 1 & -1 & 2 \\ 4 & -8 & 0 \end{bmatrix} \xrightarrow{-4R_1+R_2} \begin{bmatrix} 1 & -1 & 2 \\ 0 & -4 & -8 \end{bmatrix}$$

Example 7.4.11

Perform the needed row operation that will get the first entry in row 2 to be zero in the augmented matrix: $\left[\begin{array}{cc|c} 1 & -1 & 2 \\ 3 & -6 & 2 \end{array}\right]$

Answer

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & -3 & -4 \end{bmatrix}$$

Example 7.4.12

Perform the needed row operation that will get the first entry in row 2 to be zero in the augmented matrix: $\left[\begin{array}{cc|c} 1 & -1 & 3 \\ -2 & -3 & 2 \end{array}\right]$

Answer

$$\begin{bmatrix} 1 & -1 & 3 \\ 0 & -5 & 8 \end{bmatrix}$$

Solve Systems of Equations Using Matrices

To solve a system of equations using matrices, we transform the augmented matrix into a matrix in **row-echelon form** using row operations. For a consistent and independent system of equations, its **augmented matrix** is in row-echelon form when to the left of the vertical line, each entry on the diagonal is a 1 and all entries below the diagonal are zeros.

ROW-ECHELON FORM

For a consistent and independent system of equations, its augmented matrix is in **row-echelon form** when to the left of the vertical line, each entry on the diagonal is a 1 and all entries below the diagonal are zeros.

$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \end{bmatrix}$ $\begin{bmatrix} 1 & a & b & d \\ 0 & 1 & c & e \\ 0 & 0 & 1 & f \end{bmatrix}$ a, b, c, d, e, f are real numbers

Once we get the augmented matrix into row-echelon form, we can write the equivalent system of equations and read the value of at least one variable. We then substitute this value in another equation to continue to solve for the other variables. This process is illustrated in the next example.

Example 7.4.13: How to Solve a System of Equations Using a Matrix

Solve the system of equations using a matrix: $\begin{cases} 3x+4y=5 \\ x+2y=1 \end{cases}$

Answer

Step 1. Write the augmented matrix for the system of equations.		$\begin{matrix} 3x+4y=5 \\ x+2y=1 \end{matrix}$ $\begin{bmatrix} 3 & 4 & 5 \\ 1 & 2 & 1 \end{bmatrix}$
Step 2. Using row operations get the entry in row 1, column 1 to be 1.	Interchange the rows, so 1 will be in row 1, column 1.	$\begin{matrix} R_2 \\ R_1 \end{matrix}$ $\begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & 5 \end{bmatrix}$
Step 3. Using row operations, get zeros in column 1 below the 1.	Multiply row 1 by –3 and add it to row 2.	$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 2 \end{bmatrix}$ $-3R_1+R_2$
Step 4. Using row operations, get the entry in row 2, column 2 to be 1.	Multiply row 2 by $-\frac{1}{2}$.	$-\frac{1}{2}R_2$ $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$
Step 5. Continue the process until the matrix is in row-echelon form.	The matrix is now in row-echelon form.	$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$
Step 6. Write the corresponding system of equations.	$\begin{matrix} x & y \end{matrix}$ $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$	$\begin{matrix} x+2y=1 \\ y=-1 \end{matrix}$
Step 7. Use substitution to find the remaining variables.	Substitute $y=-1$ into $x+2y=1$.	$\begin{matrix} y=-1 \\ x+2y=1 \\ x+2(-1)=1 \\ x-2=1 \\ x=3 \end{matrix}$

Step 8. Write the solution as an ordered pair or triple.		(3, –1)
Step 9. Check that the solution makes the original equations true.		We leave the check to you.

Example 7.4.14

Solve the system of equations using a matrix: $\begin{cases} 2x+y=7 \\ x-2y=6 \end{cases}$

Answer

The solution is $(4,-1)$.

Example 7.4.15

Solve the system of equations using a matrix: $\begin{cases} 2x+y=-4 \\ x-y=-2 \end{cases}$

Answer

The solution is $(-2,0)$.

The steps are summarized here.

SOLVE A SYSTEM OF EQUATIONS USING MATRICES.

1. Write the augmented matrix for the system of equations.
2. Using row operations get the entry in row 1, column 1 to be 1.
3. Using row operations, get zeros in column 1 below the 1.
4. Using row operations, get the entry in row 2, column 2 to be 1.
5. Continue the process until the matrix is in row-echelon form.
6. Write the corresponding system of equations.
7. Use substitution to find the remaining variables.
8. Write the solution as an ordered pair or triple.
9. Check that the solution makes the original equations true.

Here is a visual to show the order for getting the 1's and 0's in the proper position for row-echelon form.

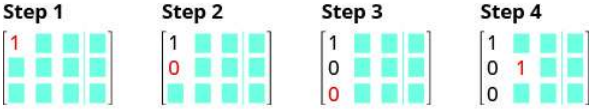
2 × 3 matrix

Step 1 **Step 2** **Step 3**

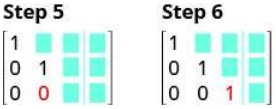


3 × 4 matrix

Step 1 **Step 2** **Step 3** **Step 4**



Step 5 **Step 6**



We use the same procedure when the system of equations has three equations.

Example 7.4.16

Solve the system of equations using a matrix: $\begin{cases} 3x+8y+2z=-5 \\ 2x+5y-3z=0 \\ x+2y-2z=-1 \end{cases}$

Answer

	$\begin{cases} 3x+8y+2z=-5 \\ 2x+5y-3z=0 \\ x+2y-2z=-1 \end{cases}$
Write the augmented matrix for the equations.	$\left[\begin{array}{ccc\|c} 3 & 8 & 2 & -5 \\ 2 & 5 & -3 & 0 \\ 1 & 2 & -2 & -1 \end{array}\right]$
Interchange row 1 and 3 to get the entry in row 1, column 1 to be 1.	$\begin{matrix} R_3 \\ \\ R_1 \end{matrix}\left[\begin{array}{ccc\|c} 1 & 2 & -2 & -1 \\ 2 & 5 & -3 & 0 \\ 3 & 8 & 2 & -5 \end{array}\right]$
Using row operations, get zeros in column 1 below the 1.	$\begin{matrix} \\ -2R_1+R_2 \\ \\ \end{matrix}\left[\begin{array}{ccc\|c} 1 & 2 & -2 & -1 \\ 0 & 1 & 1 & 2 \\ 3 & 8 & 2 & -5 \end{array}\right]$
	$\begin{matrix} \\ \\ -3R_1+R_3 \end{matrix}\left[\begin{array}{ccc\|c} 1 & 2 & -2 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 8 & -2 \end{array}\right]$
The entry in row 2, column 2 is now 1.	
Continue the process until the matrix is in row-echelon form.	$\begin{matrix} \\ \\ -2R_2+R_3 \end{matrix}\left[\begin{array}{ccc\|c} 1 & 2 & -2 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 6 & -6 \end{array}\right]$
	$\begin{matrix} \\ \\ \frac{1}{6}R_3 \end{matrix}\left[\begin{array}{ccc\|c} 1 & 2 & -2 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right]$
The matrix is now in row-echelon form.	$\left[\begin{array}{ccc\|c} 1 & 2 & -2 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right]$
Write the corresponding system of equations.	$\begin{cases} x+2y-2z=-1 \\ y+z=2 \\ z=-1 \end{cases}$
Use substitution to find the remaining variables.	$\begin{aligned} y+z&=2 \\ y+(-1)&=2 \\ y&=3 \end{aligned}$
	$\begin{aligned} x+2y-2z&=-1 \\ x+2(3)-2(-1)&=-1 \\ x+6+2&=-1 \\ x&=-9 \end{aligned}$
Write the solution as an ordered pair or triple.	$(-9, 3, -1)$
Check that the solution makes the original equations true.	We leave the check for you.

Example 7.4.17

Solve the system of equations using a matrix: $\begin{cases} 2x-5y+3z=8 \\ 3x-y+4z=7 \\ x+3y+2z=-3 \end{cases}$

Answer

$(6,-1,-3)$

Example 7.4.18

Solve the system of equations using a matrix: $\begin{cases} -3x+y+z=-4 \\ -x+2y-2z=1 \\ 2x-y-z=-1 \end{cases}$

Answer

$(5,7,4)$

So far our work with matrices has only been with systems that are consistent and independent, which means they have exactly one solution. Let's now look at what happens when we use a matrix for a dependent or inconsistent system.

Example 7.4.19

Solve the system of equations using a matrix: $\begin{cases} x+y+3z=0 \\ x+3y+5z=0 \\ 2x+4z=1 \end{cases}$

Answer

	$\begin{cases} x+y+3z=0 \\ x+3y+5z=0 \\ 2x+4z=1 \end{cases}$
Write the augmented matrix for the equations.	$\left[\begin{array}{ccc\|c} 1 & 1 & 3 & 0 \\ 1 & 3 & 5 & 0 \\ 2 & 0 & 4 & 1 \end{array}\right]$
The entry in row 1, column 1 is 1.	
Using row operations, get zeros in column 1 below the 1.	$\begin{matrix} \\ -1R_1+R_2 \\ \\ \end{matrix}\left[\begin{array}{ccc\|c} 1 & 1 & 3 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 4 & 1 \end{array}\right]$
	$\begin{matrix} \\ \\ -2R_1+R_3 \end{matrix}\left[\begin{array}{ccc\|c} 1 & 1 & 3 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & -2 & -2 & 1 \end{array}\right]$
Continue the process until the matrix is in row-echelon form.	$\begin{matrix} \\ \frac{1}{2}R_2 \\ \\ \end{matrix}\left[\begin{array}{ccc\|c} 1 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -2 & -2 & 1 \end{array}\right]$
Multiply row 2 by 2 and add it to row 3.	$\begin{matrix} \\ \\ 2R_2+R_3 \end{matrix}\left[\begin{array}{ccc\|c} 1 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$
At this point, we have all zeros on the left of row 3.	

Write the corresponding system of equations.	$\begin{cases} x+y+3z=0 \\ y+z=0 \\ 0\neq 1 \end{cases}$

Since $0 \neq 1$ we have a false statement. Just as when we solved a system using other methods, this tells us we have an inconsistent system. There is no solution.

Example 7.4.20

Solve the system of equations using a matrix: $\begin{cases} x-2y+2z=1 \\ -2x+y-z=2 \\ x-y+z=5 \end{cases}$

Answer

no solution

Example 7.4.21

Solve the system of equations using a matrix: $\begin{cases} 3x+4y-3z=-2 \\ -2x+3y-z=-1 \\ 2x+y-2z=6 \end{cases}$

Answer

no solution

The last system was inconsistent and so had no solutions. The next example is dependent and has infinitely many solutions.

Example 7.4.22

Solve the system of equations using a matrix: $\begin{cases} x-2y+3z=1 \\ x+y-3z=7 \\ 3x-4y+5z=7 \end{cases}$

Answer

	$\begin{cases} x-2y+3z=1 \\ x+y-3z=7 \\ 3x-4y+5z=7 \end{cases}$
Write the augmented matrix for the equations.	$\left[\begin{array}{ccc\|c} 1 & -2 & 3 & 1 \\ 1 & 1 & -3 & 7 \\ 3 & -4 & 5 & 7 \end{array}\right]$
The entry in row 1, column 1 is 1.	
Using row operations, get zeros in column 1 below the 1.	$\begin{matrix} \\ -1R_1+R_2 \\ \\ \end{matrix}\left[\begin{array}{ccc\|c} 1 & -2 & 3 & 1 \\ 0 & 3 & -6 & 6 \\ 3 & -4 & 5 & 7 \end{array}\right]$
	$\begin{matrix} \\ \\ -3R_1+R_3 \end{matrix}\left[\begin{array}{ccc\|c} 1 & -2 & 3 & 1 \\ 0 & 3 & -6 & 6 \\ 0 & 2 & -4 & 4 \end{array}\right]$

Continue the process until the matrix is in row-echelon form.	$\frac{1}{3}R_2 \begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -2 & 2 \\ 0 & 2 & -4 & 4 \end{bmatrix}$
Multiply row 2 by -2 and add it to row 3.	$-2R_2+R_3 \begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
At this point, we have all zeros in the bottom row.	
Write the corresponding system of equations.	$\begin{cases} x-2y+3z=1 \\ y-2z=2 \\ 0=0 \end{cases}$
Since $0=0$ we have a true statement. Just as when we solved by substitution, this tells us we have a dependent system. There are infinitely many solutions.	
Solve for y in terms of z in the second equation.	$y-2z=2$ $y=2z+2$
Solve the first equation for x in terms of z.	$x-2y+3z=1$
Substitute $y=2z+2$.	$x-2(2z+2)+3z=1$
Simplify.	$x-4z-4+3z=1$
Simplify.	$x-z-4=1$
Simplify.	$x=z+5$
The system has infinitely many solutions (x,y,z), where $x=z+5;\ y=2z+2;\ z$ is any real number.	

? Example 7.4.23

Solve the system of equations using a matrix: $\begin{cases} x+y-z=0 \\ 2x+4y-2z=6 \\ 3x+6y-3z=9 \end{cases}$

Answer

infinitely many solutions (x,y,z), where $x=z-3;\ y=3;\ z$ is any real number.

? Example 7.4.24

Solve the system of equations using a matrix: $\begin{cases} x-y-z=1 \\ -x+2y-3z=-4 \\ 3x-2y-7z=0 \end{cases}$

Answer

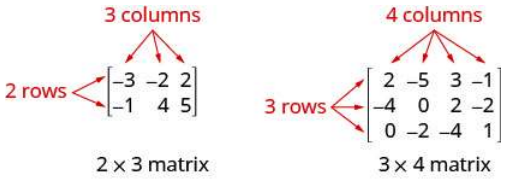
infinitely many solutions (x,y,z), where $x=5z-2;\ y=4z-3;\ z$ is any real number.

Access this online resource for additional instruction and practice with Gaussian Elimination.

- Gaussian Elimination

Key Concepts

- **Matrix:** A matrix is a rectangular array of numbers arranged in rows and columns. A matrix with *m* rows and *n* columns has *order* $m \times n$. The matrix on the left below has 2 rows and 3 columns and so it has order 2×3. We say it is a 2 by 3 matrix.



Each number in the matrix is called an *element* or *entry* in the matrix.

- **Row Operations:** In a matrix, the following operations can be performed on any row and the resulting matrix will be equivalent to the original matrix.
  - Interchange any two rows
  - Multiply a row by any real number except 0
  - Add a nonzero multiple of one row to another row
- **Row-Echelon Form:** For a consistent and independent system of equations, its augmented matrix is in row-echelon form when to the left of the vertical line, each entry on the diagonal is a 1 and all entries below the diagonal are zeros.

$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \end{bmatrix}$ $\begin{bmatrix} 1 & a & b & d \\ 0 & 1 & c & e \\ 0 & 0 & 1 & f \end{bmatrix}$ a, b, c, d, e, f are real numbers

- **How to solve a system of equations using matrices.**
  1. Write the augmented matrix for the system of equations.
  2. Using row operations get the entry in row 1, column 1 to be 1.
  3. Using row operations, get zeros in column 1 below the 1.
  4. Using row operations, get the entry in row 2, column 2 to be 1.
  5. Continue the process until the matrix is in row-echelon form.
  6. Write the corresponding system of equations.
  7. Use substitution to find the remaining variables.
  8. Write the solution as an ordered pair or triple.
  9. Check that the solution makes the original equations true.

Glossary

matrix

A matrix is a rectangular array of numbers arranged in rows and columns.

row-echelon form

A matrix is in row-echelon form when to the left of the vertical line, each entry on the diagonal is a 1 and all entries below the diagonal are zeros.

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CHAPTER OVERVIEW

8: Conic Sections

In this chapter, we will investigate the two-dimensional figures that are formed when a right circular cone is intersected by a plane. We will begin by studying each of three figures created in this manner. We will develop defining equations for each figure and then learn how to use these equations to solve a variety of problems. Conic sections are formed when a plane intersects two right circular cones aligned tip to tip and extending infinitely far in opposite directions, which we also call a cone. The way in which we slice the cone will determine the type of conic section formed at the intersection. A circle is formed by slicing a cone with a plane perpendicular to the axis of symmetry of the cone. An ellipse is formed by slicing a single cone with a slanted plane not perpendicular to the axis of symmetry.

8.1: The Parabola

8.2: The Ellipse

8.3: The Hyperbola

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8.1: The Parabola

In this section, you will:

- Graph parabolas with vertices at the origin.
- Write equations of parabolas in standard form.
- Graph parabolas with vertices not at the origin.
- Solve applied problems involving parabolas.



Figure **1** The Olympic torch concludes its journey around the world when it is used to light the Olympic cauldron during the opening ceremony. (credit: Ken Hackman, U.S. Air Force)

Did you know that the Olympic torch is lit several months before the start of the games? The ceremonial method for lighting the flame is the same as in ancient times. The ceremony takes place at the Temple of Hera in Olympia, Greece, and is rooted in Greek mythology, paying tribute to Prometheus, who stole fire from Zeus to give to all humans. One of eleven acting priestesses places the torch at the focus of a parabolic mirror (see Figure 1), which focuses light rays from the sun to ignite the flame.

Parabolic mirrors (or reflectors) are able to capture energy and focus it to a single point. The advantages of this property are evidenced by the vast list of parabolic objects we use every day: satellite dishes, suspension bridges, telescopes, microphones, spotlights, and car headlights, to name a few. Parabolic reflectors are also used in alternative energy devices, such as solar cookers and water heaters, because they are inexpensive to manufacture and need little maintenance. In this section we will explore the parabola and its uses, including low-cost, energy-efficient solar designs.

Graphing Parabolas with Vertices at the Origin

In The Ellipse, we saw that an ellipse is formed when a plane cuts through a right circular cone. If the plane is parallel to the edge of the cone, an unbounded curve is formed. This curve is a **parabola**. See Figure 2.

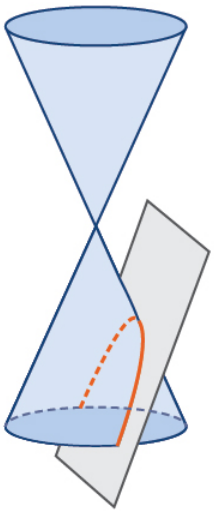


Figure **2** Parabola

Like the ellipse and hyperbola, the parabola can also be defined by a set of points in the coordinate plane. A parabola is the set of all points (x,y)(x,y) in a plane that are the same distance from a fixed line, called the **directrix**, and a fixed point (the **focus**) not on the directrix.

In Quadratic Functions, we learned about a parabola's vertex and axis of symmetry. Now we extend the discussion to include other key features of the parabola. See Figure 3. Notice that the axis of symmetry passes through the focus and vertex and is perpendicular to the directrix. The vertex is the midpoint between the directrix and the focus.

The line segment that passes through the focus and is parallel to the directrix is called the **latus rectum**. The endpoints of the latus rectum lie on the curve. By definition, the distance dd from the focus to any point PP on the parabola is equal to the distance from PP to the directrix.

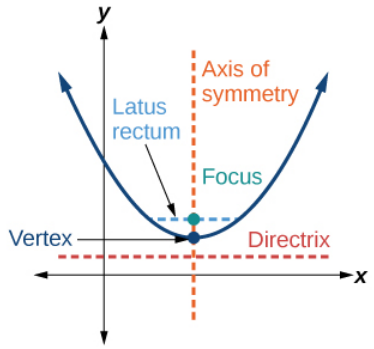


Figure **3** Key features of the parabola

To work with parabolas in the coordinate plane, we consider two cases: those with a vertex at the origin and those with a vertex at a point other than the origin. We begin with the former.

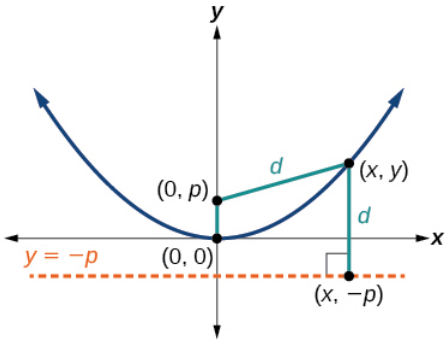


Figure **4**

Let (x,y)(x,y) be a point on the parabola with vertex (0,0),(0,0), focus (0,p),(0,p), and directrix y= −py= −p as shown in Figure 4. The distance dd from point (x,y)(x,y) to point (x,−p)(x,−p) on the directrix is the difference of the *y*-values: d=y+p.d=y+p. The distance from the focus (0,p)(0,p) to the point (x,y)(x,y) is also equal to dd and can be expressed using the distance formula.

$$d=\sqrt{(x-0)^2+(y-p)^2}=\sqrt{x^2+(y-p)^2}\qquad d=\sqrt{(x-0)^2+(y-p)^2}=\sqrt{x^2+(y-p)^2}$$

Set the two expressions for dd equal to each other and solve for yy to derive the equation of the parabola. We do this because the distance from (x,y)(x,y) to (0,p)(0,p) equals the distance from (x,y)(x,y) to (x, −p).(x, −p).

$$\sqrt{x^2+(y-p)^2}=y+p\qquad x^2+(y-p)^2=(y+p)^2\qquad x^2+y^2-2py+p^2=y^2+2py+p^2\qquad x^2-2py=2py\qquad x^2=4py$$

We then square both sides of the equation, expand the squared terms, and simplify by combining like terms.

$$x^2+(y-p)^2=(y+p)^2\qquad x^2+y^2-2py+p^2=y^2+2py+p^2\qquad x^2-2py=2py\qquad x^2=4py$$

The equations of parabolas with vertex (0,0)(0,0) are y2=4pxy2=4px when the *x*-axis is the axis of symmetry and x2=4pyx2=4py when the *y*-axis is the axis of symmetry. These standard forms are given below, along with their general graphs and key features.

Table 1 and Figure 5 summarize the standard features of parabolas with a vertex at the origin.

Axis of Symmetry	Equation	Focus	Directrix	Endpoints of Latus Rectum
x-axis	y2=4pxy2=4px	(p,0)(p,0)	x=−px=−p	(p,±2p)(p,±2p)
y-axis	x2=4pyx2=4py	(0,p)(0,p)	y=−py=−p	(±2p,p)(±2p,p)

Table 1

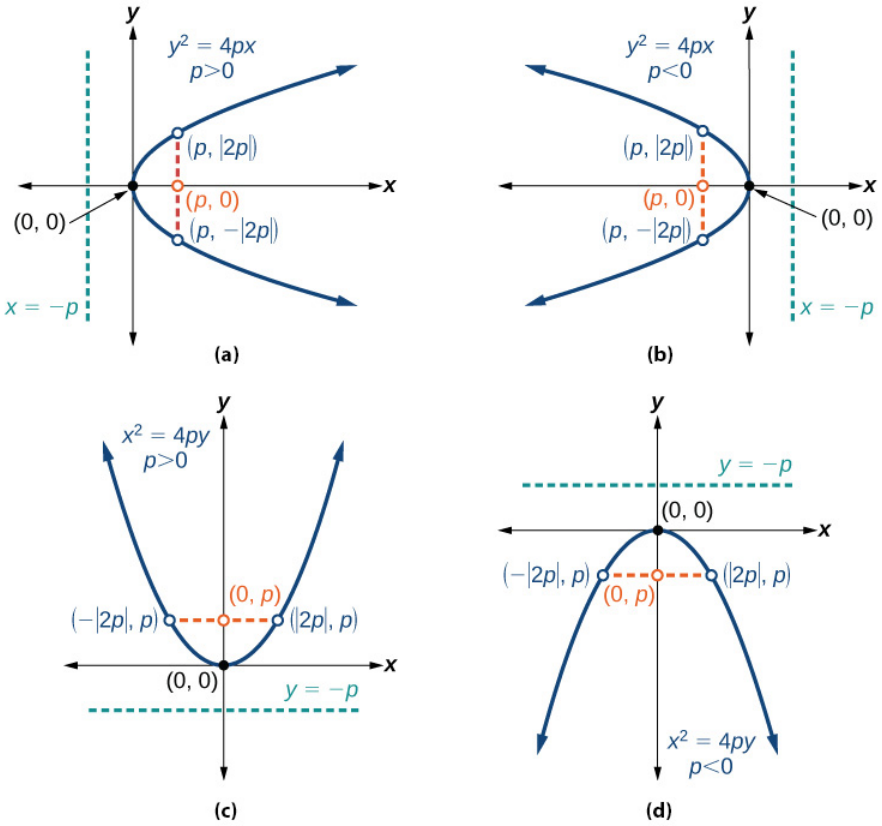


Figure **5** (a) When p>0p>0 and the axis of symmetry is the *x*-axis, the parabola opens right. (b) When p<0p<0 and the axis of symmetry is the *x*-axis, the parabola opens left. (c) When p>0p>0 and the axis of symmetry is the *y*-axis, the parabola opens up. (d) When p<0p<0 and the axis of symmetry is the *y*-axis, the parabola opens down.

The key features of a parabola are its vertex, axis of symmetry, focus, directrix, and latus rectum. See Figure 5. When given a standard equation for a parabola centered at the origin, we can easily identify the key features to graph the parabola.

A line is said to be tangent to a curve if it intersects the curve at exactly one point. If we sketch lines tangent to the parabola at the endpoints of the latus rectum, these lines intersect on the axis of symmetry, as shown in Figure 6.

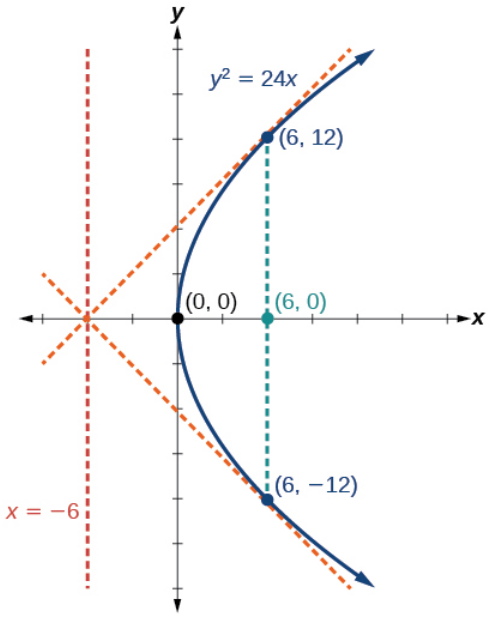


Figure **6**

Given a standard form equation for a parabola centered at (0, 0), sketch the graph.

