MCC: MTH 212 CALCULUS III

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CHAPTER OVERVIEW

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11.1: Vectors in the Plane

Learning Objectives

- Describe a plane vector, using correct notation.
- Perform basic vector operations (scalar multiplication, addition, subtraction).
- Express a vector in component form.
- Explain the formula for the magnitude of a vector.
- Express a vector in terms of unit vectors.
- Give two examples of vector quantities.

When describing the movement of an airplane in flight, it is important to communicate two pieces of information: the direction in which the plane is traveling and the plane's speed. Together these form the plane's velocity. When measuring a force, such as the thrust of the plane's engines, it is important to describe not only the strength of that force, but also the direction in which it is applied. Some quantities, such as velocity and force, are defined in terms of both size (also called magnitude) and direction. A quantity that has magnitude and direction is called a **vector**. In textbooks, vectors are often denoted by boldface letters, such as $\mathbf{v}$. Since it is difficult to write in clear boldface when we write vectors by hand, we will also include an arrow or harpoon above the letter representing the vector. To make the vectors clearer in this textbook (and to reinforce the way you should write them by hand), we will generally use arrows or harpoons above boldface (or italic) letters to represent vectors, giving us $\vec{\boldsymbol{v}}$ or $\vec{\mathbf{v}}$. Note that some figures will still only use boldface letters to denote vectors.

Definition: Vector

A vector is a quantity that has both magnitude and direction.

Vector Representation

A vector in a plane is represented by a directed line segment (an arrow). The endpoints of the segment are called the **initial point** and the **terminal point** of the vector. An arrow from the initial point to the terminal point indicates the direction of the vector. The length of the line segment represents its **magnitude**. We use the notation $\|\vec{\mathbf{v}}\|$ to denote the magnitude of the vector $\vec{\mathbf{v}}$. A vector with an initial point and terminal point that are the same is called the **zero vector**, denoted $\vec{\mathbf{0}}$. The zero vector is the only vector without a direction, and by convention can be considered to have any direction convenient to the problem at hand.

Vectors with the same magnitude and direction are called equivalent vectors. We treat equivalent vectors as equal, even if they have different initial points. Thus, if $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ are equivalent, we write

$$\vec{\mathbf{v}} = \vec{\mathbf{w}}. \tag{11.1.1}$$

Definition: Equivalent Vectors

Vectors are said to be *equivalent vectors* if they have the same magnitude and direction.

The arrows in Figure 11.1.1(b) are equivalent. Each arrow has the same length and direction. A closely related concept is the idea of parallel vectors. Two vectors are said to be parallel if they have the same or opposite directions. We explore this idea in more detail later in the chapter. A vector is defined by its magnitude and direction, regardless of where its initial point is located.

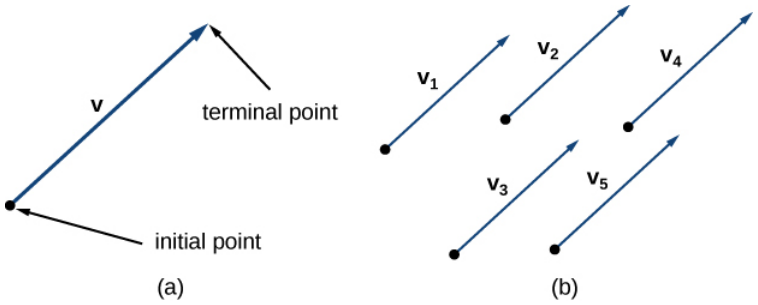


Figure 11.1.1: (a) A vector is represented by a directed line segment from its initial point to its terminal point. (b) Vectors $\vec{\mathbf{v}}_1$ through $\vec{\mathbf{v}}_5$ are equivalent.

The use of boldface, lowercase letters to name vectors is a common representation in print, but there are alternative notations. When writing the name of a vector by hand, for example, it is easier to sketch an arrow over the variable than to show it is a vector: $\vec{v}$. When a vector has initial point P and terminal point Q, the notation $\overrightarrow{PQ}$ is useful because it indicates the direction and location of the vector.

Example 11.1.1: Sketching Vectors

Sketch a vector in the plane from initial point $P(1,1)$ to terminal point $Q(8,5)$.

Solution

See Figure 11.1.2. Because the vector goes from point P to point Q, we name it $\overrightarrow{PQ}$.

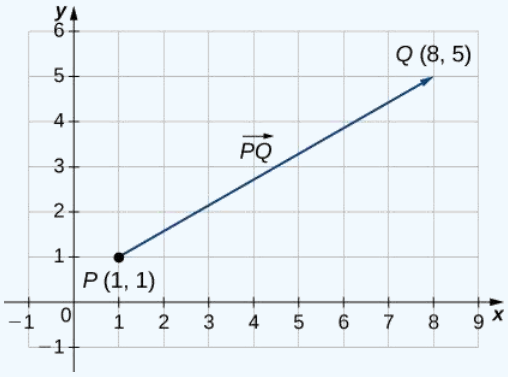


Figure 11.1.2: The vector with initial point $(1,1)$ and terminal point $(8,5)$ is named $\overrightarrow{PQ}$.

Exercise 11.1.1

Sketch the vector $\overrightarrow{ST}$ where S is point $(3,-1)$ and T is point $(-2,3)$.

Combining Vectors

Vectors have many real-life applications, including situations involving force or velocity. For example, consider the forces acting on a boat crossing a river. The boat's motor generates a force in one direction, and the current of the river generates a force in another direction. Both forces are vectors. We must take both the magnitude and direction of each force into account if we want to know where the boat will go.

A second example that involves vectors is a quarterback throwing a football. The quarterback does not throw the ball parallel to the ground; instead, he aims up into the air. The velocity of his throw can be represented by a vector. If we know how hard he throws the ball (magnitude—in this case, speed), and the angle (direction), we can tell how far the ball will travel down the field.

A real number is often called a **scalar** in mathematics and physics. Unlike vectors, scalars are generally considered to have a magnitude only, but no direction. Multiplying a vector by a scalar changes the vector's magnitude. This is called scalar multiplication. Note that changing the magnitude of a vector does not indicate a change in its direction. For example, wind blowing from north to south might increase or decrease in speed while maintaining its direction from north to south.

Definition: Scalar Multiplication

The product $k\vec{\mathbf{v}}$ of a vector $\vec{\mathbf{v}}$ and a scalar k is a vector with a magnitude that is $|k|$ times the magnitude of $\vec{\mathbf{v}}$, and with a direction that is the same as the direction of $\vec{\mathbf{v}}$ if $k > 0$, and opposite the direction of $\vec{\mathbf{v}}$ if $k < 0$. This is called *scalar multiplication*. If $k = 0$ or $\vec{\mathbf{v}} = \vec{\mathbf{0}}$, then $k\vec{\mathbf{v}} = \vec{\mathbf{0}}$.

As you might expect, if $k = -1$, we denote the product $k\vec{\mathbf{v}}$ as

$$k\vec{\mathbf{v}} = (-1)\vec{\mathbf{v}} = -\vec{\mathbf{v}}. \tag{11.1.2}$$

Note that $-\vec{\mathbf{v}}$ has the same magnitude as $\vec{\mathbf{v}}$, but has the opposite direction (Figure 11.1.3).

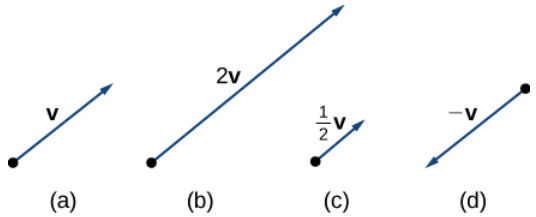


Figure 11.1.3: (a) The original vector $\vec{\mathbf{v}}$ has length n units. (b) The length of $2\vec{\mathbf{v}}$ equals $2n$ units. (c) The length of $\vec{\mathbf{v}}/2$ is $n/2$ units. (d) The vectors $\vec{\mathbf{v}}$ and $-\vec{\mathbf{v}}$ have the same length but opposite directions.

Another operation we can perform on vectors is to add them together in vector addition, but because each vector may have its own direction, the process is different from adding two numbers. The most common graphical method for adding two vectors is to place the initial point of the second vector at the terminal point of the first, as in Figure 11.1.4(a). To see why this makes sense, suppose, for example, that both vectors represent displacement. If an object moves first from the initial point to the terminal point of vector $\vec{\mathbf{v}}$, then from the initial point to the terminal point of vector $\vec{\mathbf{w}}$, the overall displacement is the same as if the object had made just one movement from the initial point to the terminal point of the vector $\vec{\mathbf{v}} + \vec{\mathbf{w}}$. For obvious reasons, this approach is called the **triangle method**. Notice that if we had switched the order, so that $\vec{\mathbf{w}}$ was our first vector and $\vec{\mathbf{v}}$ was our second vector, we would have ended up in the same place. (Again, see Figure 11.1.4(a).) Thus,

$$\vec{\mathbf{v}} + \vec{\mathbf{w}} = \vec{\mathbf{w}} + \vec{\mathbf{v}}. \tag{11.1.3}$$

A second method for adding vectors is called the **parallelogram method**. With this method, we place the two vectors so they have the same initial point, and then we draw a parallelogram with the vectors as two adjacent sides, as in Figure 11.1.4(b). The length of the diagonal of the parallelogram is the sum. Comparing Figure 11.1.4(b) and Figure 11.1.4(a), we can see that we get the same answer using either method. The vector $\vec{\mathbf{v}} + \vec{\mathbf{w}}$ is called the **vector sum.**

Definition: Vector Addition

The sum of two vectors $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ can be constructed graphically by placing the initial point of $\vec{\mathbf{w}}$ at the terminal point of $\vec{\mathbf{v}}$. Then, the vector sum, $\vec{\mathbf{v}} + \vec{\mathbf{w}}$, is the vector with an initial point that coincides with the initial point of $\vec{\mathbf{v}}$ and has a terminal point that coincides with the terminal point of $\vec{\mathbf{w}}$. This operation is known as *vector addition*.

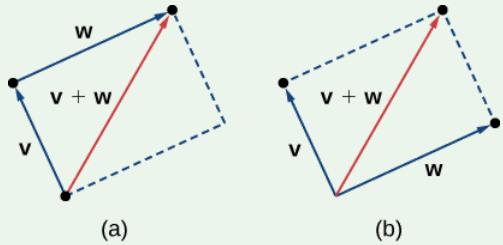


Figure 11.1.4: (a) When adding vectors by the triangle method, the initial point of $\vec{\mathbf{w}}$ is the terminal point of $\vec{\mathbf{v}}$. (b) When adding vectors by the parallelogram method, the vectors$\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ have the same initial point.

It is also appropriate here to discuss vector subtraction. We define $\vec{\mathbf{v}} - \vec{\mathbf{w}}$ as $\vec{\mathbf{v}} + (-\vec{\mathbf{w}}) = \vec{\mathbf{v}} + (-1)\vec{\mathbf{w}}$. The vector $\vec{\mathbf{v}} - \vec{\mathbf{w}}$ is called the **vector difference**. Graphically, the vector $\vec{\mathbf{v}} - \vec{\mathbf{w}}$ is depicted by drawing a vector from the terminal point of $\vec{\mathbf{w}}$ to the terminal point of $\vec{\mathbf{v}}$ (Figure 11.1.5).

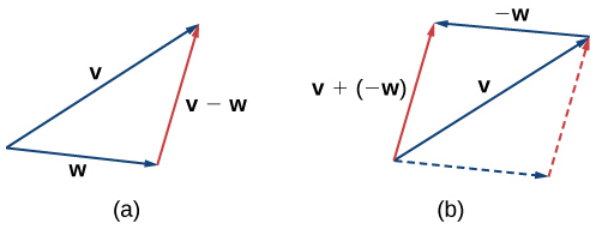


Figure 11.1.5: (a) The vector difference $\vec{v} - \vec{w}$ is depicted by drawing a vector from the terminal point of $\vec{w}$ to the terminal point of $\vec{v}$. (b) The vector $\vec{v} - \vec{w}$ is equivalent to the vector $\vec{v} + (-\vec{w})$.

In Figure 11.1.4(*a*), the initial point of $\vec{v} + \vec{w}$ is the initial point of $\vec{v}$. The terminal point of $\vec{v} + \vec{w}$ is the terminal point of $\vec{w}$. These three vectors form the sides of a triangle. It follows that the length of any one side is less than the sum of the lengths of the remaining sides. So we have

$$\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|. \tag{11.1.4}$$

This is known more generally as the **triangle inequality**. There is one case, however, when the resultant vector $\vec{u} + \vec{v}$ has the same magnitude as the sum of the magnitudes of $\vec{u}$ and $\vec{v}$. This happens only when $\vec{u}$ and $\vec{v}$ have the same direction.

Example 11.1.2: Combining Vectors

Given the vectors $\vec{v}$ and $\vec{w}$ shown in Figure 11.1.6, sketch the vectors

a. $3\vec{w}$
b. $\vec{v} + \vec{w}$
c. $2\vec{v} - \vec{w}$

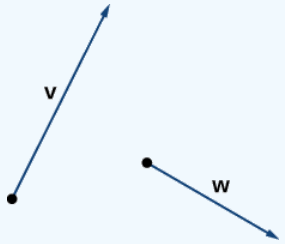
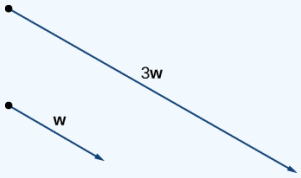


Figure 11.1.6: Vectors $\vec{v}$ and $\vec{w}$ lie in the same plane.

Solution

a. The vector $3\vec{w}$ has the same direction as $\vec{w}$; it is three times as long as $\vec{w}$.



Vector $3\vec{w}$ has the same direction as $\vec{w}$ and is three times as long.

b. Use either addition method to find $\vec{v} + \vec{w}$.

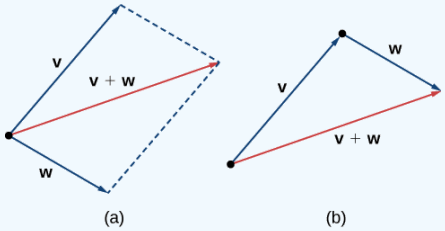


Figure 11.1.7: To find $\vec{v} + \vec{w}$, align the vectors at their initial points or place the initial point of one vector at the terminal point of the other. (a) The vector $\vec{v} + \vec{w}$ is the diagonal of the parallelogram with sides $\vec{v}$ and $\vec{w}$. (b) The vector $\vec{v} + \vec{w}$ is the third side of a triangle formed with $\vec{w}$ placed at the terminal point of $\vec{v}$.

c. To find $2\vec{v} - \vec{w}$, we can first rewrite the expression as $2\vec{v} + (-\vec{w})$. Then we can draw the vector $-\vec{w}$, then add it to the vector $2\vec{v}$.

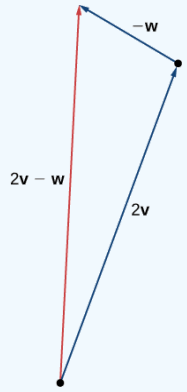


Figure 11.1.8: To find $2\vec{v} - \vec{w}$, simply add $2\vec{v} + (-\vec{w})$.

Exercise 11.1.2

Using vectors $\vec{v}$ and $\vec{w}$ from Example 11.1.2, sketch the vector $2\vec{w} - \vec{v}$.

Vector Components

Working with vectors in a plane is easier when we are working in a coordinate system. When the initial points and terminal points of vectors are given in Cartesian coordinates, computations become straightforward.

Example 11.1.3: Comparing Vectors

Are $\vec{v}$ and $\vec{w}$ equivalent vectors?

a.

- $\vec{v}$ has initial point $(3, 2)$ and terminal point $(7, 2)$
- $\vec{w}$ has initial point $(1, -4)$ and terminal point $(1, 0)$

b.

- $\vec{v}$ has initial point $(0, 0)$ and terminal point $(1, 1)$
- $\vec{w}$ has initial point $(-2, 2)$ and terminal point $(-1, 3)$

Solution

a. The vectors are each 4 units long, but they are oriented in different directions. So $\vec{v}$ and $\vec{w}$ are not equivalent (Figure 11.1.9).

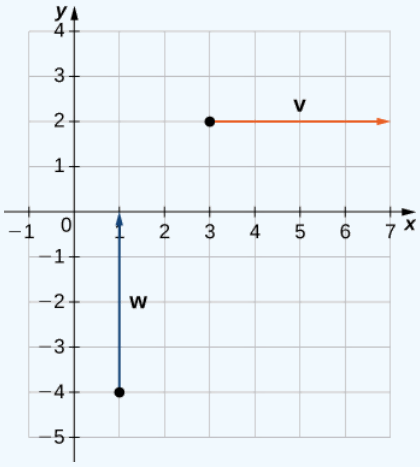


Figure 11.1.9: These vectors are not equivalent.

b. Based on Figure **11.1.10**, and using a bit of geometry, it is clear these vectors have the same length and the same direction, so $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ are equivalent.

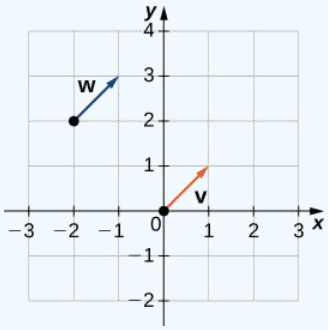
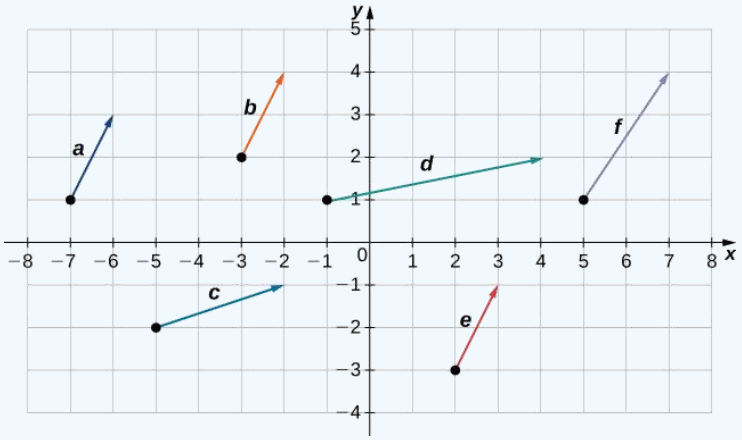


Figure **11.1.10**: These vectors are equivalent.

Exercise 11.1.3

Which of the following vectors are equivalent?



We have seen how to plot a vector when we are given an initial point and a terminal point. However, because a vector can be placed anywhere in a plane, it may be easier to perform calculations with a vector when its initial point coincides with the origin. We call a vector with its initial point at the origin a **standard-position vector.** Because the initial point of any vector in standard position is known to be $(0, 0)$, we can describe the vector by looking at the coordinates of its terminal point. Thus, if vector $\vec{\mathbf{v}}$ has its initial point at the origin and its terminal point at (x, y), we write the vector in component form as

$$\vec{\mathbf{v}} = \langle x, y \rangle. \tag{11.1.5}$$

When a vector is written in component form like this, the scalars x and y are called the **components** of $\vec{\mathbf{v}}$.

Definition: Vector components

The vector with initial point $(0, 0)$ and terminal point (x, y) can be written in component form as

$$\vec{\mathbf{v}} = \langle x, y \rangle. \tag{11.1.6}$$

The scalars x and y are called the components of $\vec{\mathbf{v}}$.

Recall that vectors are named with lowercase letters in bold type or by drawing an arrow over their name. We have also learned that we can name a vector by its component form, with the coordinates of its terminal point in angle brackets. However, when writing the component form of a vector, it is important to distinguish between $\langle x, y \rangle$ and (x, y). The first ordered pair uses angle brackets to describe a vector, whereas the second uses parentheses to describe a point in a plane. The initial point of $\langle x, y \rangle$ is $(0, 0)$; the terminal point of $\langle x, y \rangle$ is (x, y).

When we have a vector not already in standard position, we can determine its component form in one of two ways. We can use a geometric approach, in which we sketch the vector in the coordinate plane, and then sketch an equivalent standard-position vector. Alternatively, we can find it algebraically, using the coordinates of the initial point and the terminal point. To find it algebraically, we subtract the x-coordinate of the initial point from the x-coordinate of the terminal point to get the x-component, and we subtract the y-coordinate of the initial point from the y-coordinate of the terminal point to get the y-component.

Rule: Component Form of a Vector

Let $\vec{\mathbf{v}}$ be a vector with initial point (x_i, y_i) and terminal point (x_t, y_t). Then we can express $\vec{\mathbf{v}}$ in component form as $\vec{\mathbf{v}} = \langle x_t - x_i, y_t - y_i \rangle$.

Example 11.1.4: Expressing Vectors in Component Form

Express vector $\vec{\mathbf{v}}$ with initial point $(-3, 4)$ and terminal point $(1, 2)$ in component form.

Solution:

a. Geometric

1. Sketch the vector in the coordinate plane (Figure 11.1.11).
2. The terminal point is 4 units to the right and 2 units down from the initial point.
3. Find the point that is 4 units to the right and 2 units down from the origin.
4. In standard position, this vector has initial point $(0, 0)$ and terminal point $(4, -2)$:

$$\vec{\mathbf{v}} = \langle 4, -2 \rangle.$$

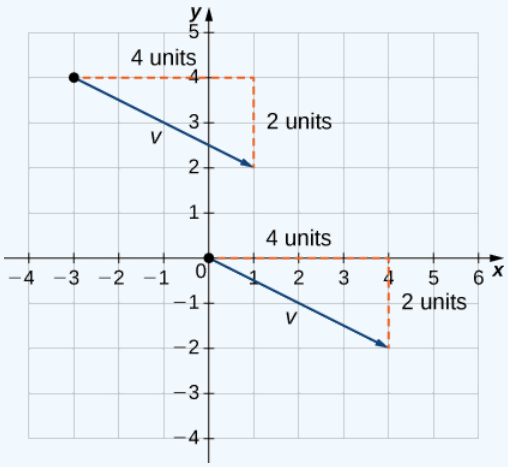


Figure 11.1.11: These vectors are equivalent.

b. Algebraic

In the first solution, we used a sketch of the vector to see that the terminal point lies 4 units to the right. We can accomplish this algebraically by finding the difference of the x-coordinates:

$$x_t - x_i = 1 - (-3) = 4.$$

Similarly, the difference of the y-coordinates shows the vertical length of the vector.

$$y_t - y_i = 2 - 4 = -2.$$

So, in component form,

$$\vec{\mathbf{v}} = \langle x_t - x_i, y_t - y_i \rangle = \langle 1 - (-3), 2 - 4 \rangle = \langle 4, -2 \rangle.$$

Exercise 11.1.4

Vector $\vec{\mathbf{w}}$ has initial point $(-4,-5)$ and terminal point $(-1,2)$. Express $\vec{\mathbf{w}}$ in component form.

To find the magnitude of a vector, we calculate the distance between its initial point and its terminal point. The **magnitude** of vector $\vec{\mathbf{v}} = \langle x, y \rangle$ is denoted $\|\vec{\mathbf{v}}\|$, or $|\vec{\mathbf{v}}|$, and can be computed using the formula

$$\|\vec{\mathbf{v}}\| = \sqrt{x^2 + y^2}. \tag{11.1.7}$$

Note that because this vector is written in component form, it is equivalent to a vector in standard position, with its initial point at the origin and terminal point (x, y). Thus, it suffices to calculate the magnitude of the vector in standard position. Using the distance formula to calculate the distance between initial point $(0,0)$ and terminal point (x, y), we have

$$\|\vec{\mathbf{v}}\| = \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2}. \tag{11.1.8}$$

Based on this formula, it is clear that for any vector $\vec{\mathbf{v}}$, $\|\vec{\mathbf{v}}\| \geq 0$, and $\|\vec{\mathbf{v}}\| = 0$ if and only if $\vec{\mathbf{v}} = \vec{\mathbf{0}}$.

The magnitude of a vector can also be derived using the Pythagorean theorem, as in the following figure.

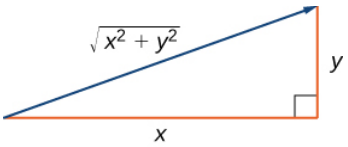


Figure **11.1.12**: If you use the components of a vector to define a right triangle, the magnitude of the vector is the length of the triangle's hypotenuse.

We have defined scalar multiplication and vector addition geometrically. Expressing vectors in component form allows us to perform these same operations algebraically.

Definition: Scalar multiplication and Vector addition

Let $\vec{\mathbf{v}} = \langle x_1, y_1 \rangle$ and $\vec{\mathbf{w}} = \langle x_2, y_2 \rangle$ be vectors, and let k be a scalar.

- *Scalar multiplication:*

$$k\vec{\mathbf{v}} = \langle kx_1, ky_1 \rangle \tag{11.1.9}$$

- *Vector addition:*

$$\vec{\mathbf{v}} + \vec{\mathbf{w}} = \langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle = \langle x_1 + x_2, y_1 + y_2 \rangle \tag{11.1.10}$$

Example 11.1.5: Performing Operations in Component Form

Let $\vec{\mathbf{v}}$ be the vector with initial point $(2,5)$ and terminal point $(8,13)$, and let $\vec{\mathbf{w}} = \langle -2, 4 \rangle$.

a. Express $\vec{\mathbf{v}}$ in component form and find $\|\vec{\mathbf{v}}\|$. Then, using algebra, find
b. $\vec{\mathbf{v}} + \vec{\mathbf{w}}$,
c. $3\vec{\mathbf{v}}$, and
d. $\vec{\mathbf{v}} - 2\vec{\mathbf{w}}$.

Solution

a. To place the initial point of $\vec{\mathbf{v}}$ at the origin, we must translate the vector 2 units to the left and 5 units down (Figure **11.1.13**). Using the algebraic method, we can express $\vec{\mathbf{v}}$ as $\vec{\mathbf{v}} = \langle 8-2, 13-5 \rangle = \langle 6, 8 \rangle$:

$$\|\vec{\mathbf{v}}\| = \sqrt{6^2 + 8^2} = \sqrt{36 + 64} = \sqrt{100} = 10.$$

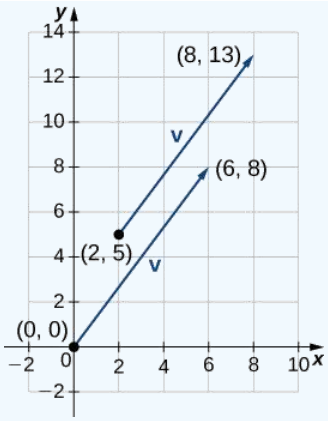


Figure 11.1.13: In component form, $\vec{\mathbf{v}} = \langle 6, 8 \rangle$.

b. To find $\vec{\mathbf{v}} + \vec{\mathbf{w}}$, add the x-components and the y-components separately:

$$\vec{\mathbf{v}} + \vec{\mathbf{w}} = \langle 6, 8 \rangle + \langle -2, 4 \rangle = \langle 4, 12 \rangle.$$

c. To find $3\vec{\mathbf{v}}$, multiply $\vec{\mathbf{v}}$ by the scalar $k = 3$:

$$3\vec{\mathbf{v}} = 3 \cdot \langle 6, 8 \rangle = \langle 3 \cdot 6, 3 \cdot 8 \rangle = \langle 18, 24 \rangle.$$

d. To find $\vec{\mathbf{v}} - 2\vec{\mathbf{w}}$, find $-2\vec{\mathbf{w}}$ and add it to $\vec{\mathbf{v}}$:

$$\vec{\mathbf{v}} - 2\vec{\mathbf{w}} = \langle 6, 8 \rangle - 2 \cdot \langle -2, 4 \rangle = \langle 6, 8 \rangle + \langle 4, -8 \rangle = \langle 10, 0 \rangle.$$

Exercise 11.1.5A

Let $\vec{\mathbf{a}} = \langle 7, 1 \rangle$ and let $\vec{\mathbf{b}}$ be the vector with initial point $(3, 2)$ and terminal point $(-1, -1)$.

a. Find $\|\vec{\mathbf{a}}\|$.
b. Express $\vec{\mathbf{b}}$ in component form.
c. Find $3\vec{\mathbf{a}} - 4\vec{\mathbf{b}}$.

Now that we have established the basic rules of vector arithmetic, we can state the properties of vector operations. We will prove two of these properties. The others can be proved in a similar manner.

Properties of Vector Operations

Let $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$ be vectors in a plane. Let r and s be scalars.

i. *Commutative property*

$$\vec{\mathbf{u}} + \vec{\mathbf{v}} = \vec{\mathbf{v}} + \vec{\mathbf{u}} \tag{11.1.11}$$

ii. *Associative property*

$$(\vec{\mathbf{u}} + \vec{\mathbf{v}}) + \vec{\mathbf{w}} = \vec{\mathbf{u}} + (\vec{\mathbf{v}} + \vec{\mathbf{w}}) \tag{11.1.12}$$

iii. *Additive identity property*

$$\vec{\mathbf{u}} + \vec{\mathbf{0}} = \vec{\mathbf{u}} \tag{11.1.13}$$

iv. *Additive inverse property*

$$\vec{\mathbf{u}} + (-\vec{\mathbf{u}}) = \vec{\mathbf{0}} \tag{11.1.14}$$

v. *Associativity of scalar multiplication*

$$r(s\vec{\mathbf{u}}) = (rs)\vec{\mathbf{u}} \tag{11.1.15}$$

vi. *Distributive property*

$$(r+s)\vec{\mathbf{u}} = r\vec{\mathbf{u}} + s\vec{\mathbf{u}} \tag{11.1.16}$$

vii. *Distributive property*

$$r(\vec{\mathbf{u}} + \vec{\mathbf{v}}) = r\vec{\mathbf{u}} + r\vec{\mathbf{v}} \tag{11.1.17}$$

viii. *Identity and zero properties*

$$1\vec{\mathbf{u}} = \vec{\mathbf{u}},\ 0\vec{\mathbf{u}} = \vec{\mathbf{0}} \tag{11.1.18}$$

Proof of Commutative Property

Let $\vec{\mathbf{u}} = \langle x_1, y_1 \rangle$ and $\vec{\mathbf{v}} = \langle x_2, y_2 \rangle$. Apply the commutative property (Equation 11.1.11) for real numbers:

$$\begin{aligned} \vec{\mathbf{u}} + \vec{\mathbf{v}} &= \langle x_1 + x_2, y_1 + y_2 \rangle \\ &= \langle x_2 + x_1, y_2 + y_1 \rangle \\ &= \vec{\mathbf{v}} + \vec{\mathbf{u}}. \end{aligned}$$

□

Proof of Distributive Property

Apply the distributive property (Equation 11.1.16) for real numbers:

$$\begin{aligned} r(\vec{\mathbf{u}} + \vec{\mathbf{v}}) &= r \cdot \langle x_1 + x_2, y_1 + y_2 \rangle \\ &= \langle r(x_1 + x_2), r(y_1 + y_2) \rangle \\ &= \langle rx_1 + rx_2, ry_1 + ry_2 \rangle \\ &= \langle rx_1, ry_1 \rangle + \langle rx_2, ry_2 \rangle \\ &= r\vec{\mathbf{u}} + r\vec{\mathbf{v}}. \end{aligned}$$

□

Exercise 11.1.5B

Prove the additive inverse property.

We have found the components of a vector given its initial and terminal points. In some cases, we may only have the magnitude and direction of a vector, not the points. For these vectors, we can identify the horizontal and vertical components using trigonometry (Figure 11.1.14).

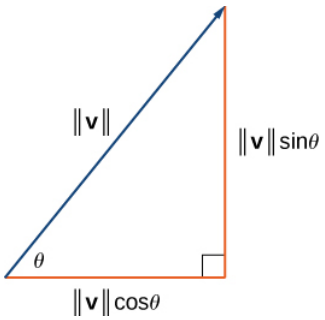


Figure 11.1.14: The components of a vector form the legs of a right triangle, with the vector as the hypotenuse.

Consider the angle θ formed by the vector $\vec{\mathbf{v}}$ and the positive x-axis. We can see from the triangle that the components of vector $\vec{\mathbf{v}}$ are $\langle \|\vec{\mathbf{v}}\| \cos\theta,\ \|\vec{\mathbf{v}}\| \sin\theta \rangle$. Therefore, given an angle and the magnitude of a vector, we can use the cosine and sine of the angle to find the components of the vector.

Example 11.1.6: Finding the Component Form of a Vector Using Trigonometry

Find the component form of a vector with magnitude 4 that forms an angle of $-45°$ with the x-axis.

Solution

Let x and y represent the components of the vector (Figure 11.1.15). Then $x = 4\cos(-45°) = 2\sqrt{2}$ and $y = 4\sin(-45°) = -2\sqrt{2}$. The component form of the vector is $\langle 2\sqrt{2}, -2\sqrt{2}\rangle$.

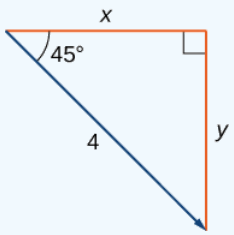


Figure 11.1.15: Use trigonometric ratios, $x = \|\vec{v}\|\cos\theta$ *and* $y = \|\vec{v}\|\sin\theta$, to identify the components of the vector.

Exercise 11.1.6

Find the component form of vector $\vec{v}$ with magnitude 10 that forms an angle of $120°$ with the positive x-axis.

Unit Vectors

A **unit vector** is a vector with magnitude 1. For any nonzero vector $\vec{v}$, we can use scalar multiplication to find a unit vector $\vec{u}$ that has the same direction as $\vec{v}$. To do this, we multiply the vector by the reciprocal of its magnitude:

$$\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v}. \tag{11.1.19}$$

Recall that when we defined scalar multiplication, we noted that $\|k\vec{v}\| = |k| \cdot \|\vec{v}\|$. For $\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v}$, it follows that $\|\vec{u}\| = \frac{1}{\|\vec{v}\|}(\|\vec{v}\|) = 1$. We say that $\vec{u}$ is the *unit vector in the direction of* $\vec{v}$ (Figure 11.1.16). The process of using scalar multiplication to find a unit vector with a given direction is called **normalization.**

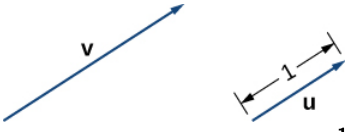


Figure 11.1.16: The vector $\vec{v}$ *and associated unit vector* $\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v}$. *In this case,* $\|\vec{v}\| > 1$.

Example 11.1.7: Finding a Unit Vector

Let $\vec{v} = \langle 1, 2\rangle$.

a. Find a unit vector with the same direction as $\vec{v}$.
b. Find a vector $\vec{w}$ with the same direction as $\vec{v}$ such that $\|\vec{w}\| = 7$.

Solution:

a. First, find the magnitude of $\vec{v}$, then divide the components of $\vec{v}$ by the magnitude:

$$\|\vec{v}\| = \sqrt{1^2 + 2^2} = \sqrt{1+4} = \sqrt{5}$$

$$\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v} = \frac{1}{\sqrt{5}}\langle 1, 2\rangle = \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle = \left\langle \frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5} \right\rangle.$$

b. The vector $\vec{u}$ is in the same direction as $\vec{v}$ and $\|\vec{u}\| = 1$. Use scalar multiplication to increase the length of $\vec{u}$ without changing direction:

$$\vec{\mathbf{w}} = 7\vec{\mathbf{u}} = 7\left\langle \frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5} \right\rangle = \left\langle \frac{7\sqrt{5}}{5}, \frac{14\sqrt{5}}{5} \right\rangle.$$

Exercise 11.1.7

Let $\vec{\mathbf{v}} = \langle 9, 2 \rangle$. Find a vector with magnitude 5 in the opposite direction as $\vec{\mathbf{v}}$.

We have seen how convenient it can be to write a vector in component form. Sometimes, though, it is more convenient to write a vector as a sum of a horizontal vector and a vertical vector. To make this easier, let's look at standard unit vectors. The **standard unit vectors** are the vectors $\hat{\mathbf{i}} = \langle 1, 0 \rangle$ and $\hat{\mathbf{j}} = \langle 0, 1 \rangle$ (Figure 11.1.17).

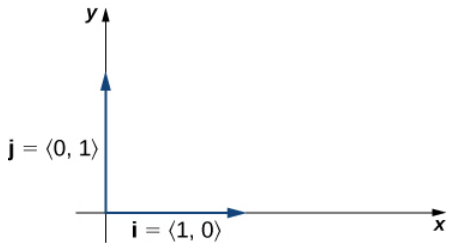


Figure 11.1.17: The standard unit vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$.

By applying the properties of vectors, it is possible to express any vector in terms of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ in what we call a *linear combination:*

$$\vec{\mathbf{v}} = \langle x, y \rangle = \langle x, 0 \rangle + \langle 0, y \rangle = x\langle 1, 0 \rangle + y\langle 0, 1 \rangle = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}. \tag{11.1.20}$$

Thus, $\vec{\mathbf{v}}$ is the sum of a horizontal vector with magnitude x, and a vertical vector with magnitude y, as in Figure 11.1.18.

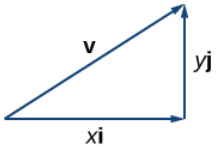


Figure 11.1.18: The vector $\vec{\mathbf{v}}$ is the sum of $x\hat{\mathbf{i}}$ and $y\hat{\mathbf{j}}$.

Now consider Figure 11.1.19 below.

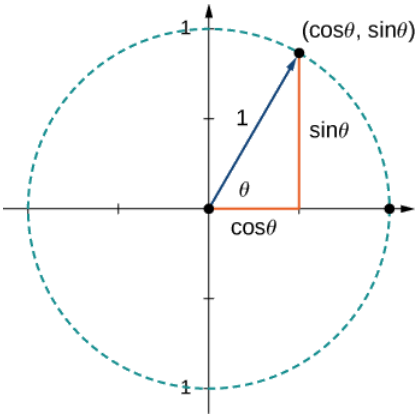


Figure 11.1.19: The terminal point of $\vec{\mathbf{u}}$ lies on the unit circle at the point $(\cos\theta, \sin\theta)$.

Note that since every vector in standard position (with its initial point at the origin) and with its terminal point on the unit circle is a unit vector! And given any angle θ with the positive x-axis, we already know the coordinates of the terminal point are $(\cos\theta, \sin\theta)$. This gives us an easy way to determine a unit vector in any given direction! The blue unit vector shown in Figure 11.1.19 above will be:

$$\vec{\mathbf{u}} = < \cos\theta, \sin\theta > \text{ or } \vec{\mathbf{u}} = \cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}.$$

If we want to scale one of these unit vectors to obtain a vector of a given length c at a given angle θ with the positive x-axis, we can just scale the unit vector by multiplying it by the scalar c. That is,

$$\vec{v} = c\vec{u} = c(\cos\theta\,\hat{i} + \sin\theta\,\hat{j}) = c\cos\theta\,\hat{i} + c\sin\theta\,\hat{j}.$$

Example 11.1.8: Using Standard Unit Vectors

a. Express the vector $\vec{w} = \langle 3, -4\rangle$ in terms of standard unit vectors.

b. Vector $\vec{u}$ is a unit vector that forms an angle of $60°$ with the positive x-axis. Use standard unit vectors to describe $\vec{u}$.

Solution:

a. Resolve vector $\vec{w}$ into a vector with a zero y-component and a vector with a zero x-component:

$$\vec{w} = \langle 3, -4\rangle = 3\hat{i} - 4\hat{j}.$$

b. Because $\vec{u}$ is a unit vector, the terminal point lies on the unit circle when the vector is placed in standard position (Figure 11.1.19).

$$\begin{aligned}\vec{u} &= \langle \cos 60°, \sin 60°\rangle \\ &= \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2}\right\rangle \\ &= \frac{1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j}.\end{aligned}$$

Exercise 11.1.8

Let $\vec{a} = \langle 16, -11\rangle$ and let $\vec{b}$ be a unit vector that forms an angle of $225°$ with the positive x-axis. Express $\vec{a}$ and $\vec{b}$ in terms of the standard unit vectors.

Example 11.1.9: Using the Unit Circle to Find Vectors of a Given Length and Angle

a. Let $\vec{u}$ be the vector of length 20 that forms an angle of $150°$ with the positive x-axis. Express $\vec{u}$ in terms of the standard unit vectors.

b. Find the vector $\vec{v}$ that has a magnitude of 4 and makes an angle of $\frac{\pi}{3}$ with the positive x-axis.

Solution

a. First we use the unit circle to write a unit vector making the given angle with the positive x-axis.

$$\vec{w} = \cos 150°\,\hat{i} + \sin 150°\,\hat{j}$$

Since we know the exact values of these trigonometric ratios for $150°$, we can now write,

$$\vec{w} = -\frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}$$

Note: Since we know that $150°$ is in the 2nd quadrant and it's $30°$ above the x-axis there, we know the coordinates of the terminal point of this unit vector on the unit circle are $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. Since we're in the 2nd quadrant, we know the x-coordinate (the cosine value) is negative and the y-coordinate (the sine value) is positive.

Then the vector $\vec{u}$ that we need is just a scaled version of the unit vector $\vec{w}$. That is, $\vec{u} = 20\vec{w}$.

So

$$\begin{aligned}\vec{u} &= -20 \cdot \frac{\sqrt{3}}{2}\hat{i} + 20 \cdot \frac{1}{2}\hat{j} \\ &= -10\sqrt{3}\hat{i} + 10\hat{j}\end{aligned}$$

b. Here we just multiply the relevant unit vector by 4.

$$\vec{v} = 4 \cdot \langle \cos\tfrac{\pi}{3}, \sin\tfrac{\pi}{3} \rangle$$
$$= 4\left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$$
$$= \langle 2, 2\sqrt{3} \rangle$$
$$= 2\hat{i} + 2\sqrt{3}\hat{j}$$

Exercise 11.1.9

Find vector $\vec{u}$ if we know that $\|\vec{u}\| = 12$ and the vector $\vec{u}$ has an angle of $\frac{7\pi}{4}$ with the positive x-axis.

Finding the Angle a Given Vector Makes with the Positive x-axis

Since we know that any given vector $\vec{v}$ in the xy-plane can be normalized to find a unit vector in the same direction, it becomes fairly easy to determine the angle between this vector $\vec{v}$ and the positive x-axis. Once we normalize the vector $\vec{v}$, we know that this unit vector could be placed on the unit circle and can be written in the form:

$$< \cos\theta, \sin\theta >= \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$

From our work above, we know the unit vector in the same direction as the vector $\vec{v}$ could be written as $\vec{w} = \frac{\vec{v}}{\|\vec{v}\|}$.

Let's say this simplifies down to a unit vector with components $\vec{w} = \langle a, b \rangle$. Since this is a unit vector, we know the first component is equal to $\cos\theta$ and the second component is equal to $\sin\theta$, where θ is the angle between this vector and the positive x-axis.

That is,

$$\cos\theta = a \text{ and } \sin\theta = b.$$

If θ is in the first quadrant, either term will give us the correct θ using either $\theta = \arccos a$ or $\theta = \arcsin b$. If we are in another quadrant (look at the signs of the two components to tell which quadrant it lies in), we need to think more carefully about the angle we obtain to be sure it is in the correct quadrant, making an adjustment as needed or using the other inverse trig function.

Example 11.1.10

a. Given $\vec{v} = 3\hat{i} + 4\hat{j}$, find the angle θ that this vector makes with the positive x-axis (to the nearest hundredth of a degree).

b. Find the angle θ between the positive x-axis and the vector $\vec{u} =< -2, 4 >$ (to the nearest thousandth of a radian and to the nearest tenth of a degree).

c. Find the angle θ between the positive x-axis and the vector $\vec{w} = 1\,\hat{i} - 3\,\hat{j}$ (to the nearest thousandth of a radian and to the nearest tenth of a degree).

Solution

a. Since this vector is in the first quadrant, this one is easiest to work out. First we normalize this vector (find a unit vector in the same direction).

Since $\|\vec{v}\| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$, we get the unit vector,

$$\vec{u} = \frac{3}{5}\,\hat{i} + \frac{4}{5}\,\hat{j}.$$

Now we can find θ with either component. That is,

$$\theta = \arccos\frac{3}{5} = \arcsin\frac{4}{5} \approx 53.13°.$$

b. The signs of the components of this vector tell us it is in the second quadrant, so we know that the arccosine function will still work for us, as it returns an angle in the first or second quadrants (between 0 and π radians). But note that arcsine would not give us the correct angle (right away) since it always returns an angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, (which effectively returns an

angle in the first or fourth quadrants). We could still use it here, knowing it would give us an angle in the first quadrant here (since the sign of the second component is positive). Because of the symmetry involved, we would have to subtract the angle we get from π radians to obtain the correct angle in the second quadrant. This is not trivial, as it requires a clear understanding and an ability to visualize the angles, but we'll show how it works below.

First let's normalize this vector.

Since $\|\vec{\mathbf{u}}\| = \sqrt{(-2)^2 + 4^2} = \sqrt{4+16} = \sqrt{20} = 2\sqrt{5}$, we get the unit vector,

$$\left\langle \frac{-2}{2\sqrt{5}}, \frac{4}{2\sqrt{5}} \right\rangle = \left\langle -\frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5} \right\rangle.$$

Now using the first component, we find,

$$\theta = \arccos -\frac{\sqrt{5}}{5} \approx 2.034 \text{ radians } \approx 116.6°.$$

Note that this angle is indeed in the second quadrant, as it needed to be!

Now let's see what would have happened if we used the second component and the arcsine function.

$$\arcsin \frac{2\sqrt{5}}{5} \approx 1.1071 \text{ radians } \approx 63.4°$$

As explained above, since this angle is in the first quadrant, we'll need to subtract it from π radians to get to the correct angle.

So here we get,

$$\theta = \pi - \arcsin \frac{2\sqrt{5}}{5} \approx 3.14159 - 1.1071 \approx 2.034 \text{ radians}$$

or in degrees,

$$\theta = 180° - \arcsin \frac{2\sqrt{5}}{5} \approx 180° - 63.4 \approx 116.6°.$$

We can see that this is the same angle!

c. Since $\|\vec{\mathbf{w}}\| = \sqrt{10}$, the unit vector in the same direction as $\vec{\mathbf{w}} = 1\,\hat{\mathbf{i}} - 3\,\hat{\mathbf{j}}$ is $\frac{\sqrt{10}}{10}\,\hat{\mathbf{i}} - \frac{3\sqrt{10}}{10}\,\hat{\mathbf{j}}$. Because this vector is clearly in the fourth quadrant, the arcsine function is our choice.

Here,

$\theta = \arcsin -\frac{3\sqrt{10}}{10} \approx -1.249$ radians $\approx -71.6°$.

This is a correct answer, but if we were asked for an angle between 0 and 2π radians, for example, we would still need to adjust it by adding 2π radians to the radian version of the angle. Then $\theta \approx 5.034$ radians.

If the vector is in the third quadrant neither the arccosine nor the arcsine give us the right angle. We could use either one and adjust the angle as we showed with arcsine in Example **11.1.10**, part b above. But there is another option. Although it also may need an adjustment, this adjustment is somewhat easier to make.

If we run into difficulty with the approach above or just want to use a different method, we can instead use the arctangent function to find the angle θ a vector $\vec{\mathbf{v}}$ makes with the positive x-axis. One advantage this approach gives us is that we don't need to normalize the vector first. Since $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, this ratio of the two vector components automatically removes any scalar that may be present and gives us the correct tangent ratio.

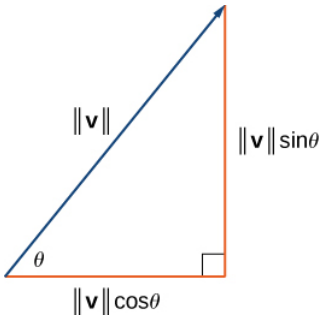


Figure 11.1.20: The components of a vector form the legs of a right triangle, with the vector as the hypotenuse.

As we observe in Figure 11.1.20, given a non-zero vector, $\vec{\mathbf{v}} =< \|\vec{\mathbf{v}}\| \cos\theta, \|\vec{\mathbf{v}}\| \sin\theta >$, we know that,

$$\tan\theta = \frac{\|\vec{\mathbf{v}}\| \sin\theta}{\|\vec{\mathbf{v}}\| \cos\theta} = \frac{\sin\theta}{\cos\theta}.$$

However, like the arcsine function, the arctangent function only returns angles between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, so we will still need to adjust the resulting angle based on which quadrant in which we know it must lie. At least in this case, the worst that we will need to do is to add π radians or $180°$ to our angle to get it into the correct quadrant. [Note that this is because the tangent will give the same results when the signs of the sine and cosine are the same, and it will give the same results when the signs of the two ratios are different.]

Example 11.1.11

Find the angle θ between the positive x-axis and the vector $\vec{\mathbf{d}} =< -5, -1 >$ (to the nearest thousandth of a radian and to the nearest tenth of a degree).

Solution

The vector $\vec{\mathbf{d}} =< -5, -1 >$ is clearly in the third quadrant, and will not be given directly by either the arccosine function or the arcsine function, as mentioned above. But we do know that the arctangent function will return an angle in the first quadrant in this situation (the two components are the same sign and thus their ratio is positive), and the correct angle in the third quadrant will be 180° or π radians greater than the angle we are given by arctangent in the first quadrant.

Remembering that we don't need to determine the unit vector this time, we instead write the tangent ratio we will need to work with.

$$\tan\theta = \frac{-1}{-5} = \frac{1}{5}$$

So,

$\theta = \pi + \arctan\frac{1}{5} \approx \pi + 0.197$ radians ≈ 3.339 radians $\approx 191.3°$.

Note that in degrees, $\arctan\frac{1}{5} \approx 11.3°$. Adding $180°$ also gives us $191.3°$.

Exercise 11.1.10

Use the arctangent function to determine the angle θ between the given vector and the positive x-axis. Give the angle accurate to the nearest tenth of a degree.

a. $\vec{\mathbf{u}} =< -4, -8 >$

b. $\vec{\mathbf{v}} =< 10, -15 >$

c. $\vec{\mathbf{w}} =< -7, 1 >$

Applications of Vectors

Because vectors have both direction and magnitude, they are valuable tools for solving problems involving such applications as motion and **force.** Recall the boat example and the quarterback example we described earlier. Here we look at two other examples in detail.

Example 11.1.12A: Finding Resultant Force

Jane's car is stuck in the mud. Lisa and Jed come along in a truck to help pull her out. They attach one end of a tow strap to the front of the car and the other end to the truck's trailer hitch, and the truck starts to pull. Meanwhile, Jane and Jed get behind the car and push. The truck generates a horizontal force of **300** lb on the car. Jane and Jed are pushing at a slight upward angle and generate a force of **150** lb on the car. These forces can be represented by vectors, as shown in Figure **11.1.21**. The angle between these vectors is **15°**. Find the resultant force (the vector sum) and give its magnitude to the nearest tenth of a pound and its direction angle from the positive x-axis.

Figure **11.1.21**: Two forces acting on a car in different directions.

Solution

To find the effect of combining the two forces, add their representative vectors. First, express each vector in component form or in terms of the standard unit vectors. For this purpose, it is easiest if we align one of the vectors with the positive x-axis. The horizontal vector, then, has initial point $(0, 0)$ and terminal point $(300, 0)$. It can be expressed as $\langle 300, 0\rangle$ or $300\hat{\mathbf{i}}$.

The second vector has magnitude 150 and makes an angle of $15°$ with the first, so we can express it as $\langle 150\cos(15°), 150\sin(15°)\rangle$, or $150\cos(15°)\hat{\mathbf{i}} + 150\sin(15°)\hat{\mathbf{j}}$. Then, the sum of the vectors, or resultant vector, is $\vec{\mathbf{r}} = \langle 300, 0\rangle + \langle 150\cos(15°), 150\sin(15°)\rangle$, and we have

$$\|\vec{\mathbf{r}}\| = \sqrt{(300 + 150\cos(15°))^2 + (150\sin(15°))^2} \approx 446.6.$$

The angle θ made by $\vec{\mathbf{r}}$ and the positive x-axis has $\tan\theta = \dfrac{150\sin 15°}{(300 + 150\cos 15°)} \approx 0.09$, so $\theta \approx \tan^{-1}(0.09) \approx 5°$, which means the resultant force $\vec{\mathbf{r}}$ has an angle of $5°$ above the horizontal axis.

Example 11.1.12B: Finding Resultant Velocity

An airplane flies due west at an airspeed of **425** mph. The wind is blowing from the northeast at **40** mph. What is the ground speed of the airplane? What is the bearing of the airplane?

Solution

Let's start by sketching the situation described (Figure **11.1.22**).

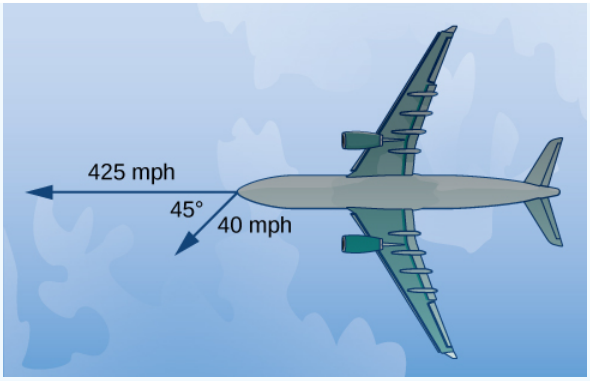


Figure 11.1.22: Initially, the plane travels due west. The wind is from the northeast, so it is blowing to the southwest. The angle between the plane's course and the wind is 45°. (Figure not drawn to scale.)

Set up a sketch so that the initial points of the vectors lie at the origin. Then, the plane's velocity vector is $\vec{\mathbf{p}} = -425\hat{\mathbf{i}}$. The vector describing the wind makes an angle of 225° with the positive x-axis:

$$\vec{\mathbf{w}} = \langle 40\cos(225°), 40\sin(225°)\rangle = \left\langle -\frac{40\sqrt{2}}{2}, -\frac{40\sqrt{2}}{2}\right\rangle = -20\sqrt{2}\,\hat{\mathbf{i}} - 20\sqrt{2}\,\hat{\mathbf{j}}.$$

When the airspeed and the wind act together on the plane, we can add their vectors to find the resultant force:

$$\vec{\mathbf{p}} + \vec{\mathbf{w}} = -425\hat{\mathbf{i}} + \left(-20\sqrt{2}\,\hat{\mathbf{i}} - 20\sqrt{2}\,\hat{\mathbf{j}}\right) = \left(-425 - 20\sqrt{2}\right)\hat{\mathbf{i}} - 20\sqrt{2}\,\hat{\mathbf{j}}.$$

The magnitude of the resultant vector shows the effect of the wind on the ground speed of the airplane:

$$\|\vec{\mathbf{p}} + \vec{\mathbf{w}}\| = \sqrt{\left(-425 - 20\sqrt{2}\right)^2 + \left(-20\sqrt{2}\right)^2} \approx 454.17 \text{ mph}$$

As a result of the wind, the plane is traveling at approximately 454 mph relative to the ground.

To determine the bearing of the airplane, we want to find the direction of the vector $\vec{\mathbf{p}} + \vec{\mathbf{w}}$:

$$\tan\theta = \frac{-20\sqrt{2}}{\left(-425 - 20\sqrt{2}\right)} \approx 0.06$$

$$\theta \approx 3.57°.$$

The overall direction of the plane is 3.57° south of west.

Exercise 11.1.11

An airplane flies due north at an airspeed of 550 mph. The wind is blowing from the northwest at 50 mph. What is the ground speed of the airplane?

Key Concepts

- Vectors are used to represent quantities that have both magnitude and direction.
- We can add vectors by using the parallelogram method or the triangle method to find the sum. We can multiply a vector by a scalar to change its length or give it the opposite direction.
- Subtraction of vectors is defined in terms of adding the negative of the vector.
- A vector is written in component form as $\vec{\mathbf{v}} = \langle x, y\rangle$.
- The magnitude of a vector is a scalar: $\|\vec{\mathbf{v}}\| = \sqrt{x^2 + y^2}$.
- A unit vector $\vec{\mathbf{u}}$ has magnitude 1 and can be found by dividing a vector by its magnitude: $\vec{\mathbf{u}} = \dfrac{1}{\|\vec{\mathbf{v}}\|}\vec{\mathbf{v}}$. The standard unit vectors are $\hat{\mathbf{i}} = \langle 1, 0\rangle$ and $\hat{\mathbf{j}} = \langle 0, 1\rangle$. A vector $\vec{\mathbf{v}} = \langle x, y\rangle$ can be expressed in terms of the standard unit vectors as $\vec{\mathbf{v}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$.
- Vectors are often used in physics and engineering to represent forces and velocities, among other quantities.

Contributors and Attributions

- Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.
- Paul Seeburger (Monroe Community College) added exposition below Figure **11.1.8** and above Example **11.1.8,** and also added Example **11.1.9,** Exercise **11.1.9,** and all of the sub-section titled **Finding the Angle a Given Vector Makes with the Positive x-axis**, ending just before the sub-section titled **Applications of Vectors**. Paul also simplified the notation in Example **11.1.12***B*.

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11.1E: Exercises for Vectors in the Plane

For exercises 1 - 10, consider points $P(-1,3)$, $Q(1,5)$, and $R(-3,7)$. Determine the requested vectors and express each of them

a. in component form and

b. by using standard unit vectors.

1) $\overrightarrow{PQ}$

2) $\overrightarrow{PR}$

3) $\overrightarrow{QP}$

4) $\overrightarrow{RP}$

5) $\overrightarrow{PQ}+\overrightarrow{PR}$

6) $\overrightarrow{PQ}-\overrightarrow{PR}$

7) $2\overrightarrow{PQ}-2\overrightarrow{PR}$

8) $2\overrightarrow{PQ}+\frac{1}{2}\overrightarrow{PR}$

9) The unit vector in the direction of $\overrightarrow{PQ}$

10) The unit vector in the direction of $\overrightarrow{PR}$

11) A vector $\vec{\mathbf{v}}$ has initial point $(-1,-3)$ and terminal point $(2,1)$. Find the unit vector in the direction of $\vec{\mathbf{v}}$. Express the answer in component form.

12) A vector $\vec{\mathbf{v}}$ has initial point $(-2,5)$ and terminal point $(3,-1)$. Find the unit vector in the direction of $\vec{\mathbf{v}}$. Express the answer in component form.

13) The vector $\vec{\mathbf{v}}$ has initial point $P(1,0)$ and terminal point Q that is on the y-axis and above the initial point. Find the coordinates of terminal point Q such that the magnitude of the vector $\vec{\mathbf{v}}$ is $\sqrt{5}$.

14) The vector $\vec{\mathbf{v}}$ has initial point $P(1,1)$ and terminal point Q that is on the x-axis and left of the initial point. Find the coordinates of terminal point Q such that the magnitude of the vector $\vec{\mathbf{v}}$ is $\sqrt{10}$.

For exercises 15 and 16, use the given vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$.

a. Determine the vector sum $\vec{\mathbf{a}}+\vec{\mathbf{b}}$ and express it in both the component form and by using the standard unit vectors.

b. Find the vector difference $\vec{\mathbf{a}}-\vec{\mathbf{b}}$ and express it in both the component form and by using the standard unit vectors.

c. Verify that the vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, and $\vec{\mathbf{a}}+\vec{\mathbf{b}}$, and, respectively, $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, and $\vec{\mathbf{a}}-\vec{\mathbf{b}}$ satisfy the triangle inequality.

d. Determine the vectors $2\vec{\mathbf{a}}$, $-\vec{\mathbf{b}}$, and $2\vec{\mathbf{a}}-\vec{\mathbf{b}}$. Express the vectors in both the component form and by using standard unit vectors.

15) $\vec{\mathbf{a}}=2\hat{\mathbf{i}}+\hat{\mathbf{j}}, \quad \vec{\mathbf{b}}=\hat{\mathbf{i}}+3\hat{\mathbf{j}}$

16) $\vec{\mathbf{a}}=2\hat{\mathbf{i}}, \quad \vec{\mathbf{b}}=-2\hat{\mathbf{i}}+2\hat{\mathbf{j}}$

17) Let $\vec{\mathbf{a}}$ be a standard-position vector with terminal point $(-2,-4)$. Let $\vec{\mathbf{b}}$ be a vector with initial point $(1,2)$ and terminal point $(-1,4)$. Find the magnitude of vector $-3\vec{\mathbf{a}}+\vec{\mathbf{b}}-4\hat{\mathbf{i}}+\hat{\mathbf{j}}$.

18) Let $\vec{\mathbf{a}}$ be a standard-position vector with terminal point at $(2,5)$. Let $\vec{\mathbf{b}}$ be a vector with initial point $(-1,3)$ and terminal point $(1,0)$. Find the magnitude of vector $\vec{\mathbf{a}}-3\vec{\mathbf{b}}+14\hat{\mathbf{i}}-14\hat{\mathbf{j}}$.

19) Let $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ be two nonzero vectors that are nonequivalent. Consider the vectors $\vec{\mathbf{a}}=4\vec{\mathbf{u}}+5\vec{\mathbf{v}}$ and $\vec{\mathbf{b}}=\vec{\mathbf{u}}+2\vec{\mathbf{v}}$ defined in terms of $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$. Find the scalar λ such that vectors $\vec{\mathbf{a}}+\lambda\vec{\mathbf{b}}$ and $\vec{\mathbf{u}}-\vec{\mathbf{v}}$ are equivalent.

20) Let $\vec{u}$ and $\vec{v}$ be two nonzero vectors that are nonequivalent. Consider the vectors $\vec{a} = 2\vec{u} - 4\vec{v}$ and $\vec{b} = 3\vec{u} - 7\vec{v}$ defined in terms of $\vec{u}$ and $\vec{v}$. Find the scalars α and β such that vectors $\alpha\vec{a} + \beta\vec{b}$ and $\vec{u} - \vec{v}$ are equivalent.

21) Consider the vector $\vec{a}(t) = \langle \cos t, \sin t \rangle$ with components that depend on a real number t. As the number t varies, the components of $\vec{a}(t)$ change as well, depending on the functions that define them.

a. Write the vectors $\vec{a}(0)$ and $\vec{a}(\pi)$ in component form.

b. Show that the magnitude $\| \vec{a}(t) \|$ of vector $\vec{a}(t)$ remains constant for any real number t.

c. As t varies, show that the terminal point of vector $\vec{a}(t)$ describes a circle centered at the origin of radius 1.

22) Consider vector $\vec{a}(x) = \langle x, \sqrt{1-x^2} \rangle$ with components that depend on a real number $x \in [-1, 1]$. As the number x varies, the components of $\vec{a}(x)$ change as well, depending on the functions that define them.

a. Write the vectors $\vec{a}(0)$ and $\vec{a}(1)$ in component form.

b. Show that the magnitude $\| \vec{a}(x) \|$ of vector $\vec{a}(x)$ remains constant for any real number x.

c. As x varies, show that the terminal point of vector $\vec{a}(x)$ describes a circle centered at the origin of radius 1.

23) Show that vectors $\vec{a}(t) = \langle \cos t, \sin t \rangle$ and $\vec{a}(x) = \langle x, \sqrt{1-x^2} \rangle$ are equivalent for $x = 1$ and $t = 2k\pi$, where k is an integer.

Answer Answers may vary

24) Show that vectors $\vec{a}(t) = \langle \cos t, \sin t \rangle$ and $\vec{a}(x) = \langle x, \sqrt{1-x^2} \rangle$ are opposite for $x = 1$ and $t = \pi + 2k\pi$, where k is an integer.

For exercises 25-28, find a vector $\vec{v}$ with the given magnitude and in the same direction as the vector $\vec{u}$.

25) $\|\vec{v}\| = 7, \quad \vec{u} = \langle 3, 4 \rangle$

26) $\|\vec{v}\| = 3, \quad \vec{u} = \langle -2, 5 \rangle$

27) $\|\vec{v}\| = 7, \quad \vec{u} = \langle 3, -5 \rangle$

28) $\|\vec{v}\| = 10, \quad \vec{u} = \langle 2, -1 \rangle$

For exercises 29-34, find the component form of vector $\vec{u}$, given its magnitude and the angle the vector makes with the positive x-axis. Give exact answers when possible.

29) $\|\vec{u}\| = 2, \theta = 30°$

30) $\|\vec{u}\| = 6, \theta = 60°$

31) $\|\vec{u}\| = 5, \theta = \frac{\pi}{2}$

32) $\|\vec{u}\| = 8, \theta = \pi$

33) $\|\vec{u}\| = 10, \theta = \frac{5\pi}{6}$

34) $\|\vec{u}\| = 50, \theta = \frac{3\pi}{4}$

For exercises 35 and 36, vector $\vec{u}$ is given. Find the angle $\theta \in [0, 2\pi)$ that vector $\vec{u}$ makes with the positive direction of the x-axis, in a counter-clockwise direction.

35) $\vec{u} = 5\sqrt{2}\hat{i} - 5\sqrt{2}\hat{j}$

36) $\vec{u} = -\sqrt{3}\hat{i} - \hat{j}$

37) Let $\vec{a} = \langle a_1, a_2 \rangle$, $\vec{b} = \langle b_1, b_2 \rangle$, and $\vec{c} = \langle c_1, c_2 \rangle$ be three nonzero vectors. If $a_1 b_2 - a_2 b_1 \neq 0$, then show there are two scalars, α and β, such that $\vec{c} = \alpha\vec{a} + \beta\vec{b}$.

Answer Answers may vary

38) Consider vectors $\vec{a} = \langle 2, -4 \rangle$, $\vec{b} = \langle -1, 2 \rangle$, and $\vec{c} = \vec{0}$ Determine the scalars α and β such that $\vec{c} = \alpha\vec{a} + \beta\vec{b}$.

39) Let $P(x_0, f(x_0))$ be a fixed point on the graph of the differentiable function f with a domain that is the set of real numbers.

a. Determine the real number z_0 such that point $Q(x_0 + 1, z_0)$ is situated on the line tangent to the graph of f at point P.

b. Determine the unit vector $\vec{\mathbf{u}}$ with initial point P and terminal point Q.

40) Consider the function $f(x) = x^4$, where $x \in R$.

a. Determine the real number z_0 such that point $Q(2, z_0)$ s situated on the line tangent to the graph of f at point $P(1, 1)$.

b. Determine the unit vector $\vec{\mathbf{u}}$ with initial point P and terminal point Q.

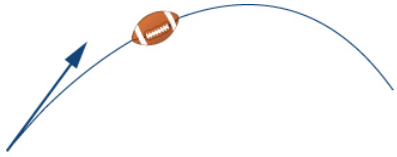
41) Consider f and g two functions defined on the same set of real numbers D. Let $\vec{\mathbf{a}} = \langle x, f(x) \rangle$ and $\vec{\mathbf{b}} = \langle x, g(x) \rangle$ be two vectors that describe the graphs of the functions, where $x \in D$. Show that if the graphs of the functions f and g do not intersect, then the vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are not equivalent.

42) Find $x \in R$ such that vectors $\vec{\mathbf{a}} = \langle x, \sin x \rangle$ and $\vec{\mathbf{b}} = \langle x, \cos x \rangle$ are equivalent.

43) Calculate the coordinates of point D such that $ABCD$ is a parallelogram, with $A(1, 1), B(2, 4)$, and $C(7, 4)$.

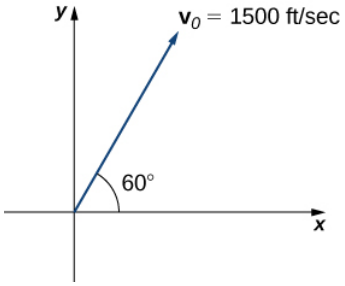
44) Consider the points $A(2, 1), B(10, 6), C(13, 4)$, and $D(16, -2)$. Determine the component form of vector $\overrightarrow{AD}$.

45) The **speed** of an object is the magnitude of its related velocity vector. A football thrown by a quarterback has an initial speed of 70 mph and an angle of elevation of $30°$. Determine the velocity vector in mph and express it in component form. (Round to two decimal places.)



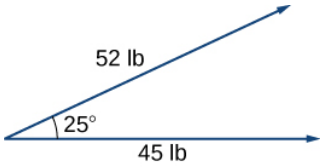
46) A baseball player throws a baseball at an angle of $30°$ with the horizontal. If the initial speed of the ball is 100 mph, find the horizontal and vertical components of the initial velocity vector of the baseball. (Round to two decimal places.)

47) A bullet is fired with an initial velocity of 1500 ft/sec at an angle of $60°$ with the horizontal. Find the horizontal and vertical components of the velocity vector of the bullet. (Round to two decimal places.)



48) [T] A 65-kg sprinter exerts a force of 798 N at a $19°$ angle with respect to the ground on the starting block at the instant a race begins. Find the horizontal component of the force. (Round to two decimal places.)

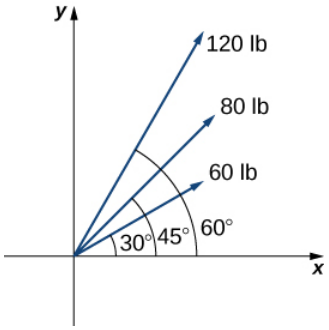
49) [T] Two forces, a horizontal force of 45 lb and another of 52 lb, act on the same object. The angle between these forces is $25°$. Find the magnitude and direction angle from the positive x-axis of the resultant force that acts on the object. (Round to two decimal places.)



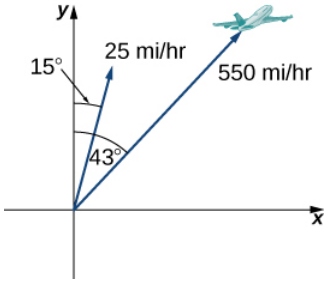
50) [T] Two forces, a vertical force of 26 lb and another of 45 lb, act on the same object. The angle between these forces is $55°$. Find the magnitude and direction angle from the positive x-axis of the resultant force that acts on the object. (Round to two decimal places.)

51) [T] Three forces act on object. Two of the forces have the magnitudes 58 N and 27 N, and make angles $53°$ and $152°$, respectively, with the positive x-axis. Find the magnitude and the direction angle from the positive x-axis of the third force such that the resultant force acting on the object is zero. (Round to two decimal places.)

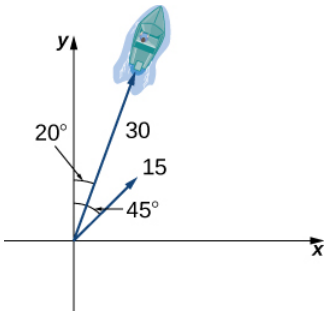
52) Three forces with magnitudes **80** lb, **120** lb, and **60** lb act on an object at angles of $45°, 60°$ and $30°$, respectively, with the positive x-axis. Find the magnitude and direction angle from the positive x-axis of the resultant force. (Round to two decimal places.)



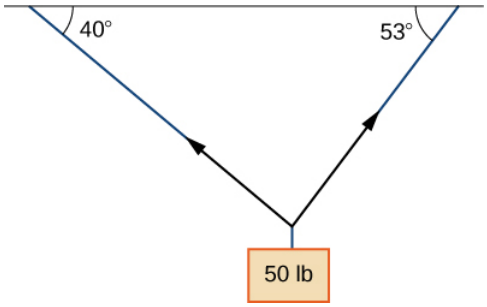
53) [T] An airplane is flying in the direction of $43°$ east of north (also abbreviated as $N43E$ at a speed of 550 mph. A wind with speed 25 mph comes from the southwest at a bearing of $N15E$. What are the ground speed and new direction of the airplane?



54) [T] A boat is traveling in the water at 30 mph in a direction of $N20E$ (that is, $20°$ east of north). A strong current is moving at 15 mph in a direction of $N45E$. What are the new speed and direction of the boat?

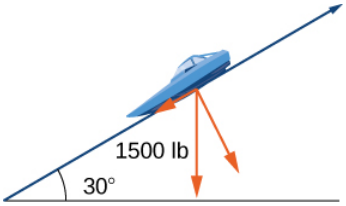


55) [T] A 50-lb weight is hung by a cable so that the two portions of the cable make angles of $40°$ and $53°$, respectively, with the horizontal. Find the magnitudes of the forces of tension $\vec{T}_1$ and $\vec{T}_2$ in the cables if the resultant force acting on the object is zero. (Round to two decimal places.)



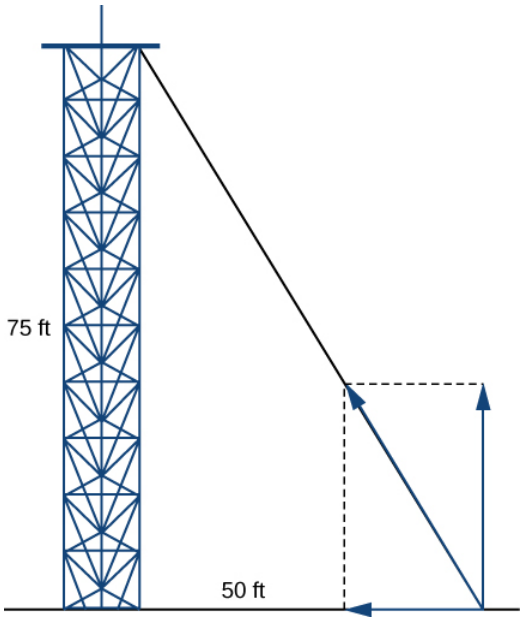
56) [T] A 62-lb weight hangs from a rope that makes the angles of $29°$ and $61°$, respectively, with the horizontal. Find the magnitudes of the forces of tension $\vec{T}_1$ and $\vec{T}_2$ in the cables if the resultant force acting on the object is zero. (Round to two decimal places.)

57) [T] A 1500-lb boat is parked on a ramp that makes an angle of $30°$ with the horizontal. The boat's weight vector points downward and is a sum of two vectors: a horizontal vector $\vec{v}_1$ that is parallel to the ramp and a vertical vector $\vec{v}_2$ that is perpendicular to the inclined surface. The magnitudes of vectors $\vec{v}_1$ and $\vec{v}_2$ are the horizontal and vertical component, respectively, of the boat's weight vector. Find the magnitudes of $\vec{v}_1$ and $\vec{v}_2$. (Round to the nearest integer.)



58) [T] An 85-lb box is at rest on a $26°$ incline. Determine the magnitude of the force parallel to the incline necessary to keep the box from sliding. (Round to the nearest integer.)

59) A guy-wire supports a pole that is 75 ft high. One end of the wire is attached to the top of the pole and the other end is anchored to the ground 50 ft from the base of the pole. Determine the horizontal and vertical components of the force of tension in the wire if its magnitude is 50 lb. (Round to the nearest integer.)



60) A telephone pole guy-wire has an angle of elevation of $35°$ with respect to the ground. The force of tension in the guy-wire is 120 lb. Find the horizontal and vertical components of the force of tension. (Round to the nearest integer.)

Contributors

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11.2: Vectors in Space

Learning Objectives

- Describe three-dimensional space mathematically.
- Locate points in space using coordinates.
- Write the distance formula in three dimensions.
- Write the equations for simple planes and spheres.
- Perform vector operations in $\mathbb{R}^3$.

Vectors are useful tools for solving two-dimensional problems. Life, however, happens in three dimensions. To expand the use of vectors to more realistic applications, it is necessary to create a framework for describing three-dimensional space. For example, although a two-dimensional map is a useful tool for navigating from one place to another, in some cases the topography of the land is important. Does your planned route go through the mountains? Do you have to cross a river? To appreciate fully the impact of these geographic features, you must use three dimensions. This section presents a natural extension of the two-dimensional Cartesian coordinate plane into three dimensions.

Three-Dimensional Coordinate Systems

As we have learned, the two-dimensional rectangular coordinate system contains two perpendicular axes: the horizontal x-axis and the vertical y-axis. We can add a third dimension, the z-axis, which is perpendicular to both the x-axis and the y-axis. We call this system the three-dimensional rectangular coordinate system. It represents the three dimensions we encounter in real life.

Definition: Three-dimensional Rectangular Coordinate System

The three-dimensional rectangular coordinate system consists of three perpendicular axes: the x-axis, the y-axis, and the z-axis. Because each axis is a number line representing all real numbers in $\mathbb{R}$, the three-dimensional system is often denoted by $\mathbb{R}^3$.

In Figure 11.2.1*a*, the positive z-axis is shown above the plane containing the x- and y-axes. The positive x-axis appears to the left and the positive y-axis is to the right. A natural question to ask is: How was this arrangement determined? The system displayed follows the **right-hand rule**. If we take our right hand and align the fingers with the positive x-axis, then curl the fingers so they point in the direction of the positive y-axis, our thumb points in the direction of the positive z-axis (Figure 11.2.1*b*). In this text, we always work with coordinate systems set up in accordance with the right-hand rule. Some systems do follow a left-hand rule, but the right-hand rule is considered the standard representation.

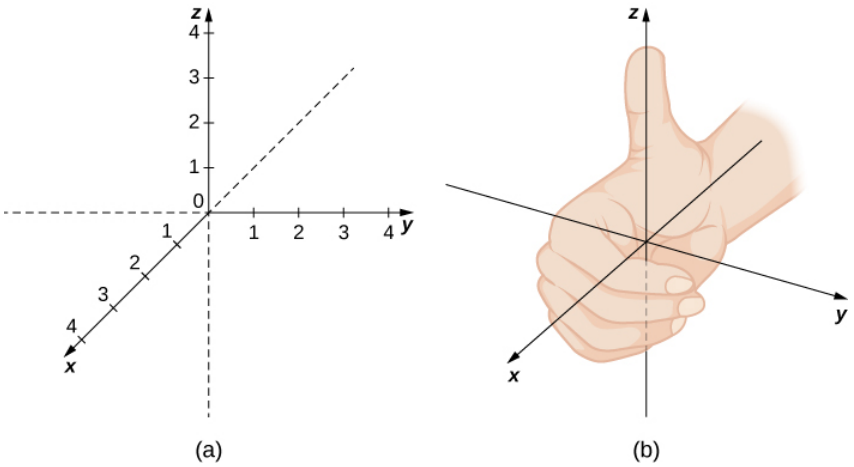


Figure 11.2.1: (a) We can extend the two-dimensional rectangular coordinate system by adding a third axis, the z-axis, that is perpendicular to both the x-axis and the y-axis. (b) The right-hand rule is used to determine the placement of the coordinate axes in the standard Cartesian plane.

In two dimensions, we describe a point in the plane with the coordinates (x, y). Each coordinate describes how the point aligns with the corresponding axis. In three dimensions, a new coordinate, z, is appended to indicate alignment with the z-axis: (x, y, z). A point

in space is identified by all three coordinates (Figure 11.2.2). To plot the point (x, y, z), go x units along the x-axis, then y units in the direction of the y-axis, then z units in the direction of the z-axis.

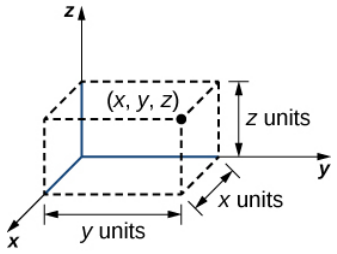


Figure 11.2.2: To plot the point (x, y, z) go x units along the x-axis, then y units in the direction of the y-axis, then z units in the direction of the z-axis.

Example 11.2.1: Locating Points in Space

Sketch the point $(1, -2, 3)$ in three-dimensional space.

Solution

To sketch a point, start by sketching three sides of a rectangular prism along the coordinate axes: one unit in the positive x direction, 2 units in the negative y direction, and 3 units in the positive z direction. Complete the prism to plot the point (Figure 11.2.3).

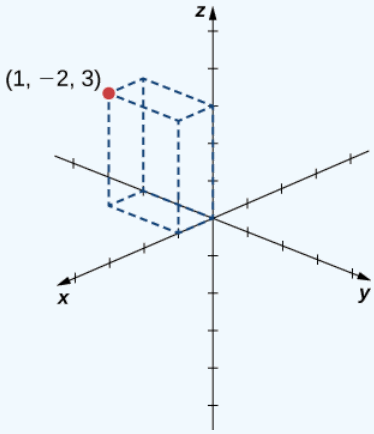


Figure 11.2.3: Sketching the point $(1, -2, 3)$.

Exercise 11.2.1

Sketch the point $(-2, 3, -1)$ in three-dimensional space.

In two-dimensional space, the coordinate plane is defined by a pair of perpendicular axes. These axes allow us to name any location within the plane. In three dimensions, we define **coordinate planes** by the coordinate axes, just as in two dimensions. There are three axes now, so there are three intersecting pairs of axes. Each pair of axes forms a coordinate plane: the xy-plane, the xz-plane, and the yz-plane (Figure 11.2.4). We define the xy-plane formally as the following set: $\{(x, y, 0) : x, y \in \mathbb{R}\}$. Similarly, the xz-plane and the yz-plane are defined as $\{(x, 0, z) : x, z \in \mathbb{R}\}$ and $\{(0, y, z) : y, z \in \mathbb{R}\}$, respectively.

To visualize this, imagine you're building a house and are standing in a room with only two of the four walls finished. (Assume the two finished walls are adjacent to each other.) If you stand with your back to the corner where the two finished walls meet, facing out into the room, the floor is the xy-plane, the wall to your right is the xz-plane, and the wall to your left is the yz-plane.

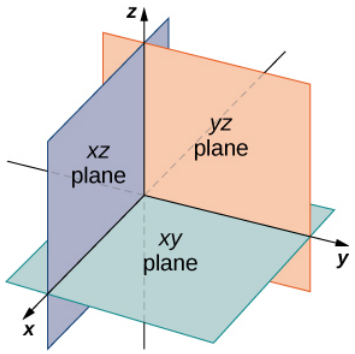


Figure **11.2.4**: The plane containing the x- *and* y-axes is called the xy-plane. The plane containing the x- and z-axes is called the xz-plane, and the y- and z-axes define the yz-plane.

In two dimensions, the coordinate axes partition the plane into four quadrants. Similarly, the coordinate planes divide space between them into eight regions about the origin, called **octants.** The octants fill $\mathbb{R}^3$ in the same way that quadrants fill $\mathbb{R}^2$, as shown in Figure **11.2.5**.

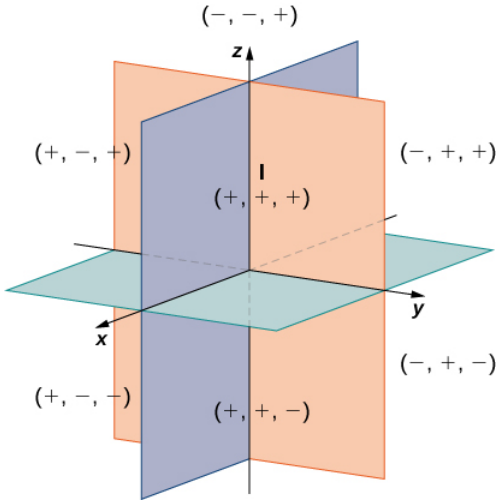


Figure **11.2.5**: Points that lie in octants have three nonzero coordinates.

Most work in three-dimensional space is a comfortable extension of the corresponding concepts in two dimensions. In this section, we use our knowledge of circles to describe spheres, then we expand our understanding of vectors to three dimensions. To accomplish these goals, we begin by adapting the distance formula to three-dimensional space.

If two points lie in the same coordinate plane, then it is straightforward to calculate the distance between them. We know that the distance d between two points (x_1, y_1) and (x_2, y_2) in the xy-coordinate plane is given by the formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \tag{11.2.1}$$

The formula for the distance between two points in space is a natural extension of this formula.

The Distance between Two Points in Space

The distance d between points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by the formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}. \tag{11.2.2}$$

The proof of this theorem is left as an exercise. (Hint: First find the distance d_1 between the points (x_1, y_1, z_1) and (x_2, y_2, z_1) as shown in Figure **11.2.6**.)

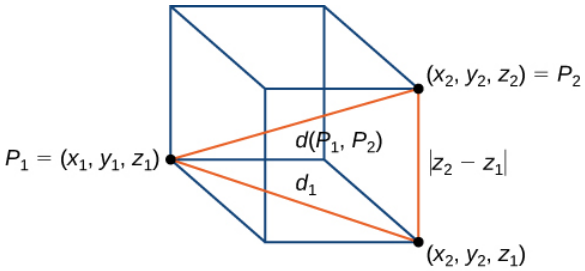


Figure **11.2.6**: The distance between P_1 and P_2 is the length of the diagonal of the rectangular prism having P_1 and P_2 as opposite corners.

Example 11.2.2: Distance in Space

Find the distance between points $P_1 = (3, -1, 5)$ and $P_2 = (2, 1, -1)$.

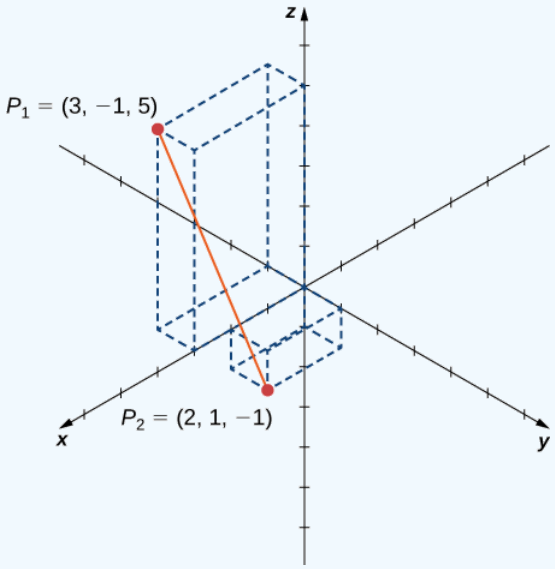


Figure **11.2.7**: Find the distance between the two points.

Solution

Substitute values directly into the distance formula (Equation 11.2.2):

$$\begin{aligned} d(P_1, P_2) &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \\ &= \sqrt{(2-3)^2 + (1-(-1))^2 + (-1-5)^2} \\ &= \sqrt{(-1)^2 + 2^2 + (-6)^2} \\ &= \sqrt{41}. \end{aligned}$$

Exercise 11.2.2

Find the distance between points $P_1 = (1, -5, 4)$ and $P_2 = (4, -1, -1)$.

Before moving on to the next section, let's get a feel for how $\mathbb{R}^3$ differs from $\mathbb{R}^2$. For example, in $\mathbb{R}^2$, lines that are not parallel must always intersect. This is not the case in $\mathbb{R}^3$. For example, consider the lines shown in Figure **11.2.8**. These two lines are not parallel, nor do they intersect.

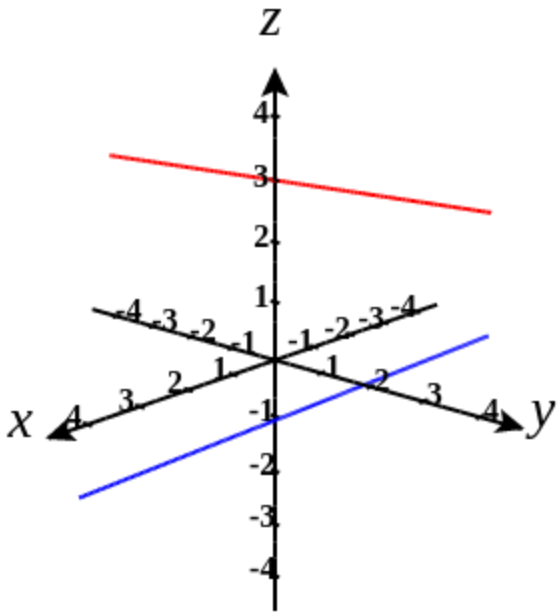


Figure 11.2.8: These two lines are not parallel, but still do not intersect.

You can also have circles that are interconnected but have no points in common, as in Figure 11.2.9.

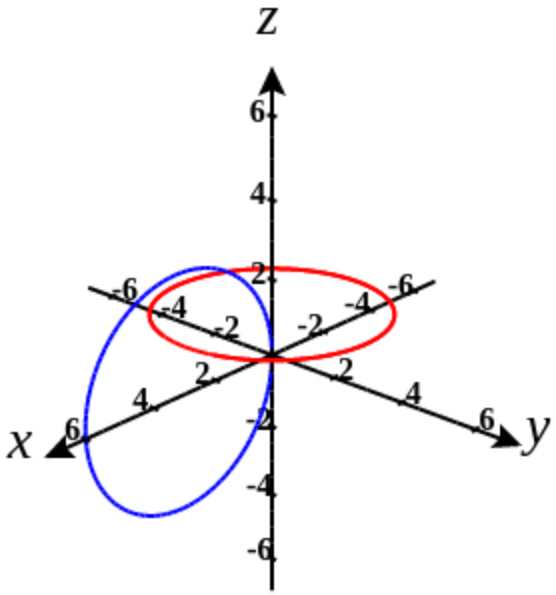


Figure 11.2.9: These circles are interconnected, but have no points in common.

We have a lot more flexibility working in three dimensions than we do if we stuck with only two dimensions.

Writing Equations in $\mathbb{R}^3$

Now that we can represent points in space and find the distance between them, we can learn how to write equations of geometric objects such as lines, planes, and curved surfaces in $\mathbb{R}^3$. First, we start with a simple equation. Compare the graphs of the equation $x = 0$ in $\mathbb{R}$, $\mathbb{R}^2$,and $\mathbb{R}^3$ (Figure 11.2.10). From these graphs, we can see the same equation can describe a point, a line, or a plane.

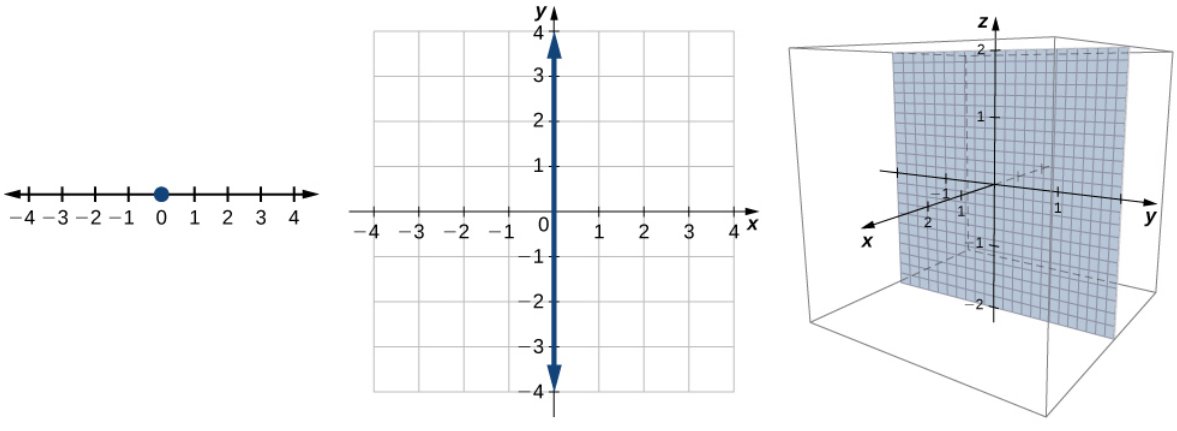
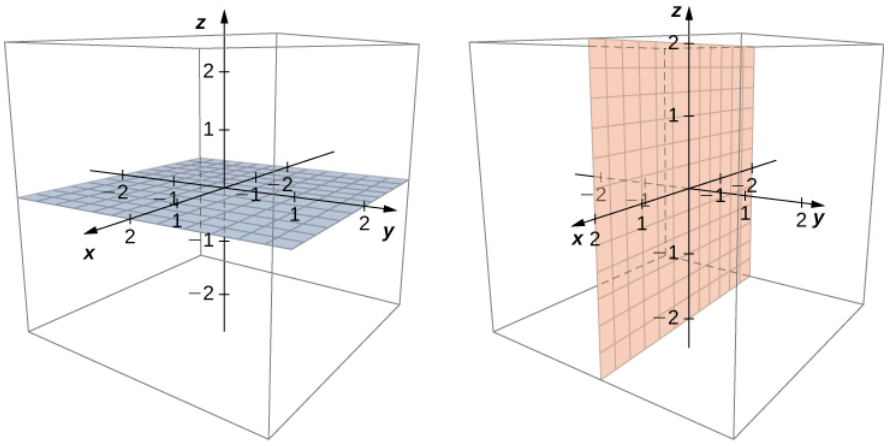


Figure 11.2.10: (a) In $\mathbb{R}$, the equation $x = 0$ describes a single point. (b) In $\mathbb{R}^2$, the equation $x = 0$ describes a line, the y-axis. (c) In $\mathbb{R}^3$, the equation $x = 0$ describes a plane, the yz-plane.

In space, the equation $x = 0$ describes all points $(0, y, z)$. This equation defines the yz-plane. Similarly, the xy-plane contains all points of the form $(x, y, 0)$. The equation $z = 0$ defines the xy-plane and the equation $y = 0$ describes the xz-plane (Figure 11.2.11).



(a) (b)

Figure 11.2.11: (a) In space, the equation $z = 0$ describes the xy-plane. (b) All points in the xz-plane satisfy the equation $y = 0$.

Understanding the equations of the coordinate planes allows us to write an equation for any plane that is parallel to one of the coordinate planes. When a plane is parallel to the xy-plane, for example, the z-coordinate of each point in the plane has the same constant value. Only the x- and y-coordinates of points in that plane vary from point to point.

Equations of Planes Parallel to Coordinate Planes

1. The plane in space that is parallel to the xy-plane and contains point (a, b, c) can be represented by the equation $z = c$.
2. The plane in space that is parallel to the xz-plane and contains point (a, b, c) can be represented by the equation $y = b$.
3. The plane in space that is parallel to the yz-plane and contains point (a, b, c) can be represented by the equation $x = a$.

Example 11.2.3: Writing Equations of Planes Parallel to Coordinate Planes

a. Write an equation of the plane passing through point $(3, 11, 7)$ that is parallel to the yz-plane.
b. Find an equation of the plane passing through points $(6, -2, 9)$, $(0, -2, 4)$, and $(1, -2, -3)$.

Solution

a. When a plane is parallel to the yz-plane, only the y- and z-coordinates may vary. The x-coordinate has the same constant value for all points in this plane, so this plane can be represented by the equation $x = 3$.
b. Each of the points $(6, -2, 9)$, $(0, -2, 4)$, and $(1, -2, -3)$ has the same y-coordinate. This plane can be represented by the equation $y = -2$.

Exercise 11.2.3

Write an equation of the plane passing through point $(1, -6, -4)$ that is parallel to the xy-plane.

As we have seen, in $\mathbb{R}^2$ the equation $x = 5$ describes the vertical line passing through point $(5, 0)$. This line is parallel to the y-axis. In a natural extension, the equation $x = 5$ in $\mathbb{R}^3$ describes the plane passing through point $(5, 0, 0)$, which is parallel to the yz-plane. Another natural extension of a familiar equation is found in the equation of a sphere.

Definition: Sphere

A sphere is the set of all points in space equidistant from a fixed point, the center of the sphere (Figure 11.2.12), just as the set of all points in a plane that are equidistant from the center represents a circle. In a sphere, as in a circle, the distance from the center to a point on the sphere is called the *radius*.

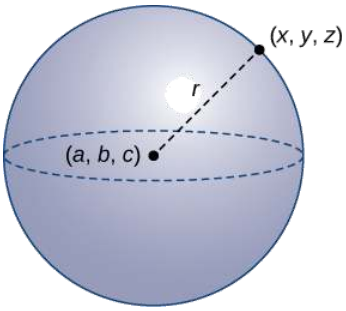


Figure 11.2.12: Each point (x, y, z) on the surface of a sphere is r units away from the center (a, b, c).

The equation of a circle is derived using the distance formula in two dimensions. In the same way, the equation of a sphere is based on the three-dimensional formula for distance.

Standard Equation of a Sphere

The sphere with center (a, b, c) and radius r can be represented by the equation

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2. \tag{11.2.3}$$

This equation is known as the *standard equation of a sphere*.

Example 11.2.4: Finding an Equation of a Sphere

Find the standard equation of the sphere with center $(10, 7, 4)$ and point $(-1, 3, -2)$, as shown in Figure 11.2.13.

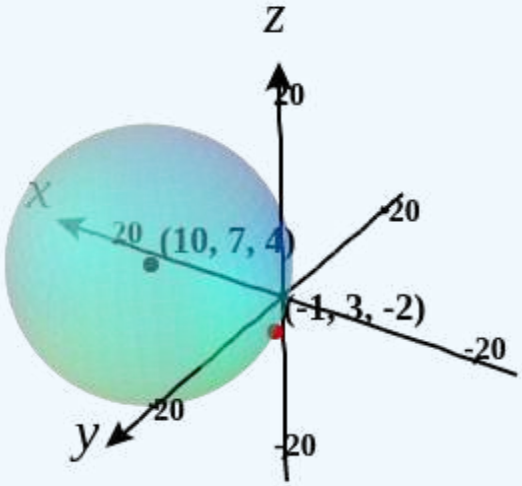


Figure 11.2.13: The sphere centered at $(10, 7, 4)$ containing point $(-1, 3, -2)$.

Solution

Use the distance formula to find the radius r of the sphere:

$$\begin{aligned} r &= \sqrt{(-1-10)^2+(3-7)^2+(-2-4)^2} \\ &= \sqrt{(-11)^2+(-4)^2+(-6)^2} \\ &= \sqrt{173} \end{aligned}$$

The standard equation of the sphere is

$$(x-10)^2+(y-7)^2+(z-4)^2 = 173.$$

Exercise 11.2.4

Find the standard equation of the sphere with center $(-2,4,-5)$ containing point $(4,4,-1)$.

Example 11.2.5: Finding the Equation of a Sphere

Let $P=(-5,2,3)$ and $Q=(3,4,-1)$, and suppose line segment $\overline{PQ}$ forms the diameter of a sphere (Figure 11.2.14). Find the equation of the sphere.

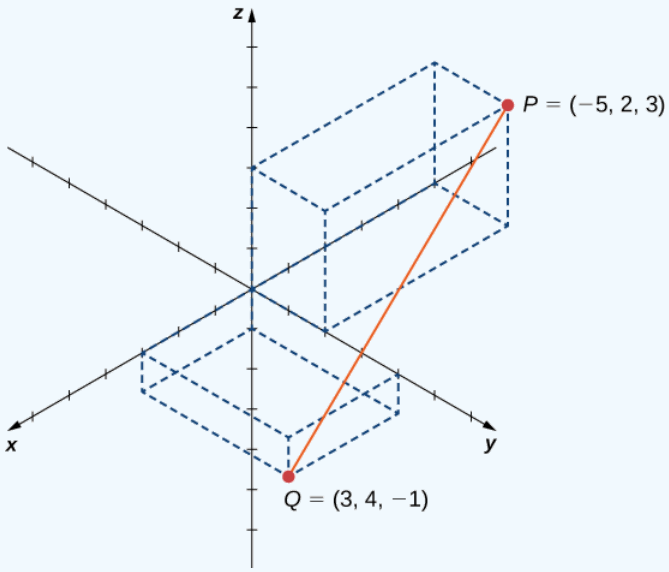


Figure 11.2.14: Line segment $\overline{PQ}$.

Solution:

Since $\overline{PQ}$ is a diameter of the sphere, we know the center of the sphere is the midpoint of $\overline{PQ}$.Then,

$$C = \left(\frac{-5+3}{2}, \frac{2+4}{2}, \frac{3+(-1)}{2}\right) = (-1,3,1).$$

Furthermore, we know the radius of the sphere is half the length of the diameter. This gives

$$\begin{aligned} r &= \frac{1}{2}\sqrt{(-5-3)^2+(2-4)^2+(3-(-1))^2} \\ &= \frac{1}{2}\sqrt{64+4+16} \\ &= \sqrt{21} \end{aligned}$$

Then, the equation of the sphere is $(x+1)^2+(y-3)^2+(z-1)^2 = 21$.

Exercise 11.2.5

Find the equation of the sphere with diameter $\overline{PQ}$, where $P=(2,-1,-3)$ and $Q=(-2,5,-1)$.

Example 11.2.6: Graphing Other Equations in Three Dimensions

Describe the set of points that satisfies $(x-4)(z-2)=0$, and graph the set.

Solution

We must have either $x-4=0$ or $z-2=0$, so the set of points forms the two planes $x=4$ and $z=2$ (Figure 11.2.15).

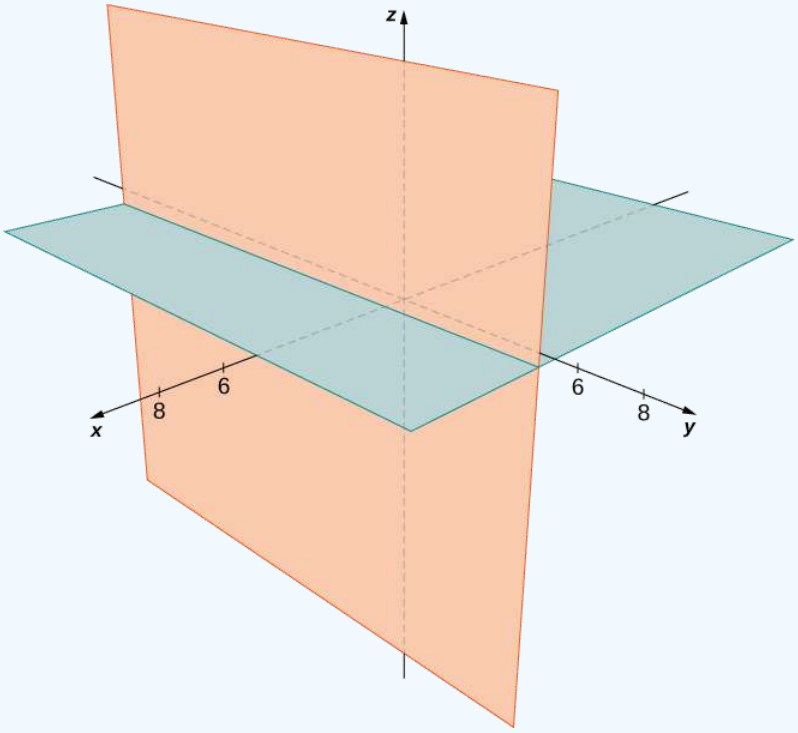


Figure 11.2.15: The set of points satisfying $(x-4)(z-2)=0$ forms the two planes $x=4$ and $z=2$.

Exercise 11.2.6

Describe the set of points that satisfies $(y+2)(z-3)=0$, and graph the set.

Example 11.2.7: Graphing Other Equations in Three Dimensions

Describe the set of points in three-dimensional space that satisfies $(x-2)^2+(y-1)^2=4$, and graph the set.

Solution

The x- and y-coordinates form a circle in the xy-plane of radius 2, centered at $(2,1)$. Since there is no restriction on the z-coordinate, the three-dimensional result is a circular cylinder of radius 2 centered on the line with $x=2$ and $y=1$. The cylinder extends indefinitely in the z-direction (Figure 11.2.16).

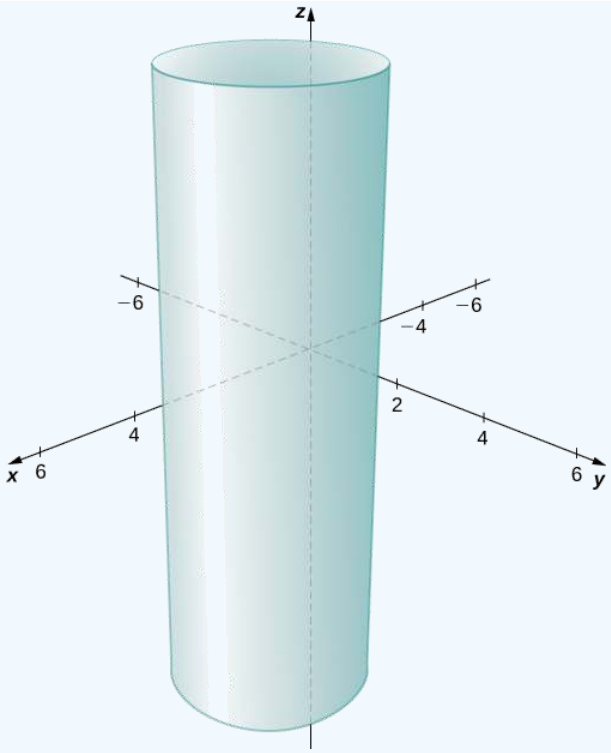


Figure 11.2.16: The set of points satisfying $(x-2)^2+(y-1)^2=4$. This is a cylinder of radius 2 centered on the line with $x=2$ and $y=1$.

Exercise 11.2.7

Describe the set of points in three dimensional space that satisfies $x^2+(z-2)^2=16$, and graph the surface.

Working with Vectors in $\mathbb{R}^3$

Just like two-dimensional vectors, three-dimensional vectors are quantities with both magnitude and direction, and they are represented by directed line segments (arrows). With a three-dimensional vector, we use a three-dimensional arrow.

Three-dimensional vectors can also be represented in component form. The notation $\vec{\mathbf{v}}=\langle x,y,z\rangle$ is a natural extension of the two-dimensional case, representing a vector with the initial point at the origin, $(0,0,0)$, and terminal point (x,y,z). The zero vector is $\vec{\mathbf{0}}=\langle 0,0,0\rangle$. So, for example, the three dimensional vector $\vec{\mathbf{v}}=\langle 2,4,1\rangle$ is represented by a directed line segment from point $(0,0,0)$ to point $(2,4,1)$ (Figure 11.2.17).

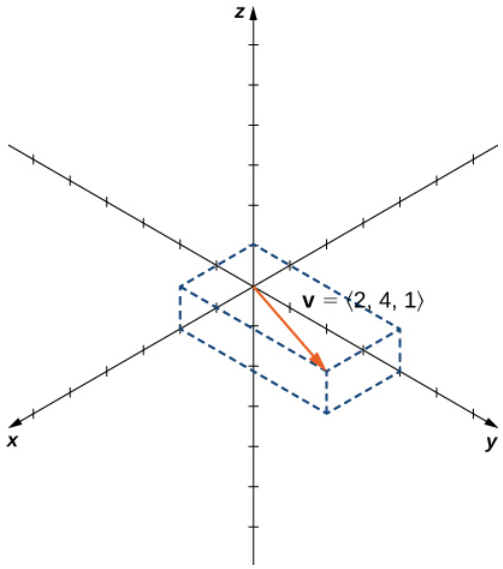


Figure 11.2.17: Vector $\vec{\mathbf{v}} = \langle 2, 4, 1 \rangle$ is represented by a directed line segment from point $(0, 0, 0)$ to point $(2, 4, 1)$.

Vector addition and scalar multiplication are defined analogously to the two-dimensional case. If $\vec{\mathbf{v}} = \langle x_1, y_1, z_1 \rangle$ and $\vec{\mathbf{w}} = \langle x_2, y_2, z_2 \rangle$ are vectors, and k is a scalar, then

$$\vec{\mathbf{v}} + \vec{\mathbf{w}} = \langle x_1 + x_2, y_1 + y_2, z_1 + z_2 \rangle \tag{11.2.4}$$

and

$$k\vec{\mathbf{v}} = \langle kx_1, ky_1, kz_1 \rangle. \tag{11.2.5}$$

If $k = -1$, then $k\vec{\mathbf{v}} = (-1)\vec{\mathbf{v}}$ is written as $-\vec{\mathbf{v}}$, and vector subtraction is defined by $\vec{\mathbf{v}} - \vec{\mathbf{w}} = \vec{\mathbf{v}} + (-\vec{\mathbf{w}}) = \vec{\mathbf{v}} + (-1)\vec{\mathbf{w}}$.

The standard unit vectors extend easily into three dimensions as well, $\hat{\mathbf{i}} = \langle 1, 0, 0 \rangle$, $\hat{\mathbf{j}} = \langle 0, 1, 0 \rangle$, and $\hat{\mathbf{k}} = \langle 0, 0, 1 \rangle$, and we use them in the same way we used the standard unit vectors in two dimensions. Thus, we can represent a vector in $\mathbb{R}^3$ in the following ways:

$$\vec{\mathbf{v}} = \langle x, y, z \rangle = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} \tag{11.2.6}$$

.

Example 11.2.8: Vector Representations

Let $\overrightarrow{PQ}$ be the vector with initial point $P = (3, 12, 6)$ and terminal point $Q = (-4, -3, 2)$ as shown in Figure 11.2.18. Express $\overrightarrow{PQ}$ in both component form and using standard unit vectors.

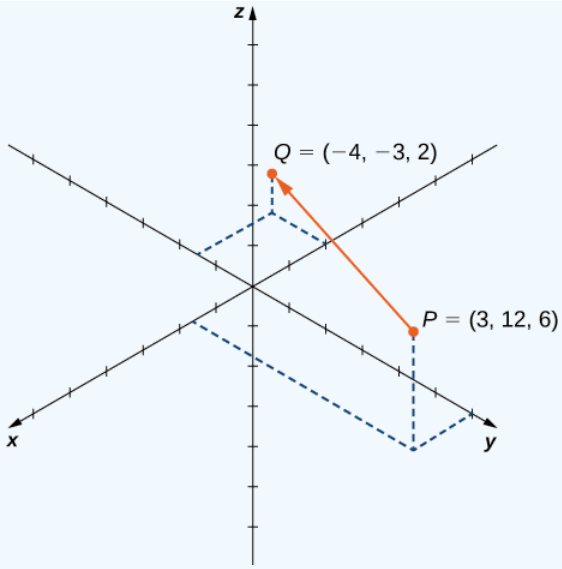


Figure 11.2.18: The vector with initial point $P = (3, 12, 6)$ and terminal point $Q = (-4, -3, 2)$.

Solution

In component form,

$$\begin{aligned}\overrightarrow{PQ} &= \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle \\ &= \langle -4 - 3, -3 - 12, 2 - 6 \rangle \\ &= \langle -7, -15, -4 \rangle.\end{aligned}$$

In standard unit form,

$$\overrightarrow{PQ} = -7\hat{\mathbf{i}} - 15\hat{\mathbf{j}} - 4\hat{\mathbf{k}}.$$

Exercise 11.2.8

Let $S = (3, 8, 2)$ and $T = (2, -1, 3)$. Express $\vec{ST}$ in component form and in standard unit form.

As described earlier, vectors in three dimensions behave in the same way as vectors in a plane. The geometric interpretation of vector addition, for example, is the same in both two- and three-dimensional space (Figure 11.2.19).

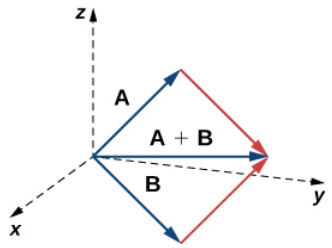


Figure 11.2.19: To add vectors in three dimensions, we follow the same procedures we learned for two dimensions.

We have already seen how some of the algebraic properties of vectors, such as vector addition and scalar multiplication, can be extended to three dimensions. Other properties can be extended in similar fashion. They are summarized here for our reference.

Properties of Vectors in Space

Let $\vec{\mathbf{v}} = \langle x_1, y_1, z_1 \rangle$ and $\vec{\mathbf{w}} = \langle x_2, y_2, z_2 \rangle$ be vectors, and let k be a scalar.

- **Scalar multiplication**:

$$k\vec{\mathbf{v}} = \langle kx_1, ky_1, kz_1 \rangle \tag{11.2.7}$$

- **Vector addition**:

$$\vec{\mathbf{v}} + \vec{\mathbf{w}} = \langle x_1, y_1, z_1 \rangle + \langle x_2, y_2, z_2 \rangle = \langle x_1 + x_2, y_1 + y_2, z_1 + z_2 \rangle \tag{11.2.8}$$

- **Vector subtraction**:

$$\vec{\mathbf{v}} - \vec{\mathbf{w}} = \langle x_1, y_1, z_1 \rangle - \langle x_2, y_2, z_2 \rangle = \langle x_1 - x_2, y_1 - y_2, z_1 - z_2 \rangle \tag{11.2.9}$$

- **Vector magnitude**:

$$\|\vec{\mathbf{v}}\| = \sqrt{x_1^2 + y_1^2 + z_1^2} \tag{11.2.10}$$

- **Unit vector in the direction of $\vec{\mathbf{v}}$**:

$$\frac{1}{\|\vec{\mathbf{v}}\|}\vec{\mathbf{v}} = \frac{1}{\|\vec{\mathbf{v}}\|}\langle x_1, y_1, z_1 \rangle = \langle \frac{x_1}{\|\vec{\mathbf{v}}\|}, \frac{y_1}{\|\vec{\mathbf{v}}\|}, \frac{z_1}{\|\vec{\mathbf{v}}\|} \rangle, \quad \text{if } \vec{\mathbf{v}} \neq \vec{\mathbf{0}} \tag{11.2.11}$$

We have seen that vector addition in two dimensions satisfies the commutative, associative, and additive inverse properties. These properties of vector operations are valid for three-dimensional vectors as well. Scalar multiplication of vectors satisfies the distributive property, and the zero vector acts as an additive identity. The proofs to verify these properties in three dimensions are straightforward extensions of the proofs in two dimensions.

Example 11.2.9: Vector Operations in Three Dimensions

Let $\vec{\mathbf{v}} = \langle -2, 9, 5 \rangle$ and $\vec{\mathbf{w}} = \langle 1, -1, 0 \rangle$ (Figure 11.2.20). Find the following vectors.

a. $3\vec{\mathbf{v}} - 2\vec{\mathbf{w}}$
b. $5\|\vec{\mathbf{w}}\|$
c. $\|5\vec{\mathbf{w}}\|$
d. A unit vector in the direction of $\vec{\mathbf{v}}$

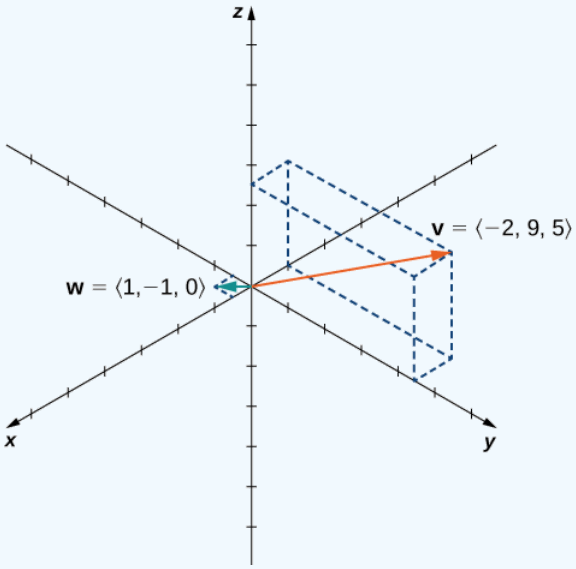


Figure 11.2.20: The vectors $\vec{\mathbf{v}} = \langle -2, 9, 5 \rangle$ and $\vec{\mathbf{w}} = \langle 1, -1, 0 \rangle$.

Solution

a. First, use scalar multiplication of each vector, then subtract:

$$\begin{aligned} 3\vec{\mathbf{v}} - 2\vec{\mathbf{w}} &= 3\langle -2, 9, 5 \rangle - 2\langle 1, -1, 0 \rangle \\ &= \langle -6, 27, 15 \rangle - \langle 2, -2, 0 \rangle \\ &= \langle -6 - 2, 27 - (-2), 15 - 0 \rangle \\ &= \langle -8, 29, 15 \rangle. \end{aligned}$$

b. Write the equation for the magnitude of the vector, then use scalar multiplication:

$$5\|\vec{w}\| = 5\sqrt{1^2 + (-1)^2 + 0^2} = 5\sqrt{2}.$$

c. First, use scalar multiplication, then find the magnitude of the new vector. Note that the result is the same as for part b.:

$$\|5\vec{w}\| = \| \langle 5, -5, 0 \rangle \| = \sqrt{5^2 + (-5)^2 + 0^2} = \sqrt{50} = 5\sqrt{2}$$

d. Recall that to find a unit vector in two dimensions, we divide a vector by its magnitude. The procedure is the same in three dimensions:

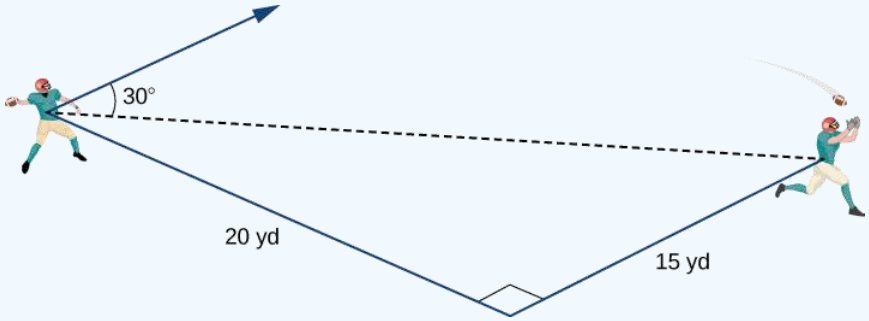
$$\begin{aligned} \frac{\vec{v}}{\|\vec{v}\|} &= \frac{1}{\|\vec{v}\|}\langle -2, 9, 5 \rangle \\ &= \frac{1}{\sqrt{(-2)^2 + 9^2 + 5^2}}\langle -2, 9, 5 \rangle \\ &= \frac{1}{\sqrt{110}}\langle -2, 9, 5 \rangle \\ &= \langle \frac{-2}{\sqrt{110}}, \frac{9}{\sqrt{110}}, \frac{5}{\sqrt{110}} \rangle. \end{aligned}$$

Exercise 11.2.9:

Let $\vec{v} = \langle -1, -1, 1 \rangle$ and $\vec{w} = \langle 2, 0, 1 \rangle$. Find a unit vector in the direction of $5\vec{v} + 3\vec{w}$.

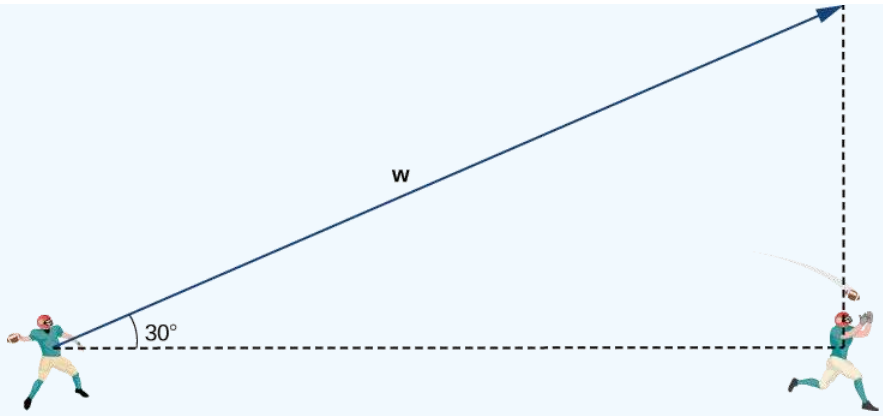
Example 11.2.10: Throwing a Forward Pass

A quarterback is standing on the football field preparing to throw a pass. His receiver is standing 20 yd down the field and 15 yd to the quarterback's left. The quarterback throws the ball at a velocity of 60 mph toward the receiver at an upward angle of $30°$ (see the following figure). Write the initial velocity vector of the ball, $\vec{v}$, in component form.



Solution

The first thing we want to do is find a vector in the same direction as the velocity vector of the ball. We then scale the vector appropriately so that it has the right magnitude. Consider the vector $\vec{w}$ extending from the quarterback's arm to a point directly above the receiver's head at an angle of $30°$ (see the following figure). This vector would have the same direction as $\vec{v}$, but it may not have the right magnitude.



The receiver is 20 yd down the field and 15 yd to the quarterback's left. Therefore, the straight-line distance from the quarterback to the receiver is

$$\textbf{Dist from QB to receiver} = \sqrt{15^2 + 20^2} = \sqrt{225 + 400} = \sqrt{625} = 25 \text{ yd.}$$

We have $\frac{25}{\|\vec{w}\|} = \cos 30°$. Then the magnitude of $\vec{w}$ is given by

$$\|\vec{w}\| = \frac{25}{\cos 30°} = \frac{25 \cdot 2}{\sqrt{3}} = \frac{50}{\sqrt{3}} \text{ yd}$$

and the vertical distance from the receiver to the terminal point of $\vec{w}$ is

$$\textbf{Vert dist from receiver to terminal point of } \vec{w} = \|\vec{w}\| \sin 30° = \frac{50}{\sqrt{3}} \cdot \frac{1}{2} = \frac{25}{\sqrt{3}} \text{ yd.}$$

Then $\vec{w} = \langle 20, 15, \frac{25}{\sqrt{3}} \rangle$, and has the same direction as $\vec{v}$.

Recall, though, that we calculated the magnitude of $\vec{w}$ to be $\|\vec{w}\| = \frac{50}{\sqrt{3}}$ yd, and $\vec{v}$ has magnitude 60 mph. So, we need to multiply vector $\vec{w}$ by an appropriate constant, k. We want to find a value of k so that $\| k\vec{w} \| = 60$ mph*. We have

$$\|k\vec{w}\| = k\|\vec{w}\| = k\frac{50}{\sqrt{3}} \text{ yd,}$$

so we want

$$k\left(\frac{50}{\sqrt{3}} \text{ yd}\right) = 60 \text{ mph}$$

$$k = \frac{60\sqrt{3}}{50} \text{ mph / yd}$$

$$k = \frac{6\sqrt{3}}{5} \text{ mph / yd.}$$

Then

$$\vec{v} = k\vec{w} = k\langle 20, 15, \frac{25}{\sqrt{3}} \rangle = \frac{6\sqrt{3}}{5} \langle 20, 15, \frac{25}{\sqrt{3}} \rangle = \langle 24\sqrt{3}, 18\sqrt{3}, 30 \rangle.$$

Let's double-check that $\|\vec{v}\| = 60$ mph. We have

$$\|\vec{v}\| = \sqrt{(24\sqrt{3})^2 + (18\sqrt{3})^2 + (30)^2} = \sqrt{1728 + 972 + 900} = \sqrt{3600} = 60 \text{ mph.}$$

So, we have found the correct components for $\vec{v}$.

Note *

Readers who have been watching the units of measurement may be wondering what exactly is going on at this point: haven't we just mixed yards and miles per hour? We haven't, but the reason is subtle. One way to understand it is to realize that there are really two parallel coordinate systems in this problem: one gives positions down the field, across the field, and up into the air in units of yards; the other gives speeds down the field, across the field, and up into the air in units of

miles per hour. The vector $\vec{\mathbf{w}}$ is calculated in the position coordinate system; vector $\vec{\mathbf{v}}$ will be in the speed system. Because corresponding axes in each system are parallel, directions in the two systems are also parallel, so the claim that $\vec{\mathbf{w}}$ and $\vec{\mathbf{v}}$ point in the same direction is correct. The constant k that we're looking for is a conversion factor between the magnitudes of these two vectors, converting from the position system to the speed one in the process. And as seen above, our calculation of k produces the right units for such a conversion, namely miles per hour per yard.

Including these units in the final calculation, we have:

$$\vec{\mathbf{v}} = k\vec{\mathbf{w}} = k\langle 20 \text{ yd}, 15 \text{ yd}, \frac{25}{\sqrt{3}} \text{ yd}\rangle = \left(\frac{6\sqrt{3}}{5} \text{ mph/yd}\right) \langle 20 \text{ yd}, 15 \text{ yd}, \frac{25}{\sqrt{3}} \text{ yd}\rangle = \langle 24\sqrt{3} \text{ mph}, 18\sqrt{3} \text{ mph}, 30 \text{ mph}\rangle.$$

For simplicity, we usually do not include units in vector calculations, but it can sometimes be useful to clearly see what is happening.

Exercise 11.2.10

Assume the quarterback and the receiver are in the same place as in the previous example. This time, however, the quarterback throws the ball at velocity of 40 mph and an angle of $45°$. Write the initial velocity vector of the ball, $\vec{\mathbf{v}}$, in component form.

Key Concepts

- The three-dimensional coordinate system is built around a set of three axes that intersect at right angles at a single point, the origin. Ordered triples (x, y, z) are used to describe the location of a point in space.
- The distance d between points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by the formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

- In three dimensions, the equations $x = a$, $y = b$, and $z = c$ describe planes that are parallel to the coordinate planes.
- The standard equation of a sphere with center (a, b, c) and radius r is

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2.$$

- In three dimensions, as in two, vectors are commonly expressed in component form, $\vec{\mathbf{v}} = \langle x, y, z\rangle$, or in terms of the standard unit vectors, $\vec{\mathbf{v}} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$.
- Properties of vectors in space are a natural extension of the properties for vectors in a plane. Let $\vec{\mathbf{v}} = \langle x_1, y_1, z_1\rangle$ and $\vec{\mathbf{w}} = \langle x_2, y_2, z_2\rangle$ be vectors, and let k be a scalar.

  Scalar multiplication:

$$k\vec{\mathbf{v}} = \langle kx_1, ky_1, kz_1\rangle$$

  Vector addition:

$$\vec{\mathbf{v}} + \vec{\mathbf{w}} = \langle x_1, y_1, z_1\rangle + \langle x_2, y_2, z_2\rangle = \langle x_1 + x_2, y_1 + y_2, z_1 + z_2\rangle$$

  Vector subtraction:

$$\vec{\mathbf{v}} - \vec{\mathbf{w}} = \langle x_1, y_1, z_1\rangle - \langle x_2, y_2, z_2\rangle = \langle x_1 - x_2, y_1 - y_2, z_1 - z_2\rangle$$

  Vector magnitude:

$$\|\vec{\mathbf{v}}\| = \sqrt{x_1^2 + y_1^2 + z_1^2}$$

  Unit vector in the direction of $\vec{\mathbf{v}}$:

$$\frac{\vec{\mathbf{v}}}{\|\vec{\mathbf{v}}\|} = \frac{1}{\|\vec{\mathbf{v}}\|}\langle x_1, y_1, z_1\rangle = \langle \frac{x_1}{\|\vec{\mathbf{v}}\|}, \frac{y_1}{\|\vec{\mathbf{v}}\|}, \frac{z_1}{\|\vec{\mathbf{v}}\|}\rangle, \ \vec{\mathbf{v}} \neq \vec{\mathbf{0}}$$

Key Equations

Distance between two points in space:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Sphere with center (a, b, c) and radius r:

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$$

Glossary

Contributors

Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.

Example **11.2.10** has been modified by Doug Baldwin and Paul Seeburger to clarify the units of measurement that it uses and how it uses them.

Paul Seeburger also created dynamic versions of Figures **11.2.8, 11.2.9** and **11.2.13** using CalcPlot3D.

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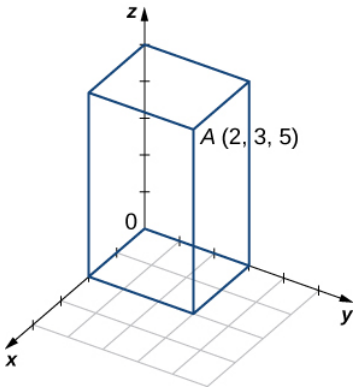
11.2E: Exercises for Vectors in Space

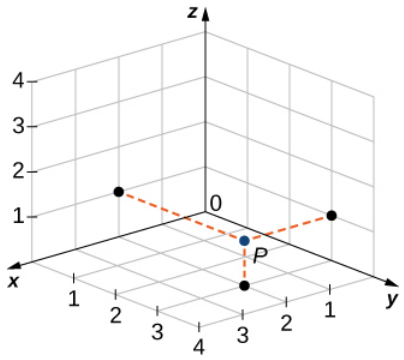
1) Consider a rectangular box with one of the vertices at the origin, as shown in the following figure. If point $A(2,3,5)$ is the opposite vertex to the origin, then find

a. the coordinates of the other six vertices of the box and

b. the length of the diagonal of the box determined by the vertices O and A.



2) Find the coordinates of point P and determine its distance to the origin.



For exercises 3-6, describe and graph the set of points that satisfies the given equation.

3) $(y-5)(z-6)=0$

4) $(z-2)(z-5)=0$

5) $(y-1)^2+(z-1)^2=1$

6) $(x-2)^2+(z-5)^2=4$

7) Write the equation of the plane passing through point $(1,1,1)$ that is parallel to the xy-plane.

8) Write the equation of the plane passing through point $(1,-3,2)$ that is parallel to the xz-plane.

9) Find an equation of the plane passing through points $(1,-3,-2)$, $(0,3,-2)$, and $(1,0,-2)$.

10) Find an equation of the plane passing through points $(1,9,2)$, $(1,3,6)$, and $(1,-7,8)$.

For exercises 11-14, find the equation of the sphere in standard form that satisfies the given conditions.

11) Center $C(-1,7,4)$ and radius 4

12) Center $C(-4,7,2)$ and radius 6

13) Diameter PQ, where $P(-1,5,7)$ and $Q(-5,2,9)$

14) Diameter PQ, where $P(-16,-3,9)$ and $Q(-2,3,5)$

For exercises 15 and 16, find the center and radius of the sphere with an equation in general form that is given.

15) $x^2+y^2+z^2-4z+3=0$

16) $x^2+y^2+z^2-6x+8y-10z+25=0$

For exercises 17-20, express vector $\overrightarrow{PQ}$ with the initial point at P and the terminal point at Q

a. in component form and

b. by using standard unit vectors.

17) $P(3,0,2)$ and $Q(-1,-1,4)$

18) $P(0,10,5)$ and $Q(1,1,-3)$

19) $P(-2,5,-8)$ and $M(1,-7,4)$, where M is the midpoint of the line segment $\overline{PQ}$

20) $Q(0,7,-6)$ and $M(-1,3,2)$, where M is the midpoint of the line segment $\overline{PQ}$

21) Find terminal point Q of vector $\overrightarrow{PQ}=\langle 7,-1,3\rangle$ with the initial point at $P(-2,3,5)$.

22) Find initial point P of vector $\overrightarrow{PQ}=\langle -9,1,2\rangle$ with the terminal point at $Q(10,0,-1)$.

For exercises 23-26, use the given vectors $\vec{a}$ and $\vec{b}$ to find and express the vectors $\vec{a}+\vec{b}$, $4\vec{a}$, and $-5\vec{a}+3\vec{b}$ in component form.

23) $\vec{a}=\langle -1,-2,4\rangle$, $\vec{b}=\langle -5,6,-7\rangle$

24) $\vec{a}=\langle 3,-2,4\rangle$, $\vec{b}=\langle -5,6,-9\rangle$

25) $\vec{a}=-\hat{k}$, $\vec{b}=-\hat{i}$

26) $\vec{a}=\hat{i}+\hat{j}+\hat{k}$, $\vec{b}=2\hat{i}-3\hat{j}+2\hat{k}$

For exercises 27-30, vectors $\vec{u}$ and $\vec{v}$ are given. Find the magnitudes of vectors $\vec{u}-\vec{v}$ and $-2\vec{u}$.

27) $\vec{u}=2\hat{i}+3\hat{j}+4\hat{k}$, $\vec{v}=-\hat{i}+5\hat{j}-\hat{k}$

28) $\vec{u}=\hat{i}+\hat{j}$, $\vec{v}=\hat{j}-\hat{k}$

29) $\vec{u}=\langle 2\cos t,-2\sin t,3\rangle$, $\vec{v}=\langle 0,0,3\rangle$, where t is a real number.

30) $\vec{u}=\langle 0,1,\sinh t\rangle$, $\vec{v}=\langle 1,1,0\rangle$, where t is a real number.

For exercises 31-36, find the unit vector in the direction of the given vector $\vec{a}$ and express it using standard unit vectors.

31) $\vec{a}=3\hat{i}-4\hat{j}$

32) $\vec{a}=\langle 4,-3,6\rangle$

33) $\vec{a}=\overrightarrow{PQ}$, where $P(-2,3,1)$ and $Q(0,-4,4)$

34) $\vec{a}=\overrightarrow{OP}$, where $P(-1,-1,1)$

35) $\vec{a}=\vec{u}-\vec{v}+\vec{w}$, where $\vec{u}=\hat{i}-\hat{j}-\hat{k}$, $\vec{v}=2\hat{i}-\hat{j}+\hat{k}$, and $\vec{w}=-\hat{i}+\hat{j}+3\hat{k}$

36) $\vec{a}=2\vec{u}+\vec{v}-\vec{w}$, where $\vec{u}=\hat{i}-\hat{k}$, $\vec{v}=2\hat{j}$, and $\vec{w}=\hat{i}-\hat{j}$

37) Determine whether $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ are equivalent vectors, where $A(1,1,1)$, $B(3,3,3)$, $P(1,4,5)$, and $Q(3,6,7)$.

38) Determine whether the vectors $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ are equivalent, where $A(1,4,1)$, $B(-2,2,0)$, $P(2,5,7)$, and $Q(-3,2,1)$.

For exercises 39-42, find vector $\vec{u}$ with a magnitude that is given and satisfies the given conditions.

39) $\vec{v}=\langle 7,-1,3\rangle$, $\|\vec{u}\|=10$, and $\vec{u}$ and $\vec{v}$ have the same direction

40) $\vec{v}=\langle 2,4,1\rangle$, $\|\vec{u}\|=15$, and $\vec{u}$ and $\vec{v}$ have the same direction

41) $\vec{v}=\langle 2\sin t,\,2\cos t,1\rangle$, $\|\vec{u}\|=2$, $\vec{u}$ and $\vec{v}$ have opposite directions for any t, where t is a real number

42) $\vec{v}=\langle 3\sinh t,0,3\rangle$, $\|\vec{u}\|=5$, and $\vec{u}$ and $\vec{v}$ have opposite directions for any t, where t is a real number

43) Determine a vector of magnitude 5 in the direction of vector $\overrightarrow{AB}$, where $A(2,1,5)$ and $B(3,4,-7)$.

44) Find a vector of magnitude 2 that points in the opposite direction than vector $\overrightarrow{AB}$, where $A(-1,-1,1)$ and $B(0,1,1)$. Express the answer in component form.

45) Consider the points $A(2,\alpha,0)$, $B(0,1,\beta)$, and $C(1,1,\beta)$, where α and β are negative real numbers. Find α and β such that $\|\overrightarrow{OA} - \overrightarrow{OB} + \overrightarrow{OC}\| = \|\overrightarrow{OB}\| = 4$.

46) Consider points $A(\alpha,0,0)$, $B(0,\beta,0)$, and $C(\alpha,\beta,\beta)$, where α and β are positive real numbers. Find α and β such that $\|\overrightarrow{OA} + \overrightarrow{OB}\| = \sqrt{2}$ and $\|\overrightarrow{OC}\| = \sqrt{3}$.

47) Let $P(x,y,z)$ be a point situated at an equal distance from points $A(1,-1,0)$ and $B(-1,2,1)$. Show that point P lies on the plane of equation $-2x + 3y + z = 2$.

48) Let $P(x,y,z)$ be a point situated at an equal distance from the origin and point $A(4,1,2)$. Show that the coordinates of point P satisfy the equation $8x + 2y + 4z = 21$.

49) The points A, B, and C are collinear (in this order) if the relation $\|\overrightarrow{AB}\| + \|\overrightarrow{BC}\| = \|\overrightarrow{AC}\|$ is satisfied. Show that $A(5,3,-1)$, $B(-5,-3,1)$, and $C(-15,-9,3)$ are collinear points.

50) Show that points $A(1,0,1)$, $B(0,1,1)$, and $C(1,1,1)$ are not collinear.

51) [T] A force $\vec{\mathbf{F}}$ of $50\,N$ acts on a particle in the direction of the vector $\overrightarrow{OP}$, where $P(3,4,0)$.

a. Express the force as a vector in component form.

b. Find the angle between force $\vec{\mathbf{F}}$ and the positive direction of the x-axis. Express the answer in degrees rounded to the nearest integer.

52) [T] A force $\vec{\mathbf{F}}$ of $40\,N$ acts on a box in the direction of the vector $\overrightarrow{OP}$, where $P(1,0,2)$.

a. Express the force as a vector by using standard unit vectors.

b. Find the angle between force $\vec{\mathbf{F}}$ and the positive direction of the x-axis.

53) If $\vec{\mathbf{F}}$ is a force that moves an object from point $P_1(x_1,y_1,z_1)$ to another point $P_2(x_2,y_2,z_2)$, then the displacement vector is defined as $\vec{\mathbf{D}} = (x_2 - x_1)\hat{\mathbf{i}} + (y_2 - y_1)\hat{\mathbf{j}} + (z_2 - z_1)\hat{\mathbf{k}}$. A metal container is lifted 10 m vertically by a constant force $\vec{\mathbf{F}}$. Express the displacement vector $\vec{\mathbf{D}}$ by using standard unit vectors.

54) A box is pulled 4 yd horizontally in the x-direction by a constant force $\vec{\mathbf{F}}$. Find the displacement vector in component form.

55) The sum of the forces acting on an object is called the resultant or net force. An object is said to be in static equilibrium if the resultant force of the forces that act on it is zero. Let $\vec{\mathbf{F}}_1 = \langle 10,6,3\rangle$, $\vec{\mathbf{F}}_2 = \langle 0,4,9\rangle$, and $\vec{\mathbf{F}}_3 = \langle 10,-3,-9\rangle$ be three forces acting on a box. Find the force $\vec{\mathbf{F}}_4$ acting on the box such that the box is in static equilibrium. Express the answer in component form.

56) [T] Let $\vec{\mathbf{F}}_k = \langle 1,k,k^2\rangle, k = 1,\ldots,n$ be n forces acting on a particle, with $n \geq 2$.

a. Find the net force $\vec{\mathbf{F}} = \sum_{k=1}^{n} \vec{\mathbf{F}}_k$. Express the answer using standard unit vectors.

b. Use a computer algebra system (CAS) to find n such that $\|\vec{\mathbf{F}}\| < 100$.

57) The force of gravity $\vec{\mathbf{F}}$ acting on an object is given by $\vec{\mathbf{F}} = m\vec{\mathbf{g}}$, where m is the mass of the object (expressed in kilograms) and $\vec{\mathbf{g}}$ is acceleration resulting from gravity, with $\|\vec{\mathbf{g}}\| = 9.8\,N/kg$. A 2-kg disco ball hangs by a chain from the ceiling of a room.

a. Find the force of gravity $\vec{\mathbf{F}}$ acting on the disco ball and find its magnitude.

b. Find the force of tension $\vec{\mathbf{T}}$ in the chain and its magnitude.

Express the answers using standard unit vectors.

Figure 18: (credit: modification of work by Kenneth Lu, Flickr)

58) A 5-kg pendant chandelier is designed such that the alabaster bowl is held by four chains of equal length, as shown in the following figure.

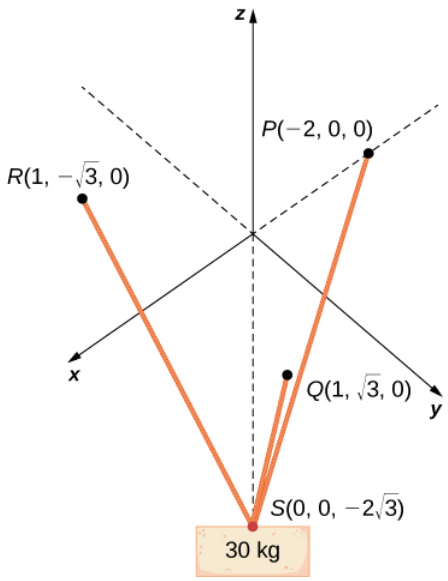
a. Find the magnitude of the force of gravity acting on the chandelier.

b. Find the magnitudes of the forces of tension for each of the four chains (assume chains are essentially vertical).

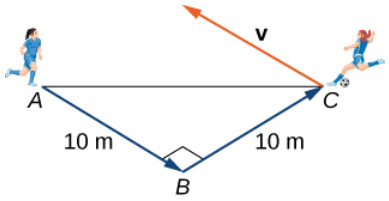
59) [T] A 30-kg block of cement is suspended by three cables of equal length that are anchored at points $P(-2,0,0), Q(1,\sqrt{3},0)$, and $R(1,-\sqrt{3},0)$. The load is located at $S(0,0,-2\sqrt{3})$, as shown in the following figure. Let $\vec{\mathbf{F}}_1, \vec{\mathbf{F}}_2$, and $\vec{\mathbf{F}}_3$ be the forces of tension resulting from the load in cables RS, QS, and PS, respectively.

a. Find the gravitational force $\vec{\mathbf{F}}$ acting on the block of cement that counterbalances the sum $\vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 + \vec{\mathbf{F}}_3$ of the forces of tension in the cables.

b. Find forces $\vec{\mathbf{F}}_1, \vec{\mathbf{F}}_2$, and $\vec{\mathbf{F}}_3$. Express the answer in component form.



60) Two soccer players are practicing for an upcoming game. One of them runs 10 m from point A to point B. She then turns left at $90°$ and runs 10 m until she reaches point C. Then she kicks the ball with a speed of 10 m/sec at an upward angle of $45°$ to her teammate, who is located at point A. Write the velocity of the ball in component form.



61) Let $\vec{\mathbf{r}}(t) = \langle x(t),\ y(t),\ z(t)\rangle$ be the position vector of a particle at the time $t \in [0, T]$, where $x, y,$ and z are smooth functions on $[0, T]$. The instantaneous velocity of the particle at time t is defined by vector $\vec{\mathbf{v}}(t) = \langle x'(t),\ y'(t),\ z'(t)\rangle$, with components that are the derivatives with respect to t, of the functions $x, y,$ and z, respectively. The magnitude $\| \vec{\mathbf{v}}(t) \|$ of the instantaneous velocity vector is called the speed of the particle at time t. Vector $\vec{\mathbf{a}}(t) = \langle x''(t),\ y''(t),\ z''(t)\rangle$, with components that are the second derivatives with respect to t, of the functions $x, y,$ and z, respectively, gives the acceleration of the particle at time t. Consider $\vec{\mathbf{r}}(t) = \langle \cos t,\ \sin t,\ 2t\rangle$ the position vector of a particle at time $t \in [0, 30]$, where the components of $\vec{\mathbf{r}}$ are expressed in centimeters and time is expressed in seconds.

a. Find the instantaneous velocity, speed, and acceleration of the particle after the first second. Round your answer to two decimal places.

b. Use a CAS to visualize the path of the particle—that is, the set of all points of coordinates $(\cos t, \sin t, 2t)$, where $t \in [0, 30]$.

62) [T] Let $\vec{\mathbf{r}}(t) = \langle t, 2t^2, 4t^2\rangle$ be the position vector of a particle at time t (in seconds), where $t \in [0, 10]$ (here the components of $\vec{\mathbf{r}}$ are expressed in centimeters).

a. Find the instantaneous velocity, speed, and acceleration of the particle after the first two seconds. Round your answer to two decimal places.

b. Use a CAS to visualize the path of the particle defined by the points $(t,\ 2t^2,\ 4t^2)$, where $t \in [0,\ 60]$.

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11.3: The Dot Product

Learning Objectives

- Calculate the dot product of two given vectors.
- Determine whether two given vectors are perpendicular.
- Find the direction cosines of a given vector.
- Explain what is meant by the vector projection of one vector onto another vector, and describe how to compute it.
- Calculate the work done by a given force.

If we apply a force to an object so that the object moves, we say that work is done by the force. Previously, we looked at a constant force and we assumed the force was applied in the direction of motion of the object. Under those conditions, work can be expressed as the product of the force acting on an object and the distance the object moves. In this chapter, however, we have seen that both force and the motion of an object can be represented by vectors.

In this section, we develop an operation called the dot product, which allows us to calculate work in the case when the force vector and the motion vector have different directions. The dot product essentially tells us how much of the force vector is applied in the direction of the motion vector. The dot product can also help us measure the angle formed by a pair of vectors and the position of a vector relative to the coordinate axes. It even provides a simple test to determine whether two vectors meet at a right angle.

The Dot Product and Its Properties

We have already learned how to add and subtract vectors. In this chapter, we investigate two types of vector multiplication. The first type of vector multiplication is called the dot product, based on the notation we use for it, and it is defined as follows:

Definition: dot product

The *dot product* of vectors $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$ is given by the sum of the products of the components

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

Note that if u and v are two-dimensional vectors, we calculate the dot product in a similar fashion. Thus, if $\vec{\mathbf{u}} = \langle u_1, u_2 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2 \rangle$, then

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = u_1 v_1 + u_2 v_2.$$

When two vectors are combined under addition or subtraction, the result is a vector. When two vectors are combined using the dot product, the result is a scalar. For this reason, the dot product is often called the *scalar product*. It may also be called the *inner product*.

Example 11.3.1: Calculating Dot Products

a. Find the dot product of $\vec{\mathbf{u}} = \langle 3, 5, 2 \rangle$ and $\vec{\mathbf{v}} = \langle -1, 3, 0 \rangle$.

b. Find the scalar product of $\vec{\mathbf{p}} = 10\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 7\hat{\mathbf{k}}$ and $\vec{\mathbf{q}} = -2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 6\hat{\mathbf{k}}$.

Solution:

a. Substitute the vector components into the formula for the dot product:

$$\begin{aligned} \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} &= u_1 v_1 + u_2 v_2 + u_3 v_3 \\ &= 3(-1) + 5(3) + 2(0) \\ &= -3 + 15 + 0 \\ &= 12. \end{aligned}$$

b. The calculation is the same if the vectors are written using standard unit vectors. We still have three components for each vector to substitute into the formula for the dot product:

$$\begin{aligned}\vec{\mathbf{p}} \cdot \vec{\mathbf{q}} &= p_1q_1 + p_2q_2 + p_3q_3 \\ &= 10(-2) + (-4)(1) + (7)(6) \\ &= -20 - 4 + 42 \\ &= 18.\end{aligned}$$

Exercise 11.3.1

Find $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}$, where $\vec{\mathbf{u}} = \langle 2, 9, -1 \rangle$ and $\vec{\mathbf{v}} = \langle -3, 1, -4 \rangle$.

Like vector addition and subtraction, the dot product has several algebraic properties. We prove three of these properties and leave the rest as exercises.

Properties of the Dot Product

Let $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$ be vectors, and let c be a scalar.

i. *Commutative property*

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = \vec{\mathbf{v}} \cdot \vec{\mathbf{u}}$$

ii. *Distributive property*

$$\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} + \vec{\mathbf{w}}) = \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} + \vec{\mathbf{u}} \cdot \vec{\mathbf{w}}$$

iii. *Associative property*

$$c(\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}) = (c\vec{\mathbf{u}}) \cdot \vec{\mathbf{v}} = \vec{\mathbf{u}} \cdot (c\vec{\mathbf{v}})$$

iv. *Property of magnitude*

$$\vec{\mathbf{v}} \cdot \vec{\mathbf{v}} = \|\vec{\mathbf{v}}\|^2$$

Proof

Let $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$. Then

$$\begin{aligned}\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} &= \langle u_1, u_2, u_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle \\ &= u_1v_1 + u_2v_2 + u_3v_3 \\ &= v_1u_1 + v_2u_2 + v_3u_3 \\ &= \langle v_1, v_2, v_3 \rangle \cdot \langle u_1, u_2, u_3 \rangle \\ &= \vec{\mathbf{v}} \cdot \vec{\mathbf{u}}.\end{aligned}$$

The associative property looks like the associative property for real-number multiplication, but pay close attention to the difference between scalar and vector objects:

$$\begin{aligned}c(\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}) &= c(u_1v_1 + u_2v_2 + u_3v_3) \\ &= c(u_1v_1) + c(u_2v_2) + c(u_3v_3) \\ &= (cu_1)v_1 + (cu_2)v_2 + (cu_3)v_3 \\ &= \langle cu_1, cu_2, cu_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle \\ &= c\langle u_1, u_2, u_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle \\ &= (c\vec{\mathbf{u}}) \cdot \vec{\mathbf{v}}.\end{aligned}$$

The proof that $c(\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}) = \vec{\mathbf{u}} \cdot (c\vec{\mathbf{v}})$ is similar.

The fourth property shows the relationship between the magnitude of a vector and its dot product with itself:

$$\begin{aligned}\vec{\mathbf{v}} \cdot \vec{\mathbf{v}} &= \langle v_1, v_2, v_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle \\ &= (v_1)^2 + (v_2)^2 + (v_3)^2 \\ &= \left[\sqrt{(v_1)^2 + (v_2)^2 + (v_3)^2}\right]^2 \\ &= \|\vec{\mathbf{v}}\|^2.\end{aligned}$$

□

Note that the definition of the dot product yields $\vec{\mathbf{0}} \cdot \vec{\mathbf{v}} = 0$. By property iv. if $\vec{\mathbf{v}} \cdot \vec{\mathbf{v}} = 0$, then $\vec{\mathbf{v}} = \vec{\mathbf{0}}$.

Example 11.3.2: Using Properties of the Dot Product

Let $\vec{\mathbf{a}} = \langle 1, 2, -3 \rangle$, $\vec{\mathbf{b}} = \langle 0, 2, 4 \rangle$, and $\vec{\mathbf{c}} = \langle 5, -1, 3 \rangle$.

Find each of the following products.

a. $(\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})\vec{\mathbf{c}}$
b. $\vec{\mathbf{a}} \cdot (2\vec{\mathbf{c}})$
c. $\|\vec{\mathbf{b}}\|^2$

Solution

a. Note that this expression asks for the scalar multiple of $\vec{\mathbf{c}}$ by $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}$:

$$\begin{aligned}(\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})\vec{\mathbf{c}} &= (\langle 1, 2, -3 \rangle \cdot \langle 0, 2, 4 \rangle)\langle 5, -1, 3 \rangle \\ &= (1(0) + 2(2) + (-3)(4))\langle 5, -1, 3 \rangle \\ &= -8\langle 5, -1, 3 \rangle \\ &= \langle -40, 8, -24 \rangle.\end{aligned}$$

b. This expression is a dot product of vector $\vec{\mathbf{a}}$ and scalar multiple $2\vec{\mathbf{c}}$:

$$\begin{aligned}\vec{\mathbf{a}} \cdot (2\vec{\mathbf{c}}) &= 2(\vec{\mathbf{a}} \cdot \vec{\mathbf{c}}) \\ &= 2(\langle 1, 2, -3 \rangle \cdot \langle 5, -1, 3 \rangle) \\ &= 2(1(5) + 2(-1) + (-3)(3)) \\ &= 2(-6) = -12.\end{aligned}$$

c. Simplifying this expression is a straightforward application of the dot product:

$$\begin{aligned}\|\vec{\mathbf{b}}\|^2 &= \vec{\mathbf{b}} \cdot \vec{\mathbf{b}} \\ &= \langle 0, 2, 4 \rangle \cdot \langle 0, 2, 4 \rangle \\ &= 0^2 + 2^2 + 4^2 \\ &= 0 + 4 + 16 \\ &= 20.\end{aligned}$$

Exercise 11.3.2

Find the following products for $\vec{\mathbf{p}} = \langle 7, 0, 2 \rangle$, $\vec{\mathbf{q}} = \langle -2, 2, -2 \rangle$, and $\vec{\mathbf{r}} = \langle 0, 2, -3 \rangle$.

a. $(\vec{\mathbf{r}} \cdot \vec{\mathbf{p}})\vec{\mathbf{q}}$
b. $\|\vec{\mathbf{p}}\|^2$

Using the Dot Product to Find the Angle between Two Vectors

When two nonzero vectors are placed in standard position, whether in two dimensions or three dimensions, they form an angle between them (Figure 11.3.1). The dot product provides a way to find the measure of this angle. This property is a result of the fact that we can express the dot product in terms of the cosine of the angle formed by two vectors.

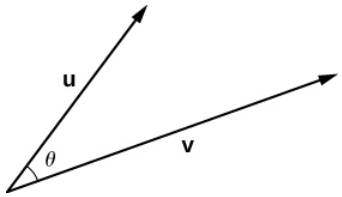


Figure 11.3.1: Let θ be the angle between two nonzero vectors $\vec{u}$ and $\vec{v}$ such that $0 \le \theta \le \pi$.

Evaluating a Dot Product

The dot product of two vectors is the product of the magnitude of each vector and the cosine of the angle between them:

$$\vec{u} \cdot \vec{v} = \|\vec{u}\|\|\vec{v}\| \cos\theta. \tag{11.3.1}$$

Proof

Place vectors $\vec{u}$ and $\vec{v}$ in standard position and consider the vector $\vec{v} - \vec{u}$ (Figure 11.3.2). These three vectors form a triangle with side lengths $\|\vec{u}\|, \|\vec{v}\|$, and $\|\vec{v} - \vec{u}\|$.

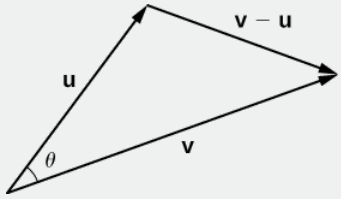


Figure 11.3.2: The lengths of the sides of the triangle are given by the magnitudes of the vectors that form the triangle.

Recall from trigonometry that the law of cosines describes the relationship among the side lengths of the triangle and the angle θ. Applying the law of cosines here gives

$$\|\vec{v} - \vec{u}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\| \cos\theta. \tag{11.3.2}$$

The dot product provides a way to rewrite the left side of Equation 11.3.2:

$$\begin{aligned} \|\vec{v} - \vec{u}\|^2 &= (\vec{v} - \vec{u}) \cdot (\vec{v} - \vec{u}) \\ &= (\vec{v} - \vec{u}) \cdot \vec{v} - (\vec{v} - \vec{u}) \cdot \vec{u} \\ &= \vec{v} \cdot \vec{v} - \vec{u} \cdot \vec{v} - \vec{v} \cdot \vec{u} + \vec{u} \cdot \vec{u} \\ &= \vec{v} \cdot \vec{v} - \vec{u} \cdot \vec{v} - \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{u} \\ &= \|\vec{v}\|^2 - 2\vec{u} \cdot \vec{v} + \|\vec{u}\|^2. \end{aligned}$$

Substituting into the law of cosines yields

$$\begin{aligned} \|\vec{v} - \vec{u}\|^2 &= \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\| \cos\theta \\ \|\vec{v}\|^2 - 2\vec{u} \cdot \vec{v} + \|\vec{u}\|^2 &= \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\| \cos\theta \\ -2\vec{u} \cdot \vec{v} &= -2\|\vec{u}\|\|\vec{v}\| \cos\theta \\ \vec{u} \cdot \vec{v} &= \|\vec{u}\|\|\vec{v}\| \cos\theta. \end{aligned}$$

□

We can use the form of the dot product in Equation 11.3.1 to find the measure of the angle between two nonzero vectors by rearranging Equation 11.3.1 to solve for the cosine of the angle:

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|}. \tag{11.3.3}$$

Using this equation, we can find the cosine of the angle between two nonzero vectors. Since we are considering the smallest angle between the vectors, we assume $0° \le \theta \le 180°$ (or $0 \le \theta \le \pi$ if we are working in radians). The inverse cosine is unique over this

range, so we are then able to determine the measure of the angle θ.

Example 11.3.3: Finding the Angle between Two Vectors

Find the measure of the angle between each pair of vectors.

a. $\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$ and $2\hat{\mathbf{i}}-\hat{\mathbf{j}}-3\hat{\mathbf{k}}$

b. $\langle 2,5,6\rangle$ and $\langle -2,-4,4\rangle$

Solution

a. To find the cosine of the angle formed by the two vectors, substitute the components of the vectors into Equation 11.3.3:

$$\begin{aligned}\cos\theta &= \frac{(\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}})\cdot(2\hat{\mathbf{i}}-\hat{\mathbf{j}}-3\hat{\mathbf{k}})}{\|\,\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}\,\|\cdot\|\,2\hat{\mathbf{i}}-\hat{\mathbf{j}}-3\hat{\mathbf{k}}\,\|} \\ &= \frac{1(2)+(1)(-1)+(1)(-3)}{\sqrt{1^2+1^2+1^2}\sqrt{2^2+(-1)^2+(-3)^2}} \\ &= \frac{-2}{\sqrt{3}\sqrt{14}} = \frac{-2}{\sqrt{42}}.\end{aligned}$$

Therefore, $\theta = \arccos\dfrac{-2}{\sqrt{42}}$ rad.

b. Start by finding the value of the cosine of the angle between the vectors:

$$\begin{aligned}\cos\theta &= \frac{\langle 2,5,6\rangle\cdot\langle -2,-4,4\rangle}{\|\,\langle 2,5,6\rangle\,\|\cdot\|\,\langle -2,-4,4\rangle\,\|} \\ &= \frac{2(-2)+(5)(-4)+(6)(4)}{\sqrt{2^2+5^2+6^2}\sqrt{(-2)^2+(-4)^2+4^2}} \\ &= \frac{0}{\sqrt{65}\sqrt{36}} = 0.\end{aligned}$$

Now, $\cos\theta = 0$ and $0 \le \theta \le \pi$, so $\theta = \pi/2$.

Exercise 11.3.3

Find the measure of the angle, in radians, formed by vectors $\vec{\mathbf{a}} = \langle 1,2,0\rangle$ and $\vec{\mathbf{b}} = \langle 2,4,1\rangle$. Round to the nearest hundredth.

The angle between two vectors can be acute ($0 < \cos\theta < 1$), obtuse ($-1 < \cos\theta < 0$), or straight ($\cos\theta = -1$). If $\cos\theta = 1$, then both vectors have the same direction. If $\cos\theta = 0$, then the vectors, when placed in standard position, form a right angle (Figure 11.3.3). We can formalize this result into a theorem regarding orthogonal (perpendicular) vectors.

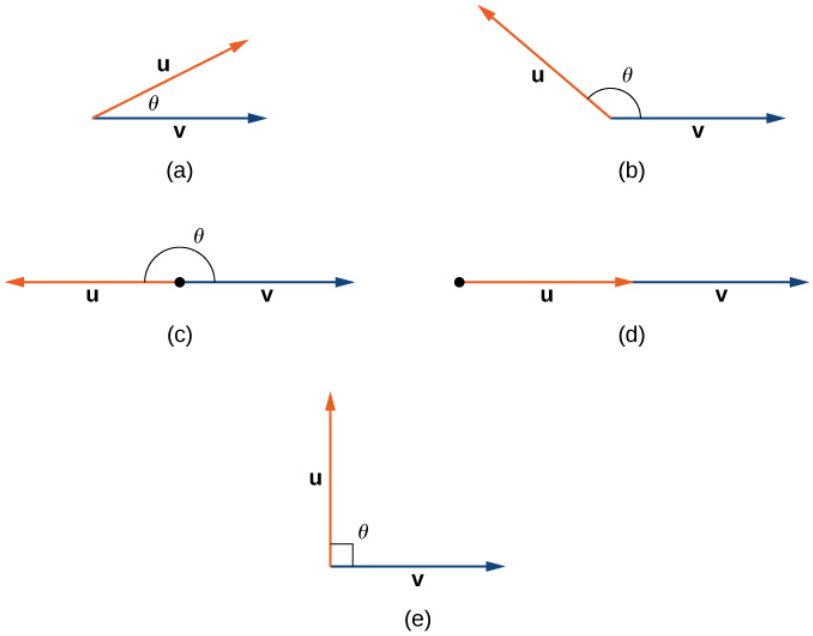


Figure 11.3.3: (a) An acute angle has $0 < \cos\theta < 1$. (b) An obtuse angle has $-1 < \cos\theta < 0$. (c) A straight line has $\cos\theta = -1$. (d) If the vectors have the same direction, $\cos\theta = 1$. (e) If the vectors are orthogonal (perpendicular), $\cos\theta = 0$.

Orthogonal Vectors

The nonzero vectors $\vec{u}$ and $\vec{v}$ are **orthogonal vectors** if and only if $\vec{u} \cdot \vec{v} = 0$.

Proof

Let $\vec{u}$ and $\vec{v}$ be nonzero vectors, and let θ denote the angle between them. First, assume $\vec{u} \cdot \vec{v} = 0$.Then

$$\|\vec{u}\|\|\vec{v}\| \cos\theta = 0.$$

However, $\|\vec{u}\| \neq 0$ and $\|\vec{v}\| \neq 0$, so we must have $\cos\theta = 0$. Hence, $\theta = 90°$, and the vectors are orthogonal.

Now assume $\vec{u}$ and $\vec{v}$ are orthogonal. Then $\theta = 90°$ and we have

$$\begin{aligned} \vec{u} \cdot \vec{v} &= \|\vec{u}\|\|\vec{v}\| \cos\theta \\ &= \|\vec{u}\|\|\vec{v}\| \cos 90° \\ &= \|\vec{u}\|\|\vec{v}\|(0) \\ &= 0. \end{aligned}$$

□

The terms **orthogonal, perpendicular,** and **normal** each indicate that mathematical objects are intersecting at right angles. The use of each term is determined mainly by its context. We say that vectors are orthogonal and lines are perpendicular. The term *normal* is used most often when measuring the angle made with a plane or other surface.

Example 11.3.4: Identifying Orthogonal Vectors

Determine whether $\vec{p} = \langle 1, 0, 5 \rangle$ and $\vec{q} = \langle 10, 3, -2 \rangle$ are orthogonal vectors.

Solution

Using the definition, we need only check the dot product of the vectors:

$$\vec{p} \cdot \vec{q} = 1(10) + (0)(3) + (5)(-2) = 10 + 0 - 10 = 0.$$

Because $\vec{p} \cdot \vec{q} = 0$, the vectors are orthogonal (Figure 11.3.4).

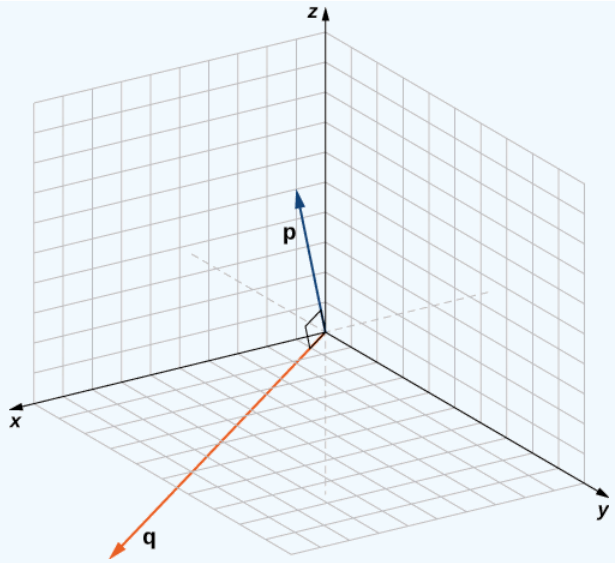


Figure 11.3.4: Vectors $\vec{\mathbf{p}}$ and $\vec{\mathbf{q}}$ form a right angle when their initial points are aligned.

Exercise 11.3.4

For which value of x is $\vec{\mathbf{p}} = \langle 2, 8, -1 \rangle$ orthogonal to $\vec{\mathbf{q}} = \langle x, -1, 2 \rangle$?

Example 11.3.5: Measuring the Angle Formed by Two Vectors

Let $\vec{\mathbf{v}} = \langle 2, 3, 3 \rangle$. Find the measures of the angles formed by the following vectors.

a. $\vec{\mathbf{v}}$ and $\hat{\mathbf{i}}$
b. $\vec{\mathbf{v}}$ and $\hat{\mathbf{j}}$
c. $\vec{\mathbf{v}}$ and $\hat{\mathbf{k}}$

Solution

a.Let α be the angle formed by $\vec{\mathbf{v}}$ and $\hat{\mathbf{i}}$:

$$\cos\alpha = \frac{\vec{\mathbf{v}} \cdot \hat{\mathbf{i}}}{\|\vec{\mathbf{v}}\| \cdot \|\hat{\mathbf{i}}\|} = \frac{\langle 2,3,3 \rangle \cdot \langle 1,0,0 \rangle}{\sqrt{2^2+3^2+3^2}\sqrt{1}} = \frac{2}{\sqrt{22}}$$

$$\alpha = \arccos \frac{2}{\sqrt{22}} \approx 1.130 \text{ rad.}$$

b. Let β represent the angle formed by $\vec{\mathbf{v}}$ and $\hat{\mathbf{j}}$:

$$\cos\beta = \frac{\vec{\mathbf{v}} \cdot \hat{\mathbf{j}}}{\|\vec{\mathbf{v}}\| \cdot \|\hat{\mathbf{j}}\|} = \frac{\langle 2,3,3 \rangle \cdot \langle 0,1,0 \rangle}{\sqrt{2^2+3^2+3^2}\sqrt{1}} = \frac{3}{\sqrt{22}}$$

$$\beta = \arccos \frac{3}{\sqrt{22}} \approx 0.877 \text{ rad.}$$

c. Let γ represent the angle formed by $\vec{\mathbf{v}}$ and $\hat{\mathbf{k}}$:

$$\cos\gamma = \frac{\vec{\mathbf{v}} \cdot \hat{\mathbf{k}}}{\|\vec{\mathbf{v}}\| \cdot \|\hat{\mathbf{k}}\|} = \frac{\langle 2,3,3 \rangle \cdot \langle 0,0,1 \rangle}{\sqrt{2^2+3^2+3^2}\sqrt{1}} = \frac{3}{\sqrt{22}}$$

$$\gamma = \arccos \frac{3}{\sqrt{22}} \approx 0.877 \text{ rad.}$$

Exercise 11.3.5

Let $\vec{\mathbf{v}} = \langle 3, -5, 1 \rangle$. Find the measure of the angles formed by each pair of vectors.

a. $\vec{\mathbf{v}}$ and $\hat{\mathbf{i}}$
b. $\vec{\mathbf{v}}$ and $\hat{\mathbf{j}}$
c. $\vec{\mathbf{v}}$ and $\hat{\mathbf{k}}$

The angle a vector makes with each of the coordinate axes, called a direction angle, is very important in practical computations, especially in a field such as engineering. For example, in astronautical engineering, the angle at which a rocket is launched must be determined very precisely. A very small error in the angle can lead to the rocket going hundreds of miles off course. Direction angles are often calculated by using the dot product and the cosines of the angles, called the direction cosines. Therefore, we define both these angles and their cosines.

Definition: direction angles

The angles formed by a nonzero vector and the coordinate axes are called the **direction angles** for the vector (Figure 11.3.5). The cosines for these angles are called the **direction cosines.**

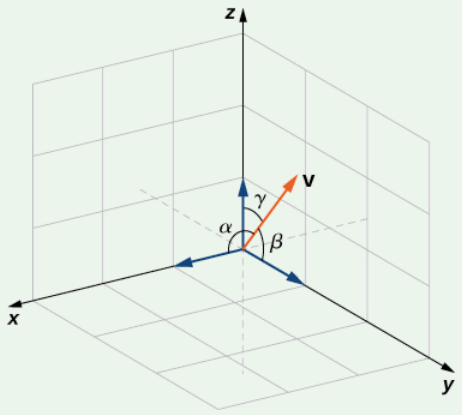


Figure 11.3.5: Angle α is formed by vector $\vec{\mathbf{v}}$ and unit vector $\hat{\mathbf{i}}$. Angle β is formed by vector $\vec{\mathbf{v}}$ and unit vector $\hat{\mathbf{j}}$. Angle γ is formed by vector $\vec{\mathbf{v}}$ and unit vector $\hat{\mathbf{k}}$.

In Example 11.3.5, the direction cosines of $\vec{\mathbf{v}} = \langle 2, 3, 3 \rangle$ are $\cos\alpha = \frac{2}{\sqrt{22}}, \cos\beta = \frac{3}{\sqrt{22}}$, and $\cos\gamma = \frac{3}{\sqrt{22}}$. The direction angles of $\vec{\mathbf{v}}$ are $\alpha = 1.130$ rad, $\beta = 0.877$ rad, and $\gamma = 0.877$ rad.

So far, we have focused mainly on vectors related to force, movement, and position in three-dimensional physical space. However, vectors are often used in more abstract ways. For example, suppose a fruit vendor sells apples, bananas, and oranges. On a given day, he sells 30 apples, 12 bananas, and 18 oranges. He might use a quantity vector, $\vec{\mathbf{q}} = \langle 30, 12, 18 \rangle$, to represent the quantity of fruit he sold that day. Similarly, he might want to use a price vector, $\vec{\mathbf{p}} = \langle 0.50, 0.25, 1 \rangle$, to indicate that he sells his apples for 50¢ each, bananas for 25¢ each, and oranges for $1 apiece. In this example, although we could still graph these vectors, we do not interpret them as literal representations of position in the physical world. We are simply using vectors to keep track of particular pieces of information about apples, bananas, and oranges.

This idea might seem a little strange, but if we simply regard vectors as a way to order and store data, we find they can be quite a powerful tool. Going back to the fruit vendor, let's think about the dot product, $\vec{\mathbf{q}} \cdot \vec{\mathbf{p}}$. We compute it by multiplying the number of apples sold (30) by the price per apple (50¢), the number of bananas sold by the price per banana, and the number of oranges sold by the price per orange. We then add all these values together. So, in this example, the dot product tells us how much money the fruit vendor had in sales on that particular day.

When we use vectors in this more general way, there is no reason to limit the number of components to three. What if the fruit vendor decides to start selling grapefruit? In that case, he would want to use four-dimensional quantity and price vectors to represent the number of apples, bananas, oranges, and grapefruit sold, and their unit prices. As you might expect, to calculate the

dot product of four-dimensional vectors, we simply add the products of the components as before, but the sum has four terms instead of three.

Example 11.3.6: Using Vectors in an Economic Context

AAA Party Supply Store sells invitations, party favors, decorations, and food service items such as paper plates and napkins. When AAA buys its inventory, it pays 25¢ per package for invitations and party favors. Decorations cost AAA 50¢ each, and food service items cost 20¢ per package. AAA sells invitations for $2.50 per package and party favors for $1.50 per package. Decorations sell for $4.50 each and food service items for $1.25 per package.

During the month of May, AAA Party Supply Store sells 1258 invitations, 342 party favors, 2426 decorations, and 1354 food service items. Use vectors and dot products to calculate how much money AAA made in sales during the month of May. How much did the store make in profit?

Solution

The cost, price, and quantity vectors are

$$\begin{aligned}\vec{\mathbf{c}} &= \langle 0.25, 0.25, 0.50, 0.20\rangle \\ \vec{\mathbf{p}} &= \langle 2.50, 1.50, 4.50, 1.25\rangle \\ \vec{\mathbf{q}} &= \langle 1258, 342, 2426, 1354\rangle.\end{aligned}$$

AAA sales for the month of May can be calculated using the dot product $\vec{\mathbf{p}} \cdot \vec{\mathbf{q}}$. We have

$$\begin{aligned}\vec{\mathbf{p}} \cdot \vec{\mathbf{q}} &= \langle 2.50, 1.50, 4.50, 1.25\rangle \cdot \langle 1258, 342, 2426, 1354\rangle \\ &= 3145 + 513 + 10917 + 1692.5 \\ &= 16267.5.\end{aligned}$$

So, AAA took in $16,267.50 during the month of May. To calculate the profit, we must first calculate how much AAA paid for the items sold. We use the dot product $\vec{\mathbf{c}} \cdot \vec{\mathbf{q}}$ to get

$$\begin{aligned}\vec{\mathbf{c}} \cdot \vec{\mathbf{q}} &= \langle 0.25, 0.25, 0.50, 0.20\rangle \cdot \langle 1258, 342, 2426, 1354\rangle \\ &= 314.5 + 85.5 + 1213 + 270.8 \\ &= 1883.8.\end{aligned}$$

So, AAA paid $1,883.80 for the items they sold. Their profit, then, is given by

$$\vec{\mathbf{p}} \cdot \vec{\mathbf{q}} - \vec{\mathbf{c}} \cdot \vec{\mathbf{q}} = 16267.5 - 1883.8 = 14383.7.$$

Therefore, AAA Party Supply Store made $14,383.70 in May.

Exercise 11.3.6

On June 1, AAA Party Supply Store decided to increase the price they charge for party favors to $2 per package. They also changed suppliers for their invitations, and are now able to purchase invitations for only 10¢ per package. All their other costs and prices remain the same. If AAA sells 1408 invitations, 147 party favors, 2112 decorations, and 1894 food service items in the month of June, use vectors and dot products to calculate their total sales and profit for June.

Projections

As we have seen, addition combines two vectors to create a resultant vector. But what if we are given a vector and we need to find its component parts? We use vector projections to perform the opposite process; they can break down a vector into its components. The magnitude of a vector projection is a scalar projection. For example, if a child is pulling the handle of a wagon at a 55° angle, we can use projections to determine how much of the force on the handle is actually moving the wagon forward (**11.3.6**). We return to this example and learn how to solve it after we see how to calculate projections.

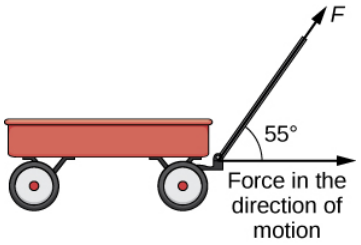


Figure **11.3.6**: When a child pulls a wagon, only the horizontal component of the force propels the wagon forward.

Definition: Vector and Projection

The **vector projection** of $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$ is the vector labeled $\text{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}$ in Figure **11.3.7**. It has the same initial point as $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ and the same direction as $\vec{\mathbf{u}}$, and represents the component of $\vec{\mathbf{v}}$ that acts in the direction of $\vec{\mathbf{u}}$. If θ represents the angle between $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$, then, by properties of triangles, we know the length of $\text{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}$ is $\|\text{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}\| = \|\vec{\mathbf{v}}\|\cos\theta$. When expressing $\cos\theta$ in terms of the dot product, this becomes

$$\|\text{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}\| = \|\vec{\mathbf{v}}\|\cos\theta = \|\vec{\mathbf{v}}\|\left(\frac{\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\|}\right) = \frac{\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|}.$$

We now multiply by a unit vector in the direction of $\vec{\mathbf{u}}$ to get $\text{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}$:

$$\text{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}} = \frac{\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|}\left(\frac{1}{\|\vec{\mathbf{u}}\|}\vec{\mathbf{u}}\right) = \frac{\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|^2}\vec{\mathbf{u}}.$$

The length of this vector is also known as the **scalar projection** of $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$ and is denoted by

$$\|\text{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}\| = \text{comp}_{\vec{\mathbf{u}}}\vec{\mathbf{v}} = \frac{\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|}.$$

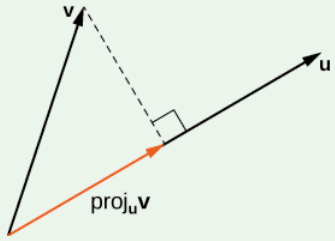


Figure **11.3.7**: The projection of $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$ shows the component of vector $\vec{\mathbf{v}}$ in the direction of $\vec{\mathbf{u}}$.

Example 11.3.7: Finding Projections

Find the projection of $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$.

a. $\vec{\mathbf{v}} = \langle 3, 5, 1\rangle$ and $\vec{\mathbf{u}} = \langle -1, 4, 3\rangle$
b. $\vec{\mathbf{v}} = 3\hat{\mathbf{i}} - 2\hat{\mathbf{j}}$ and $\vec{\mathbf{u}} = \hat{\mathbf{i}} + 6\hat{\mathbf{j}}$

Solution

a. Substitute the components of $\vec{\mathbf{v}}$ and $\vec{\mathbf{u}}$ into the formula for the projection:

$$\begin{aligned}\text{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}} &= \frac{\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|^2}\vec{\mathbf{u}} \\ &= \frac{\langle -1,4,3\rangle\cdot\langle 3,5,1\rangle}{\|\langle -1,4,3\rangle\|^2}\langle -1,4,3\rangle \\ &= \frac{-3+20+3}{(-1)^2+4^2+3^2}\langle -1,4,3\rangle \\ &= \frac{20}{26}\langle -1,4,3\rangle \\ &= \langle -\frac{10}{13},\frac{40}{13},\frac{30}{13}\rangle.\end{aligned}$$

b. To find the two-dimensional projection, simply adapt the formula to the two-dimensional case:

$$\begin{aligned}\text{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}} &= \frac{\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|^2}\vec{\mathbf{u}} \\ &= \frac{(\hat{\mathbf{i}}+6\hat{\mathbf{j}})\cdot(3\hat{\mathbf{i}}-2\hat{\mathbf{j}})}{\|\hat{\mathbf{i}}+6\hat{\mathbf{j}}\|^2}(\hat{\mathbf{i}}+6\hat{\mathbf{j}}) \\ &= \frac{1(3)+6(-2)}{1^2+6^2}(\hat{\mathbf{i}}+6\hat{\mathbf{j}}) \\ &= -\frac{9}{37}(\hat{\mathbf{i}}+6\hat{\mathbf{j}}) \\ &= -\frac{9}{37}\hat{\mathbf{i}}-\frac{54}{37}\hat{\mathbf{j}}.\end{aligned}$$

Sometimes it is useful to decompose vectors—that is, to break a vector apart into a sum. This process is called the **resolution of a vector into components.** Projections allow us to identify two orthogonal vectors having a desired sum. For example, let $\vec{\mathbf{v}} = \langle 6,-4\rangle$ and let $\vec{\mathbf{u}} = \langle 3,1\rangle$. We want to decompose the vector $\vec{\mathbf{v}}$ into orthogonal components such that one of the component vectors has the same direction as $\vec{\mathbf{u}}$.

We first find the component that has the same direction as $\vec{\mathbf{u}}$ by projecting $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$. Let $\vec{\mathbf{p}} = \text{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}$. Then, we have

$$\begin{aligned}\vec{\mathbf{p}} &= \frac{\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|^2}\vec{\mathbf{u}} \\ &= \frac{18-4}{9+1}\vec{\mathbf{u}} \\ = \frac{7}{5}\vec{\mathbf{u}} = \frac{7}{5}\langle 3,1\rangle &= \langle \frac{21}{5},\frac{7}{5}\rangle.\end{aligned}$$

Now consider the vector $\vec{\mathbf{q}} = \vec{\mathbf{v}} - \vec{\mathbf{p}}$. We have

$$\begin{aligned}\vec{\mathbf{q}} &= \vec{\mathbf{v}} - \vec{\mathbf{p}} \\ &= \langle 6,-4\rangle - \langle \frac{21}{5},\frac{7}{5}\rangle \\ &= \langle \frac{9}{5},-\frac{27}{5}\rangle.\end{aligned}$$

Clearly, by the way we defined $\vec{\mathbf{q}}$, we have $\vec{\mathbf{v}} = \vec{\mathbf{q}} + \vec{\mathbf{p}}$, and

$$\begin{aligned}\vec{\mathbf{q}}\cdot\vec{\mathbf{p}} &= \langle \frac{9}{5},-\frac{27}{5}\rangle\cdot\langle \frac{21}{5},\frac{7}{5}\rangle \\ &= \frac{9(21)}{25} + -\frac{27(7)}{25} \\ &= \frac{189}{25} - \frac{189}{25} = 0.\end{aligned}$$

Therefore, $\vec{\mathbf{q}}$ and $\vec{\mathbf{p}}$ are orthogonal.