1. Determine which of the standard forms applies to the given equation: y2=4pxy2=4px or x2=4py.x2=4py.
2. Use the standard form identified in Step 1 to determine the axis of symmetry, focus, equation of the directrix, and endpoints of the latus rectum.
   a. If the equation is in the form y2=4px,y2=4px, then
      - the axis of symmetry is the *x*-axis, y=0y=0
      - set 4p4p equal to the coefficient of *x* in the given equation to solve for p.p. If p>0,p>0, the parabola opens right. If p<0,p<0, the parabola opens left.
      - use p p to find the coordinates of the focus, (p,0)(p,0)
      - use pp to find the equation of the directrix, x=−px=−p
      - use pp to find the endpoints of the latus rectum, (p,±2p).(p,±2p). Alternately, substitute x=px=p into the original equation.
   b. If the equation is in the form x2=4py,x2=4py, then
      - the axis of symmetry is the *y*-axis, x=0x=0
      - set 4p4p equal to the coefficient of *y* in the given equation to solve for p.p. If p>0,p>0, the parabola opens up. If p<0,p<0, the parabola opens down.
      - use pp to find the coordinates of the focus, (0,p)(0,p)
      - use pp to find equation of the directrix, y=−py=−p
      - use pp to find the endpoints of the latus rectum, (±2p,p)(±2p,p)
3. Plot the focus, directrix, and latus rectum, and draw a smooth curve to form the parabola.

EXAMPLE 1

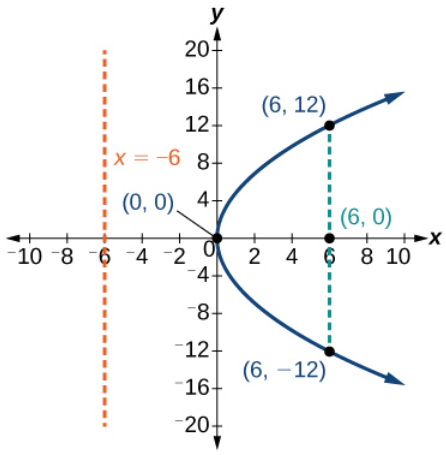
Graphing a Parabola with Vertex (0, 0) and the *x*-axis as the Axis of Symmetry

Graph y2=24x.y2=24x. Identify and label the focus, directrix, and endpoints of the latus rectum.

Answer

-
-
-

-



Graph y2=−16x.y2=−16x. Identify and label the focus, directrix, and endpoints of the latus rectum.

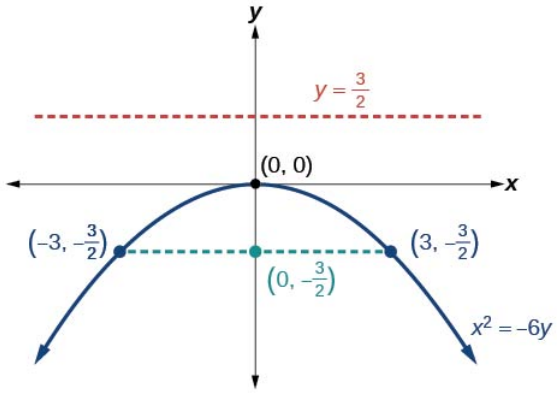
EXAMPLE 2

Graphing a Parabola with Vertex (0, 0) and the *y*-axis as the Axis of Symmetry

Graph x2=−6y.x2=−6y. Identify and label the focus, directrix, and endpoints of the latus rectum.

Answer

-
-
-
-



Graph x2=8y.x2=8y. Identify and label the focus, directrix, and endpoints of the latus rectum.

Writing Equations of Parabolas in Standard Form

In the previous examples, we used the standard form equation of a parabola to calculate the locations of its key features. We can also use the calculations in reverse to write an equation for a parabola when given its key features.

Given its focus and directrix, write the equation for a parabola in standard form.

1. Determine whether the axis of symmetry is the *x*- or *y*-axis.

a. If the given coordinates of the focus have the form (p,0),(p,0), then the axis of symmetry is the *x*-axis. Use the standard form y2=4px.y2=4px.
b. If the given coordinates of the focus have the form (0,p),(0,p), then the axis of symmetry is the *y*-axis. Use the standard form x2=4py.x2=4py.

2. Multiply 4p.4p.
3. Substitute the value from Step 2 into the equation determined in Step 1.

EXAMPLE 3

Writing the Equation of a Parabola in Standard Form Given its Focus and Directrix

What is the equation for the parabola with focus (−12,0)(−12,0) and directrix x=12?x=12?

Answer

-
-

What is the equation for the parabola with focus (0,72)(0,72) and directrix y=−72?y=−72?

Graphing Parabolas with Vertices Not at the Origin

Like other graphs we've worked with, the graph of a parabola can be translated. If a parabola is translated hh units horizontally and kk units vertically, the vertex will be (h,k).(h,k). This translation results in the standard form of the equation we saw previously with xx replaced by (x−h)(x−h) and yy replaced by (y−k).(y−k).

To graph parabolas with a vertex (h,k)(h,k) other than the origin, we use the standard form (y−k)2=4p(x−h)(y−k)2=4p(x−h) for parabolas that have an axis of symmetry parallel to the *x*-axis, and (x−h)2=4p(y−k)(x−h)2=4p(y−k) for parabolas that have an axis of symmetry parallel to the *y*-axis. These standard forms are given below, along with their general graphs and key features.

Table 2 and Figure 9 summarize the standard features of parabolas with a vertex at a point (h,k).(h,k).

Axis of Symmetry	Equation	Focus	Directrix	Endpoints of Latus Rectum
y=ky=k	(y−k)2=4p(x−h) (y−k)2=4p(x−h)	(h+p,k)(h+p,k)	x=h−px=h−p	(h+p,k±2p)(h+p,k±2p)
x=hx=h	(x−h)2=4p(y−k) (x−h)2=4p(y−k)	(h,k+p)(h,k+p)	y=k−py=k−p	(h±2p,k+p)(h±2p,k+p)

Table 2

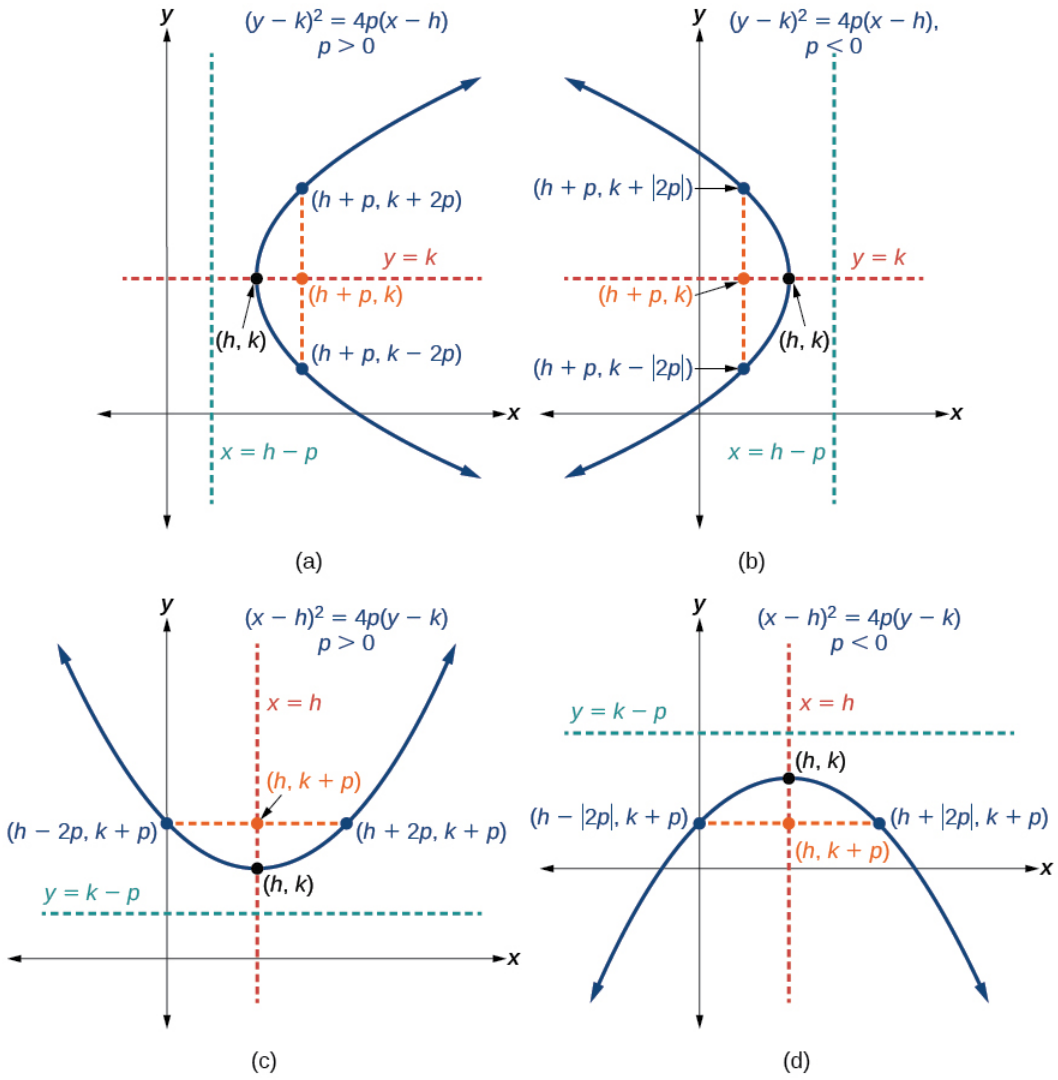


Figure **9** (a) When p>0,p>0, the parabola opens right. (b) When p<0,p<0, the parabola opens left. (c) When p>0,p>0, the parabola opens up. (d) When p<0,p<0, the parabola opens down.

Given a standard form equation for a parabola centered at (*h*, *k*), sketch the graph.

1. Determine which of the standard forms applies to the given equation: (y−k)2=4p(x−h)(y−k)2=4p(x−h) or (x−h)2=4p(y−k). (x−h)2=4p(y−k).
2. Use the standard form identified in Step 1 to determine the vertex, axis of symmetry, focus, equation of the directrix, and endpoints of the latus rectum.
   a. If the equation is in the form (y−k)2=4p(x−h),(y−k)2=4p(x−h), then:
      - use the given equation to identify h h and kk for the vertex, (h,k)(h,k)
      - use the value of kk to determine the axis of symmetry, y=ky=k
      - set 4p4p equal to the coefficient of (x−h)(x−h) in the given equation to solve for p.p. If p>0,p>0, the parabola opens right. If p<0,p<0, the parabola opens left.
      - use h,k,h,k, and pp to find the coordinates of the focus, (h+p,k)(h+p,k)
      - use hh and pp to find the equation of the directrix, x=h−px=h−p
      - use h,k,h,k, and pp to find the endpoints of the latus rectum, (h+p,k±2p)(h+p,k±2p)
   b. If the equation is in the form (x−h)2=4p(y−k),(x−h)2=4p(y−k), then:
      - use the given equation to identify hh and kk for the vertex, (h,k)(h,k)

- use the value of hh to determine the axis of symmetry, x=hx=h
- set 4p4p equal to the coefficient of (y−k)(y−k) in the given equation to solve for p.p. If p>0,p>0, the parabola opens up. If p<0,p<0, the parabola opens down.
- use h,k,h,k, and pp to find the coordinates of the focus, (h,k+p)(h,k+p)
- use kk and pp to find the equation of the directrix, y=k−py=k−p
- use h,k,h, k, and p p to find the endpoints of the latus rectum, (h±2p,k+p)(h±2p,k+p)

3. Plot the vertex, axis of symmetry, focus, directrix, and latus rectum, and draw a smooth curve to form the parabola.

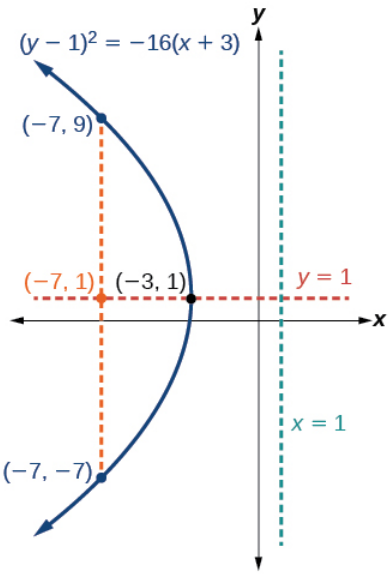
EXAMPLE 4

Graphing a Parabola with Vertex (*h*, *k*) and Axis of Symmetry Parallel to the *x*-axis

Graph (y−1)2=−16(x+3).(y−1)2=−16(x+3). Identify and label the vertex, axis of symmetry, focus, directrix, and endpoints of the latus rectum.

Answer

-
-
-
-
-
-



Graph (y+1)2=4(x−8).(y+1)2=4(x−8). Identify and label the vertex, axis of symmetry, focus, directrix, and endpoints of the latus rectum.

EXAMPLE 5

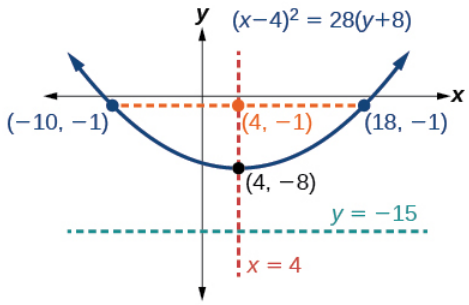
Graphing a Parabola from an Equation Given in General Form

Graph x2−8x−28y−208=0.x2−8x−28y−208=0. Identify and label the vertex, axis of symmetry, focus, directrix, and endpoints of the latus rectum.

Answer

-
-
-

-
-
-



Graph (x+2)2=−20(y−3).(x+2)2=−20(y−3). Identify and label the vertex, axis of symmetry, focus, directrix, and endpoints of the latus rectum.

Solving Applied Problems Involving Parabolas

As we mentioned at the beginning of the section, parabolas are used to design many objects we use every day, such as telescopes, suspension bridges, microphones, and radar equipment. Parabolic mirrors, such as the one used to light the Olympic torch, have a very unique reflecting property. When rays of light parallel to the parabola's axis of symmetry are directed toward any surface of the mirror, the light is reflected directly to the focus. See Figure 12. This is why the Olympic torch is ignited when it is held at the focus of the parabolic mirror.

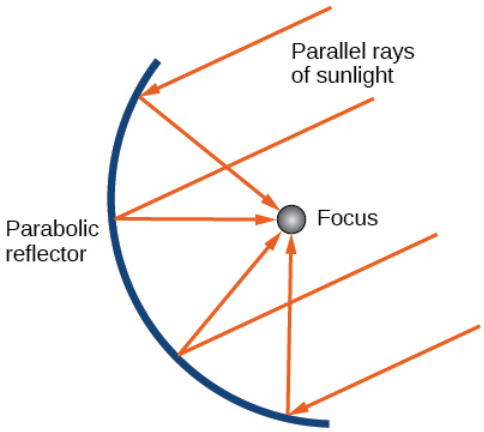


Figure **12** Reflecting property of parabolas

Parabolic mirrors have the ability to focus the sun's energy to a single point, raising the temperature hundreds of degrees in a matter of seconds. Thus, parabolic mirrors are featured in many low-cost, energy efficient solar products, such as solar cookers, solar heaters, and even travel-sized fire starters.

EXAMPLE 6

Solving Applied Problems Involving Parabolas

A cross-section of a design for a travel-sized solar fire starter is shown in Figure 13. The sun's rays reflect off the parabolic mirror toward an object attached to the igniter. Because the igniter is located at the focus of the parabola, the reflected rays cause the object to burn in just seconds.

- ⓐ Find the equation of the parabola that models the fire starter. Assume that the vertex of the parabolic mirror is the origin of the coordinate plane.
- ⓑ Use the equation found in part ⓐ to find the depth of the fire starter.

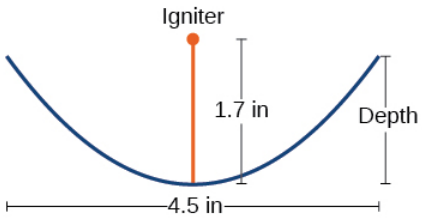


Figure **13** Cross-section of a travel-sized solar fire starter

Answer

-
-

Balcony-sized solar cookers have been designed for families living in India. The top of a dish has a diameter of 1600 mm. The sun's rays reflect off the parabolic mirror toward the "cooker," which is placed 320 mm from the base.

ⓐ Find an equation that models a cross-section of the solar cooker. Assume that the vertex of the parabolic mirror is the origin of the coordinate plane, and that the parabola opens to the right (i.e., has the *x*-axis as its axis of symmetry).

ⓑ Use the equation found in part ⓐ to find the depth of the cooker.

Access these online resources for additional instruction and practice with parabolas.

- Conic Sections: The Parabola Part 1 of 2
- Conic Sections: The Parabola Part 2 of 2
- Parabola with Vertical Axis
- Parabola with Horizontal Axis

10.3 Section Exercises

Verbal

1.

Define a parabola in terms of its focus and directrix.

2.

If the equation of a parabola is written in standard form and pp is positive and the directrix is a vertical line, then what can we conclude about its graph?

3.

If the equation of a parabola is written in standard form and pp is negative and the directrix is a horizontal line, then what can we conclude about its graph?

4.

What is the effect on the graph of a parabola if its equation in standard form has increasing values of p?p?

5.

As the graph of a parabola becomes wider, what will happen to the distance between the focus and directrix?

Algebraic

For the following exercises, determine whether the given equation is a parabola. If so, rewrite the equation in standard form.

6.

$y2=4-x2y2=4-x2$

7.

y=4x2y=4x2

8.

3x2−6y2=123x2−6y2=12

9.

(y−3)2=8(x−2)(y−3)2=8(x−2)

10.

y2+12x−6y−51=0y2+12x−6y−51=0

For the following exercises, rewrite the given equation in standard form, and then determine the vertex (V),(V), focus (F),(F), and directrix (d)(d) of the parabola.

11.

x=8y2x=8y2

12.

y=14x2y=14x2

13.

y=−4x2y=−4x2

14.

x=18y2x=18y2

15.

x=36y2x=36y2

16.

x=136y2x=136y2

17.

(x−1)2=4(y−1)(x−1)2=4(y−1)

18.

(y−2)2=45(x+4)(y−2)2=45(x+4)

19.

(y−4)2=2(x+3)(y−4)2=2(x+3)

20.

(x+1)2=2(y+4)(x+1)2=2(y+4)

21.

(x+4)2=24(y+1)(x+4)2=24(y+1)

22.

(y+4)2=16(x+4)(y+4)2=16(x+4)

23.

y2+12x−6y+21=0y2+12x−6y+21=0

24.

x2−4x−24y+28=0x2−4x−24y+28=0

25.

$5x2-50x-4y+113=05x2-50x-4y+113=0$

26.

$y2-24x+4y-68=0y2-24x+4y-68=0$

27.

$x2-4x+2y-6=0x2-4x+2y-6=0$

28.

$y2-6y+12x-3=0y2-6y+12x-3=0$

29.

$3y2-4x-6y+23=03y2-4x-6y+23=0$

30.

$x2+4x+8y-4=0x2+4x+8y-4=0$

Graphical

For the following exercises, graph the parabola, labeling the focus and the directrix.

31.

$x=18y2x=18y2$

32.

$y=36x2y=36x2$

33.

$y=136x2y=136x2$

34.

$y=-9x2y=-9x2$

35.

$(y-2)2=-43(x+2)(y-2)2=-43(x+2)$

36.

$-5(x+5)2=4(y+5)-5(x+5)2=4(y+5)$

37.

$-6(y+5)2=4(x-4)-6(y+5)2=4(x-4)$

38.

$y2-6y-8x+1=0y2-6y-8x+1=0$

39.

$x2+8x+4y+20=0x2+8x+4y+20=0$

40.

$3x2+30x-4y+95=03x2+30x-4y+95=0$

41.

$y2-8x+10y+9=0y2-8x+10y+9=0$

42.

$x2+4x+2y+2=0x2+4x+2y+2=0$

43.

$y^2+2y-12x+61=0 y^2+2y-12x+61=0$

44.

$-2x^2+8x-4y-24=0 -2x^2+8x-4y-24=0$

For the following exercises, find the equation of the parabola given information about its graph.

45.

Vertex is $(0,0);(0,0)$; directrix is $y=4, y=4$, focus is $(0,-4).(0,-4)$.

46.

Vertex is $(0,0);(0,0)$; directrix is $x=4, x=4$, focus is $(-4,0).(-4,0)$.

47.

Vertex is $(2,2);(2,2)$; directrix is $x=2-2-\sqrt{}, x=2-2$, focus is $(2+2-\sqrt{},2).(2+2,2)$.

48.

Vertex is $(-2,3);(-2,3)$; directrix is $x=-72, x=-72$, focus is $(-12,3).(-12,3)$.

49.

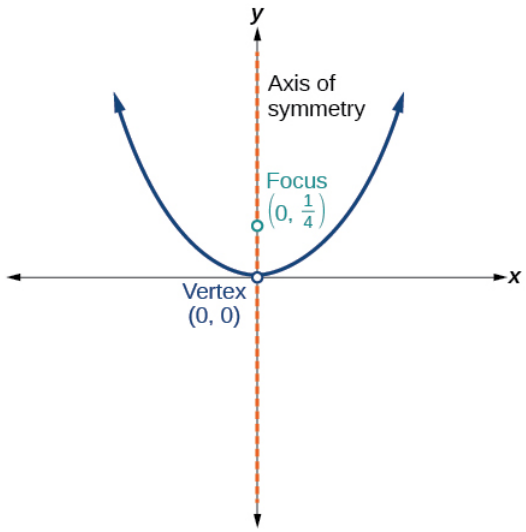
Vertex is $(2-\sqrt{},-3-\sqrt{});(2,-3)$; directrix is $x=22-\sqrt{}, x=22$, focus is $(0,-3-\sqrt{}).(0,-3)$.

50.

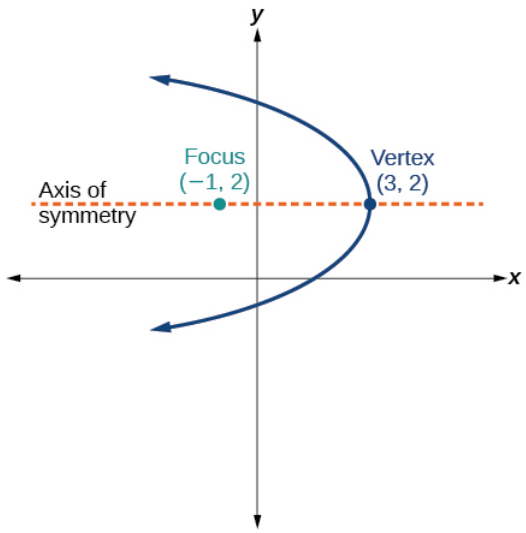
Vertex is $(1,2)$; $(1,2)$; directrix is $y=113, y=113$, focus is $(1,13).(1,13)$.

For the following exercises, determine the equation for the parabola from its graph.

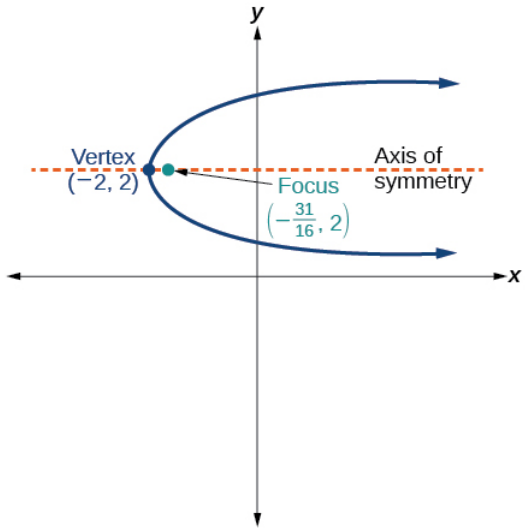
51.



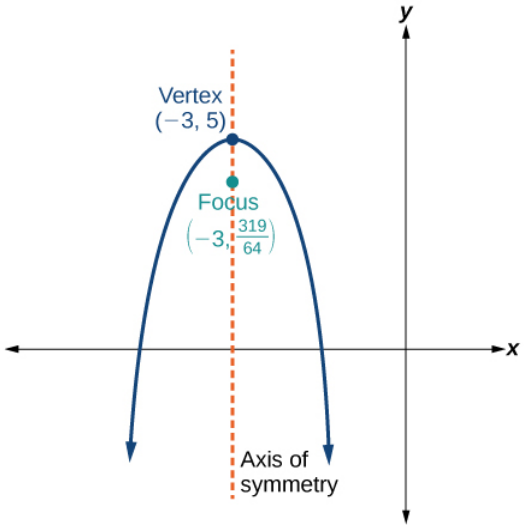
52.


y
Focus
(−1, 2)
Vertex
(3, 2)
Axis of
symmetry
x

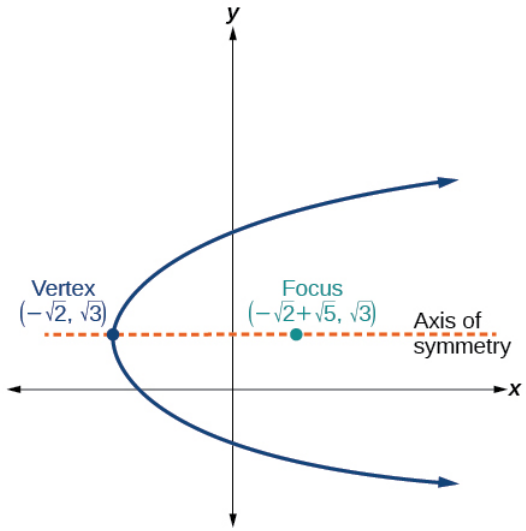
53.


y
Vertex
(−2, 2)
Axis of
symmetry
Focus
$\left(-\frac{31}{16}, 2\right)$
x

54.


y
Vertex
(−3, 5)
Focus
$\left(-3, \frac{319}{64}\right)$
x
Axis of
symmetry

55.



Extensions

For the following exercises, the vertex and endpoints of the latus rectum of a parabola are given. Find the equation.

56.

V(0,0)V(0,0), Endpoints (2,1)(2,1), (−2,1)(−2,1)

57.

V(0,0)V(0,0), Endpoints (−2,4)(−2,4), (−2,−4)(−2,−4)

58.

V(1,2)V(1,2), Endpoints (−5,5)(−5,5), (7,5)(7,5)

59.

V(−3,−1)V(−3,−1), Endpoints (0,5)(0,5), (0,−7)(0,−7)

60.

V(4,−3)V(4,−3), Endpoints (5,−72)(5,−72), (3,−72)(3,−72)

Real-World Applications

61.

The mirror in an automobile headlight has a parabolic cross-section with the light bulb at the focus. On a schematic, the equation of the parabola is given as x2=4y.x2=4y. At what coordinates should you place the light bulb?

62.

If we want to construct the mirror from the previous exercise such that the focus is located at (0,0.25),(0,0.25), what should the equation of the parabola be?

63.

A satellite dish is shaped like a paraboloid of revolution. This means that it can be formed by rotating a parabola around its axis of symmetry. The receiver is to be located at the focus. If the dish is 12 feet across at its opening and 4 feet deep at its center, where should the receiver be placed?

64.

Consider the satellite dish from the previous exercise. If the dish is 8 feet across at the opening and 2 feet deep, where should we place the receiver?

65.

The reflector in a searchlight is shaped like a paraboloid of revolution. A light source is located 1 foot from the base along the axis of symmetry. If the opening of the searchlight is 3 feet across, find the depth.

66.

If the reflector in the searchlight from the previous exercise has the light source located 6 inches from the base along the axis of symmetry and the opening is 4 feet, find the depth.

67.

An arch is in the shape of a parabola. It has a span of 100 feet and a maximum height of 20 feet. Find the equation of the parabola, and determine the height of the arch 40 feet from the center.

68.

If the arch from the previous exercise has a span of 160 feet and a maximum height of 40 feet, find the equation of the parabola, and determine the distance from the center at which the height is 20 feet.

69.

An object is projected so as to follow a parabolic path given by $y=-x2+96x,y=-x2+96x$, where xx is the horizontal distance traveled in feet and yy is the height. Determine the maximum height the object reaches.

70.

For the object from the previous exercise, assume the path followed is given by $y=-0.5x2+80x.y=-0.5x2+80x$. Determine how far along the horizontal the object traveled to reach maximum height.

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8.2: The Ellipse

Learning Objectives

In this section, you will:

- Write equations of ellipses in standard form.
- Graph ellipses centered at the origin.
- Graph ellipses not centered at the origin.
- Solve applied problems involving ellipses.



Figure 8.2.1: The National Statuary Hall in Washington, D.C. (credit: Greg Palmer, Flickr)

Can you imagine standing at one end of a large room and still being able to hear a whisper from a person standing at the other end? The National Statuary Hall in Washington, D.C., shown in Figure 8.2.1, is such a room.[1] It is an oval-shaped room called a *whispering chamber* because the shape makes it possible for sound to travel along the walls. In this section, we will investigate the shape of this room and its real-world applications, including how far apart two people in Statuary Hall can stand and still hear each other whisper.

Writing Equations of Ellipses in Standard Form

A conic section, or **conic**, is a shape resulting from intersecting a right circular cone with a plane. The angle at which the plane intersects the cone determines the shape, as shown in Figure 8.2.2

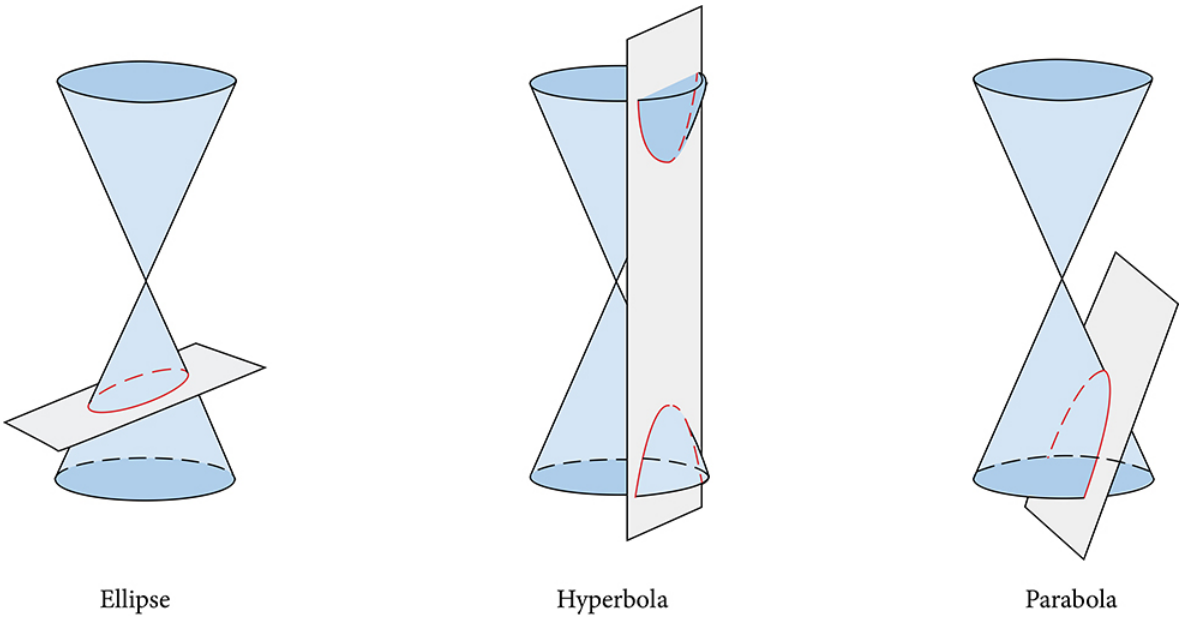


Figure 8.2.2:

Conic sections can also be described by a set of points in the coordinate plane. Later in this chapter, we will see that the graph of any quadratic equation in two variables is a conic section. The signs of the equations and the coefficients of the variable terms determine the shape. This section focuses on the four variations of the standard form of the equation for the ellipse. An **ellipse** is the set of all points (x, y) in a plane such that the sum of their distances from two fixed points is a constant. Each fixed point is called a **focus** (plural: **foci**).

We can draw an ellipse using a piece of cardboard, two thumbtacks, a pencil, and string. Place the thumbtacks in the cardboard to form the foci of the ellipse. Cut a piece of string longer than the distance between the two thumbtacks (the length of the string represents the constant in the definition). Tack each end of the string to the cardboard, and trace a curve with a pencil held taut against the string. The result is an ellipse. See Figure 8.2.3.

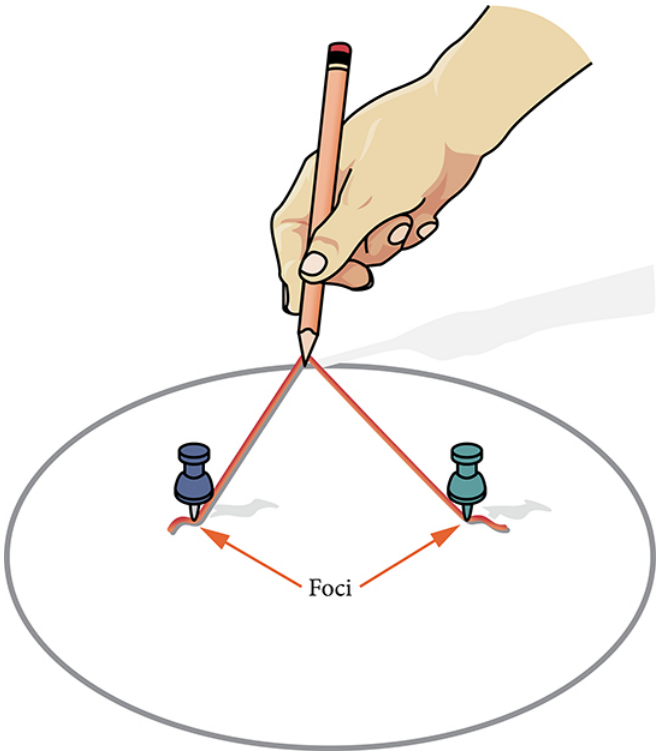


Figure 8.2.3:

Every ellipse has two axes of symmetry. The longer axis is called the **major axis**, and the shorter axis is called the **minor axis**. Each endpoint of the major axis is the **vertex** of the ellipse (plural: **vertices**), and each endpoint of the minor axis is a **co-vertex** of the ellipse. The **center of an ellipse** is the midpoint of both the major and minor axes. The axes are perpendicular at the center. The foci always lie on the major axis, and the sum of the distances from the foci to any point on the ellipse (the constant sum) is greater than the distance between the foci. See Figure 8.2.4.

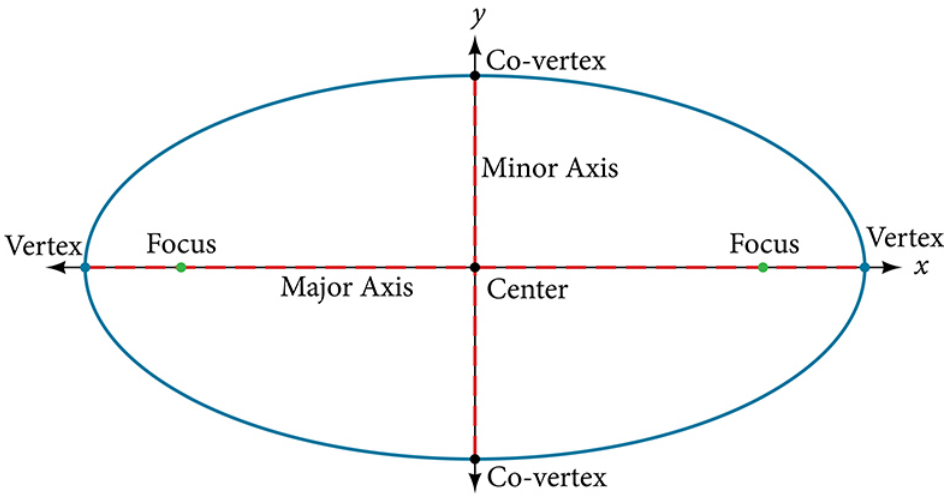


Figure 8.2.4:

In this section, we restrict ellipses to those that are positioned vertically or horizontally in the coordinate plane. That is, the axes will either lie on or be parallel to the x- and y-axes. Later in the chapter, we will see ellipses that are rotated in the coordinate plane.

To work with horizontal and vertical ellipses in the coordinate plane, we consider two cases: those that are centered at the origin and those that are centered at a point other than the origin. First we will learn to derive the equations of ellipses, and then we will

learn how to write the equations of ellipses in standard form. Later we will use what we learn to draw the graphs.

Deriving the Equation of an Ellipse Centered at the Origin

To derive the equation of an ellipse centered at the origin, we begin with the foci (−c,0) and (c,0). The ellipse is the set of all points (x,y) such that the sum of the distances from (x,y) to the foci is constant, as shown in Figure 8.2.5.

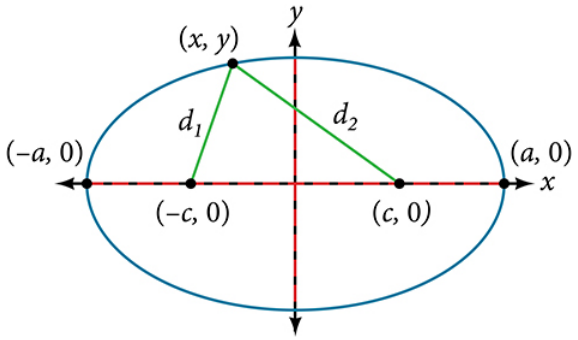


Figure 8.2.5:

If $(a,0)$ is a vertex of the ellipse, the distance from $(-c,0)$ to $(a,0)$ is $a-(-c)=a+c$. The distance from $(c,0)$ to $(a,0)$ is $a-c$. The sum of the distances from the foci to the vertex is

$$(a+c)+(a-c)=2a \tag{8.2.1}$$

If (x,y) is a point on the ellipse, then we can define the following variables:

$$_1 = \text{ the distance from } (-c,0) \text{ to } (x,y)$$

$$d_2 = \text{ the distance from } (c,0) \text{ to } (x,y)$$

By the definition of an ellipse, d_1+d_2 is constant for any point (x,y) on the ellipse. We know that the sum of these distances is $2a$ for the vertex $(a,0)$. It follows that $d_1+d_2=2a$ for any point on the ellipse. We will begin the derivation by applying the distance formula. The rest of the derivation is algebraic.

$d_1+d_2=\sqrt{(x-(-c))^2+(y-0)^2}+\sqrt{(x-c)^2+(y-0)^2}=2a$	Distance formula
$\sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2}=2a$	Simplify expressions.
$\sqrt{(x+c)^2+y^2}=2a-\sqrt{(x-c)^2+y^2}$	Move radical to opposite side.
$(x+c)^2+y^2=\left[2a-\sqrt{(x-c)^2+y^2}\right]^2$	Square both sides.
$x^2+2cx+c^2+y^2=4a^2-4a\sqrt{(x-c)^2+y^2}+(x-c)^2+y^2$	Expand the squares.
$x^2+2cx+c^2+y^2=4a^2-4a\sqrt{(x-c)^2+y^2}+x^2-2cx+c^2+y^2$	Expand remaining squares.
$2cx=4a^2-4a\sqrt{(x-c)^2+y^2}-2cx$	Combine like terms.
$4cx-4a^2=-4a\sqrt{(x-c)^2+y^2}$	Isolate the radical.
$cx-a^2=-a\sqrt{(x-c)^2+y^2}$	Divide by 4 .
$\left[cx-a^2\right]^2=a^2\left[\sqrt{(x-c)^2+y^2}\right]^2$	Square both sides.
$c^2x^2-2a^2cx+a^4=a^2\left(x^2-2cx+c^2+y^2\right)$	Expand the squares.
$c^2x^2-2a^2cx+a^4=a^2x^2-2a^2cx+a^2c^2+a^2y^2$	Distribute a^2.
$a^2x^2-c^2x^2+a^2y^2=a^4-a^2c^2$	Rewrite.
$x^2\left(a^2-c^2\right)+a^2y^2=a^2\left(a^2-c^2\right)$	Factor common terms.
$x^2b^2+a^2y^2=a^2b^2$	Set $b^2=a^2-c^2$.

$\frac{x^2b^2}{a^2b^2}+\frac{a^2y^2}{a^2b^2}=\frac{a^2b^2}{a^2b^2}$	Divide both sides by a^2b^2.
$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	Simplify.

Thus, the standard equation of an ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. This equation defines an ellipse centered at the origin. If a>b, the ellipse is stretched further in the horizontal direction, and if $b>a$, the ellipse is stretched further in the vertical direction.

Writing Equations of Ellipses Centered at the Origin in Standard Form

Standard forms of equations tell us about key features of graphs. Take a moment to recall some of the standard forms of equations we've worked with in the past: linear, quadratic, cubic, exponential, logarithmic, and so on. By learning to interpret standard forms of equations, we are bridging the relationship between algebraic and geometric representations of mathematical phenomena.

The key features of the ellipse are its center, vertices, co-vertices, foci, and lengths and positions of the major and minor axes. Just as with other equations, we can identify all of these features just by looking at the standard form of the equation. There are four variations of the standard form of the ellipse. These variations are categorized first by the location of the center (the origin or not the origin), and then by the position (horizontal or vertical). Each is presented along with a description of how the parts of the equation relate to the graph. Interpreting these parts allows us to form a mental picture of the ellipse.

Note

STANDARD FORMS OF THE EQUATION OF AN ELLIPSE WITH CENTER $(0,0)$

The standard form of the equation of an ellipse with center $(0,0)$ and major axis on the x-axis is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \tag{8.2.2}$$

where

- $a>b$
- the length of the major axis is $2a$
- the coordinates of the vertices are $(\pm a,0)$
- the length of the minor axis is $2b$
- the coordinates of the co-vertices are $(0,\pm b)$
- the coordinates of the foci are $(\pm c,0)$, where $c^2=a^2-b^2$. See Figure 6 a

The standard form of the equation of an ellipse with center $(0,0)$ and major axis on the y-axis is

$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1 \tag{8.2.3}$$

where

- $a>b$
- the length of the major axis is $2a$
- the coordinates of the vertices are $(0,\pm a)$
- the length of the minor axis is $2b$
- the coordinates of the co-vertices are $(\pm b,0)$
- the coordinates of the foci are $(0,\pm c)$, where $c^2=a^2-b^2$. See Figure $6\mathbf{b}$

Note that the vertices, co-vertices, and foci are related by the equation $c^2=a^2-b^2$. When we are given the coordinates of the foci and vertices of an ellipse, we can use this relationship to find the equation of the ellipse in standard form.

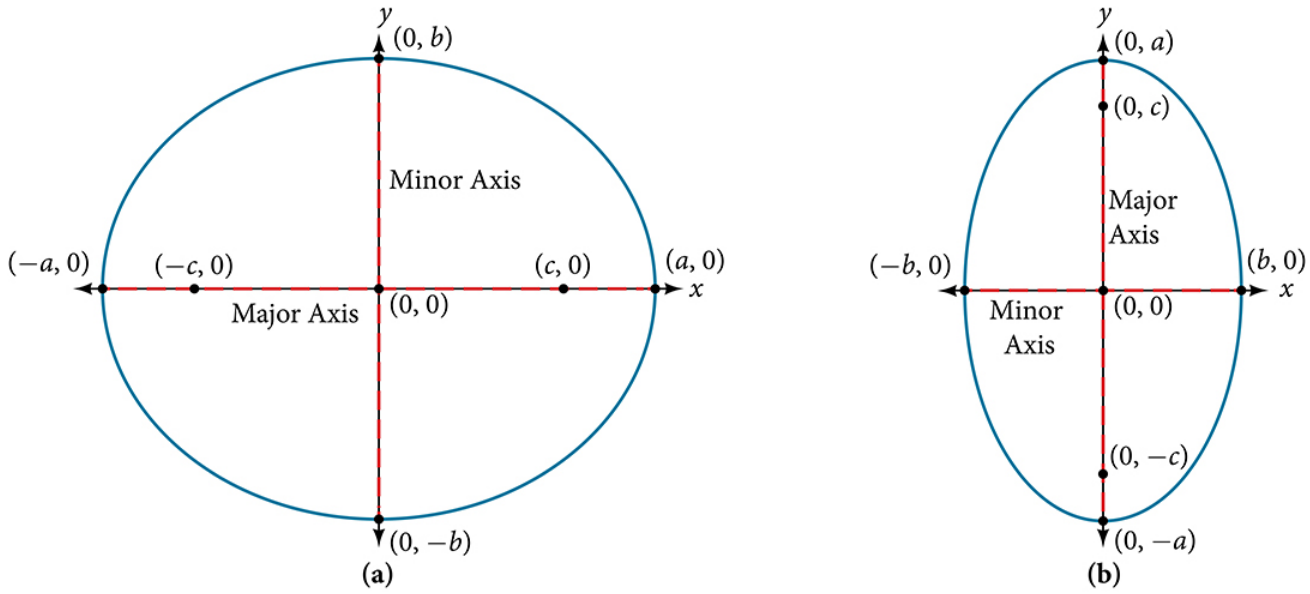


Figure 8.2.6: Horizontal ellipse with center (0,0) (b) Vertical ellipse with center (0,0)

How To

Given the vertices and foci of an ellipse centered at the origin, write its equation in standard form.

1. Determine whether the major axis lies on the x - or y-axis.

   a. If the given coordinates of the vertices and foci have the form $(\pm a, 0)$ and $(\pm c, 0)$ respectively, then the major axis is the x-axis. Use the standard form $\frac{x^2}{a^2}+d\frac{y^2}{b^2}=1$.

   b. If the given coordinates of the vertices and foci have the form $(0, \pm a)$ and $(0, \pm c)$, respectively, then the major axis is the y-axis. Use the standard form $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$.

2. Use the equation $c^2=a^2-b^2$, along with the given coordinates of the vertices and foci, to solve for b^2.

3. Substitute the values for a^2 and b^2 into the standard form of the equation determined in Step 1 .

Exercise 8.2.1

Writing the Equation of an Ellipse Centered at the Origin in Standard Form

What is the standard form equation of the ellipse that has vertices $(\pm 8, 0)$ and foci $(\pm 5, 0)$?

Answer

The foci are on the x-axis, so the major axis is the x-axis. Thus, the equation will have the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \tag{8.2.4}$$

The vertices are $(\pm 8, 0)$, so $a=8$ and $a^2=64$.
The foci are $(\pm 5, 0)$, so $c=5$ and $c^2=25$.

We know that the vertices and foci are related by the equation $c^2 = a^2 - b^2$. Solving for b^2, we have:

$$\begin{aligned} c^2 &= a^2 - b^2 \\ 25 &= 64 - b^2 \quad \text{Substitute for } c^2 \text{ and } a^2. \\ b^2 &= 39 \quad \text{Solve for } b^2. \end{aligned} \tag{8.2.5}$$

Now we need only substitute $a^2 = 64$ and $b^2 = 39$ into the standard form of the equation. The equation of the ellipse is $\frac{x^2}{64} + \frac{y^2}{39} = 1$.

Try It

What is the standard form equation of the ellipse that has vertices $(0, \pm 4)$ and foci $(0, \pm\sqrt{15})$?

Q&A

Can we write the equation of an ellipse centered at the origin given coordinates of just one focus and vertex?

Yes. Ellipses are symmetrical, so the coordinates of the vertices of an ellipse centered around the origin will always have the form $(\pm a, 0)$ or $(0, \pm a)$. Similarly, the coordinates of the foci will always have the form $(\pm c, 0)$ or $(0, \pm c)$. Knowing this, we can use a and c from the given points, along with the equation $c^2 = a^2 - b^2$, to find b^2.

Writing Equations of Ellipses Not Centered at the Origin

Like the graphs of other equations, the graph of an ellipse can be translated. If an ellipse is translated hh units horizontally and k units vertically, the center of the ellipse will be (h, k). This translation results in the standard form of the equation we saw previously, with x replaced by $(x - h)$ and y replaced by $(y - k)$.

Note

STANDARD FORMS OF THE EQUATION OF AN ELLIPSE WITH CENTER (H, K)

The standard form of the equation of an ellipse with center (h, k) and major axis parallel to the x-axis is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \tag{8.2.6}$$

where

- $a > b$
- the length of the major axis is $2a$
- the coordinates of the vertices are $(h \pm a, k)$
- the length of the minor axis is $2b$
- the coordinates of the co-vertices are $(h, k \pm b)$
- the coordinates of the foci are $(h \pm c, k)$, where $c^2 = a^2 - b^2$. See Figure $\mathbf{7a}$

The standard form of the equation of an ellipse with center (h, k) and major axis parallel to the y-axis is

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1 \tag{8.2.7}$$

where

- $a > b$
- the length of the major axis is $2a$
- the coordinates of the vertices are $(h, k \pm a)$
- the length of the minor axis is $2b$
- the coordinates of the co-vertices are $(h \pm b, k)$
- the coordinates of the foci are $(h, k \pm c)$, where $c^2 = a^2 - b^2$. See Figure $\mathbf{7b}$

Just as with ellipses centered at the origin, ellipses that are centered at a point (h, k) have vertices, covertices, and foci that are related by the equation $c^2 = a^2 - b^2$. We can use this relationship along with the midpoint and distance formulas to find the equation of the ellipse in standard form when the vertices and foci are given.

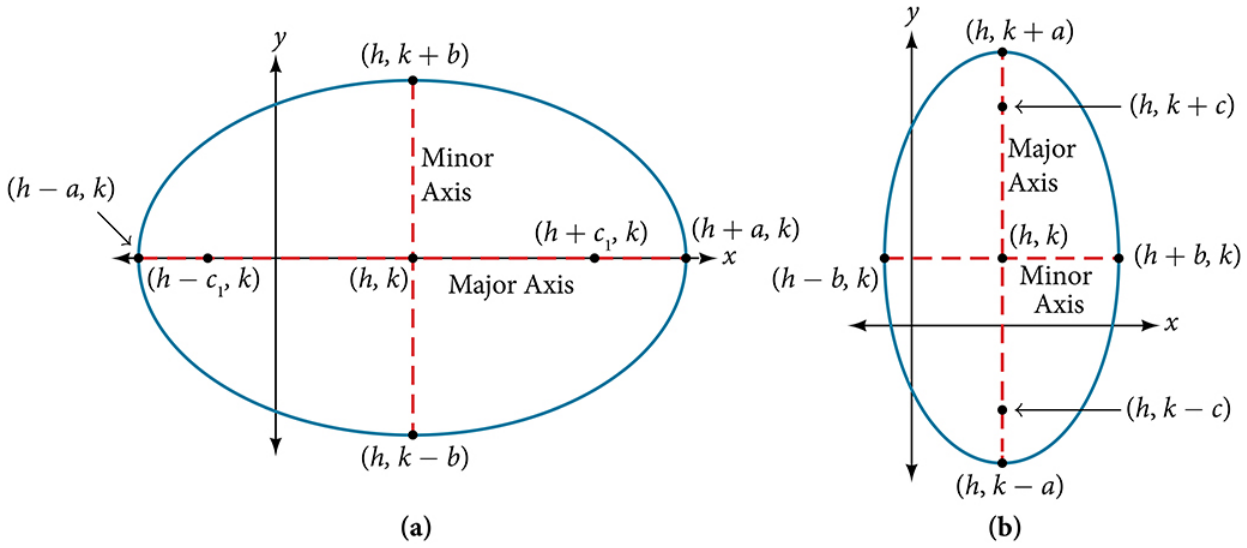


Figure 8.2.7: (a) Horizontal ellipse with center (h,k)(h,k) (b) Vertical ellipse with center (h,k)(h,k)

How To

Given the vertices and foci of an ellipse not centered at the origin, write its equation in standard form.

1. Determine whether the major axis is parallel to the x - or y-axis.
    a. If the y-coordinates of the given vertices and foci are the same, then the major axis is parallel to the x-axis. Use the standard form $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$.
    b. If the x-coordinates of the given vertices and foci are the same, then the major axis is parallel to the y-axis. Use the standard form $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$.
2. Identify the center of the ellipse (h, k) using the midpoint formula and the given coordinates for the vertices.
3. Find a^2 by solving for the length of the major axis, $2a$, which is the distance between the given vertices.
4. Find c^2 using h and k, found in Step 2, along with the given coordinates for the foci.
5. Solve for b^2 using the equation $c^2 = a^2 - b^2$.
6. Substitute the values for h, k, a^2, and b^2 into the standard form of the equation determined in Step 1.

Exercise 8.2.2

Writing the Equation of an Ellipse Centered at a Point Other Than the Origin

What is the standard form equation of the ellipse that has vertices $(-2,-8)$ and $(-2,2)$ and foci $(-2,-7)$ and $(-2,1)$?

Answer

The x-coordinates of the vertices and foci are the same, so the major axis is parallel to the y-axis. Thus, the equation of the ellipse will have the form

$$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1 \tag{8.2.8}$$

First, we identify the center, (h,k). The center is halfway between the vertices, $(-2,-8)$ and $(-2,2)$. Applying the midpoint formula, we have:

$$\begin{aligned}(h,k)&=\left(\frac{-2+(-2)}{2},\frac{-8+2}{2}\right)\\&=(-2,-3)\end{aligned}$$

Next, we find a^2. The length of the major axis, $2a$, is bounded by the vertices. We solve for a by finding the distance between the y-coordinates of the vertices.

$$\begin{aligned}2a&=2-(-8)\\2a&=10\\a&=5\end{aligned} \tag{8.2.9}$$

So $a^2=25$.
Now we find c^2. The foci are given by $(h,k\pm c)$. So, $(h,k-c)=(-2,-7)$ and $(h,k+c)=(-2,1)$. We substitute $k=-3$ using either of these points to solve for c.

$$\begin{aligned}k+c&=1\\-3+c&=1\\c&=4\end{aligned} \tag{8.2.10}$$

So $c^2=16$.
Next, we solve for b^2 using the equation $c^2=a^2-b^2$.

$$\begin{aligned}c^2&=a^2-b^2\\16&=25-b^2\\b^2&=9\end{aligned} \tag{8.2.11}$$

Finally, we substitute the values found for h,k,a^2, and b^2 into the standard form equation for an ellipse:

$$\frac{(x+2)^2}{9}+\frac{(y+3)^2}{25}=1 \tag{8.2.12}$$

Try It

What is the standard form equation of the ellipse that has vertices $(-3,3)$ and $(5,3)$ and foci $(1-2\sqrt{3},3)$ and $(1+2\sqrt{3},3)$?