Example 11.3.8: Resolving Vectors into Components

Express $\vec{\mathbf{v}} = \langle 8, -3, -3 \rangle$ as a sum of orthogonal vectors such that one of the vectors has the same direction as $\vec{\mathbf{u}} = \langle 2, 3, 2 \rangle$.

Solution

Let $\vec{\mathbf{p}}$ represent the projection of $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$:

$$\begin{aligned}\vec{\mathbf{p}} &= \text{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}} \\ &= \frac{\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|^2}\vec{\mathbf{u}} \\ &= \frac{\langle 2,3,2\rangle \cdot \langle 8,-3,-3\rangle}{\| \langle 2,3,2\rangle \|^2}\langle 2,3,2\rangle \\ &= \frac{16-9-6}{2^2+3^2+2^2}\langle 2,3,2\rangle \\ &= \frac{1}{17}\langle 2,3,2\rangle \\ &= \langle \frac{2}{17}, \frac{3}{17}, \frac{2}{17}\rangle.\end{aligned}$$

Then,

$$\begin{aligned}\vec{\mathbf{q}} &= \vec{\mathbf{v}} - \vec{\mathbf{p}} = \langle 8,-3,-3\rangle - \langle \frac{2}{17}, \frac{3}{17}, \frac{2}{17}\rangle \\ &= \langle \frac{134}{17}, -\frac{54}{17}, -\frac{53}{17}\rangle.\end{aligned}$$

To check our work, we can use the dot product to verify that $\vec{\mathbf{p}}$ and $\vec{\mathbf{q}}$ are orthogonal vectors:

$$\begin{aligned}\vec{\mathbf{p}} \cdot \vec{\mathbf{q}} &= \langle \frac{2}{17}, \frac{3}{17}, \frac{2}{17}\rangle \cdot \langle \frac{134}{17}, -\frac{54}{17}, -\frac{53}{17}\rangle \\ &= \frac{268}{289} - \frac{162}{289} - \frac{106}{289} = 0.\end{aligned}$$

Then,

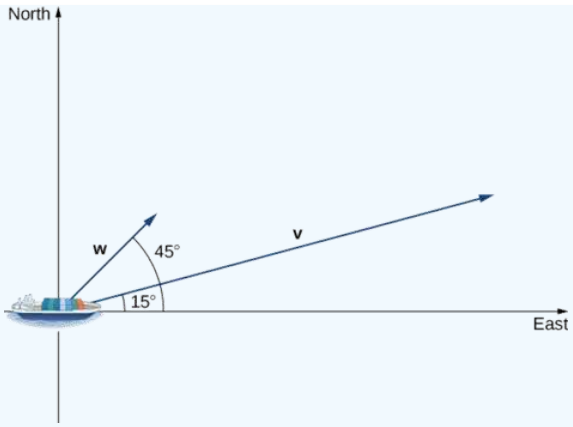
$$\vec{\mathbf{v}} = \vec{\mathbf{p}} + \vec{\mathbf{q}} = \langle \frac{2}{17}, \frac{3}{17}, \frac{2}{17}\rangle + \langle \frac{134}{17}, -\frac{54}{17}, -\frac{53}{17}\rangle.$$

Exercise 11.3.7

Express $\vec{\mathbf{v}} = 5\hat{\mathbf{i}} - \hat{\mathbf{j}}$ as a sum of orthogonal vectors such that one of the vectors has the same direction as $\vec{\mathbf{u}} = 4\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$.

Example 11.3.9: Scalar Projection of Velocity

A container ship leaves port traveling $15°$ north of east. Its engine generates a speed of 20 knots along that path (see the following figure). In addition, the ocean current moves the ship northeast at a speed of 2 knots. Considering both the engine and the current, how fast is the ship moving in the direction $15°$ north of east? Round the answer to two decimal places.



Solution

Let $\vec{\mathbf{v}}$ be the velocity vector generated by the engine, and let $\vec{\mathbf{w}}$ be the velocity vector of the current. We already know $\|\vec{\mathbf{v}}\| = 20$ along the desired route. We just need to add in the scalar projection of $\vec{\mathbf{w}}$ onto $\vec{\mathbf{v}}$. We get

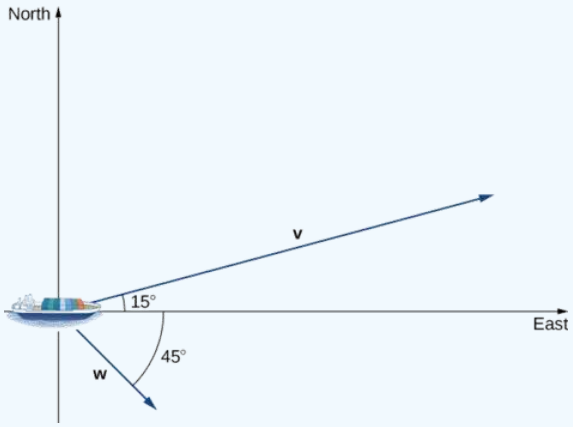
$$\text{comp}_{\vec{\mathbf{v}}}\vec{\mathbf{w}} = \frac{\vec{\mathbf{v}} \cdot \vec{\mathbf{w}}}{\|\vec{\mathbf{v}}\|}$$

$$= \frac{\|\vec{\mathbf{v}}\|\|\vec{\mathbf{w}}\| \cos(30°)}{\|\vec{\mathbf{v}}\|} = \|\vec{\mathbf{w}}\| \cos(30°) = 2\frac{\sqrt{3}}{2} = \sqrt{3} \approx 1.73 \text{ knots.}$$

The ship is moving at 21.73 knots in the direction $15°$ north of east.

? Exercise 11.3.8

Repeat the previous example, but assume the ocean current is moving southeast instead of northeast, as shown in the following figure.



Work

Now that we understand dot products, we can see how to apply them to real-life situations. The most common application of the dot product of two vectors is in the calculation of work.

From physics, we know that work is done when an object is moved by a force. When the force is constant and applied in the same direction the object moves, then we define the work done as the product of the force and the distance the object travels: $W = Fd$. We saw several examples of this type in earlier chapters. Now imagine the direction of the force is different from the direction of motion, as with the example of a child pulling a wagon. To find the work done, we need to multiply the component of the force that acts in the direction of the motion by the magnitude of the displacement. The dot product allows us to do just that. If we represent

an applied force by a vector $\vec{\mathbf{F}}$ and the displacement of an object by a vector $\vec{\mathbf{s}}$, then the **work done by the force** is the dot product of $\vec{\mathbf{F}}$ and $\vec{\mathbf{s}}$.

Definition: Constant Force

When a constant force is applied to an object so the object moves in a straight line from point P to point Q, the work W done by the force $\vec{\mathbf{F}}$, acting at an angle θ from the line of motion, is given by

$$W = \vec{\mathbf{F}} \cdot \overrightarrow{PQ} = \| \vec{\mathbf{F}} \| \| \overrightarrow{PQ} \| \cos\theta.$$

Let's revisit the problem of the child's wagon introduced earlier. Suppose a child is pulling a wagon with a force having a magnitude of 8 lb on the handle at an angle of 55°. If the child pulls the wagon 50 ft, find the work done by the force (Figure 11.3.8).

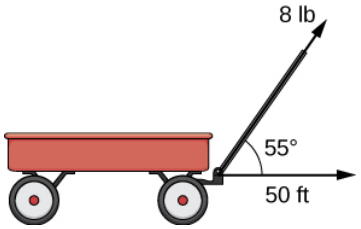


Figure 11.3.8: The horizontal component of the force is the projection of $\vec{\mathbf{F}}$ onto the positive x-axis.

We have

$$W = \| \vec{\mathbf{F}} \| \| \overrightarrow{PQ} \| \cos\theta = 8(50)(\cos(55°)) \approx 229\,\text{ft·lb}.$$

In U.S. standard units, we measure the magnitude of force $\| \vec{\mathbf{F}} \|$ in pounds. The magnitude of the displacement vector $\| \overrightarrow{PQ} \|$ tells us how far the object moved, and it is measured in feet. The customary unit of measure for work, then, is the foot-pound. One foot-pound is the amount of work required to move an object weighing 1 lb a distance of 1 ft straight up. In the metric system, the unit of measure for force is the newton (N), and the unit of measure of magnitude for work is a newton-meter (N·m), or a joule (J).

Example 11.3.10: Calculating Work

A conveyor belt generates a force $\vec{\mathbf{F}} = 5\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + \hat{\mathbf{k}}$ that moves a suitcase from point $(1, 1, 1)$ to point $(9, 4, 7)$ along a straight line. Find the work done by the conveyor belt. The distance is measured in meters and the force is measured in newtons.

Solution

The displacement vector $\overrightarrow{PQ}$ has initial point $(1, 1, 1)$ and terminal point $(9, 4, 7)$:

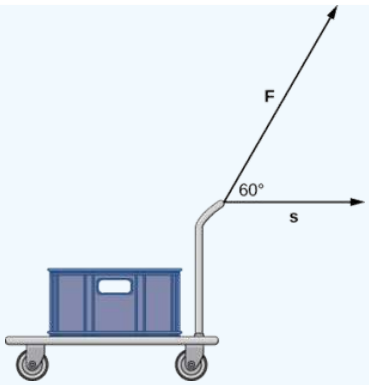
$$\overrightarrow{PQ} = \langle 9-1, 4-1, 7-1 \rangle = \langle 8, 3, 6 \rangle = 8\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}}.$$

Work is the dot product of force and displacement:

$$\begin{aligned} W &= \vec{\mathbf{F}} \cdot \overrightarrow{PQ} \\ &= (5\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + \hat{\mathbf{k}}) \cdot (8\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}}) \\ &= 5(8) + (-3)(3) + 1(6) \\ &= 37\,\text{N·m} \\ &= 37\,\text{J} \end{aligned}$$

Exercise 11.3.9

A constant force of 30 lb is applied at an angle of 60° to pull a handcart 10 ft across the ground. What is the work done by this force?



Key Concepts

- The dot product, or scalar product, of two vectors $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$ is $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = u_1 v_1 + u_2 v_2 + u_3 v_3$.
- The dot product satisfies the following properties:
  - $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = \vec{\mathbf{v}} \cdot \vec{\mathbf{u}}$
  - $\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} + \vec{\mathbf{w}}) = \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} + \vec{\mathbf{u}} \cdot \vec{\mathbf{w}}$
  - $c(\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}) = (c\vec{\mathbf{u}}) \cdot \vec{\mathbf{v}} = \vec{\mathbf{u}} \cdot (c\vec{\mathbf{v}})$
  - $\vec{\mathbf{v}} \cdot \vec{\mathbf{v}} = \|\vec{\mathbf{v}}\|^2$
- The dot product of two vectors can be expressed, alternatively, as $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = \|\vec{\mathbf{u}}\| \|\vec{\mathbf{v}}\| \cos\theta$. This form of the dot product is useful for finding the measure of the angle formed by two vectors.
- Vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are orthogonal if $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = 0$.
- The angles formed by a nonzero vector and the coordinate axes are called the *direction angles* for the vector. The cosines of these angles are known as the *direction cosines.*
- The vector projection of $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$ is the vector $\text{proj}_{\vec{\mathbf{u}}} \vec{\mathbf{v}} = \dfrac{\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|^2} \vec{\mathbf{u}}$. The magnitude of this vector is known as the *scalar projection* of $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$, given by $\text{comp}_{\vec{\mathbf{u}}} \vec{\mathbf{v}} = \dfrac{\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|}$.
- Work is done when a force is applied to an object, causing displacement. When the force is represented by the vector $\vec{\mathbf{F}}$ and the displacement is represented by the vector $\vec{\mathbf{s}}$, then the work done W is given by the formula $W = \vec{\mathbf{F}} \cdot \vec{\mathbf{s}} = \| \vec{\mathbf{F}} \| \, \| \vec{\mathbf{s}} \| \cos\theta$.

Key Equations

- **Dot product of $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$**

  $$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = u_1 v_1 + u_2 v_2 + u_3 v_3 = \|\vec{\mathbf{u}}\| \|\vec{\mathbf{v}}\| \cos\theta$$

- **Cosine of the angle formed by $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$**

  $$\cos\theta = \frac{\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\| \|\vec{\mathbf{v}}\|}$$

- **Vector projection of $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$**

  $$\text{proj}_{\vec{\mathbf{u}}} \vec{\mathbf{v}} = \frac{\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|^2} \vec{\mathbf{u}}$$

- **Scalar projection of $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$**

  $$\text{comp}_{\vec{\mathbf{u}}} \vec{\mathbf{v}} = \frac{\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|}$$

- **Work done by a force $\vec{\mathbf{F}}$ to move an object through displacement vector $\overrightarrow{PQ}$**

  $$W = \vec{\mathbf{F}} \cdot \overrightarrow{PQ} = \| \vec{\mathbf{F}} \| \, \| \overrightarrow{PQ} \| \cos\theta$$

Glossary

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11.3E: Exercises for The Dot Product

For exercises 1-4, the vectors $\vec{u}$ and $\vec{v}$ are given. Calculate the dot product $\vec{u} \cdot \vec{v}$.

1) $\vec{u} = \langle 3, 0 \rangle, \quad \vec{v} = \langle 2, 2 \rangle$

2) $\vec{u} = \langle 3, -4 \rangle, \quad \vec{v} = \langle 4, 3 \rangle$

3) $\vec{u} = \langle 2, 2, -1 \rangle, \quad \vec{v} = \langle -1, 2, 2 \rangle$

4) $\vec{u} = \langle 4, 5, -6 \rangle, \quad \vec{v} = \langle 0, -2, -3 \rangle$

For exercises 5-8, the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are given. Determine the vectors $(\vec{a} \cdot \vec{b})\vec{c}$ and $(\vec{a} \cdot \vec{c})\vec{b}$. Express the vectors in component form.

5) $\vec{a} = \langle 2, 0, -3 \rangle, \quad \vec{b} = \langle -4, -7, 1 \rangle, \quad \vec{c} = \langle 1, 1, -1 \rangle$

6) $\vec{a} = \langle 0, 1, 2 \rangle, \quad \vec{b} = \langle -1, 0, 1 \rangle, \quad \vec{c} = \langle 1, 0, -1 \rangle$

7) $\vec{a} = \hat{i} + \hat{j}, \quad \vec{b} = \hat{i} - \hat{k}, \quad \vec{c} = \hat{i} - 2\hat{k}$

8) $\vec{a} = \hat{i} - \hat{j} + \hat{k}, \quad \vec{b} = \hat{j} + 3\hat{k}, \quad \vec{c} = -\hat{i} + 2\hat{j} - 4\hat{k}$

For exercises 9-12, two vectors are given.

a. Find the measure of the angle θ between these two vectors. Express the answer in radians rounded to two decimal places, if it is not possible to express it exactly.

b. Is θ an acute angle?

9) [T] $\vec{a} = \langle 3, -1 \rangle, \quad \vec{b} = \langle -4, 0 \rangle$

10) [T] $\vec{a} = \langle 2, 1 \rangle, \quad \vec{b} = \langle -1, 3 \rangle$

11) $\vec{u} = 3\hat{i}, \quad \vec{v} = 4\hat{i} + 4\hat{j}$

12) $\vec{u} = 5\hat{i}, \quad \vec{v} = -6\hat{i} + 6\hat{j}$

For exercises 13-18, find the measure of the angle between the three-dimensional vectors $\vec{a}$ and $\vec{b}$. Express the answer in radians rounded to two decimal places, if it is not possible to express it exactly.

13) $\vec{a} = \langle 3, -1, 2 \rangle, \quad \vec{b} = \langle 1, -1, -2 \rangle$

14) $\vec{a} = \langle 0, -1, -3 \rangle, \quad \vec{b} = \langle 2, 3, -1 \rangle$

15) $\vec{a} = \hat{i} + \hat{j}, \quad \vec{b} = \hat{j} - \hat{k}$

16) $\vec{a} = \hat{i} - 2\hat{j} + \hat{k}, \quad \vec{b} = \hat{i} + \hat{j} - 2\hat{k}$

17) [T] $\vec{a} = 3\hat{i} - \hat{j} - 2\hat{k}, \quad \vec{b} = \vec{v} + \vec{w}$, where $\vec{v} = -2\hat{i} - 3\hat{j} + 2\hat{k}$ and $\vec{w} = \hat{i} + 2\hat{k}$

18) [T] $\vec{a} = 3\hat{i} - \hat{j} + 2\hat{k}, \quad \vec{b} = \vec{v} - \vec{w}$, where $\vec{v} = 2\hat{i} + \hat{j} + 4\hat{k}$ and $\vec{w} = 6\hat{i} + \hat{j} + 2\hat{k}$

For exercises 19-22, determine whether the given vectors are orthogonal.

19) $\vec{a} = \langle x, y \rangle, \quad \vec{b} = \langle -y, x \rangle$, where x and y are nonzero real numbers

20) $\vec{a} = \langle x, x \rangle, \quad \vec{b} = \langle -y, y \rangle$, where x and y are nonzero real numbers

21) $\vec{a} = 3\hat{i} - \hat{j} - 2\hat{k}, \quad \vec{b} = -2\hat{i} - 3\hat{j} + \hat{k}$

22) $\vec{a} = \hat{i} - \hat{j}, \quad \vec{b} = 7\hat{i} + 2\hat{j} - \hat{k}$

23) Find all two-dimensional vectors $\vec{a}$ orthogonal to vector $\vec{b} = \langle 3, 4 \rangle$. Express the answer in component form.

24) Find all two-dimensional vectors $\vec{a}$ orthogonal to vector $\vec{b} = \langle 5, -6 \rangle$. Express the answer by using standard unit vectors.

25) Determine all three-dimensional vectors $\vec{u}$ orthogonal to vector $\vec{v} = \langle 1, 1, 0 \rangle$. Express the answer by using standard unit vectors.

26) Determine all three-dimensional vectors $\vec{u}$ orthogonal to vector $\vec{v} = \hat{i} - \hat{j} - \hat{k}$. Express the answer in component form.

27) Determine the real number α such that vectors $\vec{a} = 2\hat{i} + 3\hat{j}$ and $\vec{b} = 9\hat{i} + \alpha\hat{j}$ are orthogonal.

28) Determine the real number α such that vectors $\vec{a} = -3\hat{i} + 2\hat{j}$ and $\vec{b} = 2\hat{i} + \alpha\hat{j}$ are orthogonal.

29) [T] Consider the points $P(4, 5)$ and $Q(5, -7)$, and note that O represents the origin.

a. Determine vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$. Express the answer by using standard unit vectors.

b. Determine the measure of angle O in triangle OPQ. Express the answer in degrees rounded to two decimal places.

30) [T] Consider points $A(1, 1), B(2, -7)$, and $C(6, 3)$.

a. Determine vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$. Express the answer in component form.

b. Determine the measure of angle B in triangle ABC. Express the answer in degrees rounded to two decimal places.

31) Determine the measure of angle A in triangle ABC, where $A(1, 1, 8), B(4, -3, -4)$, and $C(-3, 1, 5)$. Express your answer in degrees rounded to two decimal places.

32) Consider points $P(3, 7, -2)$ and $Q(1, 1, -3)$. Determine the angle between vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$. [Remember that O represents the origin.] Express the answer in degrees rounded to two decimal places.

For exercises 33-34, determine which (if any) pairs of the following vectors are orthogonal.

33) $\vec{u} = \langle 3, 7, -2 \rangle$, $\vec{v} = \langle 5, -3, -3 \rangle$, $\vec{w} = \langle 0, 1, -1 \rangle$

34) $\vec{u} = \hat{i} - \hat{k}$, $\vec{v} = 5\hat{j} - 5\hat{k}$, $\vec{w} = 10\hat{j}$

35) Use vectors to show that a parallelogram with equal diagonals is a **rectangle**.

36) Use vectors to show that the diagonals of a **rhombus** are perpendicular.

37) Show that $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$ is true for any vectors $\vec{u}, \vec{v}$, and $\vec{w}$.

38) Verify the identity $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$ for vectors $\vec{u} = \langle 1, 0, 4 \rangle, \vec{v} = \langle -2, 3, 5 \rangle$, and $\vec{w} = \langle 4, -2, 6 \rangle$.

For exercises 39-41, determine $\vec{u} \cdot \vec{v}$ using the given information.

39) $\|\vec{u}\| = 5$, $\|\vec{v}\| = 3$, and the angle between $\vec{u}$ and $\vec{v}$ is $\pi/6$ rad.

40) $\|\vec{u}\| = 20$, $\|\vec{v}\| = 15$, and the angle between $\vec{u}$ and $\vec{v}$ is $5\pi/4$ rad.

41) $\|\vec{u}\| = 7$, $\|\vec{v}\| = 12$, and the angle between $\vec{u}$ and $\vec{v}$ is $\pi/2$ rad.

42) Considering the definition of the dot product, $\vec{u} \cdot \vec{v} = \|\vec{u}\|\|\vec{v}\| \cos\theta$, where θ is the angle between $\vec{u}$ and $\vec{v}$, what can you say about the angle θ between two nonzero vectors $\vec{u}$ and $\vec{v}$ if:

a. $\vec{u} \cdot \vec{v} > 0$?

b. $\vec{u} \cdot \vec{v} < 0$?

c. $\vec{u} \cdot \vec{v} = 0$?

In exercises 43-45, you are given the vertices of a triangle. Use dot products to determine whether each triangle is *acute*, *obtuse*, or *right*.

43) $(2,3,0)$, $(3,1,-2)$, $(-1,4,5)$

44) $(5,1,0)$, $(7,1,1)$, $(6,3,2)$

45) $(6,1,4)$, $(3,2,-1)$, $(2,3,1)$

46) Which of the following operations are allowed for nonzero vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$, and which are not. Explain your reasoning.

a. $\vec{u}+(\vec{v}\cdot\vec{w})$ b. $(\vec{u}\cdot\vec{v})\cdot\vec{w}$ c. $(\vec{u}\cdot\vec{v})\,\vec{w}$
d. $(\vec{u}+\vec{v})\cdot\vec{w}$ e. $(\vec{u}+\vec{v})\cdot\|\vec{w}\|$ f. $\|\vec{u}+\vec{v}\|\|\vec{w}\|$

Projections

For the exercises 47-50, given the vectors $\vec{u}$ and $\vec{v}$:

a. Find the vector projection $\text{Proj}_{\vec{u}}\vec{v}$ of vector $\vec{v}$ onto vector $\vec{u}$ and the component of $\vec{v}$ that is orthogonal to $\vec{u}$, i.e., $\vec{v}_{\text{perp}}$. Express your answers in component form.

b. Find the scalar projection $\text{comp}_{\vec{u}}\vec{v}=\|\text{Proj}_{\vec{u}}\vec{v}\|$ of vector $\vec{v}$ onto vector $\vec{u}$.

c. Find the vector projection $\text{Proj}_{\vec{v}}\vec{u}$ of vector $\vec{u}$ onto vector $\vec{v}$ and the component of $\vec{u}$ that is orthogonal to $\vec{v}$, i.e., $\vec{u}_{\text{perp}}$. Express your answers in unit vector form.

d. Find the scalar projection $\text{comp}_{\vec{v}}\vec{u}$ of vector $\vec{u}$ onto vector $\vec{v}$.

47) $\vec{u}=5\hat{i}+2\hat{j}$, $\vec{v}=2\hat{i}+3\hat{j}$

48) $\vec{u}=\langle -4,7\rangle$, $\vec{v}=\langle 3,5\rangle$

49) $\vec{u}=3\hat{i}+2\hat{k}$, $\vec{v}=2\hat{j}+4\hat{k}$

50) $\vec{u}=\langle 4,4,0\rangle$, $\vec{v}=\langle 0,4,1\rangle$

51) Consider the vectors $\vec{u}=4\hat{i}-3\hat{j}$ and $\vec{v}=3\hat{i}+2\hat{j}$.

a. Find the component form of vector $\text{Proj}_{\vec{u}}\vec{v}$ that represents the projection of $\vec{v}$ onto $\vec{u}$.

b. Write the decomposition $\vec{v}=\vec{w}+\vec{q}$ of vector $\vec{v}$ into the orthogonal components $\vec{w}$ and $\vec{q}$, where $\vec{w}$ is the projection of $\vec{v}$ onto $\vec{u}$ and $\vec{q}$ is the vector component of $\vec{v}$ orthogonal to the direction of $\vec{u}$. That is, $\vec{q}=\vec{v}_{\text{perp}}$.

52) Consider vectors $\vec{u}=2\hat{i}+4\hat{j}$ and $\vec{v}=4\hat{j}+2\hat{k}$.

a. Find the component form of vector $\vec{w}=\text{Proj}_{\vec{u}}\vec{v}$ that represents the projection of $\vec{v}$ onto $\vec{u}$.

b. Write the decomposition $\vec{v}=\vec{w}+\vec{q}$ of vector $\vec{v}$ into the orthogonal components $\vec{w}$ and $\vec{q}$, where $\vec{w}$ is the projection of $\vec{v}$ onto $\vec{u}$ and $\vec{q}$ is a vector orthogonal to the direction of $\vec{u}$.

53) A $50,000$-pound truck is parked on a hill with a $5°$ slope above the horizontal (in the positive x-direction). Considering only the force due to gravity, find

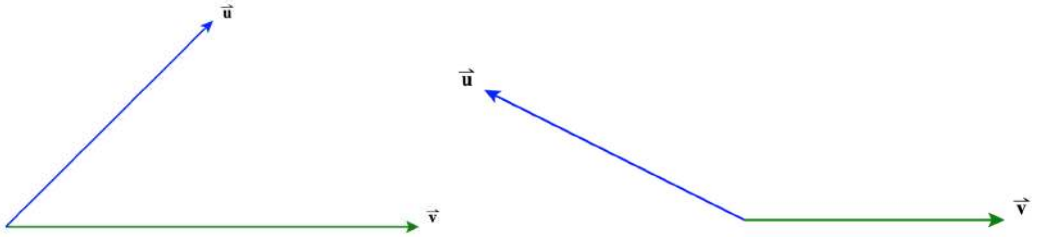
a. the component of the weight of the truck that is pulling the truck down the hill (aligned with the road)

b. the component of the weight of the truck that is perpendicular to the road.

c. the magnitude of the force needed to keep the truck from rolling down the hill

d. the magnitude of the force that is perpendicular to the road.

54) Given the vectors $\vec{u}$ and $\vec{v}$ shown in each diagram below, draw in $\text{Proj}_{\vec{v}}\vec{u}$ and $\vec{u}_{\text{perp}}$.

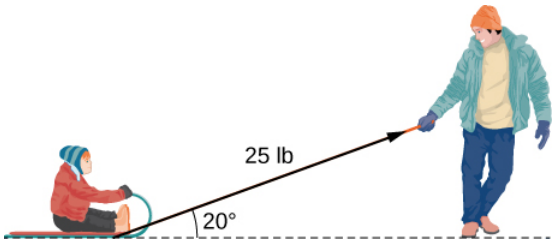


Work

55) Find the work done by force $\vec{\mathbf{F}} = \langle 5, 6, -2 \rangle$ (measured in Newtons) that moves a particle from point $P(3, -1, 0)$ to point $Q(2, 3, 1)$ along a straight line (the distance is measured in meters).

56) [T] A sled is pulled by exerting a force of 100 N on a rope that makes an angle of $25°$ with the horizontal. Find the work done in pulling the sled 40 m. (Round the answer to one decimal place.)

57) [T] A father is pulling his son on a sled at an angle of $20°$ with the horizontal with a force of 25 lb (see the following image). He pulls the sled in a straight path of 50 ft. How much work was done by the man pulling the sled? (Round the answer to the nearest integer.)



58) [T] A car is towed using a force of 1600 N. The rope used to pull the car makes an angle of 25° with the horizontal. Find the work done in towing the car 2 km. Express the answer in joules (1 J$= 1$ N·m) rounded to the nearest integer.

59) [T] A boat sails north aided by a wind blowing in a direction of $N30°E$ with a magnitude of 500 lb. How much work is performed by the wind as the boat moves 100 ft? (Round the answer to two decimal places.)

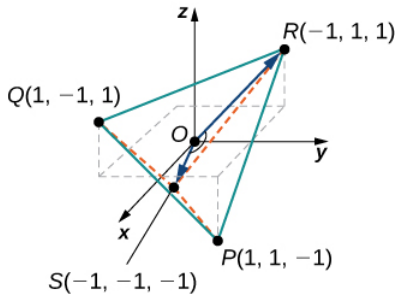
Other Applications of the Dot Product

60) [T] Find the vectors that join the center of a clock to the hours 1:00, 2:00, and 3:00. Assume the clock is circular with a radius of 1 unit.

61) A methane molecule has a carbon atom situated at the origin and four hydrogen atoms located at points $P(1, 1, -1), Q(1, -1, 1), R(-1, 1, 1)$, and $S(-1, -1, -1)$ (see figure).

a. Find the distance between the hydrogen atoms located at $P$and R.

b. Find the angle between vectors $\overrightarrow{OS}$ and $\overrightarrow{OR}$ that connect the carbon atom with the hydrogen atoms located at S and R, which is also called the *bond angle*. Express the answer in degrees rounded to two decimal places.



62) Vector $\vec{\mathbf{p}} = \langle 150, 225, 375 \rangle$ represents the price of certain models of bicycles sold by a bicycle shop. Vector $\vec{\mathbf{n}} = \langle 10, 7, 9 \rangle$ represents the number of bicycles sold of each model, respectively. Compute the dot product $\vec{\mathbf{p}} \cdot \vec{\mathbf{n}}$ and state its meaning.

63) [T] Two forces $\vec{\mathbf{F}}_1$ and $\vec{\mathbf{F}}_2$ are represented by vectors with initial points that are at the origin. The first force has a magnitude of 20 lb and passes through the point $P(1,1,0)$. The second force has a magnitude of 40 lb and passes through the point $Q(0,1,1)$. Let $\vec{\mathbf{F}}$ be the resultant force of forces $\vec{\mathbf{F}}_1$ and $\vec{\mathbf{F}}_2$.

a. Find the magnitude of $\vec{\mathbf{F}}$. (Round the answer to one decimal place.)

b. Find the direction angles of $\vec{\mathbf{F}}$. (Express the answer in degrees rounded to one decimal place.)

64) [T] Consider $\vec{\mathbf{r}}(t) = \langle \cos t, \sin t, 2t \rangle$ the position vector of a particle at time $t \in [0, 30]$, where the components of $\vec{\mathbf{r}}$ are expressed in centimeters and time in seconds. Let $\overrightarrow{OP}$ be the position vector of the particle after 1 sec.

a. Show that all vectors $\overrightarrow{PQ}$, where $Q(x,y,z)$ is an arbitrary point, orthogonal to the instantaneous velocity vector $\vec{\mathbf{v}}(1)$ of the particle after 1 sec, can be expressed as $\overrightarrow{PQ} = \langle x - \cos 1, y - \sin 1, z - 2 \rangle$, where $x \sin 1 - y \cos 1 - 2z + 4 = 0$. The set of point Q describes a plane called the *normal plane* to the path of the particle at point P.

b. Use a CAS to visualize the instantaneous velocity vector and the normal plane at point P along with the path of the particle.

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Exercises and LaTeX edited by Paul Seeburger. Problems 39-46 and 53 & 54 were created by Paul Seeburger.
Solution to Problem 59 also was added by Paul Seeburger.

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11.4: The Cross Product

Learning Objectives

- Calculate the cross product of two given vectors.
- Use determinants to calculate a cross product.
- Find a vector orthogonal to two given vectors.
- Determine areas and volumes by using the cross product.
- Calculate the torque of a given force and position vector.

Imagine a mechanic turning a wrench to tighten a bolt. The mechanic applies a force at the end of the wrench. This creates rotation, or torque, which tightens the bolt. We can use vectors to represent the force applied by the mechanic, and the distance (radius) from the bolt to the end of the wrench. Then, we can represent torque by a vector oriented along the axis of rotation. Note that the torque vector is orthogonal to both the force vector and the radius vector.

In this section, we develop an operation called the **cross product**, which allows us to find a vector orthogonal to two given vectors. Calculating torque is an important application of cross products, and we examine torque in more detail later in the section.

The Cross Product and Its Properties

The dot product is a multiplication of two vectors that results in a scalar. In this section, we introduce a product of two vectors that generates a third vector orthogonal to the first two. Consider how we might find such a vector. Let $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$ be nonzero vectors. We want to find a vector $\vec{\mathbf{w}} = \langle w_1, w_2, w_3 \rangle$ orthogonal to both $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$—that is, we want to find $\vec{\mathbf{w}}$ such that $\vec{\mathbf{u}} \cdot \vec{\mathbf{w}} = 0$ and $\vec{\mathbf{v}} \cdot \vec{\mathbf{w}} = 0$. Therefore, w_1, w_2, and w_3 must satisfy

$$u_1 w_1 + u_2 w_2 + u_3 w_3 = 0 \tag{11.4.1}$$

$$v_1 w_1 + v_2 w_2 + v_3 w_3 = 0. \tag{11.4.2}$$

If we multiply the top equation by v_3 and the bottom equation by u_3 and subtract, we can eliminate the variable w_3, which gives

$$(u_1 v_3 - v_1 u_3) w_1 + (u_2 v_3 - v_2 u_3) w_2 = 0.$$

If we select

$$w_1 = u_2 v_3 - u_3 v_2$$
$$w_2 = -(u_1 v_3 - u_3 v_1),$$

we get a possible solution vector. Substituting these values back into the original equations (Equations 11.4.1 and 11.4.2) gives

$$w_3 = u_1 v_2 - u_2 v_1.$$

That is, vector

$$\vec{\mathbf{w}} = \langle u_2 v_3 - u_3 v_2, -(u_1 v_3 - u_3 v_1), u_1 v_2 - u_2 v_1 \rangle$$

is orthogonal to both $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$, which leads us to define the following operation, called the *cross product*.

Definition: Cross Product

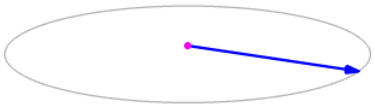
Let $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$. Then, the *cross product* $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ is vector

$$\begin{aligned} \vec{\mathbf{u}} \times \vec{\mathbf{v}} &= (u_2 v_3 - u_3 v_2)\hat{\mathbf{i}} - (u_1 v_3 - u_3 v_1)\hat{\mathbf{j}} + (u_1 v_2 - u_2 v_1)\hat{\mathbf{k}} \\ &= \langle u_2 v_3 - u_3 v_2, -(u_1 v_3 - u_3 v_1), u_1 v_2 - u_2 v_1 \rangle. \end{aligned} \tag{11.4.3}$$

From the way we have developed $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$, it should be clear that the cross product is orthogonal to both $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$. However, it never hurts to check. To show that $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ is orthogonal to $\vec{\mathbf{u}}$, we calculate the dot product of $\vec{\mathbf{u}}$ and $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$.

$$\begin{aligned}\vec{u}\cdot(\vec{u}\times\vec{v}) &= \langle u_1,u_2,u_3\rangle\cdot\langle u_2v_3-u_3v_2,-u_1v_3+u_3v_1,u_1v_2-u_2v_1\rangle\\ &= u_1(u_2v_3-u_3v_2)+u_2(-u_1v_3+u_3v_1)+u_3(u_1v_2-u_2v_1)\\ &= u_1u_2v_3-u_1u_3v_2-u_1u_2v_3+u_2u_3v_1+u_1u_3v_2-u_2u_3v_1\\ &= (u_1u_2v_3-u_1u_2v_3)+(-u_1u_3v_2+u_1u_3v_2)+(u_2u_3v_1-u_2u_3v_1)\\ &= 0\end{aligned}$$

In a similar manner, we can show that the cross product is also orthogonal to $\vec{v}$.



The cross product $\vec{a}\times\vec{b}$ (vertical, in pink) changes as the angle between the vectors $\vec{a}$ (blue) and $\vec{b}$ (red) changes. The cross product (purple) is always perpendicular to both vectors, and has magnitude zero when the vectors are parallel and maximum magnitude $\|\vec{a}\|\|\vec{b}\|$ when they are perpendicular. (Public Domain; LucasVB).

Example 11.4.1: Finding a Cross Product

Let $\vec{p}=\langle -1,2,5\rangle$ and $\vec{q}=\langle 4,0,-3\rangle$ (Figure 11.4.1). Find $\vec{p}\times\vec{q}$.

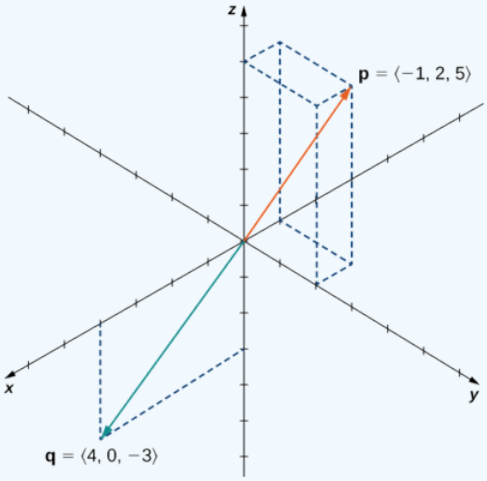


Figure 11.4.1: Finding a cross product to two given vectors.

Solution

Substitute the components of the vectors into Equation 11.4.3:

$$\begin{aligned}\vec{p}\times\vec{q} &= \langle -1,2,5\rangle\times\langle 4,0,-3\rangle\\ &= \langle p_2q_3-p_3q_2,-(p_1q_3-p_3q_1),p_1q_2-p_2q_1\rangle\\ &= \langle 2(-3)-5(0),-(-1)(-3)+5(4),(-1)(0)-2(4)\rangle\\ &= \langle -6,17,-8\rangle.\end{aligned}$$

Exercise 11.4.1

Find $\vec{p} \times \vec{q}$ for $\vec{p} = \langle 5, 1, 2 \rangle$ and $\vec{q} = \langle -2, 0, 1 \rangle$. Express the answer using standard unit vectors.

Although it may not be obvious from Equation 11.4.3, the direction of $\vec{u} \times \vec{v}$ is given by the right-hand rule. If we hold the right hand out with the fingers pointing in the direction of $\vec{u}$, then curl the fingers toward vector $\vec{v}$, the thumb points in the direction of the cross product, as shown in Figure 11.4.2.

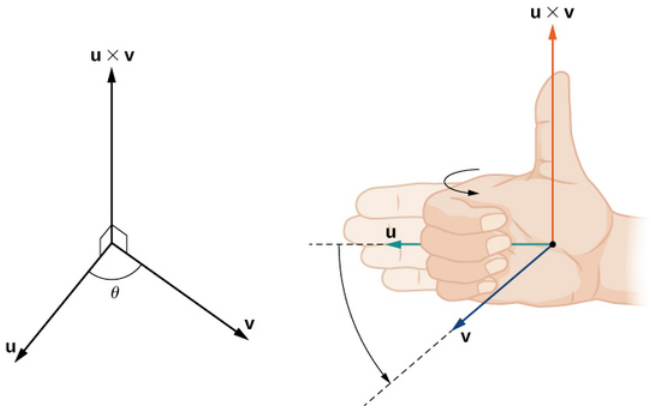


Figure 11.4.2: The direction of $\vec{u} \times \vec{v}$ is determined by the right-hand rule.

Notice what this means for the direction of $\vec{v} \times \vec{u}$. If we apply the right-hand rule to $\vec{v} \times \vec{u}$, we start with our fingers pointed in the direction of $\vec{v}$, then curl our fingers toward the vector $\vec{u}$. In this case, the thumb points in the opposite direction of $\vec{u} \times \vec{v}$. (Try it!)

Example 11.4.2: Anticommutativity of the Cross Product

Let $\vec{u} = \langle 0, 2, 1 \rangle$ and $\vec{v} = \langle 3, -1, 0 \rangle$. Calculate $\vec{u} \times \vec{v}$ and $\vec{v} \times \vec{u}$ and graph them.

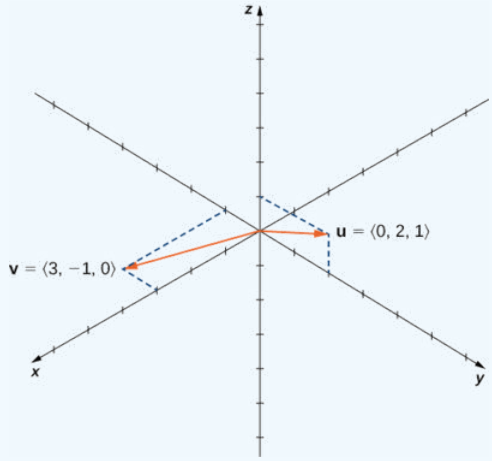


Figure 11.4.3: Are the cross products $\vec{u} \times \vec{v}$ and $\vec{v} \times \vec{u}$ in the same direction?

Solution

We have

$$\vec{u} \times \vec{v} = \langle (0+1), -(0-3), (0-6) \rangle = \langle 1, 3, -6 \rangle$$

$$\vec{v} \times \vec{u} = \langle (-1-0), -(3-0), (6-0) \rangle = \langle -1, -3, 6 \rangle.$$

We see that, in this case, $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$ (Figure 11.4.4). We prove this in general later in this section.

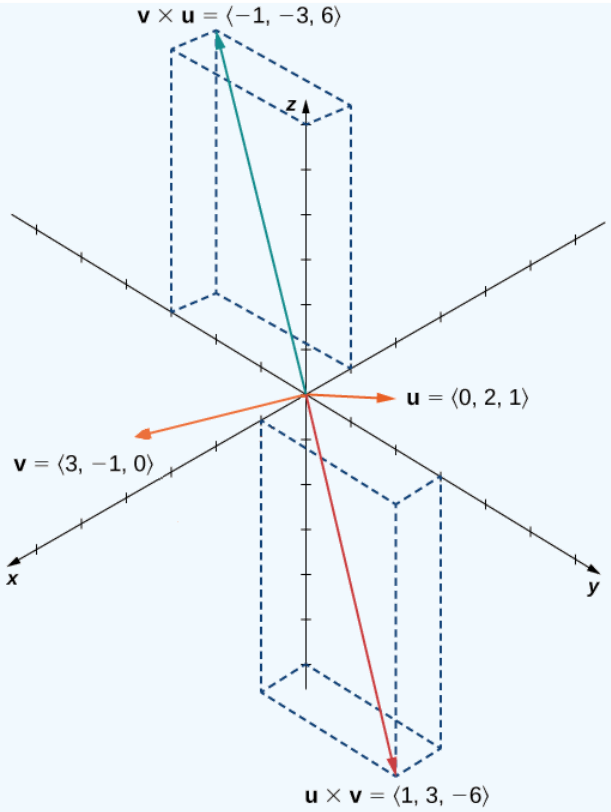


Figure 11.4.4: The cross products $\vec{u} \times \vec{v}$ and $\vec{v} \times \vec{u}$ are both orthogonal to $\vec{u}$ and $\vec{v}$, but in opposite directions.

? Exercise 11.4.2

Suppose vectors $\vec{u}$ and $\vec{v}$ lie in the xy-plane (the z-component of each vector is zero). Now suppose the x- and y-components of $\vec{u}$ and the y-component of $\vec{v}$ are all positive, whereas the x-component of $\vec{v}$ is negative. Assuming the coordinate axes are oriented in the usual positions, in which direction does $\vec{u} \times \vec{v}$ point?

The cross products of the standard unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ can be useful for simplifying some calculations, so let's consider these cross products. A straightforward application of the definition shows that

$$\hat{\mathbf{i}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}} \times \hat{\mathbf{k}} = \vec{\mathbf{0}}.$$

(The cross product of two vectors is a vector, so each of these products results in the zero vector, not the scalar 0.) It's up to you to verify the calculations on your own.

Furthermore, because the cross product of two vectors is orthogonal to each of these vectors, we know that the cross product of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ is parallel to $\hat{\mathbf{k}}$. Similarly, the vector product of $\hat{\mathbf{i}}$ and $\hat{\mathbf{k}}$ is parallel to $\hat{\mathbf{j}}$, and the vector product of $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ is parallel to $\hat{\mathbf{i}}$.

We can use the right-hand rule to determine the direction of each product. Then we have

$$\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}$$
$$\hat{\mathbf{j}} \times \hat{\mathbf{i}} = -\hat{\mathbf{k}}$$

$$\hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}}$$
$$\hat{\mathbf{k}} \times \hat{\mathbf{j}} = -\hat{\mathbf{i}}$$

$$\hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}}$$
$$\hat{\mathbf{i}} \times \hat{\mathbf{k}} = -\hat{\mathbf{j}}.$$

These formulas come in handy later.

Example 11.4.3: Cross Product of Standard Unit Vectors

Find $\hat{\mathbf{i}} \times (\hat{\mathbf{j}} \times \hat{\mathbf{k}})$.

Solution

We know that $\hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}}$. Therefore, $\hat{\mathbf{i}} \times (\hat{\mathbf{j}} \times \hat{\mathbf{k}}) = \hat{\mathbf{i}} \times \hat{\mathbf{i}} = \vec{\mathbf{0}}$.

Exercise 11.4.3

Find $(\hat{\mathbf{i}} \times \hat{\mathbf{j}}) \times (\hat{\mathbf{k}} \times \hat{\mathbf{i}})$.

As we have seen, the dot product is often called the *scalar product* because it results in a scalar. The cross product results in a vector, so it is sometimes called the **vector product.** These operations are both versions of vector multiplication, but they have very different properties and applications. Let's explore some properties of the cross product. We prove only a few of them. Proofs of the other properties are left as exercises.

Properties of the Cross Product

Let $\vec{\mathbf{u}}, \vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$ be vectors in space, and let c be a scalar.

i. Anticommutative property:

$$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = -(\vec{\mathbf{v}} \times \vec{\mathbf{u}})$$

ii. Distributive property:

$$\vec{\mathbf{u}} \times (\vec{\mathbf{v}} + \vec{\mathbf{w}}) = \vec{\mathbf{u}} \times \vec{\mathbf{v}} + \vec{\mathbf{u}} \times \vec{\mathbf{w}}$$

iii. Multiplication by a constant:

$$c(\vec{\mathbf{u}} \times \vec{\mathbf{v}}) = (c\vec{\mathbf{u}}) \times \vec{\mathbf{v}} = \vec{\mathbf{u}} \times (c\vec{\mathbf{v}})$$

iv. Cross product of the zero vector:

$$\vec{\mathbf{u}} \times \vec{\mathbf{0}} = \vec{\mathbf{0}} \times \vec{\mathbf{u}} = \vec{\mathbf{0}}$$

v. Cross product of a vector with itself:

$$\vec{\mathbf{v}} \times \vec{\mathbf{v}} = \vec{\mathbf{0}}$$

vi. Triple scalar product:

$$\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) = (\vec{\mathbf{u}} \times \vec{\mathbf{v}}) \cdot \vec{\mathbf{w}}$$

vii. Triple cross product:

$$\vec{\mathbf{u}} \times (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) = (\vec{\mathbf{u}} \cdot \vec{\mathbf{w}})\,\vec{\mathbf{v}} - (\vec{\mathbf{u}} \cdot \vec{\mathbf{v}})\,\vec{\mathbf{w}}$$

Proof

For property i, we want to show $\vec{\mathbf{u}} \times \vec{\mathbf{v}} = -(\vec{\mathbf{v}} \times \vec{\mathbf{u}})$. We have

$$\begin{aligned}
\vec{\mathbf{u}} \times \vec{\mathbf{v}} &= \langle u_1, u_2, u_3 \rangle \times \langle v_1, v_2, v_3 \rangle \\
&= \langle u_2 v_3 - u_3 v_2, -u_1 v_3 + u_3 v_1, u_1 v_2 - u_2 v_1 \rangle \\
&= -\langle u_3 v_2 - u_2 v_3, -u_3 v_1 + u_1 v_3, u_2 v_1 - u_1 v_2 \rangle \\
&= -\langle v_1, v_2, v_3 \rangle \times \langle u_1, u_2, u_3 \rangle \\
&= -(\vec{\mathbf{v}} \times \vec{\mathbf{u}}).
\end{aligned}$$

Unlike most operations we've seen, the cross product is not commutative. This makes sense if we think about the right-hand rule.

For property *iv.*, this follows directly from the definition of the cross product. We have

$$\vec{\mathbf{u}} \times \vec{\mathbf{0}} = \langle u_2(0) - u_3(0), -(u_1(0) - u_3(0)), u_1(0) - u_2(0) \rangle = \langle 0, 0, 0 \rangle = \vec{\mathbf{0}}.$$

Then, by property i., $\vec{\mathbf{0}} \times \vec{\mathbf{u}} = \vec{\mathbf{0}}$ as well. Remember that the dot product of a vector and the zero vector is the *scalar* $\mathbf{0}$, whereas the cross product of a vector with the zero vector is the *vector* $\vec{\mathbf{0}}$.

Property *vi.* looks like the associative property, but note the change in operations:

$$\begin{aligned} \vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) &= u \cdot \langle v_2 w_3 - v_3 w_2, -v_1 w_3 + v_3 w_1, v_1 w_2 - v_2 w_1 \rangle \\ &= u_1(v_2 w_3 - v_3 w_2) + u_2(-v_1 w_3 + v_3 w_1) + u_3(v_1 w_2 - v_2 w_1) \\ &= u_1 v_2 w_3 - u_1 v_3 w_2 - u_2 v_1 w_3 + u_2 v_3 w_1 + u_3 v_1 w_2 - u_3 v_2 w_1 \\ &= (u_2 v_3 - u_3 v_2) w_1 + (u_3 v_1 - u_1 v_3) w_2 + (u_1 v_2 - u_2 v_1) w_3 \\ &= \langle u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1 \rangle \cdot \langle w_1, w_2, w_3 \rangle = (\vec{\mathbf{u}} \times \vec{\mathbf{v}}) \cdot \vec{\mathbf{w}}. \end{aligned}$$

□

Example 11.4.4: Using the Properties of the Cross Product

Use the cross product properties to calculate $(2\hat{\mathbf{i}} \times 3\hat{\mathbf{j}}) \times \hat{\mathbf{j}}$.

Solution

$$\begin{aligned} (2\hat{\mathbf{i}} \times 3\hat{\mathbf{j}}) \times \hat{\mathbf{j}} &= 2(\hat{\mathbf{i}} \times 3\hat{\mathbf{j}}) \times \hat{\mathbf{j}} \\ &= 2(3)(\hat{\mathbf{i}} \times \hat{\mathbf{j}}) \times \hat{\mathbf{j}} \\ &= (6\hat{\mathbf{k}}) \times \hat{\mathbf{j}} \\ &= 6(\hat{\mathbf{k}} \times \hat{\mathbf{j}}) \\ &= 6(-\hat{\mathbf{i}}) = -6\hat{\mathbf{i}}. \end{aligned}$$

Exercise 11.4.4

Use the properties of the cross product to calculate $(\hat{\mathbf{i}} \times \hat{\mathbf{k}}) \times (\hat{\mathbf{k}} \times \hat{\mathbf{j}})$.

So far in this section, we have been concerned with the direction of the vector $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$, but we have not discussed its magnitude. It turns out there is a simple expression for the magnitude of $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ involving the magnitudes of $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$, and the sine of the angle between them.

Magnitude of the Cross Product

Let $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ be vectors, and let θ be the angle between them. Then, $\|\vec{\mathbf{u}} \times \vec{\mathbf{v}}\| = \|\vec{\mathbf{u}}\| \cdot \|\vec{\mathbf{v}}\| \cdot \sin\theta$.

Proof

Let $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$ be vectors, and let θ denote the angle between them. Then

$\|\vec{\mathbf{u}}$
$\times \vec{\mathbf{v}}$

$$\begin{aligned}\|^2 &= (u_2v_3 - u_3v_2)^2 + (u_3v_1 - u_1v_3)^2 + (u_1v_2 - u_2v_1)^2 \\ &= u_2^2v_3^2 - 2u_2u_3v_2v_3 + u_3^2v_2^2 + u_3^2v_1^2 - 2u_1u_3v_1v_3 + u_1^2v_3^2 + u_1^2v_2^2 \\ &\quad - 2u_1u_2v_1v_2 + u_2^2v_1^2 \\ &= u_1^2v_1^2 + u_1^2v_2^2 + u_1^2v_3^2 + u_2^2v_1^2 + u_2^2v_2^2 + u_2^2v_3^2 + u_3^2v_1^2 + u_3^2v_2^2 + u_3^2v_3^2 \\ &\quad - (u_1^2v_1^2 + u_2^2v_2^2 + u_3^2v_3^2 + 2u_1u_2v_1v_2 + 2u_1u_3v_1v_3 + 2u_2u_3v_2v_3) \\ &= (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1v_1 + u_2v_2 + u_3v_3)^2 \\ &= \|\vec{u}\|^2\|\vec{v}\|^2 - (\vec{u} \cdot \vec{v})^2 \\ &= \|\vec{u}\|^2\|\vec{v}\|^2 - \|\vec{u}\|^2\|\vec{v}\|^2 \cos^2\theta \\ &= \|\vec{u}\|^2\|\vec{v}\|^2(1 - \cos^2\theta) \\ &= \|\vec{u}\|^2\|\vec{v}\|^2(\sin^2\theta).\end{aligned}$$

Taking square roots and noting that $\sqrt{\sin^2\theta} = \sin\theta$ for $0 \le \theta \le 180°$, we have the desired result:

$$\|\vec{u} \times \vec{v}\| = \|\vec{u}\|\|\vec{v}\| \sin\theta.$$

□

This definition of the cross product allows us to visualize or interpret the product geometrically. It is clear, for example, that the cross product is defined only for vectors in three dimensions, not for vectors in two dimensions. In two dimensions, it is impossible to generate a vector simultaneously orthogonal to two nonparallel vectors.

✔ Example 11.4.5: Calculating the Cross Product

Use "Magnitude of the Cross Product" to find the magnitude of the cross product of $\vec{u} = \langle 0, 4, 0\rangle$ and $\vec{v} = \langle 0, 0, -3\rangle$.

Solution

We have

$$\begin{aligned}\|\vec{u} \times \vec{v}\| &= \|\vec{u}\| \cdot \|\vec{v}\| \cdot \sin\theta \\ &= \sqrt{0^2 + 4^2 + 0^2} \cdot \sqrt{0^2 + 0^2 + (-3)^2} \cdot \sin\frac{\pi}{2} \\ &= 4(3)(1) = 12\end{aligned}$$

? Exercise 11.4.5

Use "Magnitude of the Cross Product" to find the magnitude of $\vec{u} \times \vec{v}$, where $\vec{u} = \langle -8, 0, 0\rangle$ and $\vec{v} = \langle 0, 2, 0\rangle$.

Determinants and the Cross Product

Using Equation 11.4.3 to find the cross product of two vectors is straightforward, and it presents the cross product in the useful component form. The formula, however, is complicated and difficult to remember. Fortunately, we have an alternative. We can calculate the cross product of two vectors using determinant notation.

A 2×2 determinant is defined by

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - b_1a_2.$$

For example,

$$\begin{vmatrix} 3 & -2 \\ 5 & 1 \end{vmatrix} = 3(1) - 5(-2) = 3 + 10 = 13.$$

A 3×3 determinant is defined in terms of 2×2 determinants as follows:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}. \tag{11.4.4}$$

Equation 11.4.4 is referred to as the expansion of the determinant along the first row. Notice that the multipliers of each of the 2×2 determinants on the right side of this expression are the entries in the first row of the 3×3 determinant. Furthermore, each of the 2×2 determinants contains the entries from the 3×3 determinant that would remain if you crossed out the row and column containing the multiplier. Thus, for the first term on the right, a_1 is the multiplier, and the 2×2 determinant contains the entries that remain if you cross out the first row and first column of the 3×3 determinant. Similarly, for the second term, the multiplier is a_2, and the 2×2 determinant contains the entries that remain if you cross out the first row and second column of the 3×3 determinant. Notice, however, that the coefficient of the second term is negative. The third term can be calculated in similar fashion.

Example 11.4.6: Using Expansion Along the First Row to Compute a 3×3 Determinant

Evaluate the determinant $\begin{vmatrix} 2 & 5 & -1 \\ -1 & 1 & 3 \\ -2 & 3 & 4 \end{vmatrix}$.

Solution

We have

$$\begin{aligned} \begin{vmatrix} 2 & 5 & -1 \\ -1 & 1 & 3 \\ -2 & 3 & 4 \end{vmatrix} &= 2 \begin{vmatrix} 1 & 3 \\ 3 & 4 \end{vmatrix} - 5 \begin{vmatrix} -1 & 3 \\ -2 & 4 \end{vmatrix} - 1 \begin{vmatrix} -1 & 1 \\ -2 & 3 \end{vmatrix} \\ &= 2(4-9) - 5(-4+6) - 1(-3+2) \\ &= 2(-5) - 5(2) - 1(-1) = -10 - 10 + 1 \\ &= -19 \end{aligned}$$

Exercise 11.4.6

Evaluate the determinant $\begin{vmatrix} 1 & -2 & -1 \\ 3 & 2 & -3 \\ 1 & 5 & 4 \end{vmatrix}$.

Technically, determinants are defined only in terms of arrays of real numbers. However, the determinant notation provides a useful mnemonic device for the cross product formula.

Rule: Cross Product Calculated by a Determinant

Let $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$ be vectors. Then the cross product $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ is given by

$$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \hat{\mathbf{i}} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \hat{\mathbf{j}} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \hat{\mathbf{k}}.$$

Example 11.4.7: Using Determinant Notation to find $\vec{\mathbf{p}} \times \vec{\mathbf{q}}$

Let $\vec{\mathbf{p}} = \langle -1, 2, 5 \rangle$ and $\vec{\mathbf{q}} = \langle 4, 0, -3 \rangle$. Find $\vec{\mathbf{p}} \times \vec{\mathbf{q}}$.

Solution

We set up our determinant by putting the standard unit vectors across the first row, the components of $\vec{\mathbf{u}}$ in the second row, and the components of $\vec{\mathbf{v}}$ in the third row. Then, we have

$$\vec{p} \times \vec{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & 5 \\ 4 & 0 & -3 \end{vmatrix} = \begin{vmatrix} 2 & 5 \\ 0 & -3 \end{vmatrix}\hat{i} - \begin{vmatrix} -1 & 5 \\ 4 & -3 \end{vmatrix}\hat{j} + \begin{vmatrix} -1 & 2 \\ 4 & 0 \end{vmatrix}\hat{k}$$
$$= (-6-0)\hat{i} - (3-20)\hat{j} + (0-8)\hat{k}$$
$$= -6\hat{i} + 17\hat{j} - 8\hat{k}.$$

Notice that this answer confirms the calculation of the cross product in Example 11.4.1.

? Exercise 11.4.7

Use determinant notation to find $\vec{a} \times \vec{b}$, where $\vec{a} = \langle 8, 2, 3 \rangle$ and $\vec{b} = \langle -1, 0, 4 \rangle$.

Using the Cross Product

The cross product is very useful for several types of calculations, including finding a vector orthogonal to two given vectors, computing areas of triangles and parallelograms, and even determining the volume of the three-dimensional geometric shape made of parallelograms known as a *parallelepiped*. The following examples illustrate these calculations.

✓ Example 11.4.8: Finding a Unit Vector Orthogonal to Two Given Vectors

Let $\vec{a} = \langle 5, 2, -1 \rangle$ and $\vec{b} = \langle 0, -1, 4 \rangle$. Find a unit vector orthogonal to both $\vec{a}$ and $\vec{b}$.

Solution

The cross product $\vec{a} \times \vec{b}$ is orthogonal to both vectors $\vec{a}$ and $\vec{b}$. We can calculate it with a determinant:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & 2 & -1 \\ 0 & -1 & 4 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ -1 & 4 \end{vmatrix}\hat{i} - \begin{vmatrix} 5 & -1 \\ 0 & 4 \end{vmatrix}\hat{j} + \begin{vmatrix} 5 & 2 \\ 0 & -1 \end{vmatrix}\hat{k}$$
$$= (8-1)\hat{i} - (20-0)\hat{j} + (-5-0)\hat{k}$$
$$= 7\hat{i} - 20\hat{j} - 5\hat{k}.$$

Normalize this vector to find a unit vector in the same direction:

$$\|\vec{a} \times \vec{b}\| = \sqrt{(7)^2 + (-20)^2 + (-5)^2} = \sqrt{474}.$$

Thus, $\left\langle \frac{7}{\sqrt{474}}, \frac{-20}{\sqrt{474}}, \frac{-5}{\sqrt{474}} \right\rangle$ is a unit vector orthogonal to $\vec{a}$ and $\vec{b}$.

Simplified, this vector becomes $\left\langle \frac{7\sqrt{474}}{474}, \frac{-10\sqrt{474}}{237}, \frac{-5\sqrt{474}}{474} \right\rangle$.

? Exercise 11.4.8

Find a unit vector orthogonal to both $\vec{a}$ and $\vec{b}$, where $\vec{a} = \langle 4, 0, 3 \rangle$ and $\vec{b} = \langle 1, 1, 4 \rangle$.

To use the cross product for calculating areas, we state and prove the following theorem.

Area of a Parallelogram

If we locate vectors $\vec{u}$ and $\vec{v}$ such that they form adjacent sides of a parallelogram, then the area of the parallelogram is given by $\|\vec{u} \times \vec{v}\|$ (Figure 11.4.5).

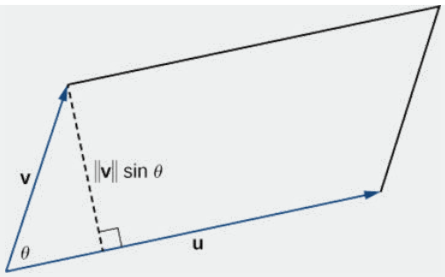


Figure 11.4.5: The parallelogram with adjacent sides $\vec{u}$ and $\vec{v}$ has base $\|\vec{u}\|$ and height $\|\vec{v}\| \sin\theta$.

Proof

We show that the magnitude of the cross product is equal to the base times height of the parallelogram.

$$\begin{aligned} \text{Area of a parallelogram} &= \text{base} \times \text{height} \\ &= \|\vec{u}\|(\|\vec{v}\| \sin\theta) \\ &= \|\vec{u} \times \vec{v}\| \end{aligned}$$

□

Example 11.4.9: Finding the Area of a Triangle

Let $P = (1,0,0), Q = (0,1,0)$, and $R = (0,0,1)$ be the vertices of a triangle (Figure 11.4.6). Find its area.

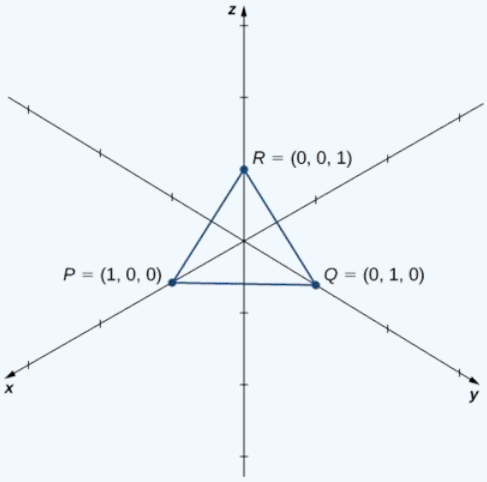


Figure 11.4.6: Finding the area of a triangle by using the cross product.

Solution

We have $\overrightarrow{PQ} = \langle 0-1, 1-0, 0-0\rangle = \langle -1,1,0\rangle$ and $\overrightarrow{PR} = \langle 0-1, 0-0, 1-0\rangle = \langle -1,0,1\rangle$. The area of the parallelogram with adjacent sides $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ is given by $\| \overrightarrow{PQ} \times \overrightarrow{PR} \|$:

$$\begin{aligned} \overrightarrow{PQ} \times \overrightarrow{PR} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} \\ &= (1-0)\hat{\mathbf{i}} - (-1-0)\hat{\mathbf{j}} + (0-(-1))\hat{\mathbf{k}} \\ &= \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}} \end{aligned}$$

$$\begin{aligned} \| \overrightarrow{PQ} \times \overrightarrow{PR} \| &= \| \langle 1,1,1\rangle \| \\ &= \sqrt{1^2+1^2+1^2} \\ &= \sqrt{3}. \end{aligned}$$

The area of ΔPQR is half the area of the parallelogram or $\sqrt{3}/2\ \text{units}^2$.

Exercise 11.4.9

Find the area of the parallelogram $PQRS$ with vertices $P(1,1,0)$, $Q(7,1,0)$, $R(9,4,2)$, and $S(3,4,2)$.

The Triple Scalar Product

Because the cross product of two vectors is a vector, it is possible to combine the dot product and the cross product. The dot product of a vector with the cross product of two other vectors is called the **triple scalar product** because the result is a scalar.

Definition: Triple Scalar Product

The triple scalar product of vectors $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$ is

$$\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}).$$

Calculating a Triple Scalar Product

The triple scalar product of vectors

$$\vec{\mathbf{u}} = u_1\hat{\mathbf{i}} + u_2\hat{\mathbf{j}} + u_3\hat{\mathbf{k}}$$

$$\vec{\mathbf{v}} = v_1\hat{\mathbf{i}} + v_2\hat{\mathbf{j}} + v_3\hat{\mathbf{k}}$$

and

$$\vec{\mathbf{w}} = w_1\hat{\mathbf{i}} + w_2\hat{\mathbf{j}} + w_3\hat{\mathbf{k}}$$

is the determinant of the 3×3 matrix formed by the components of the vectors:

$$\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}. \tag{11.4.5}$$

Proof

The calculation is straightforward.

$$\begin{aligned} \vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) &= \langle u_1, u_2, u_3 \rangle \cdot \langle v_2w_3 - v_3w_2, -v_1w_3 + v_3w_1, v_1w_2 - v_2w_1 \rangle \\ &= u_1(v_2w_3 - v_3w_2) + u_2(-v_1w_3 + v_3w_1) + u_3(v_1w_2 - v_2w_1) \\ &= u_1(v_2w_3 - v_3w_2) - u_2(v_1w_3 - v_3w_1) + u_3(v_1w_2 - v_2w_1) \\ &= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}. \end{aligned}$$

□

Example 11.4.10: Calculating the Triple Scalar Product

Let $\vec{\mathbf{u}} = \langle 1, 3, 5 \rangle$, $\vec{\mathbf{v}} = \langle 2, -1, 0 \rangle$ and $\vec{\mathbf{w}} = \langle -3, 0, -1 \rangle$. Calculate the triple scalar product $\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}})$.

Solution

Apply Equation 11.4.5 directly:

$$\begin{aligned}\vec{\mathbf{u}}\cdot(\vec{\mathbf{v}}\times\vec{\mathbf{w}}) &= \begin{vmatrix}1 & 3 & 5\\ 2 & -1 & 0\\ -3 & 0 & -1\end{vmatrix}\\ &= 1\begin{vmatrix}-1 & 0\\ 0 & -1\end{vmatrix} - 3\begin{vmatrix}2 & 0\\ -3 & -1\end{vmatrix} + 5\begin{vmatrix}2 & -1\\ -3 & 0\end{vmatrix}\\ &= (1-0) - 3(-2-0) + 5(0-3)\\ &= 1+6-15 = -8.\end{aligned}$$

? Exercise 11.4.10

Calculate the triple scalar product $\vec{\mathbf{a}}\cdot(\vec{\mathbf{b}}\times\vec{\mathbf{c}})$, where $\vec{\mathbf{a}} = \langle 2,-4,1\rangle$, $\vec{\mathbf{b}} = \langle 0,3,-1\rangle$, and $\vec{\mathbf{c}} = \langle 5,-3,3\rangle$.

When we create a matrix from three vectors, we must be careful about the order in which we list the vectors. If we list them in a matrix in one order and then rearrange the rows, the absolute value of the determinant remains unchanged. However, each time two rows switch places, the determinant changes sign:

$$\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix} = d \qquad \begin{vmatrix}b_1 & b_2 & b_3\\ a_1 & a_2 & a_3\\ c_1 & c_2 & c_3\end{vmatrix} = -d \qquad \begin{vmatrix}b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\\ a_1 & a_2 & a_3\end{vmatrix} = d \qquad \begin{vmatrix}c_1 & c_2 & c_3\\ b_1 & b_2 & b_3\\ a_1 & a_2 & a_3\end{vmatrix} = -d$$

Verifying this fact is straightforward, but rather messy. Let's take a look at this with an example:

$$\begin{aligned}\begin{vmatrix}1 & 2 & 1\\ -2 & 0 & 3\\ 4 & 1 & -1\end{vmatrix} &= \begin{vmatrix}0 & 3\\ 1 & -1\end{vmatrix} - 2\begin{vmatrix}-2 & 3\\ 4 & -1\end{vmatrix} + \begin{vmatrix}-2 & 0\\ 4 & 1\end{vmatrix}\\ &= (0-3) - 2(2-12) + (-2-0)\\ &= -3+20-2 = 15.\end{aligned}$$

Switching the top two rows we have

$$\begin{aligned}\begin{vmatrix}-2 & 0 & 3\\ 1 & 2 & 1\\ 4 & 1 & -1\end{vmatrix} &= -2\begin{vmatrix}2 & 1\\ 1 & -1\end{vmatrix} + 3\begin{vmatrix}1 & 2\\ 4 & 1\end{vmatrix}\\ &= -2(-2-1) + 3(1-8)\\ &= 6-21 = -15.\end{aligned}$$

Rearranging vectors in the triple products is equivalent to reordering the rows in the matrix of the determinant. Let $\vec{\mathbf{u}} = u_1\hat{\mathbf{i}} + u_2\hat{\mathbf{j}} + u_3\hat{\mathbf{k}}$, $\vec{\mathbf{v}} = v_1\hat{\mathbf{i}} + v_2\hat{\mathbf{j}} + v_3\hat{\mathbf{k}}$, and $\vec{\mathbf{w}} = w_1\hat{\mathbf{i}} + w_2\hat{\mathbf{j}} + w_3\hat{\mathbf{k}}$. Applying Calculating a Triple Scalar Product, we have

$$\vec{\mathbf{u}}\cdot(\vec{\mathbf{v}}\times\vec{\mathbf{w}}) = \begin{vmatrix}u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\\ w_1 & w_2 & w_3\end{vmatrix}$$

and

$$\vec{\mathbf{u}}\cdot(\vec{\mathbf{w}}\times\vec{\mathbf{v}}) = \begin{vmatrix}u_1 & u_2 & u_3\\ w_1 & w_2 & w_3\\ v_1 & v_2 & v_3\end{vmatrix}.$$

We can obtain the determinant for calculating $\vec{\mathbf{u}}\cdot(\vec{\mathbf{w}}\times\vec{\mathbf{v}})$ by switching the bottom two rows of $\vec{\mathbf{u}}\cdot(\vec{\mathbf{v}}\times\vec{\mathbf{w}})$. Therefore, $\vec{\mathbf{u}}\cdot(\vec{\mathbf{v}}\times\vec{\mathbf{w}}) = -\vec{\mathbf{u}}\cdot(\vec{\mathbf{w}}\times\vec{\mathbf{v}})$.

Following this reasoning and exploring the different ways we can interchange variables in the triple scalar product lead to the following identities:

$$\vec{\mathbf{u}}\cdot(\vec{\mathbf{v}}\times\vec{\mathbf{w}}) = -\vec{\mathbf{u}}\cdot(\vec{\mathbf{w}}\times\vec{\mathbf{v}}) \tag{11.4.6}$$

$$\vec{\mathbf{u}}\cdot(\vec{\mathbf{v}}\times\vec{\mathbf{w}}) = \vec{\mathbf{v}}\cdot(\vec{\mathbf{w}}\times\vec{\mathbf{u}}) = \vec{\mathbf{w}}\cdot(\vec{\mathbf{u}}\times\vec{\mathbf{v}}). \tag{11.4.7}$$

Let $\vec{u}$ and $\vec{v}$ be two vectors in standard position. If $\vec{u}$ and $\vec{v}$ are not scalar multiples of each other, then these vectors form adjacent sides of a parallelogram. We saw in Area of a Parallelogram that the area of this parallelogram is $\|\vec{u} \times \vec{v}\|$. Now suppose we add a third vector $\vec{w}$ that does not lie in the same plane as $\vec{u}$ and $\vec{v}$ but still shares the same initial point. Then these vectors form three edges of a **parallelepiped**, a three-dimensional prism with six faces that are each parallelograms, as shown in Figure 11.4.7. The volume of this prism is the product of the figure's height and the area of its base. The triple scalar product of $\vec{u}, \vec{v}$, and $\vec{w}$ provides a simple method for calculating the volume of the parallelepiped defined by these vectors.

Volume of a Parallelepiped

The volume of a parallelepiped with adjacent edges given by the vectors $\vec{u}, \vec{v}$, and $\vec{w}$ is the absolute value of the triple scalar product (Figure 11.4.7):

$$V = |\vec{u} \cdot (\vec{v} \times \vec{w})|.$$

Note that, as the name indicates, the triple scalar product produces a scalar. The volume formula just presented uses the absolute value of a scalar quantity.

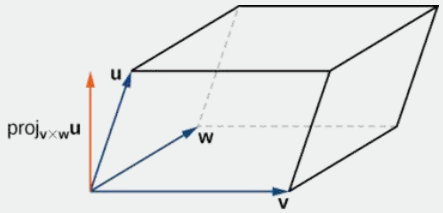


Figure 11.4.7: The height of the parallelepiped is given by $\|\text{proj}_{\vec{v}\times\vec{w}}\vec{u}\|$.

Proof

The area of the base of the parallelepiped is given by $\|\vec{v} \times \vec{w}\|$. The height of the figure is given by $\|\text{proj}_{\vec{v}\times\vec{w}}\vec{u}\|$. The volume of the parallelepiped is the product of the height and the area of the base, so we have

$$\begin{aligned} V &= \| \text{proj}_{\vec{v}\times\vec{w}}\vec{u} \| \, \|\vec{v} \times \vec{w}\| \\ &= \left| \frac{\vec{u} \cdot (\vec{v} \times \vec{w})}{\|\vec{v} \times \vec{w}\|} \right| \|\vec{v} \times \vec{w}\| \\ &= |\vec{u} \cdot (\vec{v} \times \vec{w})|. \end{aligned}$$

□

Example 11.4.11: Calculating the Volume of a Parallelepiped

Let $\vec{u} = \langle -1, -2, 1 \rangle, \vec{v} = \langle 4, 3, 2 \rangle$, and $\vec{w} = \langle 0, -5, -2 \rangle$. Find the volume of the parallelepiped with adjacent edges $\vec{u}, \vec{v}$, and $\vec{w}$ (Figure 11.4.8).

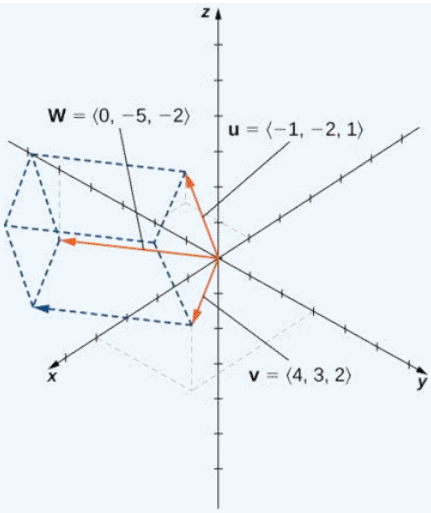


Figure 11.4.8

Solution

We have

$$\begin{aligned}\vec{u}\cdot(\vec{v}\times\vec{w}) &= \begin{vmatrix}-1 & -2 & 1\\ 4 & 3 & 2\\ 0 & -5 & -2\end{vmatrix}\\ &= (-1)\begin{vmatrix}3 & 2\\ -5 & -2\end{vmatrix} + 2\begin{vmatrix}4 & 2\\ 0 & -2\end{vmatrix} + \begin{vmatrix}4 & 3\\ 0 & -5\end{vmatrix}\\ &= (-1)(-6+10) + 2(-8-0) + (-20-0)\\ &= -4-16-20\\ &= -40.\end{aligned}$$

Thus, the volume of the parallelepiped is $|-40| = 40$ units^3

? Exercise 11.4.11

Find the volume of the parallelepiped formed by the vectors $\vec{a} = 3\hat{i} + 4\hat{j} - \hat{k}$, $\vec{b} = 2\hat{i} - \hat{j} - \hat{k}$, and $\vec{c} = 3\hat{j} + \hat{k}$.

Applications of the Cross Product

The cross product appears in many practical applications in mathematics, physics, and engineering. Let's examine some of these applications here, including the idea of torque, with which we began this section. Other applications show up in later chapters, particularly in our study of vector fields such as gravitational and electromagnetic fields (Introduction to Vector Calculus).

✓ Example 11.4.12: Using the Triple Scalar Product

Use the triple scalar product to show that vectors $\vec{u} = \langle 2, 0, 5\rangle$, $\vec{v} = \langle 2, 2, 4\rangle$, and $\vec{w} = \langle 1, -1, 3\rangle$ are coplanar—that is, show that these vectors lie in the same plane.

Solution

Start by calculating the triple scalar product to find the volume of the parallelepiped defined by $\vec{u}, \vec{v}$, and $\vec{w}$:

$$\begin{aligned}\vec{u}\cdot(\vec{v}\times\vec{w}) &= \begin{vmatrix}2 & 0 & 5\\ 2 & 2 & 4\\ 1 & -1 & 3\end{vmatrix}\\ &= [2(2)(3) + (0)(4)(1) + 5(2)(-1)] - [5(2)(1) + (2)(4)(-1) + (0)(2)(3)]\\ &= 2 - 2 = 0.\end{aligned}$$

The volume of the parallelepiped is 0 units^3, so one of the dimensions must be zero. Therefore, the three vectors all lie in the same plane.

Exercise 11.4.12

Are the vectors $\vec{a} = \hat{i} + \hat{j} - \hat{k}$, $\vec{b} = \hat{i} - \hat{j} + \hat{k}$, and $\vec{c} = \hat{i} + \hat{j} + \hat{k}$ coplanar?

Example 11.4.13: Finding an Orthogonal Vector

Only a single plane can pass through any set of three noncolinear points. Find a vector orthogonal to the plane containing points $P = (9, -3, -2), Q = (1, 3, 0)$, and $R = (-2, 5, 0)$.

Solution

The plane must contain vectors $\overrightarrow{PQ}$ and $\overrightarrow{QR}$:

$$\overrightarrow{PQ} = \langle 1 - 9, 3 - (-3), 0 - (-2) \rangle = \langle -8, 6, 2 \rangle$$

$$\overrightarrow{QR} = \langle -2 - 1, 5 - 3, 0 - 0 \rangle = \langle -3, 2, 0 \rangle.$$

The cross product $\overrightarrow{PQ} \times \overrightarrow{QR}$ produces a vector orthogonal to both $\overrightarrow{PQ}$ and $\overrightarrow{QR}$. Therefore, the cross product is orthogonal to the plane that contains these two vectors:

$$\begin{aligned} \overrightarrow{PQ} \times \overrightarrow{QR} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -8 & 6 & 2 \\ -3 & 2 & 0 \end{vmatrix} \\ &= 0\hat{i} - 6\hat{j} - 16\hat{k} - (-18\hat{k} + 4\hat{i} + 0\hat{j}) \\ &= -4\hat{i} - 6\hat{j} + 2\hat{k}. \end{aligned}$$

We have seen how to use the triple scalar product and how to find a vector orthogonal to a plane. Now we apply the cross product to real-world situations.

Sometimes a force causes an object to rotate. For example, turning a screwdriver or a wrench creates this kind of rotational effect, called torque.

Definition: Torque

Torque, $\vec{\tau}$ (the Greek letter tau), measures the tendency of a force to produce rotation about an axis of rotation. Let $\vec{r}$ be a vector with an initial point located on the axis of rotation and with a terminal point located at the point where the force is applied, and let vector $\vec{F}$ represent the force. Then torque is equal to the cross product of $\vec{r}$ and $\vec{F}$:

$$\vec{\tau} = \vec{r} \times \vec{F}.$$

See Figure 11.4.9.

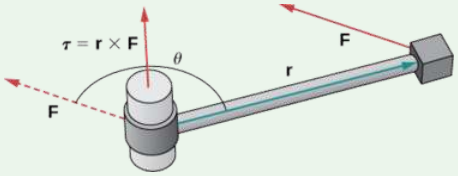


Figure 11.4.9: Torque measures how a force causes an object to rotate.

Think about using a wrench to tighten a bolt. The torque τ applied to the bolt depends on how hard we push the wrench (force) and how far up the handle we apply the force (distance). The torque increases with a greater force on the wrench at a greater distance from the bolt. Common units of torque are the newton-meter or foot-pound. Although torque is dimensionally equivalent to work (it has the same units), the two concepts are distinct. Torque is used specifically in the context of rotation, whereas work typically involves motion along a line.

Example 11.4.14: Evaluating Torque

A bolt is tightened by applying a force of 6 N to a 0.15-m wrench (Figure 11.4.10). The angle between the wrench and the force vector is 40°. Find the magnitude of the torque about the center of the bolt. Round the answer to two decimal places.

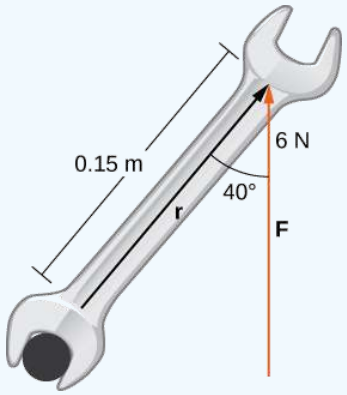


Figure 11.4.10: Torque describes the twisting action of the wrench.

Solution:

Substitute the given information into the equation defining torque:

$$\begin{aligned} \|\vec{\tau}\| &= \|\vec{r} \times \vec{F}\| \\ &= \|\vec{r}\| \, \|\vec{F}\| \sin\theta \\ &= (0.15\,\text{m})(6\,\text{N}) \sin 40° \\ &\approx 0.58\,\text{N·m}. \end{aligned}$$

Exercise 11.4.14

Calculate the force required to produce 15 N·m torque at an angle of 30° from a 150-cm rod.

Key Concepts

- The cross product $\vec{u} \times \vec{v}$ of two vectors $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ is a vector orthogonal to both $\vec{u}$ and $\vec{v}$. Its length is given by $\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \cdot \|\vec{v}\| \cdot \sin\theta$, where θ is the angle between $\vec{u}$ and $\vec{v}$. Its direction is given by the right-hand rule.
- The algebraic formula for calculating the cross product of two vectors,

  $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$, is

  $\vec{u} \times \vec{v} = (u_2 v_3 - u_3 v_2)\hat{i} - (u_1 v_3 - u_3 v_1)\hat{j} + (u_1 v_2 - u_2 v_1)\hat{k}$.
- The cross product satisfies the following properties for vectors $\vec{u}, \vec{v}$, and $\vec{w}$, and scalar c:

  $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$

  $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$

  $c(\vec{u} \times \vec{v}) = (c\vec{u}) \times \vec{v} = \vec{u} \times (c\vec{v})$

  $\vec{u} \times \vec{0} = \vec{0} \times \vec{u} = \vec{0}$

  $\vec{v} \times \vec{v} = \vec{0}$

  $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$
- The cross product of vectors $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ is the determinant $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$
- If vectors $\vec{u}$ and $\vec{v}$ form adjacent sides of a parallelogram, then the area of the parallelogram is given by $\|\vec{u} \times \vec{v}\|$.
- The triple scalar product of vectors $\vec{u}, \vec{v}$, and $\vec{w}$ is $\vec{u} \cdot (\vec{v} \times \vec{w})$.

- The volume of a parallelepiped with adjacent edges given by vectors $\vec{\mathbf{u}}, \vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$ is $V = |\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}})|$.
- If the triple scalar product of vectors $\vec{\mathbf{u}}, \vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$ is zero, then the vectors are coplanar. The converse is also true: If the vectors are coplanar, then their triple scalar product is zero.
- The cross product can be used to identify a vector orthogonal to two given vectors or to a plane.
- Torque $\vec{\boldsymbol{\tau}}$ measures the tendency of a force to produce rotation about an axis of rotation. If force $\vec{\mathbf{F}}$ is acting at a distance (displacement) $\vec{\mathbf{r}}$ from the axis, then torque is equal to the cross product of $\vec{\mathbf{r}}$ and $\vec{\mathbf{F}}$: $\vec{\boldsymbol{\tau}} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}$.

Key Equations

- **The cross product of two vectors in terms of the unit vectors**

$$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = (u_2v_3 - u_3v_2)\hat{\mathbf{i}} - (u_1v_3 - u_3v_1)\hat{\mathbf{j}} + (u_1v_2 - u_2v_1)\hat{\mathbf{k}}$$

Glossary

cross product

$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = (u_2v_3 - u_3v_2)\hat{\mathbf{i}} - (u_1v_3 - u_3v_1)\hat{\mathbf{j}} + (u_1v_2 - u_2v_1)\hat{\mathbf{k}}$, where $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$

determinant

a real number associated with a square matrix

parallelepiped

a three-dimensional prism with six faces that are parallelograms

torque

the effect of a force that causes an object to rotate

triple scalar product

the dot product of a vector with the cross product of two other vectors: $\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}})$

vector product

the cross product of two vectors

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11.4E: Exercises for The Cross Product

For exercises 1-4, the vectors $\vec{u}$ and $\vec{v}$ are given.

a. Find the cross product $\vec{u} \times \vec{v}$ of the vectors $\vec{u}$ and $\vec{v}$. Express the answer in component form.

b. Sketch the vectors $\vec{u}$, $\vec{v}$, and $\vec{u} \times \vec{v}$.

1) $\vec{u} = \langle 2, 0, 0 \rangle$, $\vec{v} = \langle 2, 2, 0 \rangle$

2) $\vec{u} = \langle 3, 2, -1 \rangle$, $\vec{v} = \langle 1, 1, 0 \rangle$

3) $\vec{u} = 2\hat{i} + 3\hat{j}$, $\vec{v} = \hat{j} + 2\hat{k}$

4) $\vec{u} = 2\hat{j} + 3\hat{k}$, $\vec{v} = 3\hat{i} + \hat{k}$

5) Simplify $(\hat{i} \times \hat{i} - 2\hat{i} \times \hat{j} - 4\hat{i} \times \hat{k} + 3\hat{j} \times \hat{k}) \times \hat{i}$.

6) Simplify $\hat{j} \times (\hat{k} \times \hat{j} + 2\hat{j} \times \hat{i} - 3\hat{j} \times \hat{j} + 5\hat{i} \times \hat{k})$.

In exercises 7-10, vectors $\vec{u}$ and $\vec{v}$ are given. Find unit vector $\vec{w}$ in the direction of the cross product vector $\vec{u} \times \vec{v}$. Express your answer using standard unit vectors.

7) $\vec{u} = \langle 3, -1, 2 \rangle$, $\vec{v} = \langle -2, 0, 1 \rangle$

8) $\vec{u} = \langle 2, 6, 1 \rangle$, $\vec{v} = \langle 3, 0, 1 \rangle$

9) $\vec{u} = \overrightarrow{AB}$, $\vec{v} = \overrightarrow{AC}$, where $A(1, 0, 1)$, $B(1, -1, 3)$, and $C(0, 0, 5)$

10) $\vec{u} = \overrightarrow{OP}$, $\vec{v} = \overrightarrow{PQ}$, where $P(-1, 1, 0)$ and $Q(0, 2, 1)$

11) Determine the real number α such that $\vec{u} \times \vec{v}$ and $\hat{i}$ are orthogonal, where $\vec{u} = 3\hat{i} + \hat{j} - 5\hat{k}$ and $\vec{v} = 4\hat{i} - 2\hat{j} + \alpha\hat{k}$.

12) Show that $\vec{u} \times \vec{v}$ and $2\hat{i} - 14\hat{j} + 2\hat{k}$ cannot be orthogonal for any α real number, where $\vec{u} = \hat{i} + 7\hat{j} - \hat{k}$ and $\vec{v} = \alpha\hat{i} + 5\hat{j} + \hat{k}$.

13) Show that $\vec{u} \times \vec{v}$ is orthogonal to $\vec{u} + \vec{v}$ and $\vec{u} - \vec{v}$, where $\vec{u}$ and $\vec{v}$ are nonzero vectors.

14) Show that $\vec{v} \times \vec{u}$ is orthogonal to $(\vec{u} \cdot \vec{v})(\vec{u} + \vec{v}) + \vec{u}$, where $\vec{u}$ and $\vec{v}$ are nonzero vectors.

15) Calculate the determinant $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 7 \\ 2 & 0 & 3 \end{vmatrix}$.

16) Calculate the determinant $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 3 & -4 \\ 1 & 6 & -1 \end{vmatrix}$.

For exercises 17-18, the vectors $\vec{u}$ and $\vec{v}$ are given. Use determinant notation to find vector $\vec{w}$ orthogonal to vectors $\vec{u}$ and $\vec{v}$.

17) $\vec{u} = \langle -1, 0, e^t \rangle$, $\vec{v} = \langle 1, e^{-t}, 0 \rangle$, where t is a real number

18) $\vec{u} = \langle 1, 0, x \rangle$, $\vec{v} = \langle \frac{2}{x}, 1, 0 \rangle$, where x is a nonzero real number

19) Find vector $(\vec{a} - 2\vec{b}) \times \vec{c}$, where $\vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 5 \\ 0 & 1 & 8 \end{vmatrix}$, $\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 1 \\ 2 & -1 & -2 \end{vmatrix}$, and $\vec{c} = \hat{i} + \hat{j} + \hat{k}$.

20) Find vector $\vec{c} \times (\vec{a} + 3\vec{b})$, where $\vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & 0 & 9 \\ 0 & 1 & 0 \end{vmatrix}$, $\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -1 & 1 \\ 7 & 1 & -1 \end{vmatrix}$, and $\vec{c} = \hat{i} - \hat{k}$.

21) [T] Use the cross product $\vec{u} \times \vec{v}$ to find the acute angle between vectors $\vec{u}$ and $\vec{v}$, where $\vec{u} = \hat{i} + 2\hat{j}$ and $\vec{v} = \hat{i} + \hat{k}$. Express the answer in degrees rounded to the nearest integer.

22) [T] Use the cross product $\vec{u} \times \vec{v}$ to find the obtuse angle between vectors $\vec{u}$ and $\vec{v}$, where $\vec{u} = -\hat{i} + 3\hat{j} + \hat{k}$ and $\vec{v} = \hat{i} - 2\hat{j}$. Express the answer in degrees rounded to the nearest integer.

23) Use the sine and cosine of the angle between two nonzero vectors $\vec{u}$ and $\vec{v}$ to prove Lagrange's identity: $\|\vec{u}\times\vec{v}\|^2 = \|\vec{u}\|^2\|\vec{v}\|^2 - (\vec{u}\cdot\vec{v})^2$.

24) Verify Lagrange's identity $\|\vec{u}\times\vec{v}\|^2 = \|\vec{u}\|^2\|\vec{v}\|^2 - (\vec{u}\cdot\vec{v})^2$ for vectors $\vec{u} = -\hat{i}+\hat{j}-2\hat{k}$ and $\vec{v} = 2\hat{i}-\hat{j}$.

25) Nonzero vectors $\vec{u}$ and $\vec{v}$ are called collinear if there exists a nonzero scalar α such that $\vec{v} = \alpha\vec{u}$. Show that $\vec{u}$ and $\vec{v}$ are collinear if and only if $\vec{u}\times\vec{v} = 0$.

26) Nonzero vectors $\vec{u}$ and $\vec{v}$ are called *collinear* if there exists a nonzero scalar α such that $\vec{v} = \alpha\vec{u}$. Show that vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are collinear, where $A(4,1,0)$, $B(6,5,-2)$, and $C(5,3,-1)$.

27) Find the area of the parallelogram with adjacent sides $\vec{u} = \langle 3,2,0\rangle$ and $\vec{v} = \langle 0,2,1\rangle$.

28) Find the area of the parallelogram with adjacent sides $\vec{u} = \hat{i}+\hat{j}$ and $\vec{v} = \hat{i}+\hat{k}$.

29) Consider points $A(3,-1,2)$, $B(2,1,5)$, and $C(1,-2,-2)$.

a. Find the area of parallelogram $ABCD$ with adjacent sides $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

b. Find the area of triangle ABC.

c. Find the distance from point A to line BC.

30) Consider points $A(2,-3,4)$, $B(0,1,2)$, and $C(-1,2,0)$.

a. Find the area of parallelogram $ABCD$ with adjacent sides $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

b. Find the area of triangle ABC.

c. Find the distance from point B to line AC.

In exercises 31-32, vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ are given.

a. Find the triple scalar product $\vec{u}\cdot(\vec{v}\times\vec{w})$.

b. Find the volume of the parallelepiped with the adjacent edges $\vec{u}$, $\vec{v}$, and $\vec{w}$.

31) $\vec{u} = \hat{i}+\hat{j}$, $\vec{v} = \hat{j}+\hat{k}$, and $\vec{w} = \hat{i}+\hat{k}$

32) $\vec{u} = \langle -3,5,-1\rangle$, $\vec{v} = \langle 0,2,-2\rangle$, and $\vec{w} = \langle 3,1,1\rangle$

33) Calculate the triple scalar products $\vec{v}\cdot(\vec{u}\times\vec{w})$ and $\vec{w}\cdot(\vec{u}\times\vec{v})$, where $\vec{u} = \langle 1,1,1\rangle$, $\vec{v} = \langle 7,6,9\rangle$, and $\vec{w} = \langle 4,2,7\rangle$.

34) Calculate the triple scalar products $\vec{w}\cdot(\vec{v}\times\vec{u})$ and $\vec{u}\cdot(\vec{w}\times\vec{v})$, where $\vec{u} = \langle 4,2,-1\rangle$, $\vec{v} = \langle 2,5,-3\rangle$, and $\vec{w} = \langle 9,5,-10\rangle$.

35) Find vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ with a triple scalar product given by the determinant $\begin{vmatrix}1&2&3\\0&2&5\\8&9&2\end{vmatrix}$. Determine their triple scalar product.

36) The triple scalar product of vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ is given by the determinant $\begin{vmatrix}0&-2&1\\0&1&4\\1&-3&7\end{vmatrix}$. Find vector $\vec{a}-\vec{b}+\vec{c}$.

37) Consider the parallelepiped with edges OA, OB, and OC, where $A(2,1,0)$, $B(1,2,0)$, and $C(0,1,\alpha)$.

a. Find the real number $\alpha > 0$ such that the volume of the parallelepiped is 3 units^3.

b. For $\alpha = 1$, find the height h from vertex C of the parallelepiped. Sketch the parallelepiped.

38) Consider points $A(\alpha,0,0)$, $B(0,\beta,0)$, and $C(0,0,\gamma)$, with α,β, and γ positive real numbers.

a. Determine the volume of the parallelepiped with adjacent sides $\overrightarrow{OA}$, $\overrightarrow{OB}$, and $\overrightarrow{OC}$.

b. Find the volume of the tetrahedron with vertices O, A, B, and C. (Hint: The volume of the tetrahedron is $1/6$ of the volume of the parallelepiped.)

c. Find the distance from the origin to the plane determined by A, B, and C. Sketch the parallelepiped and tetrahedron.

39) Let u, v, and w be three-dimensional vectors and c be a real number. Prove the following properties of the cross product.

a. $\vec{\mathbf{u}} \times \vec{\mathbf{u}} = \vec{\mathbf{0}}$

b. $\vec{\mathbf{u}} \times (\vec{\mathbf{v}} + \vec{\mathbf{w}}) = (\vec{\mathbf{u}} \times \vec{\mathbf{v}}) + (\vec{\mathbf{u}} \times \vec{\mathbf{w}})$

c. $c(\vec{\mathbf{u}} \times \vec{\mathbf{v}}) = (c\vec{\mathbf{u}}) \times \vec{\mathbf{v}} = \vec{\mathbf{u}} \times (c\vec{\mathbf{v}})$

d. $\vec{\mathbf{u}} \cdot (\vec{\mathbf{u}} \times \vec{\mathbf{v}}) = \vec{\mathbf{0}}$

40) Show that vectors $\vec{\mathbf{u}} = \langle 1, 0, -8 \rangle$, $\vec{\mathbf{v}} = \langle 0, 1, 6 \rangle$, and $\vec{\mathbf{w}} = \langle -1, 9, 3 \rangle$ satisfy the following properties of the cross product.

a. $\vec{\mathbf{u}} \times \vec{\mathbf{u}} = \vec{\mathbf{0}}$

b. $\vec{\mathbf{u}} \times (\vec{\mathbf{v}} + \vec{\mathbf{w}}) = (\vec{\mathbf{u}} \times \vec{\mathbf{v}}) + (\vec{\mathbf{u}} \times \vec{\mathbf{w}})$

c. $c(\vec{\mathbf{u}} \times \vec{\mathbf{v}}) = (c\vec{\mathbf{u}}) \times \vec{\mathbf{v}} = \vec{\mathbf{u}} \times (c\vec{\mathbf{v}})$

d. $\vec{\mathbf{u}} \cdot (\vec{\mathbf{u}} \times \vec{\mathbf{v}}) = \vec{\mathbf{0}}$

41) Nonzero vectors $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$ are said to be linearly dependent if one of the vectors is a linear combination of the other two. For instance, there exist two nonzero real numbers α and β such that $\vec{\mathbf{w}} = \alpha\vec{\mathbf{u}} + \beta\vec{\mathbf{v}}$. Otherwise, the vectors are called linearly independent. Show that $\vec{\mathbf{u}}, \vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$ could be placed on the same plane if and only if they are linear dependent.

42) Consider vectors $\vec{\mathbf{u}} = \langle 1, 4, -7 \rangle$, $\vec{\mathbf{v}} = \langle 2, -1, 4 \rangle$, $\vec{\mathbf{w}} = \langle 0, -9, 18 \rangle$, and $\vec{\mathbf{p}} = \langle 0, -9, 17 \rangle$.

a. Show that $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$ can be placed on the same plane by using their triple scalar product

b. Show that $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$ can be placed on the same plane by using the definition that there exist two nonzero real numbers α and β such that $w = \alpha u + \beta v$.

c. Show that $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, and $\vec{\mathbf{p}}$ are linearly independent—that is, none of the vectors is a linear combination of the other two.

43) Consider points $A(0, 0, 2), B(1, 0, 2), C(1, 1, 2)$, and $D(0, 1, 2)$. Are vectors $\overrightarrow{AB}, \overrightarrow{AC}$, and $\overrightarrow{AD}$ linearly dependent (that is, one of the vectors is a linear combination of the other two)?

44) Show that vectors $\hat{\mathbf{i}} + \hat{\mathbf{j}}, \hat{\mathbf{i}} - \hat{\mathbf{j}}$, and $\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ are linearly independent—that is, there exist two nonzero real numbers α and β such that $\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}} = \alpha(\hat{\mathbf{i}} + \hat{\mathbf{j}}) + \beta(\hat{\mathbf{i}} - \hat{\mathbf{j}})$.

45) Let $\vec{\mathbf{u}} = \langle u_1, u_2 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2 \rangle$ be two-dimensional vectors. The cross product of vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ is not defined. However, if the vectors are regarded as the three-dimensional vectors $\tilde{\vec{\mathbf{u}}} = \langle u_1, u_2, 0 \rangle$ and $\tilde{\vec{\mathbf{v}}} = \langle v_1, v_2, 0 \rangle$, respectively, then, in this case, we can define the cross product of $\tilde{\vec{\mathbf{u}}}$ and $\tilde{\vec{\mathbf{v}}}$. In particular, in determinant notation, the cross product of $\tilde{\vec{\mathbf{u}}}$ and $\tilde{\vec{\mathbf{v}}}$ is given by

$$\tilde{\vec{\mathbf{u}}} \times \tilde{\vec{\mathbf{v}}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ u_1 & u_2 & 0 \\ v_1 & v_2 & 0 \end{vmatrix}.$$

Use this result to compute $(\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}) \times (\sin\theta\,\hat{\mathbf{i}} - \cos\theta\,\hat{\mathbf{j}})$, where θ is a real number.

46) Consider points $P(2, 1), Q(4, 2)$, and $R(1, 2)$.

a. Find the area of triangle PQR.

b. Determine the distance from point R to the line passing through P and Q.

47) Determine a vector of magnitude 10 perpendicular to the plane passing through the x-axis and point $P(1, 2, 4)$.

48) Determine a unit vector perpendicular to the plane passing through the z-axis and point $A(3, 1, -2)$.

49) Consider $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ two three-dimensional vectors. If the magnitude of the cross product vector $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ is k times larger than the magnitude of vector $\vec{\mathbf{u}}$, show that the magnitude of $\vec{\mathbf{v}}$ is greater than or equal to k, where k is a natural number.

50) [T] Assume that the magnitudes of two nonzero vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are known. The function $f(\theta) = \|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\| \sin\theta$ defines the magnitude of the cross product vector $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$, where $\theta \in [0, \pi]$ is the angle between $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$.

a. Graph the function f.

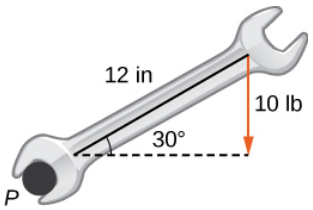
b. Find the absolute minimum and maximum of function f. Interpret the results.

c. If $\|\vec{\mathbf{u}}\| = 5$ and $\|\vec{\mathbf{v}}\| = 2$, find the angle between $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ if the magnitude of their cross product vector is equal to 9.

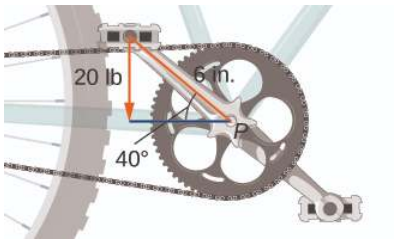
51) Find all vectors $\vec{\mathbf{w}} = \langle w_1, w_2, w_3 \rangle$ that satisfy the equation $\langle 1, 1, 1 \rangle \times \vec{\mathbf{w}} = \langle -1, -1, 2 \rangle$. Hint: You should be able to write all components of $\vec{\mathbf{w}}$ in terms of one of the constants w_1, w_2, or w_3.

52) Solve the equation $\vec{\mathbf{w}} \times \langle 1, 0, -1 \rangle = \langle 3, 0, 3 \rangle$, where $\vec{\mathbf{w}} = \langle w_1, w_2, w_3 \rangle$ is a nonzero vector with a magnitude of 3.

53) [T] A mechanic uses a 12-in. wrench to turn a bolt. The wrench makes a $30°$ angle with the horizontal. If the mechanic applies a vertical force of 10 lb on the wrench handle, what is the magnitude of the torque at point P (see the following figure)? Express the answer in foot-pounds rounded to two decimal places.



54) [T] A boy applies the brakes on a bicycle by applying a downward force of 20 lb on the pedal when the 6-in. crank makes a $40°$ angle with the horizontal (see the following figure). Find the torque at point P. Express your answer in foot-pounds rounded to two decimal places.



55) [T] Find the magnitude of the force that needs to be applied to the end of a 20-cm wrench located on the positive direction of the y-axis if the force is applied in the direction $\langle 0, 1, -2 \rangle$ and it produces a 100 N·m torque to the bolt located at the origin.

56) [T] What is the magnitude of the force required to be applied to the end of a 1-ft wrench at an angle of $35°$ to produce a torque of 20 N·m?

57) [T] The force vector $\vec{\mathbf{F}}$ acting on a proton with an electric charge of $1.6 \times 10^{-19}\ C$ (in coulombs) moving in a magnetic field $\vec{\mathbf{B}}$ where the velocity vector $\vec{\mathbf{v}}$ is given by $\vec{\mathbf{F}} = 1.6 \times 10^{-19}(\vec{\mathbf{v}} \times \vec{\mathbf{B}})$ (here, $\vec{\mathbf{v}}$ is expressed in meters per second, $\vec{\mathbf{B}}$ is in tesla [T], and $\vec{\mathbf{F}}$ is in newtons [N]). Find the force that acts on a proton that moves in the xy-plane at velocity $\vec{\mathbf{v}} = 10^5\hat{\mathbf{i}} + 10^5\hat{\mathbf{j}}$ (in meters per second) in a magnetic field given by $\vec{\mathbf{B}} = 0.3\hat{\mathbf{j}}$.

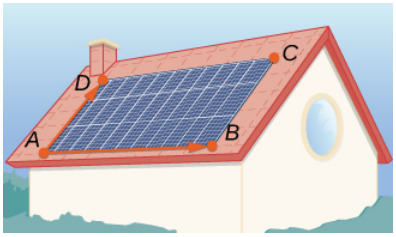
58) [T] The force vector $\vec{\mathbf{F}}$ acting on a proton with an electric charge of $1.6 \times 10^{-19}\ C$ moving in a magnetic field $\vec{\mathbf{B}}$ where the velocity vector $\vec{\mathbf{v}}$ is given by $\vec{\mathbf{F}} = 1.6 \times 10^{-19}(\vec{\mathbf{v}} \times \vec{\mathbf{B}})$ (here, $\vec{\mathbf{v}}$ is expressed in meters per second, $\vec{\mathbf{B}}$ in T, and $\vec{\mathbf{F}}$ in N). If the magnitude of force $\vec{\mathbf{F}}$ acting on a proton is $5.9 \times 10^{-17}\ N$ and the proton is moving at the speed of 300 m/sec in magnetic field $\vec{\mathbf{B}}$ of magnitude 2.4 T, find the angle between velocity vector $\vec{\mathbf{v}}$ of the proton and magnetic field $\vec{\mathbf{B}}$. Express the answer in degrees rounded to the nearest integer.

59) [T] Consider $\vec{\mathbf{r}}(t) = \langle \cos t,\ \sin t,\ 2t \rangle$ the position vector of a particle at time $t \in [0, 30]$, where the components of $\vec{\mathbf{r}}$ are expressed in centimeters and time in seconds. Let $\overrightarrow{OP}$ be the position vector of the particle after 1 sec.

a. Determine unit vector $\vec{\mathbf{B}}(t)$ (called the binormal unit vector) that has the direction of cross product vector $\vec{\mathbf{v}}(t) \times \vec{\mathbf{a}}(t)$, where $\vec{\mathbf{v}}(t)$ and $\vec{\mathbf{a}}(t)$ are the instantaneous velocity vector and, respectively, the acceleration vector of the particle after t seconds.

b. Use a CAS to visualize vectors $\vec{\mathbf{v}}(1)$, $\vec{\mathbf{a}}(1)$, and $\vec{\mathbf{B}}(1)$ as vectors starting at point P along with the path of the particle.

60) A solar panel is mounted on the roof of a house. The panel may be regarded as positioned at the points of coordinates (in meters) $A(8, 0, 0), B(8, 18, 0), C(0, 18, 8)$, and $D(0, 0, 8)$ (see the following figure).



a. Find vector $\vec{\mathbf{n}} = \overrightarrow{AB} \times \overrightarrow{AD}$ perpendicular to the surface of the solar panels. Express the answer using standard unit vectors. Note that the magnitude of this vector should give us the area of rectangle $ABCD$.

b. Assume unit vector $\vec{\mathbf{s}} = \frac{1}{\sqrt{3}}\hat{\mathbf{i}} + \frac{1}{\sqrt{3}}\hat{\mathbf{j}} + \frac{1}{\sqrt{3}}\hat{\mathbf{k}}$ points toward the Sun at a particular time of the day and the flow of solar energy is $\vec{\mathbf{F}} = 900\vec{\mathbf{s}}$ (in watts per square meter $[W/m^2]$). Find the predicted amount of electrical power the panel can produce, which is given by the dot product of vectors $\vec{\mathbf{F}}$ and $\vec{\mathbf{n}}$ (expressed in watts).

c. Determine the angle of elevation of the Sun above the solar panel. Express the answer in degrees rounded to the nearest whole number. (Hint: The angle between vectors $\vec{\mathbf{n}}$ and $\vec{\mathbf{s}}$ and the angle of elevation are complementary.)

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11.5: Equations of Lines and Planes in Space

Learning Objectives

- Write the vector, parametric, and symmetric equations of a line through a given point in a given direction, and a line through two given points.
- Find the distance from a point to a given line.
- Write the vector and scalar equations of a plane through a given point with a given normal.
- Find the distance from a point to a given plane.
- Find the angle between two planes.

By now, we are familiar with writing equations that describe a line in two dimensions. To write an equation for a line, we must know two points on the line, or we must know the direction of the line and at least one point through which the line passes. In two dimensions, we use the concept of slope to describe the orientation, or direction, of a line. In three dimensions, we describe the direction of a line using a vector parallel to the line. In this section, we examine how to use equations to describe lines and planes in space.

Equations for a Line in Space

Let's first explore what it means for two vectors to be parallel. Recall that parallel vectors must have the same or opposite directions. If two nonzero vectors, $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$, are parallel, we claim there must be a scalar, k, such that $\vec{\mathbf{u}} = k\vec{\mathbf{v}}$. If $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ have the same direction, simply choose

$$k = \frac{\|\vec{\mathbf{u}}\|}{\|\vec{\mathbf{v}}\|}. \tag{11.5.1}$$

If $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ have opposite directions, choose

$$k = -\frac{\|\vec{\mathbf{u}}\|}{\|\vec{\mathbf{v}}\|}. \tag{11.5.2}$$

Note that the converse holds as well. If $\vec{\mathbf{u}} = k\vec{\mathbf{v}}$ for some scalar k, then either $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ have the same direction $(k > 0)$ or opposite directions $(k < 0)$, so $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are parallel. Therefore, two nonzero vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are parallel if and only if $\vec{\mathbf{u}} = k\vec{\mathbf{v}}$ for some scalar k. By convention, the zero vector $\vec{\mathbf{0}}$ is considered to be parallel to all vectors.

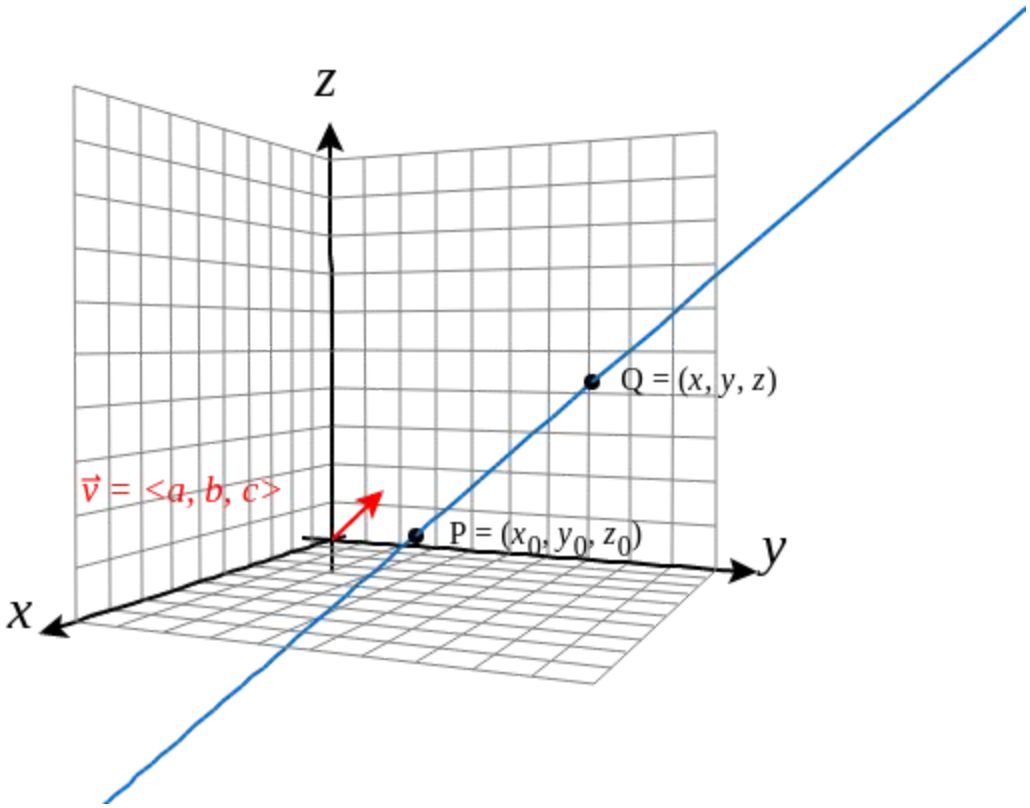


***Figure* 11.5.1:** *Vector $\vec{\mathbf{v}}$ is the direction vector for $\overrightarrow{PQ}$.*

As in two dimensions, we can describe a line in space using a point on the line and the direction of the line, or a parallel vector, which we call the **direction vector** (Figure 11.5.1). Let L be a line in space passing through point $P(x_0, y_0, z_0)$. Let $\vec{\mathbf{v}} = \langle a, b, c\rangle$ be a vector parallel to L. Then, for any point on line $Q(x, y, z)$, we know that $\overrightarrow{PQ}$ is parallel to $\vec{\mathbf{v}}$. Thus, as we just discussed, there is a scalar, t, such that $\overrightarrow{PQ} = t\vec{\mathbf{v}}$, which gives

$$\begin{aligned} \overrightarrow{PQ} &= t\vec{\mathbf{v}} \\ \langle x - x_0, y - y_0, z - z_0\rangle &= t\langle a, b, c\rangle \\ \langle x - x_0, y - y_0, z - z_0\rangle &= \langle ta, tb, tc\rangle. \end{aligned} \tag{11.5.3}$$

Using vector operations, we can rewrite Equation 11.5.3

$$\begin{aligned} \langle x - x_0, y - y_0, z - z_0\rangle &= \langle ta, tb, tc\rangle \\ \langle x, y, z\rangle - \langle x_0, y_0, z_0\rangle &= t\langle a, b, c\rangle \\ \underbrace{\langle x, y, z\rangle}_{\vec{\mathbf{r}}} &= \underbrace{\langle x_0, y_0, z_0\rangle}_{\vec{\mathbf{r}}_0} + t\underbrace{\langle a, b, c\rangle}_{\vec{\mathbf{v}}}. \end{aligned}$$

Setting $\vec{\mathbf{r}} = \langle x, y, z\rangle$ and $\vec{\mathbf{r}}_0 = \langle x_0, y_0, z_0\rangle$, we now have the **vector equation of a line:**

$$\vec{\mathbf{r}} = \vec{\mathbf{r}}_0 + t\vec{\mathbf{v}}. \tag{11.5.4}$$

Equating components, Equation 11.5.4 shows that the following equations are simultaneously true: $x - x_0 = ta, y - y_0 = tb$, and $z - z_0 = tc$. If we solve each of these equations for the component variables x, y, and z, we get a set of equations in which each

variable is defined in terms of the parameter t and that, together, describe the line. This set of three equations forms a set of **parametric equations of a line:**

$$x = x_0 + ta$$
$$y = y_0 + tb$$
$$z = z_0 + tc.$$

If we solve each of the equations for t assuming a, b, and c are nonzero, we get a different description of the same line:

$$\frac{x - x_0}{a} = t$$
$$\frac{y - y_0}{b} = t$$
$$\frac{z - z_0}{c} = t.$$

Because each expression equals t, they all have the same value. We can set them equal to each other to create **symmetric equations of a line:**

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

We summarize the results in the following theorem.

Theorem: Parametric and Symmetric Equations of a Line

A line L parallel to vector $\vec{\mathbf{v}} = \langle a, b, c \rangle$ and passing through point $P(x_0, y_0, z_0)$ can be described by the following parametric equations:

$$x = x_0 + ta, y = y_0 + tb, \tag{11.5.5}$$

and

$$z = z_0 + tc. \tag{11.5.6}$$

If the constants a, b, and c are all nonzero, then L can be described by the symmetric equation of the line:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}. \tag{11.5.7}$$

The parametric equations of a line are not unique. Using a different parallel vector or a different point on the line leads to a different, equivalent representation. Each set of parametric equations leads to a related set of symmetric equations, so it follows that a symmetric equation of a line is not unique either.

Example 11.5.1: Equations of a Line in Space

Find parametric and symmetric equations of the line passing through points $(1, 4, -2)$ and $(-3, 5, 0)$.

Solution

First, identify a vector parallel to the line:

$$\vec{\mathbf{v}} = \langle -3 - 1, 5 - 4, 0 - (-2) \rangle = \langle -4, 1, 2 \rangle.$$

Use either of the given points on the line to complete the parametric equations:

$$x = 1 - 4t$$
$$y = 4 + t,$$

and

$$z = -2 + 2t.$$

Solve each equation for t to create the symmetric equation of the line:

$$\frac{x-1}{-4} = y-4 = \frac{z+2}{2}.$$

Exercise 11.5.1

Find parametric and symmetric equations of the line passing through points $(1,-3,2)$ and $(5,-2,8)$.

Sometimes we don't want the equation of a whole line, just a line segment. In this case, we limit the values of our parameter t. For example, let $P(x_0,y_0,z_0)$ and $Q(x_1,y_1,z_1)$ be points on a line, and let $\vec{\mathbf{p}} = \langle x_0,y_0,z_0\rangle$ and $\vec{\mathbf{q}} = \langle x_1,y_1,z_1\rangle$ be the associated position vectors. In addition, let $\vec{\mathbf{r}} = \langle x,y,z\rangle$. We want to find a vector equation for the line segment between P and Q. Using P as our known point on the line, and $\overrightarrow{PQ} = \langle x_1-x_0, y_1-y_0, z_1-z_0\rangle$ as the direction vector equation, Equation 11.5.4 gives

$$\vec{\mathbf{r}} = \vec{\mathbf{p}} + t(\overrightarrow{PQ}). \tag{11.5.8}$$

Equation 11.5.8 can be expanded using properties of vectors:

$$\begin{aligned}
\vec{\mathbf{r}} &= \vec{\mathbf{p}} + t(\overrightarrow{PQ}) \\
&= \langle x_0,y_0,z_0\rangle + t\langle x_1-x_0, y_1-y_0, z_1-z_0\rangle \\
&= \langle x_0,y_0,z_0\rangle + t(\langle x_1,y_1,z_1\rangle - \langle x_0,y_0,z_0\rangle) \\
&= \langle x_0,y_0,z_0\rangle + t\langle x_1,y_1,z_1\rangle - t\langle x_0,y_0,z_0\rangle \\
&= (1-t)\langle x_0,y_0,z_0\rangle + t\langle x_1,y_1,z_1\rangle \\
&= (1-t)\vec{\mathbf{p}} + t\vec{\mathbf{q}}.
\end{aligned}$$

Thus, the vector equation of the line passing through P and Q is

$$\vec{\mathbf{r}} = (1-t)\vec{\mathbf{p}} + t\vec{\mathbf{q}}. \tag{11.5.9}$$

Remember that we did not want the equation of the whole line, just the line segment between P and Q. Notice that when $t=0$, we have $\vec{\mathbf{r}} = \vec{\mathbf{p}}$, and when $t=1$, we have $\vec{\mathbf{r}} = \vec{\mathbf{q}}$. Therefore, the vector equation of the line segment between P and Q is

$$\vec{\mathbf{r}} = (1-t)\vec{\mathbf{p}} + t\vec{\mathbf{q}}, 0 \le t \le 1. \tag{11.5.10}$$

Going back to Equation 11.5.4, we can also find parametric equations for this line segment. We have

$$\begin{aligned}
\vec{\mathbf{r}} &= \vec{\mathbf{p}} + t(\overrightarrow{PQ}) \\
\langle x,y,z\rangle &= \langle x_0,y_0,z_0\rangle + t\langle x_1-x_0, y_1-y_0, z_1-z_0\rangle \\
&= \langle x_0 + t(x_1-x_0), y_0 + t(y_1-y_0), z_0 + t(z_1-z_0)\rangle.
\end{aligned}$$

Then, the parametric equations are

$$\begin{aligned}
x &= x_0 + t(x_1-x_0) \\
y &= y_0 + t(y_1-y_0) \\
z &= z_0 + t(z_1-z_0),\ 0 \le t \le 1.
\end{aligned}$$

Example 11.5.2: Parametric Equations of a Line Segment

Find parametric equations of the line segment between the points $P(2,1,4)$ and $Q(3,-1,3)$.

Solution

Start with the parametric equations for a line (Equations ???) and work with each component separately:

$$\begin{aligned}
x &= x_0 + t(x_1-x_0) \\
&= 2 + t(3-2) \\
&= 2 + t, \\
y &= y_0 + t(y_1-y_0) \\
&= 1 + t(-1-1) \\
&= 1 - 2t,
\end{aligned}$$

and

$$\begin{aligned} z &= z_0 + t(z_1 - z_0) \\ &= 4 + t(3 - 4) \\ &= 4 - t. \end{aligned}$$

Therefore, the parametric equations for the line segment are

$$\begin{aligned} x &= 2 + t \\ y &= 1 - 2t \\ z &= 4 - t,\ 0 \le t \le 1. \end{aligned}$$

Exercise 11.5.2

Find parametric equations of the line segment between points $P(-1, 3, 6)$ and $Q(-8, 2, 4)$.

Distance between a Point and a Line

We already know how to calculate the distance between two points in space. We now expand this definition to describe the distance between a point and a line in space. Several real-world contexts exist when it is important to be able to calculate these distances. When building a home, for example, builders must consider "setback" requirements, when structures or fixtures have to be a certain distance from the property line. Air travel offers another example. Airlines are concerned about the distances between populated areas and proposed flight paths.

Let L be a line in the plane and let M be any point not on the line. Then, we define distance d from M to L as the length of line segment $\overline{MP}$, where P is a point on L such that $\overline{MP}$ is perpendicular to L (Figure 11.5.2).

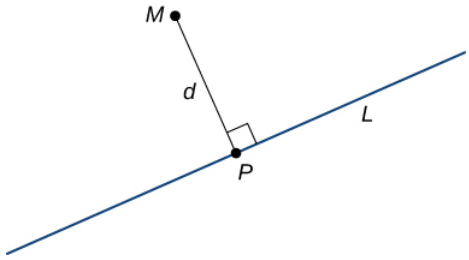


Figure 11.5.2: The distance from point M to line L is the length of $\overline{MP}$.

When we're looking for the distance between a line and a point in space, Figure 11.5.2 still applies. We still define the distance as the length of the perpendicular line segment connecting the point to the line. In space, however, there is no clear way to know which point on the line creates such a perpendicular line segment, so we select an arbitrary point on the line and use properties of vectors to calculate the distance. Therefore, let P be an arbitrary point on line L and let $\vec{\mathbf{v}}$ be a direction vector for L (Figure 11.5.3).

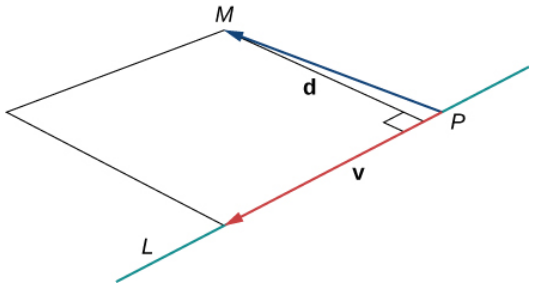


Figure 11.5.3: Vectors $\overrightarrow{PM}$ and $\vec{\mathbf{v}}$ *form two sides of a parallelogram with base* $\|\vec{\mathbf{v}}\|$ *and height* d*, which is the distance between a line and a point in space.*

Vectors $\overrightarrow{PM}$ and $\vec{\mathbf{v}}$ form two sides of a parallelogram with area $\|\overrightarrow{PM} \times \vec{\mathbf{v}}\|$. Using a formula from geometry, the area of this parallelogram can also be calculated as the product of its base and height:

$$\|\overrightarrow{PM} \times \vec{\mathbf{v}}\| = \|\vec{\mathbf{v}}\| d. \tag{11.5.11}$$

We can use this formula to find a general formula for the distance between a line in space and any point not on the line.

Distance from a Point to a Line

Let L be a line in space passing through point P with direction vector $\vec{\mathbf{v}}$. If M is any point not on L, then the distance from M to L is

$$d = \frac{\|\overrightarrow{PM} \times \vec{\mathbf{v}}\|}{\|\vec{\mathbf{v}}\|}. \tag{11.5.12}$$

Example 11.5.3: Calculating the Distance from a Point to a Line

Find the distance between the point $M = (1, 1, 3)$ and line $\frac{x-3}{4} = \frac{y+1}{2} = z - 3$.

Solution:

From the symmetric equations of the line, we know that vector $\vec{\mathbf{v}} = \langle 4, 2, 1 \rangle$ is a direction vector for the line. Setting the symmetric equations of the line equal to zero, we see that point $P(3, -1, 3)$ lies on the line. Then,

$$\begin{aligned} \overrightarrow{PM} &= \langle 1-3, 1-(-1), 3-3 \rangle \\ &= \langle -2, 2, 0 \rangle. \end{aligned}$$

To calculate the distance, we need to find $\overrightarrow{PM} \times \vec{\mathbf{v}}$:

$$\begin{aligned} \overrightarrow{PM} \times \vec{\mathbf{v}} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -2 & 2 & 0 \\ 4 & 2 & 1 \end{vmatrix} \\ &= (2-0)\hat{\mathbf{i}} - (-2-0)\hat{\mathbf{j}} + (-4-8)\hat{\mathbf{k}} \\ &= 2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 12\hat{\mathbf{k}}. \end{aligned}$$

Therefore, the distance between the point and the line is (Figure 11.5.4)

$$\begin{aligned} d &= \frac{\|\overrightarrow{PM} \times \vec{\mathbf{v}}\|}{\|\vec{\mathbf{v}}\|} \\ &= \frac{\sqrt{2^2 + 2^2 + 12^2}}{\sqrt{4^2 + 2^2 + 1^2}} \\ &= \frac{2\sqrt{38}}{\sqrt{21}} \\ &= \frac{2\sqrt{798}}{21} \text{ units} \end{aligned}$$

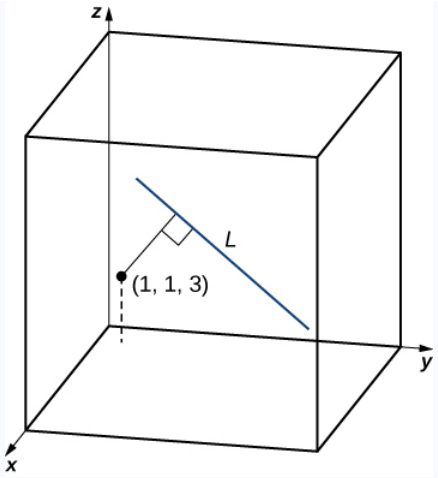


Figure 11.5.4: Point $(1,1,3)$ is approximately 2.7 units from the line with symmetric equations $\frac{x-3}{4}=\frac{y+1}{2}=z-3$.

Exercise 11.5.3

Find the distance between point $(0,3,6)$ and the line with parametric equations $x=1-t, y=1+2t, z=5+3t$.

Relationships between Lines

Given two lines in the two-dimensional plane, the lines are equal, they are parallel but not equal, or they intersect in a single point. In three dimensions, a fourth case is possible. If two lines in space are not parallel, but do not intersect, then the lines are said to be **skew lines** (Figure 11.5.5).

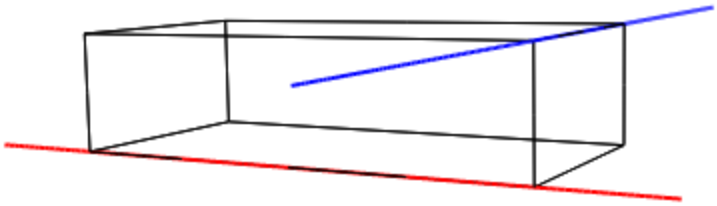


Figure* 11.5.5*: *In three dimensions, it is possible that two lines do not cross, even when they have different directions.*

To classify lines as parallel but not equal, equal, intersecting, or skew, we need to know two things: whether the direction vectors are parallel and whether the lines share a point (Figure **11.5.6**).

Direction Vectors Are Parallel? \ Lines Share A Common Point?	Yes	No
Yes	Equal	Parallel but not equal
No	Intersecting	Skew

Figure **11.5.6**: Determine the relationship between two lines based on whether their direction vectors are parallel and whether they share a point.

Example 11.5.4: Classifying Lines in Space

For each pair of lines, determine whether the lines are equal, parallel but not equal, skew, or intersecting.

a.

- $L_1 : x = 2s - 1, y = s - 1, z = s - 4$

https://math.libretexts.org/@go/page/10159

- $L_2 : x = t - 3, y = 3t + 8, z = 5 - 2t$

b.

- $L_1 : x = -y = z$
- $L_2 : \dfrac{x-3}{2} = y = z - 2$

c.

- $L_1 : x = 6s - 1, y = -2s, z = 3s + 1$
- $L_2 : \dfrac{x-4}{6} = \dfrac{y+3}{-2} = \dfrac{z-1}{3}$

Solution

a. Line L_1 has direction vector $\vec{\mathbf{v}}_1 = \langle 2, 1, 1 \rangle$; line L_2 has direction vector $\vec{\mathbf{v}}_2 = \langle 1, 3, -2 \rangle$. Because the direction vectors are not parallel vectors, the lines are either intersecting or skew. To determine whether the lines intersect, we see if there is a point, (x, y, z), that lies on both lines. To find this point, we use the parametric equations to create a system of equalities:

$$2s - 1 = t - 3; \tag{11.5.13}$$

$$s - 1 = 3t + 8; \tag{11.5.14}$$

$$s - 4 = 5 - 2t. \tag{11.5.15}$$

By the first equation, $t = 2s + 2$. Substituting into the second equation yields

$$s - 1 = 3(2s + 2) + 8$$

$$s - 1 = 6s + 6 + 8$$

$$5s = -15$$

$$s = -3.$$

Substitution into the third equation, however, yields a contradiction:

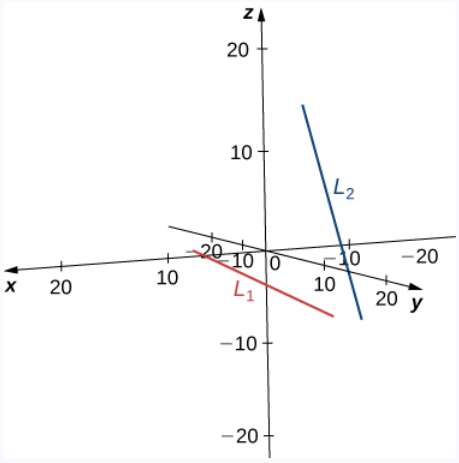
$$s - 4 = 5 - 2(2s + 2)$$

$$s - 4 = 5 - 4s - 4$$

$$5s = 5$$

$$s = 1.$$

There is no single point that satisfies the parametric equations for L_1 and L_2 simultaneously. These lines do not intersect, so they are skew (see the following figure).

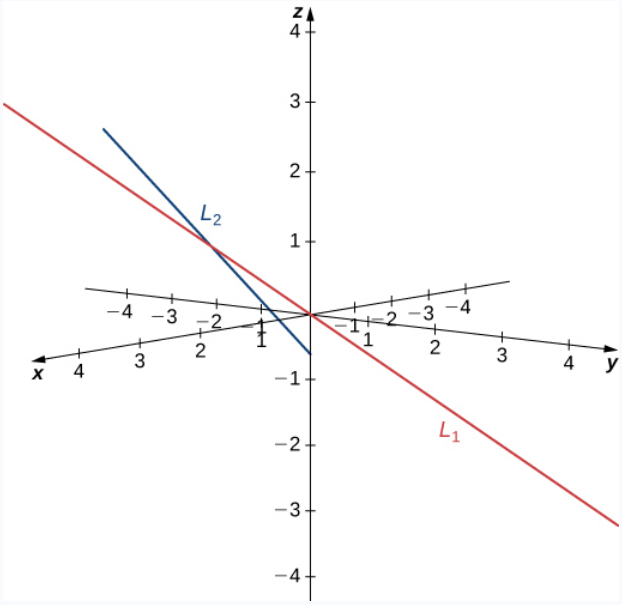


b. Line L_1 has direction vector $\vec{\mathbf{v}}_1 = \langle 1, -1, 1 \rangle$ and passes through the origin, $(0, 0, 0)$. Line L_2 has a different direction vector, $\vec{\mathbf{v}}_2 = \langle 2, 1, 1 \rangle$, so these lines are not parallel or equal. Let r represent the parameter for line L_1 and let s represent the parameter for

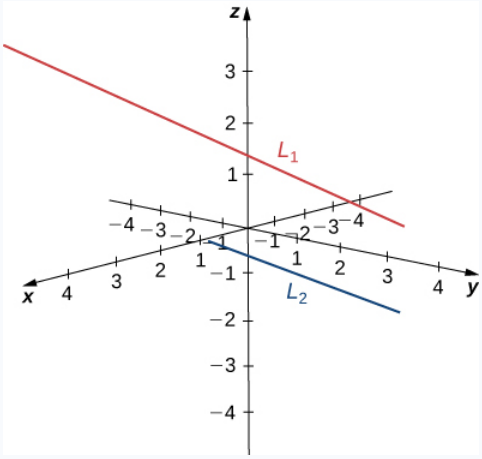
L_2:

Line L_1 :	Line L_2 :
$x = r$	$x = 2s + 3$
$y = -r$	$y = s$
$z = r$	$z = s + 2$

Solve the system of equations to find $r = 1$ and $s = -1$. If we need to find the point of intersection, we can substitute these parameters into the original equations to get $(1, -1, 1)$ (see the following figure).



c. Lines L_1 and L_2 have equivalent direction vectors: $\vec{\mathbf{v}} = \langle 6, -2, 3 \rangle$. These two lines are parallel (see the following figure).



Exercise 11.5.4

Describe the relationship between the lines with the following parametric equations:

$$x = 1 - 4t, y = 3 + t, z = 8 - 6t$$

$$x = 2 + 3s, y = 2s, z = -1 - 3s.$$

Equations for a Plane

We know that a line is determined by two points. In other words, for any two distinct points, there is exactly one line that passes through those points, whether in two dimensions or three. Similarly, given any three points that do not all lie on the same line, there is a unique plane that passes through these points. Just as a line is determined by two points, a plane is determined by three.

This may be the simplest way to characterize a plane, but we can use other descriptions as well. For example, given two distinct, intersecting lines, there is exactly one plane containing both lines. A plane is also determined by a line and any point that does not lie on the line. These characterizations arise naturally from the idea that a plane is determined by three points. Perhaps the most surprising characterization of a plane is actually the most useful.

Imagine a pair of orthogonal vectors that share an initial point. Visualize grabbing one of the vectors and twisting it. As you twist, the other vector spins around and sweeps out a plane. Here, we describe that concept mathematically. Let $\vec{\mathbf{n}} = \langle a, b, c \rangle$ be a vector and $P = (x_0, y_0, z_0)$ be a point. Then the set of all points $Q = (x, y, z)$ such that $\overrightarrow{PQ}$ is orthogonal to $\vec{\mathbf{n}}$ forms a plane (Figure 11.5.7). We say that $\vec{\mathbf{n}}$ is a **normal vector**, or perpendicular to the plane. Remember, the dot product of orthogonal vectors is zero. This fact generates the **vector equation of a plane**:

$$\vec{\mathbf{n}} \cdot \overrightarrow{PQ} = 0. \tag{11.5.16}$$

Rewriting this equation provides additional ways to describe the plane:

$$\begin{aligned} \vec{\mathbf{n}} \cdot \overrightarrow{PQ} &= 0 \\ \langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle &= 0 \\ a(x - x_0) + b(y - y_0) + c(z - z_0) &= 0. \end{aligned}$$

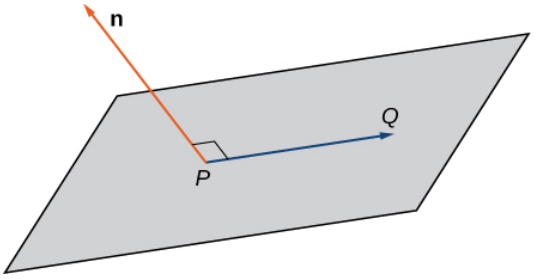


Figure 11.5.7: Given a point P and vector $\vec{\mathbf{n}}$, *the set of all points Q with $\overrightarrow{PQ}$ orthogonal to $\vec{\mathbf{n}}$ forms a plane.*

Definition: Standard Form of the Equation of a Plane

Given a point P and vector $\vec{\mathbf{n}}$, the set of all points Q satisfying the equation $\vec{\mathbf{n}} \cdot \overrightarrow{PQ} = 0$ forms a plane. The equation

$$\vec{\mathbf{n}} \cdot \overrightarrow{PQ} = 0$$

is known as the *vector equation of a plane*.

The *standard form of the equation of a plane* containing point $P = (x_0, y_0, z_0)$ with normal vector $\vec{n} = \langle a, b, c \rangle$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

This equation can be expressed as $ax + by + cz + d = 0$, where $d = -ax_0 - by_0 - cz_0$. This form of the equation is sometimes called the *general form of the equation of a plane.*

As the standard and general forms of the equation of a plane no longer contain an explicit vector, they are sometimes called *scalar equations of a plane.*

As described earlier in this section, any three points that do not all lie on the same line determine a plane. Given three such points, we can find an equation for the plane containing these points.

Example 11.5.5: Writing an Equation of a Plane Given Three Points in the Plane

Write an equation for the plane containing points $P = (1, 1, -2), Q = (0, 2, 1)$, and $R = (-1, -1, 0)$ in both standard and general forms.

Solution

To write an equation for a plane, we must find a normal vector for the plane. We start by identifying two vectors in the plane:

$$\begin{aligned}\overrightarrow{PQ} &= \langle 0-1, 2-1, 1-(-2)\rangle \\ &= \langle -1, 1, 3\rangle \\ \overrightarrow{QR} &= \langle -1-0, -1-2, 0-1\rangle \\ &= \langle -1, -3, -1\rangle.\end{aligned}$$

The cross product $\overrightarrow{PQ} \times \overrightarrow{QR}$ is orthogonal to both $\overrightarrow{PQ}$ and $\overrightarrow{QR}$, so it is normal to the plane that contains these two vectors:

$$\begin{aligned}\vec{\mathbf{n}} &= \overrightarrow{PQ} \times \overrightarrow{QR} \\ &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -1 & 1 & 3 \\ -1 & -3 & -1 \end{vmatrix} \\ &= (-1+9)\hat{\mathbf{i}} - (1+3)\hat{\mathbf{j}} + (3+1)\hat{\mathbf{k}} \\ &= 8\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 4\hat{\mathbf{k}}.\end{aligned}$$

Thus, $\vec{\mathbf{n}} = \langle 8, -4, 4\rangle$, and we can choose any of the three given points to write an equation of the plane:

$$\begin{aligned}8(x-1) - 4(y-1) + 4(z+2) = 0 \quad &\text{(Standard form)} \\ 8x - 4y + 4z + 4 = 0. \quad &\text{(General form)}\end{aligned}$$

The scalar equations of a plane vary depending on the normal vector and point chosen.

Example 11.5.6: Writing an Equation for a Plane Given a Point and a Line

Find an equation of the plane that passes through point $(1, 4, 3)$ and contains the line given by $x = \dfrac{y-1}{2} = z + 1$.

Solution

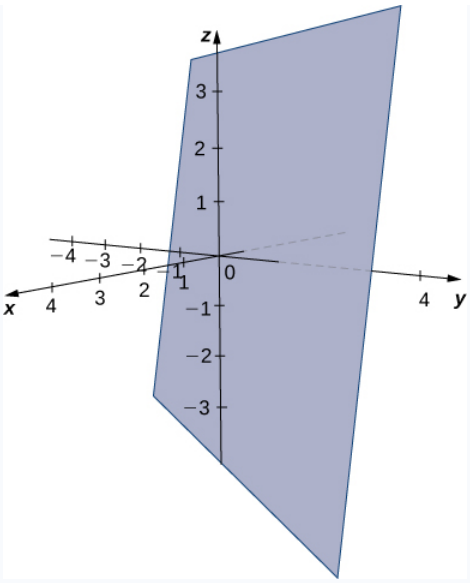
Symmetric equations describe the line that passes through point $(0, 1, -1)$ parallel to vector $\vec{\mathbf{v}}_1 = \langle 1, 2, 1\rangle$ (see the following figure). Use this point and the given point, $(1, 4, 3)$, to identify a second vector parallel to the plane:

$$\vec{\mathbf{v}}_2 = \langle 1-0, 4-1, 3-(-1)\rangle = \langle 1, 3, 4\rangle.$$

Use the cross product of these vectors to identify a normal vector for the plane:

$$\begin{aligned}\vec{\mathbf{n}} &= \vec{\mathbf{v}}_1 \times \vec{\mathbf{v}}_2 \\ &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 2 & 1 \\ 1 & 3 & 4 \end{vmatrix} \\ &= (8-3)\hat{\mathbf{i}} - (4-1)\hat{\mathbf{j}} + (3-2)\hat{\mathbf{k}} \\ &= 5\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + \hat{\mathbf{k}}.\end{aligned}$$

The scalar equations for the plane are $5x - 3(y-1) + (z+1) = 0$ and $5x - 3y + z + 4 = 0$.



Exercise 11.5.6

Find an equation of the plane containing the lines L_1 and L_2:

$$L_1 : x = -y = z$$

$$L_2 : \frac{x-3}{2} = y = z - 2.$$

Distance Between a Plane and a Point

Now that we can write an equation for a plane, we can use the equation to find the distance d between a point P and the plane. It is defined as the shortest possible distance from P to a point on the plane.

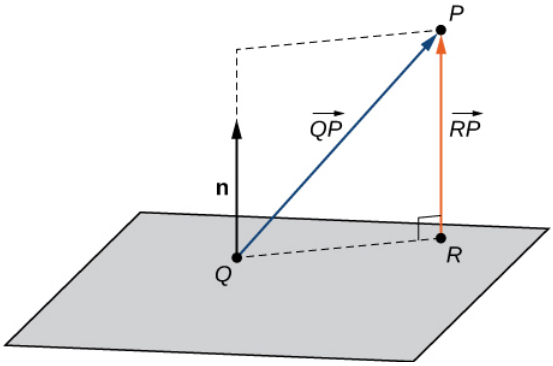


Figure 11.5.8: We want to find the shortest distance from point P to the plane. Let point R be the point in the plane such that, for any other point in the plane Q, $\|\overrightarrow{RP}\| < \|\overrightarrow{QP}\|$.

Just as we find the two-dimensional distance between a point and a line by calculating the length of a line segment perpendicular to the line, we find the three-dimensional distance between a point and a plane by calculating the length of a line segment perpendicular to the plane. Let R be the point in the plane such that $\overrightarrow{RP}$ is orthogonal to the plane, and let Q be an arbitrary point in the plane. Then the projection of vector $\overrightarrow{QP}$ onto the normal vector describes vector $\overrightarrow{RP}$, as shown in Figure 11.5.8.

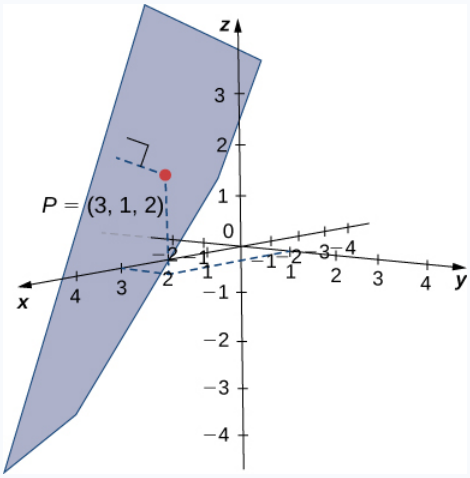
The Distance between a Plane and a Point

Suppose a plane with normal vector $\vec{\mathbf{n}}$ passes through point Q. The distance d from the plane to a point P not in the plane is given by

$$d = \|\text{proj}_{\vec{\mathbf{n}}} \overrightarrow{QP}\| = |\text{ comp}_{\vec{\mathbf{n}}} \overrightarrow{QP}| = \frac{|\overrightarrow{QP} \cdot \vec{\mathbf{n}}|}{\|\vec{\mathbf{n}}\|}. \tag{11.5.17}$$

Example 11.5.7: Distance between a Point and a Plane

Find the distance between point $P = (3, 1, 2)$ and the plane given by $x - 2y + z = 5$ (see the following figure).



Solution

The coefficients of the plane's equation provide a normal vector for the plane: $\vec{\mathbf{n}} = \langle 1, -2, 1 \rangle$. To find vector $\overrightarrow{QP}$, we need a point in the plane. Any point will work, so set $y = z = 0$ to see that point $Q = (5, 0, 0)$ lies in the plane. Find the component form of the vector from Q to P:

$$\overrightarrow{QP} = \langle 3 - 5, 1 - 0, 2 - 0 \rangle = \langle -2, 1, 2 \rangle.$$

Apply the distance formula from Equation 11.5.17:

$$\begin{aligned} d &= \frac{|\overrightarrow{QP} \cdot \vec{\mathbf{n}}|}{\|\vec{\mathbf{n}}\|} \\ &= \frac{|\langle -2, 1, 2 \rangle \cdot \langle 1, -2, 1 \rangle|}{\sqrt{1^2 + (-2)^2 + 1^2}} \\ &= \frac{|-2 - 2 + 2|}{\sqrt{6}} \\ &= \frac{2}{\sqrt{6}} = \frac{\sqrt{6}}{3} \text{ units.} \end{aligned}$$

Exercise 11.5.7

Find the distance between point $P = (5, -1, 0)$ and the plane given by $4x + 2y - z = 3$.

Intersection of a Line and a Plane

A given line and a given plane may or may not intersect. If the line does intersect with the plane, it's possible that the line is completely contained in the plane as well. How can we differentiate between these three possibilities?

Example 11.5.8: Finding the intersection of a Line and a plane

Determine whether the following line intersects with the given plane. If they do intersect, determine whether the line is contained in the plane or intersects it in a single point. Finally, if the line intersects the plane in a single point, determine this point of intersection.

$$\text{Line:}\quad \begin{aligned} x &= 2 - t \\ y &= 1 + t \\ z &= 3t \end{aligned} \qquad \text{Plane:}\quad 3x - 2y + z = 10$$

Solution

Notice that we can substitute the expressions of t given in the parametric equations of the line into the plane equation for x, y, and z.

$$3(2 - t) - 2(1 + t) + 3t = 10$$

Solving this equation for t:

$$6 - 3t - 2 - 2t + 3t = 10$$

$$4 - 2t = 10$$

$$-2t = 6$$

$$t = -3$$

Since we found a single value of t from this process, we know that the line should intersect the plane in a single point, here where $t = -3$. So the point of intersection can be determined by plugging this value in for t in the parametric equations of the line.

Here: $x = 2 - (-3) = 5, \quad y = 1 + (-3) = -2, \text{ and } \quad z = 3(-3) = -9$.

So the point of intersection of this line with this plane is $(5, -2, -9)$. We can verify this by putting the coordinates of this point into the plane equation and checking to see that it is satisfied.

Check: $3(5) - 2(-2) + (-9) = 15 + 4 - 9 = 10$ ✓

Now that we have examined what happens when there is a single point of intersection between a line and a point, let's consider how we know if the line either does not intersect the plane at all or if it lies on the plane (i.e., every point on the line is also on the plane).

Example 11.5.9: Other relationships between a line and a plane

Determine whether the following line intersects with the given plane. If they do intersect, determine whether the line is contained in the plane or intersects it in a single point. Finally, if the line intersects the plane in a single point, determine this point of intersection.

$$\text{Line:}\quad \begin{aligned} x &= 1 + 2t \\ y &= -2 + 3t \\ z &= -1 + 4t \end{aligned} \qquad \text{Plane:}\quad x + 2y - 2z = 5$$

Solution

Substituting the expressions of t given in the parametric equations of the line into the plane equation gives us:

$$(1 + 2t) + 2(-2 + 3t) - 2(-1 + 4t) = 5$$

Simplifying the left side gives us:

$$1 + 2t - 4 + 6t + 2 - 8t = 5$$

Collecting like terms on the left side causes the variable t to cancel out and leaves us with a contradiction:

$$-1 = 5$$

Since this is not true, we know that there is no value of t that makes this equation true, and thus there is no value of t that will give us a point on the line that is also on the plane. This means that **this line does not intersect with this plane** and there will be **no point of intersection**.

How can we tell if a line is contained in the plane?

What if we keep the same line, but modify the plane equation to be $x + 2y - 2z = -1$? In this case, repeating the steps above would again cause the variable t to be eliminated from the equation, but it would leave us with an identity, $-1 = -1$, rather than a contradiction. This means that **every** value of t will produce a point on the line that is also on the plane, telling us that the line is contained in the plane whose equation is $x + 2y - 2z = -1$.

Parallel and Intersecting Planes

We have discussed the various possible relationships between two lines in two dimensions and three dimensions. When we describe the relationship between two planes in space, we have only two possibilities: the two distinct planes are parallel or they intersect. When two planes are parallel, their normal vectors are parallel. When two planes intersect, the intersection is a line (Figure 11.5.9).

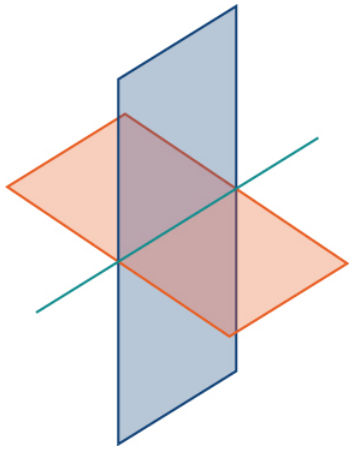
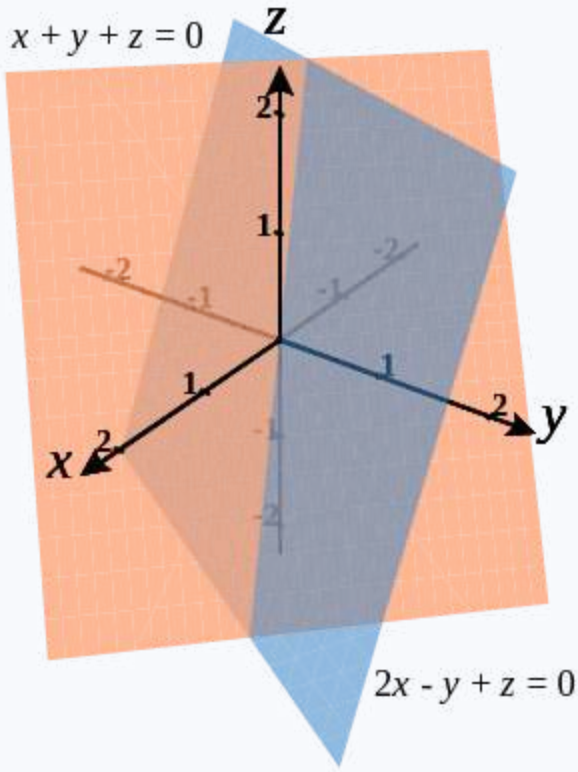


Figure 11.5.9: The intersection of two nonparallel planes is always a line.

We can use the equations of the two planes to find parametric equations for the line of intersection as shown below in Example 11.5.10.

Example 11.5.10: Finding the Line of Intersection for Two Planes

Determine the parametric equations for the line of intersection of the planes given by $x + y + z = 0$ and $2x - y + z = 0$. See the following figure.



Solution

Finding the line of intersection:

We will use two steps to solve for the line of intersection.

1. Find a point on the line of intersection (i.e., a point that lies on both planes).
2. Find a direction vector for the line of intersection.

1. To find a point that lies on both planes, we first use the elimination method for solving a system of equations to eliminate one of the variables, in this case, y. For this pair of plane equations, we just have to add the equations to eliminate a variable, but sometimes we may need to multiply one or both equations by a factor to make it easier to eliminate a variable when we add the equations.

Adding these equations gives us:

$$x + y + z = 0$$

$$2x - y + z = 0$$

$$3x + 2z = 0.$$

Solve for one of the variables. If we solve for x, we find $x = -\frac{2}{3}z$. Now choose any value for z, say $z = 3$. Then $x = -2$.

Now plugging these two values into one of the plane equations, we can solve for the corresponding value of y that will give us a point that should satisfy both planes (i.e., it will lie on the line of intersection).

Plugging into the equation $x+y+z=0$ gives us $-2+y+3=0 \quad \rightarrow \quad y=-1$. So a point on the line of intersection is $(-2,-1,3)$.

Verify that this point satisfies both plane equations: $-2+(-1)+3=0\checkmark$ and $2(-2)-(-1)+3=-4+1+3=0\checkmark$

2. Now we are ready to find a direction vector for the line of intersection. Since this line must lie in both planes, the direction vector must be orthogonal to the normal vector s of both planes. Since we need a vector that is orthogonal to both normal vectors, we use the cross product of the normal vectors to obtain one. Thus:

Direction vector of the line of intersection: $\vec{\mathbf{v}}=\vec{\mathbf{n}}_1\times\vec{\mathbf{n}}_2=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 1 & 1 & 1\\ 2 & -1 & 1\end{vmatrix}=(1-(-1))\hat{\mathbf{i}}-(1-2)\hat{\mathbf{j}}+(-1-2)\hat{\mathbf{k}}=2\hat{\mathbf{i}}+\hat{\mathbf{j}}-3\hat{\mathbf{k}}.$

Now the line of intersection will contain the point $(-2,-1,3)$ and have direction vector $\vec{\mathbf{v}}=2\hat{\mathbf{i}}+\hat{\mathbf{j}}-3\hat{\mathbf{k}}$.

So a possible set of parametric equations of the line of intersection are: $x=-2+2t, \quad y=-1+t, \quad z=3-3t.$

To check, we could either find another point on this line (for $t\neq 0$ and verify that it satisfies both plane equations, or we could substitute these expressions of t into both plane equations and show that we obtain an identity in both cases (such as $5=5$ or $0=0$).

For example, if we let $t=1$, the line's equations give us the origin $(0,0,0)$ as a second point on the intersection line. It's easy to see that this point satisfies both plane equations.

Exercise 11.5.8

Find parametric equations for the line formed by the intersection of planes $x+y-z=3$ and $3x-y+3z=5$.

In addition to finding the equation of the line of intersection between two planes, we may need to find the angle formed by the intersection of two planes. For example, builders constructing a house need to know the angle where different sections of the roof meet to know whether the roof will look good and drain properly.

When two planes intersect, note that there are two supplementary angles formed between the planes (See Figure 11.5.9). Typically the acute angle between two planes is the one desired. We can determine the acute angle between the two planes by finding the angle between their normal vectors and forcing it to be acute (by taking its supplement, if necessary). Figure 11.5.10 shows why this is true.

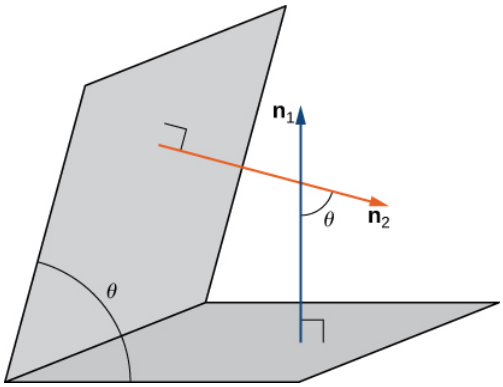


Figure 11.5.10: The acute angle between two planes has the same measure as the angle between the normal vectors for the planes or its supplement.

To be sure we obtain the acute angle between the two planes, we take the absolute value of the dot product of the two normal vectors, thus forcing the resulting angle θ to be acute.

So, to find the acute angle θ between two planes with normal vectors $\vec{\mathbf{n}}_1$ and $\vec{\mathbf{n}}_2$, we use the definition of the dot product and add the absolute values mentioned above to first find the cosine of the angle:

$$\cos\theta=\frac{|\vec{\mathbf{n}}_1\cdot\vec{\mathbf{n}}_2|}{\|\vec{\mathbf{n}}_1\|\|\vec{\mathbf{n}}_2\|}. \tag{11.5.18}$$

Example 11.5.11: Finding the Angle between Two Planes

Find the angle between the planes given by $x+y+z=0$ and $2x-y+z=0$ for which we found the line of intersection in Example 11.5.10.

Solution

Finding the angle between the planes:

Note that the two planes have nonparallel normals, so the planes intersect. They intersect forming two angles, one obtuse and one acute. We choose to find the acute angle, so we use the modified version of the definition of the dot product that we stated above.

We can read the components of a normal vector from the coefficients of the variables in each plane equation.

Thus: $\vec{\mathbf{n}}_1 =< 1,1,1 >$ and $\vec{\mathbf{n}}_2 =< 2,-1,1 >$.

Now

$$\vec{\mathbf{n}}_1 \cdot \vec{\mathbf{n}}_2 = 2+(-1)+1 = 2.$$

Their magnitudes are:

$$\|\vec{\mathbf{n}}_1\| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$$

$$\|\vec{\mathbf{n}}_2\| = \sqrt{2^2+(-1)^2+1^2} = \sqrt{6}$$

Then

$$\cos\theta = \frac{|\vec{\mathbf{n}}_1 \cdot \vec{\mathbf{n}}_2|}{\|\vec{\mathbf{n}}_1\|\|\vec{\mathbf{n}}_2\|} = \frac{|2|}{\sqrt{3}\sqrt{6}} = \frac{2}{\sqrt{18}}. \tag{11.5.19}$$

So the acute angle between these two planes is given by $\theta = \arccos\left(\frac{2}{\sqrt{18}}\right) \approx 61.9° \approx 1.08$ radians.

Once we find the angle between two planes we can use it to determine whether the two planes are parallel or orthogonal or if they intersect at some other angle.

Example 11.5.12: determine if planes are Parallel, Orthogonal, or neither

Determine whether each pair of planes is parallel, orthogonal, or neither. If the planes are intersecting, but not orthogonal, find the measure of the angle between them. Give the answer in radians and round to two decimal places.

a. $x+2y-z=8$ and $2x+4y-2z=10$
b. $2x-3y+2z=3$ and $6x+2y-3z=1$
c. $x+y+z=4$ and $x-3y+5z=1$

Solution:

a. The normal vectors for these planes are $\vec{\mathbf{n}}_1 = \langle 1,2,-1\rangle$ and $\vec{\mathbf{n}}_2 = \langle 2,4,-2\rangle$. These two vectors are scalar multiples of each other. The normal vectors are parallel, so the planes are parallel.
b. The normal vectors for these planes are $\vec{\mathbf{n}}_1 = \langle 2,-3,2\rangle$ and $\vec{\mathbf{n}}_2 = \langle 6,2,-3\rangle$. Taking the dot product of these vectors, we have

$$\begin{aligned}\vec{\mathbf{n}}_1 \cdot \vec{\mathbf{n}}_2 &= \langle 2,-3,2\rangle \cdot \langle 6,2,-3\rangle \\ &= 2(6)-3(2)+2(-3) = 0.\end{aligned}$$

The normal vectors are orthogonal, so the corresponding planes are orthogonal as well.
c. The normal vectors for these planes are $\vec{\mathbf{n}}_1 = \langle 1,1,1\rangle$ and $\vec{\mathbf{n}}_2 = \langle 1,-3,5\rangle$:

$$\begin{aligned}\cos\theta &= \frac{|\vec{\mathbf{n}}_1 \cdot \vec{\mathbf{n}}_2|}{\|\vec{\mathbf{n}}_1\|\|\vec{\mathbf{n}}_2\|} \\ &= \frac{|\langle 1,1,1\rangle \cdot \langle 1,-3,5\rangle|}{\sqrt{1^2+1^2+1^2}\sqrt{1^2+(-3)^2+5^2}} \\ &= \frac{3}{\sqrt{105}}\end{aligned}$$

Then $\theta = \arccos\frac{3}{\sqrt{105}} \approx 1.27$ rad.
Thus the angle between the two planes is about 1.27 rad, or approximately $73°$.

Exercise 11.5.9

Find the measure of the angle between planes $x + y - z = 3$ and $3x - y + 3z = 5$. Give the answer in radians and round to two decimal places.

When we find that two planes are parallel, we may need to find the distance between them. To find this distance, we simply select a point in one of the planes. The distance from this point to the other plane is the distance between the planes.

Previously, we introduced the formula for calculating this distance in Equation 11.5.17:

$$d = \frac{\overrightarrow{QP} \cdot \vec{\mathbf{n}}}{\|\vec{\mathbf{n}}\|}, \tag{11.5.20}$$

where Q is a point on the plane, P is a point not on the plane, and $\vec{n}$ is the normal vector that passes through point Q. Consider the distance from point (x_0, y_0, z_0) to plane $ax + by + cz + k = 0$. Let (x_1, y_1, z_1) be any point in the plane. Substituting into the formula yields

$$\begin{aligned}d &= \frac{|a(x_0 - x_1) + b(y_0 - y_1) + c(z_0 - z_1)|}{\sqrt{a^2+b^2+c^2}} \\ &= \frac{|ax_0 + by_0 + cz_0 + k|}{\sqrt{a^2+b^2+c^2}}.\end{aligned}$$

We state this result formally in the following theorem.

Distance from a Point to a Plane

Let $P(x_0, y_0, z_0)$ be a point. The distance from P to plane $ax + by + cz + k = 0$ is given by

$$d = \frac{|ax_0 + by_0 + cz_0 + k|}{\sqrt{a^2+b^2+c^2}}. \tag{11.5.21}$$

Example 11.5.13: Finding the Distance between Parallel Planes

Find the distance between the two parallel planes given by $2x + y - z = 2$ and $2x + y - z = 8$.

Solution

Point $(1, 0, 0)$ lies in the first plane. The desired distance, then, is

$$\begin{aligned}d &= \frac{|ax_0 + by_0 + cz_0 + k|}{\sqrt{a^2+b^2+c^2}} \\ &= \frac{|2(1) + 1(0) + (-1)(0) + (-8)|}{\sqrt{2^2+1^2+(-1)^2}} \\ &= \frac{6}{\sqrt{6}} = \sqrt{6}\,\text{units}\end{aligned}$$

Exercise 11.5.10:

Find the distance between parallel planes $5x - 2y + z = 6$ and $5x - 2y + z = -3$.

Distance between Two Skew Lines

Finding the distance from a point to a line or from a line to a plane seems like a pretty abstract procedure. But, if the lines represent pipes in a chemical plant or tubes in an oil refinery or roads at an intersection of highways, confirming that the distance between them meets specifications can be both important and awkward to measure. One way is to model the two pipes as lines, using the techniques in this chapter, and then calculate the distance between them. The calculation involves forming vectors along the directions of the lines and using both the cross product and the dot product.

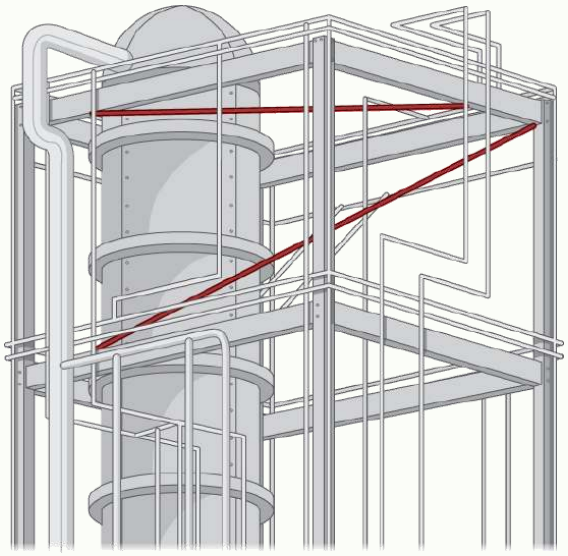


Figure 11.5.11: Industrial pipe installations often feature pipes running in different directions. How can we find the distance between two skew pipes?

The symmetric forms of two lines, L_1 and L_2, are

$$L_1 : \frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1} \tag{11.5.22}$$

$$L_2 : \frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}. \tag{11.5.23}$$

You are to develop a formula for the distance d between these two lines, in terms of the values $a_1, b_1, c_1; a_2, b_2, c_2; x_1, y_1, z_1$; and x_2, y_2, z_2. The distance between two lines is usually taken to mean the minimum distance, so this is the length of a line segment or the length of a vector that is perpendicular to both lines and intersects both lines.

1. First, write down two vectors, $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$, that lie along L_1 and L_2, respectively.

2. Find the cross product of these two vectors and call it $\vec{\mathbf{n}}$. This vector is perpendicular to $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$, and hence is perpendicular to both lines.

3. Use symmetric equations to find a convenient vector $\vec{\mathbf{v}}_{12}$ that lies between any two points, one on each line. Again, this can be done directly from the symmetric equations.

4. Find the projection of vector $\vec{\mathbf{v}}_{12}$ found in step **3** onto unit vector $\vec{\mathbf{n}}$ found in step **2**. This projection is perpendicular to both lines, and hence its length must be the perpendicular distance d between them.

5. Check that your formula gives the correct distance of $d = \frac{25}{\sqrt{198}} = \frac{25\sqrt{22}}{66}$ **units** ≈ 1.78 **units** between the following two lines:

$$L_1 : \frac{x-5}{2} = \frac{y-3}{4} = \frac{z-1}{3} \tag{11.5.24}$$

$$L_2 : \frac{x-6}{3} = \frac{y-1}{5} = \frac{z}{7}. \tag{11.5.25}$$

6. Is your general expression valid when the lines are parallel? If not, why not? (Hint: What do you know about the value of the cross product of two parallel vectors? Where would that result show up in your expression for d?)

7. Demonstrate that your expression for the distance is zero when the lines intersect. Recall that two lines intersect if they are not parallel and they are in the same plane. Hence, consider the direction of $\vec{\mathbf{n}}$ and $\vec{\mathbf{v}}_{12}$. What is the result of their dot product?

8. Consider the following application. Engineers at a refinery have determined they need to install support struts between many of the gas pipes to reduce damaging vibrations. To minimize cost, they plan to install these struts at the closest points between adjacent skewed pipes. Because they have detailed schematics of the structure, they are able to determine the correct lengths of the struts needed, and hence manufacture and distribute them to the installation crews without spending valuable time making measurements.

The rectangular frame structure has the dimensions $4.0 \times 15.0 \times 10.0\ \text{m}$ (height, width, and depth). One sector has a pipe entering the lower corner of the standard frame unit and exiting at the diametrically opposed corner (the one farthest away at the top); call this L_1. A second pipe enters and exits at the two different opposite lower corners; call this L_2 (Figure 11.5.12).

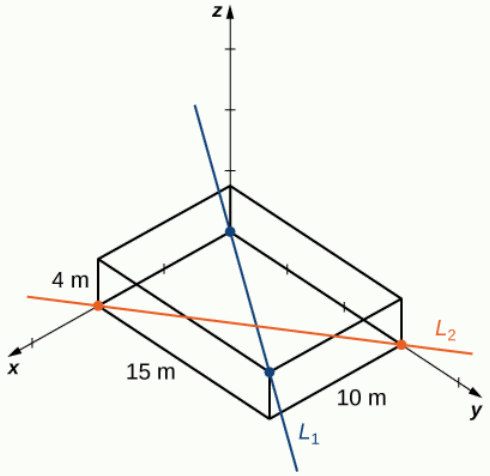


Figure 11.5.12: Two pipes cross through a standard frame unit.

Write down the vectors along the lines representing those pipes, find the cross product between them from which to create the mutually orthogonal vector $\vec{\mathbf{n}}$, define a vector that spans two points on each line, and finally determine the minimum distance between the lines. (Take the origin to be at the lower corner of the first pipe.) Similarly, you may also develop the symmetric equations for each line and substitute directly into your formula.

Key Concepts

- In three dimensions, the direction of a line is described by a direction vector. The vector equation of a line with direction vector $\vec{\mathbf{v}} = \langle a, b, c \rangle$ passing through point $P = (x_0, y_0, z_0)$ is $\vec{\mathbf{r}} = \vec{\mathbf{r}}_0 + t\vec{\mathbf{v}}$, where $\vec{\mathbf{r}}_0 = \langle x_0, y_0, z_0 \rangle$ is the position vector of point P. This equation can be rewritten to form the parametric equations of the line: $x = x_0 + ta, y = y_0 + tb$, and $z = z_0 + tc$. The line can also be described with the symmetric equations

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}. \tag{11.5.26}$$

- Let L be a line in space passing through point P with direction vector $\vec{\mathbf{v}}$. If Q is any point not on L, then the distance from Q to L is

$$d = \frac{\|\overrightarrow{PQ} \times \vec{\mathbf{v}}\|}{\|\vec{\mathbf{v}}\|}.$$

- In three dimensions, two lines may be parallel but not equal, equal, intersecting, or skew.
- Given a point P and vector $\vec{\mathbf{n}}$, the set of all points Q satisfying equation $\vec{\mathbf{n}} \cdot \overrightarrow{PQ} = 0$ forms a plane. This equation is known as the *vector equation of a plane*.
- The scalar equation of a plane containing point $P = (x_0, y_0, z_0)$ with normal vector $\vec{\mathbf{n}} = \langle a, b, c \rangle$ is

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

. This equation can be expressed as

$$ax+by+cz+d=0,$$

where $d=-ax_0-by_0-cz_0$. This form of the equation is sometimes called the general *form of the equation of a plane.*

- Suppose a plane with normal vector n passes through point Q. The distance D from the plane to point P not in the plane is given by

$$D=\|\text{proj}_{\vec{\mathbf{n}}}\overrightarrow{QP}\|=|\text{ comp}_{\vec{\mathbf{n}}}\vec{QP}|=\frac{|\vec{QP}\cdot\vec{\mathbf{n}}|}{\|\vec{\mathbf{n}}\|}.$$

- The normal vectors of parallel planes are parallel. When two planes intersect, they form a line.
- The measure of the angle θ between two intersecting planes can be found using the equation:

$$\cos\theta=\frac{|\vec{\mathbf{n}}_1\cdot\vec{\mathbf{n}}_2|}{\|\vec{\mathbf{n}}_1\|\|\vec{\mathbf{n}}_2\|}$$

where $\vec{\mathbf{n}}_1$ and $\vec{\mathbf{n}}_2$ are normal vectors to the planes.

- The distance D from point (x_0,y_0,z_0) to plane $ax+by+cz+d=0$ is given by

$$D=\frac{|a(x_0-x_1)+b(y_0-y_1)+c(z_0-z_1)|}{\sqrt{a^2+b^2+c^2}}=\frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}$$

.

Key Equations

- **Vector Equation of a Line**

$$\vec{\mathbf{r}}=\vec{\mathbf{r}}_0+t\vec{\mathbf{v}}$$

- **Parametric Equations of a Line**

$$x=x_0+ta, y=y_0+tb,$$

and

$$z=z_0+tc$$

- **Symmetric Equations of a Line**

$$\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}$$

- **Vector Equation of a Plane**

$$\vec{\mathbf{n}}\cdot\overrightarrow{PQ}=0$$

- **Scalar Equation of a Plane**

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

- **Distance between a Plane and a Point**

$$d=\|\text{proj}_{\vec{\mathbf{n}}}\overrightarrow{QP}\|=|\text{ comp}_{\vec{\mathbf{n}}}\overrightarrow{QP}|=\frac{|\overrightarrow{QP}\cdot\vec{\mathbf{n}}|}{\|\vec{\mathbf{n}}\|}$$

Glossary

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11.5E: Exercises for Equations of Lines and Planes in Space

In exercises 1 - 4, points P and Q are given. Let L be the line passing through points P and Q.

a. Find the vector equation of line L.

b. Find parametric equations of line L.

c. Find symmetric equations of line L.

d. Find parametric equations of the line segment determined by P and Q.

1) $P(-3,5,9), \quad Q(4,-7,2)$

2) $P(4,0,5), \quad Q(2,3,1)$

3) $P(-1,0,5), \quad Q(4,0,3)$

4) $P(7,-2,6), \quad Q(-3,0,6)$

For exercises 5 - 8, point P and vector $\vec{\mathbf{v}}$ are given. Let L be the line passing through point P with direction $\vec{\mathbf{v}}$.

a. Find parametric equations of line L.

b. Find symmetric equations of line L.

c. Find the intersection of the line with the xy-plane.

5) $P(1,-2,3)$, $\vec{\mathbf{v}} = \langle 1,2,3 \rangle$

6) $P(3,1,5)$, $\vec{\mathbf{v}} = \langle 1,1,1 \rangle$

7) $P(3,1,5)$, $\vec{\mathbf{v}} = \overrightarrow{QR}$, where $Q(2,2,3)$ and $R(3,2,3)$

8) $P(2,3,0)$, $\vec{\mathbf{v}} = \overrightarrow{QR}$, where $Q(0,4,5)$ and $R(0,4,6)$

For exercises 9 and 10, line L is given.

a. Find a point P that belongs to the line and a direction vector $\vec{\mathbf{v}}$ of the line. Express $\vec{\mathbf{v}}$ in component form.

b. Find the distance from the origin to line L.

9) $x = 1 + t, \quad y = 3 + t, \quad z = 5 + 4t, \quad t \in R$

10) $-x = y + 1, \quad z = 2$

11) Find the distance between point $A(-3,1,1)$ and the line of symmetric equations

$x = -y = -z.$

12) Find the distance between point $A(4,2,5)$ and the line of parametric equations

$x = -1 - t, \; y = -t, \; z = 2, \; t \in R.$

For exercises 13 - 14, lines L_1 and L_2 are given.

a. Verify whether lines L_1 and L_2 are parallel.

b. If the lines L_1 and L_2 are parallel, then find the distance between them.

13) $L_1 : x = 1 + t, \quad y = t, \quad z = 2 + t, \quad t \in R$ and $L_2 : x - 3 = y - 1 = z - 3$

14) $L_1 : x = 2, \quad y = 1, \quad z = t, \quad t \in R$ and $L_2 : x = 1, \quad y = 1, \quad z = 2 - 3t, \quad t \in R$

15) Show that the line passing through points $P(3,1,0)$ and $Q(1,4,-3)$ is perpendicular to the line with equation $x=3t, \quad y=-32+8t, \quad z=-9+6t, \quad t \in R$.

16) Are the lines of equations $x=-2+2t, \quad y=-6, \quad z=2+6t, \quad t \in R$ and $x=-1+t, \quad y=1+t, \quad z=t, \quad t \in R$, perpendicular to each other?

17) Find the point of intersection of the lines of equations $x=-2y=3z$ and $x=-5-t, \quad y=-1+t, \quad z=t-11, \quad t \in R$.

18) Find the intersection point of the x-axis with the line of parametric equations $x=10+t, \quad y=2-2t, \quad z=-3+3t, \quad t \in R$.

For exercises 19 - 22, lines L_1 and L_2 are given. Determine whether the lines are *equal*, *parallel but not equal*, *skew*, or *intersecting*.

19) $L_1: x=y-1=-z$ and $L_2: x-2=-y=\frac{z}{2}$

20) $L_1: x=2t, \quad y=0, \quad z=3, \quad t \in R$ and $L_2: x=0, \quad y=8+s, \quad z=7+s, \quad s \in R$

21) $L_1: x=-1+2t, \quad y=1+3t, \quad z=7t, \quad t \in R$ and $L_2: x-1=\frac{2}{3}(y-4)=\frac{2}{7}z-2$

22) $L_1: 3x=y+1=2z$ and $L_2: x=6+2t, \quad y=17+6t, \quad z=9+3t, \quad t \in R$

23) Consider line L of symmetric equations $x-2=-y=\frac{z}{2}$ and point $A(1,1,1)$.

a. Find parametric equations for a line parallel to L that passes through point A.

b. Find symmetric equations of a line skew to L and that passes through point A.

c. Find symmetric equations of a line that intersects L and passes through point A.

24) Consider line L of parametric equations $x=t, \quad y=2t, \quad z=3, \quad t \in R$.

a. Find parametric equations for a line parallel to L that passes through the origin.

b. Find parametric equations of a line skew to L that passes through the origin.

c. Find symmetric equations of a line that intersects L and passes through the origin.

For exercises 25 - 28, point P and vector $\vec{\mathbf{n}}$ are given.

a. Find the standard form of the equation of the plane that passes through P and has normal vector $\vec{\mathbf{n}}$.

b. Find the general form of the equation of the plane that passes through P and has normal vector $\vec{\mathbf{n}}$.

25) $P(0,0,0), \quad \vec{\mathbf{n}}=3\hat{\mathbf{i}}-2\hat{\mathbf{j}}+4\hat{\mathbf{k}}$

26) $P(3,2,2), \quad \vec{\mathbf{n}}=2\hat{\mathbf{i}}+3\hat{\mathbf{j}}-\hat{\mathbf{k}}$

27) $P(1,2,3), \quad \vec{\mathbf{n}}=\langle 1,2,3\rangle$

28) $P(0,0,0), \quad \vec{\mathbf{n}}=\langle -3,2,-1\rangle$

For exercises 29 - 32, the equation of a plane is given.

a. Find normal vector $\vec{\mathbf{n}}$ to the plane. Express $\vec{\mathbf{n}}$ using standard unit vectors.

b. Find the intersections of the plane with each of the coordinate axes (its intercepts).

c. Sketch the plane.

29) [T] $4x+5y+10z-20=0$

30) $3x+4y-12=0$

31) $3x-2y+4z=0$

32) $x+z=0$

33) Given point $P(1,2,3)$ and vector $\vec{\mathbf{n}}=\hat{\mathbf{i}}+\hat{\mathbf{j}}$, find point Q on the x-axis such that $\overrightarrow{PQ}$ and $\vec{\mathbf{n}}$ are orthogonal.

34) Show there is no plane perpendicular to $\vec{\mathbf{n}}=\hat{\mathbf{i}}+\hat{\mathbf{j}}$ that passes through points $P(1,2,3)$ and $Q(2,3,4)$.

35) Find parametric equations of the line passing through point $P(-2,1,3)$ that is perpendicular to the plane of equation $2x-3y+z=7$.

36) Find symmetric equations of the line passing through point $P(2,5,4)$ that is perpendicular to the plane of equation $2x+3y-5z=0$.

37) Show that line $\dfrac{x-1}{2}=\dfrac{y+1}{3}=\dfrac{z-2}{4}$ is parallel to plane $x-2y+z=6$.

38) Find the real number α such that the line of parametric equations $x=t,\quad y=2-t,\quad z=3+t,\quad t\in R$ is parallel to the plane of equation $\alpha x+5y+z-10=0$.

For exercises 39 - 42, the equations of two planes are given.

a. Determine whether the planes are *parallel*, *orthogonal*, or *neither*.

b. If the planes are neither parallel nor orthogonal, then find the measure of the angle between the planes. Express the answer in degrees rounded to the nearest integer.

c. If the planes intersect, find the line of intersection of the planes, providing the parametric equations of this line.

39) [T] $x+y+z=0,\quad 2x-y+z-7=0$

40) $5x-3y+z=4,\quad x+4y+7z=1$

41) $x-5y-z=1,\quad 5x-25y-5z=-3$

42) [T] $x-3y+6z=4,\quad 5x+y-z=4$

For exercises 43 - 46, determine whether the given line intersects with the given plane. If they do intersect, state the point of intersection.