Graphing Ellipses Centered at the Origin

Just as we can write the equation for an ellipse given its graph, we can graph an ellipse given its equation. To graph ellipses centered at the origin, we use the standard form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, a>b$ for horizontal ellipses and $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1, a>b$ for vertical

ellipses.

How To

Given the standard form of an equation for an ellipse centered at $(0,0)$, sketch the graph.

1. Use the standard forms of the equations of an ellipse to determine the major axis, vertices, covertices, and foci.

a. If the equation is in the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $a>b$, then

- the major axis is the x-axis
- the coordinates of the vertices are $(\pm a,0)$
- the coordinates of the co-vertices are $(0,\pm b)$
- the coordinates of the foci are $(\pm c,0)$

b. If the equation is in the form $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$, where $a>b$, then

- the major axis is the y-axis
- the coordinates of the vertices are $(0,\pm a)$
- the coordinates of the co-vertices are $(\pm b,0)$
- the coordinates of the foci are $(0,\pm c)$

2. Solve for c using the equation $c^2=a^2-b^2$.

3. Plot the center, vertices, co-vertices, and foci in the coordinate plane, and draw a smooth curve to form the ellipse.

Exercise 8.2.3

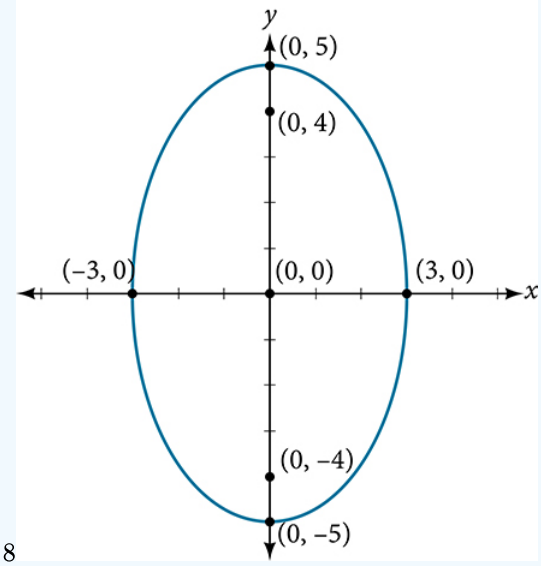
Graph the ellipse given by the equation, $\frac{x^2}{9}+\frac{y^2}{25}=1$. Identify and label the center, vertices, covertices, and foci.

Answer

First, we determine the position of the major axis. Because $25>9$, the major axis is on the y-axis. Therefore, the equation is in the form $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$, where $b^2=9$ and $a^2=25$. It follows that:
- the center of the ellipse is $(0,0)$
- the coordinates of the vertices are $(0,\pm a)=(0,\pm\sqrt{25})=(0,\pm 5)$
- the coordinates of the co-vertices are $(\pm b,0)=(\pm\sqrt{9},0)=(\pm 3,0)$
- the coordinates of the foci are $(0,\pm c)$, where $c^2=a^2-b^2$ Solving for c, we have:

$$\begin{aligned} c &= \pm\sqrt{a^2-b^2} \\ &= \pm\sqrt{25-9} \\ &= \pm\sqrt{16} \\ &= \pm 4 \end{aligned} \tag{8.2.13}$$

Therefore, the coordinates of the foci are $(0,\pm 4)$.
Next, we plot and label the center, vertices, co-vertices, and foci, and draw a smooth curve to form the ellipse. SeeFigure



8.2.8

Figure 8.2.8

Try It

Graph the ellipse given by the equation $\frac{x^2}{36}+\frac{y^2}{4}=1$. Identify and label the center, vertices, covertices, and foci.

Exercise 8.2.4

Graphing an Ellipse Centered at the Origin from an Equation Not in Standard Form

Graph the ellipse given by the equation $4x^2+25y^2=100$. Rewrite the equation in standard form. Then identify and label the center, vertices, co-vertices, and foci.

Answer

First, use algebra to rewrite the equation in standard form.

$$\begin{aligned} 4x^2+25y^2&=100\\ \frac{4x^2}{100}+\frac{25y^2}{100}&=\frac{100}{100}\\ \frac{x^2}{25}+\frac{y^2}{4}&=1 \end{aligned} \tag{8.2.14}$$

Next, we determine the position of the major axis. Because $25>4$, the major axis is on the x-axis. Therefore, the equation is in the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $a^2=25$ and $b^2=4$. It follows that:

- the center of the ellipse is $(0,0)$
- the coordinates of the vertices are $(\pm a,0)=(\pm\sqrt{25},0)=(\pm 5,0)$
- the coordinates of the co-vertices are $(0,\pm b)=(0,\pm\sqrt{4})=(0,\pm 2)$
- the coordinates of the foci are $(\pm c,0)$, where $c^2=a^2-b^2$. Solving for c, we have:

$$\begin{aligned} c &= \pm\sqrt{a^2 - b^2} \\ &= \pm\sqrt{25 - 4} \\ &= \pm\sqrt{21} \end{aligned} \tag{8.2.15}$$

Therefore the coordinates of the foci are $(\pm\sqrt{21}, 0)$.
Next, we plot and label the center, vertices, co-vertices, and foci, and draw a smooth curve to form the ellipse.

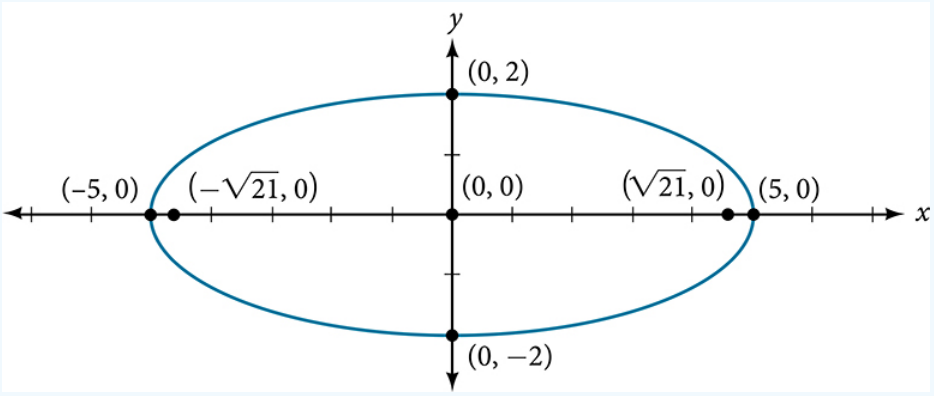


Figure 8.2.9

Try It

Graph the ellipse given by the equation $49x^2 + 16y^2 = 784$. Rewrite the equation in standard form. Then identify and label the center, vertices, co-vertices, and foci

Graphing Ellipses Not Centered at the Origin

When an ellipse is not centered at the origin, we can still use the standard forms to find the key features of the graph. When the ellipse is centered at some point, (h, k), we use the standard forms $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1, a > b$ for horizontal ellipses and $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1, a > b$ for vertical ellipses. From these standard equations, we can easily determine the center, vertices, co-vertices, foci, and positions of the major and minor axes.

How To

Given the standard form of an equation for an ellipse centered at (h, k), sketch the graph.

1. Use the standard forms of the equations of an ellipse to determine the center, position of the major axis, vertices, co-vertices, and foci.

a. If the equation is in the form $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, where $a > b$, then the center is (h, k)

- the major axis is parallel to the x-axis
- the coordinates of the vertices are $(h \pm a, k)$
- the coordinates of the co-vertices are $(h, k \pm b)$
- the coordinates of the foci are $(h \pm c, k)$

b. If the equation is in the form $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$, where $a > b$, then the center is (h, k)

- the major axis is parallel to the y-axis
- the coordinates of the vertices are $(h, k \pm a)$
- the coordinates of the co-vertices are $(h \pm b, k)$

- the coordinates of the foci are $(h, k \pm c)$

2. Solve for c using the equation $c^2 = a^2 - b^2$.

3. Plot the center, vertices, co-vertices, and foci in the coordinate plane, and draw a smooth curve to form the ellipse.

Exercise 8.2.5

Graphing an Ellipse Centered at (h, k)

Graph the ellipse given by the equation, $\frac{(x+2)^2}{4} + \frac{(y-5)^2}{9} = 1$. Identify and label the center, vertices, co-vertices, and foci.

Answer

First, we determine the position of the major axis. Because $9 > 4$, the major axis is parallel to the y axis. Therefore, the equation is in the form $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$, where $b^2 = 4$ and $a^2 = 9$. It follows that:

- the center of the ellipse is $(h, k) = (-2, 5)$
- the coordinates of the vertices are $(h, k \pm a) = (-2, 5 \pm \sqrt{9}) = (-2, 5 \pm 3)$, or $(-2, 2)$ and $(-2, 8)$
- the coordinates of the co-vertices are $(h \pm b, k) = (-2 \pm \sqrt{4}, 5) = (-2 \pm 2, 5)$, or $(-4, 5)$ and $(0, 5)$
- the coordinates of the foci are $(h, k \pm c)$, where $c^2 = a^2 - b^2$. Solving for c, we have:

$$\begin{aligned} c &= \pm\sqrt{a^2 - b^2} \\ &= \pm\sqrt{9-4} \\ &= \pm\sqrt{5} \end{aligned} \tag{8.2.16}$$

Therefore, the coordinates of the foci are $(-2, 5-\sqrt{5})$ and $(-2, 5+\sqrt{5})$.

Next, we plot and label the center, vertices, co-vertices, and foci, and draw a smooth curve to form the ellipse.

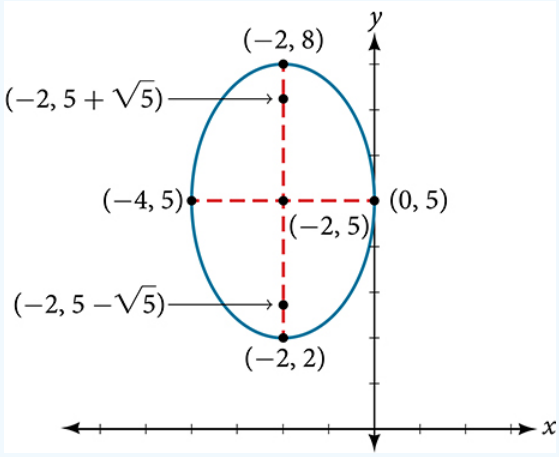


Figure 8.2.10

Try It

Graph the ellipse given by the equation $\frac{(x-4)^2}{36} + \frac{(y-2)^2}{20} = 1$. Identify and label the center, vertices, co-vertices, and foci.

How To

Given the general form of an equation for an ellipse centered at (h,k), express the equation in standard form.

1. Recognize that an ellipse described by an equation in the form $ax^2+by^2+cx+dy+e=0$ is in general form.

2. Rearrange the equation by grouping terms that contain the same variable. Move the constant term to the opposite side of the equation.

3. Factor out the coefficients of the x^2 and y^2 terms in preparation for completing the square.

4. Complete the square for each variable to rewrite the equation in the form of the sum of multiples of two binomials squared set equal to a constant, $m_1(x-h)^2+m_2(y-k)^2=m_3$, where m_1, m_2, and m_3 are constants.

5. Divide both sides of the equation by the constant term to express the equation in standard form.

Exercise 8.2.6

Graphing an Ellipse Centered at (h,k) by First Writing It in Standard FormGraph the ellipse given by the equation $4x^2+9y^2-40x+36y+100=0$. Identify and label the center, vertices, co-vertices, and foci.

Answer

We must begin by rewriting the equation in standard form.

$$4x^2+9y^2-40x+36y+100=0 \tag{8.2.17}$$

Group terms that contain the same variable, and move the constant to the opposite side of the equation.

$$\left(4x^2-40x\right)+\left(9y^2+36y\right)=-100 \tag{8.2.18}$$

Factor out the coefficients of the squared terms.

$$4\left(x^2-10x\right)+9\left(y^2+4y\right)=-100 \tag{8.2.19}$$

Complete the square twice. Remember to balance the equation by adding the same constants to each side.

$$4\left(x^2-10x+25\right)+9\left(y^2+4y+4\right)=-100+100+36 \tag{8.2.20}$$

Rewrite as perfect squares.

$$4(x-5)^2+9(y+2)^2=36 \tag{8.2.21}$$

Divide both sides by the constant term to place the equation in standard form.

$$\frac{(x-5)^2}{9}+\frac{(y+2)^2}{4}=1 \tag{8.2.22}$$

Now that the equation is in standard form, we can determine the position of the major axis. Because $9>4$, the major axis is parallel to the x-axis. Therefore, the equation is in the form $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$, where $a^2=9$ and $b^2=4$. It follows that:
the center of the ellipse is $(h,k)=(5,-2)$
the coordinates of the vertices are $(h\pm a,k)=(5\pm\sqrt{9},-2)=(5\pm3,-2)$, or $(2,-2)$ and $(8,-2)$
the coordinates of the co-vertices are $(h,k\pm b)=(5,-2\pm\sqrt{4})=(5,-2\pm2)$, or $(5,-4)$ and $(5,0)$
the coordinates of the foci are $(h\pm c,k)$, where $c^2=a^2-b^2$. Solving for c, we have:

$$\begin{aligned} c&=\pm\sqrt{a^2-b^2}\\ &=\pm\sqrt{9-4}\\ &=\pm\sqrt{5}\end{aligned}$$

Therefore, the coordinates of the foci are $(5-\sqrt{5},-2)$ and $(5+\sqrt{5},-2)$.
Next we plot and label the center, vertices, co-vertices, and foci, and draw a smooth curve to form the ellipse as shown in Figure 8.2.11

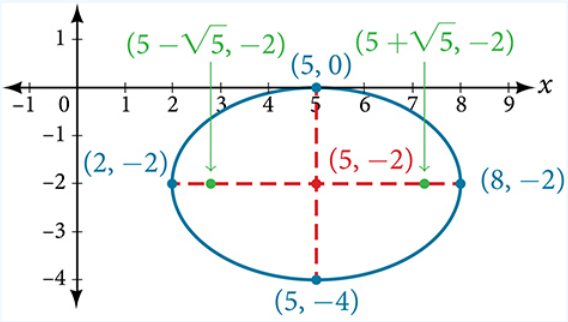


Figure 8.2.11

Try It

Graph the ellipse given by the equation $\frac{(x-4)^2}{36}+\frac{(y-2)^2}{20}=1$. Identify and label the center, vertices, co-vertices, and foci.

Solving Applied Problems Involving Ellipses

Many real-world situations can be represented by ellipses, including orbits of planets, satellites, moons and comets, and shapes of boat keels, rudders, and some airplane wings. A medical device called a lithotripter uses elliptical reflectors to break up kidney stones by generating sound waves. Some buildings, called whispering chambers, are designed with elliptical domes so that a person whispering at one focus can easily be heard by someone standing at the other focus. This occurs because of the acoustic properties of an ellipse. When a sound wave originates at one focus of a whispering chamber, the sound wave will be reflected off the elliptical dome and back to the other focus. See Figure 8.2.12 In the whisper chamber at the Museum of Science and Industry in Chicago, two people standing at the foci—about 43 feet apart—can hear each other whisper.

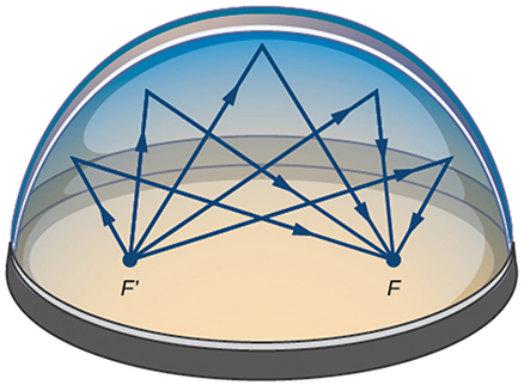


Figure 8.2.12 Sound waves are reflected between foci in an elliptical room, called a whispering chamber.

Exercise 8.2.7

Locating the Foci of a Whispering Chamber

The Statuary Hall in the Capitol Building in Washington, D.C. is a whispering chamber. Its dimensions are 46 feet wide by 96 feet long as shown in Figure 8.2.13 .

a. What is the standard form of the equation of the ellipse representing the outline of the room? Hint: assume a horizontal ellipse, and let the center of the room be the point (0,0).
b. If two senators standing at the foci of this room can hear each other whisper, how far apart are the senators? Round to the nearest foot.

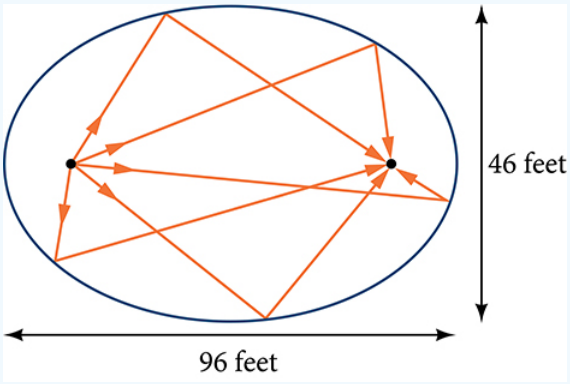


Figure 8.2.13

Answer

a. We are assuming a horizontal ellipse with center $(0,0)$, so we need to find an equation of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $a>b$. We know that the length of the major axis, $2a$, is longer than the length of the minor axis, $2b$. So the length of the room, 96 , is represented by the major axis, and the width of the room, 46 , is represented by the minor axis.

- Solving for a, we have $2a=96$, so $a=48$, and $a^2=2304$.
- Solving for b, we have $2b=46$, so $b=23$, and $b^2=529$.

Therefore, the equation of the ellipse is $\frac{x^2}{2304}+\frac{y^2}{529}=1$.

b. To find the distance between the senators, we must find the distance between the foci, $(\pm c,0)$, where $c^2=a^2-b^2$. Solving for c, we have:

$c^2=a^2-b^2$	
$c^2=2304-529$	Substitute using the values found in part (a).
$c=\pm\sqrt{2304-529}$	Take the square root of both sides.
$c=\pm\sqrt{1775}$	Subtract
$c\approx\pm 42$	Round to the nearest foot.

Try It

Suppose a whispering chamber is 480 feet long and 320 feet wide.

a. What is the standard form of the equation of the ellipse representing the room? Hint: assume a horizontal ellipse, and let the center of the room be the point (0,0).(0,0).

b. If two people are standing at the foci of this room and can hear each other whisper, how far apart are the people? Round to the nearest foot

Media

Access these online resources for additional instruction and practice with ellipses.

Conic Sections: The Ellipse



Graph an Ellipse with Center at the Origin



Graph an Ellipse with Center Not at the Origin



10.2 Section Exercises

Verbal

1. Define an ellipse in terms of its foci.

2. Where must the foci of an ellipse lie?

3. What special case of the ellipse do we have when the major and minor axis are of the same length?

4. For the special case mentioned in the previous question, what would be true about the foci of that ellipse?

5. What can be said about the symmetry of the graph of an ellipse with center at the origin and foci along the y-axis?

Algebraic

For the following exercises, determine whether the given equations represent ellipses. If yes, write in standard form.

6. $2x^2+y=4$

7. $4x^2+9y^2=36$

8. $4x^2-y^2=4$

9. $4x^2+9y^2=1$

10. $4x^2-8x+9y^2-72y+112=0$

For the following exercises, write the equation of an ellipse in standard form, and identify the end points of the major and minor axes as well as the foci.

11. $\frac{x^2}{4}+\frac{y^2}{49}=1$

12. $\frac{x^2}{100}+\frac{y^2}{64}=1$

13. $x^2+9y^2=1$

14. $4x^2+16y^2=1$

15. $\frac{(x-2)^2}{49}+\frac{(y-4)^2}{25}=1$

16. $\frac{(x-2)^2}{81}+\frac{(y+1)^2}{16}=1$

17. $\frac{(x+5)^2}{4}+\frac{(y-7)^2}{9}=1$

18. $\frac{(x-7)^2}{49}+\frac{(y-7)^2}{49}=1$

19. $4x^2-8x+9y^2-72y+112=0$
20. $9x^2-54x+9y^2-54y+81=0$
21. $4x^2-24x+36y^2-360y+864=0$
22. $4x^2+24x+16y^2-128y+228=0$
23. $4x^2+40x+25y^2-100y+100=0$
24. $x^2+2x+100y^2-1000y+2401=0$
25. $4x^2+24x+25y^2+200y+336=0$
26. $9x^2+72x+16y^2+16y+4=0$

For the following exercises, find the foci for the given ellipses.

27. $\frac{(x+3)^2}{25}+\frac{(y+1)^2}{36}=1$
28. $\frac{(x+1)^2}{100}+\frac{(y-2)^2}{4}=1$
29. $x^2+y^2=1$
30. $x^2+4y^2+4x+8y=1$
31. $10x^2+y^2+200x=0$

Graphical

For the following exercises, graph the given ellipses, noting center, vertices, and foci.

32. $\frac{x^2}{25}+\frac{y^2}{36}=1$
33. $\frac{x^2}{16}+\frac{y^2}{9}=1$
34. $4x^2+9y^2=1$
35. $81x^2+49y^2=1$
36. $\frac{(x-2)^2}{64}+\frac{(y-4)^2}{16}=1$
37. $\frac{(x+3)^2}{9}+\frac{(y-3)^2}{9}=1$
38. $\frac{x^2}{2}+\frac{(y+1)^2}{5}=1$
39. $4x^2-8x+16y^2-32y-44=0$
40. $x^2-8x+25y^2-100y+91=0$
41. $x^2+8x+4y^2-40y+112=0$
42. $64x^2+128x+9y^2-72y-368=0$
43. $16x^2+64x+4y^2-8y+4=0$
44. $100x^2+1000x+y^2-10y+2425=0$
45. $4x^2+16x+4y^2+16y+16=0$

For the following exercises, use the given information about the graph of each ellipse to determine its equation.

46. Center at the origin, symmetric with respect to the x - and y-axes, focus at $(4,0)$, and point on graph $(0,3)$.
47. Center at the origin, symmetric with respect to the x - and y-axes, focus at $(0,-2)$, and point on graph $(5,0)$.
48. Center at the origin, symmetric with respect to the x - and y-axes, focus at $(3,0)$, and major axis is twice as long as minor axis.

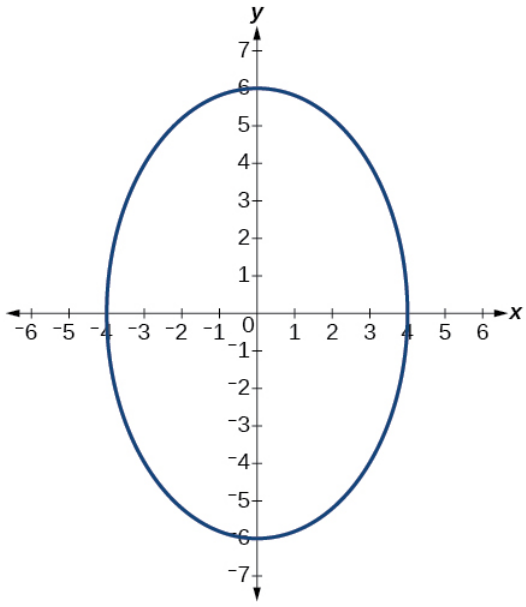
49. Center $(4, 2)$; vertex $(9, 2)$; one focus: $(4+2\sqrt{6}, 2)$.

50. Center $(3, 5)$; vertex $(3, 11)$; one focus: $(3, 5+4\sqrt{2})$

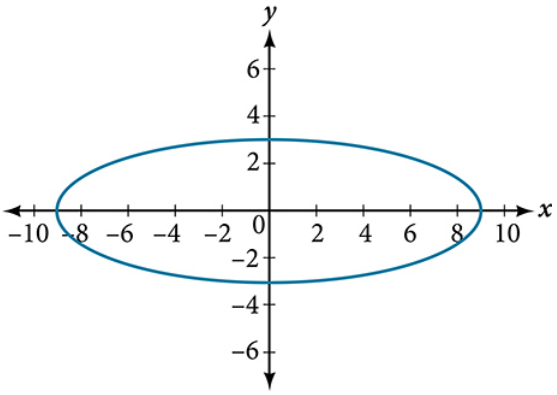
51. Center $(-3, 4)$; vertex $(1, 4)$; one focus: $(-3+2\sqrt{3}, 4)$

For the following exercises, given the graph of the ellipse, determine its equation.

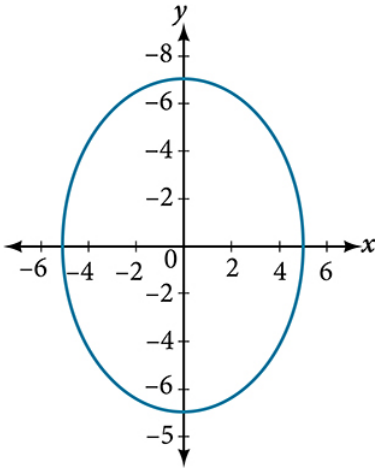
52.



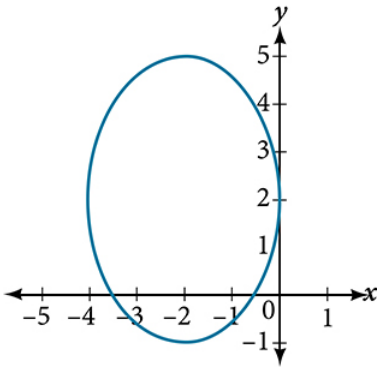
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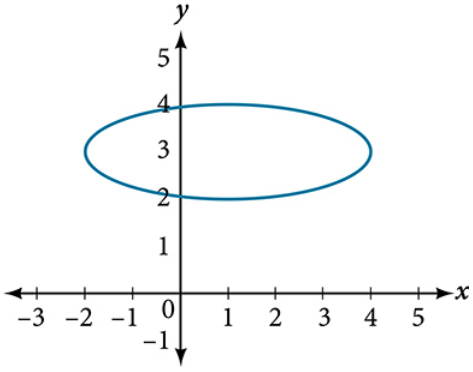
54.



55.



56.



Extensions

For the following exercises, find the area of the ellipse. The area of an ellipse is given by the formula Area $= a \cdot b \cdot \pi$.

57. $\frac{(x-3)^2}{9} + \frac{(y-3)^2}{16} = 1$

58. $\frac{(x+6)^2}{16} + \frac{(y-6)^2}{36} = 1$

59. $\frac{(x+1)^2}{4} + \frac{(y-2)^2}{5} = 1$

60. $4x^2 - 8x + 9y^2 - 72y + 112 = 0$

61. $9x^2 - 54x + 9y^2 - 54y + 81 = 0$

Real-World Applications

62. Find the equation of the ellipse that will just fit inside a box that is 8 units wide and 4 units high.

63. Find the equation of the ellipse that will just fit inside a box that is four times as wide as it is high. Express in terms of h, the height.

64. An arch has the shape of a semi-ellipse (the top half of an ellipse). The arch has a height of 8 feet and a span of 20 feet. Find an equation for the ellipse, and use that to find the height to the nearest 0.01 foot of the arch at a distance of 4 feet from the center.

65. An arch has the shape of a semi-ellipse. The arch has a height of 12 feet and a span of 40 feet. Find an equation for the ellipse, and use that to find the distance from the center to a point at which the height is 6 feet. Round to the nearest hundredth.

66. A bridge is to be built in the shape of a semi-elliptical arch and is to have a span of 120 feet. The height of the arch at a distance of 40 feet from the center is to be 8 feet. Find the height of the arch at its center.

67. A person in a whispering gallery standing at one focus of the ellipse can whisper and be heard by a person standing at the other focus because all the sound waves that reach the ceiling are reflected to the other person. If a whispering gallery has a length of 120 feet, and the foci are located 30 feet from the center, find the height of the ceiling at the center.

68. A person is standing 8 feet from the nearest wall in a whispering gallery. If that person is at one focus, and the other focus is 80 feet away, what is the length and height at the center of the gallery?

Footnotes

1. Architect of the Capitol. http://www.aoc.gov. Accessed April 15, 2014.

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8.3: The Hyperbola

In this section, you will:

- Locate a hyperbola's vertices and foci.
- Write equations of hyperbolas in standard form.
- Graph hyperbolas centered at the origin.
- Graph hyperbolas not centered at the origin.
- Solve applied problems involving hyperbolas.

What do paths of comets, supersonic booms, ancient Grecian pillars, and natural draft cooling towers have in common? They can all be modeled by the same type of conic. For instance, when something moves faster than the speed of sound, a shock wave in the form of a cone is created. A portion of a conic is formed when the wave intersects the ground, resulting in a sonic boom. See Figure 1.

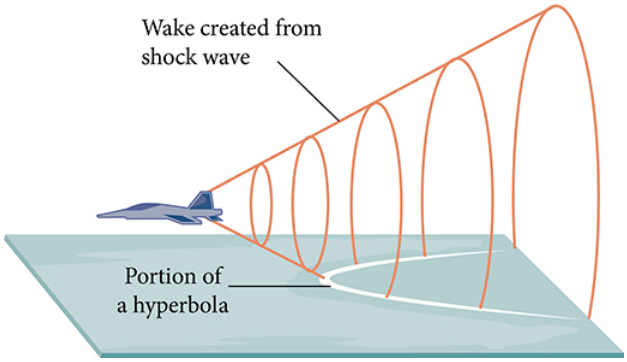


Figure **1** A shock wave intersecting the ground forms a portion of a conic and results in a sonic boom.

Most people are familiar with the sonic boom created by supersonic aircraft, but humans were breaking the sound barrier long before the first supersonic flight. The crack of a whip occurs because the tip is exceeding the speed of sound. The bullets shot from many firearms also break the sound barrier, although the bang of the gun usually supersedes the sound of the sonic boom.

Locating the Vertices and Foci of a Hyperbola

In analytic geometry, a **hyperbola** is a conic section formed by intersecting a right circular cone with a plane at an angle such that both halves of the cone are intersected. This intersection produces two separate unbounded curves that are mirror images of each other. See Figure 2.

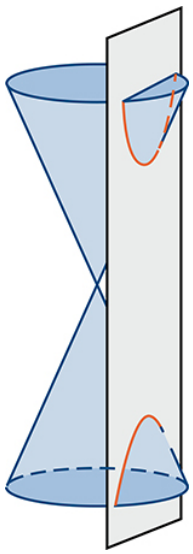


Figure **2** A hyperbola

Like the ellipse, the hyperbola can also be defined as a set of points in the coordinate plane. A hyperbola is the set of all points (x,y)(x,y) in a plane such that the difference of the distances between (x,y)(x,y) and the foci is a positive constant.

Notice that the definition of a hyperbola is very similar to that of an ellipse. The distinction is that the hyperbola is defined in terms of the *difference* of two distances, whereas the ellipse is defined in terms of the *sum* of two distances.

As with the ellipse, every hyperbola has two axes of symmetry. The **transverse axis** is a line segment that passes through the center of the hyperbola and has vertices as its endpoints. The foci lie on the line that contains the transverse axis. The **conjugate axis** is perpendicular to the transverse axis and has the co-vertices as its endpoints. The **center of a hyperbola** is the midpoint of both the transverse and conjugate axes, where they intersect. Every hyperbola also has two **asymptotes** that pass through its center. As a hyperbola recedes from the center, its branches approach these asymptotes. The **central rectangle** of the hyperbola is centered at the origin with sides that pass through each vertex and co-vertex; it is a useful tool for graphing the hyperbola and its asymptotes. To sketch the asymptotes of the hyperbola, simply sketch and extend the diagonals of the central rectangle. See Figure 3.

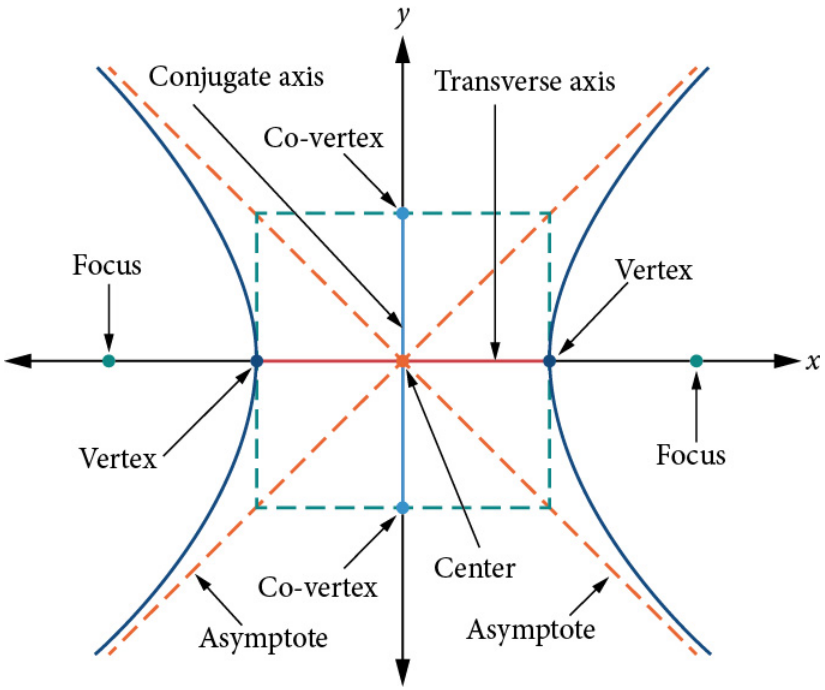


Figure **3** Key features of the hyperbola

In this section, we will limit our discussion to hyperbolas that are positioned vertically or horizontally in the coordinate plane; the axes will either lie on or be parallel to the *x*- and *y*-axes. We will consider two cases: those that are centered at the origin, and those that are centered at a point other than the origin.

Deriving the Equation of a Hyperbola Centered at the Origin

Let (−c,0)(−c,0) and (c,0)(c,0) be the foci of a hyperbola centered at the origin. The hyperbola is the set of all points (x,y)(x,y) such that the difference of the distances from (x,y)(x,y) to the foci is constant. See Figure 4.

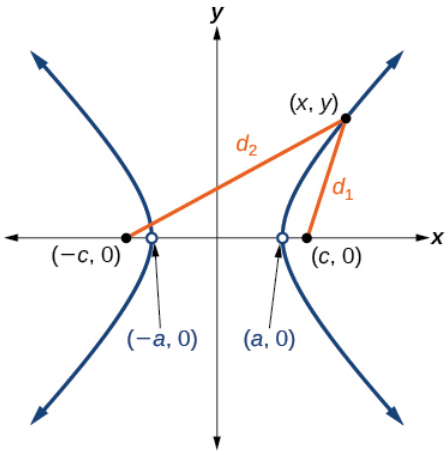


Figure **4**

If (a,0)(a,0) is a vertex of the hyperbola, the distance from (−c,0)(−c,0) to (a,0)(a,0) is a−(−c)=a+c.a−(−c)=a+c. The distance from (c,0)(c,0) to (a,0)(a,0) is c−a.c−a. The difference of the distances from the foci to the vertex is

(a+c)−(c−a)=2a(a+c)−(c−a)=2a

If (x,y)(x,y) is a point on the hyperbola, we can define the following variables:

d2=the distance from (−c,0)to (x,y)d1=the distance from (c,0)to (x,y)d2=the distance from (−c,0)to (x,y)d1=the distance from (c,0)to (x,y)

By definition of a hyperbola, d2−d1d2−d1 is constant for any point (x,y)(x,y) on the hyperbola. We know that the difference of these distances is 2a2a for the vertex (a,0).(a,0). It follows that d2−d1=2ad2−d1=2a for any point on the hyperbola. As with the derivation of the equation of an ellipse, we will begin by applying the distance formula. The rest of the derivation is algebraic. Compare this derivation with the one from the previous section for ellipses.

d2−d1=(x−(−c))2+(y−0)2−−−−−−−−−−−−−−−−−−√−(x−c)2+(y−0)2−−−−−−−−−−−−−−−−√=2a(x+c)2+y2−−−−−−−−−−√−(x−c)2+y2−−−−−−−−−−√=2a (x+c)2+y2−−−−−−−−−−√=2a+(x−c)2+y2−−−−−−−−−−√ (x+c)2+y2=(2a+(x−c)2+y2−−−−−−−−−−√)2 x2+2cx+c2+y2=4a2+4a(x−c)2+y2−−−−−−−−−−√+(x−c)2+y2 x2+2cx+c2+y2=4a2+4a(x−c)2+y2−−−−−−−−−−√+x2−2cx+c2+y2 2cx=4a2+4a(x−c)2+y2−−−−−−−−−−√−2cx 4cx−4a2=4a(x−c)2+y2−−−−−−−−−−√ cx−a2=a(x−c)2+y2−−−−−−−−−√ (cx−a2)2=a2((x−c)2+y2−−−−−−−−−−√)2 c2x2−2a2cx+a4=a2(x2−2cx+c2+y2) c2x2−2a2cx+a4=a2x2−2a2cx+a2c2+a2y2 a4+c2x2=a2x2+a2c2+a2y2 c2x2−a2x2−a2y2=a2c2−a4 x2(c2−a2)−a2y2=a2(c2−a2) x2b2−a2y2=a2b2 x2b2a2b2−a2y2a2b2=a2b2a2b2 x2a2−y2b2=1Distance FormulaSimplify expressions.Move radical to opposite side.Square both sides.Expand the squares.Expand remaining square.Combine like terms.Isolate the radical.Divide by 4.Square both sides.Expand the squares.Distribute a2.Combine like terms.Rearrange terms.Factor common terms.Set b2=c2−a2.Divide both sides by a2b2 d2−d1=(x−(−c))2+(y−0)2−(x−c)2+(y−0)2=2aDistance Formula(x+c)2+y2−(x−c)2+y2=2aSimplify expressions. (x+c)2+y2=2a+(x−c)2+y2Move radical to opposite side. (x+c)2+y2=(2a+(x−c)2+y2)2Square both sides. x2+2cx+c2+y2=4a2+4a(x−c)2+y2+(x−c)2+y2Expand the squares. x2+2cx+c2+y2=4a2+4a(x−c)2+y2+x2−2cx+c2+y2Expand remaining square. 2cx=4a2+4a(x−c)2+y2−2cxCombine like terms. 4cx−4a2=4a(x−c)2+y2Isolate the radical. cx−a2=a(x−c)2+y2Divide by 4. (cx−a2)2=a2((x−c)2+y2)2Square both sides. c2x2−2a2cx+a4=a2(x2−2cx+c2+y2)Expand the squares. c2x2−2a2cx+a4=a2x2−2a2cx+a2c2+a2y2Distribute a2. a4+c2x2=a2x2+a2c2+a2y2Combine like terms. c2x2−a2x2−a2y2=a2c2−a4Rearrange terms. x2(c2−a2)−a2y2=a2(c2−a2)Factor common terms. x2b2−a2y2=a2b2Set b2=c2−a2. x2b2a2b2−a2y2a2b2=a2b2a2b2Divide both sides by a2b2 x2a2−y2b2=1

This equation defines a hyperbola centered at the origin with vertices (±a,0)(±a,0) and co-vertices (0±b).(0±b).

The standard form of the equation of a hyperbola with center (0,0)(0,0) and transverse axis on the *x*-axis is

x2a2−y2b2=1x2a2−y2b2=1

where

- the length of the transverse axis is 2a2a

- the coordinates of the vertices are (±a,0)(±a,0)
- the length of the conjugate axis is 2b2b
- the coordinates of the co-vertices are (0,±b)(0,±b)
- the distance between the foci is 2c,2c, where c2=a2+b2c2=a2+b2
- the coordinates of the foci are (±c,0)(±c,0)
- the equations of the asymptotes are y=±baxy=±bax

See Figure 5**a**.

The standard form of the equation of a hyperbola with center (0,0)(0,0) and transverse axis on the *y*-axis is

y2a2−x2b2=1y2a2−x2b2=1

where

- the length of the transverse axis is 2a2a
- the coordinates of the vertices are (0,±a)(0,±a)
- the length of the conjugate axis is 2b2b
- the coordinates of the co-vertices are (±b,0)(±b,0)
- the distance between the foci is 2c,2c, where c2=a2+b2c2=a2+b2
- the coordinates of the foci are (0,±c)(0,±c)
- the equations of the asymptotes are y=±abxy=±abx

See Figure 5**b**.

Note that the vertices, co-vertices, and foci are related by the equation c2=a2+b2.c2=a2+b2. When we are given the equation of a hyperbola, we can use this relationship to identify its vertices and foci.

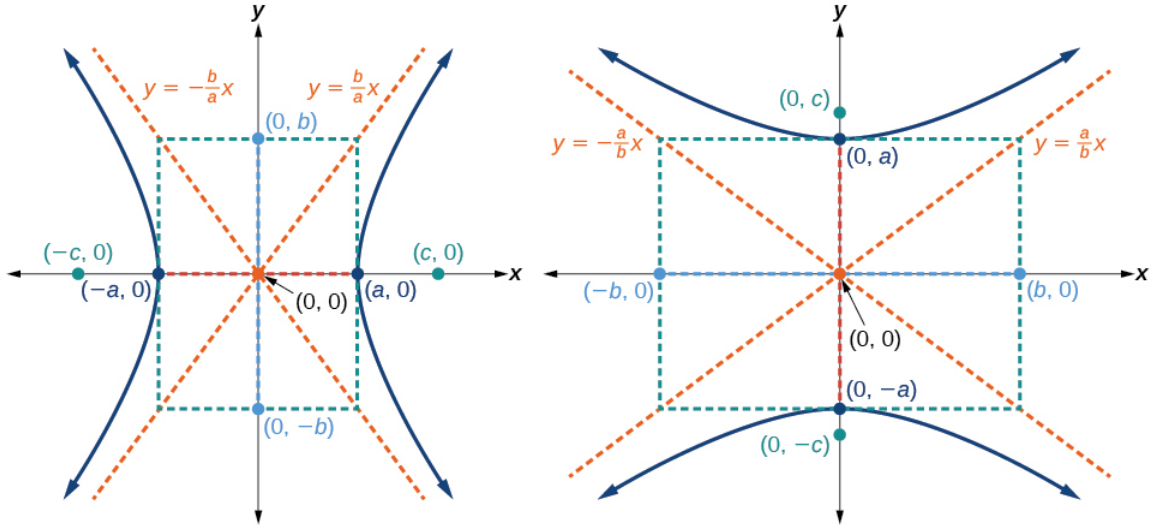


Figure **5** (a) Horizontal hyperbola with center (0,0)(0,0) (b) Vertical hyperbola with center (0,0)(0,0)

Given the equation of a hyperbola in standard form, locate its vertices and foci.

1. Determine whether the transverse axis lies on the *x*- or *y*-axis. Notice that a2a2 is always under the variable with the positive coefficient. So, if you set the other variable equal to zero, you can easily find the intercepts. In the case where the hyperbola is centered at the origin, the intercepts coincide with the vertices.
   a. If the equation has the form x2a2−y2b2=1,x2a2−y2b2=1, then the transverse axis lies on the *x*-axis. The vertices are located at (±a,0),(±a,0), and the foci are located at (±c,0).(±c,0).
   b. If the equation has the form y2a2−x2b2=1,y2a2−x2b2=1, then the transverse axis lies on the *y*-axis. The vertices are located at (0,±a),(0,±a), and the foci are located at (0,±c).(0,±c).
2. Solve for aa using the equation a=a2−−√.a=a2.
3. Solve for cc using the equation c=a2+b2−−−−−−√.c=a2+b2.

EXAMPLE 1

Locating a Hyperbola's Vertices and Foci

Identify the vertices and foci of the hyperbola with equation y249−x232=1.y249−x232=1.

Answer

Identify the vertices and foci of the hyperbola with equation x29−y225=1.x29−y225=1.

Writing Equations of Hyperbolas in Standard Form

Just as with ellipses, writing the equation for a hyperbola in standard form allows us to calculate the key features: its center, vertices, co-vertices, foci, asymptotes, and the lengths and positions of the transverse and conjugate axes. Conversely, an equation for a hyperbola can be found given its key features. We begin by finding standard equations for hyperbolas centered at the origin. Then we will turn our attention to finding standard equations for hyperbolas centered at some point other than the origin.

Hyperbolas Centered at the Origin

Reviewing the standard forms given for hyperbolas centered at (0,0),(0,0), we see that the vertices, co-vertices, and foci are related by the equation c2=a2+b2.c2=a2+b2. Note that this equation can also be rewritten as b2=c2−a2.b2=c2−a2. This relationship is used to write the equation for a hyperbola when given the coordinates of its foci and vertices.

Given the vertices and foci of a hyperbola centered at (0,0),(0,0), write its equation in standard form.

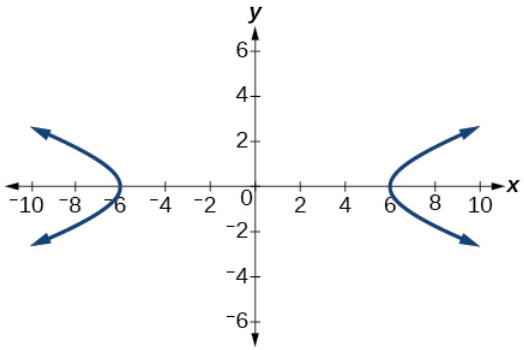
1. Determine whether the transverse axis lies on the *x*- or *y*-axis.
   a. If the given coordinates of the vertices and foci have the form (±a,0)(±a,0) and (±c,0),(±c,0), respectively, then the transverse axis is the *x*-axis. Use the standard form x2a2−y2b2=1.x2a2−y2b2=1.
   b. If the given coordinates of the vertices and foci have the form (0,±a)(0,±a) and (0,±c),(0,±c), respectively, then the transverse axis is the *y*-axis. Use the standard form y2a2−x2b2=1.y2a2−x2b2=1.
2. Find b2b2 using the equation b2=c2−a2.b2=c2−a2.
3. Substitute the values for a2a2 and b2b2 into the standard form of the equation determined in Step 1.

EXAMPLE 2

Finding the Equation of a Hyperbola Centered at (0,0) Given its Foci and Vertices

What is the standard form equation of the hyperbola that has vertices (±6,0)(±6,0) and foci (±210−−√,0)?(±210,0)?

Answer



What is the standard form equation of the hyperbola that has vertices (0,±2)(0,±2) and foci (0,±25–√)?(0,±25)?

Hyperbolas Not Centered at the Origin

Like the graphs for other equations, the graph of a hyperbola can be translated. If a hyperbola is translated hh units horizontally and kk units vertically, the center of the hyperbola will be (h,k).(h,k). This translation results in the standard form of the equation we saw previously, with xx replaced by (x−h)(x−h) and yy replaced by (y−k).(y−k).

The standard form of the equation of a hyperbola with center (h,k)(h,k) and transverse axis parallel to the *x*-axis is

(x−h)2a2−(y−k)2b2=1(x−h)2a2−(y−k)2b2=1

where

- the length of the transverse axis is 2a2a
- the coordinates of the vertices are (h±a,k)(h±a,k)
- the length of the conjugate axis is 2b2b
- the coordinates of the co-vertices are (h,k±b)(h,k±b)
- the distance between the foci is 2c,2c, where c2=a2+b2c2=a2+b2
- the coordinates of the foci are (h±c,k)(h±c,k)

The asymptotes of the hyperbola coincide with the diagonals of the central rectangle. The length of the rectangle is 2a2a and its width is 2b.2b. The slopes of the diagonals are ±ba,±ba, and each diagonal passes through the center (h,k).(h,k). Using the **point-slope formula**, it is simple to show that the equations of the asymptotes are y=±ba(x−h)+k.y=±ba(x−h)+k. See Figure 7**a**

The standard form of the equation of a hyperbola with center (h,k)(h,k) and transverse axis parallel to the *y*-axis is

(y−k)2a2−(x−h)2b2=1(y−k)2a2−(x−h)2b2=1

where

- the length of the transverse axis is 2a2a
- the coordinates of the vertices are (h,k±a)(h,k±a)
- the length of the conjugate axis is 2b2b
- the coordinates of the co-vertices are (h±b,k)(h±b,k)
- the distance between the foci is 2c,2c, where c2=a2+b2c2=a2+b2
- the coordinates of the foci are (h,k±c)(h,k±c)

Using the reasoning above, the equations of the asymptotes are y=±ab(x−h)+k.y=±ab(x−h)+k. See Figure 7**b**.

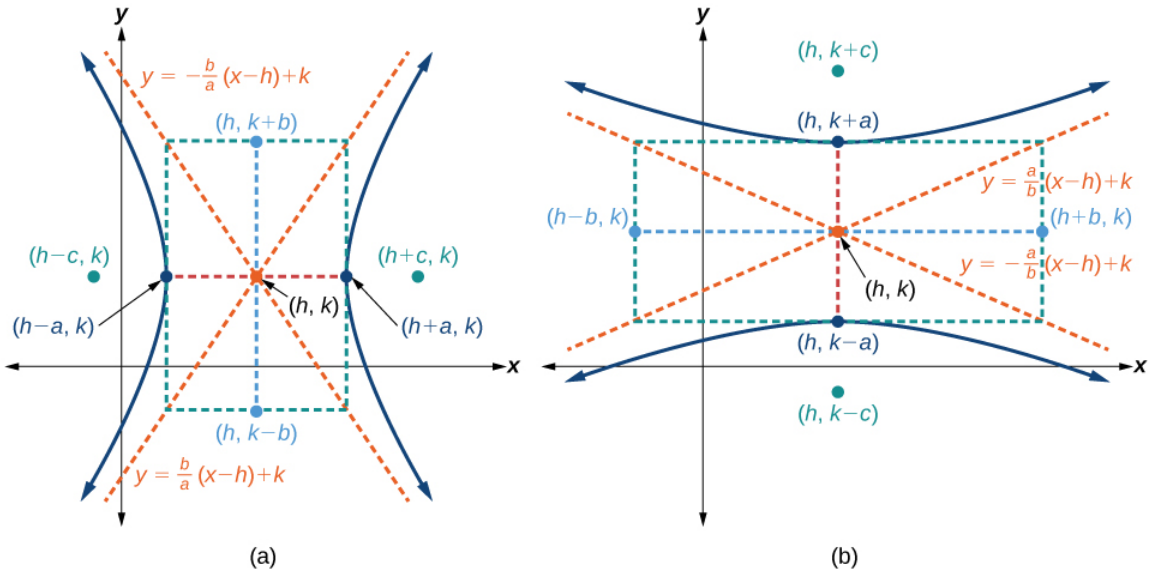


Figure **7** (a) Horizontal hyperbola with center (h,k)(h,k) (b) Vertical hyperbola with center (h,k)(h,k)

Like hyperbolas centered at the origin, hyperbolas centered at a point (h,k)(h,k) have vertices, co-vertices, and foci that are related by the equation c2=a2+b2.c2=a2+b2. We can use this relationship along with the midpoint and distance formulas to find the

standard equation of a hyperbola when the vertices and foci are given.

Given the vertices and foci of a hyperbola centered at (h,k),(h,k), write its equation in standard form.

1. Determine whether the transverse axis is parallel to the *x*- or *y*-axis.
   a. If the *y*-coordinates of the given vertices and foci are the same, then the transverse axis is parallel to the *x*-axis. Use the standard form (x−h)2a2−(y−k)2b2=1.(x−h)2a2−(y−k)2b2=1.
   b. If the *x*-coordinates of the given vertices and foci are the same, then the transverse axis is parallel to the *y*-axis. Use the standard form (y−k)2a2−(x−h)2b2=1.(y−k)2a2−(x−h)2b2=1.
2. Identify the center of the hyperbola, (h,k),(h,k), using the midpoint formula and the given coordinates for the vertices.
3. Find a2a2 by solving for the length of the transverse axis, 2a2a , which is the distance between the given vertices.
4. Find c2c2 using hh and kk found in Step 2 along with the given coordinates for the foci.
5. Solve for b2b2 using the equation b2=c2−a2.b2=c2−a2.
6. Substitute the values for h,k,a2,h, k, a2, and b2b2 into the standard form of the equation determined in Step 1.

EXAMPLE 3

Finding the Equation of a Hyperbola Centered at (*h*, *k*) Given its Foci and Vertices

What is the standard form equation of the hyperbola that has vertices at (0,−2)(0,−2) and (6,−2)(6,−2) and foci at (−2,−2)(−2,−2) and (8,−2)?(8,−2)?

Answer

What is the standard form equation of the hyperbola that has vertices (1,−2)(1,−2) and (1,8)(1,8) and foci (1,−10)(1,−10) and (1,16)?(1,16)?