43) Plane: $2x+y-z=11$ Line: $x=1+t,\quad y=3-2t,\quad z=2+4t$

44) Plane: $-x+2y+z=2$ Line: $x=1+2t,\quad y=-2+t,\quad z=5-3t$

45) Plane: $x-3y+2z=4$ Line: $x=2-t,\quad y=t,\quad z=4+2t$

46) Plane: $x-3y+2z=10$ Line: $x=2-t,\quad y=t,\quad z=4+2t$

47) Show that the lines of equations $x=t,\quad y=1+t,\quad z=2+t,\quad t\in R$, and $\dfrac{x}{2}=\dfrac{y-1}{3}=z-3$ are skew, and find the distance between them.

48) Show that the lines of equations $x=-1+t,\quad y=-2+t,\quad z=3t,\quad t\in R$, and $x=5+s,\quad y=-8+2s,\quad z=7s,\quad s\in R$ are skew, and find the distance between them.

49) Consider point $C(-3,2,4)$ and the plane of equation $2x+4y-3z=8$.

a. Find the radius of the sphere with center C tangent to the given plane.

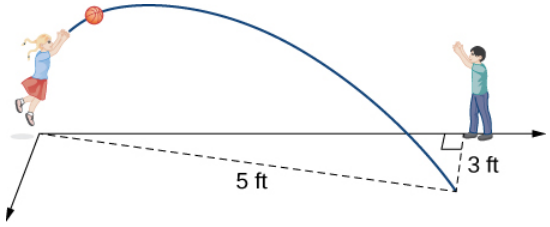
b. Find point P of tangency.

50) Consider the plane of equation $x-y-z-8=0$.

a. Find the equation of the sphere with center C at the origin that is tangent to the given plane.

b. Find parametric equations of the line passing through the origin and the point of tangency.

51) Two children are playing with a ball. The girl throws the ball to the boy. The ball travels in the air, curves 3 ft to the right, and falls 5 ft away from the girl (see the following figure). If the plane that contains the trajectory of the ball is perpendicular to the ground, find its equation.



52) [T] John allocates d dollars to consume monthly three goods of prices a, b, and c. In this context, the budget equation is defined as $ax + by + cz = d$, where $x \geq 0,\ y \geq 0$, and $z \geq 0$ represent the number of items bought from each of the goods. The budget set is given by $\{(x, y, z) \mid ax + by + cz \leq d,\ x \geq 0,\ y \geq 0,\ z \geq 0\}$, and the budget plane is the part of the plane of equation $ax + by + cz = d$ for which $x \geq 0,\ y \geq 0$, and $z \geq 0$. Consider $a = \$8,\ b = \$5,\ c = \$10$, and $d = \$500$.

a. Use a CAS to graph the budget set and budget plane.

b. For $z = 25$, find the new budget equation and graph the budget set in the same system of coordinates.

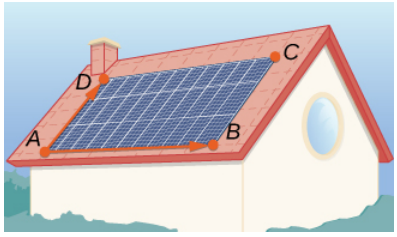
53) [T] Consider $\vec{\mathbf{r}}(t) = \langle \sin t, \cos t, 2t \rangle$ the position vector of a particle at time $t \in [0, 3]$, where the components of $\vec{\mathbf{r}}$ are expressed in centimeters and time is measured in seconds. Let $\overrightarrow{OP}$ be the position vector of the particle after 1 sec.

a. Determine the velocity vector $\vec{\mathbf{v}}(1)$ of the particle after 1 sec.

b. Find the scalar equation of the plane that is perpendicular to $v(1)$ and passes through point P. This plane is called the normal plane to the path of the particle at point P.

c. Use a CAS to visualize the path of the particle along with the velocity vector and normal plane at point P.

54) [T] A solar panel is mounted on the roof of a house. The panel may be regarded as positioned at the points of coordinates (in meters) $A(8, 0, 0),\ B(8, 18, 0),\ C(0, 18, 8)$, and $D(0, 0, 8)$ (see the following figure).



a. Find the general form of the equation of the plane that contains the solar panel by using points A, B, and C, and show that its normal vector is equivalent to $\overrightarrow{AB} \times \overrightarrow{AD}$.

b. Find parametric equations of line L_1 that passes through the center of the solar panel and has direction vector $\vec{\mathbf{s}} = \frac{1}{\sqrt{3}}\hat{\mathbf{i}} + \frac{1}{\sqrt{3}}\hat{\mathbf{j}} + \frac{1}{\sqrt{3}}\hat{\mathbf{k}}$, which points toward the position of the Sun at a particular time of day.

c. Find symmetric equations of line L_2 that passes through the center of the solar panel and is perpendicular to it.

d. Determine the angle of elevation of the Sun above the solar panel by using the angle between lines L_1 and L_2.

Contributors

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Problems 15 and 43 - 46 created by Paul Seeburger

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11.6: Quadric Surfaces

Learning Objectives

- Identify a cylinder as a type of three-dimensional surface.
- Recognize the main features of ellipsoids, paraboloids, and hyperboloids.
- Use traces to draw the intersections of quadric surfaces with the coordinate planes.

We have been exploring vectors and vector operations in three-dimensional space, and we have developed equations to describe lines, planes, and spheres. In this section, we use our knowledge of planes and spheres, which are examples of three-dimensional figures called *surfaces*, to explore a variety of other surfaces that can be graphed in a three-dimensional coordinate system.

Identifying Cylinders

The first surface we'll examine is the cylinder. Although most people immediately think of a hollow pipe or a soda straw when they hear the word cylinder, here we use the broad mathematical meaning of the term. As we have seen, cylindrical surfaces don't have to be circular. A rectangular heating duct is a cylinder, as is a rolled-up yoga mat, the cross-section of which is a spiral shape.

In the two-dimensional coordinate plane, the equation $x^2 + y^2 = 9$ describes a circle centered at the origin with radius 3. In three-dimensional space, this same equation represents a surface. Imagine copies of a circle stacked on top of each other centered on the z-axis (Figure 11.6.1), forming a hollow tube. We can then construct a cylinder from the set of lines parallel to the z-axis passing through the circle $x^2 + y^2 = 9$ in the xy-plane, as shown in the figure. In this way, any curve in one of the coordinate planes can be extended to become a surface.

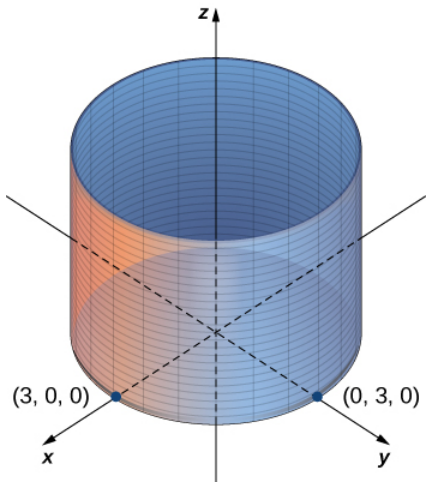


Figure 11.6.1: In three-dimensional space, the graph of equation $x^2 + y^2 = 9$ is a cylinder with radius 3 centered on the z-axis. It continues indefinitely in the positive and negative directions.

Definition: cylinders and rulings

A set of lines parallel to a given line passing through a given curve is known as a cylindrical surface, or *cylinder*. The parallel lines are called rulings.

From this definition, we can see that we still have a cylinder in three-dimensional space, even if the curve is not a circle. Any curve can form a cylinder, and the rulings that compose the cylinder may be parallel to any given line (Figure 11.6.2).

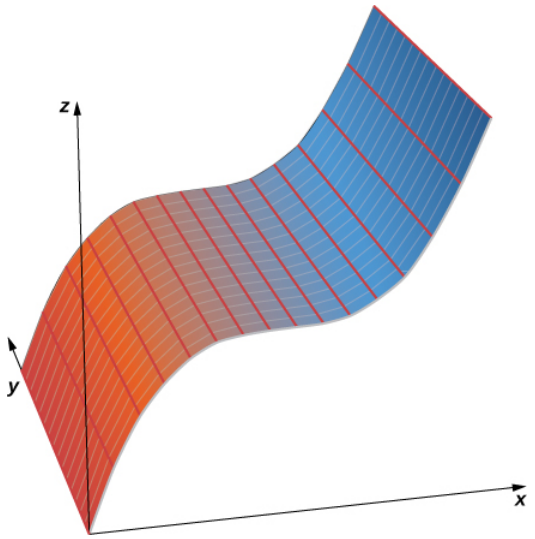


Figure 11.6.2: In three-dimensional space, the graph of equation $z = x^3$ is a cylinder, or a cylindrical surface with rulings parallel to the y-axis.

Example 11.6.1: Graphing Cylindrical Surfaces

Sketch the graphs of the following cylindrical surfaces.

a. $x^2 + z^2 = 25$
b. $z = 2x^2 - y$
c. $y = \sin x$

Solution

a. The variable y can take on any value without limit. Therefore, the lines ruling this surface are parallel to the y-axis. The intersection of this surface with the xz-plane forms a circle centered at the origin with radius 5 (see Figure 11.6.3).

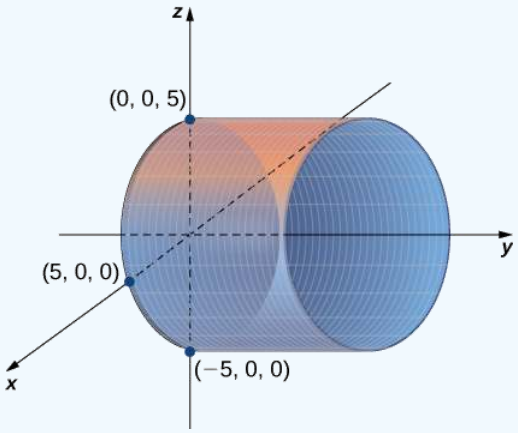


Figure 11.6.3: The graph of equation $x^2 + z^2 = 25$ is a cylinder with radius 5 centered on the y-axis.

b. In this case, the equation contains all three variables —$x, y,$ and z— so none of the variables can vary arbitrarily. The easiest way to visualize this surface is to use a computer graphing utility (Figure 11.6.4).

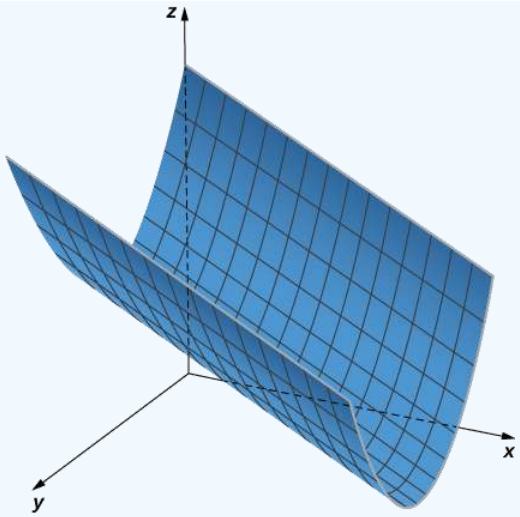


Figure 11.6.4

c. In this equation, the variable z can take on any value without limit. Therefore, the lines composing this surface are parallel to the z-axis. The intersection of this surface with the xy-plane outlines curve $y = \sin x$ (Figure 11.6.5).

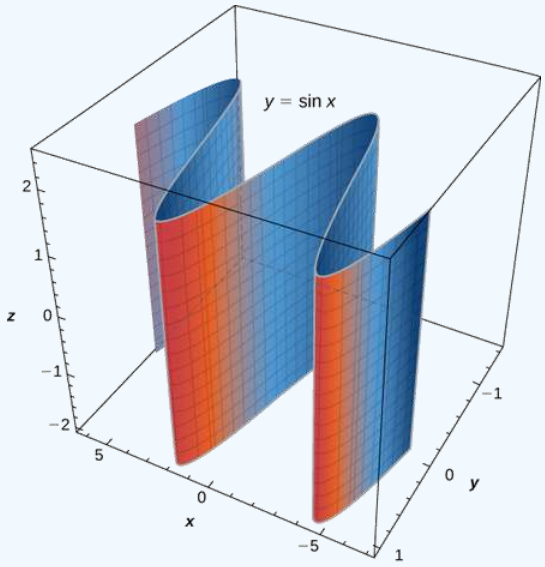


Figure 11.6.5: The graph of equation $y = \sin x$ is formed by a set of lines parallel to the z-axis passing through curve $y = \sin x$ in the xy-plane.

? Exercise 11.6.1:

Sketch or use a graphing tool to view the graph of the cylindrical surface defined by equation $z = y^2$.

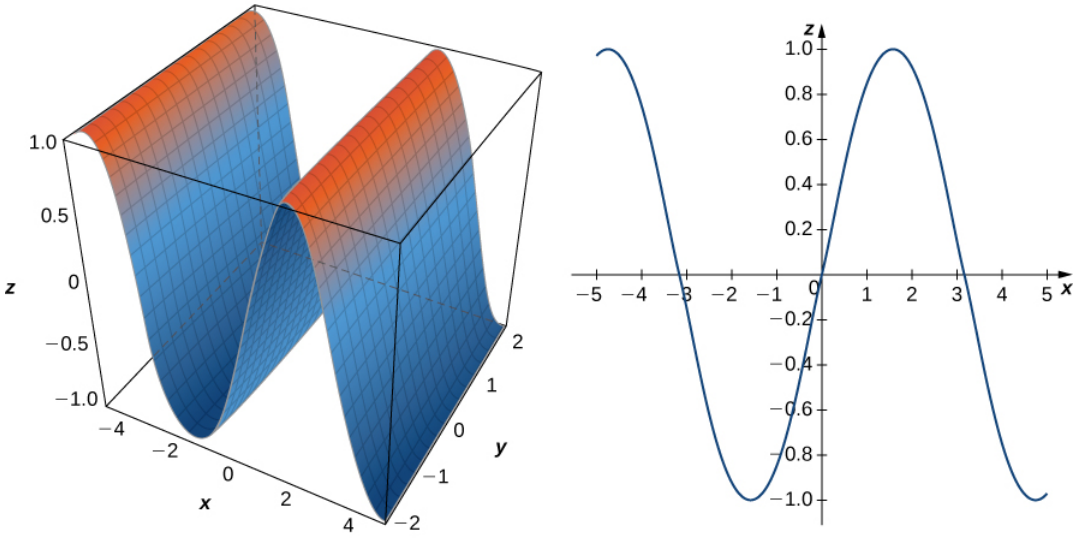
When sketching surfaces, we have seen that it is useful to sketch the intersection of the surface with a plane parallel to one of the coordinate planes. These curves are called traces. We can see them in the plot of the cylinder in Figure 11.6.6.

Definition: traces

The *traces* of a surface are the cross-sections created when the surface intersects a plane parallel to one of the coordinate planes.

Traces are useful in sketching cylindrical surfaces. For a cylinder in three dimensions, though, only one set of traces is useful. Notice, in Figure 11.6.6, that the trace of the graph of $z = \sin x$ in the xz-plane is useful in constructing the graph. The trace in the xy

-plane, though, is just a series of parallel lines, and the trace in the yz-plane is simply one line.



(a) (b)

Figure 11.6.6: (a) This is one view of the graph of equation $z = \sin x$. (b) To find the trace of the graph in the xz-plane, set $y = 0$. The trace is simply a two-dimensional sine wave.

Cylindrical surfaces are formed by a set of parallel lines. Not all surfaces in three dimensions are constructed so simply, however. We now explore more complex surfaces, and traces are an important tool in this investigation.

Quadric Surfaces

We have learned about surfaces in three dimensions described by first-order equations; these are planes. Some other common types of surfaces can be described by second-order equations. We can view these surfaces as three-dimensional extensions of the conic sections we discussed earlier: the ellipse, the parabola, and the hyperbola. We call these graphs quadric surfaces

Definition: Quadric surfaces and conic sections

Quadric surfaces are the graphs of equations that can be expressed in the form

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Jz + K = 0.$$

When a quadric surface intersects a coordinate plane, the trace is a conic section.

An *ellipsoid* is a surface described by an equation of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. Set $x = 0$ to see the trace of the ellipsoid in the yz-plane. To see the traces in the xy- and xz-planes, set $z = 0$ and $y = 0$, respectively. Notice that, if $a = b$, the trace in the xy-plane is a circle. Similarly, if $a = c$, the trace in the xz-plane is a circle and, if $b = c$, then the trace in the yz-plane is a circle. A sphere, then, is an ellipsoid with $a = b = c$.

Example 11.6.2: Sketching an Ellipsoid

Sketch the ellipsoid

$$\frac{x^2}{2^2} + \frac{y^2}{3^2} + \frac{z^2}{5^2} = 1.$$

Solution

Start by sketching the traces. To find the trace in the xy-plane, set $z = 0$: $\frac{x^2}{2^2} + \frac{y^2}{3^2} = 1$ (Figure 11.6.7). To find the other traces, first set $y = 0$ and then set $x = 0$.

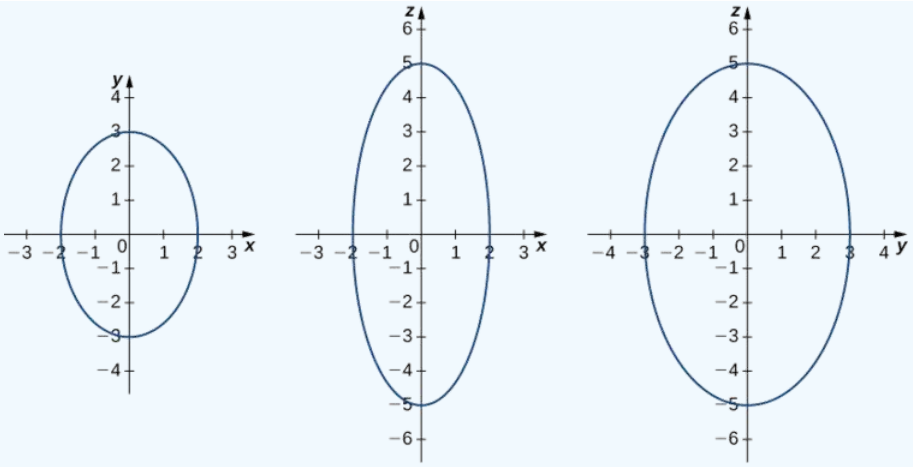


Figure 11.6.7: (a) This graph represents the trace of equation $\frac{x^2}{2^2}+\frac{y^2}{3^2}+\frac{z^2}{5^2}=1$ in the xy-plane, when we set $z=0$. (b) When we set $y=0$, we get the trace of the ellipsoid in the xz-plane, which is an ellipse. (c) When we set $x=0$, we get the trace of the ellipsoid in the yz-plane, which is also an ellipse.

Now that we know what traces of this solid look like, we can sketch the surface in three dimensions (Figure 11.6.8).

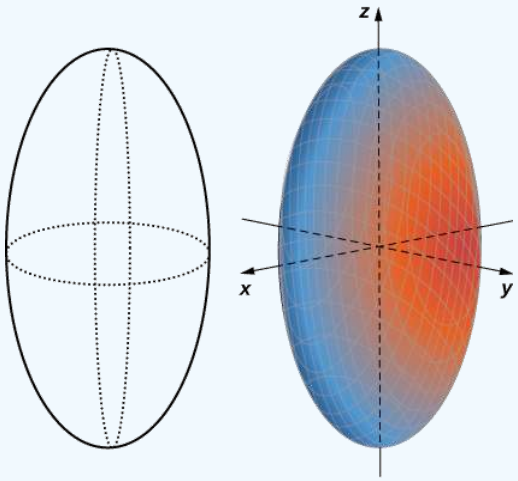


Figure 11.6.8: (a) The traces provide a framework for the surface. (b) The center of this ellipsoid is the origin.

The trace of an ellipsoid is an ellipse in each of the coordinate planes. However, this does not have to be the case for all quadric surfaces. Many quadric surfaces have traces that are different kinds of conic sections, and this is usually indicated by the name of the surface. For example, if a surface can be described by an equation of the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z}{c}$$

then we call that surface an **elliptic paraboloid**. The trace in the *xy*-plane is an ellipse, but the traces in the xz-plane and yz-plane are parabolas (Figure 11.6.9). Other elliptic paraboloids can have other orientations simply by interchanging the variables to give us a different variable in the linear term of the equation $\frac{x^2}{a^2}+\frac{z^2}{c^2}=\frac{y}{b}$ or $\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{x}{a}$.

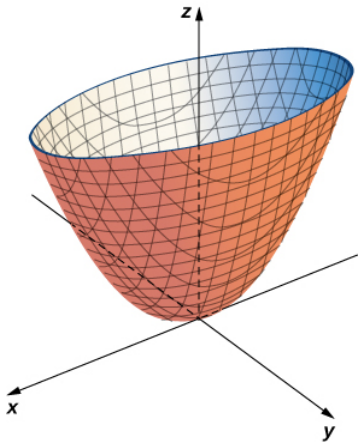


Figure 11.6.9: This quadric surface is called an elliptic paraboloid.

Example 11.6.3: Identifying Traces of Quadric Surfaces

Describe the traces of the elliptic paraboloid $x^2 + \frac{y^2}{2^2} = \frac{z}{5}$.

Solution

To find the trace in the xy-plane, set $z = 0 : x^2 + \frac{y^2}{2^2} = 0$. The trace in the plane $z = 0$ is simply one point, the origin. Since a single point does not tell us what the shape is, we can move up the z-axis to an arbitrary plane to find the shape of other traces of the figure.

The trace in plane $z = 5$ is the graph of equation $x^2 + \frac{y^2}{2^2} = 1$, which is an ellipse. In the xz-plane, the equation becomes $z = 5x^2$. The trace is a parabola in this plane and in any plane with the equation $y = b$.

In planes parallel to the yz-plane, the traces are also parabolas, as we can see in Figure 11.6.10.

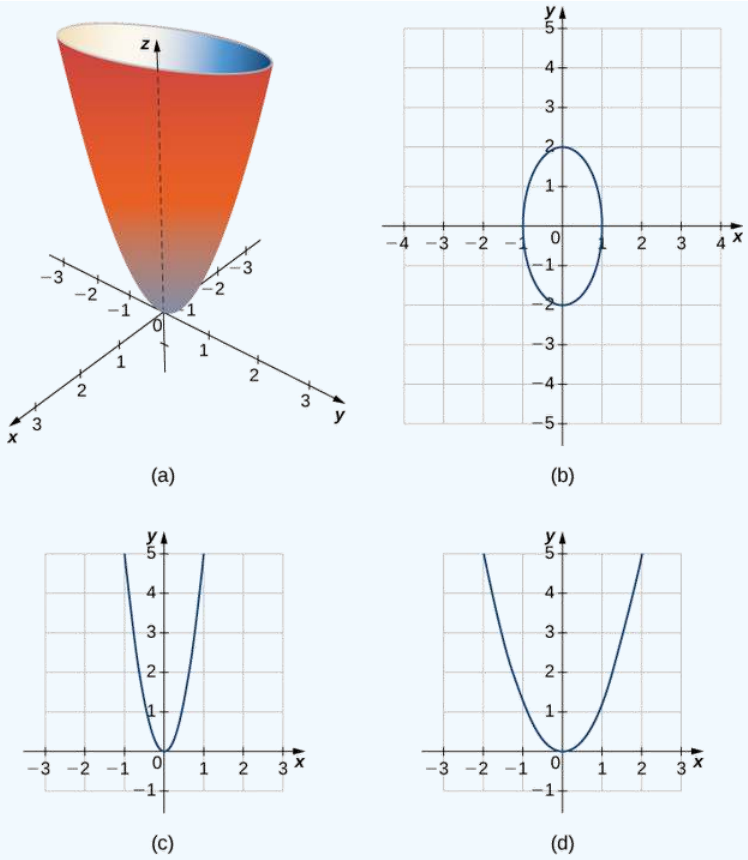


Figure 11.6.10: (a) The paraboloid $x^2+\frac{y^2}{2^2}=\frac{z}{5}$. (b) The trace in plane $z=5$. (c) The trace in the xz-plane. (d) The trace in the yz-plane.

? Exercise 11.6.2:

A hyperboloid of one sheet is any surface that can be described with an equation of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$. Describe the traces of the hyperboloid of one sheet given by equation $\frac{x^2}{3^2}+\frac{y^2}{2^2}-\frac{z^2}{5^2}=1$.

Hyperboloids of one sheet have some fascinating properties. For example, they can be constructed using straight lines, such as in the sculpture in Figure 11.6.11*a*. In fact, cooling towers for nuclear power plants are often constructed in the shape of a hyperboloid. The builders are able to use straight steel beams in the construction, which makes the towers very strong while using relatively little material (Figure 11.6.11*b*).

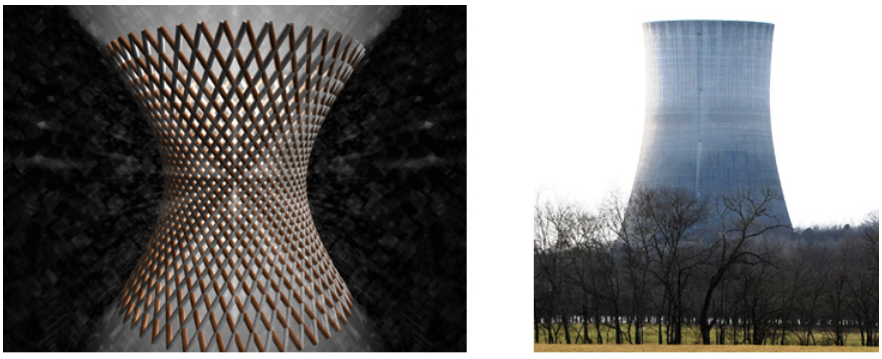


Figure 11.6.11: (a) A sculpture in the shape of a hyperboloid can be constructed of straight lines. (b) Cooling towers for nuclear power plants are often built in the shape of a hyperboloid.

Example 11.6.4: Chapter Opener: Finding the Focus of a Parabolic Reflector

Energy hitting the surface of a parabolic reflector is concentrated at the focal point of the reflector (Figure 11.6.12). If the surface of a parabolic reflector is described by equation $\frac{x^2}{100}+\frac{y^2}{100}=\frac{z}{4}$, where is the focal point of the reflector?

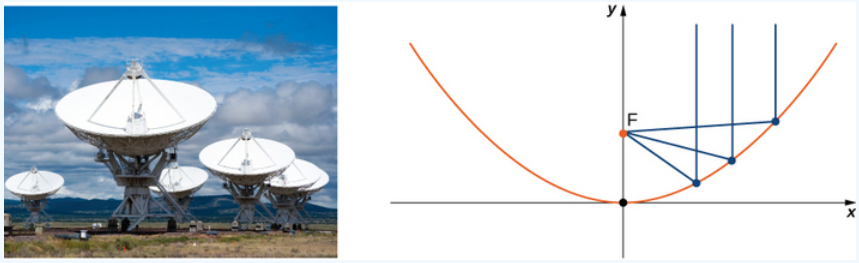


Figure 11.6.12: Energy reflects off of the parabolic reflector and is collected at the focal point. (credit: modification of CGP Grey, Wikimedia Commons)

Solution

Since z is the first-power variable, the axis of the reflector corresponds to the z-axis. The coefficients of x^2 and y^2 are equal, so the cross-section of the paraboloid perpendicular to the z-axis is a circle. We can consider a trace in the xz-plane or the yz-plane; the result is the same. Setting $y=0$, the trace is a parabola opening up along the z-axis, with standard equation $x^2=4pz$, where p is the focal length of the parabola. In this case, this equation becomes $x^2=100\cdot\frac{z}{4}=4pz$ or $25=4p$. So p is 6.25 m, which tells us that the focus of the paraboloid is 6.25 m up the axis from the vertex. Because the vertex of this surface is the origin, the focal point is $(0,0,6.25)$.

Seventeen standard quadric surfaces can be derived from the general equation

$$Ax^2+By^2+Cz^2+Dxy+Exz+Fyz+Gx+Hy+Jz+K=0.$$

The following figures summarize the most important ones.

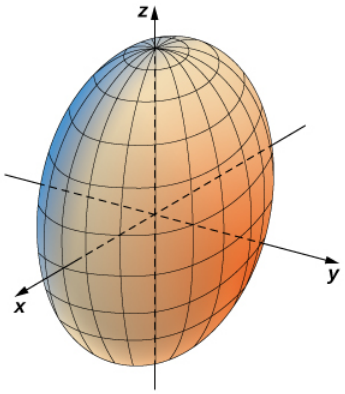
Characteristics of Common Quadric Surfaces

Ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Traces
In plane $z = p$: an ellipse
In plane $y = q$: an ellipse
In plane $x = r$: an ellipse

If $a = b = c$, then this surface is a sphere.

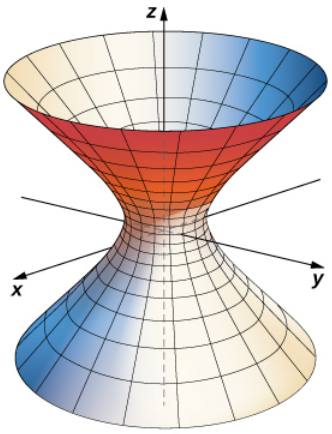


Hyperboloid of One Sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

Traces
In plane $z = p$: an ellipse
In plane $y = q$: a hyperbola
In plane $x = r$: a hyperbola

In the equation for this surface, two of the variables have positive coefficients and one has a negative coefficient. The axis of the surface corresponds to the variable with the negative coefficient.



Hyperboloid of Two Sheets

$$\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Traces
In plane $z = p$: an ellipse or the empty set (no trace)
In plane $y = q$: a hyperbola
In plane $x = r$: a hyperbola

In the equation for this surface, two of the variables have negative coefficients and one has a positive coefficient. The axis of the surface corresponds to the variable with a positive coefficient. The surface does not intersect the coordinate plane perpendicular to the axis.

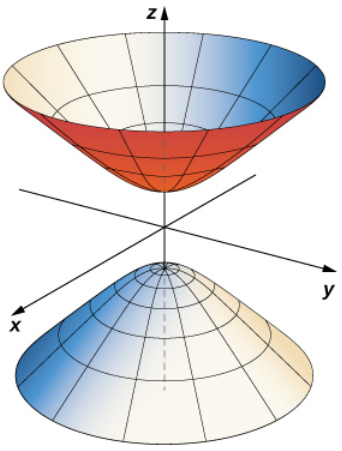


Figure **11.6.13**: Characteristics of Common Quadric Surfaces: Ellipsoid, Hyperboloid of One Sheet, Hyperboloid of Two Sheets.

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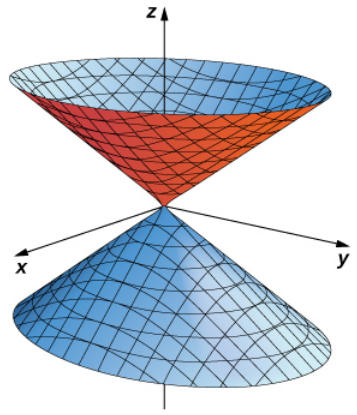
Characteristics of Common Quadric Surfaces

Elliptic Cone

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$

Traces
In plane $z = p$: an ellipse
In plane $y = q$: a hyperbola
In plane $x = r$: a hyperbola
In the xz – plane: a pair of lines that intersect at the origin
In the yz – plane: a pair of lines that intersect at the origin

The axis of the surface corresponds to the variable with a negative coefficient. The traces in the coordinate planes parallel to the axis are intersecting lines.

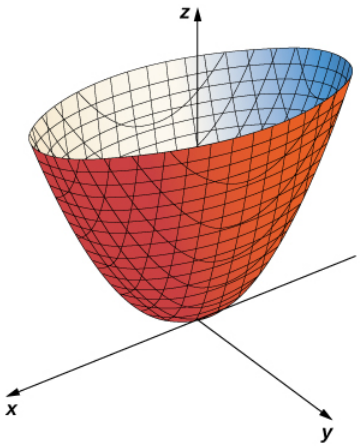


Elliptic Paraboloid

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

Traces
In plane $z = p$: an ellipse
In plane $y = q$: a parabola
In plane $x = r$: a parabola

The axis of the surface corresponds to the linear variable.



Hyperbolic Paraboloid

$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

Traces
In plane $z = p$: a hyperbola
In plane $y = q$: a parabola
In plane $x = r$: a parabola

The axis of the surface corresponds to the linear variable.

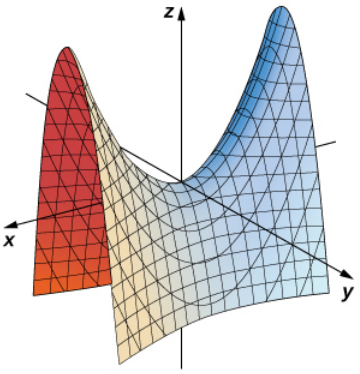


Figure **11.6.14**: Characteristics of Common Quadric Surfaces: Elliptic Cone, Elliptic Paraboloid, Hyperbolic Paraboloid.

Example 11.6.5: Identifying Equations of Quadric Surfaces

Identify the surfaces represented by the given equations.

a. $16x^2 + 9y^2 + 16z^2 = 144$
b. $9x^2 - 18x + 4y^2 + 16y - 36z + 25 = 0$

Solution

a. The $x, y,$ and z terms are all squared, and are all positive, so this is probably an ellipsoid. However, let's put the equation into the standard form for an ellipsoid just to be sure. We have

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$$16x^2 + 9y^2 + 16z^2 = 144.$$

Dividing through by 144 gives

$$\frac{x^2}{9} + \frac{y^2}{16} + \frac{z^2}{9} = 1.$$

So, this is, in fact, an ellipsoid, centered at the origin.

b. We first notice that the z term is raised only to the first power, so this is either an elliptic paraboloid or a hyperbolic paraboloid. We also note there are x terms and y terms that are not squared, so this quadric surface is not centered at the origin. We need to complete the square to put this equation in one of the standard forms. We have

$$\begin{aligned} 9x^2 - 18x + 4y^2 + 16y - 36z + 25 &= 0 \\ 9x^2 - 18x + 4y^2 + 16y + 25 &= 36z \\ 9(x^2 - 2x) + 4(y^2 + 4y) + 25 &= 36z \\ 9(x^2 - 2x + 1 - 1) + 4(y^2 + 4y + 4 - 4) + 25 &= 36z \\ 9(x-1)^2 - 9 + 4(y+2)^2 - 16 + 25 &= 36z \\ 9(x-1)^2 + 4(y+2)^2 &= 36z \\ \frac{(x-1)^2}{4} + \frac{(y-2)^2}{9} &= z. \end{aligned}$$

This is an elliptic paraboloid centered at $(1, 2, 0)$.

? Exercise 11.6.3

Identify the surface represented by equation $9x^2 + y^2 - z^2 + 2z - 10 = 0$.

Key Concepts

- A set of lines parallel to a given line passing through a given curve is called a *cylinder*, or a cylindrical surface. The parallel lines are called *rulings*.
- The intersection of a three-dimensional surface and a plane is called a trace. To find the trace in the xy-, yz-, or xz-planes, set $z = 0$, $x = 0$, or $y = 0$, respectively.
- Quadric surfaces are three-dimensional surfaces with traces composed of conic sections. Every quadric surface can be expressed with an equation of the form

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Jz + K = 0.$$

- To sketch the graph of a quadric surface, start by sketching the traces to understand the framework of the surface.
- Important quadric surfaces are summarized in Figures 11.6.13 and 11.6.14.

Glossary

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11.6E: Exercises for Quadric Surfaces

For exercises 1 - 6, sketch and describe the cylindrical surface of the given equation.

1) [T] $x^2 + z^2 = 1$

2) [T] $x^2 + y^2 = 9$

3) [T] $z = \cos\left(\frac{\pi}{2} + x\right)$

4) [T] $z = e^x$

5) [T] $z = 9 - y^2$

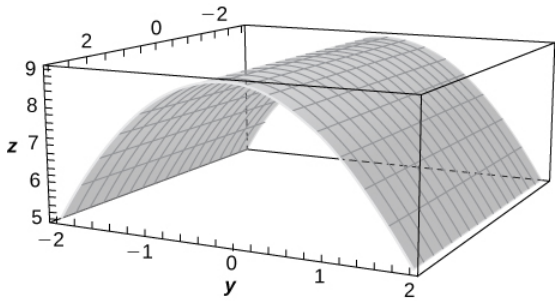
6) [T] $z = \ln x$

For exercises 7 - 10, the graph of a quadric surface is given.

a. Specify the name of the quadric surface.

b. Determine the axis of symmetry of the quadric surface.

7)



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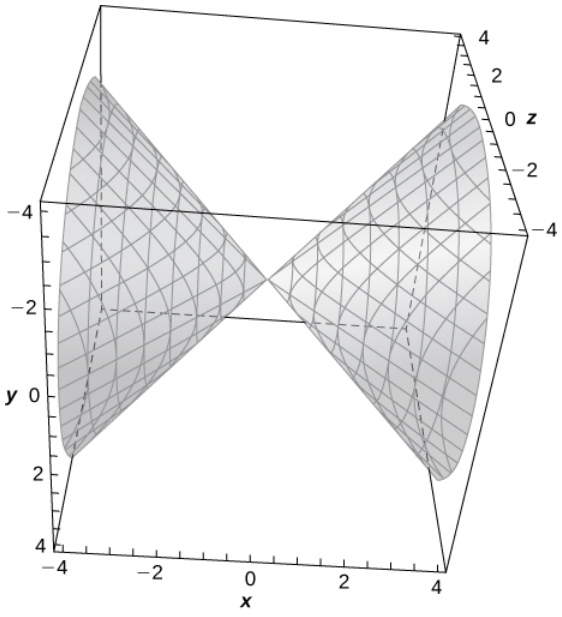

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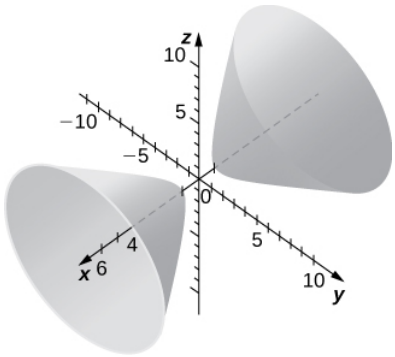

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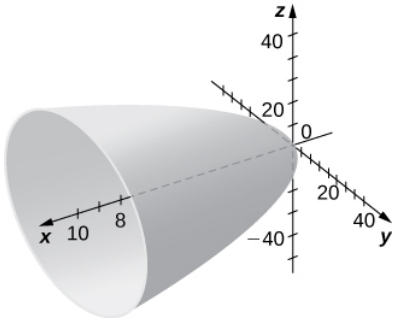


For exercises 11 - 16, match the given quadric surface with its corresponding equation in standard form.

a. $\frac{x^2}{4}+\frac{y^2}{9}-\frac{z^2}{12}=1$

b. $\frac{x^2}{4}-\frac{y^2}{9}-\frac{z^2}{12}=1$

c. $\frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{12}=1$

d. $z=4x^2+3y^2$

e. $z=4x^2-y^2$

f. $4x^2+y^2-z^2=0$

11) Hyperboloid of two sheets

12) Ellipsoid

13) Elliptic paraboloid

14) Hyperbolic paraboloid

15) Hyperboloid of one sheet

16) Elliptic cone

For exercises 17 - 28, rewrite the given equation of the quadric surface in standard form. Identify the surface.

17) $-x^2+36y^2+36z^2=9$

18) $-4x^2+25y^2+z^2=100$

19) $-3x^2+5y^2-z^2=10$

20) $3x^2-y^2-6z^2=18$

21) $5y=x^2-z^2$

22) $8x^2-5y^2-10z=0$

23) $x^2 + 5y^2 + 3z^2 - 15 = 0$

24) $63x^2 + 7y^2 + 9z^2 - 63 = 0$

25) $x^2 + 5y^2 - 8z^2 = 0$

26) $5x^2 - 4y^2 + 20z^2 = 0$

27) $6x = 3y^2 + 2z^2$

28) $49y = x^2 + 7z^2$

For exercises 29 - 34, find the trace of the given quadric surface in the specified plane of coordinates and sketch it.

29) [T] $x^2 + z^2 + 4y = 0, \quad z = 0$

30) [T] $x^2 + z^2 + 4y = 0, \quad x = 0$

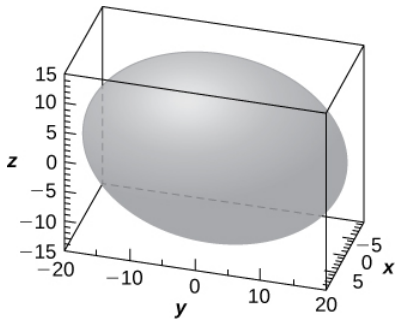
31) [T] $-4x^2 + 25y^2 + z^2 = 100, \quad x = 0$

32) [T] $-4x^2 + 25y^2 + z^2 = 100, \quad y = 0$

33) [T] $x^2 + \frac{y^2}{4} + \frac{z^2}{100} = 1, \quad x = 0$

34) [T] $x^2 - y - z^2 = 1, \quad y = 0$

35) Use the graph of the given quadric surface to answer the questions.

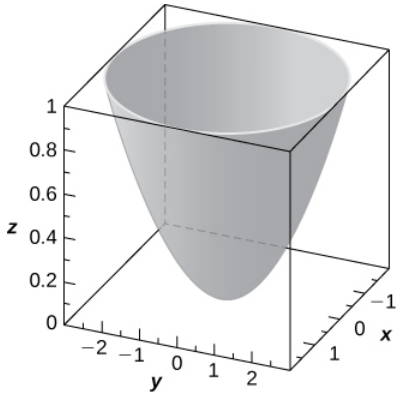


a. Specify the name of the quadric surface.

b. Which of the equations—$16x^2 + 9y^2 + 36z^2 = 3600$, $9x^2 + 36y^2 + 16z^2 = 3600$, or $36x^2 + 9y^2 + 16z^2 = 3600$ —corresponds to the graph?

c. Use b. to write the equation of the quadric surface in standard form.

36) Use the graph of the given quadric surface to answer the questions.



a. Specify the name of the quadric surface.

b. Which of the equations—$36z = 9x^2 + y^2$, $9x^2 + 4y^2 = 36z$, or $-36z = -81x^2 + 4y^2$ —corresponds to the graph above?

c. Use b. to write the equation of the quadric surface in standard form.

For exercises 37 - 42, the equation of a quadric surface is given.

a. Use the method of completing the square to write the equation in standard form.

b. Identify the surface.

37) $x^2+2z^2+6x-8z+1=0$

38) $4x^2-y^2+z^2-8x+2y+2z+3=0$

39) $x^2+4y^2-4z^2-6x-16y-16z+5=0$

40) $x^2+z^2-4y+4=0$

41) $x^2+\frac{y^2}{4}-\frac{z^2}{3}+6x+9=0$

42) $x^2-y^2+z^2-12z+2x+37=0$

43) Write the standard form of the equation of the ellipsoid centered at the origin that passes through points $A(2,0,0)$, $B(0,0,1)$, and $C\left(12,\sqrt{11},\frac{1}{2}\right)$.

44) Write the standard form of the equation of the ellipsoid centered at point $P(1,1,0)$ that passes through points $A(6,1,0)$, $B(4,2,0)$ and $C(1,2,1)$.

45) Determine the intersection points of elliptic cone $x^2-y^2-z^2=0$ with the line of symmetric equations $\frac{x-1}{2}=\frac{y+1}{3}=z$.

46) Determine the intersection points of parabolic hyperboloid $z=3x^2-2y^2$ with the line of parametric equations $x=3t$, $y=2t$, $z=19t$, where $t\in R$.

47) Find the equation of the quadric surface with points $P(x,y,z)$ that are equidistant from point $Q(0,-1,0)$ and plane of equation $y=1$. Identify the surface.

48) Find the equation of the quadric surface with points $P(x,y,z)$ that are equidistant from point $Q(0,2,0)$ and plane of equation $y=-2$. Identify the surface.

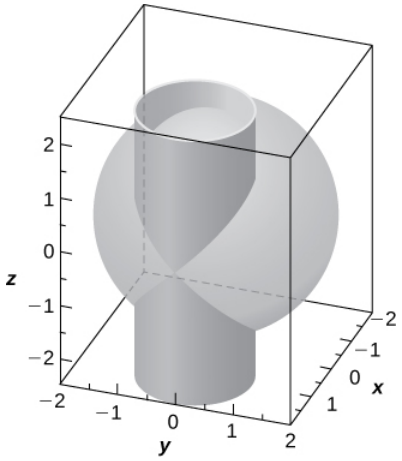
49) If the surface of a parabolic reflector is described by equation $400z=x^2+y^2$, find the focal point of the reflector.

50) Consider the parabolic reflector described by equation $z=20x^2+20y^2$. Find its focal point.

51) Show that quadric surface $x^2+y^2+z^2+2xy+2xz+2yz+x+y+z=0$ reduces to two parallel planes.

52) Show that quadric surface $x^2+y^2+z^2-2xy-2xz+2yz-1=0$ reduces to two parallel planes passing.

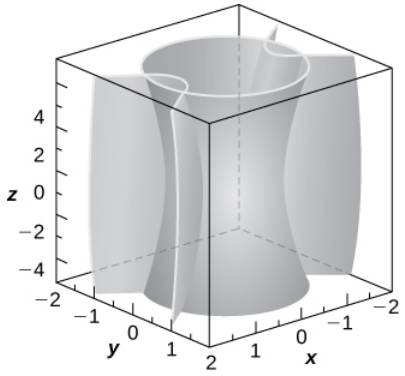
53) [T] The intersection between cylinder $(x-1)^2+y^2=1$ and sphere $x^2+y^2+z^2=4$ is called a *Viviani curve*.



a. Solve the system consisting of the equations of the surfaces to find the equation of the intersection curve. (Hint: Find x and y in terms of z.)

b. Use a computer algebra system (CAS) or CalcPlot3D to visualize the intersection curve on sphere $x^2+y^2+z^2=4$.

54) Hyperboloid of one sheet $25x^2+25y^2-z^2=25$ and elliptic cone $-25x^2+75y^2+z^2=0$ are represented in the following figure along with their intersection curves. Identify the intersection curves and find their equations (Hint: Find y from the system consisting of the equations of the surfaces.)



55) [T] Use a CAS or CalcPlot3D to create the intersection between cylinder $9x^2+4y^2=18$ and ellipsoid $36x^2+16y^2+9z^2=144$, and find the equations of the intersection curves.

56) [T] A spheroid is an ellipsoid with two equal semiaxes. For instance, the equation of a spheroid with the z-axis as its axis of symmetry is given by $\frac{x^2}{a^2}+\frac{y^2}{a^2}+\frac{z^2}{c^2}=1$, where a and c are positive real numbers. The spheroid is called oblate if $c<a$, and prolate for $c>a$.

a. The eye cornea is approximated as a prolate spheroid with an axis that is the eye, where $a=8.7mm$ and $c=9.6mm$.Write the equation of the spheroid that models the cornea and sketch the surface.

b. Give two examples of objects with prolate spheroid shapes.

57) [T] In cartography, Earth is approximated by an oblate spheroid rather than a sphere. The radii at the equator and poles are approximately 3963mi and 3950mi, respectively.

a. Write the equation in standard form of the ellipsoid that represents the shape of Earth. Assume the center of Earth is at the origin and that the trace formed by plane $z=0$ corresponds to the equator.

b. Sketch the graph.

c. Find the equation of the intersection curve of the surface with plane $z=1000$ that is parallel to the xy-plane. The intersection curve is called a *parallel.*

d. Find the equation of the intersection curve of the surface with plane $x+y=0$ that passes through the z-axis. The intersection curve is called a meridian.

58) [T] A set of buzzing stunt magnets (or “rattlesnake eggs”) includes two sparkling, polished, superstrong spheroid-shaped magnets well-known for children’s entertainment. Each magnet is 1.625 in. long and 0.5 in. wide at the middle. While tossing them into the air, they create a buzzing sound as they attract each other.

a. Write the equation of the prolate spheroid centered at the origin that describes the shape of one of the magnets.

b. Write the equations of the prolate spheroids that model the shape of the buzzing stunt magnets. Use a CAS or CalcPlot3D to create the graphs.

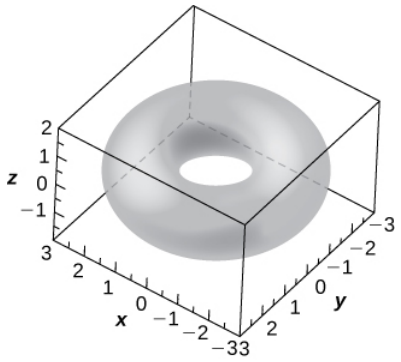
59) [T] A heart-shaped surface is given by equation $(x^2+\frac{9}{4}y^2+z^2-1)^3-x^2z^3-\frac{9}{80}y^2z^3=0$.

a. Use a CAS or CalcPlot3D to graph the surface that models this shape.

b. Determine and sketch the trace of the heart-shaped surface on the xz-plane.

60) [T] The ring torus symmetric about the z-axis is a special type of surface in topology and its equation is given by $(x^2+y^2+z^2+R^2-r^2)^2=4R^2(x^2+y^2)$, where $R>r>0$. The numbers R and r are called are the major and minor radii,

respectively, of the surface. The following figure shows a ring torus for which $R = 2$ and $r = 1$.



a. Write the equation of the ring torus with $R = 2$ and $r = 1$, and use a CAS or CalcPlot3D to graph the surface. Compare the graph with the figure given.

b. Determine the equation and sketch the trace of the ring torus from a. on the xy-plane.

c. Give two examples of objects with ring torus shapes.

Contributors

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11.7: Cylindrical and Spherical Coordinates

Learning Objectives

- Convert from cylindrical to rectangular coordinates.
- Convert from rectangular to cylindrical coordinates.
- Convert from spherical to rectangular coordinates.
- Convert from rectangular to spherical coordinates.

The Cartesian coordinate system provides a straightforward way to describe the location of points in space. Some surfaces, however, can be difficult to model with equations based on the Cartesian system. This is a familiar problem; recall that in two dimensions, polar coordinates often provide a useful alternative system for describing the location of a point in the plane, particularly in cases involving circles. In this section, we look at two different ways of describing the location of points in space, both of them based on extensions of polar coordinates. As the name suggests, cylindrical coordinates are useful for dealing with problems involving cylinders, such as calculating the volume of a round water tank or the amount of oil flowing through a pipe. Similarly, spherical coordinates are useful for dealing with problems involving spheres, such as finding the volume of domed structures.

Cylindrical Coordinates

When we expanded the traditional Cartesian coordinate system from two dimensions to three, we simply added a new axis to model the third dimension. Starting with polar coordinates, we can follow this same process to create a new three-dimensional coordinate system, called the cylindrical coordinate system. In this way, cylindrical coordinates provide a natural extension of polar coordinates to three dimensions.

Definition: The Cylindrical Coordinate System

In the cylindrical coordinate system, a point in space (Figure 11.7.1) is represented by the ordered triple (r, θ, z), where

- (r, θ) are the polar coordinates of the point's projection in the xy-plane
- z is the usual **z-coordinate** in the Cartesian coordinate system

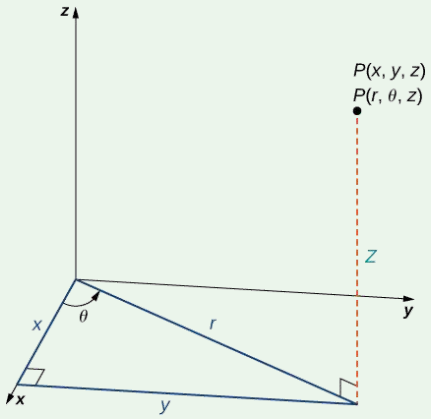


Figure 11.7.1: The right triangle lies in the xy-plane. The length of the hypotenuse is r and θ is the measure of the angle formed by the positive x-axis and the hypotenuse. The z-coordinate describes the location of the point above or below the xy-plane.

In the xy-plane, the right triangle shown in Figure 11.7.1 provides the key to transformation between cylindrical and Cartesian, or rectangular, coordinates.

Conversion between Cylindrical and Cartesian Coordinates

The rectangular coordinates (x, y, z) and the cylindrical coordinates (r, θ, z) of a point are related as follows:

These equations are used to convert from cylindrical coordinates to rectangular coordinates.

- $x = r\cos\theta$
- $y = r\sin\theta$
- $z = z$

These equations are used to convert from rectangular coordinates to cylindrical coordinates

1. $r^2 = x^2 + y^2$
2. $\tan\theta = \dfrac{y}{x}$
3. $z = z$

As when we discussed conversion from rectangular coordinates to polar coordinates in two dimensions, it should be noted that the equation $\tan\theta = \frac{y}{x}$ has an infinite number of solutions. However, if we restrict θ to values between 0 and 2π, then we can find a unique solution based on the quadrant of the xy-plane in which original point (x, y, z) is located. Note that if $x = 0$, then the value of θ is either $\frac{\pi}{2}$, $\frac{3\pi}{2}$, or 0, depending on the value of y.

Notice that these equations are derived from properties of right triangles. To make this easy to see, consider point P in the xy-plane with rectangular coordinates $(x, y, 0)$ and with cylindrical coordinates $(r, \theta, 0)$, as shown in Figure 11.7.2.

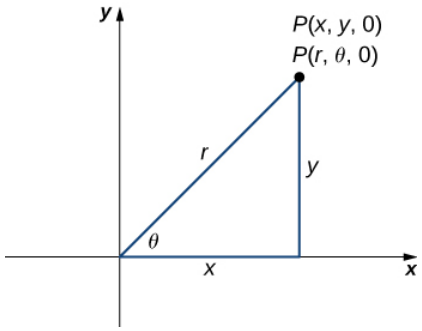


Figure 11.7.2: The Pythagorean theorem provides equation $r^2 = x^2 + y^2$. Right-triangle relationships tell us that $x = r\cos\theta, y = r\sin\theta$, and $\tan\theta = y/x$.

Let's consider the differences between rectangular and cylindrical coordinates by looking at the surfaces generated when each of the coordinates is held constant. If c is a constant, then in rectangular coordinates, surfaces of the form $x = c, y = c$, or $z = c$ are all planes. Planes of these forms are parallel to the yz-plane, the xz-plane, and the xy-plane, respectively. When we convert to cylindrical coordinates, the z-coordinate does not change. Therefore, in cylindrical coordinates, surfaces of the form $z = c$ are planes parallel to the xy-plane. Now, let's think about surfaces of the form $r = c$. The points on these surfaces are at a fixed distance from the z-axis. In other words, these surfaces are vertical circular cylinders. Last, what about $\theta = c$? The points on a surface of the form $\theta = c$ are at a fixed angle from the x-axis, which gives us a half-plane that starts at the z-axis (Figures 11.7.3 and 11.7.4).

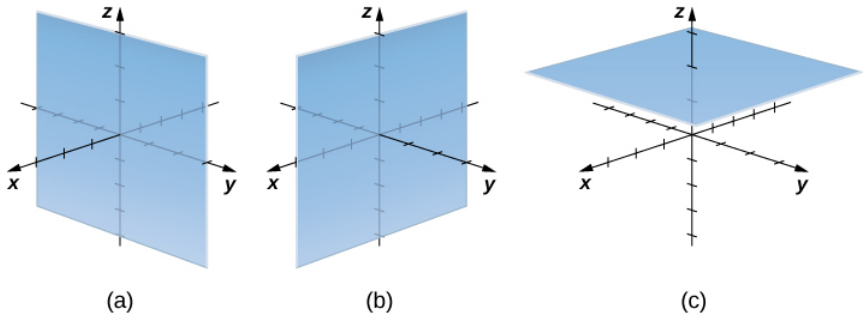


Figure 11.7.3: In rectangular coordinates, (a) surfaces of the form $x = c$ are planes parallel to the yz-plane, (b) surfaces of the form $y = c$ are planes parallel to the xz-plane, and (c) surfaces of the form $z = c$ are planes parallel to the xy-plane.

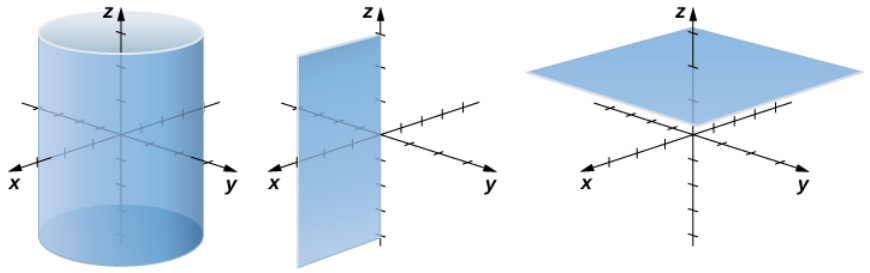


Figure 11.7.4: In cylindrical coordinates, (a) surfaces of the form $r = c$ are vertical cylinders of radius r, (b) surfaces of the form $\theta = c$ are half-planes at angle θ from the x-axis, and (c) surfaces of the form $z = c$ are planes parallel to the xy-plane.

Example 11.7.1: Converting from Cylindrical to Rectangular Coordinates

Plot the point with cylindrical coordinates $(4, \frac{2\pi}{3}, -2)$ and express its location in rectangular coordinates.

Solution

Conversion from cylindrical to rectangular coordinates requires a simple application of the equations listed in Conversion between Cylindrical and Cartesian Coordinates:

$$x = r\cos\theta = 4\cos\frac{2\pi}{3} = -2$$
$$y = r\sin\theta = 4\sin\frac{2\pi}{3} = 2\sqrt{3}.$$
$$z = -2$$

The point with cylindrical coordinates $(4, \frac{2\pi}{3}, -2)$ has rectangular coordinates $(-2, 2\sqrt{3}, -2)$ (Figure 11.7.5).

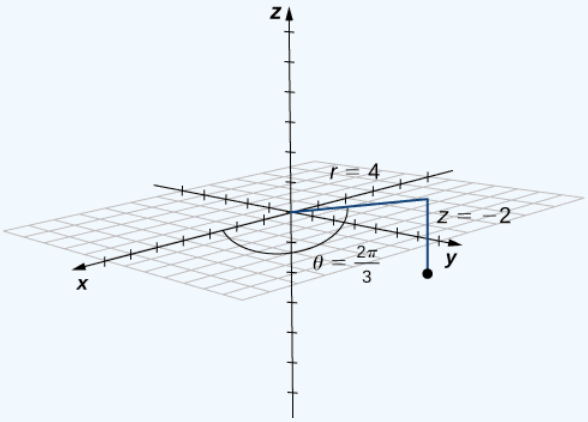


Figure 11.7.5: The projection of the point in the xy-plane is 4 units from the origin. The line from the origin to the point's projection forms an angle of $\frac{2\pi}{3}$ with the positive x-axis. The point lies 2 units below the xy-plane.

Exercise 11.7.1

Point R has cylindrical coordinates $(5, \frac{\pi}{6}, 4)$. Plot R and describe its location in space using rectangular, or Cartesian, coordinates.

If this process seems familiar, it is with good reason. This is exactly the same process that we followed in Introduction to Parametric Equations and Polar Coordinates to convert from polar coordinates to two-dimensional rectangular coordinates.

Example 11.7.2: Converting from Rectangular to Cylindrical Coordinates

Convert the rectangular coordinates $(1, -3, 5)$ to cylindrical coordinates.

Solution

Use the second set of equations from Conversion between Cylindrical and Cartesian Coordinates to translate from rectangular to cylindrical coordinates:

$$\begin{aligned} r^2 &= x^2 + y^2 \\ r &= \pm\sqrt{1^2 + (-3)^2} \\ &= \pm\sqrt{10}. \end{aligned}$$

We choose the positive square root, so $r = \sqrt{10}$.Now, we apply the formula to find θ. In this case, y is negative and x is positive, which means we must select the value of θ between $\frac{3\pi}{2}$ and 2π:

$$\begin{aligned} \tan\theta &= \frac{y}{x} & &= \frac{-3}{1} \\ \theta &= \arctan(-3) & &\approx 5.03\,\text{rad}. \end{aligned}$$

In this case, the z-coordinates are the same in both rectangular and cylindrical coordinates:

$$z = 5.$$

The point with rectangular coordinates $(1, -3, 5)$ has cylindrical coordinates approximately equal to $(\sqrt{10}, 5.03, 5)$.

Exercise 11.7.2

Convert point $(-8, 8, -7)$ from Cartesian coordinates to cylindrical coordinates.

The use of cylindrical coordinates is common in fields such as physics. Physicists studying electrical charges and the capacitors used to store these charges have discovered that these systems sometimes have a cylindrical symmetry. These systems have complicated modeling equations in the Cartesian coordinate system, which make them difficult to describe and analyze. The equations can often be expressed in more simple terms using cylindrical coordinates. For example, the cylinder described by equation $x^2 + y^2 = 25$ in the Cartesian system can be represented by cylindrical equation $r = 5$.

Example 11.7.3: Identifying Surfaces in the Cylindrical Coordinate System

Describe the surfaces with the given cylindrical equations.

a. $\theta = \frac{\pi}{4}$
b. $r^2 + z^2 = 9$
c. $z = r$

Solution

a. When the angle θ is held constant while r and z are allowed to vary, the result is a half-plane (Figure 11.7.6).

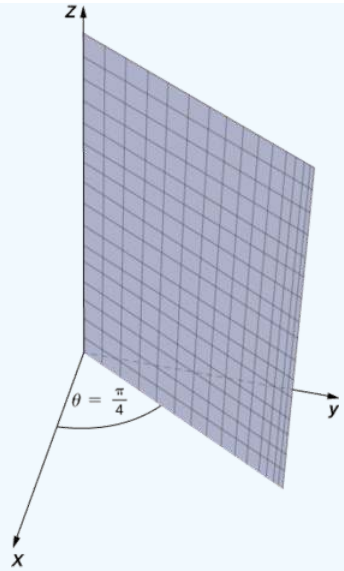


Figure 11.7.6: In polar coordinates, the equation $\theta = \pi/4$ describes the ray extending diagonally through the first quadrant. In three dimensions, this same equation describes a half-plane.

b. Substitute $r^2 = x^2 + y^2$ into equation $r^2 + z^2 = 9$ to express the rectangular form of the equation: $x^2 + y^2 + z^2 = 9$. This equation describes a sphere centered at the origin with radius 3 (Figure 11.7.7).

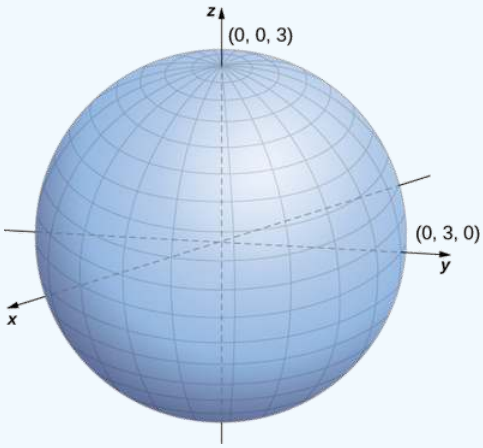


Figure 11.7.7: The sphere centered at the origin with radius 3 can be described by the cylindrical equation $r^2 + z^2 = 9$.

c. To describe the surface defined by equation $z = r$, is it useful to examine traces parallel to the xy-plane. For example, the trace in plane $z = 1$ is circle $r = 1$, the trace in plane $z = 3$ is circle $r = 3$, and so on. Each trace is a circle. As the value of z increases, the radius of the circle also increases. The resulting surface is a cone (Figure 11.7.8).

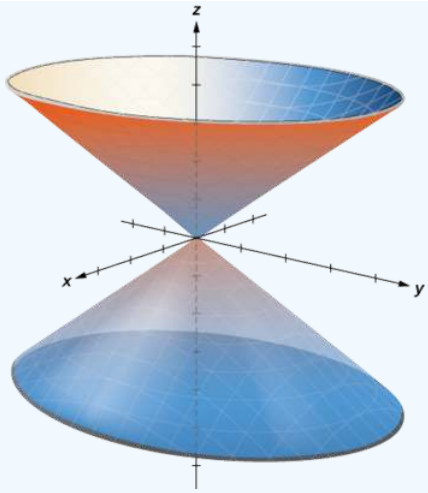


Figure **11.7.8**: The traces in planes parallel to the xy-plane are circles. The radius of the circles increases as z increases.

? Exercise 11.7.3

Describe the surface with cylindrical equation $r = 6$.

Spherical Coordinates

In the Cartesian coordinate system, the location of a point in space is described using an ordered triple in which each coordinate represents a distance. In the cylindrical coordinate system, the location of a point in space is described using two distances (r and z) and an angle measure (θ). In the spherical coordinate system, we again use an ordered triple to describe the location of a point in space. In this case, the triple describes one distance and two angles. Spherical coordinates make it simple to describe a sphere, just as cylindrical coordinates make it easy to describe a cylinder. Grid lines for spherical coordinates are based on angle measures, like those for polar coordinates.

Definition: spherical coordinate system

In the *spherical coordinate system*, a point P in space (Figure **11.7.9**) is represented by the ordered triple (ρ, θ, φ) where

- ρ (the Greek letter rho) is the distance between P and the origin ($\rho \neq 0$);
- θ is the same angle used to describe the location in cylindrical coordinates;
- φ (the Greek letter phi) is the angle formed by the positive z-axis and line segment $\overline{OP}$, where O is the origin and $0 \leq \varphi \leq \pi$.

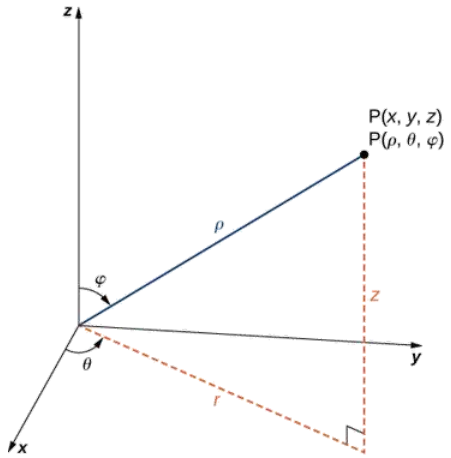


Figure **11.7.9**: The relationship among spherical, rectangular, and cylindrical coordinates.

By convention, the origin is represented as $(0, 0, 0)$ in spherical coordinates.

HOW TO: Converting among Spherical, Cylindrical, and Rectangular Coordinates

Rectangular coordinates (x, y, z), cylindrical coordinates (r, θ, z), and spherical coordinates (ρ, θ, φ) of a point are related as follows:

Convert from spherical coordinates to rectangular coordinates

These equations are used to convert from spherical coordinates to rectangular coordinates.

- $x = \rho \sin \varphi \cos \theta$
- $y = \rho \sin \varphi \sin \theta$
- $z = \rho \cos \varphi$

Convert from rectangular coordinates to spherical coordinates

These equations are used to convert from rectangular coordinates to spherical coordinates.

- $\rho^2 = x^2 + y^2 + z^2$
- $\tan \theta = \dfrac{y}{x}$
- $\varphi = \arccos\left(\dfrac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$.

Convert from spherical coordinates to cylindrical coordinates

These equations are used to convert from spherical coordinates to cylindrical coordinates.

- $r = \rho \sin \varphi$
- $\theta = \theta$
- $z = \rho \cos \varphi$

Convert from cylindrical coordinates to spherical coordinates

These equations are used to convert from cylindrical coordinates to spherical coordinates.

- $\rho = \sqrt{r^2 + z^2}$
- $\theta = \theta$
- $\varphi = \arccos\left(\dfrac{z}{\sqrt{r^2 + z^2}}\right)$

The formulas to convert from spherical coordinates to rectangular coordinates may seem complex, but they are straightforward applications of trigonometry. Looking at Figure **11.7.10**, it is easy to see that $r = \rho \sin \varphi$. Then, looking at the triangle in the xy-plane with r as its hypotenuse, we have $x = r \cos \theta = \rho \sin \varphi \cos \theta$. The derivation of the formula for y is similar. Figure **11.7.10** also shows that $\rho^2 = r^2 + z^2 = x^2 + y^2 + z^2$ and $z = \rho \cos \varphi$. Solving this last equation for φ and then substituting $\rho = \sqrt{r^2 + z^2}$ (from the first equation) yields $\varphi = \arccos\left(\dfrac{z}{\sqrt{r^2 + z^2}}\right)$. Also, note that, as before, we must be careful when using the formula $\tan \theta = \dfrac{y}{x}$ to choose the correct value of θ.

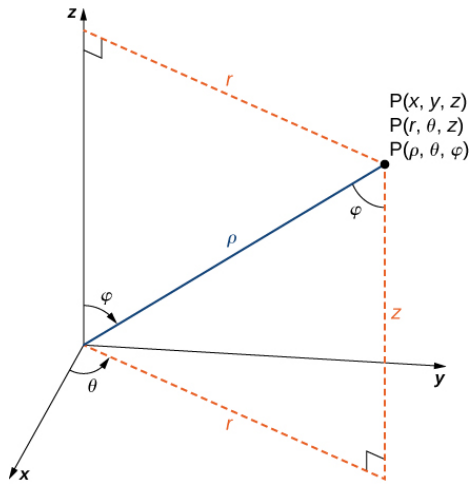


Figure 11.7.10: The equations that convert from one system to another are derived from right-triangle relationships.

As we did with cylindrical coordinates, let's consider the surfaces that are generated when each of the coordinates is held constant. Let c be a constant, and consider surfaces of the form $\rho = c$. Points on these surfaces are at a fixed distance from the origin and form a sphere. The coordinate θ in the spherical coordinate system is the same as in the cylindrical coordinate system, so surfaces of the form $\theta = c$ are half-planes, as before. Last, consider surfaces of the form $\varphi = c$. The points on these surfaces are at a fixed angle from the z-axis and form a half-cone (Figure 11.7.11).

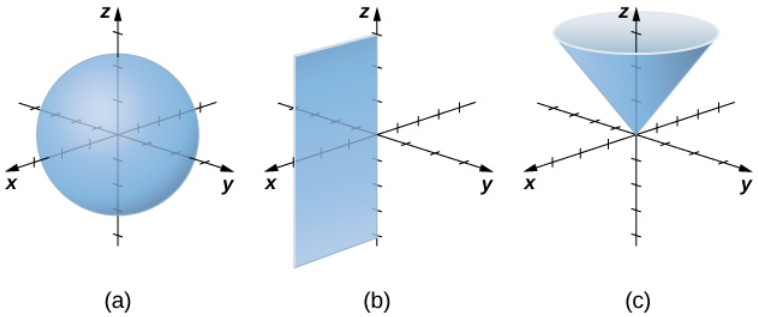


Figure 11.7.11: In spherical coordinates, surfaces of the form $\rho = c$ are spheres of radius ρ (a), surfaces of the form $\theta = c$ are half-planes at an angle θ from the x-axis (b), and surfaces of the form $\phi = c$ are half-cones at an angle ϕ from the z-axis (c).

Example 11.7.4: Converting from Spherical Coordinates

Plot the point with spherical coordinates $(8, \frac{\pi}{3}, \frac{\pi}{6})$ and express its location in both rectangular and cylindrical coordinates.

Solution

Use the equations in Converting among Spherical, Cylindrical, and Rectangular Coordinates to translate between spherical and cylindrical coordinates (Figure 11.7.12):

https://math.libretexts.org/@go/page/8627

$$
\begin{aligned}
x &= \rho \sin\varphi \cos\theta \\
&= 8 \sin\left(\frac{\pi}{6}\right) \cos\left(\frac{\pi}{3}\right) \\
&= 8\left(\frac{1}{2}\right)\frac{1}{2} \\
&= 2 \\
y &= \rho \sin\varphi \sin\theta \\
&= 8 \sin\left(\frac{\pi}{6}\right) \sin\left(\frac{\pi}{3}\right) \\
&= 8\left(\frac{1}{2}\right)\frac{\sqrt{3}}{2} \\
&= 2\sqrt{3} \\
z &= \rho \cos\varphi \\
&= 8 \cos\left(\frac{\pi}{6}\right) \\
&= 8\left(\frac{\sqrt{3}}{2}\right) \\
&= 4\sqrt{3}
\end{aligned}
$$

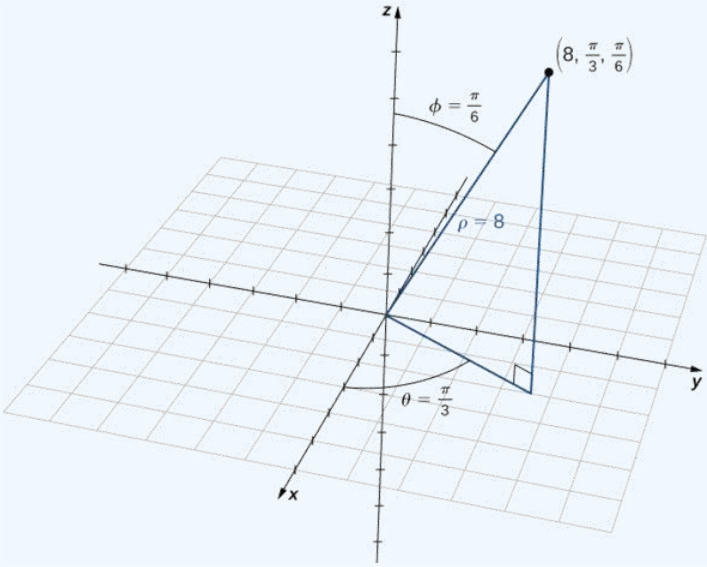


Figure 11.7.12: The projection of the point in the xy-plane is 4 units from the origin. The line from the origin to the point's projection forms an angle of $\pi/3$ with the positive x-axis. The point lies $4\sqrt{3}$ units above the xy-plane.

The point with spherical coordinates $(8, \frac{\pi}{3}, \frac{\pi}{6})$ has rectangular coordinates $(2, 2\sqrt{3}, 4\sqrt{3})$.

Finding the values in cylindrical coordinates is equally straightforward:

$$
\begin{aligned}
r &= \rho \sin\varphi \\
&= 8 \sin\frac{\pi}{6} \\
&= 4 \\
\theta &= \theta \\
z &= \rho \cos\varphi \\
&= 8 \cos\frac{\pi}{6} \\
&= 4\sqrt{3}.
\end{aligned}
$$

Thus, cylindrical coordinates for the point are $(4, \frac{\pi}{3}, 4\sqrt{3})$.

Exercise 11.7.4

Plot the point with spherical coordinates $(2, -\frac{5\pi}{6}, \frac{\pi}{6})$ and describe its location in both rectangular and cylindrical coordinates.

Example 11.7.5: Converting from Rectangular Coordinates

Convert the rectangular coordinates $(-1, 1, \sqrt{6})$ to both spherical and cylindrical coordinates.

Solution

Start by converting from rectangular to spherical coordinates:

$$\rho^2 = x^2 + y^2 + z^2 = (-1)^2 + 1^2 + (\sqrt{6})^2 = 8$$

$$\tan\theta = \frac{1}{-1}$$

$$\rho = 2\sqrt{2} \text{ and } \theta = \arctan(-1) = \frac{3\pi}{4}.$$

Because $(x, y) = (-1, 1)$, then the correct choice for θ is $\frac{3\pi}{4}$.

There are actually two ways to identify φ. We can use the equation $\varphi = \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$. A more simple approach, however, is to use equation $z = \rho\cos\varphi$. We know that $z = \sqrt{6}$ and $\rho = 2\sqrt{2}$, so

$$\sqrt{6} = 2\sqrt{2}\cos\varphi, \text{ so } \cos\varphi = \frac{\sqrt{6}}{2\sqrt{2}} = \frac{\sqrt{3}}{2}$$

and therefore $\varphi = \frac{\pi}{6}$. The spherical coordinates of the point are $(2\sqrt{2}, \frac{3\pi}{4}, \frac{\pi}{6})$.

To find the cylindrical coordinates for the point, we need only find r:

$$r = \rho\sin\varphi = 2\sqrt{2}\sin(\frac{\pi}{6}) = \sqrt{2}.$$

The cylindrical coordinates for the point are $(\sqrt{2}, \frac{3\pi}{4}, \sqrt{6})$.

Example 11.7.6: Identifying Surfaces in the Spherical Coordinate System

Describe the surfaces with the given spherical equations.

a. $\theta = \frac{\pi}{3}$

b. $\varphi = \frac{5\pi}{6}$

c. $\rho = 6$

d. $\rho = \sin\theta\sin\varphi$

Solution

a. The variable θ represents the measure of the same angle in both the cylindrical and spherical coordinate systems. Points with coordinates $(\rho, \frac{\pi}{3}, \varphi)$ lie on the plane that forms angle $\theta = \frac{\pi}{3}$ with the positive x-axis. Because $\rho > 0$, the surface described by equation $\theta = \frac{\pi}{3}$ is the half-plane shown in Figure 11.7.13.

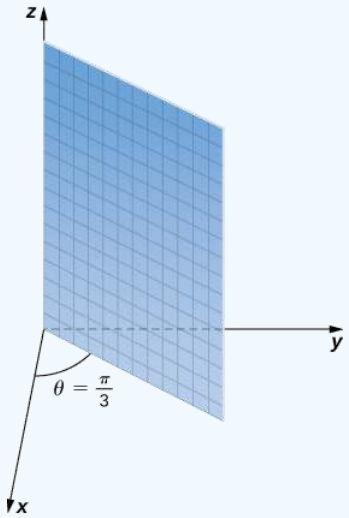


Figure 11.7.13: The surface described by equation $\theta = \frac{\pi}{3}$ is a half-plane.

b. Equation $\varphi = \frac{5\pi}{6}$ describes all points in the spherical coordinate system that lie on a line from the origin forming an angle measuring $\frac{5\pi}{6}$ rad with the positive z-axis. These points form a half-cone (Figure 11.7.14). Because there is only one value for φ that is measured from the positive z-axis, we do not get the full cone (with two pieces).

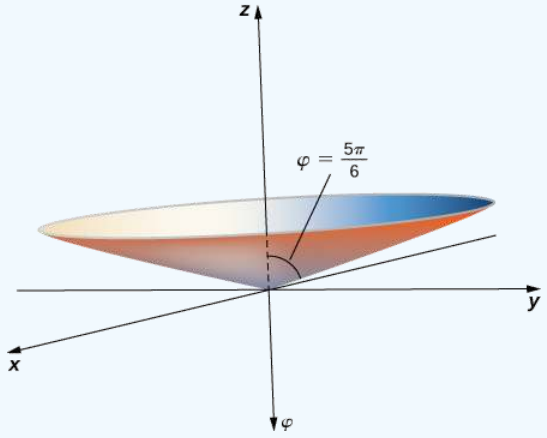


Figure 11.7.14: The equation $\varphi = \frac{5\pi}{6}$ describes a cone.

To find the equation in rectangular coordinates, use equation $\varphi = \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$.

$$\frac{5\pi}{6} = \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$$

$$\cos\frac{5\pi}{6} = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$-\frac{\sqrt{3}}{2} = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$\frac{3}{4} = \frac{z^2}{x^2+y^2+z^2}$$

$$\frac{3x^2}{4} + \frac{3y^2}{4} + \frac{3z^2}{4} = z^2$$

$$\frac{3x^2}{4} + \frac{3y^2}{4} - \frac{z^2}{4} = 0.$$

This is the equation of a cone centered on the z-axis.

c. Equation $\rho = 6$ describes the set of all points 6 units away from the origin—a sphere with radius 6 (Figure $11.7.15$).

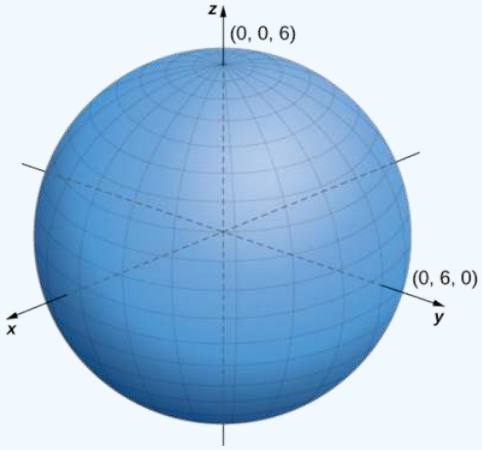


Figure $11.7.15$: Equation $\rho = 6$ describes a sphere with radius 6.

d. To identify this surface, convert the equation from spherical to rectangular coordinates, using equations $y = \rho \sin \varphi \sin \theta$ and $\rho^2 = x^2 + y^2 + z^2$:

$\rho = \sin \theta \sin \varphi$

$\rho^2 = \rho \sin \theta \sin \varphi$ Multiply both sides of the equation by ρ.

$x^2 + y^2 + z^2 = y$ Substitute rectangular variables using the equations above.

$x^2 + y^2 - y + z^2 = 0$ Subtract y from both sides of the equation.

$x^2 + y^2 - y + \frac{1}{4} + z^2 = \frac{1}{4}$ Complete the square.

$x^2 + \left(y - \frac{1}{2}\right)^2 + z^2 = \frac{1}{4}$. Rewrite the middle terms as a perfect square.

The equation describes a sphere centered at point $(0, \frac{1}{2}, 0)$ with radius $\frac{1}{2}$.

? Exercise 11.7.5

Describe the surfaces defined by the following equations.

a. $\rho = 13$
b. $\theta = \frac{2\pi}{3}$
c. $\varphi = \frac{\pi}{4}$

Spherical coordinates are useful in analyzing systems that have some degree of symmetry about a point, such as the volume of the space inside a domed stadium or wind speeds in a planet's atmosphere. A sphere that has Cartesian equation $x^2 + y^2 + z^2 = c^2$ has the simple equation $\rho = c$ in spherical coordinates.

In geography, latitude and longitude are used to describe locations on Earth's surface, as shown in Figure $11.7.16$. Although the shape of Earth is not a perfect sphere, we use spherical coordinates to communicate the locations of points on Earth. Let's assume Earth has the shape of a sphere with radius 4000 mi. We express angle measures in degrees rather than radians because latitude and longitude are measured in degrees.

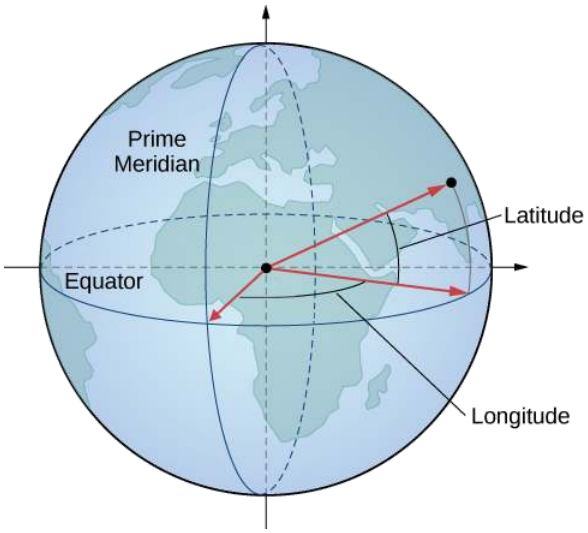


Figure 11.7.16: In the latitude–longitude system, angles describe the location of a point on Earth relative to the equator and the prime meridian.

Let the center of Earth be the center of the sphere, with the ray from the center through the North Pole representing the positive z-axis. The prime meridian represents the trace of the surface as it intersects the xz-plane. The equator is the trace of the sphere intersecting the xy-plane.

✔ Example 11.7.7: Converting Latitude and Longitude to Spherical Coordinates

The latitude of Columbus, Ohio, is $40°$ N and the longitude is $83°$ W, which means that Columbus is $40°$ north of the equator. Imagine a ray from the center of Earth through Columbus and a ray from the center of Earth through the equator directly south of Columbus. The measure of the angle formed by the rays is $40°$. In the same way, measuring from the prime meridian, Columbus lies $83°$ to the west. Express the location of Columbus in spherical coordinates.

Solution

The radius of Earth is 4000mi, so $\rho = 4000$. The intersection of the prime meridian and the equator lies on the positive x-axis. Movement to the west is then described with negative angle measures, which shows that $\theta = -83°$, Because Columbus lies $40°$ north of the equator, it lies $50°$ south of the North Pole, so $\varphi = 50°$. In spherical coordinates, Columbus lies at point $(4000, -83°, 50°)$.

? Exercise 11.7.6

Sydney, Australia is at $34°S$ and $151°E$. Express Sydney's location in spherical coordinates.

Cylindrical and spherical coordinates give us the flexibility to select a coordinate system appropriate to the problem at hand. A thoughtful choice of coordinate system can make a problem much easier to solve, whereas a poor choice can lead to unnecessarily complex calculations. In the following example, we examine several different problems and discuss how to select the best coordinate system for each one.

✔ Example 11.7.8: Choosing the Best Coordinate System

In each of the following situations, we determine which coordinate system is most appropriate and describe how we would orient the coordinate axes. There could be more than one right answer for how the axes should be oriented, but we select an orientation that makes sense in the context of the problem. Note: There is not enough information to set up or solve these problems; we simply select the coordinate system (Figure 11.7.17).

a. Find the center of gravity of a bowling ball.
b. Determine the velocity of a submarine subjected to an ocean current.
c. Calculate the pressure in a conical water tank.
d. Find the volume of oil flowing through a pipeline.
e. Determine the amount of leather required to make a football.

(a) (b) (c)

(d) (e)

Figure 11.7.17: (credit: (a) modification of work by scl hua, Wikimedia, (b) modification of work by DVIDSHUB, Flickr, (c) modification of work by Michael Malak, Wikimedia, (d) modification of work by Sean Mack, Wikimedia, (e) modification of work by Elvert Barnes, Flickr)

Solution

a. Clearly, a bowling ball is a sphere, so spherical coordinates would probably work best here. The origin should be located at the physical center of the ball. There is no obvious choice for how the x-, y- and z-axes should be oriented. Bowling balls normally have a weight block in the center. One possible choice is to align the z-axis with the axis of symmetry of the weight block.

b. A submarine generally moves in a straight line. There is no rotational or spherical symmetry that applies in this situation, so rectangular coordinates are a good choice. The z-axis should probably point upward. The x- and y-axes could be aligned to point east and north, respectively. The origin should be some convenient physical location, such as the starting position of the submarine or the location of a particular port.

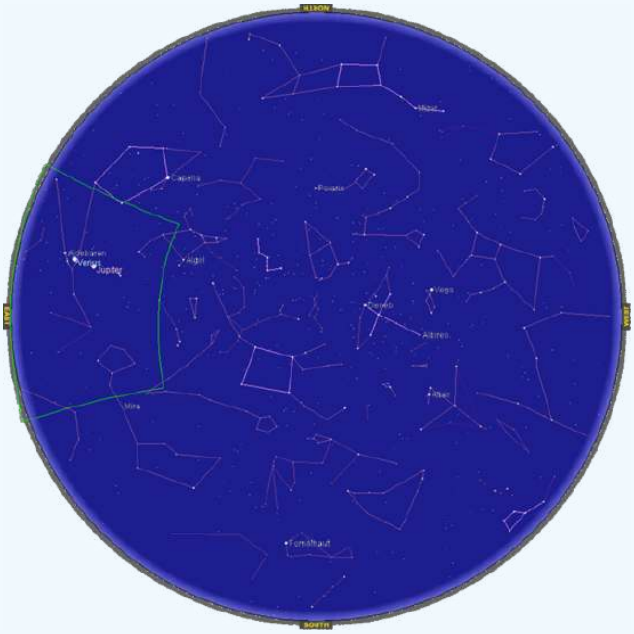
c. A cone has several kinds of symmetry. In cylindrical coordinates, a cone can be represented by equation $z = kr$, where k is a constant. In spherical coordinates, we have seen that surfaces of the form $\varphi = c$ are half-cones. Last, in rectangular coordinates, elliptic cones are quadric surfaces and can be represented by equations of the form $z^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$. In this case, we could choose any of the three. However, the equation for the surface is more complicated in rectangular coordinates than in the other two systems, so we might want to avoid that choice. In addition, we are talking about a water tank, and the depth of the water might come into play at some point in our calculations, so it might be nice to have a component that represents height and depth directly. Based on this reasoning, cylindrical coordinates might be the best choice. Choose the z-axis to align with the axis of the cone. The orientation of the other two axes is arbitrary. The origin should be the bottom point of the cone.

d. A pipeline is a cylinder, so cylindrical coordinates would be best the best choice. In this case, however, we would likely choose to orient our z-axis with the center axis of the pipeline. The x-axis could be chosen to point straight downward or to some other logical direction. The origin should be chosen based on the problem statement. Note that this puts the z-axis in a horizontal orientation, which is a little different from what we usually do. It may make sense to choose an unusual orientation for the axes if it makes sense for the problem.

e. A football has rotational symmetry about a central axis, so cylindrical coordinates would work best. The z-axis should align with the axis of the ball. The origin could be the center of the ball or perhaps one of the ends. The position of the x-axis is arbitrary.

? Exercise 11.7.7

Which coordinate system is most appropriate for creating a star map, as viewed from Earth (see the following figure)?



How should we orient the coordinate axes?

Key Concepts

- In the cylindrical coordinate system, a point in space is represented by the ordered triple (r, θ, z), where (r, θ) represents the polar coordinates of the point's projection in the xy-plane and z represents the point's projection onto the z-axis.
- To convert a point from cylindrical coordinates to Cartesian coordinates, use equations $x = r\cos\theta, y = r\sin\theta$, and $z = z$.
- To convert a point from Cartesian coordinates to cylindrical coordinates, use equations $r^2 = x^2 + y^2, \tan\theta = \frac{y}{x}$, and $z = z$.
- In the spherical coordinate system, a point P in space is represented by the ordered triple (ρ, θ, φ), where ρ is the distance between P and the origin $(\rho \neq 0), \theta$ is the same angle used to describe the location in cylindrical coordinates, and φ is the angle formed by the positive z-axis and line segment $\overline{OP}$, where O is the origin and $0 \leq \varphi \leq \pi$.
- To convert a point from spherical coordinates to Cartesian coordinates, use equations $x = \rho\sin\varphi\cos\theta, y = \rho\sin\varphi\sin\theta$, and $z = \rho\cos\varphi$.
- To convert a point from Cartesian coordinates to spherical coordinates, use equations $\rho^2 = x^2 + y^2 + z^2, \tan\theta = \frac{y}{x}$, and $\varphi = \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$.
- To convert a point from spherical coordinates to cylindrical coordinates, use equations $r = \rho\sin\varphi, \theta = \theta$, and $z = \rho\cos\varphi$.
- To convert a point from cylindrical coordinates to spherical coordinates, use equations $\rho = \sqrt{r^2+z^2}, \theta = \theta$, and $\varphi = \arccos\left(\frac{z}{\sqrt{r^2+z^2}}\right)$.

Glossary

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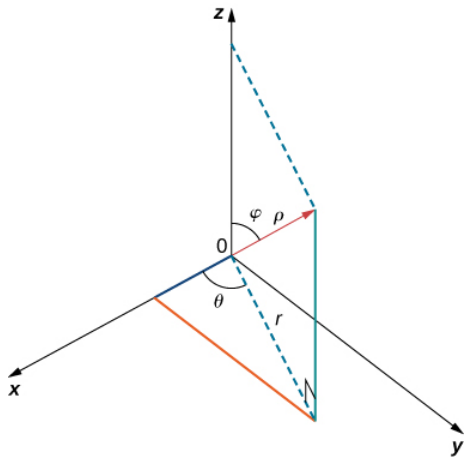
11.7E: Exercises for Cylindrical and Spherical Coordinates

Use the following figure as an aid in identifying the relationship between the rectangular, cylindrical, and spherical coordinate systems.



For exercises 1 - 4, the cylindrical coordinates (r, θ, z) of a point are given. Find the rectangular coordinates (x, y, z) of the point.

1) $\left(4, \frac{\pi}{6}, 3\right)$

2) $\left(3, \frac{\pi}{3}, 5\right)$

3) $\left(4, \frac{7\pi}{6}, 3\right)$

4) $(2, \pi, -4)$

For exercises 5 - 8, the rectangular coordinates (x, y, z) of a point are given. Find the cylindrical coordinates (r, θ, z) of the point.

5) $(1, \sqrt{3}, 2)$

6) $(1, 1, 5)$

7) $(3, -3, 7)$

8) $(-2\sqrt{2}, 2\sqrt{2}, 4)$

For exercises 9 - 16, the equation of a surface in cylindrical coordinates is given. Find the equation of the surface in rectangular coordinates. Identify and graph the surface.

9) [T] $r = 4$

10) [T] $z = r^2 \cos^2 \theta$

11) [T] $r^2 \cos(2\theta) + z^2 + 1 = 0$

12) [T] $r = 3 \sin \theta$

13) [T] $r = 2 \cos \theta$

14) [T] $r^2 + z^2 = 5$

15) [T] $r = 2 \sec \theta$

16) [T] $r = 3 \csc \theta$

For exercises 17 - 22, the equation of a surface in rectangular coordinates is given. Find the equation of the surface in cylindrical coordinates.

17) $z = 3$

18) $x = 6$

19) $x^2 + y^2 + z^2 = 9$

20) $y = 2x^2$

21) $x^2 + y^2 - 16x = 0$

22) $x^2 + y^2 - 3\sqrt{x^2 + y^2} + 2 = 0$

For exercises 23 - 26, the spherical coordinates (ρ, θ, φ) of a point are given. Find the rectangular coordinates (x, y, z) of the point.

23) $(3, 0, \pi)$

24) $\left(1, \frac{\pi}{6}, \frac{\pi}{6}\right)$

25) $\left(12, -\frac{\pi}{4}, \frac{\pi}{4}\right)$

26) $\left(3, \frac{\pi}{4}, \frac{\pi}{6}\right)$

For exercises 27 - 30, the rectangular coordinates (x, y, z) of a point are given. Find the spherical coordinates (ρ, θ, φ) of the point. Express the measure of the angles in degrees rounded to the nearest integer.

27) $(4, 0, 0)$

28) $(-1, 2, 1)$

29) $(0, 3, 0)$

30) $(-2, 2\sqrt{3}, 4)$

For exercises 31 - 36, the equation of a surface in spherical coordinates is given. Find the equation of the surface in rectangular coordinates. Identify and graph the surface.

31) [T] $\rho = 3$

32) [T] $\varphi = \frac{\pi}{3}$

33) [T] $\rho = 2\cos\varphi$

34) [T] $\rho = 4\csc\varphi$

35) [T] $\varphi = \frac{\pi}{2}$

36) [T] $\rho = 6\csc\varphi\sec\theta$

For exercises 37 - 40, the equation of a surface in rectangular coordinates is given. Find the equation of the surface in spherical coordinates. Identify the surface.

37) $x^2 + y^2 - 3z^2 = 0, \quad z \neq 0$

38) $x^2 + y^2 + z^2 - 4z = 0$

39) $z = 6$

40) $x^2 + y^2 = 9$

For exercises 41 - 44, the cylindrical coordinates of a point are given. Find its associated spherical coordinates, with the measure of the angle φ in radians rounded to four decimal places.

41) [T] $\left(1, \frac{\pi}{4}, 3\right)$

42) [T] $(5, \pi, 12)$

43) $\left(3, \frac{\pi}{2}, 3\right)$

44) $\left(3, -\frac{\pi}{6}, 3\right)$

For exercises 45 - 48, the spherical coordinates of a point are given. Find its associated cylindrical coordinates.

45) $(2, -\frac{\pi}{4}, \frac{\pi}{2})$

46) $(4, \frac{\pi}{4}, \frac{\pi}{6})$

47) $(8, \frac{\pi}{3}, \frac{\pi}{2})$

48) $(9, -\frac{\pi}{6}, \frac{\pi}{3})$

For exercises 49 - 52, find the most suitable system of coordinates to describe the solids.

49) The solid situated in the first octant with a vertex at the origin and enclosed by a cube of edge length a, where $a > 0$

50) A spherical shell determined by the region between two concentric spheres centered at the origin, of radii of a and b, respectively, where $b > a > 0$

51) A solid inside sphere $x^2 + y^2 + z^2 = 9$ and outside cylinder $\left(x - \frac{3}{2}\right)^2 + y^2 = \frac{9}{4}$

52) A cylindrical shell of height 10 determined by the region between two cylinders with the same center, parallel rulings, and radii of 2 and 5, respectively

53) [T] Use a CAS or CalcPlot3D to graph in cylindrical coordinates the region between elliptic paraboloid $z = x^2 + y^2$ and cone $x^2 + y^2 - z^2 = 0$.

54) [T] Use a CAS or CalcPlot3D to graph in spherical coordinates the "ice cream-cone region" situated above the xy-plane between sphere $x^2 + y^2 + z^2 = 4$ and elliptical cone $x^2 + y^2 - z^2 = 0$.

55) Washington, DC, is located at $39°$ N and $77°$ W (see the following figure). Assume the radius of Earth is 4000 mi. Express the location of Washington, DC, in spherical coordinates.

56) San Francisco is located at $37.78°N$ and $122.42°W$. Assume the radius of Earth is 4000mi. Express the location of San Francisco in spherical coordinates.

57) Find the latitude and longitude of Rio de Janeiro if its spherical coordinates are $(4000, -43.17°, 102.91°)$.

58) Find the latitude and longitude of Berlin if its spherical coordinates are $(4000, 13.38°, 37.48°)$.

59) [T] Consider the torus of equation $\left(x^2 + y^2 + z^2 + R^2 - r^2\right)^2 = 4R^2(x^2 + y^2)$, where $R \geq r > 0$.

a. Write the equation of the torus in spherical coordinates.

b. If $R = r$, the surface is called a horn torus. Show that the equation of a horn torus in spherical coordinates is $\rho = 2R \sin \varphi$.

c. Use a CAS or CalcPlot3D to graph the horn torus with $R = r = 2$ in spherical coordinates.

60) [T] The "bumpy sphere" with an equation in spherical coordinates is $\rho = a + b\cos(m\theta)\sin(n\varphi)$, with $\theta \in [0, 2\pi]$ and $\varphi \in [0, \pi]$, where a and b are positive numbers and m and n are positive integers, may be used in applied mathematics to model tumor growth.

a. Show that the "bumpy sphere" is contained inside a sphere of equation $\rho = a + b$. Find the values of θ and φ at which the two surfaces intersect.

b. Use a CAS or CalcPlot3D to graph the surface for $a = 14, b = 2, m = 4$, and $n = 6$ along with sphere $\rho = a + b$.

c. Find the equation of the intersection curve of the surface at b. with the cone $\varphi = \frac{\pi}{12}$. Graph the intersection curve in the plane of intersection.

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Links to CalcPlot3D added by Paul Seeburger (Monroe Community College).

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11R: Chapter 11 Review Exercises

For exercises 1 - 4, determine whether the statement is *true* or *false*. Justify the answer with a proof or a counterexample.

1) For vectors $\vec{a}$ and $\vec{b}$ and any given scalar c, $c(\vec{a} \cdot \vec{b}) = (c\vec{a}) \cdot \vec{b}$.

2) For vectors $\vec{a}$ and $\vec{b}$ and any given scalar c, $c(\vec{a} \times \vec{b}) = (c\vec{a}) \times \vec{b}$.

3) The symmetric equation for the line of intersection between two planes $x + y + z = 2$ and $x + 2y - 4z = 5$ is given by $-\frac{x-1}{6} = \frac{y-1}{5} = z$.

4) If $\vec{a} \cdot \vec{b} = 0$, then $\vec{a}$ is perpendicular to $\vec{b}$.

For exercises 5 and 6, use the given vectors to find the quantities.

5) $\vec{a} = 9\hat{i} - 2\hat{j}, \quad \vec{b} = -3\hat{i} + \hat{k}$

a. $3\vec{a} + \vec{b}$

b. $\|\vec{a}\|$

c. $\vec{a} \times \|\vec{b} \times \vec{a}\|$

d. $\|\vec{b} \times \vec{a}\|$

6) $\vec{a} = 2\hat{i} + \hat{j} - 9\hat{k}, \quad \vec{b} = -\hat{i} + 2\hat{k}, \quad \vec{c} = 4\hat{i} - 2\hat{j} + \hat{k}$

a. $2\vec{a} - \vec{b}$

b. $\|\vec{b} \times \vec{c}\|$

c. $\vec{b} \times \left(\vec{b} \times \vec{c}\right)$

d. $\vec{c} \times \|\vec{b} \times \vec{a}\|$

e. $\text{Proj}_{\vec{a}}\, \vec{b}$

7) Find the values of a such that vectors $\langle 2, 4, a\rangle$ and $\langle 0, -1, a\rangle$ are orthogonal.

For exercises 8 and 9, find the unit vectors.

8) Find the unit vector that has the same direction as vector $\vec{v}$ that begins at $(0, -3)$ and ends at $(4, 10)$.

9) Find the unit vector that has the same direction as vector $\vec{v}$ that begins at $(1, 4, 10)$ and ends at $(3, 0, 4)$.

For exercises 10 and 11, find the area or volume of the given shapes.

10) The parallelogram spanned by vectors $\vec{a} = \langle 1, 13\rangle$ and $\vec{b} = \langle 3, 21\rangle$

11) The parallelepiped formed by $\vec{a} = \langle 1, 4, 1\rangle$ and $\vec{b} = \langle 3, 6, 2\rangle$, and $\vec{c} = \langle -2, 1, -5\rangle$

For exercises 12 and 13, find the parametric equations and the vector equation of the line with the given properties.

12) The line that passes through point $(2, -3, 7)$ that is parallel to vector $\langle 1, 3, -2\rangle$

13) The line that passes through points $(1, 3, 5)$ and $(-2, 6, -3)$

For exercises 14 and 15, find the equation of the plane with the given properties.

14) The plane that passes through point $(4, 7, -1)$ and has normal vector $\vec{n} = \langle 3, 4, 2\rangle$

15) The plane that passes through points $(0, 1, 5), (2, -1, 6)$, and $(3, 2, 5)$.

For exercises 16 and 17, find the traces for the surfaces in planes $x = k$, $y = k$, and $z = k$. Then, describe and draw the surfaces.

16) $9x^2 + 4y^2 - 16y + 36z^2 = 20$

17) $x^2 = y^2 + z^2$

For exercises 18 and 19, write the given equation in cylindrical coordinates and spherical coordinates.

18) $x^2 + y^2 + z^2 = 144$

19) $z = x^2 + y^2 - 1$

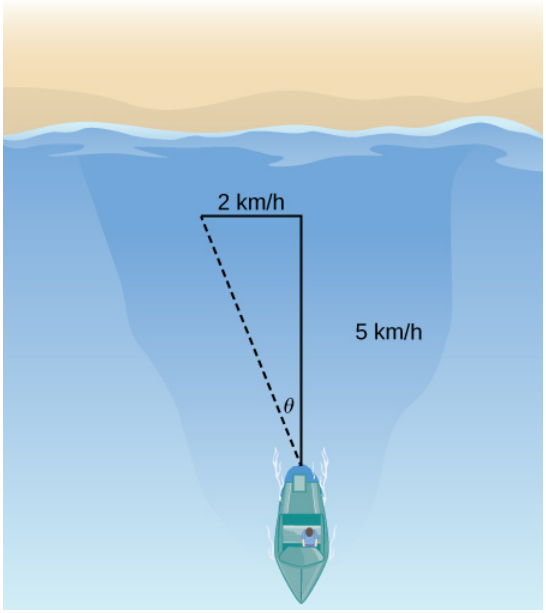
For exercises 20 and 21, convert the given equations from cylindrical or spherical coordinates to rectangular coordinates. Identify the given surface.

20) $\rho^2(\sin^2(\varphi) - \cos^2(\varphi)) = 1$

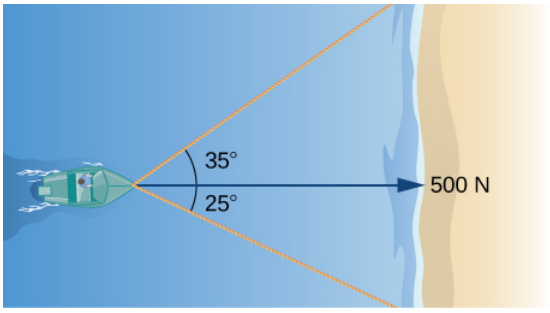
21) $r^2 - 2r\cos(\theta) + z^2 = 1$

For exercises 22 and 23, consider a small boat crossing a river.

22) If the boat velocity is 5 km/h due north in still water and the water has a current of 2 km/h due west (see the following figure), what is the velocity of the boat relative to shore? What is the angle θ that the boat is actually traveling?



23) When the boat reaches the shore, two ropes are thrown to people to help pull the boat ashore. One rope is at an angle of $25°$ and the other is at $35°$. If the boat must be pulled straight and at a force of 500 N, find the magnitude of force for each rope (see the following figure).



24) An airplane is flying in the direction of 52° east of north with a speed of 450 mph. A strong wind has a bearing 33° east of north with a speed of 50 mph. What is the resultant ground speed and bearing of the airplane?

25) Calculate the work done by moving a particle from position $(1, 2, 0)$ to $(8, 4, 5)$ along a straight line with a force $\vec{\mathbf{F}} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}}$.

In problems 26 and 27, consider your unsuccessful attempt to take the tire off your car using a wrench to loosen the bolts. Assume the wrench is 0.3 m long and you are able to apply a 200-N force.

26) Because your tire is flat, you are only able to apply your force at a $60°$ angle. What is the torque at the center of the bolt? Assume this force is not enough to loosen the bolt.

27) Someone lends you a tire jack and you are now able to apply a 200-N force at an $80°$ angle. Is your resulting torque going to be more or less? What is the new resulting torque at the center of the bolt? Assume this force is not enough to loosen the bolt.

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CHAPTER OVERVIEW

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12.1: Vector-Valued Functions and Space Curves

Our study of vector-valued functions combines ideas from our earlier examination of single-variable calculus with our description of vectors in three dimensions from the preceding chapter. In this section, we extend concepts from earlier chapters and also examine new ideas concerning curves in three-dimensional space. These definitions and theorems support the presentation of material in the rest of this chapter and also in the remaining chapters of the text.

Definition of a Vector-Valued Function

Our first step in studying the calculus of vector-valued functions is to define what exactly a vector-valued function is. We can then look at graphs of vector-valued functions and see how they define curves in both two and three dimensions.

Definition: Vector-valued Functions

A vector-valued function is a function of the form

$$\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} \quad \text{or} \quad \vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} + h(t)\,\hat{\mathbf{k}}, \tag{12.1.1}$$

where the component functions f, g, and h, are real-valued functions of the parameter t.

Vector-valued functions can also be written in the form

$$\vec{\mathbf{r}}(t) = \langle f(t),\, g(t)\rangle \text{ or } \vec{\mathbf{r}}(t) = \langle f(t),\, g(t),\, h(t)\rangle. \tag{12.1.2}$$

In both cases, the first form of the function defines a two-dimensional vector-valued function in the plane; the second form describes a three-dimensional vector-valued function in space.

We often use t as a parameter because t can represent time.

The parameter t may lie between two real numbers: $a \le t \le b$, or its value may range over the entire set of real numbers.

Each of the component functions that make up a vector-valued function may have domain restrictions that enforce restrictions on the value of t.

The domain of a vector-valued function $\vec{\mathbf{r}}$ is the intersection of the domains of its component functions, i.e., it is the set of all values of t for which the vector-valued function is defined.

Example 12.1.1: Finding the domain of a vector-valued function

State the domain of the vector-valued function $\vec{\mathbf{r}}(t) = \sqrt{2-t}\,\hat{\mathbf{i}} + \ln(t+3)\,\hat{\mathbf{j}} + e^t\,\hat{\mathbf{k}}$.

Solution

We first consider the natural domain of each component function. Note that we list the domains in both **set-builder notation** and **interval notation**.

Function:	**Domain**:
$\sqrt{2-t}$	$\{t \mid t \le 2\}$ or $(-\infty, 2]$
$\ln(t+3)$	$\{t \mid t > -3\}$ or $(-3, \infty)$
e^t	$(-\infty, \infty)$

The domain of $\vec{\mathbf{r}}$ is the intersection of these domains, so it must contain all values of t that work in all three, but no value of t that does not work in any one of these functions.

Hence, the domain of $\vec{\mathbf{r}}$ is: $D_{\vec{\mathbf{r}}} : \{t \mid -3 < t \le 2\}$ or $(-3, 2]$.

Note that only one form of the domain of $\vec{\mathbf{r}}$ need be given. The first, $\{t \mid -3 < t \le 2\}$, is in ***set-builder notation***, while the second, $(-3, 2]$, is in ***interval notation***.

Example 12.1.2: Evaluating Vector-Valued Functions and Determining Domains

For each of the following vector-valued functions, evaluate $\vec{\mathbf{r}}(0)$, $\vec{\mathbf{r}}\left(\frac{\pi}{2}\right)$, and $\vec{\mathbf{r}}\left(\frac{2\pi}{3}\right)$. Do any of these functions have domain restrictions?

a. $\vec{\mathbf{r}}(t) = 4\cos t\,\hat{\mathbf{i}} + 3\sin t\,\hat{\mathbf{j}}$
b. $\vec{\mathbf{r}}(t) = 3\tan t\,\hat{\mathbf{i}} + 4\sec t\,\hat{\mathbf{j}} + 5t\,\hat{\mathbf{k}}$

Solution

a. To calculate each of the function values, substitute the appropriate value of t into the function:

$$\begin{aligned}
\vec{\mathbf{r}}(0) &= 4\cos(0)\hat{\mathbf{i}} + 3\sin(0)\hat{\mathbf{j}} \\
&= 4\hat{\mathbf{i}} + 0\hat{\mathbf{j}} = 4\hat{\mathbf{i}} \\
\vec{\mathbf{r}}\left(\frac{\pi}{2}\right) &= 4\cos\left(\frac{\pi}{2}\right)\hat{\mathbf{i}} + 3\sin\left(\frac{\pi}{2}\right)\hat{\mathbf{j}} \\
&= 0\hat{\mathbf{i}} + 3\hat{\mathbf{j}} = 3\hat{\mathbf{j}} \\
\vec{\mathbf{r}}\left(\frac{2\pi}{3}\right) &= 4\cos\left(\frac{2\pi}{3}\right)\hat{\mathbf{i}} + 3\sin\left(\frac{2\pi}{3}\right)\hat{\mathbf{j}} \\
&= 4\left(-\tfrac{1}{2}\right)\hat{\mathbf{i}} + 3\left(\tfrac{\sqrt{3}}{2}\right)\hat{\mathbf{j}} = -2\hat{\mathbf{i}} + \tfrac{3\sqrt{3}}{2}\hat{\mathbf{j}}
\end{aligned}$$

To determine whether this function has any domain restrictions, consider the component functions separately. The first component function is $f(t) = 4\cos t$ and the second component function is $g(t) = 3\sin t$. Neither of these functions has a domain restriction, so the domain of $\vec{\mathbf{r}}(t) = 4\cos t\,\hat{\mathbf{i}} + 3\sin t\,\hat{\mathbf{j}}$ is all real numbers.

b. To calculate each of the function values, substitute the appropriate value of t into the function:

$$\begin{aligned}
\vec{\mathbf{r}}(0) &= 3\tan(0)\hat{\mathbf{i}} + 4\sec(0)\hat{\mathbf{j}} + 5(0)\hat{\mathbf{k}} \\
&= 0\hat{\mathbf{i}} + 4j + 0\hat{\mathbf{k}} = 4\hat{\mathbf{j}} \\
\vec{\mathbf{r}}\left(\frac{\pi}{2}\right) &= 3\tan\left(\frac{\pi}{2}\right)\hat{\mathbf{i}} + 4\sec\left(\frac{\pi}{2}\right)\hat{\mathbf{j}} + 5\left(\frac{\pi}{2}\right)\hat{\mathbf{k}}, \text{ which does not exist} \\
\vec{\mathbf{r}}\left(\frac{2\pi}{3}\right) &= 3\tan\left(\frac{2\pi}{3}\right)\hat{\mathbf{i}} + 4\sec\left(\frac{2\pi}{3}\right)\hat{\mathbf{j}} + 5\left(\frac{2\pi}{3}\right)\hat{\mathbf{k}} \\
&= 3(-\sqrt{3})\hat{\mathbf{i}} + 4(-2)\hat{\mathbf{j}} + \frac{10\pi}{3}\hat{\mathbf{k}} \\
&= (-3\sqrt{3})\hat{\mathbf{i}} - 8\hat{\mathbf{j}} + \frac{10\pi}{3}\hat{\mathbf{k}}
\end{aligned}$$

To determine whether this function has any domain restrictions, consider the component functions separately. The first component function is $f(t) = 3\tan t$, the second component function is $g(t) = 4\sec t$, and the third component function is $h(t) = 5t$. The first two functions are not defined for odd multiples of $\frac{\pi}{2}$, so the function is not defined for odd multiples of $\frac{\pi}{2}$. Therefore,

$$\mathrm{D}_{\vec{\mathbf{r}}} = \left\{t \,|\, t \neq \frac{(2n+1)\pi}{2}\right\},$$

where n is any integer.

Exercise 12.1.1

For the vector-valued function $\vec{\mathbf{r}}(t) = (t^2 - 3t)\,\hat{\mathbf{i}} + (4t + 1)\,\hat{\mathbf{j}}$, evaluate $\vec{\mathbf{r}}(0)$, $\vec{\mathbf{r}}(1)$, and $\vec{\mathbf{r}}(-4)$. Does this function have any domain restrictions?

Example 12.1.1 illustrates an important concept. The domain of a vector-valued function consists of real numbers. The domain can be all real numbers or a subset of the real numbers. The range of a vector-valued function consists of vectors. Each real number in the domain of a vector-valued function is mapped to either a two- or a three-dimensional vector.

Graphing Vector-Valued Functions

Recall that a plane vector consists of two quantities: direction and magnitude. Given any point in the plane (the *initial point*), if we move in a specific direction for a specific distance, we arrive at a second point. This represents the *terminal point* of the vector. We calculate the components of the vector by subtracting the coordinates of the initial point from the coordinates of the terminal point.

A vector is considered to be in *standard position* if the initial point is located at the origin. When graphing a vector-valued function, we typically graph the vectors in the domain of the function in standard position, because doing so guarantees the uniqueness of the graph. This convention applies to the graphs of three-dimensional vector-valued functions as well. The graph of a vector-valued function of the form

$$\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}}$$

consists of the set of all points $(f(t),\, g(t))$, and the path it traces is called a **plane curve**. The graph of a vector-valued function of the form

$$\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} + h(t)\,\hat{\mathbf{k}}$$

consists of the set of all points $(f(t),\, g(t),\, h(t))$, and the path it traces is called a **space curve**. Any representation of a plane curve or space curve using a vector-valued function is called a **vector parameterization** of the curve.

Each plane curve and space curve has an **orientation**, indicated by arrows drawn in on the curve, that shows the direction of motion along the curve as the value of the parameter t increases.

Example 12.1.3: Graphing a Vector-Valued Function

Create a graph of each of the following vector-valued functions:

a. The plane curve represented by $\vec{\mathbf{r}}(t) = 4\cos t\,\hat{\mathbf{i}} + 3\sin t\,\hat{\mathbf{j}},\ 0 \le t \le 2\pi$
b. The plane curve represented by $\vec{\mathbf{r}}(t) = 4\cos(t^3)\,\hat{\mathbf{i}} + 3\sin(t^3)\,\hat{\mathbf{j}},\ 0 \le t \le \sqrt[3]{2\pi}$
c. The space curve represented by $\vec{\mathbf{r}}(t) = 4\cos t\,\hat{\mathbf{i}} + 4\sin t\,\hat{\mathbf{j}} + t\,\hat{\mathbf{k}},\ 0 \le t \le 4\pi$

Solution

a. As with any graph, we start with a table of values. We then graph each of the vectors in the second column of the table in standard position and connect the terminal points of each vector to form a curve (Figure 12.1.1). This curve turns out to be an ellipse centered at the origin.

Table 12.1.1: Table of Values for $\vec{\mathbf{r}}(t) = 4\cos t\,\hat{\mathbf{i}} + 3\sin t\,\hat{\mathbf{j}},\ 0 \le t \le 2\pi$

t	$\vec{\mathbf{r}}(t)$	t	$\vec{\mathbf{r}}(t)$
0	$4\hat{\mathbf{i}}$	π	$-4\hat{\mathbf{i}}$
$\frac{\pi}{4}$	$2\sqrt{2}\hat{\mathbf{i}} + \frac{3\sqrt{2}}{2}\hat{\mathbf{j}}$	$\frac{5\pi}{4}$	$-2\sqrt{2}\hat{\mathbf{i}} - \frac{3\sqrt{2}}{2}\hat{\mathbf{j}}$
$\frac{\pi}{2}$	$3\hat{\mathbf{j}}$	$\frac{3\pi}{2}$	$-3\hat{\mathbf{j}}$
$\frac{3\pi}{4}$	$-2\sqrt{2}\hat{\mathbf{i}} + \frac{3\sqrt{2}}{2}\hat{\mathbf{j}}$	$\frac{7\pi}{4}$	$2\sqrt{2}\hat{\mathbf{i}} - \frac{3\sqrt{2}}{2}\hat{\mathbf{j}}$
2π	$4\hat{\mathbf{i}}$		

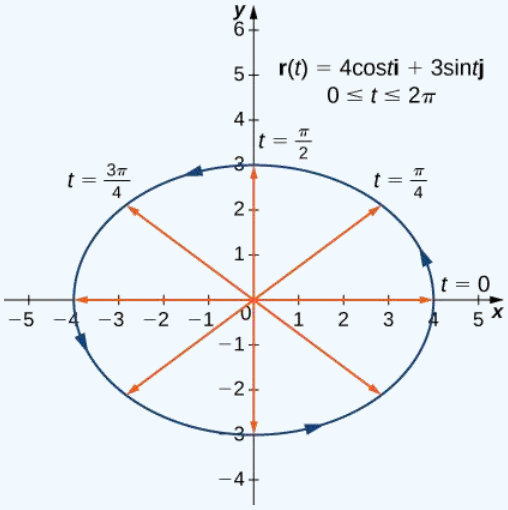


Figure 12.1.1: The graph of the first vector-valued function is an ellipse.

b. The table of values for $\vec{\mathbf{r}}(t) = 4\cos(t^3)\,\hat{\mathbf{i}} + 3\sin(t^3)\,\hat{\mathbf{j}},\ 0 \le t \le \sqrt[3]{2\pi}$ is as follows:

Table of Values for $\vec{\mathbf{r}}(t) = 4\cos(t^3)\,\hat{\mathbf{i}} + 3\sin(t^3)\,\hat{\mathbf{j}},\ 0 \le t \le \sqrt[3]{2\pi}$

t	$\vec{\mathbf{r}}(t)$	t	$\vec{\mathbf{r}}(t)$
0	$4\hat{\mathbf{i}}$	$\sqrt[3]{\pi}$	$-4\hat{\mathbf{i}}$
$\sqrt[3]{\frac{\pi}{4}}$	$2\sqrt{2}\hat{\mathbf{i}} + \frac{3\sqrt{2}}{2}\hat{\mathbf{j}}$	$\sqrt[3]{\frac{5\pi}{4}}$	$-2\sqrt{2}\hat{\mathbf{i}} - \frac{3\sqrt{2}}{2}\hat{\mathbf{j}}$
$\sqrt[3]{\frac{\pi}{2}}$	$3\hat{\mathbf{j}}$	$\sqrt[3]{\frac{3\pi}{2}}$	$-3\hat{\mathbf{j}}$
$\sqrt[3]{\frac{3\pi}{4}}$	$-2\sqrt{2}\hat{\mathbf{i}} + \frac{3\sqrt{2}}{2}\hat{\mathbf{j}}$	$\sqrt[3]{\frac{7\pi}{4}}$	$2\sqrt{2}\hat{\mathbf{i}} - \frac{3\sqrt{2}}{2}\hat{\mathbf{j}}$
$\sqrt[3]{2\pi}$	$4\hat{\mathbf{i}}$		

The graph of this curve is also an ellipse centered at the origin.

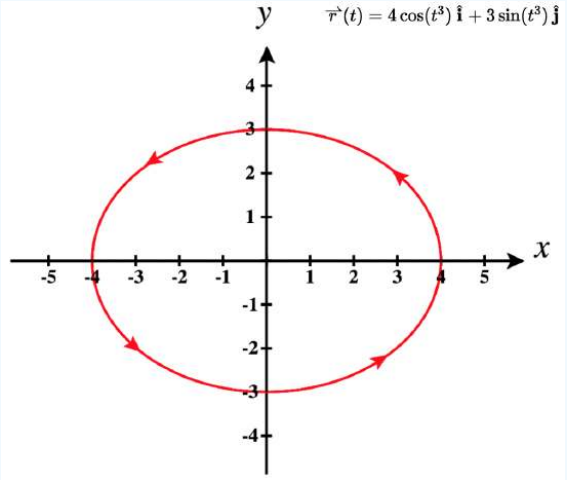


Figure 12.1.2: The graph of the second vector-valued function is also an ellipse.

c. We go through the same procedure for a three-dimensional vector function.

Table of Values for $\mathbf{r}(t) = 4\cos t\,\hat{\mathbf{i}} + 4\sin t\,\hat{\mathbf{j}} + t\hat{\mathbf{k}},\ 0 \le t \le 4\pi$

t	$\vec{\mathbf{r}}(t)$	t	$\vec{\mathbf{r}}(t)$
0	$4\hat{\mathbf{i}}$	π	$-4\hat{\mathbf{i}}+\pi\hat{\mathbf{k}}$
$\frac{\pi}{4}$	$2\sqrt{2}\hat{\mathbf{i}}+2\sqrt{2}\hat{\mathbf{j}}+\frac{\pi}{4}\hat{\mathbf{k}}$	$\frac{5\pi}{4}$	$-2\sqrt{2}\hat{\mathbf{i}}-2\sqrt{2}\hat{\mathbf{j}}+\frac{5\pi}{4}\hat{\mathbf{k}}$
$\frac{\pi}{2}$	$4\hat{\mathbf{j}}+\frac{\pi}{2}\hat{\mathbf{k}}$	$\frac{3\pi}{2}$	$-4\hat{\mathbf{j}}+\frac{3\pi}{2}\hat{\mathbf{k}}$
$\frac{3\pi}{4}$	$-2\sqrt{2}\hat{\mathbf{i}}+2\sqrt{2}\hat{\mathbf{j}}+\frac{3\pi}{4}\hat{\mathbf{k}}$	$\frac{7\pi}{4}$	$2\sqrt{2}\hat{\mathbf{i}}-2\sqrt{2}\hat{\mathbf{j}}+\frac{7\pi}{4}\hat{\mathbf{k}}$
2π	$4\hat{\mathbf{j}}+2\pi\hat{\mathbf{k}}$		

The values then repeat themselves, except for the fact that the coefficient of $\hat{\mathbf{k}}$ is always increasing (12.1.3). This curve is called a helix. Notice that if the $\hat{\mathbf{k}}$ component is eliminated, then the function becomes $\vec{\mathbf{r}}(t)=4\cos t\hat{\mathbf{i}}+4\sin t\hat{\mathbf{j}}$, which is a circle of radius 4 centered at the origin.

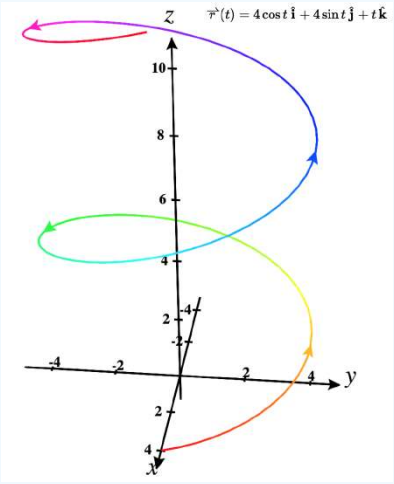


Figure 12.1.3: The graph of the third vector-valued function is a helix.

You may notice that the graphs in parts a. and b. are identical. This happens because the function describing the curve in part b. is a so-called reparameterization of the function describing the curve in part a. In fact, any curve has an infinite number of reparameterizations; for example, we can replace t with $2t$ in any of the three previous curves without changing the shape of the curve. The interval over which t is defined may change, but that is all. We return to this idea later in this chapter when we study arc-length parameterization. As mentioned, the name of the shape of the curve of the graph in 12.1.3 is a *helix*. The curve resembles a spring, with a circular cross-section looking down along the z-axis. It is possible for a helix to be elliptical in cross-section as well. For example, the vector-valued function $\vec{\mathbf{r}}(t)=4\cos t\hat{\mathbf{i}}+3\sin t\hat{\mathbf{j}}+t\hat{\mathbf{k}}$ describes an elliptical helix. The projection of this helix into the xy-plane is an ellipse. Last, the arrows in the graph of this helix indicate the orientation of the curve as t progresses from 0 to 4π.

? Exercise 12.1.2

Create a graph of the vector-valued function $\vec{\mathbf{r}}(t)=t\hat{\mathbf{i}}+t^3\hat{\mathbf{j}}$.

At this point, you may notice a similarity between vector-valued functions and parameterized curves. Indeed, given a vector-valued function $\vec{\mathbf{r}}(t)=f(t)\hat{\mathbf{i}}+g(t)\hat{\mathbf{j}}$ we can define $x=f(t)$ and $y=g(t)$. The graph of the parameterized function would then agree with the graph of the vector-valued function, except that the vector-valued function's graph would be traced out by vectors rather than just being a collection of points. Since we can parameterize a curve defined by a function $y=f(x)$, it is also possible to represent an arbitrary plane curve by a vector-valued function.

Finding a Vector-Valued Function to Trace out the Graph of a Function y = f(x)

As you can see in the examples above, a vector-valued function traces out a curve in the plane or in space. What if we wish to write a vector-valued function that traces out the graph of a particular curve in the xy-plane?

What function's graph is traced out by the vector-valued function in Exercise **12.1.2** above: $\vec{\mathbf{r}}(t) = t\,\hat{\mathbf{i}} + t^3\,\hat{\mathbf{j}}$? It looks like the graph of $y = x^3$, doesn't it?

Remembering what was just said about the components of the vector-valued function corresponding to the parametic equations of a parameterized curve, we see that here we have:

$$x = t$$
$$y = t^3$$

Since $x = t$, we can replace t in the equation $y = t^3$ with x, giving us the function: $y = x^3$.

So we were correct in our guess.

How could we write a vector-valued function to trace out the graph of a function, $y = f(x)$?

Well, there are two orientations to consider: **left-to-right** and **right-to-left**.

Tracing a Function from Left-To-Right:

To trace out the graph of $y = f(x)$ from left-to-right, use: $\vec{\mathbf{r}}(t) = t\,\hat{\mathbf{i}} + f(t)\,\hat{\mathbf{j}}$

Note that what's important here is to have the x component be an increasing function. *Any increasing function will work.* We could use $x = t^3$, for example. But then we would need to remember to replace x in the function $f(x)$ with this expression t^3, giving us $y = f(t^3)$. This means the function $y = f(x)$ could also be parameterized from left-to-right by the vector-valued function: $\vec{\mathbf{r}}(t) = t^3\,\hat{\mathbf{i}} + f(t^3)\,\hat{\mathbf{j}}$

Tracing a Function from Right-To-Left:

To trace out the graph of $y = f(x)$ from right-to-left, use: $\vec{\mathbf{r}}(t) = -t\,\hat{\mathbf{i}} + f(-t)\,\hat{\mathbf{j}}$

Again note that we could use any decreasing function of t for the x component and obtain a vector-valued function that traces out the graph of $y = f(x)$ from right-to-left. Using $x = -t$ is just the simplest decreasing function we can choose.

Example 12.1.4: Finding a vector-valued function to trace the graph of $y = f(x)$

Determine a vector-valued function that will trace out the graph of $y = \cos x$ from left-to-right, and another one to trace it out from right-to-left.

Solution

Left-to-right: $\vec{\mathbf{r}}(t) = t\,\hat{\mathbf{i}} + \cos t\,\hat{\mathbf{j}}$

Right-to-left: $\vec{\mathbf{r}}(t) = -t\,\hat{\mathbf{i}} + \cos(-t)\,\hat{\mathbf{j}}$

Finding a Vector-Valued Function to Trace out the Graph of an Equation in x and y and Vice Versa

What if we wish to find a vector-valued function to trace out the graph of a circle, an ellipse, or a hyperbola, given its implicit equation?

Well, note that in Example **12.1.3**, the vector-valued function $\vec{\mathbf{r}}(t) = 4\cos t\,\hat{\mathbf{i}} + 3\sin t\,\hat{\mathbf{j}}$ traced out the graph of the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$.

In this vector-valued function we see that:

$$x = 4\cos t \quad \text{and} \quad y = 3\sin t \tag{12.1.3}$$

What we need now is a way to convert this to an implicit equation involving x and y. To accomplish this, remember the Pythagorean identity,

$$\cos^2 t + \sin^2 t = 1 \tag{12.1.4}$$

.

Now all we need to do is solve the above equations for $\cos t$ and $\sin t$ and we can substitute into this identity to obtain an equation in x and y.

So:

$$\cos t = \frac{x}{4} \quad \text{and} \quad \sin t = \frac{y}{3} \tag{12.1.5}$$

Substituting into the identity gives us:

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2 = 1 \tag{12.1.6}$$

Simplifying this implicit equation gives us the implicit equation of the ellipse in Example 12.1.3 that we wrote above:

$$\frac{x^2}{16} + \frac{y^2}{9} = 1 \tag{12.1.7}$$

To go the other way and find a vector-valued function that traces out an ellipse requires us to simply take these steps in the opposite direction!

✔ Example 12.1.5: Writing a vector-valued function for a given circle, ellipse, or Hyperbola

Write a vector-valued function that traces out each of the following implicit curves:

a. The ellipse: $\frac{x^2}{16} + \frac{y^2}{9} = 1$
b. The circle: $x^2 + y^2 = 4$
c. The hyperbola: $\frac{x^2}{25} - \frac{y^2}{16} = 1$

Solution

a. Let's just use the process shown above in reverse. First, let's rewrite the implicit equation so it shows a sum of quantities squared equals one.

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$

Now we need the identity we used above, $\cos^2 t + \sin^2 t = 1$.

Equating the parts being squared (note we actually have a choice here about which to make $\cos t$ and which to make $\sin t$), we get:

$$\frac{x}{4} = \cos t \quad \text{and} \quad \frac{y}{3} = \sin t$$

Now we just need to solve for x and y.

$$x = 4\cos t \quad \text{and} \quad y = 3\sin t$$

We can now write a vector-valued function that traces out this ellipse: $\vec{\mathbf{r}}(t) = 4\cos t\,\hat{\mathbf{i}} + 3\sin t\,\hat{\mathbf{j}}$

Note that we could also have written $\vec{\mathbf{r}}(t) = 4\sin t\,\hat{\mathbf{i}} + 3\cos t\,\hat{\mathbf{j}}$, since we could have chosen to switch the $\sin t$ and $\cos t$ above. It will trace out the same ellipse, but with the opposite orientation.

b. Tracing out a circle is fairly straightforward, not really needing the process we showed above, although it still may be helpful at first. Remember that all the vectors on the unit circle can be represented in the form: $\vec{\mathbf{v}} = \cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}$.

So the vector-valued function, $\vec{\mathbf{r}}(t) = \cos t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}}$, will trace out the unit circle with equation, $x^2 + y^2 = 1$.

To obtain a circle of radius 2 centered at the origin (which is the graph of $x^2 + y^2 = 4$), we just need to multiply through this vector-valued function by a scalar factor of 2.

Thus, a vector-valued function that will trace out this circle is: $\vec{\mathbf{r}}(t) = 2\cos t\,\hat{\mathbf{i}} + 2\sin t\,\hat{\mathbf{j}}$.

Note again that another possibility is: $\vec{\mathbf{r}}(t) = 2\sin t\,\hat{\mathbf{i}} + 2\cos t\,\hat{\mathbf{j}}$. It will trace out the same circle, but with the opposite orientation.

To use the technique above, you start by dividing each term in the equation by the square of the radius, here 4, thus putting the circle equation in "ellipse form". The rest of the steps follow the pattern shown in part a.

c. To trace out a hyperbola of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ or $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, we need to locate a trigonometric identity showing the difference of two squares equals 1. If you don't already have such an identity memorized, we can obtain one from the Pythagorean identity used above. That is,

$$\cos^2 t + \sin^2 t = 1$$

Dividing through each term by $\cos^2 t$

$$\frac{\cos^2 t}{\cos^2 t} + \frac{\sin^2 t}{\cos^2 t} = \frac{1}{\cos^2 t}$$

yields,

$$1 + \tan^2 t = \sec^2 t$$

Rewriting this equation gives us the identity we need:

$$\sec^2 t - \tan^2 t = 1$$

Now, the equation of this hyperbola is:

$$\frac{x^2}{25} - \frac{y^2}{16} = 1$$

Rewriting the left side to show the quantities that are squared:

$$\left(\frac{x}{5}\right)^2 - \left(\frac{y}{4}\right)^2 = 1$$

We can then equate the squared expressions corresponding terms:

$$\frac{x}{5} = \sec t \quad \text{and} \quad \frac{y}{4} = \tan t$$

Solving for x and y, we have:

$$x = 5\sec t \quad \text{and} \quad y = 4\tan t$$

So a vector-valued function that will trace out the hyperbola

$$\frac{x^2}{25} - \frac{y^2}{16} = 1 \tag{12.1.8}$$

is

$$\vec{\mathbf{r}}(t) = 5\sec t\,\hat{\mathbf{i}} + 4\tan t\,\hat{\mathbf{j}}.$$

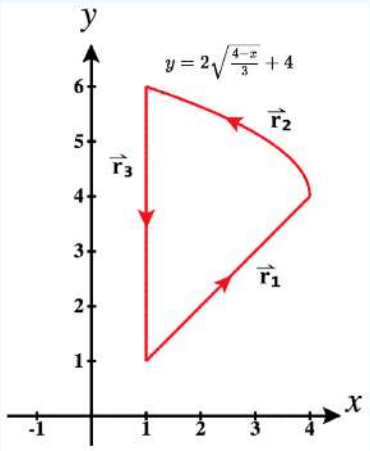
Parameterizing a Piecewise Path

There are times when it is necessary to parameterize a path made up of pieces of different curves. This piecewise path may be open or form the boundary of a closed region as does the example shown in Figure 12.1.4. In addition to determining a vector-valued function to trace out each piece separately, with the indicated orientation, we also need to determine a suitable range of values for the parameter t.

Note that there are many ways to parameterize any one piece, so there are many correct ways to parameterize a path in this way.

✔ Example 12.1.6: Parameterizing a piecewise path

Determine a piecewise parameterization of the path shown in Figure 12.1.4, starting with $t = 0$ and continuing on through each piece.



Solution

Our first task is to identify the three pieces in this piecewise path.

Note how we labeled these sequentially as $\vec{\mathbf{r}}_1$, $\vec{\mathbf{r}}_2$, and $\vec{\mathbf{r}}_3$. Now we need to identify the function for each and write the corresponding vector-valued function with the correct orientation (left-to-right or right-to-left).

Determining $\vec{\mathbf{r}}_1$: The equation of the linear function in this piece is $y = x$.

Since it is oriented from left-to-right between $t = 1$ and $t = 4$, we can write:

$$\vec{\mathbf{r}}_{1a}(t) = t\,\hat{\mathbf{i}} + t\,\hat{\mathbf{j}} \quad \text{for} \quad 1 \le t \le 4$$

If we wish to begin this piece at $t = 0$, we just need to shift the value of t one unit to the left. One way to do this is to write $\vec{\mathbf{r}}_{1a}$ in terms of t_1 instead of t to make the translation easier to see.

Thus, we have $\vec{\mathbf{r}}_{1a}(t_1) = t_1\,\hat{\mathbf{i}} + t_1\,\hat{\mathbf{j}}$ for $1 \le t_1 \le 4$.

Figure* 12.1.4*: *A closed piecewise path*

Subtracting 1 from each part of this range of parameter values, we have: $0 \le t_1 - 1 \le 3$.

Now we let $t = t_1 - 1$. Solving for t_1, we obtain: $t_1 = t + 1$.

Replacing t_1 with the expression $t + 1$ will effectively shift the range of parameter values one unit to the left.

So, starting with $t = 0$, we have:

$$\vec{\mathbf{r}}_1(t) = (t+1)\,\hat{\mathbf{i}} + (t+1)\,\hat{\mathbf{j}} \quad \text{for} \quad 0 \le t \le 3$$

Double-check that this vector-valued function will trace out this segment in the correct direction before going on to r_2.

Determining $\vec{\mathbf{r}}_2$: This piece has a label showing the function whose graph it traces along. If it were oriented from left-to-right, we would have:

$$\text{Left-to-right:} \quad \vec{\mathbf{r}}_{2a}(t) = t\,\hat{\mathbf{i}} + \left(2\sqrt{\frac{4-t}{3}} + 4\right)\hat{\mathbf{j}} \quad \text{for} \quad 1 \le t \le 4$$

But since we need it to be oriented from right-to-left, we need to replace t with $-t$ in the function and we need to divide through the range inequality by -1 to obtain the corresponding range. Thus we obtain:

$$\vec{\mathbf{r}}_{2b}(t) = -t\,\hat{\mathbf{i}} + \left(2\sqrt{\frac{4-(-t)}{3}} + 4\right)\hat{\mathbf{j}} \quad \text{for} \quad -4 \le t \le -1$$

Check that it works!

Now we wish to have this piece start at $t = 3$ just after the first one finishes. Again let's make this easier to see by writing r_{2b} in terms on t_2.

$$\vec{\mathbf{r}}_{2b}(t_2) = -t_2\,\hat{\mathbf{i}} + \left(2\sqrt{\frac{4-(-t_2)}{3}} + 4\right)\hat{\mathbf{j}} \quad \text{for} \quad -4 \le t_2 \le -1$$

To force r_2 to start with $t = 3$ instead of $t = -4$, we need to add 7 to each part of the inequality. This yields: $3 \le t_2 + 7 \le 6$.

Let $t = t_2 + 7$. Then solving for $-t_2$ (since this is what we need to replace in r_{2b}), we have: $-t_2 = 7 - t$.

Replacing $-t_2$ with $(7 - t)$ in $\vec{\mathbf{r}}_{2b}$, we obtain:

$$\vec{\mathbf{r}}_2(t) = (7-t)\,\hat{\mathbf{i}} + \left(2\sqrt{\frac{4-(7-t)}{3}} + 4\right)\hat{\mathbf{j}} \quad \text{for} \quad 3 \le t \le 6$$

This can be combined with our earlier result for r_1 to write a piecewise-defined vector-valued function that traces out the first two pieces, starting at $t = 0$:

$$\vec{\mathbf{r}}(t) = \begin{cases} (t+1)\,\hat{\mathbf{i}} + (t+1)\,\hat{\mathbf{j}}, & 0 \le t \le 3 \\ (7-t)\,\hat{\mathbf{i}} + \left(2\sqrt{\frac{t-3}{3}} + 4\right)\hat{\mathbf{j}}, & 3 < t \le 6 \end{cases}$$

Note that one small modification was made to the second range so that when $t = 3$, there is no confusion about which piece to evaluate.

Determining $\vec{\mathbf{r}}_3$: To determine this last piece we need to think a little differently. This is because it is a vertical segment, which cannot be represented with a function of the form, $y = f(x)$. Note that it could be represented by a function of the form $x = f(y)$. Letting $y = t$, we can write $x = f(t)$ and writing a parameterization in increasing y values (bottom-to-top), we'd get: $\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + t\,\hat{\mathbf{j}}$.

The equation of this line is $x = 1$. Thus, if we wished to parameterize this segment with upward orientation (increasing values of y), we have:

$$\vec{\mathbf{r}}_{3a}(t) = 1\,\hat{\mathbf{i}} + t\,\hat{\mathbf{j}} \quad \text{for} \quad 1 \le t \le 6$$

But since we wish to use a downward orientation (decreasing values of y), we need to use a decreasing function of t for y. As before, the simplest case is to use $y = -t$. Then, in the general case, we'd trace a function $x = f(y)$ in a downwards orientation with $\vec{\mathbf{r}}(t) = f(-t)\,\hat{\mathbf{i}} - t\,\hat{\mathbf{j}}$.

In the case of r_3, this gives us:

$$\vec{\mathbf{r}}_{3b}(t) = 1\,\hat{\mathbf{i}} - t\,\hat{\mathbf{j}} \quad \text{for} \quad -6 \le t \le -1$$

Note that since $x = 1,\ f(-t) = 1$, that is, it did not change the first component since it was constant and not a variable function of the parameter t.

Also note that since we negated t, we also had to negate the range, dividing it through by -1.

As above, to facilitate the translation, we'll replace t with t_3, giving us:

$$\vec{\mathbf{r}}_{3b}(t_3) = 1\,\hat{\mathbf{i}} - t_3\,\hat{\mathbf{j}} \quad \text{for} \quad -6 \le t_3 \le -1$$

Now, we wish this final piece to start at $t = 6$ where the second piece we formed above leaves off. We see that we need to add 12 to the range of paramater t to accomplish this, giving us a new range of $6 \le t_3 + 12 \le 11$.

Let $t = t_3 + 12$. Then solving for $-t_3$ (since this is what we need to replace in r_{3b}), we have: $-t_3 = 12 - t$.

Replacing $-t_3$ with $(12 - t)$ in $\vec{\mathbf{r}}_{3b}$, we obtain:

$$\vec{\mathbf{r}}_3(t) = 1\,\hat{\mathbf{i}} + (12-t)\,\hat{\mathbf{j}} \quad \text{for} \quad 6 \le t \le 11$$

Check that this still traces out this vertical segment from top-to-bottom.

We can now state the final answer as a single piecewise-defined vector-valued function that traces out this entire path, starting when $t = 0$.

$$\vec{\mathbf{r}}(t) = \begin{cases} (t+1)\,\hat{\mathbf{i}} + (t+1)\,\hat{\mathbf{j}}, & 0 \le t \le 3 \\ (7-t)\,\hat{\mathbf{i}} + \left(2\sqrt{\frac{t-3}{3}} + 4\right)\hat{\mathbf{j}}, & 3 < t \le 6 \\ 1\,\hat{\mathbf{i}} + (12-t)\,\hat{\mathbf{j}} & 6 < t \le 11 \end{cases}$$

Be sure to verify that this single vector-valued function does indeed trace out the entire path!

Limits and Continuity of a Vector-Valued Function

We now take a look at the limit of a vector-valued function. This is important to understand to study the calculus of vector-valued functions.

Definition: limit of a vector-valued function

A vector-valued function $\vec{\mathbf{r}}$ approaches the limit $\vec{\mathbf{L}}$ as t approaches a, written

$$\lim_{t\to a}\vec{\mathbf{r}}(t)=\vec{\mathbf{L}}, \tag{12.1.9}$$

provided

$$\lim_{t\to a}\|\vec{\mathbf{r}}(t)-\vec{\mathbf{L}}\|=0. \tag{12.1.10}$$

This is a rigorous definition of the limit of a vector-valued function. In practice, we use the following theorem:

Theorem: Limit of a vector-valued function

Let f, g, and h be functions of t. Then the limit of the vector-valued function $\vec{\mathbf{r}}(t)=f(t)\hat{\mathbf{i}}+g(t)\hat{\mathbf{j}}$ as t approaches a is given by

$$\lim_{t\to a}\vec{\mathbf{r}}(t)=[\lim_{t\to a}f(t)]\hat{\mathbf{i}}+[\lim_{t\to a}g(t)]\hat{\mathbf{j}}, \tag{12.1.11}$$

provided the limits $\lim_{t\to a}f(t)$ and $\lim_{t\to a}g(t)$ exist.

Similarly, the limit of the vector-valued function $\vec{\mathbf{r}}(t)=f(t)\hat{\mathbf{i}}+g(t)\hat{\mathbf{j}}+h(t)\hat{\mathbf{k}}$ as t approaches a is given by

$$\lim_{t\to a}\vec{\mathbf{r}}(t)=[\lim_{t\to a}f(t)]\hat{\mathbf{i}}+[\lim_{t\to a}g(t)]\hat{\mathbf{j}}+[\lim_{t\to a}h(t)]\hat{\mathbf{k}}, \tag{12.1.12}$$

provided the limits $\lim_{t\to a}f(t)$, $\lim_{t\to a}g(t)$ and $\lim_{t\to a}h(t)$ exist.

In the following example, we show how to calculate the limit of a vector-valued function.

Example 12.1.7: Evaluating the Limit of a Vector-Valued Function

For each of the following vector-valued functions, calculate $\lim_{t\to 3}\vec{\mathbf{r}}(t)$ for

a. $\vec{\mathbf{r}}(t)=(t^2-3t+4)\hat{\mathbf{i}}+(4t+3)\hat{\mathbf{j}}$
b. $\vec{\mathbf{r}}(t)=\frac{2t-4}{t+1}\hat{\mathbf{i}}+\frac{t}{t^2+1}\hat{\mathbf{j}}+(4t-3)\hat{\mathbf{k}}$

Solution

a. Use Equation 12.1.11 and substitute the value $t=3$ into the two component expressions:

$$\begin{aligned}\lim_{t\to 3}\vec{\mathbf{r}}(t) &=\lim_{t\to 3}\left[(t^2-3t+4)\hat{\mathbf{i}}+(4t+3)\hat{\mathbf{j}}\right]\\ &=\left[\lim_{t\to 3}(t^2-3t+4)\right]\hat{\mathbf{i}}+\left[\lim_{t\to 3}(4t+3)\right]\hat{\mathbf{j}}\\ &=4\hat{\mathbf{i}}+15\hat{\mathbf{j}}\end{aligned}$$

b. Use Equation 12.1.12 and substitute the value $t=3$ into the three component expressions:

$$\begin{aligned}\lim_{t\to 3}\vec{\mathbf{r}}(t) &=\lim_{t\to 3}\left(\frac{2t-4}{t+1}\hat{\mathbf{i}}+\frac{t}{t^2+1}\hat{\mathbf{j}}+(4t-3)\hat{\mathbf{k}}\right)\\ &=\left[\lim_{t\to 3}\left(\frac{2t-4}{t+1}\right)\right]\hat{\mathbf{i}}+\left[\lim_{t\to 3}\left(\frac{t}{t^2+1}\right)\right]\hat{\mathbf{j}}+\left[\lim_{t\to 3}(4t-3)\right]\hat{\mathbf{k}}\\ &=\tfrac{1}{2}\hat{\mathbf{i}}+\tfrac{3}{10}\hat{\mathbf{j}}+9\hat{\mathbf{k}}\end{aligned}$$

Exercise 12.1.3

Calculate $\lim_{t\to 2}\vec{\mathbf{r}}(t)$ for the function $\vec{\mathbf{r}}(t)=\sqrt{t^2+3t-1}\,\hat{\mathbf{i}}-(4t-3)\,\hat{\mathbf{j}}-\sin\frac{(t+1)\pi}{2}\,\hat{\mathbf{k}}$

Now that we know how to calculate the limit of a vector-valued function, we can define **continuity at a point** for such a function.

Definitions

Let f, g, and h be functions of t. Then, the vector-valued function $\vec{\mathbf{r}}(t) = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}}$ is **continuous at point** $t = a$ if the following three conditions hold:

1. $\vec{\mathbf{r}}(a)$ exists
2. $\lim\limits_{t \to a} \vec{\mathbf{r}}(t)$ exists
3. $\lim\limits_{t \to a} \vec{\mathbf{r}}(t) = \vec{\mathbf{r}}(a)$

Similarly, the vector-valued function $\vec{\mathbf{r}}(t) = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}} + h(t)\hat{\mathbf{k}}$ is **continuous at point** $t = a$ if the following three conditions hold:

1. $\vec{\mathbf{r}}(a)$ exists
2. $\lim\limits_{t \to a} \vec{\mathbf{r}}(t)$ exists
3. $\lim\limits_{t \to a} \vec{\mathbf{r}}(t) = \vec{\mathbf{r}}(a)$

Summary

- A vector-valued function is a function of the form $\vec{\mathbf{r}}(t) = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}}$ or $\vec{\mathbf{r}}(t) = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}} + h(t)\hat{\mathbf{k}}$, where the component functions f, g, and h are real-valued functions of the parameter t.
- The graph of a vector-valued function of the form $\vec{\mathbf{r}}(t) = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}}$ is called a *plane curve*. The graph of a vector-valued function of the form $\vec{\mathbf{r}}(t) = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}} + h(t)\hat{\mathbf{k}}$ is called a *space curve*.
- It is possible to represent an arbitrary plane curve by a vector-valued function.
- To calculate the limit of a vector-valued function, calculate the limits of the component functions separately.

Key Equations

- **Vector-valued function**
  $\vec{\mathbf{r}}(t) = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}}$ or $\vec{\mathbf{r}}(t) = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}} + h(t)\hat{\mathbf{k}}$,or $\vec{\mathbf{r}}(t) = \langle f(t), g(t) \rangle$ or $\vec{\mathbf{r}}(t) = \langle f(t), g(t), h(t) \rangle$
- **Limit of a vector-valued function**
  $\lim\limits_{t \to a} \vec{\mathbf{r}}(t) = [\lim\limits_{t \to a} f(t)]\hat{\mathbf{i}} + [\lim\limits_{t \to a} g(t)]\hat{\mathbf{j}}$ or $\lim\limits_{t \to a} \vec{\mathbf{r}}(t) = [\lim\limits_{t \to a} f(t)]\hat{\mathbf{i}} + [\lim\limits_{t \to a} g(t)]\hat{\mathbf{j}} + [\lim\limits_{t \to a} h(t)]\hat{\mathbf{k}}$

Glossary

Contributors

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- Edited by Paul Seeburger (Monroe Community College)
- Paul Seeburger created Example **12.1.1**, Exercise **12.1.1**, and the subsections titled: Finding a Vector-Valued Function to Trace out the Graph of a Function $y = f(x)$, Finding a Vector-Valued Function to Trace out the Graph of an Equation in x and y and Vice Versa, and Parameterizing a Piecewise Path.

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12.1E: Exercises for Section 12.1

Introduction to Vector-Valued Functions

1) Give the component functions $x = f(t)$ and $y = g(t)$ for the vector-valued function $\vec{\mathbf{r}}(t) = 3\sec t\,\hat{\mathbf{i}} + 2\tan t\,\hat{\mathbf{j}}$.

2) Given $\vec{\mathbf{r}}(t) = 3\sec t\hat{\mathbf{i}} + 2\tan t\hat{\mathbf{j}}$, find the following values (if possible).

a. $\vec{\mathbf{r}}\left(\frac{\pi}{4}\right)$
b. $\vec{\mathbf{r}}(\pi)$
c. $\vec{\mathbf{r}}\left(\frac{\pi}{2}\right)$

3) Sketch the curve of the vector-valued function $\vec{\mathbf{r}}(t) = 3\sec t\hat{\mathbf{i}} + 2\tan t\hat{\mathbf{j}}$ and give the orientation of the curve. Sketch asymptotes as a guide to the graph.

Limits of Vector-Valued Functions

4) Evaluate $\lim\limits_{t\to 0}\left(e^t\hat{\mathbf{i}} + \frac{\sin t}{t}\hat{\mathbf{j}} + e^{-t}\hat{\mathbf{k}}\right)$

5) Given the vector-valued function $\vec{\mathbf{r}}(t) = \langle \cos t, \sin t\rangle$ find the following values:

a. $\lim\limits_{t\to\frac{\pi}{4}} \vec{\mathbf{r}}(t)$
b. $\vec{\mathbf{r}}\left(\frac{\pi}{3}\right)$
c. Is $\vec{\mathbf{r}}(t)$ continuous at $t = \frac{\pi}{3}$?
d. Graph $\vec{\mathbf{r}}(t)$.

6) Given the vector-valued function $\vec{\mathbf{r}}(t) = \langle t, t^2 + 1\rangle$, find the following values:

a. $\lim\limits_{t\to -3} \vec{\mathbf{r}}(t)$
b. $\vec{\mathbf{r}}(-3)$
c. Is $\vec{\mathbf{r}}(t)$ continuous at $x = -3$?
d. $\vec{\mathbf{r}}(t+2) - \vec{\mathbf{r}}(t)$

7) Let $\vec{\mathbf{r}}(t) = e^t\hat{\mathbf{i}} + \sin t\hat{\mathbf{j}} + \ln t\hat{\mathbf{k}}$. Find the following values:

a. $\vec{\mathbf{r}}\left(\frac{\pi}{4}\right)$
b. $\lim\limits_{t\to\frac{\pi}{4}} \vec{\mathbf{r}}(t)$
c. Is $\vec{\mathbf{r}}(t)$ continuous at $t = \frac{\pi}{4}$?

For exercises 8 - 13, find the limit of the following vector-valued functions at the indicated value of t.

8) $\lim\limits_{t\to 4}\langle\sqrt{t-3}, \frac{\sqrt{t}-2}{t-4}, \tan\left(\frac{\pi}{t}\right)\rangle$

9) $\lim\limits_{t\to\frac{\pi}{2}} \vec{\mathbf{r}}(t)$ for $\vec{\mathbf{r}}(t) = e^t\hat{\mathbf{i}} + \sin t\hat{\mathbf{j}} + \ln t\hat{\mathbf{k}}$

10) $\lim\limits_{t\to\infty}\langle e^{-2t}, \frac{2t+3}{3t-1}, \arctan(2t)\rangle$

11) $\lim\limits_{t\to e^2}\langle t\ln(t), \frac{\ln t}{t^2}, \sqrt{\ln(t^2)}\rangle$

12) $\lim\limits_{t\to\frac{\pi}{6}}\langle\cos 2t, \sin 2t, 1\rangle$

13) $\lim\limits_{t\to\infty} \vec{\mathbf{r}}(t)$ for $\vec{\mathbf{r}}(t) = 2e^{-t}\mathbf{i} + e^{-t}\hat{\mathbf{j}} + \ln(t-1)\hat{\mathbf{k}}$

Domain of a Vector-Valued Function

For problems 14 - 17, find the domain of the vector-valued functions.

14) Domain: $\vec{\mathbf{r}}(t) = \langle t^2, t, \sin t\rangle$

15) Domain: $\vec{\mathbf{r}}(t) = \langle t^2, \tan t, \ln t\rangle$

16) Domain: $\vec{\mathbf{r}}(t) = \langle t^2, \sqrt{t-3}, \frac{3}{2t+1}\rangle$

17) Domain: $\vec{\mathbf{r}}(t) = \langle \csc(t), \frac{1}{\sqrt{t-3}}, \ln(t-2) \rangle$

18) a. Find the domain of $\vec{\mathbf{r}}(t) = 2e^{-t}\hat{\mathbf{i}} + e^{-t}\hat{\mathbf{j}} + \ln(t-1)\hat{\mathbf{k}}$.

b. For what values of t is $\vec{\mathbf{r}}(t) = 2e^{-t}\hat{\mathbf{i}} + e^{-t}\hat{\mathbf{j}} + \ln(t-1)\hat{\mathbf{k}}$ continuous?

19) Domain: $\vec{\mathbf{r}}(t) = (\arccos t)\,\hat{\mathbf{i}} + \sqrt{2t-1}\,\hat{\mathbf{j}} + \ln(t)\,\hat{\mathbf{k}}$

Visualizing Vector-Valued Functions

20) Describe the curve defined by the vector-valued function $\vec{\mathbf{r}}(t) = (1+t)\hat{\mathbf{i}} + (2+5t)\hat{\mathbf{j}} + (-1+6t)\hat{\mathbf{k}}$.

21) Let $\vec{\mathbf{r}}(t) = \langle \cos t, t, \sin t \rangle$ and use it to answer the following questions.

a. For what values of t is $\vec{\mathbf{r}}(t)$ continuous?
b. Sketch the graph of $\vec{\mathbf{r}}(t)$.

22) Produce a careful sketch of the graph of $\vec{\mathbf{r}}(t) = t^2\,\hat{\mathbf{i}} + t\,\hat{\mathbf{j}}$.

In questions 23 - 25, use a graphing utility to sketch each of the vector-valued functions:

23) **[T]** $\vec{\mathbf{r}}(t) = 2\cos^2 t\hat{\mathbf{i}} + (2-\sqrt{t})\hat{\mathbf{j}}$

24) **[T]** $\vec{\mathbf{r}}(t) = \langle e^{\cos(3t)}, e^{-\sin(t)} \rangle$

25) **[T]** $\vec{\mathbf{r}}(t) = \langle 2 - \sin(2t), 3 + 2\cos t \rangle$

Finding Equations in x and y for the Path Traced out by Vector-Valued Functions

For questions 26-33, eliminate the parameter t, write the equation in Cartesian coordinates, then sketch the graph of the vector-valued functions.

26) $\vec{\mathbf{r}}(t) = 2t\hat{\mathbf{i}} + t^2\hat{\mathbf{j}}$
(*Hint:* Let $x = 2t$ and $y = t^2$. Solve the first equation for t in terms of x and substitute this result into the second equation.)

27) $\vec{\mathbf{r}}(t) = t^3\hat{\mathbf{i}} + 2t\hat{\mathbf{j}}$

28) $\vec{\mathbf{r}}(t) = \sin t\,\hat{\mathbf{i}} + \cos t\,\hat{\mathbf{j}}$

29) $\vec{\mathbf{r}}(t) = 3\cos t\,\hat{\mathbf{i}} + 3\sin t\,\hat{\mathbf{j}}$

30) $\vec{\mathbf{r}}(t) = \langle \sin t, 4\cos t \rangle$

31) $\vec{\mathbf{r}}(t) = 2\sin t\,\hat{\mathbf{i}} - 3\cos t\,\hat{\mathbf{j}}$

32) $\vec{\mathbf{r}}(t) = \tan t\,\hat{\mathbf{i}} - 2\sec t\,\hat{\mathbf{j}}$

33) $\vec{\mathbf{r}}(t) = 3\sec t\,\hat{\mathbf{i}} + 4\tan t\,\hat{\mathbf{j}}$

Finding a Vector-Valued Function to Trace out the Graph of an Equation in x and y

For questions 34 - 40, find a vector-valued function that traces out the given curve in the indicated direction.

34) $4x^2 + 9y^2 = 36$; clockwise and counterclockwise

35) $y = x^2$; from left to right

36) The line through P and Q where P is $(1, 4, -2)$ and Q is $(3, 9, 6)$

37) The circle, $x^2 + y^2 = 36$, oriented clockwise, with position $(-6, 0)$ at time $t = 0$.

38) The ellipse, $x^2 + \frac{y^2}{36} = 1$, oriented counterclockwise

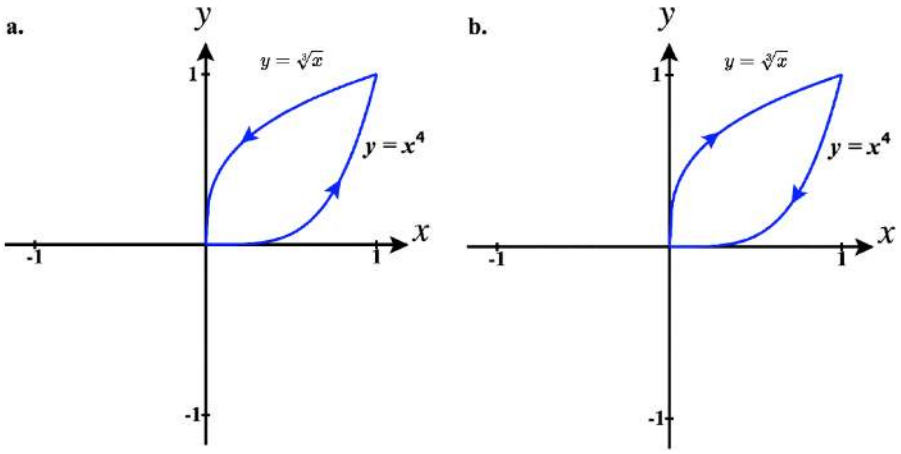
39) The hyperbola, $\frac{y^2}{36} - x^2 = 1$, top piece is oriented from left-to-right

40) The hyperbola, $\frac{x^2}{49} - \frac{y^2}{64} = 1$, right piece is oriented from bottom-to-top

Parameterizing a Piecewise Path

For questions 41 - 44, provide a parameterization for each piecewise path. Try to write a parameterization that starts with $t = 0$ and progresses on through values of t as you move from one piece to another.

41)



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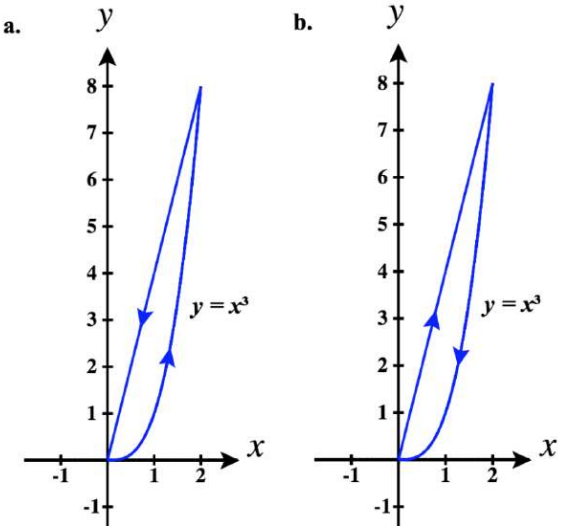

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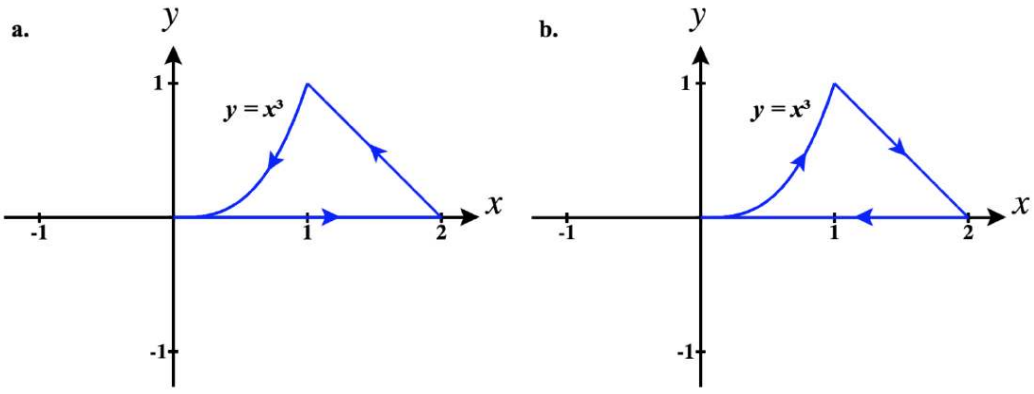

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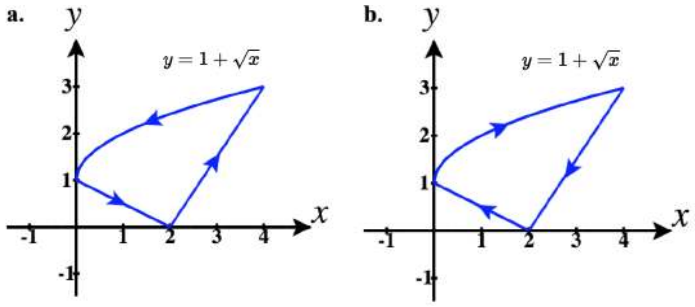

Additional Vector-Valued Function Questions

For questions 45 - 48, consider the curve described by the vector-valued function $\vec{\mathbf{r}}(t) = (50e^{-t}\cos t)\hat{\mathbf{i}} + (50e^{-t}\sin t)\hat{\mathbf{j}} + (5 - 5e^{-t})\hat{\mathbf{k}}$.

45) What is the initial point of the path corresponding to $\vec{\mathbf{r}}(0)$?

46) What is $\lim_{t\to\infty} \vec{\mathbf{r}}(t)$?

47) **[T]** Use technology to sketch the curve.

48) Eliminate the parameter t to show that $z = 5 - \frac{r}{10}$ where $r^2 = x^2 + y^2$.

49) **[T]** Let $\vec{\mathbf{r}}(t) = \cos t\hat{\mathbf{i}} + \sin t\hat{\mathbf{j}} + 0.3\sin(2t)\hat{\mathbf{k}}$. Use technology to graph the curve (called the *roller-coaster curve*) over the interval $[0, 2\pi)$. Choose at least two views to determine the peaks and valleys.

50) **[T]** Use the result of the preceding problem to construct an equation of a roller coaster with a steep drop from the peak and steep incline from the "valley." Then, use technology to graph the equation.

51) Use the results of the preceding two problems to construct an equation of a path of a roller coaster with more than two turning points (peaks and valleys).

52) Complete the following investigation.

a. Graph the curve $\vec{\mathbf{r}}(t) = (4 + \cos(18t))\cos(t)\hat{\mathbf{i}} + (4 + \cos(18t)\sin(t))\hat{\mathbf{j}} + 0.3\sin(18t)\hat{\mathbf{k}}$ using two viewing angles of your choice to see the overall shape of the curve.
b. Does the curve resemble a "slinky"?
c. What changes to the equation should be made to increase the number of coils of the slinky?

Contributors

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Paul Seeburger (Monroe Community College) created problems 12, 14, 19, 22, 30-33, 37- 44.

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12.2: The Calculus of Vector-Valued Functions

To study the calculus of vector-valued functions, we follow a similar path to the one we took in studying real-valued functions. First, we define the derivative, then we examine applications of the derivative, then we move on to defining integrals. However, we will find some interesting new ideas along the way as a result of the vector nature of these functions and the properties of space curves.

Derivatives of Vector-Valued Functions

Now that we have seen what a vector-valued function is and how to take its limit, the next step is to learn how to differentiate a vector-valued function. The definition of the derivative of a vector-valued function is nearly identical to the definition of a real-valued function of one variable. However, because the range of a vector-valued function consists of vectors, the same is true for the range of the derivative of a vector-valued function.

Definition: Derivative of Vector-Valued Functions

The derivative of a vector-valued function $\vec{\mathbf{r}}(t)$ is

$$\vec{\mathbf{r}}'(t) = \lim_{\Delta t \to 0} \frac{\vec{\mathbf{r}}(t + \Delta t) - \vec{\mathbf{r}}(t)}{\Delta t} \tag{12.2.1}$$

provided the limit exists. If $\vec{\mathbf{r}}'(t)$ exists, then $\vec{\mathbf{r}}(t)$ is differentiable at t. If $\vec{\mathbf{r}}'(t)$ exists for all t in an open interval (a, b) then $\vec{\mathbf{r}}(t)$ is differentiable over the interval (a, b). For the function to be differentiable over the closed interval $[a, b]$, the following two limits must exist as well:

$$\vec{\mathbf{r}}'(a) = \lim_{\Delta t \to 0^+} \frac{\vec{\mathbf{r}}(a + \Delta t) - \vec{\mathbf{r}}(a)}{\Delta t} \tag{12.2.2}$$

and

$$\vec{\mathbf{r}}'(b) = \lim_{\Delta t \to 0^-} \frac{\vec{\mathbf{r}}(b + \Delta t) - \vec{\mathbf{r}}(b)}{\Delta t} \tag{12.2.3}$$

Graphical Interpretation of the Derivative: Recall that the derivative of a real-valued function can be interpreted as the slope of a tangent line or the instantaneous rate of change of the function. The derivative of a vector-valued function can be understood to be an instantaneous rate of change as well; for example, when the function represents the position of an object at a given point in time, the derivative represents its velocity at that same point in time.

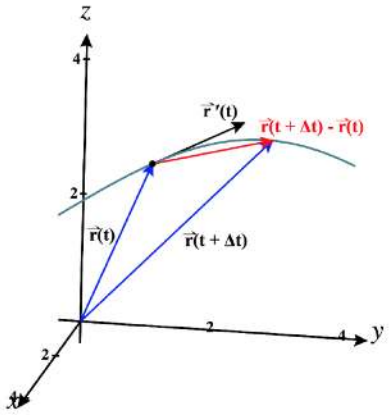

Figure 12.2.1

To understand what this derivative represents geometrically, it helps to consider the diagram shown in Figure 12.2.1. For the purposes of this illustration, consider $\vec{\mathbf{r}}(t)$ to the represent position of an object.

We see vectors $\vec{\mathbf{r}}(t)$ and $\vec{\mathbf{r}}(t + \Delta t)$, where Δt is a small change in time producing a second vector that is farther along in the direction of motion.

The difference of these two vectors gives us the **displacement vector,** $\vec{\mathbf{r}}(t + \Delta t) - \vec{\mathbf{r}}(t)$, shown here in red.

As $\Delta t \to 0$, $\vec{\mathbf{r}}(t+\Delta t)$ will approach $\vec{\mathbf{r}}(t)$, and the displacement vector $\vec{\mathbf{r}}(t+\Delta t) - \vec{\mathbf{r}}(t)$ will become closer and closer to being tangent to the curve at the terminal point of $\vec{\mathbf{r}}(t)$.

Although the length of the displacement vector is approaching 0, we are actually taking the limit not of the displacement vector itself, but of the average velocity over the time Δt, found by dividing the displacement vector by the scalar Δt.

$$\text{Average Velocity} \quad = \quad \frac{\vec{\mathbf{r}}(t+\Delta t) - \vec{\mathbf{r}}(t)}{\Delta t} \tag{12.2.4}$$

Since Δt is also approaching 0, dividing by it will tend to lengthen the average velocity vector (as long as $\Delta t < 1$).

The limit gives us the vector $\vec{\mathbf{r}}'(t)$, the velocity of the motion along the curve, which is tangent to the curve at this point.

We now demonstrate taking the derivative of a vector-valued function.

Example 12.2.1: Finding the Derivative of a Vector-Valued Function

Use the definition to calculate the derivative of the function

$$\vec{\mathbf{r}}(t) = (3t+4)\,\hat{\mathbf{i}} + (t^2 - 4t + 3)\,\hat{\mathbf{j}}.$$

Solution

Let's use Equation 12.2.1:

$$\begin{aligned}
\vec{\mathbf{r}}'(t) &= \lim_{\Delta t \to 0} \frac{\vec{\mathbf{r}}(t+\Delta t) - \vec{\mathbf{r}}(t)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{[(3(t+\Delta t)+4)\,\hat{\mathbf{i}} + ((t+\Delta t)^2 - 4(t+\Delta t) + 3)\,\hat{\mathbf{j}}] - [(3t+4)\,\hat{\mathbf{i}} + (t^2 - 4t + 3)\,\hat{\mathbf{j}}]}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{(3t + 3\Delta t + 4)\,\hat{\mathbf{i}} - (3t+4)\,\hat{\mathbf{i}} + (t^2 + 2t\Delta t + (\Delta t)^2 - 4t - 4\Delta t + 3)\,\hat{\mathbf{j}} - (t^2 - 4t + 3)\,\hat{\mathbf{j}}}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{(3\Delta t)\,\hat{\mathbf{i}} + (2t\Delta t + (\Delta t)^2 - 4\Delta t)\,\hat{\mathbf{j}}}{\Delta t} \\
&= \lim_{\Delta t \to 0} (3\,\hat{\mathbf{i}} + (2t + \Delta t - 4)\,\hat{\mathbf{j}}) \\
&= 3\,\hat{\mathbf{i}} + (2t-4)\,\hat{\mathbf{j}}
\end{aligned}$$

Exercise 12.2.1

Use the definition to calculate the derivative of the function $\vec{\mathbf{r}}(t) = (2t^2+3)\,\hat{\mathbf{i}} + (5t-6)\,\hat{\mathbf{j}}$.

Notice that in the calculations in Example 12.2.1, we could also obtain the answer by first calculating the derivative of each component function, then putting these derivatives back into the vector-valued function. This is always true for calculating the derivative of a vector-valued function, whether it is in two or three dimensions. We state this in the following theorem. The proof of this theorem follows directly from the definitions of the limit of a vector-valued function and the derivative of a vector-valued function.

Theorem 12.2.1: Differentiation of Vector-Valued Functions

Let f, g, and h be differentiable functions of t.

1. If $\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}}$ then

$$\vec{\mathbf{r}}'(t) = f'(t)\,\hat{\mathbf{i}} + g'(t)\,\hat{\mathbf{j}}. \tag{12.2.5}$$

2. If $\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} + h(t)\,\hat{\mathbf{k}}$ then

$$\vec{\mathbf{r}}'(t) = f'(t)\,\hat{\mathbf{i}} + g'(t)\,\hat{\mathbf{j}} + h'(t)\,\hat{\mathbf{k}}. \tag{12.2.6}$$

Example 12.2.2: Calculating the Derivative of Vector-Valued Functions

Use Theorem **12.2.1** to calculate the derivative of each of the following functions.

a. $\vec{\mathbf{r}}(t) = (6t+8)\,\hat{\mathbf{i}} + (4t^2+2t-3)\,\hat{\mathbf{j}}$
b. $\vec{\mathbf{r}}(t) = 3\cos t\,\hat{\mathbf{i}} + 4\sin t\,\hat{\mathbf{j}}$
c. $\vec{\mathbf{r}}(t) = e^t \sin t\,\hat{\mathbf{i}} + e^t \cos t\,\hat{\mathbf{j}} - e^{2t}\,\hat{\mathbf{k}}$

Solution

We use Theorem **12.2.1** and what we know about differentiating functions of one variable.

a. The first component of

$$\vec{\mathbf{r}}(t) = (6t+8)\,\hat{\mathbf{i}} + (4t^2+2t-3)\,\hat{\mathbf{j}}$$

is $f(t) = 6t+8$. The second component is $g(t) = 4t^2+2t-3$. We have $f'(t) = 6$ and $g'(t) = 8t+2$, so Theorem **12.2.1** gives

$$\vec{\mathbf{r}}'(t) = 6\,\hat{\mathbf{i}} + (8t+2)\,\hat{\mathbf{j}}$$

b. The first component is $f(t) = 3\cos t$ and the second component is $g(t) = 4\sin t$. We have $f'(t) = -3\sin t$ and $g'(t) = 4\cos t$, so we obtain

$$\vec{\mathbf{r}}'(t) = -3\sin t\,\hat{\mathbf{i}} + 4\cos t\,\hat{\mathbf{j}}$$

c. The first component of $\vec{\mathbf{r}}(t) = e^t \sin t\,\hat{\mathbf{i}} + e^t \cos t\,\hat{\mathbf{j}} - e^{2t}\,\hat{\mathbf{k}}$ is $f(t) = e^t \sin t$, the second component is $g(t) = e^t \cos t$, and the third component is $h(t) = -e^{2t}$. We have $f'(t) = e^t(\sin t + \cos t)$, $g'(t) = e^t(\cos t - \sin t)$, and $h'(t) = -2e^{2t}$, so the theorem gives

$$\vec{\mathbf{r}}'(t) = e^t(\sin t + \cos t)\,\hat{\mathbf{i}} + e^t(\cos t - \sin t)\,\hat{\mathbf{j}} - 2e^{2t}\,\hat{\mathbf{k}}$$

Exercise 12.2.2

Calculate the derivative of the function

$$\vec{\mathbf{r}}(t) = (t\ln t)\,\hat{\mathbf{i}} + (5e^t)\,\hat{\mathbf{j}} + (\cos t - \sin t)\,\hat{\mathbf{k}}.$$

Describing Motion with Vector-Valued Functions

As has been mentioned already in this section, vector-valued functions are often used to describe motion in the plane or in space. Now that we understand the derivative of a vector-valued function, let's make the relationships clear.

First, since the components of a vector-valued function represent the coordinates of a point on the corresponding curve, it may help to write

$$\vec{\mathbf{r}}(t) = x(t)\,\hat{\mathbf{i}} + y(t)\,\hat{\mathbf{j}} + z(t)\,\hat{\mathbf{k}} \tag{12.2.7}$$

This approach has already been used in Section 12.1.

Now we can define how to calculate the velocity, speed, acceleration, and jerk of the object's motion.

Definition: Velocity, speed, Acceleration, and Jerk

Let $\vec{\mathbf{r}}(t)$ be a twice-differentiable vector-valued function of the parameter t that represents the position of an object as a function of time.

The **velocity vector** $\vec{\mathbf{v}}(t)$ of the object is given by

$$\text{Velocity} = \vec{\mathbf{v}}(t) = \vec{\mathbf{r}}'(t). \tag{12.2.8}$$

The ***speed*** is defined to be

$$\text{Speed} = \|\vec{\mathbf{v}}(t)\| = \|\vec{\mathbf{r}}'(t)\| = \frac{ds}{dt}. \tag{12.2.9}$$

The **acceleration vector** $\vec{\mathbf{a}}(t)$ is defined to be

$$\text{Acceleration } = \vec{\mathbf{a}}(t) = \vec{\mathbf{v}}'(t) = \vec{\mathbf{r}}''(t). \tag{12.2.10}$$

If $\vec{\mathbf{r}}(t)$ has a third derivative, we can also define the **jerk vector** $\overrightarrow{\text{jerk}}(t)$ to be

$$\text{Jerk } = \overrightarrow{\text{jerk}}(t) = \vec{\mathbf{a}}'(t) = \vec{\mathbf{r}}'''(t). \tag{12.2.11}$$

Now let's see how this looks in the context of a particular position function.

Example 12.2.3: Finding velocity, speed, acceleration, and jerk

Consider the position function,

$$\vec{\mathbf{r}}(t) = (1-t^2)\,\hat{\mathbf{i}} + t\,\hat{\mathbf{j}}, \text{ where time is in seconds}$$

a. Determine the velocity, speed, acceleration, and jerk as functions of time for this position function.

b. Evaluate each of these functions, including the position function, at times $t = 0$ sec and $t = 2$ sec.

c. Sketch a graph of this position function and draw in each of the non-zero vectors from part b on the graph of $\vec{\mathbf{r}}$, starting at the location of the object on the curve at the corresponding time.