Graphing Hyperbolas Centered at the Origin

When we have an equation in standard form for a hyperbola centered at the origin, we can interpret its parts to identify the key features of its graph: the center, vertices, co-vertices, asymptotes, foci, and lengths and positions of the transverse and conjugate axes. To graph hyperbolas centered at the origin, we use the standard form x2a2−y2b2=1x2a2−y2b2=1 for horizontal hyperbolas and the standard form y2a2−x2b2=1y2a2−x2b2=1 for vertical hyperbolas.

Given a standard form equation for a hyperbola centered at (0,0),(0,0), sketch the graph.

1. Determine which of the standard forms applies to the given equation.
2. Use the standard form identified in Step 1 to determine the position of the transverse axis; coordinates for the vertices, co-vertices, and foci; and the equations for the asymptotes.
   a. If the equation is in the form x2a2−y2b2=1, x2a2−y2b2=1, then
      - the transverse axis is on the *x*-axis
      - the coordinates of the vertices are (±a,0)(±a,0)
      - the coordinates of the co-vertices are (0,±b)(0,±b)
      - the coordinates of the foci are (±c,0)(±c,0)
      - the equations of the asymptotes are y=±baxy=±bax
   b. If the equation is in the form y2a2−x2b2=1,y2a2−x2b2=1, then
      - the transverse axis is on the *y*-axis
      - the coordinates of the vertices are (0,±a)(0,±a)
      - the coordinates of the co-vertices are (±b,0)(±b,0)
      - the coordinates of the foci are (0,±c)(0,±c)
      - the equations of the asymptotes are y=±abxy=±abx
3. Solve for the coordinates of the foci using the equation c=±a2+b2−−−−−−√.c=±a2+b2.

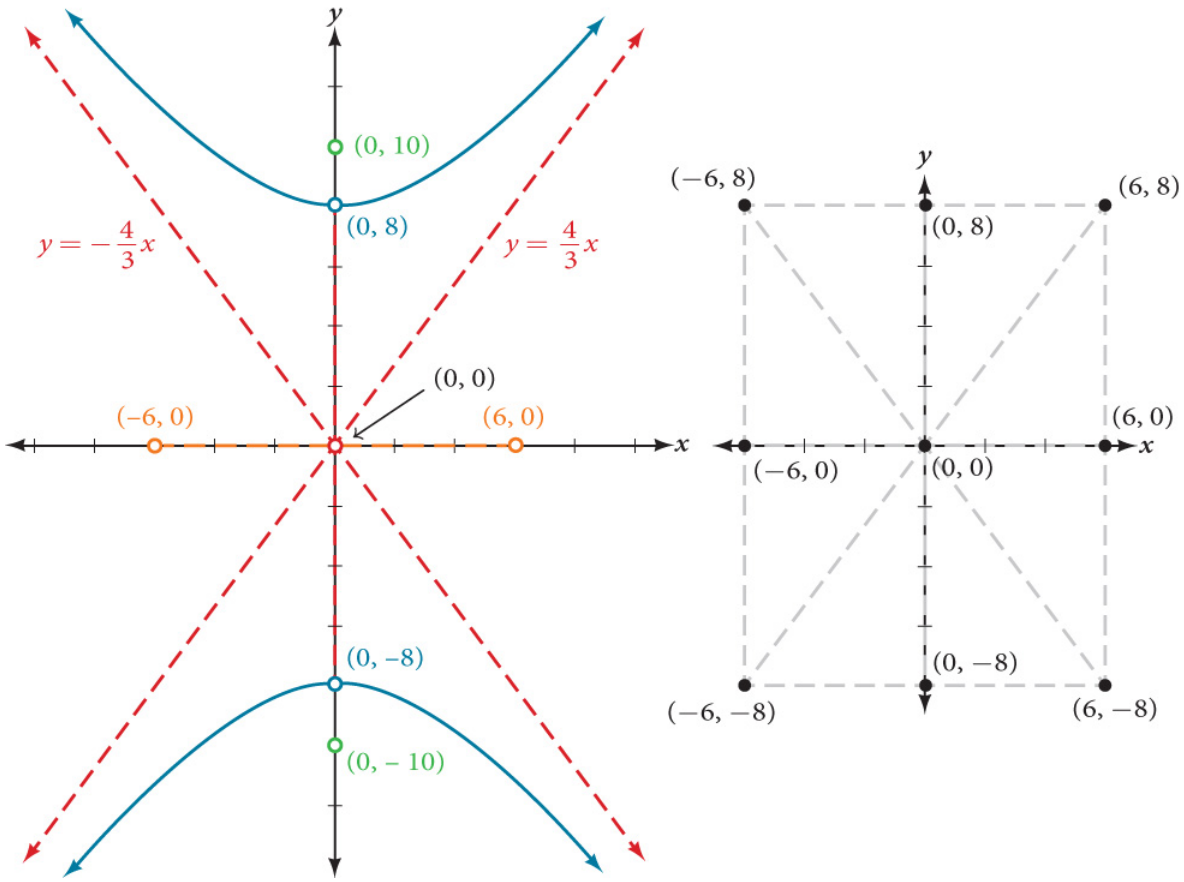
4. Plot the vertices, co-vertices, foci, and asymptotes in the coordinate plane, and draw a smooth curve to form the hyperbola.

EXAMPLE 4

Graphing a Hyperbola Centered at (0, 0) Given an Equation in Standard Form

Graph the hyperbola given by the equation y264−x236=1. y264−x236=1. Identify and label the vertices, co-vertices, foci, and asymptotes.

Answer



Graph the hyperbola given by the equation x2144−y281=1.x2144−y281=1. Identify and label the vertices, co-vertices, foci, and asymptotes.

Graphing Hyperbolas Not Centered at the Origin

Graphing hyperbolas centered at a point (h,k)(h,k) other than the origin is similar to graphing ellipses centered at a point other than the origin. We use the standard forms (x−h)2a2−(y−k)2b2=1(x−h)2a2−(y−k)2b2=1 for horizontal hyperbolas, and (y−k)2a2−(x−h)2b2=1(y−k)2a2−(x−h)2b2=1 for vertical hyperbolas. From these standard form equations we can easily calculate and plot key features of the graph: the coordinates of its center, vertices, co-vertices, and foci; the equations of its asymptotes; and the positions of the transverse and conjugate axes.

Given a general form for a hyperbola centered at (h,k),(h, k), sketch the graph.

1. Convert the general form to that standard form. Determine which of the standard forms applies to the given equation.
2. Use the standard form identified in Step 1 to determine the position of the transverse axis; coordinates for the center, vertices, co-vertices, foci; and equations for the asymptotes.
   a. If the equation is in the form (x−h)2a2−(y−k)2b2=1,(x−h)2a2−(y−k)2b2=1, then

- the transverse axis is parallel to the *x*-axis
- the center is (h,k)(h,k)
- the coordinates of the vertices are (h±a,k)(h±a,k)
- the coordinates of the co-vertices are (h,k±b)(h,k±b)
- the coordinates of the foci are (h±c,k)(h±c,k)
- the equations of the asymptotes are y=±ba(x−h)+ky=±ba(x−h)+k

b. If the equation is in the form (y−k)2a2−(x−h)2b2=1,(y−k)2a2−(x−h)2b2=1, then

- the transverse axis is parallel to the *y*-axis
- the center is (h,k)(h,k)
- the coordinates of the vertices are (h,k±a)(h,k±a)
- the coordinates of the co-vertices are (h±b,k)(h±b,k)
- the coordinates of the foci are (h,k±c)(h,k±c)
- the equations of the asymptotes are y=±ab(x−h)+ky=±ab(x−h)+k

3. Solve for the coordinates of the foci using the equation c=±a2+b2−−−−−−√.c=±a2+b2.
4. Plot the center, vertices, co-vertices, foci, and asymptotes in the coordinate plane and draw a smooth curve to form the hyperbola.

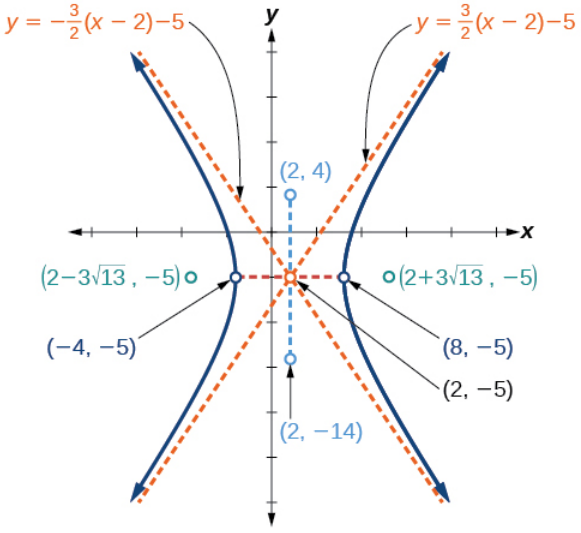
EXAMPLE 5

Graphing a Hyperbola Centered at (*h*, *k*) Given an Equation in General Form

Graph the hyperbola given by the equation 9x2−4y2−36x−40y−388=0.9x2−4y2−36x−40y−388=0. Identify and label the center, vertices, co-vertices, foci, and asymptotes.

Answer

-
-
-
-



Graph the hyperbola given by the standard form of an equation (y+4)2100−(x−3)264=1.(y+4)2100−(x−3)264=1. Identify and label the center, vertices, co-vertices, foci, and asymptotes.

Solving Applied Problems Involving Hyperbolas

As we discussed at the beginning of this section, hyperbolas have real-world applications in many fields, such as astronomy, physics, engineering, and architecture. The design efficiency of hyperbolic cooling towers is particularly interesting. Cooling towers are used to transfer waste heat to the atmosphere and are often touted for their ability to generate power efficiently. Because

of their hyperbolic form, these structures are able to withstand extreme winds while requiring less material than any other forms of their size and strength. See Figure 10. For example, a 500-foot tower can be made of a reinforced concrete shell only 6 or 8 inches wide!



Figure **10** Cooling towers at the Drax power station in North Yorkshire, United Kingdom (credit: Les Haines, Flickr)

The first hyperbolic towers were designed in 1914 and were 35 meters high. Today, the tallest cooling towers are in France, standing a remarkable 170 meters tall. In Example 6 we will use the design layout of a cooling tower to find a hyperbolic equation that models its sides.

EXAMPLE 6

Solving Applied Problems Involving Hyperbolas

The design layout of a cooling tower is shown in Figure 11. The tower stands 179.6 meters tall. The diameter of the top is 72 meters. At their closest, the sides of the tower are 60 meters apart.

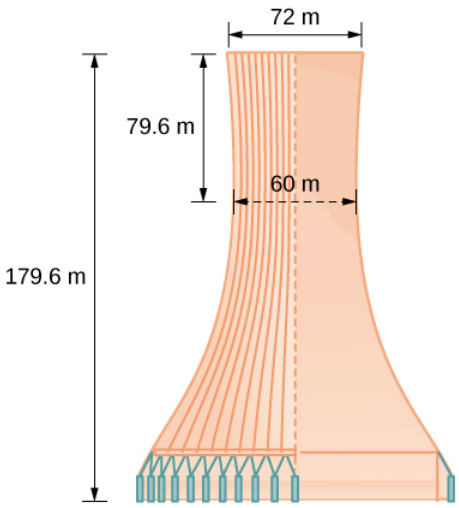


Figure **11** Project design for a natural draft cooling tower

Find the equation of the hyperbola that models the sides of the cooling tower. Assume that the center of the hyperbola—indicated by the intersection of dashed perpendicular lines in the figure—is the origin of the coordinate plane. Round final values to four decimal places.

Answer

A design for a cooling tower project is shown in Figure 12. Find the equation of the hyperbola that models the sides of the cooling tower. Assume that the center of the hyperbola—indicated by the intersection of dashed perpendicular lines in the figure—is the origin of the coordinate plane. Round final values to four decimal places.

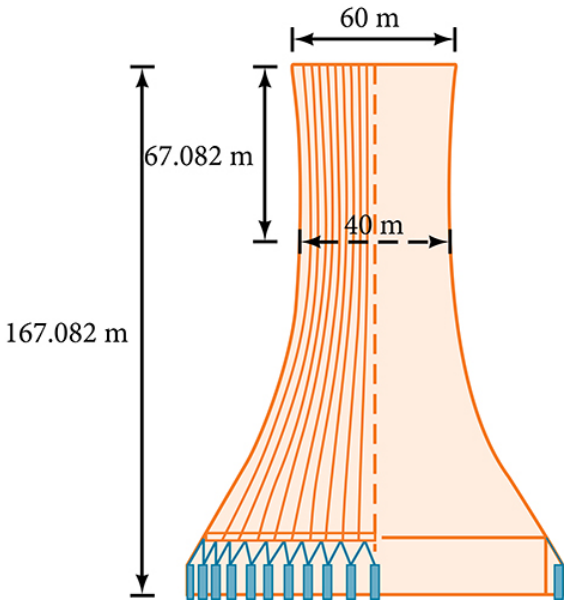


Figure **12**

Access these online resources for additional instruction and practice with hyperbolas.

- Conic Sections: The Hyperbola Part 1 of 2
- Conic Sections: The Hyperbola Part 2 of 2
- Graph a Hyperbola with Center at Origin
- Graph a Hyperbola with Center not at Origin

10.2 Section Exercises

Verbal

1.

Define a hyperbola in terms of its foci.

2.

What can we conclude about a hyperbola if its asymptotes intersect at the origin?

3.

What must be true of the foci of a hyperbola?

4.

If the transverse axis of a hyperbola is vertical, what do we know about the graph?

5.

Where must the center of hyperbola be relative to its foci?

Algebraic

For the following exercises, determine whether the following equations represent hyperbolas. If so, write in standard form.

6.

3y2+2x=63y2+2x=6

7.

x236−y29=1x236−y29=1

8.

5y2+4x2=6x5y2+4x2=6x

9.

25x2−16y2=40025x2−16y2=400

10.

−9x2+18x+y2+4y−14=0−9x2+18x+y2+4y−14=0

For the following exercises, write the equation for the hyperbola in standard form if it is not already, and identify the vertices and foci, and write equations of asymptotes.

11.

x225−y236=1x225−y236=1

12.

x2100−y29=1x2100−y29=1

13.

y24−x281=1y24−x281=1

14.

9y2−4x2=19y2−4x2=1

15.

(x−1)29−(y−2)216=1(x−1)29−(y−2)216=1

16.

(y−6)236−(x+1)216=1(y−6)236−(x+1)216=1

17.

(x−2)249−(y+7)249=1(x−2)249−(y+7)249=1

18.

4x2−8x−9y2−72y+112=04x2−8x−9y2−72y+112=0

19.

−9x2−54x+9y2−54y+81=0−9x2−54x+9y2−54y+81=0

20.

4x2−24x−36y2−360y+864=04x2−24x−36y2−360y+864=0

21.

−4x2+24x+16y2−128y+156=0−4x2+24x+16y2−128y+156=0

22.

−4x2+40x+25y2−100y+100=0−4x2+40x+25y2−100y+100=0

23.

x2+2x−100y2−1000y+2401=0x2+2x−100y2−1000y+2401=0

24.

−9x2+72x+16y2+16y+4=0−9x2+72x+16y2+16y+4=0

25.

4x2+24x−25y2+200y−464=04x2+24x−25y2+200y−464=0

For the following exercises, find the equations of the asymptotes for each hyperbola.

26.

y232−x232=1y232−x232=1

27.

(x−3)252−(y+4)222=1(x−3)252−(y+4)222=1

28.

(y−3)232−(x+5)262=1(y−3)232−(x+5)262=1

29.

9x2−18x−16y2+32y−151=09x2−18x−16y2+32y−151=0

30.

16y2+96y−4x2+16x+112=016y2+96y−4x2+16x+112=0

Graphical

For the following exercises, sketch a graph of the hyperbola, labeling vertices and foci.

31.

x249−y216=1x249−y216=1

32.

x264−y24=1x264−y24=1

33.

y29−x225=1y29−x225=1

34.

81x2−9y2=181x2−9y2=1

35.

(y+5)29−(x−4)225=1(y+5)29−(x−4)225=1

36.

(x−2)28−(y+3)227=1(x−2)28−(y+3)227=1

37.

(y−3)29−(x−3)29=1(y−3)29−(x−3)29=1

38.

−4x2−8x+16y2−32y−52=0−4x2−8x+16y2−32y−52=0

39.

x2−8x−25y2−100y−109=0x2−8x−25y2−100y−109=0

40.

−x2+8x+4y2−40y+88=0−x2+8x+4y2−40y+88=0

41.

64x2+128x−9y2−72y−656=064x2+128x−9y2−72y−656=0

42.

$16x^2+64x-4y^2-8y-4=016x^2+64x-4y^2-8y-4=0$

43.

$-100x^2+1000x+y^2-10y-2575=0-100x^2+1000x+y^2-10y-2575=0$

44.

$4x^2+16x-4y^2+16y+16=04x^2+16x-4y^2+16y+16=0$

For the following exercises, given information about the graph of the hyperbola, find its equation.

45.

Vertices at $(3,0)(3,0)$ and $(-3,0)(-3,0)$ and one focus at $(5,0).(5,0)$.

46.

Vertices at $(0,6)(0,6)$ and $(0,-6)(0,-6)$ and one focus at $(0,-8).(0,-8)$.

47.

Vertices at $(1,1)(1,1)$ and $(11,1)(11,1)$ and one focus at $(12,1).(12,1)$.

48.

Center: $(0,0);(0,0)$; vertex: $(0,-13);(0,-13)$; one focus: $(0,313---\sqrt{}).(0,313)$.

49.

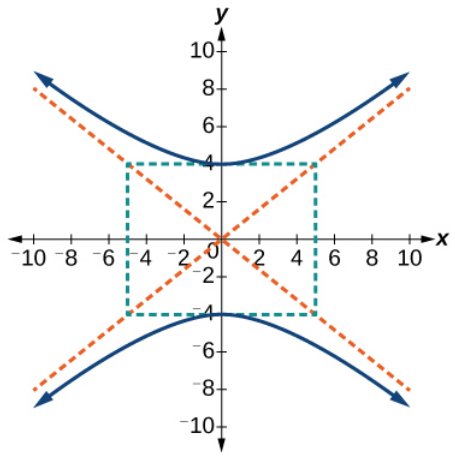
Center: $(4,2);(4,2)$; vertex: $(9,2);(9,2)$; one focus: $(4+26--\sqrt{},2).(4+26,2)$.

50.

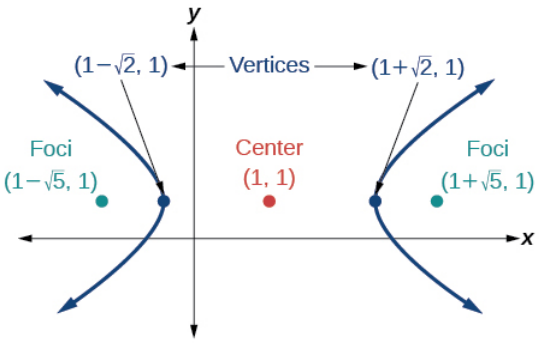
Center: $(3,5);(3,5)$; vertex: $(3,11);(3,11)$; one focus: $(3,5+210--\sqrt{}).(3,5+210)$.

For the following exercises, given the graph of the hyperbola, find its equation.

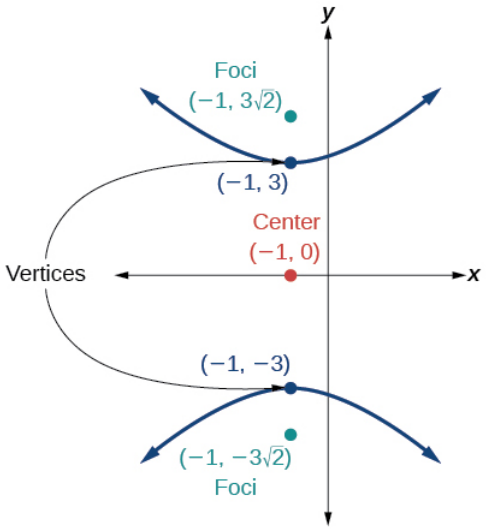
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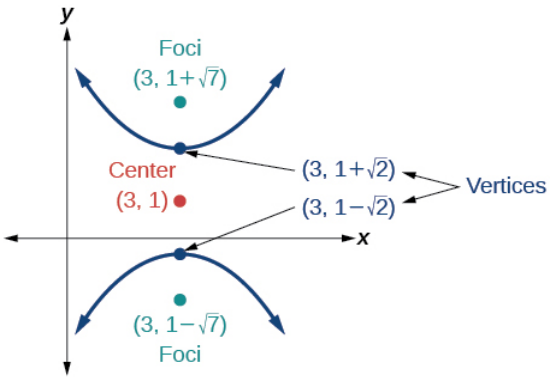
52.



53.

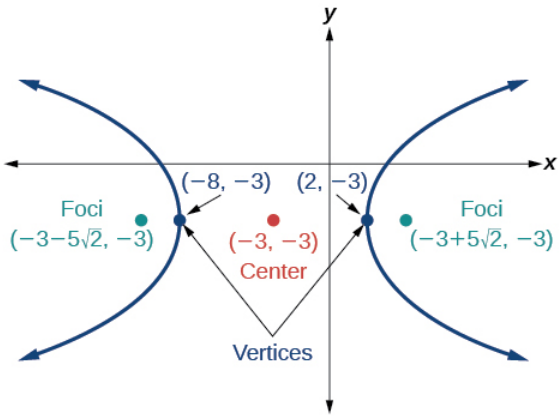


54.



55.

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Extensions

For the following exercises, express the equation for the hyperbola as two functions, with yy as a function of x.x. Express as simply as possible. Use a graphing calculator to sketch the graph of the two functions on the same axes.

56.

x24−y29=1x24−y29=1

57.

y29−x21=1y29−x21=1

58.

(x−2)216−(y+3)225=1(x−2)216−(y+3)225=1

59.

−4x2−16x+y2−2y−19=0−4x2−16x+y2−2y−19=0

60.

4x2−24x−y2−4y+16=04x2−24x−y2−4y+16=0

Real-World Applications

For the following exercises, a hedge is to be constructed in the shape of a hyperbola near a fountain at the center of the yard. Find the equation of the hyperbola and sketch the graph.

61.

The hedge will follow the asymptotes y=xand y=−x,y=xand y=−x, and its closest distance to the center fountain is 5 yards.

62.

The hedge will follow the asymptotes y=2xand y=−2x,y=2xand y=−2x, and its closest distance to the center fountain is 6 yards.

63.

The hedge will follow the asymptotes y=12x y=12x and y=−12x,y=−12x, and its closest distance to the center fountain is 10 yards.

64.

The hedge will follow the asymptotes y=23xy=23x and y=−23x,y=−23x, and its closest distance to the center fountain is 12 yards.

65.

The hedge will follow the asymptotes y=34xy=34x and y=−34x,y=−34x, and its closest distance to the center fountain is 20 yards.

For the following exercises, assume an object enters our solar system and we want to graph its path on a coordinate system with the sun at the origin and the x-axis as the axis of symmetry for the object's path. Give the equation of the flight path of each object using the given information.

66.

The object enters along a path approximated by the line $y=x-2y=x-2$ and passes within 1 au (astronomical unit) of the sun at its closest approach, so that the sun is one focus of the hyperbola. It then departs the solar system along a path approximated by the line $y=-x+2.y=-x+2$.

67.

The object enters along a path approximated by the line $y=2x-2y=2x-2$ and passes within 0.5 au of the sun at its closest approach, so the sun is one focus of the hyperbola. It then departs the solar system along a path approximated by the line $y=-2x+2.y=-2x+2$.

68.

The object enters along a path approximated by the line $y=0.5x+2y=0.5x+2$ and passes within 1 au of the sun at its closest approach, so the sun is one focus of the hyperbola. It then departs the solar system along a path approximated by the line $y=-0.5x-2.y=-0.5x-2$.

69.

The object enters along a path approximated by the line $y=13x-1y=13x-1$ and passes within 1 au of the sun at its closest approach, so the sun is one focus of the hyperbola. It then departs the solar system along a path approximated by the line $y=-13x+1.y=-13x+1$.

70.

The object enters along a path approximated by the line $y=3x-9y=3x-9$ and passes within 1 au of the sun at its closest approach, so the sun is one focus of the hyperbola. It then departs the solar system along a path approximated by the line $y=-3x+9$.

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CHAPTER OVERVIEW

9: Sequences, Series, and the Binomial Theorem

9.1: Introduction to Sequences and Series
9.2: Arithmetic Sequences and Series
9.3: Geometric Sequences and Series
9.4: Binomial Theorem

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9.1: Introduction to Sequences and Series

Learning Objectives

- Find any element of a sequence given a formula for its general term.
- Use sigma notation and expand corresponding series.
- Distinguish between a sequence and a series.
- Calculate the nth partial sum of sequence.

Sequences

A **sequence**[1] is a function whose domain is a set of consecutive natural numbers beginning with 1. For example, the following equation with domain $\{1, 2, 3, \ldots\}$defines an **infinite sequence**[2]:

$$a(n) = 5n - 3 \text{ or } a_n = 5n - 3$$

The elements in the range of this function are called terms of the sequence. It is common to define the nth term, or the **general term of a sequence**[3], using the subscripted notation a_n, which reads "a sub n." Terms can be found using substitution as follows:

$$\begin{aligned} General\ term: &\quad a_n = 5n - 3 \\ First\ term (n = 1): &\quad a_1 = 5(1) - 3 = 2 \\ Second\ term (n = 2): &\quad a_2 = 5(2) - 3 = 7 \\ Third\ term (n = 3): &\quad a_3 = 5(3) - 3 = 12 \\ Fourth\ term (n = 4): &\quad a_4 = 5(4) - 3 = 17 \\ Fifth\ term (n = 5): &\quad a_5 = 5(5) - 3 = 22 \\ &\quad \vdots \end{aligned}$$

This produces an ordered list,

$$2, 7, 12, 17, 22, \ldots$$

The ellipsis $(\ldots)$ indicates that this sequence continues forever. Unlike a set, order matters. If the domain of a sequence consists of natural numbers that end, such as $\{1, 2, 3, \ldots, k\}$, then it is called a **finite sequence**[4].

Example 9.1.1:

Given the general term of a sequence, find the first 5 terms as well as the 100th term: $a_n = \frac{n(n-1)}{2}$.

Solution:

To find the first 5 terms, substitute $1, 2, 3, 4$, and 5 for n and then simplify.

$$\begin{aligned} a_1 &= \frac{1(1-1)}{2} = \frac{1(0)}{2} = \frac{0}{2} = 0 \\ a_2 &= \frac{2(2-1)}{2} = \frac{2(1)}{2} = \frac{2}{2} = 1 \\ a_3 &= \frac{3(3-1)}{2} = \frac{3(2)}{2} = \frac{6}{2} = 3 \\ a_4 &= \frac{4(4-1)}{2} = \frac{4(3)}{2} = \frac{12}{2} = 6 \\ a_5 &= \frac{5(5-1)}{2} = \frac{5(4)}{2} = \frac{20}{2} = 10 \end{aligned}$$

Use $n = 100$ to determine the 100th term in the sequence.

$$a_{100} = \frac{100(100-1)}{2} = \frac{100(99)}{2} = \frac{9{,}900}{2} = 4,950$$

Answer:

First five terms: $0, 1, 3, 6, 10$; $a_{100} = 4,950$

Sometimes the general term of a sequence will alternate in sign and have a variable other than n.