Solution

a. Since $\vec{\mathbf{r}}(t) = (1-t^2)\,\hat{\mathbf{i}} + t\,\hat{\mathbf{j}}$, we have

$$\begin{aligned}
\vec{\mathbf{v}}(t) &= \vec{\mathbf{r}}\,'(t) = -2t\,\hat{\mathbf{i}} + 1\hat{\mathbf{j}},\\
\text{speed}(t) &= \|\vec{\mathbf{v}}(t)\| = \sqrt{(-2t)^2 + (1)^2} = \sqrt{4t^2+1},\\
\vec{\mathbf{a}}(t) &= \vec{\mathbf{v}}'(t) = -2\,\hat{\mathbf{i}},\\
\overrightarrow{\text{jerk}}(t) &= \vec{\mathbf{0}}
\end{aligned}$$

b. Evaluating these functions at time $t = 0$, we get:

$$\begin{aligned}
\vec{\mathbf{r}}(0) &= (1-0^2)\,\hat{\mathbf{i}} + 0\hat{\mathbf{j}} = \hat{\mathbf{i}}\\
\vec{\mathbf{v}}(0) &= -2(0)\,\hat{\mathbf{i}} + 1\hat{\mathbf{j}} = \hat{\mathbf{j}}\\
\text{speed}(0) &= \sqrt{4(0)^2+1} = \sqrt{1} = 1 \text{ unit/sec}\\
\vec{\mathbf{a}}(0) &= -2\,\hat{\mathbf{i}},\\
\overrightarrow{\text{jerk}}(0) &= \vec{\mathbf{0}}
\end{aligned}$$

And for time $t = 2$, we have

$$\begin{aligned}
\vec{\mathbf{r}}(2) &= (1-2^2)\,\hat{\mathbf{i}} + 2\hat{\mathbf{j}} = -3\,\hat{\mathbf{i}} + 2\hat{\mathbf{j}}\\
\vec{\mathbf{v}}(2) &= -2(2)\,\hat{\mathbf{i}} + 1\hat{\mathbf{j}} = -4\,\hat{\mathbf{i}} + \hat{\mathbf{j}}\\
\text{speed}(2) &= \sqrt{4(2)^2+1} = \sqrt{17} \text{ units/sec}\\
\vec{\mathbf{a}}(0) &= -2\,\hat{\mathbf{i}},\\
\overrightarrow{\text{jerk}}(0) &= \vec{\mathbf{0}}
\end{aligned}$$

c. The graph of this curve is shown below in Figure 12.2.2. We then drew in and labeled the vectors $\vec{\mathbf{v}}(0)$, $\vec{\mathbf{a}}(0)$, $\vec{\mathbf{v}}(2)$, and $\vec{\mathbf{a}}(2)$.

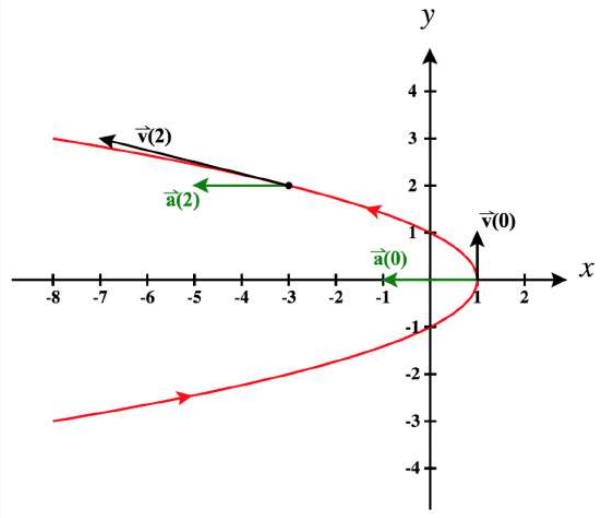

Figure 12.2.2: Graphing velocity and acceleration vectors

Smooth Vector-Valued Functions

Definition: smoothness

Let $\vec{\mathbf{r}}(t) = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}} + h(t)\hat{\mathbf{k}}$ be the parameterization of a curve that is differentiable on an open interval I.

Then $\vec{\mathbf{r}}(t)$ is **smooth** on the open interval I, if

$$\vec{\mathbf{r}}\,'(t) \neq \vec{\mathbf{0}}, \text{ for any value of } t \text{ in the interval } I.$$

To put this another way, $\vec{\mathbf{r}}(t)$ is **smooth** on the open interval I if:

1. $\vec{\mathbf{r}}(t)$ is defined and continuous on I, and
2. f', g' and h' are all continuous on I, that is, $\vec{\mathbf{r}}\,'(t)$ is defined and continuous on I, and
3. $\vec{\mathbf{r}}\,'(t) \neq \vec{\mathbf{0}}$ for all values of t in the interval I.

The concept of smoothness is very closely connected to the application of vector-valued functions to motion. If you have ever ridden rides at an amusement park, you can probably identify some where the ride felt smooth, and others where there are moments during the ride where it was not smooth. These moments often correspond to points in the motion where the relevant parameterization is not smooth by the definition above.

To identify open intervals in the domain of $\vec{\mathbf{r}}$ on which $\vec{\mathbf{r}}$ is smooth, we usually identify values of t for which $\vec{\mathbf{r}}$ is **not smooth**.

In addition to using the definition above to identify these points, it is often apparent from the graph of the vector-valued function $\vec{\mathbf{r}}(t)$. A **cusp**, i.e., a sharp turning point, is a common occurrence at points where a curve is not smooth. In Figure 12.2.3 below, you can quickly identify points at which the parameterization fails to be smooth by locating cusps in the graph.

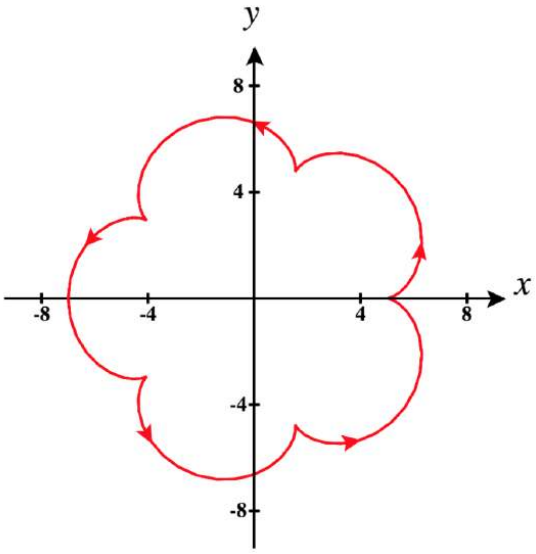

Figure **12.2.3**: This epicycloid is not smooth at the points between it's petals.

When we consider the motion described by $\vec{\mathbf{r}}(t)$, there are at least two additional events that can occur at points where the parameterization fails to be smooth: a reversal of motion or a momentary pause in the motion. To summarize these events:

If a vector-valued function $\vec{\mathbf{r}}(t)$ is **not smooth at time** t, we will observe that:

1. There is a cusp at the associated point on the graph of $\vec{\mathbf{r}}(t)$, or
2. The motion reverses itself at the associated point, causing the motion to travel back along the same path in the opposite direction, or
3. The motion actually stops and starts up again, with no visual cue, that is, where the curve appears smooth.

Let's consider a couple examples to see how this can look.

Example 12.2.4: Determining open intervals on which a vector-valued function is smooth

Determine the open intervals on which each of the following vector-valued functions is smooth:

a. $\vec{\mathbf{r}}(t) = (2t^3 - 3t^2)\ \hat{\mathbf{i}} + (t^2 - 2t)\ \hat{\mathbf{j}} + 2\ \hat{\mathbf{k}}$
b. $\vec{\mathbf{r}}(t) = t^3\ \hat{\mathbf{i}} + \sqrt{t^2 - 1}\,\hat{\mathbf{j}}$
c. $\vec{\mathbf{r}}(t) = t^4\ \hat{\mathbf{i}} + \ln(t^2 + 1)\,\hat{\mathbf{j}}$
d. $\vec{\mathbf{r}}(t) = (6\cos t - \cos(6t))\ \hat{\mathbf{i}} + (6\sin t - \sin(6t))\ \hat{\mathbf{j}}$, for $0 \le t \le 2\pi$

Solution

a. First we need to consider the domain of $\vec{\mathbf{r}}$. In this case, $\mathbf{D}_{\vec{\mathbf{r}}} : (-\infty, \infty)$.

Then we need to find $\vec{\mathbf{r}}\,'(t)$. Since $\vec{\mathbf{r}}(t) = (2t^3 - 3t^2)\ \hat{\mathbf{i}} + (t^2 - 2t)\ \hat{\mathbf{j}} + 2\ \hat{\mathbf{k}}$,

$$\vec{\mathbf{r}}\,'(t) = (6t^2 - 6t)\ \hat{\mathbf{i}} + (2t - 2)\ \hat{\mathbf{j}}$$

Next we need to consider the domain of $\vec{\mathbf{r}}\,'$. Here, $\mathbf{D}_{\vec{\mathbf{r}}\,'} : (-\infty, \infty)$.

Finally we need to determine if there are any values of t in $\mathbf{D}_{\vec{\mathbf{r}}}$ where $\vec{\mathbf{r}}\,'(t) = \vec{\mathbf{0}}$.

To do this, we set each component of $\vec{\mathbf{r}}\,'(t)$ equal to 0 and solve for t.

$$\begin{aligned} 6t^2 - 6t &= 0 \quad \text{and} & 2t - 2 &= 0 \\ 6t(t-1) &= 0 & 2(t-1) &= 0 \\ t = 0 \quad \text{or} \quad t &= 1 & t &= 1 \end{aligned}$$

Since $t = 1$ makes both components of $\vec{\mathbf{r}}\,'$ equal 0, we have $\vec{\mathbf{r}}\,'(1) = \vec{\mathbf{0}}$, so we know that $\vec{\mathbf{r}}$ is **not smooth** at $t = 1$.

Note however that $\vec{\mathbf{r}}$ is smooth at $t = 0$, because although one of the components of $\vec{\mathbf{r}}\,'$ is 0, the other one is not.

Therefore we can conclude that $\vec{\mathbf{r}}$ is smooth on the open intervals:

$$(-\infty, 1) \quad \text{and} \quad (1, \infty)$$

Note in the figure that there is a **cusp** in the curve when $t = 1$.

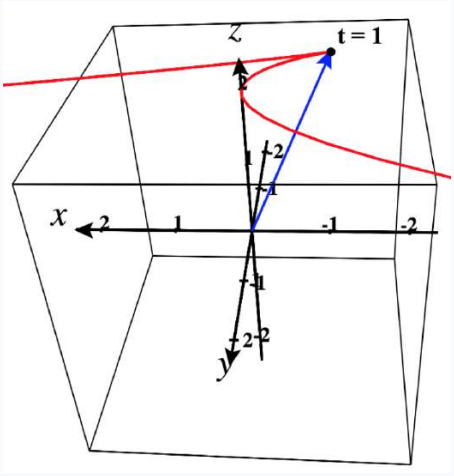

Figure 12.2.4: Plot of $\vec{\mathbf{r}}(t) = (2t^3 - 3t^2)\ \hat{\mathbf{i}} + (t^2 - 2t)\ \hat{\mathbf{j}} + 2\ \hat{\mathbf{k}}$

b. Next let $\vec{\mathbf{r}}(t) = t^3\ \hat{\mathbf{i}} + \sqrt{t^2 - 1}\hat{\mathbf{j}}$.

The domain of $\vec{\mathbf{r}}$ is $\mathbf{D}_{\vec{\mathbf{r}}} : (-\infty, -1] \cup [1, \infty)$, since $t^2 - 1$ is less than 0 between -1 and 1.

Now we calculate $\vec{\mathbf{r}}\,'(t)$. Since $\vec{\mathbf{r}}(t) = t^3\ \hat{\mathbf{i}} + \sqrt{t^2 - 1}\hat{\mathbf{j}}$, we have

$$\vec{\mathbf{r}}\,'(t) = 3t^2\ \hat{\mathbf{i}} + \frac{t}{\sqrt{t^2 - 1}}\ \hat{\mathbf{j}}$$

Then, since we cannot have zero in the denominator, $t = -1$ and $t = 1$ are not in the domain of $\vec{\mathbf{r}}\,'$. Thus, $\mathbf{D}_{\vec{\mathbf{r}}\,'} : (-\infty, -1) \cup (1, \infty)$.

Lastly we determine if there are any values of t in $\mathbf{D}_{\vec{\mathbf{r}}}$ where $\vec{\mathbf{r}}\,'(t) = \vec{\mathbf{0}}$.

Setting each component of $\vec{\mathbf{r}}\,'(t)$ equal to 0 and solving for t we get:

$$\begin{array}{ll} 3t^2 = 0 & \text{and} \qquad \dfrac{t}{\sqrt{t^2 - 1}} = 0 \\ t = 0 & \qquad\qquad\quad t = 0 \end{array}$$

If $t = 0$ were in the domain of $\vec{\mathbf{r}}\,'$, then $\vec{\mathbf{r}}$ would not be smooth there, but since it is not in the domain, we have nothing to remove, and we can conclude that $\vec{\mathbf{r}}$ is smooth on the entire domain of its derivative $\vec{\mathbf{r}}\,'$, that is,

$$\vec{\mathbf{r}} \text{ is smooth on the open intervals:} \quad (-\infty, -1) \quad \text{and} \quad (1, \infty).$$

Observe the two separate smooth pieces of this function in the graph below.

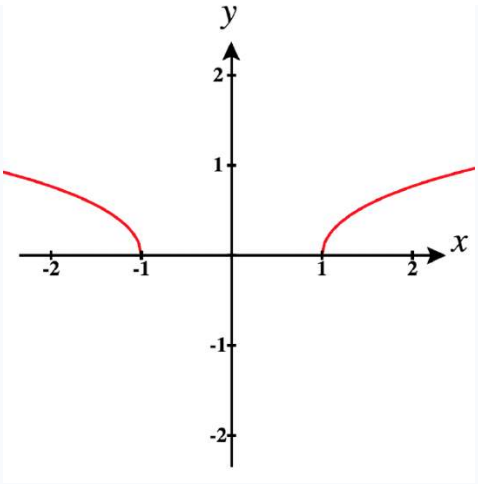

Figure 12.2.5: Plot of $\vec{\mathbf{r}}(t) = t^3\,\hat{\mathbf{i}} + \sqrt{t^2-1}\,\hat{\mathbf{j}}$

c. Now let's consider, $\vec{\mathbf{r}}(t) = t^4\,\hat{\mathbf{i}} + \ln(t^2+1)\,\hat{\mathbf{j}}$.

The domain of $\vec{\mathbf{r}}$ is $\mathbf{D}_{\vec{\mathbf{r}}} : (-\infty, \infty)$, since the argument of the natural log function, t^2+1, is always greater than 0 (In fact, it's always at least 1).

Now we calculate $\vec{\mathbf{r}}\,'(t)$. Here

$$\vec{\mathbf{r}}\,'(t) = 4t^3\,\hat{\mathbf{i}} + \frac{2t}{t^2+1}\,\hat{\mathbf{j}}$$

Then, since t^2+1 cannot be 0, the domain of $\vec{\mathbf{r}}\,'$ is also all real numbers, $\mathbf{D}_{\vec{\mathbf{r}}\,'} : (-\infty, \infty)$.

Lastly we determine if there are any values of t in $\mathbf{D}_{\vec{\mathbf{r}}}$ where $\vec{\mathbf{r}}\,'(t) = \vec{\mathbf{0}}$.

Setting each component of $\vec{\mathbf{r}}\,'(t)$ equal to 0 and solving for t we get:

$$\begin{array}{ccc} 4t^3 = 0 & \text{and} & \dfrac{2t}{t^2+1} = 0 \\ t = 0 & & t = 0 \end{array}$$

Note that in this example, we have $\vec{\mathbf{r}}\,'(0) = \vec{\mathbf{0}}$, so $\vec{\mathbf{r}}$ is **not smooth** when $t = 0$.

Therefore we conclude that $\vec{\mathbf{r}}$ is smooth on the open intervals:

$$(-\infty, 0) \quad \text{and} \quad (0, \infty)$$

In this example, note that the motion reverses itself when $t = 0$. We can't see this very well just by looking at its graph (see Figure 12.2.6). But if you evaluate $\vec{\mathbf{r}}(t)$ for various values of t from -2 to 2, you'll see that the motion comes in towards the origin when t is negative, arrives at the origin when $t = 0$, and then moves away from the origin as t becomes more positive.

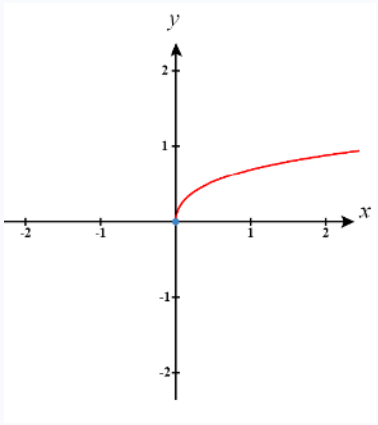

Figure 12.2.6: Plot of $\vec{\mathbf{r}}(t) = t^4\,\hat{\mathbf{i}} + \ln(t^2+1)\,\hat{\mathbf{j}}$

d. Finally, let's look at a really interesting example:

$$\vec{\mathbf{r}}(t) = (6\cos t - \cos(6t))\,\hat{\mathbf{i}} + (6\sin t - \sin(6t))\,\hat{\mathbf{j}}.$$

Its graph is the epicycloid shown above in Figure 12.2.3.

Again, $\vec{\mathbf{r}}$ has no issues with its domain. We have $\mathbf{D}_{\vec{\mathbf{r}}} : (-\infty, \infty)$.

Calculating $\vec{\mathbf{r}}\,'(t)$, we get:

$$\vec{\mathbf{r}}\,'(t) = (-6\sin t + 6\sin(6t))\,\hat{\mathbf{i}} + (6\cos t - 6\cos(6t))\,\hat{\mathbf{j}}$$

There are still no issues, so $\mathbf{D}_{\vec{\mathbf{r}}\,'} : (-\infty, \infty)$.

Now for the most interesting part! For what values of t is $\vec{\mathbf{r}}\,'(t) = \vec{\mathbf{0}}$?

Setting the two components equal to 0, we get:

$$-6\sin t + 6\sin(6t) = 0 \quad \text{and} \quad 6\cos t - 6\cos(6t) = 0$$

This time it's not as easy to quickly solve each of these equations for all values of t that make these component functions equal to 0. But if you read carefully, you'll see that it's really not that difficult to work out.

Rewriting the first equation gives us

$$\sin(6t) = \sin t \tag{12.2.12}$$

To solve this equation, note that the periodic nature of the sine function means that its value is the same every 2π radians.

That is, for any integer n, we know

$$\sin(t + 2n\pi) = \sin t$$

This means that one way for Equation 12.2.12 to be true would be for us to have

$$t + 2n\pi = 6t$$

Solving this equation for t, we have:

$$t = \frac{2n\pi}{5} \text{ for any integer } n$$

On the interval $0 \le t \le 2\pi$, this would give us the following solutions of Equation 12.2.12:

$$t = 0,\ \frac{2\pi}{5},\ \frac{4\pi}{5},\ \frac{6\pi}{5},\ \frac{8\pi}{5},\ \frac{10\pi}{5} = 2\pi$$

Verifying that $t = \frac{2\pi}{5}$ is a solution, we have:

$$\sin\left(6 \cdot \frac{2\pi}{5}\right) = \sin\left(2\pi + \frac{2\pi}{5}\right) = \sin\left(\frac{2\pi}{5}\right). \quad \checkmark$$

Note that the sine value (as the y-coordinate of a point on the unit circle) is the same for reflected angles about the \(y)-axis (e.g., in the first and second quadrants). That is,

$$\sin(\pi - t) = \sin t$$

Incorporating the periodic nature of the sine function again, we obtain

$$\sin\Big((2n+1)\pi - t\Big) = \sin t$$

Applying this identity to Equation 12.2.12, yields

$$\sin\Big((2n+1)\pi - t\Big) = \sin(6t)$$

So, another way for Equation 12.2.12 to be true would be for us to have

$$(2n+1)\pi - t = 6t$$

Solving this equation for t, we have:

$$t = \frac{(2n+1)\pi}{7} \text{ for any integer } n$$

On the interval $0 \le t \le 2\pi$, this would give us the following solutions of Equation 12.2.12:

$$t = \frac{\pi}{7}, \frac{3\pi}{7}, \frac{5\pi}{7}, \frac{7\pi}{7} = \pi, \frac{9\pi}{7}, \frac{11\pi}{7}, \frac{13\pi}{7}$$

Verifying that $t = \frac{\pi}{7}$ is a solution, we have:

$$\sin\left(6 \cdot \frac{\pi}{7}\right) = \sin\left(\pi - \frac{\pi}{7}\right) = \sin\left(\frac{\pi}{7}\right). \quad \checkmark$$

Next we need to solve the equation from the setting the other component of $\vec{\mathbf{r}}\,'(t)$ to 0. That is

$$6\cos t - 6\cos(6t) = 0$$

Remember that not all values of t that make one of these components equal to 0 will cause $\vec{\mathbf{r}}$ not to be smooth, but only those values of t that make both components equal to 0 **at the same time**. Once we solve this equation, we'll need to look for any values of t that are solutions to both of these equations.

Rewriting the equation above gives us

$$\cos(6t) = \cos t \tag{12.2.13}$$

We can solve this equation using a similar process to what was shown for Equation 12.2.12 above.

As the cosine function is periodic in the same way that sine is, we have

$$\cos(t + 2n\pi) = \cos t$$

which tells us Equation 12.2.13 will be true when

$$t + 2n\pi = 6t$$

Solving this equation for t, we have:

$$t = \frac{2n\pi}{5} \text{ for any integer } n$$

Note that these values for t also made the first component of $\vec{\mathbf{r}}\,'(t)$ equal to 0! Try verifying that one of these solutions satisfies Equation 12.2.13, as we showed for Equation 12.2.12 above.

Since the cosine values (as the x-coordinates of points on the unit circle) are the same for reflected angles about the x-axis (e.g. in the first and fourth quadrants). This is represented in the identity:

$$\cos(-t) = \cos(t)$$

Incorporating the periodic nature of the cosine function (as we did before), we obtain

$$\cos(2n\pi - t) = \cos(t)$$

Applying this identity to Equation 12.2.13, yields

$$\cos(2n\pi - t) = \cos(6t)$$

This is true when

$$2n\pi - t = 6t$$

Solving this equation for t, we have:

$$t = \frac{2n\pi}{7} \text{ for any integer } n$$

Although these may at first look like the same values as we found solved Equation 12.2.12 above, they are not.

On the interval $0 \le t \le 2\pi$, this would give us the following solutions of Equation 12.2.13:

$$t = 0, \frac{2\pi}{7}, \frac{4\pi}{7}, \frac{6\pi}{7}, \frac{8\pi}{7}, \frac{10\pi}{7}, \frac{12\pi}{7}, \frac{14\pi}{7} = 2\pi$$

Except for $t = 0$ and $t = 2\pi$ that are already contained in the first set of solutions $t = \frac{2n\pi}{5}$ above, this second set of solutions does not add any additional values of t that would make both components of $\vec{\mathbf{r}}\,'(t)$ equal to 0 at the same time.

We have now determined that $\vec{\mathbf{r}}\,'(t) = \vec{0}$ and therefore $\vec{\mathbf{r}}$ is not smooth when $t = \frac{2n\pi}{5}$ for any integer n.

Removing these values from the interval $0 \le t \le 2\pi$, we can say that $\vec{\mathbf{r}}$ is smooth on the open intervals:

$$\left(0, \frac{2\pi}{5}\right), \left(\frac{2\pi}{5}, \frac{4\pi}{5}\right), \left(\frac{4\pi}{5}, \frac{6\pi}{5}\right), \left(\frac{6\pi}{5}, \frac{8\pi}{5}\right), \text{ and } \left(\frac{8\pi}{5}, 2\pi\right).$$

More generally, we can state that r is smooth on the open intervals:

$$\left(\frac{2n\pi}{5}, \frac{(2n+2)\pi}{5}\right) \text{ for any integer } n$$

Note in Figure 12.2.7, how there are cusps on the plot of this vector-valued function for each value of t at which $\vec{\mathbf{r}}$ is not smooth.

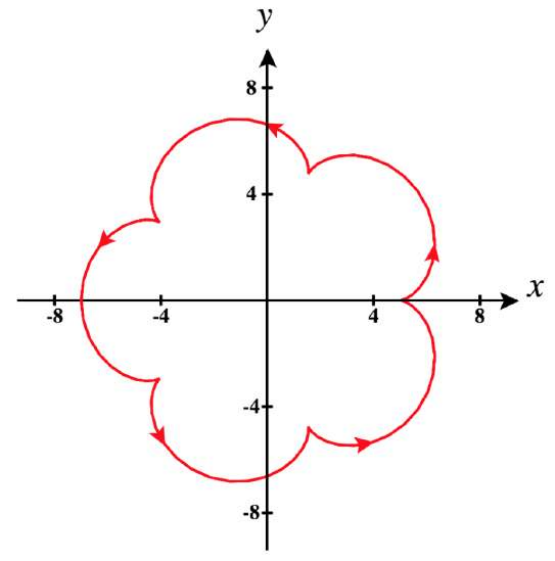

Figure 12.2.7: This epicycloid is not smooth at the points between it's petals. These occur when $t = 0, \frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5}, \frac{10\pi}{5} = 2\pi$.

Properties of the Derivative

We can extend to vector-valued functions the properties of the derivative that we presented previously for real-valued functions of a single variable. In particular, the constant multiple rule, the sum and difference rules, the product rule, and the chain rule all extend to vector-valued functions. However, in the case of the product rule, there are actually three extensions:

1. for a real-valued function multiplied by a vector-valued function,
2. for the dot product of two vector-valued functions, and
3. for the cross product of two vector-valued functions.

Theorem: Properties of the Derivative of Vector-Valued Functions

Let $\vec{\mathbf{r}}$ and $\vec{\mathbf{u}}$ be differentiable vector-valued functions of t, let f be a differentiable real-valued function of t, and let c be a scalar.

i.	$\frac{d}{dt}[c\vec{\mathbf{r}}(t)]$	$=$	$c\vec{\mathbf{r}}'(t)$	Scalar multiple
ii.	$\frac{d}{dt}[\vec{\mathbf{r}}(t) \pm \vec{\mathbf{u}}(t)]$	$=$	$\vec{\mathbf{r}}'(t) \pm \vec{\mathbf{u}}'(t)$	Sum and difference
iii.	$\frac{d}{dt}[f(t)\vec{\mathbf{u}}(t)]$	$=$	$f'(t)\vec{\mathbf{u}}(t) + f(t)\vec{\mathbf{u}}'(t)$	Scalar product
iv.	$\frac{d}{dt}[\vec{\mathbf{r}}(t) \cdot \vec{\mathbf{u}}(t)]$	$=$	$\vec{\mathbf{r}}'(t) \cdot \vec{\mathbf{u}}(t) + \vec{\mathbf{r}}(t) \cdot \vec{\mathbf{u}}'(t)$	Dot product
v.	$\frac{d}{dt}[\vec{\mathbf{r}}(t) \times \vec{\mathbf{u}}(t)]$	$=$	$\vec{\mathbf{r}}'(t) \times \vec{\mathbf{u}}(t) + \vec{\mathbf{r}}(t) \times \vec{\mathbf{u}}'(t)$	Cross product
vi.	$\frac{d}{dt}[\vec{\mathbf{r}}(f(t))]$	$=$	$\vec{\mathbf{r}}'(f(t)) \cdot f'(t)$	Chain rule
vii.	If $\vec{\mathbf{r}}(t) \cdot \vec{\mathbf{r}}(t)$	$=$	c, then $\vec{\mathbf{r}}(t) \cdot \vec{\mathbf{r}}'(t) = 0$.	

Proof

The proofs of the first two properties follow directly from the definition of the derivative of a vector-valued function. The third property can be derived from the first two properties, along with the product rule. Let $\vec{\mathbf{u}}(t) = g(t)\,\hat{\mathbf{i}} + h(t)\,\hat{\mathbf{j}}$. Then

$$\begin{aligned}\frac{d}{dt}[f(t)\vec{\mathbf{u}}(t)] &= \frac{d}{dt}[f(t)(g(t)\,\hat{\mathbf{i}} + h(t)\,\hat{\mathbf{j}})] \\ &= \frac{d}{dt}[f(t)g(t)\,\hat{\mathbf{i}} + f(t)h(t)\,\hat{\mathbf{j}}] \\ &= \frac{d}{dt}[f(t)g(t)]\,\hat{\mathbf{i}} + \frac{d}{dt}[f(t)h(t)]\,\hat{\mathbf{j}} \\ &= (f'(t)g(t) + f(t)g'(t))\,\hat{\mathbf{i}} + (f'(t)h(t) + f(t)h'(t))\,\hat{\mathbf{j}} \\ &= f'(t)\vec{\mathbf{u}}(t) + f(t)\vec{\mathbf{u}}'(t).\end{aligned}$$

To prove property iv. let $\vec{\mathbf{r}}(t) = f_1(t)\,\hat{\mathbf{i}} + g_1(t)\,\hat{\mathbf{j}}$ and $\vec{\mathbf{u}}(t) = f_2(t)\,\hat{\mathbf{i}} + g_2(t)\,\hat{\mathbf{j}}$. Then

$$\begin{aligned}\frac{d}{dt}[\vec{\mathbf{r}}(t) \cdot \vec{\mathbf{u}}(t)] &= \frac{d}{dt}[f_1(t)f_2(t) + g_1(t)g_2(t)] \\ &= f_1{}'(t)f_2(t) + f_1(t)f_2{}'(t) + g_1{}'(t)g_2(t) + g_1(t)g_2{}'(t) = f_1{}'(t)f_2(t) + g_1 \\ &\quad {}'(t)g_2(t) + f_1(t)f_2{}'(t) + g_1(t)g_2{}'(t) \\ &= (f_1{}'\,\hat{\mathbf{i}} + g_1{}'\,\hat{\mathbf{j}}) \cdot (f_2\,\hat{\mathbf{i}} + g_2\,\hat{\mathbf{j}}) + (f_1\,\hat{\mathbf{i}} + g_1\,\hat{\mathbf{j}}) \cdot (f_2{}'\,\hat{\mathbf{i}} + g_2{}'\,\hat{\mathbf{j}}) \\ &= \vec{\mathbf{r}}'(t) \cdot \vec{\mathbf{u}}(t) + \vec{\mathbf{r}}(t) \cdot \vec{\mathbf{u}}'(t).\end{aligned}$$

The proof of property v. is similar to that of property iv. Property vi. can be proved using the chain rule. Last, property vii. follows from property iv:

$$\begin{aligned}\frac{d}{dt}[\vec{\mathbf{r}}(t) \cdot \vec{\mathbf{r}}(t)] &= \frac{d}{dt}[c] \\ \vec{\mathbf{r}}'(t) \cdot \vec{\mathbf{r}}(t) + \vec{\mathbf{r}}(t) \cdot \vec{\mathbf{r}}'(t) &= 0 \\ 2\vec{\mathbf{r}}(t) \cdot \vec{\mathbf{r}}'(t) &= 0 \\ \vec{\mathbf{r}}(t) \cdot \vec{\mathbf{r}}'(t) &= 0\end{aligned}$$

Now for some examples using these properties.

Example 12.2.5: Using the Properties of Derivatives of Vector-Valued Functions

Given the vector-valued functions

$$\vec{\mathbf{r}}(t) = (6t + 8)\,\hat{\mathbf{i}} + (4t^2 + 2t - 3)\,\hat{\mathbf{j}} + 5t\,\hat{\mathbf{k}}$$

and

$$\vec{\mathbf{u}}(t) = (t^2 - 3)\,\hat{\mathbf{i}} + (2t + 4)\,\hat{\mathbf{j}} + (t^3 - 3t)\,\hat{\mathbf{k}},$$

calculate each of the following derivatives using the properties of the derivative of vector-valued functions.

a. $\frac{d}{dt}[\vec{\mathbf{r}}(t) \cdot \vec{\mathbf{u}}(t)]$

b. $\frac{d}{dt}[\vec{\mathbf{u}}(t) \times \vec{\mathbf{u}}'(t)]$

Solution

We have $\vec{\mathbf{r}}'(t) = 6\,\hat{\mathbf{i}} + (8t+2)\,\hat{\mathbf{j}} + 5\,\hat{\mathbf{k}}$ and $\vec{\mathbf{u}}'(t) = 2t\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + (3t^2-3)\,\hat{\mathbf{k}}$. Therefore, according to **property iv**:

1.
$$\begin{aligned}\frac{d}{dt}[\vec{\mathbf{r}}(t)\cdot\vec{\mathbf{u}}(t)] &= \vec{\mathbf{r}}'(t)\cdot\vec{\mathbf{u}}(t) + \vec{\mathbf{r}}(t)\cdot\vec{\mathbf{u}}'(t)\\ &= (6\,\hat{\mathbf{i}} + (8t+2)\,\hat{\mathbf{j}} + 5\,\hat{\mathbf{k}})\cdot((t^2-3)\,\hat{\mathbf{i}} + (2t+4)\,\hat{\mathbf{j}} + (t^3-3t)\,\hat{\mathbf{k}})\\ &+((6t+8)\,\hat{\mathbf{i}} + (4t^2+2t-3)\,\hat{\mathbf{j}} + 5t\,\hat{\mathbf{k}})\cdot(2t\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + (3t^2-3)\,\hat{\mathbf{k}})\\ &= 6(t^2-3) + (8t+2)(2t+4) + 5(t^3-3t)\\ &+2t(6t+8) + 2(4t^2+2t-3) + 5t(3t^2-3)\\ &= 20t^3 + 42t^2 + 26t - 16.\end{aligned}$$

2. First, we need to adapt **property v** for this problem:

$$\frac{d}{dt}[\vec{\mathbf{u}}(t)\times\vec{\mathbf{u}}'(t)] = \vec{\mathbf{u}}'(t)\times\vec{\mathbf{u}}'(t) + \vec{\mathbf{u}}(t)\times\vec{\mathbf{u}}''(t).$$

Recall that the cross product of any vector with itself is zero. Furthermore, $\vec{\mathbf{u}}''(t)$ represents the second derivative of $\vec{\mathbf{u}}(t)$:

$$\vec{\mathbf{u}}''(t) = \frac{d}{dt}[\vec{\mathbf{u}}'(t)] = \frac{d}{dt}[2t\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + (3t^2-3)\,\hat{\mathbf{k}}] = 2\,\hat{\mathbf{i}} + 6t\,\hat{\mathbf{k}}.$$

Therefore,

$$\begin{aligned}\frac{d}{dt}[\vec{\mathbf{u}}(t)\times\vec{\mathbf{u}}'(t)] &= 0 + ((t^2-3)\,\hat{\mathbf{i}} + (2t+4)\,\hat{\mathbf{j}} + (t^3-3t)\,\hat{\mathbf{k}})\times(2\,\hat{\mathbf{i}} + 6t\,\hat{\mathbf{k}})\\ &= \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ t^2-3 & 2t+4 & t^3-3t\\ 2 & 0 & 6t\end{vmatrix}\\ &= 6t(2t+4)\,\hat{\mathbf{i}} - (6t(t^2-3) - 2(t^3-3t))\,\hat{\mathbf{j}} - 2(2t+4)\,\hat{\mathbf{k}}\\ &= (12t^2+24t)\,\hat{\mathbf{i}} + (12t-4t^3)\,\hat{\mathbf{j}} - (4t+8)\,\hat{\mathbf{k}}.\end{aligned}$$

Exercise 12.2.3

Calculate $\frac{d}{dt}[\vec{\mathbf{r}}(t)\cdot\vec{\mathbf{r}}'(t)]$ and $\frac{d}{dt}[\vec{\mathbf{u}}(t)\times\vec{\mathbf{r}}(t)]$ for the vector-valued functions:

- $\vec{\mathbf{r}}(t) = \cos t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}} - e^{2t}\,\hat{\mathbf{k}}$
- $\vec{\mathbf{u}}(t) = t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}} + \cos t\,\hat{\mathbf{k}}$,

Tangent Vectors and Unit Tangent Vectors

Recall that the derivative at a point can be interpreted as the slope of the tangent line to the graph at that point. In the case of a vector-valued function, the derivative provides a tangent vector to the curve represented by the function. Consider the vector-valued function

$$\vec{\mathbf{r}}(t) = \cos t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}} \tag{12.2.14}$$

The derivative of this function is

$$\vec{\mathbf{r}}'(t) = -\sin t\,\hat{\mathbf{i}} + \cos t\,\hat{\mathbf{j}}$$

If we substitute the value $t = \pi/6$ into both functions we get

$$\vec{\mathbf{r}}\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}\,\hat{\mathbf{i}} + \frac{1}{2}\,\hat{\mathbf{j}}$$

and

$$\vec{\mathbf{r}}'\left(\frac{\pi}{6}\right) = -\frac{1}{2}\hat{\mathbf{i}} + \frac{\sqrt{3}}{2}\hat{\mathbf{j}}.$$

The graph of this function appears in Figure 12.2.8, along with the vectors $\vec{\mathbf{r}}\left(\frac{\pi}{6}\right)$ and $\vec{\mathbf{r}}'\left(\frac{\pi}{6}\right)$.

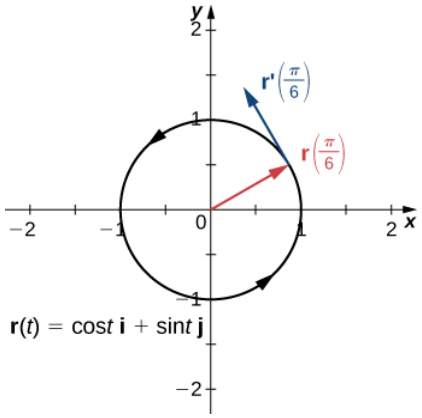

Figure 12.2.8: The tangent line at a point is calculated from the derivative of the vector-valued function $\vec{\mathbf{r}}(t)$.

Notice that the vector $\vec{\mathbf{r}}'\left(\frac{\pi}{6}\right)$ is tangent to the circle at the point corresponding to $t = \frac{\pi}{6}$. This is an example of a tangent vector to the plane curve defined by Equation 12.2.14.

Definition: principal unit tangent vector

Let C be a curve defined by a vector-valued function $\vec{\mathbf{r}}$, and assume that $\vec{\mathbf{r}}'(t)$ exists when $t = t_0$ A tangent vector $\vec{\mathbf{r}}$ at $t = t_0$ is any vector such that, when the tail of the vector is placed at point $\vec{\mathbf{r}}(t_0)$ on the graph, vector $\vec{\mathbf{r}}$ is tangent to curve C. Vector $\vec{\mathbf{r}}'(t_0)$ is an example of a tangent vector at point $t = t_0$. Furthermore, assume that $\vec{\mathbf{r}}'(t) \neq \mathbf{0}$. The principal unit tangent vector at t is defined to be

$$\vec{\mathbf{T}}(t) = \frac{\vec{\mathbf{r}}'(t)}{\|\vec{\mathbf{r}}'(t)\|}, \tag{12.2.15}$$

provided $\|\vec{\mathbf{r}}'(t)\| \neq 0$.

The unit tangent vector is exactly what it sounds like: a unit vector that is tangent to the curve. To calculate a unit tangent vector, first find the derivative $\vec{\mathbf{r}}'(t)$. Second, calculate the magnitude of the derivative. The third step is to divide the derivative by its magnitude.

Example 12.2.6: Finding a Unit Tangent Vector

Find the unit tangent vector for each of the following vector-valued functions:

a. $\vec{\mathbf{r}}(t) = \cos t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}}$
b. $\vec{\mathbf{u}}(t) = (3t^2 + 2t)\,\hat{\mathbf{i}} + (2 - 4t^3)\,\hat{\mathbf{j}} + (6t + 5)\,\hat{\mathbf{k}}$

Solution

a.

$$\begin{aligned}
\text{First step:} \quad \vec{\mathbf{r}}'(t) &= -\sin t\,\hat{\mathbf{i}} + \cos t\,\hat{\mathbf{j}} \\
\text{Second step:} \quad \|\vec{\mathbf{r}}'(t)\| &= \sqrt{(-\sin t)^2 + (\cos t)^2} = 1 \\
\text{Third step:} \quad \vec{\mathbf{T}}(t) &= \frac{\vec{\mathbf{r}}'(t)}{\|\vec{\mathbf{r}}'(t)\|} = \frac{-\sin t\,\hat{\mathbf{i}} + \cos t\,\hat{\mathbf{j}}}{1} = -\sin t\,\hat{\mathbf{i}} + \cos t\,\hat{\mathbf{j}}
\end{aligned}$$

b.

$$\begin{aligned}
\textbf{First step:} \quad \vec{\mathbf{r}}'(t) &= (6t+2)\,\hat{\mathbf{i}} - 12t^2\,\hat{\mathbf{j}} + 6\,\hat{\mathbf{k}} \\
\textbf{Second step:} \quad \|\vec{\mathbf{r}}'(t)\| &= \sqrt{(6t+2)^2 + (-12t^2)^2 + 6^2} \\
&= \sqrt{144t^4 + 36t^2 + 24t + 40} \\
&= 2\sqrt{36t^4 + 9t^2 + 6t + 10} \\
\textbf{Third step:} \quad \vec{\mathbf{T}}(t) &= \frac{\vec{\mathbf{r}}'(t)}{\|\vec{\mathbf{r}}'(t)\|} = \frac{(6t+2)\,\hat{\mathbf{i}} - 12t^2\,\hat{\mathbf{j}} + 6\,\hat{\mathbf{k}}}{2\sqrt{36t^4 + 9t^2 + 6t + 10}} \\
&= \frac{3t+1}{\sqrt{36t^4 + 9t^2 + 6t + 10}}\,\hat{\mathbf{i}} - \frac{6t^2}{\sqrt{36t^4 + 9t^2 + 6t + 10}}\,\hat{\mathbf{j}} + \frac{3}{\sqrt{36t^4 + 9t^2 + 6t + 10}}\,\hat{\mathbf{k}}
\end{aligned}$$

Exercise 12.2.4

Find the unit tangent vector for the vector-valued function

$$\vec{\mathbf{r}}(t) = (t^2 - 3)\,\hat{\mathbf{i}} + (2t+1)\,\hat{\mathbf{j}} + (t-2)\,\hat{\mathbf{k}}.$$

Integrals of Vector-Valued Functions

We introduced antiderivatives of real-valued functions in Antiderivatives and definite integrals of real-valued functions in The Definite Integral. Each of these concepts can be extended to vector-valued functions. Also, just as we can calculate the derivative of a vector-valued function by differentiating the component functions separately, we can calculate the antiderivative in the same manner. Furthermore, the Fundamental Theorem of Calculus applies to vector-valued functions as well.

The antiderivative of a vector-valued function appears in applications. For example, if a vector-valued function represents the velocity of an object at time t, then its antiderivative represents position. Or, if the function represents the acceleration of the object at a given time, then the antiderivative represents its velocity.

Definition: Definite and indefinite integrals of vector-valued functions

Let f, g, and h be integrable real-valued functions over the closed interval $[a, b]$.

1. The indefinite integral of a vector-valued function $\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}}$ is

$$\int [f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}}]\,dt = \left[\int f(t)\,dt\right]\hat{\mathbf{i}} + \left[\int g(t)\,dt\right]\hat{\mathbf{j}}. \tag{12.2.16}$$

   The definite integral of a vector-valued function is

$$\int_a^b [f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}}]\,dt = \left[\int_a^b f(t)\,dt\right]\hat{\mathbf{i}} + \left[\int_a^b g(t)\,dt\right]\hat{\mathbf{j}}. \tag{12.2.17}$$

2. The indefinite integral of a vector-valued function $\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} + h(t)\,\hat{\mathbf{k}}$ is

$$\begin{aligned}
&\int [f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} + h(t)\,\hat{\mathbf{k}}]\,dt \\
&= \left[\int f(t)\,dt\right]\hat{\mathbf{i}} + \left[\int g(t)\,dt\right]\hat{\mathbf{j}} + \left[\int h(t)\,dt\right]\hat{\mathbf{k}}.
\end{aligned} \tag{12.2.18}$$

   The definite integral of the vector-valued function is

$$\begin{aligned}
&\int_a^b [f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} + h(t)\,\hat{\mathbf{k}}]\,dt \\
&= \left[\int_a^b f(t)\,dt\right]\hat{\mathbf{i}} + \left[\int_a^b g(t)\,dt\right]\hat{\mathbf{j}} + \left[\int_a^b h(t)\,dt\right]\hat{\mathbf{k}}.
\end{aligned} \tag{12.2.19}$$

Since the indefinite integral of a vector-valued function involves indefinite integrals of the component functions, each of these component integrals contains an integration constant. They can all be different. For example, in the two-dimensional case, we can have

$$\int f(t)\,dt = F(t) + C_1 \text{ and } \int g(t)\,dt = G(t) + C_2,$$

where F and G are antiderivatives of f and g, respectively. Then

$$\begin{aligned}\int [f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}}]\,dt &= \left[\int f(t)\,dt\right]\hat{\mathbf{i}} + \left[\int g(t)\,dt\right]\hat{\mathbf{j}} \\ &= (F(t)+C_1)\,\hat{\mathbf{i}} + (G(t)+C_2)\,\hat{\mathbf{j}} \\ &= F(t)\,\hat{\mathbf{i}} + G(t)\,\hat{\mathbf{j}} + C_1\,\hat{\mathbf{i}} + C_2\,\hat{\mathbf{j}} \\ &= F(t)\,\hat{\mathbf{i}} + G(t)\,\hat{\mathbf{j}} + \vec{\mathbf{C}}\end{aligned}$$

where $\vec{\mathbf{C}} = C_1\,\hat{\mathbf{i}} + C_2\,\hat{\mathbf{j}}$. Therefore, the *integration constants* becomes a *constant vector*.

Example 12.2.7: Integrating Vector-Valued Functions

Calculate each of the following integrals:

a. $\displaystyle\int [(3t^2+2t)\,\hat{\mathbf{i}} + (3t-6)\,\hat{\mathbf{j}} + (6t^3+5t^2-4)\,\hat{\mathbf{k}}]\,dt$

b. $\displaystyle\int [\langle t, t^2, t^3\rangle \times \langle t^3, t^2, t\rangle]\,dt$

c. $\displaystyle\int_0^{\frac{\pi}{3}} [\sin 2t\,\hat{\mathbf{i}} + \tan t\,\hat{\mathbf{j}} + e^{-2t}\,\hat{\mathbf{k}}]\,dt$

Solution

a. We use the first part of the definition of the integral of a space curve:

$$\begin{aligned}\int [(3t^2+2t)\,\hat{\mathbf{i}} + (3t-6)\,\hat{\mathbf{j}} &+ (6t^3+5t^2-4)\,\hat{\mathbf{k}}]\,dt = \left[\int 3t^2+2t\,dt\right]\hat{\mathbf{i}} + \left[\int 3t-6\,dt\right]\hat{\mathbf{j}} \\ &+ \left[\int 6t^3+5t^2-4\,dt\right]\hat{\mathbf{k}} \\ &= (t^3+t^2)\,\hat{\mathbf{i}} + \left(\frac{3}{2}t^2-6t\right)\hat{\mathbf{j}} \\ &+ \left(\frac{3}{2}t^4+\frac{5}{3}t^3-4t\right)\hat{\mathbf{k}} + \vec{\mathbf{C}}.\end{aligned}$$

b. First calculate $\langle t, t^2, t^3\rangle \times \langle t^3, t^2, t\rangle$:

$$\begin{aligned}\langle t, t^2, t^3\rangle \times \langle t^3, t^2, t\rangle &= \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ t & t^2 & t^3 \\ t^3 & t^2 & t\end{vmatrix} \\ &= (t^2(t) - t^3(t^2))\,\hat{\mathbf{i}} - (t^2 - t^3(t^3))\,\hat{\mathbf{j}} + (t(t^2) - t^2(t^3))\,\hat{\mathbf{k}} \\ &= (t^3 - t^5)\,\hat{\mathbf{i}} + (t^6 - t^2)\,\hat{\mathbf{j}} + (t^3 - t^5)\,\hat{\mathbf{k}}.\end{aligned}$$

Next, substitute this back into the integral and integrate:

$$\begin{aligned}\int [\langle t, t^2, t^3\rangle \times \langle t^3, t^2, t\rangle]\,dt &= \int (t^3 - t^5)\,\hat{\mathbf{i}} + (t^6 - t^2)\,\hat{\mathbf{j}} + (t^3 - t^5)\,\hat{\mathbf{k}}\,dt \\ &= \left(\frac{t^4}{4} - \frac{t^6}{6}\right)\hat{\mathbf{i}} + \left(\frac{t^7}{7} - \frac{t^3}{3}\right)\hat{\mathbf{j}} + \left(\frac{t^4}{4} - \frac{t^6}{6}\right)\hat{\mathbf{k}} + \vec{\mathbf{C}}.\end{aligned}$$

c. Use the second part of the definition of the integral of a space curve:

$$\int_0^{\frac{\pi}{3}} [\sin 2t\,\hat{\mathbf{i}} + \tan t\,\hat{\mathbf{j}} + e^{-2t}\hat{\mathbf{k}}]\,dt = \left[\int_0^{\frac{\pi}{3}} \sin 2t\,dt\right]\hat{\mathbf{i}} + \left[\int_0^{\frac{\pi}{3}} \tan t\,dt\right]\hat{\mathbf{j}} + \left[\int_0^{\frac{\pi}{3}} e^{-2t}\,dt\right]\hat{\mathbf{k}}$$

$$= \left(-\tfrac{1}{2}\cos 2t\right)\Big|_0^{\pi/3}\hat{\mathbf{i}} - (\ln|\cos t|)\Big|_0^{\pi/3}\hat{\mathbf{j}} - \left(\tfrac{1}{2}e^{-2t}\right)\Big|_0^{\pi/3}\hat{\mathbf{k}}$$

$$= \left(-\tfrac{1}{2}\cos\tfrac{2\pi}{3} + \tfrac{1}{2}\cos 0\right)\hat{\mathbf{i}} - \left(\ln\left(\cos\tfrac{\pi}{3}\right) - \ln(\cos 0)\right)\hat{\mathbf{j}} - \left(\tfrac{1}{2}e^{-2\pi/3} - \tfrac{1}{2}e^{-2(0)}\right)\hat{\mathbf{k}}$$

$$= \left(\tfrac{1}{4} + \tfrac{1}{2}\right)\hat{\mathbf{i}} - (-\ln 2)\hat{\mathbf{j}} - \left(\tfrac{1}{2}e^{-2\pi/3} - \tfrac{1}{2}\right)\hat{\mathbf{k}}$$

$$= \tfrac{3}{4}\hat{\mathbf{i}} + (\ln 2)\hat{\mathbf{j}} + \left(\tfrac{1}{2} - \tfrac{1}{2}e^{-2\pi/3}\right)\hat{\mathbf{k}}.$$

Exercise 12.2.5

Calculate the following integral:

$$\int_1^3 [(2t+4)\,\hat{\mathbf{i}} + (3t^2 - 4t)\,\hat{\mathbf{j}}]\,dt$$

Integration in the Context of Motion

Above, we defined velocity as the derivative of position and acceleration as the derivative of velocity. Integration allows us to go the other way! As you learned in first semester calculus, integration allows us to generate an object's velocity as a function of time given its acceleration and an initial velocity. We can also find the object's position function by integrating the velocity and using an initial position.

Let's consider an example.

Example 12.2.8: Finding an object's velocity from its acceleration

An object has an acceleration over time given by $\vec{\mathbf{a}}(t) = \sin t\,\hat{\mathbf{i}} - 2t\,\hat{\mathbf{j}}$, and it's initial velocity was $\vec{\mathbf{v}}(0) = 2\,\hat{\mathbf{i}} + \hat{\mathbf{j}}$.

a. Find the object's velocity as a function of time.

b. Assuming the object was located at the point (2, 3, -1) when time $t = 0$, determine the object's position function and find its location at time $t = 3$ sec.

Solution

a. First, we find the velocity as the antiderivative of the acceleration.

$$\vec{\mathbf{v}}(t) = \int \vec{\mathbf{a}}(t)\,dt = \int \left(\sin t\,\hat{\mathbf{i}} - 2t\,\hat{\mathbf{j}}\right)\,dt = -\cos t\,\hat{\mathbf{i}} - t^2\,\hat{\mathbf{j}} + \vec{\mathbf{C}}_1$$

Now we use the initial velocity to determine $\vec{\mathbf{C}}_1$.

$$\vec{\mathbf{v}}(0) = -\cos 0\,\hat{\mathbf{i}} - (0)^2\,\hat{\mathbf{j}} + \vec{\mathbf{C}}_1 = 2\,\hat{\mathbf{i}} + \hat{\mathbf{j}}$$

$$-\hat{\mathbf{i}} + \vec{\mathbf{C}}_1 = 2\,\hat{\mathbf{i}} + \hat{\mathbf{j}}$$

$$\text{And so} \quad \vec{\mathbf{C}}_1 = 3\,\hat{\mathbf{i}} + \hat{\mathbf{j}}$$

Incorporating this constant vector into our velocity function from above, we obtain the velocity describing this object's motion over time:

$$\vec{\mathbf{v}}(t) = (3 - \cos t)\,\hat{\mathbf{i}} + (1 - t^2)\,\hat{\mathbf{j}} \tag{12.2.20}$$

b. Since the object's position is at the point (2, 3, -1) when time $t = 0$, we know $\vec{\mathbf{r}}(0) = 2\,\hat{\mathbf{i}} + 3\,\hat{\mathbf{j}} - \hat{\mathbf{k}}$.

Now, to determine the object's position function, we integrate it's velocity.

$$\begin{aligned}\vec{\mathbf{r}}(t) = \int \vec{\mathbf{v}}(t)\, dt &= \int \left[(3 - \cos t)\, \hat{\mathbf{i}} + (1 - t^2)\, \hat{\mathbf{j}} \right] dt \\ &= (3t - \sin t)\, \hat{\mathbf{i}} + \left(t - \frac{t^3}{3} \right) \hat{\mathbf{j}} + \vec{\mathbf{C}}_2\end{aligned}$$

Now we use the initial position to determine $\vec{\mathbf{C}}_2$.

$$\vec{\mathbf{r}}(0) = (3(0) - \sin 0)\, \hat{\mathbf{i}} + \left(0 - \frac{(0)^3}{3} \right) \hat{\mathbf{j}} + \vec{\mathbf{C}}_2 = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}}$$

$$\text{And so} \quad \vec{\mathbf{C}}_2 = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}}$$

Incorporating this constant vector into our position function from above, we obtain:

$$\vec{\mathbf{r}}(t) = (2 + 3t - \sin t)\, \hat{\mathbf{i}} + \left(3 + t - \frac{t^3}{3} \right) \hat{\mathbf{j}} - \hat{\mathbf{k}} \tag{12.2.21}$$

To find the object's position at time $t = 3$ seconds, we just evaluate this position function at $t = 3$.

$$\begin{aligned}\vec{\mathbf{r}}(3) &= (2 + 3(3) - \sin 3)\, \hat{\mathbf{i}} + \left(3 + 3 - \frac{(3)^3}{3} \right) \hat{\mathbf{j}} - \hat{\mathbf{k}} \\ &= (11 - \sin 3)\, \hat{\mathbf{i}} - 3\hat{\mathbf{j}} - \hat{\mathbf{k}}\end{aligned}$$

This position vector indicates that the object will be located at the point $(11 - \sin 3, -3, -1)$ at time $t = 3$ seconds.

The Meaning of Definite Integrals of Vector-Valued Functions

Now let's turn our attention to the meaning of a definite integral of a vector-valued function. The context in which this will make the most sense is where the function we integrate is a velocity function. That is,

$$\int_a^b \vec{\mathbf{v}}(t)\, dt \tag{12.2.22}$$

We know that the antiderivative of velocity is position and that this definite integral gives us the change in position over the time interval, $a \le t \le b$. In other words,

$$\int_a^b \vec{\mathbf{v}}(t)\, dt = \vec{\mathbf{r}}(b) - \vec{\mathbf{r}}(a) \tag{12.2.23}$$

Thus, the definite integral of velocity over a time interval $a \le t \le b$ gives us the **displacement vector** that indicates the change in position over this time interval.

In general, the definite integral

$$\int_a^b \vec{\mathbf{r}}(t)\, dt = \vec{\mathbf{q}}(b) - \vec{\mathbf{q}}(a), \text{ where } \vec{\mathbf{q}}(t) \text{ is the antiderivative of } \vec{\mathbf{r}}(t),$$

gives us a change in the antiderivative of our vector-valued function over the given inteval $[a, b]$. This will always be a constant vector that would fit from tip to tip of the vectors given by the antiderivative function at $t = a$ and $t = b$, respectively (assuming the vectors were placed in standard position).

Summary

- To calculate the derivative of a vector-valued function, calculate the derivatives of the component functions, then put them back into a new vector-valued function.
- Many of the properties of differentiation of scalar functions also apply to vector-valued functions.
- The derivative of a vector-valued function $\vec{\mathbf{r}}(t)$ is also a tangent vector to the curve. The unit tangent vector $\vec{\mathbf{T}}(t)$ is calculated by dividing the derivative of a vector-valued function by its magnitude.
- The antiderivative of a vector-valued function is found by finding the antiderivatives of the component functions, then putting them back together in a vector-valued function.

- The definite integral of a vector-valued function is found by finding the definite integrals of the component functions, then putting them back together in a vector-valued function.

Key Equations

- **Derivative of a vector-valued function**

$$\vec{\mathbf{r}}'(t) = \lim_{\Delta t \to 0} \frac{\vec{\mathbf{r}}(t + \Delta t) - \vec{\mathbf{r}}(t)}{\Delta t}$$

- **Principal unit tangent vector**

$$\vec{\mathbf{T}}(t) = \frac{\vec{\mathbf{r}}'(t)}{\|\vec{\mathbf{r}}'(t)\|} \tag{12.2.24}$$

- **Indefinite integral of a vector-valued function**

$$\int [f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} + h(t)\,\hat{\mathbf{k}}]\,dt = \left[\int f(t)\,dt\right]\hat{\mathbf{i}} + \left[\int g(t)\,dt\right]\hat{\mathbf{j}} + \left[\int h(t)\,dt\right]\hat{\mathbf{k}}$$

- **Definite integral of a vector-valued function**

$$\int_a^b [f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} + h(t)\,\hat{\mathbf{k}}]\,dt = \left[\int_a^b f(t)\,dt\right]\hat{\mathbf{i}} + \left[\int_a^b g(t)\,dt\right]\hat{\mathbf{j}} + \left[\int_a^b h(t)\,dt\right]\hat{\mathbf{k}}$$

Glossary

Contributors

- Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.
- Paul Seeburger (Monroe Community College) edited this section and added content on visualizing the derivative of a vector-valued function and content on smooth vector-valued functions.

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12.2E: Exercises for Section 12.2

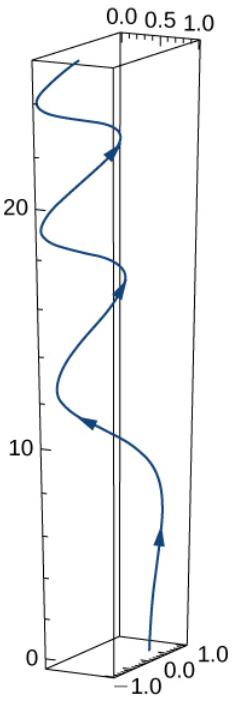
Taking Derivatives of Vector-Valued Functions

In questions 1 - 10, compute the derivative of each vector-valued function.

1) $\vec{\mathbf{r}}(t) = t^3\,\hat{\mathbf{i}} + 3t^2\,\hat{\mathbf{j}} + \dfrac{t^3}{6}\,\hat{\mathbf{k}}$

2) $\vec{\mathbf{r}}(t) = \sin(t)\,\hat{\mathbf{i}} + \cos(t)\,\hat{\mathbf{j}} + e^t\,\hat{\mathbf{k}}$

3) $\vec{\mathbf{r}}(t) = e^{-t}\,\hat{\mathbf{i}} + \sin(3t)\,\hat{\mathbf{j}} + 10\sqrt{t}\,\hat{\mathbf{k}}$. A sketch of the graph is shown here. Notice the varying periodic nature of the graph.



4) $\vec{\mathbf{r}}(t) = e^t\,\hat{\mathbf{i}} + 2e^t\,\hat{\mathbf{j}} + \hat{\mathbf{k}}$

5) $\vec{\mathbf{r}}(t) = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$

6) $\vec{\mathbf{r}}(t) = te^t\,\hat{\mathbf{i}} + t\ln(t)\,\hat{\mathbf{j}} + \sin(3t)\,\hat{\mathbf{k}}$

7) $\vec{\mathbf{r}}(t) = \left\langle \dfrac{1}{t+1},\ \arctan(t),\ \ln t^3 \right\rangle$

8) $\vec{\mathbf{r}}(t) = \langle \tan 2t,\ \sec 2t,\ \sin^2 t \rangle$

9) $\vec{\mathbf{r}}(t) = \langle 3,\ 4\sin(3t),\ t\cos(t) \rangle$

10) $\vec{\mathbf{r}}(t) = t^2\,\hat{\mathbf{i}} + te^{-2t}\,\hat{\mathbf{j}} - 5e^{-4t}\,\hat{\mathbf{k}}$

11) a. Describe and sketch the curve represented by the vector-valued function $\vec{\mathbf{r}}(t) = \langle 6t, 6t - t^2 \rangle$.

b. Locate the highest point on the curve $\vec{\mathbf{r}}(t) = \langle 6t, 6t - t^2 \rangle$ and give the value of the function at this point.

12) Find the parametric equations of the tangent line to the curve $\vec{\mathbf{r}}(t) = \langle t, t^2, t \rangle$ at $t = 2$.

13) Find the parametric equations of the tangent line to the curve $\vec{\mathbf{r}}(t) = \langle e^t, e^{-t}, 0 \rangle$ at $t = 0$.

14) Compute the first, second, and third derivatives of $\vec{\mathbf{r}}(t) = 3t\,\hat{\mathbf{i}} + 6\ln(t)\,\hat{\mathbf{j}} + 5e^{-3t}\,\hat{\mathbf{k}}$.

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Describing Motion with Vector-Valued Functions

In questions 15 - 17, find the velocity and acceleration at the given times, plot the graph of the position function, and draw in the velocity and acceleration vectors at the corresponding locations on the curve.

15) $\vec{\mathbf{r}}(t) = t\,\hat{\mathbf{i}} + t^3\,\hat{\mathbf{j}}$, at $t = 0$ and at $t = 1$

16) $\vec{\mathbf{r}}(t) = \cos t\,\hat{\mathbf{i}} + \sin 3t\,\hat{\mathbf{j}}$, at $t = 0$, at $t = \frac{\pi}{4}$, and at $t = \frac{\pi}{2}$

17) $\vec{\mathbf{r}}(t) = \ln t\,\hat{\mathbf{i}} + (t-2)\,\hat{\mathbf{j}}$, at $t = 1$ and at $t = 2$

In questions 18 - 24, find the velocity, speed, and acceleration of a particle with the given position function. Remember that the speed is the magnitude of the velocity represented by $\|\vec{\mathbf{v}}(t)\|$ or $\|\vec{\mathbf{r}}'(t)\|$.

18) $\vec{\mathbf{r}}(t) = e^{2t}\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}}$

19) $\vec{\mathbf{r}}(t) = \cos t^3\,\hat{\mathbf{i}} + \sin t^3\,\hat{\mathbf{j}}$

20) $\vec{\mathbf{r}}(t) = \langle e^t, e^{-t}, 0\rangle$

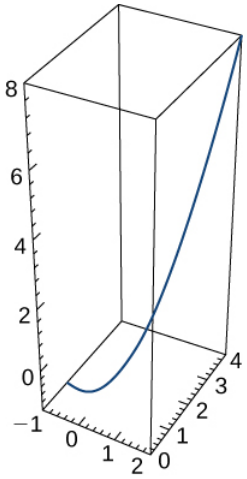
21) $\vec{\mathbf{r}}(t) = \langle t + \cos t, t - \sin t\rangle$

22) $\vec{\mathbf{r}}(t) = \dfrac{2t-1}{2t+1}\,\hat{\mathbf{i}} + \ln(1-4t^2)\,\hat{\mathbf{j}}$

23) $\vec{\mathbf{r}}(t) = \cos 3t\,\hat{\mathbf{i}} + \sin 3t\,\hat{\mathbf{j}} + 0.5t\,\hat{\mathbf{k}}$

24) $\vec{\mathbf{r}}(t) = e^{-t}\,\hat{\mathbf{i}} + (\ln t)\,\hat{\mathbf{j}} + (\sin 7t)\,\hat{\mathbf{k}}$

25) Consider the position vector for a particle to be $\vec{\mathbf{r}}(t) = t\,\hat{\mathbf{i}} + t^2\,\hat{\mathbf{j}} + t^3\,\hat{\mathbf{k}}$. The graph is shown here:



a. Find the velocity vector at any time.

b. Find the speed of the particle at time $t = 2$ sec.

c. Find the acceleration at time $t = 2$ sec.

26) A particle travels along the path of an ellipse with the equation $\vec{\mathbf{r}}(t) = \cos t\,\hat{\mathbf{i}} + 2\sin t\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}$. Find the following:

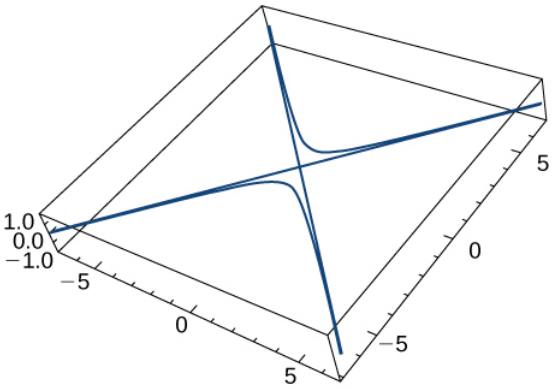
a. Velocity of the particle

b. Speed of the particle at $t = \frac{\pi}{4}$

c. Acceleration of the particle at $t = \frac{\pi}{4}$

27) Show that if the speed of a particle traveling along a curve represented by a vector-valued function is constant, then the velocity function is always perpendicular to the acceleration function.

28) Given the vector-valued function $\vec{\mathbf{r}}(t) = \langle \tan t, \sec t, 0\rangle$ (graph is shown here), find the following:



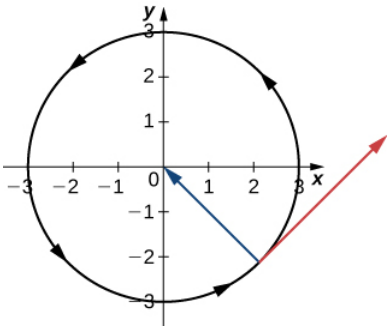
a. Velocity

b. Speed

c. Acceleration

29) Find the minimum speed of a particle traveling along the curve $\vec{\mathbf{r}}(t) = \langle t + \cos t, t - \sin t \rangle$, where $t \in [0, 2\pi)$. Then also find its maximum speed on this interval.

For questions 30 - 31, consider a particle that moves on a circular path of radius b according to the function $\vec{\mathbf{r}}(t) = b\cos(\omega t)\,\hat{\mathbf{i}} + b\sin(\omega t)\,\hat{\mathbf{j}}$, where ω is the angular velocity, $\frac{d\theta}{dt}$.



30) Show that the speed of the particle is proportional to the angular velocity.

31) Find the velocity function and show that $\vec{\mathbf{v}}(t)$ is always orthogonal to $\vec{\mathbf{r}}(t)$.

Smoothness of Vector-Valued Functions

For questions 32 - 40,

a. Determine any values of t at which $\vec{\mathbf{r}}$ is not smooth.

b. Determine the open intervals on which $\vec{\mathbf{r}}$ is smooth.

c. Graph the vector-valued function and describe its behavior at the points where it is not smooth.

32) $\vec{\mathbf{r}}(t) = \langle 3t, 5t^2 - 1 \rangle$

33) $\vec{\mathbf{r}}(t) = t^3\,\hat{\mathbf{i}} + 5t^2\,\hat{\mathbf{j}}$

34) $\vec{\mathbf{r}}(t) = \langle 5,\, 2\sin(t),\, \cos(t) \rangle$

35) $\vec{\mathbf{r}}(t) = \langle t^3 - 3t^2, 7 \rangle$

36) $\vec{\mathbf{r}}(t) = t^2\,\hat{\mathbf{i}} + t^3\,\hat{\mathbf{j}} - 5e^{-4t}\,\hat{\mathbf{k}}$

37) $\vec{\mathbf{r}}(t) = \left\langle \ln(t^2 + 4t + 5),\, \frac{t^3}{3} - 4t,\, 5 \right\rangle$

38) $\vec{\mathbf{r}}(t) = (5\cos t - \cos 5t)\,\hat{\mathbf{i}} + (5\sin t - \sin 5t)\,\hat{\mathbf{j}}$, for $0 \le t \le 2\pi$

39) $\vec{\mathbf{r}}(t) = \sqrt{t^3 + 9t^2}\,\hat{\mathbf{i}} + (t^2 + 12t)\,\hat{\mathbf{j}} + 7\hat{\mathbf{k}}$

40) $\vec{\mathbf{r}}(t) = \cos^3 t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}}$, for $0 \le t \le 2\pi$

Properties of the Derivative

For questions, 41 - 43, evaluate each expression given that $\vec{\mathbf{r}}(t) = t\,\hat{\mathbf{i}} + t^2\,\hat{\mathbf{j}} - t^4\,\hat{\mathbf{k}}$ and $\vec{\mathbf{s}}(t) = \sin(t)\,\hat{\mathbf{i}} + e^t\,\hat{\mathbf{j}} + \cos(t)\,\hat{\mathbf{k}}$

41) $\dfrac{d}{dt}\left[\vec{\mathbf{r}}(t^2)\right]$

42) $\dfrac{d}{dt}\left[t^2 \cdot \vec{\mathbf{s}}(t)\right]$

43) $\dfrac{d}{dt}\left[\vec{\mathbf{r}}(t) \cdot \vec{\mathbf{s}}(t)\right]$

44) Find $\vec{\mathbf{r}}'(t) \cdot \vec{\mathbf{r}}''(t)$ *for* $\vec{\mathbf{r}}(t) = -3t^5\,\hat{\mathbf{i}} + 5t\,\hat{\mathbf{j}} + 2t^2\,\hat{\mathbf{k}}$.

45) Given $\vec{\mathbf{r}}(t) = t\,\hat{\mathbf{i}} + 3t\,\hat{\mathbf{j}} + t^2\,\hat{\mathbf{k}}$ and $\vec{\mathbf{u}}(t) = 4t\,\hat{\mathbf{i}} + t^2\,\hat{\mathbf{j}} + t^3\,\hat{\mathbf{k}}$, find $\frac{d}{dt}(\vec{\mathbf{r}}(t) \times \vec{\mathbf{u}}(t))$.

46) Evaluate $\dfrac{d}{dt}\left[\vec{\mathbf{u}}(t) \times \vec{\mathbf{u}}'(t)\right]$ given $\vec{\mathbf{u}}(t) = t^2\,\hat{\mathbf{i}} - 2t\,\hat{\mathbf{j}} + \hat{\mathbf{k}}$.

47) Given $\vec{\mathbf{r}}(t) = t\,\hat{\mathbf{i}} + 2\sin t\,\hat{\mathbf{j}} + 2\cos t\,\hat{\mathbf{k}}$ and $\vec{\mathbf{u}}(t) = \dfrac{1}{t}\,\hat{\mathbf{i}} + 2\sin t\,\hat{\mathbf{j}} + 2\cos t\,\hat{\mathbf{k}}$, find the following:

a. $\vec{\mathbf{r}}(t) \times \vec{\mathbf{u}}(t)$

b. $\frac{d}{dt}\left(\vec{\mathbf{r}}(t) \times \vec{\mathbf{u}}(t)\right)$

c. Now, use the product rule for the derivative of the cross product of two vectors and show this result is the same as the answer for the preceding problem.

Unit Tangent Vectors

For questions 48 - 51, find a *unit tangent vector* at the indicated value of t.

48) $\vec{\mathbf{r}}(t) = 3t^3\,\hat{\mathbf{i}} + 2t^2\,\hat{\mathbf{j}} + \dfrac{1}{t}\,\hat{\mathbf{k}}; \quad t = 1$

49) $\vec{\mathbf{r}}(t) = t\,\hat{\mathbf{i}} + \sin(2t)\,\hat{\mathbf{j}} + \cos(3t)\,\hat{\mathbf{k}}; \quad t = \frac{\pi}{3}$

50) $\vec{\mathbf{r}}(t) = \cos(2t)\,\hat{\mathbf{i}} + 2\sin t\,\hat{\mathbf{j}} + t^2\,\hat{\mathbf{k}}; \quad t = \frac{\pi}{2}$

51) $\vec{\mathbf{r}}(t) = 3e^t\,\hat{\mathbf{i}} + 2e^{-3t}\,\hat{\mathbf{j}} + 4e^{2t}\,\hat{\mathbf{k}}; \quad t = \ln(2)$

For questions 52 - 58, find the unit tangent vector $\vec{\mathbf{T}}(t)$ for the following parameterized curves.

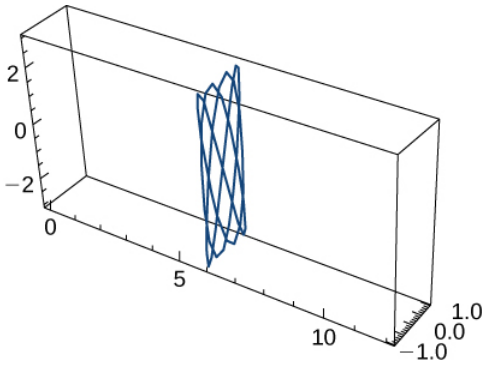
52) $\vec{\mathbf{r}}(t) = t\,\hat{\mathbf{i}} + 3t\,\hat{\mathbf{j}} + t^2\,\hat{\mathbf{k}}$

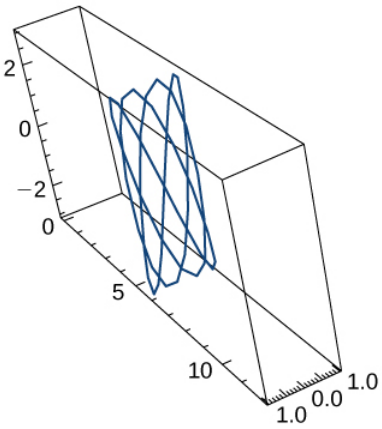
53) $\vec{\mathbf{r}}(t) = 6\,\hat{\mathbf{i}} + \cos(3t)\,\hat{\mathbf{j}} + 3\sin(4t)\,\hat{\mathbf{k}}, \quad 0 \le t < 2\pi$

54) $\vec{\mathbf{r}}(t) = \langle t\cos t, t\sin t\rangle$

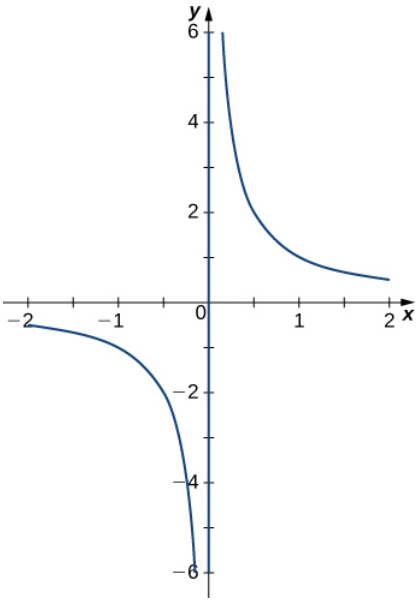
55) $\vec{\mathbf{r}}(t) = \langle t+1, 2t+1, 2t+2\rangle$

56) $\vec{\mathbf{r}}(t) = \cos t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}} + \sin t\,\hat{\mathbf{k}}, \quad 0 \le t < 2\pi$. Two views of this curve are presented here:





57) $\vec{\mathbf{r}}(t) = \left\langle t, \frac{1}{t} \right\rangle$. The graph is shown here:



58) $\vec{\mathbf{r}}(t) = 3\cos(4t)\,\hat{\mathbf{i}} + 3\sin(4t)\,\hat{\mathbf{j}} + 5t\,\hat{\mathbf{k}}, \quad 1 \le t \le 2$

59) A particle travels along the path of a helix with the equation $\vec{\mathbf{r}}(t) = \cos(t)\,\hat{\mathbf{i}} + \sin(t)\,\hat{\mathbf{j}} + t\,\hat{\mathbf{k}}$. See the graph presented here:

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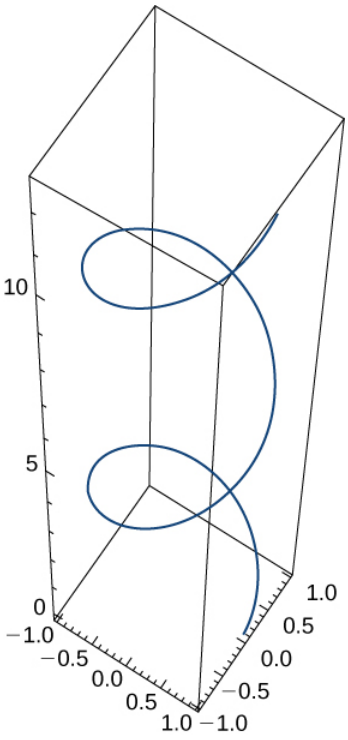

Find the following:

a. Velocity of the particle at any time

b. Speed of the particle at any time

c. Acceleration of the particle at any time

d. Find the unit tangent vector for the helix.

Integration of Vector-Valued Functions

Evaluate the following integrals:

60) $\int \left(e^t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}} + \dfrac{1}{2t-1}\,\hat{\mathbf{k}}\right)\,dt$

61) $\int_0^1 \vec{\mathbf{r}}(t)\,dt$, where $\vec{\mathbf{r}}(t) = \left\langle \sqrt[3]{t}, \dfrac{1}{t+1}, e^{-t} \right\rangle$

62) Evaluate $\int_0^3 \|t\,\hat{\mathbf{i}} + t^2\,\hat{\mathbf{j}}\|dt$.

63) The acceleration function, initial velocity, and initial position of a particle are

$$\vec{\mathbf{a}}(t) = -5\cos t\,\hat{\mathbf{i}} - 5\sin t\,\hat{\mathbf{j}},$$
$$\vec{\mathbf{v}}(0) = 9\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}}, \quad \text{and}$$
$$\vec{\mathbf{r}}(0) = 5\,\hat{\mathbf{i}}$$

Find $\vec{\mathbf{v}}(t)$ and $\vec{\mathbf{r}}(t)$.

64) Find the antiderivative of $\vec{\mathbf{r}}'(t) = \cos(2t)\,\hat{\mathbf{i}} - 2\sin t\,\hat{\mathbf{j}} + \dfrac{1}{1+t^2}\,\hat{\mathbf{k}}$ that satisfies the initial condition $\vec{\mathbf{r}}(0) = 3\,\hat{\mathbf{i}} - 2\,\hat{\mathbf{j}} + \hat{\mathbf{k}}$.

65) An object starts from rest at point $P(1,2,0)$ and moves with an acceleration of $\vec{\mathbf{a}}(t) = \hat{\mathbf{j}} + 2\,\hat{\mathbf{k}}$, where $\|\vec{\mathbf{a}}(t)\|$ is measured in feet per second per second. Find the location of the object after $t = 2$ sec.

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- Paul Seeburger (Monroe Community College) created questions 11 - 19.

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12.2B: The Calculus of Vector-Valued Functions II

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The previous section introduced us to a new mathematical object, the vector-valued function. We now apply calculus concepts to these functions. We start with the limit, then work our way through derivatives to integrals.

Limits of Vector-Valued Functions

The initial definition of the limit of a vector-valued function is a bit intimidating, as was the definition of the limit in Definition 1. The theorem following the definition shows that in practice, taking limits of vector-valued functions is no more difficult than taking limits of real-valued functions.

Definition 68: Limits of Vector-Valued Functions

Let I be an open interval containing c, and let $\vec{\mathbf{r}}(t)$ be a vector-valued function defined on I, except possibly at c. The **limit of $\vec{\mathbf{r}}(t)$, as t approaches c, is $\vec{\mathbf{L}}$**, expressed as

$$\lim_{t\to c}\vec{\mathbf{r}}(t)=\vec{\mathbf{L}}, \tag{12.2B.1}$$

means that given any $\epsilon>0$, there exists a $\delta>0$ such that for all $t\neq c$, if $|t-c|<\delta$, we have $\|\vec{\mathbf{r}}(t)-\vec{\mathbf{L}}\|<\epsilon$.

Note how the measurement of distance between real numbers is the absolute value of their difference; the measure of distance between vectors is the vector norm, or magnitude, of their difference.

theorem 89: Limits of Vector-Valued Functions

1. Let $\vec{\mathbf{r}}(t)=\langle f(t),g(t)\rangle$ be a vector-valued function in $\mathbb{R}^2$ defined on an open interval I containing c. Then
$$\lim_{t\to c}\vec{\mathbf{r}}(t)=\langle\lim_{t\to c}f(t),\lim_{t\to c}g(t)\rangle. \tag{12.2B.2}$$
2. Let $\vec{\mathbf{r}}(t)=\langle f(t),g(t),h(t)\rangle$ be a vector-valued function in $\mathbb{R}^3$ defined on an open interval I containing c. Then
$$\lim_{t\to c}\vec{\mathbf{r}}(t)=\langle\lim_{t\to c}f(t),\lim_{t\to c}g(t),\lim_{t\to c}h(t)\rangle \tag{12.2B.3}$$

Theorem 89 states that we compute limits component-wise.

Example 12.2B.1: Finding limits of vector-valued functions

Let $\vec{\mathbf{r}}(t)=\langle\frac{\sin t}{t},t^2-3t+3,\cos t\rangle$. Find $\lim_{t\to 0}\vec{\mathbf{r}}(t)$.

Solution

We apply the theorem and compute limits component-wise.

$$\begin{aligned}\lim_{t\to 0}\vec{\mathbf{r}}(t)&=\langle\lim_{t\to 0}\frac{\sin t}{t},\lim_{t\to 0}t^2-3t+3,\lim_{t\to 0}\cos t\rangle\\&=\langle 1,3,1\rangle.\end{aligned}$$

Continuity

Definition 69: Continuity of Vector-Valued Functions

Let $\vec{\mathbf{r}}(t)$ be a vector-valued function defined on an open interval I containing c.

1. $\vec{\mathbf{r}}(t)$ is **continuous at c** if $\lim_{t\to c}\vec{\mathbf{r}}(t)=r(c)$.
2. If $\vec{\mathbf{r}}(t)$ is continuous at all c in I, then $\vec{\mathbf{r}}(t)$ is **continuous on I**.

We again have a theorem that lets us evaluate continuity component-wise.

THEOREM 90: Continuity of Vector-Valued Functions

Let $\vec{\mathbf{r}}(t)$ be a vector-valued function defined on an open interval I containing c. $\vec{\mathbf{r}}(t)$ is continuous at c if, and only if, each of its component functions is continuous at c.

Example 12.2B.2: Evaluating continuity of vector-valued functions

Let $\vec{\mathbf{r}}(t) = \langle \frac{\sin t}{t}, t^2 - 3t + 3, \cos t \rangle$. Determine whether $\vec{\mathbf{r}}$ is continuous at $t = 0$ and $t = 1$.

Solution

While the second and third components of $\vec{\mathbf{r}}(t)$ are defined at $t = 0$, the first component, $(\sin t)/t$, is not. Since the first component is not even defined at $t = 0$, $\vec{\mathbf{r}}(t)$ is not defined at $t = 0$, and hence it is not continuous at $t = 0$.

At $t = 1$ each of the component functions is continuous. Therefore $\vec{\mathbf{r}}(t)$ is continuous at $t = 1$.

Derivatives

Consider a vector-valued function $\vec{\mathbf{r}}$ defined on an open interval I containing t_0 and t_1. We can compute the displacement of $\vec{\mathbf{r}}$ on $[t_0, t_1]$, as shown in Figure 12.2B.1a. Recall that dividing the displacement vector by $t_1 - t_0$ gives the average rate of change on $[t_0, t_1]$, as shown in 12.2B.1b.

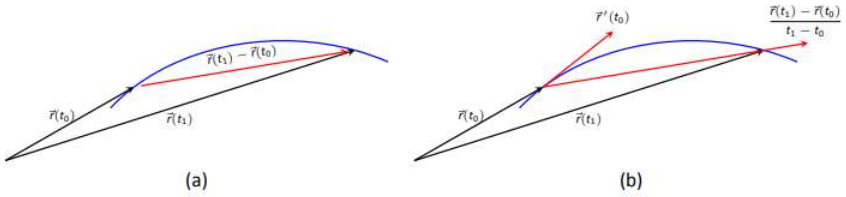

Figure 12.2B.1: Illustrating displacement, leading to an understanding of the derivative of vector-valued functions.

The **derivative** of a vector-valued function is a measure of the *instantaneous* rate of change, measured by taking the limit as the length of $[t_0, t_1]$ goes to 0. Instead of thinking of an interval as $[t_0, t_1]$, we think of it as $[c, c + h]$ for some value of h (hence the interval has length h). The *average* rate of change is

$$\frac{\vec{\mathbf{r}}(c + h) - \vec{\mathbf{r}}(c)}{h} \tag{12.2B.4}$$

for any value of $h \neq 0$. We take the limit as $h \to 0$ to measure the instantaneous rate of change; this is the derivative of $\vec{\mathbf{r}}$.

Definition 70: Derivative of a Vector-Valued Function

Let $\vec{\mathbf{r}}(t)$ be continuous on an open interval I containing c.

1. The derivative of $\vec{\mathbf{r}}$ at $t = c$ is

$$\vec{\mathbf{r}}'(c) = \lim_{h \to 0} \frac{\vec{\mathbf{r}}(c + h) - \vec{\mathbf{r}}(c)}{h}. \tag{12.2B.5}$$

2. The derivative of $\vec{\mathbf{r}}$ is

$$\vec{\mathbf{r}}'(t) = \lim_{h \to 0} \frac{\vec{\mathbf{r}}(t + h) - \vec{\mathbf{r}}(t)}{h}. \tag{12.2B.6}$$

Note: Alternate notations for the derivative of $\vec{\mathbf{r}}$ include:

$$\vec{\mathbf{r}}'(t) = \frac{d}{dt}\left(\vec{\mathbf{r}}(t)\right) = \frac{d\vec{\mathbf{r}}}{dt}. \tag{12.2B.7}$$

If a vector-valued function has a derivative for all c in an open interval I, we say that $\vec{\mathbf{r}}(t)$ is **differentiable** on I.

Once again we might view this definition as intimidating, but recall that we can evaluate limits component-wise. The following theorem verifies that this means we can compute derivatives component-wise as well, making the task not too difficult.

theorem 91: Derivatives of Vector-Valued Functions

1. Let $\vec{\mathbf{r}}(t) = \langle f(t), g(t) \rangle$. Then

$$\vec{\mathbf{r}}'(t) = \langle f'(t), g'(t) \rangle. \tag{12.2B.8}$$

2. Let $\vec{\mathbf{r}}(t) = \langle f(t), g(t), h(t) \rangle$. Then

$$\vec{\mathbf{r}}'(t) = \langle f'(t), g'(t), h'(t) \rangle. \tag{12.2B.9}$$

Example 12.2B.3: Derivatives of vector-valued functions

Let $\vec{\mathbf{r}}(t) = \langle t^2, t \rangle$.

1. Sketch $\vec{\mathbf{r}}(t)$ and $\vec{\mathbf{r}}'(t)$ on the same axes.
2. Compute $\vec{\mathbf{r}}'(1)$ and sketch this vector with its initial point at the origin and at $\vec{\mathbf{r}}(1)$.

Solution

1. Theorem 91 allows us to compute derivatives component-wise, so

$$\vec{\mathbf{r}}'(t) = \langle 2t, 1 \rangle. \tag{12.2B.10}$$

   $\vec{\mathbf{r}}(t)$ and $\vec{\mathbf{r}}'(t)$ are graphed together in Figure 11.9(a). Note how plotting the two of these together, in this way, is not very illuminating. When dealing with real-valued functions, plotting $f(x)$ with $f'(x)$ gave us useful information as we were able to compare f and f' at the same x-values. When dealing with vector-valued functions, it is hard to tell which points on the graph of $\vec{\mathbf{r}}'$ correspond to which points on the graph of $\vec{\mathbf{r}}$.
2. We easily compute $\vec{\mathbf{r}}'(1) = \langle 2, 1 \rangle$, which is drawn in Figure 12.2B.2a with its initial point at the origin, as well as at $\vec{\mathbf{r}}(1) = \langle 1, 1 \rangle$. These are sketched in Figure 12.2B.2b.

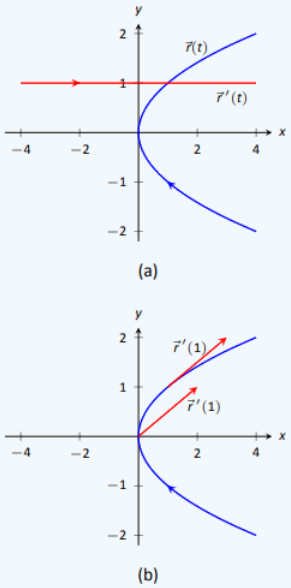

Figure 12.2B.2: Graphing the derivative of a vector-valued function in Example 11.2.3

Example 12.2B.4: Derivatives of vector-valued functions

Let $\vec{\mathbf{r}}(t) = \langle \cos t, \sin t, t \rangle$. Compute $\vec{\mathbf{r}}'(t)$ and $\vec{\mathbf{r}}'(\pi/2)$. Sketch $\vec{\mathbf{r}}'(\pi/2)$ with its initial point at the origin and at $\vec{\mathbf{r}}(\pi/2)$.

Solution

We compute $\vec{\mathbf{r}}'$ as $\vec{\mathbf{r}}'(t) = \langle -\sin t, \cos t, 1 \rangle$. At $t = \pi/2$, we have $\vec{\mathbf{r}}'(\pi/2) = \langle -1, 0, 1 \rangle$. Figure 12.2B.3 shows a graph of $\vec{\mathbf{r}}(t)$, with $\vec{\mathbf{r}}'(\pi/2)$ plotted with its initial point at the origin and at $\vec{\mathbf{r}}(\pi/2)$.

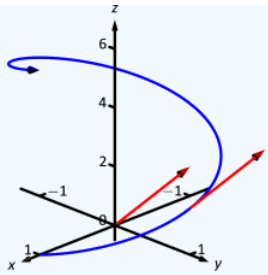

Figure 12.2B.3: Viewing a vector-valued function and its derivative at one point.

In Examples 12.2B.3 and 12.2B.4, sketching a particular derivative with its initial point at the origin did not seem to reveal anything significant. However, when we sketched the vector with its initial point on the corresponding point on the graph, we did see something significant: the vector appeared to be *tangent* to the graph. We have not yet defined what "tangent" means in terms of curves in space; in fact, we use the derivative to define this term.

Definition 71: Tangent Vector, Tangent Line

Let $\vec{\mathbf{r}}(t)$ be a differentiable vector-valued function on an open interval I containing c, where $\vec{\mathbf{r}}'(c) \neq \vec{\mathbf{0}}$.

1. A vector $\vec{\mathbf{v}}$ is **tangent to the graph of** $\vec{\mathbf{r}}(t)$ at $t = c$ if $\vec{\mathbf{v}}$ is parallel to $\vec{\mathbf{r}}'(c)$.
2. The tangent line to the graph of $\vec{\mathbf{r}}(t)$ at $t = c$ is the line through $\vec{\mathbf{r}}(c)$ with direction parallel to $\vec{\mathbf{r}}'(c)$. An equation of the tangent line is

$$\vec{\ell}(t) = \vec{\mathbf{r}}(c) + t\,\vec{\mathbf{r}}'(c). \tag{12.2B.11}$$

Example 12.2B.5: Finding tangent lines to curves in space

Let $\vec{\mathbf{r}}(t) = \langle t, t^2, t^3 \rangle$ on $[-1.5, 1.5]$. Find the vector equation of the line tangent to the graph of $\vec{\mathbf{r}}$ at $t = -1$.

Solution

To find the equation of a line, we need a point on the line and the line's direction. The point is given by $\vec{\mathbf{r}}(-1) = \langle -1, 1, -1 \rangle$. (To be clear, $\langle -1, 1, -1 \rangle$ is a *vector*, not a point, but we use the point "pointed to" by this vector.)

The direction comes from $\vec{\mathbf{r}}'(-1)$. We compute, component-wise, $\vec{\mathbf{r}}'(t) = \langle 1, 2t, 3t^2 \rangle$. Thus $\vec{\mathbf{r}}'(-1) = \langle 1, -2, 3 \rangle$.

The vector equation of the line is $\ell(t) = \langle -1, 1, -1 \rangle + t\langle 1, -2, 3 \rangle$. This line and $\vec{\mathbf{r}}(t)$ are sketched, from two perspectives, in Figure 12.2B.4*a* and 12.2B.4*b*.

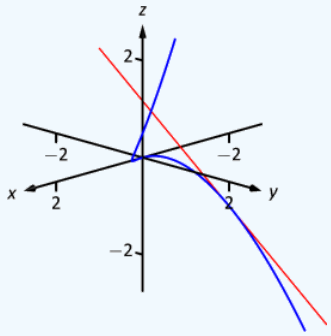

Figure 12.2B.4: Graphing a curve in space with its tangent line.

Example 12.2B.6: Finding tangent lines to curves

Find the equations of the lines tangent to $\vec{\mathbf{r}}(t) = \langle t^3, t^2 \rangle$ at $t = -1$ and $t = 0$.

Solution

We find that $\vec{\mathbf{r}}'(t) = \langle 3t^2, 2t \rangle$. At $t = -1$, we have

$$\vec{\mathbf{r}}(-1) = \langle -1, 1\rangle \quad \text{and} \quad \vec{\mathbf{r}}\,'(-1) = \langle 3, -2\rangle,$$

so the equation of the line tangent to the graph of $\vec{\mathbf{r}}(t)$ at $t = -1$ is

$$\vec{\ell}(t) = \langle -1, 1\rangle + t\langle 3, -2\rangle.$$

This line is graphed with $\vec{\mathbf{r}}(t)$ in Figure 12.2B.5.

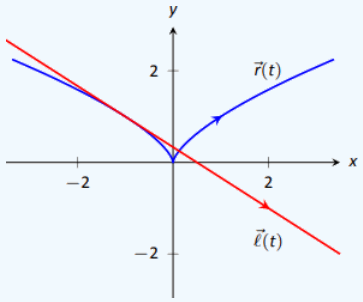

Figure 12.2B.5: Graphing $\vec{\mathbf{r}}(t)$ and its tangent line in Example 12.2B.6.

At $t = 0$, we have $\vec{\mathbf{r}}\,'(0) = \langle 0, 0\rangle = \vec{\mathbf{0}}$! This implies that the tangent line "has no direction." We cannot apply Definition 71, hence cannot find the equation of the tangent line.

We were unable to compute the equation of the tangent line to $\vec{\mathbf{r}}(t) = \langle t^3, t^2\rangle$ at $t = 0$ because $\vec{\mathbf{r}}\,'(0) = \vec{\mathbf{0}}$. The graph in Figure 11.12 shows that there is a cusp at this point. This leads us to another definition of **smooth**, previously defined by Definition 46 in Section 9.2.

Definition 72: Smooth Vector-Valued Functions

Let $\vec{\mathbf{r}}(t)$ be a differentiable vector-valued function on an open interval I. $\vec{\mathbf{r}}(t)$ is **smooth** on I if $\vec{\mathbf{r}}\,'(t) \neq \vec{\mathbf{0}}$ on I.

Having established derivatives of vector-valued functions, we now explore the relationships between the derivative and other vector operations. The following theorem states how the derivative interacts with vector addition and the various vector products.

THEOREM 92: Properties of Derivatives of Vector-Valued Functions

Let $\vec{\mathbf{r}}$ and $\vec{\mathbf{s}}$ be differentiable vector-valued functions, let f be a differentiable real-valued function, and let c be a real number.

1. $\frac{d}{dt}\left(\vec{\mathbf{r}}(t) \pm \vec{\mathbf{s}}(t)\right) = \vec{\mathbf{r}}\,'(t) \pm \vec{\mathbf{s}}\,'(t)$
2. $\frac{d}{dt}\left(c\vec{\mathbf{r}}(t)\right) = c\vec{\mathbf{r}}\,'(t)$
3. $\frac{d}{dt}\left(f(t)\vec{\mathbf{r}}(t)\right) = f'(t)\vec{\mathbf{r}}(t) + f(t)\vec{\mathbf{r}}\,'(t)$ **Product Rule**
4. $\frac{d}{dt}\left(\vec{\mathbf{r}}(t) \cdot \vec{\mathbf{s}}(t)\right) = \vec{\mathbf{r}}\,'(t) \cdot \vec{\mathbf{s}}(t) + \vec{\mathbf{r}}(t) \cdot \vec{\mathbf{s}}\,'(t)$ **Product Rule**
5. $\frac{d}{dt}\left(\vec{\mathbf{r}}(t) \times \vec{\mathbf{s}}(t)\right) = \vec{\mathbf{r}}\,'(t) \times \vec{\mathbf{s}}(t) + \vec{\mathbf{r}}(t) \times \vec{\mathbf{s}}\,'(t)$ **Product Rule**
6. $\frac{d}{dt}\left(\vec{\mathbf{r}}\left(f(t)\right)\right) = \vec{\mathbf{r}}\,'\left(f(t)\right)f'(t)$ **Chain Rule**

Example 12.2B.7: Using derivative properties of vector-valued functions

Let $\vec{\mathbf{r}}(t) = \langle t, t^2 - 1\rangle$ and let $\vec{\mathbf{u}}(t)$ be the unit vector that points in the direction of $\vec{\mathbf{r}}(t)$.

1. Graph $\vec{\mathbf{r}}(t)$ and $\vec{\mathbf{u}}(t)$ on the same axes, on $[-2, 2]$.
2. Find $\vec{\mathbf{u}}\,'(t)$ and sketch $\vec{\mathbf{u}}\,'(-2)$, $\vec{\mathbf{u}}\,'(-1)$ and $\vec{\mathbf{u}}\,'(0)$. Sketch each with initial point the corresponding point on the graph of $\vec{\mathbf{u}}$.

Solution

1. To form the unit vector that points in the direction of $\vec{\mathbf{r}}$, we need to divide $\vec{\mathbf{r}}(t)$ by its magnitude.

$$\|\vec{\mathbf{r}}(t)\| = \sqrt{t^2 + (t^2 - 1)^2} \quad \Rightarrow \qquad (12.2B.12)$$
$$\vec{\mathbf{u}}(t) = \frac{1}{\sqrt{t^2 + (t^2 - 1)^2}}\langle t, t^2 - 1\rangle.$$

$\vec{\mathbf{r}}(t)$ and $\vec{\mathbf{u}}(t)$ are graphed in Figure 11.13. Note how the graph of $\vec{\mathbf{u}}(t)$ forms part of a circle; this must be the case, as the length of $\vec{\mathbf{u}}(t)$ is 1 for all t.

2. To compute $\vec{\mathbf{u}}\,'(t)$, we use Theorem 92, writing

$$\vec{\mathbf{u}}(t) = f(t)\vec{\mathbf{r}}(t), \quad \text{where} \qquad f(t) = \frac{1}{\sqrt{t^2 + (t^2-1)^2}} = \left(t^2 + (t^2-1)^2\right)^{-1/2}. \tag{12.2B.13}$$

(We *could* write

$$\vec{\mathbf{u}}(t) = \left\langle \frac{t}{\sqrt{t^2 + (t^2-1)^2}}, \frac{t^2-1}{\sqrt{t^2 + (t^2-1)^2}} \right\rangle \tag{12.2B.14}$$

and then take the derivative. It is a matter of preference; this latter method requires two applications of the Quotient Rule where our method uses the Product and Chain Rules.)

We find $f'(t)$ using the Chain Rule:

$$\begin{aligned} f'(t) &= -\frac{1}{2}\left(t^2 + (t^2-1)^2\right)^{-3/2}\left(2t + 2(t^2-1)(2t)\right) \\ &= -\frac{2t(2t^2-1)}{2\left(\sqrt{t^2 + (t^2-1)^2}\right)^3} \end{aligned}$$

We now find $\vec{\mathbf{u}}\,'(t)$ using part 3 of Theorem 92:

$$\begin{aligned} \vec{\mathbf{u}}\,'(t) &= f'(t)\vec{\mathbf{u}}(t) + f(t)\vec{\mathbf{u}}\,'(t) \\ &= -\frac{2t(2t^2-1)}{2\left(\sqrt{t^2 + (t^2-1)^2}\right)^3}\langle t, t^2-1\rangle + \frac{1}{\sqrt{t^2 + (t^2-1)^2}}\langle 1, 2t\rangle. \end{aligned}$$

This is admittedly very "messy;" such is usually the case when we deal with unit vectors. We can use this formula to compute $\vec{\mathbf{u}}\,(-2)$, $\vec{\mathbf{u}}\,(-1)$ and $\vec{\mathbf{u}}\,(0)$:

$$\begin{aligned} \vec{\mathbf{u}}\,(-2) &= \left\langle -\frac{15}{13\sqrt{13}}, -\frac{10}{13\sqrt{13}} \right\rangle \approx \langle -0.320, -0.213\rangle \\ \vec{\mathbf{u}}\,(-1) &= \langle 0, -2\rangle \\ \vec{\mathbf{u}}\,(0) &= \langle 1, 0\rangle \end{aligned}$$

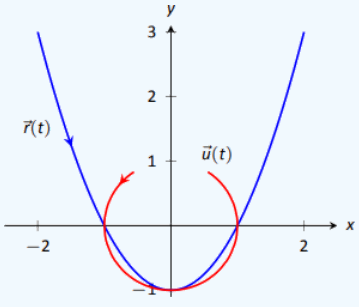

Figure 12.2B.6: Graphing $\vec{\mathbf{r}}(t)$ and $\vec{\mathbf{u}}(t)$ in Example 12.2B.7

Each of these is sketched in Figure 12.2B.6. Note how the length of the vector gives an indication of how quickly the circle is being traced at that point. When $t = -2$, the circle is being drawn relatively slow; when $t = -1$, the circle is being traced much

more quickly.

It is a basic geometric fact that a line tangent to a circle at a point P is perpendicular to the line passing through the center of the circle and P. This is illustrated in Figure 11.14; each tangent vector is perpendicular to the line that passes through its initial point and the center of the circle. Since the center of the circle is the origin, we can state this another way: $\vec{\mathbf{u}}\,'(t)$ is orthogonal to $\vec{\mathbf{u}}(t)$.

Recall that the dot product serves as a test for orthogonality: if $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = 0$, then $\vec{\mathbf{u}}$ is orthogonal to $\vec{\mathbf{v}}$. Thus in the above example, $\vec{\mathbf{u}}(t) \cdot \vec{\mathbf{u}}\,'(t) = 0$.

This is true of any vector-valued function that has a constant length, that is, that traces out part of a circle. It has important implications later on, so we state it as a theorem (and leave its formal proof as an Exercise.)

THEOREM 93 Vector-Valued Functions of Constant Length

Let $\vec{\mathbf{r}}(t)$ be a differentiable vector-valued function on an open interval I of constant length. That is, $\|\vec{\mathbf{r}}(t)\| = c$ for all t in I (equivalently, $\vec{\mathbf{r}}(t) \cdot \vec{\mathbf{r}}(t) = c^2$ for all t in I). Then $\vec{\mathbf{r}}(t) \cdot \vec{\mathbf{r}}'(t) = 0$ for all t in I.

Integration

Indefinite and definite integrals of vector-valued functions are also evaluated component-wise.

THEOREM 94: Indefinite and Definite Integrals of Vector-Valued Functions

Let $\vec{\mathbf{r}}(t) = \langle f(t), g(t)\rangle$ be a vector-valued function in $\mathbb{R}^2$.

1. $\int \vec{\mathbf{r}}(t)\,dt = \langle \int f(t)\,dt, \int g(t)\,dt\rangle$
2. $\int_a^b \vec{\mathbf{r}}(t)\,dt = \langle \int_a^b f(t)\,dt, \int_a^b g(t)\,dt\rangle$

A similar statement holds for vector-valued functions in $\mathbb{R}^3$.

Example 12.2B.8: Evaluating a definite integral of a vector-valued function

Let $\vec{\mathbf{r}}(t) = \langle e^{2t}, \sin t\rangle$. Evaluate $\int_0^1 \vec{\mathbf{r}}(t)\,dt$.

Solution

We follow Theorem 94.

$$\begin{aligned}\int_0^1 \vec{\mathbf{r}}(t)\,dt &= \int_0^1 \langle e^{2t}, \sin t\rangle\,dt \\ &= \langle \int_0^1 e^{2t}\,dt\,,\, \int_0^1 \sin t\,dt\rangle \\ &= \langle \frac{1}{2}e^{2t}\Big|_0^1\,,\, -\cos t\Big|_0^1\rangle \\ &= \langle \frac{1}{2}(e^2-1)\,,\, -\cos(1)+1\rangle \\ &\approx \langle 3.19, 0.460\rangle.\end{aligned}$$

Example 12.2B.9: Solving an initial value problem

Let $\vec{\mathbf{r}}''(t) = \langle 2, \cos t, 12t\rangle$. Find $\vec{\mathbf{r}}(t)$ where:

1. $\vec{\mathbf{r}}(0) = \langle -7, -1, 2\rangle$ and
2. $\vec{\mathbf{r}}'(0) = \langle 5, 3, 0\rangle$.

Solution

Knowing $\vec{\mathbf{r}}''(t) = \langle 2, \cos t, 12t\rangle$, we find $\vec{\mathbf{r}}'(t)$ by evaluating the indefinite integral.

$$\begin{aligned}\int \vec{\mathbf{r}}''(t)dt &= \langle \int 2dt\ , \int \cos t dt\ , \int 12t dt\rangle \\ &= \langle 2t + C_1, \sin t + C_2, 6t^2 + C_3\rangle \\ &= \langle 2t, \sin t, 6t^2\rangle + \langle C_1, C_2, C_3\rangle \\ &= \langle 2t, \sin t, 6t^2\rangle + \vec{\mathbf{C}}_1.\end{aligned}$$

Note how each indefinite integral creates its own constant which we collect as one constant vector $\vec{\mathbf{C}}_1$. Knowing $\vec{\mathbf{r}}'(0) = \langle 5, 3, 0\rangle$ allows us to solve for $\vec{\mathbf{C}}_1$:

$$\begin{aligned}\vec{\mathbf{r}}'(t) &= \langle 2t, \sin t, 6t^2\rangle + \vec{\mathbf{C}}_1 \\ \vec{\mathbf{r}}'(0) &= \langle 0, 0, 0\rangle + \vec{\mathbf{C}}_1 \\ \langle 5, 3, 0\rangle &= \vec{\mathbf{C}}_1.\end{aligned}$$

So $\vec{\mathbf{r}}'(t) = \langle 2t, \sin t, 6t^2\rangle + \langle 5, 3, 0\rangle = \langle 2t + 5, \sin t + 3, 6t^2\rangle$. To find $\vec{\mathbf{r}}(t)$, we integrate once more.

$$\begin{aligned}\int \vec{\mathbf{r}}'(t)\ dt &= \langle \int 2t + 5\ dt, \int \sin t + 3\ dt, \int 6t^2 dt\rangle \\ &= \langle t^2 + 5t, -\cos t + 3t, 2t^3\rangle + \vec{\mathbf{C}}_2.\end{aligned}$$

With $\vec{\mathbf{r}}(0) = \langle -7, -1, 2\rangle$, we solve for $\vec{\mathbf{C}}_2$:

$$\begin{aligned}\vec{\mathbf{r}}(t) &= \langle t^2 + 5t, -\cos t + 3t, 2t^3\rangle + \vec{\mathbf{C}}_2 \\ \vec{\mathbf{r}}(0) &= \langle 0, -1, 0\rangle + \vec{\mathbf{C}}_2 \\ \langle -7, -1, 2\rangle &= \langle 0, -1, 0\rangle + \vec{\mathbf{C}}_2 \\ \langle -7, 0, 2\rangle &= \vec{\mathbf{C}}_2.\end{aligned}$$

So $\vec{\mathbf{r}}(t) = \langle t^2 + 5t, -\cos t + 3t, 2t^3\rangle + \langle -7, 0, 2\rangle = \langle t^2 + 5t - 7, -\cos t + 3t, 2t^3 + 2\rangle$.

What does the integration of a vector-valued function *mean*? There are many applications, but none as direct as "the area under the curve" that we used in understanding the integral of a real-valued function.

A key understanding for us comes from considering the integral of a derivative:

$$\int_a^b \vec{\mathbf{r}}'(t)dt = \vec{\mathbf{r}}(t)\Big|_a^b = \vec{\mathbf{r}}(b) - \vec{\mathbf{r}}(a). \tag{12.2B.15}$$

Integrating a *rate of change* function gives *displacement*.

Noting that vector-valued functions are closely related to parametric equations, we can describe the arc length of the graph of a vector-valued function as an integral. Given parametric equations $x = f(t)$, $y = g(t)$, the arc length on $[a, b]$ of the graph is

$$\text{Arc Length} = \int_a^b \sqrt{f'(t)^2 + g'(t)^2}dt, \tag{12.2B.16}$$

as stated in Theorem 82 in Section 9.3. If $\vec{\mathbf{r}}(t) = \langle f(t), g(t)\rangle$, note that $\sqrt{f'(t)^2 + g'(t)^2} = \|\vec{\mathbf{r}}'(t)\|$. Therefore we can express the arc length of the graph of a vector-valued function as an integral of the magnitude of its derivative.

THEOREM 95: Arc Length of a Vector-Valued Function

Let $\vec{\mathbf{r}}(t)$ be a vector-valued function where $\vec{\mathbf{r}}'(t)$ is continuous on $[a, b]$. The arc length L of the graph of $\vec{\mathbf{r}}(t)$ is

$$L = \int_a^b \|\vec{\mathbf{r}}'(t)\|\ dt. \tag{12.2B.17}$$

Note that we are actually integrating a scalar-function here, not a vector-valued function.

The next section takes what we have established thus far and applies it to objects in motion. We will let $\vec{\mathbf{r}}(t)$ describe the path of an object in the plane or in space and will discover the information provided by $\vec{\mathbf{r}}'(t)$ and $\vec{\mathbf{r}}''(t)$.

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12.3: Motion in Space

We have now seen how to describe curves in the plane and in space, and how to determine their properties, including their derivatives, where they are smooth, and their unit tangent vectors. All of this leads to the main goal of this chapter, which is the description of motion along plane curves and space curves. We now have all the tools we need; in this section, we put these ideas together and look at how to use them.

Motion Vectors in the Plane and in Space

Our starting point is using vector-valued functions to represent the position of an object as a function of time. All of the following material can be applied either to curves in the plane or to space curves. For example, when we look at the orbit of the planets, the curves defining these orbits all lie in a plane because they are elliptical. However, a particle traveling along a helix moves on a curve in three dimensions.

Definition: Speed, Velocity, and Acceleration

Let $\vec{\mathbf{r}}(t)$ be a twice-differentiable vector-valued function of the parameter t that represents the position of an object as a function of time.

The velocity vector $\vec{\mathbf{v}}(t)$ of the object is given by

$$\text{Velocity} = \vec{\mathbf{v}}(t) = \vec{\mathbf{r}}\,'(t). \tag{12.3.1}$$

The acceleration vector $\vec{\mathbf{a}}(t)$ is defined to be

$$\text{Acceleration} = \vec{\mathbf{a}}(t) = \vec{\mathbf{v}}\,'(t) = \vec{\mathbf{r}}\,''(t). \tag{12.3.2}$$

The *speed* is defined to be

$$\text{Speed} = \|\vec{\mathbf{v}}(t)\| = \|\vec{\mathbf{r}}\,'(t)\| = \frac{ds}{dt}. \tag{12.3.3}$$

Since $\vec{\mathbf{r}}(t)$ can be in either two or three dimensions, these vector-valued functions can have either two or three components. In two dimensions, we define $\vec{\mathbf{r}}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}}$ and in three dimensions $\vec{\mathbf{r}}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}$. Then the velocity, acceleration, and speed can be written as shown in the following table.

Table* 12.3.1*: *Formulas for Position, Velocity, Acceleration, and Speed*

Quantity	Two Dimensions	Three Dimensions
Position	$\vec{\mathbf{r}}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}}$	$\vec{\mathbf{r}}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}$
Velocity	$\vec{\mathbf{v}}(t) = x\,'(t)\hat{\mathbf{i}} + y'(t)\hat{\mathbf{j}}$	$\vec{\mathbf{v}}(t) = x\,'(t)\hat{\mathbf{i}} + y'(t)\hat{\mathbf{j}} + z\,'(t)\hat{\mathbf{k}}$
Acceleration	$\vec{\mathbf{a}}(t) = x\,''(t)\hat{\mathbf{i}} + y''(t)\hat{\mathbf{j}}$	$\vec{\mathbf{a}}(t) = x\,''(t)\hat{\mathbf{i}} + y''(t)\hat{\mathbf{j}} + z\,''(t)\hat{\mathbf{k}}$
Speed	$\|\vec{\mathbf{v}}(t)\| = \sqrt{(x\,'(t))^2 + (y'(t))^2}$	$\|\vec{\mathbf{v}}(t)\| = \sqrt{(x\,'(t))^2 + (y'(t))^2 + (z\,'(t))^2}$

Example 12.3.1: Studying Motion Along a Parabola

A particle moves in a parabolic path defined by the vector-valued function $\vec{\mathbf{r}}(t) = t^2\hat{\mathbf{i}} + \sqrt{5 - t^2}\hat{\mathbf{j}}$, where t measures time in seconds.

1. Find the velocity, acceleration, and speed as functions of time.
2. Sketch the curve along with the velocity vector at time $t = 1$.
3. Write the parametric equations of the tangent line to this curve at the point specified by $t = 1$.

Solution

1. We use Equations 12.3.1, 12.3.2, and 12.3.3:

$$\vec{\mathbf{v}}(t) = \vec{\mathbf{r}}\,'(t) = 2t\hat{\mathbf{i}} - \frac{t}{\sqrt{5-t^2}}\hat{\mathbf{j}}$$
$$\vec{\mathbf{a}}(t) = \vec{\mathbf{v}}\,'(t) = 2\hat{\mathbf{i}} - 5(5-t^2)^{-\frac{3}{2}}\hat{\mathbf{j}}$$
$$\begin{aligned} ||\vec{\mathbf{v}}(t)|| &= ||\vec{\mathbf{r}}\,'(t)|| \\ &= \sqrt{(2t)^2 + \left(-\frac{t}{\sqrt{5-t^2}}\right)^2} \\ &= \sqrt{4t^2 + \frac{t^2}{5-t^2}} \\ &= \sqrt{\frac{21t^2 - 4t^4}{5-t^2}}. \end{aligned}$$

2. The graph of $\vec{\mathbf{r}}(t) = t^2\hat{\mathbf{i}} + \sqrt{5-t^2}\hat{\mathbf{j}}$ is a portion of a parabola (Figure 12.3.1).

   When $t = 1$, $\vec{\mathbf{r}}(1) = (1)^2\hat{\mathbf{i}} + \sqrt{5-(1)^2}\hat{\mathbf{j}} = \hat{\mathbf{i}} + \sqrt{4}\hat{\mathbf{j}} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}}$.

   Thus the particle would be located at the point $(1, 2)$ when $t = 1$.

   The velocity vector at $t = 1$ is

   $$\vec{\mathbf{v}}(1) = \vec{\mathbf{r}}\,'(1) = 2(1)\hat{\mathbf{i}} - \frac{1}{\sqrt{5-1^2}}\hat{\mathbf{j}} = 2\hat{\mathbf{i}} - \frac{1}{2}\hat{\mathbf{j}} \tag{12.3.4}$$

   and the acceleration vector at $t = 1$ is

   $$\vec{\mathbf{a}}(1) = \vec{\mathbf{v}}\,'(1) = 2\hat{\mathbf{i}} - 5(5-1^2)^{-3/2}\hat{\mathbf{j}} = 2\hat{\mathbf{i}} - \frac{5}{8}\hat{\mathbf{j}}. \tag{12.3.5}$$

   Notice that the velocity vector is tangent to the path, as is always the case.

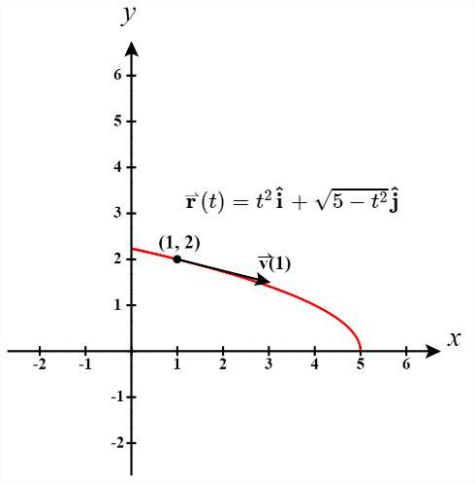

Figure 12.3.1: This graph depicts the velocity vector at time $t = 1$ *for a particle moving in a parabolic path.*

3. To find the parametric equations of the tangent line, we need the point at which it is tangent to the curve and a direction vector for the line. The point is just what we found above, $(1, 2)$. And it turns out that we can use the velocity vector as a direction vector for the tangent line. This is always true, unless the velocity is the zero vector at this point.

   So as we found above, $\vec{\mathbf{v}}(1) = 2\hat{\mathbf{i}} - \frac{1}{2}\hat{\mathbf{j}}$. This is the direction vector for this tangent line so we can write the parametric equations of the tangent line as:

$$\begin{aligned} x &= 1 + 2t \\ y &= 2 - \frac{1}{2}t \\ z &= 0 \end{aligned}$$

A corresponding vector parameterization of this tangent line would be:

$$\vec{\mathbf{L}}(t) = (1+2t)\,\hat{\mathbf{i}} + \left(2 - \frac{1}{2}t\right)\hat{\mathbf{j}}. \tag{12.3.6}$$

Exercise 12.3.1

A particle moves in a path defined by the vector-valued function $\vec{\mathbf{r}}(t) = (t^2 - 3t)\,\hat{\mathbf{i}} + (2t-4)\,\hat{\mathbf{j}} + (t+2)\,\hat{\mathbf{k}}$, where t measures time in seconds and where distance is measured in feet. Find the velocity, acceleration, and speed as functions of time.

To gain a better understanding of the velocity and acceleration vectors, imagine you are driving along a curvy road. If you do not turn the steering wheel, you would continue in a straight line and run off the road. The speed at which you are traveling when you run off the road, coupled with the direction, gives a vector representing your velocity, as illustrated in Figure 12.3.2.

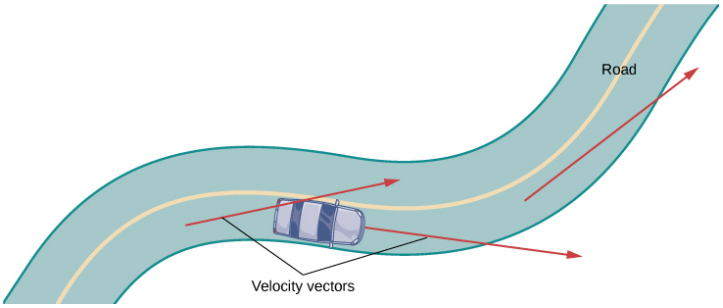

***Figure* 12.3.2:** *At each point along a road traveled by a car, the velocity vector of the car is tangent to the path traveled by the car.*

However, the fact that you must turn the steering wheel to stay on the road indicates that your velocity is always changing (even if your speed is not) because your *direction* is constantly changing to keep you on the road. As you turn to the right, your acceleration vector also points to the right. As you turn to the left, your acceleration vector points to the left. This indicates that your velocity and acceleration vectors are constantly changing, regardless of whether your actual speed varies (Figure 12.3.3).

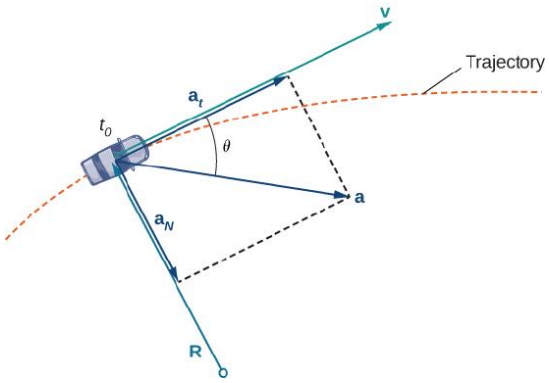

Figure 12.3.3: The dashed line represents the trajectory of an object (a car, for example). The acceleration vector points toward the inside of the turn at all times.

Projectile Motion

Now let's look at an application of vector functions. In particular, let's consider the effect of gravity on the motion of an object as it travels through the air, and how it determines the resulting trajectory of that object. In the following, we ignore the effect of air resistance. This situation, with an object moving with an initial velocity but with no forces acting on it other than gravity, is known as projectile motion. It describes the motion of objects from golf balls to baseballs, and from arrows to cannonballs.

First we need to choose a coordinate system. If we are standing at the origin of this coordinate system, then we choose the positive y-axis to be up, the negative y-axis to be down, and the positive x-axis to be forward (i.e., away from the thrower of the object). The

effect of gravity is in a downward direction, so Newton's second law tells us that the force on the object resulting from gravity is equal to the mass of the object times the acceleration resulting from gravity, or $\vec{\mathbf{F}}_g = m\vec{\mathbf{a}}$, where $\vec{\mathbf{F}}_g$ represents the force from gravity and $\vec{\mathbf{a}} = -g\hat{\mathbf{j}}$ represents the acceleration resulting from gravity at Earth's surface. The value of g in the English system of measurement is approximately 32 ft/sec^2 and it is approximately 9.8 m/sec^2 in the metric system. This is the only force acting on the object. Since gravity acts in a downward direction, we can write the force resulting from gravity in the form $\vec{\mathbf{F}}_g = -mg\hat{\mathbf{j}}$, as shown in Figure 12.3.4.

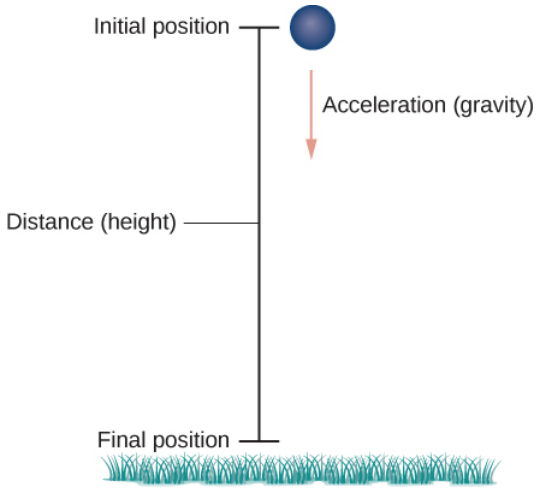

Figure 12.3.4: An object is falling under the influence of gravity.

Newton's second law also tells us that $F = m\vec{\mathbf{a}}$, where $\vec{\mathbf{a}}$ represents the acceleration vector of the object. This force must be equal to the force of gravity at all times, so we therefore know that

$$\begin{aligned} \vec{\mathbf{F}} &= \vec{\mathbf{F}}_g \\ m\vec{\mathbf{a}} &= -mg\hat{\mathbf{j}} \\ \vec{\mathbf{a}} &= -g\hat{\mathbf{j}}. \end{aligned}$$

Now we use the fact that the acceleration vector is the first derivative of the velocity vector. Therefore, we can rewrite the last equation in the form

$$\vec{\mathbf{v}}\,'(t) = -g\hat{\mathbf{j}} \tag{12.3.8}$$

By taking the antiderivative of each side of this equation we obtain

$$\vec{\mathbf{v}}(t) = \int -g\hat{\mathbf{j}}\, dt = -gt\hat{\mathbf{j}} + \vec{\mathbf{C}}_1 \tag{12.3.9}$$

for some constant vector $\vec{\mathbf{C}}_1$. To determine the value of this vector, we can use the velocity of the object at a fixed time, say at time $t = 0$. We call this velocity the *initial velocity*: $\vec{\mathbf{v}}(0) = \vec{\mathbf{v}}_0$. Therefore, $\vec{\mathbf{v}}(0) = -g(0)\,\hat{\mathbf{j}} + \vec{\mathbf{C}}_1 = \vec{\mathbf{v}}_0$ and $\vec{\mathbf{C}}_1 = \vec{\mathbf{v}}_0$. This gives the velocity vector as $\vec{\mathbf{v}}(t) = -gt\hat{\mathbf{j}} + \vec{\mathbf{v}}_0$.

Next we use the fact that velocity $\vec{\mathbf{v}}(t)$ is the derivative of position $\vec{\mathbf{s}}(t)$. This gives the equation

$$\vec{\mathbf{s}}\,'(t) = -gt\hat{\mathbf{j}} + \vec{\mathbf{v}}_0. \tag{12.3.10}$$

Taking the antiderivative of both sides of this equation leads to

$$\begin{aligned} \vec{\mathbf{s}}(t) &= \int -gt\hat{\mathbf{j}} + \vec{\mathbf{v}}_0\, dt \\ &= -\frac{1}{2}gt^2\hat{\mathbf{j}} + \vec{\mathbf{v}}_0 t + \vec{\mathbf{C}}_2 \end{aligned}$$

with another unknown constant vector $\vec{\mathbf{C}}_2$. To determine the value of $\vec{\mathbf{C}}_2$, we can use the position of the object at a given time, say at time $t = 0$. We call this position the *initial position*: $\vec{\mathbf{s}}(0) = \vec{\mathbf{s}}_0$. Therefore, $\vec{\mathbf{s}}(0) = -(1/2)g(0)^2\,\hat{\mathbf{j}} + \vec{\mathbf{v}}_0(0) + \vec{\mathbf{C}}_2 = \vec{\mathbf{s}}_0$. This gives the position of the object at any time as

$$\vec{\mathbf{s}}(t) = -\frac{1}{2}gt^2\hat{\mathbf{j}} + \vec{\mathbf{v}}_0 t + \vec{\mathbf{s}}_0. \tag{12.3.11}$$

Let's take a closer look at the initial velocity and initial position. In particular, suppose the object is thrown upward from the origin at an angle θ to the horizontal, with initial speed $\vec{\mathbf{v}}_0$. How can we modify the previous result to reflect this scenario? First, we can assume it is thrown from the origin. If not, then we can move the origin to the point from where it is thrown. Therefore, $\vec{\mathbf{s}}_0 = \vec{\mathbf{0}}$, as shown in Figure 12.3.5.

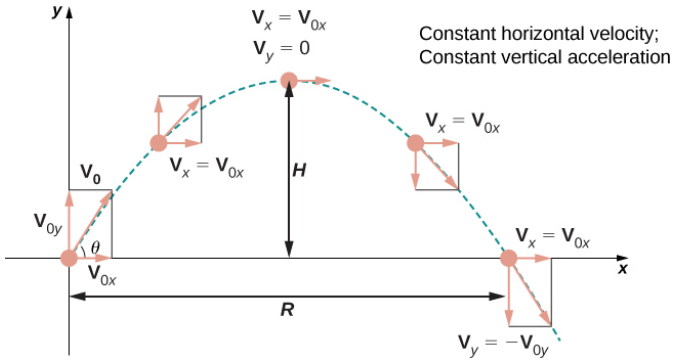

Figure 12.3.5: Projectile motion when the object is thrown upward at an angle θ. The horizontal motion is at constant velocity and the vertical motion is at constant acceleration.

We can rewrite the initial velocity vector in the form $\vec{\mathbf{v}}_0 = v_0\cos\theta\,\hat{\mathbf{i}} + v_0\sin\theta\,\hat{\mathbf{j}}$. Then the equation for the position function $\vec{\mathbf{s}}(t)$ becomes

$$\begin{aligned}\vec{\mathbf{s}}(t) &= -\frac{1}{2}gt^2\hat{\mathbf{j}} + v_0 t\cos\theta\,\hat{\mathbf{i}} + v_0 t\sin\theta\,\hat{\mathbf{j}} \\ &= v_0 t\cos\theta\,\hat{\mathbf{i}} + v_0 t\sin\theta\,\hat{\mathbf{j}} - \frac{1}{2}gt^2\hat{\mathbf{j}} \\ &= v_0 t\cos\theta\,\hat{\mathbf{i}} + \left(v_0 t\sin\theta - \frac{1}{2}gt^2\right)\hat{\mathbf{j}}.\end{aligned}$$

The coefficient of $\hat{\mathbf{i}}$ represents the horizontal component of $\vec{\mathbf{s}}(t)$ and is the horizontal distance of the object from the origin at time t. The maximum value of the horizontal distance (measured at the same initial and final altitude) is called the range R. The coefficient of $\hat{\mathbf{j}}$ represents the vertical component of $\vec{\mathbf{s}}(t)$ and is the altitude of the object at time t. The maximum value of the vertical distance is the height H.

Example 12.3.2: Motion of a Cannonball

During an Independence Day celebration, a cannonball is fired from a cannon on a cliff toward the water. The cannon is aimed at an angle of 30° above horizontal and the initial speed of the cannonball is 600 ft/sec. The cliff is 100 ft above the water (Figure 12.3.6).

a. Find the maximum height of the cannonball.
b. How long will it take for the cannonball to splash into the sea?
c. How far out to sea will the cannonball hit the water?

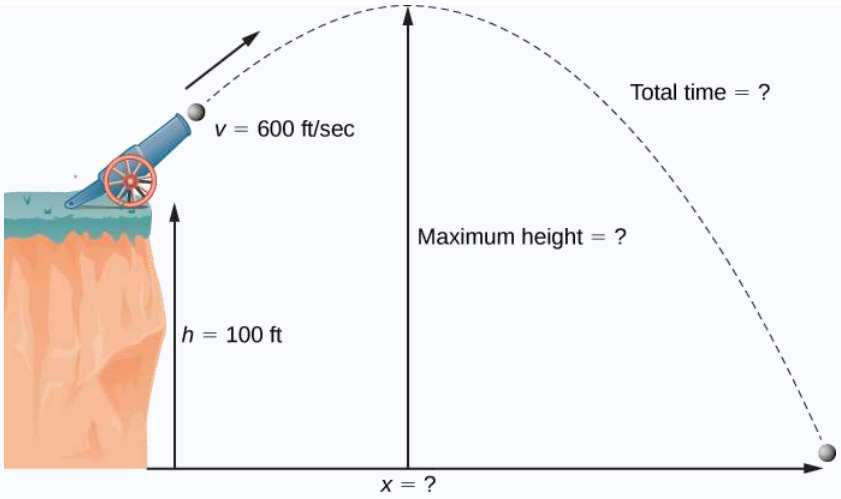

Figure 12.3.6: The flight of a cannonball (ignoring air resistance) is projectile motion.

Solution

We use the equation

$$\vec{\mathbf{s}}(t) = v_0 t\cos\theta\,\hat{\mathbf{i}} + \left(v_0 t\sin\theta - \frac{1}{2}gt^2\right)\hat{\mathbf{j}} \tag{12.3.12}$$

with $\theta = 30°$, $g = 32\frac{\text{ft}}{\text{sec}^2}$, and $v_0 = 600\frac{\text{ft}}{\text{sec}}$. Then the position equation becomes

$$\begin{aligned}\vec{\mathbf{s}}(t) &= 600t(\cos 30°)\,\hat{\mathbf{i}} + \left(600t\sin 30° - \frac{1}{2}(32)t^2\right)\hat{\mathbf{j}}\\ &= 300t\sqrt{3}\,\hat{\mathbf{i}} + \left(300t - 16t^2\right)\hat{\mathbf{j}}\end{aligned}$$

a. The cannonball reaches its maximum height when the vertical component of its velocity is zero, because the cannonball is neither rising nor falling at that point. The velocity vector is

$$\begin{aligned}\vec{\mathbf{v}}(t) &= \vec{\mathbf{s}}\,'(t)\\ &= 300\sqrt{3}\,\hat{\mathbf{i}} + (300 - 32t)\,\hat{\mathbf{j}}\end{aligned}$$

Therefore, the vertical component of velocity is given by the expression $300 - 32t$. Setting this expression equal to zero and solving for t gives $t = 9.375$ sec. The height of the cannonball at this time is given by the vertical component of the position vector, evaluated at $t = 9.375$.

$$\vec{\mathbf{s}}(9.375) = 300(9.375)\sqrt{3}\,\hat{\mathbf{i}} + (300(9.375) - 16(9.375)^2)\,\hat{\mathbf{j}} = 4871.39\,\hat{\mathbf{i}} + 1406.25\,\hat{\mathbf{j}}$$

Therefore, the maximum height of the cannonball is 1406.39 ft above the cannon, or 1506.39 ft above sea level.

b. When the cannonball lands in the water, it is 100 ft below the cannon. Therefore, the vertical component of the position vector is equal to -100. Setting the vertical component of $\vec{\mathbf{s}}(t)$ equal to -100 and solving, we obtain

$$\begin{aligned}300t - 16t^2 &= -100\\ 16t^2 - 300t - 100 &= 0\\ 4t^2 - 75 - 25 &= 0\\ t &= \frac{75 \pm \sqrt{(-75)^2 - 4(4)(-25)}}{2(4)}\\ &= \frac{75 \pm \sqrt{6025}}{8}\\ &= \frac{75 \pm 5\sqrt{241}}{8}\end{aligned}$$

The positive value of t that solves this equation is approximately 19.08. Therefore, the cannonball hits the water after approximately 19.08 sec.

c. To find the distance out to sea, we simply substitute the answer from part (b) into $\vec{\mathbf{s}}(t)$:

$$\begin{aligned}\vec{\mathbf{s}}(19.08) &= 300(19.08)\sqrt{3}\,\hat{\mathbf{i}} + \left(300(19.08) - 16(19.08)^2\right)\hat{\mathbf{j}} \\ &= 9914.26\,\hat{\mathbf{i}} - 100.7424\,\hat{\mathbf{j}}\end{aligned}$$

Therefore, the ball hits the water about 9914.26 ft away from the base of the cliff. Notice that the vertical component of the position vector is very close to −100, which tells us that the ball just hit the water. Note that 9914.26 feet is not the true range of the cannon since the cannonball lands in the ocean at a location below the cannon. The range of the cannon would be determined by finding how far out the cannonball is when its height is 100 ft above the water (the same as the altitude of the cannon).

Exercise 12.3.2

An archer fires an arrow at an angle of 40° above the horizontal with an initial speed of 98 m/sec. The height of the archer is 171.5 cm. Find the horizontal distance the arrow travels before it hits the ground.

One final question remains: In general, what is the maximum distance a projectile can travel, given its initial speed? To determine this distance, we assume the projectile is fired from ground level and we wish it to return to ground level. In other words, we want to determine an equation for the range. In this case, the equation of projectile motion is

$$\vec{\mathbf{s}} = v_0 t\cos\theta\,\hat{\mathbf{i}} + \left(v_0 t\sin\theta - \frac{1}{2}gt^2\right)\hat{\mathbf{j}}. \tag{12.3.13}$$

Setting the second component equal to zero and solving for t yields

$$\begin{aligned}v_0 t\sin\theta - \frac{1}{2}gt^2 &= 0 \\ t\left(v_0\sin\theta - \frac{1}{2}gt\right) &= 0\end{aligned}$$

Therefore, either $t = 0$ or $t = \dfrac{2v_0\sin\theta}{g}$. We are interested in the second value of t, so we substitute this into $\vec{\mathbf{s}}(t)$, which gives

$$\begin{aligned}\vec{\mathbf{s}}\left(\frac{2v_0\sin\theta}{g}\right) &= v_0\left(\frac{2v_0\sin\theta}{g}\right)\cos\theta\,\hat{\mathbf{i}} + \left(v_0\left(\frac{2v_0\sin\theta}{g}\right)\sin\theta - \frac{1}{2}g\left(\frac{2v_0\sin\theta}{g}\right)^2\right)\hat{\mathbf{j}} \\ &= \left(\frac{2v_0^2\sin\theta\cos\theta}{g}\right)\hat{\mathbf{i}} \\ &= \frac{v_0^2\sin 2\theta}{g}\,\hat{\mathbf{i}}.\end{aligned}$$

Thus, the expression for the range of a projectile fired at an angle θ is

$$R = \frac{v_0^2\sin 2\theta}{g}\,\hat{\mathbf{i}}. \tag{12.3.14}$$

The only variable in this expression is θ. To maximize the distance traveled, take the derivative of the coefficient of $\hat{\mathbf{i}}$ with respect to θ and set it equal to zero:

$$\begin{aligned}\frac{d}{d\theta}\left(\frac{v_0^2\sin 2\theta}{g}\right) &= 0 \\ \frac{2v_0^2\cos 2\theta}{g} &= 0 \\ \theta &= 45^\circ\end{aligned}$$

This value of θ) is the smallest positive value that makes the derivative equal to zero. Therefore, in the absence of air resistance, the best angle to fire a projectile (to maximize the range) is at a 45° angle. The distance it travels is given by

$$\vec{\mathbf{s}}\left(\frac{2v_0\sin 45^\circ}{g}\right) = \frac{v_0^2\sin 90^\circ}{g}\,\hat{\mathbf{i}} = \frac{v_0^2}{g}\,\hat{\mathbf{i}} \tag{12.3.15}$$

Therefore, the range for an angle of 45° is $\dfrac{v_0^2}{g}$ units.

Key Concepts

- If $\vec{\mathbf{r}}(t)$ represents the position of an object at time t, then $\vec{\mathbf{r}}\,'(t)$ represents the velocity and $\vec{\mathbf{r}}\,''(t)$ represents the acceleration of the object at time t. The magnitude of the velocity vector is speed.
- Projectile motion can be represented by the vector-valued function

$$\vec{\mathbf{r}}(t) = v_0 t\cos\theta\,\hat{\mathbf{i}} + \left(v_0 t\sin\theta - \frac{1}{2}gt^2\right)\hat{\mathbf{j}}, \tag{12.3.16}$$

  where the initial velocity is given by $\vec{\mathbf{v}}_0 = v_0\cos\theta\,\hat{\mathbf{i}} + v_0\sin\theta\,\hat{\mathbf{j}}$.

Key Equations

- **Velocity**

$$\vec{\mathbf{v}}(t) = \vec{\mathbf{r}}\,'(t)$$

- **Acceleration**

$$\vec{\mathbf{a}}(t) = \vec{\mathbf{v}}\,'(t) = \vec{\mathbf{r}}\,''(t)$$

- **Speed**

$$v(t) = ||\vec{\mathbf{v}}(t)|| = ||\vec{\mathbf{r}}\,'(t)|| = \frac{ds}{dt}$$

- **Projectile Motion**

$$\vec{\mathbf{r}}(t) = v_0 t\cos\theta\,\hat{\mathbf{i}} + \left(v_0 t\sin\theta - \frac{1}{2}gt^2\right)\hat{\mathbf{j}} \tag{12.3.17}$$

Glossary

Contributors

- Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.
- Edited by Paul Seeburger
  Paul Seeburger added finding point $(1, 2)$ when $t = 1$ and added part 3 (finding the equations of the tangent line) in Example 12.3.1.
  He also created Figure 12.3.1.

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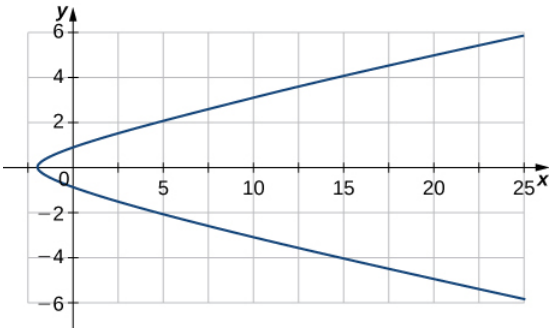
12.3E: Exercises for Section 12.3

Motion in Space

1) Given $\vec{\mathbf{r}}(t) = (3t^2 - 2)\,\hat{\mathbf{i}} + (2t - \sin t)\,\hat{\mathbf{j}}$,

a. find the velocity of a particle moving along this curve.

b. find the acceleration of a particle moving along this curve.



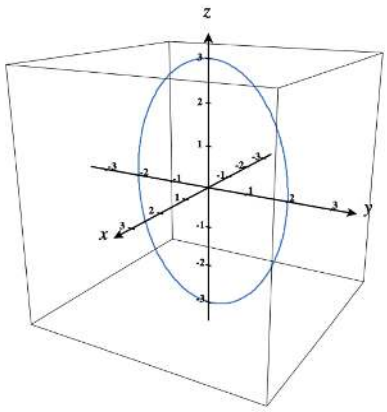
In questions 2 - 5, given the position function, find the velocity, acceleration, and speed in terms of the parameter t.

2) $\vec{\mathbf{r}}(t) = e^{-t}\,\hat{\mathbf{i}} + t^2\,\hat{\mathbf{j}} + \tan t\,\hat{\mathbf{k}}$

3) $\vec{\mathbf{r}}(t) = \langle 3\cos t,\ 3\sin t,\ t^2 \rangle$

4) $\vec{\mathbf{r}}(t) = t^5\,\hat{\mathbf{i}} + (3t^2 + 2t - 5)\,\hat{\mathbf{j}} + (3t - 1)\,\hat{\mathbf{k}}$

5) $\vec{\mathbf{r}}(t) = 2\cos t\,\hat{\mathbf{j}} + 3\sin t\,\hat{\mathbf{k}}$. The graph is shown here:

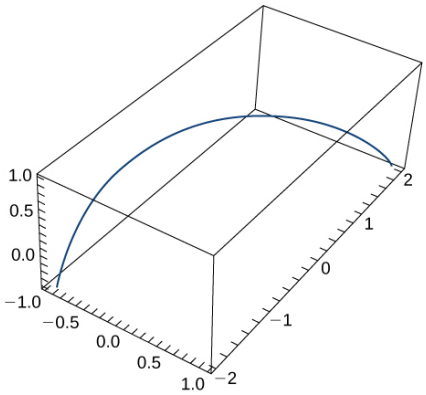


In questions 6 - 8, find the velocity, acceleration, and speed of a particle with the given position function.

6) $\vec{\mathbf{r}}(t) = \langle t^2 - 1, t \rangle$

7) $\vec{\mathbf{r}}(t) = \langle e^t, e^{-t} \rangle$

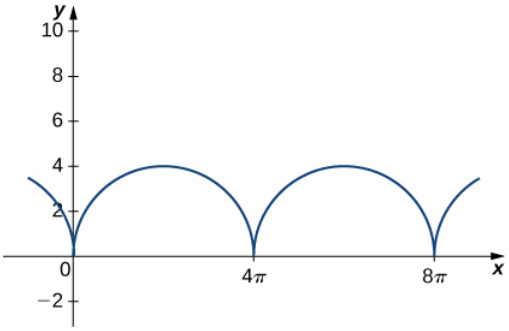
8) $\vec{\mathbf{r}}(t) = \langle \sin t, t, \cos t \rangle$. The graph is shown here:



9) The position function of an object is given by $\vec{\mathbf{r}}(t) = \langle t^2, 5t, t^2 - 16t \rangle$. At what time is the speed a minimum?

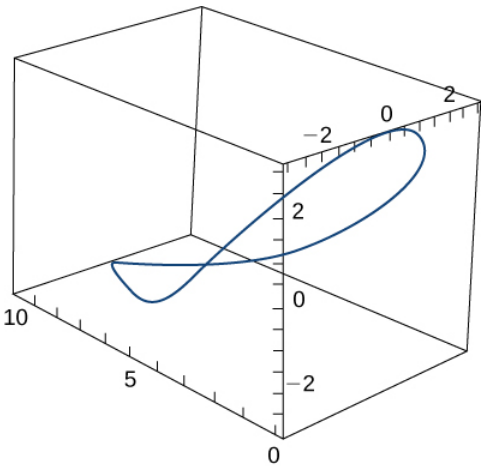
10) Let $\vec{\mathbf{r}}(t) = r\cosh(\omega t)\,\hat{\mathbf{i}} + r\sinh(\omega t)\,\hat{\mathbf{j}}$. Find the velocity and acceleration vectors and show that the acceleration is proportional to $\vec{\mathbf{r}}(t)$.

11) Consider the motion of a point on the circumference of a rolling circle. As the circle rolls, it generates the cycloid $\vec{\mathbf{r}}(t) = (\omega t - \sin(\omega t))\,\hat{\mathbf{i}} + (1 - \cos(\omega t))\,\hat{\mathbf{j}}$, where ω is the angular velocity of the circle and b is the radius of the circle:



Find the equations for the velocity, acceleration, and speed of the particle at any time.

12) A person on a hang glider is spiraling upward as a result of the rapidly rising air on a path having position vector $\vec{\mathbf{r}}(t) = (3\cos t)\,\hat{\mathbf{i}} + (3\sin t)\,\hat{\mathbf{j}} + t^2\,\hat{\mathbf{k}}$. The path is similar to that of a helix, although it is not a helix. The graph is shown here:



Find the following quantities:

a. The velocity and acceleration vectors

b. The glider's speed at any time

c. The times, if any, at which the glider's acceleration is orthogonal to its velocity

13) Given that $\vec{\mathbf{r}}(t) = \langle e^{-5t}\sin t,\ e^{-5t}\cos t,\ 4e^{-5t}\rangle$ is the position vector of a moving particle, find the following quantities:

a. The velocity of the particle

b. The speed of the particle

c. The acceleration of the particle

14) Find the maximum speed of a point on the circumference of an automobile tire of radius 1 ft when the automobile is traveling at 55 mph.

15) Find the position vector-valued function $\vec{\mathbf{r}}(t)$, given that $\vec{\mathbf{a}}(t) = \hat{\mathbf{i}} + e^t\,\hat{\mathbf{j}}$, $\vec{\mathbf{v}}(0) = 2\,\hat{\mathbf{j}}$, and $\vec{\mathbf{r}}(0) = 2\,\hat{\mathbf{i}}$.

16) Find $\vec{\mathbf{r}}(t)$ given that $\vec{\mathbf{a}}(t) = -32\,\hat{\mathbf{j}}, \vec{\mathbf{v}}(0) = 600\sqrt{3}\,\hat{\mathbf{i}} + 600\,\hat{\mathbf{j}}$, and $\vec{\mathbf{r}}(0) = \vec{\mathbf{0}}$.

17) The acceleration of an object is given by $\vec{\mathbf{a}}(t) = t\,\hat{\mathbf{j}} + t\,\hat{\mathbf{k}}$. The velocity at $t = 1$ sec is $\vec{\mathbf{v}}(1) = 5\,\hat{\mathbf{j}}$ and the position of the object at $t = 1$ sec is $\vec{\mathbf{r}}(1) = 0\,\hat{\mathbf{i}} + 0\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}$. Find the object's position at any time.

Projectile Motion

18) A projectile is shot in the air from ground level with an initial velocity of 500 m/sec at an angle of 60° with the horizontal.

a. At what time does the projectile reach maximum height?

b. What is the approximate maximum height of the projectile?

c. At what time is the maximum range of the projectile attained?

d. What is the maximum range?

e. What is the total flight time of the projectile?

19) A projectile is fired at a height of 1.5 m above the ground with an initial velocity of 100 m/sec and at an angle of 30° above the horizontal. Use this information to answer the following questions:

a. Determine the maximum height of the projectile.

b. Determine the range of the projectile.

20) A golf ball is hit in a horizontal direction off the top edge of a building that is 100 ft tall. How fast must the ball be launched to land 450 ft away?

21) A projectile is fired from ground level at an angle of 8° with the horizontal. The projectile is to have a range of 50 m. Find the minimum velocity (speed) necessary to achieve this range.

22) Prove that an object moving in a straight line at a constant speed has an acceleration of zero.

Contributors

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12.4: Arc Length and Curvature

In this section, we study formulas related to curves in both two and three dimensions, and see how they are related to various properties of the same curve. For example, suppose a vector-valued function describes the motion of a particle in space. We would like to determine how far the particle has traveled over a given time interval, which can be described by the arc length of the path it follows. Or, suppose that the vector-valued function describes a road we are building and we want to determine how sharply the road curves at a given point. This is described by the curvature of the function at that point. We explore each of these concepts in this section.

Arc Length for Vector Functions

We have seen how a vector-valued function describes a curve in either two or three dimensions. Recall that the formula for the arc length of a curve defined by the parametric functions $x = x(t)$ and $y = y(t)$, for $t_1 \leq t \leq t_2$ is given by

$$s = \int_{t_1}^{t_2} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt. \tag{12.4.1}$$

In a similar fashion, if we define a smooth curve using a vector-valued function $\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}}$, where $a \leq t \leq b$, the arc length is given by the formula

$$s = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}\, dt. \tag{12.4.2}$$

In three dimensions, if the vector-valued function is described by $\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} + h(t)\,\hat{\mathbf{k}}$ over the same interval $a \leq t \leq b$, the arc length is given by

$$s = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}\, dt. \tag{12.4.3}$$

Theorem: Arc-Length Formulas for Plane and Space curves

Plane curve: Given a smooth curve C defined by the function $\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}}$, where t lies within the interval $[a, b]$, the arc length of C over the interval is

$$s = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt \tag{12.4.4}$$

$$= \int_a^b \|\vec{\mathbf{r}}\,'(t)\|\, dt. \tag{12.4.5}$$

Space curve: Given a smooth curve C defined by the function $\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} + h(t)\,\hat{\mathbf{k}}$, where t lies within the interval $[a, b]$, the arc length of C over the interval is

$$s = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt \tag{12.4.6}$$

$$= \int_a^b \|\vec{\mathbf{r}}\,'(t)\|\, dt. \tag{12.4.7}$$

The two formulas are very similar; they differ only in the fact that a space curve has three component functions instead of two. Note that the formulas are defined for smooth curves: curves where the vector-valued function $\vec{\mathbf{r}}(t)$ is differentiable with a non-zero derivative. The smoothness condition guarantees that the curve has no cusps (or corners) that could make the formula problematic.

Example 12.4.1: Finding the Arc Length

Calculate the arc length for each of the following vector-valued functions:

a. $\vec{\mathbf{r}}(t) = (3t - 2)\,\hat{\mathbf{i}} + (4t + 5)\,\hat{\mathbf{j}}, \quad 1 \leq t \leq 5$
b. $\vec{\mathbf{r}}(t) = \langle t\cos t, t\sin t, 2t\rangle, 0 \leq t \leq 2\pi$

Solution

a. Using Equation 12.4.5, $\vec{\mathbf{r}}\,'(t) = 3\,\hat{\mathbf{i}} + 4\,\hat{\mathbf{i}}$, so

$$\begin{aligned} s &= \int_a^b \|\vec{\mathbf{r}}\,'(t)\|\,dt \\ &= \int_1^5 \sqrt{3^2+4^2}\,dt \\ &= \int_1^5 5\,dt = 5t\big|_1^5 = 20. \end{aligned}$$

b. Using Equation 12.4.7, $\vec{\mathbf{r}}\,'(t) = \langle \cos t - t\sin t, \sin t + t\cos t, 2\rangle$, so

$$\begin{aligned} s &= \int_a^b \| \vec{\mathbf{r}}\,'(t) \|\,dt \\ &= \int_0^{2\pi} \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + 2^2}\,dt \\ &= \int_0^{2\pi} \sqrt{(\cos^2 t - 2t\sin t\cos t + t^2\sin^2 t) + (\sin^2 t + 2t\sin t\cos t + t^2\cos^2 t) + 4}\,dt \\ &= \int_0^{2\pi} \sqrt{\cos^2 t + \sin^2 t + t^2(\cos^2 t + \sin^2 t) + 4}\,dt \\ &= \int_0^{2\pi} \sqrt{t^2+5}\,dt \end{aligned}$$

Here we can use a table integration formula

$$\int \sqrt{u^2+a^2}du = \frac{u}{2}\sqrt{u^2+a^2} + \frac{a^2}{2}\ln\left|u+\sqrt{u^2+a^2}\right| + C,$$

so we obtain

$$\begin{aligned} \int_0^{2\pi} \sqrt{t^2+5}\,dt &= \frac{1}{2}\left(t\sqrt{t^2+5} + 5\ln\left|t+\sqrt{t^2+5}\right|\right)_0^{2\pi} \\ &= \frac{1}{2}\left(2\pi\sqrt{4\pi^2+5} + 5\ln\left(2\pi+\sqrt{4\pi^2+5}\right)\right) - \frac{5}{2}\ln\sqrt{5} \\ &\approx 25.343 \text{ units.} \end{aligned}$$

Exercise 12.4.1

Calculate the arc length of the parameterized curve

$$\vec{\mathbf{r}}(t) = \langle 2t^2+1, 2t^2-1, t^3\rangle, \quad 0 \le t \le 3.$$

We now return to the helix introduced earlier in this chapter. A vector-valued function that describes a helix can be written in the form

$$\vec{\mathbf{r}}(t) = R\cos\left(\frac{2\pi Nt}{h}\right)\hat{\mathbf{i}} + R\sin\left(\frac{2\pi Nt}{h}\right)\hat{\mathbf{j}} + t\hat{\mathbf{k}}, 0 \le t \le h, \tag{12.4.8}$$

where R represents the radius of the helix, h represents the height (distance between two consecutive turns), and the helix completes N turns. Let's derive a formula for the arc length of this helix using Equation 12.4.7. First of all,

$$\vec{\mathbf{r}}\,'(t) = -\frac{2\pi NR}{h}\sin\left(\frac{2\pi Nt}{h}\right)\hat{\mathbf{i}} + \frac{2\pi NR}{h}\cos\left(\frac{2\pi Nt}{h}\right)\hat{\mathbf{j}} + \hat{\mathbf{k}}. \tag{12.4.9}$$

Therefore,

$$s = \int_a^b \|\vec{\mathbf{r}}\,'(t)\|\,dt \tag{12.4.10}$$

$$= \int_0^h \sqrt{\left(-\frac{2\pi NR}{h}\sin\left(\frac{2\pi Nt}{h}\right)\right)^2 + \left(\frac{2\pi NR}{h}\cos\left(\frac{2\pi Nt}{h}\right)\right)^2 + 1^2}\,dt \tag{12.4.11}$$

$$= \int_0^h \sqrt{\frac{4\pi^2N^2R^2}{h^2}\left(\sin^2\left(\frac{2\pi Nt}{h}\right) + \cos^2\left(\frac{2\pi Nt}{h}\right)\right) + 1}\,dt \tag{12.4.12}$$

$$= \int_0^h \sqrt{\frac{4\pi^2N^2R^2}{h^2} + 1}\,dt \tag{12.4.13}$$

$$= \left[t\sqrt{\frac{4\pi^2N^2R^2}{h^2} + 1}\right]_0^h \tag{12.4.14}$$

$$= h\sqrt{\frac{4\pi^2N^2R^2 + h^2}{h^2}} \tag{12.4.15}$$

$$= \sqrt{4\pi^2N^2R^2 + h^2}. \tag{12.4.16}$$

This gives a formula for the length of a wire needed to form a helix with N turns that has radius R and height h.

Arc-Length Parameterization

We now have a formula for the arc length of a curve defined by a vector-valued function. Let's take this one step further and examine what an **arc-length function** is.

If a vector-valued function represents the position of a particle in space as a function of time, then the arc-length function measures how far that particle travels as a function of time. The formula for the arc-length function follows directly from the formula for arc length:

$$s = \int_a^t \sqrt{(f\,'(u))^2 + (g\,'(u))^2 + (h\,'(u))^2}\,du. \tag{12.4.17}$$

If the curve is in two dimensions, then only two terms appear under the square root inside the integral. The reason for using the independent variable u is to distinguish between time and the variable of integration. Since $s(t)$ measures distance traveled as a function of time, $s\,'(t)$ measures the speed of the particle at any given time. Since we have a formula for $s(t)$ in Equation 12.4.17, we can differentiate both sides of the equation:

$$s\,'(t) = \frac{d}{dt}\left[\int_a^t \sqrt{(f\,'(u))^2 + (g\,'(u))^2 + (h\,'(u))^2}\,du\right] \tag{12.4.18}$$

$$= \frac{d}{dt}\left[\int_a^t \|\vec{\mathbf{r}}\,'(u)\|\,du\right] \tag{12.4.19}$$

$$= \|\vec{\mathbf{r}}\,'(t)\|. \tag{12.4.20}$$

If we assume that $\vec{\mathbf{r}}(t)$ defines a smooth curve, then the arc length is always increasing, so $s\,'(t) > 0$ for $t > a$. Last, if $\vec{\mathbf{r}}(t)$ is a curve on which $\|\vec{\mathbf{r}}\,'(t)\| = 1$ for all t, then

$$s(t) = \int_a^t \|\vec{\mathbf{r}}\,'(u)\|\,du = \int_a^t 1\,du = t - a, \tag{12.4.21}$$

which means that t represents the arc length as long as $a = 0$.

Theorem: Arc-Length Function

Let $\vec{\mathbf{r}}(t)$ describe a smooth curve for $t \geq a$. Then the arc-length function is given by

$$s(t) = \int_a^t \|\vec{\mathbf{r}}\,'(u)\|\,du \tag{12.4.22}$$

Furthermore, $\frac{ds}{dt} = \|\vec{\mathbf{r}}'(t)\| > 0$. If $\|\vec{\mathbf{r}}'(t)\| = 1$ for all $t \geq a$, then the parameter t represents the arc length from the starting point at $t = a$.

A useful application of this theorem is to find an alternative parameterization of a given curve, called an **arc-length parameterization**. Recall that any vector-valued function can be reparameterized via a change of variables. For example, if we have a function $\vec{\mathbf{r}}(t) = \langle 3\cos t, 3\sin t\rangle, 0 \leq t \leq 2\pi$ that parameterizes a circle of radius 3, we can change the parameter from t to $4t$, obtaining a new parameterization $\vec{\mathbf{r}}(t) = \langle 3\cos 4t, 3\sin 4t\rangle$. The new parameterization still defines a circle of radius 3, but now we need only use the values $0 \leq t \leq \pi/2$ to traverse the circle once.

Suppose that we find the arc-length function $s(t)$ and are able to solve this function for t as a function of s. We can then reparameterize the original function $\vec{\mathbf{r}}(t)$ by substituting the expression for t back into $\vec{\mathbf{r}}(t)$. The vector-valued function is now written in terms of the parameter s. Since the variable s represents the arc length, we call this an *arc-length parameterization* of the original function $\vec{\mathbf{r}}(t)$. One advantage of finding the arc-length parameterization is that the distance traveled along the curve starting from $s = 0$ is now equal to the parameter s. The arc-length parameterization also appears in the context of curvature (which we examine later in this section) and line integrals.

Example 12.4.2: Finding an Arc-Length Parameterization

Find the arc-length parameterization for each of the following curves:

a. $\vec{\mathbf{r}}(t) = 4\cos t\,\hat{\mathbf{i}} + 4\sin t\,\hat{\mathbf{j}}, \quad t \geq 0$
b. $\vec{\mathbf{r}}(t) = \langle t+3, 2t-4, 2t\rangle, \quad t \geq 3$

Solution

a. First we find the arc-length function using Equation 12.4.17:

$$\begin{aligned} s(t) &= \int_a^t \|\vec{\mathbf{r}}'(u)\|\,du \\ &= \int_0^t \|\langle -4\sin u, 4\cos u\rangle\|\,du \\ &= \int_0^t \sqrt{(-4\sin u)^2 + (4\cos u)^2}\,du \\ &= \int_0^t \sqrt{16\sin^2 u + 16\cos^2 u}\,du \\ &= \int_0^t 4\,du = 4t, \end{aligned}$$

which gives the relationship between the arc length s and the parameter t as $s = 4t$; so, $t = s/4$. Next we replace the variable t in the original function $\vec{\mathbf{r}}(t) = 4\cos t\,\hat{\mathbf{i}} + 4\sin t\,\hat{\mathbf{j}}$ with the expression $s/4$ to obtain

$$\vec{\mathbf{r}}(s) = 4\cos\left(\frac{s}{4}\right)\hat{\mathbf{i}} + 4\sin\left(\frac{s}{4}\right)\hat{\mathbf{j}}.$$

This is the arc-length parameterization of $\vec{\mathbf{r}}(t)$. Since the original restriction on t was given by $t \geq 0$, the restriction on s becomes $s/4 \geq 0$, or $s \geq 0$.

b. The arc-length function is given by Equation 12.4.17:

$$\begin{aligned} s(t) &= \int_a^t \|\vec{\mathbf{r}}\,'(u)\|\,du \\ &= \int_3^t \|\langle 1,2,2\rangle\|\,du \\ &= \int_3^t \sqrt{1^2+2^2+2^2}\,du \\ &= \int_3^t 3\,du \\ &= 3t-9. \end{aligned}$$

Therefore, the relationship between the arc length s and the parameter t is $s=3t-9$, so $t=\frac{s}{3}+3$. Substituting this into the original function $\vec{\mathbf{r}}(t)=\langle t+3,2t-4,2t\rangle$ yields

$$\vec{\mathbf{r}}(s)=\langle\left(\frac{s}{3}+3\right)+3,\,2\left(\frac{s}{3}+3\right)-4,\,2\left(\frac{s}{3}+3\right)\rangle=\langle\frac{s}{3}+6,\frac{2s}{3}+2,\frac{2s}{3}+6\rangle.$$

This is an arc-length parameterization of $\vec{\mathbf{r}}(t)$. The original restriction on the parameter t was $t\geq 3$, so the restriction on s is $(s/3)+3\geq 3$, or $s\geq 0$.

Exercise 12.4.2

Find the arc-length function for the helix

$$\vec{\mathbf{r}}(t)=\langle 3\cos t,3\sin t,4t\rangle,\quad t\geq 0.$$

Then, use the relationship between the arc length and the parameter t to find an arc-length parameterization of $\vec{\mathbf{r}}(t)$.

Curvature

An important topic related to arc length is curvature. The concept of curvature provides a way to measure how sharply a smooth curve turns. A circle has constant curvature. The smaller the radius of the circle, the greater the curvature.

Think of driving down a road. Suppose the road lies on an arc of a large circle. In this case you would barely have to turn the wheel to stay on the road. Now suppose the radius is smaller. In this case you would need to turn more sharply to stay on the road. In the case of a curve other than a circle, it is often useful first to inscribe a circle to the curve at a given point so that it is tangent to the curve at that point and "hugs" the curve as closely as possible in a neighborhood of the point (Figure 12.4.1). The curvature of the graph at that point is then defined to be the same as the curvature of the inscribed circle.

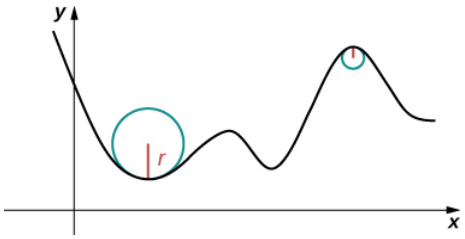


Figure 12.4.1: The graph represents the curvature of a function $y=f(x)$. *The sharper the turn in the graph, the greater the curvature, and the smaller the radius of the inscribed circle.*

Definition: curvature

Let C be a smooth curve in the plane or in space given by $\vec{\mathbf{r}}(s)$, where s is the arc-length parameter. The curvature κ at s is

$$\kappa=\left\|\frac{d\vec{\mathbf{T}}}{ds}\right\|=\|\vec{\mathbf{T}}\,'(s)\|. \tag{12.4.23}$$

As the unit vector $\vec{\mathbf{T}}(t)$ is not changing in length, the curvature could be described as **a measure of the speed at which the *direction* of the motion is changing**. Since the parameter here is s, we have forced the varying speed of travel along the path out of this measurement.

The formula in the definition of curvature is not very useful in terms of calculation. In particular, recall that $\vec{\mathbf{T}}(t)$ represents the unit tangent vector to a given vector-valued function $\vec{\mathbf{r}}(t)$, and the formula for $\vec{\mathbf{T}}(t)$ is

$$\vec{\mathbf{T}}(t) = \frac{\vec{\mathbf{r}}\,'(t)}{\| \vec{\mathbf{r}}\,'(t) \|}. \tag{12.4.24}$$

To use the formula for curvature, it is first necessary to express $\vec{\mathbf{r}}(t)$ in terms of the arc-length parameter s, then find the unit tangent vector $\vec{\mathbf{T}}(s)$ for the function $\vec{\mathbf{r}}(s)$, then take the derivative of $\vec{\mathbf{T}}(s)$ with respect to s. This is a tedious process. Fortunately, there are equivalent formulas for curvature.

Theorem: Alternate Formulas of Curvature

If C is a smooth curve given by $\vec{\mathbf{r}}(t)$, then the curvature κ of C at t is given by

$$\kappa = \frac{\|\vec{\mathbf{T}}\,'(t)\|}{\|\vec{\mathbf{r}}\,'(t)\|}. \tag{12.4.25}$$

If C is a three-dimensional curve, then the curvature can be given by the formula

$$\kappa = \frac{\|\vec{\mathbf{r}}\,'(t) \times \vec{\mathbf{r}}\,''(t)\|}{\|\vec{\mathbf{r}}\,'(t)\|^3}. \tag{12.4.26}$$

Note that in the context of motion, this formula could also be written

$$\kappa = \frac{\|\vec{\mathbf{v}}(t) \times \vec{\mathbf{a}}(t)\|}{\|\vec{\mathbf{v}}(t)\|^3}.$$

If C is the graph of a function $y = f(x)$ and both y' and y'' exist, then the curvature κ at point (x, y) is given by

$$\kappa = \frac{|y''|}{\left[1 + (y')^2\right]^{3/2}}. \tag{12.4.27}$$

Proof

The first formula follows directly from the chain rule:

$$\frac{d\vec{\mathbf{T}}}{dt} = \frac{d\vec{\mathbf{T}}}{ds}\frac{ds}{dt},$$

where s is the arc length along the curve C. Dividing both sides by ds/dt, and taking the magnitude of both sides gives

$$\left\| \frac{d\vec{\mathbf{T}}}{ds} \right\| = \left\| \frac{\vec{\mathbf{T}}\,'(t)}{\frac{ds}{dt}} \right\|.$$

Since $ds/dt = \|\vec{\mathbf{r}}\,'(t)\|$, this gives the formula for the curvature κ of a curve C in terms of any parameterization of C:

$$\kappa = \frac{\|\vec{\mathbf{T}}\,'(t)\|}{\|\vec{\mathbf{r}}\,'(t)\|}.$$

In the case of a three-dimensional curve, we start with the formulas $\vec{\mathbf{T}}(t) = (\vec{\mathbf{r}}\,'(t))/\|\vec{\mathbf{r}}\,'(t)\|$ and $ds/dt = \|\vec{\mathbf{r}}\,'(t)\|$. Therefore, $\vec{\mathbf{r}}\,'(t) = (ds/dt)\vec{\mathbf{T}}(t)$. We can take the derivative of this function using the scalar product formula:

$$\vec{\mathbf{r}}\,''(t) = \frac{d^2s}{dt^2}\vec{\mathbf{T}}(t) + \frac{ds}{dt}\vec{\mathbf{T}}\,'(t).$$

Using these last two equations we get

$$\begin{aligned} \vec{\mathbf{r}}\,'(t) \times \vec{\mathbf{r}}\,''(t) &= \frac{ds}{dt}\vec{\mathbf{T}}(t) \times \left(\frac{d^2s}{dt^2}\vec{\mathbf{T}}(t) + \frac{ds}{dt}\vec{\mathbf{T}}\,'(t) \right) \\ &= \frac{ds}{dt}\frac{d^2s}{dt^2}\vec{\mathbf{T}}(t) \times \vec{\mathbf{T}}(t) + \left(\frac{ds}{dt}\right)^2 \vec{\mathbf{T}}(t) \times \vec{\mathbf{T}}\,'(t). \end{aligned}$$

Since $\vec{\mathbf{T}}(t) \times \vec{\mathbf{T}}(t) = \mathbf{0}$, this reduces to

$$\vec{\mathbf{r}}\,'(t) \times \vec{\mathbf{r}}\,''(t) = \left(\frac{ds}{dt}\right)^2 \vec{\mathbf{T}}(t) \times \vec{\mathbf{T}}\,'(t).$$

Since $\vec{\mathbf{T}}'$ is parallel to $\vec{\mathbf{N}}$, and $\vec{\mathbf{T}}$ is orthogonal to $\vec{\mathbf{N}}$, it follows that $\vec{\mathbf{T}}$ and $\vec{\mathbf{T}}'$ are orthogonal. This means that $\|\vec{\mathbf{T}} \times \vec{\mathbf{T}}'\| = \|\vec{\mathbf{T}}\|\|\vec{\mathbf{T}}\,'\| \sin(\pi/2) = \|\vec{\mathbf{T}}'\|$, so

$$\|\vec{\mathbf{r}}\,'(t) \times \vec{\mathbf{r}}\,''(t)\| = \left(\frac{ds}{dt}\right)^2 \|\vec{\mathbf{T}}\,'(t)\|.$$

Now we solve this equation for $\|\vec{\mathbf{T}}\,'(t)\|$ and use the fact that $ds/\,dt = \|\vec{\mathbf{r}}\,'(t)\|$:

$$\|\vec{\mathbf{T}}\,'(t)\| = \frac{\|\vec{\mathbf{r}}\,'(t) \times \vec{\mathbf{r}}\,''(t)\|}{\|\vec{\mathbf{r}}\,'(t)\|^2}.$$

Then, we divide both sides by $\|\vec{\mathbf{r}}\,'(t)\|$. This gives

$$\kappa = \frac{\|\vec{\mathbf{T}}\,'(t)\|}{\|\vec{\mathbf{r}}\,'(t)\|} = \frac{\|\vec{\mathbf{r}}\,'(t) \times \vec{\mathbf{r}}\,''(t)\|}{\|\vec{\mathbf{r}}\,'(t)\|^3}.$$

This proves 12.4.26. To prove 12.4.27, we start with the assumption that curve C is defined by the function $y = f(x)$. Then, we can define $\vec{\mathbf{r}}(t) = x\,\hat{\mathbf{i}} + f(x)\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}$. Using the previous formula for curvature:

$$\begin{aligned}
\vec{\mathbf{r}}\,'(t) &= \hat{\mathbf{i}} + f\,'(x)\,\hat{\mathbf{j}} \\
\vec{\mathbf{r}}\,''(t) &= f\,''(x)\,\hat{\mathbf{j}} \\
\vec{\mathbf{r}}\,'(t) \times \vec{\mathbf{r}}\,''(t) &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & f\,'(x) & 0 \\ 0 & f\,''(x) & 0 \end{vmatrix} = f\,''(x)\,\hat{\mathbf{k}}.
\end{aligned}$$

Therefore,

$$\kappa = \frac{\|\vec{\mathbf{r}}\,'(t) \times \vec{\mathbf{r}}\,''(t)\|}{\|\vec{\mathbf{r}}\,'(t)\|^3} = \frac{|f\,''(x)|}{(1 + [f\,'(x)]^2)^{3/2}}$$

Note that in the context of motion, the first alternate formula can be rewritten

$$\kappa = \frac{\|\vec{\mathbf{T}}\,'(t)\|}{\|\vec{\mathbf{v}}(t)\|}. \tag{12.4.28}$$

Considering what was stated above about the curvature being a measure of the speed at which the direction of motion is changing at a point, notice that we have normalized this measurement this time by dividing out the actual speed of travel along the curve at the given instant.

As indicated above, the second alternate formula can be rewritten as

$$\kappa = \frac{\|\vec{\mathbf{v}}(t) \times \vec{\mathbf{a}}(t)\|}{\|\vec{\mathbf{v}}(t)\|^3}. \tag{12.4.29}$$

Remember that the cross product measures the extent to which the two vectors are aligned, returning its largest value when they are orthogonal. Note also that the component of acceleration that is orthogonal to the velocity is the only part we care about here, as it is the component that will affect the change of direction of the motion. The component of acceleration that is aligned with the velocity (and tangent to the curve) will only affect the speed of the motion, which is being ignored in our measurement of curvature. Note that one factor of $\frac{1}{\|\vec{\mathbf{v}}(t)\|}$ will multiply into the cross product to make the first vector be $\vec{\mathbf{T}}(t)$.

Example 12.4.3: Finding Curvature

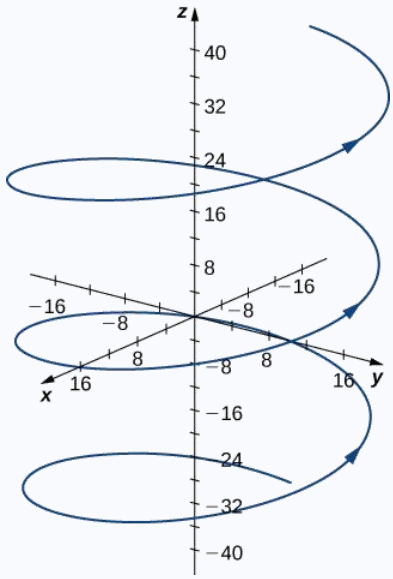
Find the curvature for each of the following curves at the given point:

a. $\vec{\mathbf{r}}(t) = 4\cos t\,\hat{\mathbf{i}} + 4\sin t\,\hat{\mathbf{j}} + 3t\,\hat{\mathbf{k}}, \quad t = \dfrac{4\pi}{3}$

b. $f(x) = \sqrt{4x - x^2}, \quad x = 2$

Solution

a. This function describes a helix.



The curvature of the helix at $t = (4\pi)/3$ can be found by using 12.4.25. First, calculate $\vec{\mathbf{T}}(t)$:

$$\begin{aligned}\vec{\mathbf{T}}(t) &= \frac{\vec{\mathbf{r}}\,'(t)}{\|\vec{\mathbf{r}}\,'(t)\|} \\ &= \frac{\langle -4\sin t, 4\cos t, 3\rangle}{\sqrt{(-4\sin t)^2 + (4\cos t)^2 + 3^2}} \\ &= \langle -\frac{4}{5}\sin t, \frac{4}{5}\cos t, \frac{3}{5}\rangle.\end{aligned}$$

Next, calculate $\vec{\mathbf{T}}\,'(t)$:

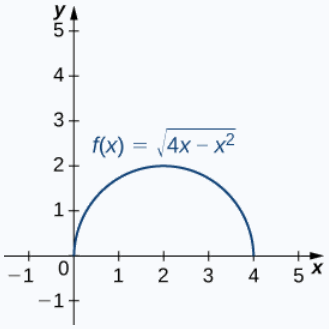
$$\vec{\mathbf{T}}\,'(t) = \langle -\frac{4}{5}\cos t, -\frac{4}{5}\sin t, 0\rangle.$$

Last, apply 12.4.25:

$$\begin{aligned}\kappa &= \frac{\|\vec{\mathbf{T}}\,'(t)\|}{\|\vec{\mathbf{r}}\,'(t)\|} = \frac{\|\langle -\frac{4}{5}\cos t, -\frac{4}{5}\sin t, 0\rangle\|}{\|\langle -4\sin t, 4\cos t, 3\rangle\|} \\ &= \frac{\sqrt{(-\frac{4}{5}\cos t)^2 + (-\frac{4}{5}\sin t)^2 + 0^2}}{\sqrt{(-4\sin t)^2 + (4\cos t)^2 + 3^2}} \\ &= \frac{4/5}{5} = \frac{4}{25}.\end{aligned}$$

The curvature of this helix is constant at all points on the helix.

2. This function describes a semicircle.



To find the curvature of this graph, we must use 12.4.27. First, we calculate y' and y'':

$$\begin{aligned} y &= \sqrt{4x-x^2} = (4x-x^2)^{1/2} \\ y' &= \frac{1}{2}(4x-x^2)^{-1/2}(4-2x) = (2-x)(4x-x^2)^{-1/2} \\ y'' &= -(4x-x^2)^{-1/2} + (2-x)(-\frac{1}{2})(4x-x^2)^{-3/2}(4-2x) \\ &= -\frac{4x-x^2}{(4x-x^2)^{3/2}} - \frac{(2-x)^2}{(4x-x^2)^{3/2}} \\ &= \frac{x^2-4x-(4-4x+x^2)}{(4x-x^2)^{3/2}} \\ &= -\frac{4}{(4x-x^2)^{3/2}}. \end{aligned}$$

Then, we apply 12.4.27:

$$\begin{aligned} \kappa &= \frac{|y''|}{[1+(y')^2]^{3/2}} \\ &= \frac{\left|-\frac{4}{(4x-x^2)^{3/2}}\right|}{\left[1+((2-x)(4x-x^2)^{-1/2})^2\right]^{3/2}} = \frac{\left|\frac{4}{(4x-x^2)^{3/2}}\right|}{\left[1+\frac{(2-x)^2}{4x-x^2}\right]^{3/2}} \\ &= \frac{\left|\frac{4}{(4x-x^2)^{3/2}}\right|}{\left[\frac{4x-x^2+x^2-4x+4}{4x-x^2}\right]^{3/2}} = \left|\frac{4}{(4x-x^2)^{3/2}}\right| \cdot \frac{(4x-x^2)^{3/2}}{8} \\ &= \frac{1}{2}. \end{aligned}$$

The curvature of this circle is equal to the reciprocal of its radius. There is a minor issue with the absolute value in 12.4.27; however, a closer look at the calculation reveals that the denominator is positive for any value of x.

Exercise 12.4.3

Find the curvature of the curve defined by the function

$$y = 3x^2 - 2x + 4$$

at the point $x = 2$.

The Normal and Binormal Vectors

We have seen that the derivative $\vec{\mathbf{r}}\,'(t)$ of a vector-valued function is a tangent vector to the curve defined by $\vec{\mathbf{r}}(t)$, and the unit tangent vector $\vec{\mathbf{T}}(t)$ can be calculated by dividing $\vec{\mathbf{r}}\,'(t)$ by its magnitude. When studying motion in three dimensions, two other vectors are useful in describing the motion of a particle along a path in space: the principal unit normal vector and the **binormal vector**.

Definition: binormal vectors

Let C be a three-dimensional **smooth** curve represented by $\vec{\mathbf{r}}$ over an open interval I. If $\vec{\mathbf{T}}'(t) \neq \vec{\mathbf{0}}$, then the principal unit normal vector at t is defined to be

$$\vec{\mathbf{N}}(t) = \frac{\vec{\mathbf{T}}'(t)}{\|\vec{\mathbf{T}}'(t)\|}. \tag{12.4.30}$$

The binormal vector at t is defined as

$$\vec{\mathbf{B}}(t) = \vec{\mathbf{T}}(t) \times \vec{\mathbf{N}}(t), \tag{12.4.31}$$

where $\vec{\mathbf{T}}(t)$ is the unit tangent vector.

Note that, by definition, the binormal vector is orthogonal to both the unit tangent vector and the normal vector. Furthermore, $\vec{\mathbf{B}}(t)$ is always a unit vector. This can be shown using the formula for the magnitude of a cross product:

$$\|\vec{\mathbf{B}}(t)\| = \|\vec{\mathbf{T}}(t) \times \vec{\mathbf{N}}(t)\| = \|\vec{\mathbf{T}}(t)\|\|\vec{\mathbf{N}}(t)\| \sin\theta, \tag{12.4.32}$$

where θ is the angle between $\vec{\mathbf{T}}(t)$ and $\vec{\mathbf{N}}(t)$. Since $\vec{\mathbf{N}}(t)$ is the derivative of a unit vector, property (vii) of the derivative of a vector-valued function tells us that $\vec{\mathbf{T}}(t)$ and $\vec{\mathbf{N}}(t)$ are orthogonal to each other, so $\theta = \pi/2$. Furthermore, they are both unit vectors, so their magnitude is 1. Therefore, $\|\vec{\mathbf{T}}(t)\|\|\vec{\mathbf{N}}(t)\| \sin\theta = (1)(1)\sin(\pi/2) = 1$ and $\vec{\mathbf{B}}(t)$ is a unit vector.