Example 9.1.2:

Find the first 5 terms of the sequence: $a_n = (-1)^n x^{n+1}$.

Solution:

Here we take care to replace n with the first 5 natural numbers and not x.

$$a_1 = (-1)^1 x^{1+1} = -x^2$$
$$a_2 = (-1)^2 x^{2+1} = x^3$$
$$a_3 = (-1)^3 x^{3+1} = -x^4$$
$$a_4 = (-1)^4 x^{4+1} = x^5$$
$$a_5 = (-1)^5 x^{5+1} = -x^6$$

Answer:

$-x^2, x^3, -x^4, x^5, -x^6$

Exercise 9.1.1

Find the first 5 terms of the sequence: $a_n = (-1)^{n+1} 2^n$.

Answer

$2, -4, 8, -16, 32.$

http://www.youtube.com/v/uuQ3jYL-g_I

One interesting example is the Fibonacci sequence. The first two numbers in the Fibonacci sequence are 1, and each successive term is the sum of the previous two. Therefore, the general term is expressed in terms of the previous two as follows:

$$F_n = F_{n-2} + F_{n-1}$$

Here $F_1 = 1, F_2 = 1$, and $n > 2$. A formula that describes a sequence in terms of its previous terms is called a **recurrence relation**[5].

Example 9.1.3:

Find the first 7 Fibonacci numbers.

Solution:

Given that $F_1 = 1$ and $F_2 = 1$, use the recurrence relation $F_n = F_{n-2} + F_{n-1}$ where n is an integer starting with $n = 3$ to find the next 5 terms:

$$F_3 = F_{3-2} + F_{3-1} = F_1 + F_2 = 1 + 1 = 2$$
$$F_4 = F_{4-2} + F_{4-1} = F_2 + F_3 = 1 + 2 = 3$$
$$F_5 = F_{5-2} + F_{5-1} = F_3 + F_4 = 2 + 3 = 5$$
$$F_6 = F_{6-2} + F_{6-1} = F_4 + F_5 = 3 + 5 = 8$$
$$F_7 = F_{7-2} + F_{7-1} = F_5 + F_6 = 5 + 8 = 13$$

Answer:

$1, 1, 2, 3, 5, 8, 13$



***Figure 9.1.1**: Leonardo Fibonacci (1170–1250)*

Fibonacci numbers appear in applications ranging from art to computer science and biology. The beauty of this sequence can be visualized by constructing a Fibonacci spiral. Consider a tiling of squares where each side has a length that matches each Fibonacci number:

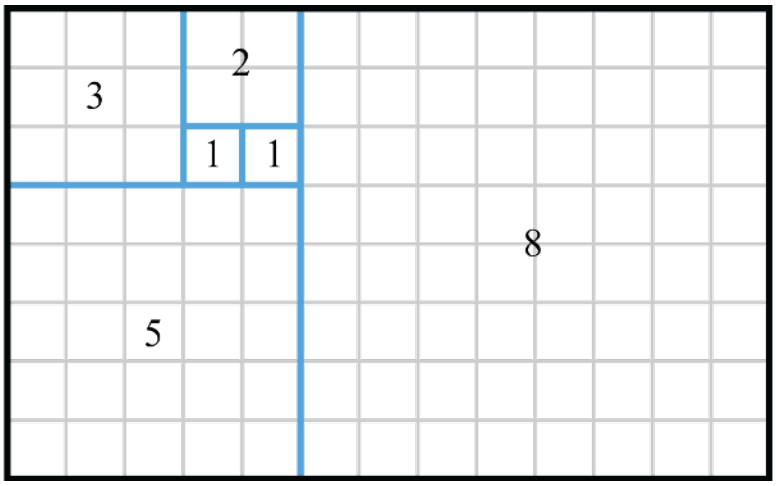


Figure 9.1.2

Connecting the opposite corners of the squares with an arc produces a special spiral shape.

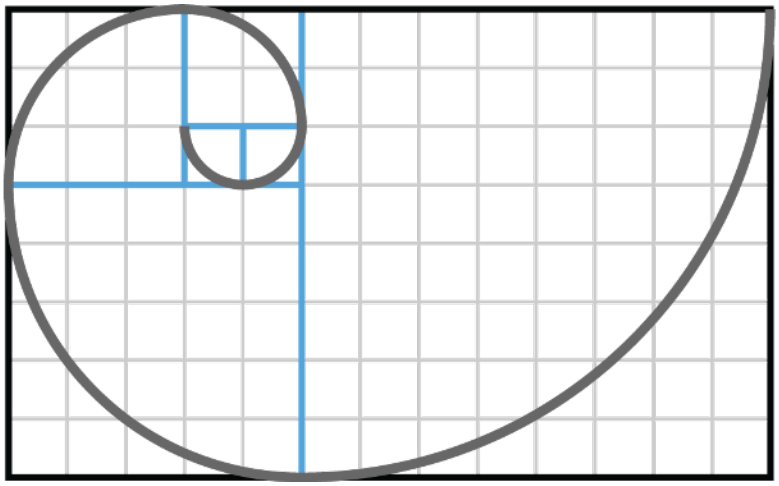


Figure 9.1.3

This shape is called the Fibonacci spiral and approximates many spiral shapes found in nature.

Series

A **series**[6] is the sum of the terms of a sequence. The sum of the terms of an infinite sequence results in an **infinite series**[7], denoted S_∞. The sum of the first n terms in a sequence is called a **partial sum**[8], denoted S_n. For example, given the sequence of positive odd integers $1, 3, 5, \ldots$ we can write:

$$S_\infty = 1+3+5+7+9+\ldots \quad \textit{Infinite series}$$
$$S_5 = 1+3+5+7+9 = 25 \quad \textit{5th partial sum}$$

Example 9.1.4:

Determine the 3rd and 5th partial sums of the sequence: $3, -6, 12, -24, 48, \ldots$

Solution:

$$S_3 = 3+(-6)+12 = 9$$
$$S_5 = 3+(-6)+12+(-24)+48 = 33$$

Answer:

$S_3 = 9; S_5 = 33$

If the general term is known, then we can express a series using **sigma**[9] (or **summation**[10]) **notation**:

$$S_\infty = \sum_{n=1}^{\infty} n^2 = 1^2+2^2+3^2+\ldots \quad \textit{Infinite series}$$
$$S_3 = \sum_{n=1}^{3} n^2 = 1^2+2^2+3^2 \quad \textit{3rd partial sum}$$

The symbol Σ (upper case Greek letter sigma) is used to indicate a series. The expressions above and below indicate the range of the **index of summation**[11], in this case represented by n. The lower number indicates the starting integer and the upper value indicates the ending integer. The nth partial sum S_n can be expressed using sigma notation as follows:

$$S_n = \textstyle\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$$

This is read, "the sum of a_k as k goes from 1 to n." Replace n with ∞ to indicate an infinite sum.

Example 9.1.5:

Evaluate: $\sum_{k=1}^{5}(-3)^{n-1}$.

$$\begin{aligned}\sum_{k=1}^{5}(-3)^{k-1} &= (-3)^{1-1}+(-3)^{2-1}+(-3)^{3-1}+(-3)^{4-1}+(-3)^{5-1}\\ &= (-3)^0+(-3)^1+(-3)^2+(-3)^3+(-3)^4\\ &= 1-3+9-27+81\\ &= 61\end{aligned}$$

Answer:

61

When working with sigma notation, the index does not always start at 1.

Example 9.1.1:

Evaluate: $\sum_{k=2}^{5}(-1)^k(3k)$

Solution:

Here the index is expressed using the variable k, which ranges from 2 to 5.

Exercise 9.1.2

Evaluate: $\sum_{n=1}^{5}(15-9n)$.

Answer

-60

http://www.youtube.com/v/aZ3sPd8N1TU

Infinity is used as the upper bound of a sum to indicate an infinite series.

Example 9.1.7:

Write in expanded form: $\sum_{i=0}^{\infty}\frac{n}{n+1}$.

Solution:

In this case we begin with $n=0$ and add three dots to indicate that this series continues forever.

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{n}{n+1} &= \frac{0}{0+1}+\frac{1}{1+1}+\frac{2}{2+1}+\frac{3}{3+1}+\cdots\\ &= \frac{0}{1}+\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\cdots\\ &= 0+\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\cdots\end{aligned}$$

Answer:

$0+\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\cdots$

When expanding a series, take care to replace only the variable indicated by the index.

Example 9.1.8:

Write in expanded form: $\sum_{i=1}^{\infty}(-1)^{i-1}x^{2i}$.

Solution:

$$\begin{aligned}\sum_{i=1}^{\infty}(-1)^{i-1}x^{2i} &= (-1)^{1-1}x^{2(1)}+(-1)^{2-1}x^{2(2)}+(-1)^{3-1}x^{2(3)}+\cdots\\ &= (-1)^0x^{2(1)}+(-1)^1x^{2(2)}+(-1)^2x^{2(3)}+\cdots\\ &= x^2-x^4+x^6-\cdots\end{aligned}$$

Answer:

$x^2-x^4+x^6-\cdots$

Key Takeaways

- A sequence is a function whose domain consists of a set of natural numbers beginning with 1. In addition, a sequence can be thought of as an ordered list.
- Formulas are often used to describe the nth term, or general term, of a sequence using the subscripted notation a_n.

- A series is the sum of the terms in a sequence. The sum of the first n terms is called the nth partial sum and is denoted S_n.
- Use sigma notation to denote summations in a compact manner. The nth partial sum, using sigma notation, can be written $S_n = \sum_{k=1}^{n} a_k$. The symbol Σ denotes a summation where the expression below indicates that the index k starts at 1 and iterates through the natural numbers ending with the value n above.

Footnotes

[1]A function whose domain is a set of consecutive natural numbers starting with 1.

[2]A sequence whose domain is the set of natural numbers $\{1, 2, 3, \ldots\}$.

[3]An equation that defines the nth term of a sequence commonly denoted using subscripts a_n.

[4]A sequence whose domain is $\{1, 2, 3, \ldots, k\}$where k is a natural number.

[5]A formula that uses previous terms of a sequence to describe subsequent terms.

[6]The sum of the terms of a sequence.

[7]The sum of the terms of an infinite sequence denoted S_∞.

[8]The sum of the first n terms in a sequence denoted S_n.

[9]A sum denoted using the symbol Σ (upper case Greek letter sigma).

[10]Used when referring to sigma notation.

[11]The variable used in sigma notation to indicate the lower and upper bounds of the summation.

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9.2: Arithmetic Sequences and Series

Learning Objectives

- Identify the common difference of an arithmetic sequence.
- Find a formula for the general term of an arithmetic sequence.
- Calculate the nth partial sum of an arithmetic sequence.

Arithmetic Sequences

An **arithmetic sequence**[12], or **arithmetic progression**[13], is a sequence of numbers where each successive number is the sum of the previous number and some constant d.

$$a_n = a_{n-1} + d \quad \textit{Arithmetic sequence}$$

And because $a_n - a_{n-1} = d$, the constant d is called the **common difference**[14]. For example, the sequence of positive odd integers is an arithmetic sequence,

$$1, 3, 5, 7, 9, \ldots$$

Here $a_1 = 1$ and the difference between any two successive terms is 2. We can construct the general term $a_n = a_{n-1} + 2$ where,

$$\begin{aligned} a_1 &= 1 \\ a_2 &= a_1 + 2 = 1 + 2 = 3 \\ a_3 &= a_2 + 2 = 3 + 2 = 5 \\ a_4 &= a_3 + 2 = 5 + 2 = 7 \\ a_5 &= a_4 + 2 = 7 + 2 = 9 \\ &\vdots \end{aligned}$$

In general, given the first term a_1 of an arithmetic sequence and its common difference d, we can write the following:

$$\begin{aligned} a_2 &= a_1 + d \\ a_3 &= a_2 + d = (a_1 + d) + d = a_1 + 2d \\ a_4 &= a_3 + d = (a_1 + 2d) + d = a_1 + 3d \\ a_5 &= a_4 + d = (a_1 + 3d) + d = a_1 + 4d \\ &\vdots \end{aligned}$$

From this we see that any arithmetic sequence can be written in terms of its first element, common difference, and index as follows:

$$a_n = a_1 + (n-1)d \quad \textit{Arithmetic Sequence}$$

In fact, any general term that is linear in n defines an arithmetic sequence.

Example 9.2.1:

Find an equation for the general term of the given arithmetic sequence and use it to calculate its 100th term: $7, 10, 13, 16, 19, \ldots$

Solution

Begin by finding the common difference,

$$d = 10 - 7 = 3$$

Note that the difference between any two successive terms is 3. The sequence is indeed an arithmetic progression where $a_1 = 7$ and $d = 3$.

$$\begin{aligned} a_n &= a_1 + (n-1)d \\ &= 7 + (n-1) \cdot 3 \\ &= 7 + 3n - 3 \\ &= 3n + 4 \end{aligned}$$

Therefore, we can write the general term $a_n = 3n + 4$. Take a minute to verify that this equation describes the given sequence. Use this equation to find the 100th term:

$$a_{100} = 3(100) + 4 = 304$$

Answer:

$a_n = 3n + 4; a_{100} = 304$

The common difference of an arithmetic sequence may be negative.

Example 9.2.2:

Find an equation for the general term of the given arithmetic sequence and use it to calculate its 75th term: $6, 4, 2, 0, -2, \ldots$

Solution

Begin by finding the common difference,

$$d = 4 - 6 = -2$$

Next find the formula for the general term, here $a_1 = 6$ and $d = -2$.

$$\begin{aligned} a_n &= a_1 + (n-1)d \\ &= 6 + (n-1) \cdot (-2) \\ &= 6 - 2n + 2 \\ &= 8 - 2n \end{aligned}$$

Therefore, $a_n = 8 - 2n$ and the 75th term can be calculated as follows:

$$\begin{aligned} a_{75} &= 8 - 2(75) \\ &= 8 - 150 \\ &= -142 \end{aligned}$$

Answer:

$a_n = 8 - 2n; a_{100} = -142$

The terms between given terms of an arithmetic sequence are called **arithmetic means**[15].

Example 9.2.3:

Find all terms in between $a_1 = -8$ and $a_7 = 10$ of an arithmetic sequence. In other words, find all arithmetic means between the 1st and 7th terms.

Solution

Begin by finding the common difference d. In this case, we are given the first and seventh term:

$$\begin{aligned} a_n &= a_1 + (n-1)d \quad \textit{Use } n = 7. \\ a_7 &= a_1 + (7-1)d \\ a_7 &= a_1 + 6d \end{aligned}$$

Substitute $a_1 = -8$ and $a_7 = 10$ into the above equation and then solve for the common difference d.

$$\begin{aligned} 10 &= -8 + 6d \\ 18 &= 6d \\ 3 &= d \end{aligned}$$

Next, use the first term $a_1 = -8$ and the common difference $d = 3$ to find an equation for the nth term of the sequence.

$$\begin{aligned} a_n &= -8 + (n-1) \cdot 3 \\ &= -8 + 3n - 3 \\ &= -11 + 3n \end{aligned}$$

With $a_n = 3n - 11$, where n is a positive integer, find the missing terms.

$$\left.\begin{aligned} a_1 &= 3(1)-11 = 3-11 = -8 \\ a_2 &= 3(2)-11 = 6-11 = -5 \\ a_3 &= 3(3)-11 = 9-11 = -2 \\ a_4 &= 3(4)-11 = 12-11 = 1 \\ a_5 &= 3(5)-11 = 15-11 = 4 \\ a_6 &= 3(6)-11 = 18-11 = 7 \\ a_7 &= 3(7)-11 = 21-111 = 10 \end{aligned}\right\} \textit{arithmetic means}$$

Answer:

$-5, -2, 1, 4, 7$

In some cases, the first term of an arithmetic sequence may not be given.

Example 9.2.4:

Find the general term of an arithmetic sequence where $a_3 = -1$ and $a_{10} = 48$.

Solution

To determine a formula for the general term we need a_1 and d. A linear system with these as variables can be formed using the given information and $a_n = a_1 + (n-1)d$:

$$\begin{cases} a_3 = a_1 + (3-1)d \\ a_{10} = a_1 + (10-1)d \end{cases} \Rightarrow \begin{cases} -1 = a_1 + 2d & Use\ a_3 = -1. \\ 48 = a_1 + 9d & Use\ a_{10} = 48. \end{cases}$$

Eliminate a_1 by multiplying the first equation by -1 and add the result to the second equation.

$$\begin{cases} -1 = a_1 + 2d \\ 48 = a_1 + 9d \end{cases} \overset{x(-1)}{\Longrightarrow} + \begin{cases} 1 = -a_1 - 2d \\ 48 = a_1 + 9d \end{cases}$$

$$\begin{aligned} 49 &= 7d \\ 7 &= d \end{aligned}$$

Substitute $d = 7$ into $-1 = a_1 + 2d$ to find a_1.

$$\begin{aligned} -1 &= a_1 + 2(7) \\ -1 &= a_1 + 14 \\ -15 &= a_1 \end{aligned}$$

Next, use the first term $a_1 = -15$ and the common difference $d = 7$ to find a formula for the general term.

$$\begin{aligned} a_n &= a_1 + (n-1)d \\ &= -15 + (n-1)\cdot 7 \\ &= -15 + 7n - 7 \\ &= -22 + 7n \end{aligned}$$

Answer:

$a_n = 7n - 22$

Exercise 9.2.1

Find an equation for the general term of the given arithmetic sequence and use it to calculate its 100th term: $\frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, \ldots$

Answer

$a_n = \frac{1}{2}n + 1; a_{100} = 51$

www.youtube.com/v/_ovjvVKtKpQ

Arithmetic Series

An **arithmetic series**[16] is the sum of the terms of an arithmetic sequence. For example, the sum of the first 5 terms of the sequence defined by $a_n = 2n-1$ follows:

$$\begin{aligned} S_5 &= \sum_{n=1}^{5}(2n-1) \\ &= [2(1)-1]+[2(2)-1]+[2(3)-1]+[2(4)-1]+[2(5)-1] \\ &= 1+3+5+7+9 \\ &= 25 \end{aligned}$$

Adding 5 positive odd integers, as we have done above, is managable. However, consider adding the first 100 positive odd integers. This would be very tedious. Therefore, we next develop a formula that can be used to calculate the sum of the first n terms, denoted S_n, of any arithmetic sequence. In general,

$$S_n = a_1 + (a_1+d) + (a_1+2d) + \ldots + a_n$$

Writing this series in reverse we have,

$$S_n = a_n + (a_n-d) + (a_n-2d) + \ldots + a_1$$

And adding these two equations together, the terms involving d add to zero and we obtain n factors of a_1+a_n :

$$\begin{aligned} 2S_n &= (a_1+a_n)+(a_1+a_n)+\ldots+(a_n+a_1) \\ 2S_n &= n\,(a_1+a_n) \end{aligned}$$

Dividing both sides by 2 leads us the formula for the nth partial sum of an **arithmetic sequence**[17]:

$$S_n = \tfrac{n(a_1+a_n)}{2}$$

Use this formula to calculate the sum of the first 100 terms of the sequence defined by $a_n = 2n-1$. Here $a_1 = 1$ and $a_{100} = 199$.

$$\begin{aligned} S_{100} &= \frac{100\,(a_1+a_{100})}{2} \\ &= \frac{100(1+199)}{2} \\ &= 10,000 \end{aligned}$$

Example 9.2.5:

Find the sum of the first 50 terms of the given sequence: $4, 9, 14, 19, 24, \ldots$

Solution

Determine whether or not there is a common difference between the given terms.

$$d = 9-4 = 5$$

Note that the difference between any two successive terms is 5. The sequence is indeed an arithmetic progression and we can write

$$\begin{aligned} a_n &= a_1+(n-1)d \\ &= 4+(n-1)\cdot 5 \\ &= 4+5n-5 \\ &= 5n-1 \end{aligned}$$

Therefore, the general term is $a_n = 5n-1$. To calculate the 50th partial sum of this sequence we need the 1st and the 50th terms:

$$\begin{aligned} a_1 &= 4 \\ a_{50} &= 5(50)-1 = 249 \end{aligned}$$

Next use the formula to determine the 50th partial sum of the given arithmetic sequence.

$$\begin{aligned} S_n &= \frac{n\left(a_1+a_n\right)}{2} \\ S_{50} &= \frac{50\cdot\left(a_1+a_{50}\right)}{2} \\ &= \frac{50(4+249)}{2} \\ &= 25(253) \\ &= 6,325 \end{aligned}$$

Answer:

$S_{50}=6,325$

Example 9.2.6:

Evaluate: $\sum_{n=1}^{35}(10-4n)$.

Solution

In this case, we are asked to find the sum of the first 35 terms of an arithmetic sequence with general term $a_n=10-4n$. Use this to determine the 1st and the 35th term.

$$\begin{aligned} a_1 &= 10-4(1)=6 \\ a_{35} &= 10-4(35)=-130 \end{aligned}$$

Next use the formula to determine the 35th partial sum.

$$\begin{aligned} S_n &= \frac{n\left(a_1+a_n\right)}{2} \\ S_{35} &= \frac{35\cdot\left(a_1+a_{35}\right)}{2} \\ &= \frac{35[6+(-130)]}{2} \\ &= \frac{35(-124)}{2} \\ &= -2,170 \end{aligned}$$

Answer:

$-2,170$

Example 9.2.7:

The first row of seating in an outdoor amphitheater contains 26 seats, the second row contains 28 seats, the third row contains 30 seats, and so on. If there are 18 rows, what is the total seating capacity of the theater?



Figure 9.2.1: Roman Theater

Solution

Begin by finding a formula that gives the number of seats in any row. Here the number of seats in each row forms a sequence:

$$26, 28, 30, \ldots$$

Note that the difference between any two successive terms is 2. The sequence is an arithmetic progression where $a_1 = 26$ and $d = 2$.

$$\begin{aligned} a_n &= a_1 + (n-1)d \\ &= 26 + (n-1)\cdot 2 \\ &= 26 + 2n - 2 \\ &= 2n + 24 \end{aligned}$$

Therefore, the number of seats in each row is given by $a_n = 2n + 24$. To calculate the total seating capacity of the 18 rows we need to calculate the 18th partial sum. To do this we need the 1st and the 18th terms:

$$\begin{aligned} a_1 &= 26 \\ a_{18} &= 2(18) + 24 = 60 \end{aligned}$$

Use this to calculate the 18th partial sum as follows:

$$\begin{aligned} S_n &= \frac{n\,(a_1 + a_n)}{2} \\ S_{18} &= \frac{18\cdot(a_1 + a_{18})}{2} \\ &= \frac{18(26+60)}{2} \\ &= 9(86) \\ &= 774 \end{aligned}$$

Answer:

There are 774 seats total.

? Exercise 9.2.2

Find the sum of the first 60 terms of the given sequence: $5, 0, -5, -10, -15, \ldots$

Answer

$S_{60} = -8,550$

www.youtube.com/v/baYq2a_kBKo

Key Takeaways

- An arithmetic sequence is a sequence where the difference d between successive terms is constant.
- The general term of an arithmetic sequence can be written in terms of its first term a_1, common difference d, and index n as follows: $a_n = a_1 + (n-1)d$.
- An arithmetic series is the sum of the terms of an arithmetic sequence.
- The nth partial sum of an arithmetic sequence can be calculated using the first and last terms as follows: $S_n = \frac{n(a_1+a_n)}{2}$.

Footnotes

[12]A sequence of numbers where each successive number is the sum of the previous number and some constant d.

[13]Used when referring to an arithmetic sequence.

[14]The constant d that is obtained from subtracting any two successive terms of an arithmetic sequence; $a_n - a_{n-1} = d$.

[15]The terms between given terms of an arithmetic sequence.

[16]The sum of the terms of an arithmetic sequence.

[17]The sum of the first n terms of an arithmetic sequence given by the formula: $S_n = \frac{n(a_1+a_n)}{2}$.

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9.3: Geometric Sequences and Series

Learning Objectives

- Identify the common ratio of a geometric sequence.
- Find a formula for the general term of a geometric sequence.
- Calculate the nth partial sum of a geometric sequence.
- Calculate the sum of an infinite geometric series when it exists.

Geometric Sequences

A **geometric sequence**[18], or **geometric progression**[19], is a sequence of numbers where each successive number is the product of the previous number and some constant r.

$$a_n = ra_{n-1} \quad \textit{Geometric Sequence}$$

And because $\frac{a_n}{a_{n-1}} = r$, the constant factor r is called the **common ratio**[20]. For example, the following is a geometric sequence,

$$9, 27, 81, 243, 729 \ldots$$

Here $a_1 = 9$ and the ratio between any two successive terms is 3. We can construct the general term $a_n = 3a_{n-1}$ where,

$$\begin{aligned} a_1 &= 9 \\ a_2 &= 3a_1 = 3(9) = 27 \\ a_3 &= 3a_2 = 3(27) = 81 \\ a_4 &= 3a_3 = 3(81) = 243 \\ a_5 &= 3a_4 = 3(243) = 729 \\ &\vdots \end{aligned}$$

In general, given the first term a_1 and the common ratio r of a geometric sequence we can write the following:

$$\begin{aligned} a_2 &= ra_1 \\ a_3 &= ra_2 = r\left(a_1 r\right) = a_1 r^2 \\ a_4 &= ra_3 = r\left(a_1 r^2\right) = a_1 r^3 \\ a_5 &= ra_3 = r\left(a_1 r^3\right) = a_1 r^4 \\ &\vdots \end{aligned}$$

From this we see that any geometric sequence can be written in terms of its first element, its common ratio, and the index as follows:

$$a_n = a_1 r^{n-1} \quad \textit{Geometric Sequence}$$

In fact, any general term that is exponential in n is a geometric sequence.

Example 9.3.1:

Find an equation for the general term of the given geometric sequence and use it to calculate its 10th term: $3, 6, 12, 24, 48 \ldots$

Solution

Begin by finding the common ratio,

$$r = \frac{6}{3} = 2$$

Note that the ratio between any two successive terms is 2. The sequence is indeed a geometric progression where $a_1 = 3$ and $r = 2$.

$$\begin{aligned} a_n &= a_1 r^{n-1} \\ &= 3(2)^{n-1} \end{aligned}$$

Therefore, we can write the general term $a_n = 3(2)^{n-1}$ and the 10th term can be calculated as follows:

$$\begin{aligned} a_{10} &= 3(2)^{10-1} \\ &= 3(2)^9 \\ &= 1,536 \end{aligned}$$

Answer:

$a_n = 3(2)^{n-1}; a_{10} = 1,536$

The terms between given terms of a geometric sequence are called **geometric means**[21].