The principal unit normal vector can be challenging to calculate because the unit tangent vector involves a quotient, and this quotient often has a square root in the denominator. In the three-dimensional case, finding the cross product of the unit tangent vector and the unit normal vector can be even more cumbersome. Fortunately, we have alternative formulas for finding these two vectors, and they are presented in **Motion in Space**.

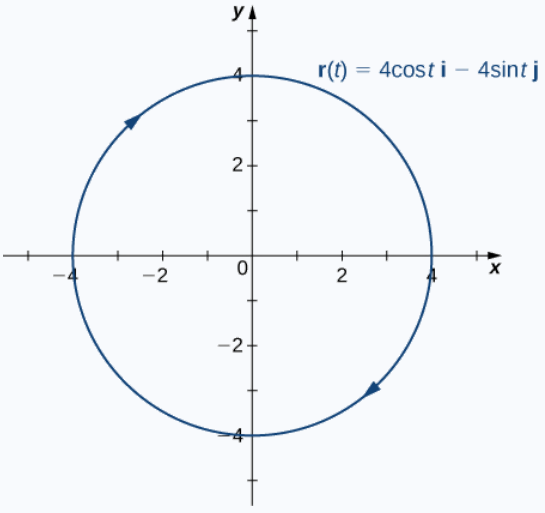
Example 12.4.4: Finding the Principal Unit Normal Vector and Binormal Vector

For each of the following vector-valued functions, find the principal unit normal vector. Then, if possible, find the binormal vector.

1. $\vec{\mathbf{r}}(t) = 4\cos t\,\hat{\mathbf{i}} - 4\sin t\,\hat{\mathbf{j}}$
2. $\vec{\mathbf{r}}(t) = (6t+2)\,\hat{\mathbf{i}} + 5t^2\,\hat{\mathbf{j}} - 8t\,\hat{\mathbf{k}}$

Solution

1. This function describes a circle.



To find the principal unit normal vector, we first must find the unit tangent vector $\vec{\mathbf{T}}(t)$:

$$\begin{aligned}\vec{\mathbf{T}}(t) &= \frac{\vec{\mathbf{r}}\,'(t)}{\|\vec{\mathbf{r}}\,'(t)\|} \\ &= \frac{-4\sin t\,\hat{\mathbf{i}} - 4\cos t\,\hat{\mathbf{j}}}{\sqrt{(-4\sin t)^2 + (-4\cos t)^2}} \\ &= \frac{-4\sin t\,\hat{\mathbf{i}} - 4\cos t\,\hat{\mathbf{j}}}{\sqrt{16\sin^2 t + 16\cos^2 t}} \\ &= \frac{-4\sin t\,\hat{\mathbf{i}} - 4\cos t\,\hat{\mathbf{j}}}{\sqrt{16(\sin^2 t + \cos^2 t)}} \\ &= \frac{-4\sin t\,\hat{\mathbf{i}} - 4\cos t\,\hat{\mathbf{j}}}{4} \\ &= -\sin t\,\hat{\mathbf{i}} - \cos t\,\hat{\mathbf{j}}.\end{aligned}$$

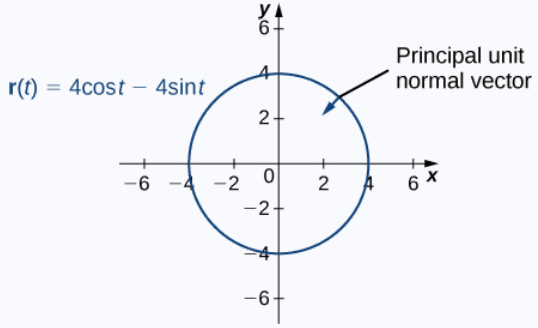
Next, we use 12.4.30:

$$\begin{aligned}\vec{\mathbf{N}}(t) &= \frac{\vec{\mathbf{T}}\,'(t)}{\|\vec{\mathbf{T}}\,'(t)\|} \\ &= \frac{-\cos t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}}}{\sqrt{(-\cos t)^2 + (\sin t)^2}} \\ &= \frac{-\cos t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}}}{\sqrt{\cos^2 t + \sin^2 t}} \\ &= -\cos t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}}.\end{aligned}$$

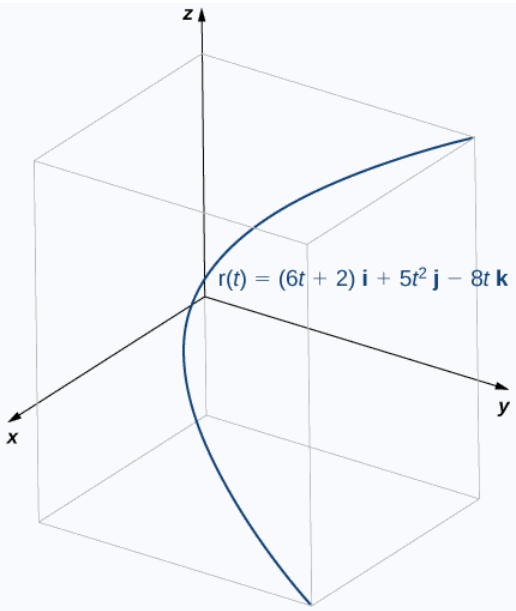
Notice that the unit tangent vector and the principal unit normal vector are orthogonal to each other for all values of t:

$$\begin{aligned}\vec{\mathbf{T}}(t) \cdot \vec{\mathbf{N}}(t) &= \langle -\sin t, -\cos t\rangle \cdot \langle -\cos t, \sin t\rangle \\ &= \sin t\cos t - \cos t\sin t \\ &= 0.\end{aligned}$$

Furthermore, the principal unit normal vector points toward the center of the circle from every point on the circle. Since $\vec{\mathbf{r}}(t)$ defines a curve in two dimensions, we cannot calculate the binormal vector.



2. This function looks like this:



To find the principal unit normal vector, we first find the unit tangent vector $\vec{\mathbf{T}}(t)$:

$$\begin{aligned}\vec{\mathbf{T}}(t) &= \frac{\vec{\mathbf{r}}\,'(t)}{\|\vec{\mathbf{r}}\,'(t)\|} \\ &= \frac{6\hat{\mathbf{i}} + 10t\hat{\mathbf{j}} - 8\hat{\mathbf{k}}}{\sqrt{6^2 + (10t)^2 + (-8)^2}} \\ &= \frac{6\hat{\mathbf{i}} + 10t\hat{\mathbf{j}} - 8\hat{\mathbf{k}}}{\sqrt{36 + 100t^2 + 64}} \\ &= \frac{6\hat{\mathbf{i}} + 10t\hat{\mathbf{j}} - 8\hat{\mathbf{k}}}{\sqrt{100(t^2 + 1)}} \\ &= \frac{3\hat{\mathbf{i}} + 5t\hat{\mathbf{j}} - 4\hat{\mathbf{k}}}{5\sqrt{t^2 + 1}} \\ &= \frac{3}{5}(t^2 + 1)^{-1/2}\hat{\mathbf{i}} + t(t^2 + 1)^{-1/2}\hat{\mathbf{j}} - \frac{4}{5}(t^2 + 1)^{-1/2}\hat{\mathbf{k}}.\end{aligned}$$

Next, we calculate $\vec{\mathbf{T}}\,'(t)$ and $\|\vec{\mathbf{T}}\,'(t)\|$:

$$\begin{aligned}\vec{\mathbf{T}}'(t) &= \frac{3}{5}\left(-\frac{1}{2}\right)(t^2+1)^{-3/2}(2t)\,\hat{\mathbf{i}} + \left((t^2+1)^{-1/2} - t\left(\frac{1}{2}\right)(t^2+1)^{-3/2}(2t)\right)\hat{\mathbf{j}} \\ &\quad - \frac{4}{5}\left(-\frac{1}{2}\right)(t^2+1)^{-3/2}(2t)\,\hat{\mathbf{k}} \\ &= -\frac{3t}{5(t^2+1)^{3/2}}\,\hat{\mathbf{i}} + \frac{1}{(t^2+1)^{3/2}}\,\hat{\mathbf{j}} + \frac{4t}{5(t^2+1)^{3/2}}\,\hat{\mathbf{k}}\end{aligned}$$

$$\begin{aligned}\|\vec{\mathbf{T}}'(t)\| &= \sqrt{\left(-\frac{3t}{5(t^2+1)^{3/2}}\right)^2 + \left(\frac{1}{(t^2+1)^{3/2}}\right)^2 + \left(\frac{4t}{5(t^2+1)^{3/2}}\right)^2} \\ &= \sqrt{\frac{9t^2}{25(t^2+1)^3} + \frac{1}{(t^2+1)^3} + \frac{16t^2}{25(t^2+1)^3}} \\ &= \sqrt{\frac{25t^2+25}{25(t^2+1)^3}} \\ &= \sqrt{\frac{1}{(t^2+1)^2}} \\ &= \frac{1}{t^2+1}.\end{aligned}$$

Therefore, according to 12.4.30:

$$\begin{aligned}\vec{\mathbf{N}}(t) &= \frac{\vec{\mathbf{T}}'(t)}{\|\vec{\mathbf{T}}'(t)\|} \\ &= \left(-\frac{3t}{5(t^2+1)^{3/2}}\,\hat{\mathbf{i}} - \frac{1}{(t^2+1)^{3/2}}\,\hat{\mathbf{j}} + \frac{4t}{5(t^2+1)^{3/2}}\,\hat{\mathbf{k}}\right)(t^2+1) \\ &= -\frac{3t}{5(t^2+1)^{1/2}}\,\hat{\mathbf{i}} - \frac{5}{5(t^2+1)^{1/2}}\,\hat{\mathbf{j}} + \frac{4t}{5(t^2+1)^{1/2}}\,\hat{\mathbf{k}} \\ &= -\frac{3t\,\hat{\mathbf{i}} + 5\,\hat{\mathbf{j}} - 4t\,\hat{\mathbf{k}}}{5\sqrt{t^2+1}}.\end{aligned}$$

Once again, the unit tangent vector and the principal unit normal vector are orthogonal to each other for all values of t:

$$\begin{aligned}\vec{\mathbf{T}}(t)\cdot\vec{\mathbf{N}}(t) &= \left(\frac{3\,\hat{\mathbf{i}} - 5t\,\hat{\mathbf{j}} - 4\,\hat{\mathbf{k}}}{5\sqrt{t^2+1}}\right)\cdot\left(-\frac{3t\,\hat{\mathbf{i}} + 5\,\hat{\mathbf{j}} - 4t\,\hat{\mathbf{k}}}{5\sqrt{t^2+1}}\right) \\ &= \frac{3(-3t) - 5t(-5) - 4(4t)}{25(t^2+1)} \\ &= \frac{-9t + 25t - 16t}{25(t^2+1)} \\ &= 0.\end{aligned}$$

Last, since $\vec{\mathbf{r}}(t)$ represents a three-dimensional curve, we can calculate the binormal vector using Equation 12.4.31:

$$
\begin{aligned}
\vec{\mathbf{B}}(t) &= \vec{\mathbf{T}}(t) \times \vec{\mathbf{N}}(t) \\
&= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{3}{5\sqrt{t^2+1}} & -\frac{5t}{5\sqrt{t^2+1}} & -\frac{4}{5\sqrt{t^2+1}} \\ -\frac{3t}{5\sqrt{t^2+1}} & -\frac{5}{5\sqrt{t^2+1}} & \frac{4t}{5\sqrt{t^2+1}} \end{vmatrix} \\
&= \left(\left(-\frac{5t}{5\sqrt{t^2+1}}\right)\left(\frac{4t}{5\sqrt{t^2+1}}\right) - \left(-\frac{4}{5\sqrt{t^2+1}}\right)\left(-\frac{5}{5\sqrt{t^2+1}}\right)\right)\hat{\mathbf{i}} \\
&\quad - \left(\left(\frac{3}{5\sqrt{t^2+1}}\right)\left(\frac{4t}{5\sqrt{t^2+1}}\right) - \left(-\frac{4}{5\sqrt{t^2+1}}\right)\left(-\frac{3t}{5\sqrt{t^2+1}}\right)\right)\hat{\mathbf{j}} \\
&\quad + \left(\left(\frac{3}{5\sqrt{t^2+1}}\right)\left(-\frac{5}{5\sqrt{t^2+1}}\right) - \left(-\frac{5t}{5\sqrt{t^2+1}}\right)\left(-\frac{3t}{5\sqrt{t^2+1}}\right)\right)\hat{\mathbf{k}} \\
&= \left(\frac{-20t^2-20}{25(t^2+1)}\right)\hat{\mathbf{i}} + \left(\frac{-15-15t^2}{25(t^2+1)}\right)\hat{\mathbf{k}} \\
&= -20\left(\frac{t^2+1}{25(t^2+1)}\right)\hat{\mathbf{i}} - 15\left(\frac{t^2+1}{25(t^2+1)}\right)\hat{\mathbf{k}} \\
&= -\frac{4}{5}\hat{\mathbf{i}} - \frac{3}{5}\hat{\mathbf{k}}.
\end{aligned}
$$

Exercise 12.4.4

Find the unit normal vector for the vector-valued function $\vec{\mathbf{r}}(t) = (t^2 - 3t)\,\hat{\mathbf{i}} + (4t+1)\,\hat{\mathbf{j}}$ and evaluate it at $t = 2$.

For any smooth curve in three dimensions that is defined by a vector-valued function, we now have formulas for the unit tangent vector $\vec{\mathbf{T}}$, the unit normal vector $\vec{\mathbf{N}}$, and the binormal vector $\vec{\mathbf{B}}$. The unit normal vector and the binormal vector form a plane that is perpendicular to the curve at any point on the curve, called the normal plane. In addition, these three vectors form a frame of reference in three-dimensional space called the *Frenet frame of reference* (also called the **TNB** frame) (Figure 12.4.2). Last, the plane determined by the vectors $\vec{\mathbf{T}}$ and $\vec{\mathbf{N}}$ forms the osculating plane of C at any point P on the curve.

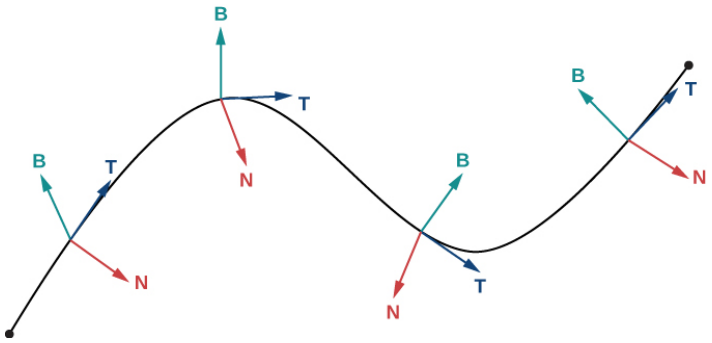


Figure 12.4.2: This figure depicts a Frenet frame of reference. At every point P *on a three-dimensional curve, the unit tangent, unit normal, and binormal vectors form a three-dimensional frame of reference.*

Suppose we form a circle in the osculating plane of C at point P on the curve. Assume that the circle has the same curvature as the curve does at point P and let the circle have radius r. Then, the curvature of the circle is given by $\frac{1}{r}$. We call r the radius of curvature of the curve, and it is equal to the reciprocal of the curvature. If this circle lies on the concave side of the curve and is tangent to the curve at point P, then this circle is called the *osculating circle* of C at P, as shown in Figure 12.4.3.

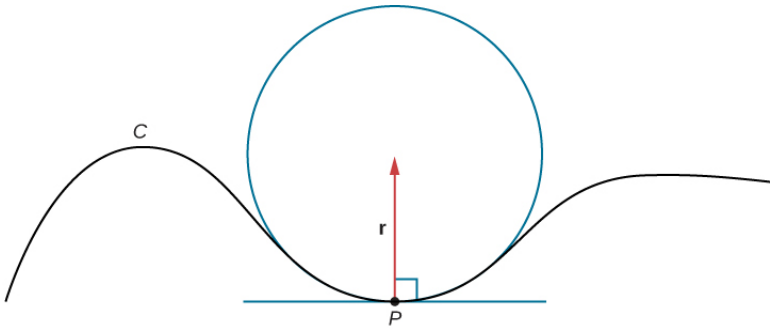


Figure 12.4.3: In this osculating circle, the circle is tangent to curve C at point P *and shares the same curvature.*

For more information on osculating circles, see this demonstration on curvature and torsion, this article on osculating circles, and this discussion of Serret formulas.

To find the equation of an osculating circle in two dimensions, we need find only the center and radius of the circle.

Example 12.4.5: Finding the Equation of an Osculating Circle

Find the equation of the osculating circle of the curve defined by the function $y = x^3 - 3x + 1$ at $x = 1$.

Solution

Figure 12.4.4 shows the graph of $y = x^3 - 3x + 1$.

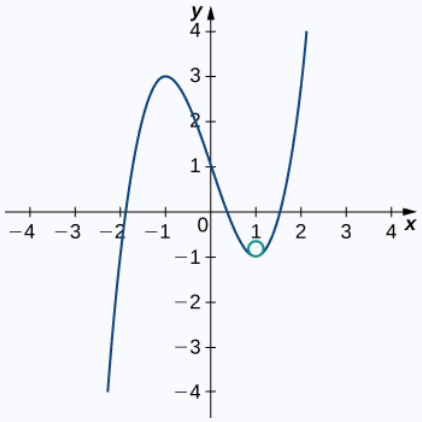


Figure 12.4.4: We want to find the osculating circle of this graph at the point where $x = 1$.

First, let's calculate the curvature at $x = 1$:

$$\kappa = \frac{|f''(x)|}{\left(1 + [f'(x)]^2\right)^{3/2}} = \frac{|6x|}{(1 + [3x^2 - 3]^2)^{3/2}}. \tag{12.4.33}$$

This gives $\kappa = 6$. Therefore, the radius of the osculating circle is given by $R = \frac{1}{\kappa} = \frac{1}{6}$. Next, we then calculate the coordinates of the center of the circle. When $x = 1$, the slope of the tangent line is zero. Therefore, the center of the osculating circle is directly above the point on the graph with coordinates $(1, -1)$. The center is located at $(1, -\frac{5}{6})$. The formula for a circle with radius r and center (h, k) is given by $(x - h)^2 + (y - k)^2 = r^2$. Therefore, the equation of the osculating circle is $(x - 1)^2 + (y + \frac{5}{6})^2 = \frac{1}{36}$. The graph and its osculating circle appears in the following graph.

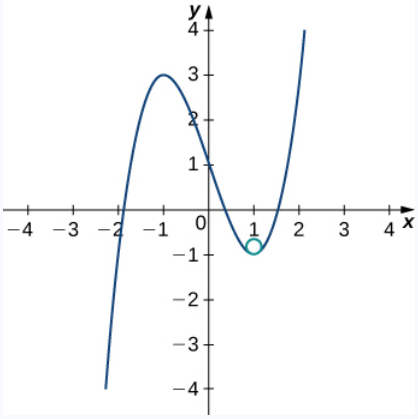


Figure 12.4.5: The osculating circle has radius $R = \frac{1}{6}$.

Exercise 12.4.5

Find the equation of the osculating circle of the curve defined by the vector-valued function $y = 2x^2 - 4x + 5$ at $x = 1$.

Key Concepts

- The arc-length function for a vector-valued function is calculated using the integral formula $s(t) = \int_a^b \|\vec{\mathbf{r}}\,'(t)\|\,dt$. This formula is valid in both two and three dimensions.
- The curvature of a curve at a point in either two or three dimensions is defined to be the curvature of the inscribed circle at that point. The arc-length parameterization is used in the definition of curvature.
- There are several different formulas for curvature. The curvature of a circle is equal to the reciprocal of its radius.
- The principal unit normal vector at t is defined to be

$$\vec{\mathbf{N}}(t) = \frac{\vec{\mathbf{T}}\,'(t)}{\|\vec{\mathbf{T}}\,'(t)\|}.$$

- The binormal vector at t is defined as $\vec{\mathbf{B}}(t) = \vec{\mathbf{T}}(t) \times \vec{\mathbf{N}}(t)$, where $\vec{\mathbf{T}}(t)$ is the unit tangent vector.
- The Frenet frame of reference is formed by the unit tangent vector, the principal unit normal vector, and the binormal vector.
- The osculating circle is tangent to a curve at a point and has the same curvature as the tangent curve at that point.

Key Equations

- **Arc length of space curve**
  $s = \int_a^b \sqrt{[f\,'(t)]^2 + [g\,'(t)]^2 + [h\,'(t)]^2}\,dt = \int_a^b \|\vec{\mathbf{r}}\,'(t)\|\,dt$
- **Arc-length function**
  $s(t) = \int_a^t \sqrt{f\,'(u))^2 + (g\,'(u))^2 + (h\,'(u))^2}\,du \text{ or } s(t) = \int_a^t \|\vec{\mathbf{r}}\,'(u)\|\,du$
- **Curvature**
  $\kappa = \frac{\|\vec{\mathbf{T}}\,'(t)\|}{\|\vec{\mathbf{r}}\,'(t)\|} \text{ or } \kappa = \frac{\|\vec{\mathbf{r}}\,'(t) \times \vec{\mathbf{r}}\,''(t)\|}{\|\vec{\mathbf{r}}\,'(t)\|^3} \text{ or } \kappa = \frac{|y''|}{[1+(y')^2]^{3/2}}$
- **Principal unit normal vector**
  $\vec{\mathbf{N}}(t) = \frac{\vec{\mathbf{T}}\,'(t)}{\|\vec{\mathbf{T}}\,'(t)\|}$
- **Binormal vector**
  $\vec{\mathbf{B}}(t) = \vec{\mathbf{T}}(t) \times \vec{\mathbf{N}}(t)$

Glossary

Contributors

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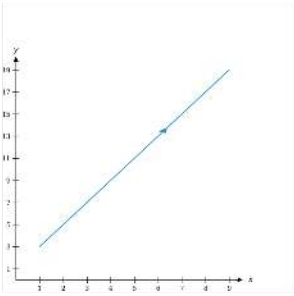
12.4E: Exercises for Section 12.4

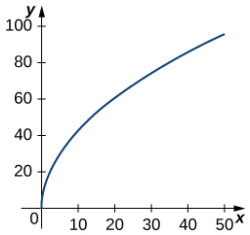
Determining Arc Length

In questions 1 - 5, find the arc length of the curve on the given interval.

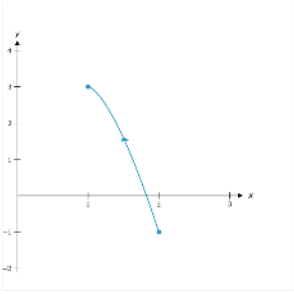
1) $\vec{\mathbf{r}}(t) = t^2\,\hat{\mathbf{i}} + (2t^2+1)\,\hat{\mathbf{j}}, \quad 1 \le t \le 3$. This portion of the graph is shown here:



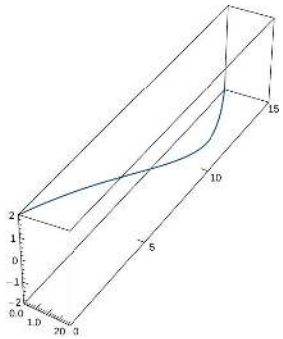
2) $\vec{\mathbf{r}}(t) = t^2\,\hat{\mathbf{i}} + 14t\,\hat{\mathbf{j}}, \quad 0 \le t \le 7$. This portion of the graph is shown here:



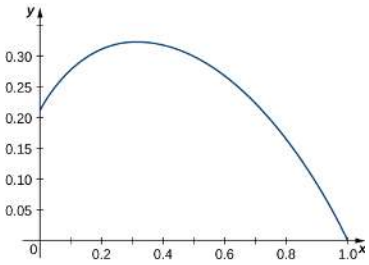
3) $\vec{\mathbf{r}}(t) = \langle t^2+1, 4t^3+3\rangle, \quad -1 \le t \le 0$. This portion of the graph is shown here:



4) $\vec{\mathbf{r}}(t) = \langle 2\sin t, 5t, 2\cos t\rangle, \quad 0 \le t \le \pi$. This portion of the graph is shown here:



5) $\vec{\mathbf{r}}(t) = \langle e^{-t\cos t}, e^{-t\sin t}\rangle$ over the interval $[0, \frac{\pi}{2}]$. Here is the portion of the graph on the indicated interval:



6) Set up an integral to represent the arc length from $t = 0$ to $t = 2$ along the curve traced out by $\vec{\mathbf{r}}(t) = \langle t, t^4 \rangle$. Then use technology to approximate this length to the nearest thousandth of a unit.

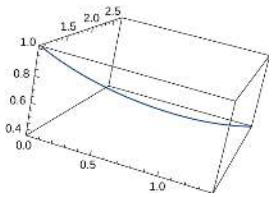
7) Find the length of one turn of the helix given by $\vec{\mathbf{r}}(t) = \frac{1}{2}\cos t\,\hat{\mathbf{i}} + \frac{1}{2}\sin t\,\hat{\mathbf{j}} + \frac{\sqrt{3}}{2}\,t\,\hat{\mathbf{k}}$.

8) Find the arc length of the vector-valued function $\vec{\mathbf{r}}(t) = -t\,\hat{\mathbf{i}} + 4t\,\hat{\mathbf{j}} + 3t\,\hat{\mathbf{k}}$ over $[0, 1]$.

9) A particle travels in a circle with the equation of motion $\vec{\mathbf{r}}(t) = 3\cos t\,\hat{\mathbf{i}} + 3\sin t\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}$. Find the distance traveled around the circle by the particle.

10) Set up an integral to find the circumference of the ellipse with the equation $\vec{\mathbf{r}}(t) = \cos t\,\hat{\mathbf{i}} + 2\sin t\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}$.

11) Find the length of the curve $\vec{\mathbf{r}}(t) = \langle \sqrt{2}t, e^t, e^{-t} \rangle$ over the interval $0 \le t \le 1$. The graph is shown here:



12) Find the length of the curve $\vec{\mathbf{r}}(t) = \langle 2\sin t,\ 5t,\ 2\cos t \rangle$ for $t \in [-10, 10]$.

Unit Tangent Vectors and Unit Normal Vectors

13) The position function for a particle is $\vec{\mathbf{r}}(t) = a\cos(\omega t)\,\hat{\mathbf{i}} + b\sin(\omega t)\,\hat{\mathbf{j}}$. Find the unit tangent vector and the unit normal vector at $t = 0$.

14) Given $\vec{\mathbf{r}}(t) = a\cos(\omega t)\,\hat{\mathbf{i}} + b\sin(\omega t)\,\hat{\mathbf{j}}$, find the binormal vector $\vec{\mathbf{B}}(0)$.

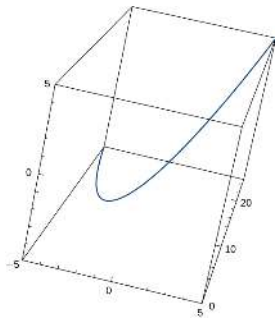
15) Given $\vec{\mathbf{r}}(t) = \langle 2e^t, e^t\cos t, e^t\sin t \rangle$, determine the unit tangent vector $\vec{\mathbf{T}}(t)$.

16) Given $\vec{\mathbf{r}}(t) = \langle 2e^t,\ e^t\cos t,\ e^t\sin t \rangle$, find the unit tangent vector $\vec{\mathbf{T}}(t)$ evaluated at $t = 0$, $\vec{\mathbf{T}}(0)$.

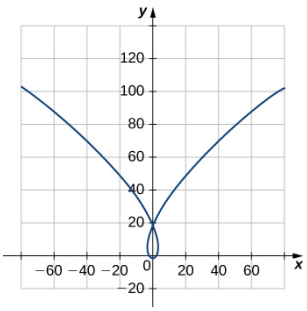
17) Given $\vec{\mathbf{r}}(t) = \langle 2e^t,\ e^t\cos t,\ e^t\sin t \rangle$, determine the unit normal vector $\vec{\mathbf{N}}(t)$.

18) Given $\vec{\mathbf{r}}(t) = \langle 2e^t,\ e^t\cos t,\ e^t\sin t \rangle$, find the unit normal vector $\vec{\mathbf{N}}(t)$ evaluated at $t = 0$, $\vec{\mathbf{N}}(0)$.

19) Given $\vec{\mathbf{r}}(t) = t\,\hat{\mathbf{i}} + t^2\,\hat{\mathbf{j}} + t\,\hat{\mathbf{k}}$, find the unit tangent vector $\vec{\mathbf{T}}(t)$. The graph is shown here:



20) Find the unit tangent vector $\vec{\mathbf{T}}(t)$ and unit normal vector $\vec{\mathbf{N}}(t)$ at $t = 0$ for the plane curve $\vec{\mathbf{r}}(t) = \langle t^3 - 4t, 5t^2 - 2 \rangle$. The graph is shown here:



21) Find the unit tangent vector $\vec{\mathbf{T}}(t)$ for $\vec{\mathbf{r}}(t) = 3t\,\hat{\mathbf{i}} + 5t^2\,\hat{\mathbf{j}} + 2t\,\hat{\mathbf{k}}$.

22) Find the principal normal vector to the curve $\vec{\mathbf{r}}(t) = \langle 6\cos t, 6\sin t\rangle$ at the point determined by $t = \frac{\pi}{3}$.

23) Find $\vec{\mathbf{T}}(t)$ for the curve $\vec{\mathbf{r}}(t) = (t^3 - 4t)\,\hat{\mathbf{i}} + (5t^2 - 2)\,\hat{\mathbf{j}}$.

24) Find $\vec{\mathbf{N}}(t)$ for the curve $\vec{\mathbf{r}}(t) = (t^3 - 4t)\,\hat{\mathbf{i}} + (5t^2 - 2)\,\hat{\mathbf{j}}$.

25) Find the unit tangent vector $\vec{\mathbf{T}}(t)$ for $\vec{\mathbf{r}}(t) = \langle 2\sin t,\ 5t,\ 2\cos t\rangle$.

26) Find the unit normal vector $\vec{\mathbf{N}}(t)$ for $\vec{\mathbf{r}}(t) = \langle 2\sin t,\ 5t,\ 2\cos t\rangle$.

Arc Length Parameterizations

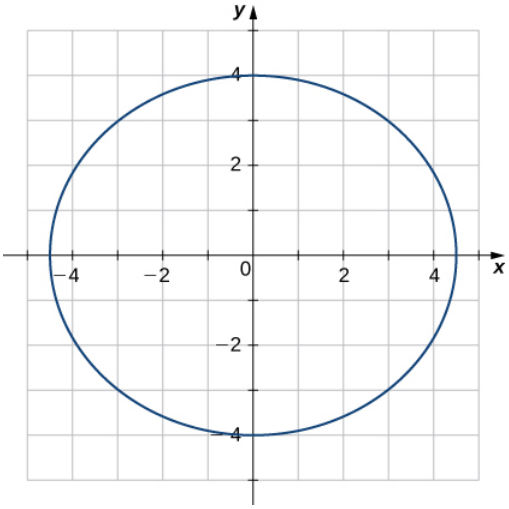
27) Find the arc-length function $\vec{\mathbf{s}}(t)$ for the line segment given by $\vec{\mathbf{r}}(t) = \langle 3 - 3t,\ 4t\rangle$. Then write the arc-length parameterization of r with s as the parameter.

28) Parameterize the helix $\vec{\mathbf{r}}(t) = \cos t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}} + t\,\hat{\mathbf{k}}$ using the arc-length parameter s, from $t = 0$.

29) Parameterize the curve using the arc-length parameter s, at the point at which $t = 0$ for $\vec{\mathbf{r}}(t) = e^t \sin t\,\hat{\mathbf{i}} + e^t \cos t\,\hat{\mathbf{j}}$

Curvature and the Osculating Circle

30) Find the curvature of the curve $\vec{\mathbf{r}}(t) = 5\cos t\,\hat{\mathbf{i}} + 4\sin t\,\hat{\mathbf{j}}$ at $t = \pi/3$. (*Note:* The graph is an ellipse.)



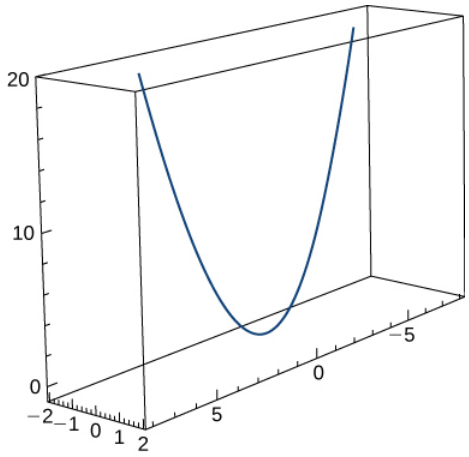
31) Find the x-coordinate at which the curvature of the curve $y = 1/x$ is a maximum value.

32) Find the curvature of the curve $\vec{\mathbf{r}}(t) = 5\cos t\,\hat{\mathbf{i}} + 5\sin t\,\hat{\mathbf{j}}$. Does the curvature depend upon the parameter t?

33) Find the curvature κ for the curve $y = x - \frac{1}{4}x^2$ at the point $x = 2$.

34) Find the curvature κ for the curve $y = \frac{1}{3}x^3$ at the point $x = 1$.

35) Find the curvature κ of the curve $\vec{\mathbf{r}}(t) = t\,\hat{\mathbf{i}} + 6t^2\,\hat{\mathbf{j}} + 4t\,\hat{\mathbf{k}}$. The graph is shown here:



36) Find the curvature of $\vec{\mathbf{r}}(t) = \langle 2\sin t, 5t, 2\cos t\rangle$.

37) Find the curvature of $\vec{\mathbf{r}}(t) = \sqrt{2}t\,\hat{\mathbf{i}} + e^t\,\hat{\mathbf{j}} + e^{-t}\,\hat{\mathbf{k}}$ at point $P(0,1,1)$.

38) At what point does the curve $y = e^x$ have maximum curvature?

39) What happens to the curvature as $x \to \infty$ for the curve $y = e^x$?

40) Find the point of maximum curvature on the curve $y = \ln x$.

41) Find the equations of the normal plane and the osculating plane of the curve $\vec{\mathbf{r}}(t) = \langle 2\sin(3t), t, 2\cos(3t)\rangle$ at point $(0, \pi, -2)$.

42) Find equations of the osculating circles of the ellipse $4y^2 + 9x^2 = 36$ at the points $(2,0)$ and $(0,3)$.

43) Find the equation for the osculating plane at point $t = \pi/4$ on the curve $\vec{\mathbf{r}}(t) = \cos(2t)\,\hat{\mathbf{i}} + \sin(2t)\,\hat{\mathbf{j}} + t\,\hat{\mathbf{k}}$.

44) Find the radius of curvature of $6y = x^3$ at the point $(2, \frac{4}{3})$.

45) Find the curvature at each point (x, y) on the hyperbola $\vec{\mathbf{r}}(t) = \langle a\cosh(t), b\sinh(t)\rangle$.

46) Calculate the curvature of the circular helix $\vec{\mathbf{r}}(t) = r\sin(t)\,\hat{\mathbf{i}} + r\cos(t)\,\hat{\mathbf{j}} + t\,\hat{\mathbf{k}}$.

47) Find the radius of curvature of $y = \ln(x+1)$ at point $(2, \ln 3)$.

48) Find the radius of curvature of the hyperbola $xy = 1$ at point $(1,1)$.

A particle moves along the plane curve C described by $\vec{\mathbf{r}}(t) = t\,\hat{\mathbf{i}} + t^2\,\hat{\mathbf{j}}$. Use this parameterization to answer questions 49 - 51.

49) Find the length of the curve over the interval $[0,2]$.

50) Find the curvature of the plane curve at $t = 0, 1, 2$.

51) Describe the curvature as t increases from $t = 0$ to $t = 2$.

The surface of a large cup is formed by revolving the graph of the function $y = 0.25x^{1.6}$ from $x = 0$ to $x = 5$ about the y-axis (measured in centimeters).

52) **[T]** Use technology to graph the surface.

53) Find the curvature κ of the generating curve as a function of x.

54) **[T]** Use technology to graph the curvature function.

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12.5: Acceleration and Kepler's Laws

Components of the Acceleration Vector

We can combine some of the concepts discussed in Arc Length and Curvature with the acceleration vector to gain a deeper understanding of how this vector relates to motion in the plane and in space. Recall that the unit tangent vector $\vec{\mathbf{T}}$ and the unit normal vector $\vec{\mathbf{N}}$ form an osculating plane at any point P on the curve defined by a vector-valued function $\vec{\mathbf{r}}(t)$. The following theorem shows that the acceleration vector $\vec{\mathbf{a}}(t)$ lies in the osculating plane and can be written as a linear combination of the unit tangent and the unit normal vectors.

Theorem 12.5.1: The Plane of the Acceleration Vector

The acceleration vector $\vec{\mathbf{a}}(t)$ of an object moving along a curve traced out by a twice-differentiable function $\vec{\mathbf{r}}(t)$ lies in the plane formed by the unit tangent vector $\vec{\mathbf{T}}(t)$ and the principal unit normal vector $\vec{\mathbf{N}}(t)$ to C.

Furthermore,

$$\vec{\mathbf{a}}(t) = v'(t)\vec{\mathbf{T}}(t) + [v(t)]^2\kappa\vec{\mathbf{N}}(t) \tag{12.5.1}$$

Here, $v(t) = \|\vec{\mathbf{v}}(t)\|$ is the speed of the object and κ is the curvature of C traced out by $\vec{\mathbf{r}}(t)$.

Proof

Because $\vec{\mathbf{v}}(t) = \vec{\mathbf{r}}'(t)$ and $\vec{\mathbf{T}}(t) = \dfrac{\vec{\mathbf{r}}'(t)}{||\vec{\mathbf{r}}'(t)||}$, we have $\vec{\mathbf{v}}(t) = ||\vec{\mathbf{r}}'(t)||\vec{\mathbf{T}}(t) = v(t)\vec{\mathbf{T}}(t)$.

Now we differentiate this equation:

$$\vec{\mathbf{a}}(t) = \vec{\mathbf{v}}'(t) = \frac{d}{dt}\left(v(t)\vec{\mathbf{T}}(t)\right) = v'(t)\vec{\mathbf{T}}(t) + v(t)\vec{\mathbf{T}}'(t) \tag{12.5.2}$$

Since $\vec{\mathbf{N}}(t) = \dfrac{\vec{\mathbf{T}}'(t)}{||\vec{\mathbf{T}}'(t)||}$, we know $\vec{\mathbf{T}}'(t) = ||\vec{\mathbf{T}}'(t)||\vec{\mathbf{N}}(t)$, so

$$\vec{\mathbf{a}}(t) = v'(t)\vec{\mathbf{T}}(t) + v(t)||\vec{\mathbf{T}}'(t)||\vec{\mathbf{N}}(t). \tag{12.5.3}$$

A formula for curvature is $\kappa = \dfrac{||\vec{\mathbf{T}}'(t)||}{||\vec{\mathbf{r}}'(t)||}$, so $\vec{\mathbf{T}}'(t) = \kappa||\vec{\mathbf{r}}'(t)|| = \kappa v(t)$.

This gives $\vec{\mathbf{a}}(t) = v'(t)\vec{\mathbf{T}}(t) + \kappa(v(t))^2\vec{\mathbf{N}}(t)$.

□

The real-valued coefficients of $\vec{\mathbf{T}}(t)$ and $\vec{\mathbf{N}}(t)$ (possibly real-valued functions of t) are referred to as the **tangential component of acceleration** and the **normal component of acceleration**, respectively. We write $a_{\vec{\mathbf{T}}}$ to denote the tangential component and $a_{\vec{\mathbf{N}}}$ to denote the normal component. Note that these names are a little misleading, as these functions of t are **not** vector components, but the magnitudes of the corresponding vector components of acceleration.

Theorem 12.5.2: Tangential and Normal Components of Acceleration (as scalar magnitudes)

Let $\vec{\mathbf{r}}(t)$ be a vector-valued function that denotes the position of an object as a function of time. Then $\vec{\mathbf{a}}(t) = \vec{\mathbf{r}}\,''(t)$ is the acceleration vector. The tangential and normal components of acceleration $a_{\vec{\mathbf{T}}}$ and $a_{\vec{\mathbf{N}}}$ (the magnitudes of the corresponding vector components of acceleration) are given by the formulas

$$a_{\vec{\mathbf{T}}} = \vec{\mathbf{a}} \cdot \vec{\mathbf{T}} = \frac{\vec{\mathbf{v}} \cdot \vec{\mathbf{a}}}{||\vec{\mathbf{v}}||} \tag{12.5.4}$$

and

$$a_{\vec{\mathbf{N}}} = \vec{\mathbf{a}} \cdot \vec{\mathbf{N}} = \frac{||\vec{\mathbf{v}} \times \vec{\mathbf{a}}||}{||\vec{\mathbf{v}}||} = \sqrt{||\vec{\mathbf{a}}||^2 - \left(a_{\vec{\mathbf{T}}}\right)^2}. \tag{12.5.5}$$

The vector components of acceleration are related by the formula

$$\vec{\mathbf{a}}(t) = a_{\vec{\mathbf{T}}}\vec{\mathbf{T}}(t) + a_{\vec{\mathbf{N}}}\vec{\mathbf{N}}(t). \tag{12.5.6}$$

Here $\vec{\mathbf{T}}(t)$ is the unit tangent vector to the curve defined by $\vec{\mathbf{r}}(t)$, and $\vec{\mathbf{N}}(t)$ is the unit normal vector to the curve defined by $\vec{\mathbf{r}}(t)$.

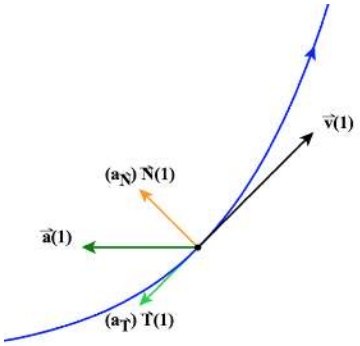

Figure 12.5.1: The acceleration can be broken into components parallel to $\vec{\mathbf{N}}(t)$ and $\vec{\mathbf{T}}(t)$. Here $t = 1$.

At a given point on the curve, breaking the acceleration vector up into these two components, a tangential component parallel to the velocity vector (and $\vec{\mathbf{T}}$) and a normal component parallel to the normal vector $\vec{\mathbf{N}}$, allows us to more clearly see the effect of the acceleration vector on the velocity at that point. See Figure 12.5.1.

Only the component of acceleration that is parallel to the velocity will change the speed of the motion (affecting the magnitude of the velocity). And only the component that is normal to the curve will affect the direction of the velocity (and the corresponding bending of the curve at this point). Note that for the example in Figure 12.5.1, the acceleration is clearly slowing the velocity and turning the curve to the left.

This means that if acceleration is orthogonal to the velocity, it will only affect the direction and will not change the speed at this point. And if the acceleration is parallel to the velocity, it will only affect the speed and not change the direction at this point. If the acceleration vector is always parallel to the velocity vector for a given position function, the motion must be along a straight line.

Exercise 12.5.1

What can you say about the speed of the motion along a curve if the acceleration vector is always orthogonal to the velocity vector for the corresponding position function?

Note that in this context, the unit vectors $\vec{\mathbf{T}}(t)$ and $\vec{\mathbf{N}}(t)$ are filling a role very similar to $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$. But instead of representing a vector in the xy-plane where the vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ reside, the linear combination of $\vec{\mathbf{T}}(t)$ and $\vec{\mathbf{N}}(t)$ represents an acceleration vector in the plane formed by the vectors $\vec{\mathbf{T}}(t)$ and $\vec{\mathbf{N}}(t)$.

The normal component of acceleration is also called the *centripetal component of acceleration* or sometimes the *radial component of acceleration*. To understand centripetal acceleration, suppose you are traveling in a car on a circular track at a constant speed. Then, as we saw earlier, the acceleration vector points toward the center of the track at all times. As a rider in the car, you feel a pull toward the *outside* of the track because you are constantly turning. This sensation acts in the opposite direction of centripetal acceleration. The same holds true for non-circular paths. The reason is that your body tends to travel in a straight line and resists the force resulting from acceleration that push it toward the side. Note that at point B in Figure 12.5.2 the acceleration vector is pointing backward. This is because the car is decelerating as it goes into the curve.

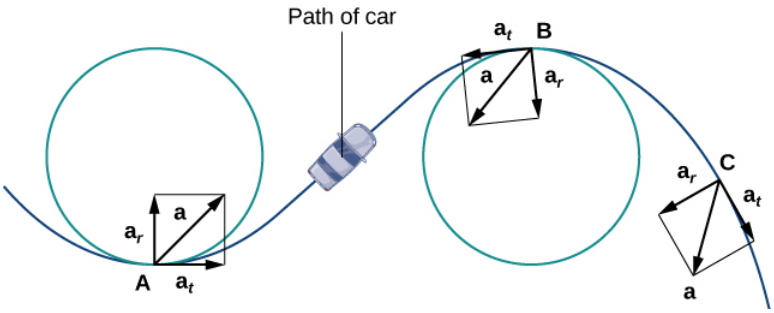

Figure 12.5.2: The tangential and normal components of acceleration can be used to describe the acceleration vector.

The tangential and normal unit vectors at any given point on the curve provide a frame of reference at that point. The tangential and normal components of acceleration (as vectors) are the projections of the acceleration vector onto $\vec{\mathbf{T}}$ and $\vec{\mathbf{N}}$, respectively.

Note that there are two ways for us to calculate the vector components of acceleration that are tangent and normal to the curve. We can either use the formulas in Theorem 12.5.2 above and multiply them by the unit tangent vector and the unit normal vector, respectively, **or** we can calculate these vectors directly (with a lot less effort) as the projection of the acceleration vector onto the velocity vector and the corresponding component of acceleration that is perpendicular to the velocity vector and that together with the projection sums to the original acceleration vector. We can thus use the simpler approach shown in Theorem 12.5.3 below.

Theorem 12.5.3: Tangential and Normal Components of Acceleration (as vector components)

Let $\vec{\mathbf{r}}(t)$ be a vector-valued function that denotes the position of an object as a function of time. Then $\vec{\mathbf{a}}(t) = \vec{\mathbf{r}}\,''(t)$ is the acceleration vector. The tangential and normal components of acceleration $a_{\vec{\mathbf{T}}}\vec{\mathbf{T}}(t)$ and $a_{\vec{\mathbf{N}}}\vec{\mathbf{N}}(t)$ are given by the formulas

$$a_{\vec{\mathbf{T}}}\vec{\mathbf{T}}(t) = \text{Proj}_{\vec{\mathbf{v}}(t)}\vec{\mathbf{a}}(t) \qquad \text{The vector component of acceleration tangent to the curve} \tag{12.5.7}$$

and

$$a_{\vec{\mathbf{N}}}\vec{\mathbf{N}}(t) = \vec{\mathbf{a}}(t)_{\text{perp}} = \vec{\mathbf{a}}(t) - \text{Proj}_{\vec{\mathbf{v}}(t)}\vec{\mathbf{a}}(t) \tag{12.5.8}$$
$$\text{The vector component of acceleration normal to the curve}$$

If we have already calculated $\vec{\mathbf{T}}(t)$ and $\vec{\mathbf{N}}(t)$, we can also determine these vector components of acceleration using these and the values of $a_{\vec{\mathbf{T}}}$ and $a_{\vec{\mathbf{N}}}$ from above.

Example 12.5.1: Finding Components of Acceleration (as magnitudes)

A particle moves in a path defined by the vector-valued function $\vec{\mathbf{r}}(t) = t^2\,\hat{\mathbf{i}} + (2t-3)\,\hat{\mathbf{j}} + (3t^2 - 3t)\,\hat{\mathbf{k}}$, where t measures time in seconds and distance is measured in feet.

a. Find $a_{\vec{\mathbf{T}}}$ and $a_{\vec{\mathbf{N}}}$ as functions of t.
b. Find $a_{\vec{\mathbf{T}}}$ and $a_{\vec{\mathbf{N}}}$ at time $t = 2$.

Solution

a. Let's start deriving the velocity and acceleration functions

$$\vec{\mathbf{v}}(t) = \vec{\mathbf{r}}\,'(t) = 2t\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + (6t-3)\,\hat{\mathbf{k}}$$

$$\vec{\mathbf{a}}(t) = \vec{\mathbf{v}}\,'(t) = 2\,\hat{\mathbf{i}} + 6\,\hat{\mathbf{k}}$$

Now we use Equation 12.5.4:

$$
\begin{aligned}
a_{\vec{\mathbf{T}}} &= \frac{\vec{\mathbf{v}} \cdot \vec{\mathbf{a}}}{||\vec{\mathbf{v}}||} \\
&= \frac{(2t\,\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + (6t-3)\,\hat{\mathbf{k}}) \cdot (2\,\hat{\mathbf{i}} + 6\,\hat{\mathbf{k}})}{||2t\,\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + (6t-3)\,\hat{\mathbf{k}}||} \\
&= \frac{4t + 6(6t-3)}{\sqrt{(2t)^2 + 2^2 + (6t-3)^2}} \\
&= \frac{40t - 18}{40t^2 - 36t + 13}
\end{aligned}
$$

Then we apply Equation 12.5.5:

$$
\begin{aligned}
a_{\vec{\mathbf{N}}} &= \sqrt{||\vec{\mathbf{a}}||^2 - \left(a_{\vec{\mathbf{T}}}\right)^2} \\
&= \sqrt{||2\,\hat{\mathbf{i}} + 6\,\hat{\mathbf{k}}||^2 - \left(\frac{40t-18}{\sqrt{40t^2-36t+13}}\right)^2} \\
&= \sqrt{4 + 36 - \frac{(40t-18)^2}{40t^2-36t+13}} \\
&= \sqrt{\frac{40(40t^2-36t+13) - (1600t^2 - 1440t + 324)}{40t^2-36t+13}} \\
&= \sqrt{\frac{196}{40t^2-36t+13}} \\
&= \frac{14}{\sqrt{40t^2-36t+13}}
\end{aligned}
$$

b. We must evaluate each of the answers from part a at $t = 2$:

$$
\begin{aligned}
a_{\vec{\mathbf{T}}}(2) &= \frac{40(2)-18}{\sqrt{40(2)^2-36(2)+13}} \\
&= \frac{80-18}{\sqrt{160-72+13}} \\
&= \frac{62}{\sqrt{101}} = \frac{62\sqrt{101}}{101} \\
a_{\vec{\mathbf{N}}}(2) &= \frac{14}{\sqrt{40(2)^2-36(2)+13}} \\
&= \frac{14}{\sqrt{160-72+13}} = \frac{14}{\sqrt{101}} = \frac{14\sqrt{101}}{101}.
\end{aligned}
$$

The units of acceleration are feet per second squared, as are the units of the normal and tangential components of acceleration.

Exercise 12.5.2

An object moves in a path defined by the vector-valued function $\vec{\mathbf{r}}(t) = 4t\,\hat{\mathbf{i}} + t^2\,\hat{\mathbf{j}}$, where t measures time in seconds.

a. Find $a_{\vec{\mathbf{T}}}$ and $a_{\vec{\mathbf{N}}}$ as functions of t.
b. Find $a_{\vec{\mathbf{T}}}$ and $a_{\vec{\mathbf{N}}}$ at time $t = -3$.

Now let's calculate the vector components of acceleration for the vector-valued function in Example **12.5.1**. We'll start by calculating these vector components at a time $t = 2$ using the projection of the acceleration vector onto the velocity vector at that time. Then we'll show the same approach to finding the vector components of acceleration as functions of t.

Example 12.5.2

A particle moves in a path defined by the vector-valued function $\vec{\mathbf{r}}(t) = t^2\,\hat{\mathbf{i}} + (2t-3)\,\hat{\mathbf{j}} + (3t^2-3t)\,\hat{\mathbf{k}}$, where t measures time in seconds and distance is measured in feet.

a. Find the vector components of acceleration at time $t = 2$.
b. Find the vector components of the acceleration tangent and normal to the curve as functions of t.

Solution

From our work above, we have

$$\vec{\mathbf{v}}(t) = \vec{\mathbf{r}}'(t) = 2t\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + (6t-3)\,\hat{\mathbf{k}}$$

$$\vec{\mathbf{a}}(t) = \vec{\mathbf{v}}'(t) = 2\,\hat{\mathbf{i}} + 6\,\hat{\mathbf{k}}$$

a. At time $t = 2$, we have

$$\begin{aligned}\vec{\mathbf{v}}(2) &= 2(2)\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + \big(6(2)-3\big)\,\hat{\mathbf{k}} \\ &= 4\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + 9\,\hat{\mathbf{k}}\end{aligned}$$

and

$$\vec{\mathbf{a}}(2) = 2\,\hat{\mathbf{i}} + 6\,\hat{\mathbf{k}}$$

Then the component of acceleration tangent to the curve at time $t = 2$ is:

$$a_{\vec{\mathbf{T}}}\vec{\mathbf{T}}(2) = \text{Proj}_{\vec{\mathbf{v}}(2)}\vec{\mathbf{a}}(2) = \underbrace{\frac{\vec{\mathbf{a}}(2)\cdot\vec{\mathbf{v}}(2)}{\|\vec{\mathbf{v}}(2)\|}}_{a_{\vec{\mathbf{T}}}(2)} \cdot \underbrace{\frac{\vec{\mathbf{v}}(2)}{\|\vec{\mathbf{v}}(2)\|}}_{\vec{\mathbf{T}}(2)}$$

Since $\vec{\mathbf{a}}(2)\cdot\vec{\mathbf{v}}(2) = 2(4)+0(2)+6(9) = 8+0+54 = 62$,

and $\|\vec{\mathbf{v}}(2)\| = \sqrt{4^2+2^2+9^2} = \sqrt{16+4+81} = \sqrt{101}$,

$$\begin{aligned}a_{\vec{\mathbf{T}}}\vec{\mathbf{T}}(2) = \text{Proj}_{\vec{\mathbf{v}}(2)}\vec{\mathbf{a}}(2) &= \frac{\vec{\mathbf{a}}(2)\cdot\vec{\mathbf{v}}(2)}{\|\vec{\mathbf{v}}(2)\|}\cdot\frac{\vec{\mathbf{v}}(2)}{\|\vec{\mathbf{v}}(2)\|} \\ &= \frac{62}{101}\vec{\mathbf{v}}(2) \\ &= \frac{248}{101}\,\hat{\mathbf{i}} + \frac{124}{101}\,\hat{\mathbf{j}} + \frac{558}{101}\,\hat{\mathbf{k}}\end{aligned}$$

Since the sum of these two components is $\vec{\mathbf{a}}(2)$, the component of acceleration normal to the curve at time $t = 2$ is:

$$\begin{aligned}a_{\vec{\mathbf{N}}}\vec{\mathbf{N}}(2) = \vec{\mathbf{a}}(2)_{\text{perp}} &= \vec{\mathbf{a}}(2) - \text{Proj}_{\vec{\mathbf{v}}(2)}\vec{\mathbf{a}}(2) \\ &= 2\,\hat{\mathbf{i}} + 6\,\hat{\mathbf{k}} - \left(\frac{248}{101}\,\hat{\mathbf{i}} + \frac{124}{101}\,\hat{\mathbf{j}} + \frac{558}{101}\,\hat{\mathbf{k}}\right) \\ &= \left(2 - \frac{248}{101}\right)\hat{\mathbf{i}} - \frac{124}{101}\,\hat{\mathbf{j}} + \left(6 - \frac{558}{101}\right)\hat{\mathbf{k}} \\ &= -\frac{46}{101}\,\hat{\mathbf{i}} - \frac{124}{101}\,\hat{\mathbf{j}} + \frac{48}{101}\,\hat{\mathbf{k}}\end{aligned}$$

See the vector $\vec{\mathbf{a}}(2)$ and its component vectors in Figure 12.5.3 below.

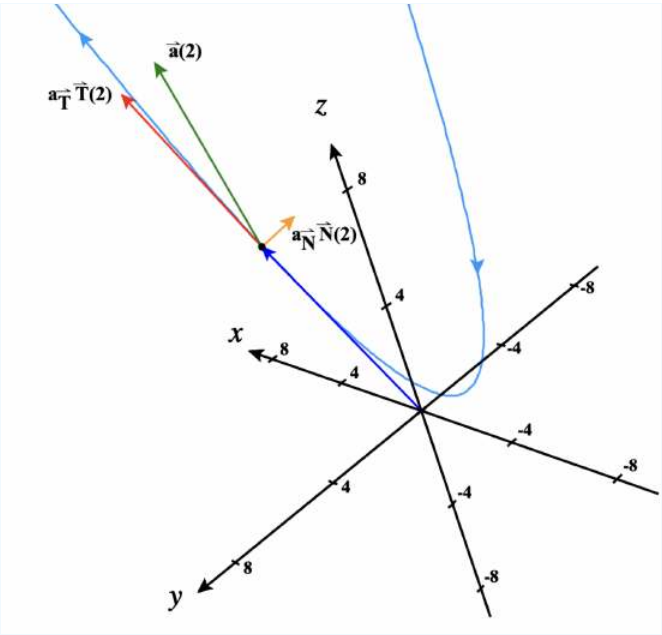

Figure 12.5.3: The components of acceleration at $t = 2$. The component of acceleration tangent to the curve is shown in red and the component normal to the curve is shown in orange.

b. The more general vector components of acceleration (as functions of t) can be found in the same way with somewhat less work than taking time to first directly determine $\vec{\mathbf{T}}(t)$ and $\vec{\mathbf{N}}(t)$.

Since

$$\vec{\mathbf{v}}(t) = \vec{\mathbf{r}}'(t) = 2t\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + (6t-3)\,\hat{\mathbf{k}}$$

and

$$\vec{\mathbf{a}}(t) = \vec{\mathbf{v}}'(t) = 2\,\hat{\mathbf{i}} + 6\,\hat{\mathbf{k}}$$

we have

$$a_{\vec{\mathbf{T}}}\vec{\mathbf{T}}(t) = \text{Proj}_{\vec{\mathbf{v}}(t)}\vec{\mathbf{a}}(t) = \underbrace{\frac{\vec{\mathbf{a}}(t)\cdot\vec{\mathbf{v}}(t)}{\|\vec{\mathbf{v}}(t)\|}}_{a_{\vec{\mathbf{T}}}(t)} \cdot \underbrace{\frac{\vec{\mathbf{v}}(t)}{\|\vec{\mathbf{v}}(t)\|}}_{\vec{\mathbf{T}}(t)}$$

As in part (a), we calculate $\vec{\mathbf{a}}(t)\cdot\vec{\mathbf{v}}(t) = 2(2t) + 0(2) + 6(6t-3) = 4t + 0 + 36t - 18 = 40t - 18$,

$$\|\vec{\mathbf{v}}(t)\| = \sqrt{(2t)^2 + 2^2 + (6t-3)^2}$$

and

$$= \sqrt{4t^2 + 4 + 36t^2 - 36t + 9}$$

$$= \sqrt{40t^2 - 36t + 13}.$$

Then the vector component of acceleration tangent to the curve at time t is

$$\begin{aligned} a_{\vec{\mathbf{T}}}\vec{\mathbf{T}}(t) = \text{Proj}_{\vec{\mathbf{v}}(t)}\vec{\mathbf{a}}(t) &= \frac{\vec{\mathbf{a}}(t)\cdot\vec{\mathbf{v}}(t)}{\|\vec{\mathbf{v}}(t)\|} \cdot \frac{\vec{\mathbf{v}}(t)}{\|\vec{\mathbf{v}}(t)\|} \\ &= \frac{40t-18}{40t^2-36t+13}\vec{\mathbf{v}}(t) \\ &= \frac{2t(40t-18)}{40t^2-36t+13}\hat{\mathbf{i}} + \frac{2(40t-18)}{40t^2-36t+13}\hat{\mathbf{j}} + \frac{(6t-3)(40t-18)}{40t^2-36t+13}\hat{\mathbf{k}} \\ &= \frac{4t(20t-9)}{40t^2-36t+13}\hat{\mathbf{i}} + \frac{4(20t-9)}{40t^2-36t+13}\hat{\mathbf{j}} + \frac{6(40t^2-38t+9)}{40t^2-36t+13}\hat{\mathbf{k}} \end{aligned}$$

And the vector component of acceleration normal to the curve at time t is

$$
\begin{aligned}
a_{\vec{\mathbf{N}}}\vec{\mathbf{N}}(t) &= \vec{\mathbf{a}}(t)_{\text{perp}} = \vec{\mathbf{a}}(t) - \text{Proj}_{\vec{\mathbf{v}}(t)}\vec{\mathbf{a}}(t) \\
&= 2\hat{\mathbf{i}} + 6\hat{\mathbf{k}} - \left(\frac{4t(20t-9)}{40t^2-36t+13}\hat{\mathbf{i}} + \frac{4(20t-9)}{40t^2-36t+13}\hat{\mathbf{j}} + \frac{6(40t^2-38t+9)}{40t^2-36t+13}\hat{\mathbf{k}}\right) \\
&= \left(2 - \frac{4t(20t-9)}{40t^2-36t+13}\right)\hat{\mathbf{i}} - \frac{4(20t-9)}{40t^2-36t+13}\hat{\mathbf{j}}. \\
&\quad + \left(6 - \frac{6(40t^2-38t+9)}{40t^2-36t+13}\right)\hat{\mathbf{k}} \\
&= \frac{2(13-18t)}{40t^2-36t+13}\hat{\mathbf{i}} - \frac{4(20t-9)}{40t^2-36t+13}\hat{\mathbf{j}} + \frac{12(t+2)}{40t^2-36t+13}\hat{\mathbf{k}}
\end{aligned}
$$

Checking these functions at $t = 2$, we first evaluate the expression $40t^2 - 36t + 13$ at $t = 2$.

This gives us $40(2)^2 - 36(2) + 13 = 160 - 72 + 13 = 101$.

Then

$$
\begin{aligned}
a_{\vec{\mathbf{T}}}\vec{\mathbf{T}}(2) &= \frac{4(2)(20(2)-9)}{101}\hat{\mathbf{i}} + \frac{4(20(2)-9)}{101}\hat{\mathbf{j}} + \frac{6(40(2)^2-38(2)+9)}{101}\hat{\mathbf{k}} \\
&= \frac{248}{101}\hat{\mathbf{i}} + \frac{124}{101}\hat{\mathbf{j}} + \frac{558}{101}\hat{\mathbf{k}} \quad \checkmark
\end{aligned}
$$

and

$$
\begin{aligned}
a_{\vec{\mathbf{N}}}\vec{\mathbf{N}}(2) &= \frac{2(13-18(2))}{101}\hat{\mathbf{i}} - \frac{4(20(2)-9)}{101}\hat{\mathbf{j}} + \frac{12(2+2)}{101}\hat{\mathbf{k}} \\
&= -\frac{46}{101}\hat{\mathbf{i}} - \frac{124}{101}\hat{\mathbf{j}} + \frac{48}{101}\hat{\mathbf{k}} \quad \checkmark
\end{aligned}
$$

.

Note that we could even use the component vectors we found in Example 12.5.2 part (a) to determine $a_{\vec{\mathbf{T}}}(2)$ and $a_{\vec{\mathbf{N}}}(2)$, and even $\vec{\mathbf{N}}(2)$.

All we need to do to find the two scalar components of acceleration is find the magnitudes of the two vector components.

Thus

$$
\begin{aligned}
a_{\vec{\mathbf{T}}}(2) = \|a_{\vec{\mathbf{T}}}\vec{\mathbf{T}}(2)\| &= \sqrt{\left(\frac{248}{101}\right)^2 + \left(\frac{124}{101}\right)^2 + \left(\frac{558}{101}\right)^2} \\
&= \frac{\sqrt{248^2 + 124^2 + 558^2}}{101} \\
&= \frac{\sqrt{62^2(4^2 + 2^2 + 9^2}}{101} \\
&= \frac{62\sqrt{101}}{101}
\end{aligned}
$$

and

$$
\begin{aligned}
a_{\vec{\mathbf{N}}}(2) = \|a_{\vec{\mathbf{N}}}\vec{\mathbf{N}}(2)\| &= \sqrt{\frac{46^2 + 124^2 + 48^2}{101}} \\
&= \frac{\sqrt{2^2(23^2 + 62^2 + 24^2}}{101} \\
&= \frac{2\sqrt{4949}}{101} \\
&= \frac{2\sqrt{49 \cdot 101}}{101} \\
&= \frac{14\sqrt{101}}{101}
\end{aligned}
$$

Now all we need to do to find the unit normal vector for this curve at $t = 2$ is to normalize the vector component of acceleration that is normal to the curve at that location (and value of t).

So

$$\vec{\mathbf{N}}(2) = \frac{a_{\vec{\mathbf{N}}}\vec{\mathbf{N}}(2)}{\|a_{\vec{\mathbf{N}}}\vec{\mathbf{N}}(2)\|} = \frac{101}{14}\left[-\frac{46}{101}\,\hat{\mathbf{i}} - \frac{124}{101}\,\hat{\mathbf{j}} + \frac{48}{101}\,\hat{\mathbf{k}}\right]$$
$$= -\frac{23\sqrt{101}}{707}\,\hat{\mathbf{i}} - \frac{62\sqrt{101}}{707}\,\hat{\mathbf{j}} + \frac{24\sqrt{101}}{707}\,\hat{\mathbf{k}}$$

Kepler's Laws

During the early 1600s, Johannes Kepler was able to use the amazingly accurate data from his mentor Tycho Brahe to formulate his three laws of planetary motion, now known as Kepler's laws of planetary motion. These laws also apply to other objects in the solar system in orbit around the Sun, such as comets (e.g., Halley's comet) and asteroids. Variations of these laws apply to satellites in orbit around Earth.

Theorem 12.5.3: Kepler's Laws of Planetary Motion

1. The path of any planet about the Sun is elliptical in shape, with the center of the Sun located at one focus of the ellipse (the law of ellipses).
2. A line drawn from the center of the Sun to the center of a planet sweeps out equal areas in equal time intervals (the law of equal areas) (Figure 12.5.4).
3. The ratio of the squares of the periods of any two planets is equal to the ratio of the cubes of the lengths of their semimajor orbital axes (the Law of Harmonies).

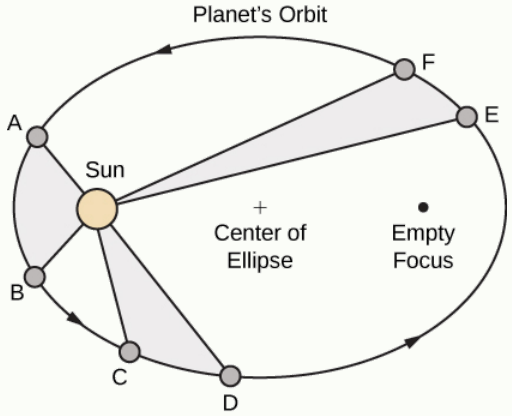

Figure 12.5.4: Kepler's first and second laws are pictured here. The Sun is located at a focus of the elliptical orbit of any planet. Furthermore, the shaded areas are all equal, assuming that the amount of time measured as the planet moves is the same for each region.

Kepler's third law is especially useful when using appropriate units. In particular, *1 astronomical unit* is defined to be the average distance from Earth to the Sun, and is now recognized to be 149,597,870,700 m or, approximately 93,000,000 mi. We therefore write 1 A.U. = 93,000,000 mi. Since the time it takes for Earth to orbit the Sun is 1 year, we use Earth years for units of time. Then, substituting 1 year for the period of Earth and 1 A.U. for the average distance to the Sun, Kepler's third law can be written as

$$T_p^2 = D_p^3 \tag{12.5.9}$$

for any planet in the solar system, where T_P is the period of that planet measured in Earth years and D_P is the average distance from that planet to the Sun measured in astronomical units. Therefore, if we know the average distance from a planet to the Sun (in astronomical units), we can then calculate the length of its year (in Earth years), and vice versa.

Kepler's laws were formulated based on observations from Brahe; however, they were not proved formally until Sir Isaac Newton was able to apply calculus. Furthermore, Newton was able to generalize Kepler's third law to other orbital systems, such as a moon orbiting around a planet. Kepler's original third law only applies to objects orbiting the Sun.

Proof

Let's now prove Kepler's first law using the calculus of vector-valued functions. First we need a coordinate system. Let's place the Sun at the origin of the coordinate system and let the vector-valued function $\vec{\mathbf{r}}(t)$ represent the location of a planet as a function of time. Newton proved Kepler's law using his second law of motion and his law of universal gravitation. Newton's second law of motion can be written as $\vec{\mathbf{F}} = m\vec{\mathbf{a}}$, where $\vec{\mathbf{F}}$ represents the net force acting on the planet. His law of universal gravitation can be written in the form $\vec{\mathbf{F}} = -\dfrac{GmM}{||\vec{\mathbf{r}}||^2} \cdot \dfrac{\vec{\mathbf{r}}}{||\vec{\mathbf{r}}||}$, which indicates that the force resulting from the gravitational attraction of the Sun points back toward the Sun, and has magnitude $\dfrac{GmM}{||\vec{\mathbf{r}}||^2}$ (Figure 12.5.5).

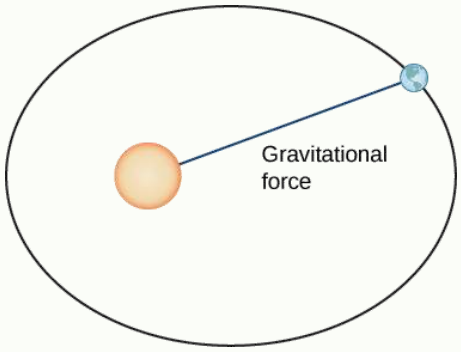

Figure 12.5.5: The gravitational force between Earth and the Sun is equal to the mass of the earth times its acceleration.

Setting these two forces equal to each other, and using the fact that $\vec{\mathbf{a}}(t) = \vec{\mathbf{v}}'(t)$, we obtain

$$m\vec{\mathbf{v}}'(t) = -\frac{GmM}{\|\vec{\mathbf{r}}\|^2} \cdot \frac{\vec{\mathbf{r}}}{\|\vec{\mathbf{r}}\|}, \tag{12.5.10}$$

which can be rewritten as

$$\frac{d\vec{\mathbf{v}}}{dt} = -\frac{GM}{||\vec{\mathbf{r}}||^3}\vec{\mathbf{r}}. \tag{12.5.11}$$

This equation shows that the vectors $d\vec{\mathbf{v}}/dt$ and $\vec{\mathbf{r}}$ are parallel to each other, so $d\vec{\mathbf{v}}/dt \times \vec{\mathbf{r}} = \vec{\mathbf{0}}$. Next, let's differentiate $\vec{\mathbf{r}} \times \vec{\mathbf{v}}$ with respect to time:

$$\frac{d}{dt}(\vec{\mathbf{r}} \times \vec{\mathbf{v}}) = \frac{d\vec{\mathbf{r}}}{dt} \times \vec{\mathbf{v}} + \vec{\mathbf{r}} \times \frac{d\vec{\mathbf{v}}}{dt} = \vec{\mathbf{v}} \times \vec{\mathbf{v}} + \vec{\mathbf{0}} = \vec{\mathbf{0}}. \tag{12.5.12}$$

This proves that $\vec{\mathbf{r}} \times \vec{\mathbf{v}}$ is a constant vector, which we call $\vec{\mathbf{C}}$. Since $\vec{\mathbf{r}}$ and $\vec{\mathbf{v}}$ are both perpendicular to $\vec{\mathbf{C}}$ for all values of t, they must lie in a plane perpendicular to $\vec{\mathbf{C}}$. Therefore, the motion of the planet lies in a plane.

Next we calculate the expression $d\vec{\mathbf{v}}/dt \times \vec{\mathbf{C}}$:

$$\frac{d\vec{\mathbf{v}}}{dt} \times \vec{\mathbf{C}} = -\frac{GM}{||\vec{\mathbf{r}}||^3}\vec{\mathbf{r}} \times (\vec{\mathbf{r}} \times \vec{\mathbf{v}}) = -\frac{GM}{||\vec{\mathbf{r}}||^3}[(\vec{\mathbf{r}} \cdot \vec{\mathbf{v}})\vec{\mathbf{r}} - (\vec{\mathbf{r}} \cdot \vec{\mathbf{r}})\vec{\mathbf{v}}]. \tag{12.5.13}$$

The last equality in Equation 12.5.13 is from the triple cross product formula (The Cross Product). We need an expression for $\vec{\mathbf{r}} \cdot \vec{\mathbf{v}}$. To calculate this, we differentiate $\vec{\mathbf{r}} \cdot \vec{\mathbf{r}}$ with respect to time:

$$\frac{d}{dt}(\vec{\mathbf{r}} \cdot \vec{\mathbf{r}}) = \frac{d\vec{\mathbf{r}}}{dt} \cdot \vec{\mathbf{r}} + \vec{\mathbf{r}} \cdot \frac{d\vec{\mathbf{r}}}{dt} = 2\vec{\mathbf{r}} \cdot \frac{d\vec{\mathbf{r}}}{dt} = 2\vec{\mathbf{r}} \cdot \vec{\mathbf{v}}. \tag{12.5.14}$$

Since $\vec{\mathbf{r}} \cdot \vec{\mathbf{r}} = ||\vec{\mathbf{r}}||^2$, we also have

$$\frac{d}{dt}(\vec{\mathbf{r}} \cdot \vec{\mathbf{r}}) = \frac{d}{dt}||\vec{\mathbf{r}}||^2 = 2||\vec{\mathbf{r}}||\frac{d}{dt}||\vec{\mathbf{r}}||. \tag{12.5.15}$$

Combining Equation 12.5.14 and Equation 12.5.15, we get

$$2\vec{\mathbf{r}}\cdot\vec{\mathbf{v}} = 2||\vec{\mathbf{r}}||\frac{d}{dt}||\vec{\mathbf{r}}||$$

$$\vec{\mathbf{r}}\cdot\vec{\mathbf{v}} = ||\vec{\mathbf{r}}\|\frac{d}{dt}||\vec{\mathbf{r}}||.$$

Substituting this into Equation 12.5.13 gives us

$$\begin{aligned}\frac{d\vec{\mathbf{v}}}{dt}\times\vec{\mathbf{C}} &= -\frac{GM}{||\vec{\mathbf{r}}||^3}[(\vec{\mathbf{r}}\cdot\vec{\mathbf{v}})\vec{\mathbf{r}} - (\vec{\mathbf{r}}\cdot\vec{\mathbf{r}})\vec{\mathbf{v}}] \\ &= -\frac{GM}{||\vec{\mathbf{r}}||^3}\left[||\vec{\mathbf{r}}||\left(\frac{d}{dt}||\vec{\mathbf{r}}||\right)\vec{\mathbf{r}} - ||\vec{\mathbf{r}}||^2\vec{\mathbf{v}}\right] \\ &= -GM\left[\frac{1}{||\vec{\mathbf{r}}||^2}\left(\frac{d}{dt}||\vec{\mathbf{r}}||\right)\vec{\mathbf{r}} - \frac{1}{||\vec{\mathbf{r}}||}\vec{\mathbf{v}}\right] \\ &= GM\left[\frac{\vec{\mathbf{v}}}{||\vec{\mathbf{r}}||} - \frac{\vec{\mathbf{r}}}{||\vec{\mathbf{r}}||^2}\left(\frac{d}{dt}||\vec{\mathbf{r}}||\right)\right]. \end{aligned} \tag{12.5.16}$$

However,

$$\begin{aligned}\frac{d}{dt}\frac{\vec{\mathbf{r}}}{||\vec{\mathbf{r}}||} &= \frac{\frac{d}{dt}(\vec{\mathbf{r}})||\vec{\mathbf{r}}|| - \vec{\mathbf{r}}\frac{d}{dt}||\vec{\mathbf{r}}||}{||\vec{\mathbf{r}}||^2} \\ &= \frac{\frac{d\vec{\mathbf{r}}}{dt}}{||\vec{\mathbf{r}}||} - \frac{\vec{\mathbf{r}}}{||\vec{\mathbf{r}}||^2}\frac{d}{dt}||\vec{\mathbf{r}}|| \\ &= \frac{\vec{\mathbf{v}}}{||\vec{\mathbf{r}}||} - \frac{\vec{\mathbf{r}}}{||\vec{\mathbf{r}}||^2}\frac{d}{dt}||\vec{\mathbf{r}}||.\end{aligned}$$

Therefore, Equation 12.5.16 becomes

$$\frac{d\vec{\mathbf{v}}}{dt}\times\vec{\mathbf{C}} = GM\left(\frac{d}{dt}\frac{\vec{\mathbf{r}}}{||\vec{\mathbf{r}}||}\right). \tag{12.5.17}$$

Since $\vec{\mathbf{C}}$ is a constant vector, we can integrate both sides and obtain

$$\vec{\mathbf{v}}\times\vec{\mathbf{C}} = GM\frac{\vec{\mathbf{r}}}{||\vec{\mathbf{r}}||} + \vec{\mathbf{D}}, \tag{12.5.18}$$

where $\vec{\mathbf{D}}$ is a constant vector. Our goal is to solve for $||\vec{\mathbf{r}}||$. Let's start by calculating $\vec{\mathbf{r}}\cdot\left(\vec{\mathbf{v}}\times\vec{\mathbf{C}}\right)$:

$$\vec{\mathbf{r}}\cdot\left(\vec{\mathbf{v}}\times\vec{\mathbf{C}}\right) = GM\frac{||\vec{\mathbf{r}}||^2}{||\vec{\mathbf{r}}||} + \vec{\mathbf{r}}\cdot\vec{\mathbf{D}} = GM||\vec{\mathbf{r}}|| + \vec{\mathbf{r}}\cdot\vec{\mathbf{D}}. \tag{12.5.19}$$

However, $\vec{\mathbf{r}}\cdot(\vec{\mathbf{v}}\times\vec{\mathbf{C}}) = (\vec{\mathbf{r}}\times\vec{\mathbf{v}})\cdot\vec{\mathbf{C}}$, so

$$(\vec{\mathbf{r}}\times\vec{\mathbf{v}})\cdot\vec{\mathbf{C}} = GM||\vec{\mathbf{r}}|| + \vec{\mathbf{r}}\cdot\vec{\mathbf{D}}. \tag{12.5.20}$$

Since $\vec{\mathbf{r}}\times\vec{\mathbf{v}} = \vec{\mathbf{C}}$, we have

$$||\vec{\mathbf{C}}||^2 = GM||\vec{\mathbf{r}}|| + \vec{\mathbf{r}}\cdot\vec{\mathbf{D}}. \tag{12.5.21}$$

Note that $\vec{\mathbf{r}}\cdot\vec{\mathbf{D}} = ||\vec{\mathbf{r}}||||\vec{\mathbf{D}}||\cos\theta$, where θ is the angle between $\vec{\mathbf{r}}$ and $\vec{\mathbf{D}}$. Therefore,

$$||\vec{\mathbf{C}}||^2 = GM||\vec{\mathbf{r}}|| + ||\vec{\mathbf{r}}||||\vec{\mathbf{D}}||\cos\theta \tag{12.5.22}$$

Solving for $||\vec{\mathbf{r}}||$,

$$||\vec{\mathbf{r}}|| = \frac{||\vec{\mathbf{C}}||^2}{GM + ||\vec{\mathbf{D}}||\cos\theta} = \frac{||\vec{\mathbf{C}}||^2}{GM}\left(\frac{1}{1+e\cos\theta}\right). \tag{12.5.23}$$

where $e = ||\vec{\mathbf{D}}||/GM$. This is the polar equation of a conic with a focus at the origin, which we set up to be the Sun. It is a hyperbola if $e > 1$, a parabola if $e = 1$, or an ellipse if $e < 1$. Since planets have closed orbits, the only possibility is an ellipse.

However, at this point it should be mentioned that hyperbolic comets do exist. These are objects that are merely passing through the solar system at speeds too great to be trapped into orbit around the Sun. As they pass close enough to the Sun, the gravitational field of the Sun deflects the trajectory enough so the path becomes hyperbolic.

□

Kepler's third law of planetary motion can be modified to the case of one object in orbit around an object other than the Sun, such as the Moon around the Earth. In this case, Kepler's third law becomes

$$P^2 = \frac{4\pi^2 a^3}{G(m+M)}, \tag{12.5.24}$$

where m is the mass of the Moon and M is the mass of Earth, a represents the length of the major axis of the elliptical orbit, and P represents the period.

Example 12.5.3: Using Kepler's Third Law for Nonheliocentric Orbits

Given that the mass of the Moon is 7.35×10^{22} kg, the mass of Earth is 5.97×10^{24} kg, $G = 6.67 \times 10^{-11} \text{m/kg} \cdot \text{sec}^2$, and the period of the moon is 27.3 days, let's find the length of the major axis of the orbit of the Moon around Earth.

Solution

It is important to be consistent with units. Since the universal gravitational constant contains seconds in the units, we need to use seconds for the period of the Moon as well:

$$27.3 \text{ days} \times \frac{24 \text{ hr}}{1 \text{ day}} \times \frac{3600 \text{ sec}}{1 \text{ hour}} = 2,358,720 \text{ sec} \tag{12.5.25}$$

Substitute all the data into Equation 12.5.24 and solve for a:

$$(2,358,720 \text{ sec})^2 = \frac{4\pi^2 a^3}{\left(6.67 \times 10^{-11} \frac{m}{\text{kg} \times \text{sec}^2}\right)(7.35 \times 10^{22}\text{kg} + 5.97 \times 10^{24}\text{kg})}$$

$$5.563 \times 10^{12} = \frac{4\pi^2 a^3}{(6.67 \times 10^{-11}\text{m}^3)(6.04 \times 10^{24})}$$

$$(5.563 \times 10^{12})(6.67 \times 10^{-11}\text{m}^3)(6.04 \times 10^{24}) = 4\pi^2 a^3$$

$$a^3 = \frac{2.241 \times 10^{27}}{4\pi^2}\text{m}^3$$

$$a = 3.84 \times 10^8 \text{m}$$

$$\approx 384,000 \text{ km}.$$

Analysis

According to solarsystem.nasa.gov, the actual average distance from the Moon to Earth is 384,400 km. This is calculated using reflectors left on the Moon by Apollo astronauts back in the 1960s.

Exercise 12.5.3

Titan is the largest moon of Saturn. The mass of Titan is approximately $1.35 \times 10^{23} kg$. The mass of Saturn is approximately 5.68×10^{26} kg. Titan takes approximately 16 days to orbit Saturn. Use this information, along with the universal gravitation constant $G = 6.67 \times 10^{-11} \text{m/kg} \cdot \text{sec}^2$ to estimate the distance from Titan to Saturn.

Example 12.5.4: Halley's Comet

We now return to the chapter opener, which discusses the motion of Halley's comet around the Sun. Kepler's first law states that Halley's comet follows an elliptical path around the Sun, with the Sun as one focus of the ellipse. The period of Halley's comet is approximately 76.1 years, depending on how closely it passes by Jupiter and Saturn as it passes through the outer solar system. Let's use $T = 76.1$ years. What is the average distance of Halley's comet from the Sun?

Figure 12.5.6: Halley's comet

Solution

Using the equation $T^2 = D^3$ with $T = 76.1$, we obtain $D^3 = 5791.21$, so $D \approx 17.96$ A.U. This comes out to approximately 1.67×10^9 mi.

A natural question to ask is: What are the maximum (aphelion) and minimum (perihelion) distances from Halley's Comet to the Sun? The eccentricity of the orbit of Halley's Comet is 0.967 (Source: http://nssdc.gsfc.nasa.gov/planetary...cometfact.html). Recall that the formula for the eccentricity of an ellipse is $e = c/a$, where a is the length of the semimajor axis and c is the distance from the center to either focus. Therefore, $0.967 = c/17.96$ and $c \approx 17.37$ A.U. Subtracting this from a gives the perihelion distance $p = a - c = 17.96 - 17.37 = 0.59$ A.U. According to the National Space Science Data Center (Source: http://nssdc.gsfc.nasa.gov/planetary...cometfact.html), the perihelion distance for Halley's comet is 0.587 A.U. To calculate the aphelion distance, we add

$$P = a + c = 17.96 + 17.37 = 35.33 \text{ A.U.} \tag{12.5.27}$$

This is approximately 3.3×10^9 mi. The average distance from Pluto to the Sun is 39.5 A.U. (Source: http://www.oarval.org/furthest.htm), so it would appear that Halley's Comet stays just within the orbit of Pluto.

NAVIGATING A BANKED TURN

How fast can a racecar travel through a circular turn without skidding and hitting the wall? The answer could depend on several factors:

- The weight of the car;
- The friction between the tires and the road;
- The radius of the circle;
- The "steepness" of the turn.

In this project we investigate this question for NASCAR racecars at the Bristol Motor Speedway in Tennessee. Before considering this track in particular, we use vector functions to develop the mathematics and physics necessary for answering questions such as this.

A car of mass m moves with constant angular speed ω around a circular curve of radius R (Figure 12.5.7). The curve is banked at an angle θ. If the height of the car off the ground is h, then the position of the car at time t is given by the function $\vec{\mathbf{r}}(t) = < R\cos(\omega t), R\sin(\omega t), h >$.

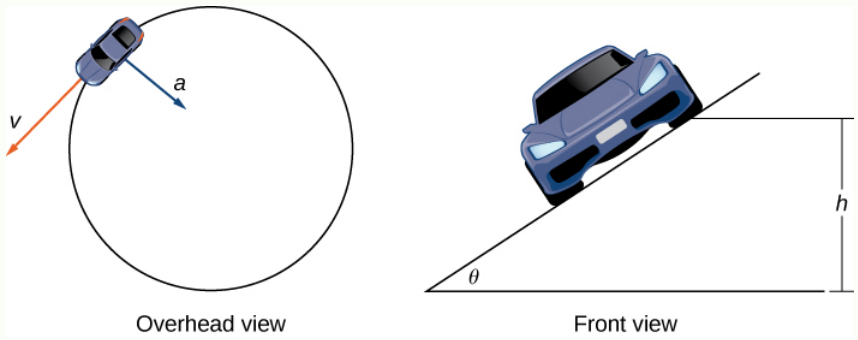

Figure 12.5.7: Views of a race car moving around a track.

1. Find the velocity function $\vec{\mathbf{v}}(t)$ of the car. Show that $\vec{\mathbf{v}}$ is tangent to the circular curve. This means that, without a force to keep the car on the curve, the car will shoot off of it.
2. Show that the speed of the car is ωR. Use this to show that $(2\pi 4)/\|\vec{\mathbf{v}}\| = (2\pi)/\omega$.
3. Find the acceleration $\vec{\mathbf{a}}$. Show that this vector points toward the center of the circle and that $\|\vec{\mathbf{a}}\| = R\omega^2$.
4. The force required to produce this circular motion is called the *centripetal force*, and it is denoted $\vec{\mathbf{F}}_{cent}$. This force points toward the center of the circle (not toward the ground). Show that $\|\vec{\mathbf{F}}_{cent}\| = (m|\vec{\mathbf{v}}|^2)/R$.

As the car moves around the curve, three forces act on it: gravity, the force exerted by the road (this force is perpendicular to the ground), and the friction force (Figure 12.5.8). Because describing the frictional force generated by the tires and the road is complex, we use a standard approximation for the frictional force. Assume that $\vec{\mathbf{f}} = \mu\vec{\mathbf{N}}$ for some positive constant μ. The constant μ is called the *coefficient of friction*.

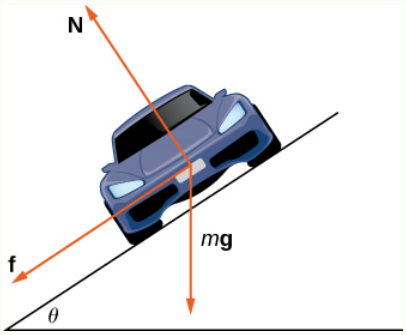

Figure 12.5.8: The car has three forces acting on it: gravity (denoted by $m\vec{\mathbf{g}}$), *the friction force* $\vec{\mathbf{f}}$, *and the force exerted by the road* $\vec{\mathbf{N}}$.

Let v_{max} denote the maximum speed the car can attain through the curve without skidding. In other words, v_{max} is the fastest speed at which the car can navigate the turn. When the car is traveling at this speed, the magnitude of the centripetal force is

$$\|\vec{\mathbf{F}}_{cent}\| = \frac{m(v_{max})^2}{R}. \tag{12.5.28}$$

The next three questions deal with developing a formula that relates the speed v_{max} to the banking angle θ.

5. Show that $\vec{\mathbf{N}}\cos\theta = m\vec{\mathbf{g}} + \vec{\mathbf{f}}\sin\theta$. Conclude that $\vec{\mathbf{N}} = (m\vec{\mathbf{g}})/(\cos\theta - \mu\sin\theta)$.
6. The centripetal force is the sum of the forces in the horizontal direction, since the centripetal force points toward the center of the circular curve. Show that

$$\vec{\mathbf{F}}_{cent} = \vec{\mathbf{N}}\sin\theta + \vec{\mathbf{f}}\cos\theta. \tag{12.5.29}$$

   Conclude that

$$\vec{\mathbf{F}}_{cent} = \frac{\sin\theta + \mu\cos\theta}{cos\theta - \mu\sin\theta} m\vec{\mathbf{g}}. \tag{12.5.30}$$

7. Show that $(v_{max})^2 = ((\sin\theta + \mu\, cos\theta)/(\cos\theta - \mu\sin\theta))gR$. Conclude that the maximum speed does not actually depend on the mass of the car.
   Now that we have a formula relating the maximum speed of the car and the banking angle, we are in a position to answer the questions like the one posed at the beginning of the project.
   The Bristol Motor Speedway is a NASCAR short track in Bristol, Tennessee. The track has the approximate shape shown in Figure 12.5.9. Each end of the track is approximately semicircular, so when cars make turns they are traveling along an approximately circular curve. If a car takes the inside track and speeds along the bottom of turn 1, the car travels along a semicircle of radius approximately 211 ft with a banking angle of 24°. If the car decides to take the outside track and speeds along the top of turn 1, then the car travels along a semicircle with a banking angle of 28°. (The track has variable angle banking.)

Figure 12.5.9: At the Bristol Motor Speedway, Bristol, Tennessee (a), the turns have an inner radius of about 211 ft and a width of 40 ft (b). (credit: part (a) photo by Raniel Diaz, Flickr)

The coefficient of friction for a normal tire in dry conditions is approximately 0.7. Therefore, we assume the coefficient for a NASCAR tire in dry conditions is approximately 0.98.

Before answering the following questions, note that it is easier to do computations in terms of feet and seconds, and then convert the answers to miles per hour as a final step.

8. In dry conditions, how fast can the car travel through the bottom of the turn without skidding?
9. In dry conditions, how fast can the car travel through the top of the turn without skidding?
10. In wet conditions, the coefficient of friction can become as low as 0.1. If this is the case, how fast can the car travel through the bottom of the turn without skidding?
11. Suppose the measured speed of a car going along the outside edge of the turn is 105 mph. Estimate the coefficient of friction for the car's tires.

Key Concepts

- The acceleration vector always points toward the concave side of the curve defined by $\vec{\mathbf{r}}(t)$. The tangential and normal components of acceleration $a_{\vec{\mathbf{T}}}$ and $a_{\vec{\mathbf{N}}}$ are the projections of the acceleration vector onto the unit tangent and unit normal vectors to the curve.
- Kepler's three laws of planetary motion describe the motion of objects in orbit around the Sun. His third law can be modified to describe motion of objects in orbit around other celestial objects as well.
- Newton was able to use his law of universal gravitation in conjunction with his second law of motion and calculus to prove Kepler's three laws.

Key Equations

- **Tangential component of acceleration**

$$a_{\vec{\mathbf{T}}} = \vec{\mathbf{a}} \cdot \vec{\mathbf{T}} = \frac{\vec{\mathbf{v}} \cdot \vec{\mathbf{a}}}{||\vec{\mathbf{v}}||}$$

- **Normal component of acceleration**

$$a_{\vec{\mathbf{N}}} = \vec{\mathbf{a}} \cdot \vec{\mathbf{N}} = \frac{||\vec{\mathbf{v}} \times \vec{\mathbf{a}}||}{||\vec{\mathbf{v}}||} = \sqrt{||\vec{\mathbf{a}}||^2 - \left(a_{\vec{\mathbf{T}}}\right)^2}$$

Glossary

Contributors

- Gilbert Strang (MIT) and Edwin "Jed" Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.
- Paul Seeburger added Figure 12.5.1, Exercise 12.5.1, the exposition below Figure 12.5.1 through Theorem 12.5.3, and from Example 12.5.2 until just before Kepler's Laws.

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12.5E: Exercises for Section 12.5

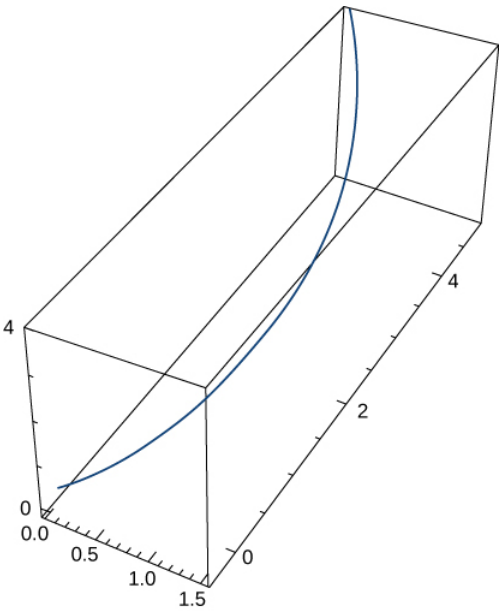
Finding Components of Acceleration & Kepler's Laws

1) Find the tangential and normal components of acceleration for $\vec{\mathbf{r}}(t) = t^2\,\hat{\mathbf{i}} + 2t\,\hat{\mathbf{j}}$ when $t = 1$.

In questions 2 - 8, find the tangential and normal components of acceleration.

2) $\vec{\mathbf{r}}(t) = \langle \cos(2t),\ \sin(2t),\ 1 \rangle$

3) $\vec{\mathbf{r}}(t) = \langle e^t \cos t,\ e^t \sin t,\ e^t \rangle$. The graph is shown here:



4) $\vec{\mathbf{r}}(t) = \langle \frac{2}{3}(1+t)^{3/2},\ \frac{2}{3}(1-t)^{3/2},\ \sqrt{2}t \rangle$

5) $\vec{\mathbf{r}}(t) = \left\langle 2t,\ t^2,\ \dfrac{t^3}{3} \right\rangle$

6) $\vec{\mathbf{r}}(t) = t^2\,\hat{\mathbf{i}} + t^2\,\hat{\mathbf{j}} + t^3\,\hat{\mathbf{k}}$

7) $\vec{\mathbf{r}}(t) = \langle 6t,\ 3t^2,\ 2t^3 \rangle$

8) $\vec{\mathbf{r}}(t) = 3\cos(2\pi t)\,\hat{\mathbf{i}} + 3\sin(2\pi t)\,\hat{\mathbf{j}}$

9) Find the tangential and normal components of acceleration for $\vec{\mathbf{r}}(t) = a\cos(\omega t)\,\hat{\mathbf{i}} + b\sin(\omega t)\,\hat{\mathbf{j}}$ at $t = 0$.

10) Suppose that the position function for an object in three dimensions is given by the equation $\vec{\mathbf{r}}(t) = t\cos(t)\,\hat{\mathbf{i}} + t\sin(t)\,\hat{\mathbf{j}} + 3t\,\hat{\mathbf{k}}$.

a. Show that the particle moves on a circular cone.

b. Find the angle between the velocity and acceleration vectors when $t = 1.5$.

c. Find the tangential and normal components of acceleration when $t = 1.5$.

11) The force on a particle is given by $\vec{\mathbf{f}}(t) = (\cos t)\,\hat{\mathbf{i}} + (\sin t)\,\hat{\mathbf{j}}$. The particle is located at point $(c, 0)$ at $t = 0$. The initial velocity of the particle is given by $\vec{\mathbf{v}}(0) = v_0\,\hat{\mathbf{j}}$. Find the path of the particle of mass m. (Recall, $\vec{\mathbf{F}} = m\vec{\mathbf{a}}$.)

12) An automobile that weighs 2700 lb makes a turn on a flat road while traveling at 56 ft/sec. If the radius of the turn is 70 ft, what is the required frictional force to keep the car from skidding?

13) Using Kepler's laws, it can be shown that $v_0 = \sqrt{\dfrac{2GM}{r_0}}$ is the minimum speed needed when $\theta = 0$ so that an object will escape from the pull of a central force resulting from mass M. Use this result to find the minimum speed when $\theta = 0$ for a space capsule to escape from the gravitational pull of Earth if the probe is at an altitude of 300 km above Earth's surface.

14) Find the time in years it takes the dwarf planet Pluto to make one orbit about the Sun given that $a = 39.5$ A.U.

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12R: Review Exercises for Chapter 12

***True or False*? Justify your answer with a proof or a counterexample.**

1. A parametric equation that passes through points P and Q can be given by $\vec{\mathbf{r}}(t) = \langle t^2,\ 3t+1,\ t-2\rangle$, where $P(1,4,-1)$ and $Q(16,11,2)$.

2. $\dfrac{d}{dt}\left[\vec{\mathbf{u}}(t)\times\vec{\mathbf{u}}(t)\right] = 2\vec{\mathbf{u}}'(t)\times\vec{\mathbf{u}}(t)$

3. The curvature of a circle of radius r is constant everywhere. Furthermore, the curvature is equal to $1/r$.

4. The speed of a particle with a position function $\vec{\mathbf{r}}(t)$ is $\dfrac{\vec{\mathbf{r}}'(t)}{\|\vec{\mathbf{r}}'(t)\|}$.

Find the domains of the vector-valued functions.

5. $\vec{\mathbf{r}}(t) = \langle \sin(t),\ \ln(t),\ \sqrt{t}\rangle$

6. $\vec{\mathbf{r}}(t) = \left\langle e^t,\ \dfrac{1}{\sqrt{4-t}},\ \sec t\right\rangle$

Sketch the curves for the following vector equations. Use a calculator if needed.

7. **[T]** $\vec{\mathbf{r}}(t) = \langle t^2,\ t^3\rangle$

8. **[T]** $\vec{\mathbf{r}}(t) = \langle \sin(20t)e^{-t},\ \cos(20t)e^{-t},\ e^{-t}\rangle$

Find a vector function that describes the following curves.

9. Intersection of the cylinder $x^2+y^2=4$ with the plane $x+z=6$

10. Intersection of the cone $z=\sqrt{x^2+y^2}$ and plane $z=y-4$

Find the derivatives of $\vec{\mathbf{u}}(t)$, $\vec{\mathbf{u}}'(t)$, $\vec{\mathbf{u}}'(t)\times\vec{\mathbf{u}}(t)$, $\vec{\mathbf{u}}(t)\times\vec{\mathbf{u}}'(t)$, and $\vec{\mathbf{u}}(t)\cdot\vec{\mathbf{u}}'(t)$. Find the unit tangent vector.

11. $\vec{\mathbf{u}}(t) = \langle e^t,\ e^{-t}\rangle$

12. $\vec{\mathbf{u}}(t) = \langle t^2,\ 2t+6,\ 4t^5-12\rangle$

Evaluate the following integrals.

13. $\displaystyle\int \left(\tan(t)\sec(t)\,\hat{\mathbf{i}} - te^{3t}\,\hat{\mathbf{j}}\right)\,dt$

14. $\displaystyle\int_1^4 \vec{\mathbf{u}}(t)\,dt$, with $\vec{\mathbf{u}}(t) = \left\langle \dfrac{\ln t}{t},\ \dfrac{1}{\sqrt{t}},\ \sin\left(\tfrac{t\pi}{4}\right)\right\rangle$

Find the length for the following curves.

15. $\vec{\mathbf{r}}(t) = \langle 3t,\ 4\cos t,\ 4\sin t\rangle$ for $1\le t\le 4$

16. $\vec{\mathbf{r}}(t) = 2\,\hat{\mathbf{i}} + t\,\hat{\mathbf{j}} + 3t^2\,\hat{\mathbf{k}}$ for $0\le t\le 1$

Reparameterize the following functions with respect to their arc length measured from $t=0$ in direction of increasing t.

17. $\vec{\mathbf{r}}(t) = 2t\,\hat{\mathbf{i}} + (4t-5)\,\hat{\mathbf{j}} + (1-3t)\,\hat{\mathbf{k}}$

18. $\vec{\mathbf{r}}(t) = \cos(2t)\,\hat{\mathbf{i}} + 8t\,\hat{\mathbf{j}} - \sin(2t)\,\hat{\mathbf{k}}$

Find the curvature for the following vector functions.

19. $\vec{\mathbf{r}}(t) = (2\sin t)\,\hat{\mathbf{i}} - 4t\,\hat{\mathbf{j}} + (2\cos t)\,\hat{\mathbf{k}}$

20. $\vec{\mathbf{r}}(t) = \sqrt{2}e^t\,\hat{\mathbf{i}} + \sqrt{2}e^{-t}\,\hat{\mathbf{j}} + 2t\,\hat{\mathbf{k}}$

21. Find the unit tangent vector, the unit normal vector, and the binormal vector for $\vec{\mathbf{r}}(t) = 2\cos t\,\hat{\mathbf{i}} + 3t\,\hat{\mathbf{j}} + 2sint\,\hat{\mathbf{k}}$.

22. Find the tangential and normal acceleration components with the position vector $\vec{\mathbf{r}}(t) = \langle \cos t,\ \sin t,\ e^t\rangle$.

23. A Ferris wheel car is moving at a constant speed v and has a constant radius r. Find the tangential and normal acceleration of the Ferris wheel car.

24. The position of a particle is given by $\vec{\mathbf{r}}(t) = \langle t^2, \ln t, \sin(\pi t)\rangle$, where t is measured in seconds and r is measured in meters. Find the velocity, acceleration, and speed functions. What are the position, velocity, speed, and acceleration of the particle at 1 sec?

The following problems consider launching a cannonball out of a cannon. The cannonball is shot out of the cannon with an angle θ and initial velocity $\vec{\mathbf{v}}_0$. The only force acting on the cannonball is gravity, so we begin with a constant acceleration $\vec{\mathbf{a}}(t) = -g\hat{\mathbf{j}}$.

25. Find the velocity vector function $\vec{\mathbf{v}}(t)$.

26. Find the position vector $\vec{\mathbf{r}}(t)$ and the parametric representation for the position.

27. At what angle do you need to fire the cannonball for the horizontal distance to be greatest? What is the total distance it would travel?

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More Review Exercises for Chapter 12

12.1: Vector-Valued Functions

Terms and Concepts

1. Vector-valued functions are closely related to _____ _____ of graphs.

2. When sketching vector-valued functions, technically one isn't graphing points, but rather _____.

3. It can be useful to think of _____ as a vector that points from a starting position to an ending position.

Problems

In Exercises 4-11, sketch the vector-valued function on the given interval.

4. $\vec{r}(t) = \langle t^2, t^2 - 1\rangle$, for $-2 \le t \le 2$.

5. $\vec{r}(t) = \langle t^2, t^3\rangle$, for $-2 \le t \le 2$.

6. $\vec{r}(t) = \langle 1/t, t/t^2\rangle$, for $-2 \le t \le 2$.

7. $\vec{r}(t) = \langle \frac{1}{10}t^2, \sin t\rangle$, for $-2\pi \le t \le 2\pi$.

8. $\vec{r}(t) = \langle \frac{1}{10}t^2, \sin t\rangle$, for $-2\pi \le t \le 2\pi$.

9. $\vec{r}(t) = \langle 3\sin(\pi t), 2\cos(\pi t)\rangle$, on $[0, 2]$.

10. $\vec{r}(t) = \langle 3\cos t, 2\sin(2t)\rangle$, on $[0, 2\pi]$.

11. $\vec{r}(t) = \langle 2\sec t, \tan t\rangle$, on $[-\pi, \pi]$.

In Exercises 12-15, sketch the vector-valued function on the given interval in $\mathbb{R}^3$. Technology may be useful in creating the sketch.

12. $\vec{r}(t) = \langle 2\cos t, t, 2\sin t\rangle$, on $[0, 2\pi]$.

13. $\vec{r}(t) = \langle 3\cos t, \sin t, t/\pi\rangle$, on $[0, 2\pi]$.

14. $\vec{r}(t) = \langle \cos t, \sin t, \sin t\rangle$, on $[0, 2\pi]$.

15. $\vec{r}(t) = \langle \cos t, \sin t, \sin(2t)\rangle$, on $[0, 2\pi]$.

In Exercises 16-19, find $\|\vec{r}(t)\|$.

16. $\vec{r}(t) = \langle t, t^2\rangle$.

17. $\vec{r}(t) = \langle 5\cos t, 3\sin t\rangle$.

18. $\vec{r}(t) = \langle 2\cos t, 2\sin t, t\rangle$.

19. $\vec{r}(t) = \langle \cos t, t, t^2\rangle$.

In Exercises 20-27, create a vector-valued function whose graph matches the given description.

20. A circle of radius 2, centered $(1, 2)$, traced counter-clockwise once on $[0, 2\pi]$.

21. A circle of radius 3, centered $(5, 5)$, traced clockwise once on $[0, 2\pi]$.

22. An ellipse, centered at $(0, 0)$ with vertical major axis length 10 and minor axis of length 3, traced once counter-clockwise on $[0, 2\pi]$.

23. An ellipse, centered at $(3, -2)$ with horizontal major axis length 6 and minor axis of length 4, traced once clockwise on $[0, 2\pi]$.

24. A line through $(2, 3)$ with a slope of 5.

25. A line through $(1, 5)$ with a slope of -1/2.

26. A vertically oriented helix with radius of 2 that starts at $(2, 0, 0)$ and ends at $(2, 0, 4\pi)$ after 1 revolution on $[0, 2\pi]$.

27. A vertically oriented helix with radius of 3 that starts at $(3,0,0)$ and ends at $(3,0,3)$ after 2 revolutions on $[0,1]$.

In Exercises 28-31, find the average rate of change of $\vec{r}(t)$ on the given interval.

28. $\vec{r}(t) = \langle t, t^2 \rangle$ on $[-2,2]$.

29. $\vec{r}(t) = \langle t, t + \sin t \rangle$ on $[0, 2\pi]$.

30. $\vec{r}(t) = \langle 3\cos t, 2\sin t, t \rangle$ on $[0, 2\pi]$.

31. $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ on $[-1,3]$.

12.2: Calculus and Vector-Valued Functions

Terms and Concepts

1. Limits, derivatives and integrals of vector-valued functions are all evaluated _____ -wise.

2. The definite integral of a rate of change function gives _____.

3. Why is it generally not useful to graph both $\vec{r}(t)$ and $\vec{r}\,'(t)$ on the same axes?

Problems

In Exercises 4-7, evaluate the given limit.

4. $\lim\limits_{t\to 5} \langle 2t+1, 3t^2-1, \sin t \rangle$

5. $\lim\limits_{t\to 3} \langle e^t, \frac{t^2-9}{t+3} \rangle$

6. $\lim\limits_{t\to 0} \langle \frac{t}{\sin t}, (1+t)^{\frac{1}{t}} \rangle$

7. $\lim\limits_{h\to 0} \frac{\vec{r}(t+h)-\vec{r}(t)}{h}$, where $\vec{r}(t) = \langle t^2, t, 1 \rangle$

In Exercises 8-9, identify the interval(s) on which $\vec{r}(t)$ is continuous.

8. $\vec{r}(t) = \langle t^2, 1/t \rangle$

9. $\vec{r}(t) = \langle \cos t, e^t, \ln t \rangle$

In Exercises 10-14, find the derivative of the given function.

10. $\vec{r}(t) = \langle \cos t, e^t, \ln t \rangle$

11. $\vec{r}(t) = \langle \frac{1}{t}, \frac{2t-1}{3t+1}, \tan t \rangle$

12. $\vec{r}(t) = (t^2)\langle \sin t, 2t+5 \rangle$

13. $\vec{r}(t) = \langle t^2+1, t-1 \rangle \cdot \langle \sin t, 2t+5 \rangle$

14. $\vec{r}(t) = \langle t^2+1, t-1, 1 \rangle \times \langle \sin t, 2t+5, 1 \rangle$

In Exercises 15-18, find $\vec{r}\,'(t)$. Sketch $\vec{r}(t)$ and $\vec{r}\,'(1)$, with the initial point of $\vec{r}\,'(1)$ at $\vec{r}(1)$.

15. $\vec{r}(t) = \langle t^2+t, t^2-t \rangle$

16. $\vec{r}(t) = \langle t^2-2t+2, t^3-3t^2+2t \rangle$

17. $\vec{r}(t) = \langle t^2+1, t^3-t \rangle$

18. $\vec{r}(t) = \langle t^2-4t+5, t^3-6t^2+11t-6 \rangle$

In Exercises 19-22, give the equation of the line tangent to the graph of $\vec{r}(t)$ at the given t value.

19. $\vec{r}(t) = \langle t^2+t, t^2-t \rangle$ at $t=1$.

20. $\vec{r}(t) = \langle 3\cos t, \sin t \rangle$ at $t = \pi/4$.

21. $\vec{r}(t) = \langle 3\cos t, 3\sin t, t \rangle$ at $t = \pi$.

22. $\vec{r}(t) = \langle e^t, \tan t, t \rangle$ at $t = 0$.

In Exercises 23-26, find the value(s) of t for which $\vec{r}(t)$ is not smooth.

23. $\vec{r}(t) = \langle \cos t, \sin t - t \rangle$

24. $\vec{r}(t) = \langle t^2 - 2t + 1, t^3 + t^2 - 5t + 3 \rangle$

25. $\vec{r}(t) = \langle \cos t - \sin t, -\cos t, \cos(4t) \rangle$

26. $\vec{r}(t) = \langle t^3 - 3t + 2, -\cos(\pi t), \sin^2(\pi t) \rangle$

Exercises 27-29 ask you to verify parts of Theorem 92. In each let $f(t) = t^3, \vec{r}(t) = \langle t^2, t-1, 1 \rangle$ and $\vec{s}(t) = \langle \sin t, e^t, t \rangle$. Compute the various derivatives as indicated.

27. Simplify $f(t)\vec{r}(t)$, then find its derivative; show this is the same as $f'(t)\vec{r}'(t) + f(t)\vec{r}'(t)$.

28. Simplify $\vec{r}(t) \cdot \vec{s}(t)$, then find its derivative; show this is the same as $\vec{r}'(t) \cdot \vec{s}'(t) + \vec{r}(t) \cdot \vec{s}'(t)$.

29. Simplify $\vec{r}(t) \times \vec{s}(t)$, then find its derivative; show this is the same as $\vec{r}'(t) \times \vec{s}'(t) + \vec{r}(t) \times \vec{s}'(t)$.

In Exercises 30-33, evaluate the given definite or indefinite integral.

30. $\int \langle t^3, \cos t, te^t \rangle \, dt$

31. $\int \left\langle \frac{1}{1+t^2}, \sec^2 t \right\rangle \, dt$

32. $\int_0^{\pi} \langle -\sin t, \cos t \rangle \, dt$

33. $\int_{-2}^{2} \langle 2t + 1, 2t - 1 \rangle \, dt$

In Exercises 34-37, solve the given initial value problems.

34. Find $\vec{r}(t)$, given that $\vec{r}'(t) = \langle t, \sin t \rangle$ and $\vec{r}(0) = \langle 2, 2 \rangle$.

35. Find $\vec{r}(t)$, given that $\vec{r}'(t) = \langle 1, (t+1), \tan t \rangle$ and $\vec{r}(0) = \langle 1, 2 \rangle$.

36. Find $\vec{r}(t)$, given that $\vec{r}''(t) = \langle t^2, t, 1 \rangle$, $\vec{r}'(0) = \langle 1, 2, 3 \rangle$ and $\vec{r}(0) = \langle 4, 5, 6 \rangle$.

37. Find $\vec{r}(t)$, given that $\vec{r}''(t) = \langle \cos t, \sin t, e^t \rangle$, $\vec{r}'(0) = \langle 0, 0, 0 \rangle$ and $\vec{r}(0) = \langle 0, 0, 0 \rangle$.

In Exercises 38-41, find the arc length of $\vec{r}(t)$ on the indicated interval.

38. $\vec{r}(t) = \langle 2\cos t, 2\sin t, 3t \rangle$ on $[0, 2\pi]$

39. $\vec{r}(t) = \langle 5\cos t, 3\sin t, 4\sin t \rangle$ on $[0, 2\pi]$.

40. $\vec{r}(t) = \langle t^3, t^2, t^3 \rangle$ on $[0, 1]$.

41. $\vec{r}(t) = \langle e^{-t}\cos t, e^{-t}\sin t \rangle$ on $[0, 1]$.

42. Prove Theorem 93; that is, show if $\vec{r}(t)$ has constant length and is differentiable, then $\vec{r}(t) \cdot \vec{r}'(t) = 0$. (Hint: use the Product Rule to compute $\frac{d}{dt}(\vec{r}(t) \cdot \vec{r}(t))$.)

12.3: The Calculus of Motion

Terms and Concepts

1. How is velocity different from speed?

2. What is the difference between displacement and distance traveled?

3. What is the difference between average velocity and average speed?

4. Distance traveled is the same as _____ _____, just viewed in a different context.

5. Describe a scenario where an object's average speed is a large number, but the magnitude of the average velocity is not a large number.

6. Explain why it is not possible to have an average velocity with a large magnitude but a small average speed.

Problems

In Exercises 7-10, a position function $\vec{r}(t)$ is given. Find $\vec{v}(t)$ and $\vec{a}(t)$.

7. $\vec{r}(t) = \langle 2t+1, 5t-2, 7\rangle$

8. $\vec{r}(t) = \langle 3t^2-2t+1, -t^2+t+14\rangle$

9. $\vec{r}(t) = \langle \cos t, \sin t\rangle$

10. $\vec{r}(t) = \langle t/10, -\cos t, \sin t\rangle$

In Exercises 11-14, a position function $\vec{r}(t)$ is given. Sketch $\vec{r}(t)$ and $\vec{a}(t)$, then add $\vec{r}(t_0)$ and $\vec{a}(t_0)$ to your sketch, with their initial points at $\vec{r}(t_0)$, for the given value of t_0.

11. $\vec{r}(t) = \langle t, \sin t\rangle$ on $[0, \pi/2]$; $t_0 = \pi/4$

12. $\vec{r}(t) = \langle t^2, \sin t^2\rangle$ on $[0, \pi/2]$; $t_0 = \sqrt{\pi/4}$

13. $\vec{r}(t) = \langle t^2+t, -t^2+2t\rangle$ on $[-2, 2]$; $t_0 = 1$

14. $\vec{r}(t) = \langle \frac{2t+3}{t^2+1}, t^2\rangle$ on $[-1, 1]$; $t_0 = 0$

In Exercises 15-24, a position function $\vec{r}(t)$ of an object is given. Find the speed of the object in terms of t, and find where the speed is minimized/maximized on the indicated interval.

15. $\vec{r}(t) = \langle t^2, t\rangle$ on $[-1, 1]$

16. $\vec{r}(t) = \langle t^2, t^2-t^3\rangle$ on $[-1, 1]$

17. $\vec{r}(t) = \langle 5\cos t, 5\sin t\rangle$ on $[0, 2\pi]$

18. $\vec{r}(t) = \langle 2\cos t, 5\sin t\rangle$ on $[0, 2\pi]$

19. $\vec{r}(t) = \langle \sec t, \tan t\rangle$ on $[0, \pi/4]$

20. $\vec{r}(t) = \langle t+\cos t, 1-\sin t\rangle$ on $[0, 2\pi]$

21. $\vec{r}(t) = \langle 12t, 5\cos t, 5\sin t\rangle$ on $[0, 4\pi]$

22. $\vec{r}(t) = \langle t^2-t, t^2+t, t\rangle$ on $[0, 1]$

23. $\vec{r}(t) = \left\langle t, t^2, \sqrt{1-t^2}\right\rangle$ on $[-1, 1]$

24. **Projectile Motion**: $\vec{r}(t) = \left\langle (v_0\cos\theta)t, -\frac{1}{2}gt^2 + (v_0\sin\theta)t\right\rangle$ on $\left[0, \frac{2v_0\sin\theta}{g}\right]$.

In Exercises 25-28, position functions $\vec{r}_1(t)$ and $\vec{r}_2(s)$ for two objects are given that follow the same path on the respective intervals.
(a) Show that the positions are the same at the indicated t_0 and s_0 values; ie., show $\vec{r}_1(t_0) = \vec{r}_2(s_0)$.
(b) Find the velocity, speed and acceleration of the two objects at t_0 and s_0, respectively.

25.
$\vec{r}_1(t) = \langle t, t^2\rangle$ on $[0, 1]; t_0 = 1$
$\vec{r}_2(t) = \langle s^2, s^4\rangle$ on $[0, 1]; s_0 = 1$

26.
$\vec{r}_1(t) = \langle 3\cos t, 3\sin t\rangle$ on $[0, 2\pi]; t_0 = \pi/2$
$\vec{r}_2(t) = \langle 3\cos(4s), 3\sin(4s)\rangle$ on $[0, \pi/2]; s_0 = \pi/8$

27.
$\vec{r}_1(t) = \langle 3t, 2t\rangle$ on $[0, 2]; t_0 = 2$
$\vec{r}_2(t) = \langle 6t-6, 4t-4\rangle$ on $[1, 2]; s_0 = 2$

28.
$\vec{r}_1(t) = \langle t, \sqrt{t}\rangle$ on $[0, 1]; t_0 = 1$
$\vec{r}_2(t) = \langle \sin t, \sqrt{\sin t}\rangle$ on $[0, \pi/2]; s_0 = \pi/2$

In Exercises 29-32, find the position function of an object given its acceleration and initial velocity and position.

29. $\vec{a}(t) = \langle 2, 3\rangle; \quad \vec{v}(0) = \langle 1, 2\rangle, \quad \vec{r}(0) = \langle 5, -2\rangle$

30. $\vec{a}(t) = \langle 2, 3\rangle; \quad \vec{v}(1) = \langle 1, 2\rangle, \quad \vec{r}(1) = \langle 5, -2\rangle$

31. $\vec{a}(t) = \langle \cos t, -\sin t\rangle; \quad \vec{v}(0) = \langle 0, 1\rangle, \quad \vec{r}(0) = \langle 0, 0\rangle$

32. $\vec{a}(t) = \langle 0, -32\rangle; \quad \vec{v}(0) = \langle 10, 50\rangle, \quad \vec{r}(0) = \langle 0, 0\rangle$

In Exercises 33-36, find the displacement, distance traveled, average velocity, and average speed of the described object on the given interval.

33. An object with position function $\vec{r}(t) = \langle 2\cos t, 2\sin t, 3t\rangle$, where distances are measured in feet and time is in seconds, on $[0, 2\pi]$.

34. An object with position function $\vec{r}(t) = \langle 5\cos t, -5\sin t\rangle$, where distances are measured in feet and time is in seconds, on $[0, \pi]$.

35. An object with velocity function $\vec{v}(t) = \langle \cos t, \sin t\rangle$, where distances are measured in feet and time is in seconds, on $[0, 2\pi]$.

36. An object with velocity function $\vec{r}(t) = \langle 1, 2, -1\rangle$, where distances are measured in feet and time is in seconds, on $[0, 10]$.

Exercises 37-42 ask you to solve a variety of problems based on the principles of projectile motion.

37. A boy whirls a ball, attached to a 3 ft string, above his head in a counter–clockwise circle. The ball makes 2 revolutions per second.
At what *t*-values should the boy release the string so that the ball heads directly for a tree standing 10 ft in front of him?

38. David faces Goliath with only a stone in a 3 ft sling, which he whirls above his head at 4 revolutions per second. They stand 20 ft apart.
(a) At what *t*-values must David release the stone in his sling in order to hit Goliath?
(b) What is the speed at which the stone is traveling when released?
(c) Assume David releases the stone from a height of 6 ft and Goliath's forehead is 9 ft above the ground. What angle of elevation must David apply to the stone to hit Goliath's head?

39. A hunter aims at a deer which is 40 yards away. Her cross-bow is at a height of 5 ft, and she aims for a spot on the deer 4 ft above the ground. The crossbow fires her arrows at 300 ft/s.
(a) At what angle of elevation should she hold the crossbow to hit her target?
(b) If the deer is moving perpendicularly to her line of sight at a rate of 20 mph, by approximately how much should she lead the deer in order to hit in the desired location?

40. A baseball player hits a ball at 100 mph, with an initial height of 3 ft and an angle of elevation of 20°, at Boston's Fenway Park. The ball flies towards the famed "Green Monster," a wall 37 ft high located 310 ft from home plate.
(a) Show that as hit, the ball hits the wall.
(b) Show that if the angle of elevation is 21°, the ball clears the Green Monster.

41. A Cessna flies at 1000 ft at 150 mph and drops a box of supplies to the professor (and his wife) on an island. Ignoring wind resistance, how far horizontally will the supplies travel before they land?

42. A football quarterback throws a pass from a height of 6 ft, intending to hit his receiver 20 yds away at a height of 5 ft.
(a) If the ball is thrown at a rate of 50 mph, what angle of elevation is needed to hit his intended target?
(b) If the ball is thrown at with an angle of elevation of 8°, what initial ball speed is needed to hit his target?

12.4: Unit Tangent and Normal Vectors

Terms and Concepts

1. If $\vec{T}(t)$ is a unit tangent vector, what is $\|\vec{T}(t)\|$?

2. If $\vec{N}(t)$ is a unit normal vector, what is $\vec{N}(t) \cdot \vec{r}\,'(t)$?

3. The acceleration vector $\vec{a}(t)$ lies in the plane defined by what two vectors?

4. a_T measures how much the acceleration is affecting the _____ of an object.

Problems

In Exercises 5-8, given $\vec{r}(t)$, find $\vec{T}(t)$ and evaluate it at the indicated value of t.

5. $\vec{r}(t) = \langle 2t^2, t^2 - 1\rangle, \quad t = 1$

6. $\vec{r}(t) = \langle t, \cos t\rangle, \quad t = \pi/4$

7. $\vec{r}(t) = \langle \cos^3 t, \sin^3 t\rangle, \quad t = \pi/4$

8. $\vec{r}(t) = \langle \cos t, \sin t\rangle, \quad t = \pi$

In Exercises 9-12, find the equation of the line tangent to the curve at the indicated t-value using the unit tangent vector. Note: these are the same problems as in Exercises 5-8.

9. $\vec{r}(t) = \langle 2t^2, t^2 - 1\rangle, \quad t = 1$

10. $\vec{r}(t) = \langle t, \cos t\rangle, \quad t = \pi/4$

11. $\vec{r}(t) = \langle \cos^3 t, \sin^3 t\rangle, \quad t = \pi/4$

12. $\vec{r}(t) = \langle \cos t, \sin t\rangle, \quad t = \pi$

In Exercises 13-16, find $\vec{N}(t)$ using Definition 75. Confirm the result using the Theorem 97.

13. $\vec{r}(t) = \langle 3\cos t, 3\sin t\rangle$

14. $\vec{r}(t) = \langle t, t^2\rangle$

15. $\vec{r}(t) = \langle \cos t, 2\sin t\rangle$

16. $\vec{r}(t) = \langle e^t, e^{-t}\rangle$

In Exercises 17-20, a position function $\vec{r}(t)$ is given along with its unit tangent vector $\vec{T}(t)$ evaluated at $t = a$, for some value of a.
(a) Confirm that $\vec{T}(a)$ is as stated.
(b) Using a graph of $\vec{r}(t)$ and Theorem 97, find $\vec{N}(a)$.

17. $\vec{r}(t) = \langle 3\cos t, 5\sin t\rangle; \quad \vec{T}(\pi/4) = \left\langle -\frac{3}{\sqrt{34}}, \frac{5}{\sqrt{34}}\right\rangle$

18. $\vec{r}(t) = \left\langle t, \frac{1}{t^2+1}\right\rangle; \quad \vec{T}(1) = \left\langle \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right\rangle$

19. $\vec{r}(t) = (1 + 2\sin t)\langle \cos t, \sin t\rangle; \quad \vec{T}(0) = \left\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right\rangle$

20. $\vec{r}(t) = \langle \cos^3 t, \sin^3 t\rangle; \quad \vec{T}(\pi/4) = \left\langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right\rangle$

In Exercises 21-24, find $\vec{N}(t)$.

21. $\vec{r}(t) = \langle 4t, 2\sin t, 2\cos t\rangle$

22. $\vec{r}(t) = \langle 5\cos t, 3\sin t, 4\sin t\rangle$

23. $\vec{r}(t) = \langle a\cos t, a\sin t, bt\rangle$

24. $\vec{r}(t) = \langle \cos(at), \sin(at), t\rangle$

In Exercises 25-30, find a_T and a_N given $\vec{r}(t)$. Sketch $\vec{r}(t)$ on the indicated interval, and comment on the relative sizes of a_T and a_N at the indicated t values.

25. $\vec{r}(t) = \langle t, t^2\rangle$ on $[-1, 1]$; consider $t = 0$ and $t = 1$.

26. $\vec{r}(t) = \langle t, 1/t\rangle$ on $(0, 4$; consider $t = 1$ and $t = 2$.

27. $\vec{r}(t) = \langle 2\cos t, 2\sin t\rangle$ on $[0, 2\pi]$; consider $t = 1$ and $t = \pi/2$.

28. $\vec{r}(t) = \langle \cos(t^2), \sin(t^2)\rangle$ on $(0, 2\pi]$; consider $t = \sqrt{\pi/2}$ and $t = \sqrt{\pi}$.

29. $\vec{r}(t) = \langle a\cos t, a\sin t, bt\rangle$ on $[0, 2\pi]$; where $a, b > 0$; consider $t = 0$ and $t = \pi/2$.

30. $\vec{r}(t) = \langle 5\cos t, 4\sin t, 3\sin t\rangle$ on $[0, 2\pi]$; consider $t = 0$ and $t = \pi/2$.

12.5: The Arc Length Parameter and Curvature

Terms and Concepts

1. It is common to describe position in terms of both _____ and/or _____.

2. A measure of the "curviness" of a curve is _____.

3. Give two shapes with constant curvature.

4. Describe in your own words what an "osculating circle" is.

5. Complete the identity: $\vec{T}'(s) =$_____$\vec{N}(s)$.

6. Given a position function $\vec{r}(t)$, how are a_T and a_N affected by the curvature?

Problems

In Exercises 7-10, a position function $\vec{r}(t)$ is given, where $t = 0$ corresponds to the initial position. Find the arc length parameter s, and rewrite $\vec{r}(t)$ in terms of s; that is, find $\vec{r}(s)$.

7. $\vec{r}(t) = \langle 2t, t, -2t \rangle$

8. $\vec{r}(t) = \langle 7\cos t, 7\sin t \rangle$

9. $\vec{r}(t) = \langle 3\cos t, 3\sin t, 2t \rangle$

10. $\vec{r}(t) = \langle 5\cos t, 13\sin t, 12\cos t \rangle$

In Exercises 11-22, a curve C is described along with 2 points on C.
(a) Using a sketch, determine at which of these points the curvature is greater.
(b) Find the curvature κ of C, and evaluate κ at each of the 2 given points.

11. C is defined by $y = x^3 - x$; points given at $x = 0$ and $x = 1/2$.

12. C is defined by $y = \frac{1}{x^2+1}$; points given at $x = 0$ and $x = 2$.

13. C is defined by $y = \cos x$; points given at $x = 0$ and $x = \pi/2$.

14. C is defined by $y = \sqrt{1 - x^2}$ on $(-1, 1)$; points given at $x = 0$ and $x = 1/2$.

15. C is defined by $\vec{r}(t) = \langle \cos t, \sin(2t) \rangle$; points given at $t = 0$ and $t = \pi/4$.

16. C is defined by $\vec{r}(t) = \langle \cos^2(t), \sin t \cos t \rangle$; points given at $t = 0$ and $t = \pi/3$.

17. C is defined by $\vec{r}(t) = \langle t^2 - 1, t^3 - t \rangle$; points given at $t = 0$ and $t = \pi/6$.

18. C is defined by $\vec{r}(t) = \langle \tan t, \sec t \rangle$; points given at $t = 0$ and $t = \pi/6$.

19. C is defined by $\vec{r}(t) = \langle 4t + 2, 3t - 1, 2t + 5 \rangle$; points given at $t = 0$ and $t = 1$.

20. C is defined by $\vec{r}(t) = \langle t^3 - t, t^3 - 4, t^2 - 1 \rangle$; points given at $t = 0$ and $t = 1$.

21. C is defined by $\vec{r}(t) = \langle 3\cos t, 3\sin t, 2t \rangle$; points given at $t = 0$ and $t = \pi/2$.

22. C is defined by $\vec{r}(t) = \langle 5\cos t, 13\sin t, 12\cos t \rangle$; points given at $t = 0$ and $t = \pi/2$.

In Exercises 23-26, find the value of x or t where curvature is maximized.

23. $y = \frac{1}{6}x^3$

24. $y = \sin x$

25. $\vec{r}(t) = \langle t^2 + 2t, 3t - t^2 \rangle$

26. $\vec{r}(t) = \langle t, 4/t, 3/t \rangle$

In Exercises 27-30, find the radius of curvature at the indicated value.

27. $y = \tan x, \text{ at } x = \pi/4$

28. $y = x^2 + x - 3, \text{ at } x = \pi/4$

29. $\vec{r}(t) = \langle \cos t, \sin(3t) \rangle, \text{ at } t = 0$

30. $\vec{r}(t) = \langle 5\cos(3t), t \rangle, \text{ at } t = 0$

In Exercises 31-34, find the equation of the osculating circle to the curve at the indicated t-value.

31. $\vec{r}(t) = \langle t, t^2 \rangle, \text{ at } t = 0.$

32. $\vec{r}(t) = \langle 3\cos t, \sin t \rangle, \text{ at } t = 0.$

33. $\vec{r}(t) = \langle 3\cos t, \sin t \rangle, \text{ at } t = \pi/2.$

34. $\vec{r}(t) = \langle t^2 - t, t^2 + t \rangle, \text{ at } t = 0.$

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CHAPTER OVERVIEW

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13.0: Introduction to Functions of Multiple Variables

In Introduction to Applications of Derivatives, we studied how to determine the maximum and minimum of a function of one variable over a closed interval. This function might represent the temperature over a given time interval, the position of a car as a function of time, or the altitude of a jet plane as it travels from New York to San Francisco. In each of these examples, the function has one independent variable.

Figure **13.0.1**: Americans use (and lose) millions of golf balls a year, which keeps golf ball manufacturers in business. In this chapter, we study a profit model and learn methods for calculating optimal production levels for a typical golf ball manufacturing company. (credit: modification of work by oatsy40, Flickr)

Suppose, however, that we have a quantity that depends on more than one variable. For example, temperature can depend on location and the time of day, or a company's profit model might depend on the number of units sold and the amount of money spent on advertising. In this chapter, we look at a company that produces golf balls. We develop a profit model and, under various restrictions, we find that the optimal level of production and advertising dollars spent determines the maximum possible profit. Depending on the nature of the restrictions, both the method of solution and the solution itself changes.

When dealing with a function of more than one independent variable, several questions naturally arise. For example, how do we calculate limits of functions of more than one variable? The definition of derivative we used before involved a limit. Does the new definition of derivative involve limits as well? Do the rules of differentiation apply in this context? Can we find relative extrema of functions using derivatives? All these questions are answered in this chapter.

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13.1: Functions of Multiple Variables

Our first step is to explain what a function of more than one variable is, starting with functions of two independent variables. This step includes identifying the domain and range of such functions and learning how to graph them. We also examine ways to relate the graphs of functions in three dimensions to graphs of more familiar planar functions.

Functions of Two Variables

The definition of a function of two variables is very similar to the definition for a function of one variable. The main difference is that, instead of mapping values of one variable to values of another variable, we map ordered pairs of variables to another variable.

Definition: Function of Two Variables

A *function of two variables* $z = f(x, y)$ maps each ordered pair (x, y) in a subset D of the real plane $\mathbb{R}^2$ to a unique real number z. The set D is called the domain of the function. The range of f is the set of all real numbers z that has at least one ordered pair $(x, y) \in D$ such that $f(x, y) = z$ as shown in Figure 13.1.1.

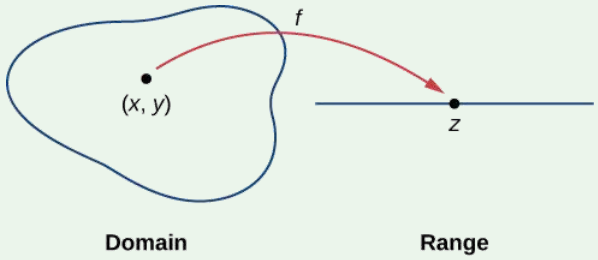


Figure 13.1.1: The domain of a function of two variables consists of ordered pairs (x, y).

Determining the domain of a function of two variables involves taking into account any domain restrictions that may exist. Let's take a look.

Example 13.1.1: Domains and Ranges for Functions of Two Variables

Find the domain and range of each of the following functions:

a. $f(x, y) = 3x + 5y + 2$
b. $g(x, y) = \sqrt{9 - x^2 - y^2}$

Solution

a. This is an example of a linear function in two variables. There are no values or combinations of x and y that cause $f(x, y)$ to be undefined, so the domain of f is $\mathbb{R}^2$. Written in set-builder notation, this could be written as, $\{(x, y) | x \in \mathbb{R}, y \in \mathbb{R}\}$.

To determine the range, first pick a value for z. We need to find a solution to the equation $f(x, y) = z$, or $3x - 5y + 2 = z$. One such solution can be obtained by first setting $y = 0$, which yields the equation $3x + 2 = z$. The solution to this equation is $x = \frac{z-2}{3}$, which gives the ordered pair $\left(\frac{z-2}{3}, 0\right)$ as a solution to the equation $f(x, y) = z$ for any value of z. Therefore, the range of the function is all real numbers, or $(-\infty, \infty)$.

b. For the function $g(x, y)$ to have a real value, the quantity under the square root must be nonnegative:

$$9 - x^2 - y^2 \geq 0.$$

This inequality can be written in the form

$$x^2 + y^2 \leq 9.$$

Therefore, the domain of $g(x, y)$ is $\{(x, y) \in R^2 \mid x^2 + y^2 \leq 9\}$. The graph of this set of points can be described as a disk of radius 3 centered at the origin. The domain includes the boundary circle as shown in the following graph.

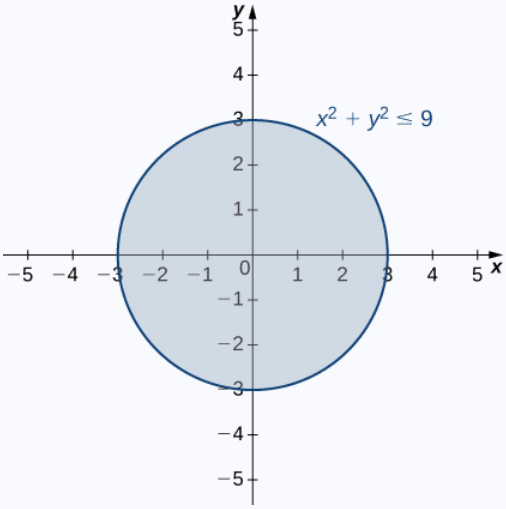


Figure **13.1.2**: The domain of the function $g(x,y) = \sqrt{9 - x^2 - y^2}$ is a closed disk of radius 3.

To determine the range of $g(x,y) = \sqrt{9 - x^2 - y^2}$ we start with a point (x_0, y_0) on the boundary of the domain, which is defined by the relation $x^2 + y^2 = 9$. It follows that $x_0^2 + y_0^2 = 9$ and

$$\begin{aligned} g(x_0, y_0) &= \sqrt{9 - x_0^2 - y_0^2} \\ &= \sqrt{9 - (x_0^2 + y_0^2)} \\ &= \sqrt{9 - 9} \\ &= 0. \end{aligned}$$

If $x_0^2 + y_0^2 = 0$ (in other words, $x_0 = y_0 = 0$), then

$$\begin{aligned} g(x_0, y_0) &= \sqrt{9 - x_0^2 - y_0^2} \\ &= \sqrt{9 - (x_0^2 + y_0^2)} \\ &= \sqrt{9 - 0} = 3. \end{aligned}$$

This is the maximum value of the function. Given any value c between 0 and 3, we can find an entire set of points inside the domain of g such that $g(x,y) = c$:

$$\begin{aligned} \sqrt{9 - x^2 - y^2} &= c \\ 9 - x^2 - y^2 &= c^2 \\ x^2 + y^2 &= 9 - c^2. \end{aligned}$$

Since $9 - c^2 > 0$, this describes a circle of radius $\sqrt{9 - c^2}$ centered at the origin. Any point on this circle satisfies the equation $g(x,y) = c$. Therefore, the range of this function can be written in interval notation as $[0, 3]$.

Exercise 13.1.1

Find the domain and range of the function $f(x,y) = \sqrt{36 - 9x^2 - 9y^2}$.

Graphing Functions of Two Variables

Suppose we wish to graph the function $z = f(x,y)$. This function has two independent variables (x and y) and one dependent variable (z). When graphing a function $y = f(x)$ of one variable, we use the Cartesian plane. We are able to graph any ordered pair (x,y) in the plane, and every point in the plane has an ordered pair (x,y) associated with it. With a function of two variables, each ordered pair (x,y) in the domain of the function is mapped to a real number z. Therefore, the graph of the function f consists of ordered triples (x,y,z). The graph of a function $z = f(x,y)$ of two variables is called a surface.

To understand more completely the concept of plotting a set of ordered triples to obtain a surface in three-dimensional space, imagine the (x, y) coordinate system laying flat. Then, every point in the domain of the function f has a unique z-value associated with it. If z is positive, then the graphed point is located above the xy-plane, if z is negative, then the graphed point is located below the xy-plane. The set of all the graphed points becomes the two-dimensional surface that is the graph of the function f.

Example 13.1.2: Graphing Functions of Two Variables

Create a graph of each of the following functions:

a. $g(x, y) = \sqrt{9 - x^2 - y^2}$
b. $f(x, y) = x^2 + y^2$

Solution

a. In Example 13.1.2, we determined that the domain of $g(x, y) = \sqrt{9 - x^2 - y^2}$ is $\{(x, y) \in R^2 \mid x^2 + y^2 \leq 9\}$ and the range is $\{z \in R^2 \mid 0 \leq z \leq 3\}$. When $x^2 + y^2 = 9$ we have $g(x, y) = 0$. Therefore any point on the circle of radius 3 centered at the origin in the xy-plane maps to $z = 0$ in R^3. If $x^2 + y^2 = 8$, then $g(x, y) = 1$, so any point on the circle of radius $2\sqrt{2}$ centered at the origin in the xy-plane maps to $z = 1$ in R^3. As $x^2 + y^2$ gets closer to zero, the value of z approaches 3. When $x^2 + y^2 = 0$, then $g(x, y) = 3$. This is the origin in the xy-plane If $x^2 + y^2$ is equal to any other value between 0 and 9, then $g(x, y)$ equals some other constant between 0 and 3. The surface described by this function is a hemisphere centered at the origin with radius 3 as shown in the following graph.

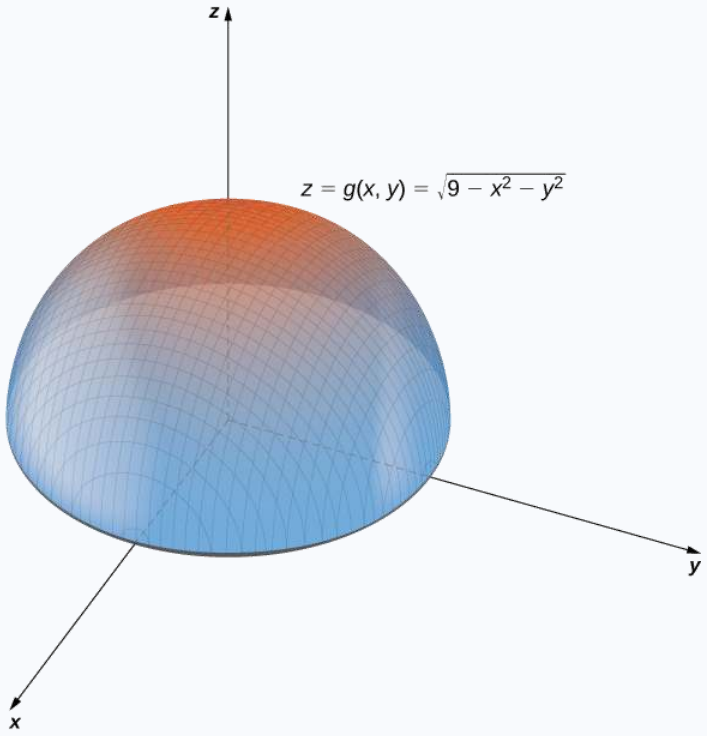


Figure 13.1.3: Graph of the hemisphere represented by the given function of two variables.

b. This function also contains the expression $x^2 + y^2$. Setting this expression equal to various values starting at zero, we obtain circles of increasing radius. The minimum value of $f(x, y) = x^2 + y^2$ is zero (attained when $x = y = 0$.. When $x = 0$, the function becomes $z = y^2$, and when $y = 0$, then the function becomes $z = x^2$. These are cross-sections of the graph, and are parabolas. Recall from Introduction to Vectors in Space that the name of the graph of $f(x, y) = x^2 + y^2$ is a **paraboloid**. The graph of f appears in the following graph.

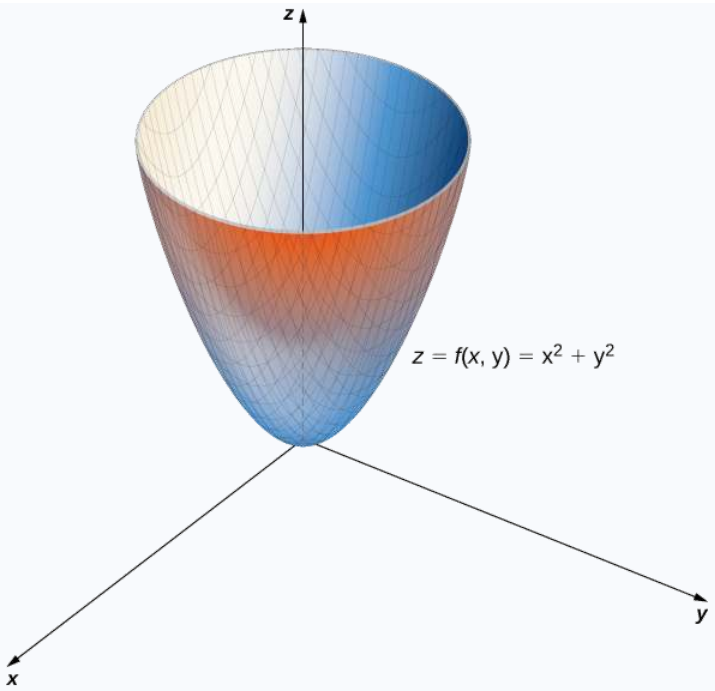


Figure 13.1.4: A paraboloid is the graph of the given function of two variables.

Example 13.1.3: Nuts and Bolts

A profit function for a hardware manufacturer is given by

$$f(x,y) = 16 - (x-3)^2 - (y-2)^2,$$

where x is the number of nuts sold per month (measured in thousands) and y represents the number of bolts sold per month (measured in thousands). Profit is measured in thousands of dollars. Sketch a graph of this function.

Solution

This function is a polynomial function in two variables. The domain of f consists of (x,y) coordinate pairs that yield a nonnegative profit:

$$16 - (x-3)^2 - (y-2)^2 \geq 0$$
$$(x-3)^2 + (y-2)^2 \leq 16.$$

This is a disk of radius 4 centered at $(3,2)$. A further restriction is that both x and y must be nonnegative. When $x = 3$ and $y = 2$, $f(x,y) = 16$. Note that it is possible for either value to be a noninteger; for example, it is possible to sell 2.5 thousand nuts in a month. The domain, therefore, contains thousands of points, so we can consider all points within the disk. For any $z < 16$, we can solve the equation $f(x,y) = 16$:

$$16 - (x-3)^2 - (y-2)^2 = z$$
$$(x-3)^2 + (y-2)^2 = 16 - z.$$

Since $z < 16$, we know that $16 - z > 0$, so the previous equation describes a circle with radius $\sqrt{16-z}$ centered at the point $(3,2)$. Therefore. the range of $f(x,y)$ is $\{z \in \mathbb{R} | z \leq 16\}$. The graph of $f(x,y)$ is also a paraboloid, and this paraboloid points downward as shown.

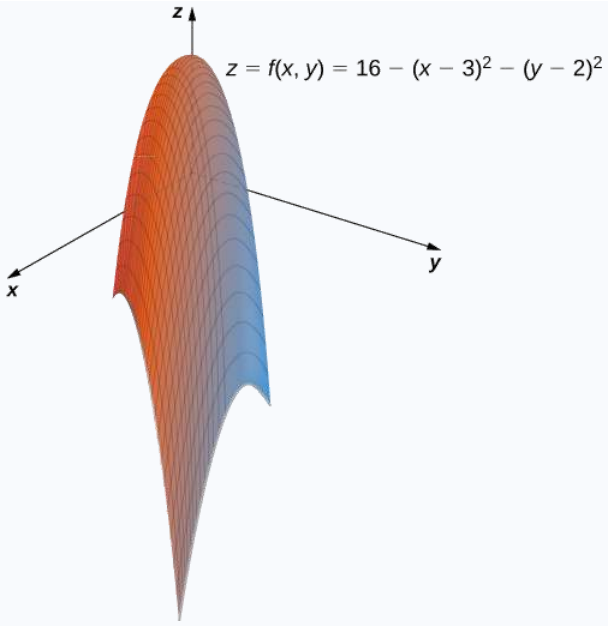
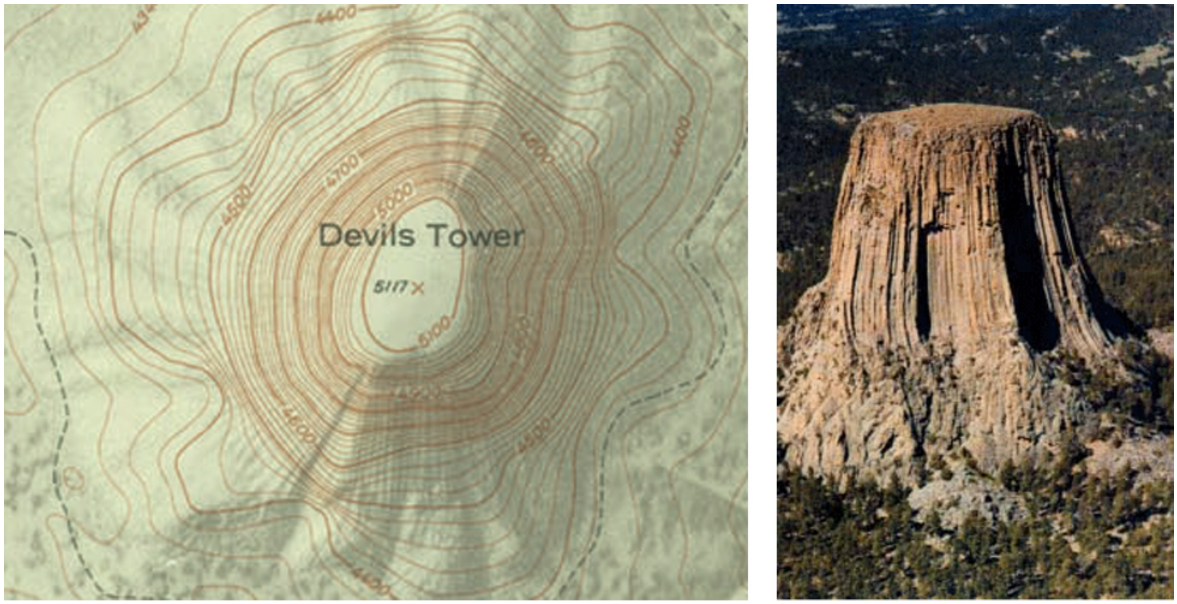


Figure 13.1.5: The graph of the given function of two variables is also a paraboloid.

Level Curves

If hikers walk along rugged trails, they might use a topographical map that shows how steeply the trails change. A *topographical map* contains curved lines called contour lines. Each contour line corresponds to the points on the map that have equal elevation (Figure 13.1.6). A level curve of a function of two variables $f(x, y)$ is completely analogous to a contour line on a topographical map.



(a) (b)

Figure 13.1.6: (a) A topographical map of Devil's Tower, Wyoming. Lines that are close together indicate very steep terrain. (b) A perspective photo of Devil's Tower shows just how steep its sides are. Notice the top of the tower has the same shape as the center of the topographical map.

Definition: Level Curves

Given a function $f(x, y)$ and a number c in the range of f, a level curve of a function of two variables for the value c is defined to be the set of points satisfying the equation $f(x, y) = c$.

Returning to the function $g(x,y)=\sqrt{9-x^2-y^2}$, we can determine the level curves of this function. The range of g is the closed interval $[0,3]$. First, we choose any number in this closed interval—say, $c=2$. The level curve corresponding to $c=2$ is described by the equation

$$\sqrt{9-x^2-y^2}=2. \tag{13.1.1}$$

To simplify, square both sides of this equation:

$$9-x^2-y^2=4. \tag{13.1.2}$$

Now, multiply both sides of the equation by -1 and add 9 to each side:

$$x^2+y^2=5. \tag{13.1.3}$$

This equation describes a circle centered at the origin with radius $\sqrt{5}$. Using values of c between 0 and 3 yields other circles also centered at the origin. If $c=3$, then the circle has radius 0, so it consists solely of the origin. Figure 13.1.7 is a graph of the level curves of this function corresponding to $c=0,1,2,$ and 3. Note that in the previous derivation it may be possible that we introduced extra solutions by squaring both sides. This is not the case here because the range of the square root function is nonnegative.

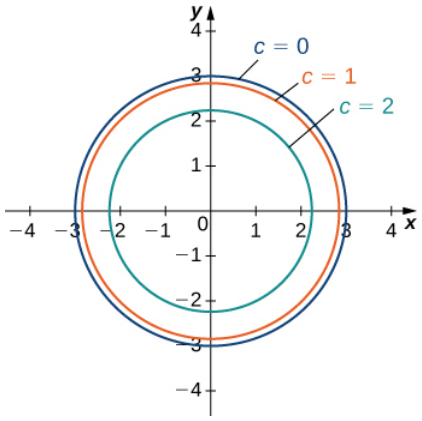


Figure 13.1.7: Level curves of the function $g(x,y)=\sqrt{9-x^2-y^2}$, using $c=0,1,2,$ and $3(c=3$ corresponds to the origin).

A graph of the various level curves of a function is called a *contour map*.

Example 13.1.4: Making a Contour Map

Given the function $f(x,y)=\sqrt{8+8x-4y-4x^2-y^2}$, find the level curve corresponding to $c=0$. Then create a contour map for this function. What are the domain and range of f?

Solution

To find the level curve for $c=0$, we set $f(x,y)=0$ and solve. This gives

$$0=\sqrt{8+8x-4y-4x^2-y^2}.$$

We then square both sides and multiply both sides of the equation by -1:

$$4x^2+y^2-8x+4y-8=0.$$

Now, we rearrange the terms, putting the x terms together and the y terms together, and add 8 to each side:

$$4x^2-8x+y^2+4y=8.$$

Next, we group the pairs of terms containing the same variable in parentheses, and factor 4 from the first pair:

$$4(x^2-2x)+(y^2+4y)=8.$$

Then we complete the square in each pair of parentheses and add the correct value to the right-hand side:

$$4(x^2-2x+1)+(y^2+4y+4)=8+4(1)+4.$$

Next, we factor the left-hand side and simplify the right-hand side:

$$4(x-1)^2+(y+2)^2=16.$$

Last, we divide both sides by 16:

$$\frac{(x-1)^2}{4}+\frac{(y+2)^2}{16}=1. \tag{13.1.4}$$

This equation describes an ellipse centered at $(1,-2)$. The graph of this ellipse appears in the following graph.

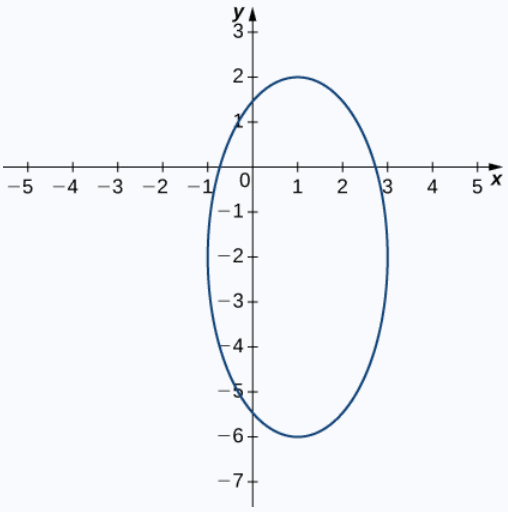


Figure 13.1.8: Level curve of the function $f(x,y)=\sqrt{8+8x-4y-4x^2-y^2}$ corresponding to $c=0$

We can repeat the same derivation for values of c less than 4. Then, Equation 13.1.4 becomes

$$\frac{4(x-1)^2}{16-c^2}+\frac{(y+2)^2}{16-c^2}=1$$

for an arbitrary value of c.

Setting the function equal to c, we get:

$$\sqrt{8+8x-4y-4x^2-y^2}=c$$

We then square both sides and multiply both sides of the equation by -1:

$$4x^2+y^2-8x+4y-8=-c^2.$$

Now, we rearrange the terms, putting the x terms together and the y terms together, and add 8 to each side:

$$4x^2-8x+y^2+4y=8-c^2.$$

Next, we group the pairs of terms containing the same variable in parentheses, and factor 4 from the first pair:

$$4(x^2-2x)+(y^2+4y)=8-c^2.$$

Then we complete the square in each pair of parentheses and add the correct value to the right-hand side:

$$4(x^2-2x+1)+(y^2+4y+4)=8-c^2+4(1)+4.$$

Next, we factor the left-hand side and simplify the right-hand side:

$$4(x-1)^2+(y+2)^2=16-c^2.$$

Last, we divide both sides by $16-c^2$:

$$\frac{4(x-1)^2}{16-c^2}+\frac{(y+2)^2}{16-c^2}=1$$

Figure 13.1.9 shows a contour map for $f(x,y)$ using the values $c=0,1,2$, and 3. When $c=4$, the level curve is the point $(-1,2)$.

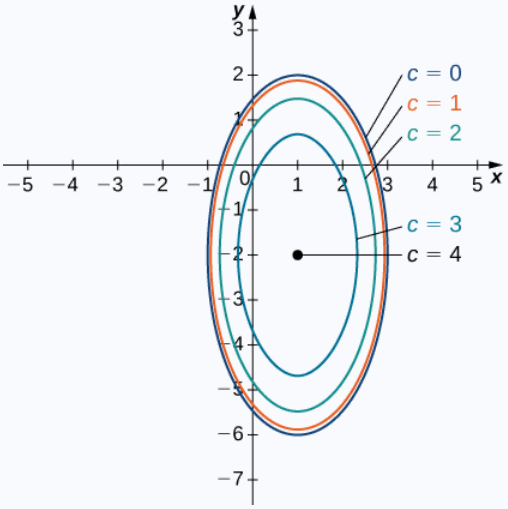


Figure 13.1.9: Contour map for the function $f(x,y) = \sqrt{8+8x-4y-4x^2-y^2}$ using the values $c = 0, 1, 2, 3$, and 4.

Finding the Domain & Range

Since this is a square root function, the radicand must not be negative. So we have

$$8+8x-4y-4x^2-y^2 \geq 0$$

Recognizing that the boundary of the domain is an ellipse, we repeat the steps we showed above to obtain

$$\frac{(x-1)^2}{4} + \frac{(y+2)^2}{16} \leq 1$$

So the domain of f can be written: $\left\{(x,y) \mid \frac{(x-1)^2}{4} + \frac{(y+2)^2}{16} \leq 1\right\}$.

To find the range of f, we need to consider the possible outputs of this square root function. We know the output cannot be negative, so we need to next check if its output is ever 0. From the work we completed above to find the level curve for $c = 0$, we know the value of f is 0 for any point on that level curve (on the ellipse, $\frac{(x-1)^2}{4} + \frac{(y+2)^2}{16} = 1$). So we know the lower bound of the range of this function is 0.

To determine the upper bound for the range of the function in this problem, it's easier if we first complete the square under the radical.

$$\begin{aligned} f(x,y) &= \sqrt{8+8x-4y-4x^2-y^2} \\ &= \sqrt{8-4(x^2-2x\quad)-(y^2+4y\quad)} \\ &= \sqrt{8-4(x^2-2x+1-1)-(y^2+4y+4-4)} \\ &= \sqrt{8-4(x^2-2x+1)+4-(y^2+4y+4)+4} \\ &= \sqrt{16-4(x-1)^2-(y+2)^2} \end{aligned}$$

Now that we have f in this form, we can see how large the radicand can be. Since we are subtracting two perfect squares from 16, we know that the value of the radicand cannot be greater than 16. At the point $(1,-2)$, we can see the radicand will be 16 (since we will be subtracting 0 from 16 at this point. This gives us the maximum value of f, that is $f(1,-2) = \sqrt{16} = 4$.

So the range of this function is $[0,4]$.

Exercise 13.1.2

Find and graph the level curve of the function $g(x,y) = x^2 + y^2 - 6x + 2y$ corresponding to $c = 15$.

Another useful tool for understanding the **graph of a function of two variables** is called a vertical trace. Level curves are always graphed in the xy-plane, but as their name implies, vertical traces are graphed in the xz– or yz– planes.

Definition: Vertical Traces

Consider a function $z = f(x, y)$ with domain $D \subseteq \mathbb{R}^2$. A *vertical trace* of the function can be either the set of points that solves the equation $f(a, y) = z$ for a given constant $x = a$ or $f(x, b) = z$ for a given constant $y = b$.

Example 13.1.5: Finding Vertical Traces

Find vertical traces for the function $f(x, y) = \sin x \cos y$ corresponding to $x = -\frac{\pi}{4}, 0$, and $\frac{\pi}{4}$, and $y = -\frac{\pi}{4}, 0$, and $\frac{\pi}{4}$.

Solution

First set $x = -\frac{\pi}{4}$ in the equation $z = \sin x \cos y$:

$$z = \sin\left(-\frac{\pi}{4}\right)\cos y = -\frac{\sqrt{2}\cos y}{2} \approx -0.7071 \cos y.$$

This describes a cosine graph in the plane $x = -\frac{\pi}{4}$. The other values of z appear in the following table.

Vertical Traces Parallel to the xz-Plane for the Function $f(x, y) = \sin x \cos y$

c	Vertical Trace for $x = c$
$-\frac{\pi}{4}$	$z = -\frac{\sqrt{2}\cos y}{2}$
0	$z = 0$
$\frac{\pi}{4}$	$z = \frac{\sqrt{2}\cos y}{2}$

In a similar fashion, we can substitute the y-values in the equation $f(x, y)$ to obtain the traces in the yz-plane, as listed in the following table.

Vertical Traces Parallel to the yz-Plane for the Function $f(x, y) = \sin x \cos y$

d	Vertical Trace for $y = d$
$\frac{\pi}{4}$	$z = \frac{\sqrt{2}\sin x}{2}$
0	$z = \sin x$
$-\frac{\pi}{4}$	$z = \frac{\sqrt{2}\sin x}{2}$

The three traces in the xz-plane are cosine functions; the three traces in the yz-plane are sine functions. These curves appear in the intersections of the surface with the planes $x = -\frac{\pi}{4}$, $x = 0$, $x = \frac{\pi}{4}$ and $y = -\frac{\pi}{4}$, $y = 0$, $y = \frac{\pi}{4}$ as shown in the following figure.

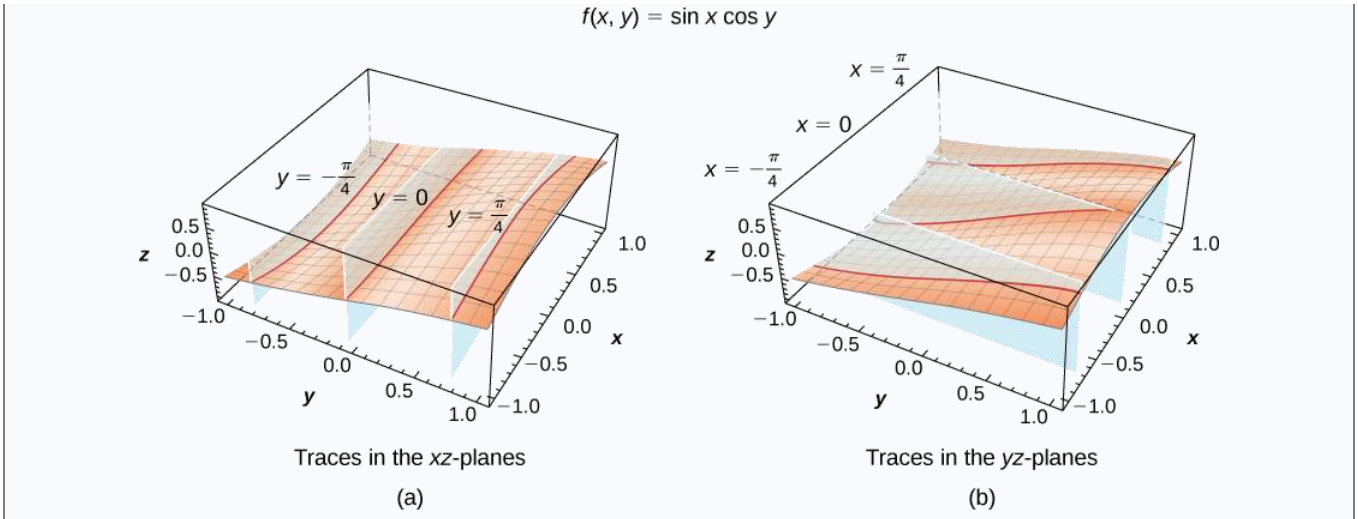


Figure **13.1.10**: Vertical traces of the function $f(x,y)$ are cosine curves in the xz-planes (a) and sine curves in the yz-planes (b).

Exercise 13.1.3

Determine the equation of the vertical trace of the function $g(x,y) = -x^2 - y^2 + 2x + 4y - 1$ corresponding to $y = 3$, and describe its graph.

Functions of two variables can produce some striking-looking surfaces. Figure **13.1.11** shows two examples.

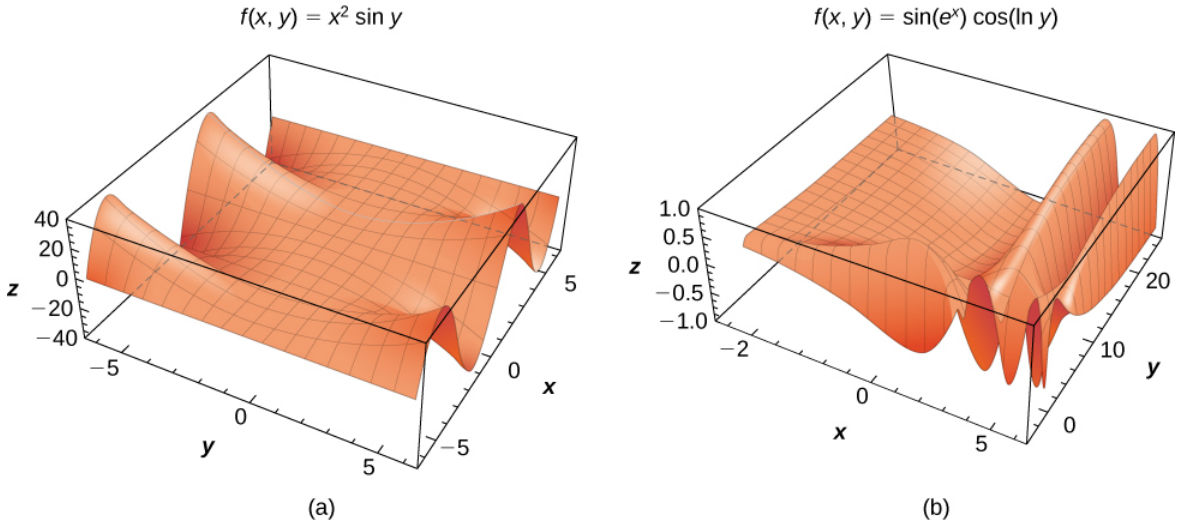


Figure **13.1.11**: Examples of surfaces representing functions of two variables: (a) a combination of a power function and a sine function and (b) a combination of trigonometric, exponential, and logarithmic functions.

Functions of More Than Two Variables

So far, we have examined only functions of two variables. However, it is useful to take a brief look at functions of more than two variables. Two such examples are

$$\underbrace{f(x,y,z) = x^2 - 2xy + y^2 + 3yz - z^2 + 4x - 2y + 3x - 6}_{\text{a polynomial in three variables}} \tag{13.1.5}$$

and

$$g(x,y,t) = (x^2 - 4xy + y^2)\sin t - (3x + 5y)\cos t. \tag{13.1.6}$$

In the first function, (x,y,z) represents a point in space, and the function f maps each point in space to a fourth quantity, such as temperature or wind speed. In the second function, (x,y) can represent a point in the plane, and t can represent time. The function

might map a point in the plane to a third quantity (for example, pressure) at a given time t. The method for finding the domain of a function of more than two variables is analogous to the method for functions of one or two variables.

Example 13.1.6: Domains for Functions of Three Variables

Find the domain of each of the following functions:

a. $f(x,y,z)=\dfrac{3x-4y+2z}{\sqrt{9-x^2-y^2-z^2}}$

b. $g(x,y,t)=\dfrac{\sqrt{2t-4}}{x^2-y^2}$

Solution:

a. For the function $f(x,y,z)=\dfrac{3x-4y+2z}{\sqrt{9-x^2-y^2-z^2}}$ to be defined (and be a real value), two conditions must hold:

1. The denominator **cannot** be zero.
2. The radicand **cannot** be negative.

Combining these conditions leads to the inequality

$$9-x^2-y^2-z^2>0.$$

Moving the variables to the other side and reversing the inequality gives the domain as

$$domain(f)=\{(x,y,z)\in R^3 \mid x^2+y^2+z^2<9\},$$

which describes a ball of radius 3 centered at the origin. (Note: The surface of the ball is not included in this domain.)

b. For the function $g(x,y,t)=\dfrac{\sqrt{2t-4}}{x^2-y^2}$ to be defined (and be a real value), two conditions must hold:

1. The radicand cannot be negative.
2. The denominator cannot be zero.

Since the radicand cannot be negative, this implies $2t-4\geq 0$, and therefore that $t\geq 2$. Since the denominator cannot be zero, $x^2-y^2\neq 0$, or $x^2\neq y^2$, Which can be rewritten as $y=\pm x$, which are the equations of two lines passing through the origin. Therefore, the domain of g is

$$domain(g)=\{(x,y,t)|y\neq\pm x, t\geq 2\}.$$

Exercise 13.1.4

Find the domain of the function $h(x,y,t)=(3t-6)\sqrt{y-4x^2+4}$.

Functions of two variables have level curves, which are shown as curves in the $xy-plane$. However, when the function has three variables, the curves become surfaces, so we can define level surfaces for functions of three variables.

Definition: Level Surface of a Function of Three Variables

Given a function $f(x,y,z)$ and a number c in the range of f, a *level surface of a function of three variables* is defined to be the set of points satisfying the equation $f(x,y,z)=c$.

Example 13.1.7: Finding a Level Surface

Find the level surface for the function $f(x,y,z)=4x^2+9y^2-z^2$ corresponding to $c=1$.

Solution

The level surface is defined by the equation $4x^2+9y^2-z^2=1$. This equation describes a hyperboloid of one sheet as shown in Figure **13.1.12**.

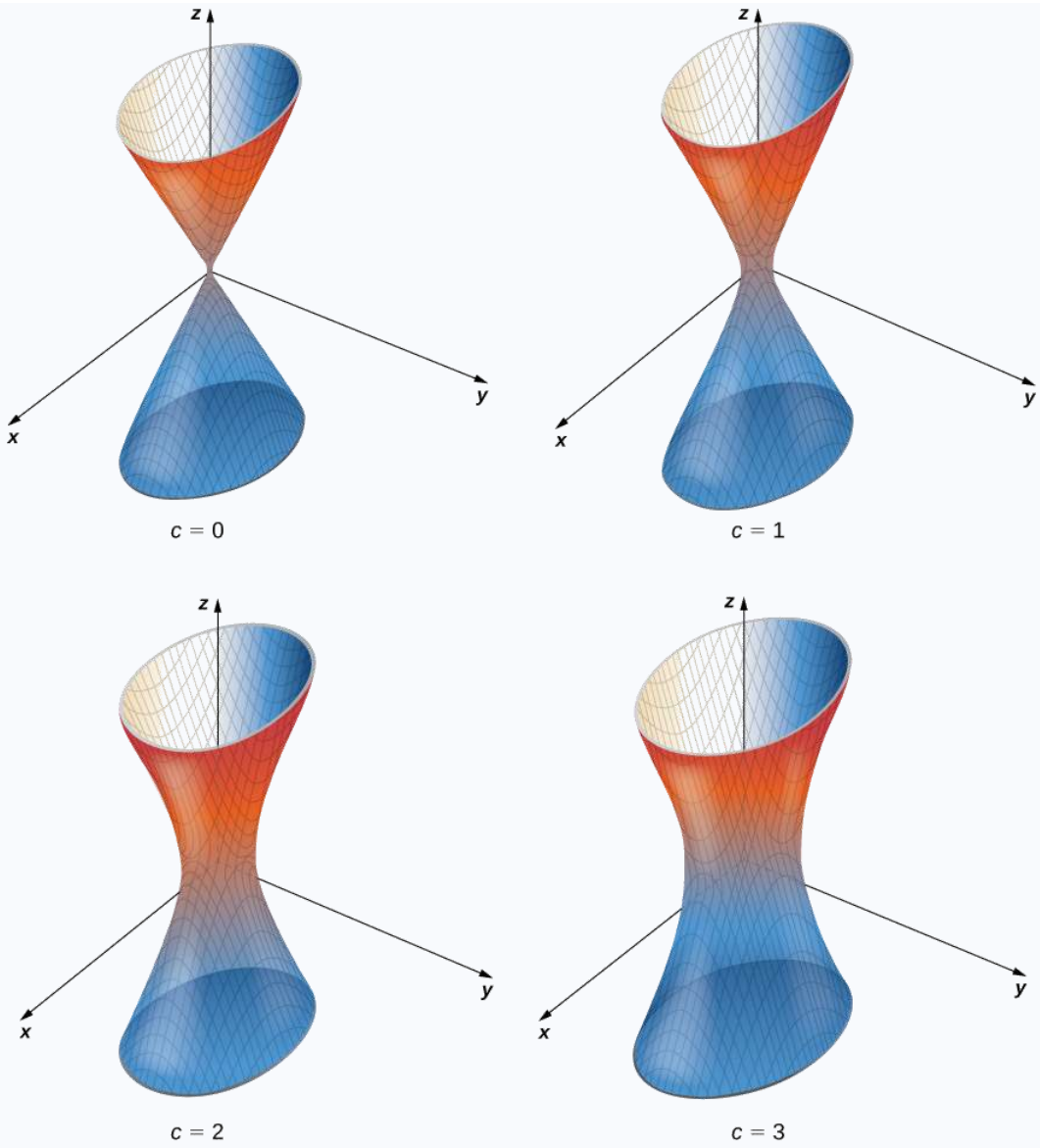


Figure **13.1.12**: A hyperboloid of one sheet with some of its level surfaces.

Exercise 13.1.5

Find the equation of the level surface of the function

$$g(x,y,z) = x^2 + y^2 + z^2 - 2x + 4y - 6z$$

corresponding to $c = 2$, and describe the surface, if possible.

Summary

- The graph of a function of two variables is a surface in $\mathbb{R}^3$ and can be studied using level curves and vertical traces.
- A set of level curves is called a contour map.

Key Equations

- **Vertical trace**

  $f(a,y) = z$ for $x = a$ or $f(x,b) = z$ for $y = b$

- **Level surface of a function of three variables**

$f(x, y, z) = c$

Glossary

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13.1E: Functions of Multiple Variables (Exercises)

For the following exercises, evaluate each function at the indicated values.

1) $W(x,y) = 4x^2 + y^2$. Find $W(2,-1), W(-3,6)$.

2) $W(x,y) = 4x^2 + y^2$. Find $W(2+h, 3+h)$.

3) The volume of a right circular cylinder is calculated by a function of two variables, $V(x,y) = \pi x^2 y$, where x is the radius of the right circular cylinder and y represents the height of the cylinder. Evaluate $V(2,5)$ and explain what this means.

4) An oxygen tank is constructed of a right cylinder of height y and radius x with two hemispheres of radius x mounted on the top and bottom of the cylinder. Express the volume of the cylinder as a function of two variables, x and y, find $V(10,2)$, and explain what this means.

For exercises 5 - 10, find the domain and range of the given function. State the domain in set-builder notation and the range in interval notation.

5) $V(x,y) = 4x^2 + y^2$

6) $f(x,y) = \sqrt{x^2 + y^2 - 4}$

7) $f(x,y) = 4\ln(y^2 - x)$

8) $g(x,y) = \sqrt{16 - 4x^2 - y^2}$

9) $z = \arccos(y - x)$

10) $f(x,y) = \dfrac{y+2}{x^2}$

Find the range of the functions.

11) $g(x,y) = \sqrt{16 - 4x^2 - y^2}$

12) $V(x,y) = 4x^2 + y^2$

13) $z = y^2 - x^2$

In exercises 14 - 29, find the level curves of each function at the indicated values of c to visualize the given function. Sketch a contour plot for those exercises where you are asked for more than 3 values of c.

14) $z(x,y) = y^2 - x^2, \quad c = 1$

15) $z(x,y) = y^2 - x^2, \quad c = 4$

16) $g(x,y) = x^2 + y^2; \quad c = 0, 1, 2, 3, 4, 9$

17) $g(x,y) = 4 - x - y; \quad c = 0, 1, 2, 3, 4$

18) $f(x,y) = xy; c = 1; \quad c = -1$

19) $h(x,y) = 2x - y; \quad c = -2, 0, 2$

20) $f(x,y) = x^2 - y; \quad c = 1, 2$

21) $g(x,y) = \dfrac{x}{x+y}; c = -1, 0, 1, 2$

22) $g(x,y) = x^3 - y; \quad c = -1, 0, 2$

23) $g(x,y) = e^{xy}; \quad c = \frac{1}{2}, 3$

24) $f(x,y) = x^2; \quad c = 4, 9$

25) $f(x,y) = xy - x; \quad c = -2, 0, 2$

26) $h(x,y) = \ln(x^2 + y^2); \quad c = -1, 0, 1$

27) $g(x,y) = \ln\left(\dfrac{y}{x^2}\right); \quad c = -2, 0, 2$

28) $z = f(x,y) = \sqrt{x^2+y^2}, \quad c = 3$

29) $f(x,y) = \dfrac{y+2}{x^2}, \quad c =$ any constant

In exercises 30-32, find the vertical traces of the functions at the indicated values of x and y, and plot the traces.

30) $z = 4 - x - y, \quad x = 2$

31) $f(x,y) = 3x + y^3, \quad x = 1$

32) $z = \cos\sqrt{x^2+y^2}, \quad x = 1$

In exercises 33 - 38, find the domain and range of each function.

33) $z = \sqrt{100 - 4x^2 - 25y^2}$

34) $z = \ln(x - y^2)$

35) $f(x,y,z) = \dfrac{1}{\sqrt{36 - 4x^2 - 9y^2 - z^2}}$

36) $f(x,y,z) = \sqrt{49 - x^2 - y^2 - z^2}$

37) $f(x,y,z) = \sqrt[3]{16 - x^2 - y^2 - z^2}$

38) $f(x,y) = \cos\sqrt{x^2+y^2}$

In exercises 39 - 40, plot a graph of the function.

39) $z = f(x,y) = \sqrt{x^2+y^2}$

40) $z = x^2 + y^2$

41) Use technology to graph $z = x^2 y$.

In exercises 42 - 46, sketch the function by finding its level curves. Verify the graph using technology, such as CalcPlot3D.

42) $f(x,y) = \sqrt{4 - x^2 - y^2}$

43) $f(x,y) = 2 - \sqrt{x^2+y^2}$

44) $z = 1 + e^{-x^2-y^2}$

45) $z = \cos\sqrt{x^2+y^2}$

46) $z = y^2 - x^2$

47) Describe the contour lines for several values of c for $z = x^2 + y^2 - 2x - 2y$.

In exercises, 48 - 52, find the level surface for the given value of c for each function of three variables and describe it.

48) $w(x,y,z) = x - 2y + z, \quad c = 4$

49) $w(x,y,z) = x^2 + y^2 + z^2, \quad c = 9$

50) $w(x,y,z) = x^2 + y^2 - z^2, \quad c = -4$

51) $w(x,y,z) = x^2 + y^2 - z^2, \quad c = 4$

52) $w(x,y,z) = 9x^2 - 4y^2 + 36z^2, \quad c = 0$

In exercises 53 - 55, find an equation of the level curve of f that contains the point P.

53) $f(x,y) = 1 - 4x^2 - y^2, \quad P(0,1)$

54) $g(x,y) = y^2 \arctan x, \quad P(1,2)$

55) $g(x,y) = e^{xy}(x^2+y^2), \quad P(1,0)$

56) The strength E of an electric field at point (x,y,z) resulting from an infinitely long charged wire lying along the y-axis is given by $E(x,y,z) = k/\sqrt{x^2+y^2}$, where k is a positive constant. For simplicity, let $k = 1$ and find the equations of the level surfaces for $E = 10$ and $E = 100$.

57) A thin plate made of iron is located in the xy-plane The temperature T in degrees Celsius at a point $P(x, y)$ is inversely proportional to the square of its distance from the origin. Express T as a function of x and y.

58) Refer to the preceding problem. Using the temperature function found there, determine the proportionality constant if the temperature at point $P(1, 2)$ is $50°C$. Use this constant to determine the temperature at point $Q(3, 4)$.

59) Refer to the preceding problem. Find the level curves for $T = 40\,°C$ and $T = 100\,°C$, and describe what the level curves represent.

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- Paul Seeburger (Monroe Community College) added the contour plots to answers for problems 17, 21 and 29.