Example 9.3.2:

Find all terms between $a_1 = -5$ and $a_4 = -135$ of a geometric sequence. In other words, find all geometric means between the 1st and 4th terms.

Solution

Begin by finding the common ratio r. In this case, we are given the first and fourth terms:

$$\begin{aligned} a_n &= a_1 r^{n-1} \quad Use\ n = 4 \\ a_4 &= a_1 r^{4-1} \\ a_4 &= a_1 r^3 \end{aligned}$$

Substitute $a_1 = -5$ and $a_4 = -135$ into the above equation and then solve for r.

$$\begin{aligned} -135 &= -5r^3 \\ 27 &= r^3 \\ 3 &= r \end{aligned}$$

Next use the first term $a_1 = -5$ and the common ratio $r = 3$ to find an equation for the nth term of the sequence.

$$\begin{aligned} a_n &= a_1 r^{n-1} \\ a_n &= -5(3)^{n-1} \end{aligned}$$

Now we can use $a_n = -5(3)^{n-1}$ where n is a positive integer to determine the missing terms.

$$\left.\begin{aligned} a_1 &= -5(3)^{1-1} = -5 \cdot 3^0 = -5 \\ a_2 &= -5(3)^{2-1} = -5 \cdot 3^1 = -15 \\ a_3 &= -5(3)^{3-1} = -5 \cdot 3^2 = -45 \\ a_4 &= -5(3)^{4-1} = -5 \cdot 3^3 = -135 \end{aligned}\right\} geometric\ means$$

Answer:

$-15, -45$

The first term of a geometric sequence may not be given.

Example 9.3.3:

Find the general term of a geometric sequence where $a_2 = -2$ and $a_5 = \frac{2}{125}$.

Solution

To determine a formula for the general term we need a_1 and r. A nonlinear system with these as variables can be formed using the given information and $a_n = a_1 r^{n-1}$: :

$$\begin{cases} a_2 = a_1 r^{2-1} \\ a_5 = a_1 r^{5-1} \end{cases} \Longrightarrow \begin{cases} -2 = a_1 r & Use\ a_2 = -2. \\ \frac{2}{125} = a_1 r^4 & Use\ a_5 = \frac{2}{125}. \end{cases}$$

Solve for a_1 in the first equation,

$$\begin{aligned} -2 = a_1 r \quad &\Rightarrow \quad \frac{-2}{r} = a_1 \\ \frac{2}{125} &= a_1 r^4 \end{aligned}$$

Substitute $a_1 = \frac{-2}{r}$ into the second equation and solve for r.

$$\begin{aligned}\frac{2}{125} &= a_1 r^4 \\ \frac{2}{125} &= \left(\frac{-2}{r}\right) r^4 \\ \frac{2}{125} &= -2r^3 \\ -\frac{1}{125} &= r^3 \\ -\frac{1}{5} &= r\end{aligned}$$

Back substitute to find a_1:

$$\begin{aligned}a_1 &= \frac{-2}{r} \\ &= \frac{-2}{\left(-\frac{1}{5}\right)} \\ &= 10\end{aligned}$$

Therefore, $a_1 = 10$ and $r = -\frac{1}{5}$.

Answer:

$a_n = 10\left(-\frac{1}{5}\right)^{n-1}$

? Exercise 9.3.1

Find an equation for the general term of the given geometric sequence and use it to calculate its 6th term: $2, \frac{4}{3}, \frac{8}{9}, \ldots$

Answer

$a_n = 2\left(\frac{2}{3}\right)^{n-1}; a_6 = \frac{64}{243}$

www.youtube.com/v/IGPEl9vloLY

Geometric Series

A **geometric series**[22] is the sum of the terms of a geometric sequence. For example, the sum of the first 5 terms of the geometric sequence defined by $a_n = 3^{n+1}$ follows:

$$\begin{aligned}S_5 &= \sum_{n=1}^{5} 3^{n+1} \\ &= 3^{1+1} + 3^{2+1} + 3^{3+1} + 3^{4+1} + 3^{5+1} \\ &= 3^2 + 3^3 + 3^4 + 3^5 + 3^6 \\ &= 9 + 27 + 81 + 3^5 + 3^6 \\ &= 1,089\end{aligned}$$

Adding 5 positive integers is manageable. However, the task of adding a large number of terms is not. Therefore, we next develop a formula that can be used to calculate the sum of the first n terms of any geometric sequence. In general,

$$S_n = a_1 + a_1 r + a_1 r^2 + \ldots + a_1 r^{n-1}$$

Multiplying both sides by r we can write,

$$rS_n = a_1 r + a_1 r^2 + a_1 r^3 + \ldots + a_1 r^n$$

Subtracting these two equations we then obtain,

$$\begin{aligned}S_n - rS_n &= a_1 - a_1 r^n \\ S_n(1-r) &= a_1\left(1 - r^n\right)\end{aligned}$$

Assuming $r \neq 1$ dividing both sides by $(1-r)$ leads us to the formula for the ***n*th partial sum of a geometric sequence**[23]:

$$S_n = \frac{a_1(1-r^n)}{1-r} (r \neq 1)$$

In other words, the nth partial sum of any geometric sequence can be calculated using the first term and the common ratio. For example, to calculate the sum of the first 15 terms of the geometric sequence defined by $a_n = 3^{n+1}$, use the formula with $a_1 = 9$ and $r = 3$.

$$\begin{aligned} S_{15} &= \frac{a_1\left(1-r^{15}\right)}{1-r} \\ &= \frac{9\cdot\left(1-3^{15}\right)}{1-3} \\ &= \frac{9(-14,348,906)}{-2} \\ &= 64,570,077 \end{aligned}$$

Example 9.3.4:

Find the sum of the first 10 terms of the given sequence: $4, -8, 16, -32, 64, \ldots$

Solution

Determine whether or not there is a common ratio between the given terms.

$$r = \tfrac{-8}{4} = -2$$

Note that the ratio between any two successive terms is -2; hence, the given sequence is a geometric sequence. Use $r = -2$ and the fact that $a_1 = 4$ to calculate the sum of the first 10 terms,

$$\begin{aligned} S_n &= \frac{a_1\left(1-r^n\right)}{1-r} \\ S_{10} &= \frac{4\left[1-(-2)^{10}\right]}{1-(-2)}] \\ &= \frac{4(1-1,024)}{1+2} \\ &= \frac{4(-1,023)}{3} \\ &= -1,364 \end{aligned}$$

Answer:

$S_{10} = -1,364$

Example 9.3.5:

Evaluate: $\sum_{n=1}^{6} 2(-5)^n$.

Solution

In this case, we are asked to find the sum of the first 6 terms of a geometric sequence with general term $a_n = 2(-5)^n$. Use this to determine the 1st term and the common ratio r:

$$a_1 = 2(-5)^1 = -10$$

To show that there is a common ratio we can use successive terms in general as follows:

$$\begin{aligned} r &= \frac{a_n}{a_{n-1}} \\ &= \frac{2(-5)^n}{2(-5)^{n-1}} \\ &= (-5)^{n-(n-1)} \\ &= (-5)^1 \\ &= -5 \end{aligned}$$

Use $a_1 = -10$ and $r = -5$ to calculate the 6th partial sum.

$$S_n = \frac{a_1\left(1-r^n\right)}{1-r}$$
$$S_6 = \frac{-10\left[1-(-5)^6\right]}{1-(-5)}$$
$$= \frac{-10(1-15,625)}{1+5}$$
$$= \frac{-10(-15,624)}{6}$$
$$= 26,040$$

Answer:

$26,040$

Exercise 9.3.2

Find the sum of the first 9 terms of the given sequence: $-2, 1, -1/2, \ldots$

Answer

$S_9 = -\frac{171}{128}$

www.youtube.com/v/v-t3P95rWe8

If the common ratio r of an infinite geometric sequence is a fraction where $|r| < 1$ (that is $-1 < r < 1$), then the factor $\left(1-r^n\right)$ found in the formula for the nth partial sum tends toward 1 as n increases. For example, if $r = \frac{1}{10}$ and $n = 2, 4, 6$ we have,

$$1-\left(\tfrac{1}{10}\right)^2 = 1-0.01 = 0.99$$
$$1-\left(\tfrac{1}{10}\right)^4 = 1-0.0001 = 0.9999$$
$$1-\left(\tfrac{1}{10}\right)^6 = 1-0.00001 = 0.999999$$

Here we can see that this factor gets closer and closer to 1 for increasingly larger values of n. This illustrates the idea of a limit, an important concept used extensively in higher-level mathematics, which is expressed using the following notation:

$$\lim_{n\to\infty}\left(1-r^n\right) = 1 \text{ where } |r| < 1$$

This is read, "the limit of $\left(1-r^n\right)$ as n approaches infinity equals 1." While this gives a preview of what is to come in your continuing study of mathematics, at this point we are concerned with developing a formula for special infinite geometric series. Consider the nth partial sum of any geometric sequence,

$$S_n = \frac{a_1(1-r^n)}{1-r} = \frac{a_1}{1-r}(1-r^n)$$

If $|r| < 1$ then the limit of the partial sums as n approaches infinity exists and we can write,

$$S_n = \frac{a_1}{1-r}(1-r^n) \quad n \xrightarrow{} \infty \quad S_\infty = \frac{a_1}{1-4}\cdot 1$$

Therefore, a **convergent geometric series**[24] is an infinite geometric series where $|r| < 1$; its sum can be calculated using the formula:

$$S_\infty = \frac{a_1}{1-r}$$

Example 9.3.6:

Find the sum of the infinite geometric series: $\frac{3}{2} + \frac{1}{2} + \frac{1}{6} + \frac{1}{18} + \frac{1}{54} + \ldots$

Solution

Determine the common ratio, Since the common ratio $r = \frac{1}{3}$ is a fraction between -1 and 1, this is a convergent geometric series. Use the first term $a_1 = \frac{3}{2}$ and the common ratio to calculate its sum

$$\begin{aligned} S_\infty &= \frac{a_1}{1-r} \\ &= \frac{\frac{3}{2}}{1-\left(\frac{1}{3}\right)} \\ &= \frac{\frac{3}{3}}{\frac{2}{3}} \\ &= \frac{3}{2}\cdot\frac{3}{2} \\ &= \frac{9}{4} \end{aligned}$$

Answer:

$S_\infty = \frac{9}{4}$

Note

In the case of an infinite geometric series where $|r| \geq 1$, the series diverges and we say that there is no sum. For example, if $a_n = (5)^{n-1}$ then $r = 5$ and we have

$$S_\infty = \sum_{n=1}^{\infty}(5)^{n-1} = 1+5+25+\cdots$$

We can see that this sum grows without bound and has no sum.

Exercise 9.3.3

Find the sum of the infinite geometric series: $\sum_{n=1}^{\infty} -2\left(\frac{5}{9}\right)^{n-1}$

Answer

$-\frac{9}{2}$

www.youtube.com/v/KxsPVUyle_A

A repeating decimal can be written as an infinite geometric series whose common ratio is a power of $1/10$. Therefore, the formula for a convergent geometric series can be used to convert a repeating decimal into a fraction.

Example 9.3.7:

Write as a fraction: $1.181818\ldots$

Solution

Begin by identifying the repeating digits to the right of the decimal and rewrite it as a geometric progression.

$$\begin{aligned} 0.181818\ldots &= 0.18+0.0018+0.000018+\ldots \\ &= \frac{18}{100}+\frac{18}{10,000}+\frac{18}{1,000,000}+\ldots \end{aligned}$$

In this form we can determine the common ratio,

$$\begin{aligned} r &= \frac{\frac{18}{10,000}}{\frac{18}{100}} \\ &= \frac{18}{10,000}\times\frac{100}{18} \\ &= \frac{1}{100} \end{aligned}$$

Note that the ratio between any two successive terms is $\frac{1}{100}$. Use this and the fact that $a_1 = \frac{18}{100}$ to calculate the infinite sum:

$$\begin{aligned} S_\infty &= \frac{a_1}{1-r} \\ &= \frac{\frac{18}{100}}{1-\left(\frac{1}{100}\right)} \\ &= \frac{\frac{18}{100}}{\frac{90}{100}} \\ &= \frac{18}{100} \cdot \frac{100}{99} \\ &= \frac{2}{11} \end{aligned}$$

Therefore, $0.181818\ldots = \frac{2}{11}$ and we have,

$$1.181818\ldots = 1 + \tfrac{2}{11} = 1\tfrac{2}{11}$$

Answer:

$1\frac{2}{11}$

Example 9.3.8:

A certain ball bounces back to two-thirds of the height it fell from. If this ball is initially dropped from 27 feet, approximate the total distance the ball travels.

Solution

We can calculate the height of each successive bounce:

$27 \cdot \frac{2}{3} = 18$ feet *Height of the first bounce*

$18 \cdot \frac{2}{3} = 12$ feet *Height of the second bounce*

$12 \cdot \frac{2}{3} = 8$ feet *Height of the third bounce*

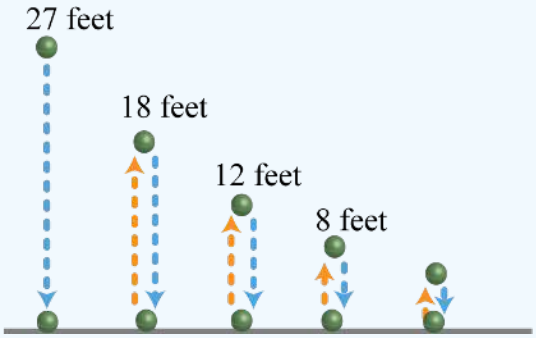


Figure 9.3.1

The total distance that the ball travels is the sum of the distances the ball is falling and the distances the ball is rising. The distances the ball falls forms a geometric series,

$27 + 18 + 12 + \ldots$ *Distance the ball is falling*

where $a_1 = 27$ and $r = \frac{2}{3}$. Because r is a fraction between -1 and 1, this sum can be calculated as follows:

$$\begin{aligned} S_\infty &= \frac{a_1}{1-r} \\ &= \frac{27}{1-\frac{2}{3}} \\ &= \frac{27}{\frac{1}{3}} \\ &= 81 \end{aligned}$$

Therefore, the ball is falling a total distance of 81 feet. The distances the ball rises forms a geometric series,

$$18+12+8+\cdots \quad \textit{Distance the ball is rising}$$

where $a_1=18$ and $r=\frac{2}{3}$. Calculate this sum in a similar manner:

$$\begin{aligned} S_\infty &= \frac{a_1}{1-r} \\ &= \frac{18}{1-\frac{2}{3}} \\ &= \frac{18}{\frac{1}{3}} \\ &= 54 \end{aligned}$$

Therefore, the ball is rising a total distance of 54 feet. Approximate the total distance traveled by adding the total rising and falling distances:

$$81+54=135 \text{ feet}$$

Answer:

135 feet

Key Takeaways

- A geometric sequence is a sequence where the ratio r between successive terms is constant.
- The general term of a geometric sequence can be written in terms of its first term a_1, common ratio r, and index n as follows: $a_n=a_1r^{n-1}$.
- A geometric series is the sum of the terms of a geometric sequence.
- The nth partial sum of a geometric sequence can be calculated using the first term a_1 and common ratio r as follows: $S_n=\frac{a_1(1-r^n)}{1-r}$.
- The infinite sum of a geometric sequence can be calculated if the common ratio is a fraction between -1 and 1 (that is $|r|<1$) as follows: $S_\infty=\frac{a_1}{1-r}$. If $|r|\geq 1$, then no sum exists.

Footnotes

[18]A sequence of numbers where each successive number is the product of the previous number and some constant r.

[19]Used when referring to a geometric sequence.

[20]The constant r that is obtained from dividing any two successive terms of a geometric sequence; $\frac{a_n}{a_{n-1}}=r$.

[21]The terms between given terms of a geometric sequence.

[22]The sum of the terms of a geometric sequence.

[23]The sum of the first n terms of a geometric sequence, given by the formula: $S_n=\frac{a_1(1-r^n)}{1-r}, r\neq 1$.

[24]An infinite geometric series where $|r|<1$ whose sum is given by the formula:$S_\infty=\frac{a_1}{1-r}$.

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9.4: Binomial Theorem

Learning Objectives

- Evaluate expressions involving factorials.
- Calculate binomial coefficients.
- Expand powers of binomials using the binomial theorem.

Factorials and the Binomial Coefficient

We begin by defining the *factorial*[25] of a natural number n, denoted $n!$, as the product of all natural numbers less than or equal to n.

$$n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$$

For example,

$$\begin{aligned} 7! &= 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1 = 5,040 && \textit{Seven factorial} \\ 5! &= 5\cdot 4\cdot 3\cdot 2\cdot 1 = 120 && \textit{Five factorial} \\ 3! &= 3\cdot 2\cdot 1 = 6 && \textit{Three factorial} \\ 1! &= 1 = 1 && \textit{One factorial} \end{aligned}$$

We define *zero factorial*[26] to be equal to 1,

$$0! = 1 \quad \textit{Zero factorial}$$

The factorial of a negative number is not defined.

On most modern calculators you will find a factorial function. Some calculators do not provide a button dedicated to it. However, it usually can be found in the menu system if one is provided.

The factorial can also be expressed using the following recurrence relation,

$$n! = n(n-1)!$$

For example, the factorial of 8 can be expressed as the product of 8 and 7!:

$$\begin{aligned} 8! &= 8\cdot 7! \\ &= 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1 \\ &= 40,320 \end{aligned}$$

When working with ratios involving factorials, it is often the case that many of the factors cancel.

Example 9.4.1

Evaluate: $\frac{12!}{6!}$.

Solution

$$\begin{aligned} \frac{12!}{6!} &= \frac{12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} \\ &= \frac{12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7\cdot \cancel{6!}}{\cancel{6!}} \\ &= 12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7 \\ &= 665,280 \end{aligned}$$

Answer

$665,280$

The *binomial coefficient*[27], denoted ${}_nC_k = \binom{n}{k}$, is read "n choose k" and is given by the following formula:

$$ {}_nC_k = \binom{n}{k} = \frac{n!}{k!(n-k)!} $$

This formula is very important in a branch of mathematics called combinatorics. It gives the number of ways k elements can be chosen from a set of n elements where order does not matter. In this section, we are concerned with the ability to calculate this quantity.

Example 9.4.2

Calculate $\binom{7}{3}$.

Solution

Use the formula for the binomial coefficient,

$$ \binom{n}{k} = \frac{n!}{k!(n-k)!} $$

where $n = 7$ and $k = 3$. After substituting, look for factors to cancel.

$$ \begin{aligned} \binom{7}{3} &= \frac{7!}{3!(7-3)!} \\ &= \frac{7!}{3!4!} \\ &= \frac{7 \cdot 6 \cdot 5 \cdot \cancel{4!}}{3! \cancel{4!}} \\ &= \frac{210}{6} \\ &= 35 \end{aligned} $$

Answer:

35

Check the menu system of your calculator for a function that calculates this quantity. Look for the notation ${}_nC_k$ in the probability subsection.

Exercise 9.4.1

Calculate $\binom{8}{5}$.

Answer

56

www.youtube.com/v/Rpb8KD1HQGc

Consider the following binomial raised to the 3rd power in its expanded form:

$$ (x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 $$

Compare it to the following calculations,

$$\binom{3}{0} = \frac{3!}{0!(3-0)!} = \frac{3!}{1 \cdot 3!} = 1$$
$$\binom{3}{1} = d\frac{3!}{1!(3-1)!} = \frac{3 \cdot 2!}{1 \cdot 2!} = 3$$
$$\binom{3}{2} = \frac{3!}{2!(3-2)!} = \frac{3 \cdot 2!}{2!} = 3$$
$$\binom{3}{3} = \frac{3!}{3!(3-3)!} = \frac{3!}{3!0!} = 1$$

Notice that there appears to be a connection between these calculations and the coefficients of the expanded binomial. This observation is generalized in the next section.

Binomial Theorem

Consider expanding $(x+2)^5$:

$$(x+2)^5 = (x+2)(x+2)(x+2)(x+2)(x+2)$$

One quickly realizes that this is a very tedious calculation involving multiple applications of the distributive property. The **binomial theorem**[28] provides a method of expanding binomials raised to powers without directly multiplying each factor:

$$(x+y)^n = \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y^1 + \binom{n}{2} x^{n-2} y^2 + \ldots + \binom{n}{n-1} x^1 y^{n-1}$$

More compactly we can write,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \quad \textit{Binomial theorem}$$

Example 9.4.3

Expand using the binomial theorem: $(x+2)^5$.

Solution

Use the binomial theorem where $n = 5$ and $y = 2$.

$$(x+2)^5 = \binom{5}{0} x^5 2^0 + \binom{5}{1} x^4 2^1 + \binom{5}{2} x^3 2^2 + \binom{5}{3} x^2 2^3 + \binom{5}{4} x^1 2^4$$

Sometimes it is helpful to identify the pattern that results from applying the binomial theorem. Notice that powers of the variable x start at 5 and decrease to zero. The powers of the constant term start at 0 and increase to 5. The binomial coefficients can be calculated off to the side and are left to the reader as an exercise.

$$\begin{aligned}(x+2)^5 &= \binom{5}{0} x^5 2^0 + \binom{5}{1} x^4 2^1 + \binom{5}{2} x^3 2^2 + \binom{5}{3} x^2 2^3 + \binom{5}{4} x^1 2^4 \\ &= 1x^5 \times 1 + 5x^4 \times 2 + 10x^3 \times 4 + 10x^2 \times 8 + 5x^1 \times 16 + 1 \times 1 \\ &= x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32\end{aligned}$$

Answer

$x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32$

The binomial may have negative terms, in which case we will obtain an alternating series.

Example 9.4.4

Expand using the binomial theorem: $(u-2v)^4$.

Solution

Use the binomial theorem where $n=4, x=u$, and $y=-2v$ and then simplify each term.

$$\begin{aligned}(u-2v)^4 &= \binom{4}{0}u^4(-2v)^0+\binom{4}{1}u^3(-2v)^1+\binom{4}{2}u^2(-2v)^2+\binom{4}{3}u^1(-2v)^3+\binom{4}{4}u^0(-2v)^4\\ &= 1\times u^4\times 1+4u^3(-2v)+6u^2\left(4v^2\right)+4u\left(-8v^3\right)+16v^4\\ &= u^4-8u^3v+24u^2v^2-32uv^3+16v^4\end{aligned}$$

Answer

$u^4-8u^3v+24u^2v^2-32uv^3+16v^4$

Exercise 9.4.2

Expand using the binomial theorem: $\left(a^2-3\right)^4$

Answer

$a^8-12a^6+54a^4-108a^2+81$

www.youtube.com/v/wICbqmoa4T4

Next we study the coefficients of the expansions of $(x+y)^n$ starting with $n=0$:

$$\begin{aligned}(x+y)^0 &= 1\\ (x+y)^1 &= x+y\\ (x+y)^2 &= x+y\\ (x+y)^3 &= x^2+2xy+y^2\\ (x+y)^3 &= x^3+3x^2y+3xy^2+y^3\\ (x+y)^4 &= x^4+4x^3y+6x^2y^2+4xy^3+y^4\end{aligned}$$

Write the coefficients in a triangular array and note that each number below is the sum of the two numbers above it, always leaving a 1 on either end.

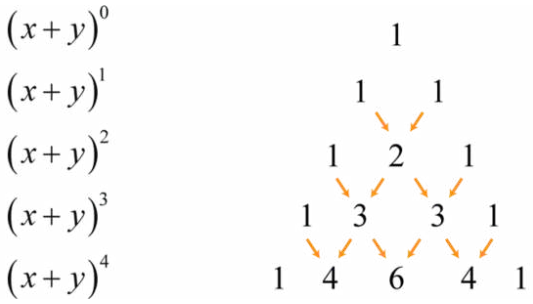


Figure 9.4.1

This is **Pascal's triangle**[29]; it provides a quick method for calculating the binomial coefficients. Use this in conjunction with the binomial theorem to streamline the process of expanding binomials raised to powers. For example, to expand $(x-1)^6$ we would need two more rows of Pascal's triangle,

$$
\begin{array}{ll}
(x+y)^0 & 1 \\
(x+y)^1 & 1 \quad 1 \\
(x+y)^2 & 1 \quad 2 \quad 1 \\
(x+y)^3 & 1 \quad 3 \quad 3 \quad 1 \\
(x+y)^4 & 1 \quad 4 \quad 6 \quad 4 \quad 1 \\
(x+y)^5 & 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
(x+y)^6 & 1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1
\end{array}
$$

Figure 9.4.2

The binomial coefficients that we need are in blue. Use these numbers and the binomial theorem to quickly expand $(x-1)^6$ as follows:

$$
\begin{aligned}
(x-1)^6 &= 1x^6(-1)^0 + 6x^5(-1)^1 + 15x^4(-1)^2 + 20x^3(-1)^3 + 15x^2(-1)^4 + 6x(-1)^5 \\
&= x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1
\end{aligned}
$$

Example 9.4.5

Expand using the binomial theorem and Pascal's triangle: $(2x-5)^4$.

Solution

From Pascal's triangle we can see that when $n=4$ the binomial coefficients are $1, 4, 6, 4$, and 1.Use these numbers and the binomial theorem as follows:

$$
\begin{aligned}
(2x-5)^4 &= 1(2x)^4(-5)^0 + 4(2x)^3(-5)^1 + 6(2x)^2(-5)^2 + 4(2x)^1(-5)^3 + (2x)^0(-5)^4 \\
&= 16x^4 \cdot 1 + 4 \cdot 8x^3(-5) + 6 \cdot 4x^2 \cdot 25 + 4 \cdot 2x(-125) + 1 \cdot 625 \\
&= 16x^4 - 160x^3 + 600x^2 - 1,000x + 625
\end{aligned}
$$

Answer:

$16x^4 - 160x^3 + 600x^2 - 1,000x + 625$

Key Takeaways

- To calculate the factorial of a natural number, multiply that number by all natural numbers less than it: $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$. Remember that we have defined $0! = 1$.
- The binomial coefficients are the integers calculated using the formula:

$$
\binom{n}{k} = \frac{n!}{k!(n-k)!}.
$$

- The binomial theorem provides a method for expanding binomials raised to powers without directly multiplying each factor:

$$
(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k
$$

- Use Pascal's triangle to quickly determine the binomial coefficients.

Footnotes

[25] The product of all natural numbers less than or equal to a given natural number, denoted $n!$.

[26] The factorial of zero is defined to be equal to $1; 0! = 1$.

[27] An integer that is calculated using the formula: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

[28] Describes the algebraic expansion of binomials raised to powers: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$.

[29] A triangular array of numbers that correspond to the binomial coefficients.

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