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13.2: Limits and Continuity

Learning Objectives

- Calculate the limit of a function of two variables.
- Learn how a function of two variables can approach different values at a boundary point, depending on the path of approach.
- State the conditions for continuity of a function of two variables.
- Verify the continuity of a function of two variables at a point.
- Calculate the limit of a function of three or more variables and verify the continuity of the function at a point.

We have now examined functions of more than one variable and seen how to graph them. In this section, we see how to take the limit of a function of more than one variable, and what it means for a function of more than one variable to be continuous at a point in its domain. It turns out these concepts have aspects that just don't occur with functions of one variable.

Limit of a Function of Two Variables

Recall from Section 2.5 that the definition of a limit of a function of one variable:

Let $f(x)$ be defined for all $x \neq a$ in an open interval containing a. Let L be a real number. Then

$$\lim_{x \to a} f(x) = L$$

if for every $\varepsilon > 0$, there exists a $\delta > 0$, such that if $0 < |x - a| < \delta$ for all x in the domain of f, then

$$|f(x) - L| < \varepsilon.$$

Before we can adapt this definition to define a limit of a function of two variables, we first need to see how to extend the idea of an open interval in one variable to an open interval in two variables.

Definition: δ Disks

Consider a point $(a, b) \in \mathbb{R}^2$. A δ **disk** centered at point (a, b) is defined to be an open disk of radius δ centered at point (a, b) — that is,

$$\{(x, y) \in \mathbb{R}^2 \mid (x - a)^2 + (y - b)^2 < \delta^2\}$$

as shown in Figure 13.2.1.

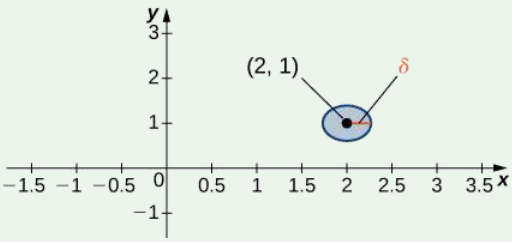

Figure 13.2.1: A δ disk centered around the point $(2, 1)$.

The idea of a δ disk appears in the definition of the limit of a function of two variables. If δ is small, then all the points (x, y) in the δ disk are close to (a, b). This is completely analogous to x being close to a in the definition of a limit of a function of one variable. In one dimension, we express this restriction as

$$a - \delta < x < a + \delta.$$

In more than one dimension, we use a δ disk.

Definition: limit of a function of two variables

Let f be a function of two variables, x and y. The limit of $f(x,y)$ as (x,y) approaches (a,b) is L, written

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

if for each $\varepsilon > 0$ there exists a small enough $\delta > 0$ such that for all points (x,y) in a δ disk around (a,b), except possibly for (a,b) itself, the value of $f(x,y)$ is no more than ε away from L (Figure 13.2.2).

Using symbols, we write the following: For any $\varepsilon > 0$, there exists a number $\delta > 0$ such that

$$|f(x,y) - L| < \varepsilon$$

whenever

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta.$$

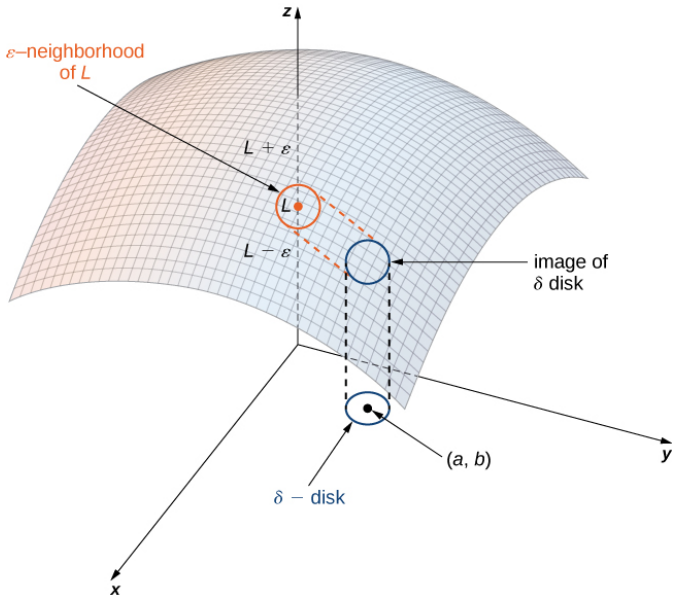

Figure 13.2.2: The limit of a function involving two variables requires that $f(x,y)$ be within ε of L whenever (x,y) is within δ of (a,b). The smaller the value of ε, the smaller the value of δ.

Proving that a limit exists using the definition of a **limit of a function of two variables** can be challenging. Instead, we use the following theorem, which gives us shortcuts to finding limits. The formulas in this theorem are an extension of the formulas in the limit laws theorem in the section titled The Limit Laws in single-variable calculus.

Theorem 13.2.1: Limit Laws for Functions of Two Variables

Let $f(x,y)$ and $g(x,y)$ be defined for all $(x,y) \neq (a,b)$ in a neighborhood around (a,b), and assume the neighborhood is contained completely inside the domain of f. Assume that L and M are real numbers such that

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

and

$$\lim_{(x,y)\to(a,b)} g(x,y) = M,$$

and let c be a constant. Then each of the following statements holds:

Constant Law:

$$\lim_{(x,y)\to(a,b)} c = c$$

Identity Laws:

$$\lim_{(x,y)\to(a,b)} x = a$$

$$\lim_{(x,y)\to(a,b)} y = b$$

Sum Law:

$$\lim_{(x,y)\to(a,b)} (f(x,y) + g(x,y)) = L + M$$

Difference Law:

$$\lim_{(x,y)\to(a,b)} (f(x,y) - g(x,y)) = L - M$$

Constant Multiple Law:

$$\lim_{(x,y)\to(a,b)} (cf(x,y)) = cL$$

Product Law:

$$\lim_{(x,y)\to(a,b)} (f(x,y)g(x,y)) = LM$$

Quotient Law:

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M} \text{ for } M \neq 0$$

Power Law:

$$\lim_{(x,y)\to(a,b)} (f(x,y))^n = L^n$$

for any positive integer n.

Root Law:

$$\lim_{(x,y)\to(a,b)} \sqrt[n]{f(x,y)} = \sqrt[n]{L}$$

for all L if n is odd and positive, and for $L \geq 0$ if n is even and positive.

The proofs of these properties are similar to those for the limits of functions of one variable. We can apply these laws to finding limits of various functions.

Example 13.2.1: Finding the Limit of a Function of Two Variables

Find each of the following limits:

a. $\lim\limits_{(x,y)\to(2,-1)} (x^2 - 2xy + 3y^2 - 4x + 3y - 6)$

b. $\lim\limits_{(x,y)\to(2,-1)} \dfrac{2x + 3y}{4x - 3y}$

Solution

a. First use the sum and difference laws to separate the terms:

$$\lim_{(x,y)\to(2,-1)} (x^2 - 2xy + 3y^2 - 4x + 3y - 6) = \left(\lim_{(x,y)\to(2,-1)} x^2\right) - \left(\lim_{(x,y)\to(2,-1)} 2xy\right) + \left(\lim_{(x,y)\to(2,-1)} 3y^2\right) - \left(\lim_{(x,y)\to(2,-1)} 4x\right) + \left(\lim_{(x,y)\to(2,-1)} 3y\right) - \left(\lim_{(x,y)\to(2,-1)} 6\right).$$

Next, use the constant multiple law on the second, third, fourth, and fifth limits:

$$= (\lim_{(x,y)\to(2,-1)} x^2) - 2(\lim_{(x,y)\to(2,-1)} xy) + 3(\lim_{(x,y)\to(2,-1)} y^2) - 4(\lim_{(x,y)\to(2,-1)} x)$$
$$+3(\lim_{(x,y)\to(2,-1)} y) - \lim_{(x,y)\to(2,-1)} 6.$$

Now, use the power law on the first and third limits, and the product law on the second limit:

$$\left(\lim_{(x,y)\to(2,-1)} x\right)^2 - 2\left(\lim_{(x,y)\to(2,-1)} x\right)\left(\lim_{(x,y)\to(2,-1)} y\right) + 3\left(\lim_{(x,y)\to(2,-1)} y\right)^2$$
$$-4\left(\lim_{(x,y)\to(2,-1)} x\right) + 3\left(\lim_{(x,y)\to(2,-1)} y\right) - \lim_{(x,y)\to(2,-1)} 6.$$

Last, use the identity laws on the first six limits and the constant law on the last limit:

$$\lim_{(x,y)\to(2,-1)} (x^2 - 2xy + 3y^2 - 4x + 3y - 6)$$
$$= (2)^2 - 2(2)(-1) + 3(-1)^2 - 4(2) + 3(-1) - 6$$
$$= -6.$$

b. Before applying the quotient law, we need to verify that the limit of the denominator is nonzero. Using the difference law, constant multiple law, and identity law,

$$\lim_{(x,y)\to(2,-1)} (4x - 3y) = \lim_{(x,y)\to(2,-1)} 4x - \lim_{(x,y)\to(2,-1)} 3y$$
$$= 4(\lim_{(x,y)\to(2,-1)} x) - 3(\lim_{(x,y)\to(2,-1)} y)$$
$$= 4(2) - 3(-1) = 11.$$

Since the limit of the denominator is nonzero, the quotient law applies. We now calculate the limit of the numerator using the difference law, constant multiple law, and identity law:

$$\lim_{(x,y)\to(2,-1)} (2x + 3y) = \lim_{(x,y)\to(2,-1)} 2x + \lim_{(x,y)\to(2,-1)} 3y$$
$$= 2(\lim_{(x,y)\to(2,-1)} x) + 3(\lim_{(x,y)\to(2,-1)} y)$$
$$= 2(2) + 3(-1) = 1.$$

Therefore, according to the quotient law we have

$$\lim_{(x,y)\to(2,-1)} \frac{2x + 3y}{4x - 3y} = \frac{\lim_{(x,y)\to(2,-1)} (2x + 3y)}{\lim_{(x,y)\to(2,-1)} (4x - 3y)}$$
$$= \frac{1}{11}.$$

? Exercise 13.2.1:

Evaluate the following limit:

$$\lim_{(x,y)\to(5,-2)} \sqrt[3]{\frac{x^2 - y}{y^2 + x - 1}}.$$

Since we are taking the limit of a function of two variables, the point (a, b) is in $\mathbb{R}^2$, and it is possible to approach this point from an infinite number of directions. Sometimes when calculating a limit, the answer varies depending on the path taken toward (a, b). If this is the case, then the limit fails to exist. In other words, the limit must be unique, regardless of path taken.

✓ Example 13.2.2: Limits That Fail to Exist

Show that neither of the following limits exist:

a. $\lim_{(x,y)\to(0,0)} \frac{2xy}{3x^2 + y^2}$

b. $\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2 + 3y^4}$

Solution

a. The domain of the function $f(x,y) = \dfrac{2xy}{3x^2+y^2}$ consists of all points in the xy-plane except for the point $(0,0)$ (Figure **13.2.3**). To show that the limit does not exist as (x,y) approaches $(0,0)$, we note that it is impossible to satisfy the definition of a limit of a function of two variables because of the fact that the function takes different values along different lines passing through point $(0,0)$. First, consider the line $y=0$ in the xy-plane. Substituting $y=0$ into $f(x,y)$ gives

$$f(x,0) = \frac{2x(0)}{3x^2+0^2} = 0$$

for any value of x. Therefore the value of f remains constant for any point on the x-axis, and as y approaches zero, the function remains fixed at zero.

Next, consider the line $y=x$. Substituting $y=x$ into $f(x,y)$ gives

$$f(x,x) = \frac{2x(x)}{3x^2+x^2} = \frac{2x^2}{4x^2} = \tfrac{1}{2}.$$

This is true for any point on the line $y=x$. If we let x approach zero while staying on this line, the value of the function remains fixed at $\frac{1}{2}$, regardless of how small x is.

Choose a value for ε that is less than $1/2$—say, $1/4$. Then, no matter how small a δ disk we draw around $(0,0)$, the values of $f(x,y)$ for points inside that δ disk will include both 0 and $\frac{1}{2}$. Therefore, the definition of limit at a point is never satisfied and the limit fails to exist.

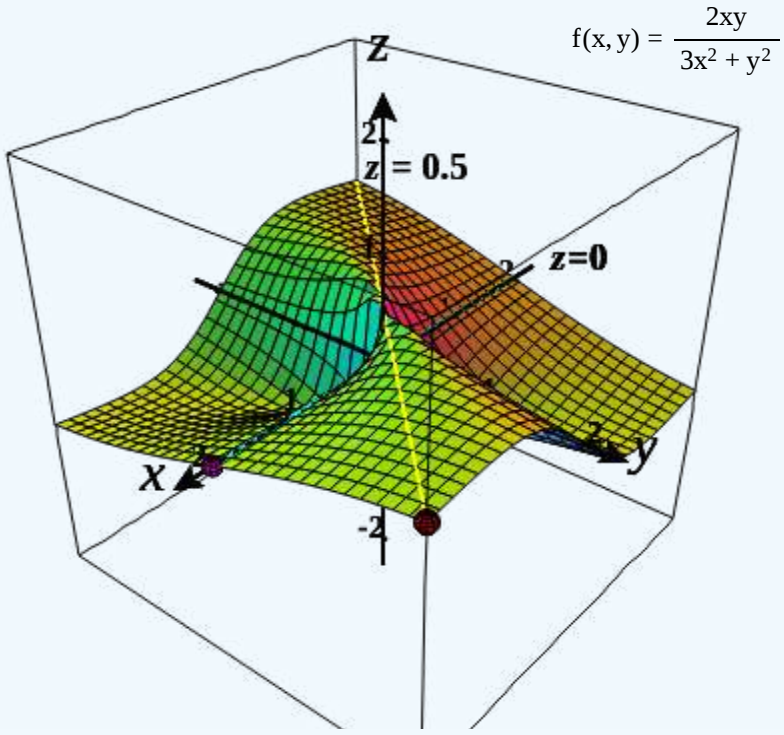

Figure **13.2.3**: Graph of the function $f(x,y) = \dfrac{2xy}{3x^2+y^2}$. Along the line $y=0$, the function is equal to zero; along the line $y=x$, the function is equal to $\frac{1}{2}$.

b. In a similar fashion to a., we can approach the origin along any straight line passing through the origin. If we try the x-axis (i.e., $y=0$), then the function remains fixed at zero. The same is true for the y-axis. Suppose we approach the origin along a straight line of slope k. The equation of this line is $y=kx$. Then the limit becomes

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+3y^4} &= \lim_{(x,y)\to(0,0)} \frac{4x(kx)^2}{x^2+3(kx)^4} \\ &= \lim_{(x,y)\to(0,0)} \frac{4k^2x^3}{x^2+3k^4x^4} \\ &= \lim_{(x,y)\to(0,0)} \frac{4k^2x}{1+3k^4x^2} \\ &= \frac{\lim\limits_{(x,y)\to(0,0)}(4k^2x)}{\lim\limits_{(x,y)\to(0,0)}(1+3k^4x^2)} \\ &= 0.\end{aligned}$$

regardless of the value of k. It would seem that the limit is equal to zero. What if we chose a curve passing through the origin instead? For example, we can consider the parabola given by the equation $x = y^2$. Substituting y^2 in place of x in $f(x,y)$ gives

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+3y^4} &= \lim_{(x,y)\to(0,0)} \frac{4(y^2)y^2}{(y^2)^2+3y^4} \\ &= \lim_{(x,y)\to(0,0)} \frac{4y^4}{y^4+3y^4} \\ &= \lim_{(x,y)\to(0,0)} 1 \\ &= 1.\end{aligned}$$

By the same logic in part a, it is impossible to find a δ disk around the origin that satisfies the definition of the limit for any value of $\varepsilon < 1$. Therefore,

$$\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+3y^4}$$

does **not** exist.

? Exercise 13.2.2:

Show that

$$\lim_{(x,y)\to(2,1)} \frac{(x-2)(y-1)}{(x-2)^2+(y-1)^2}$$

does not exist.

Interior Points and Boundary Points

To study continuity and differentiability of a function of two or more variables, we first need to learn some new terminology.

Definition: interior and boundary points

Let S be a subset of $\mathbb{R}^2$ (Figure 13.2.4).

1. A point P_0 is called an *interior point* of S if there is a δ disk centered around P_0 contained completely in S.
2. A point P_0 is called a *boundary point* of S if every δ disk centered around P_0 contains points both inside and outside S.

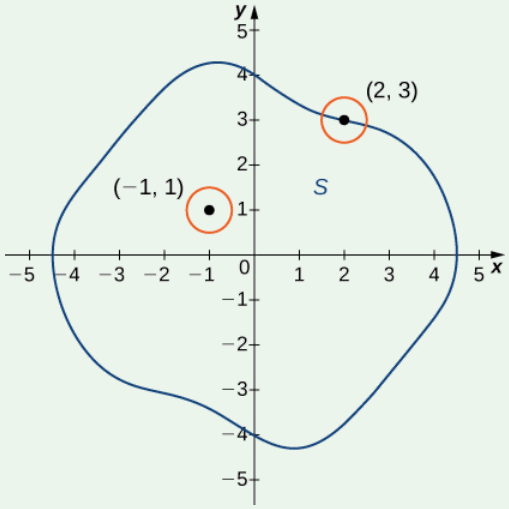

Figure 13.2.4: In the set S shown, $(-1,1)$ is an interior point and $(2,3)$ is a boundary point.

Definition: Open and closed sets

Let S be a subset of $\mathbb{R}^2$ (Figure 13.2.4).

1. S is called an *open set* if every point of S is an interior point.
2. S is called a *closed set* if it contains all its boundary points.

An example of an open set is a δ disk. If we include the boundary of the disk, then it becomes a closed set. A set that contains some, but not all, of its boundary points is neither open nor closed. For example if we include half the boundary of a δ disk but not the other half, then the set is neither open nor closed.

Definition: connected sets and Regions

Let S be a subset of $\mathbb{R}^2$ (Figure 13.2.4).

1. An open set S is a *connected set* if it cannot be represented as the union of two or more disjoint, nonempty open subsets.
2. A set S is a *region* if it is open, connected, and nonempty.

The definition of a limit of a function of two variables requires the δ disk to be contained inside the domain of the function. However, if we wish to find the limit of a function at a boundary point of the domain, the δ **disk** is not contained inside the domain. By definition, some of the points of the δ **disk** are inside the domain and some are outside. Therefore, we need only consider points that are inside both the δ disk and the domain of the function. This leads to the definition of the limit of a function at a boundary point.

Definition

Let f be a function of two variables, x and y, and suppose (a,b) is on the boundary of the domain of f. Then, the limit of $f(x,y)$ as (x,y) approaches (a,b) is L, written

$$\lim_{(x,y)\to(a,b)} f(x,y) = L,$$

if for any $\varepsilon > 0$, there exists a number $\delta > 0$ such that for any point (x,y) inside the domain of f and within a suitably small distance positive δ of (a,b), the value of $f(x,y)$ is no more than ε away from L (Figure 13.2.2). Using symbols, we can write: For any $\varepsilon > 0$, there exists a number $\delta > 0$ such that

$$|f(x,y) - L| < \varepsilon \text{ whenever } 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta.$$

Example 13.2.3: Limit of a Function at a Boundary Point

Prove

$$\lim_{(x,y)\to(4,3)} \sqrt{25-x^2-y^2} = 0.$$

Solution

The domain of the function $f(x,y) = \sqrt{25-x^2-y^2}$ is $\{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le 25\}$, which is a circle of radius 5 centered at the origin, along with its interior as shown in Figure 13.2.5.

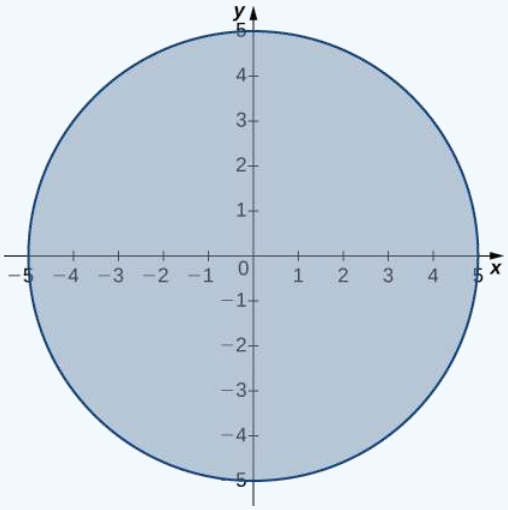

Figure 13.2.5: Domain of the function $f(x,y) = \sqrt{25-x^2-y^2}$.

We can use the limit laws, which apply to limits at the boundary of domains as well as interior points:

$$\begin{aligned}\lim_{(x,y)\to(4,3)} \sqrt{25-x^2-y^2} &= \sqrt{\lim_{(x,y)\to(4,3)} (25-x^2-y^2)} \\ &= \sqrt{\lim_{(x,y)\to(4,3)} 25 - \lim_{(x,y)\to(4,3)} x^2 - \lim_{(x,y)\to(4,3)} y^2} \\ &= \sqrt{25-4^2-3^2} \\ &= 0\end{aligned}$$

See the following graph.

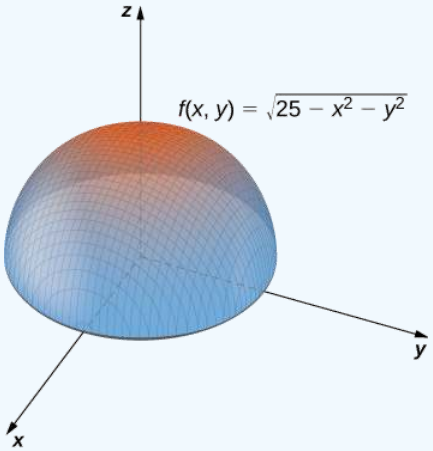

Figure 13.2.6: Graph of the function $f(x,y) = \sqrt{25-x^2-y^2}$.

? Exercise 13.2.3

Evaluate the following limit:

$$\lim_{(x,y)\to(5,-2)} \sqrt{29-x^2-y^2}.$$

Continuity of Functions of Two Variables

In Continuity, we defined the continuity of a function of one variable and saw how it relied on the limit of a function of one variable. In particular, three conditions are necessary for $f(x)$ to be continuous at point $x=a$

1. $f(a)$ exists.
2. $\lim\limits_{x\to a} f(x)$ exists.
3. $\lim\limits_{x\to a} f(x) = f(a)$.

These three conditions are necessary for continuity of a function of two variables as well.

Definition: continuous Functions

A function $f(x,y)$ is continuous at a point (a,b) in its domain if the following conditions are satisfied:

1. $f(a,b)$ exists.
2. $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ exists.
3. $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.

Example 13.2.4: Demonstrating Continuity for a Function of Two Variables

Show that the function

$$f(x,y) = \frac{3x+2y}{x+y+1}$$

is continuous at point $(5,-3)$.

Solution

There are three conditions to be satisfied, per the definition of continuity. In this example, $a=5$ and $b=-3$.

1. $f(a,b)$ exists. This is true because the domain of the function f consists of those ordered pairs for which the denominator is nonzero (i.e., $x+y+1 \neq 0$). Point $(5,-3)$ satisfies this condition. Furthermore,

$$f(a,b) = f(5,-3) = \frac{3(5)+2(-3)}{5+(-3)+1} = \frac{15-6}{2+1} = 3.$$

2. $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ exists. This is also true:

$$\begin{aligned}\lim_{(x,y)\to(a,b)} f(x,y) &= \lim_{(x,y)\to(5,-3)} \frac{3x+2y}{x+y+1} \\ &= \frac{\lim\limits_{(x,y)\to(5,-3)} (3x+2y)}{\lim\limits_{(x,y)\to(5,-3)} (x+y+1)} \\ &= \frac{15-6}{5-3+1} \\ &= 3.\end{aligned}$$

3. $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$. This is true because we have just shown that both sides of this equation equal three.

Exercise 13.2.4

Show that the function

$$f(x,y)=\sqrt{26-2x^2-y^2}$$

is continuous at point $(2,-3)$.

Continuity of a function of any number of variables can also be defined in terms of delta and epsilon. A function of two variables is continuous at a point (x_0,y_0) in its domain if for every $\varepsilon>0$ there exists a $\delta>0$ such that, whenever $\sqrt{(x-x_0)^2+(y-y_0)^2}<\delta$ it is true, $|f(x,y)-f(a,b)|<\varepsilon$. This definition can be combined with the formal definition (that is, the *epsilon–delta definition*) of continuity of a function of one variable to prove the following theorems:

Theorem 13.2.2: The Sum of Continuous Functions is Continuous

If $f(x,y)$ is continuous at (x_0,y_0), and $g(x,y)$ is continuous at (x_0,y_0), then $f(x,y)+g(x,y)$ is continuous at (x_0,y_0).

Theorem 13.2.3: The Product of Continuous Functions is Continuous

If $g(x)$ is continuous at x_0 and $h(y)$ is continuous at y_0, then $f(x,y)=g(x)h(y)$ is continuous at (x_0,y_0).

Theorem 13.2.4: The Composition of Continuous Functions is Continuous

Let g be a function of two variables from a domain $D\subseteq\mathbb{R}^2$ to a range $R\subseteq R$. Suppose g is continuous at some point $(x_0,y_0)\in D$ and define $z_0=g(x_0,y_0)$. Let f be a function that maps R to R such that z_0 is in the domain of f. Last, assume f is continuous at z_0. Then $f\circ g$ is continuous at (x_0,y_0) as shown in Figure 13.2.7.

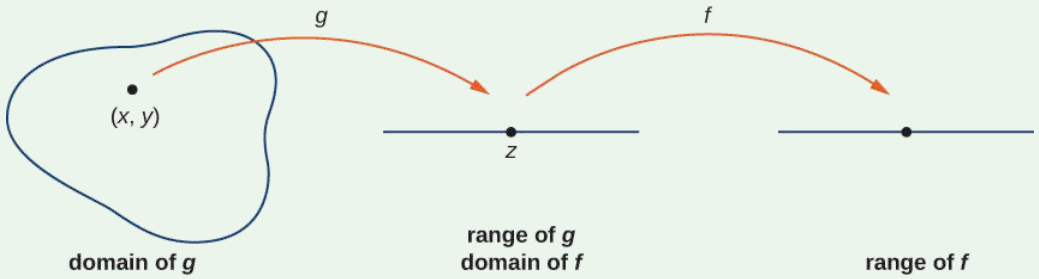

Figure 13.2.7: The composition of two continuous functions is continuous.

Let's now use the previous theorems to show continuity of functions in the following examples.

Example 13.2.5: More Examples of Continuity of a Function of Two Variables

Show that the functions $f(x,y)=4x^3y^2$ and $g(x,y)=\cos(4x^3y^2)$ are continuous everywhere.

Solution

The polynomials $g(x)=4x^3$ and $h(y)=y^2$ are continuous at every real number, and therefore by the product of continuous functions theorem, $f(x,y)=4x^3y^2$ is continuous at every point (x,y) in the xy-plane. Since $f(x,y)=4x^3y^2$ is continuous at every point (x,y) in the xy-plane and $g(x)=\cos x$ is continuous at every real number x, the continuity of the composition of functions tells us that $g(x,y)=\cos(4x^3y^2)$ is continuous at every point (x,y) in the xy-plane.

Exercise 13.2.5

Show that the functions $f(x,y)=2x^2y^3+3$ and $g(x,y)=(2x^2y^3+3)^4$ are continuous everywhere.

Functions of Three or More Variables

The limit of a function of three or more variables occurs readily in applications. For example, suppose we have a function $f(x,y,z)$ that gives the temperature at a physical location (x,y,z) in three dimensions. Or perhaps a function $g(x,y,z,t)$ can indicate air pressure at a location (x,y,z) at time t. How can we take a limit at a point in $\mathbb{R}^3$? What does it mean to be continuous at a point in four dimensions?

The answers to these questions rely on extending the concept of a δ disk into more than two dimensions. Then, the ideas of the limit of a function of three or more variables and the continuity of a function of three or more variables are very similar to the definitions given earlier for a function of two variables.

Definition: δ-balls

Let (x_0,y_0,z_0) be a point in $\mathbb{R}^3$. Then, a δ-ball in three dimensions consists of all points in $\mathbb{R}^3$ lying at a distance of less than δ from (x_0,y_0,z_0) —that is,

$$\{(x,y,z)\in\mathbb{R}^3 \mid \sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}<\delta\}.$$

To define a δ-ball in higher dimensions, add additional terms under the radical to correspond to each additional dimension. For example, given a point $P=(w_0,x_0,y_0,z_0)$ in $\mathbb{R}^4$, a δ ball around P can be described by

$$\{(w,x,y,z)\in\mathbb{R}^4 \mid \sqrt{(w-w_0)^2+(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}<\delta\}.$$

To show that a limit of a function of three variables exists at a point (x_0,y_0,z_0), it suffices to show that for any point in a δ ball centered at (x_0,y_0,z_0), the value of the function at that point is arbitrarily close to a fixed value (the limit value). All the limit laws for functions of two variables hold for functions of more than two variables as well.

Example 13.2.6: Finding the Limit of a Function of Three Variables

Find

$$\lim_{(x,y,z)\to(4,1,-3)}\frac{x^2y-3z}{2x+5y-z}.$$

Solution

Before we can apply the quotient law, we need to verify that the limit of the denominator is nonzero. Using the difference law, the identity law, and the constant law,

$$\begin{aligned}\lim_{(x,y,z)\to(4,1,-3)}(2x+5y-z) &= 2\left(\lim_{(x,y,z)\to(4,1,-3)}x\right)+5\left(\lim_{(x,y,z)\to(4,1,-3)}y\right)-\left(\lim_{(x,y,z)\to(4,1,-3)}z\right)\\ &= 2(4)+5(1)-(-3)\\ &= 16.\end{aligned}$$

Since this is nonzero, we next find the limit of the numerator. Using the product law, power law, difference law, constant multiple law, and identity law,

$$\begin{aligned}\lim_{(x,y,z)\to(4,1,-3)}(x^2y-3z) &= \left(\lim_{(x,y,z)\to(4,1,-3)}x\right)^2\left(\lim_{(x,y,z)\to(4,1,-3)}y\right)-3\lim_{(x,y,z)\to(4,1,-3)}z\\ &= (4^2)(1)-3(-3)\\ &= 16+9\\ &= 25\end{aligned}$$

Last, applying the quotient law:

$$\lim_{(x,y,z)\to(4,1,-3)}\frac{x^2y-3z}{2x+5y-z}=\frac{\lim\limits_{(x,y,z)\to(4,1,-3)}(x^2y-3z)}{\lim\limits_{(x,y,z)\to(4,1,-3)}(2x+5y-z)}=\frac{25}{16}$$

? Exercise 13.2.6

Find

$$\lim_{(x,y,z)\to(4,-1,3)} \sqrt{13-x^2-2y^2+z^2}$$

Key Concepts

- To study limits and continuity for functions of two variables, we use a δ disk centered around a given point.
- A function of several variables has a limit if for any point in a δ ball centered at a point P, the value of the function at that point is arbitrarily close to a fixed value (the limit value).
- The limit laws established for a function of one variable have natural extensions to functions of more than one variable.
- A function of two variables is continuous at a point if the limit exists at that point, the function exists at that point, and the limit and function are equal at that point.

Glossary

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13.2E: Exercises for Limits and Continuity

1) Use the limit laws for functions of two variables to evaluate each limit below, given that $\lim_{(x,y)\to(a,b)} f(x,y) = 5$ and $\lim_{(x,y)\to(a,b)} g(x,y) = 2$.

a. $\lim_{(x,y)\to(a,b)} [f(x,y) + g(x,y)]$

b. $\lim_{(x,y)\to(a,b)} [f(x,y)g(x,y)]$

c. $\lim_{(x,y)\to(a,b)} \left[\dfrac{7f(x,y)}{g(x,y)}\right]$

d. $\lim_{(x,y)\to(a,b)} \left[\dfrac{2f(x,y) - 4g(x,y)}{f(x,y) - g(x,y)}\right]$

In exercises 2 - 4, find the limit of the function.

2) $\lim_{(x,y)\to(1,2)} x$

3) $\lim_{(x,y)\to(1,2)} \dfrac{5x^2y}{x^2+y^2}$

4) Show that the limit $\lim_{(x,y)\to(0,0)} \dfrac{5x^2y}{x^2+y^2}$ exists and is the same along the paths: y-axis and x-axis, and along $y = x$.

In exercises 5 - 19, evaluate the limits at the indicated values of x and y. If the limit does not exist, state this and explain why the limit does not exist.

5) $\lim_{(x,y)\to(0,0)} \dfrac{4x^2+10y^2+4}{4x^2-10y^2+6}$

6) $\lim_{(x,y)\to(11,13)} \sqrt{\dfrac{1}{xy}}$

7) $\lim_{(x,y)\to(0,1)} \dfrac{y^2 \sin x}{x}$

8) $\lim_{(x,y)\to(0,0)} \sin\left(\dfrac{x^8+y^7}{x-y+10}\right)$

9) $\lim_{(x,y)\to(\pi/4,1)} \dfrac{y \tan x}{y+1}$

10) $\lim_{(x,y)\to(0,\pi/4)} \dfrac{\sec x + 2}{3x - \tan y}$

11) $\lim_{(x,y)\to(2,5)} \left(\dfrac{1}{x} - \dfrac{5}{y}\right)$

12) $\lim_{(x,y)\to(4,4)} x \ln y$

13) $\lim_{(x,y)\to(4,4)} e^{-x^2-y^2}$

14) $\lim_{(x,y)\to(0,0)} \sqrt{9-x^2-y^2}$

15) $\lim_{(x,y)\to(1,2)} (x^2y^3 - x^3y^2 + 3x + 2y)$

16) $\lim_{(x,y)\to(\pi,\pi)} x \sin\left(\dfrac{x+y}{4}\right)$

17) $\lim_{(x,y)\to(0,0)} \dfrac{xy+1}{x^2+y^2+1}$

18) $\lim_{(x,y)\to(0,0)} \dfrac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}$

19) $\lim_{(x,y)\to(0,0)} \ln(x^2+y^2)$

In exercises 20 - 21, complete the statement.

20) A point (x_0, y_0) in a plane region R is an interior point of R if _____.

21) A point (x_0, y_0) in a plane region R is called a boundary point of R if _____.

In exercises 22 - 25, use algebraic techniques to evaluate the limit.

22) $\lim\limits_{(x,y)\to(2,1)} \dfrac{x-y-1}{\sqrt{x-y}-1}$

23) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^4-4y^4}{x^2+2y^2}$

24) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3-y^3}{x-y}$

25) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-xy}{\sqrt{x}-\sqrt{y}}$

In exercises 26 - 27, evaluate the limits of the functions of three variables.

26) $\lim\limits_{(x,y,z)\to(1,2,3)} \dfrac{xz^2-y^2z}{xyz-1}$

27) $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{x^2-y^2-z^2}{x^2+y^2-z^2}$

In exercises 28 - 31, evaluate the limit of the function by determining the value the function approaches along the indicated paths. If the limit does not exist, explain why not.

28) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy+y^3}{x^2+y^2}$

a. Along the x-axis $(y=0)$

b. Along the y-axis $(x=0)$

c. Along the path $y=2x$

29) Evaluate $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy+y^3}{x^2+y^2}$ using the results of previous problem.

30) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^4+y^2}$

a. Along the x-axis $(y=0)$

b. Along the y-axis $(x=0)$

c. Along the path $y=x^2$

31) Evaluate $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^4+y^2}$ using the results of previous problem.

In exercises 32 - 35, discuss the continuity of each function. Find the largest region in the xy-plane in which each function is continuous.

32) $f(x,y)=\sin(xy)$

33) $f(x,y)=\ln(x+y)$

34) $f(x,y)=e^{3xy}$

35) $f(x,y)=\dfrac{1}{xy}$

In exercises 36 - 38, determine the region in which the function is continuous. Explain your answer.

36) $f(x,y)=\dfrac{x^2y}{x^2+y^2}$

37) $f(x,y)=\begin{cases}\dfrac{x^2y}{x^2+y^2} & if(x,y)\neq(0,0)\\ 0 & if(x,y)=(0,0)\end{cases}$

38) $f(x,y)=\dfrac{\sin(x^2+y^2)}{x^2+y^2}$

39) Determine whether $g(x,y)=\dfrac{x^2-y^2}{x^2+y^2}$ is continuous at $(0,0)$.

40) Create a plot using graphing software to determine where the limit does not exist. Determine the region of the coordinate plane in which $f(x,y)=\dfrac{1}{x^2-y}$ is continuous.

41) Determine the region of the xy-plane in which the composite function $g(x,y)=\arctan(\frac{xy^2}{x+y})$ is continuous. Use technology to support your conclusion.

42) Determine the region of the xy-plane in which $f(x,y)=\ln(x^2+y^2-1)$ is continuous. Use technology to support your conclusion. (Hint: Choose the range of values for x and y carefully!)

43) At what points in space is $g(x,y,z)=x^2+y^2-2z^2$ continuous?

44) At what points in space is $g(x,y,z)=\dfrac{1}{x^2+z^2-1}$ continuous?

45) Show that $\lim\limits_{(x,y)\to(0,0)}\dfrac{1}{x^2+y^2}$ does not exist at $(0,0)$ by plotting the graph of the function.

46) [T] Evaluate $\lim\limits_{(x,y)\to(0,0)}\dfrac{-xy^2}{x^2+y^4}$ by plotting the function using a CAS. Determine analytically the limit along the path $x=y^2$.

47) [T]

a. Use a CAS to draw a contour map of $z=\sqrt{9-x^2-y^2}$.

b. What is the name of the geometric shape of the level curves?

c. Give the general equation of the level curves.

d. What is the maximum value of z?

e. What is the domain of the function?

f. What is the range of the function?

48) True or False: If we evaluate $\lim\limits_{(x,y)\to(0,0)}f(x)$ along several paths and each time the limit is 1, we can conclude that $\lim\limits_{(x,y)\to(0,0)}f(x)=1$.

49) Use polar coordinates to find $\lim\limits_{(x,y)\to(0,0)}\dfrac{\sin\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}$. You can also find the limit using L'Hôpital's rule.

50) Use polar coordinates to find $\lim\limits_{(x,y)\to(0,0)}\cos(x^2+y^2)$.

51) Discuss the continuity of $f(g(x,y))$ where $f(t)=1/t$ and $g(x,y)=2x-5y$.

52) Given $f(x,y)=x^2-4y$, find $\lim\limits_{h\to0}\dfrac{f(x+h,y)-f(x,y)}{h}$.

53) Given $f(x,y)=x^2-4y$, find $\lim\limits_{h\to0}\dfrac{f(1+h,y)-f(1,y)}{h}$.

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13.3: Partial Derivatives

Learning Objectives

- Calculate the partial derivatives of a function of two variables.
- Calculate the partial derivatives of a function of more than two variables.
- Determine the higher-order derivatives of a function of two variables.
- Explain the meaning of a partial differential equation and give an example.

Now that we have examined limits and continuity of functions of two variables, we can proceed to study derivatives. Finding derivatives of functions of two variables is the key concept in this chapter, with as many applications in mathematics, science, and engineering as differentiation of single-variable functions. However, we have already seen that limits and continuity of multivariable functions have new issues and require new terminology and ideas to deal with them. This carries over into differentiation as well.

Derivatives of a Function of Two Variables

When studying derivatives of functions of one variable, we found that one interpretation of the derivative is an instantaneous rate of change of y as a function of x. Leibniz notation for the derivative is dy/dx, which implies that y is the dependent variable and x is the independent variable. For a function $z = f(x, y)$ of two variables, x and y are the independent variables and z is the dependent variable. This raises two questions right away: How do we adapt Leibniz notation for functions of two variables? Also, what is an interpretation of the derivative? The answer lies in partial derivatives.

Definition: Partial Derivatives

Let $f(x, y)$ be a function of two variables. Then the *partial derivative* of f with respect to x, written as $\partial f/\partial x$,, or f_x, is defined as

$$\frac{\partial f}{\partial x} = f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h} \tag{13.3.1}$$

The partial derivative of f with respect to y, written as $\partial f/\partial y$, or f_y, is defined as

$$\frac{\partial f}{\partial y} = f_y(x, y) = \lim_{k \to 0} \frac{f(x, y + k) - f(x, y)}{k}. \tag{13.3.2}$$

This definition shows two differences already. First, the notation changes, in the sense that we still use a version of Leibniz notation, but the d in the original notation is replaced with the symbol ∂. (This rounded "d" is usually called "partial," so $\partial f/\partial x$ is spoken as the "partial of f with respect to x.") This is the first hint that we are dealing with partial derivatives. Second, we now have two different derivatives we can take, since there are two different independent variables. Depending on which variable we choose, we can come up with different partial derivatives altogether, and often do.

Example 13.3.1: Calculating Partial Derivatives from the Definition

Use the definition of the partial derivative as a limit to calculate $\partial f/\partial x$ and $\partial f/\partial y$ for the function

$$f(x, y) = x^2 - 3xy + 2y^2 - 4x + 5y - 12.$$

Solution

First, calculate $f(x + h, y)$.

$$\begin{aligned} f(x + h, y) &= (x + h)^2 - 3(x + h)y + 2y^2 - 4(x + h) + 5y - 12 \\ &= x^2 + 2xh + h^2 - 3xy - 3hy + 2y^2 - 4x - 4h + 5y - 12. \end{aligned}$$

Next, substitute this into Equation 13.3.1 and simplify:

$$\begin{aligned}\frac{\partial f}{\partial x} &= \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{(x^2+2xh+h^2-3xy-3hy+2y^2-4x-4h+5y-12)}\\ &= \lim_{h\to 0}\frac{-(x^2-3xy+2y^2-4x+5y-12)}{x^2+2xh+h^2-3xy-3hy+2y^2-4x-4h+5y-12-x^2+3xy-2y^2}\\ &= \lim_{h\to 0}\frac{+4x-5y+12}{h}\\ &= \lim_{h\to 0}\frac{2xh+h^2-3hy-4h}{h}\\ &= \lim_{h\to 0}\frac{h(2x+h-3y-4)}{h}\\ &= \lim_{h\to 0}(2x+h-3y-4)\\ &= 2x-3y-4.\end{aligned}$$

To calculate $\frac{\partial f}{\partial y}$, first calculate $f(x,y+h)$:

$$\begin{aligned}f(x+h,y) &= x^2-3x(y+h)+2(y+h)^2-4x+5(y+h)-12\\ &= x^2-3xy-3xh+2y^2+4yh+2h^2-4x+5y+5h-12.\end{aligned}$$

Next, substitute this into Equation 13.3.2 and simplify:

$$\begin{aligned}\frac{\partial f}{\partial y} &= \lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{(x^2-3xy-3xh+2y^2+4yh+2h^2-4x+5y+5h-12)}\\ &= \lim_{h\to 0}\frac{-(x^2-3xy+2y^2-4x+5y-12)}{x^2-3xy-3xh+2y^2+4yh+2h^2-4x+5y+5h-12-x^2+3xy-2y^2}\\ &= \lim_{h\to 0}\frac{+4x-5y+12}{h}\\ &= \lim_{h\to 0}\frac{-3xh+4yh+2h^2+5h}{h}\\ &= \lim_{h\to 0}\frac{h(-3x+4y+2h+5)}{h}\\ &= \lim_{h\to 0}(-3x+4y+2h+5)\\ &= -3x+4y+5\end{aligned}$$

? Exercise 13.3.1

Use the definition of the partial derivative as a limit to calculate $\partial f/\partial x$ and $\partial f/\partial y$ for the function

$$f(x,y)=4x^2+2xy-y^2+3x-2y+5.$$

The idea to keep in mind when calculating partial derivatives is to treat all independent variables, other than the variable with respect to which we are differentiating, as constants. Then proceed to differentiate as with a function of a single variable. To see why this is true, first fix y and define $g(x)=f(x,y)$ as a function of x. Then

$$\begin{aligned}g'(x) &= \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\ &= \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}\\ &= \frac{\partial f}{\partial x}.\end{aligned}$$

The same is true for calculating the partial derivative of f with respect to y. This time, fix x and define $h(y)=f(x,y)$ as a function of y. Then

$$\begin{aligned}h'(x) &= \lim_{k\to 0}\frac{h(x+k)-h(x)}{k}\\ &= \lim_{k\to 0}\frac{f(x,y+k)-f(x,y)}{k}\\ &= \frac{\partial f}{\partial y}.\end{aligned}$$

All differentiation rules apply.

Example 13.3.2: Calculating Partial Derivatives

Calculate $\partial f/\partial x$ and $\partial f/\partial y$ for the following functions by holding the opposite variable constant then differentiating:

a. $f(x,y) = x^2 - 3xy + 2y^2 - 4x + 5y - 12$
b. $g(x,y) = \sin(x^2y - 2x + 4)$

Solution:

a. To calculate $\partial f/\partial x$, treat the variable y as a constant. Then differentiate $f(x,y)$ with respect to x using the sum, difference, and power rules:

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\left[x^2 - 3xy + 2y^2 - 4x + 5y - 12\right] \\ &= \frac{\partial}{\partial x}[x^2] - \frac{\partial}{\partial x}[3xy] + \frac{\partial}{\partial x}[2y^2] - \frac{\partial}{\partial x}[4x] + \frac{\partial}{\partial x}[5y] - \frac{\partial}{\partial x}[12] \\ &= 2x - 3y + 0 - 4 + 0 - 0 \\ &= 2x - 3y - 4.\end{aligned}$$

The derivatives of the third, fifth, and sixth terms are all zero because they do not contain the variable x, so they are treated as constant terms. The derivative of the second term is equal to the coefficient of x, which is $-3y$. Calculating $\partial f/\partial y$:

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left[x^2 - 3xy + 2y^2 - 4x + 5y - 12\right] \\ &= \frac{\partial}{\partial y}[x^2] - \frac{\partial}{\partial y}[3xy] + \frac{\partial}{\partial y}[2y^2] - \frac{\partial}{\partial y}[4x] + \frac{\partial}{\partial y}[5y] - \frac{\partial}{\partial y}[12] \\ &= 0 - 3x + 4y - 0 + 5 - 0 \\ &= -3x + 4y + 5.\end{aligned}$$

These are the same answers obtained in Example 13.3.1.

b. To calculate $\partial g/\partial x$, treat the variable y as a constant. Then differentiate $g(x,y)$ with respect to x using the chain rule and power rule:

$$\begin{aligned}\frac{\partial g}{\partial x} &= \frac{\partial}{\partial x}\left[\sin(x^2y - 2x + 4)\right] \\ &= \cos(x^2y - 2x + 4)\frac{\partial}{\partial x}[x^2y - 2x + 4] \\ &= (2xy - 2)\cos(x^2y - 2x + 4).\end{aligned}$$

To calculate $\partial g/\partial y$, treat the variable x as a constant. Then differentiate $g(x,y)$ with respect to y using the chain rule and power rule:

$$\begin{aligned}\frac{\partial g}{\partial y} &= \frac{\partial}{\partial y}\left[\sin(x^2y - 2x + 4)\right] \\ &= \cos(x^2y - 2x + 4)\frac{\partial}{\partial y}[x^2y - 2x + 4] \\ &= x^2\cos(x^2y - 2x + 4).\end{aligned}$$

Exercise 13.3.2

Calculate $\partial f/\partial x$ and $\partial f/\partial y$ for the function

$$f(x,y) = \tan(x^3 - 3x^2y^2 + 2y^4)$$

by holding the opposite variable constant, then differentiating.

How can we interpret these partial derivatives? Recall that the graph of a function of two variables is a surface in R^3. If we remove the limit from the definition of the partial derivative with respect to x, the difference quotient remains:

$$\frac{f(x+h,y)-f(x,y)}{h}.$$

This resembles the difference quotient for the derivative of a function of one variable, except for the presence of the y variable. Figure 13.3.1 illustrates a surface described by an arbitrary function $z=f(x,y)$.

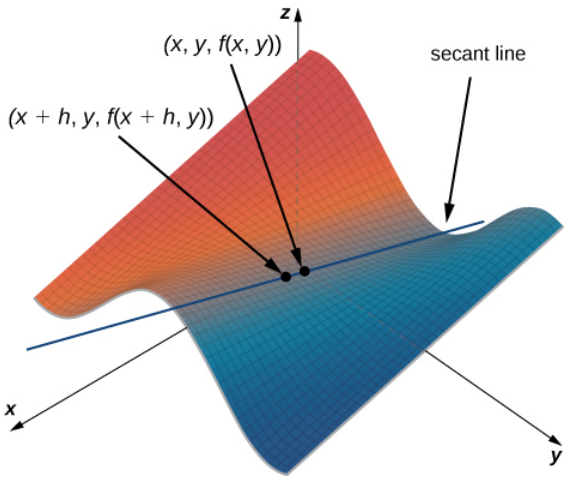


Figure 13.3.1: Secant line passing through the points $(x,y,f(x,y))$ and $(x+h,y,f(x+h,y))$.

In Figure 13.3.1, the value of h is positive. If we graph $f(x,y)$ and $f(x+h,y)$ for an arbitrary point (x,y), then the slope of the secant line passing through these two points is given by

$$\frac{f(x+h,y)-f(x,y)}{h}.$$

This line is parallel to the x-axis. Therefore, the slope of the secant line represents an average rate of change of the function f as we travel parallel to the x-axis. As h approaches zero, the slope of the secant line approaches the slope of the tangent line.

If we choose to change y instead of x by the same incremental value h, then the secant line is parallel to the y-axis and so is the tangent line. Therefore, $\partial f/\partial x$ represents the slope of the tangent line passing through the point $(x,y,f(x,y))$ parallel to the x-axis and $\partial f/\partial y$ represents the slope of the tangent line passing through the point $(x,y,f(x,y))$ parallel to the y-axis. If we wish to find the slope of a tangent line passing through the same point in any other direction, then we need what are called directional derivatives.

We now return to the idea of contour maps, which we introduced in Functions of Several Variables. We can use a **contour map** to estimate partial derivatives of a function $g(x,y)$.

Example 13.3.3: Partial Derivatives from a Contour Map

Use a contour map to estimate $\partial g/\partial x$ at the point $(\sqrt{5},0)$ for the function

$$g(x,y)=\sqrt{9-x^2-y^2}.$$

Solution

Figure 13.3.2 represents a contour map for the function $g(x,y)$.

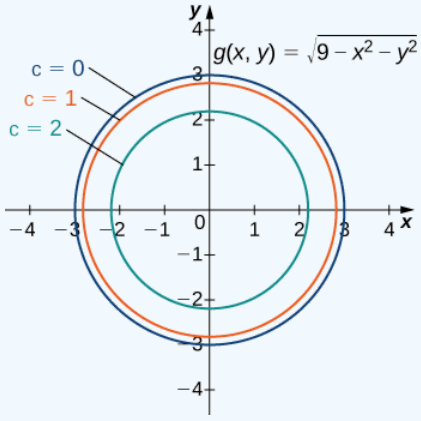


Figure 13.3.2: Contour map for the function $g(x,y) = \sqrt{9-x^2-y^2}$, using $c = 0, 1, 2,$ and 3($c = 3$ corresponds to the origin).

The inner circle on the contour map corresponds to $c = 2$ and the next circle out corresponds to $c = 1$. The first circle is given by the equation $2 = \sqrt{9-x^2-y^2}$; the second circle is given by the equation $1 = \sqrt{9-x^2-y^2}$. The first equation simplifies to $x^2 + y^2 = 5$ and the second equation simplifies to $x^2 + y^2 = 8$. The x-intercept of the first circle is $(\sqrt{5}, 0)$ and the x-intercept of the second circle is $(2\sqrt{2}, 0)$. We can estimate the value of $\partial g/\partial x$ evaluated at the point $(\sqrt{5}, 0)$ using the slope formula:

$$\begin{aligned} \frac{\partial g}{\partial x}\Big|_{(x,y)=(\sqrt{5},0)} &\approx \frac{g(\sqrt{5},0) - g(2\sqrt{2},0)}{\sqrt{5} - 2\sqrt{2}} \\ &= \frac{2-1}{\sqrt{5}-2\sqrt{2}} \\ &= \frac{1}{\sqrt{5}-2\sqrt{2}} \approx -1.688. \end{aligned}$$

To calculate the exact value of $\partial g/\partial x$ evaluated at the point $(\sqrt{5}, 0)$, we start by finding $\partial g/\partial x$ using the chain rule. First, we rewrite the function as

$$g(x,y) = \sqrt{9-x^2-y^2} = (9-x^2-y^2)^{1/2}$$

and then differentiate with respect to x while holding y constant:

$$\begin{aligned} \frac{\partial g}{\partial x} &= \frac{1}{2}(9-x^2-y^2)^{-1/2}(-2x) \\ &= -\frac{x}{\sqrt{9-x^2-y^2}}. \end{aligned}$$

Next, we evaluate this expression using $x = \sqrt{5}$ and $y = 0$:

$$\begin{aligned} \frac{\partial g}{\partial x}\Big|_{(x,y)=(\sqrt{5},0)} &= -\frac{\sqrt{5}}{\sqrt{9-(\sqrt{5})^2-(0)^2}} \\ &= -\frac{\sqrt{5}}{\sqrt{4}} \\ &= -\frac{\sqrt{5}}{2} \approx -1.118. \end{aligned}$$

The estimate for the partial derivative corresponds to the slope of the secant line passing through the points $(\sqrt{5}, 0, g(\sqrt{5}, 0))$ and $(2\sqrt{2}, 0, g(2\sqrt{2}, 0))$. It represents an approximation to the slope of the tangent line to the surface through the point $(\sqrt{5}, 0, g(\sqrt{5}, 0))$, which is parallel to the x-axis.

? Exercise 13.3.3

Use a contour map to estimate $\partial f/\partial y$ at point $(0, \sqrt{2})$ for the function

$$f(x,y) = x^2 - y^2.$$

Compare this with the exact answer.

Functions of More Than Two Variables

Suppose we have a function of three variables, such as $w = f(x, y, z)$. We can calculate partial derivatives of w with respect to any of the independent variables, simply as extensions of the definitions for partial derivatives of functions of two variables.

Definition: Partial Derivatives

Let $f(x, y, z)$ be a function of three variables. Then, the partial derivative of f with respect to x, written as $\partial f/\partial x$, or f_x, is defined to be

$$\frac{\partial f}{\partial x} = f_x(x, y, z) = \lim_{h \to 0} \frac{f(x + h, y, z) - f(x, y, z)}{h}. \tag{13.3.3}$$

The *partial derivative* of f with *respect* to y, written as $\partial f/\partial y$, or f_y, is defined to be

$$\frac{\partial f}{\partial y} = f_y(x, y, z) = \lim_{k \to 0} \frac{f(x, y + k, z) - f(x, y, z)}{k.} \tag{13.3.4}$$

The *partial derivative* of f with respect to z, written as $\partial f/\partial z$, or f_z, is defined to be

$$\frac{\partial f}{\partial z} = f_z(x, y, z) = \lim_{m \to 0} \frac{f(x, y, z + m) - f(x, y, z)}{m}. \tag{13.3.5}$$

We can calculate a partial derivative of a function of three variables using the same idea we used for a function of two variables. For example, if we have a function f of x, y, and z, and we wish to calculate $\partial f/\partial x$, then we treat the other two independent variables as if they are constants, then differentiate with respect to x.

Example 13.3.4: Calculating Partial Derivatives for a Function of Three Variables

Use the limit definition of partial derivatives to calculate $\partial f/\partial x$ for the function

$$f(x, y, z) = x^2 - 3xy + 2y^2 - 4xz + 5yz^2 - 12x + 4y - 3z.$$

Then, find $\partial f/\partial y$ and $\partial f/\partial z$ by setting the other two variables constant and differentiating accordingly.

Solution:

We first calculate $\partial f/\partial x$ using Equation 13.3.3, then we calculate the other two partial derivatives by holding the remaining variables constant. To use the equation to find $\partial f/\partial x$, we first need to calculate $f(x + h, y, z)$:

$$\begin{aligned} f(x + h, y, z) &= (x + h)^2 - 3(x + h)y + 2y^2 - 4(x + h)z + 5yz^2 - 12(x + h) + 4y - 3z \\ &= x^2 + 2xh + h^2 - 3xy - 3xh + 2y^2 - 4xz - 4hz + 5yz^2 \\ &\quad - 12x - 12h + 4y - 3z \end{aligned}$$

and recall that $f(x, y, z) = x^2 - 3xy + 2y^2 - 4zx + 5yz^2 - 12x + 4y - 3z$. Next, we substitute these two expressions into the equation:

$$\begin{aligned} \frac{\partial f}{\partial x} &= \lim_{h \to 0} \\ &\quad \left[\frac{x^2 + 2xh + h^2 - 3xy - 3hy + 2y^2 - 4xz - 4hz + 5yz^2 - 12x - 12h + 4y - 3zh - x^2}{h} \right. \\ &\quad \left. \frac{- 3xy + 2y^2 - 4xz + 5yz^2 - 12x + 4y - 3z}{h} \right] \\ &= \lim_{h \to 0} \left[\frac{2xh + h^2 - 3hy - 4hz - 12h}{h} \right] \\ &= \lim_{h \to 0} \left[\frac{h(2x + h - 3y - 4z - 12)}{h} \right] \\ &= \lim_{h \to 0} (2x + h - 3y - 4z - 12) \\ &= 2x - 3y - 4z - 12. \end{aligned}$$

Then we find $\partial f/\partial y$ by holding x and z constant. Therefore, any term that does not include the variable y is constant, and its derivative is zero. We can apply the sum, difference, and power rules for functions of one variable:

$$\begin{aligned}&\frac{\partial}{\partial y}\left[x^2-3xy+2y^2-4xz+5yz^2-12x+4y-3z\right]\\&=\frac{\partial}{\partial y}[x^2]-\frac{\partial}{\partial y}[3xy]+\frac{\partial}{\partial y}[2y^2]-\frac{\partial}{\partial y}[4xz]+\frac{\partial}{\partial y}[5yz^2]-\frac{\partial}{\partial y}[12x]\\&\quad+\frac{\partial}{\partial y}[4y]-\frac{\partial}{\partial z}[3z]\\&=0-3x+4y-0+5z^2-0+4-0\\&=-3x+4y+5z^2+4.\end{aligned}$$

To calculate $\partial f/\partial z$, we hold x and y constant and apply the sum, difference, and power rules for functions of one variable:

$$\begin{aligned}&\frac{\partial}{\partial z}[x^2-3xy+2y^2-4xz+5yz^2-12x+4y-3z]\\&=\frac{\partial}{\partial z}[x^2]-\frac{\partial}{\partial z}[3xy]+\frac{\partial}{\partial z}[2y^2]-\frac{\partial}{\partial z}[4xz]+\frac{\partial}{\partial z}[5yz^2]-\frac{\partial}{\partial z}[12x]+\frac{\partial}{\partial z}[4y]-\frac{\partial}{\partial z}[3z]\\&=0-0+0-4x+10yz-0+0-3\\&=-4x+10yz-3\end{aligned}$$

? Exercise 13.3.4

Use the limit definition of partial derivatives to calculate $\partial f/\partial x$ for the function

$$f(x,y,z)=2x^2-4x^2y+2y^2+5xz^2-6x+3z-8.$$

Then find $\partial f/\partial y$ and $\partial f/\partial z$ by setting the other two variables constant and differentiating accordingly.

✓ Example 13.3.5: Calculating Partial Derivatives for a Function of Three Variables

Calculate the three partial derivatives of the following functions.

a. $f(x,y,z)=\dfrac{x^2y-4xz+y^2}{x-3yz}$

b. $g(x,y,z)=\sin(x^2y-z)+\cos(x^2-yz)$

Solution

In each case, treat all variables as constants except the one whose partial derivative you are calculating.

a.

$$\begin{aligned}\frac{\partial f}{\partial x}&=\frac{\partial}{\partial x}\left[\frac{x^2y-4xz+y^2}{x-3yz}\right]\\&=\frac{\frac{\partial}{\partial x}(x^2y-4xz+y^2)(x-3yz)-(x^2y-4xz+y^2)\frac{\partial}{\partial x}(x-3yz)}{(x-3yz)^2}\\&=\frac{(2xy-4z)(x-3yz)-(x^2y-4xz+y^2)(1)}{(x-3yz)^2}\\&=\frac{2x^2y-6xy^2z-4xz+12yz^2-x^2y+4xz-y^2}{(x-3yz)^2}\\&=\frac{x^2y-6xy^2z-4xz+12yz^2+4xz-y^2}{(x-3yz)^2}\end{aligned}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[\frac{x^2y - 4xz + y^2}{x - 3yz}\right]$$

$$= \frac{\frac{\partial}{\partial y}(x^2y - 4xz + y^2)(x - 3yz) - (x^2y - 4xz + y^2)\frac{\partial}{\partial y}(x - 3yz)}{(x - 3yz)^2}$$

$$= \frac{(x^2 + 2y)(x - 3yz) - (x^2y - 4xz + y^2)(-3z)}{(x - 3yz)^2}$$

$$= \frac{x^3 - 3x^2yz + 2xy - 6y^2z + 3x^2yz - 12xz^2 + 3y^2z}{(x - 3yz)^2}$$

$$= \frac{x^3 + 2xy - 3y^2z - 12xz^2}{(x - 3yz)^2}$$

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left[\frac{x^2y - 4xz + y^2}{x - 3yz}\right]$$

$$= \frac{\frac{\partial}{\partial z}(x^2y - 4xz + y^2)(x - 3yz) - (x^2y - 4xz + y^2)\frac{\partial}{\partial z}(x - 3yz)}{(x - 3yz)^2}$$

$$= \frac{(-4x)(x - 3yz) - (x^2y - 4xz + y^2)(-3y)}{(x - 3yz)^2}$$

$$= \frac{-4x^2 + 12xyz + 3x^2y^2 - 12xyz + 3y^3}{(x - 3yz)^2}$$

$$= \frac{-4x^2 + 3x^2y^2 + 3y^3}{(x - 3yz)^2}$$

b.

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[\sin(x^2y - z) + \cos(x^2 - yz)\right]$$

$$= (\cos(x^2y - z))\frac{\partial}{\partial x}(x^2y - z) - (\sin(x^2 - yz))\frac{\partial}{\partial x}(x^2 - yz)$$

$$= 2xy\cos(x^2y - z) - 2x\sin(x^2 - yz)$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[\sin(x^2y - z) + \cos(x^2 - yz)]$$

$$= (\cos(x^2y - z))\frac{\partial}{\partial y}(x^2y - z) - (\sin(x^2 - yz))\frac{\partial}{\partial y}(x^2 - yz)$$

$$= x^2\cos(x^2y - z) + z\sin(x^2 - yz)$$

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}[\sin(x^2y - z) + \cos(x^2 - yz)]$$

$$= (\cos(x^2y - z))\frac{\partial}{\partial z}(x^2y - z) - (\sin(x^2 - yz))\frac{\partial}{\partial z}(x^2 - yz)$$

$$= -\cos(x^2y - z) + y\sin(x^2 - yz)$$

? Exercise 13.3.5

Calculate $\partial f/\partial x, \partial f/\partial y$, and $\partial f/\partial z$ for the function

$$f(x, y, z) = \sec(x^2y) - \tan(x^3yz^2).$$

Higher-Order Partial Derivatives

Consider the function

$$f(x, y) = 2x^3 - 4xy^2 + 5y^3 - 6xy + 5x - 4y + 12.$$

Its partial derivatives are

$$\frac{\partial f}{\partial x} = 6x^2 - 4y^2 - 6y + 5$$

and

$$\frac{\partial f}{\partial y} = -8xy + 15y^2 - 6x - 4.$$

Each of these partial derivatives is a function of two variables, so we can calculate partial derivatives of these functions. Just as with derivatives of single-variable functions, we can call these *second-order derivatives*, third-order derivatives, and so on. In general, they are referred to as **higher-order partial derivatives.** There are four second-order partial derivatives for any function (provided they all exist):

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right]$$
$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right]$$
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right]$$
$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right].$$

An alternative notation for each is f_{xx}, f_{xy}, f_{yx}, and f_{yy}, respectively. Higher-order partial derivatives calculated with respect to different variables, such as f_{xy} and f_{yx}, are commonly called **mixed partial derivatives.**

Example 13.3.6: Calculating Second Partial Derivatives

Calculate all four second partial derivatives for the function

$$f(x, y) = xe^{-3y} + \sin(2x - 5y). \tag{13.3.6}$$

Solution:

To calculate $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial y \partial x}$, we first calculate $\partial f/\partial x$:

$$\frac{\partial f}{\partial x} = e^{-3y} + 2\cos(2x - 5y). \tag{13.3.7}$$

To calculate $\frac{\partial^2 f}{\partial x^2}$, differentiate $\partial f/\partial x$ (Equation 13.3.7) with respect to x:

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right] \\ &= \frac{\partial}{\partial x}[e^{-3y} + 2\cos(2x - 5y)] \\ &= -4\sin(2x - 5y).\end{aligned}$$

To calculate $\frac{\partial^2 f}{\partial y \partial x}$, differentiate $\partial f/\partial x$ (Equation 13.3.7) with respect to y:

$$\begin{aligned}\frac{\partial^2 f}{\partial y\,\partial x} &= \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] \\ &= \frac{\partial}{\partial y}[e^{-3y} + 2\cos(2x - 5y)] \\ &= -3e^{-3y} + 10\sin(2x - 5y).\end{aligned}$$

To calculate $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y^2}$, first calculate $\partial f/\partial y$:

$$\frac{\partial f}{\partial y} = -3xe^{-3y} - 5\cos(2x - 5y). \tag{13.3.8}$$

To calculate $\frac{\partial^2 f}{\partial x \partial y}$, differentiate $\partial f/\partial y$ (Equation 13.3.8) with respect to x:

$$\begin{aligned}\frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] \\ &= \frac{\partial}{\partial x}[-3xe^{-3y} - 5\cos(2x - 5y)] \\ &= -3e^{-3y} + 10\sin(2x - 5y).\end{aligned}$$

To calculate $\frac{\partial^2 f}{\partial y^2}$, differentiate $\partial f/\partial y$ (Equation 13.3.8) with respect to y:

$$\begin{aligned}\frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right] \\ &= \frac{\partial}{\partial y}[-3xe^{-3y} - 5\cos(2x - 5y)] \\ &= 9xe^{-3y} - 25\sin(2x - 5y).\end{aligned}$$

? Exercise 13.3.6

Calculate all four second partial derivatives for the function

$$f(x, y) = \sin(3x - 2y) + \cos(x + 4y).$$

At this point we should notice that, in both Example **13.3.6** and the checkpoint, it was true that $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$. Under certain conditions, this is always true. In fact, it is a direct consequence of the following theorem.

Theorem 13.3.1: Equality of Mixed Partial Derivatives (Clairaut's Theorem)

Suppose that $f(x, y)$ is defined on an open disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are continuous on D, then $f_{xy}(a, b) = f_{yx}(a, b)$.

Clairaut's theorem guarantees that as long as mixed second-order derivatives are continuous, the order in which we choose to differentiate the functions (i.e., which variable goes first, then second, and so on) does not matter. It can be extended to higher-order derivatives as well. The proof of Clairaut's theorem can be found in most advanced calculus books.

Two other second-order partial derivatives can be calculated for any function $f(x, y)$. The partial derivative f_{xx} is equal to the partial derivative of f_x with respect to x, and f_{yy} is equal to the partial derivative of f_y with respect to y.

Partial Differential Equations

Previously, we studied differential equations in which the unknown function had one independent variable. A partial differential equation is an equation that involves an unknown function of more than one independent variable and one or more of its partial derivatives. Examples of partial differential equations are

$$u_t = c^2(u_{xx} + u_{yy})$$
heat equation in two dimensions

$$u_{tt} = c^2(u_{xx} + u_{yy})$$
wave equation in two dimensions

$$u_{xx} + u_{yy} = 0$$
Laplace's equation in two dimensions

In the heat and wave equations, the unknown function u has three independent variables: t, x, and y with c is an arbitrary constant. The independent variables x and y are considered to be spatial variables, and the variable t represents time. In Laplace's equation, the unknown function u has two independent variables x and y.

Example 13.3.7: A Solution to the Wave Equation

Verify that

$$u(x,y,t) = 5\sin(3\pi x)\sin(4\pi y)\cos(10\pi t)$$

is a solution to the wave equation

$$u_{tt} = 4(u_{xx} + u_{yy}). \tag{13.3.9}$$

Solution

First, we calculate u_{tt}, u_{xx}, and u_{yy} :

$$\begin{aligned} u_{tt}(x,y,t) &= \frac{\partial}{\partial t}\left[\frac{\partial u}{\partial t}\right] \\ &= \frac{\partial}{\partial t}[5\sin(3\pi x)\sin(4\pi y)(-10\pi\sin(10\pi t))] \\ &= \frac{\partial}{\partial t}[-50\pi\sin(3\pi x)\sin(4\pi y)\sin(10\pi t)] \\ &= -500\pi^2\sin(3\pi x)\sin(4\pi y)\cos(10\pi t) \end{aligned}$$

$$\begin{aligned} u_{xx}(x,y,t) &= \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x}\right] \\ &= \frac{\partial}{\partial x}[15\pi\cos(3\pi x)\sin(4\pi y)\cos(10\pi t)] \\ &= -45\pi^2\sin(3\pi x)\sin(4\pi y)\cos(10\pi t) \end{aligned}$$

$$\begin{aligned} u_{yy}(x,y,t) &= \frac{\partial}{\partial y}\left[\frac{\partial u}{\partial y}\right] \\ &= \frac{\partial}{\partial y}[5\sin(3\pi x)(4\pi\cos(4\pi y))\cos(10\pi t)] \\ &= \frac{\partial}{\partial y}[20\pi\sin(3\pi x)\cos(4\pi y)\cos(10\pi t)] \\ &= -80\pi^2\sin(3\pi x)\sin(4\pi y)\cos(10\pi t). \end{aligned}$$

Next, we substitute each of these into the right-hand side of Equation 13.3.9 and simplify:

$$\begin{aligned} 4(u_{xx}+u_{yy}) &= 4(-45\pi^2\sin(3\pi x)\sin(4\pi y)\cos(10\pi t) \\ &\quad + -80\pi^2\sin(3\pi x)\sin(4\pi y)\cos(10\pi t)) \\ &= 4(-125\pi^2\sin(3\pi x)\sin(4\pi y)\cos(10\pi t)) \\ &= -500\pi^2\sin(3\pi x)\sin(4\pi y)\cos(10\pi t) \\ &= u_{tt}. \end{aligned}$$

This verifies the solution.

Exercise 13.3.7: A Solution to the Heat Equation

Verify that

$$u(x,y,t) = 2\sin\left(\frac{x}{3}\right)\sin\left(\frac{y}{4}\right)e^{-25t/16}$$

is a solution to the heat equation

$$u_t = 9(u_{xx} + u_{yy}).$$

Since the solution to the two-dimensional heat equation is a function of three variables, it is not easy to create a visual representation of the solution. We can graph the solution for fixed values of t, which amounts to snapshots of the heat distributions at fixed times. These snapshots show how the heat is distributed over a two-dimensional surface as time progresses. The graph of the preceding solution at time $t = 0$ appears in Figure **13.3.3**. As time progresses, the extremes level out, approaching zero as t approaches infinity.

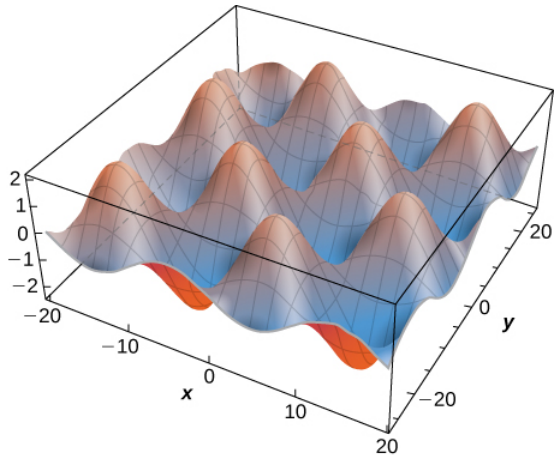


Figure **13.3.3**

If we consider the heat equation in one dimension, then it is possible to graph the solution over time. The heat equation in one dimension becomes

$$u_t = c^2 u_{xx},$$

where c^2 represents the thermal diffusivity of the material in question. A solution of this differential equation can be written in the form

$$u_m(x,t) = e^{-\pi^2 m^2 c^2 t} \sin(m\pi x)$$

where m is any positive integer. A graph of this solution using $m = 1$ appears in Figure **13.3.4**, where the initial temperature distribution over a wire of length 1 is given by $u(x,0) = \sin \pi x$. Notice that as time progresses, the wire cools off. This is seen because, from left to right, the highest temperature (which occurs in the middle of the wire) decreases and changes color from red to blue.

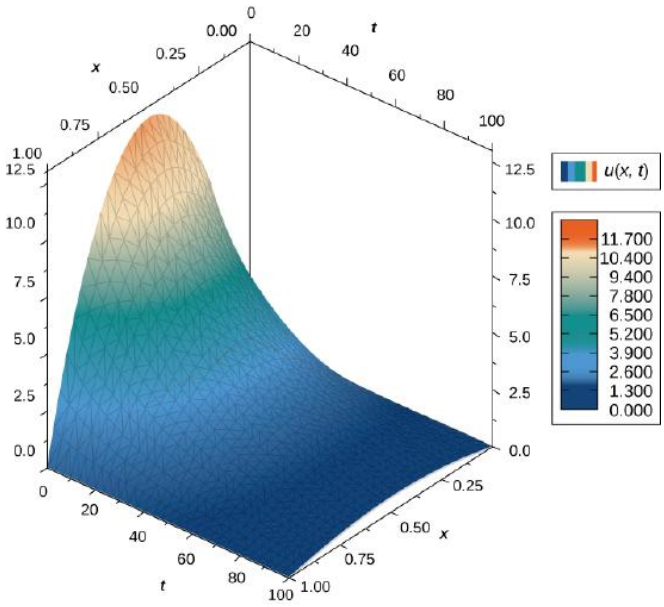
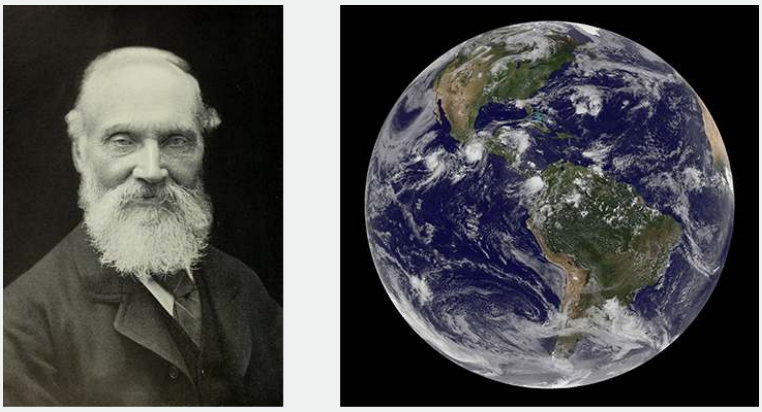


Figure **13.3.4**: Graph of a solution of the heat equation in one dimension over time.

Lord Kelvin and the Age of Earth

During the late 1800s, the scientists of the new field of geology were coming to the conclusion that Earth must be “millions and millions” of years old. At about the same time, Charles Darwin had published his treatise on evolution. Darwin’s view was that evolution needed many millions of years to take place, and he made a bold claim that the Weald chalk fields, where important fossils were found, were the result of **300** million years of erosion.



(a) (b)

Figure **13.3.5**: (a) William Thomson (Lord Kelvin), 1824-1907, was a British physicist and electrical engineer; (b) Kelvin used the heat diffusion equation to estimate the age of Earth (credit: modification of work by NASA).

At that time, eminent physicist William Thomson (Lord Kelvin) used an important partial differential equation, known as the heat diffusion equation, to estimate the age of Earth by determining how long it would take Earth to cool from molten rock to what we had at that time. His conclusion was a range of **20** to **400** million years, but most likely about **50** million years. For many decades, the proclamations of this irrefutable icon of science did not sit well with geologists or with Darwin.

- Read Kelvin’s paper on estimating the age of the Earth.

Kelvin made reasonable assumptions based on what was known in his time, but he also made several assumptions that turned out to be wrong. One incorrect assumption was that Earth is solid and that the cooling was therefore via conduction only, hence justifying the use of the diffusion equation. But the most serious error was a forgivable one—omission of the fact that Earth contains radioactive elements that continually supply heat beneath Earth’s mantle. The discovery of radioactivity came near the end of Kelvin’s life and he acknowledged that his calculation would have to be modified.

Kelvin used the simple one-dimensional model applied only to Earth’s outer shell, and derived the age from graphs and the roughly known temperature gradient near Earth’s surface. Let’s take a look at a more appropriate version of the diffusion equation in radial coordinates, which has the form

$$\frac{\partial T}{\partial t} = K\left[\frac{\partial^2 T}{\partial^2 r} + \frac{2}{r}\frac{\partial T}{\partial r}\right] \tag{13.3.10}$$

.

Here, $T(r,t)$ is temperature as a function of r (measured from the center of Earth) and time t. K is the heat conductivity—for molten rock, in this case. The standard method of solving such a partial differential equation is by separation of variables, where we express the solution as the product of functions containing each variable separately. In this case, we would write the temperature as

$$T(r,t) = R(r)f(t).$$

1. Substitute this form into Equation 13.3.10 and, noting that $f(t)$ is constant with respect to distance (r) and $R(r)$ is constant with respect to time (t), show that

$$\frac{1}{f}\frac{\partial f}{\partial t} = \frac{K}{R}\left[\frac{\partial^2 R}{\partial r^2} + \frac{2}{r}\frac{\partial R}{\partial r}\right].$$

2. This equation represents the separation of variables we want. The left-hand side is only a function of t and the right-hand side is only a function of r, and they must be equal for all values of r and t. Therefore, they both must be equal to a constant. Let's call that constant $-\lambda^2$. (The convenience of this choice is seen on substitution.) So, we have

$$\frac{1}{f}\frac{\partial f}{\partial t} = -\lambda^2 \text{and} \frac{K}{R}\left[\frac{\partial^2 R}{\partial r^2} + \frac{2}{r}\frac{\partial R}{\partial r}\right] = -\lambda^2.$$

3. Now, we can verify through direct substitution for each equation that the solutions are $f(t) = Ae^{-\lambda^2 t}$ and $R(r) = B\left(\frac{\sin \alpha r}{r}\right) + C\left(\frac{\cos \alpha r}{r}\right)$, where $\alpha = \lambda/\sqrt{K}$. Note that $f(t) = Ae^{+\lambda n^2 t}$ is also a valid solution, so we could have chosen $+\lambda^2$for our constant. Can you see why it would not be valid for this case as time increases?
4. Let's now apply boundary conditions.
   a. The temperature must be finite at the center of Earth, $r = 0$. Which of the two constants, B or C, must therefore be zero to keep R finite at $r = 0$? (Recall that $\sin(\alpha r)/r \to \alpha =$ as $r \to 0$, but $\cos(\alpha r)/r$ behaves very differently.)
   b. Kelvin argued that when magma reaches Earth's surface, it cools very rapidly. A person can often touch the surface within weeks of the flow. Therefore, the surface reached a moderate temperature very early and remained nearly constant at a surface temperature T_s. For simplicity, let's set $T = 0$ at $r = R_E$ and find α such that this is the temperature there for all time t. (Kelvin took the value to be $300K \approx 80°F$. We can add this $300K$ constant to our solution later.) For this to be true, the sine argument must be zero at $r = R_E$. Note that α has an infinite series of values that satisfies this condition. Each value of α represents a valid solution (each with its own value for A). The total or general solution is the sum of all these solutions.
   c. At $t = 0$, we assume that all of Earth was at an initial hot temperature T_0 (Kelvin took this to be about $7000K$.) The application of this boundary condition involves the more advanced application of Fourier coefficients. As noted in part b. each value of α_n represents a valid solution, and the general solution is a sum of all these solutions. This results in a series solution:

$$T(r,t) = \left(\frac{T_0 R_E}{\pi}\right)\sum_n \frac{(-1)^{n-1}}{n} e^{-\lambda n^2 t}\frac{\sin(\alpha_n r)}{r}$$

   where $\alpha_n = n\pi/R_E$.

Note how the values of α_n come from the boundary condition applied in part b. The term $\frac{-1^{n-1}}{n}$ is the constant A_n for each term in the series, determined from applying the Fourier method. Letting $\beta = \frac{\pi}{R_E}$, examine the first few terms of this solution shown here and note how λ^2 in the exponential causes the higher terms to decrease quickly as time progresses:

$$T(r,t) = \frac{T_0 R_E}{\pi r}$$
$$\left(e^{-K\beta^2 t}(\sin \beta r) - \frac{1}{2}e^{-4K\beta^2 t}(\sin 2\beta r) + \frac{1}{3}e^{-9K\beta^2 t}(\sin 3\beta r) - \frac{1}{4}e^{-16K\beta^2 t}(\sin 4\beta r)\right.$$
$$\left. + \frac{1}{5}e^{-25K\beta^2 t}(\sin 5\beta r)\ldots\right).$$

Near time $t = 0$, many terms of the solution are needed for accuracy. Inserting values for the conductivity K and $\beta = \pi/R_E$ for time approaching merely thousands of years, only the first few terms make a significant contribution. Kelvin only needed to look at the solution near Earth's surface (Figure 13.3.6) and, after a long time, determine what time best yielded the estimated temperature gradient known during his era ($1°F$ increase per $50ft$). He simply chose a range of times with a gradient close to this value. In Figure 13.3.6, the solutions are plotted and scaled, with the $300 - K$ surface temperature added. Note that the center of Earth would be relatively cool. At the time, it was thought Earth must be solid.

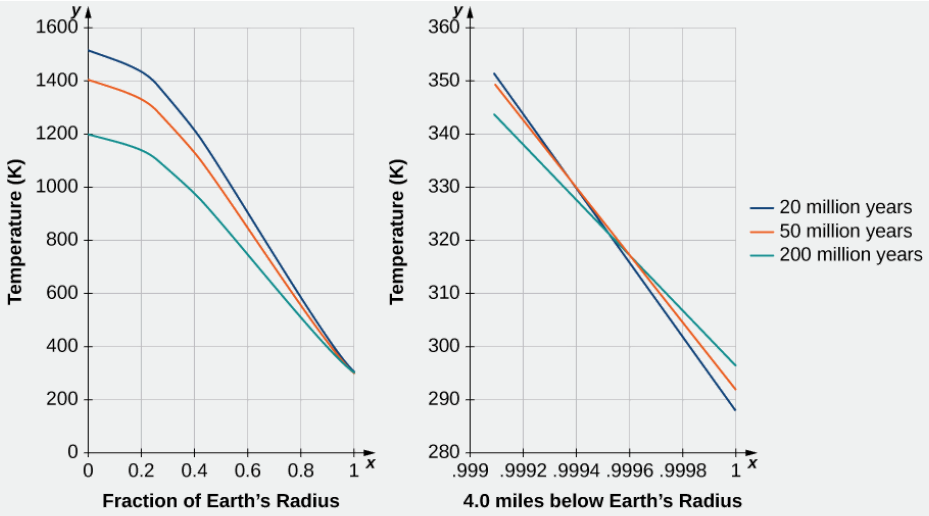


Figure **13.3.6**: Temperature versus radial distance from the center of Earth. (a) Kelvin's results, plotted to scale. (b) A close-up of the results at a depth of **4.0** miles below Earth's surface.

Epilog

On May **20, 1904,** physicist **Ernest Rutherford** spoke at the Royal Institution to announce a revised calculation that included the contribution of radioactivity as a source of Earth's heat. In Rutherford's own words:

> "I came into the room, which was half-dark, and presently spotted Lord Kelvin in the audience, and realized that I was in for trouble at the last part of my speech dealing with the age of the Earth, where my views conflicted with his. To my relief, Kelvin fell fast asleep, but as I came to the important point, I saw the old bird sit up, open an eye and cock a baleful glance at me.
>
> Then a sudden inspiration came, and I said Lord Kelvin had limited the age of the Earth, provided no new source [of heat] was discovered. That prophetic utterance referred to what we are now considering tonight, radium! Behold! The old boy beamed upon me."

Rutherford calculated an age for Earth of about **500** million years. Today's accepted value of Earth's age is about **4.6** billion years.

Key Concepts

- A partial derivative is a derivative involving a function of more than one independent variable.
- To calculate a partial derivative with respect to a given variable, treat all the other variables as constants and use the usual differentiation rules.
- Higher-order partial derivatives can be calculated in the same way as higher-order derivatives.

Key Equations

Partial derivative of f with respect to x

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

Partial derivative of f with respect to y

$$\frac{\partial f}{\partial y} = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}$$

Glossary

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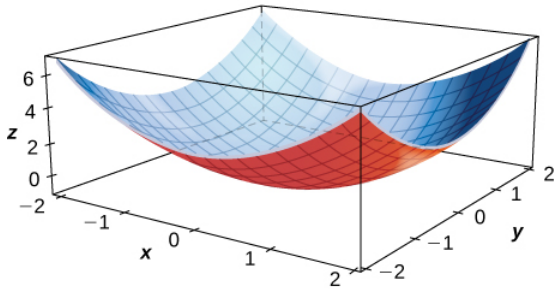
13.3E: Partial Derivatives (Exercises)

In the following exercise, calculate the partial derivative using the limit definitions only.

1) $\frac{\partial z}{\partial y}$ for $z = x^2 - 3xy + y^2$

For exercises 2 - 5, calculate the sign of the partial derivative using the graph of the surface.



2) $f_x(1,1)$

3) $f_x(-1,1)$

4) $f_y(1,1)$

5) $f_x(0,0)$

In exercises 6 - 16, calculate the requested partial derivatives.

6) $\frac{\partial z}{\partial x}$ for $z = \sin(3x)\cos(3y)$

7) $\frac{\partial z}{\partial y}$ for $z = \sin(3x)\cos(3y)$

8) $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for $z = x^8 e^3 y$

9) $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for $z = \ln(x^6 + y^4)$

10) Find $f_y(x,y)$ for $f(x,y) = e^{xy}\cos(x)\sin(y)$.

11) Let $z = e^{xy}$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

12) Let $z = \ln(\frac{x}{y})$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

13) Let $z = \tan(2x - y)$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

14) Let $z = \sinh(2x + 3y)$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

15) Let $f(x,y) = \arctan(\frac{y}{x})$. Evaluate $f_x(2,-2)$ and $f_y(2,-2)$.

16) Let $f(x,y) = \frac{xy}{x-y}$. Find $f_x(2,-2)$ and $f_y(2,-2)$.

17) Find $\frac{\partial z}{\partial x}$ at $(0,1)$ for $z = e^{-x}cos(y)$.

18) Given $f(x,y,z) = x^3yz^2$, find $\frac{\partial^2 f}{\partial x \partial y}$ and $f_z(1,1,1)$.

19) Given $f(x,y,z) = 2\sin(x+y)$, find $f_x(0,\frac{\pi}{2},-4)$, $f_y(0,\frac{\pi}{2},-4)$, and $f_z(0,\frac{\pi}{2},-4)$.

20) The area of a parallelogram with adjacent side lengths that are a and b, and in which the angle between these two sides is θ, is given by the function $A(a,b,\theta) = ba\sin(\theta)$.Find the rate of change of the area of the parallelogram with respect to the following:

a. Side a

b. Side b

c. Angle θ

21) Express the volume of a right circular cylinder as a function of two variables:

a. its radius r and its height h.

b. Show that the rate of change of the volume of the cylinder with respect to its radius is the product of its circumference multiplied by its height.

c. Show that the rate of change of the volume of the cylinder with respect to its height is equal to the area of the circular base.

22) Calculate $\dfrac{\partial w}{\partial z}$ for $w = z\sin(xy^2 + 2z)$.

In exercises 23 - 39, find the indicated higher-order partial derivatives.

23) $f_{xy}(x,y)$ for $f(x,y) = \ln(x-y)$

24) $f_{yx}(x,y)$ for $f(x,y) = \ln(x-y)$

25) Let $z = x^2 + 3xy + 2y^2$. Find $\dfrac{\partial^2 z}{\partial x^2}$ and $\dfrac{\partial^2 z}{\partial y^2}$.

26) Given $z = e^x \tan y$, find $\dfrac{\partial^2 z}{\partial x \partial y}$ and $\dfrac{\partial^2 z}{\partial y \partial x}$.

27) Given $f(x,y,z) = xyz$, find $f_{xyy}(x,y,z)$, $f_{yxy}(x,y,z)$, and $f_{yyx}(x,y,z)$.

28) Given $f(x,y,z) = e^{-2x}\sin(z^2 y)$, show that $f_{xyy}(x,y,z) = f_{yxy}(x,y,z)$.

29) Show that $z = \frac{1}{2}(e^y - e^{-y})\sin x$ is a solution of the differential equation $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0$.

30) Find $f_{xx}(x,y)$ for $f(x,y) = \frac{4x^2}{y} + \frac{y^2}{2x}$.

31) Let $f(x,y,z) = x^2y^3z - 3xy^2z^3 + 5x^2z - y^3z$. Find f_{xyz}.

32) Let $F(x,y,z) = x^3yz^2 - 2x^2yz + 3xz - 2y^3z$. Find $F_{xyz}(x,y,z)$.

33) Given $f(x,y) = x^2 + x - 3xy + y^3 - 5$, find all points at which $f_x(x,y) = f_y(x,y) = 0$ simultaneously.

34) Given $f(x,y) = 2x^2 + 2xy + y^2 + 2x - 3$, find all points at which $\dfrac{\partial f}{\partial x} = 0$ and $\dfrac{\partial f}{\partial y} = 0$ simultaneously.

35) Given $f(x,y) = y^3 - 3yx^2 - 3y^2 - 3x^2 + 1$, find all points on f at which $f_x(x,y) = f_y(x,y) = 0$ simultaneously.

36) Given $f(x,y) = 15x^3 - 3xy + 15y^3$, find all points at which $f_x(x,y) = f_y(x,y) = 0$ simultaneously.

37) Show that $z = e^x \sin y$ satisfies the equation $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0$.

38) Show that $f(x,y) = \ln(x^2 + y^2)$ solves Laplace's equation $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0$.

39) Show that $z = e^{-t}\cos(\frac{x}{c})$ satisfies the heat equation $\dfrac{\partial z}{\partial t} = -e^{-t}\cos(\frac{x}{c})$.

40) Find $\lim\limits_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x,y)}{\Delta x}$ for $f(x,y) = -7x - 2xy + 7y$.

41) Find $\lim\limits_{\Delta y \to 0} \dfrac{f(x,y+\Delta y) - f(x,y)}{\Delta y}$ for $f(x,y) = -7x - 2xy + 7y$.

42) Find $\lim\limits_{\Delta x \to 0} \dfrac{\Delta f}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{f(x+\Delta x,y) - f(x,y)}{\Delta x}$ for $f(x,y) = x^2y^2 + xy + y$.

43) Find $\lim\limits_{\Delta x \to 0} \dfrac{\Delta f}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{f(x+\Delta x,y) - f(x,y)}{\Delta x}$ for $f(x,y) = \sin(xy)$.

44) The function $P(T,V) = \dfrac{nRT}{V}$ gives the pressure at a point in a gas as a function of temperature T and volume V. The letters n and R are constants. Find $\dfrac{\partial P}{\partial V}$ and $\dfrac{\partial P}{\partial T}$, and explain what these quantities represent.

45) The equation for heat flow in the xy-plane is $\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$. Show that $f(x,y,t) = e^{-2t}\sin x \sin y$ is a solution.

46) The basic wave equation is $f_{tt} = f_{xx}$. Verify that $f(x,t) = \sin(x+t)$ and $f(x,t) = \sin(x-t)$ are solutions.

47) The law of cosines can be thought of as a function of three variables. Let x, y, and θ be two sides of any triangle where the angle θ is the included angle between the two sides. Then, $F(x,y,\theta) = x^2 + y^2 - 2xy\cos\theta$ gives the square of the third side of the triangle. Find $\frac{\partial F}{\partial \theta}$ and $\frac{\partial F}{\partial x}$ when $x = 2$, $y = 3$, and $\theta = \frac{\pi}{6}$.

48) Suppose the sides of a rectangle are changing with respect to time. The first side is changing at a rate of 2in./sec whereas the second side is changing at the rate of 4 in/sec. How fast is the diagonal of the rectangle changing when the first side measures 16 in. and the second side measures 20 in.? (Round answer to three decimal places.)

49) A **Cobb-Douglas production function** is $f(x,y) = 200x^{0.7}y^{0.3}$, where x and y represent the amount of labor and capital available. Let $x = 500$ and $y = 1000$. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at these values, which represent the marginal productivity of labor and capital, respectively.

50) The apparent temperature index is a measure of how the temperature feels, and it is based on two variables: h, which is relative humidity, and t, which is the air temperature.

$A = 0.885t - 22.4h + 1.20th - 0.544$. Find $\frac{\partial A}{\partial t}$ and $\frac{\partial A}{\partial h}$ when $t = 20°F$ and $h = 0.90$.

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13.4: Tangent Planes, Linear Approximations, and the Total Differential

Learning Objectives

- Determine the equation of a plane tangent to a given surface at a point.
- Use the tangent plane to approximate a function of two variables at a point.
- Explain when a function of two variables is differentiable.
- Use the total differential to approximate the change in a function of two variables.

In this section, we consider the problem of finding the tangent plane to a surface, which is analogous to finding the equation of a tangent line to a curve when the curve is defined by the graph of a function of one variable, $y = f(x)$. The slope of the tangent line at the point $x = a$ is given by $m = f'(a)$; what is the slope of a tangent plane? We learned about the equation of a plane in Equations of Lines and Planes in Space; in this section, we see how it can be applied to the problem at hand.

Tangent Planes

Intuitively, it seems clear that, in a plane, only one line can be tangent to a curve at a point. However, in three-dimensional space, many lines can be tangent to a given point. If these lines lie in the same plane, they determine the tangent plane at that point. A more intuitive way to think of a tangent plane is to assume the surface is smooth at that point (no corners). Then, a tangent line to the surface at that point in any direction does not have any abrupt changes in slope because the direction changes smoothly. Therefore, in a small-enough neighborhood around the point, a tangent plane touches the surface at that point only.

Definition: Tangent Lines

Let $P_0 = (x_0, y_0, z_0)$ be a point on a surface S, and let C be any curve passing through P_0 and lying entirely in S. If the tangent lines to all such curves C at P_0 lie in the same plane, then this plane is called the *tangent plane* to S at P_0 (Figure 13.4.1).

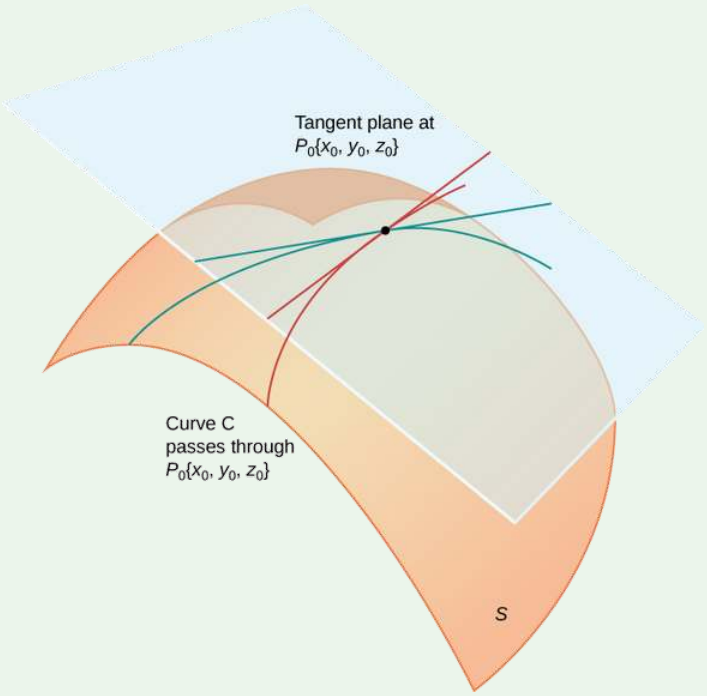

Figure 13.4.1: The tangent plane to a surface S at a point P_0 contains all the tangent lines to curves in S *that pass through* P_0.

For a tangent plane to a surface to exist at a point on that surface, it is sufficient for the function that defines the surface to be differentiable at that point. We define the term tangent plane here and then explore the idea intuitively.

Definition: Tangent Planes

Let S be a surface defined by a differentiable function $z = f(x, y)$, and let $P_0 = (x_0, y_0)$ be a point in the domain of f. Then, the equation of the tangent plane to S at P_0 is given by

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0). \tag{13.4.1}$$

To see why this formula is correct, let's first find two tangent lines to the surface S. The equation of the tangent line to the curve that is represented by the intersection of S with the vertical trace given by $x = x_0$ is $z = f(x_0, y_0) + f_y(x_0, y_0)(y - y_0)$. Similarly, the equation of the tangent line to the curve that is represented by the intersection of S with the vertical trace given by $y = y_0$ is $z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0)$. A parallel vector to the first tangent line is $\vec{\mathbf{a}} = \hat{\mathbf{j}} + f_y(x_0, y_0)\,\hat{\mathbf{k}}$; a parallel vector to the second tangent line is $\vec{\mathbf{b}} = \hat{\mathbf{i}} + f_x(x_0, y_0)\,\hat{\mathbf{k}}$. We can take the cross product of these two vectors:

$$\begin{aligned}
\vec{\mathbf{a}} \times \vec{\mathbf{b}} &= (\hat{\mathbf{j}} + f_y(x_0, y_0)\,\hat{\mathbf{k}}) \times (\hat{\mathbf{i}} + f_x(x_0, y_0)\,\hat{\mathbf{k}}) \\
&= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 1 & f_y(x_0, y_0) \\ 1 & 0 & f_x(x_0, y_0) \end{vmatrix} \\
&= f_x(x_0, y_0)\,\hat{\mathbf{i}} + f_y(x_0, y_0)\,\hat{\mathbf{j}} - \hat{\mathbf{k}}.
\end{aligned}$$

This vector is perpendicular to both lines and is therefore perpendicular to the tangent plane. We can use this vector as a normal vector to the tangent plane, along with the point $P_0 = (x_0, y_0, f(x_0, y_0))$ in the equation for a plane:

$$\begin{aligned}
\vec{\mathbf{n}} \cdot ((x - x_0)\,\hat{\mathbf{i}} + (y - y_0)\,\hat{\mathbf{j}} + (z - f(x_0, y_0))\,\hat{\mathbf{k}}) &= 0 \\
(f_x(x_0, y_0)\,\hat{\mathbf{i}} + f_y(x_0, y_0)\,\hat{\mathbf{j}} - \hat{\mathbf{k}}) \cdot ((x - x_0)\,\hat{\mathbf{i}} + (y - y_0)\,\hat{\mathbf{j}} + (z - f(x_0, y_0))\,\hat{\mathbf{k}}) &= 0 \\
f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - f(x_0, y_0)) &= 0.
\end{aligned}$$

Solving this equation for z gives Equation 13.4.1.

Example 13.4.1: Finding a Tangent Plane

Find the equation of the tangent plane to the surface defined by the function $f(x, y) = 2x^2 - 3xy + 8y^2 + 2x - 4y + 4$ at point $(2, -1)$.

Solution

First, we must calculate $f_x(x, y)$ and $f_y(x, y)$, then use Equation with $x_0 = 2$ and $y_0 = -1$:

$$\begin{aligned}
f_x(x, y) &= 4x - 3y + 2 \\
f_y(x, y) &= -3x + 16y - 4 \\
f(2, -1) &= 2(2)^2 - 3(2)(-1) + 8(-1)^2 + 2(2) - 4(-1) + 4 = 34 \\
f_x(2, -1) &= 4(2) - 3(-1) + 2 = 13 \\
f_y(2, -1) &= -3(2) + 16(-1) - 4 = -26.
\end{aligned}$$

Then Equation 13.4.1 becomes

$$\begin{aligned}
z &= f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\
z &= 34 + 13(x - 2) - 26(y - (-1)) \\
z &= 34 + 13x - 26 - 26y - 26 \\
z &= 13x - 26y - 18.
\end{aligned}$$

(See the following figure).

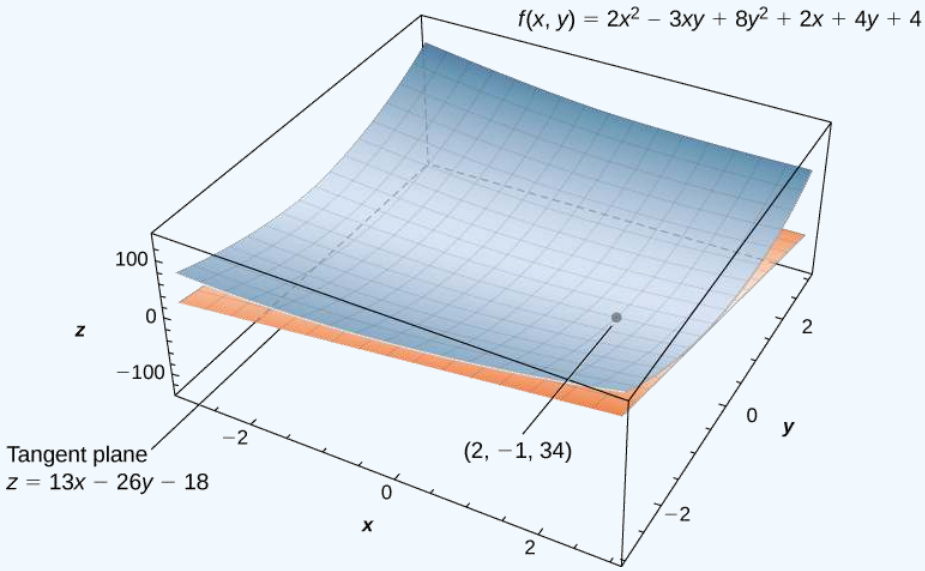

Figure 13.4.2: Calculating the equation of a tangent plane to a given surface at a given point.

? Exercise 13.4.1

Find the equation of the tangent plane to the surface defined by the function $f(x,y) = x^3 - x^2y + y^2 - 2x + 3y - 2$ at point $(-1,3)$.

✓ Example 13.4.2: Finding Another Tangent Plane

Find the equation of the tangent plane to the surface defined by the function $f(x,y) = \sin(2x)\cos(3y)$ at the point $(\pi/3, \pi/4)$.

Solution

First, calculate $f_x(x,y)$ and $f_y(x,y)$, then use Equation 13.4.1 with $x_0 = \pi/3$ and $y_0 = \pi/4$:

$$f_x(x,y) = 2\cos(2x)\cos(3y)$$
$$f_y(x,y) = -3\sin(2x)\sin(3y)$$
$$f\left(\frac{\pi}{3},\frac{\pi}{4}\right) = \sin\left(2\left(\frac{\pi}{3}\right)\right)\cos\left(3\left(\frac{\pi}{4}\right)\right) = \left(\frac{\sqrt{3}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\sqrt{6}}{4}$$
$$f_x\left(\frac{\pi}{3},\frac{\pi}{4}\right) = 2\cos\left(2\left(\frac{\pi}{3}\right)\right)\cos\left(3\left(\frac{\pi}{4}\right)\right) = 2\left(-\frac{1}{2}\right)\left(-\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}$$
$$f_y\left(\frac{\pi}{3},\frac{\pi}{4}\right) = -3\sin\left(2\left(\frac{\pi}{3}\right)\right)\sin\left(3\left(\frac{\pi}{4}\right)\right) = -3\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = -\frac{3\sqrt{6}}{4}.$$

Then Equation 13.4.1 becomes

$$\begin{aligned} z &= f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) \\ &= -\frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{2}\left(x-\frac{\pi}{3}\right) - \frac{3\sqrt{6}}{4}\left(y-\frac{\pi}{4}\right) \\ &= \frac{\sqrt{2}}{2}x - \frac{3\sqrt{6}}{4}y - \frac{\sqrt{6}}{4} - \frac{\pi\sqrt{2}}{6} + \frac{3\pi\sqrt{6}}{16} \end{aligned}$$

A tangent plane to a surface does not always exist at every point on the surface. Consider the piecewise function

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}. \tag{13.4.2}$$

The graph of this function follows.

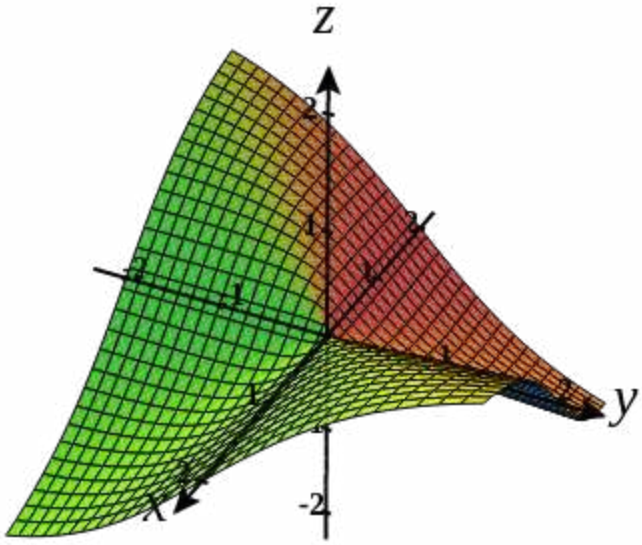

Figure* 13.4.3*: *Graph of a function that does not have a tangent plane at the origin. Dynamic figure powered by CalcPlot3D.*

If either $x = 0$ or $y = 0$, then $f(x, y) = 0$, so the value of the function does not change on either the x- or y-axis. Therefore, $f_x(x, 0) = f_y(0, y) = 0$, so as either x or y approach zero, these partial derivatives stay equal to zero. Substituting them into Equation gives $z = 0$ as the equation of the tangent line. However, if we approach the origin from a different direction, we get a different story. For example, suppose we approach the origin along the line $y = x$. If we put $y = x$ into the original function, it becomes

$$f(x, x) = \frac{x(x)}{\sqrt{x^2 + (x)^2}} = \frac{x^2}{\sqrt{2x^2}} = \frac{|x|}{\sqrt{2}}.$$

When $x > 0$, the slope of this curve is equal to $\sqrt{2}/2$; when $x < 0$, the slope of this curve is equal to $-(\sqrt{2}/2)$. This presents a problem. In the definition of tangent plane, we presumed that all tangent lines through point P (in this case, the origin) lay in the same plane. This is clearly not the case here. When we study differentiable functions, we will see that this function is not differentiable at the origin.

Linear Approximations

Recall from Linear Approximations and Differentials that the formula for the linear approximation of a function $f(x)$ at the point $x = a$ is given by

$$y \approx f(a) + f'(a)(x - a).$$

The diagram for the linear approximation of a function of one variable appears in the following graph.

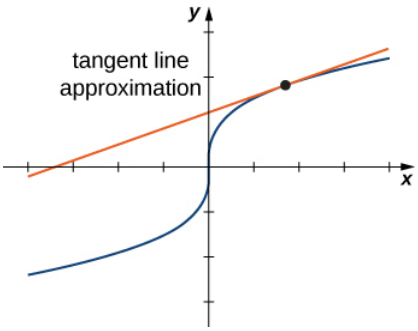

Figure 13.4.4: Linear approximation of a function in one variable.

The tangent line can be used as an approximation to the function $f(x)$ for values of x reasonably close to $x = a$. When working with a function of two variables, the tangent line is replaced by a tangent plane, but the approximation idea is much the same.

Definition: Linear Approximation

Given a function $z = f(x, y)$ with continuous partial derivatives that exist at the point (x_0, y_0), the linear approximation of f at the point (x_0, y_0) is given by the equation

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0). \tag{13.4.3}$$

Notice that this equation also represents the tangent plane to the surface defined by $z = f(x, y)$ at the point (x_0, y_0). The idea behind using a linear approximation is that, if there is a point (x_0, y_0) at which the precise value of $f(x, y)$ is known, then for values of (x, y) reasonably close to (x_0, y_0), the linear approximation (i.e., tangent plane) yields a value that is also reasonably close to the exact value of $f(x, y)$ (Figure 13.4.5). Furthermore the plane that is used to find the linear approximation is also the tangent plane to the surface at the point (x_0, y_0).

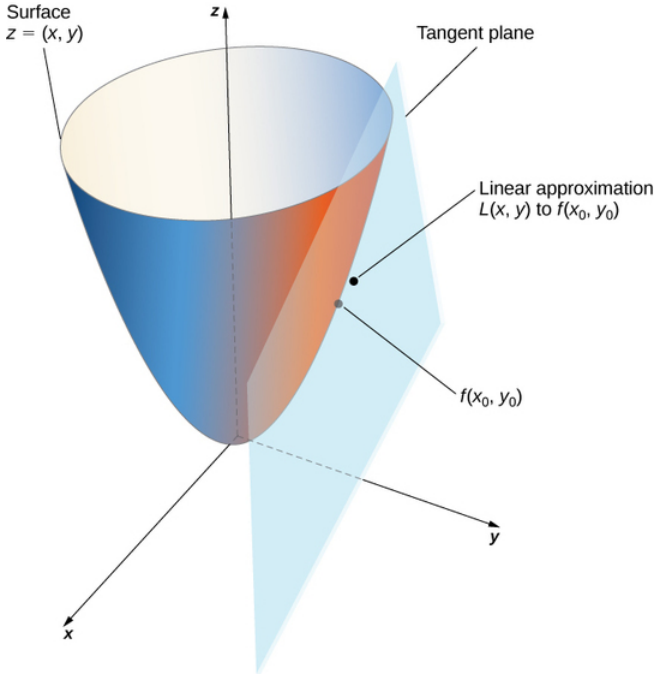

Figure 13.4.5: Using a tangent plane for linear approximation at a point.

Example 13.4.3: Using a Tangent Plane Approximation

Given the function $f(x, y) = \sqrt{41 - 4x^2 - y^2}$, approximate $f(2.1, 2.9)$ using point $(2, 3)$ for (x_0, y_0). What is the approximate value of $f(2.1, 2.9)$ to four decimal places?

Solution

To apply Equation 13.4.3, we first must calculate $f(x_0, y_0)$, $f_x(x_0, y_0)$, and $f_y(x_0, y_0)$ using $x_0 = 2$ and $y_0 = 3$:

$$f(x_0, y_0) = f(2, 3) = \sqrt{41 - 4(2)^2 - (3)^2} = \sqrt{41 - 16 - 9} = \sqrt{16} = 4$$

$$f_x(x, y) = -\frac{4x}{\sqrt{41 - 4x^2 - y^2}} \text{ so } f_x(x_0, y_0) = -\frac{4(2)}{\sqrt{41 - 4(2)^2 - (3)^2}} = -2$$

$$f_y(x, y) = -\frac{y}{\sqrt{41 - 4x^2 - y^2}} \text{ so } f_y(x_0, y_0) = -\frac{3}{\sqrt{41 - 4(2)^2 - (3)^2}} = -\frac{3}{4}.$$

Now we substitute these values into Equation 13.4.3:

$$\begin{aligned} L(x, y) &= f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\ &= 4 - 2(x - 2) - \frac{3}{4}(y - 3) \\ &= \frac{41}{4} - 2x - \frac{3}{4}y. \end{aligned}$$

Last, we substitute $x = 2.1$ and $y = 2.9$ into $L(x, y)$:

$$L(2.1, 2.9) = \frac{41}{4} - 2(2.1) - \frac{3}{4}(2.9) = 10.25 - 4.2 - 2.175 = 3.875.$$

The approximate value of $f(2.1, 2.9)$ to four decimal places is

$$f(2.1, 2.9) = \sqrt{41 - 4(2.1)^2 - (2.9)^2} = \sqrt{14.95} \approx 3.8665,$$

which corresponds to a 0.2 error in approximation.

? Exercise 13.4.2

Given the function $f(x, y) = e^{5-2x+3y}$, approximate $f(4.1, 0.9)$ using point $(4, 1)$ for (x_0, y_0). What is the approximate value of $f(4.1, 0.9)$ to four decimal places?

Differentiability

When working with a function $y = f(x)$ of one variable, the function is said to be differentiable at a point $x = a$ if $f'(a)$ exists. Furthermore, if a function of one variable is differentiable at a point, the graph is "smooth" at that point (i.e., no corners exist) and a tangent line is well-defined at that point.

The idea behind differentiability of a function of two variables is connected to the idea of smoothness at that point. In this case, a surface is considered to be smooth at point P if a tangent plane to the surface exists at that point. If a function is differentiable at a point, then a tangent plane to the surface exists at that point. Recall the formula (Equation 13.4.1) for a tangent plane at a point (x_0, y_0) is given by

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

For a tangent plane to exist at the point (x_0, y_0), the partial derivatives must therefore exist at that point. However, this is not a sufficient condition for smoothness, as was illustrated in Figure 13.4.3. In that case, the partial derivatives existed at the origin, but the function also had a corner on the graph at the origin.

Definition: Differentiable Functions

A function $f(x, y)$ is *differentiable* at a point $P(x_0, y_0)$ if, for all points (x, y) in a δ disk around P, we can write

$$f(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + E(x, y), \tag{13.4.4}$$

where the error term E satisfies

$$\lim_{(x,y)\to(x_0,y_0)} \frac{E(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 0. \tag{13.4.5}$$

The last term in Equation 13.4.4 is referred to as the **error term.** It represents how closely the tangent plane comes to the surface in a small neighborhood (δ disk) of point P. For the function f to be differentiable at P, the function must be smooth—that is, the graph of f must be close to the tangent plane for points near P.

Example 13.4.4: Demonstrating Differentiability

Show that the function $f(x,y) = 2x^2 - 4y$ is differentiable at point $(2,-3)$.

Solution

First, we calculate $f(x_0,y_0)$, $f_x(x_0,y_0)$, and $f_y(x_0,y_0)$ using $x_0 = 2$ and $y_0 = -3$, then we use Equation 13.4.4:

$$\begin{aligned} f(2,-3) &= 2(2)^2 - 4(-3) = 8 + 12 = 20 \\ f_x(2,-3) &= 4(2) = 8 \\ f_y(2,-3) &= -4. \end{aligned}$$

Therefore $m_1 = 8$ and $m_2 = -4$, and Equation 13.4.4 becomes

$$\begin{aligned} f(x,y) &= f(2,-3) + f_x(2,-3)(x-2) + f_y(2,-3)(y+3) + E(x,y) \\ 2x^2 - 4y &= 20 + 8(x-2) - 4(y+3) + E(x,y) \\ 2x^2 - 4y &= 20 + 8x - 16 - 4y - 12 + E(x,y) \\ 2x^2 - 4y &= 8x - 4y - 8 + E(x,y) \\ E(x,y) &= 2x^2 - 8x + 8. \end{aligned}$$

Next, we calculate the limit in Equation 13.4.5:

$$\begin{aligned} \lim_{(x,y)\to(x_0,y_0)} \frac{E(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} &= \lim_{(x,y)\to(2,-3)} \frac{2x^2-8x+8}{\sqrt{(x-2)^2+(y+3)^2}} \\ &= \lim_{(x,y)\to(2,-3)} \frac{2(x^2-4x+4)}{\sqrt{(x-2)^2+(y+3)^2}} \\ &= \lim_{(x,y)\to(2,-3)} \frac{2(x-2)^2}{\sqrt{(x-2)^2+(y+3)^2}} \\ &\le \lim_{(x,y)\to(2,-3)} \frac{2((x-2)^2+(y+3)^2)}{\sqrt{(x-2)^2+(y+3)^2}} \\ &= \lim_{(x,y)\to(2,-3)} 2\sqrt{(x-2)^2+(y+3)^2} \\ &= 0. \end{aligned}$$

Since $E(x,y) \ge 0$ for any value of x or y, the original limit must be equal to zero. Therefore, $f(x,y) = 2x^2 - 4y$ is differentiable at point $(2,-3)$.

Exercise 13.4.3

Show that the function $f(x,y) = 3x - 4y^2$ is differentiable at point $(-1,2)$.

This function from (Equation 13.4.2)

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

is not differentiable at the origin (Figure 13.4.3). We can see this by calculating the partial derivatives. This function appeared earlier in the section, where we showed that $f_x(0,0) = f_y(0,0) = 0$. Substituting this information into Equations 13.4.4 and 13.4.5

using $x_0 = 0$ and $y_0 = 0$, we get

$$f(x,y) = f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) + E(x,y)$$
$$E(x,y) = \frac{xy}{\sqrt{x^2+y^2}}.$$

Calculating

$$\lim_{(x,y)\to(x_0,y_0)} \frac{E(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}}$$

gives

$$\lim_{(x,y)\to(x_0,y_0)} \frac{E(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = \lim_{(x,y)\to(0,0)} \frac{\frac{xy}{\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}}$$
$$= \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}.$$

Depending on the path taken toward the origin, this limit takes different values. Therefore, the limit does not exist and the function f is not differentiable at the origin as shown in the following figure.

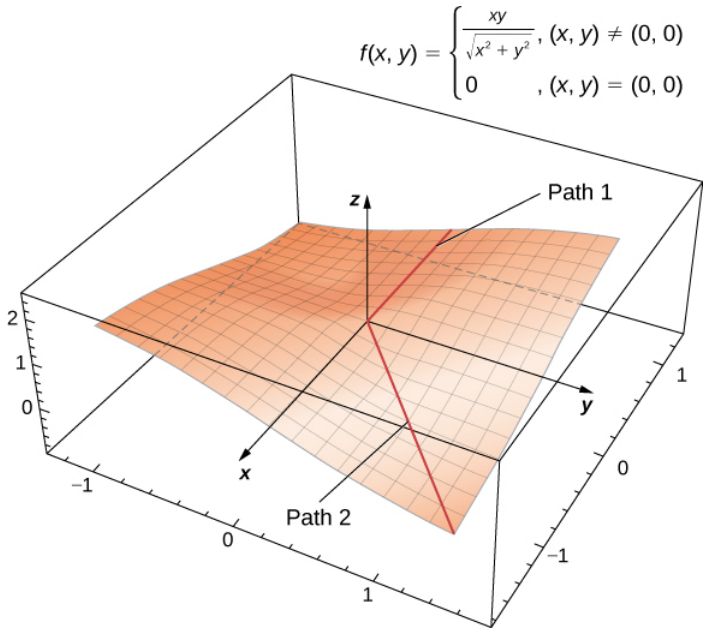

Figure 13.4.6: This function $f(x,y)$ (Equation 13.4.2) is not differentiable at the origin.

Differentiability and continuity for functions of two or more variables are connected, the same as for functions of one variable. In fact, with some adjustments of notation, the basic theorem is the same.

Theorem 13.4.1: Differentiability Implies Continuity

Let $z = f(x,y)$ be a function of two variables with (x_0,y_0) in the domain of f. If $f(x,y)$ is differentiable at (x_0,y_0), then $f(x,y)$ is continuous at (x_0,y_0).

Theorem 13.4.1 shows that if a function is differentiable at a point, then it is continuous there. However, if a function is continuous at a point, then it is not necessarily differentiable at that point. For example, the function discussed above (Equation 13.4.2)

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

is *continuous* at the origin, but it is *not differentiable* at the origin. This observation is also similar to the situation in single-variable calculus.

We can further explore the connection between continuity and differentiability at a point. This next theorem says that if the function and its partial derivatives are continuous at a point, the function is differentiable.

Theorem 13.4.2: Continuity of First Partials Implies Differentiability

Let $z = f(x, y)$ be a function of two variables with (x_0, y_0) in the domain of f. If $f(x, y)$, $f_x(x, y)$, and $f_y(x, y)$ all exist in a neighborhood of (x_0, y_0) and are continuous at (x_0, y_0), then $f(x, y)$ is differentiable there.

Recall that earlier we showed that the function in Equation 13.4.2 was not differentiable at the origin. Let's calculate the partial derivatives f_x and f_y:

$$\frac{\partial f}{\partial x} = \frac{y^3}{(x^2 + y^2)^{3/2}}$$

and

$$\frac{\partial f}{\partial y} = \frac{x^3}{(x^2 + y^2)^{3/2}}.$$

The contrapositive of the preceding theorem states that if a function is not differentiable, then at least one of the hypotheses must be false. Let's explore the condition that $f_x(0, 0)$ must be continuous. For this to be true, it must be true that

$$\lim_{(x,y)\to(0,0)} f_x(x, y) = f_x(0, 0)$$

therefor

$$\lim_{(x,y)\to(0,0)} f_x(x, y) = \lim_{(x,y)\to(0,0)} \frac{y^3}{(x^2 + y^2)^{3/2}}.$$

Let $x = ky$. Then

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{y^3}{(x^2 + y^2)^{3/2}} &= \lim_{y\to 0} \frac{y^3}{((ky)^2 + y^2)^{3/2}} \\
&= \lim_{y\to 0} \frac{y^3}{(k^2y^2 + y^2)^{3/2}} \\
&= \lim_{y\to 0} \frac{y^3}{|y|^3(k^2 + 1)^{3/2}} \\
&= \frac{1}{(k^2 + 1)^{3/2}} \lim_{y\to 0} \frac{|y|}{y}.
\end{aligned}$$

If $y > 0$, then this expression equals $1/(k^2 + 1)^{3/2}$; if $y < 0$, then it equals $-(1/(k^2 + 1)^{3/2})$. In either case, the value depends on k, so the limit fails to exist.

Differentials

In Linear Approximations and Differentials we first studied the concept of differentials. The differential of y, written dy, is defined as $f'(x)dx$. The differential is used to approximate $\Delta y = f(x + \Delta x) - f(x)$, where $\Delta x = dx$. Extending this idea to the linear approximation of a function of two variables at the point (x_0, y_0) yields the formula for the total differential for a function of two variables.

Definition: Total Differential

Let $z = f(x, y)$ be a function of two variables with (x_0, y_0) in the domain of f, and let Δx and Δy be chosen so that $(x_0 + \Delta x, y_0 + \Delta y)$ is also in the domain of f. If f is differentiable at the point (x_0, y_0), then the differentials dx and dy are defined as

$$dx = \Delta x$$

and

$$dy = \Delta y.$$

The differential dz, also called the ***total differential*** of $z = f(x, y)$ at (x_0, y_0), is defined as

$$dz = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy. \tag{13.4.6}$$

Notice that the symbol ∂ is not used to denote the total differential; rather, d appears in front of z. Now, let's define $\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$. We use dz to approximate Δz, so

$$\Delta z \approx dz = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy.$$

Therefore, the differential is used to approximate the change in the function $z = f(x_0, y_0)$ at the point (x_0, y_0) for given values of Δx and Δy. Since $\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$, this can be used further to approximate $f(x + \Delta x, y + \Delta y)$:

$$f(x + \Delta x, y + \Delta y) = f(x, y) + \Delta z \approx f(x, y) + f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y.$$

See the following figure.

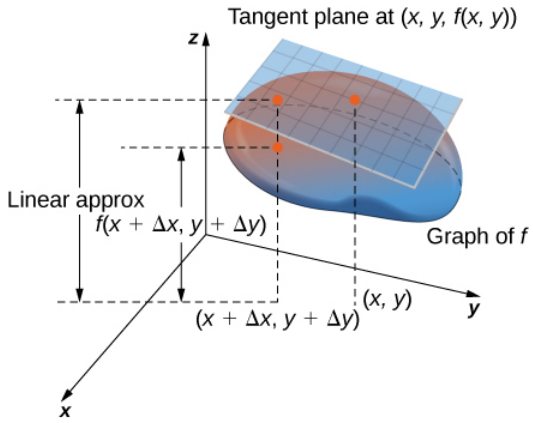

Figure 13.4.7: The linear approximation is calculated via the formula $f(x + \Delta x, y + \Delta y) \approx f(x, y) + f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y$.

One such application of this idea is to determine error propagation. For example, if we are manufacturing a gadget and are off by a certain amount in measuring a given quantity, the differential can be used to estimate the error in the total volume of the gadget.

✔ Example 13.4.5: Approximation by Differentials

Find the differential dz of the function $f(x, y) = 3x^2 - 2xy + y^2$ and use it to approximate Δz at point $(2, -3)$. Use $\Delta x = 0.1$ and $\Delta y = -0.05$. What is the exact value of Δz?

Solution

First, we must calculate $f(x_0, y_0)$, $f_x(x_0, y_0)$, and $f_y(x_0, y_0)$ using $x_0 = 2$ and $y_0 = -3$:

$$\begin{aligned} f(x_0, y_0) &= f(2, -3) = 3(2)^2 - 2(2)(-3) + (-3)^2 = 12 + 12 + 9 = 33 \\ f_x(x, y) &= 6x - 2y \\ f_y(x, y) &= -2x + 2y \\ f_x(x_0, y_0) &= fx(2, -3) \\ &= 6(2) - 2(-3) = 12 + 6 = 18 \\ f_y(x_0, y_0) &= f_y(2, -3) \\ &= -2(2) + 2(-3) \\ &= -4 - 6 = -10. \end{aligned}$$

Then, we substitute these quantities into Equation 13.4.6:

$$dz = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy$$
$$dz = 18(0.1) - 10(-0.05) = 1.8 + 0.5 = 2.3.$$

This is the approximation to $\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$. The exact value of Δz is given by

$$\begin{aligned}\Delta z &= f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)\\ &= f(2 + 0.1, -3 - 0.05) - f(2, -3)\\ &= f(2.1, -3.05) - f(2, -3)\\ &= 2.3425.\end{aligned}$$

? Exercise 13.4.4

Find the differential dz of the function $f(x, y) = 4y^2 + x^2y - 2xy$ and use it to approximate Δz at point $(1, -1)$. Use $\Delta x = 0.03$ and $\Delta y = -0.02$. What is the exact value of Δz?

Differentiability of a Function of Three Variables

All of the preceding results for differentiability of functions of two variables can be generalized to functions of three variables. First, the definition:

Definition: Differentiability at a Point

A function $f(x, y, z)$ is differentiable at a point $P(x_0, y_0, z_0)$ if for all points (x, y, z) in a δ disk around P we can write

$$\begin{aligned}f(x, y) = f(x_0, y_0, z_0) + f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0)\\ + f_z(x_0, y_0, z_0)(z - z_0) + E(x, y, z),\end{aligned}$$

where the error term E satisfies

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} \frac{E(x, y, z)}{\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}} = 0.$$

If a function of three variables is differentiable at a point (x_0, y_0, z_0), then it is continuous there. Furthermore, continuity of first partial derivatives at that point guarantees differentiability.

Key Concepts

- The analog of a tangent line to a curve is a tangent plane to a surface for functions of two variables.
- Tangent planes can be used to approximate values of functions near known values.
- A function is differentiable at a point if it is ”smooth” at that point (i.e., no corners or discontinuities exist at that point).
- The total differential can be used to approximate the change in a function $z = f(x_0, y_0)$ at the point (x_0, y_0) for given values of Δx and Δy.

Key Equations

- **Tangent plane**

  $z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$

- **Linear approximation**

  $L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$

- **Total differential**

  $dz = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy.$

- **Differentiability (two variables)**

  $f(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + E(x, y),$

  where the error term E satisfies

$$\lim_{(x,y)\to(x_0,y_0)} \frac{E(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 0.$$

- **Differentiability (three variables)**

  $$f(x,y) = f(x_0,y_0,z_0) + f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0) + E(x,y,z),$$

  where the error term E satisfies

  $$\lim_{(x,y,z)\to(x_0,y_0,z_0)} \frac{E(x,y,z)}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}} = 0.$$

Glossary

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13.4E: Tangent Planes, Linear Approximations, and the Total Differential (Exercises)

In exercises 1 - 2, find a unit normal vector to the surface at the indicated point.

1) $f(x,y) = x^3, \quad (2,-1,8)$

2) $\ln\left(\frac{x}{y-z}\right) = 0$ when $x = y = 1$

In exercises 3 - 7, find a normal vector and a tangent vector at point P.

3) $x^2 + xy + y^2 = 3, \quad P(-1,-1)$

4) $(x^2+y^2)^2 = 9(x^2-y^2), \quad P(\sqrt{2},1)$

5) $xy^2 - 2x^2 + y + 5x = 6, \quad P(4,2)$

6) $2x^3 - x^2y^2 = 3x - y - 7, \quad P(1,-2)$

7) $ze^{x^2-y^2} - 3 = 0, \quad P(2,2,3)$

In exercises 8 - 19, find the equation for the tangent plane to the surface at the indicated point. (Hint: If the given function is not already solved for z, start by solving it for z in terms of x and y.)

8) $-8x - 3y - 7z = -19, \quad P(1,-1,2)$

9) $z = -9x^2 - 3y^2, \quad P(2,1,-39)$

10) $x^2 + 10xyz + y^2 + 8z^2 = 0, \quad P(-1,-1,-1)$

11) $z = \ln(10x^2 + 2y^2 + 1), \quad P(0,0,0)$

12) $z = e^{7x^2+4y^2}, \quad P(0,0,1)$

13) $xy + yz + zx = 11, \quad P(1,2,3)$

14) $x^2 + 4y^2 = z^2, \quad P(3,2,5)$

15) $x^3 + y^3 = 3xyz, \quad P(1,2,\frac{3}{2})$

16) $z = axy, \quad P(1,\frac{1}{a},1)$

17) $z = \sin x + \sin y + \sin(x+y), \quad P(0,0,0)$

18) $h(x,y) = \ln\sqrt{x^2+y^2}, \quad P(3,4)$

19) $z = x^2 - 2xy + y^2, \quad P(1,2,1)$

In exercises 20 - 25, find parametric equations for the normal line to the surface at the indicated point. (Recall that to find the equation of a line in space, you need a point on the line, $P_0(x_0,y_0,z_0)$, and a vector $\vec{\mathbf{v}} = \langle a,b,c\rangle$ that is parallel to the line. Then the equations of the line are: $x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct.$)

20) $-3x + 9y + 4z = -4, \quad P(1,-1,2)$

21) $z = 5x^2 - 2y^2, \quad P(2,1,18)$

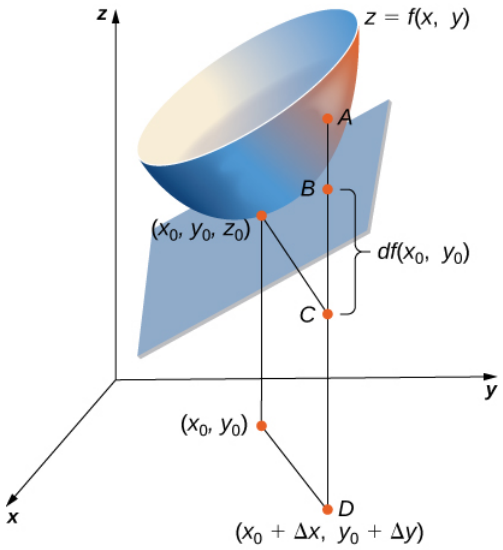
22) $x^2 - 8xyz + y^2 + 6z^2 = 0, \quad P(1,1,1)$

23) $z = \ln(3x^2 + 7y^2 + 1), \quad P(0,0,0)$

24) $z = e^{4x^2+6y^2}, \quad P(0,0,1)$

25) $z = x^2 - 2xy + y^2$ at point $P(1,2,1)$

In exercises 26 - 28, use the figure shown here.



26) The length of line segment AC is equal to what mathematical expression?

27) The length of line segment BC is equal to what mathematical expression?

28) Using the figure, explain what the length of line segment AB represents.

29) Show that $f(x,y) = e^{xy}x$ is differentiable at point $(1,0)$.

30) Show that $f(x,y) = x^2 + 3y$ is differentiable at every point. In other words, show that $\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y) = f_x\Delta x + f_y\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y$, where both ε_1 and ε_2 approach zero as $(\Delta x, \Delta y)$ approaches $(0,0)$.

31) Find the total differential of each function:

a. $z = x^3 + y^3 - 5$
b. $z = e^{xy}$
c. $z = y\cos x + \sin y$
d. $P = t^2 + 3t + tu^3$
e. $w = e^y\cos(x) + z^2$

32) a. Find the total differential dz of the function $z = \dfrac{xy}{y+x}$ and then
b. State its value where x changes from 10 to 10.5 and y changes from 15 to 13.

33) Let $z = f(x,y) = xe^y$. State its total differential. Then compute Δz from $P(1,2)$ to $Q(1.05, 2.1)$ and then find the approximate change in z, dz, from point P to point Q. Recall $\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y)$, and dz and Δz should be approximately equal, if dx and dy are both reasonably small.

34) The volume of a right circular cylinder is given by $V(r,h) = \pi r^2 h$. Find the differential dV. Interpret the formula geometrically.

35) See the preceding problem. Use differentials to estimate the amount of aluminum in an enclosed aluminum can with diameter $8.0cm$ and height $12cm$ if the aluminum is 0.04 cm thick.

36) Use the differential dz to approximate the change in $z = \sqrt{4 - x^2 - y^2}$ as (x,y) moves from point $(1,1)$ to point $(1.01, 0.97)$. Compare this approximation with the actual change in the function.

37) Let $z = f(x,y) = x^2 + 3xy - y^2$. Find the exact change in the function and the approximate change in the function as x changes from 2.00 to 2.05 and y changes from 3.00 to 2.96.

38) The centripetal acceleration of a particle moving in a circle is given by $a(r,v) = \frac{v^2}{r}$, where v is the velocity and r is the radius of the circle. Approximate the maximum percent error in measuring the acceleration resulting from errors of 3% in v and 2% in r. (Recall that the percentage error is the ratio of the amount of error over the original amount. So, in this case, the percentage error in a is given by $\frac{da}{a}$.)

39) The radius r and height h of a right circular cylinder are measured with possible errors of 4% and 5%, respectively. Approximate the maximum possible percentage error in measuring the volume (Recall that the percentage error is the ratio of the amount of error over the original amount. So, in this case, the percentage error in V is given by $\frac{dV}{V}$.)

40) The base radius and height of a right circular cone are measured as 10 in. and 25 in., respectively, with a possible error in measurement of as much as 0.1 in. each. Use differentials to estimate the maximum error in the calculated volume of the cone.

41) The **electrical resistance** R produced by wiring resistors R_1 and R_2 in parallel can be calculated from the formula $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$. If R_1 and R_2 are measured to be 7Ω and 6Ω, respectively, and if these measurements are accurate to within 0.05Ω, estimate the maximum possible error in computing R. (The symbol Ω represents an ohm, the unit of electrical resistance.)

42) The area of an ellipse with axes of length $2a$ and $2b$ is given by the formula $A = \pi ab$. Approximate the percent change in the area when a increases by 2% and b increases by 1.5%.

43) The period T of a **simple pendulum** with small oscillations is calculated from the formula $T = 2\pi\sqrt{\frac{L}{g}}$, where L is the length of the pendulum and g is the acceleration resulting from gravity. Suppose that L and g have errors of, at most, 0.5% and 0.1%, respectively. Use differentials to approximate the maximum percentage error in the calculated value of T.

44) **Electrical power** P is given by $P = \frac{V^2}{R}$, where V is the voltage and R is the resistance. Approximate the maximum percentage error in calculating power if $120V$ is applied to a $2000 - \Omega$ resistor and the possible percent errors in measuring V and R are 3% and 4%, respectively.

For exercises 45 - 49, find the linear approximation of each function at the indicated point.

45) $f(x, y) = x\sqrt{y}, \quad P(1, 4)$

46) $f(x, y) = e^x \cos y; \quad P(0, 0)$

47) $f(x, y) = \arctan(x + 2y), \quad P(1, 0)$

48) $f(x, y) = \sqrt{20 - x^2 - 7y^2}, \quad P(2, 1)$

49) $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}, \quad P(3, 2, 6)$

50) [T] Find the equation of the tangent plane to the surface $f(x, y) = x^2 + y^2$ at point $(1, 2, 5)$, and graph the surface and the tangent plane at the point.

51) [T] Find the equation for the tangent plane to the surface at the indicated point, and graph the surface and the tangent plane: $z = \ln(10x^2 + 2y^2 + 1), \quad P(0, 0, 0)$.

52) [T] Find the equation of the tangent plane to the surface $z = f(x, y) = \sin(x + y^2)$ at point $\left(\frac{\pi}{4}, 0, \frac{\sqrt{2}}{2}\right)$, and graph the surface and the tangent plane.

Contributors

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- Paul Seeburger (Monroe Community College) created all but part e of exercise 31.

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13.5: The Chain Rule for Functions of Multiple Variables

Learning Objectives

- State the chain rules for one or two independent variables.
- Use tree diagrams as an aid to understanding the chain rule for several independent and intermediate variables.
- Perform implicit differentiation of a function of two or more variables.

In single-variable calculus, we found that one of the most useful differentiation rules is the chain rule, which allows us to find the derivative of the composition of two functions. The same thing is true for multivariable calculus, but this time we have to deal with more than one form of the chain rule. In this section, we study extensions of the chain rule and learn how to take derivatives of compositions of functions of more than one variable.

Chain Rules for One or Two Independent Variables

Recall that the chain rule for the derivative of a composite of two functions can be written in the form

$$\frac{d}{dx}\Big(f(g(x))\Big) = f'\big(g(x)\big)g'(x).$$

In this equation, both $f(x)$ and $g(x)$ are functions of one variable. Now suppose that f is a function of two variables and g is a function of one variable. Or perhaps they are both functions of two variables, or even more. How would we calculate the derivative in these cases? The following theorem gives us the answer for the case of one independent variable.

Theorem 13.5.1: Chain Rule for One Independent Variable

Suppose that $x = g(t)$ and $y = h(t)$ are differentiable functions of t and $z = f(x, y)$ is a differentiable function of x and y. Then $z = f(x(t), y(t))$ is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}, \tag{13.5.1}$$

where the ordinary derivatives are evaluated at t and the partial derivatives are evaluated at (x, y).

Proof

The proof of this theorem uses the definition of differentiability of a function of two variables. Suppose that f is differentiable at the point $P(x_0, y_0)$, where $x_0 = g(t_0)$ and $y_0 = h(t_0)$ for a fixed value of t_0. We wish to prove that $z = f\big(x(t), y(t)\big)$ is differentiable at $t = t_0$ and that Equation 13.5.1 holds at that point as well.

Since f is differentiable at P, we know that

$$z(t) = f(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + E(x, y),$$

where

$$\lim_{(x,y)\to(x_0,y_0)} \frac{E(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0.$$

We then subtract $z_0 = f(x_0, y_0)$ from both sides of this equation:

$$\begin{aligned} z(t) - z(t_0) &= f(x(t), y(t)) - f(x(t_0), y(t_0)) \\ &= f_x(x_0, y_0)(x(t) - x(t_0)) + f_y(x_0, y_0)(y(t) - y(t_0)) + E(x(t), y(t)). \end{aligned}$$

Next, we divide both sides by $t - t_0$:

$$\frac{z(t) - z(t_0)}{t - t_0} = f_x(x_0, y_0)\frac{x(t) - x(t_0)}{t - t_0} + f_y(x_0, y_0)\frac{y(t) - y(t_0)}{t - t_0} + \frac{E(x(t), y(t))}{t - t_0}.$$

Then we take the limit as t approaches t_0:

$$\lim_{t\to t_0}\frac{z(t)-z(t_0)}{t-t_0} = f_x(x_0,y_0)\lim_{t\to t_0}\left(\frac{x(t)-x(t_0)}{t-t_0}\right) + f_y(x_0,y_0)\lim_{t\to t_0}\left(\frac{y(t)-y(t_0)}{t-t_0}\right) + \lim_{t\to t_0}\frac{E(x(t),y(t))}{t-t_0}.$$

The left-hand side of this equation is equal to dz/dt, which leads to

$$\frac{dz}{dt} = f_x(x_0,y_0)\frac{dx}{dt} + f_y(x_0,y_0)\frac{dy}{dt} + \lim_{t\to t_0}\frac{E(x(t),y(t))}{t-t_0}.$$

The last term can be rewritten as

$$\lim_{t\to t_0}\frac{E(x(t),y(t))}{t-t_0} = \lim_{t\to t_0}\frac{E(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}}\frac{\sqrt{(x-x_0)^2+(y-y_0)^2}}{t-t_0}) = \lim_{t\to t_0}\left(\frac{E(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}}\right) \lim_{t\to t_0}\left(\frac{\sqrt{(x-x_0)^2+(y-y_0)^2}}{t-t_0}\right).$$

As t approaches t_0, $\big(x(t),y(t)\big)$ approaches $\big(x(t_0),y(t_0)\big)$, so we can rewrite the last product as

$$\lim_{(x,y)\to(x_0,y_0)}\frac{E(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}}\lim_{(x,y)\to(x_0,y_0)}\left(\frac{\sqrt{(x-x_0)^2+(y-y_0)^2}}{t-t_0}\right).$$

Since the first limit is equal to zero, we need only show that the second limit is finite:

$$\lim_{(x,y)\to(x_0,y_0)}\frac{\sqrt{(x-x_0)^2+(y-y_0)^2}}{t-t_0} = \lim_{(x,y)\to(x_0,y_0)}\sqrt{\frac{(x-x_0)^2+(y-y_0)^2}{(t-t_0)^2}} = \lim_{(x,y)\to(x_0,y_0)}\sqrt{\left(\frac{x-x_0}{t-t_0}\right)^2+\left(\frac{y-y_0}{t-t_0}\right)^2} = \sqrt{\left[\lim_{(x,y)\to(x_0,y_0)}\left(\frac{x-x_0}{t-t_0}\right)\right]^2+\left[\lim_{(x,y)\to(x_0,y_0)}\left(\frac{y-y_0}{t-t_0}\right)\right]^2}.$$

Since $x(t)$ and $y(t)$ are both differentiable functions of t, both limits inside the last radical exist. Therefore, this value is finite. This proves the chain rule at $t=t_0$; the rest of the theorem follows from the assumption that all functions are differentiable over their entire domains.

□

Closer examination of Equation 13.5.1 reveals an interesting pattern. The first term in the equation is $\frac{\partial f}{\partial x}\cdot\frac{dx}{dt}$ and the second term is $\frac{\partial f}{\partial y}\cdot\frac{dy}{dt}$. Recall that when multiplying fractions, cancelation can be used. If we treat these derivatives as fractions, then each product "simplifies" to something resembling $\partial f/dt$. The variables x and y that disappear in this simplification are often called **intermediate variables**: they are independent variables for the function f, but are dependent variables for the variable t. Two terms appear on the right-hand side of the formula, and f is a function of two variables. This pattern works with functions of more than two variables as well, as we see later in this section.

Example 13.5.1: Using the Chain Rule

Calculate dz/dt for each of the following functions:

a. $z=f(x,y)=4x^2+3y^2, \quad x=x(t)=\sin t, \quad y=y(t)=\cos t$
b. $z=f(x,y)=\sqrt{x^2-y^2}, \quad x=x(t)=e^{2t}, \quad y=y(t)=e^{-t}$

Solution

a. To use the chain rule, we need four quantities—$\partial z/\partial x$, $\partial z/\partial y$, dx/dt, and dy/dt:

- $\dfrac{\partial z}{\partial x} = 8x$
- $\dfrac{dx}{dt} = \cos t$
- $\dfrac{\partial z}{\partial y} = 6y$
- $\dfrac{dy}{dt} = -\sin t$

Now, we substitute each of these into Equation 13.5.1:

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt} \\ &= (8x)(\cos t) + (6y)(-\sin t) \\ &= 8x\cos t - 6y\sin t.\end{aligned}$$

This answer has three variables in it. To reduce it to one variable, use the fact that $x(t) = \sin t$ and $y(t) = \cos t$. We obtain

$$\begin{aligned}\frac{dz}{dt} &= 8x\cos t - 6y\sin t \\ &= 8(\sin t)\cos t - 6(\cos t)\sin t \\ &= 2\sin t\cos t.\end{aligned}$$

This derivative can also be calculated by first substituting $x(t)$ and $y(t)$ into $f(x, y)$, then differentiating with respect to t:

$$\begin{aligned}z = f(x,y) &= f\big(x(t), y(t)\big) \\ &= 4(x(t))^2 + 3(y(t))^2 \\ &= 4\sin^2 t + 3\cos^2 t.\end{aligned}$$

Then

$$\begin{aligned}\frac{dz}{dt} &= 2(4\sin t)(\cos t) + 2(3\cos t)(-\sin t) \\ &= 8\sin t\cos t - 6\sin t\cos t \\ &= 2\sin t\cos t,\end{aligned}$$

which is the same solution. However, it may not always be this easy to differentiate in this form.

b. To use the chain rule, we again need four quantities—$\partial z/\partial x, \partial z/dy, dx/dt$, and dy/dt :

- $\dfrac{\partial z}{\partial x} = \dfrac{x}{\sqrt{x^2 - y^2}}$
- $\dfrac{dx}{dt} = 2e^{2t}$
- $\dfrac{\partial z}{\partial y} = \dfrac{-y}{\sqrt{x^2 - y^2}}$
- $\dfrac{dy}{dt} = -e^{-t}.$

We substitute each of these into Equation 13.5.1:

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt} \\ &= \left(\frac{x}{\sqrt{x^2 - y^2}}\right)(2e^{2t}) + \left(\frac{-y}{\sqrt{x^2 - y^2}}\right)(-e^{-t}) \\ &= \frac{2xe^{2t} - ye^{-t}}{\sqrt{x^2 - y^2}}.\end{aligned}$$

To reduce this to one variable, we use the fact that $x(t) = e^{2t}$ and $y(t) = e^{-t}$. Therefore,

$$\begin{aligned}\frac{dz}{dt} &= \frac{2xe^2t + ye^{-t}}{\sqrt{x^2 - y^2}} \\ &= \frac{2(e^{2t})e^{2t} + (e^{-t})e^{-t}}{\sqrt{e^{4t} - e^{-2t}}} \\ &= \frac{2e^{4t} + e^{-2t}}{\sqrt{e^{4t} - e^{-2t}}}.\end{aligned}$$

To eliminate negative exponents, we multiply the top by e^{2t} and the bottom by $\sqrt{e^{4t}}$:

$$\begin{aligned}\frac{dz}{dt} &= \frac{2e^{4t} + e^{-2t}}{\sqrt{e^{4t} - e^{-2t}}} \cdot \frac{e^{2t}}{\sqrt{e^{4t}}} \\ &= \frac{2e^{6t} + 1}{\sqrt{e^{8t} - e^{2t}}} \\ &= \frac{2e^{6t} + 1}{\sqrt{e^{2t}(e^{6t} - 1)}} \\ &= \frac{2e^{6t} + 1}{e^t\sqrt{e^{6t} - 1}}.\end{aligned}$$

Again, this derivative can also be calculated by first substituting $x(t)$ and $y(t)$ into $f(x, y)$, then differentiating with respect to t:

$$\begin{aligned}z &= f(x, y) \\ &= f(x(t), y(t)) \\ &= \sqrt{(x(t))^2 - (y(t))^2} \\ &= \sqrt{e^{4t} - e^{-2t}} \\ &= (e^{4t} - e^{-2t})^{1/2}.\end{aligned}$$

Then

$$\begin{aligned}\frac{dz}{dt} &= \frac{1}{2}(e^{4t} - e^{-2t})^{-1/2}\left(4e^{4t} + 2e^{-2t}\right) \\ &= \frac{2e^{4t} + e^{-2t}}{\sqrt{e^{4t} - e^{-2t}}}.\end{aligned}$$

This is the same solution.

? Exercise 13.5.1

Calculate dz/dt given the following functions. Express the final answer in terms of t.

$$\begin{aligned}z = f(x, y) &= x^2 - 3xy + 2y^2, \\ x = x(t) &= 3\sin 2t, \\ y = y(t) &= 4\cos 2t\end{aligned}$$

It is often useful to create a visual representation of Equation 13.5.1 for the chain rule. This is called a **tree diagram** for the chain rule for functions of one variable and it provides a way to remember the formula (Figure **13.5.1**). This diagram can be expanded for functions of more than one variable, as we shall see very shortly.

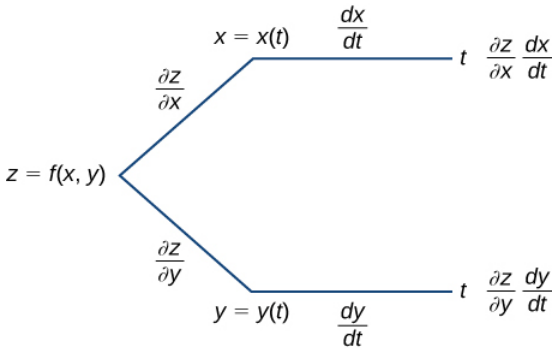

Figure 13.5.1: Tree diagram for the case $\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$.

In this diagram, the leftmost corner corresponds to $z = f(x, y)$. Since f has two **independent variables**, there are two lines coming from this corner. The upper branch corresponds to the variable x and the lower branch corresponds to the variable y. Since each of these variables is then dependent on one variable t, one branch then comes from x and one branch comes from y. Last, each of the branches on the far right has a label that represents the path traveled to reach that branch. The top branch is reached by following the x branch, then the t branch; therefore, it is labeled $(\partial z/\partial x) \times (dx/dt)$. The bottom branch is similar: first the y branch, then the t branch. This branch is labeled $(\partial z/\partial y) \times (dy/dt)$. To get the formula for dz/dt, add all the terms that appear on the rightmost side of the diagram. This gives us Equation.

In Chain Rule for Two Independent Variables, $z = f(x, y)$ is a function of x and y, and both $x = g(u, v)$ and $y = h(u, v)$ are functions of the independent variables u and v.

Theorem 13.5.2: Chain Rule for Two Independent Variables

Suppose $x = g(u, v)$ and $y = h(u, v)$ are differentiable functions of u and v, and $z = f(x, y)$ is a differentiable function of x and y. Then, $z = f(g(u, v), h(u, v))$ is a differentiable function of u and v, and

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \tag{13.5.2}$$

and

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}. \tag{13.5.3}$$

We can draw a tree diagram for each of these formulas as well as follows.

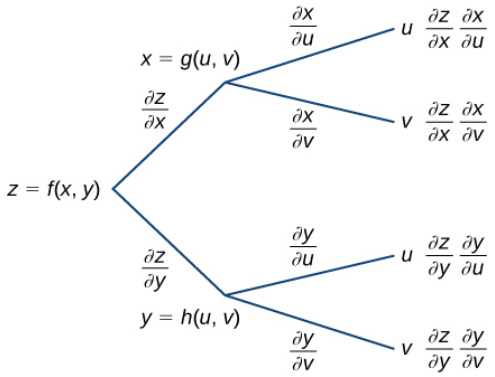

Figure 13.5.2: Tree diagram for $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$ and $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$.

To derive the formula for $\partial z/\partial u$, start from the left side of the diagram, then follow only the branches that end with u and add the terms that appear at the end of those branches. For the formula for $\partial z/\partial v$, follow only the branches that end with v and add the terms that appear at the end of those branches.

There is an important difference between these two chain rule theorems. In Chain Rule for One Independent Variable, the left-hand side of the formula for the derivative is not a partial derivative, but in Chain Rule for Two Independent Variables it is. The reason is that, in Chain Rule for One Independent Variable, z is ultimately a function of t alone, whereas in Chain Rule for Two Independent Variables, z is a function of both u and v.

Example 13.5.2: Using the Chain Rule for Two Variables

Calculate $\partial z/\partial u$ and $\partial z/\partial v$ using the following functions:

$$z = f(x,y) = 3x^2 - 2xy + y^2,\ x = x(u,v) = 3u + 2v,\ y = y(u,v) = 4u - v.$$

Solution

To implement the chain rule for two variables, we need six partial derivatives—$\partial z/\partial x$, $\partial z/\partial y$, $\partial x/\partial u$, $\partial x/\partial v$, $\partial y/\partial u$, and $\partial y/\partial v$:

$$\frac{\partial z}{\partial x} = 6x - 2y \qquad \frac{\partial z}{\partial y} = -2x + 2y$$
$$\frac{\partial x}{\partial u} = 3 \qquad \frac{\partial x}{\partial v} = 2$$
$$\frac{\partial y}{\partial u} = 4 \qquad \frac{\partial y}{\partial v} = -1.$$

To find $\partial z/\partial u$, we use Equation 13.5.2:

$$\begin{aligned}\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} \\ &= 3(6x - 2y) + 4(-2x + 2y) \\ &= 10x + 2y.\end{aligned}$$

Next, we substitute $x(u,v) = 3u + 2v$ and $y(u,v) = 4u - v$:

$$\begin{aligned}\frac{\partial z}{\partial u} &= 10x + 2y \\ &= 10(3u + 2v) + 2(4u - v) \\ &= 38u + 18v.\end{aligned}$$

To find $\partial z/\partial v$, we use Equation 13.5.3:

$$\begin{aligned}\frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \\ &= 2(6x - 2y) + (-1)(-2x + 2y) \\ &= 14x - 6y.\end{aligned}$$

Then we substitute $x(u,v) = 3u + 2v$ and $y(u,v) = 4u - v$:

$$\begin{aligned}\frac{\partial z}{\partial v} &= 14x - 6y \\ &= 14(3u + 2v) - 6(4u - v) \\ &= 18u + 34v\end{aligned}$$

Exercise 13.5.2

Calculate $\partial z/\partial u$ and $\partial z/\partial v$ given the following functions:

$$z = f(x,y) = \frac{2x - y}{x + 3y},\quad x(u,v) = e^{2u}\cos 3v,\quad y(u,v) = e^{2u}\sin 3v.$$

The Generalized Chain Rule

Now that we've see how to extend the original chain rule to functions of two variables, it is natural to ask: Can we extend the rule to more than two variables? The answer is yes, as the *generalized chain rule* states.

Theorem 13.5.3: Generalized Chain Rule

Let $w = f(x_1, x_2, \dots, x_m)$ be a differentiable function of m independent variables, and for each $i \in 1, \dots, m$, let $x_i = x_i(t_1, t_2, \dots, t_n)$ be a differentiable function of n independent variables. Then

$$\frac{\partial w}{\partial t_j} = \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_j} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_j} + \cdots + \frac{\partial w}{\partial x_m}\frac{\partial x_m}{\partial t_j}$$

for any $j \in 1, 2, \dots, n$.

In the next example we calculate the derivative of a function of three independent variables in which each of the three variables is dependent on two other variables.

Example 13.5.3: Using the Generalized Chain Rule

Calculate $\partial w/\partial u$ and $\partial w/\partial v$ using the following functions:

$$\begin{aligned} w &= f(x, y, z) = 3x^2 - 2xy + 4z^2 \\ x &= x(u, v) = e^u \sin v \\ y &= y(u, v) = e^u \cos v \\ z &= z(u, v) = e^u. \end{aligned}$$

Solution

The formulas for $\partial w/\partial u$ and $\partial w/\partial v$ are

$$\begin{aligned} \frac{\partial w}{\partial u} &= \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial u} \\ \frac{\partial w}{\partial v} &= \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial v}. \end{aligned}$$

Therefore, there are nine different partial derivatives that need to be calculated and substituted. We need to calculate each of them:

$$\begin{aligned} \frac{\partial w}{\partial x} &= 6x - 2y \quad \frac{\partial w}{\partial y} = -2x \quad \frac{\partial w}{\partial z} = 8z \\ \frac{\partial x}{\partial u} &= e^u \sin v \quad \frac{\partial y}{\partial u} = e^u \cos v \quad \frac{\partial z}{\partial u} = e^u \\ \frac{\partial x}{\partial v} &= e^u \cos v \quad \frac{\partial y}{\partial v} = -e^u \sin v \quad \frac{\partial z}{\partial v} = 0. \end{aligned}$$

Now, we substitute each of them into the first formula to calculate $\partial w/\partial u$:

$$\begin{aligned} \frac{\partial w}{\partial u} &= \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial u} \\ &= (6x - 2y)e^u \sin v - 2xe^u \cos v + 8ze^u, \end{aligned}$$

then substitute $x(u, v) = e^u \sin v$, $y(u, v) = e^u \cos v$, and $z(u, v) = e^u$ into this equation:

$$\begin{aligned} \frac{\partial w}{\partial u} &= (6x - 2y)e^u \sin v - 2xe^u \cos v + 8ze^u \\ &= (6e^u \sin v - 2eu \cos v)e^u \sin v - 2(e^u \sin v)e^u \cos v + 8e^{2u} \\ &= 6e^{2u}\sin^2 v - 4e^{2u}\sin v \cos v + 8e^{2u} \\ &= 2e^{2u}(3\sin^2 v - 2\sin v \cos v + 4). \end{aligned}$$

Next, we calculate $\partial w/\partial v$:

$$\begin{aligned}\frac{\partial w}{\partial v} &= \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v}+\frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial v}\\ &= (6x-2y)e^u\cos v-2x(-e^u\sin v)+8z(0),\end{aligned}$$

then we substitute $x(u,v)=e^u\sin v$, $y(u,v)=e^u\cos v$, and $z(u,v)=e^u$ into this equation:

$$\begin{aligned}\frac{\partial w}{\partial v} &= (6x-2y)e^u\cos v-2x(-e^u\sin v)\\ &= (6e^u\sin v-2e^u\cos v)e^u\cos v+2(e^u\sin v)(e^u\sin v)\\ &= 2e^{2u}\sin^2 v+6e^{2u}\sin v\cos v-2e^{2u}\cos^2 v\\ &= 2e^{2u}(\sin^2 v+\sin v\cos v-\cos^2 v).\end{aligned}$$

Exercise 13.5.3

Calculate $\partial w/\partial u$ and $\partial w/\partial v$ given the following functions:

$$\begin{aligned}w=f(x,y,z)&=\frac{x+2y-4z}{2x-y+3z}\\ x=x(u,v)&=e^{2u}\cos 3v\\ y=y(u,v)&=e^{2u}\sin 3v\\ z=z(u,v)&=e^{2u}.\end{aligned}$$

Example 13.5.4: Drawing a Tree Diagram

Create a tree diagram for the case when

$$w=f(x,y,z),\quad x=x(t,u,v),\quad y=y(t,u,v),\quad z=z(t,u,v)$$

and write out the formulas for the three partial derivatives of w.

Solution

Starting from the left, the function f has three independent variables: x, y, and z. Therefore, three branches must be emanating from the first node. Each of these three branches also has three branches, for each of the variables t, u, and v.

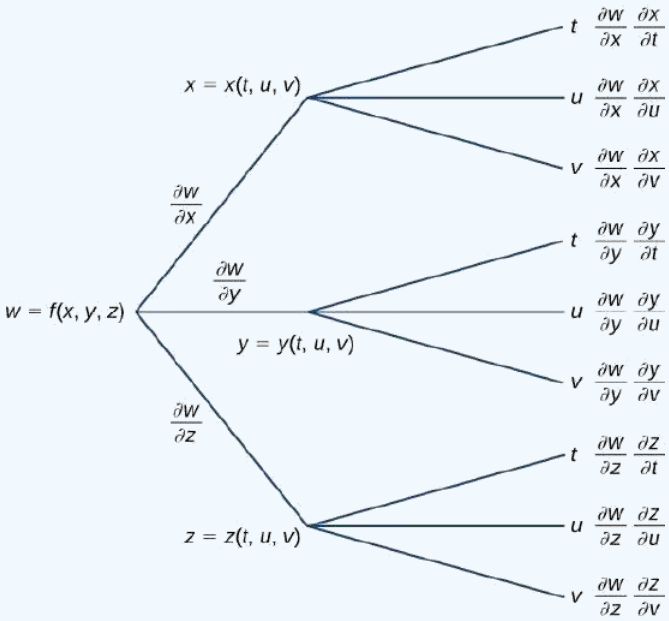

Figure 13.5.3: Tree diagram for a function of three variables, each of which is a function of three independent variables.

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The three formulas are

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$$
$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$$
$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}.$$

? Exercise 13.5.4

Create a tree diagram for the case when

$$w = f(x,y), \quad x = x(t,u,v), \quad y = y(t,u,v)$$

and write out the formulas for the three partial derivatives of w.

Implicit Differentiation

Recall from implicit differentiation provides a method for finding dy/dx when y is defined implicitly as a function of x. The method involves differentiating both sides of the equation defining the function with respect to x, then solving for dy/dx. Partial derivatives provide an alternative to this method.

Consider the ellipse defined by the equation $x^2 + 3y^2 + 4y - 4 = 0$ as follows.

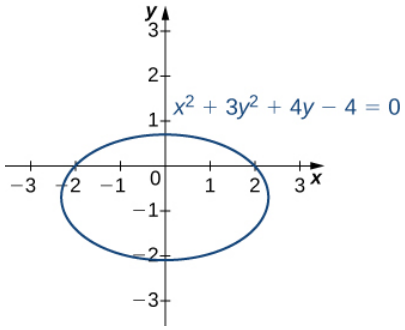

Figure 13.5.4: Graph of the ellipse defined by $x^2 + 3y^2 + 4y - 4 = 0$.

This equation implicitly defines y as a function of x. As such, we can find the derivative dy/dx using the method of implicit differentiation:

$$\frac{d}{dx}(x^2 + 3y^2 + 4y - 4) = \frac{d}{dx}(0)$$
$$2x + 6y\frac{dy}{dx} + 4\frac{dy}{dx} = 0$$
$$(6y + 4)\frac{dy}{dx} = -2x$$
$$\frac{dy}{dx} = -\frac{x}{3y + 2}$$

We can also define a function $z = f(x,y)$ by using the left-hand side of the equation defining the ellipse. Then $f(x,y) = x^2 + 3y^2 + 4y - 4$. The ellipse $x^2 + 3y^2 + 4y - 4 = 0$ can then be described by the equation $f(x,y) = 0$. Using this function and the following theorem gives us an alternative approach to calculating dy/dx.

Theorem 13.5.4: Implicit Differentiation of a Function of Two or More Variables

Suppose the function $z = f(x,y)$ defines y implicitly as a function $y = g(x)$ of x via the equation $f(x,y) = 0$. Then

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} \tag{13.5.4}$$

provided $f_y(x,y) \neq 0$.

If the equation $f(x,y,z) = 0$ defines z implicitly as a differentiable function of x and y, then

$$\frac{dz}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial z} \quad \text{and} \quad \frac{dz}{dy} = -\frac{\partial f/\partial y}{\partial f/\partial z} \tag{13.5.5}$$

as long as $f_z(x,y,z) \neq 0$.

Equation 13.5.4 is a direct consequence of Equation 13.5.2. In particular, if we assume that y is defined implicitly as a function of x via the equation $f(x,y) = 0$, we can apply the chain rule to find dy/dx :

$$\begin{aligned} \frac{d}{dx} f(x,y) &= \frac{d}{dx}(0) \\ \frac{\partial f}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} &= 0 \\ \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} &= 0. \end{aligned}$$

Solving this equation for dy/dx gives Equation 13.5.4. Equation 13.5.4 can be derived in a similar fashion.

Let's now return to the problem that we started before the previous theorem. Using Theorem 13.5.4 and the function $f(x,y) = x^2 + 3y^2 + 4y - 4$, we obtain

$$\begin{aligned} \frac{\partial f}{\partial x} &= 2x \\ \frac{\partial f}{\partial y} &= 6y + 4. \end{aligned}$$

Then Equation 13.5.4 gives

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{2x}{6y+4} = -\frac{x}{3y+2},$$

which is the same result obtained by the earlier use of implicit differentiation.

Example 13.5.5: Implicit Differentiation by Partial Derivatives

a. Calculate dy/dx if y is defined implicitly as a function of x via the equation $3x^2 - 2xy + y^2 + 4x - 6y - 11 = 0$. What is the equation of the tangent line to the graph of this curve at point $(2,1)$?

b. Calculate $\partial z/\partial x$ and $\partial z/\partial y$, given $x^2 e^y - yze^x = 0$.

Solution

a. Set $f(x,y) = 3x^2 - 2xy + y^2 + 4x - 6y - 11 = 0$, then calculate f_x and f_y : $f_x(x,y) = 6x - 2y + 4$ and $f_y(x,y) = -2x + 2y - 6$.

The derivative is given by

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = \frac{6x - 2y + 4}{-2x + 2y - 6} = \frac{3x - y + 2}{x - y + 3}.$$

The slope of the tangent line at point $(2,1)$ is given by

$$\left.\frac{dy}{dx}\right|_{(x,y)=(2,1)} = \frac{3(2) - 1 + 2}{2 - 1 + 3} = \frac{7}{4}$$

To find the equation of the tangent line, we use the point-slope form (Figure 13.5.5):

$$y - y_0 = m(x - x_0)$$
$$y - 1 = \frac{7}{4}(x - 2)$$
$$y = \frac{7}{4}x - \frac{7}{2} + 1$$
$$y = \frac{7}{4}x - \frac{5}{2}.$$

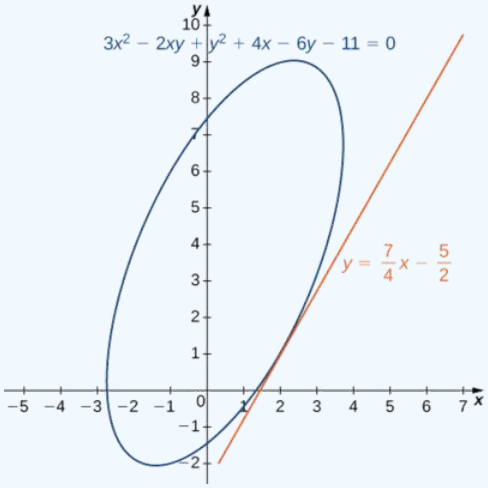

Figure 13.5.5: Graph of the rotated ellipse defined by $3x^2 - 2xy + y^2 + 4x - 6y - 11 = 0$.

b. We have $f(x, y, z) = x^2e^y - yze^x$. Therefore,

$$\frac{\partial f}{\partial x} = 2xe^y - yze^x$$
$$\frac{\partial f}{\partial y} = x^2e^y - ze^x$$
$$\frac{\partial f}{\partial z} = -ye^x$$

Using Equation 13.5.5,

$$\frac{\partial z}{\partial x} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{2xe^y - yze^x}{-ye^x} = \frac{2xe^y - yze^x}{ye^x} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{\partial f/\partial y}{\partial f/\partial z} = -\frac{x^2e^y - ze^x}{-ye^x} = \frac{x^2e^y - ze^x}{ye^x}$$

? Exercise 13.5.5

Find dy/dx if y is defined implicitly as a function of x by the equation $x^2 + xy - y^2 + 7x - 3y - 26 = 0$. What is the equation of the tangent line to the graph of this curve at point $(3, -2)$?

Key Concepts

- The chain rule for functions of more than one variable involves the partial derivatives with respect to all the independent variables.
- Tree diagrams are useful for deriving formulas for the chain rule for functions of more than one variable, where each independent variable also depends on other variables.

Key Equations

- **Chain rule, one independent variable**

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

- **Chain rule, two independent variables**

$$\frac{dz}{du} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u} \quad \frac{dz}{dv} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$$

- **Generalized chain rule**

$$\frac{\partial w}{\partial t_j} = \frac{\partial w}{\partial x_1} \frac{\partial x_1}{\partial t_j} + \frac{\partial w}{\partial x_2} \frac{\partial x_1}{\partial t_j} + \cdots + \frac{\partial w}{\partial x_m} \frac{\partial x_m}{\partial t_j}$$

Glossary

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13.5E: The Chain Rule for Functions of Multiple Variables (Exercises)

In exercises 1 - 6, use the information provided to solve the problem.

1) Let $w(x,y,z) = xy\cos z$, where $x = t, y = t^2$, and $z = \arcsin t$. Find $\frac{dw}{dt}$.

2) Let $w(t,v) = e^{tv}$ where $t = r + s$ and $v = rs$. Find $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$.

3) If $w = 5x^2 + 2y^2, \quad x = -3u + v$, and $y = u - 4v$, find $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$.

4) If $w = xy^2, x = 5\cos(2t)$, and $y = 5\sin(2t)$, find $\frac{\partial w}{\partial t}$.

5) If $f(x,y) = xy, x = r\cos\theta$, and $y = r\sin\theta$, find $\frac{\partial f}{\partial r}$ and express the answer in terms of r and θ.

6) Suppose $f(x,y) = x + y, u = e^x \sin y, \quad x = t^2$ and $y = \pi t$, where $x = r\cos\theta$ and $y = r\sin\theta$. Find $\frac{\partial f}{\partial \theta}$.

In exercises 7 - 12, find $\frac{dz}{dt}$ in two ways, first using the chain rule and then by direct substitution.

7) $z = x^2 + y^2, \quad x = t, y = t^2$

8) $z = \sqrt{x^2 + y^2}, \quad y = t^2, x = t$

9) $z = xy, \quad x = 1 - \sqrt{t}, y = 1 + \sqrt{t}$

10) $z = \frac{x}{y}, \quad x = e^t, y = 2e^t$

11) $z = \ln(x + y), \quad x = e^t, y = e^t$

12) $z = x^4, \quad x = t, y = t$

13) Let $w(x,y,z) = x^2 + y^2 + z^2, \quad x = cost, y = sint$, and $z = e^t$. Express w as a function of t and find $\frac{dw}{dt}$ directly. Then, find $\frac{dw}{dt}$ using the chain rule.

14) Let $z = x^2y$, where $x = t^2$ and $y = t^3$. Find $\frac{dz}{dt}$.

15) Let $u = e^x \sin y$, where $x = -\ln 2t$ and $y = \pi t$. Find $\frac{du}{dt}$ when $x = \ln 2$ and $y = \frac{\pi}{4}$.

In exercises 16 - 33, find $\frac{dy}{dx}$ using partial derivatives.

16) $\sin(6x) + \tan(8y) + 5 = 0$

17) $x^3 + y^2x - 3 = 0$

18) $\sin(x + y) + \cos(x - y) = 4$

19) $x^2 - 2xy + y^4 = 4$

20) $xe^y + ye^x - 2x^2y = 0$

21) $x^{2/3} + y^{2/3} = a^{2/3}$

22) $x\cos(xy) + y\cos x = 2$

23) $e^{xy} + ye^y = 1$

24) $x^2y^3 + \cos y = 0$

25) Find $\frac{dz}{dt}$ using the chain rule where $z = 3x^2y^3, \ x = t^4$, and $y = t^2$.

26) Let $z = 3\cos x - \sin(xy), x = \frac{1}{t}$, and $y = 3t$. Find $\frac{dz}{dt}$.

27) Let $z = e^{1-xy}, \ x = t^{1/3}$, and $y = t^3$. Find $\frac{dz}{dt}$.

28) Find $\frac{dz}{dt}$ by the chain rule where $z = \cosh^2(xy), \ x = \frac{1}{2}t$, and $y = e^t$.

29) Let $z = \frac{x}{y}$, $x = 2\cos u$, and $y = 3\sin v$. Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$.

30) Let $z = e^{x^2y}$, where $x = \sqrt{uv}$ and $y = \frac{1}{v}$. Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$.

31) If $z = xye^{x/y}$, $x = r\cos\theta$, and $y = r\sin\theta$, find $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$ when $r = 2$ and $\theta = \frac{\pi}{6}$.

32) Find $\frac{\partial w}{\partial s}$ if $w = 4x + y^2 + z^3$, $x = e^{rs^2}$, $y = \ln(\frac{r+s}{t})$, and $z = rst^2$.

33) If $w = \sin(xyz)$, $x = 1 - 3t$, $y = e^{1-t}$, and $z = 4t$, find $\frac{\partial w}{\partial t}$.

In exercises 34 - 36, use this information: A function $f(x,y)$ is said to be homogeneous of degree n if $f(tx,ty) = t^n f(x,y)$. For all homogeneous functions of degree n, the following equation is true: $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x,y)$. Show that the given function is homogeneous and verify that $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x,y)$.

34) $f(x,y) = 3x^2 + y^2$

35) $f(x,y) = \sqrt{x^2 + y^2}$

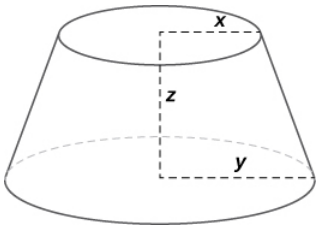
36) $f(x,y) = x^2y - 2y^3$

37) The volume of a right circular cylinder is given by $V(x,y) = \pi x^2 y$, where x is the radius of the cylinder and y is the cylinder height. Suppose x and y are functions of t given by $x = \frac{1}{2}t$ and $y = \frac{1}{3}t$ so that x and y are both increasing with time. How fast is the volume increasing when $x = 2$ and $y = 5$? Assume time is measured in seconds.

38) The pressure P of a gas is related to the volume and temperature by the formula $PV = kT$, where temperature is expressed in kelvins. Express the pressure of the gas as a function of both V and T. Find $\frac{dP}{dt}$ when $k = 1$, $\frac{dV}{dt} = 2$ cm^3/min, $\frac{dT}{dt} = 12$ K/min, $V = 20cm^3$, and $T = 20°F$.

39) The radius of a right circular cone is increasing at 3 cm/min whereas the height of the cone is decreasing at 2 cm/min. Find the rate of change of the volume of the cone when the radius is 13 cm and the height is 18 cm.

40) The volume of a frustum of a cone is given by the formula $V = \frac{1}{3}\pi z(x^2 + y^2 + xy)$, where x is the radius of the smaller circle, y is the radius of the larger circle, and z is the height of the frustum (see figure). Find the rate of change of the volume of this frustum when $x = 10$ in., $y = 12$ in., and $z = 18$ in.



41) A closed box is in the shape of a rectangular solid with dimensions x, y, and z. (Dimensions are in inches.) Suppose each dimension is changing at the rate of 0.5 in./min. Find the rate of change of the total surface area of the box when $x = 2$ in., $y = 3$ in., and $z = 1$ in.

42) The total resistance in a circuit that has three individual resistances represented by x, y, and z is given by the formula $R(x,y,z) = \frac{xyz}{yz + xz + xy}$. Suppose at a given time the x resistance is 100 Ω, the y resistance is 200 Ω, and the z resistance is 300 Ω. Also, suppose the x resistance is changing at a rate of 2 Ω/min, the y resistance is changing at the rate of 1 Ω/min, and the z resistance has no change. Find the rate of change of the total resistance in this circuit at this time.

43) The temperature T at a point (x,y) is $T(x,y)$ and is measured using the Celsius scale. A fly crawls so that its position after t seconds is given by $x = \sqrt{1+t}$ and $y = 2 + \frac{1}{3}t$, where x and y are measured in centimeters. The temperature function satisfies $T_x(2,3) = 4$ and $T_y(2,3) = 3$. How fast is the temperature increasing on the fly's path after 3 sec?

44) The x and y components of a fluid moving in two dimensions are given by the following functions: $u(x,y) = 2y$ and $v(x,y) = -2x$ with $x \geq 0$ and $y \geq 0$. The speed of the fluid at the point (x,y) is $s(x,y) = \sqrt{u(x,y)^2 + v(x,y)^2}$. Find $\frac{\partial s}{\partial x}$ and $\frac{\partial s}{\partial y}$ using the chain rule.

45) Let $u = u(x,y,z)$, where $x = x(w,t)$, $y = y(w,t)$, $z = z(w,t)$, $w = w(r,s)$, and $t = t(r,s)$. Use a tree diagram and the chain rule to find an expression for $\frac{\partial u}{\partial r}$.

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13.6: Directional Derivatives and the Gradient

In Partial Derivatives, we introduced the partial derivative. A function $z = f(x, y)$ has two partial derivatives: $\partial z/\partial x$ and $\partial z/\partial y$. These derivatives correspond to each of the independent variables and can be interpreted as instantaneous rates of change (that is, as slopes of a tangent line). For example, $\partial z/\partial x$ represents the slope of a tangent line passing through a given point on the surface defined by $z = f(x, y)$, assuming the tangent line is parallel to the x-**axis**. Similarly, $\partial z/\partial y$ represents the slope of the tangent line parallel to the y-**axis.** Now we consider the possibility of a tangent line parallel to neither axis.

Directional Derivatives

We start with the graph of a surface defined by the equation $z = f(x, y)$. Given a point (a, b) in the domain of f, we choose a direction to travel from that point. We measure the direction using an angle θ, which is measured counterclockwise in the xy-plane, starting at zero from the positive x-axis (Figure **13.6.1**). The distance we travel is h and the direction we travel is given by the unit vector $\vec{\mathbf{u}} = (\cos\theta)\,\hat{\mathbf{i}} + (\sin\theta)\,\hat{\mathbf{j}}$. Therefore, the z-coordinate of the second point on the graph is given by $z = f(a + h\cos\theta, b + h\sin\theta)$.

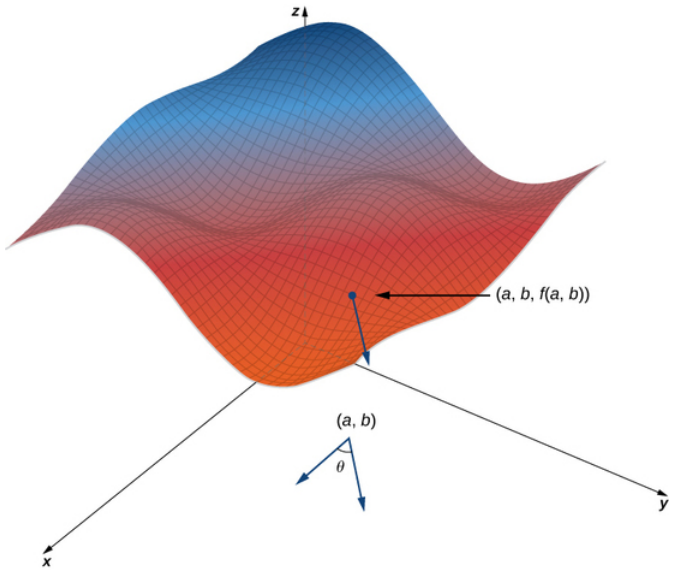

Figure **13.6.1**: Finding the directional derivative at a point on the graph of $z = f(x, y)$. The slope of the blue arrow on the graph indicates the value of the directional derivative at that point.

We can calculate the slope of the secant line by dividing the difference in z-**values** by the length of the line segment connecting the two points in the domain. The length of the line segment is h. Therefore, the slope of the secant line is

$$m_{sec} = \frac{f(a + h\cos\theta, b + h\sin\theta) - f(a, b)}{h} \tag{13.6.1}$$

To find the slope of the tangent line in the same direction, we take the limit as h approaches zero.

Definition: Directional Derivatives

Suppose $z = f(x, y)$ is a function of two variables with a domain of D. Let $(a, b) \in D$ and define $\vec{\mathbf{u}} = (\cos\theta)\,\hat{\mathbf{i}} + (\sin\theta)\,\hat{\mathbf{j}}$. Then the directional derivative of f in the direction of $\vec{\mathbf{u}}$ is given by

$$D_{\vec{\mathbf{u}}} f(a, b) = \lim_{h \to 0} \frac{f(a + h\cos\theta, b + h\sin\theta) - f(a, b)}{h} \tag{13.6.2}$$

provided the limit exists.

Equation 13.6.2 provides a formal definition of the directional derivative that can be used in many cases to calculate a directional derivative.

Note that since the point (a,b) is chosen randomly from the domain D of the function f, we can use this definition to find the directional derivative as a function of x and y.

That is,

$$D_{\vec{\mathbf{u}}}f(x,y)=\lim_{h\to 0}\frac{f(x+h\cos\theta,y+h\sin\theta)-f(x,y)}{h} \tag{13.6.3}$$

Example 13.6.1: Finding a Directional Derivative from the Definition

Let $\theta=\arccos(3/5)$. Find the directional derivative $D_{\vec{\mathbf{u}}}f(x,y)$ of $f(x,y)=x^2-xy+3y^2$ in the direction of $\vec{\mathbf{u}}=(\cos\theta)\,\hat{\mathbf{i}}+(\sin\theta)\,\hat{\mathbf{j}}$.

Then determine $D_{\vec{\mathbf{u}}}f(-1,2)$.

Solution

First of all, since $\cos\theta=3/5$ and θ is acute, this implies

$$\sin\theta=\sqrt{1-\left(\frac{3}{5}\right)^2}=\sqrt{\frac{16}{25}}=\frac{4}{5}.$$

Using $f(x,y)=x^2-xy+3y^2$, we first calculate $f(x+h\cos\theta,y+h\sin\theta)$:

$$\begin{aligned}f(x+h\cos\theta,\\ y+h\sin\theta)&=(x+h\cos\theta)^2-(x+h\cos\theta)(y+h\sin\theta)+3(y+h\sin\theta)^2\\&=x^2+2xh\cos\theta+h^2\cos^2\theta-xy-xh\sin\theta-yh\cos\theta-h^2\sin\theta\cos\theta\\&\quad+3y^2+6yh\sin\theta+3h^2\sin^2\theta\\&=x^2+2xh\left(\frac{3}{5}\right)+\frac{9h^2}{25}-xy-\frac{4xh}{5}-\frac{3yh}{5}-\frac{12h^2}{25}\\&\quad+3y^2+6yh\left(\frac{4}{5}\right)+3h^2\left(\frac{16}{25}\right)\\&=x^2-xy+3y^2+\frac{2xh}{5}+\frac{9h^2}{5}+\frac{21yh}{5}.\end{aligned}$$

We substitute this expression into Equation 13.6.2 with $a=x$ and $b=y$:

$$\begin{aligned}D_{\vec{\mathbf{u}}}f(x,y)&=\lim_{h\to 0}\frac{f(x+h\cos\theta,y+h\sin\theta)-f(x,y)}{h}\\&=\lim_{h\to 0}\frac{\left(x^2-xy+3y^2+\frac{2xh}{5}+\frac{9h^2}{5}+\frac{21yh}{5}\right)-\left(x^2-xy+3y^2\right)}{h}\\&=\lim_{h\to 0}\frac{\frac{2xh}{5}+\frac{9h^2}{5}+\frac{21yh}{5}}{h}\\&=\lim_{h\to 0}\frac{2x}{5}+\frac{9h}{5}+\frac{21y}{5}\\&=\frac{2x+21y}{5}.\end{aligned}$$

To calculate $D_{\vec{\mathbf{u}}}f(-1,2)$, we substitute $x=-1$ and $y=2$ into this answer (Figure 13.6.2):

$$D_{\vec{\mathbf{u}}}f(-1,2)=\frac{2(-1)+21(2)}{5}=\frac{-2+42}{5}=8.$$

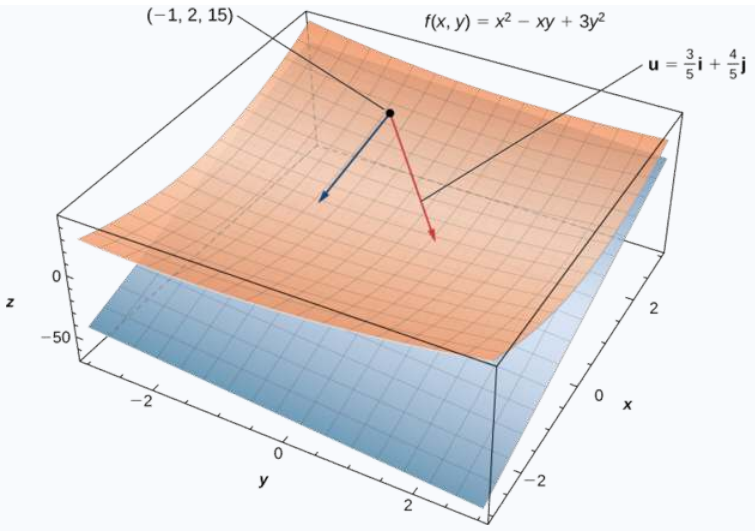

Figure **13.6.2**: Finding the directional derivative in a given direction $\vec{\mathbf{u}}$ *at a given point on a surface. The plane is tangent to the surface at the given point* $(-1, 2, 15)$.

An easier approach to calculating directional derivatives that involves partial derivatives is outlined in the following theorem.

Theorem 13.6.1: Directional Derivative of a Function of Two Variables

Let $z = f(x, y)$ be a function of two variables x and y, and assume that f_x and f_y exist. Then the directional derivative of f in the direction of $\vec{\mathbf{u}} = (\cos\theta)\,\hat{\mathbf{i}} + (\sin\theta)\,\hat{\mathbf{j}}$ is given by

$$D_{\vec{\mathbf{u}}} f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta. \tag{13.6.4}$$

Proof

Applying the definition of a directional derivative stated above in Equation 13.6.2, the directional derivative of f in the direction of $\vec{\mathbf{u}} = (\cos\theta)\,\hat{\mathbf{i}} + (\sin\theta)\,\hat{\mathbf{j}}$ at a point (x_0, y_0) in the domain of f can be written

$$D_{\vec{\mathbf{u}}} f((x_0, y_0)) = \lim_{t\to 0} \frac{f(x_0 + t\cos\theta, y_0 + t\sin\theta) - f(x_0, y_0)}{t}. \tag{13.6.5}$$

Let $x = x_0 + t\cos\theta$ and $y = y_0 + t\sin\theta$, and define $g(t) = f(x, y)$. Since f_x and f_y both exist, we can use the chain rule for functions of two variables to calculate $g'(t)$:

$$g'(t) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta. \tag{13.6.6}$$

If $t = 0$, then $x = x_0$ and $y = y_0$, so

$$g'(0) = f_x(x_0, y_0)\cos\theta + f_y(x_0, y_0)\sin\theta \tag{13.6.7}$$

By the definition of $g'(t)$, it is also true that

$$g'(0) = \lim_{t\to 0}\frac{g(t) - g(0)}{t} = \lim_{t\to 0}\frac{f(x_0 + t\cos\theta, y_0 + t\sin\theta) - f(x_0, y_0)}{t}. \tag{13.6.8}$$

Therefore, $D_{\vec{\mathbf{u}}} f(x_0, y_0) = f_x(x_0, y_0)\cos\theta + f_y(x_0, y_0)\sin\theta$.

Since the point (x_0, y_0) is an arbitrary point from the domain of f, this result holds for all points in the domain of f for which the partials f_x and f_y exist.

Therefore,

$$D_{\vec{\mathbf{u}}} f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta. \tag{13.6.9}$$

□

Example 13.6.2: Finding a Directional Derivative: Alternative Method

Let $\theta = \arccos(3/5)$. Find the directional derivative $D_{\vec{\mathbf{u}}} f(x,y)$ of $f(x,y) = x^2 - xy + 3y^2$ in the direction of $\vec{\mathbf{u}} = (\cos\theta)\,\hat{\mathbf{i}} + (\sin\theta)\,\hat{\mathbf{j}}$.

Then determine $D_{\vec{\mathbf{u}}} f(-1,2)$.

Solution

First, we must calculate the partial derivatives of f:

$$\begin{aligned} f_x(x,y) &= 2x - y \\ f_y(x,y) &= -x + 6y, \end{aligned}$$

Then we use Equation 13.6.4 with $\theta = \arccos(3/5)$:

$$\begin{aligned} D_{\vec{\mathbf{u}}} f(x,y) &= f_x(x,y)\cos\theta + f_y(x,y)\sin\theta \\ &= (2x-y)\frac{3}{5} + (-x+6y)\frac{4}{5} \\ &= \frac{6x}{5} - \frac{3y}{5} - \frac{4x}{5} + \frac{24y}{5} \\ &= \frac{2x+21y}{5}. \end{aligned}$$

To calculate $D_{\vec{\mathbf{u}}} f(-1,2)$, let $x = -1$ and $y = 2$:

$$D_{\vec{\mathbf{u}}} f(-1,2) = \frac{2(-1)+21(2)}{5} = \frac{-2+42}{5} = 8. \tag{13.6.10}$$

This is the same answer obtained in Example 13.6.1.

Exercise 13.6.1:

Find the directional derivative $D_{\vec{\mathbf{u}}} f(x,y)$ of $f(x,y) = 3x^2y - 4xy^3 + 3y^2 - 4x$ in the direction of $\vec{\mathbf{u}} = (\cos\frac{\pi}{3})\,\hat{\mathbf{i}} + (\sin\frac{\pi}{3})\,\hat{\mathbf{j}}$ using Equation 13.6.4.

What is $D_{\vec{\mathbf{u}}} f(3,4)$?

If the vector that is given for the direction of the derivative is not a unit vector, then it is only necessary to divide by the norm of the vector. For example, if we wished to find the directional derivative of the function in Example 13.6.2 in the direction of the vector $\langle -5, 12\rangle$, we would first divide by its magnitude to get $\vec{\mathbf{u}}$. This gives us $\vec{\mathbf{u}} = \langle -\frac{5}{13}, \frac{12}{13}\rangle$.

Then

$$\begin{aligned} D_{\vec{\mathbf{u}}} f(x,y) &= f_x(x,y)\cos\theta + f_y(x,y)\sin\theta \\ &= -\frac{5}{13}(2x-y) + \frac{12}{13}(-x+6y) \\ &= -\frac{22}{13}x + \frac{17}{13}y \end{aligned}$$

Gradient

The right-hand side of Equation 13.6.4 is equal to $f_x(x,y)\cos\theta + f_y(x,y)\sin\theta$, which can be written as the dot product of two vectors. Define the first vector as $\vec{\nabla} f(x,y) = f_x(x,y)\,\hat{\mathbf{i}} + f_y(x,y)\,\hat{\mathbf{j}}$ and the second vector as $\vec{\mathbf{u}} = (\cos\theta)\,\hat{\mathbf{i}} + (\sin\theta)\,\hat{\mathbf{j}}$. Then the right-hand side of the equation can be written as the dot product of these two vectors:

$$D_{\vec{\mathbf{u}}} f(x,y) = \vec{\nabla} f(x,y) \cdot \vec{\mathbf{u}}. \tag{13.6.11}$$

The first vector in Equation 13.6.11 has a special name: the gradient of the function f. The symbol ∇ is called **nabla** and the vector $\vec{\nabla} f$ is read "**del** f."

Definition: The Gradient

Let $z = f(x,y)$ be a function of x and y such that f_x and f_y exist. The vector $\vec{\nabla} f(x,y)$ is called the **gradient** of f and is defined as

$$\vec{\nabla} f(x,y) = f_x(x,y)\,\hat{\mathbf{i}} + f_y(x,y)\,\hat{\mathbf{j}}. \tag{13.6.12}$$

The vector $\vec{\nabla} f(x,y)$ is also written as "**grad** f."

Example 13.6.3: Finding Gradients

Find the gradient $\vec{\nabla} f(x,y)$ of each of the following functions:

a. $f(x,y) = x^2 - xy + 3y^2$
b. $f(x,y) = \sin 3x \cos 3y$

Solution

For both parts a. and b., we first calculate the partial derivatives f_x and f_y, then use Equation 13.6.12.

a. $f_x(x,y) = 2x - y$ and $f_y(x,y) = -x + 6y$, so

$$\begin{aligned}\vec{\nabla} f(x,y) &= f_x(x,y)\,\hat{\mathbf{i}} + f_y(x,y)\,\hat{\mathbf{j}} \\ &= (2x - y)\,\hat{\mathbf{i}} + (-x + 6y)\,\hat{\mathbf{j}}.\end{aligned}$$

b. $f_x(x,y) = 3\cos 3x \cos 3y$ and $f_y(x,y) = -3\sin 3x \sin 3y$, so

$$\begin{aligned}\vec{\nabla} f(x,y) &= f_x(x,y)\,\hat{\mathbf{i}} + f_y(x,y)\,\hat{\mathbf{j}} \\ &= (3\cos 3x \cos 3y)\,\hat{\mathbf{i}} - (3\sin 3x \sin 3y)\,\hat{\mathbf{j}}.\end{aligned}$$

Exercise 13.6.2

Find the gradient $\vec{\nabla} f(x,y)$ of $f(x,y) = \dfrac{x^2 - 3y^2}{2x + y}$.

The gradient has some important properties. We have already seen one formula that uses the gradient: the formula for the directional derivative. Recall from The Dot Product that if the angle between two vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ is φ, then $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = \|\vec{\mathbf{a}}\|\|\vec{\mathbf{b}}\| \cos\varphi$. Therefore, if the angle between $\vec{\nabla} f(x_0,y_0)$ and $\vec{\mathbf{u}} = (cos\theta)\,\hat{\mathbf{i}} + (sin\theta)\,\hat{\mathbf{j}}$ is φ, we have

$$\begin{aligned}D_{\vec{\mathbf{u}}} f(x_0,y_0) &= \vec{\nabla} f(x_0,y_0) \cdot \vec{\mathbf{u}} = \|\vec{\nabla} f(x_0,y_0)\|\|\vec{\mathbf{u}}\| \cos\varphi \\ &= \|\vec{\nabla} f(x_0,y_0)\| \cos\varphi.\end{aligned} \tag{13.6.13}$$

The $\|\vec{\mathbf{u}}\|$ disappears because $\vec{\mathbf{u}}$ is a unit vector. Therefore, the directional derivative is equal to the magnitude of the gradient evaluated at (x_0,y_0) multiplied by $\cos\varphi$. Recall that $\cos\varphi$ ranges from -1 to 1.

If $\varphi = 0$, then $\cos\varphi = 1$ and $\vec{\nabla} f(x_0,y_0)$ and $\vec{\mathbf{u}}$ both point in the same direction.

If $\varphi = \pi$, then $\cos\varphi = -1$ and $\vec{\nabla} f(x_0,y_0)$ and $\vec{\mathbf{u}}$ point in opposite directions.

In the first case, the value of $D_{\vec{\mathbf{u}}} f(x_0,y_0)$ is maximized and in the second case, the value of $D_{\vec{\mathbf{u}}} f(x_0,y_0)$ is minimized.

We can also see that if $\vec{\nabla} f(x_0,y_0) = \vec{\mathbf{0}}$, then

$$D_{\vec{\mathbf{u}}} f(x_0,y_0) = \vec{\nabla} f(x_0,y_0) \cdot \vec{\mathbf{u}} = 0 \tag{13.6.14}$$

for any vector $\vec{\mathbf{u}}$. These three cases are outlined in the following theorem.

Theorem 13.6.2: Properties of the Gradient

Suppose the function $z = f(x,y)$ is differentiable at (x_0,y_0) (Figure 13.6.3).

i. If $\vec{\nabla} f(x_0,y_0) = \vec{\mathbf{0}}$, then $D_{\vec{\mathbf{u}}} f(x_0,y_0) = 0$ for any unit vector $\vec{\mathbf{u}}$.

ii. If $\vec{\nabla} f(x_0, y_0) \neq \vec{0}$, then $D_{\vec{u}} f(x_0, y_0)$ is **maximized** when $\vec{u}$ points in the same direction as $\vec{\nabla} f(x_0, y_0)$. The maximum value of $D_{\vec{u}} f(x_0, y_0)$ is $\|\vec{\nabla} f(x_0, y_0)\|$.

iii. If $\vec{\nabla} f(x_0, y_0) \neq \vec{0}$, then $D_{\vec{u}} f(x_0, y_0)$ is **minimized** when $\vec{u}$ points in the opposite direction from $\vec{\nabla} f(x_0, y_0)$. The minimum value of $D_{\vec{u}} f(x_0, y_0)$ is $-\|\vec{\nabla} f(x_0, y_0)\|$.

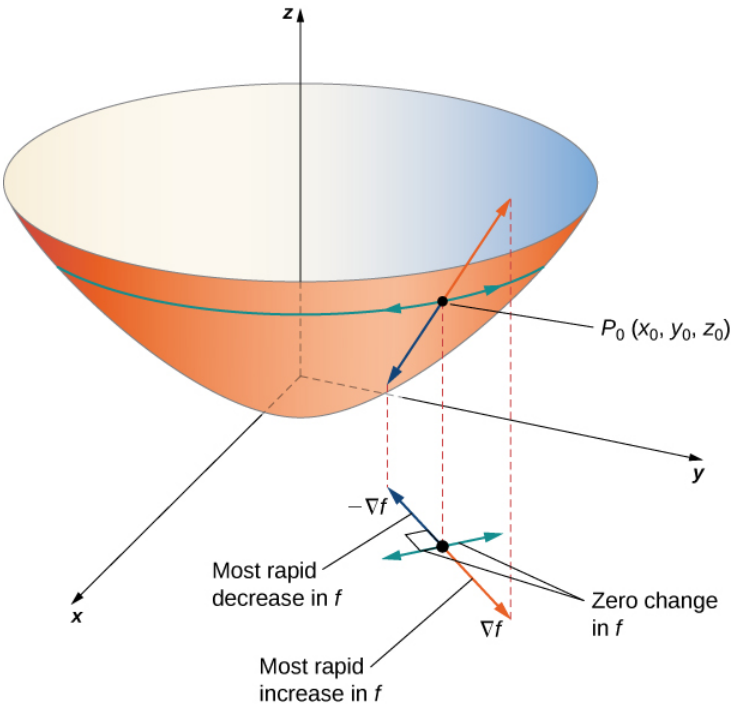

Figure **13.6.3**: The gradient indicates the maximum and minimum values of the directional derivative at a point.

Note: Gradient indicates direction of steepest ascent

Since the gradient vector points in the direction within the domain of f that corresponds to the maximum value of the directional derivative, $D_{\vec{u}} f(x_0, y_0)$, we say that the gradient vector points in the direction of **steepest ascent** or **most rapid increase in** f, that is, at any given point, the gradient points in the direction with the steepest uphill slope.

Example 13.6.4: Finding a Maximum Directional Derivative

Find the direction for which the directional derivative of $f(x, y) = 3x^2 - 4xy + 2y^2$ at $(-2, 3)$ is a maximum. What is the maximum value?

Solution:

The maximum value of the directional derivative occurs when $\vec{\nabla} f$ and the unit vector point in the same direction. Therefore, we start by calculating $\vec{\nabla} f(x, y)$:

$$f_x(x, y) = 6x - 4y \text{ and } f_y(x, y) = -4x + 4y$$

so

$$\vec{\nabla} f(x, y) = f_x(x, y)\,\hat{\mathbf{i}} + f_y(x, y)\,\hat{\mathbf{j}} = (6x - 4y)\,\hat{\mathbf{i}} + (-4x + 4y)\,\hat{\mathbf{j}}.$$

Next, we evaluate the gradient at $(-2, 3)$:

$$\vec{\nabla} f(-2, 3) = (6(-2) - 4(3))\,\hat{\mathbf{i}} + (-4(-2) + 4(3))\,\hat{\mathbf{j}} = -24\,\hat{\mathbf{i}} + 20\,\hat{\mathbf{j}}.$$

The gradient vector gives the direction of the maximum value of the directional derivative.

The maximum value of the directional derivative at $(-2, 3)$ is $\|\vec{\nabla} f(-2, 3)\| = 4\sqrt{61}$ (see the Figure **13.6.4**).

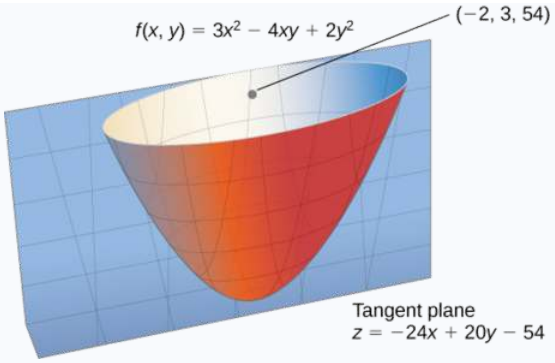

Figure **13.6.4**: The maximum value of the directional derivative at $(-2, 3)$ is in the direction of the gradient.

Exercise 13.6.3:

Find the direction for which the directional derivative of $g(x, y) = 4x - xy + 2y^2$ at $(-2, 3)$ is a maximum. What is the maximum value?

Figure **13.6.5** shows a portion of the graph of the function $f(x, y) = 3 + \sin x \sin y$. Given a point (a, b) in the domain of f, the maximum value of the directional derivative at that point is given by $\|\vec{\nabla} f(a, b)\|$. This would equal the rate of greatest ascent if the surface represented a topographical map. If we went in the opposite direction, it would be the rate of greatest descent.

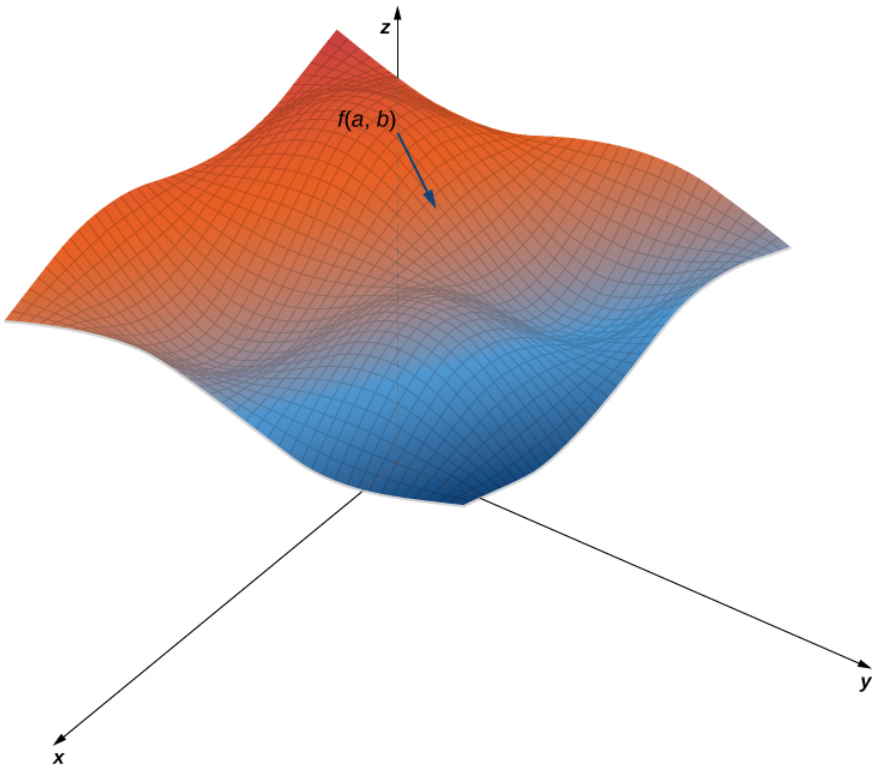

Figure **13.6.5**: A typical surface in $\mathbb{R}^3$. Given a point on the surface, the directional derivative can be calculated using the gradient.

When using a topographical map, the steepest slope is always in the direction where the contour lines are closest together (Figure **13.6.6**). This is analogous to the contour map of a function, assuming the level curves are obtained for equally spaced values throughout the range of that function.

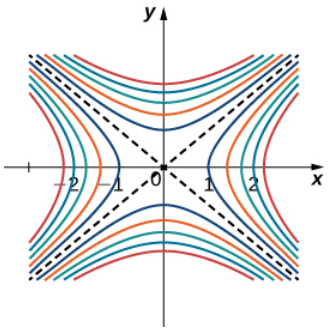

Figure 13.6.6: Contour map for the function $f(x,y) = x^2 - y^2$ using level values between -5 and 5.

Gradients and Level Curves

Recall that if a curve is defined parametrically by the function pair $(x(t), y(t))$, then the vector $x'(t)\,\hat{\mathbf{i}} + y'(t)\,\hat{\mathbf{j}}$ is tangent to the curve for every value of t in the domain. Now let's assume $z = f(x,y)$ is a differentiable function of x and y, and (x_0, y_0) is in its domain. Let's suppose further that $x_0 = x(t_0)$ and $y_0 = y(t_0)$ for some value of t, and consider the level curve $f(x,y) = k$. Define $g(t) = f(x(t), y(t))$ and calculate $g'(t)$ on the level curve. By the chain Rule,

$$g'(t) = f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t). \tag{13.6.15}$$

But $g'(t) = 0$ because $g(t) = k$ for all t. Therefore, on the one hand,

$$f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t) = 0; \tag{13.6.16}$$

on the other hand,

$$f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t) = \vec{\nabla} f(x,y) \cdot \langle x'(t), y'(t)\rangle. \tag{13.6.17}$$

Therefore,

$$\vec{\nabla} f(x,y) \cdot \langle x'(t), y'(t)\rangle = 0. \tag{13.6.18}$$

Thus, the dot product of these vectors is equal to zero, which implies they are orthogonal. However, the second vector is tangent to the level curve, which implies the gradient must be normal to the level curve, which gives rise to the following theorem.

Theorem 13.6.3: Gradient Is Normal to the Level Curve

Suppose the function $z = f(x,y)$ has continuous first-order partial derivatives in an open disk centered at a point (x_0, y_0). If $\vec{\nabla} f(x_0, y_0) \neq \vec{0}$, then $\vec{\nabla} f(x_0, y_0)$ is **normal** to the level curve of f at (x_0, y_0).

We can use this theorem to find tangent and normal vectors to level curves of a function.

Example 13.6.5: Finding Tangents to Level Curves

For the function $f(x,y) = 2x^2 - 3xy + 8y^2 + 2x - 4y + 4$, find a tangent vector to the level curve at point $(-2, 1)$. Graph the level curve corresponding to $f(x,y) = 18$ and draw in $\vec{\nabla} f(-2,1)$ and a tangent vector.

Solution:

First, we must calculate $\vec{\nabla} f(x,y)$:

$$\begin{aligned} &f_x(x,y) = 4x - 3y + 2 \text{ and } f_y = -3x + 16y - 4 \text{ so} \\ &\vec{\nabla} f(x,y) = (4x - 3y + 2)\,\hat{\mathbf{i}} + (-3x + 16y - 4)\,\hat{\mathbf{j}}. \end{aligned} \tag{13.6.19}$$

Next, we evaluate $\vec{\nabla} f(x,y)$ at $(-2,1)$:

$$\begin{aligned} &\vec{\nabla} f(-2,1) \\ &= (4(-2) - 3(1) + 2)\,\hat{\mathbf{i}} + (-3(-2) + 16(1) - 4)\,\hat{\mathbf{j}} = -9\,\hat{\mathbf{i}} + 18\,\hat{\mathbf{j}}. \end{aligned} \tag{13.6.20}$$

This vector is orthogonal to the curve at point $(-2,1)$. We can obtain a tangent vector by reversing the components and multiplying either one by -1. Thus, for example, $-18\hat{\mathbf{i}} - 9\hat{\mathbf{j}}$ is a tangent vector (see the following graph).

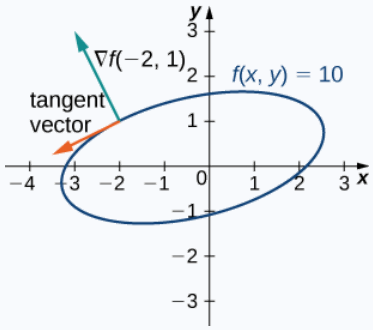

Figure 13.6.7: Tangent and normal vectors to $2x^2 - 3xy + 8y^2 + 2x - 4y + 4 = 18$ at point $(-2,1)$.

Exercise 13.6.4:

For the function $f(x,y) = x^2 - 2xy + 5y^2 + 3x - 2y + 3$, find the tangent to the level curve at point $(1,1)$. Draw the graph of the level curve corresponding to $f(x,y) = 8$ and draw $\vec{\nabla} f(1,1)$ and a tangent vector.

The fact that the gradient of a surface always points in the direction of steepest increase/decrease is very useful, as illustrated in the following example.

Example 13.6.6: The flow of water downhill

Consider the surface given by $f(x,y) = 20 - x^2 - 2y^2$. Water is poured on the surface at $(1, \frac{1}{4})$. What path does it take as it flows downhill?

SOLUTION

Let $\vec{\mathbf{r}}(t) = \langle x(t), y(t)\rangle$ be the vector--valued function describing the path of the water in the xy-plane; we seek $x(t)$ and $y(t)$. We know that water will always flow downhill in the steepest direction; therefore, at any point on its path, it will be moving in the direction of $-\vec{\nabla} f$. (We ignore the physical effects of momentum on the water.) Thus $\vec{\mathbf{r}}'(t)$ will be parallel to $\vec{\nabla} f$, and there is some constant c such that $c\vec{\nabla} f(t) = \vec{r}\,'(t) = \langle x'(t), y'(t)\rangle$.

We find $\vec{\nabla} f(x,y) = \langle -2x, -4y\rangle$ and write $x'(t)$ as $\frac{dx}{dt}$ and $y'(t)$ as $\frac{dy}{dt}$. Then

$$c\vec{\nabla} f(t) = \langle x'(t), y'(t)\rangle$$
$$\langle -2cx, -4cy\rangle = \left\langle \frac{dx}{dt}, \frac{dy}{dt}\right\rangle.$$

This implies

$$-2cx = \frac{dx}{dt} \quad \text{and} \quad -4cy = \frac{dy}{dt}, \text{ i.e.,} \tag{13.6.21}$$

$$c = -\frac{1}{2x}\frac{dx}{dt} \quad \text{and} \quad c = -\frac{1}{4y}\frac{dy}{dt}. \tag{13.6.22}$$

As c equals both expressions, we have

$$\frac{1}{4y}\frac{dy}{dt} = \frac{1}{2x}\frac{dx}{dt}. \tag{13.6.23}$$

To find an explicit relationship between x and y, we can integrate both sides with respect to t. Recall from our study of differentials that $\frac{dx}{dt}dt = dx$. Thus:

$$\int \frac{1}{4y}\frac{dy}{dt}\,dt = \int \frac{1}{2x}\frac{dx}{dt}\,dt$$
$$\int \frac{1}{4y}\,dy = \int \frac{1}{2x}\,dx$$
$$\frac{1}{4}\ln|y| = \frac{1}{2}\ln|x| + C_1$$
$$\ln|y| = 2\ln|x| + C_1$$
$$\ln|y| = \ln|x^2| + C_1$$

Now raise both sides as a power of e:

$$|y| = e^{\ln|x^2|+C_1}$$
$$|y| = e^{\ln x^2}e^{C_1}$$
$$y = \pm e^{C_1}x^2$$

Then

$$y = Cx^2, \quad \text{where } C = \pm e^{C_1} \text{ or } C = 0. \tag{13.6.24}$$

As the water started at the point $(1, \frac{1}{4})$, we can now solve for C:

$$C(1)^2 = \frac{1}{4} \quad \Rightarrow \quad C = \frac{1}{4}. \tag{13.6.25}$$

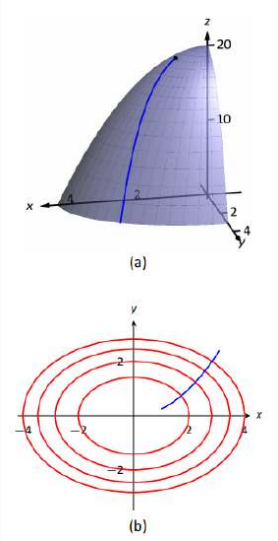

***Figure* 13.6.8:** *A graph of the surface along with the path in the xy-plane with the level curves.*

Thus the water follows the curve $y = \frac{x^2}{4}$ in the xy-plane. The surface and the path of the water is graphed in Figure 13.6.8. In part (b) of the figure, the level curves of the surface are plotted in the xy-plane, along with the curve $y = \frac{x^2}{4}$. Notice how the path intersects the level curves at right angles. As the path follows the gradient downhill, this reinforces the fact that the gradient is orthogonal to level curves.

Three-Dimensional Gradients and Directional Derivatives

The definition of a gradient can be extended to functions of more than two variables.

Definition: Gradients in 3D

Let $w = f(x, y, z)$ be a function of three variables such that f_x, f_y, and f_z exist. The vector $\vec{\nabla} f(x, y, z)$ is called the gradient of f and is defined as

$$\vec{\nabla} f(x, y, z) = f_x(x, y, z)\,\hat{\mathbf{i}} + f_y(x, y, z)\,\hat{\mathbf{j}} + f_z(x, y, z)\,\hat{\mathbf{k}}. \tag{13.6.26}$$

$\vec{\nabla} f(x,y,z)$ can also be written as **grad** $f(x,y,z)$.

Calculating the gradient of a function in three variables is very similar to calculating the gradient of a function in two variables. First, we calculate the partial derivatives f_x, f_y, and f_z, and then we use Equation 13.6.26.

Example 13.6.7: Finding Gradients in Three Dimensions

Find the gradient $\vec{\nabla} f(x,y,z)$ of each of the following functions:

a. $f(x,y,z) = 5x^2 - 2xy + y^2 - 4yz + z^2 + 3xz$
b. $f(x,y,z) = e^{-2z} \sin 2x \cos 2y$

Solution:

For both parts a. and b., we first calculate the partial derivatives f_x, f_y, and f_z, then use Equation 13.6.26.

a. $f_x(x,y,z) = 10x - 2y + 3z$, $f_y(x,y,z) = -2x + 2y - 4z$, and $f_z(x,y,z) = 3x - 4y + 2z$, so

$$\begin{aligned}\vec{\nabla} f(x,y,z) &= f_x(x,y,z)\,\hat{\mathbf{i}} + f_y(x,y,z)\,\hat{\mathbf{j}} + f_z(x,y,z)\,\hat{\mathbf{k}} \\ &= (10x - 2y + 3z)\,\hat{\mathbf{i}} + (-2x + 2y - 4z)\,\hat{\mathbf{j}} + (3x - 4y + 2z)\,\hat{\mathbf{k}}.\end{aligned}$$

b. $f_x(x,y,z) = 2e^{-2z} \cos 2x \cos 2y$, $f_y(x,y,z) = -2e^{-2z} \sin 2x \sin 2y$, and $f_z(x,y,z) = -2e^{-2z} \sin 2x \cos 2y$, so

$$\begin{aligned}\vec{\nabla} f(x,y,z) &= f_x(x,y,z)\,\hat{\mathbf{i}} + f_y(x,y,z)\,\hat{\mathbf{j}} + f_z(x,y,z)\,\hat{\mathbf{k}} \\ &= (2e^{-2z} \cos 2x \cos 2y)\,\hat{\mathbf{i}} + (-2e^{-2z} \sin 2x \sin 2y)\,\hat{\mathbf{j}} \\ &\quad + (-2e^{-2z} \sin 2x \cos 2y)\,\hat{\mathbf{k}} \\ &= 2e^{-2z}(\cos 2x \cos 2y\,\hat{\mathbf{i}} - \sin 2x \sin 2y\,\hat{\mathbf{j}} - \sin 2x \cos 2y\,\hat{\mathbf{k}}).\end{aligned}$$

Exercise 13.6.5:

Find the gradient $\vec{\nabla} f(x,y,z)$ of $f(x,y,z) = \dfrac{x^2 - 3y^2 + z^2}{2}$ and find its gradient vector at the point $(-1,2,3)$.

Theorem 13.6.4: The Gradient of a Function of Three Variables is Normal to the Level Surface

Suppose the function $z = f(x,y,z)$ has continuous first-order partial derivatives in an open sphere centered at a point (x_0,y_0,z_0). If $\vec{\nabla} f(x_0,y_0,z_0) \neq \vec{0}$, then $\vec{\nabla} f(x_0,y_0,z_0)$ is **normal** to the level surface of f at (x_0,y_0,z_0).

Figure 13.6.9 shows the gradient vectors at various points on a level surface of the function in Exercise 13.6.5. The points shown on the level surface are: $(-1,2,3)$, $(3,-2,-1)$, $(0,\sqrt{\frac{2}{3}},0)$, $(2,\sqrt{\frac{10}{3}},2)$, and $(2,\sqrt{\frac{10}{3}},-2)$.

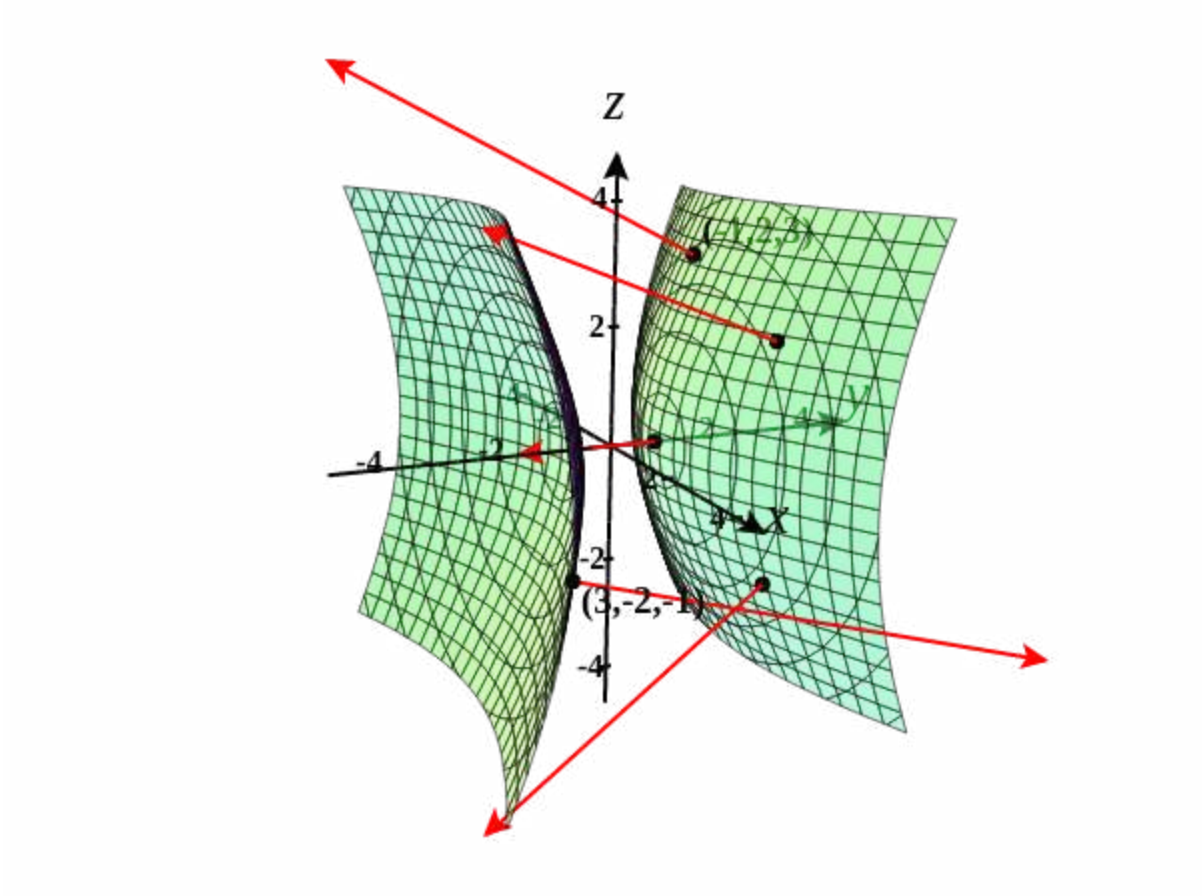

***Figure* 13.6.9:** *A level surface of the function* $f(x,y,z)=\dfrac{x^2-3y^2+z^2}{2}$ for $C=-1$ along with various points on this level surface and the corresponding gradient vectors. Note how these gradient vectors are normal to this level surface.

The directional derivative can also be generalized to functions of three variables. To determine a direction in three dimensions, a vector with three components is needed. This vector is a unit vector, and the components of the unit vector are called *directional cosines*. Given a three-dimensional unit vector $\vec{\mathbf{u}}$ in standard form (i.e., the initial point is at the origin), this vector forms three different angles with the positive x-, y-, and z-axes. Let's call these angles α, β, and γ. Then the directional cosines are given by $\cos\alpha, \cos\beta$, and $\cos\gamma$. These are the components of the unit vector $\vec{\mathbf{u}}$; since $\vec{\mathbf{u}}$ is a unit vector, it is true that $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$.

Definition: Directional Derivative of a Function of Three variables

Suppose $w=f(x,y,z)$ is a function of three variables with a domain of D. Let $(x_0,y_0,z_0)\in D$ and let $\vec{\mathbf{u}}=\cos\alpha\,\hat{\mathbf{i}}+\cos\beta\,\hat{\mathbf{j}}+\cos\gamma\,\hat{\mathbf{k}}$ be a unit vector. Then, the directional derivative of f in the direction of u is given by

$$D_{\vec{\mathbf{u}}}f(x_0,y_0,z_0) = \lim_{t\to 0}\frac{f(x_0+t\cos\alpha, y_0+t\cos\beta, z_0+t\cos\gamma)-f(x_0,y_0,z_0)}{t} \tag{13.6.27}$$

provided the limit exists.

We can calculate the directional derivative of a function of three variables by using the gradient, leading to a formula that is analogous to Equation 13.6.4.

Theorem \(\PageIndex{5}): Directional Derivative of a Function of Three Variables

Let $f(x,y,z)$ be a differentiable function of three variables and let $\vec{\mathbf{u}} = \cos\alpha\,\hat{\mathbf{i}} + \cos\beta\,\hat{\mathbf{j}} + \cos\gamma\,\hat{\mathbf{k}}$ be a unit vector. Then, the directional derivative of f in the direction of $\vec{\mathbf{u}}$ is given by

$$\begin{aligned} D_{\vec{\mathbf{u}}} f(x,y,z) &= \vec{\nabla} f(x,y,z) \cdot \vec{\mathbf{u}} \\ &= f_x(x,y,z)\cos\alpha + f_y(x,y,z)\cos\beta + f_z(x,y,z)\cos\gamma. \end{aligned} \tag{13.6.28}$$

The three angles α, β, and γ determine the unit vector $\vec{\mathbf{u}}$. In practice, we can use an arbitrary (nonunit) vector, then divide by its magnitude to obtain a unit vector in the desired direction.

Example 13.6.8: Finding a Directional Derivative in Three Dimensions

Calculate $D_{\vec{\mathbf{v}}} f(1,-2,3)$ in the direction of $\vec{\mathbf{v}} = -\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ for the function

$$f(x,y,z) = 5x^2 - 2xy + y^2 - 4yz + z^2 + 3xz.$$

Solution:

First, we find the magnitude of v:

$$\|\vec{\mathbf{v}}\| = \sqrt{(-1)^2 + (2)^2} = 3.$$

Therefore, $\dfrac{\vec{\mathbf{v}}}{\|\vec{\mathbf{v}}\|} = \dfrac{-\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}}{3} = -\dfrac{1}{3}\,\hat{\mathbf{i}} + \dfrac{2}{3}\,\hat{\mathbf{j}} + \dfrac{2}{3}\,\hat{\mathbf{k}}$ is a unit vector in the direction of $\vec{\mathbf{v}}$, so $\cos\alpha = -\dfrac{1}{3}, \cos\beta = \dfrac{2}{3}$, and $\cos\gamma = \dfrac{2}{3}$. Next, we calculate the partial derivatives of f:

$$\begin{aligned} f_x(x,y,z) &= 10x - 2y + 3z \\ f_y(x,y,z) &= -2x + 2y - 4z \\ f_z(x,y,z) &= -4y + 2z + 3x, \end{aligned}$$

then substitute them into Equation 13.6.28:

$$\begin{aligned} D_{\vec{\mathbf{v}}} f(x,y,z) &= f_x(x,y,z)\cos\alpha + f_y(x,y,z)\cos\beta + f_z(x,y,z)\cos\gamma \\ &= (10x - 2y + 3z)\left(-\tfrac{1}{3}\right) + (-2x + 2y - 4z)\left(\tfrac{2}{3}\right) + (-4y + 2z + 3x)\left(\tfrac{2}{3}\right) \\ &= -\frac{10x}{3} + \frac{2y}{3} - \frac{3z}{3} - \frac{4x}{3} + \frac{4y}{3} - \frac{8z}{3} - \frac{8y}{3} + \frac{4z}{3} + \frac{6x}{3} \\ &= -\frac{8x}{3} - \frac{2y}{3} - \frac{7z}{3}. \end{aligned}$$

Last, to find $D_{\vec{\mathbf{v}}} f(1,-2,3)$, we substitute $x = 1$, $y = -2$, and $z = 3$:

$$\begin{aligned} D_{\vec{\mathbf{v}}} f(1,-2,3) &= -\frac{8(1)}{3} - \frac{2(-2)}{3} - \frac{7(3)}{3} \\ &= -\frac{8}{3} + \frac{4}{3} - \frac{21}{3} \\ &= -\frac{25}{3}. \end{aligned}$$

Exercise 13.6.6:

Calculate $D_{\vec{\mathbf{v}}} f(x,y,z)$ and $D_{\vec{\mathbf{v}}} f(0,-2,5)$ in the direction of $\vec{\mathbf{v}} = -3\hat{\mathbf{i}} + 12\hat{\mathbf{j}} - 4\hat{\mathbf{k}}$ for the function

$$f(x,y,z) = 3x^2 + xy - 2y^2 + 4yz - z^2 + 2xz.$$

Summary

- A directional derivative represents a rate of change of a function in any given direction.
- The gradient can be used in a formula to calculate the directional derivative.
- The gradient indicates the direction of greatest change of a function of more than one variable.

Key Equations

- **directional derivative (two dimensions)**

  $$D_{\vec{\mathbf{u}}} f(a,b) = \lim_{h \to 0} \frac{f(a + h\cos\theta, b + h\sin\theta) - f(a,b)}{h}$$

  or

  $$D_{\vec{\mathbf{u}}} f(x,y) = f_x(x,y)\cos\theta + f_y(x,y)\sin\theta$$

  or

  $D_{\vec{\mathbf{u}}} f(x,y) = \vec{\nabla} f(x,y) \cdot \vec{\mathbf{u}}$, where $\vec{\mathbf{u}}$ is a unit vector in the xy-plane

- **gradient (two dimensions)**

  $$\vec{\nabla} f(x,y) = f_x(x,y)\,\hat{\mathbf{i}} + f_y(x,y)\,\hat{\mathbf{j}}$$

- **gradient (three dimensions)**

  $$\vec{\nabla} f(x,y,z) = f_x(x,y,z)\,\hat{\mathbf{i}} + f_y(x,y,z)\,\hat{\mathbf{j}} + f_z(x,y,z)\,\hat{\mathbf{k}}$$

- **directional derivative (three dimensions)**

  $$D_{\vec{\mathbf{u}}} f(x,y,z) = \vec{\nabla} f(x,y,z) \cdot \vec{\mathbf{u}} = f_x(x,y,z)\cos\alpha + f_y(x,y,z)\cos\beta + f_x(x,y,z)\cos\gamma$$

Glossary

Contributors

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- Example **13.6.6** is adapted from Apex Calculus by Gregory Hartman (Virginia Military Institute)
- Edited and expanded by Paul Seeburger (Monroe Community College)

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13.6E: Directional Derivatives and the Gradient (Exercises)

13.6: Directional Derivatives and the Gradient

In exercise 1, find the directional derivative using the limit definition only.

1) a. $f(x,y) = 5 - 2x^2 - \frac{1}{2}y^2$ at point $P(3,4)$ in the direction of $\vec{\mathbf{u}} = (\cos\frac{\pi}{4})\,\hat{\mathbf{i}} + (\sin\frac{\pi}{4})\,\hat{\mathbf{j}}$

1) b. $f(x,y) = y^2\cos(2x)$ at point $P(\frac{\pi}{3},2)$ in the direction of $\vec{\mathbf{u}} = (\cos\frac{\pi}{4})\,\hat{\mathbf{i}} + (\sin\frac{\pi}{4})\,\hat{\mathbf{j}}$

2) Find the directional derivative of $f(x,y) = y^2\sin(2x)$ at point $P(\frac{\pi}{4},2)$ in the direction of $\vec{\mathbf{u}} = 5\,\hat{\mathbf{i}} + 12\,\hat{\mathbf{j}}$.

In exercises 3 - 13, find the directional derivative of the function in the direction of $\vec{\mathbf{v}}$ as a function of x and y. Remember that you first need to find a unit vector in the direction of the direction vector. Then find the value of the directional derivative at point P.

3) $f(x,y) = xy, \quad P(-2,0), \quad \vec{\mathbf{v}} = \frac{1}{2}\,\hat{\mathbf{i}} + \frac{\sqrt{3}}{2}\,\hat{\mathbf{j}}$

4) $h(x,y) = e^x\sin y, \quad P(1,\frac{\pi}{2}), \quad \vec{\mathbf{v}} = -\hat{\mathbf{i}}$

5) $f(x,y) = x^2y, \quad P(-5,5), \quad \vec{\mathbf{v}} = 3\,\hat{\mathbf{i}} - 4\,\hat{\mathbf{j}}$

6) $f(x,y) = xy, \quad P(1,1), \quad \vec{\mathbf{u}} = \langle\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\rangle$

7) $f(x,y) = x^2 - y^2, \quad P(1,0), \quad \vec{\mathbf{u}} = \langle\frac{\sqrt{3}}{2},\frac{1}{2}\rangle$

8) $f(x,y) = 3x + 4y + 7, \quad P(0,\frac{\pi}{2}), \quad \vec{\mathbf{u}} = \langle\frac{3}{5},\frac{4}{5}\rangle$

9) $f(x,y) = e^x\cos y, \quad P = (0,\frac{\pi}{2}), \quad \vec{\mathbf{u}} = \langle 0,5\rangle$

10) $f(x,y) = y^{10}, \quad \vec{\mathbf{u}} = \langle 0,-3\rangle, \quad P = (1,-1)$

11) $f(x,y) = \ln(x^2+y^2), \quad \vec{\mathbf{u}} = \langle 2,-5\rangle, \quad P(1,2)$

12) $h(x,y,z) = xyz, \quad P(2,1,1), \quad \vec{\mathbf{v}} = 2\,\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$

13) $f(x,y,z) = y^2 + xz, \quad P(1,2,2), \quad \vec{\mathbf{v}} = \langle 2,-1,2\rangle$

In exercises 14 - 19, find the directional derivative of the function in the direction of the unit vector $\vec{\mathbf{u}} = \cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}$.

14) $f(x,y) = x^2 + 2y^2, \quad \theta = \frac{\pi}{6}$

15) $f(x,y) = \dfrac{y}{x+2y}, \quad \theta = -\frac{\pi}{4}$

16) $f(x,y) = \cos(3x+y), \quad \theta = \frac{\pi}{4}$

17) $w(x,y) = ye^x, \quad \theta = \frac{\pi}{3}$

18) $f(x,y) = x\arctan(y), \quad \theta = \frac{\pi}{2}$

19) $f(x,y) = \ln(x+2y), \quad \theta = \frac{\pi}{3}$

In exercises 20 - 23, find the gradient $\vec{\nabla}f$.

20) Find the gradient of $f(x,y) = 3x^2 + y^3 - 3x + y$. Then find it's value at the point $P(2,3)$.

21) Find the gradient of $f(x,y) = \dfrac{14 - x^2 - y^2}{3}$. Then, find the gradient at point $P(1,2)$.

22) Find the gradient of $f(x,y) = \ln(4x^3 - 3y)$. Then, find the gradient at point $P(1,1)$.

23) Find the gradient of $f(x,y,z) = xy + yz + xz$. Then find the gradient at point $P(1,2,3)$.

In exercises 24 - 25, find the directional derivative of the function at point P in the direction of Q.

24) $f(x,y) = x^2 + 3y^2, \quad P(1,1), \quad Q(4,5)$

25) $f(x,y,z) = \dfrac{y}{x+z}, \quad P(2,1,-1), \quad Q(-1,2,0)$

26) Find the directional derivative of $f(x,y,z)$) at P and in the direction of $\vec{\mathbf{u}}$: $\quad f(x,y,z) = \ln(x^2 + 2y^2 + 3z^2), \quad P(2,1,4), \quad \vec{\mathbf{u}} = \frac{-3}{13}\,\hat{\mathbf{i}} - \frac{4}{13}\,\hat{\mathbf{j}} - \frac{12}{13}\,\hat{\mathbf{k}}$.

In exercises 27 - 29, find the directional derivative of the function at P in the direction of $\vec{\mathbf{u}}$.

27) $f(x,y) = \ln(5x + 4y), \quad P(3,9), \quad \vec{\mathbf{u}} = 6\,\hat{\mathbf{i}} + 8\,\hat{\mathbf{j}}$

28) $f(x,y) = -7x + 2y, \quad P(2,-4), \quad \vec{\mathbf{u}} = 4\,\hat{\mathbf{i}} - 3\,\hat{\mathbf{j}}$

29) $f(x,y,z) = 4x^5y^2z^3, \quad P(2,-1,1), \quad \vec{\mathbf{u}} = \frac{1}{3}\,\hat{\mathbf{i}} + \frac{2}{3}\,\hat{\mathbf{j}} - \frac{2}{3}\,\hat{\mathbf{k}}$

30) [T] Use technology to sketch the level curve of $f(x,y) = 4x - 2y + 3$ that passes through $P(1,2)$ and draw the gradient vector at P.

31) [T] Use technology to sketch the level curve of $f(x,y) = x^2 + 4y^2$ that passes through $P(-2,0)$ and draw the gradient vector at P.

In exercises 32 - 35, find the gradient vector at the indicated point.

32) $f(x,y) = xy^2 - yx^2, \quad P(-1,1)$

33) $f(x,y) = xe^y - \ln(x), \quad P(-3,0)$

34) $f(x,y,z) = xy - \ln(z), \quad P(2,-2,2)$

35) $f(x,y,z) = x\sqrt{y^2 + z^2}, \quad P(-2,-1,-1)$

In exercises 36 - 40, find the indicated directional derivative of the function.

36) $f(x,y) = x^2 + xy + y^2$ at point $(-5,-4)$ in the direction the function increases most rapidly.

37) $f(x,y) = e^{xy}$ at point $(6,7)$ in the direction the function increases most rapidly.

38) $f(x,y) = \arctan\left(\dfrac{y}{x}\right)$ at point $(-9,9)$ in the direction the function increases most rapidly.

39) $f(x,y,z) = \ln(xy + yz + zx)$ at point $(-9,-18,-27)$ in the direction the function increases most rapidly.

40) $f(x,y,z) = \dfrac{x}{y} + \dfrac{y}{z} + \dfrac{z}{x}$ at point $(5,-5,5)$ in the direction the function increases most rapidly.

In exercises 41 - 43, find the maximum rate of change of f at the given point and the direction in which it occurs.

41) $f(x,y) = xe^{-y}, \quad (-2,0)$

42) $f(x,y) = \sqrt{x^2 + 2y}, \quad (4,10)$

43) $f(x,y) = \cos(3x + 2y), \quad \left(\frac{\pi}{6}, -\frac{\pi}{8}\right)$

In exercises 44 - 47, find equations of

a. the tangent plane and

b. the normal line to the given surface at the given point.

44) The level curve $f(x,y,z)=12$ for $f(x,y,z)=4x^2-2y^2+z^2$ at point $(2,2,2)$.

45) $f(x,y,z)=xy+yz+xz=3$ at point $(1,1,1)$

46) $f(x,y,z)=xyz=6$ at point $(1,2,3)$

47) $f(x,y,z)=xe^y\cos z-z=1$ at point $(1,0,0)$

In exercises 48 - 51, solve the stated problem.

48) The temperature T in a metal sphere is inversely proportional to the distance from the center of the sphere (the origin: $(0,0,0)$). The temperature at point $(1,2,2)$ is $120°$C.

a. Find the rate of change of the temperature at point $(1,2,2)$ in the direction toward point $(2,1,3)$.

b. Show that, at any point in the sphere, the direction of greatest increase in temperature is given by a vector that points toward the origin.

49) The **electrical potential** (voltage) in a certain region of space is given by the function $V(x,y,z)=5x^2-3xy+xyz$.

a. Find the rate of change of the voltage at point $(3,4,5)$ in the direction of the vector $\langle 1,1,-1\rangle$.

b. In which direction does the voltage change most rapidly at point $(3,4,5)$?

c. What is the maximum rate of change of the voltage at point $(3,4,5)$?

50) If the electric potential at a point (x,y) in the xy-plane is $V(x,y)=e^{-2x}\cos(2y)$, then the electric intensity vector at (x,y) is $E=-\vec{\nabla}V(x,y)$.

a. Find the electric intensity vector at $(\frac{\pi}{4},0)$.

b. Show that, at each point in the plane, the electric potential decreases most rapidly in the direction of the vector E.

51) In two dimensions, the motion of an ideal fluid is governed by a velocity potential φ. The velocity components of the fluid u in the x-direction and v in the y-direction, are given by $\langle u,v\rangle=\vec{\nabla}\varphi$. Find the velocity components associated with the velocity potential $\varphi(x,y)=\sin\pi x\sin 2\pi y$.

Contributors

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- Paul Seeburger (Monroe Community College) created problems 20 and 22 and added a solution to problem 41.

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13.7: Taylor Polynomials of Functions of Two Variables

Earlier this semester, we saw how to approximate a function $f(x,y)$ by a linear function, that is, by its tangent plane. The tangent plane equation just happens to be the 1^{st}-degree Taylor Polynomial of f at (x,y), as the tangent line equation was the 1^{st}-degree Taylor Polynomial of a function $f(x)$.

Now we will see how to improve this approximation of $f(x,y)$ using a quadratic function: the 2^{nd}-degree Taylor polynomial for f at (x,y).

Review of Taylor Polynomials for a Function of One Variable

Do you remember Taylor Polynomials from Calculus II?

Definition: Taylor polynomials for a function of one variable, $y = f(x)$

If f has n derivatives at $x = c$, then the polynomial,

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n \tag{13.7.1}$$

is called the **n^{th}-degree Taylor Polynomial** for f at c.

Now a function of one variable $f(x)$ can be approximated for x near c using its 1^{st}-degree Taylor Polynomial (i.e., using the equation of its **tangent line** at the point $(c, f(c))$. This 1^{st}-degree Taylor Polynomial is also called the **linear approximation** of $f(x)$ for x near c.

That is:

$$f(x) \approx f(c) + f'(c)(x-c) \tag{13.7.2}$$

Note

Remember that the first-derivative of this 1^{st}-degree Taylor polynomial at $x = c$ is equal to the first derivative of f at $x = c$. That is:

Since $P_1(x) = f(c) + f'(c)(x-c)$,

$$P_1'(c) = f'(c)$$

A better approximation of $f(x)$ for x near c is the **quadratic approximation** (i.e., the 2^{nd}-degree Taylor polynomial of f at $x = c$):

$$f(x) \approx f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 \tag{13.7.3}$$

Note

Remember that both the first and second derivatives of the 2^{nd}-degree Taylor polynomial of f at $x = c$ are the same as those for f at $x = c$. That is:

Since $P_2(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2$,

$$P_2'(c) = f'(c) \quad \text{and} \quad P_2''(c) = f''(c)$$

1st and 2nd-Degree Taylor Polynomials for Functions of Two Variables

Taylor Polynomials work the same way for functions of two variables. (There are just more of each derivative!)

Definition: first-degree Taylor polynomial of a function of two variables, $f(x,y)$

For a function of two variables $f(x,y)$ whose first partials exist at the point (a,b), the **1^{st}-degree Taylor polynomial** of f for (x,y) near the point (a,b) is:

$$f(x,y) \approx L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \quad (13.7.4)$$

$L(x,y)$ is also called the **linear** (or **tangent plane**) **approximation** of f for (x,y) near the point (a,b).

Note that this is really just the equation of the function f's tangent plane.

Also note that the first partial derivatives of this polynomial function are f_x and f_y!

We can obtain an even better approximation of f for (x,y) near the point (a,b) by using the **quadratic approximation** of f for (x,y) near the point (a,b). This is just another name for the 2^{nd}-degree Taylor polynomial of f.

Definition: Second-degree Taylor Polynomial of a function of two variables, $f(x,y)$

For a function of two variables $f(x,y)$ whose first and second partials exist at the point (a,b), the **2^{nd}-degree Taylor polynomial** of f for (x,y) near the point (a,b) is:

$$f(x,y) \approx Q(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + \frac{f_{xx}(a,b)}{2}(x-a)^2 + f_{xy}(a,b)(x-a)(y-b) + \frac{f_{yy}(a,b)}{2}(y-b)^2 \quad (13.7.5)$$

If we have already determined $L(x,y)$, we can simplify this formula as:

$$f(x,y) \approx Q(x,y) = L(x,y) + \frac{f_{xx}(a,b)}{2}(x-a)^2 + f_{xy}(a,b)(x-a)(y-b) + \frac{f_{yy}(a,b)}{2}(y-b)^2 \quad (13.7.6)$$

Note: Since both mixed partials are equal, they combine to form the middle term. Originally there were four terms for the second partials, all divided by 2.

Observe that the power on the factor $(x-a)$ corresponds to the number of times the partial is taken with respect to x and the power on the factor $y-b$ corresponds to the number of times the partial is take with respect to y. For example, in the term with $f_{xx}(a,b)$, you have the factor $(x-a)^2$, since the partial is taken with respect to x twice, and in the term with $f_{xy}(a,b)$, you have the factors $(x-a)$ and $(y-b)$ (both raised to the first power), since the partial is taken with respect to x once and with respect to y once.

Also note that both the first and second partial derivatives of this polynomial function are the same as those for the function f !

Example 13.7.1: Finding 1st and 2nd degree Taylor Polynomials

Determine the 1^{st}- and 2^{nd}-degree Taylor polynomial approximations, $L(x,y)$ & $Q(x,y)$, for the following functions of x and y near the given point.

a. $f(x,y) = \sin 2x + \cos y$ for (x,y) near the point $(0,0)$

b. $f(x,y) = xe^y + 1$ for (x,y) near the point $(1,0)$

Solution

a. To determine the first-degree Taylor polynomial linear approximation, $L(x,y)$, we first compute the partial derivatives of f.

$$f_x(x,y) = 2\cos 2x \quad \text{and} \quad f_y(x,y) = -\sin y$$

Then evaluating these partials and the function itself at the point $(0,0)$ we have:

$$\begin{aligned} f(0,0) &= \sin 2(0) + \cos 0 = 1 \\ f_x(0,0) &= 2\cos 2(0) = 2 \\ f_y(0,0) &= -\sin 0 = 0 \end{aligned}$$

Now,

$$\begin{aligned} L(x,y) &= f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) \\ &= 1 + 2x \end{aligned}$$

See the plot of this function and its linear approximation (the 1^{st}-degree Taylor polynomial) in Figure 13.7.1.

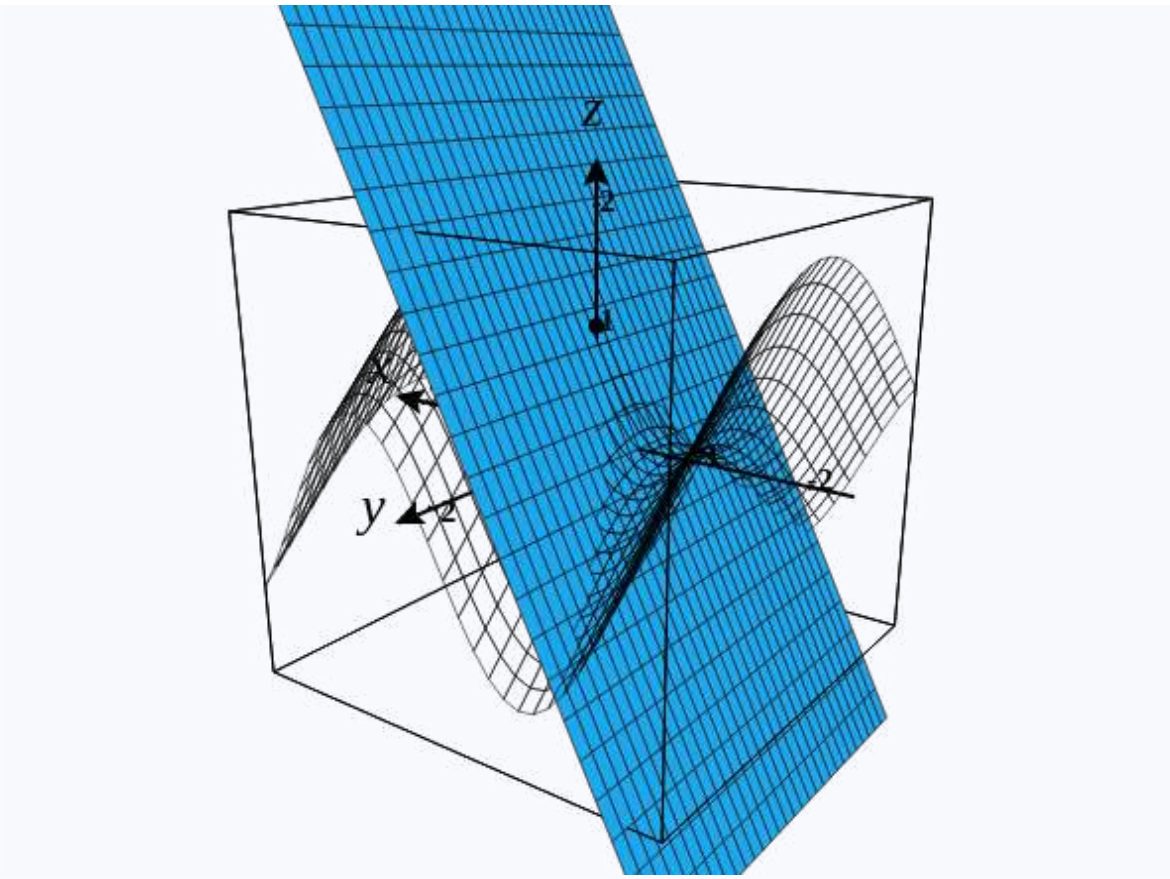

Figure 13.7.1: Graph of $f(x,y) = \sin 2x + \cos y$ and its 1^{st}-degree Taylor polynomial, $L(x,y) = 1 + 2x$

To determine the second-degree Taylor polynomial (quadratic) approximation, $Q(x,y)$, we need the second partials of f:

$$\begin{aligned} f_{xx}(x,y) &= -4\sin 2x \\ f_{xy}(x,y) &= 0 \\ f_{yy}(x,y) &= -\cos y \end{aligned}$$

Evaluating these 2nd partials at the point $(0,0)$:

$$\begin{aligned} f_{xx}(0,0) &= -4\sin 2(0) = 0 \\ f_{xy}(0,0) &= 0 \\ f_{yy}(0,0) &= -\cos 0 = -1 \end{aligned}$$

Then,

$$
\begin{aligned}
Q(x,y) &= L(x,y) + \frac{f_{xx}(0,0)}{2}(x-0)^2 + f_{xy}(0,0)(x-0)(y-0) + \frac{f_{yy}(0,0)}{2}(y-0)^2 \\
&= 1 + 2x + \frac{0}{2}x^2 + (0)xy + \frac{-1}{2}y^2 \\
&= 1 + 2x - \frac{y^2}{2}
\end{aligned}
$$

See the plot of the function f along with its quadratic approximation (the 2^{nd}-degree Taylor polynomial) in Figure 13.7.2.

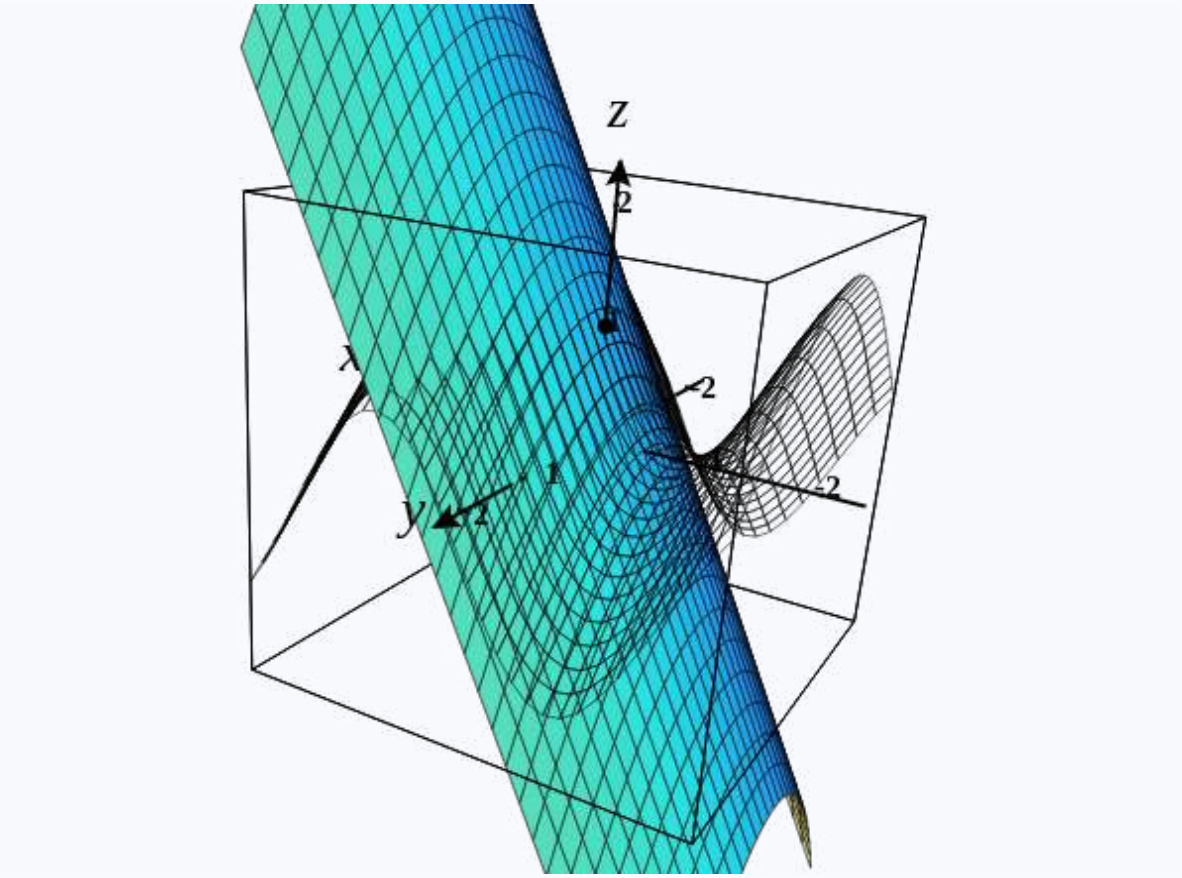

Figure 13.7.2: Graph of $f(x,y) = \sin 2x + \cos y$ and its 2^{nd}-degree Taylor polynomial, $Q(x,y) = 1 + 2x - \frac{y^2}{2}$

b. To determine the first-degree Taylor polynomial linear approximation, $L(x,y)$, we first compute the partial derivatives of $f(x,y) = xe^y + 1$.

$$f_x(x,y) = e^y \quad \text{and} \quad f_y(x,y) = xe^y$$

Then evaluating these partials and the function itself at the point $(1,0)$ we have:

$$f(1,0) = (1)e^0 + 1 = 2$$
$$f_x(1,0) = e^0 = 1$$
$$f_y(1,0) = (1)e^0 = 1$$

Now,

$$\begin{aligned} L(x,y) &= f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0) \\ &= 2 + 1(x-1) + 1y \\ &= 1 + x + y \end{aligned}$$

See the plot of this function and its linear approximation (the 1^{st}-degree Taylor polynomial) in Figure 13.7.3.

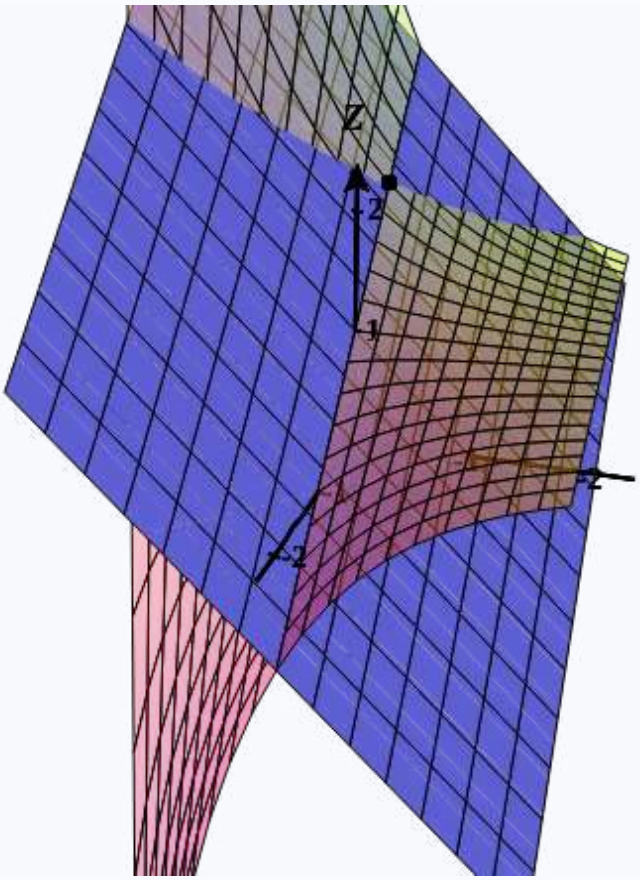

Figure 13.7.3: Graph of $f(x,y) = xe^y + 1$ and its 1^{st}-degree Taylor polynomial, $L(x,y) = 1 + x + y$

To determine the second-degree Taylor polynomial (quadratic) approximation, $Q(x,y)$, we need the second partials of f:

$$f_{xx}(x,y) = 0$$
$$f_{xy}(x,y) = e^y$$
$$f_{yy}(x,y) = xe^y$$

Evaluating these 2nd partials at the point $(1,0)$:

$$f_{xx}(1,0) = 0$$
$$f_{xy}(1,0) = e^0 = 1$$
$$f_{yy}(1,0) = (1)e^0 = 1$$

Then,

$$\begin{aligned} Q(x,y) &= L(x,y) + \frac{f_{xx}(1,0)}{2}(x-1)^2 + f_{xy}(1,0)(x-1)(y-0) + \frac{f_{yy}(1,0)}{2}(y-0)^2 \\ &= 1 + x + y + \frac{0}{2}(x-1)^2 + (1)(x-1)y + \frac{1}{2}y^2 \\ &= 1 + x + y + xy - y + \frac{y^2}{2} \\ &= 1 + x + xy + \frac{y^2}{2} \end{aligned}$$

See the plot of the function f along with its quadratic approximation (the 2^{nd}-degree Taylor polynomial) in Figure 13.7.4.

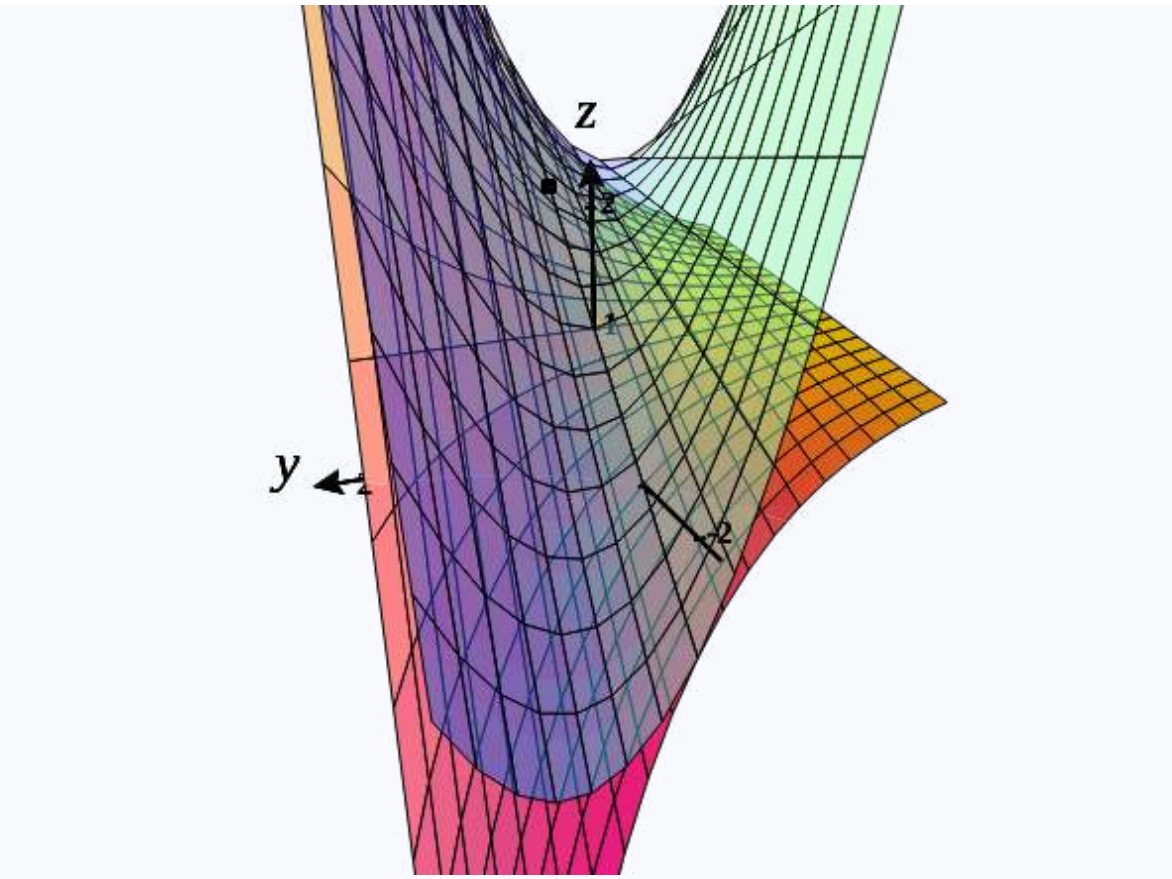

Figure 13.7.4: Graph of $f(x,y) = xe^y + 1$ and its 2^{nd}-degree Taylor polynomial, $Q(x,y) = 1 + x + xy + \frac{y^2}{2}$

Higher-Degree Taylor Polynomials of a Function of Two Variables

To calculate the Taylor polynomial of degree n for functions of two variables beyond the second degree, we need to work out the pattern that allows all the partials of the polynomial to be equal to the partials of the function being approximated at the point (a,b), up to the given degree. That is, for $P_3(x,y)$ we will need its first, second and third partials to all match those of $f(x,y)$ at the point (a,b). For $P_{10}(x,y)$ we would need all its partials up to the tenth partials to all match those of $f(x,y)$ at the point (a,b).

If you work out this pattern, it gives us the following interesting formula for the n^{th}-degree Taylor polynomial of $f(x,y)$, assuming all these partials exist.

Definition: n^{th}-degree Taylor Polynomial for a function of two variables

For a function of two variables $f(x,y)$ whose partials all exist to the n^{th} partials at the point (a,b), the **n^{th}-degree Taylor polynomial** of f for (x,y) near the point (a,b) is:

$$P_n(x,y) = \sum_{i=0}^{n} \sum_{j=0}^{n-i} \frac{\frac{d^{(i+j)}f}{\partial x^i \partial y^j}(a,b)}{i!j!}(x-a)^i(y-b)^j \tag{13.7.7}$$

Let's verify this formula for the second-degree Taylor polynomial. (We'll leave it to you to verify it for the first-degree Taylor polynomial.)

For $n = 2$, we have:

$$P_2(x,y) = \sum_{i=0}^{2} \sum_{j=0}^{2-i} \frac{\frac{d^{(i+j)}f}{\partial x^i \partial y^j}(a,b)}{i!j!}(x-a)^i(y-b)^j \tag{13.7.8}$$

Since i will start at 0 and continue to increase up to 2, while the value of j will start at 0 and increase to $2-i$ for each value of i, we would see the following values for i and j:

$$\begin{aligned} i &= 0, \quad & j &= 0 \\ i &= 0, \quad & j &= 1 \\ i &= 0, \quad & j &= 2 \\ i &= 1, \quad & j &= 0 \\ i &= 1, \quad & j &= 1 \\ i &= 2, \quad & j &= 0 \end{aligned}$$

Then by the formula:

$$\begin{aligned} P_2(x,y) &= \frac{f(a,b)}{0!0!}(x-a)^0(y-b)^0 + \frac{f_y(a,b)}{0!1!}(x-a)^0(y-b)^1 + \frac{f_{yy}(a,b)}{0!2!}(x-a)^0(y-b)^2 \\ &\quad + \frac{f_x(a,b)}{1!0!}(x-a)^1(y-b)^0 + \frac{f_{xy}(a,b)}{1!1!}(x-a)^1(y-b)^1 + \frac{f_{xx}(a,b)}{2!0!}(x-a)^2(y-b)^0 \\ &= f(a,b) + f_y(a,b)(y-b) + \frac{f_{yy}(a,b)}{2}(y-b)^2 + f_x(a,b)(x-a) \\ &\quad + f_{xy}(a,b)(x-a)(y-b) + \frac{f_{xx}(a,b)}{2}(x-a)^2 \\ &= f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + \frac{f_{xx}(a,b)}{2}(x-a)^2 \\ &\quad + f_{xy}(a,b)(x-a)(y-b) + \frac{f_{yy}(a,b)}{2}(y-b)^2 \end{aligned}$$

This equation is the same as Equation 13.7.5 above.

Note that $P_2(x,y)$ is the more formal notation for the second-degree Taylor polynomial $Q(x,y)$.

Exercise 13.7.1: Finding a third-degree Taylor polynomial for a function of two variables

Now try to find the new terms you would need to find $P_3(x,y)$ and use this new formula to calculate the third-degree Taylor polynomial for one of the functions in Example 13.7.1 above. Verify your result using a 3D function grapher like CalcPlot3D.

Contributors

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13.7E: Taylor Polynomials of Functions of Two Variables (Exercises)

13.7: Taylor Polynomials of Functions of Two Variables

In the exercises 1 - 8, find the linear approximation $L(x,y)$ and the quadratic approximation $Q(x,y)$ of each function at the indicated point. These are the 1^{st}- and 2^{nd}-degree Taylor Polynomials of these functions at these points. Use a 3D grapher like CalcPlot3D to verify that each linear approximation is tangent to the given surface at the given point and that each quadratic approximation is not only tangent to the surface at the given point, but also shares the same concavity as the surface at this point.

1) $f(x,y) = x\sqrt{y}, \quad P(1,4)$

2) $f(x,y) = e^x \cos y; \quad P(0,0)$

3) $f(x,y) = \arctan(x+2y), \quad P(1,0)$

4) $f(x,y) = \sqrt{20 - x^2 - 7y^2}, \quad P(2,1)$

5) $f(x,y) = x^2y + y^2, \quad P(1,3)$

6) $f(x,y) = \cos x \cos 3y, \quad P(0,0)$

7) $f(x,y) = \ln(x^2 + y^2 + 1), \quad P(0,0)$

8) $f(x,y) = \sqrt{2x - y}, \quad P(1,-2)$

9) Verify that the formula for higher-degree Taylor polynomials works for the first-degree Taylor polynomial $L(x,y) = P_1(x,y)$. For convenience, the formula is given below.

$$P_n(x,y) = \sum_{i=0}^{n} \sum_{j=0}^{n-i} \frac{\frac{d^{(i+j)}f}{\partial x^i \partial y^j}(a,b)}{i!j!}(x-a)^i(y-b)^j$$

10) Determine the new terms that would be added to $P_3(x,y)$ (which you found in Exercise 13.7.1) to form $P_4(x,y)$ and determine the fourth-degree Taylor polynomial for one of the functions we've considered and graph it together with the surface plot of the corresponding function in a 3D grapher like CalcPlot3D to verify that it continues to fit the surface better.

Contributors

- Paul Seeburger (Monroe Community College)
- Exercises 1-4 were adapted from problems provided in the section on Tangent Planes & Differentials from the OpenStax Calculus 3 textbook.

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13.8: Optimization of Functions of Several Variables

One of the most useful applications for derivatives of a function of one variable is the determination of maximum and/or minimum values. This application is also important for functions of two or more variables, but as we have seen in earlier sections of this chapter, the introduction of more independent variables leads to more possible outcomes for the calculations. The main ideas of finding critical points and using derivative tests are still valid, but new wrinkles appear when assessing the results.

Critical Points

For functions of a single variable, we defined critical points as the values of the function when the derivative equals zero or does not exist. For functions of two or more variables, the concept is essentially the same, except for the fact that we are now working with partial derivatives.

Definition: Critical Points

Let $z = f(x, y)$ be a function of two variables that is differentiable on an open set containing the point (x_0, y_0). The point (x_0, y_0) is called a *critical point* of a function of two variables f if one of the two following conditions holds:

1. $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$
2. Either $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ does not exist.

Example 13.8.1: Finding Critical Points

Find the critical points of each of the following functions:

a. $f(x, y) = \sqrt{4y^2 - 9x^2 + 24y + 36x + 36}$
b. $g(x, y) = x^2 + 2xy - 4y^2 + 4x - 6y + 4$

Solution:

a. First, we calculate $f_x(x, y)$ and $f_y(x, y)$:

$$\begin{aligned} f_x(x, y) &= \frac{1}{2}(-18x + 36)(4y^2 - 9x^2 + 24y + 36x + 36)^{-1/2} \\ &= \frac{-9x + 18}{\sqrt{4y^2 - 9x^2 + 24y + 36x + 36}} \end{aligned}$$

$$\begin{aligned} f_y(x, y) &= \frac{1}{2}(8y + 24)(4y^2 - 9x^2 + 24y + 36x + 36)^{-1/2} \\ &= \frac{4y + 12}{\sqrt{4y^2 - 9x^2 + 24y + 36x + 36}} \end{aligned}.$$

Next, we set each of these expressions equal to zero:

$$\begin{aligned} \frac{-9x + 18}{\sqrt{4y^2 - 9x^2 + 24y + 36x + 36}} &= 0 \\ \frac{4y + 12}{\sqrt{4y^2 - 9x^2 + 24y + 36x + 36}} &= 0. \end{aligned}$$

Then, multiply both sides of each equation by its denominator (to clear the denominators):

$$\begin{aligned} -9x + 18 &= 0 \\ 4y + 12 &= 0. \end{aligned}$$

Therefore, $x = 2$ and $y = -3$, so $(2, -3)$ is a critical point of f.

We must also check for the possibility that the denominator of each partial derivative can equal zero, thus causing the partial derivative not to exist. Since the denominator is the same in each partial derivative, we need only do this once:

$$4y^2 - 9x^2 + 24y + 36x + 36 = 0.$$

This equation represents a hyperbola. We should also note that the domain of f consists of points satisfying the inequality

$$4y^2 - 9x^2 + 24y + 36x + 36 \geq 0.$$

Therefore, any points on the hyperbola are not only critical points, they are also on the boundary of the domain. To put the hyperbola in standard form, we use the method of completing the square:

$$\begin{aligned} 4y^2 - 9x^2 + 24y + 36x + 36 &= 0 \\ 4y^2 - 9x^2 + 24y + 36x &= -36 \\ 4y^2 + 24y - 9x^2 + 36x &= -36 \\ 4(y^2 + 6y) - 9(x^2 - 4x) &= -36 \\ 4(y^2 + 6y + 9) - 9(x^2 - 4x + 4) &= -36 - 36 + 36 \\ 4(y+3)^2 - 9(x-2)^2 &= -36. \end{aligned}$$

Dividing both sides by -36 puts the equation in standard form:

$$\begin{aligned} \frac{4(y+3)^2}{-36} - \frac{9(x-2)^2}{-36} &= 1 \\ \frac{(x-2)^2}{4} - \frac{(y+3)^2}{9} &= 1. \end{aligned}$$

Notice that point $(2, -3)$ is the center of the hyperbola.

Thus, the critical points of the function f are $(2, -3)$ and all points on the hyperbola, $\frac{(x-2)^2}{4} - \frac{(y+3)^2}{9} = 1$.

b. First, we calculate $g_x(x, y)$ and $g_y(x, y)$:

$$\begin{aligned} g_x(x, y) &= 2x + 2y + 4 \\ g_y(x, y) &= 2x - 8y - 6. \end{aligned}$$

Next, we set each of these expressions equal to zero, which gives a system of equations in x and y:

$$\begin{aligned} 2x + 2y + 4 &= 0 \\ 2x - 8y - 6 &= 0. \end{aligned}$$

Subtracting the second equation from the first gives $10y + 10 = 0$, so $y = -1$. Substituting this into the first equation gives $2x + 2(-1) + 4 = 0$, so $x = -1$.

Therefore $(-1, -1)$ is a critical point of g. There are no points in $\mathbb{R}^2$ that make either partial derivative not exist.

Figure 13.8.1 shows the behavior of the surface at the critical point.

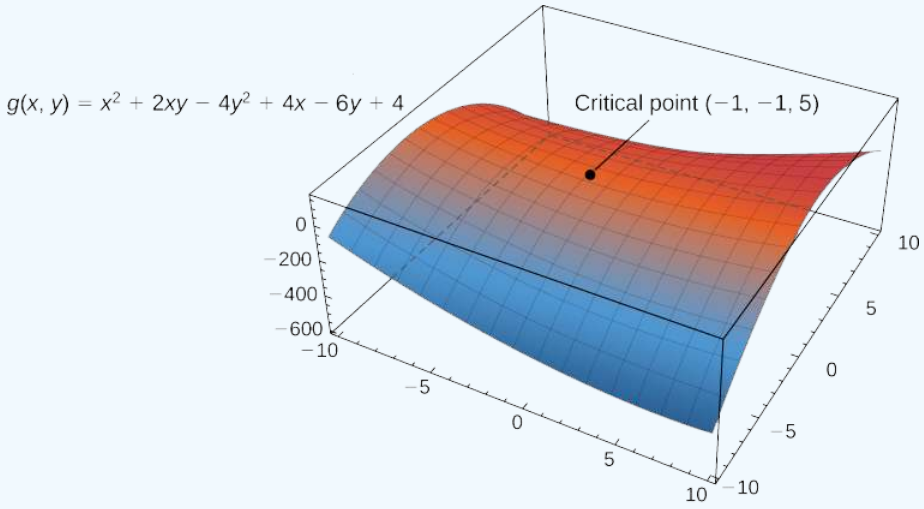

Figure 13.8.1: The function $g(x, y)$ has a critical point at $(-1, -1, 5)$.

Exercise 13.8.1:

Find the critical point of the function $f(x,y) = x^3 + 2xy - 2x - 4y$.

Determining Global and Local Extrema

The main purpose for determining critical points is to locate relative maxima and minima, as in single-variable calculus. When working with a function of one variable, the definition of a local extremum involves finding an interval around the critical point such that the function value is either greater than or less than all the other function values in that interval. When working with a function of two or more variables, we work with an open disk around the point.

Definition: Global and Local Extrema

Let $z = f(x,y)$ be a function of two variables that is defined and continuous on an open set containing the point (x_0, y_0). Then f has a **local maximum** at (x_0, y_0) if

$$f(x_0, y_0) \geq f(x,y) \tag{13.8.1}$$

for all points (x,y) within some disk centered at (x_0, y_0). The number $f(x_0, y_0)$ is called a local maximum value. If the preceding inequality holds for every point (x,y) in the domain of f, then f has a **global maximum** (also called an **absolute maximum**) at (x_0, y_0).

The function f has a local minimum at (x_0, y_0) if

$$f(x_0, y_0) \leq f(x,y) \tag{13.8.2}$$

for all points (x,y) within some disk centered at (x_0, y_0). The number $f(x_0, y_0)$ is called a local minimum value. If the preceding inequality holds for every point (x,y) in the domain of f, then f has a **global minimum** (also called an **absolute minimum**) at (x_0, y_0).

If $f(x_0, y_0)$ is either a local maximum or local minimum value, then it is called a **local extremum** (Figure 13.8.2).

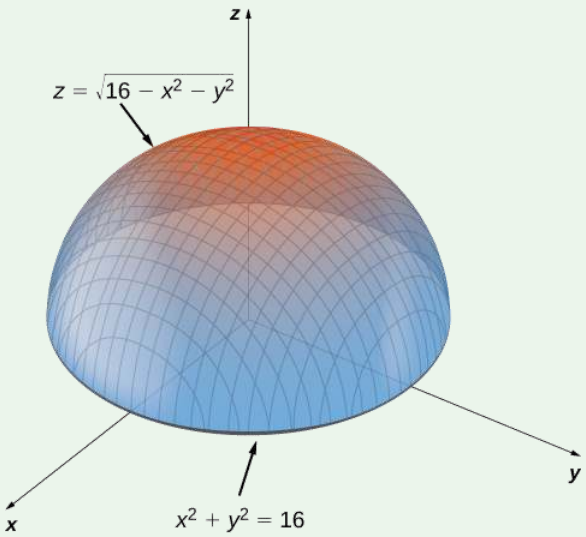

Figure 13.8.2: The graph of $z = \sqrt{16 - x^2 - y^2}$ has a maximum value when $(x,y) = (0,0)$. It attains its minimum value at the boundary of its domain, which is the circle $x^2 + y^2 = 16$.

In Calculus 1, we showed that extrema of functions of one variable occur at critical points. The same is true for functions of more than one variable, as stated in the following theorem.

Theorem 13.8.1: Fermat's Theorem for Functions of Two Variables

Let $z = f(x, y)$ be a function of two variables that is defined and continuous on an open set containing the point (x_0, y_0). Suppose f_x and f_y each exists at (x_0, y_0). If f has a local extremum at (x_0, y_0), then (x_0, y_0) is a critical point of f.

Consider the function $f(x) = x^3$. This function has a critical point at $x = 0$, since $f'(0) = 3(0)^2 = 0$. However, f does not have an extreme value at $x = 0$. Therefore, the existence of a critical value at $x = x_0$ does not guarantee a local extremum at $x = x_0$. The same is true for a function of two or more variables. One way this can happen is at a **saddle point**. An example of a saddle point appears in the following figure.

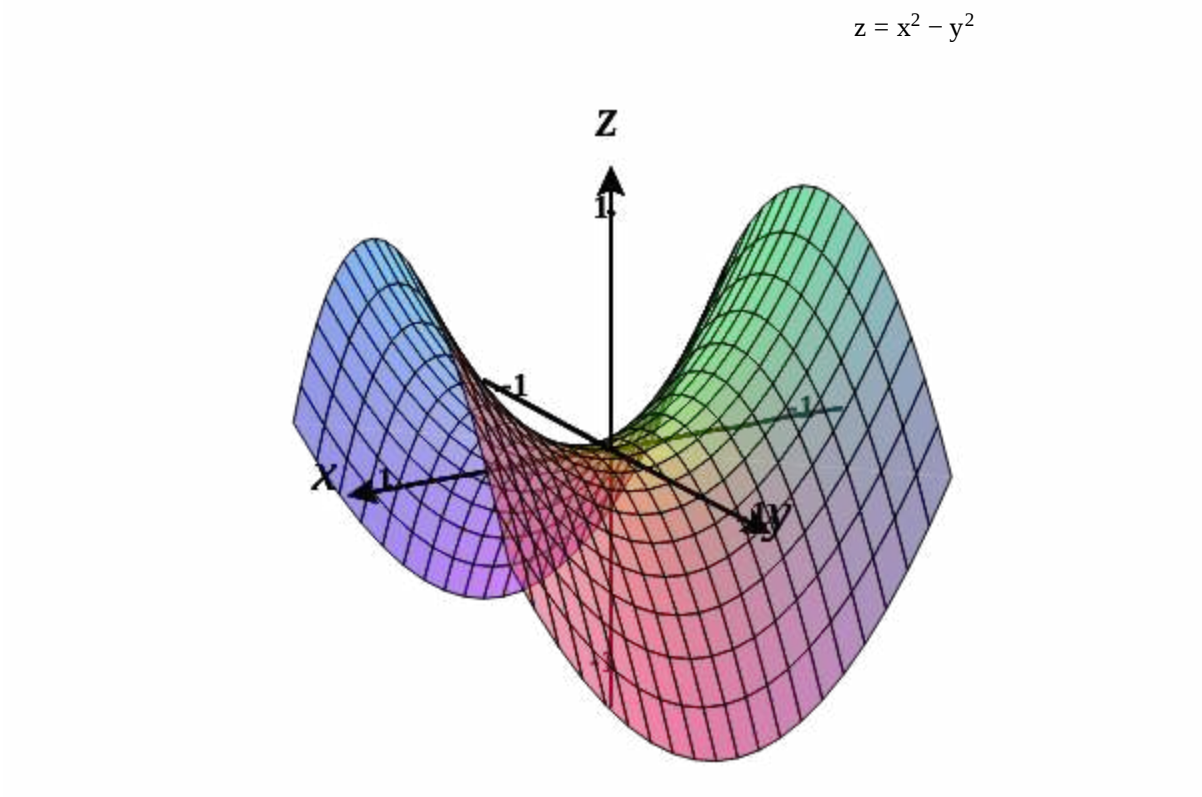

Figure* 13.8.3*: *Graph of the function $z = x^2 - y^2$. This graph has a saddle point at the origin.*

In this graph, the origin is a saddle point. This is because the first partial derivatives of $f(x, y) = x^2 - y^2$ are both equal to zero at this point, but it is neither a maximum nor a minimum for the function. Furthermore the vertical trace corresponding to $y = 0$ is $z = x^2$ (a parabola opening upward), but the vertical trace corresponding to $x = 0$ is $z = -y^2$ (a parabola opening downward). Therefore, it is both a global maximum for one trace and a global minimum for another.

Definition: Saddle Point

Given the function $z = f(x, y)$, the point $\big(x_0, y_0, f(x_0, y_0)\big)$ is a saddle point if both $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$, but f does not have a local extremum at (x_0, y_0).

Classifying Critical Points

In order to develop a general method for classifying the behavior of a function of two variables at its critical points, we need to begin by classifying the behavior of quadratic polynomial functions of two variables at their critical points.

To see why this will help us, consider that the quadratic approximation of a function of two variables (its 2nd-degree Taylor polynomial) shares the same first and second partials as the function it approximates at the chosen point of tangency (or center point). Since sharing the same second partials means the two surfaces will share the same concavity (or curvature) at the critical

point, this causes these quadratic approximation surfaces to share the same behavior as the function $z = f(x, y)$ that they approximate at the point of tangency. In other words, if the original function has a relative maximum at this point, so will the quadratic approximation. If the original function has a relative minimum at this point, so will the quadratic approximation, and if the original function has a saddle point at this point, so will the quadratic approximation.

Now there are really three basic behaviors of a quadratic polynomial in two variables at a point where it has a critical point. It will fit one of the following three forms, often being a transformation of one of the following functions.

1. A sum of two squared terms, like $z = x^2 + y^2$, producing a paraboloid that opens up and has a relative (absolute) minimum at its vertex. See the plot on the left side of Figure **13.8.4**.
2. The negative of a sum of two squared terms, like $z = -(x^2 + y^2)$, producing a paraboloid that opens down and has a relative (absolute) maximum at its vertex. See the plot on the right side of Figure **13.8.4**.
3. The difference of two squared terms, like $z = f(x, y) = x^2 - y^2$ or $z = f(x, y) = y^2 - x^2$, producing a saddle with a saddle point at its critical point. See Figure **13.8.3**.

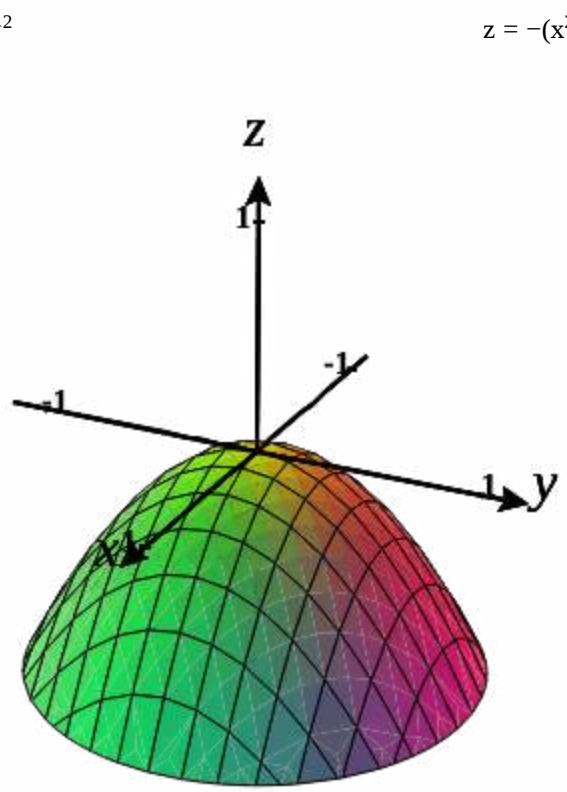

***Figure* 13.8.4:** *$z = x^2 + y^2$ has an absolute minimum of 0 at $(0, 0)$, while $z = -(x^2 + y^2)$ has an absolute maximum of 0 at $(0, 0)$,*

Example 13.8.1: Classifying the critical points of a function

Use completing the square to identify local extrema or saddle points of the following quadratic polynomial functions:

a. $f(x, y) = x^2 - 6x + y^2 + 10y + 20$
b. $f(x, y) = 12 - 3x^2 - 6x - y^2 + 12y$
c. $f(x, y) = x^2 + 8x - 2y^2 + 16y$

d. $f(x,y) = x^2 + 6xy + y^2$

Solution

a. To determine the critical points of this function, we start by setting the partials of f equal to 0.

$$\begin{array}{ll} \text{Set} & f_x(x,y) = 2x - 6 = 0 \quad \Longrightarrow \; x = 3 \\ \text{and} & f_y(x,y) = 2y + 10 = 0 \quad \Longrightarrow \; y = -5 \end{array}$$

We obtain a single critical point with coordinates $(3,-5)$. Next we need to determine the behavior of the function f at this point.

Completing the square, we get:

$$\begin{aligned} f(x,y) &= x^2 - 6x + y^2 + 10y + 20 \\ &= x^2 - 6x + 9 + y^2 + 10y + 25 + 20 - 9 - 25 \\ &= (x-3)^2 + (y+5)^2 - 14 \end{aligned}$$

Notice that this function is really just a translated version of $z = x^2 + y^2$, so it is a paraboloid that opens up with its vertex (minimum point) at the critical point $(3,-5)$. We can argue that it has an absolute minimum value of -14 at the point $(3,-5)$, since we are adding squared terms to -14 and thus cannot get a value less than -14 for any values of x and y, while we do obtain this minimum value of -14 at the vertex point $(3,-5)$.

b. Setting the partials of f equal to 0, we obtain:

$$\begin{array}{ll} \text{Set} & f_x(x,y) = -6x - 6 = 0 \quad \Longrightarrow \; x = -1 \\ \text{and} & f_y(x,y) = -2y + 12 = 0 \quad \Longrightarrow \; y = 6 \end{array}$$

We obtain a single critical point with coordinates $(-1,6)$. Next we need to determine the behavior of the function f at this point.

To complete the square here, we first need to factor out the factors of the squared terms. Doing this and reordering the terms some gives us:

$$\begin{aligned} f(x,y) &= 12 - 3x^2 - 6x - y^2 + 12y \\ &= -3\left(x^2 + 2x \quad\right) - 1\left(y^2 - 12y \quad\right) + 12 \\ &= -3\left(x^2 + 2x + 1\right) - 1\left(y^2 - 12y + 36\right) + 12 + 3 + 36 \\ &= 51 - 3(x+1)^2 - (y-6)^2 \end{aligned}$$

Notice that this function is an elliptic paraboloid that opens down with its vertex (maximum point) at the critical point $(-1,6)$. We can argue that it has an absolute maximum value of 51 at the point $(-1,6)$, since we are subtracting squared terms from 51 and thus cannot get a value more than 51 for any values of x and y, while we do obtain this minimum value of 51 at the vertex point $(-1,6)$.

c. Setting the partials of f equal to 0, we obtain:

$$\begin{array}{ll} \text{Set} & f_x(x,y) = 2x + 8 = 0 \quad \Longrightarrow \; x = -4 \\ \text{and} & f_y(x,y) = -4y + 16 = 0 \quad \Longrightarrow \; y = 4 \end{array}$$

This gives us a critical point with coordinates $(-4,4)$. To determine if f has a local extremum or saddle point at this point, we complete the square.

Factoring out -2 from the y-squared term gives us:

$$\begin{aligned} f(x,y) &= x^2 + 8x - 2y^2 + 16y \\ &= x^2 + 8x + 16 - 2\left(y^2 - 8y + 16\right) - 16 + 32 \\ &= (x+4)^2 - 2(y-4)^2 + 16 \end{aligned}$$

Since one squared term is positive and one is negative, we see that this function has the form of $z = x^2 - y^2$ and so it has a saddle point at its critical point. That is, f has a saddle point at $(-4,4,16)$.

d. Setting the partials of f equal to 0, we get:

$$\begin{aligned} \text{Set} \quad & f_x(x,y) = 2x + 6y = 0 \\ \text{and} \quad & f_y(x,y) = 6x + 2y = 0 \end{aligned} \qquad \Longrightarrow \; y = -3x$$

Substituting $-3x$ into the first equation for y gives us,

$$\begin{aligned} 2x + 6(-3x) &= 0 \\ -16x &= 0 \\ x &= 0 \end{aligned}$$

Since $y = -3x$, we have $y = -3(0) = 0$, so the critical point of f is $(0,0)$. To determine the behavior of f at this critical point, we complete the square.

$$\begin{aligned} f(x,y) &= x^2 + 6xy + y^2 \\ &= (x^2 + 6xy + 9y^2) + y^2 - 9y^2 \\ &= (x+3y)^2 - 8y^2 \end{aligned}$$

As this produces a difference of squares with one positive squared term and the other a negative squared term, we see that f takes a form similar to $z = x^2 - y^2$ and will have a saddle point at $(0,0,0)$.

Now let's consider the quadratic approximation to a function $z = f(x,y)$ centered at a critical point (x_0, y_0) of this function.

$$\begin{aligned} Q(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + \frac{f_{xx}(x_0,y_0)}{2}(x-x_0)^2 \\ + f_{xy}(x_0,y_0)(x-x_0)(y-y_0) + \frac{f_{yy}(x_0,y_0)}{2}(y-y_0)^2 \end{aligned}$$

But, since the point (x_0, y_0), in this case, is a critical point of f, we know that $f_x(x_0,y_0) = 0$ and $f_y(x_0,y_0) = 0$.

This allows us to simplify $Q(x,y)$ to just:

$$\begin{aligned} & Q(x,y) \\ = f(x_0,y_0) + \frac{f_{xx}(x_0,y_0)}{2}(x-x_0)^2 + f_{xy}(x_0,y_0)(x-x_0)(y-y_0) + \frac{f_{yy}(x_0,y_0)}{2}(y-y_0)^2 \end{aligned}$$

Now we need to complete the square on this quadratic polynomial in two variables to learn how we can classify the behavior of this function at this critical point. Remember that the original function will share the same behavior (max, min, saddle point) as this 2nd-degree Taylor polynomial at this critical point.

To make this process easier, let's make some substitutions. Let's choose to let $u = x - x_0$ and $v = y - y_0$,

and let

$$\begin{aligned} a &= \frac{f_{xx}(x_0,y_0)}{2}, \\ b &= f_{xy}(x_0,y_0), \\ c &= \frac{f_{yy}(x_0,y_0)}{2} \text{ and} \\ d &= f(x_0,y_0) \end{aligned}$$

Then we need to complete the square on the polynomial:

$$Q(x,y) = au^2 + buv + cv^2 + d$$

Completing the square:

First we factor out the coefficient of u^2:

$$= a\left[u^2 + \frac{b}{a}uv + \frac{c}{a}v^2\right] + d$$

Next, we complete the square using the first two terms:

$$= a\left[\left(u^2 + \frac{b}{a}uv + \left(\frac{b}{2a}v\right)^2\right) + \frac{c}{a}v^2 - \left(\frac{b}{2a}v\right)^2\right] + d$$

Rewriting the perfect square trinomial as the square of a binomial and combining the v^2 terms yields:

$$\begin{aligned} &= a\left[\left(u+\frac{b}{2a}v\right)^2+\left(\frac{c}{a}-\frac{b^2}{4a^2}\right)v^2\right]+d \\ &= a\left[\left(u+\frac{b}{2a}v\right)^2+\left(\frac{4ac}{4a^2}-\frac{b^2}{4a^2}\right)v^2\right]+d \\ &= a\left[\left(u+\frac{b}{2a}v\right)^2+\left(\frac{4ac-b^2}{4a^2}\right)v^2\right]+d \end{aligned}$$

Note that the shape of this function's graph depends on the sign of the coefficient of v^2. And the sign of this coefficient is determined only by its numerator, as the denominator is always positive (being a perfect square). This expression, $4ac-b^2$, is called the discriminant, as it helps us discriminate (tell the difference between) which behavior the function has at this critical point.

If $D=4ac-b^2>0$, then the two squared terms inside the brackets are both positive, and

- if $a=\frac{f_{xx}(x_0,y_0)}{2}>0$, the function f opens upwards with a local minimum at the critical point (x_0,y_0). Note it would be similar to the form, $z=x^2+y^2$.
- if $a=\frac{f_{xx}(x_0,y_0)}{2}<0$, the function f opens downwards with a local maximum at the critical point (x_0,y_0). Note it would be similar to the form, $z=-(x^2+y^2)$.

If $D=4ac-b^2<0$, then either

- the two squared terms inside the brackets have opposite signs (meaning f is concave up along a line parallel to the x-axis and concave down along a line parallel to the y-axis, or vice-versa) or
- the b^2 term, representing the square of the mixed partial $f_{xy}(x_0,y_0)$, is larger than the positive product of the two 2nd-partials $f_{xx}(x_0,y_0)$ and $f_{yy}(x_0,y_0)$. This means that even if the surface is concave up in both x- and y-directions, or concave down in both x- and y-directions, a large mixed partial can offset these and cause the surface to have a saddle point at the point (x_0,y_0).

In either case, the quadratic polynomial will be in the form of $z=x^2-y^2$ or $z=y^2-x^2$ (i.e., it will be the difference of two squared terms), so we get a saddle point at the critical point (x_0,y_0).

But if $D=4ac-b^2=0$, the quadratic polynomial reduces to $Q(x,y)=a\left(u+\frac{b}{2a}v\right)^2+d$, whose graph is a parabolic cylinder, so the behavior of the function is not clear at the critical point (x_0,y_0).

Now remembering the values of the constants a, b, and c from above, we see that:

$$\begin{aligned} D(x_0,y_0) &= 4\frac{f_{xx}(x_0,y_0)}{2}\frac{f_{yy}(x_0,y_0)}{2}-\left(f_{xy}(x_0,y_0)\right)^2 \\ &= f_{xx}(x_0,y_0)f_{yy}(x_0,y_0)-\left(f_{xy}(x_0,y_0)\right)^2 \end{aligned}$$

This formula is called the **Second Partials Test**, and it can be used to classify the behavior of any function at its critical points, as long as its second partials exist there and as long as the value of this discriminate is not zero.

The Second Partials Test

The second derivative test for a function of one variable provides a method for determining whether an extremum occurs at a critical point of a function. When extending this result to a function of two variables, an issue arises related to the fact that there are, in fact, four different second-order partial derivatives, although equality of mixed partials reduces this to three. The second partials test for a function of two variables, stated in the following theorem, uses a **discriminant** D that replaces $f''(x_0)$ in the second derivative test for a function of one variable.

Theorem 13.8.2: The Second Partials Test

Let $z=f(x,y)$ be a function of two variables for which the first- and second-order partial derivatives are continuous on some disk containing the point (x_0,y_0). Suppose $f_x(x_0,y_0)=0$ and $f_y(x_0,y_0)=0$. Define the quantity

$$D=f_{xx}(x_0,y_0)f_{yy}(x_0,y_0)-\left(f_{xy}(x_0,y_0)\right)^2. \tag{13.8.3}$$

Then:

i. If $D > 0$ and $f_{xx}(x_0, y_0) > 0$, then f is concave up at this critical point, so f has a local minimum at (x_0, y_0).
ii. If $D > 0$ and $f_{xx}(x_0, y_0) < 0$, then f is concave down at this critical point, so f has a local maximum at (x_0, y_0).
iii. If $D < 0$, then f has a saddle point at (x_0, y_0).
iv. If $D = 0$, then the test is inconclusive.

See Figure 13.8.4.

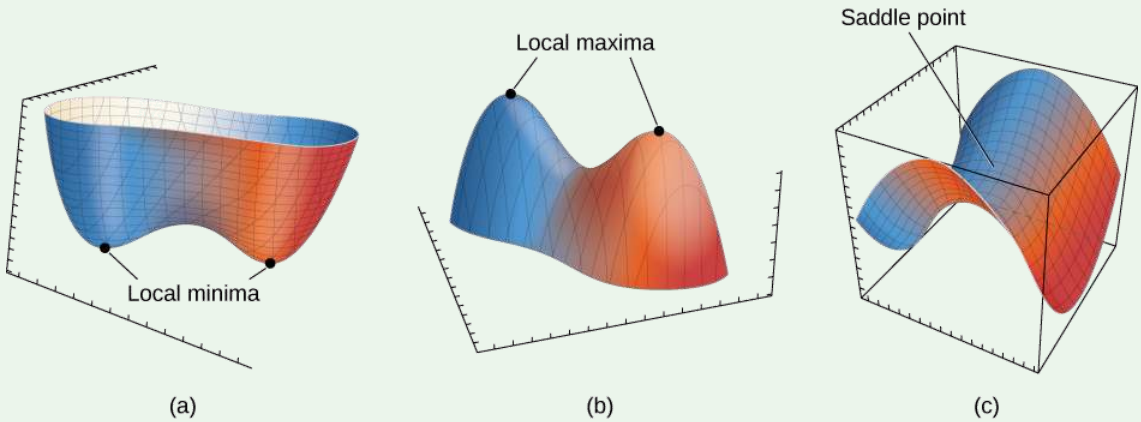

Figure 13.8.4: The second partials test can often determine whether a function of two variables has a local minima (a), a local maxima (b), or a saddle point (c).

To apply the second partials test, it is necessary that we first find the critical points of the function. There are several steps involved in the entire procedure, which are outlined in a problem-solving strategy.

Problem-Solving Strategy: Using the second partials Test for Functions of Two Variables

Let $z = f(x, y)$ be a function of two variables for which the first- and second-order partial derivatives are continuous on some disk containing the point (x_0, y_0). To apply the second partials test to find local extrema, use the following steps:

1. Determine the critical points (x_0, y_0) of the function f where $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$. If you find any critical points where at least one of the partial derivatives does not exist, you will need to find and justify extrema in another way, as you can't use the second partials test.
2. Calculate the discriminant $D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - \left(f_{xy}(x_0, y_0)\right)^2$ for each critical point of f.
3. Apply the four cases of the test to determine whether each critical point is a local maximum, local minimum, or saddle point, or whether the test is inconclusive. If the test is inconclusive, you will need to analyze and classify the behavior at the critical point another way.

Example 13.8.2: Using the Second Partials Test

Find the critical points for each of the following functions, and use the second partials test to find any local extrema or saddle points.

a. $f(x, y) = 4x^2 + 9y^2 + 8x - 36y + 24$
b. $g(x, y) = \frac{1}{3}x^3 + y^2 + 2xy - 6x - 3y + 4$

Solution:

a. **Step 1** of the problem-solving strategy requires us to find the critical points of f. To do this, we first calculate $f_x(x, y)$ and $f_y(x, y)$ and then set each of them equal to zero:

$$f_x(x, y) = 8x + 8$$
$$f_y(x, y) = 18y - 36.$$

Setting them equal to zero yields the system of equations

$$8x + 8 = 0$$
$$18y - 36 = 0.$$

The solution to this system is $x = -1$ and $y = 2$. Therefore $(-1, 2)$ is the only critical point of f.

Step 2 of the problem-solving strategy involves calculating D. To do this, we first calculate the second partial derivatives of f :

$$f_{xx}(x, y) = 8$$
$$f_{xy}(x, y) = 0$$
$$f_{yy}(x, y) = 18.$$

Therefore, $D(-1, 2) = f_{xx}(-1, 2) f_{yy}(-1, 2) - \left(f_{xy}(-1, 2)\right)^2 = (8)(18) - (0)^2 = 144 > 0$.

Step 3 tells us to apply the four cases of the test to classify the function's behavior at this critical point.

Since $D > 0$ and $f_{xx}(-1, 2) = 8 > 0$, f is concave up, so f has a local minimum of $f(-1, 2) = -16$ at $(-1, 2)$, as shown in the following figure. (Note that this corresponds to case 1 of the second partials test.)

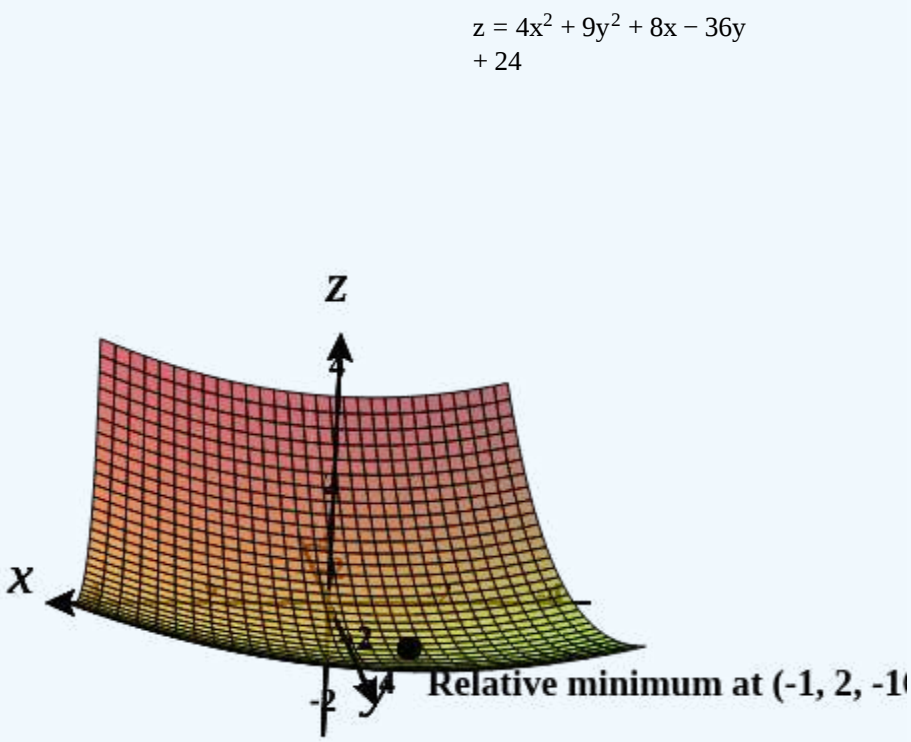

***Figure* 13.8.5:** *The function $f(x, y)$ has a local minimum at $(-1, 2, -16)$. Note the scale on the y-axis in this plot is in thousands.*

b. For **step 1**, we first calculate $g_x(x, y)$ and $g_y(x, y)$, then set each of them equal to zero:

$$g_x(x, y) = x^2 + 2y - 6$$
$$g_y(x, y) = 2y + 2x - 3.$$

Setting them equal to zero yields the system of equations

$$x^2 + 2y - 6 = 0$$
$$2y + 2x - 3 = 0.$$

To solve this system, first solve the second equation for y. This gives $y = \dfrac{3 - 2x}{2}$. Substituting this into the first equation gives

$$x^2 + 3 - 2x - 6 = 0$$
$$x^2 - 2x - 3 = 0$$
$$(x-3)(x+1) = 0.$$

Therefore, $x = -1$ or $x = 3$. Substituting these values into the equation $y = \dfrac{3-2x}{2}$ yields the critical points $\left(-1, \frac{5}{2}\right)$ and $\left(3, -\frac{3}{2}\right)$.

Step 2 involves calculating the second partial derivatives of g:

$$g_{xx}(x,y) = 2x$$
$$g_{xy}(x,y) = 2$$
$$g_{yy}(x,y) = 2.$$

Next, we substitute each critical point into the discriminant formula:

$$D\left(-1, \tfrac{5}{2}\right) = (2(-1))(2) - (2)^2 = -4 - 4 = -8$$
$$D\left(3, -\tfrac{3}{2}\right) = (2(3))(2) - (2)^2 = 12 - 4 = 8.$$

In **step 3**, we use the second partials test to classify the behavior of the function at each of its critical points.

At point $\left(-1, \frac{5}{2}\right)$, we see that $D\left(-1, \frac{5}{2}\right) = -8 < 0$ (case 3 of the test), which means that f has a saddle point at the point $\left(-1, \frac{5}{2}\right)$. The coordinates of this saddle point are $\left(-1, \frac{5}{2}, \frac{41}{12}\right)$.

Applying the theorem to point $\left(3, -\frac{3}{2}\right)$ leads to case 1. That is, since $D\left(3, -\frac{3}{2}\right) = 8 > 0$ and $g_{xx}\left(3, -\frac{3}{2}\right) = 2(3) = 6 > 0$, we know that g is concave up at this critical point, so g has a local minimum of $-\frac{29}{4}$ at the point $\left(3, -\frac{3}{2}\right)$, as shown in the following figure.

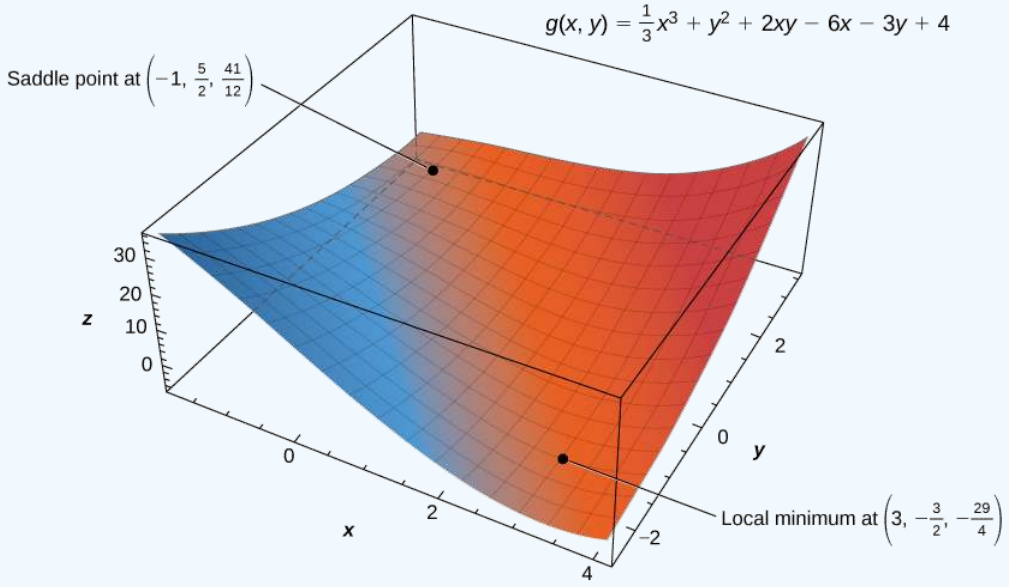

Figure **13.8.6**: The function $g(x,y)$ has a local minimum and a saddle point.

Note: Sometimes it can be helpful to find a general formula for D. For example, here we could have used the following formula:

$$\begin{aligned} D(x_0, y_0) &= g_{xx}(x_0, y_0) g_{yy}(x_0, y_0) - \left(g_{xy}(x_0, y_0)\right)^2 \\ &= (2x_0)(2) - 2^2 \\ &= 4x_0 - 4. \end{aligned}$$

Then we would have:

$$D\left(-1, \tfrac{5}{2}\right) = 4(-1) - 4 = -4 - 4 = -8$$
$$D\left(3, -\tfrac{3}{2}\right) = 4(3) - 4 = 12 - 4 = 8.$$

Note that the final values of the discriminant at each critical point are the same.

? Exercise 13.8.2

Use the second partials to find the local extrema of the function

$$f(x,y) = x^3 + 2xy - 6x - 4y^2.$$

Key Concepts

- A critical point of the function $f(x,y)$ is any point (x_0,y_0) where either $f_x(x_0,y_0) = f_y(x_0,y_0) = 0$, or at least one of $f_x(x_0,y_0)$ and $f_y(x_0,y_0)$ do not exist.
- A saddle point is a point (x_0,y_0) where $f_x(x_0,y_0) = f_y(x_0,y_0) = 0$, but (x_0,y_0) is neither a maximum nor a minimum at that point.
- To find extrema of functions of two variables, first find the critical points, then calculate the discriminant and apply the second partials test.

Key Equations

- **Discriminant**

$$D = f_{xx}(x_0,y_0)f_{yy}(x_0,y_0) - (f_{xy}(x_0,y_0))^2$$

Glossary

Contributors

- Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.
- Paul Seeburger (Monroe Community College) edited and adapted this section extensively.
  Paul also wrote the entire subsection titled Classifying Critical Points.

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13.8E: Optimization of Functions of Several Variables (Exercises)

13.8: Optimization of Functions of Several Variables

Finding Critical Points

In exercises 1 - 5, find all critical points.

1) $f(x,y) = 1 + x^2 + y^2$

2) $f(x,y) = 1 - (x-2)^2 + (y+3)^2$

3) $f(x,y) = (3x-2)^2 + (y-4)^2$

4) $f(x,y) = x^4 + y^4 - 16xy$

5) $f(x,y) = 15x^3 - 3xy + 15y^3$

Finding Extrema & the Second Partials Test

In exercises 6 - 9, find the critical points of the function and test for extrema or saddle points by using algebraic techniques (completing the square) or by examining the form of the equation. Where possible, verify your results using the Second Partials Test.

6) $f(x,y) = -\sqrt{x^2 + y^2}$

7) $f(x,y) = -x^2 - 5y^2 + 8x - 10y - 13$

8) $f(x,y) = x^2 + y^2 + 2x - 6y + 6$

9) $f(x,y) = \sqrt{x^2 + y^2} + 1$

In exercises 10 - 34, identify any critical points and use the Second Partials Test to determine the behavior of the function at each critical point, whether there is a maximum, minimum, saddle point, or none of these. If the Second Partials Test fails, determine the function's behavior at the point using another method and justify your answer clearly.

10) $f(x,y) = -x^3 + 4xy - 2y^2 + 1$

11) $f(x,y) = x^2y^2$

12) $f(x,y) = x^2 - 6x + y^2 + 4y - 8$

13) $f(x,y) = 2xy + 3x + 4y$

14) $f(x,y) = 8xy(x+y) + 7$

15) $f(x,y) = x^2 + 4xy + y^2$

16) $f(x,y) = x^3 + y^3 - 300x - 75y - 3$

17) $f(x,y) = 9 - x^4y^4$

18) $f(x,y) = x^2 + 10xy + y^2$

19) $f(x,y) = x^4 + y^2 + 2xy + 3$

20) $f(x,y) = 7x^2y + 9xy^2$

21) $f(x,y) = 3x^2 - 2xy + y^2 - 8y$

22) $f(x,y) = 3x^2 + 2xy + y^2$

23) $f(x,y) = y^2 + xy + 3y + 2x + 3$

24) $f(x,y) = x^2 + xy + y^2 - 3x$

25) $f(x,y) = x^2 + 2y^2 - x^2y$

26) $f(x,y) = x^2 + y - e^y$

27) $f(x,y) = e^{-(x^2+y^2+2x)}$

28) $f(x,y) = x^2 + xy + y^2 - x - y + 1$

29) $f(x,y) = x^2y(9 - x + y)$

30) $f(x,y) = -x^2 - 5y^2 + 10x - 30y - 62$

31) $f(x,y) = 120x + 120y - xy - x^2 - y^2$

32) $f(x,y) = 2x^2 + 2xy + y^2 + 2x - 3$

33) $f(x,y) = x^2 + x - 3xy + y^3 - 5$

34) $f(x,y) = 2xye^{-x^2-y^2}$

In exercises 35 - 37, determine the extreme values and the saddle points. Use a CAS to graph the function.

35) [T] $f(x,y) = ye^x - e^y$

36) [T] $f(x,y) = x\sin(y)$

37) [T] $f(x,y) = \sin(x)\sin(y), \quad x \in (0, 2\pi), \quad y \in (0, 2\pi)$

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- Paul Seeburger (Monroe Community College) created problems 19 and 29, and added dynamic figures for problems 27 and 35.

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13.9: Applications of Optimization, Constrained Optimization, and Absolute Extrema

In this section, we will consider some applications of optimization. Applications of optimization almost always involve some kind of constraints or boundaries.

Anytime we have a closed region or have constraints in an optimization problem the process we'll use to solve it is called ***constrained optimization***. In this section we will explore how to use what we've already learned to solve constrained optimization problems in two ways. In the next section we will learn a different approach called the Lagrange multiplier method that can be used to solve many of the same (and some additional) constrained optimization problems.

We will first look at a way to rewrite a constrained optimization problem in terms of a function of two variables, allowing us to find its critical points and determine optimal values of the function using the second partials test. We will then consider another approach to determining absolute extrema of a function of two variables on a closed region, where we can write each boundary as a function of x.

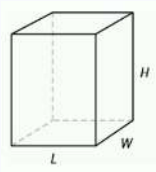
Applications of Optimization - Approach 1: Using the Second Partials Test

Example 13.9.1: A Box Problem

A closed cardboard box is constructed from different materials. If the material for the base of the box costs \$2 per square foot, and the side/top material costs \$1 per square foot, find the dimensions of the largest box that can be created for \$36.

Solution

Consider the box shown in the diagram.



Our first task will be to come up with the objective function (what we are trying to optimize). Here this is the volume of the box (see that we were asked for the "largest box"). Using the variables from the diagram, we have:

$$\text{Objective Function}: \quad V = LWH \tag{13.9.1}$$

Now we need to write the constraint using the same variables. The constraint here is the cost of the materials for the box. Since the cost of the material in the bottom of the box is twice the cost of the materials in the rest of the box, we'll need to account for this in the constraint. We have that the cost of the sides is \$1 per square foot, and so the cost of the bottom of the box must be \$2 per square foot. Now we need to write an expression for the area of the bottom of the box and another for the area of the sides and top of the box.

$$\begin{aligned} &\text{Area of bottom of box} = LW \\ &\text{Area of sides of box and the top} = 2LH + 2WH + LW \end{aligned} \tag{13.9.2}$$

Now we can put the costs of the respective material together with these areas to write the cost constraint:

$$\begin{aligned} \text{Cost Constraint}: \quad & 2LW + (2LH + 2WH + LW) = 36 \quad \rightarrow \\ & 3LW + 2LH + 2WH = 36 \end{aligned} \tag{13.9.3}$$

Note that there are also some additional implied constraints in this problem. That is, we know that $L \geq 0, W \geq 0$, and $H \geq 0$.

This constraint can be used to reduce the number of variables in the objective function, $V = LWH$, from three to two. We can choose to solve the constraint for any convenient variable, so let's solve it for H. Then,

$$\begin{aligned} & 3LW + 2LH + 2WH = 36 \\ \rightarrow \quad & 2H(L + W) = 36 - 3LW \\ \rightarrow \quad & H = \frac{36 - 3LW}{2(L + W)} \end{aligned}$$

Substituting this result into the objective function (replacing H), we obtain:

$$V(L, W) = LW\left(\frac{36 - 3LW}{2(L + W)}\right) = \frac{36LW - 3L^2W^2}{2(L + W)} \tag{13.9.4}$$

Now that we have the volume expressed as a function of just two variables, we can find its critical points (feasible for this situation) and then use the second partials test to confirm that we obtain a relative maximum volume at one of these points.

Finding Critical Points:

First we find the partial derivatives of V:

$$\begin{aligned} V_L(L, W) &= \frac{2(L + W)(36W - 6LW^2) - 2(36LW - 3L^2W^2)}{(L + W)^2} && \text{by the Quotient Rule} \\ &= \frac{-(36LW - 3L^2W^2)}{2(L + W)^2} && \text{Canceling a common factor of 2} \\ &= \frac{36LW - 6L^2W^2 + 36W^2 - 6LW^3 - 36LW + 3L^2W^2}{2(L + W)^2} && \text{Simplifying the numerator} \\ &= \frac{36W^2 - 6LW^3 - 3L^2W^2}{2(L + W)^2} && \text{Collecting like terms} \\ &= \frac{W^2(36 - 6LW - 3L^2)}{2(L + W)^2} && \text{Factoring out } W^2 \end{aligned}$$

Similarly, we find that

$$V_W(L, W) = \frac{L^2(36 - 6LW - 3W^2)}{2(L + W)^2} \tag{13.9.5}$$

Setting these partial derivatives both equal to zero, we note that the denominators cannot make either partial equal zero. We can then write the nonlinear system:

$$\begin{aligned} W^2(36 - 6LW - 3L^2) &= 0 \\ L^2(36 - 6LW - 3W^2) &= 0 \end{aligned}$$

Note that if not for the fact that in this application we know that the variables L and W are non-negative, we would find that this function has critical points anywhere the denominator of these partials is equal to 0, i.e., when $L = -W$. That is, all points on the line $L = -W$ would be critical points of this function. But since this is not possible here, we can ignore it and go on.

To solve the system above, we first note that both equations are factored and set equal to zero. Using the zero-product property, we know the first equation tells us that either

$$W = 0 \quad \text{or} \quad 36 - 6LW - 3L^2 = 0 \tag{13.9.6}$$

The second equation tells us that

$$L = 0 \quad \text{or} \quad 36 - 6LW - 3W^2 = 0 \tag{13.9.7}$$

Taking the various combinations here, we find

$W = 0$ and $L = 0 \quad \rightarrow$ Critical point: (0, 0). However, we see that this point also makes the denominator of the partials zero, making it a critical point of the second kind. We also see that with these dimensions the other constraint could not be true. We'd end up with a cost of \$0, not \$36.

$W = 0$ and $36 - 6LW - 3W^2 = 0$ is impossible, since when we put zero in for W in the second equation, we obtain the contradiction, $36 = 0$.

$36 - 6LW - 3L^2 = 0$ and $L = 0$ is also impossible, since when we put zero in for L in the first equation, we again obtain the contradiction, $36 = 0$.

Finally, considering the combination of the last two equations we have the system:

$$36 - 6LW - 3W^2 = 0$$
$$36 - 6LW - 3L^2 = 0$$

Multiplying through the bottom equation by -1, we have

$$36 - 6LW - 3W^2 = 0$$
$$-36 + 6LW + 3L^2 = 0$$

Adding the two equations eliminates the first two terms on the left side of each equation, yielding

$$3L^2 - 3W^2 = 0 \quad \rightarrow \quad 3(L - W)(L + W) = 0 \tag{13.9.8}$$

So we have that

$$L = W \quad \text{or} \quad L = -W \tag{13.9.9}$$

Since we know that none of the dimensions can be negative we can reject $L = -W$. This leaves us with the result that $L = W$.

Substituting this into the first equation of our system above (we could have used either one), gives us

$$36 - 6W^2 - 3W^2 = 0$$
$$36 - 9W^2 = 0$$
$$9(4 - W^2) = 0$$
$$9(2 - W)(2 + W) = 0$$

From this equation we see that either $W = 2$ or $W = -2$. Since the width cannot be negative, we reject $W = -2$ and conclude that $W = 2$ ft.

Then, since $L = W$, we know that $L = 2$ ft also.

Using the formula we found for H above, we find that the corresponding height is

$$H = \frac{36 - 3(2)(2)}{2(2 + 2)} = \frac{24}{8} = 3 \text{ ft} \tag{13.9.10}$$

This gives a volume of $V = 2 \cdot 2 \cdot 3 = 12 \text{ ft}^3$.

Note that the critical point we just found was actually $(2, 2)$. This is the only feasible critical point of the function V.

Now we just have to use the second partials test to prove that we obtain a relative maximum here.

Second Partials Test:

First we need to determine the second partials of $V(L, W)$, i.e., V_{LL}, V_{WW}, and V_{LW}.

$$\begin{aligned} V_{LL}(L, W) &= \frac{2(L + W)^2(-6W^3 - 6LW^2) - 4(L + W)(36W^2 - 6LW^3 - 3L^2W^2)}{4(L + W)^4} \\ &= \frac{4(L + W)\left[(L + W)(-3W^3 - 3LW^2) - 36W^2 + 6LW^3 + 3L^2W^2\right]}{4(L + W)^4} \\ &= \frac{-3LW^3 - 3L^2W^2 - 3W^4 - 3LW^3 - 36W^2 + 6LW^3 + 3L^2W^2}{(L + W)^3} \\ &= \frac{-3W^4 - 36W^2}{(L + W)^3} \end{aligned}$$

Similarly, we find

$$V_{WW}(L, W) = \frac{-3L^4 - 36L^2}{(L + W)^3} \tag{13.9.11}$$

And the mixed partial

$$\begin{aligned} V_{LW}(L, W) &= \frac{2(L + W)^2(72W - 18LW^2 - 6L^2W) - 4(L + W)(36W^2 - 6LW^3 - 3L^2W^2)}{4(L + W)^4} \\ &= \frac{4(L + W)\left[(L + W)(36W - 9LW^2 - 3L^2W) - 36W^2 + 6LW^3 + 3L^2W^2\right]}{4(L + W)^4} \\ &= \frac{\begin{array}{l}36LW + 36W^2 - 9L^2W^2 - 9LW^3 - 3L^3W - 3L^2W^2 - 36W^2 \\ + 6LW^3 + 3L^2W^2\end{array}}{(L + W)^3} \\ &= \frac{36LW - 3LW^3 - 9L^2W^2 - 3L^3W}{(L + W)^3} \end{aligned}$$

Now we can calculate the discriminant at the critical point:

$$D(2,2) = V_{LL}(2,2) \cdot V_{WW}(2,2) - [V_{LW}(2,2)]^2$$
$$= (-3)(-3) - \left(-\frac{3}{2}\right)^2 = 9 - \frac{9}{4} = \frac{27}{4} > 0$$

Since $D > 0$ and $V_{LL}(2,2) = -3 < 0$, we know that $V(L,W)$ is concave down at this critical point and thus has a relative maximum of 12 there.

Therefore, we know that the box of maximum volume that costs $\$36$ to make will have a volume of $V = 2\,\text{ft} \cdot 2\,\text{ft} \cdot 3\,\text{ft} = 12\,\text{ft}^3$.

Finding Absolute Extrema on a Closed, Bounded Region

When optimizing functions of one variable such as $y = f(x)$, we used the Extreme Value Theorem. It said that over a closed interval I, a continuous function has both an absolute maximum value and an absolute minimum value. To find these maximum and minimum values, we evaluated f at all critical points in the interval, as well as at the endpoints (the "boundaries") of the interval.

A similar theorem and procedure applies to functions of two variables. When working with a function of two variables, the closed interval is replaced by a closed, bounded region. A region is bounded if all the points in that region can be contained within a disk (or ball) of finite radius. Just as a continuous function of one variable was guaranteed to have an absolute maximum and an absolute minimum on a closed interval, a continuous function of two variables will attain an absolute maximum and absolute minimum value over a closed region (see the following theorem).

We can find these values by evaluating the function at its critical points in the region and at the critical values and endpoints of traces formed on the boundaries of the region. First, we need to find the critical points of the function that lie inside the region and calculate the corresponding function values. Then, it is necessary to find the maximum and minimum values of the function on the boundaries of the region. When we have all these values, the largest function value corresponds to the absolute (global) maximum and the smallest function value corresponds to the absolute (global) minimum.

First, however, we need to be assured that such values exist. The following theorem does this.

Theorem 13.9.1: Extreme Value Theorem

Let $z = f(x,y)$ be a continuous function on a closed, bounded region S. Then f has an absolute maximum value at some point in S and an absolute minimum value at some point in S.

The proof of this theorem is a direct consequence of the extreme value theorem and Fermat's theorem. In particular, if either extremum is not located on the boundary of S, then it is located at an interior point of S. But an interior point (x_0, y_0) of S that's an absolute extremum is also a local extremum; hence, (x_0, y_0) is a critical point of f by Fermat's theorem. Therefore the only possible values for the global extrema of f on S are the extreme values of f on the interior or boundary of S.

Now that we know any continuous function f defined on a closed, bounded region will have global extreme values on that region, let's summarize how to find them.

Finding Extreme Values of a Function of Two Variables

Assume $z = f(x,y)$ is a differentiable function of two variables defined on a closed, bounded region S. Then f will attain the absolute maximum value and the absolute minimum value, which are, respectively, the largest and smallest values found among the following:

1. The values of f at the critical points of f in S.
2. The values of f on the boundary of S.

Now let's see how this works in an example.

Example 13.9.2: Finding Extrema on a Closed, Bounded Region

Let $f(x,y) = x^2 - y^2 + 5$ and let the region S be the triangle with vertices $(-1,-2)$, $(0,1)$ and $(2,-2)$, including its interior. Find the absolute maximum and absolute minimum values of f on S.

SOLUTION

It can help to see a graph of f along with the region S. In Figure 13.9.1(a) the triangle defining S is shown in the xy-plane (using dashed red lines). Above it is the surface of f. Here we are only concerned with the portion of f enclosed by the "triangle" on its surface. Note that the triangular region forming our constrained region S in the xy-plane has been projected up onto the surface above it.

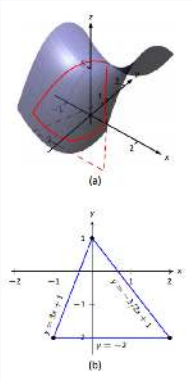


***Figure* 13.9.1:** *Plotting the surface of f along with the restricted domain S.*

We begin by finding the critical points of f. With $f_x(x,y) = 2x$ and $f_y(x,y) = -2y$, we find only one critical point, at $(0,0)$.

We now find the maximum and minimum values that f attains along the boundary of S, that is, along the edges of the triangle. In Figure 13.9.1(b) we see the triangle sketched in the plane with the equations of the lines forming its edges labeled.

Start with the bottom edge, along the line $y = -2$. If y is -2, then on the surface, we are considering points $f(x,-2)$; that is, our function reduces to $f(x,-2) = x^2 - (-2)^2 + 5 = x^2 + 1 = f_1(x)$. We want to maximize/minimize $f_1(x) = x^2 + 1$ on the interval $[-1,2]$. To do so, we evaluate $f_1(x)$ at its critical points and at the endpoints.

The critical points of f_1 are found by setting its derivative equal to 0:

$$f_1'(x) = 0 \quad \Rightarrow x = 0. \tag{13.9.12}$$

Evaluating f_1 at this critical point, and at the endpoints of $[-1,2]$ gives:

$$\begin{aligned} f_1(-1) = 2 \quad &\Rightarrow \quad f(-1,-2) = 2 \\ f_1(0) = 1 \quad &\Rightarrow \quad f(0,-2) = 1 \\ f_1(2) = 5 \quad &\Rightarrow \quad f(2,-2) = 5. \end{aligned}$$

Notice how evaluating f_1 at a point is the same as evaluating f at its corresponding point.

We need to do this process twice more, for the other two edges of the triangle.

Along the left edge, along the line $y = 3x + 1$, we substitute $3x + 1$ in for y in $f(x, y)$:

$$\begin{aligned} f(x,y) = f(x, 3x+1) &= x^2 - (3x+1)^2 + 5 \\ &= -8x^2 - 6x + 4 = f_2(x). \end{aligned} \tag{13.9.13}$$

We want the maximum and minimum values of f_2 on the interval $[-1, 0]$, so we evaluate f_2 at its critical points and the endpoints of the interval. We find the critical points:

$$f_2'(x) = -16x - 6 = 0 \quad \Rightarrow \quad x = -\tfrac{3}{8}. \tag{13.9.14}$$

Evaluate f_2 at its critical point and the endpoints of $[-1, 0]$:

$$\begin{aligned} f_2(-1) = 2 \quad &\Rightarrow \quad f(-1,-2) = 2 \\ f_2\left(-\tfrac{3}{8}\right) = \tfrac{41}{8} = 5.125 \quad &\Rightarrow \quad f\left(-\tfrac{3}{8}, -0.125\right) = 5.125 \\ f_2(0) = 1 \quad &\Rightarrow \quad f(0,1) = 4. \end{aligned}$$

Finally, we evaluate f along the right edge of the triangle, where $y = -\frac{3}{2}x + 1$.

$$\begin{aligned} f(x,y) = f\left(x, -\tfrac{3}{2}x + 1\right) &= x^2 - \left(-\tfrac{3}{2}x + 1\right)^2 + 5 \\ &= -\tfrac{5}{4}x^2 + 3x + 4 = f_3(x). \end{aligned} \tag{13.9.15}$$

The critical points of $f_3(x)$ are:

$$f_3'(x) = 0 \quad \Rightarrow \quad x = \tfrac{6}{5} = 1.2. \tag{13.9.16}$$

We evaluate f_3 at this critical point and at the endpoints of the interval $[0, 2]$:

$$\begin{aligned} f_3(0) = 4 \quad &\Rightarrow \quad f(0,1) = 4 \\ f_3(1.2) = 5.8 \quad &\Rightarrow \quad f(1.2,-0.8) = 5.8 \\ f_3(2) = 5 \quad &\Rightarrow \quad f(2,-2) = 5. \end{aligned}$$

One last point to test: the critical point of f, the point $(0, 0)$. We find $f(0, 0) = 5$.

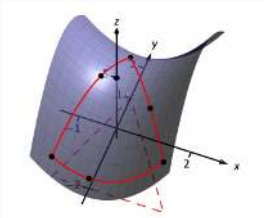


***Figure* 13.9.2:** *The surface of f along with important points along the boundary of S and the interior.*

We have evaluated f at a total of **7** different places, all shown in Figure **13.9.2**. We checked each vertex of the triangle twice, as each showed up as the endpoint of an interval twice. Of all the z-values found, the maximum is **5.8**, found at $(1.2, -0.8)$; the minimum is **1**, found at $(0, -2)$.

Notice that in this example we did not need to use the **Second Partials Test** to analyze the function's behavior at any critical points within the closed, bounded region. All we needed to do was evaluate the function at these critical points and then to find and evaluate the function at any critical points on the boundaries of this region. The largest output gives us the absolute maximum value of the function on the region, and the smallest output gives us the absolute minimum value of the function on the region.

Let's summarize the process we'll use for these problems.

Problem-Solving Strategy: Finding Absolute Maximum and Minimum Values on a Closed, Bounded Region

Let $z = f(x, y)$ be a continuous function of two variables defined on a closed, bounded region D, and assume f is differentiable on D. To find the absolute maximum and minimum values of f on D, do the following:

1. Determine the critical points of f in D.
2. Calculate f at each of these critical points.
3. Determine the maximum and minimum values of f on the boundary of its domain.
4. The maximum and minimum values of f will occur at one of the values obtained in steps **2** and **3**.

This portion of the text is entitled "Constrained Optimization" because we want to optimize a function (i.e., find its maximum and/or minimum values) subject to a *constraint* -- limits on which input points are considered. In the previous example, we constrained ourselves by considering a function only within the boundary of a triangle. This was largely arbitrary; the function and the boundary were chosen just as an example, with no real "meaning" behind the function or the chosen constraint.

However, solving constrained optimization problems is a very important topic in applied mathematics. The techniques developed here are the basis for solving larger problems, where more than two variables are involved.

We illustrate the technique again with a classic problem.

Example **13.9.3**: Constrained Optimization of a Package

The U.S. Postal Service states that the girth plus the length of Standard Post Package must not exceed 130". Given a rectangular box, the "length" is the longest side, and the "girth" is twice the sum of the width and the height.

Given a rectangular box where the width and height are equal, what are the dimensions of the box that give the maximum volume subject to the constraint of the size of a Standard Post Package?

SOLUTION

Let w, h and ℓ denote the width, height and length of a rectangular box; we assume here that $w = h$. The girth is then $2(w + h) = 4w$. The volume of the box is $V(w, \ell) = wh\ell = w^2\ell$. We wish to maximize this volume subject to the constraint $4w + \ell \leq 130$, or $\ell \leq 130 - 4w$. (Common sense also indicates that $\ell > 0, w > 0$.)

We begin by finding the critical values of V. We find that $V_w(w, \ell) = 2w\ell$ and $V_\ell(w, \ell) = w^2$; these are simultaneously 0 only at $(0, 0)$. This gives a volume of 0, so we can ignore this critical point.

We now consider the volume along the constraint $\ell = 130 - 4w$. Along this line, we have:

$$V(w, \ell) = V(w, 130 - 4w) = w^2(130 - 4w) = 130w^2 - 4w^3 = V_1(w). \tag{13.9.17}$$

The constraint restricts w to the interval $[0, 32.5]$, as indicated in the figure. Thus we want to maximize V_1 on $[0, 32.5]$.

Finding the critical values of V_1, we take the derivative and set it equal to 0:

$$V_1'(w) = 260w - 12w^2 = 0 \quad \Rightarrow \quad w(260 - 12w) = 0 \quad \Rightarrow \tag{13.9.18}$$
$$w = 0 \text{ or } w = \tfrac{260}{12} \approx 21.67.$$

We found two critical values: when $w = 0$ and when $w = 21.67$. We again ignore the $w = 0$ solution. The maximum volume, subject to the constraint, comes at $w = h = 21.67$, $\ell = 130 - 4(21.6) = 43.33$. This gives a maximum volume of $V(21.67, 43.33) \approx 19,408$ in^3.

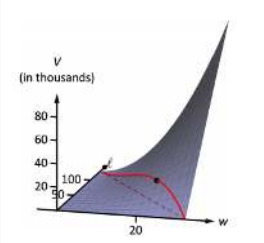


***Figure* 13.9.3:** *Graphing the volume of a box with girth $4w$ and length ℓ, subject to a size constraint.*

The volume function $V(w, \ell)$ is shown in Figure 13.9.3 along with the constraint $\ell = 130 - 4w$. As done previously, the constraint is drawn dashed in the xy-plane and also projected up onto the surface of the function. The point where the volume is maximized is indicated.

Finding the maximum and minimum values of f on the boundary of D can be challenging. If the boundary is a rectangle or set of straight lines, then it is possible to parameterize the line segments and determine the maxima on each of these segments, as seen in Example 13.9.1. The same approach can be used for other shapes such as circles and ellipses.

If the boundary of the region D is a more complicated curve defined by a function $g(x, y) = c$ for some constant c, and the first-order partial derivatives of g exist, then the method of Lagrange multipliers can prove useful for determining the extrema of f on the boundary.

Before we get to the method of Lagrange Multipliers (in the next section), let's consider a few additional examples of doing constrained optimization problems in this way.

Example 13.9.4: Finding Absolute Extrema

Use the problem-solving strategy for finding absolute extrema of a function to determine the absolute extrema of each of the following functions:

a. $f(x, y) = x^2 - 2xy + 4y^2 - 4x - 2y + 24$ on the domain defined by $0 \leq x \leq 4$ and $0 \leq y \leq 2$
b. $g(x, y) = x^2 + y^2 + 4x - 6y$ on the domain defined by $x^2 + y^2 \leq 16$

Solution:

a. Using the problem-solving strategy, step 1 involves finding the critical points of f on its domain. Therefore, we first calculate $f_x(x, y)$ and $f_y(x, y)$, then set them each equal to zero:

$$f_x(x, y) = 2x - 2y - 4$$
$$f_y(x, y) = -2x + 8y - 2.$$

Setting them equal to zero yields the system of equations

$$2x - 2y - 4 = 0$$
$$-2x + 8y - 2 = 0.$$

The solution to this system is $x = 3$ and $y = 1$. Therefore $(3, 1)$ is a critical point of f. Calculating $f(3, 1)$ gives $f(3, 1) = 17$.

The next step involves finding the extrema of f on the boundary of its domain. The boundary of its domain consists of four line segments as shown in the following graph:

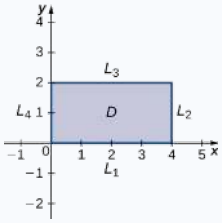


Figure 13.9.4: Graph of the domain of the function $f(x, y) = x^2 - 2xy + 4y^2 - 4x - 2y + 24$.

L_1 is the line segment connecting $(0, 0)$ and $(4, 0)$, and it can be parameterized by the equations $x(t) = t, y(t) = 0$ for $0 \leq t \leq 4$. Define $g(t) = f\big(x(t), y(t)\big)$. This gives $g(t) = t^2 - 4t + 24$. Differentiating g leads to $g'(t) = 2t - 4$. Therefore, g has a critical value at $t = 2$, which corresponds to the point $(2, 0)$. Calculating $f(2, 0)$ gives the z-value 20.

L_2 is the line segment connecting $(4, 0)$ and $(4, 2)$, and it can be parameterized by the equations $x(t) = 4, y(t) = t$ for $0 \leq t \leq 2$. Again, define $g(t) = f\big(x(t), y(t)\big)$. This gives $g(t) = 4t^2 - 10t + 24$. Then, $g'(t) = 8t - 10$. g has a critical value at $t = \frac{5}{4}$, which corresponds to the point $\left(4, \frac{5}{4}\right)$. Calculating $f\left(4, \frac{5}{4}\right)$ gives the z-value 17.75.

L_3 is the line segment connecting $(0, 2)$ and $(4, 2)$, and it can be parameterized by the equations $x(t) = t, y(t) = 2$ for $0 \leq t \leq 4$. Again, define $g(t) = f\big(x(t), y(t)\big)$. This gives $g(t) = t^2 - 8t + 36$. The critical value corresponds to the point $(4, 2)$. So, calculating $f(4, 2)$ gives the z-value 20.

L_4 is the line segment connecting $(0, 0)$ and $(0, 2)$, and it can be parameterized by the equations $x(t) = 0, y(t) = t$ for $0 \leq t \leq 2$. This time, $g(t) = 4t^2 - 2t + 24$ and the critical value $t = \frac{1}{4}$ correspond to the point $\left(0, \frac{1}{4}\right)$. Calculating $f\left(0, \frac{1}{4}\right)$ gives the z-value 23.75.

We also need to find the values of $f(x, y)$ at the corners of its domain. These corners are located at $(0, 0), (4, 0), (4, 2)$ and $(0, 2)$:

$$f(0,0) = (0)^2 - 2(0)(0) + 4(0)^2 - 4(0) - 2(0) + 24 = 24$$
$$f(4,0) = (4)^2 - 2(4)(0) + 4(0)^2 - 4(4) - 2(0) + 24 = 24$$
$$f(4,2) = (4)^2 - 2(4)(2) + 4(2)^2 - 4(4) - 2(2) + 24 = 20$$
$$f(0,2) = (0)^2 - 2(0)(2) + 4(2)^2 - 4(0) - 2(2) + 24 = 36.$$

The absolute maximum value is 36, which occurs at $(0,2)$, and the global minimum value is 20, which occurs at both $(4,2)$ and $(2,0)$ as shown in the following figure.

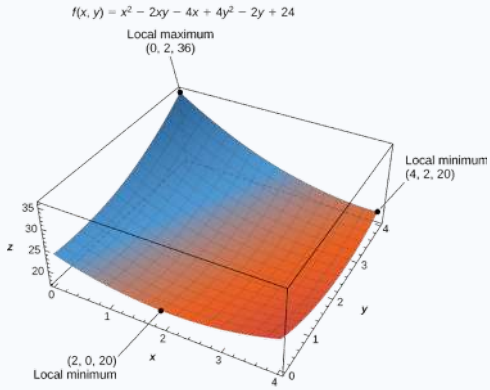


Figure 13.9.5: The function $f(x,y)$ has two global minima and one global maximum over its domain.

b. Using the problem-solving strategy, step 1 involves finding the critical points of g on its domain. Therefore, we first calculate $g_x(x,y)$ and $g_y(x,y)$, then set them each equal to zero:

$$g_x(x,y) = 2x + 4$$
$$g_y(x,y) = 2y - 6.$$

Setting them equal to zero yields the system of equations

$$2x + 4 = 0$$
$$2y - 6 = 0.$$

The solution to this system is $x = -2$ and $y = 3$. Therefore, $(-2,3)$ is a critical point of g. Calculating $g(-2,3)$, we get

$$g(-2,3) = (-2)^2 + 3^2 + 4(-2) - 6(3) = 4 + 9 - 8 - 18 = -13.$$

The next step involves finding the extrema of g on the boundary of its domain. The boundary of its domain consists of a circle of radius 4 centered at the origin as shown in the following graph.

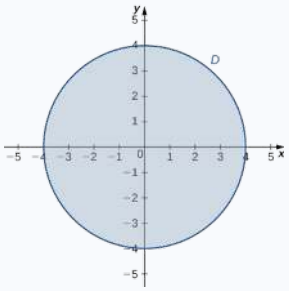


Figure 13.9.6: Graph of the domain of the function $g(x,y) = x^2 + y^2 + 4x - 6y$.

The boundary of the domain of g can be parameterized using the functions $x(t) = 4\cos t,\ y(t) = 4\sin t$ for $0 \le t \le 2\pi$. Define $h(t) = g\big(x(t), y(t)\big)$:

$$\begin{aligned} h(t) &= g\big(x(t), y(t)\big) \\ &= (4\cos t)^2 + (4\sin t)^2 + 4(4\cos t) - 6(4\sin t) \\ &= 16\cos^2 t + 16\sin^2 t + 16\cos t - 24\sin t \\ &= 16 + 16\cos t - 24\sin t. \end{aligned}$$

Setting $h'(t) = 0$ leads to

$$\begin{aligned} -16\sin t - 24\cos t &= 0 \\ -16\sin t &= 24\cos t \\ \frac{-16\sin t}{-16\cos t} &= \frac{24\cos t}{-16\cos t} \\ \tan t &= -\frac{4}{3}. \end{aligned}$$

$$\begin{aligned} -16\sin t - 24\cos t &= 0 \\ -16\sin t &= 24\cos t \\ \frac{-16\sin t}{-16\cos t} &= \frac{24\cos t}{-16\cos t} \\ \tan t &= -\frac{3}{2}. \end{aligned}$$

This equation has two solutions over the interval $0 \le t \le 2\pi$. One is $t = \pi - \arctan\left(\frac{3}{2}\right)$ and the other is $t = 2\pi - \arctan\left(\frac{3}{2}\right)$. For the first angle,

$$\sin t = \sin\left(\pi - \arctan\left(\tfrac{3}{2}\right)\right) = \sin\left(\arctan\left(\tfrac{3}{2}\right)\right) = \frac{3\sqrt{13}}{13}$$
$$\cos t = \cos\left(\pi - \arctan\left(\tfrac{3}{2}\right)\right) = -\cos\left(\arctan\left(\tfrac{3}{2}\right)\right) = -\frac{2\sqrt{13}}{13}.$$

Therefore, $x(t) = 4\cos t = -\frac{8\sqrt{13}}{13}$ and $y(t) = 4\sin t = \frac{12\sqrt{13}}{13}$, so $\left(-\frac{8\sqrt{13}}{13}, \frac{12\sqrt{13}}{13}\right)$ is a critical point on the boundary and

$$g\left(-\tfrac{8\sqrt{13}}{13}, \tfrac{12\sqrt{13}}{13}\right) = \left(-\tfrac{8\sqrt{13}}{13}\right)^2 + \left(\tfrac{12\sqrt{13}}{13}\right)^2 + 4\left(-\tfrac{8\sqrt{13}}{13}\right) - 6\left(\tfrac{12}{\sqrt{13}}13\right)$$
$$= \frac{144}{13} + \frac{64}{13} - \frac{32\sqrt{13}}{13} - \frac{72\sqrt{13}}{13}$$
$$= \frac{208 - 104\sqrt{13}}{13} \approx -12.844.$$

For the second angle,

$$\sin t = \sin(2\pi - \arctan(\tfrac{3}{2})) = -\sin(\arctan(\tfrac{3}{2})) = -\frac{3\sqrt{13}}{13}$$
$$\cos t = \cos(2\pi - \arctan(\tfrac{3}{2})) = \cos(\arctan(\tfrac{3}{2})) = \frac{2\sqrt{13}}{13}.$$

Therefore, $x(t) = 4\cos t = \frac{8\sqrt{13}}{13}$ and $y(t) = 4\sin t = -\frac{12\sqrt{13}}{13}$, so $\left(\frac{8\sqrt{13}}{13}, -\frac{12\sqrt{13}}{13}\right)$ is a critical point on the boundary and

$$g\left(\tfrac{8\sqrt{13}}{13}, -\tfrac{12\sqrt{13}}{13}\right) = \left(\tfrac{8\sqrt{13}}{13}\right)^2 + \left(-\tfrac{12\sqrt{13}}{13}\right)^2 + 4\left(\tfrac{8\sqrt{13}}{13}\right) - 6\left(-\tfrac{12\sqrt{13}}{13}\right)$$
$$= \frac{144}{13} + \frac{64}{13} + \frac{32\sqrt{13}}{13} + \frac{72\sqrt{13}}{13}$$
$$= \frac{208 + 104\sqrt{13}}{13} \approx 44.844.$$

The absolute minimum of g is -13, which is attained at the point $(-2, 3)$, which is an interior point of D. The absolute maximum of g is approximately equal to 44.844, which is attained at the boundary point $\left(\frac{8\sqrt{13}}{13}, -\frac{12\sqrt{13}}{13}\right)$. These are the absolute extrema of g on D as shown in the following figure.

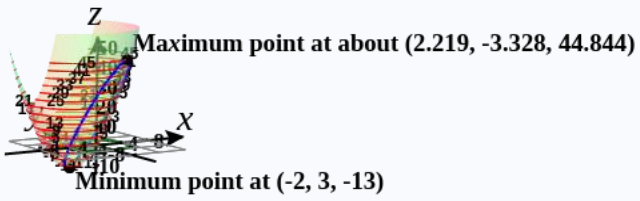


Figure 13.9.7: The function $f(x, y)$ has a local minimum and a local maximum.

Exercise 13.9.1

Use the problem-solving strategy for finding absolute extrema of a function to find the absolute extrema of the function

$$f(x, y) = 4x^2 - 2xy + 6y^2 - 8x + 2y + 3$$

on the domain defined by $0 \le x \le 2$ and $-1 \le y \le 3$.

Example 13.9.5: Profitable Golf Balls

Pro-T company has developed a profit model that depends on the number x of golf balls sold per month (measured in thousands), and the number of hours per month of advertising y, according to the function

$$z = f(x, y) = 48x + 96y - x^2 - 2xy - 9y^2, \tag{13.9.19}$$

where z is measured in thousands of dollars. The maximum number of golf balls that can be produced and sold is $50,000$, and the maximum number of hours of advertising that can be purchased is 25. Find the values of x and y that maximize profit, and find the maximum profit.

Figure 13.9.8: (credit: modification of work by oatsy40, Flickr)

Solution

Using the problem-solving strategy, step 1 involves finding the critical points of f on its domain. Therefore, we first calculate $f_x(x, y)$ and $f_y(x, y)$, then set them each equal to zero:

$$f_x(x,y) = 48 - 2x - 2y$$
$$f_y(x,y) = 96 - 2x - 18y.$$

Setting them equal to zero yields the system of equations

$$48 - 2x - 2y = 0$$
$$96 - 2x - 18y = 0.$$

The solution to this system is $x = 21$ and $y = 3$. Therefore $(21, 3)$ is a critical point of f. Calculating $f(21, 3)$ gives $f(21, 3) = 48(21) + 96(3) - 21^2 - 2(21)(3) - 9(3)^2 = 648$.

The domain of this function is $0 \le x \le 50$ and $0 \le y \le 25$ as shown in the following graph.

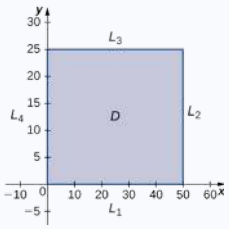


Figure 13.9.9: Graph of the domain of the function $f(x, y) = 48x + 96y - x^2 - 2xy - 9y^2$.

L_1 is the line segment connecting $(0, 0)$ and $(50, 0)$, and it can be parameterized by the equations $x(t) = t, y(t) = 0$ for $0 \le t \le 50$. We then define $g(t) = f(x(t), y(t))$:

$$\begin{aligned} g(t) &= f(x(t), y(t)) \\ &= f(t, 0) \\ &= 48t + 96(0) - y^2 - 2(t)(0) - 9(0)^2 \\ &= 48t - t^2. \end{aligned}$$

Setting $g'(t) = 0$ yields the critical point $t = 24$, which corresponds to the point $(24, 0)$ in the domain of f. Calculating $f(24, 0)$ gives 576.

L_2 is the line segment connecting and $(50, 25)$, and it can be parameterized by the equations $x(t) = 50, y(t) = t$ for $0 \le t \le 25$. Once again, we define $g(t) = f\big(x(t), y(t)\big)$:

$$\begin{aligned} g(t) &= f\big(x(t), y(t)\big) \\ &= f(50, t) \\ &= 48(50) + 96t - 50^2 - 2(50)t - 9t^2 \\ &= -9t^2 - 4t - 100. \end{aligned}$$

This function has a critical point at $t = -\frac{2}{9}$, which corresponds to the point $(50, -29)$. This point is not in the domain of f.

L_3 is the line segment connecting $(0, 25)$ and $(50, 25)$, and it can be parameterized by the equations $x(t) = t, y(t) = 25$ for $0 \le t \le 50$. We define $g(t) = f\big(x(t), y(t)\big)$:

$$\begin{aligned} g(t) &= f\big(x(t), y(t)\big) \\ &= f(t, 25) \\ &= 48t + 96(25) - t^2 - 2t(25) - 9(25^2) \\ &= -t^2 - 2t - 3225. \end{aligned}$$

This function has a critical point at $t = -1$, which corresponds to the point $(-1, 25)$, which is not in the domain.

L_4 is the line segment connecting $(0, 0)$ to $(0, 25)$, and it can be parameterized by the equations $x(t) = 0, y(t) = t$ for $0 \le t \le 25$. We define $g(t) = f\big(x(t), y(t)\big)$:

$$\begin{aligned} g(t) &= f\big(x(t), y(t)\big) \\ &= f(0, t) \\ &= 48(0) + 96t - (0)^2 - 2(0)t - 9t^2 \\ &= 96t - t^2. \end{aligned}$$

This function has a critical point at $t = \frac{16}{3}$, which corresponds to the point $\left(0, \frac{16}{3}\right)$, which is on the boundary of the domain. Calculating $f\left(0, \frac{16}{3}\right)$ gives 256.

We also need to find the values of $f(x, y)$ at the corners of its domain. These corners are located at $(0, 0), (50, 0), (50, 25)$ and $(0, 25)$:

$$f(0, 0) = 48(0) + 96(0) - (0)^2 - 2(0)(0) - 9(0)^2 = 0$$
$$f(50, 0) = 48(50) + 96(0) - (50)^2 - 2(50)(0) - 9(0)^2 = -100$$
$$f(50, 25) = 48(50) + 96(25) - (50)^2 - 2(50)(25) - 9(25)^2 = -5825$$
$$f(0, 25) = 48(0) + 96(25) - (0)^2 - 2(0)(25) - 9(25)^2 = -3225.$$

The maximum critical value is 648, which occurs at $(21, 3)$. Therefore, a maximum profit of $\$648,000$ is realized when $21,000$ golf balls are sold and 3 hours of advertising are purchased per month as shown in the following figure.

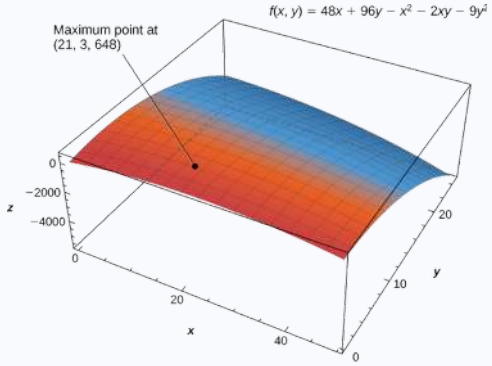


Figure 13.9.10: The profit function $f(x, y)$ has a maximum at $(21, 3, 648)$.

It is hard to overemphasize the importance of optimization. In "the real world," we routinely seek to make *something* better. By expressing the *something* as a mathematical function, "making *something* better" means to "optimize *some function*."

The techniques shown here are only the beginning of a very important field. But be aware that many functions that we will want to optimize are incredibly complex, making the step to "find the gradient, set it equal to $\vec{0}$, and find all points that make this true" highly nontrivial. Mastery of the principles on simpler problems here is key to being able to tackle these more complicated problems.

The Method of Least Squares Regression

Our world is full of data, and to interpret and extrapolate based on this data, we often try to find a function to model this data in a particular situation. You've likely heard about a **line of best fit**, also known as a **least squares regression line**. This linear model, in the form $f(x) = ax + b$, assumes the value of the output changes at a roughly constant rate with respect to the input, i.e., that these values are related linearly. And for many situations this linear model gives us a powerful way to make predictions about the value of the function's output for inputs other than those in the data we collected. For other situations, like most population models, a linear model is not sufficient, and we need to find a quadratic, cubic, exponential, logistic, or trigonometric model (or possibly something else).

Consider the data points given in the table below. Figures 13.9.11 - 13.9.14 show various best fit regression models for this data.

$$\qquad \begin{array}{c | c} x & y \\ \hline 2 & 1 \\ 4 & 2 \\ 6 & 4 \\ 8 & 7 \end{array}$$

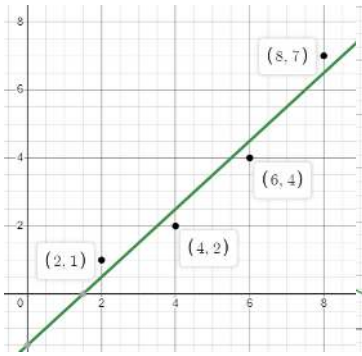


Figure 13.9.11: Linear model

$y = x - 1.5$

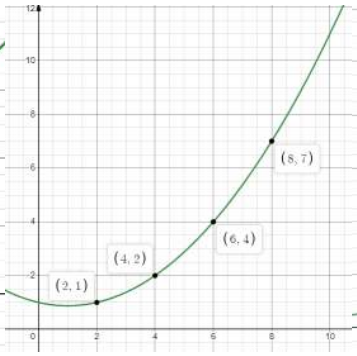


Figure 13.9.12: Quadratic model

$y = 0.125x^2 - 0.25x + 1$

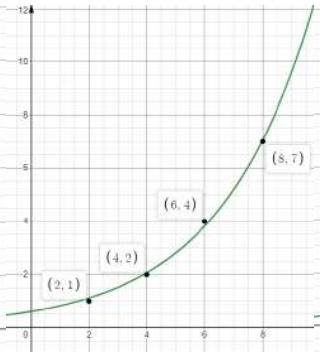


Figure 13.9.13: Exponential model

$y = 0.6061 \cdot (1.359)^x$

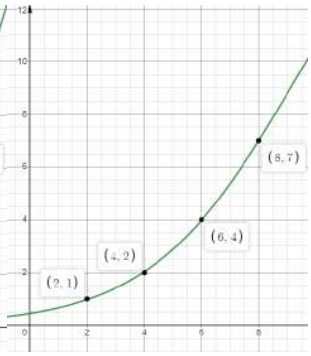


Figure 13.9.14: Logistic model

$y = \frac{18.31}{1 + 39.67e^{-0.4002x}}$

Definition: Sum of the Squared Errors Function

One way to measure how well a particular model $y = f(x)$ fits a given set of data points $\{(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots, (x_n, y_n)\}$ is to consider the squares of the differences between the values given by the model and the actual y-values of the data points.

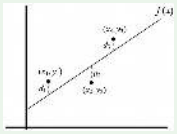


Figure 13.9.15: Differences between model $y = f(x)$ and y-coordinates of data points, where $d_i = f(x_i) - y_i$.

The differences (or errors), $d_i = f(x_i) - y_i$, are squared and added up so that all errors are considered in the measure of how well the model "fits" the data. This **sum of the squared errors** is given by

$$S = \sum_{i=1}^{n} \left(f(x_i) - y_i\right)^2$$

Squaring the errors in this sum also tends to magnify the weight of larger errors and minimize the weight of small errors in the measurement of how well the model fits the data.

Note that S is a function of the constant parameters in the particular model $y = f(x)$. For example, if we desire a linear model, then $f(x) = ax + b$, and

$$\begin{aligned} S(a, b) &= \sum_{i=1}^{n} \left(f(x_i) - y_i\right)^2 \\ &= \sum_{i=1}^{n} \left(ax_i + b - y_i\right)^2 \end{aligned}$$

To obtain the best fit version of the model, we will seek the values of the constant parameters in the model (a and b in the linear model) that give us the **minimum value** of this sum of the squared errors function S for the given set of data points.

When calculating a line of best fit in previous classes, you were likely either given the formulas for the coefficients or shown how to use features on your calculator or other device to find them.

Using our understanding of the optimization of functions of multiple variables in this class, we are now able to derive the formulas for these coefficients directly. In fact, we can use the same methods to determine the constant parameters needed to form a best fit model of any kind (quadratic, cubic, exponential, logistic, sine, cosine, etc), although these can be a bit more difficult to work out than the linear case.

Theorem 13.9.2: Least Squares Regression Line

The **least squares regression line** (or **line of best fit**) for the data points $\{(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots, (x_n, y_n)\}$ is given by the formula

$$f(x) = ax + b$$

where

$$a = \frac{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n\sum_{i=1}^{n} {x_i}^2 - \left(\sum_{i=1}^{n} x_i\right)^2} \quad \text{and} \quad b = \frac{1}{n}\left(\sum_{i=1}^{n} y_i - a\sum_{i=1}^{n} x_i\right).$$

Proof (part 1): To obtain a line that best fits the data points we need to minimize the sum of the squared errors for this model. This is a function of the two variables a and b as shown below. Note that the x_i and y_i are numbers in this function and are **not** the variables.

$$S(a,b)=\sum_{i=1}^{n}\left(ax_i+b-y_i\right)^2$$

To minimize this function of the two variables a and b, we need to determine the critical point(s) of this function. Let's start by stating the partial derivatives of S with respect to a and b and simplifying them so that the sums only include the numerical coordinates from the data points.

$$\begin{aligned}S_a(a,b)&=\sum_{i=1}^{n}2x_i\left(ax_i+b-y_i\right)\\&=\sum_{i=1}^{n}\left(2a{x_i}^2+2bx_i-2x_iy_i\right)\\&=\sum_{i=1}^{n}2a{x_i}^2+\sum_{i=1}^{n}2bx_i-\sum_{i=1}^{n}2x_iy_i\\&=2a\sum_{i=1}^{n}{x_i}^2+2b\sum_{i=1}^{n}x_i-2\sum_{i=1}^{n}x_iy_i\end{aligned}$$

$$\begin{aligned}S_b(a,b)&=\sum_{i=1}^{n}2\left(ax_i+b-y_i\right)\\&=\sum_{i=1}^{n}\left(2ax_i+2b-2y_i\right)\\&=\sum_{i=1}^{n}2ax_i+\sum_{i=1}^{n}2b-\sum_{i=1}^{n}2y_i\\&=2a\sum_{i=1}^{n}x_i+2b\sum_{i=1}^{n}1-2\sum_{i=1}^{n}y_i\\&=2a\sum_{i=1}^{n}x_i+2bn-2\sum_{i=1}^{n}y_i\end{aligned}$$

Next we set these partial derivatives equal to 0 and solve the resulting system of equations.

Set $2a\sum_{i=1}^{n}{x_i}^2+2b\sum_{i=1}^{n}x_i-2\sum_{i=1}^{n}x_iy_i=0,$

and $2a\sum_{i=1}^{n}x_i+2bn-2\sum_{i=1}^{n}y_i=0.$

Since the linear case is simple enough to make this process straightforward, we can use substitution to actually solve this system for a and b. Solving the second equation for b gives us

$$2bn=2\sum_{i=1}^{n}y_i-2a\sum_{i=1}^{n}x_i$$

$$\boxed{b=\frac{1}{n}\left(\sum_{i=1}^{n}y_i-a\sum_{i=1}^{n}x_i\right).}$$

We now substitute this expression into the first equation for b.

$$2a\sum_{i=1}^{n}{x_i}^2+2\left[\frac{1}{n}\left(\sum_{i=1}^{n}y_i-a\sum_{i=1}^{n}x_i\right)\right]\sum_{i=1}^{n}x_i-2\sum_{i=1}^{n}x_iy_i=0$$

Multiplying this equation through by n and dividing out the 2 yields

$$an\sum_{i=1}^{n}{x_i}^2+\left[\sum_{i=1}^{n}y_i-a\sum_{i=1}^{n}x_i\right]\sum_{i=1}^{n}x_i-n\sum_{i=1}^{n}x_iy_i=0.$$

Simplifying,

$$an\sum_{i=1}^{n}{x_i}^2+\sum_{i=1}^{n}x_i\sum_{i=1}^{n}y_i-a\sum_{i=1}^{n}x_i\sum_{i=1}^{n}x_i-n\sum_{i=1}^{n}x_iy_i=0.$$

Next we isolate the terms containing a on the left side of the equation, moving the other two terms to the right side of the equation.

$$an\sum_{i=1}^{n}{x_i}^2-a\left(\sum_{i=1}^{n}x_i\right)^2=n\sum_{i=1}^{n}x_iy_i-\sum_{i=1}^{n}x_i\sum_{i=1}^{n}y_i.$$

Factoring out the a,

$$a\left(n\sum_{i=1}^{n}{x_i}^2-\left(\sum_{i=1}^{n}x_i\right)^2\right)=n\sum_{i=1}^{n}x_iy_i-\sum_{i=1}^{n}x_i\sum_{i=1}^{n}y_i.$$

Finally, we solve for a to obtain

$$\boxed{a=\frac{n\sum_{i=1}^{n}x_iy_i-\sum_{i=1}^{n}x_i\sum_{i=1}^{n}y_i}{n\sum_{i=1}^{n}{x_i}^2-\left(\sum_{i=1}^{n}x_i\right)^2}.}$$

■ QED

Thus we have found formulas for the coordinates of the single critical point of the sum of squared errors function for the linear model, i.e.,

$$S(a,b)=\sum_{i=1}^{n}\left(ax_i+b-y_i\right)^2.$$

Proof (part 2): To prove that this linear model gives us a minimum of S, we will need the **Second Partials Test**.

The second partials of S are

$$S_{aa}(a,b) = \sum_{i=1}^{n} 2{x_i}^2$$
$$S_{bb}(a,b) = \sum_{i=1}^{n} 2 = 2n$$
$$S_{ab}(a,b) = \sum_{i=1}^{n} 2x_i.$$

Then the discriminant evaluated at the critical point (a,b) is

$$\begin{aligned} D(a,b) &= S_{aa}(a,b)S_{bb}(a,b) - \left[S_{ab}(a,b)\right]^2 \\ &= 2n\sum_{i=1}^{n} 2{x_i}^2 - \left(2\sum_{i=1}^{n} x_i\right)^2 \\ &= 4n\sum_{i=1}^{n} {x_i}^2 - 4\left(\sum_{i=1}^{n} x_i\right)^2 \\ &= 4\left[n\sum_{i=1}^{n} {x_i}^2 - \left(\sum_{i=1}^{n} x_i\right)^2\right]. \end{aligned}$$

Now to prove that this discriminant is positive (and thus guarantees a local max or min) requires us to use the **Cauchy-Schwarz Inequality**. This part of the proof is non-trivial, but not too hard, if we don't worry about proving the Cauchy-Schwarz Inequality itself here.

The **Cauchy-Schwarz Inequality** guarantees

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} {a_i}^2\right)\left(\sum_{i=1}^{n} {b_i}^2\right)$$

for any real-valued sequences a_i and b_i, with equality only if these two sequences are dependent (e.g., if the terms were the same or some multiple of each other).

Here we choose $a_i = 1$ and $b_i = x_i$. Since these sequences are linearly independent, we will not have equality and substituting into the Cauchy-Schwarz Inequality gives us

$$\left(\sum_{i=1}^{n} 1 \cdot x_i\right)^2 < \left(\sum_{i=1}^{n} 1^2\right)\left(\sum_{i=1}^{n} {x_i}^2\right).$$

Since $\sum_{i=1}^{n} 1 = n$, we have

$$\left(\sum_{i=1}^{n} x_i\right)^2 < n\sum_{i=1}^{n} {x_i}^2.$$

This result proves

$$D(a,b) = 4\left[n\sum_{i=1}^{n} {x_i}^2 - \left(\sum_{i=1}^{n} x_i\right)^2\right] > 0.$$

Now, since it is clear that $S_{aa}(a,b) = \sum_{i=1}^{n} 2{x_i}^2 > 0$, we know that S is concave up at the point (a,b) and thus has a relative minimum there. This concludes the proof that the parameters a and b, as shown in Theorem **13.9.2**, give us the **least squares linear regression model** or **line of best fit**,

$$f(x) = ax + b. \qquad \blacksquare \quad \text{QED}$$

Note

The linear case $f(x) = ax + b$ is special in that it is simple enough to allow us to actually solve for the constant parameters a and b directly as formulas (as shown above in Theorem **13.9.2**. For other best fit models we typically just obtain the system of equations needed to solve for the corresponding constant parameters that give us the critical point and minimum value of S.

To leave the quadratic regression case for you to try as an exercise, let's consider a cubic best fit model here.

Example 13.9.6: Finding the System of Equations for the Best-fit Cubic Model

Determine the system of equations needed to determine the cubic best fit regression model of the form, $f(x) = ax^3 + bx^2 + cx + d$, for a given set of data points, $\{(x_1,y_1),(x_2,y_2),(x_3,y_3),\ldots,(x_n,y_n)\}$.

Solution

Here consider the sum of least squares function

$$S(a,b,c,d) = \sum_{i=1}^{n} \left(a{x_i}^3 + b{x_i}^2 + cx_i + d - y_i\right)^2.$$

To find the minimum value of this function (and the corresponding values of the parameters $a, b, c,$ and d needed for the best fit cubic regression model), we need to find the critical point of this function. (Yes, even for a function of four variables!) To begin this process, we find the first partial derivatives of this function with respect to each of the parameters $a, b, c,$ and d.

$$S_a(a,b,c,d) = 2\sum_{i=1}^{n} x_i\left(a{x_i}^3 + b{x_i}^2 + cx_i + d - y_i\right)$$
$$S_b(a,b,c,d) = 2\sum_{i=1}^{n} {x_i}^2\left(a{x_i}^3 + b{x_i}^2 + cx_i + d - y_i\right)$$
$$S_c(a,b,c,d) = 2\sum_{i=1}^{n} x_i\left(a{x_i}^3 + b{x_i}^2 + cx_i + d - y_i\right)$$
$$S_d(a,b,c,d) = 2\sum_{i=1}^{n} \left(a{x_i}^3 + b{x_i}^2 + cx_i + d - y_i\right)$$

Now we set these partials equal to 0, divide out the 2 from each, and split the terms into separate sums. We also factor out the $a, b, c,$ and d which don't change as the index changes.

$$a\sum_{i=1}^{n}x_i^6+b\sum_{i=1}^{n}x_i^5+c\sum_{i=1}^{n}x_i^4+d\sum_{i=1}^{n}x_i^3-\sum_{i=1}^{n}x_i^3y_i=0$$
$$a\sum_{i=1}^{n}x_i^5+b\sum_{i=1}^{n}x_i^4+c\sum_{i=1}^{n}x_i^3+d\sum_{i=1}^{n}x_i^2-\sum_{i=1}^{n}x_i^2y_i=0$$
$$a\sum_{i=1}^{n}x_i^4+b\sum_{i=1}^{n}x_i^3+c\sum_{i=1}^{n}x_i^2+d\sum_{i=1}^{n}x_i-\sum_{i=1}^{n}x_iy_i=0$$
$$a\sum_{i=1}^{n}x_i^3+b\sum_{i=1}^{n}x_i^2+c\sum_{i=1}^{n}x_i+d\sum_{i=1}^{n}1-\sum_{i=1}^{n}y_i=0$$

This system of equations can be rewritten by moving the negative terms to the right side of the equation. We'll also emphasize the coefficients and variables and replace $\sum_{i=1}^{n}1$ with n.

$$\left(\sum_{i=1}^{n}x_i^6\right)a+\left(\sum_{i=1}^{n}x_i^5\right)b+\left(\sum_{i=1}^{n}x_i^4\right)c+\left(\sum_{i=1}^{n}x_i^3\right)d=\sum_{i=1}^{n}x_i^3y_i$$
$$\left(\sum_{i=1}^{n}x_i^5\right)a+\left(\sum_{i=1}^{n}x_i^4\right)b+\left(\sum_{i=1}^{n}x_i^3\right)c+\left(\sum_{i=1}^{n}x_i^2\right)d=\sum_{i=1}^{n}x_i^2y_i$$
$$\left(\sum_{i=1}^{n}x_i^4\right)a+\left(\sum_{i=1}^{n}x_i^3\right)b+\left(\sum_{i=1}^{n}x_i^2\right)c+\left(\sum_{i=1}^{n}x_i\right)d=\sum_{i=1}^{n}x_iy_i$$
$$\left(\sum_{i=1}^{n}x_i^3\right)a+\left(\sum_{i=1}^{n}x_i^2\right)b+\left(\sum_{i=1}^{n}x_i\right)c+nd=\sum_{i=1}^{n}y_i$$

This is a system of four linear equations with the four variables $a, b, c,$ and d. To solve for the values of these parameters that would give us the best fit cubic regression model, we would need to solve this system using the given data points. We would first evaluate each of the sums using the x- and y-coordinates of the data points and place these numerical coefficients into the equations. We would then only need to solve the resulting system of linear equations.

Let's now use this process to find the best fit cubic regression model for a set of data points.

Example 13.9.7: Finding a Best-fit Cubic Model

a. Find the best fit cubic regression model of the form, $f(x)=ax^3+bx^2+cx+d$, for the set of data points below.

\qquad \begin{array}{c | c} x & y \\ \hline 1 & 1 \\ 4 & 3 \\ 6 & 4 \\ 8 & 7 \\ 9 & 6 \end{array}

b. Determine the actual sum of the squared errors for this cubic model and these data points.

Solution

a. First we need to calculate each of the sums that appear in the system of linear equations we derived in Example 13.9.6.Note first that since we have five points, $n=5$.

$$\sum_{i=1}^{5}x_i^6=1^6+4^6+6^6+8^6+9^6=844338$$
$$\sum_{i=1}^{5}x_i^5=1^5+4^5+6^5+8^5+9^5=100618$$
$$\sum_{i=1}^{5}x_i^4=1^4+4^4+6^4+8^4+9^4=12210$$
$$\sum_{i=1}^{5}x_i^3=1^3+4^3+6^3+8^3+9^3=1522$$
$$\sum_{i=1}^{5}x_i^2=1^2+4^2+6^2+8^2+9^2=198$$

Plugging these values into the system of equations from Example 13.9.6 gives us

$$\begin{aligned}844338a+100618b+12210c+1522d&=9015\\100618a+12210b+1522c+198d&=1127\\12210a+1522b+198c+28d&=147\\1522a+198b+28c+5d&=21\end{aligned}$$

Solving this system by elimination (or using row-reduction in matrix form) gives us

$$a=-0.0254167,\ b=0.382981,\ c=-0.852949,\ d=1.547308$$

which results in the following cubic best fit regression model for this set of data points:

$$f(x)=-0.0254167x^3+0.382981x^2-0.852949x+1.547308.$$

See this best fit cubic regression model along with these data points in Figure 13.9.16 below.

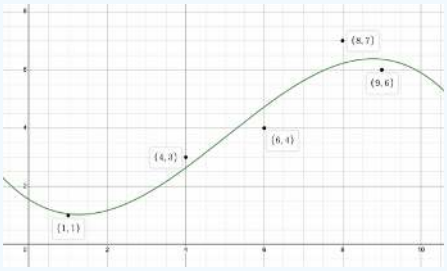


Figure 13.9.16: Cubic best fit model for these data points

b. Now that we have the model, let's consider what the errors are for each data point and calculate the sum of the squared errors. This is the absolute minimum value of the function

$$S(a,b,c,d)=\sum_{i=1}^{n}\left(ax_i^3+bx_i^2+cx_i+d-y_i\right)^2.$$

\begin{array}{|c | c | c | c|} \hline x_i & y_i & f(x_i) & \mbox{Error: }f(x_i)-y_i & \mbox{Squared Error: }\left(f(x_i)-y_i \right)^2 \\ \hline 1 & 1 & 1.0519231 & 0.0519231 & 0.002696 \\ 4 & 3 & 2.6365385 & -0.3634615 & 0.132104 \\ 6 & 4 & 4.7269231 & 0.72692

Contributors

- Edited, Mixed, and expanded by Paul Seeburger (Monroe Community College)
  (Applications of Optimization - Approach 1 and Example **13.9.1**)
  (The Method of Least Squares Regression)
- Gregory Hartman (Virginia Military Institute). Contributions were made by Troy Siemers and Dimplekumar Chalishajar of VMI and Brian Heinold of Mount Saint Mary's University. This content is copyrighted by a Creative Commons Attribution - Noncommercial (BY-NC) License. http://www.apexcalculus.com/

  (Examples **13.9.2** and **13.9.3** and some exposition at top and bottom of section.)
- Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.

  (Examples **13.9.4** and **13.9.5** and Exercise **13.9.1**.)

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13.9E: Optimization of Functions of Several Variables (Exercises)

Absolute Extrema on Closed and Bounded Regions

In exercises 1 - 4, find the absolute extrema of the given function on the indicated closed and bounded set R.

1) $f(x,y) = xy - x - 3y$; R is the triangular region with vertices $(0,0), (0,4)$, and $(5,0)$.

2) Find the absolute maximum and minimum values of $f(x,y) = x^2 + y^2 - 2y + 1$ on the region $R = \{(x,y) \mid x^2 + y^2 \leq 4\}$.

3) $f(x,y) = x^3 - 3xy - y^3$ on $R = \{(x,y) : -2 \leq x \leq 2, -2 \leq y \leq 2\}$

4) $f(x,y) = \frac{-2y}{x^2+y^2+1}$ on $R = \{(x,y) : x^2 + y^2 \leq 4\}$

5) To be created...

6) Find the absolute minimum and absolute maximum of the function $f(x,y) = xy^2 - 2x + 3$ on the region between $y = x/2$ and $y = \sqrt{x}$, for $0 \leq x \leq 4$, showing all work, including work to find any critical points of f. See the plot of the region in the xy-plane below.

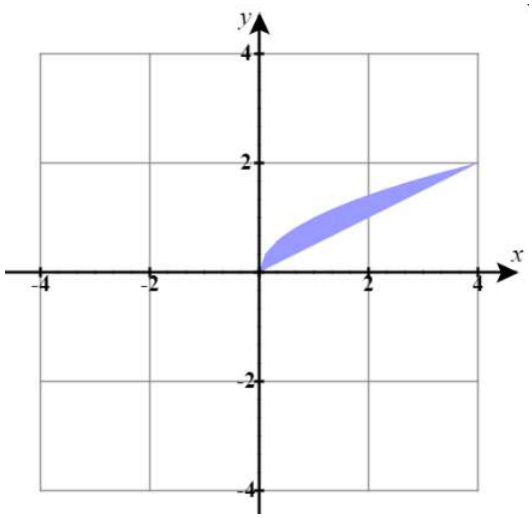

Figure 13.9E.1: Closed and bounded region in the xy-plane for Problem 6.

Applications

7) Find three positive numbers the sum of which is 27, such that the sum of their squares is as small as possible.

8) Find the points on the surface $x^2 - yz = 5$ that are closest to the origin.

9) Find the maximum volume of a rectangular box with three faces in the coordinate planes and a vertex in the first octant on the plane $x + y + z = 1$.

10) The sum of the length and the girth (perimeter of a cross-section) of a package carried by a delivery service cannot exceed 108 in. Find the dimensions of the rectangular package of largest volume that can be sent.

11) A cardboard box without a lid is to be made with a volume of 4 ft^3. Find the dimensions of the box that requires the least amount of cardboard.

12) Find the point on the surface $f(x,y) = x^2 + y^2 + 10$ nearest the plane $x + 2y - z = 0$. Identify the point on the plane.

13) Find the point in the plane $2x - y + 2z = 16$ that is closest to the origin.

14) A company manufactures two types of athletic shoes: jogging shoes and cross-trainers. The total revenue from x units of jogging shoes and y units of cross-trainers is given by $R(x,y) = -5x^2 - 8y^2 - 2xy + 42x + 102y$, where x and y are in thousands of units. Find the values of x and y to maximize the total revenue.

15) A shipping company handles rectangular boxes provided the sum of the length, width, and height of the box does not exceed 96 in. Find the dimensions of the box that meets this condition and has the largest volume.

16) Find the maximum volume of a cylindrical soda can such that the sum of its height and circumference is 120 cm.

Contributors

- Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.
- Paul Seeburger (Monroe Community College) edited the LaTeX , added all of Problem 6 and added Hints to Problems 8 and 12 and expanded the Answer to problem 12.

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13.10: Lagrange Multipliers

Solving optimization problems for functions of two or more variables can be similar to solving such problems in single-variable calculus. However, techniques for dealing with multiple variables allow us to solve more varied optimization problems for which we need to deal with additional conditions or constraints. In this section, we examine one of the more common and useful methods for solving optimization problems with constraints.

Lagrange Multipliers

In the previous section, an applied situation was explored involving maximizing a profit function, subject to certain **constraints.** In that example, the constraints involved a maximum number of golf balls that could be produced and sold in 1 month (x), and a maximum number of advertising hours that could be purchased per month (y). Suppose these were combined into a single budgetary constraint, such as $20x + 4y \leq 216$, that took into account both the cost of producing the golf balls and the number of advertising hours purchased per month. The goal is still to maximize profit, but now there is a different type of constraint on the values of x and y. This constraint and the corresponding profit function

$$f(x, y) = 48x + 96y - x^2 - 2xy - 9y^2$$

is an example of an **optimization problem**, and the function $f(x, y)$ is called the **objective function**. A graph of various level curves of the function $f(x, y)$ follows.

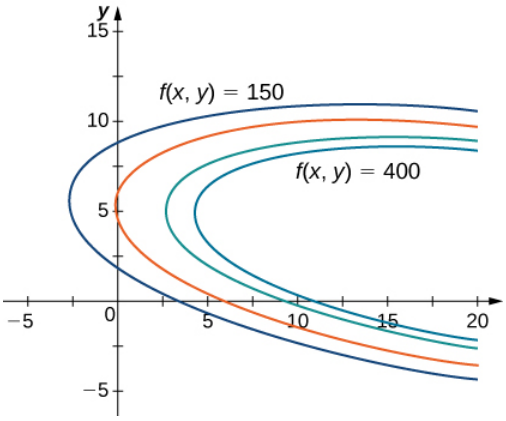

Figure $13.10.1$: Graph showing level curves of the function $f(x, y) = 48x + 96y - x^2 - 2xy - 9y^2$ *corresponding to* $c = 150, 250, 350,$ *and* 400.

In Figure $13.10.1$, the value c represents different profit levels (i.e., values of the function f). As the value of c increases, the curve shifts to the right. Since our goal is to maximize profit, we want to choose a curve as far to the right as possible. If there were no restrictions on the number of golf balls the company could produce or the number of units of advertising available, then we could produce as many golf balls as we want, and advertise as much as we want, and there would be not be a maximum profit for the company. Unfortunately, we have a budgetary constraint that is modeled by the inequality $20x + 4y \leq 216$. To see how this constraint interacts with the profit function, Figure $PageIndex2$ shows the graph of the line $20x + 4y = 216$ superimposed on the previous graph.

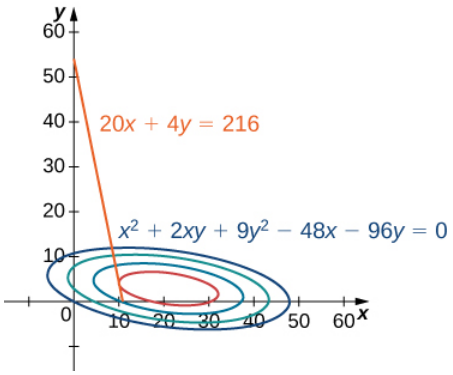

Figure 13.10.2: Graph of level curves of the function $f(x,y) = 48x + 96y - x^2 - 2xy - 9y^2$ *corresponding to* $c = 150, 250, 350,$ *and* 395. *The red graph is the constraint function.*

As mentioned previously, the maximum profit occurs when the level curve is as far to the right as possible. However, the level of production corresponding to this maximum profit must also satisfy the budgetary constraint, so the point at which this profit occurs must also lie on (or to the left of) the red line in Figure 13.10.2. Inspection of this graph reveals that this point exists where the line is tangent to the level curve of f. Trial and error reveals that this profit level seems to be around 395, when x and y are both just less than 5. We return to the solution of this problem later in this section. From a theoretical standpoint, at the point where the profit curve is tangent to the constraint line, the gradient of both of the functions evaluated at that point must point in the same (or opposite) direction. Recall that the gradient of a function of more than one variable is a vector. If two vectors point in the same (or opposite) directions, then one must be a constant multiple of the other. This idea is the basis of the **method of Lagrange multipliers.**

Theorem 13.10.1: Method of Lagrange Multipliers with One Constraint

Let f and g be functions of two variables with continuous partial derivatives at every point of some open set containing the smooth curve $g(x,y) = k$, where k is a constant. Suppose that f, when restricted to points on the curve $g(x,y) = k$, has a local extremum at the point (x_0, y_0) and that $\vec{\nabla} g(x_0, y_0) \neq 0$. Then there is a number λ called a **Lagrange multiplier,** for which

$$\vec{\nabla} f(x_0, y_0) = \lambda \vec{\nabla} g(x_0, y_0). \tag{13.10.1}$$

Proof

Assume that a constrained extremum occurs at the point (x_0, y_0). Furthermore, we assume that the equation $g(x,y) = k$ can be smoothly parameterized as

$$x = x(s) \text{ and } y = y(s)$$

where s is an arc length parameter with reference point (x_0, y_0) at $s = 0$. Therefore, the quantity $z = f(x(s), y(s))$ has a relative maximum or relative minimum at $s = 0$, and this implies that $\frac{dz}{ds} = 0$ at that point. From the chain rule,

$$\begin{aligned} \frac{dz}{ds} &= \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s} \\ &= \left(\frac{\partial f}{\partial x} \hat{\mathbf{i}} + \frac{\partial f}{\partial y} \hat{\mathbf{j}} \right) \cdot \left(\frac{\partial x}{\partial s} \hat{\mathbf{i}} + \frac{\partial y}{\partial s} \hat{\mathbf{j}} \right) \\ &= 0, \end{aligned}$$

where the derivatives are all evaluated at $s = 0$. However, the first factor in the dot product is the gradient of f, and the second factor is the unit tangent vector $\vec{\mathbf{T}}(0)$ to the constraint curve. Since the point (x_0, y_0) corresponds to $s = 0$, it follows from this equation that

$$\vec{\nabla} f(x_0, y_0) \cdot \vec{\mathbf{T}}(0) = 0,$$

which implies that the gradient is either the zero vector $\vec{\mathbf{0}}$ or it is normal to the constraint curve at a constrained relative extremum. However, the constraint curve $g(x,y) = k$ is a level curve for the function $g(x,y)$ so that if $\vec{\nabla} g(x_0, y_0) \neq 0$ then

$\vec{\nabla} g(x_0, y_0)$ is normal to this curve at (x_0, y_0) It follows, then, that there is some scalar λ such that

$$\vec{\nabla} f(x_0, y_0) = \lambda \vec{\nabla} g(x_0, y_0)$$

□

To apply **Theorem 13.10.1** to an optimization problem similar to that for the golf ball manufacturer, we need a problem-solving strategy.

Problem-Solving Strategy: Steps for Using Lagrange Multipliers

1. Determine the objective function $f(x, y)$ and the constraint function $g(x, y)$. Does the optimization problem involve maximizing or minimizing the objective function?
2. Set up a system of equations using the following template:

$$\vec{\nabla} f(x, y) = \lambda \vec{\nabla} g(x, y) \tag{13.10.2}$$

$$g(x, y) = k \tag{13.10.3}$$

3. Solve for x and y to determine the **Lagrange points**, i.e., points that satisfy the Lagrange multiplier equation.
4. If the objective function is continuous on the constraint and the constraint is a closed curve (like a circle or an ellipse), then the largest of the values of f at the solutions found in step 3 maximizes f, subject to the constraint; the smallest of those values minimizes f, subject to the constraint.

   But in other cases, we need to evaluate the objective functions f at points from the constraint on either side of each Lagrange point to determine whether we have obtained a relative maximum or a relative minimum.

Note that it is possible that our objective function will not have a relative maximum or a relative minimum at a given Lagrange point. This can occur in a couple situations, but most often when the Lagrange point is also a critical point of the objective function giving us a saddle point. Most of the time we will still get a relative extremum at a saddle point subject to a constraint, but sometimes we will not. See Figure **13.10.3** for an example of this case.

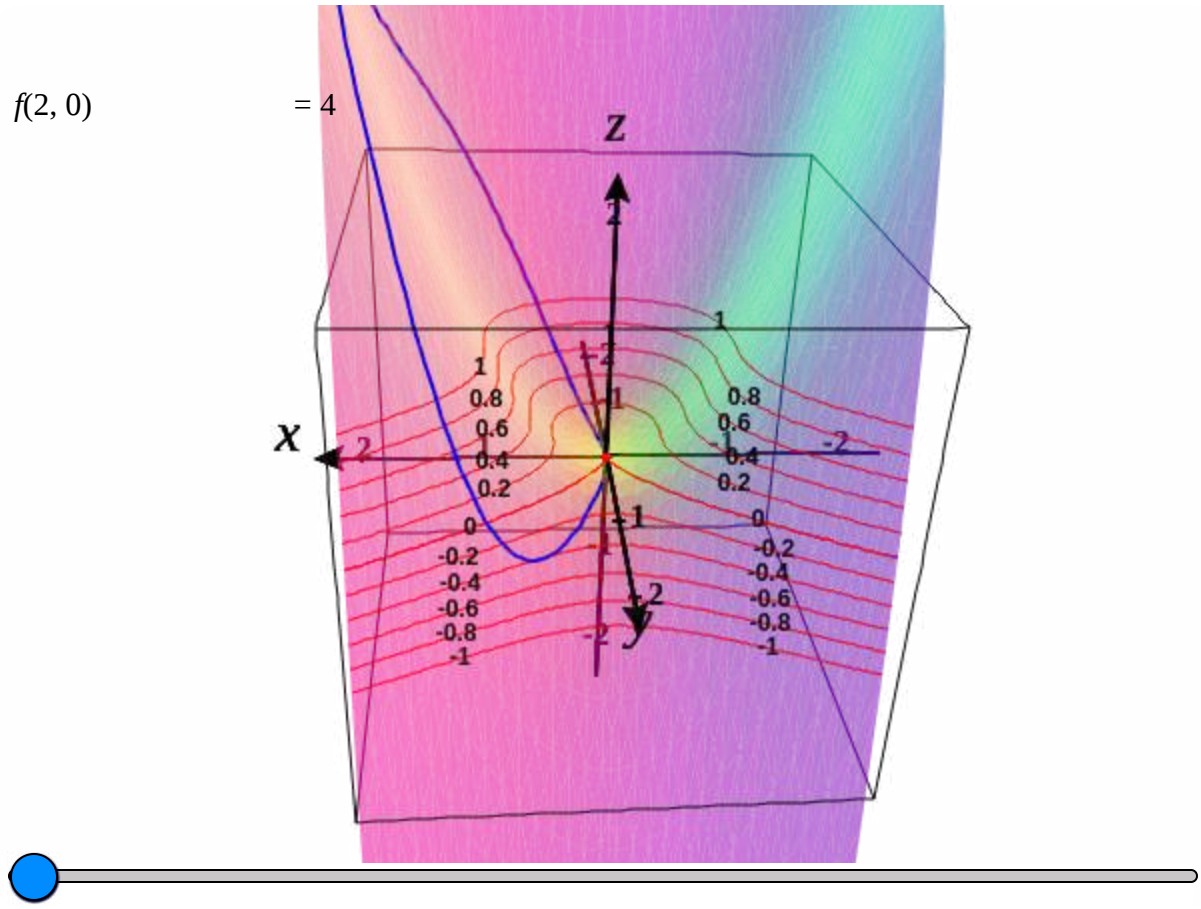

***Figure* 13.10.3:** *Graph of $f(x,y) = x^2 - y^3$ along with the constraint $(x-1)^2 + y^2 = 1$. Note that there is no relative extremum at $(0,0)$, although this point will satisfy the Lagrange Multiplier equation with $\lambda = 0$.*

Example 13.10.1: Using Lagrange Multipliers

Use the method of Lagrange multipliers to find the minimum value of $f(x,y) = x^2 + 4y^2 - 2x + 8y$ subject to the constraint $x + 2y = 7$.

Solution

Let's follow the problem-solving strategy:

1. The objective function is $f(x,y) = x^2 + 4y^2 - 2x + 8y$. The constraint function is equal to the left-hand side of the constraint equation when only a constant is on the right-hand side. So here $g(x,y) = x + 2y$. The problem asks us to solve for the minimum value of f, subject to the constraint (Figure 13.10.4).

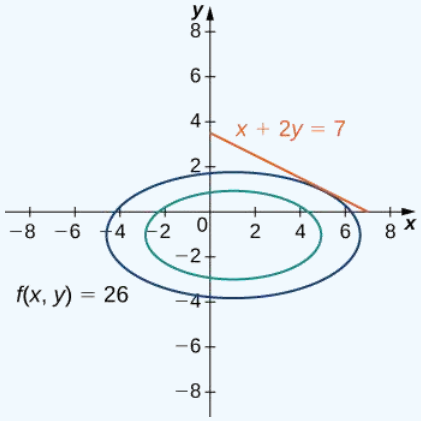

Figure 13.10.4: Graph of level curves of the function $f(x,y)=x^2+4y^2-2x+8y$ corresponding to $c=10$ and 26. The red graph is the constraint function.

2. We then must calculate the gradients of both f and g:

$$\vec{\nabla} f(x,y)=(2x-2)\,\hat{\mathbf{i}}+(8y+8)\,\hat{\mathbf{j}} \tag{13.10.4}$$
$$\vec{\nabla} g(x,y)=\hat{\mathbf{i}}+2\hat{\mathbf{j}}.$$

The equation $\vec{\nabla} f(x,y)=\lambda\vec{\nabla} g(x,y)$ becomes

$$(2x-2)\,\hat{\mathbf{i}}+(8y+8)\,\hat{\mathbf{j}}=\lambda\left(\hat{\mathbf{i}}+2\hat{\mathbf{j}}\right), \tag{13.10.5}$$

which can be rewritten as

$$(2x-2)\,\hat{\mathbf{i}}+(8y+8)\,\hat{\mathbf{j}}=\lambda\hat{\mathbf{i}}+2\lambda\hat{\mathbf{j}}. \tag{13.10.6}$$

Next, we set the coefficients of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ equal to each other:

$$2x-2=\lambda \tag{13.10.7}$$
$$8y+8=2\lambda. \tag{13.10.8}$$

The equation $g(x,y)=k$ becomes $x+2y=7$. Therefore, the system of equations that needs to be solved is

$$2x-2=\lambda \tag{13.10.9}$$
$$8y+8=2\lambda \tag{13.10.10}$$
$$x+2y=7. \tag{13.10.11}$$

3. This is a linear system of three equations in three variables. We start by solving the second equation for λ and substituting it into the first equation. This gives $\lambda=4y+4$, so substituting this into the first equation gives

$$2x-2=4y+4.$$

Solving this equation for x gives $x=2y+3$. We then substitute this into the third equation:

$$\begin{aligned}(2y+3)+2y&=7\\ 4y&=4\\ y&=1.\end{aligned}$$

Since $x=2y+3$, this gives $x=5$.

4. Next, we evaluate $f(x,y)=x^2+4y^2-2x+8y$ at the point $(5,1)$,

$$f(5,1)=5^2+4(1)^2-2(5)+8(1)=27. \tag{13.10.12}$$

To ensure this corresponds to a minimum value on the constraint function, let's try some other points on the constraint from either side of the point $(5,1)$, such as the intercepts of $g(x,y)=0$, Which are $(7,0)$ and $(0,3.5)$.

We get $f(7,0)=35>27$ and $f(0,3.5)=77>27$.

So it appears that f has a relative minimum of 27 at $(5,1)$, subject to the given constraint.

? Exercise 13.10.1

Use the method of Lagrange multipliers to find the maximum value of

$$f(x,y) = 9x^2 + 36xy - 4y^2 - 18x - 8y$$

subject to the constraint $3x + 4y = 32$.

Let's now return to the problem posed at the beginning of the section.

✓ Example 13.10.2: Golf Balls and Lagrange Multipliers

The golf ball manufacturer, Pro-T, has developed a profit model that depends on the number x of golf balls sold per month (measured in thousands), and the number of hours per month of advertising y, according to the function

$$z = f(x,y) = 48x + 96y - x^2 - 2xy - 9y^2,$$

where z is measured in thousands of dollars. The budgetary constraint function relating the cost of the production of thousands golf balls and advertising units is given by $20x + 4y = 216$. Find the values of x and y that maximize profit, and find the maximum profit.

Solution:

Again, we follow the problem-solving strategy:

1. The objective function is $f(x,y) = 48x + 96y - x^2 - 2xy - 9y^2$. To determine the constraint function, we divide both sides by 4, which gives $5x + y = 54$. The constraint function is equal to the left-hand side, so $g(x,y) = 5x + y$. The problem asks us to solve for the maximum value of f, subject to this constraint.
2. So, we calculate the gradients of both f and g:

$$\vec{\nabla} f(x,y) = (48 - 2x - 2y)\hat{\mathbf{i}} + (96 - 2x - 18y)\hat{\mathbf{j}}$$
$$\vec{\nabla} g(x,y) = 5\hat{\mathbf{i}} + \hat{\mathbf{j}}.$$

The equation $\vec{\nabla} f(x,y) = \lambda \vec{\nabla} g(x,y)$ becomes

$$(48 - 2x - 2y)\hat{i} + (96 - 2x - 18y)\hat{\mathbf{j}} = \lambda(5\hat{\mathbf{i}} + \hat{\mathbf{j}}),$$

which can be rewritten as

$$(48 - 2x - 2y)\hat{\mathbf{i}} + (96 - 2x - 18y)\hat{\mathbf{j}} = \lambda 5\hat{\mathbf{i}} + \lambda\hat{\mathbf{j}}.$$

We then set the coefficients of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ equal to each other:

$$48 - 2x - 2y = 5\lambda$$
$$96 - 2x - 18y = \lambda.$$

The equation $g(x,y) = k$ becomes $5x + y = 54$. Therefore, the system of equations that needs to be solved is

$$48 - 2x - 2y = 5\lambda$$
$$96 - 2x - 18y = \lambda$$
$$5x + y = 54.$$

3. We use the left-hand side of the second equation to replace λ in the first equation:

$$48 - 2x - 2y = 5(96 - 2x - 18y)$$
$$48 - 2x - 2y = 480 - 10x - 90y$$
$$8x = 432 - 88y$$
$$x = 54 - 11y.$$

Then we substitute this into the third equation:

$$\begin{aligned} 5(54-11y)+y &= 54 \\ 270-55y+y &= 54 \\ 216 &= 54y \\ y &= 4. \end{aligned}$$

Since $x = 54-11y$, this gives $x = 10$.

4. We then substitute $(10,4)$ into $f(x,y) = 48x+96y-x^2-2xy-9y^2$, which gives

$$\begin{aligned} f(10,4) &= 48(10)+96(4)-(10)^2-2(10)(4)-9(4)^2 \\ &= 480+384-100-80-144 = 540. \end{aligned}$$

Therefore the maximum profit that can be attained, subject to budgetary constraints, is $\$540,000$ with a production level of $10,000$ golf balls and 4 hours of advertising bought per month. Let's check to make sure this truly is a maximum. The endpoints of the line that defines the constraint are $(10.8,0)$ and $(0,54)$ Let's evaluate f at both of these points:

$$\begin{aligned} f(10.8,0) &= 48(10.8)+96(0)-10.8^2-2(10.8)(0)-9(0^2) \\ &= 401.76 \\ f(0,54) &= 48(0)+96(54)-0^2-2(0)(54)-9(54^2) \\ &= -21,060. \end{aligned}$$

The second value represents a loss, since no golf balls are produced. Neither of these values exceed 540, so it seems that our extremum is a maximum value of f, subject to the given constraint.

? Exercise 13.10.2: Optimizing the Cobb-Douglas function

A company has determined that its production level is given by the Cobb-Douglas function $f(x,y) = 2.5x^{0.45}y^{0.55}$ where x represents the total number of labor hours in 1 year and y represents the total capital input for the company. Suppose 1 unit of labor costs $\$40$ and 1 unit of capital costs $\$50$. Use the method of Lagrange multipliers to find the maximum value of $f(x,y) = 2.5x^{0.45}y^{0.55}$ subject to a budgetary constraint of $\$500,000$ per year.

In the case of an objective function with three variables and a single constraint function, it is possible to use the method of Lagrange multipliers to solve an optimization problem as well. An example of an objective function with three variables could be the **Cobb-Douglas function** in Exercise 13.10.2: $f(x,y,z) = x^{0.2}y^{0.4}z^{0.4}$, where x represents the cost of labor, y represents capital input, and z represents the cost of advertising. The method is the same as for the method with a function of two variables; the equations to be solved are

$$\begin{aligned} \vec{\nabla} f(x,y,z) &= \lambda \vec{\nabla} g(x,y,z) \\ g(x,y,z) &= k. \end{aligned}$$

✓ Example 13.10.3: Lagrange Multipliers with a Three-Variable objective function

Maximize the function $f(x,y,z) = x^2+y^2+z^2$ subject to the constraint $x+y+z = 1$.

Solution:

1. The objective function is $f(x,y,z) = x^2+y^2+z^2$. To determine the constraint function, we set it equal to the variable expression on the left-hand side of the constraint equation: $x+y+z = 1$ which gives the constraint function as $g(x,y,z) = x+y+z$.

2. Next, we calculate $\vec{\nabla} f(x,y,z)$ and $\vec{\nabla} g(x,y,z)$:

$$\begin{aligned} \vec{\nabla} f(x,y,z) &= \langle 2x, 2y, 2z \rangle \\ \vec{\nabla} g(x,y,z) &= \langle 1,1,1 \rangle. \end{aligned}$$

This leads to the equations

$$\langle 2x, 2y, 2z\rangle = \lambda\langle 1, 1, 1\rangle$$
$$x + y + z = 1$$

which can be rewritten in the following form:

$$2x = \lambda$$
$$2y = \lambda$$
$$2z = \lambda$$
$$x + y + z = 1.$$

3. Since each of the first three equations has λ on the right-hand side, we know that $2x = 2y = 2z$ and all three variables are equal to each other. Substituting $y = x$ and $z = x$ into the last equation yields $3x = 1$, so $x = \frac{1}{3}$ and $y = \frac{1}{3}$ and $z = \frac{1}{3}$ which corresponds to a critical point on the constraint curve.

4. Then, we evaluate f at the point $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$:

$$f\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 = \frac{3}{9} = \frac{1}{3} \tag{13.10.13}$$

Therefore, a possible extremum of the function is $\frac{1}{3}$. To verify it is a minimum, choose other points that satisfy the constraint from either side of the point we obtained above and calculate f at those points. For example,

$$f(1, 0, 0) = 1^2 + 0^2 + 0^2 = 1$$
$$f(0, -2, 3) = 0^2 + (-2)^2 + 3^2 = 13.$$

Both of these values are greater than $\frac{1}{3}$, leading us to believe the extremum is a minimum, subject to the given constraint.

? Exercise 13.10.3:

Use the method of Lagrange multipliers to find the minimum value of the function

$$f(x, y, z) = x + y + z$$

subject to the constraint $x^2 + y^2 + z^2 = 1$.

Problems with Two Constraints

The method of Lagrange multipliers can be applied to problems with more than one constraint. In this case the objective function, w is a function of three variables:

$$w = f(x, y, z) \tag{13.10.14}$$

and it is subject to two constraints:

$$g(x, y, z) = 0 \text{ and } h(x, y, z) = 0. \tag{13.10.15}$$

There are two Lagrange multipliers, λ_1 and λ_2, and the system of equations becomes

$$\vec{\nabla} f(x_0, y_0, z_0) = \lambda_1 \vec{\nabla} g(x_0, y_0, z_0) + \lambda_2 \vec{\nabla} h(x_0, y_0, z_0)$$
$$g(x_0, y_0, z_0) = 0$$
$$h(x_0, y_0, z_0) = 0$$

✓ Example 13.10.4: Lagrange Multipliers with Two Constraints

Find the maximum and minimum values of the function

$$f(x, y, z) = x^2 + y^2 + z^2$$

subject to the constraints $z^2 = x^2 + y^2$ and $x + y - z + 1 = 0$.

Solution:

Let's follow the problem-solving strategy:

1. The objective function is $f(x,y,z) = x^2 + y^2 + z^2$. To determine the constraint functions, we first subtract z^2 from both sides of the first constraint, which gives $x^2 + y^2 - z^2 = 0$, so $g(x,y,z) = x^2 + y^2 - z^2$. The second constraint function is $h(x,y,z) = x + y - z + 1$.
2. We then calculate the gradients of f, g, and h:

$$\vec{\nabla} f(x,y,z) = 2x\hat{\mathbf{i}} + 2y\hat{\mathbf{j}} + 2z\hat{\mathbf{k}}$$
$$\vec{\nabla} g(x,y,z) = 2x\hat{\mathbf{i}} + 2y\hat{\mathbf{j}} - 2z\hat{\mathbf{k}}$$
$$\vec{\nabla} h(x,y,z) = \hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}.$$

The equation $\vec{\nabla} f(x,y,z) = \lambda_1 \vec{\nabla} g(x,y,z) + \lambda_2 \vec{\nabla} h(x,y,z)$ becomes

$$2x\hat{\mathbf{i}} + 2y\hat{\mathbf{j}} + 2z\hat{\mathbf{k}} = \lambda_1(2x\hat{\mathbf{i}} + 2y\hat{\mathbf{j}} - 2z\hat{\mathbf{k}}) + \lambda_2(\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}), \tag{13.10.16}$$

which can be rewritten as

$$2x\hat{\mathbf{i}} + 2y\hat{\mathbf{j}} + 2z\hat{\mathbf{k}} = (2\lambda_1 x + \lambda_2)\hat{\mathbf{i}} + (2\lambda_1 y + \lambda_2)\hat{\mathbf{j}} - (2\lambda_1 z + \lambda_2)\hat{\mathbf{k}}. \tag{13.10.17}$$

Next, we set the coefficients of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ equal to each other:

$$2x = 2\lambda_1 x + \lambda_2$$
$$2y = 2\lambda_1 y + \lambda_2$$
$$2z = -2\lambda_1 z - \lambda_2.$$

The two equations that arise from the constraints are $z^2 = x^2 + y^2$ and $x + y - z + 1 = 0$. Combining these equations with the previous three equations gives

$$\begin{aligned} 2x &= 2\lambda_1 x + \lambda_2 \\ 2y &= 2\lambda_1 y + \lambda_2 \\ 2z &= -2\lambda_1 z - \lambda_2 \\ z^2 &= x^2 + y^2 \\ x + y - z + 1 &= 0. \end{aligned}$$

3. The first three equations contain the variable λ_2. Solving the third equation for λ_2 and replacing into the first and second equations reduces the number of equations to four:

$$\begin{aligned} 2x &= 2\lambda_1 x - 2\lambda_1 z - 2z \\ 2y &= 2\lambda_1 y - 2\lambda_1 z - 2z \\ z^2 &= x^2 + y^2 \\ x + y - z + 1 &= 0. \end{aligned}$$

Next, we solve the first and second equation for λ_1. The first equation gives $\lambda_1 = \dfrac{x+z}{x-z}$, the second equation gives $\lambda_1 = \dfrac{y+z}{y-z}$. We set the right-hand side of each equation equal to each other and cross-multiply:

$$\frac{x+z}{x-z} = \frac{y+z}{y-z}$$
$$(x+z)(y-z) = (x-z)(y+z)$$
$$xy - xz + yz - z^2 = xy + xz - yz - z^2$$
$$2yz - 2xz = 0$$
$$2z(y-x) = 0.$$

Therefore, either $z=0$ or $y=x$. If $z=0$, then the first constraint becomes $0=x^2+y^2$. The only real solution to this equation is $x=0$ and $y=0$, which gives the ordered triple $(0,0,0)$. This point does not satisfy the second constraint, so it is not a solution. Next, we consider $y=x$, which reduces the number of equations to three:

$$\begin{aligned} y &= x \\ z^2 &= x^2+y^2 \\ x+y-z+1 &= 0. \end{aligned}$$

We substitute the first equation into the second and third equations:

$$\begin{aligned} z^2 &= x^2+x^2 \\ &= x+x-z+1=0. \end{aligned}$$

Then, we solve the second equation for z, which gives $z=2x+1$. We then substitute this into the first equation,

$$\begin{aligned} z^2 &= 2x^2 \\ (2x+1)^2 &= 2x^2 \\ 4x^2+4x+1 &= 2x^2 \\ 2x^2+4x+1 &= 0, \end{aligned}$$

and use the quadratic formula to solve for x:

$$\begin{aligned} x &= \frac{-4\pm\sqrt{4^2-4(2)(1)}}{2(2)} = \frac{-4\pm\sqrt{8}}{4} \\ &= \frac{-4\pm 2\sqrt{2}}{4} = -1\pm\frac{\sqrt{2}}{2}. \end{aligned} \tag{13.10.18}$$

Recall $y=x$, so this solves for y as well. Then, $z=2x+1$, so

$$z=2x+1=2\left(-1\pm\frac{\sqrt{2}}{2}\right)+1=-2+1\pm\sqrt{2}=-1\pm\sqrt{2}. \tag{13.10.19}$$

Therefore, there are two ordered triplet solutions:

$$\begin{aligned} &\left(-1+\frac{\sqrt{2}}{2},-1+\frac{\sqrt{2}}{2},-1+\sqrt{2}\right) \\ &\text{and } \left(-1-\frac{\sqrt{2}}{2},-1-\frac{\sqrt{2}}{2},-1-\sqrt{2}\right). \end{aligned} \tag{13.10.20}$$

4. We substitute $\left(-1+\frac{\sqrt{2}}{2},-1+\frac{\sqrt{2}}{2},-1+\sqrt{2}\right)$ into $f(x,y,z)=x^2+y^2+z^2$, which gives

$$\begin{aligned} f\left(-1+\frac{\sqrt{2}}{2},-1+\frac{\sqrt{2}}{2},-1+\sqrt{2}\right) &= \left(-1+\frac{\sqrt{2}}{2}\right)^2+\left(-1+\frac{\sqrt{2}}{2}\right)^2 \\ &\quad +(-1+\sqrt{2})^2 \\ &= \left(1-\sqrt{2}+\frac{1}{2}\right)+\left(1-\sqrt{2}+\frac{1}{2}\right) \\ &\quad +(1-2\sqrt{2}+2) \\ &= 6-4\sqrt{2}. \end{aligned}$$

Then, we substitute $\left(-1-\frac{\sqrt{2}}{2},-1+\frac{\sqrt{2}}{2},-1+\sqrt{2}\right)$ into $f(x,y,z)=x^2+y^2+z^2$, which gives

$$f\left(-1-\frac{\sqrt{2}}{2},-1+\frac{\sqrt{2}}{2},-1+\sqrt{2}\right)=\left(-1-\frac{\sqrt{2}}{2}\right)^2+\left(-1-\frac{\sqrt{2}}{2}\right)^2+(-1-\sqrt{2})^2$$
$$=\left(1+\sqrt{2}+\frac{1}{2}\right)+\left(1+\sqrt{2}+\frac{1}{2}\right)+(1+2\sqrt{2}+2)$$
$$=6+4\sqrt{2}.$$

$6+4\sqrt{2}$ is the maximum value and $6-4\sqrt{2}$ is the minimum value of $f(x,y,z)$, subject to the given constraints.

? Exercise 13.10.4

Use the method of Lagrange multipliers to find the minimum value of the function

$$f(x,y,z)=x^2+y^2+z^2 \tag{13.10.21}$$

subject to the constraints $2x+y+2z=9$ and $5x+5y+7z=29$.

Key Concepts

- An objective function combined with one or more constraints is an example of an optimization problem.
- To solve optimization problems, we apply the method of Lagrange multipliers using a four-step problem-solving strategy.

Key Equations

- **Method of Lagrange multipliers, one constraint**

  $\vec{\nabla}f(x,y)=\lambda\vec{\nabla}g(x,y)$

  $g(x,y)=k$

- **Method of Lagrange multipliers, two constraints**

  $\vec{\nabla}f(x_0,y_0,z_0)=\lambda_1\vec{\nabla}g(x_0,y_0,z_0)+\lambda_2\vec{\nabla}h(x_0,y_0,z_0)$

  $g(x_0,y_0,z_0)=0$

  $h(x_0,y_0,z_0)=0$

Glossary

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13.10E: Exercises for Lagrange Multipliers

In exercises 1-15, use the method of Lagrange multipliers to find the maximum and minimum values of the function subject to the given constraint.

1) Objective function: $f(x,y) = 4xy$ Constraint: $\frac{x^2}{9} + \frac{y^2}{16} = 1$

2) Objective function: $f(x,y) = x^2y$ Constraint: $x^2 + 2y^2 = 6$

3) Objective function: $f(x,y) = x^2 + y^2 + 2x - 2y + 1$ Constraint: $g(x,y) = x^2 + y^2 = 2$

4) Objective function: $f(x,y) = xy$ Constraint: $4x^2 + 8y^2 = 16$

5) Objective function: $f(x,y) = x^2 + y^2$ Constraint: $xy = 1$

6) Objective function: $f(x,y) = x^2 - y^2$ Constraint: $x - 2y + 6 = 0$

7) Objective function: $f(x,y) = x^2 + y^2$ Constraint: $x + 2y - 5 = 0$

8) Objective function: $f(x,y) = x^2 + y^2$ Constraint: $(x-1)^2 + 4y^2 = 4$

9) Objective function: $f(x,y) = 4x^3 + y^2$ Constraint: $2x^2 + y^2 = 1$

10) Objective function: $f(x,y) = 2x^2 + y^2$ Constraint: $g(x,y) = x^2 + y^2 = 1$

11) Objective function: $f(x,y,z) = x + 3y - z$ Constraint: $x^2 + y^2 + z^2 = 4$

12) Objective function: $f(x,y,z) = x + y + z$ Constraint: $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$

13) Objective function: $f(x,y,z) = xyz$ Constraint: $x^2 + 2y^2 + 3z^2 = 6$

14) Objective function: $f(x,y,z) = x^2 + y^2 + z^2$ Constraint: $x^4 + y^4 + z^4 = 1$

15) Objective function: $f(x,y,z) = x^2 + y^2 + z^2$ Constraint: $xyz = 4$

In exercises 16-21, use the method of Lagrange multipliers to find the requested extremum of the given function subject to the given constraint.

16) Maximize $f(x,y) = \sqrt{6 - x^2 - y^2}$ subject to the constraint, $x + y - 2 = 0$.

17) Maximize $f(x,y) = x^2 - y^2$ subject to the constraints, $g(x,y) = y - x^2 = 0, \quad x > 0, \quad y > 0$.

18) Maximize $U(x,y) = 8x^{4/5}y^{1/5}$ subject to the constraint, $4x + 2y = 12$.

19) Minimize $f(x,y,z) = x^2 + y^2 + z^2$ subject to the constraint, $x + y + z = 1$.

20) Minimize $f(x,y) = xy$ on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

21) Maximize $f(x,y,z) = 2x + 3y + 5z$ on the sphere $x^2 + y^2 + z^2 = 19$.

In exercises 22-23, use the method of Lagrange multipliers with two constraints.

22) Optimize $f(x,y,z) = yz + xy$ subject to the constraints: $xy = 1, \quad y^2 + z^2 = 1$.

23) Minimize $f(x,y,z) = x^2 + y^2 + z^2$ when $x + y + z = 9$ and $x + 2y + 3z = 20$.

Use the method of Lagrange multipliers to solve the following applied problems.

24) A large container in the shape of a rectangular solid must have a volume of 480 m^3. The bottom of the container costs \$5/m^2 to construct whereas the top and sides cost \$3/m^2 to construct. Use Lagrange multipliers to find the dimensions of the container of this size that has the minimum cost.

25) A rectangular box without a top (a topless box) is to be made from 12 ft^2 of cardboard. Find the maximum volume of such a box.

26) Find the minimum distance from the parabola $y = x^2$ to point $(0,3)$.

27) Find the point on the line $y = 2x + 3$ that is closest to point $(4,2)$.

29) Find the minimum distance from point $(0,1)$ to the parabola $x^2 = 4y$.

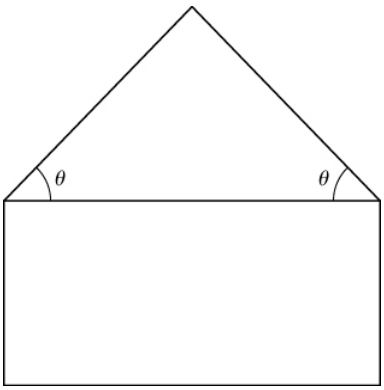
30) Find the minimum and maximum distances between the ellipse $x^2 + xy + 2y^2 = 1$ and the origin.

31) Find the minimum distance from the plane $x + y + z = 1$ to point $(2,1,1)$.

32) Find the point on the plane $4x + 3y + z = 2$ that is closest to the point $(1,-1,1)$.

33) Find the point on the surface $x^2 - 2xy + y^2 - x + y = 0$ closest to the point $(1,2,-3)$.

34) A pentagon is formed by placing an isosceles triangle on a rectangle, as shown in the diagram. If the perimeter of the pentagon is 10 in., find the lengths of the sides of the pentagon that will maximize the area of the pentagon.

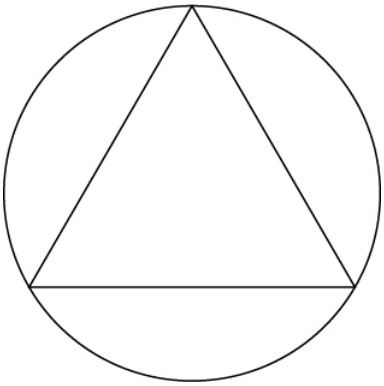


35) **[T]** By investing x units of labor and y units of capital, a watch manufacturer can produce $P(x,y) = 50x^{0.4}y^{0.6}$ watches. Find the maximum number of watches that can be produced on a budget of $20,000 if labor costs $100/unit and capital costs $200/unit. Use a grapher like CalcPlot3D to sketch a contour plot of the function.

36) A rectangular solid is contained within a tetrahedron with vertices at $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, and the origin. The base of the box has dimensions x and y, and the height of the box is z. If the sum of x, y, and z is 1, find the dimensions that maximizes the volume of the rectangular solid.

37) Find the maximum value of $f(x,y) = \sin x \sin y$, where x and y denote the acute angles of a right triangle. Draw the surface plot and contour plot of the function using a CAS.

38) Show that, of all the triangles inscribed in a circle of radius R (see diagram), the equilateral triangle has the largest perimeter.



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- Paul Seeburger (Monroe Community College) reordered these problems, adding problems 3 and 10 and answers for problems 15 and 17. He also added a full worked-out solution for problem 9 and a link to CalcPlot3D in problems 15, 21 and 35. He also

created new images for Problem 37 and expanded the answers for many problems.

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13R: Chapter 13 Review Exercises

For the following exercises, determine whether the statement is *true or false*. Justify your answer with a proof or a counterexample.

1. The domain of $f(x,y) = x^3 \arcsin(y)$ is $\{(x,y) \mid x \in \mathbb{R} \text{ and } -\pi \leq y \leq \pi\}$.

2. If the function $f(x,y)$ is continuous everywhere, then $f_{xy}(x,y) = f_{yx}(x,y)$.

Answer

True, by Clairaut's theorem

3. The linear approximation to the function of $f(x,y) = 5x^2 + x\tan y$ at the point $(2,\pi)$ is given by $L(x,y) = 22 + 21(x-2) + (y-\pi)$.

4. $(34, 916)$ is a critical point of $g(x,y) = 4x^3 - 2x^2y + y^2 - 2$.

Answer

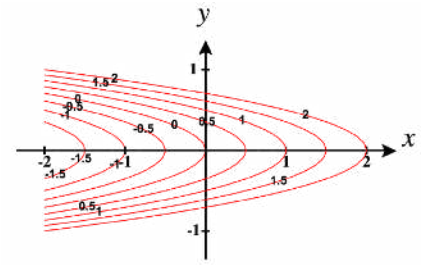
False

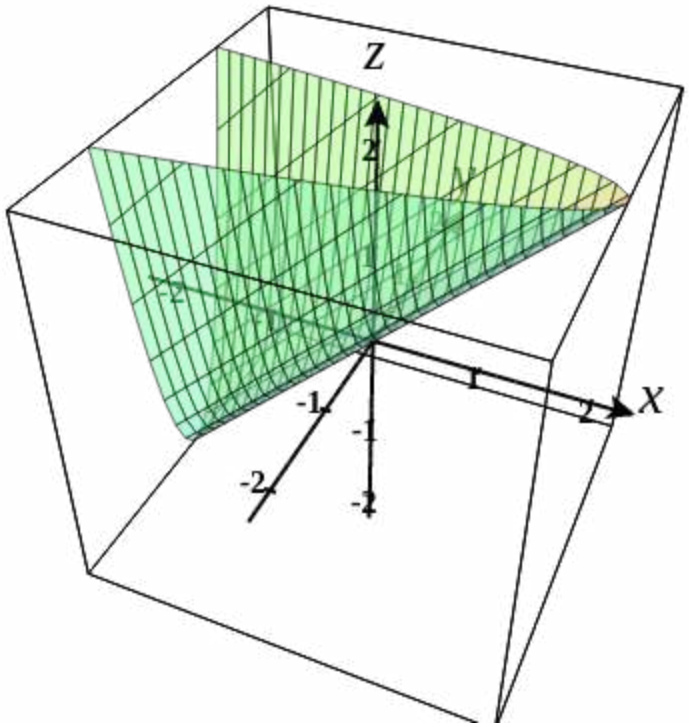
For the following exercises, sketch the function in one graph and, in a second, sketch several level curves.

5. $f(x,y) = e^{-(x^2+2y^2)}$

6. $f(x,y) = x + 4y^2$

Answer





For the following exercises, evaluate the following limits, if they exist. If they do not exist, prove it.

7. $\lim\limits_{(x,y)\to(1,1)} \dfrac{4xy}{x-2y^2}$

8. $\lim\limits_{(x,y)\to(0,0)} \dfrac{4xy}{x-2y^2}$

Answer

Does not exist.

For the following exercises, find the largest interval of continuity for the function.

9. $f(x,y)=x^3 \arcsin y$

10. $g(x,y)=\ln(4-x^2-y^2)$

Answer

Continuous at all points on the xy-plane, except where $x^2+y^2>4$.

For the following exercises, find all first partial derivatives.

11. $f(x,y)=x^2-y^2$

12. $u(x,y)=x^4-3xy+1$, with $x=2t$ and $y=t^3$

Answer

$$\frac{\partial u}{\partial x}=4x^3-3y,$$

$$\frac{\partial u}{\partial y}=-3x,$$

$$\frac{dx}{dt} = 2 \text{ and } \frac{dy}{dt} = 3t^2$$

$$\begin{aligned}\frac{du}{dt} &= \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} \\ &= 8x^3 - 6y - 9xt^2 \\ &= 8\left(2t\right)^3 - 6(t^3) - 9(2t)t^2 \\ &= 64t^3 - 6t^3 - 18t^3 \\ &= 40t^3\end{aligned}$$

For the following exercises, find all second partial derivatives.

13. $g(t,x) = 3t^2 - \sin(x+t)$

14. $h(x,y,z) = \dfrac{x^3e^{2y}}{z}$

Answer

$$\begin{aligned}h_{xx}(x,y,z) &= \frac{6xe^{2y}}{z}, \\ h_{xy}(x,y,z) &= \frac{6x^2e^{2y}}{z}, \\ h_{xz}(x,y,z) &= -\frac{3x^2e^{2y}}{z^2}, \\ h_{yx}(x,y,z) &= \frac{6x^2e^{2y}}{z}, \\ h_{yy}(x,y,z) &= \frac{4x^3e^{2y}}{z}, \\ h_{yz}(x,y,z) &= -\frac{2x^3e^{2y}}{z^2}, \\ h_{zx}(x,y,z) &= -\frac{3x^2e^{2y}}{z^2}, \\ h_{zy}(x,y,z) &= -\frac{2x^3e^{2y}}{z^2}, \\ h_{zz}(x,y,z) &= \frac{2x^3e^{2y}}{z^3}\end{aligned}$$

For the following exercises, find the equation of the tangent plane to the specified surface at the given point.

15. $z = x^3 - 2y^2 + y - 1$ at point $(1,1,-1)$

16. $z = e^x + \dfrac{2}{y}$ at point $(0,1,3)$

Answer

$z = x - 2y + 5$

17. Approximate $f(x,y) = e^{x^2} + \sqrt{y}$ at $(0.1, 9.1)$. Write down your linear approximation function $L(x,y)$. How accurate is the approximation to the exact answer, rounded to four digits?

18. Find the differential dz of $h(x,y) = 4x^2 + 2xy - 3y$ and approximate Δz at the point $(1,-2)$. Let $\Delta x = 0.1$ and $\Delta y = 0.01$.

Answer

$dz = 4\,dx - dy,\ dz(0.1, 0.01) = 0.39,\ \Delta z = 0.432$

19. Find the directional derivative of $f(x,y) = x^2 + 6xy - y^2$ in the direction $\vec{\mathbf{v}} = \hat{\mathbf{i}} + 4\hat{\mathbf{j}}$.

20. Find the maximal directional derivative magnitude and direction for the function $f(x,y) = x^3 + 2xy - \cos(\pi y)$ at point $(3,0)$.

Answer

$3\sqrt{85}\langle 27, 6\rangle$

For the following exercises, find the gradient.

21. $c(x,t) = e(t-x)^2 + 3\cos t$

22. $f(x,y) = \dfrac{\sqrt{x}+y^2}{xy}$

Answer

$$\vec{\nabla} f(x,y) = -\frac{\sqrt{x}+2y^2}{2x^2y}\,\hat{\mathbf{i}} + \left(\frac{1}{x} + \frac{1}{\sqrt{x}y^2}\right)\hat{\mathbf{j}}$$

For the following exercise, find and classify the critical points.

23. $z = x^3 - xy + y^2 - 1$

For the following exercises, use Lagrange multipliers to find the maximum and minimum values for the functions with the given constraints.

24. $f(x,y) = x^2y$, subject to the constraint: $x^2 + y^2 = 4$

Answer

maximum: $\dfrac{16}{3\sqrt{3}}$, minimum: $-\dfrac{16}{3\sqrt{3}}$,

25. $f(x,y) = x^2 - y^2$, subject to the constraint: $x + 6y = 4$

26. A machinist is constructing a right circular cone out of a block of aluminum. The machine gives an error of 5% in height and 2% in radius. Find the maximum error in the volume of the cone if the machinist creates a cone of height 6 cm and radius 2 cm.

Answer

2.3228 cm^3

27. A trash compactor is in the shape of a cuboid. Assume the trash compactor is filled with incompressible liquid. The length and width are decreasing at rates of 2 ft/sec and 3 ft/sec, respectively. Find the rate at which the liquid level is rising when the length is 14 ft, the width is 10 ft, and the height is 4 ft.

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CHAPTER OVERVIEW

Chapter 14: Multiple Integration

14.1: Iterated Integrals and Area

14.1E: Iterated Integrals and Area (Exercises)

14.2a: Double Integrals Over Rectangular Regions

14.2aE: Double Integrals Part 1 (Exercises)

14.2b: Double Integrals Over General Regions

14.2bE: Double Integrals Part 2 (Exercises)

14.3: Double Integration with Polar Coordinates

14.3b: Double Integrals in Polar Coordinates

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More Chapter 14 Review Exercises

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14.1: Iterated Integrals and Area

In the previous chapter we found that we could differentiate functions of several variables with respect to one variable, while treating all the other variables as constants or coefficients. We can integrate functions of several variables in a similar way. For instance, if we are told that $f_x(x,y) = 2xy$, we can treat y as staying constant and integrate to obtain $f(x,y)$:

$$\begin{aligned} f(x,y) &= \int f_x(x,y)\,dx \\ &= \int 2xy\,dx \\ &= x^2y + C. \end{aligned}$$

Make a careful note about the constant of integration, C. This "constant" is something with a derivative of 0 with respect to x, so it could be any expression that contains only constants and functions of y. For instance, if $f(x,y) = x^2y + \sin y + y^3 + 17$, then $f_x(x,y) = 2xy$. To signify that C is actually a function of y, we write:

$$f(x,y) = \int f_x(x,y)\,dx = x^2y + C(y). \tag{14.1.1}$$

Using this process we can even evaluate definite integrals.

Example 14.1.1: Integrating functions of more than one variable

Evaluate the integral $\int_1^{2y} 2xy\,dx$.

Solution

We find the indefinite integral as before, then apply the Fundamental Theorem of Calculus to evaluate the definite integral:

$$\begin{aligned} \int_1^{2y} 2xy\,dx &= x^2y\Big|_1^{2y} \\ &= (2y)^2y - (1)^2y \\ &= 4y^3 - y. \end{aligned}$$

We can also integrate with respect to y. In general,

$$\int_{h_1(y)}^{h_2(y)} f_x(x,y)\,dx = f(x,y)\Big|_{h_1(y)}^{h_2(y)} = f\big(h_2(y),y\big) - f\big(h_1(y),y\big), \tag{14.1.2}$$

and

$$\int_{g_1(x)}^{g_2(x)} f_y(x,y)\,dy = f(x,y)\Big|_{g_1(x)}^{g_2(x)} = f\big(x,g_2(x)\big) - f\big(x,g_1(x)\big). \tag{14.1.3}$$

Note that when integrating with respect to x, the bounds are functions of y (of the form $x = h_1(y)$ and $x = h_2(y)$) and the final result is also a function of y. When integrating with respect to y, the bounds are functions of x (of the form $y = g_1(x)$ and $y = g_2(x)$) and the final result is a function of x. Another example will help us understand this.

Example 14.1.2: Integrating functions of more than one variable

Evaluate $\int_1^x \left(5x^3y^{-3} + 6y^2\right)dy$.

Solution

We consider x as staying constant and integrate with respect to y:

$$\begin{aligned}\int_1^x \left(5x^3y^{-3}+6y^2\right)dy &= \left(\frac{5x^3y^{-2}}{-2}+\frac{6y^3}{3}\right)\Big|_1^x \\ &= \left(-\frac{5}{2}x^3x^{-2}+2x^3\right)-\left(-\frac{5}{2}x^3+2\right) \\ &= \frac{9}{2}x^3-\frac{5}{2}x-2.\end{aligned}$$

Note how the bounds of the integral are from $y=1$ to $y=x$ and that the final answer is a function of x.

In the previous example, we integrated a function with respect to y and ended up with a function of x. We can integrate this as well. This process is known as **iterated integration**, or **multiple integration**.

Example 14.1.3: Integrating an integral

Evaluate $\int_1^2 \left(\int_1^x \left(5x^3y^{-3}+6y^2\right)dy\right)dx$.

Solution

We follow a standard "order of operations" and perform the operations inside parentheses first (which is the integral evaluated in Example 14.1.2.)

$$\begin{aligned}\int_1^2 \left(\int_1^x \left(5x^3y^{-3}+6y^2\right)dy\right)dx &= \int_1^2 \left(\left[\frac{5x^3y^{-2}}{-2}+\frac{6y^3}{3}\right]\Big|_1^x\right)dx \\ &= \int_1^2 \left(\frac{9}{2}x^3-\frac{5}{2}x-2\right)dx \\ &= \left(\frac{9}{8}x^4-\frac{5}{4}x^2-2x\right)\Big|_1^2 \\ &= \frac{89}{8}.\end{aligned}$$

Note how the bounds of x were $x=1$ to $x=2$ and the final result was a number.

The previous example showed how we could perform something called an iterated integral; we do not yet know why we would be interested in doing so nor what the result, such as the number $89/8$, *means*. Before we investigate these questions, we offer some definitions.

Definition: Iterated Integration

Iterated integration is the process of repeatedly integrating the results of previous integrations. Integrating one integral is denoted as follows.

Let a, b, c and d be numbers and let $g_1(x)$, $g_2(x)$, $h_1(y)$ and $h_2(y)$ be functions of x and y, respectively. Then:

1. $\displaystyle\int_c^d\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy = \int_c^d \left(\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\right)dy.$
2. $\displaystyle\int_a^b\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\right)dx.$

Again make note of the bounds of these iterated integrals.

With $\displaystyle\int_c^d\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$, x varies from $h_1(y)$ to $h_2(y)$, whereas y varies from c to d. That is, the bounds of x are *curves*, the curves $x=h_1(y)$ and $x=h_2(y)$, whereas the bounds of y are *constants*, $y=c$ and $y=d$. It is useful to remember that when setting up and evaluating such iterated integrals, we integrate "from curve to curve, then from point to point."

We now begin to investigate *why* we are interested in iterated integrals and *what* they mean.

Area of a plane region

Consider the plane region R bounded by $a \le x \le b$ and $g_1(x) \le y \le g_2(x)$, shown in Figure 14.1.1. We learned in Section 7.1 (in Calculus I) that the area of R is given by

$$\int_a^b \big(g_2(x) - g_1(x)\big)\,dx. \tag{14.1.4}$$

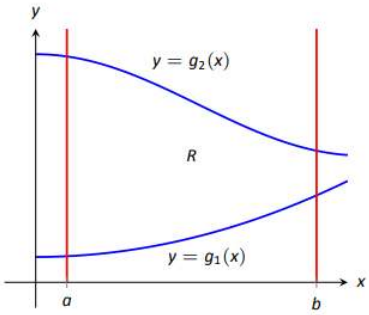

Figure 14.1.1: Calculating the area of a plane region R with an iterated integral.

We can view the expression $\big(g_2(x) - g_1(x)\big)$ as

$$\big(g_2(x) - g_1(x)\big) = \int_{g_1(x)}^{g_2(x)} 1\,dy = \int_{g_1(x)}^{g_2(x)} dy,$$

meaning we can express the area of R as an iterated integral:

$$\begin{aligned} \text{area of } R &= \int_a^b \big(g_2(x) - g_1(x)\big)\,dx \\ &= \int_a^b \left(\int_{g_1(x)}^{g_2(x)} dy\right) dx = \int_a^b \int_{g_1(x)}^{g_2(x)} dy\,dx. \end{aligned} \tag{14.1.5}$$

In short: a certain iterated integral can be viewed as giving the area of a plane region.

A region R could also be defined by $c \le y \le d$ and $h_1(y) \le x \le h_2(y)$, as shown in Figure 14.1.2. Using a process similar to that above, we have

$$\text{the area of } R = \int_c^d \int_{h_1(y)}^{h_2(y)} dx\,dy. \tag{14.1.6}$$

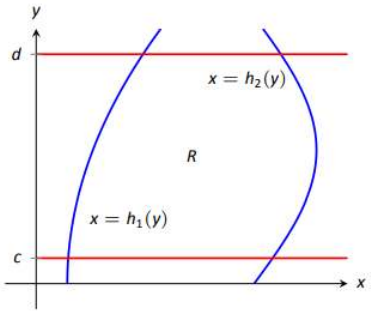

Figure 14.1.2: Calculating the area of a plane region R with an iterated integral.

We state this formally in a theorem.

THEOREM 14.1.1: Area of a plane region

1. Let R be a plane region bounded by $a \le x \le b$ and $g_1(x) \le y \le g_2(x)$, where g_1 and g_2 are continuous functions on $[a, b]$. The area A of R is

$$A = \int_a^b \int_{g_1(x)}^{g_2(x)} dy\, dx. \tag{14.1.7}$$

2. Let R be a plane region bounded by $c \le y \le d$ and $h_1(y) \le x \le h_2(y)$, where h_1 and h_2 are continuous functions on $[c, d]$. The area A of R is

$$A = \int_c^d \int_{h_1(y)}^{h_2(y)} dx\, dy. \tag{14.1.8}$$

The following examples should help us understand this theorem.

Example 14.1.4: Area of a rectangle

Find the area A of the rectangle with corners $(-1, 1)$ and $(3, 3)$, as shown in Figure 14.1.3.

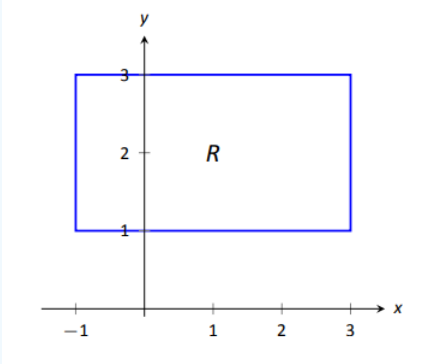

Figure 14.1.3: Calculating the area of a rectangle with an iterated integral in Example 14.1.4.

Solution

Multiple integration is obviously overkill in this situation, but we proceed to establish its use.

The region R is bounded by $x = -1$, $x = 3$, $y = 1$ and $y = 3$. Choosing to integrate with respect to y first, we have

$$A = \int_{-1}^{3} \int_{1}^{3} 1\, dy\, dx = \int_{-1}^{3} \left(y \Big|_1^3 \right) dx = \int_{-1}^{3} 2\, dx = 2x\Big|_{-1}^{3} = 8.$$

We could also integrate with respect to x first, giving:

$$A = \int_{1}^{3} \int_{-1}^{3} 1\, dx\, dy = \int_{1}^{3} \left(x \Big|_{-1}^3 \right) dy = \int_{1}^{3} 4\, dy = 4y\Big|_{1}^{3} = 8.$$

Clearly there are simpler ways to find this area, but it is interesting to note that this method works.

Example 14.1.5: Area of a triangle

Find the area A of the triangle with vertices at $(1, 1)$, $(3, 1)$ and $(5, 5)$, as shown in Figure 14.1.4.

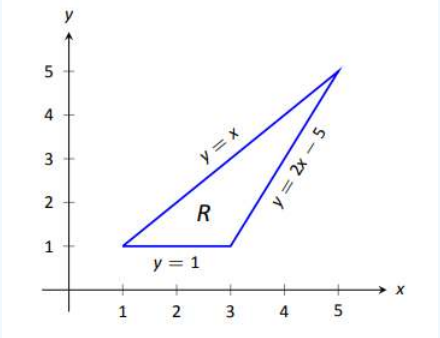

Figure 14.1.4: Calculating the area of a triangle with iterated integrals in Example 14.1.5.

Solution

The triangle is bounded by the lines as shown in the figure. Choosing to integrate with respect to x first gives that x is bounded by $x = y$ to $x = \frac{y+5}{2}$, while y is bounded by $y = 1$ to $y = 5$. (Recall that since x-values increase from left to right, the leftmost curve, $x = y$, is the lower bound and the rightmost curve, $x = (y+5)/2$, is the upper bound.) The area is

$$\begin{aligned} A &= \int_1^5 \int_y^{\frac{y+5}{2}} dx\, dy \\ &= \int_1^5 \left(x \Big|_y^{\frac{y+5}{2}} \right) dy \\ &= \int_1^5 \left(-\frac{1}{2}y + \frac{5}{2} \right) dy \\ &= \left(-\frac{1}{4}y^2 + \frac{5}{2}y \right) \Big|_1^5 \\ &= 4. \end{aligned}$$

We can also find the area by integrating with respect to y first. In this situation, though, we have two functions that act as the lower bound for the region R, $y = 1$ and $y = 2x - 5$. This requires us to use two iterated integrals. Note how the x-bounds are different for each integral:

$$\begin{aligned} A &= \int_1^3 \int_1^x 1\, dy\, dx \quad + \quad \int_3^5 \int_{2x-5}^x 1\, dy\, dx \\ &= \int_1^3 \left(y \right) \Big|_1^x dx \quad + \quad \int_3^5 \left(y \right) \Big|_{2x-5}^x dx \\ &= \int_1^3 \left(x - 1 \right) dx \quad + \quad \int_3^5 \left(-x + 5 \right) dx \\ &= 2 \quad + \quad 2 \\ &= 4. \end{aligned}$$

As expected, we get the same answer both ways.

Example 14.1.6: Area of a plane region

Find the area of the region enclosed by $y = 2x$ and $y = x^2$, as shown in Figure 14.1.5.

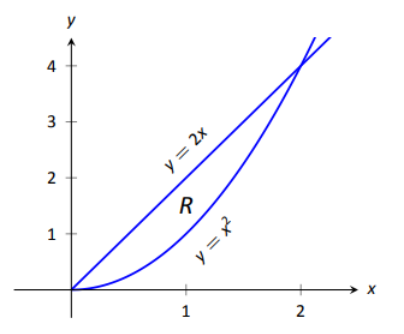

Figure 14.1.5: Calculating the area of a plane region with iterated integrals in Example 14.1.6.

Solution

Once again we'll find the area of the region using both orders of integration.

Using $dy\, dx$:

$$\int_0^2 \int_{x^2}^{2x} 1\, dy\, dx = \int_0^2 (2x - x^2)\, dx = \left(x^2 - \frac{1}{3}x^3 \right) \Big|_0^2 = \frac{4}{3}. \tag{14.1.9}$$

Using $dx\, dy$:

$$\int_0^4 \int_{y/2}^{\sqrt{y}} 1\, dx\, dy = \int_0^4 (\sqrt{y} - y/2)\, dy = \left(\frac{2}{3}y^{3/2} - \frac{1}{4}y^2 \right) \Big|_0^4 = \frac{4}{3}. \tag{14.1.10}$$

Changing Order of Integration

In each of the previous examples, we have been given a region R and found the bounds needed to find the area of R using both orders of integration. We integrated using both orders of integration to demonstrate their equality.

We now approach the skill of describing a region using both orders of integration from a different perspective. Instead of starting with a region and creating iterated integrals, we will start with an iterated integral and rewrite it in the other integration order. To do so, we'll need to understand the region over which we are integrating.

The simplest of all cases is when both integrals are bound by constants. The region described by these bounds is a rectangle (see Example 14.1.4), and so:

$$\int_a^b \int_c^d 1\,dy\,dx = \int_c^d \int_a^b 1\,dx\,dy. \tag{14.1.11}$$

When the inner integral's bounds are not constants, it is generally very useful to sketch the bounds to determine what the region we are integrating over looks like. From the sketch we can then rewrite the integral with the other order of integration.

Examples will help us develop this skill.

Example 14.1.7: Changing the order of integration

Rewrite the iterated integral $\int_0^6 \int_0^{x/3} 1\,dy\,dx$ with the order of integration $dx\,dy$.

Solution

We need to use the bounds of integration to determine the region we are integrating over.

The bounds tell us that y is bounded by 0 and $x/3$; x is bounded by 0 and 6. We plot these four curves: $y = 0$, $y = x/3$, $x = 0$ and $x = 6$ to find the region described by the bounds. Figure 14.1.6 shows these curves, indicating that R is a triangle.

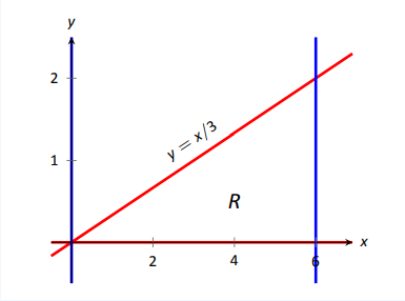

Figure 14.1.6: Sketching the region R described by the iterated integral in Example 14.1.7.

To change the order of integration, we need to consider the curves that bound the x-values. We see that the lower bound is $x = 3y$ and the upper bound is $x = 6$. The bounds on y are 0 to 2. Thus we can rewrite the integral as $\int_0^2 \int_{3y}^6 1\,dx\,dy$.

Example 14.1.8: Changing the order of integration

Change the order of integration of $\int_0^4 \int_{y^2/4}^{(y+4)/2} 1\,dx\,dy$.

Solution

We sketch the region described by the bounds to help us change the integration order. x is bounded below and above (i.e., to the left and right) by $x = y^2/4$ and $x = (y + 4)/2$ respectively, and y is bounded between 0 and 4. Graphing the previous curves, we find the region R to be that shown in Figure 14.1.7.

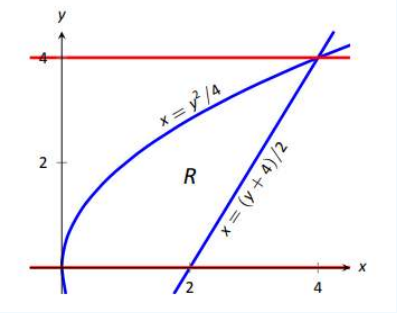

Figure **14.1.7**: Drawing the region determined by the bounds of integration in Example **14.1.8**.

To change the order of integration, we need to establish curves that bound y. The figure makes it clear that there are two lower bounds for y: $y = 0$ on $0 \le x \le 2$, and $y = 2x - 4$ on $2 \le x \le 4$. Thus we need two double integrals. The upper bound for each is $y = 2\sqrt{x}$. Thus we have

$$\int_0^4 \int_{y^2/4}^{(y+4)/2} 1\,dx\,dy = \int_0^2 \int_0^{2\sqrt{x}} 1\,dy\,dx + \int_2^4 \int_{2x-4}^{2\sqrt{x}} 1\,dy\,dx.$$

This section has introduced a new concept, the iterated integral. We developed one application for iterated integration: area between curves. However, this is not new, for we already know how to find areas bounded by curves.

In the next section we apply iterated integration to solve problems we currently do not know how to handle. The "real" goal of this section was not to learn a new way of computing area. Rather, our goal was to learn how to define a region in the plane using the bounds of an iterated integral. That skill is very important in the following sections.

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14.1E: Iterated Integrals and Area (Exercises)

Terms and Concepts

1. When integrating $f_x(x,y)$ with respect to x, the constant of integration C is really which: $C(x)$ or $C(y)$? What does this mean?

2. Integrating an iterated integral is called _____ _____.

3. When evaluating an iterated integral of the form $\int_a^b \int_{g_1(x)}^{g_2(x)} dy\,dx$, we integrate from _____ to _____, then from _____ to _____.

4. One understanding of an iterated integral is that $\int_a^b \int_{g_1(x)}^{g_2(x)} dy\,dx$ gives the _____ of a plane region.

Problems

In Exercises 5-10, evaluate the integral and subsequent iterated integral.

5. (a) $\int_2^5 (6x^2+4xy-3y^2)\,dy$

(b) $\int_{-3}^{2}\int_2^5 (6x^2+4xy-3y^2)\,dy\,dx$

6. (a) $\int_0^{\pi} (2x\cos y+\sin x)\,dx$

(b) $\int_0^{\pi/2}\int_0^{\pi} (2x\cos y+\sin x)\,dx\,dy$

7. (a) $\int_1^x (x^2y-y+2)\,dy$

(b) $\int_0^2\int_1^x (x^2y-y+2)\,dy\,dx$

8. (a) $\int_y^{y^2} (x-y)\,dx$

(b) $\int_{-1}^{1}\int_y^{y^2} (x-y)\,dx\,dy$

9. (a) $\int_0^y (\cos x\sin y)\,dx$

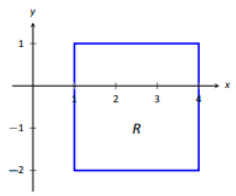
(b) $\int_0^{\pi}\int_0^y (\cos x\sin y)\,dx\,dy$

10. (a) $\int_0^x \left(\frac{1}{1+x^2}\right) dy$

(b) $\int_1^2\int_0^x \left(\frac{1}{1+x^2}\right) dy\,dx$

In Exercises 11-16, a graph of a planar region R is given. Give the iterated integrals, with both orders of integration $dy\,dx$ and $dx\,dy$, that give the area of R. Evaluate one of the iterated integrals to find the area.

11.



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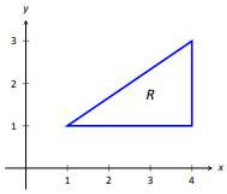

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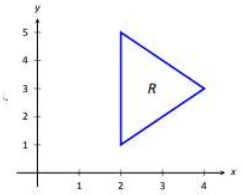

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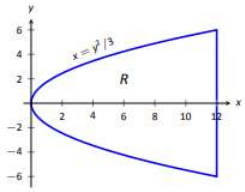

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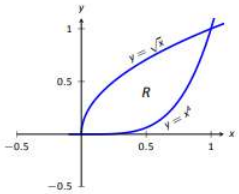

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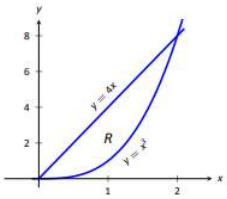

In Exercises 17-22, iterated integrals are given that compute the area of a region R in the xy-plane. Sketch the region R, and give the iterated integral(s) that give the area of R with the opposite order of integration.

17. $\int_{-2}^{2}\int_{0}^{4-x^2} dy\,dx$
18. $\int_{0}^{1}\int_{5-5x}^{5-5x^2} dy\,dx$
19. $\int_{-2}^{2}\int_{0}^{2\sqrt{4-y^2}} dx\,dy$
20. $\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dy\,dx$

21. $\int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dx\, dy + \int_1^4 \int_{y-2}^{\sqrt{y}} dx\, dy$

22. $\int_{-1}^1 \int_{(x-1)/2}^{(1-x)/2} dy\, dx$

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14.2a: Double Integrals Over Rectangular Regions

Learning Objectives

- Recognize when a function of two variables is integrable over a rectangular region.
- Recognize and use some of the properties of double integrals.
- Evaluate a double integral over a rectangular region by writing it as an iterated integral.
- Use a double integral to calculate the area of a region, volume under a surface, or average value of a function over a plane region.

In this section we investigate double integrals and show how we can use them to find the volume of a solid over a rectangular region in the xy-plane. Many of the properties of double integrals are similar to those we have already discussed for single integrals.

Volumes and Double Integrals

We begin by considering the space above a rectangular region R. Consider a continuous function $f(x,y) \geq 0$ of two variables defined on the closed rectangle R:

$$R = [a,b] \times [c,d] = \{(x,y) \in \mathbb{R}^2 | \, a \leq x \leq b, \, c \leq y \leq d\}$$

Here $[a,b] \times [c,d]$ denotes the Cartesian product of the two closed intervals $[a,b]$ and $[c,d]$. It consists of rectangular pairs (x,y) such that $a \leq x \leq b$ and $c \leq y \leq d$. The graph of f represents a surface above the xy-plane with equation $z = f(x,y)$ where z is the height of the surface at the point (x,y). Let S be the solid that lies above R and under the graph of f (Figure 14.2a.1). The base of the solid is the rectangle R in the xy-plane. We want to find the volume V of the solid S.

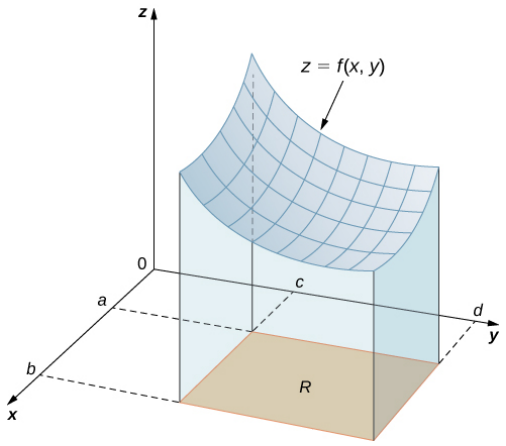

Figure 14.2a.1: The graph of $f(x,y)$ *over the rectangle R in the* xy*-plane is a curved surface.*

We divide the region R into small rectangles R_{ij}, each with area ΔA and with sides Δx and Δy (Figure 14.2a.2). We do this by dividing the interval $[a,b]$ into m subintervals and dividing the interval $[c,d]$ into n subintervals. Hence $\Delta x = \frac{b-a}{m}$, $\Delta y = \frac{d-c}{n}$, and $\Delta A = \Delta x \Delta y$.

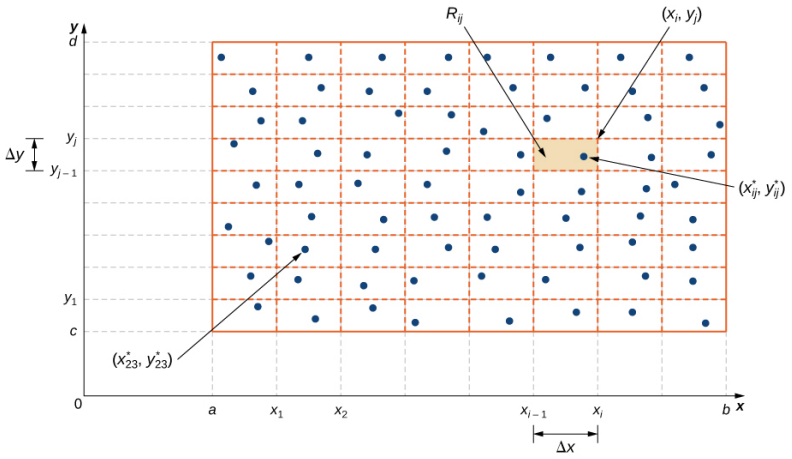

Figure **14.2a.2**: Rectangle R is divided into small rectangles R_{ij} each with area ΔA.

The volume of a thin rectangular box above R_{ij} is $f(x_{ij}^*, y_{ij}^*)\,\Delta A$, where (x_{ij}^*, y_{ij}^*) is an arbitrary sample point in each R_{ij} as shown in the following figure, $f(x_{ij}^*, y_{ij}^*)$ is the height of the corresponding thin rectangular box, and ΔA is the area of each rectangle R_{ij}.

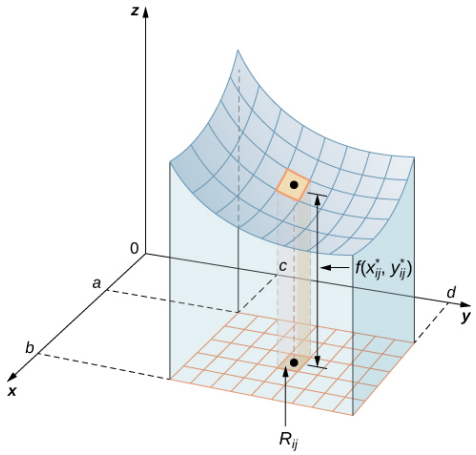

Figure **14.2a.3**: A thin rectangular box above R_{ij} with height $f(x_{ij}^*, y_{ij}^*)$.

Using the same idea for all the subrectangles, we obtain an approximate volume of the solid S as

$$V \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A.$$

This sum is known as a **double Riemann sum** and can be used to approximate the value of the volume of the solid. Here the double sum means that for each subrectangle we evaluate the function at the chosen point, multiply by the area of each rectangle, and then add all the results.

As we have seen in the single-variable case, we obtain a better approximation to the actual volume if m and n become larger.

$$V = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

or

$$V = \lim_{\Delta x,\, \Delta y\to 0} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A.$$

Note that the sum approaches a limit in either case and the limit is the volume of the solid with the base R. Now we are ready to define the double integral.

Definition: Double Integral over a Rectangular Region R

The double integral of the function $f(x, y)$ over the rectangular region R in the xy-plane is defined as

$$\iint_R f(x, y)dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A.$$

If $f(x,y) \geq 0$, then the volume V of the solid S, which lies above R in the xy-plane and under the graph of f, is the double integral of the function $f(x,y)$ over the rectangle R. If the function is ever negative, then the double integral can be considered a "signed" volume in a manner similar to the way we defined net signed area in The Definite Integral.

Example 14.2a.1: Setting up a Double Integral and Approximating It by Double Sums

Consider the function $z = f(x, y) = 3x^2 - y$ over the rectangular region $R = [0, 2] \times [0, 2]$ (Figure 14.2a.4).

a. Set up a double integral for finding the value of the signed volume of the solid S that lies above R and "under" the graph of f.

b. Divide R into four squares with $m = n = 2$, and choose the sample point as the upper right corner point of each square (1,1), (2,1),(1,2), and (2,2) (Figure 14.2a.4) to approximate the signed volume of the solid S that lies above R and "under" the graph of f.

c. Divide R into four squares with $m = n = 2$, and choose the sample point as the midpoint of each square: (1/2, 1/2), (3/2, 1/2), (1/2,3/2), and (3/2, 3/2) to approximate the signed volume.

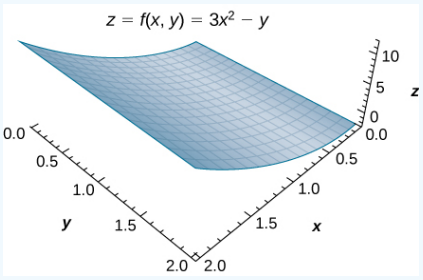

Figure 14.2a.4: The function $z = f(x, y)$ graphed over the rectangular region $R = [0, 2] \times [0, 2]$.

Solution

a. As we can see, the function $z = f(x, y) = 3x^2 - y$ is above the plane. To find the signed volume of S, we need to divide the region R into small rectangles R_{ij}, each with area ΔA and with sides Δx and Δy, and choose (x_{ij}^*, y_{ij}^*) as sample points in each R_{ij}. Hence, a double integral is set up as

$$V = \iint_R (3x^2 - y)dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} [3(x_{ij}^*)^2 - y_{ij}^*]\Delta A.$$

b. Approximating the signed volume using a Riemann sum with $m = n = 2$ we have $\Delta A = \Delta x \Delta y = 1 \times 1 = 1$. Also, the sample points are (1, 1), (2, 1), (1, 2), and (2, 2) as shown in the following figure.

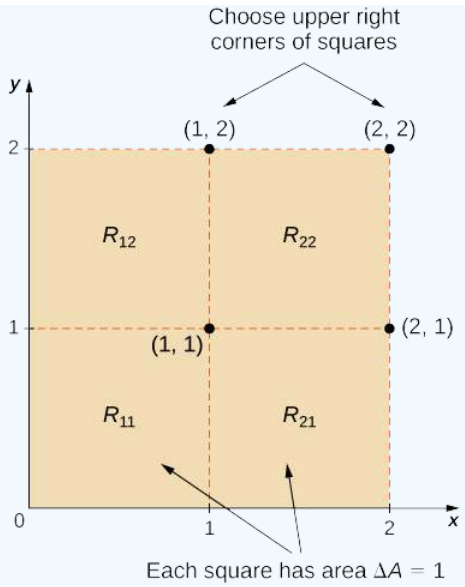

Figure 14.2a.5: Subrectangles for the rectangular region $R = [0,2] \times [0,2]$.

Hence,

$$\begin{aligned} V &\approx \sum_{i=1}^{2}\sum_{j=1}^{2} f(x_{ij}^*, y_{ij}^*)\Delta A \\ &= \sum_{i=1}^{2}(f(x_{i1}^*, y_{i1}^*) + f(x_{i2}^*, y_{i2}^*))\Delta A \\ &= f(x_{11}^*, y_{11}^*)\Delta A + f(x_{21}^*, y_{21}^*)\Delta A + f(x_{12}^*, y_{12}^*)\Delta A + f(x_{22}^*, y_{22}^*)\Delta A \\ &= f(1,1)(1) + f(2,1)(1) + f(1,2)(1) + f(2,2)(1) \\ &= (3-1)(1) + (12-1)(1) + (3-2)(1) + (12-2)(1) \\ &= 2 + 11 + 1 + 10 = 24. \end{aligned}$$

c. Approximating the signed volume using a Riemann sum with $m = n = 2$ we have$\Delta A = \Delta x \Delta y = 1 \times 1 = 1$. In this case the sample points are (1/2, 1/2), (3/2, 1/2), (1/2, 3/2), and (3/2, 3/2).
Hence,

$$\begin{aligned} V &\approx \sum_{i=1}^{2}\sum_{j=1}^{2} f(x_{ij}^*, y_{ij}^*)\Delta A \\ &= f(x_{11}^*, y_{11}^*)\Delta A + f(x_{21}^*, y_{21}^*)\Delta A + f(x_{12}^*, y_{12}^*)\Delta A + f(x_{22}^*, y_{22}^*)\Delta A \\ &= f(1/2,1/2)(1) + f(3/2,1/2)(1) + f(1/2,3/2)(1) + f(3/2,3/2)(1) \\ &= \left(\frac{3}{4} - \frac{1}{4}\right)(1) + \left(\frac{27}{4} - \frac{1}{2}\right)(1) + \left(\frac{3}{4} - \frac{3}{2}\right)(1) + \left(\frac{27}{4} - \frac{3}{2}\right)(1) \\ &= \frac{2}{4} + \frac{25}{4} + \left(-\frac{3}{4}\right) + \frac{21}{4} = \frac{45}{4} = 11. \end{aligned}$$

Analysis

Notice that the approximate answers differ due to the choices of the sample points. In either case, we are introducing some error because we are using only a few sample points. Thus, we need to investigate how we can achieve an accurate answer.

? Exercise 14.2a.1

Use the same function$z = f(x,y) = 3x^2 - y$ over the rectangular region $R = [0,2] \times [0,2]$.

Divide R into the same four squares with $m = n = 2$, and choose the sample points as the upper left corner point of each square (0,1), (1,1), (0,2), and (1,2) (Figure **14.2a.5**) to approximate the signed volume of the solid S that lies above R and "under" the graph of f.

Note that we developed the concept of double integral using a rectangular region R. This concept can be extended to any general region. However, when a region is not rectangular, the subrectangles may not all fit perfectly into R, particularly if the base area is curved. We examine this situation in more detail in the next section, where we study regions that are not always rectangular and subrectangles may not fit perfectly in the region R. Also, the heights may not be exact if the surface $z = f(x, y)$ is curved. However, the errors on the sides and the height where the pieces may not fit perfectly within the solid S approach 0 as m and n approach infinity. Also, the double integral of the function $z = f(x, y)$ exists provided that the function f is not too discontinuous. If the function is bounded and continuous over R except on a finite number of smooth curves, then the double integral exists and we say that ff is integrable over R.

Since $\Delta A = \Delta x \Delta y = \Delta y \Delta x$, we can express dA as $dx\,dy$ or $dy\,dx$. This means that, when we are using rectangular coordinates, the double integral over a region R denoted by

$$\iint_R f(x, y)\, dA$$

can be written as

$$\iint_R f(x, y)\, dx\, dy$$

or

$$\iint_R f(x, y)\, dy\, dx.$$

Now let's list some of the properties that can be helpful to compute double integrals.

Properties of Double Integrals

The properties of double integrals are very helpful when computing them or otherwise working with them. We list here six properties of double integrals. Properties 1 and 2 are referred to as the linearity of the integral, property 3 is the additivity of the integral, property 4 is the monotonicity of the integral, and property 5 is used to find the bounds of the integral. Property 6 is used if $f(x, y)$ is a product of two functions $g(x)$ and $h(y)$.

Theorem: Properties of Double Integrals

Assume that the functions $f(x, y)$ and $g(x, y)$ are integrable over the rectangular region R; S and T are subregions of R; and assume that m and M are real numbers.

i. The sum $f(x, y) + g(x, y)$ is integrable and

$$\iint_R [f(x, y) + g(x, y)]\, dA = \iint_R f(x, y)\, dA + \iint_R g(x, y)\, dA.$$

ii. If c is a constant, then $cf(x, y)$ is integrable and

$$\iint_R cf(x, y)\, dA = c \iint_R f(x, y)\, dA.$$

iii. If $R = S \cup T$ and $S \cap T = \emptyset$ except an overlap on the boundaries, then

$$\iint_R f(x, y)\, dA = \iint_S f(x, y)\, dA + \iint_T f(x, y)\, dA.$$

iv. If $f(x, y) \geq g(x, y)$ for (x, y) in R, then

$$\iint_R f(x, y)\, dA \geq \iint_R g(x, y)\, dA.$$

v. If $m \le f(x,y) \le M$ and $A(R) = \text{the area of } R$, then

$$m \cdot A(R) \le \iint_R f(x,y)\,dA \le M \cdot A(R).$$

vi. In the case where $f(x,y)$ can be factored as a product of a function $g(x)$ of x only and a function $h(y)$ of y only, then over the region $R = \{(x,y) \mid a \le x \le b,\ c \le y \le d\}$, the double integral can be written as

$$\iint_R f(x,y)\,dA = \left(\int_a^b g(x)\,dx\right)\left(\int_c^d h(y)\,dy\right).$$

These properties are used in the evaluation of double integrals, as we will see later. We will become skilled in using these properties once we become familiar with the computational tools of double integrals. So let's get to that now.

Iterated Integrals

So far, we have seen how to set up a double integral and how to obtain an approximate value for it. We can also imagine that evaluating double integrals by using the definition can be a very lengthy process if we choose larger values for m and n.Therefore, we need a practical and convenient technique for computing double integrals. In other words, we need to learn how to compute double integrals without employing the definition that uses limits and double sums.

The basic idea is that the evaluation becomes easier if we can break a double integral into single integrals by integrating first with respect to one variable and then with respect to the other. The key tool we need is called an iterated integral.

Definitions: Iterated Integrals

Assume a, b, c, and d are real numbers. We define an *iterated integral* for a function $f(x,y)$ over the rectangular region $R = [a,b] \times [c,d]$ as

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx$$

or

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy.$$

The notation $\int_a^b \left[\int_c^d f(x,y)\,dy\right] dx$ means that we integrate $f(x,y)$ with respect to y while holding x constant. Similarly, the notation $\int_c^d \left[\int_a^b f(x,y)\,dx\right] dy$ means that we integrate $f(x,y)$ with respect to x while holding y constant. The fact that double integrals can be split into iterated integrals is expressed in Fubini's theorem. Think of this theorem as an essential tool for evaluating double integrals.

Theorem: Fubini's Theorem

Suppose that $f(x,y)$ is a function of two variables that is continuous over a rectangular region $R = \{(x,y) \in \mathbb{R}^2 \mid a \le x \le b,\ c \le y \le d\}$. Then we see from Figure **14.2a.6** that the double integral of f over the region equals an iterated integral,

$$\iint_R f(x,y)\,dA = \iint_R f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy.$$

More generally, Fubini's theorem is true if f is bounded on R and f is discontinuous only on a finite number of continuous curves. In other words, f has to be integrable over R.

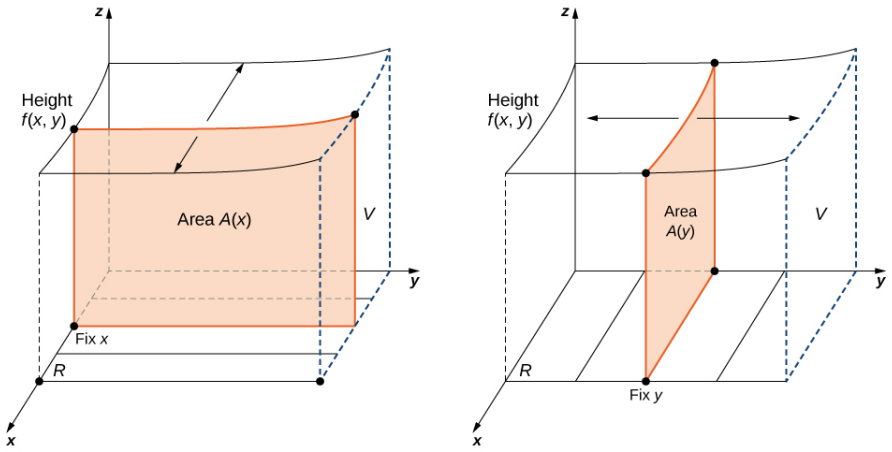

Figure **14.2a.6**: (a) Integrating first with respect to y and then with respect to x to find the area $A(x)$ and then the volume V; (b) integrating first with respect to x and then with respect to y to find the area $A(y)$ and then the volume V.

✔ Example 14.2a.2: Using Fubini's Theorem

Use Fubini's theorem to compute the double integral $\iint_R f(x,y)\,dA$ where $f(x,y) = x$ and $R = [0,2] \times [0,1]$.

Solution

Fubini's theorem offers an easier way to evaluate the double integral by the use of an iterated integral. Note how the boundary values of the region R become the upper and lower limits of integration.

$$\begin{aligned}\iint_R f(x,y)\,dA &= \iint_R f(x,y)\,dx\,dy \\ &= \int_{y=0}^{y=1}\int_{x=0}^{x=2} x\,dx\,dy \\ &= \int_{y=0}^{y=1}\left[\frac{x^2}{2}\Big|_{x=0}^{x=2}\right] dy \\ &= \int_{y=0}^{y=1} 2\,dy = 2y\Big|_{y=0}^{y=1} = 2\end{aligned}$$

The double integration in this example is simple enough to use Fubini's theorem directly, allowing us to convert a double integral into an iterated integral. Consequently, we are now ready to convert all double integrals to iterated integrals and demonstrate how the properties listed earlier can help us evaluate double integrals when the function $f(x,y)$ is more complex. Note that the order of integration can be changed (see Example 7).

✔ Example 14.2a.3: Illustrating Properties i and ii

Evaluate the double integral

$$\iint_R (xy - 3xy^2)\,dA, \text{ where } R = \{(x,y) \mid 0 \le x \le 2,\ 1 \le y \le 2\}.$$

Solution

This function has two pieces: one piece is xy and the other is $3xy^2$. Also, the second piece has a constant 3. Notice how we use properties i and ii to help evaluate the double integral.

$$\iint_{(xy}$$

$$\begin{aligned}
\iint_R (xy - 3xy^2)\,dA &= \iint_R xy\,dA + \iint_R (-3xy^2)\,dA && \text{Property i: Integral of a sum is the sum of the integrals.}\\
&= \int_{y=1}^{y=2}\int_{x=0}^{x=2} xy\,dx\,dy && \text{Convert double integrals to iterated integrals.}\\
&\quad - \int_{y=1}^{y=2}\int_{x=0}^{x=2} 3xy^2\,dx\,dy\\
&= \int_{y=1}^{y=2} \left(\frac{x^2}{2}y\right)\Big|_{x=0}^{x=2} dy && \text{Integrate with respect to } x, \text{ holding } y \text{ constant.}\\
&\quad - 3\int_{y=1}^{y=2} \left(\frac{x^2}{2}y^2\right)\Big|_{x=0}^{x=2} dy\\
&= \int_{y=1}^{y=2} 2y\,dy - \int_{y=1}^{y=2} 6y^2 dy && \text{Property ii: Placing the constant before the integral.}\\
&= 2\int_1^2 y\,dy - 6\int_1^2 y^2\,dy && \text{Integrate with respect to y.}\\
&= 2\frac{y^2}{2}\Big|_1^2 - 6\frac{y^3}{3}\Big|_1^2\\
&= y^2\Big|_1^2 - 2y^3\Big|_1^2\\
&= (4-1) - 2(8-1) = 3 - 2(7)\\
&= 3 - 14 = -11.
\end{aligned}$$

Example 14.2a.4: Illustrating Property v.

Over the region $R = \{(x,y)\,|\,1 \le x \le 3,\ 1 \le y \le 2\}$, we have $2 \le x^2 + y^2 \le 13$. Find a lower and an upper bound for the integral $\iint_R (x^2 + y^2)\,dA$.

Solution

For a lower bound, integrate the constant function 2 over the region R. For an upper bound, integrate the constant function 13 over the region R.

$$\int_1^2\int_1^3 2\,dx\,dy = \int_1^2 [2x\Big|_1^3]\,dy = \int_1^2 2(2)dy = 4y\Big|_1^2 = 4(2-1) = 4$$
$$\int_1^2\int_1^3 13dx\,dy = \int_1^2 [13x\Big|_1^3]\,dy = \int_1^2 13(2)\,dy = 26y\Big|_1^2 = 26(2-1) = 26.$$

Hence, we obtain $4 \le \iint_R (x^2 + y^2)\,dA \le 26$.

Example 14.2a.5: Illustrating Property vi

Evaluate the integral $\iint_R e^y \cos x\,dA$ over the region $R = \{(x,y)\,|\,0 \le x \le \frac{\pi}{2},\ 0 \le y \le 1\}$.

Solution

This is a great example for property vi because the function $f(x,y)$ is clearly the product of two single-variable functions e^y and $\cos x$. Thus we can split the integral into two parts and then integrate each one as a single-variable integration problem.

$$\begin{aligned}\iint_R e^y \cos x \, dA &= \int_0^1 \int_0^{\pi/2} e^y \cos x \, dx \, dy \\ &= \left(\int_0^1 e^y dy\right)\left(\int_0^{\pi/2} \cos x \, dx\right) \\ &= (e^y\Big|_0^1)(\sin x\Big|_0^{\pi/2}) \\ &= e - 1.\end{aligned}$$

? Exercise 14.2a.2

a. Use the properties of the double integral and Fubini's theorem to evaluate the integral

$$\int_0^1 \int_{-1}^3 (3 - x + 4y) \, dy \, dx.$$

b. Show that $0 \le \iint_R \sin \pi x \, \cos \pi y \, dA \le \frac{1}{32}$ where $R = \left(0, \frac{1}{4}\right)\left(\frac{1}{4}, \frac{1}{2}\right)$.

As we mentioned before, when we are using rectangular coordinates, the double integral over a region R denoted by $\iint_R f(x,y) \, dA$ can be written as $\iint_R f(x,y) \, dx \, dy$ or $\iint_R f(x,y) \, dy \, dx$. The next example shows that the results are the same regardless of which order of integration we choose.

✓ Example 14.2a.6: Evaluating an Iterated Integral in Two Ways

Let's return to the function $f(x,y) = 3x^2 - y$ from Example 1, this time over the rectangular region $R = [0,2] \times [0,3]$. Use Fubini's theorem to evaluate $\iint_R f(x,y) \, dA$ in two different ways:

a. First integrate with respect to y and then with respect to x;
b. First integrate with respect to x and then with respect to y.

Solution

Figure 14.2a.6 shows how the calculation works in two different ways.

a. First integrate with respect to y and then integrate with respect to x:

$$\begin{aligned}\iint_R f(x,y) \, dA &= \int_{x=0}^{x=2} \int_{y=0}^{y=3} (3x^2 - y) \, dy \, dx \\ &= \int_{x=0}^{x=2} \left(\int_{y=0}^{y=3} (3x^2 - y) \, dy\right) dx = \int_{x=0}^{x=2} \left[3x^2 y - \frac{y^2}{2}\Big|_{y=0}^{y=3}\right] dx \\ &= \int_{x=0}^{x=2} \left(9x^2 - \frac{9}{2}\right) dx = 3x^3 - \frac{9}{2}x\Big|_{x=0}^{x=2} = 15.\end{aligned}$$

b. First integrate with respect to x and then integrate with respect to y:

$$\begin{aligned}\iint_R f(x,y) \, dA &= \int_{y=0}^{y=3} \int_{x=0}^{x=2} (3x^2 - y) \, dx \, dy \\ &= \int_{y=0}^{y=3} \left(\int_{x=0}^{x=2} (3x^2 - y) \, dx\right) dy \\ &= \int_{y=0}^{y=3} \left[x^3 - xy\Big|_{x=0}^{x=2}\right] dy \\ &= \int_{y=0}^{y=3} (8 - 2y) \, dy = 8y - y^2\Big|_{y=0}^{y=3} = 15.\end{aligned}$$

Analysis

With either order of integration, the double integral gives us an answer of 15. We might wish to interpret this answer as a volume in cubic units of the solid S below the function $f(x,y) = 3x^2 - y$ over the region $R = [0,2] \times [0,3]$. However, remember

that the interpretation of a double integral as a (non-signed) volume works only when the integrand f is a nonnegative function over the base region R.

? Exercise 14.2a.3

Evaluate

$$\int_{y=-3}^{y=2}\int_{x=3}^{x=5}(2-3x^2+y^2)\,dx\,dy.$$

In the next example we see that it can actually be beneficial to switch the order of integration to make the computation easier. We will come back to this idea several times in this chapter.

✓ Example 14.2a.7: Switching the Order of Integration

Consider the double integral $\iint_R x\,\sin(xy)\,dA$ over the region $R=\{(x,y)\,|\,0\le x\le\pi,\ 1\le y\le 2\}$ (Figure 14.2a.7).

a. Express the double integral in two different ways.
b. Analyze whether evaluating the double integral in one way is easier than the other and why.
c. Evaluate the integral.

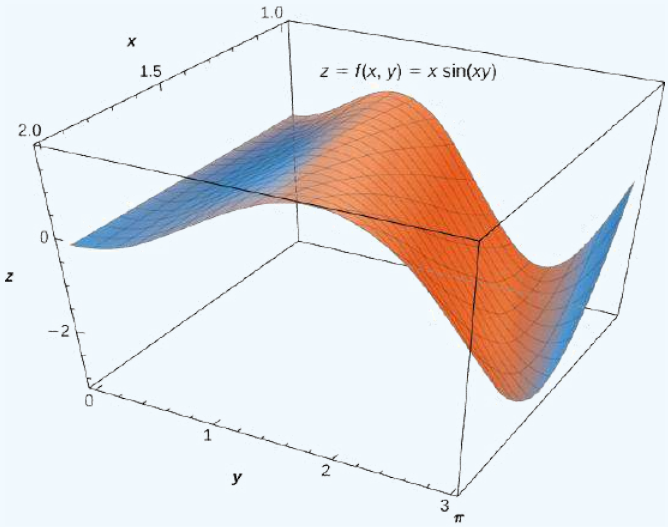

Figure 14.2a.7: The function $z=f(x,y)=x\,\sin(xy)$ over the rectangular region $R=[0,\pi]\times[1,2]$.

a. We can express $\iint_R x\,\sin(xy)\,dA$ in the following two ways: first by integrating with respect to y and then with respect to x; second by integrating with respect to x and then with respect to y.

$$\iint_R x\,\sin(xy)\,dA=\int_{x=0}^{x=\pi}\int_{y=1}^{y=2}x\,\sin(xy)\,dy\,dx$$

Integrate first with respect to y.

$$=\int_{y=1}^{y=2}\int_{x=0}^{x=\pi}x\,\sin(xy)\,dx\,dy$$

Integrate first with respect to x.

b. If we want to integrate with respect to y first and then integrate with respect to x, we see that we can use the substitution $u=xy$, which gives $du=x\,dy$. Hence the inner integral is simply $\int\sin u\,du$ and we can change the limits to be functions of x,

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$$\iint_R x\,\sin(xy)\,dA = \int_{x=0}^{x=\pi}\int_{y=1}^{y=2} x\,\sin(xy)\,dy\,dx = \int_{x=0}^{x=\pi}\left[\int_{u=x}^{u=2x}\sin(u)\,du\right]dx.$$

However, integrating with respect to x first and then integrating with respect to y requires integration by parts for the inner integral, with $u = x$ and $dv = \sin(xy)dx$

Then $du = dx$ and $v = -\frac{\cos(xy)}{y}$, so

$$\iint_R x\sin(xy)\,dA = \int_{y=1}^{y=2}\int_{x=0}^{x=\pi} x\sin(xy)\,dx\,dy$$
$$= \int_{y=1}^{y=2}\left[-\frac{x\,\cos(xy)}{y}\Big|_{x=0}^{x=\pi} + \frac{1}{y}\int_{x=0}^{x=\pi}\cos(xy)\,dx\right]dy.$$

Since the evaluation is getting complicated, we will only do the computation that is easier to do, which is clearly the first method.

c. Evaluate the double integral using the easier way.

$$\iint_R x\,\sin(xy)\,dA = \int_{x=0}^{x=\pi}\int_{y=1}^{y=2} x\,\sin(xy)\,dy\,dx$$
$$= \int_{x=0}^{x=\pi}\left[\int_{u=x}^{u=2x}\sin(u)\,du\right]dx = \int_{x=0}^{x=\pi}\left[-\cos u\Big|_{u=x}^{u=2x}\right]dx$$
$$= \int_{x=0}^{x=\pi}(-\cos 2x + \cos x)\,dx$$
$$= \left(-\frac{1}{2}\sin 2x + \sin x\right)\Big|_{x=0}^{x=\pi} = 0.$$

? Exercise 14.2a.4

Evaluate the integral $\iint_R xe^{xy}\,dA$ where $R = [0,1]\times[0,\ln 5]$.

Applications of Double Integrals

Double integrals are very useful for finding the area of a region bounded by curves of functions. We describe this situation in more detail in the next section. However, if the region is a rectangular shape, we can find its area by integrating the constant function $f(x,y) = 1$ over the region R.

Definition: Area of a Region R

The area of the region R is given by

$$A(R) = \iint_R 1\,dA.$$

This definition makes sense because using $f(x,y) = 1$ and evaluating the integral make it a product of length and width. Let's check this formula with an example and see how this works.

Example 14.2a.8: Finding Area Using a Double Integral

Find the area of the region $R = \{(x,y)\,|\,0 \le x \le 3,\ 0 \le y \le 2\}$ by using a double integral, that is, by integrating 1 over the region R.

Solution

The region is rectangular with length 3 and width 2, so we know that the area is 6. We get the same answer when we use a double integral:

$$A(R) = \int_0^2 \int_0^3 1\,dx\,dy = \int_0^2 \left[x\big|_0^3\right] dy = \int_0^2 3dy = 3\int_0^2 dy = 3y\Big|_0^2 = 3(2) = 6 \text{ units}^2.$$

We have already seen how double integrals can be used to find the volume of a solid bounded above by a function $f(x,y) \geq 0$ over a region R provided $f(x,y) \geq 0$ for all (x,y) in R. Here is another example to illustrate this concept.

Example 14.2a.9: Volume of an Elliptic Paraboloid

Find the volume V of the solid S that is bounded by the elliptic paraboloid $2x^2 + y^2 + z = 27$, the planes $x = 3$ and $y = 3$, and the three coordinate planes.

Solution

First notice the graph of the surface $z = 27 - 2x^2 - y^2$ in Figure 14.2a.8(a) and above the square region $R_1 = [-3,3] \times [-3,3]$. However, we need the volume of the solid bounded by the elliptic paraboloid $2x^2 + y^2 + z = 27$, the planes $x = 3$ and $y = 3$, and the three coordinate planes.

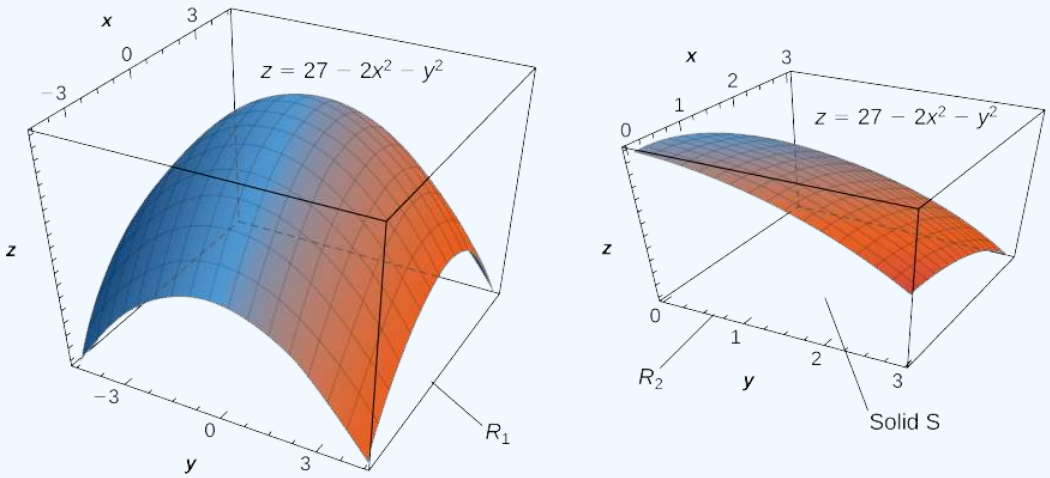


Figure 14.2a.8: (a) The surface $z = 27 - 2x^2 - y^2$ above the square region $R1 = [-3,3] \times [-3,3]$. (b) The solid S lies under the surface $z = 27 - 2x^2 - y^2$ above the square region $R1 = [0,3] \times [0,3]$.

Now let's look at the graph of the surface in Figure 14.2a.8(b). We determine the volume V by evaluating the double integral over R_2:

$$\begin{aligned} V &= \iint_R z\,dA = \iint_R (27 - 2x^2 - y^2)\,dA \\ &= \int_{y=0}^{y=3} \int_{x=0}^{x=3} (27 - 2x^2 - y^2)\,dx\,dy && \text{Convert to literal integral.} \\ &= \int_{y=0}^{y=3} \left[27x - \frac{2}{3}x^3 - y^2x\right]\Big|_{x=0}^{x=3} dy && \text{Integrate with respect to } x. \\ &= \int_{y=0}^{y=3} (63 - 3y^2)dy = 63y - y^3\Big|_{y=0}^{y=3} = 162. \end{aligned}$$

Exercise 14.2a.5

Find the volume of the solid bounded above by the graph of $f(x,y) = xy\sin(x^2y)$ and below by the xy-plane on the rectangular region $R = [0,1] \times [0,\pi]$.

Recall that we defined the average value of a function of one variable on an interval $[a,b]$ as

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Similarly, we can define the average value of a function of two variables over a region R. The main difference is that we divide by an area instead of the width of an interval.

Definition: Average Value of a Function

The **average value of a function** of two variables over a region R is

$$F_{ave} = \frac{1}{\text{Area of } R} \iint_R f(x,y)\,dx\,dy.$$

In the next example we find the average value of a function over a rectangular region. This is a good example of obtaining useful information for an integration by making individual measurements over a grid, instead of trying to find an algebraic expression for a function.

Example 14.2a.10: Calculating Average Storm Rainfall

The weather map in Figure 14.2a.9 shows an unusually moist storm system associated with the remnants of Hurricane Karl, which dumped 4–8 inches (100–200 mm) of rain in some parts of the Midwest on September 22–23, 2010. The area of rainfall measured 300 miles east to west and 250 miles north to south. Estimate the average rainfall over the entire area in those two days.

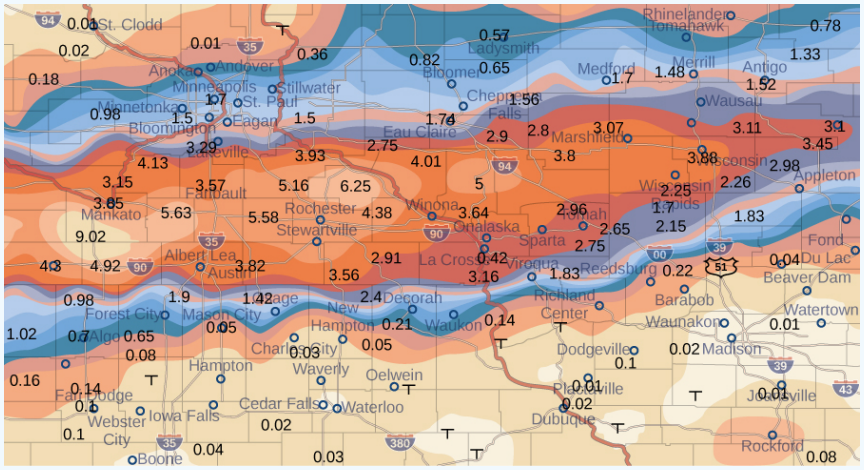


Figure 14.2a.9: Effects of Hurricane Karl, which dumped 4–8 inches (100–200 mm) of rain in some parts of southwest Wisconsin, southern Minnesota, and southeast South Dakota over a span of 300 miles east to west and 250 miles north to south.

Solution

Place the origin at the southwest corner of the map so that all the values can be considered as being in the first quadrant and hence all are positive. Now divide the entire map into six rectangles ($m = 2$ and $n = 3$), as shown in Figure 14.2a.9. Assume $f(x,y)$ denotes the storm rainfall in inches at a point approximately x miles to the east of the origin and y miles to the north of the origin. Let R represent the entire area of $250 \times 300 = 75000$ square miles. Then the area of each subrectangle is

$$\Delta A = \frac{1}{6}(75000) = 12500.$$

Assume $(x_{ij}*, y_{ij}*)$ are approximately the midpoints of each subrectangle R_{ij}. Note the color-coded region at each of these points, and estimate the rainfall. The rainfall at each of these points can be estimated as:

- At (x_{11}, y_{11}), the rainfall is 0.08.
- At (x_{12}, y_{12}), the rainfall is 0.08.
- At (x_{13}, y_{13}), the rainfall is 0.01.
- At (x_{21}, y_{21}), the rainfall is 1.70.
- At (x_{22}, y_{22}), the rainfall is 1.74.

- At (x_{23}, y_{23}), the rainfall is 3.00.

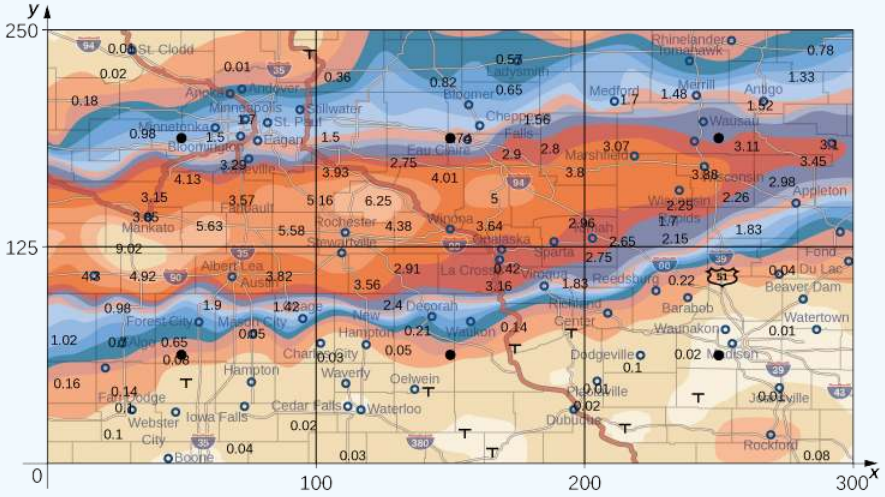


Figure 14.2a.10: Storm rainfall with rectangular axes and showing the midpoints of each subrectangle.

According to our definition, the average storm rainfall in the entire area during those two days was

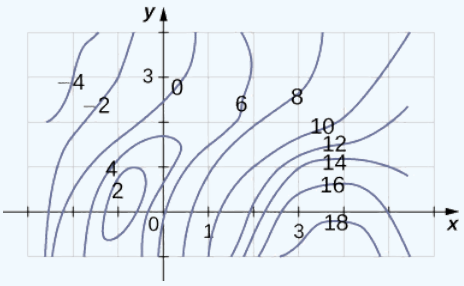
$$f_{ave} = \frac{1}{Area\ R}$$

$$\begin{aligned}
\iint_R f(x,y)\,dx\,dy &= \frac{1}{75000}\iint_R f(x,y)\,dx\,dy \\
&\approx \frac{1}{75000}\sum_{i=1}^{3}\sum_{j=1}^{2} f(x_{ij}^*, y_{ij}^*)\Delta A \\
&= \frac{1}{75000}\Big[f(x_{11}^*, y_{11}^*)\Delta A + f(x_{12}^*, y_{12}^*)\Delta A + f(x_{13}^*, y_{13}^*)\Delta A \\
&\quad + f(x_{21}^*, y_{21}^*)\Delta A + f(x_{22}^*, y_{22}^*)\Delta A + f(x_{23}^*, y_{23}^*)\Delta A\Big] \\
&\approx \frac{1}{75000}\Big[0.08 + 0.08 + 0.01 + 1.70 + 1.74 + 3.00\Big]\Delta A \\
&= \frac{1}{75000}\Big[0.08 + 0.08 + 0.01 + 1.70 + 1.74 + 3.00\Big]12500 \\
&= \frac{1}{6}\Big[0.08 + 0.08 + 0.01 + 1.70 + 1.74 + 3.00\Big] \\
&\approx 1.10 \text{ in.}
\end{aligned}$$

During September 22–23, 2010 this area had an average storm rainfall of approximately 1.10 inches.

? Exercise 14.2a.6

A contour map is shown for a function $f(x,y)$ on the rectangle $R = [-3,6] \times [-1,4]$.



a. Use the midpoint rule with $m = 3$ and $n = 2$ to estimate the value of $\iint_R f(x,y)\,dA$.

b. Estimate the average value of the function $f(x,y)$.

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Key Concepts

- We can use a double Riemann sum to approximate the volume of a solid bounded above by a function of two variables over a rectangular region. By taking the limit, this becomes a double integral representing the volume of the solid.
- Properties of double integral are useful to simplify computation and find bounds on their values.
- We can use Fubini's theorem to write and evaluate a double integral as an iterated integral.
- Double integrals are used to calculate the area of a region, the volume under a surface, and the average value of a function of two variables over a rectangular region.

Key Equations

- $$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i j*, y_i j*)\,\Delta A$$
- $$\int_a^b \int_c^d f(x,y)\,dx\,dy = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx$$

  or

  $$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy$$
- $$f_{ave} = \frac{1}{\text{Area of } R} \iint_R f(x,y)\,dx\,dy$$

Glossary

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14.2aE: Double Integrals Part 1 (Exercises)

In exercises 1 and 2, use the midpoint rule with $m = 4$ and $n = 2$ to estimate the volume of the solid bounded by the surface $z = f(x, y)$, the vertical planes $x = 1$, $x = 2$, $y = 1$, and $y = 2$, and the horizontal plane $z = 0$.

1) $f(x, y) = 4x + 2y + 8xy$

2) $f(x, y) = 16x^2 + \frac{y}{2}$

In exercises 3 and 4, estimate the volume of the solid under the surface $z = f(x, y)$ and above the rectangular region R by using a Riemann sum with $m = n = 2$ and the sample points to be the lower left corners of the sub-rectangles of the partition.

3) $f(x, y) = \sin x - \cos y$, $R = [0, \pi] \times [0, \pi]$

4) $f(x, y) = \cos x + \cos y$, $R = [0, \pi] \times [0, \frac{\pi}{2}]$

5) Use the midpoint rule with $m = n = 2$ to estimate $\iint_R f(x, y)\, dA$, where the values of the function f on $R = [8, 10] \times [9, 11]$ are given in the following table.

	y				
x	9	9.5	10	10.5	11
8	9.8	5	6.7	5	5.6
8.5	9.4	4.5	8	5.4	3.4
9	8.7	4.6	6	5.5	3.4
9.5	6.7	6	4.5	5.4	6.7
10	6.8	6.4	5.5	5.7	6.8

6) The values of the function f on the rectangle $R = [0, 2] \times [7, 9]$ are given in the following table. Estimate the double integral $\iint_R f(x, y)\, dA$ by using a Riemann sum with $m = n = 2$. Select the sample points to be the upper right corners of the sub-squares of R.

	$y_0 = 7$	$y_1 = 8$	$y_2 = 9$
$x_0 = 0$	10.22	10.21	9.85
$x_1 = 1$	6.73	9.75	9.63
$x_2 = 2$	5.62	7.83	8.21

7) The depth of a children's 4-ft by 4-ft swimming pool, measured at 1-ft intervals, is given in the following table.

a. Estimate the volume of water in the swimming pool by using a Riemann sum with $m = n = 2$. Select the sample points using the midpoint rule on $R = [0, 4] \times [0, 4]$.

b. Find the average depth of the swimming pool.

	y				
x	0	1	2	3	4
0	1	1.5	2	2.5	3
1	1	1.5	2	2.5	3
2	1	1.5	1.5	2.5	3
3	1	1	1.5	2	2.5
4	1	1	1	1.5	2

8) The depth of a 3-ft by 3-ft hole in the ground, measured at 1-ft intervals, is given in the following table.

a. Estimate the volume of the hole by using a Riemann sum with $m = n = 3$ and the sample points to be the upper left corners of the sub-squares of R.

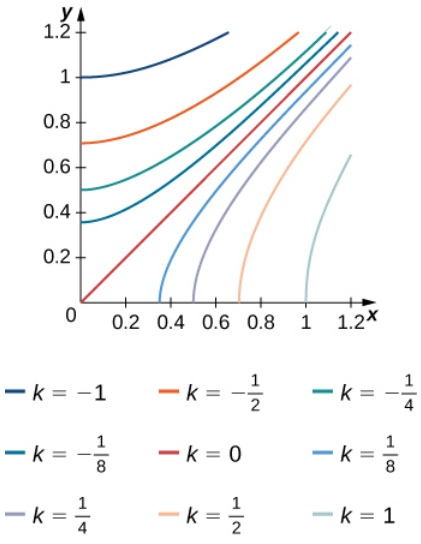
b. Find the average depth of the hole.

	y			
x	0	1	2	3
0	6	6.5	6.4	6
1	6.5	7	7.5	6.5
2	6.5	6.7	6.5	6
3	6	6.5	5	5.6

9) The level curves $f(x, y) = k$ of the function f are given in the following graph, where k is a constant.

a. Apply the midpoint rule with $m = n = 2$ to estimate the double integral $\iint_R f(x, y)\, dA$, where $R = [0.2, 1] \times [0, 0.8]$.

b. Estimate the average value of the function f on R.



10) The level curves $f(x, y) = k$ of the function f are given in the following graph, where k is a constant.

a. Apply the midpoint rule with $m = n = 2$ to estimate the double integral $\iint_R f(x, y)\, dA$, where $R = [0.1, 0.5] \times [0.1, 0.5]$.

b. Estimate the average value of the function f on R.

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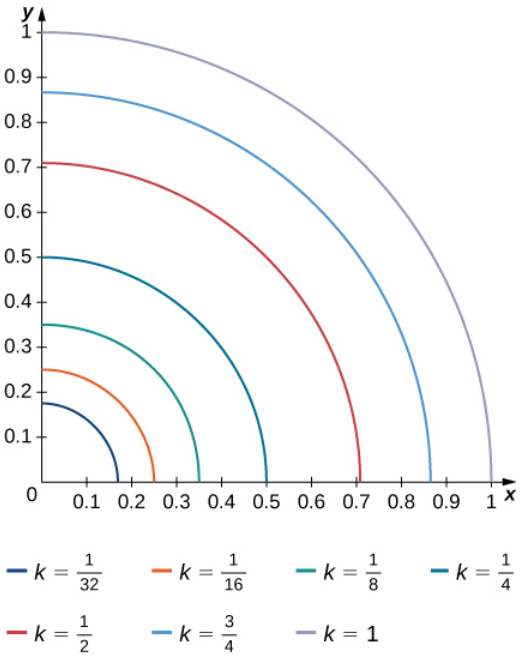
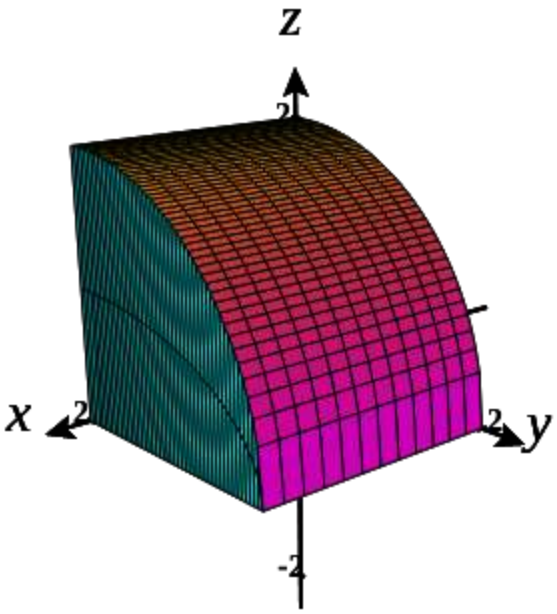


11) The solid lying under the surface $z = \sqrt{4 - y^2}$ and above the rectangular region $R = [0, 2] \times [0, 2]$ is illustrated in the following graph. Evaluate the double integral $\iint_R f(x, y)$, where $f(x, y) = \sqrt{4 - y^2}$ by finding the volume of the corresponding solid.



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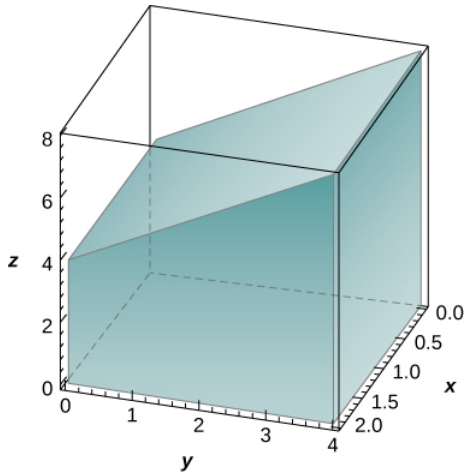
12) The solid lying under the plane $z = y + 4$ and above the rectangular region $R = [0, 2] \times [0, 4]$ is illustrated in the following graph. Evaluate the double integral $\iint_R f(x, y)\, dA$, where $f(x, y) = y + 4$, by finding the volume of the corresponding solid.



In the exercises 13 - 20, calculate the integrals by reversing the order of integration.

13) $\int_{-1}^{1} \left(\int_{-2}^{2} (2x + 3y + 5)\, dx \right) dy$

14) $\int_{0}^{2} \left(\int_{0}^{1} (x + 2e^{y} + 3)\, dx \right) dy$

15) $\int_{1}^{27} \left(\int_{1}^{2} (\sqrt[3]{x} + \sqrt[3]{y})\, dy \right) dx$

16) $\int_{1}^{16} \left(\int_{1}^{8} (\sqrt[4]{x} + 2\sqrt[3]{y})\, dy \right) dx$

17) $\int_{\ln 2}^{\ln 3} \left(\int_{0}^{1} e^{x+y}\, dy \right) dx$

18) $\int_{0}^{2} \left(\int_{0}^{1} 3^{x+y}\, dy \right) dx$

19) $\int_{1}^{6} \left(\int_{2}^{9} \frac{\sqrt{y}}{x^2}\, dy \right) dx$

20) $\int_{1}^{9} \left(\int_{4}^{2} \frac{\sqrt{x}}{y^2}\, dy \right) dx$

In exercises 21 - 34, evaluate the iterated integrals by choosing the order of integration.

21) $\int_{0}^{\pi} \int_{0}^{\pi/2} \sin(2x) \cos(3y)\, dx\, dy$

22) $\int_{\pi/12}^{\pi/8} \int_{\pi/4}^{\pi/3} [\cot x + \tan(2y)]\, dx\, dy$

23) $\int_{1}^{e} \int_{1}^{e} \left[\frac{1}{x} \sin(\ln x) + \frac{1}{y} \cos(\ln y) \right] dx\, dy$

24) $\int_{1}^{e} \int_{1}^{e} \frac{\sin(\ln x) \cos(\ln y)}{xy}\, dx\, dy$

25) $\int_{1}^{2} \int_{1}^{2} \left(\frac{\ln y}{x} + \frac{x}{2y + 1} \right) dy\, dx$

26) $\int_{1}^{e} \int_{1}^{2} x^2 \ln(x)\, dy\, dx$

27) $\int_{1}^{\sqrt{3}} \int_{1}^{2} y \arctan\left(\frac{1}{x} \right) dy\, dx$

28) $\int_{0}^{1} \int_{0}^{1/2} (\arcsin x + \arcsin y)\, dy\, dx$

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29) $\int_0^1 \int_1^2 xe^{x+4y}\,dy\,dx$

30) $\int_1^2 \int_0^1 xe^{x-y}\,dy\,dx$

31) $\int_1^e \int_1^e \left(\frac{\ln y}{\sqrt{y}} + \frac{\ln x}{\sqrt{x}}\right) dy\,dx$

32) $\int_1^e \int_1^e \left(\frac{x\ \ln y}{\sqrt{y}} + \frac{y\ \ln x}{\sqrt{x}}\right) dy\,dx$

33) $\int_0^1 \int_1^2 \left(\frac{x}{x^2+y^2}\right) dy\,dx$

34) $\int_0^1 \int_1^2 \frac{y}{x+y^2}\,dy\,dx$

In exercises 35 - 38, find the average value of the function over the given rectangles.

35)$f(x,y) = -x + 2y$, $R = [0,1] \times [0,1]$

36) $f(x,y) = x^4 + 2y^3$, $R = [1,2] \times [2,3]$

37) $f(x,y) = \sinh x + \sinh y$, $R = [0,1] \times [0,2]$

38) $f(x,y) = \arctan(xy)$, $R = [0,1] \times [0,1]$

39) Let f and g be two continuous functions such that $0 \le m_1 \le f(x) \le M_1$ for any $x \in [a,b]$ and $0 \le m_2 \le g(y) \le M_2$ for any$y \in [c,d]$. Show that the following inequality is true:

$$m_1 m_2 (b-a)(c-d) \le \int_a^b \int_c^d f(x)g(y)\,dydx \le M_1 M_2 (b-a)(c-d).$$

In exercises 40 - 43, use property v. of double integrals and the answer from the preceding exercise to show that the following inequalities are true.

40) $\frac{1}{e^2} \le \iint_R e^{-x^2-y^2}\,dA \le 1$, where $R = [0,1] \times [0,1]$

41) $\frac{\pi^2}{144} \le \iint_R \sin x \cos y\,dA \le \frac{\pi^2}{48}$, where $R = [\frac{\pi}{6}, \frac{\pi}{3}] \times [\frac{\pi}{6}, \frac{\pi}{3}]$

42) $0 \le \iint_R e^{-y}\ \cos x\,dA \le \frac{\pi}{2}$, where $R = [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]$

43) $0 \le \iint_R (\ln x)(\ln y)\,dA \le (e-1)^2$, where $R = [1,e] \times [1,e]$

44) Let f and g be two continuous functions such that $0 \le m_1 \le f(x) \le M_1$ for any $x \in [a,b]$ and $0 \le m_2 \le g(y) \le M_2$ for any $y \in [c,d]$. Show that the following inequality is true:

$$(m_1+m_2)(b-a)(c-d) \le \int_a^b \int_c^d |f(x)+g(y)|\,dy\,dx \le (M_1+M_2)(b-a)(c-d)$$

In exercises 45 - 48, use property v. of double integrals and the answer from the preceding exercise to show that the following inequalities are true.

45) $\frac{2}{e} \le \iint_R (e^{-x^2} + e^{-y^2})\,dA \le 2$, where $R = [0,1] \times [0,1]$

46) $\frac{\pi^2}{36} \iint_R (\sin x + \cos y)\,dA \le \frac{\pi^2\sqrt{3}}{36}$, where $R = [\frac{\pi}{6}, \frac{\pi}{3}] \times [\frac{\pi}{6}, \frac{\pi}{3}]$

47) $\frac{\pi}{2} e^{-\pi/2} \le \iint_R (\cos x + e^{-y})\,dA \le \pi$, where $R = [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]$

48) $\frac{1}{e} \le \iint_R (e^{-y} - \ln x)\,dA \le 2$, where $R = [0,1] \times [0,1]$

In exercises 49 - 50, the function f is given in terms of double integrals.

a. Determine the explicit form of the function f.
b. Find the volume of the solid under the surface $z = f(x,y)$ and above the region R.
c. Find the average value of the function f on R.
d. Use a computer algebra system (CAS) to plot $z = f(x,y)$ and $z = f_{ave}$ in the same system of coordinates.

49) [T] $f(x,y) = \int_0^y \int_0^x (xs + yt)ds\ dt$, where $(x,y) \in R = [0,1] \times [0,1]$

50) [T] $f(x,y) = \int_0^x \int_0^y [\cos(s) + \cos(t)]\, dt\ ds$, where $(x,y) \in R = [0,3] \times [0,3]$

51) Show that if f and g are continuous on $[a,b]$ and $[c,d]$, respectively, then

$$\int_a^b \int_c^d |f(x) + g(y)| dy\ dx = (d-c) \int_a^b f(x)\, dx$$

$$+ \int_a^b \int_c^d g(y)\, dy\ dx = (b-a) \int_c^d g(y)\, dy + \int_c^d \int_a^b f(x)\, dx\ dy.$$

52) Show that $\int_a^b \int_c^d yf(x) + xg(y)\, dy\ dx = \frac{1}{2}(d^2 - c^2) \left(\int_a^b f(x)\, dx \right) + \frac{1}{2}(b^2 - a^2) \left(\int_c^d g(y)\, dy \right)$.

53) [T] Consider the function $f(x,y) = e^{-x^2-y^2}$, where $(x,y) \in R = [-1,1] \times [-1,1]$.

a. Use the midpoint rule with $m = n = 2, 4, \ldots, 10$ to estimate the double integral $I = \iint_R e^{-x^2-y^2} dA$. Round your answers to the nearest hundredths.
b. For $m = n = 2$, find the average value of f over the region R. Round your answer to the nearest hundredths.
c. Use a CAS to graph in the same coordinate system the solid whose volume is given by $\iint_R e^{-x^2-y^2} dA$ and the plane $z = f_{ave}$.

54) [T] Consider the function $f(x,y) = \sin(x^2)\ \cos(y^2)$, where $(x, y \in R = [-1,1] \times [-1,1]$.

a. Use the midpoint rule with $m = n = 2, 4, \ldots, 10$ to estimate the double integral $I = \iint_R \sin(x^2) \cos(y^2)\ dA$. Round your answers to the nearest hundredths.
b. For $m = n = 2$, find the average value of $f$over the region R. Round your answer to the nearest hundredths.
c. Use a CAS to graph in the same coordinate system the solid whose volume is given by $\iint_R \sin(x^2) \cos(y^2)\ dA$ and the plane $z = f_{ave}$.

In exercises 55 - 56, the functions f_n are given, where $n \geq 1$ is a natural number.

a. Find the volume of the solids S_n under the surfaces $z = f_n(x,y)$ and above the region R.
b. Determine the limit of the volumes of the solids S_n as n increases without bound.

55) $f(x,y) = x^n + y^n + xy,\ (x,y) \in R = [0,1] \times [0,1]$

56) $f(x,y) = \frac{1}{x^n} + \frac{1}{y^n},\ (x,y) \in R = [1,2] \times [1,2]$

57) Show that the average value of a function f on a rectangular region $R = [a,b] \times [c,d]$ is $f_{ave} \approx \frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*)$,where (x_{ij}^*, y_{ij}^*) are the sample points of the partition of R, where $1 \leq i \leq m$ and $1 \leq j \leq n$.

58) Use the midpoint rule with $m = n$ to show that the average value of a function f on a rectangular region $R = [a,b] \times [c,d]$ is approximated by

$$f_{ave} \approx \frac{1}{n^2} \sum_{i,j=1}^n f\left(\frac{1}{2}(x_{i=1} + x_i),\ \frac{1}{2}(y_{j=1} + y_j) \right).$$

59) An isotherm map is a chart connecting points having the same temperature at a given time for a given period of time. Use the preceding exercise and apply the midpoint rule with $m = n = 2$ to find the average temperature over the region given in the following figure.

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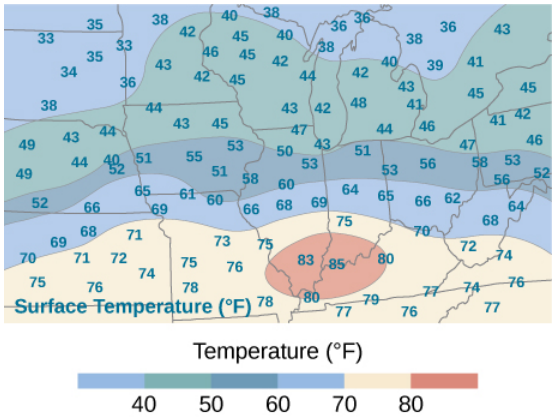

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14.2b: Double Integrals Over General Regions

Learning Objectives

- Recognize when a function of two variables is integrable over a general region.
- Evaluate a double integral by computing an iterated integral over a region bounded by two vertical lines and two functions of x, or two horizontal lines and two functions of y.
- Simplify the calculation of an iterated integral by changing the order of integration.
- Use double integrals to calculate the volume of a region between two surfaces or the area of a plane region.
- Solve problems involving double improper integrals.

Previously, we studied the concept of double integrals and examined the tools needed to compute them. We learned techniques and properties to integrate functions of two variables over rectangular regions. We also discussed several applications, such as finding the volume bounded above by a function over a rectangular region, finding area by integration, and calculating the average value of a function of two variables.

In this section we consider double integrals of functions defined over a general bounded region D on the plane. Most of the previous results hold in this situation as well, but some techniques need to be extended to cover this more general case.

General Regions of Integration

An example of a general bounded region D on a plane is shown in Figure 14.2b.1. Since D is bounded on the plane, there must exist a rectangular region R on the same plane that encloses the region D that is, a rectangular region R exists such that D is a subset of $R (D \subseteq R)$.

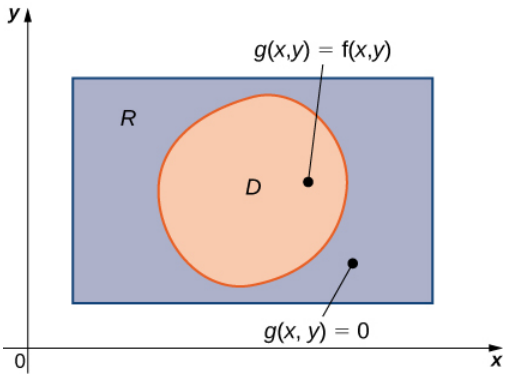

Figure 14.2b.1: For a region D that is a subset of R, we can define a function $g(x,y)$ to equal $f(x,y)$ at every point in D and 0 at every point of R not in D.

Suppose $z = f(x,y)$ is defined on a general planar bounded region D as in Figure 14.2b.1. In order to develop double integrals of f over D we extend the definition of the function to include all points on the rectangular region R and then use the concepts and tools from the preceding section. But how do we extend the definition of f to include all the points on R? We do this by defining a new function $g(x,y)$ on R as follows:

$$g(x,y) = \begin{cases} f(x,y), & \text{if } (x,y) \text{ is in } D \\ 0, & \text{if } (x,y) \text{ is in } R \text{ but not in } D \end{cases}$$

Note that we might have some technical difficulties if the boundary of D is complicated. So we assume the boundary to be a piecewise smooth and continuous simple closed curve. Also, since all the results developed in the section on Double Integrals over Rectangular Regions used an integrable function $f(x,y)$ we must be careful about $g(x,y)$ and verify that $g(x,y)$ is an integrable function over the rectangular region R. This happens as long as the region D is bounded by simple closed curves. For now we will concentrate on the descriptions of the regions rather than the function and extend our theory appropriately for integration.

We consider two types of planar bounded regions.

Definition: Type I and Type II regions

A region D in the (x, y)-plane is of Type I if it lies between two vertical lines and the graphs of two continuous functions $g_1(x)$ and $g_2(x)$. That is (Figure 14.2b.2),

$$D = \{(x, y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}.$$

A region D in the xy-plane is of Type II if it lies between two horizontal lines and the graphs of two continuous functions $h_1(y)$ and $h_2(y)$. That is (Figure 14.2b.3),

$$D = \{(x, y) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y)\}.$$

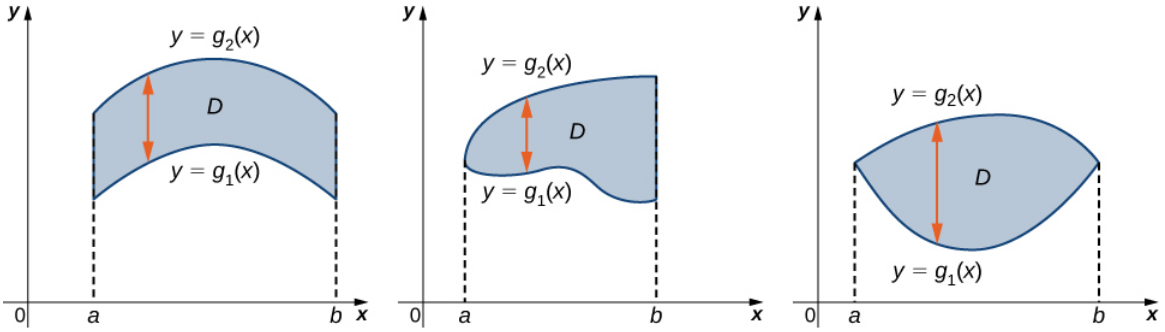

Figure 14.2b.2. A Type I region lies between two vertical lines and the graphs of two functions of x.

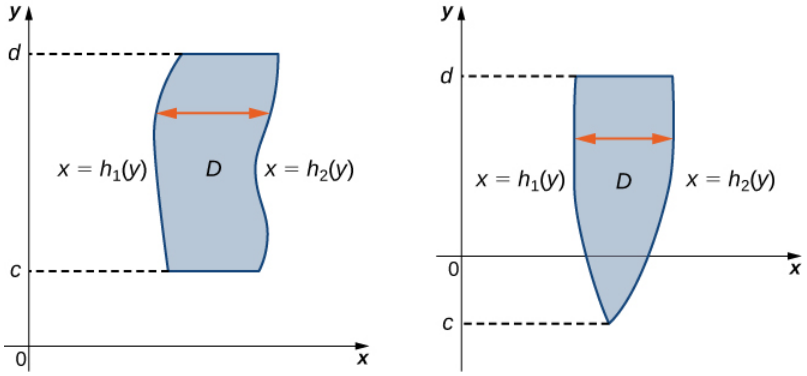

Figure 14.2b.3: A Type II region lies between two horizontal lines and the graphs of two functions of y.

Example 14.2b.1: Describing a Region as Type I and Also as Type II

Consider the region in the first quadrant between the functions $y = \sqrt{x}$ and $y = x^3$ (Figure 14.2b.4). Describe the region first as Type I and then as Type II.

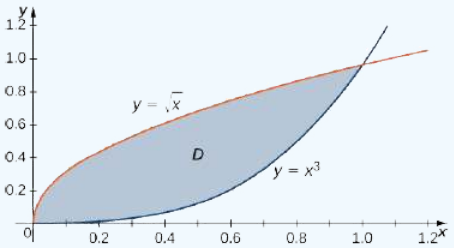

Figure 14.2b.4: Region D can be described as Type I or as Type II.

When describing a region as Type I, we need to identify the function that lies above the region and the function that lies below the region. Here, region D is bounded above by $y = \sqrt{x}$ and below by $y = x^3$ in the interval for x in $[0, 1]$. Hence, as Type I, D is described as the set $\{(x, y) \mid 0 \le x \le 1,\ x^3 \le y \le \sqrt[3]{x}\}$.

However, when describing a region as Type II, we need to identify the function that lies on the left of the region and the function that lies on the right of the region. Here, the region D is bounded on the left by $x = y^2$ and on the right by $x = \sqrt[3]{y}$ in

the interval for y in $[0,1]$. Hence, as Type II, D is described as the set $\{(x,y)\,|\,0\le y\le 1,\ y^2\le x\le \sqrt[3]{y}\}$.

? Exercise 14.2b.1

Consider the region in the first quadrant between the functions $y=2x$ and $y=x^2$. Describe the region first as Type I and then as Type II.

Double Integrals over Non-rectangular Regions

To develop the concept and tools for evaluation of a double integral over a general, nonrectangular region, we need to first understand the region and be able to express it as Type I or Type II or a combination of both. Without understanding the regions, we will not be able to decide the limits of integrations in double integrals. As a first step, let us look at the following theorem.

Theorem: Double Integrals over Nonrectangular Regions

Suppose $g(x,y)$ is the extension to the rectangle R of the function $f(x,y)$ defined on the regions D and R as shown in Figure 14.2b.1 inside R. Then $g(x,y)$ is integrable and we define the double integral of $f(x,y)$ over D by

$$\iint_D f(x,y)\,dA=\iint_R g(x,y)\,dA.$$

The right-hand side of this equation is what we have seen before, so this theorem is reasonable because R is a rectangle and $\iint_R g(x,y)dA$ has been discussed in the preceding section. Also, the equality works because the values of $g(x,y)$ are 0 for any point (x,y) that lies outside D and hence these points do not add anything to the integral. However, it is important that the rectangle R contains the region D.

As a matter of fact, if the region D is bounded by smooth curves on a plane and we are able to describe it as Type I or Type II or a mix of both, then we can use the following theorem and not have to find a rectangle R containing the region.

Theorem: Fubini's Theorem (Strong Form)

For a function $f(x,y)$ that is continuous on a region D of Type I, we have

$$\iint_D f(x,y)\,dA=\iint_D f(x,y)\,dy\,dx=\int_a^b\left[\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\right]dx.$$

Similarly, for a function $f(x,y)$ that is continuous on a region D of Type II, we have

$$\iint_D f(x,y)\,dA=\iint_D f(x,y)\,dx\,dy=\int_c^d\left[\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\right]dy.$$

The integral in each of these expressions is an iterated integral, similar to those we have seen before. Notice that, in the inner integral in the first expression, we integrate $f(x,y)$ with x being held constant and the limits of integration being $g_1(x)$ and $g_2(x)$. In the inner integral in the second expression, we integrate $f(x,y)$ with y being held constant and the limits of integration are $h_1(x)$ and $h_2(x)$.

✓ Example 14.2b.2: Evaluating an Iterated Integral over a Type I Region

Evaluate the integral $\iint_D x^2e^{xy}\,dA$ where D is shown in Figure 14.2b.5.

Solution

First construct the region as a Type I region (Figure 14.2b.5). Here $D=\{(x,y)\,|\,0\le x\le 2,\ \frac{1}{2}x\le y\le 1\}$. Then we have

$$\iint_D x^2 e^{xy}\,dA = \int_{x=0}^{x=2}\int_{y=1/2x}^{y=1} x^2 e^{xy}\,dy\,dx.$$

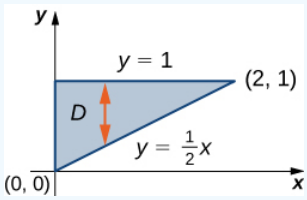

Figure 14.2b.5: We can express region D as a Type I region and integrate from $y = \frac{1}{2}x$ to $y = 1$ between the lines $x = 0$ and $x = 2$.

Therefore, we have

$$\begin{aligned}\int_{x=0}^{x=2}\int_{y=\frac{1}{2}x}^{y=1} x^2 e^{xy}\,dy\,dx &= \int_{x=0}^{x=2}\left[\text{Iterated integral for a Type I region.}\right]dx\\ &= \int_{x=0}^{x=2}\left[\text{Integrate with respect to } y\right]\Big|_{y=1/2x}^{y=1} dx\\ &= \int_{x=0}^{x=2}\left[\text{Integrate with respect to } x\right] dx\\ &= \left[xe^x - e^x - e^{\frac{1}{2}x^2}\right]\Big|_{x=0}^{x=2} = 2.\end{aligned}$$

In Example 14.2b.2, we could have looked at the region in another way, such as $D = \{(x,y)\,|\,0 \le y \le 1,\ 0 \le x \le 2y\}$ (Figure 14.2b.6).

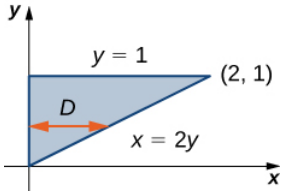

Figure 14.2b.6.

This is a Type II region and the integral would then look like

$$\iint_D x^2 e^{xy}\,dA = \int_{y=0}^{y=1}\int_{x=0}^{x=2y} x^2 e^{xy}\,dx\,dy.$$

However, if we integrate first with respect to x this integral is lengthy to compute because we have to use integration by parts twice.

✔ Example 14.2b.3: Evaluating an Iterated Integral over a Type II Region

Evaluate the integral

$$\iint_D (3x^2 + y^2)\,dA$$

where $D = \{(x,y)\,|\,-2 \le y \le 3,\ y^2 - 3 \le x \le y + 3\}$.

Solution

Notice that D can be seen as either a Type I or a Type II region, as shown in Figure 14.2b.7. However, in this case describing D as Type I is more complicated than describing it as Type II. Therefore, we use D as a Type II region for the integration.

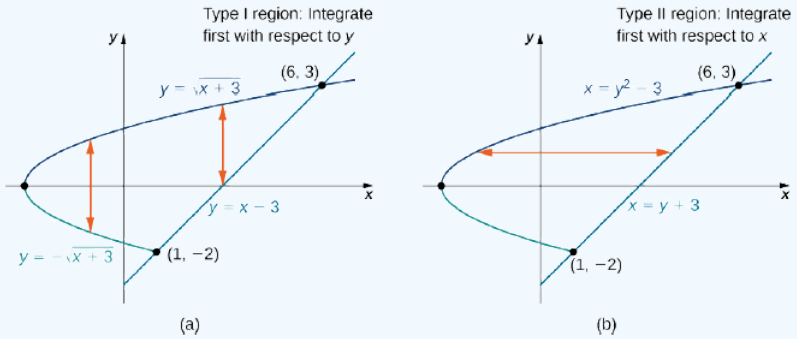

Figure **14.2b.7**: The region D in this example can be either (a) Type I or (b) Type II.

Choosing this order of integration, we have

$$\begin{aligned}
\iint_D (3x^2 + y^2)\,dA &= \int_{y=-2}^{y=3} \int_{x=y^2-3}^{x=y+3} (3x^2 + y^2)\,dx\,dy \\
&= \int_{y=-2}^{y=3} (x^3 + xy^2)\Big|_{y^2-3}^{y+3} dy && \text{Iterated integral, Type II region} \\
&= \int_{y=-2}^{y=3} \big((y+3)^3 + (y+3)y^2 \\
&\quad - (y^2-3)^3 - (y^2-3)y^2\big)\,dy \\
&= \int_{-2}^{3} (54 + 27y - 12y^2 + 2y^3 + 8y^4 \\
&\quad - y^6)\,dy && \text{Integrate with respect to } x. \\
&= \left[54y + \frac{27y^2}{2} - 4y^3 + \frac{y^4}{2} + \frac{8y^5}{5}\right. \\
&\quad \left. - \frac{y^7}{7}\right]_{-2}^{3} \\
&= \frac{2375}{7}.
\end{aligned}$$

? Exercise 14.2b.2

Sketch the region D and evaluate the iterated integral

$$\iint_D xy\,dy\,dx$$

where D is the region bounded by the curves $y = \cos\ x$ and $y = \sin\ x$ in the interval $[-3\pi/4,\ \pi/4]$.

Recall from Double Integrals over Rectangular Regions the properties of double integrals. As we have seen from the examples here, all these properties are also valid for a function defined on a non-rectangular bounded region on a plane. In particular, property 3 states:

If $R = S \cup T$ and $S \cap T = 0$ except at their boundaries, then

$$\iint_R f(x,y)\,dA = \iint_S f(x,y)\,dA + \iint_T f(x,y)\,dA.$$

Similarly, we have the following property of double integrals over a non-rectangular bounded region on a plane.

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Theorem: Decomposing Regions into Smaller Regions

Suppose the region D can be expressed as $D = D_1 \cup D_2$ where D_1 and D_2 do not overlap except at their boundaries. Then

$$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA.$$

This theorem is particularly useful for non-rectangular regions because it allows us to split a region into a union of regions of Type I and Type II. Then we can compute the double integral on each piece in a convenient way, as in the next example.

Example 14.2b.4: Decomposing Regions

Express the region D shown in Figure 14.2b.8 as a union of regions of Type I or Type II, and evaluate the integral

$$\iint_D (2x + 5y)\,dA.$$

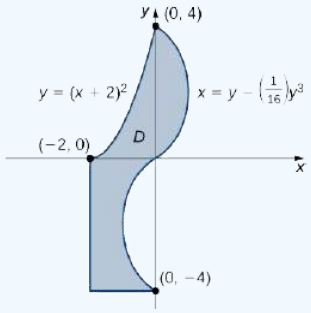

Figure 14.2b.8: This region can be decomposed into a union of three regions of Type I or Type II.

Solution

The region D is not easy to decompose into any one type; it is actually a combination of different types. So we can write it as a union of three regions D_1, D_2, and D_3 where, $D_1 = \{(x,y) \mid -2 \le x \le 0,\ 0 \le y \le (x+2)^2\}$, $D_2 = \{(x,y) \mid 0 \le y \le 4,\ 0 \le x \le \left(y - \frac{1}{16}y^3\right)\}$, and $D_3 = \{(x,y) \mid -4 \le y \le 0,\ -2 \le x \le \left(y - \frac{1}{16}y^3\right)\}$. These regions are illustrated more clearly in Figure 14.2b.9.

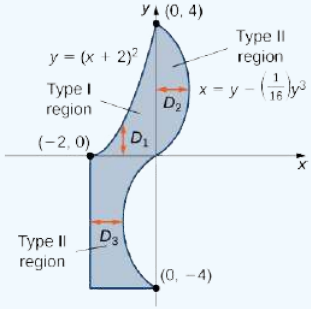

Figure 14.2b.9: Breaking the region into three subregions makes it easier to set up the integration.

Here D_1 is Type I and D_2 and D_3 are both of Type II. Hence,

$$\iint_{(2x}$$

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$$\begin{aligned}\iint_D (2x+5y)\,dA &= \iint_{D_1} (2x+5y)\,dA + \iint_{D_2} (2x+5y)\,dA + \iint_{D_3} (2x+5y)\,dA \\ &= \int_{x=-2}^{x=0}\int_{y=0}^{y=(x+2)^2} (2x+5y)\,dy\,dx + \int_{y=0}^{y=4}\int_{x=0}^{x=y-(1/16)y^3} (2x+5y)\,dx\,dy \\ &\quad + \int_{y=-4}^{y=0}\int_{x=-2}^{x=y-(1/16)y^3} (2x+5y)\,dx\,dy \\ &= \int_{x=-2}^{x=0}\left[\frac{1}{2}(2+x)^2(20+24x+5x^2)\right]dx \\ &\quad + \int_{y=0}^{y=4}\left[\frac{1}{256}y^6 - \frac{7}{16}y^4 + 6y^2\right] \\ &\quad dy + \int_{y=-4}^{y=0}\left[\frac{1}{256}y^6 - \frac{7}{16}y^4 + 6y^2 + 10y - 4\right]dy \\ &= \frac{40}{3} + \frac{1664}{35} - \frac{1696}{35} = \frac{1304}{105}.\end{aligned}$$

Now we could redo this example using a union of two Type II regions (see Exercise **14.2b.4** below).

? Exercise 14.2b.3

Consider the region bounded by the curves $y = \ln x$ and $y = e^x$ in the interval $[1, 2]$. Decompose the region into smaller regions of Type II.

? Exercise 14.2b.4

Redo Example **14.2b.4** using a union of two Type II regions.

Changing the Order of Integration

As we have already seen when we evaluate an iterated integral, sometimes one order of integration leads to a computation that is significantly simpler than the other order of integration. Sometimes the order of integration does not matter, but it is important to learn to recognize when a change in order will simplify our work.

✓ Example 14.2b.5: Changing the Order of Integration

Reverse the order of integration in the iterated integral

$$\int_{x=0}^{x=\sqrt{2}}\int_{y=0}^{y=2-x^2} xe^{x^2}\,dy\,dx.$$

Then evaluate the new iterated integral.

Solution

The region as presented is of Type I. To reverse the order of integration, we must first express the region as Type II. Refer to Figure **14.2b.10**.

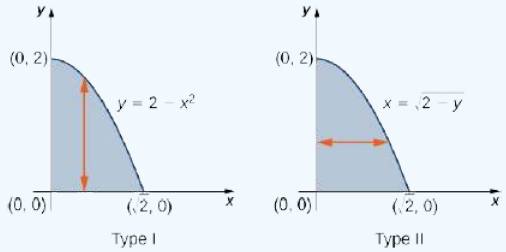

Figure **14.2b.10**: Converting a region from Type I to Type II.

We can see from the limits of integration that the region is bounded above by $y = 2 - x^2$ and below by $y = 0$ where x is in the interval $[0, \sqrt{2}]$. By reversing the order, we have the region bounded on the left by $x = 0$ and on the right by $x = \sqrt{2-y}$ where y

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is in the interval $[0, 2]$. We solved $y = 2 - x^2$ in terms of x to obtain $x = \sqrt{2-y}$.

Hence

$$\int_0^{\sqrt{2}} \int_0^{2-x^2} x e^{x^2}\, dy\, dx = \int_0^2 \int_0^{\sqrt{2-y}} x e^{x^2}\, dx\, dy \qquad \text{Reverse the order of integration then use substitution.}$$

$$= \int_0^2 \left[\frac{1}{2} e^{x^2} \Big|_0^{\sqrt{2-y}} \right] dy$$

$$= \int_0^2 \frac{1}{2}(e^{2-y} - 1)\, dy$$

$$= -\frac{1}{2}(e^{2-y} + y)\Big|_0^2$$

$$= \frac{1}{2}(e^2 - 3).$$

Example 14.2b.6: Evaluating an Iterated Integral by Reversing the Order of Integration

Consider the iterated integral

$$\iint_R f(x, y)\, dx\, dy$$

where $z = f(x, y) = x - 2y$ over a triangular region R that has sides on $x = 0$, $y = 0$, and the line $x + y = 1$. Sketch the region, and then evaluate the iterated integral by

a. integrating first with respect to y and then
b. integrating first with respect to x.

Solution

A sketch of the region appears in Figure 14.2b.11.

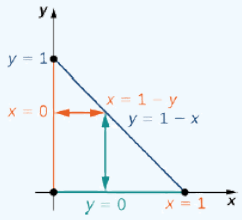

Figure 14.2b.11: A triangular region R for integrating in two ways.

We can complete this integration in two different ways.

a. One way to look at it is by first integrating y from $y = 0$ to $y = 1 - x$ vertically and then integrating x from $x = 0$ to $x = 1$:

$$\iint_R f(x, y)\, dx\, dy = \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} (x - 2y)\, dy\, dx = \int_{x=0}^{x=1} (xy - y^2)\Big|_{y=0}^{y=1-x} dx$$

$$= \int_{x=0}^{x=1} \left[x(1-x) - (1-x)^2\right] dx = \int_{x=0}^{x=1} [-1 + 3x - 2x^2] dx$$

$$= \left[-x + \frac{3}{2}x^2 - \frac{2}{3}x^3\right]\Big|_{x=0}^{x=1} = -\frac{1}{6}.$$

b. The other way to do this problem is by first integrating x from $x = 0$ to $x = 1 - y$ horizontally and then integrating y from $y = 0$ to $y = 1$:

$$\iint_R f(x,y)\,dx\,dy = \int_{y=0}^{y=1}\int_{x=0}^{x=1-y}(x-2y)\,dx\,dy = \int_{y=0}^{y=1}\left[\frac{1}{2}x^2 - 2xy\right]_{x=0}^{x=1-y}dy$$
$$= \int_{y=0}^{y=1}\left[\frac{1}{2}(1-y)^2 - 2y(1-y)\right]dy = \int_{y=0}^{y=1}\left[\frac{1}{2} - 3y + \frac{5}{2}y^2\right]dy$$
$$= \left[\frac{1}{2}y - \frac{3}{2}y^2 + \frac{5}{6}y^3\right]\Big|_{y=0}^{y=1} = -\frac{1}{6}.$$

? Exercise 14.2b.5

Evaluate the iterated integral $\iint_D (x^2+y^2)\,dA$ over the region D in the first quadrant between the functions $y = 2x$ and $y = x^2$.

Evaluate the iterated integral by integrating first with respect to y and then integrating first with resect to x.

Calculating Volumes, Areas, and Average Values

We can use double integrals over general regions to compute volumes, areas, and average values. The methods are the same as those in Double Integrals over Rectangular Regions, but without the restriction to a rectangular region, we can now solve a wider variety of problems.

✓ Example 14.2b.7: Finding the Volume of a Tetrahedron

Find the volume of the solid bounded by the planes $x = 0,\ y = 0,\ z = 0$, and $2x + 3y + z = 6$.

Solution

The solid is a tetrahedron with the base on the xy-plane and a height $z = 6 - 2x - 3y$. The base is the region D bounded by the lines, $x = 0$, $y = 0$ and $2x + 3y = 6$ where $z = 0$ (Figure 14.2b.12). Note that we can consider the region D as Type I or as Type II, and we can integrate in both ways.

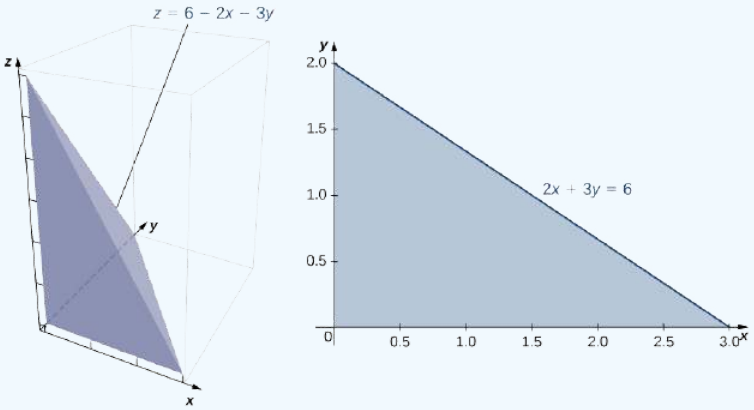

Figure 14.2b.12: A tetrahedron consisting of the three coordinate planes and the plane $z = 6 - 2x - 3y$, with the base bound by $x = 0,\ y = 0$, and $2x + 3y = 6$.

First, consider D as a Type I region, and hence $D = \{(x,y)\,|\,0 \le x \le 3,\ 0 \le y \le 2 - \frac{2}{3}x\}$.

Therefore, the volume is

$$V = \int_{x=0}^{x=3}\int_{y=0}^{y=2-(2x/3)}(6 - 2x - 3y)\,dy\,dx$$
$$= \int_{x=0}^{x=3}\left[\left(6y - 2xy - \frac{3}{2}y^2\right)\Big|_{y=0}^{y=2-(2x/3)}\right]dx$$
$$= \int_{x=0}^{x=3}\left[\frac{2}{3}(x-3)^2\right]dx = 6.$$

Now consider D as a Type II region, so $D = \{(x,y)\,|\,0 \le y \le 2,\ 0 \le x \le 3 - \frac{3}{2}y\}$. In this calculation, the volume is

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$$V = \int_{y=0}^{y=2} \int_{x=0}^{x=3-(3y/2)} (6 - 2x - 3y)\, dx\, dy = \int_{y=0}^{y=2} \left[(6x - x^2 - 3xy) \Big|_{x=0}^{x=3-(3y/2)} \right] dy$$
$$= \int_{y=0}^{y=2} \left[\frac{9}{4}(y-2)^2 \right] dy = 6.$$

Therefore, the volume is 6 cubic units.

? Exercise 14.2b.6

Find the volume of the solid bounded above by $f(x, y) = 10 - 2x + y$ over the region enclosed by the curves $y = 0$ and $y = e^x$ where x is in the interval $[0, 1]$.

Finding the area of a rectangular region is easy, but finding the area of a non-rectangular region is not so easy. As we have seen, we can use double integrals to find a rectangular area. As a matter of fact, this comes in very handy for finding the area of a general non-rectangular region, as stated in the next definition.

Definition: Double Integrals

The area of a plane-bounded region D is defined as the double integral

$$\iint_D 1\, dA.$$

We have already seen how to find areas in terms of single integration. Here we are seeing another way of finding areas by using double integrals, which can be very useful, as we will see in the later sections of this chapter.

Example 14.2b.8: Finding the Area of a Region

Find the area of the region bounded below by the curve $y = x^2$ and above by the line $y = 2x$ in the first quadrant (Figure 14.2b.13).

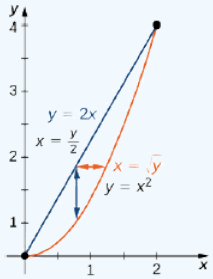

Figure 14.2b.13: The region bounded by $y = x^2$ and $y = 2x$.

Solution

We just have to integrate the constant function $f(x, y) = 1$ over the region. Thus, the area A of the bounded region is $\int_{x=0}^{x=2} \int_{y=x^2}^{y=2x} dy\, dx$ or $\int_{y=0}^{y=4} \int_{x=y/2}^{x=\sqrt{y}} dx\, dy$:

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$$
\begin{aligned}
A &= \iint_D 1\,dx\,dy \\
&= \int_{x=0}^{x=2} \int_{y=x^2}^{y=2x} 1\,dy\,dx \\
&= \int_{x=0}^{x=2} \left(y\Big|_{y=x^2}^{y=2x} \right) dx \\
&= \int_{x=0}^{x=2} (2x - x^2)\,dx \\
&= \left(x^2 - \frac{x^3}{3} \right) \Big|_0^2 = \frac{4}{3}.
\end{aligned}
$$

? Exercise 14.2b.7

Find the area of a region bounded above by the curve $y = x^3$ and below by $y = 0$ over the interval $[0, 3]$.

We can also use a double integral to find the average value of a function over a general region. The definition is a direct extension of the earlier formula.

Definition: The Average Value of a Function

If $f(x, y)$ is integrable over a plane-bounded region D with positive area $A(D)$, then the average value of the function is

$$f_{ave} = \frac{1}{A(D)} \iint_D f(x, y)\,dA.$$

Note that the area is $A(D) = \iint_D 1\,dA$.

Example 14.2b.9: Finding an Average Value

Find the average value of the function $f(x, y) = 7xy^2$ on the region bounded by the line $x = y$ and the curve $x = \sqrt{y}$ (Figure 14.2b.14).

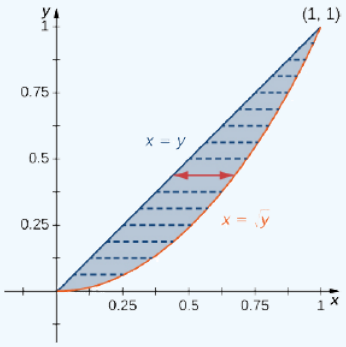

Figure 14.2b.14: The region bounded by $x = y$ and $x = \sqrt{y}$.

Solution

First find the area $A(D)$ where the region D is given by the figure. We have

$$
\begin{aligned}
A(D) = \iint_D 1\,dA &= \int_{y=0}^{y=1} \int_{x=y}^{x=\sqrt{y}} 1\,dx\,dy = \int_{y=0}^{y=1} \left[x\Big|_{x=y}^{x=\sqrt{y}} \right] dy \\
&= \int_{y=0}^{y=1} (\sqrt{y} - y)\,dy = \frac{2}{3} y^{2/3} - \frac{y^2}{2} \Big|_0^1 = \frac{1}{6}
\end{aligned}
$$

Then the average value of the given function over this region is

$$f_{ave} = \frac{1}{A(D)} \iint_D f(x, y)\,dA = \frac{1}{A(D)} \int_{y=0}^{y=1} \int_{x=y}^{x=\sqrt{y}} 7xy^2\,dx\,dy$$

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$$= 6\int_{y=0}^{y=1}\left[\frac{7}{2}y^2(y-y^2)\right]dy = 6\int_{y=0}^{y=1}\left[\frac{7}{2}x^2y^2\Big|_{x=y}^{x=\sqrt{y}}\right]dy$$

$$= \frac{42}{2}\left(\frac{y^4}{4}-\frac{y^5}{5}\right)\Big|_0^1 = \frac{42}{40} = \frac{21}{20}.$$

? Exercise 14.2b.8

Find the average value of the function $f(x,y) = xy$ over the triangle with vertices $(0,0)$, $(1,0)$ and $(1,3)$.

Improper Double Integrals

An **improper double integral** is an integral $\iint_D f\,dA$ where either D is an unbounded region or f is an unbounded function. For example, $D = \{(x,y)\,||x-y| \geq 2\}$ is an unbounded region, and the function $f(x,y) = 1/(1-x^2-2y^2)$ over the ellipse $x^2+3y^2 \geq 1$ is an unbounded function. Hence, both of the following integrals are improper integrals:

i. $$\iint_D xy\,dA \text{ where } D = \{(x,y)||\,x-y| \geq 2\};$$

ii. $$\iint_D \frac{1}{1-x^2-2y^2}\,dA \text{ where } D = \{(x,y)|\,x^2+3y^2 \leq 1\}.$$

In this section we would like to deal with improper integrals of functions over rectangles or simple regions such that f has only finitely many discontinuities. Not all such improper integrals can be evaluated; however, a form of Fubini's theorem does apply for some types of improper integrals.

Theorem: Fubini's Theorem for Improper Integrals

If D is a bounded rectangle or simple region in the plane defined by

$\{(x,y)\ : a \leq x \leq b,\ g(x) \leq y \leq h(x)\}$ and also by

$\{(x,y)\ : c \leq y \leq d,\ j(y) \leq x \leq k(y)\}$ and f is a nonnegative function on D with finitely many discontinuities in the interior of D then

$$\iint_D f\,dA = \int_{x=a}^{x=b}\int_{y=g(x)}^{y=h(x)} f(x,y)\,dy\,dx = \int_{y=c}^{y=d}\int_{x=j(y)}^{x=k(y)} f(x,y)\,dx\,dy$$

It is very important to note that we required that the function be nonnegative on D for the theorem to work. We consider only the case where the function has finitely many discontinuities inside D.

✓ Example 14.2b.10: Evaluating a Double Improper Integral

Consider the function $f(x,y) = \frac{e^y}{y}$ over the region $D = \{(x,y)\ : 0 \leq x \leq 1,\ x \leq y \leq \sqrt{x}\}$.

Notice that the function is nonnegative and continuous at all points on D except $(0,0)$. Use Fubini's theorem to evaluate the improper integral.

Solution

First we plot the region D (Figure 14.2b.15); then we express it in another way.

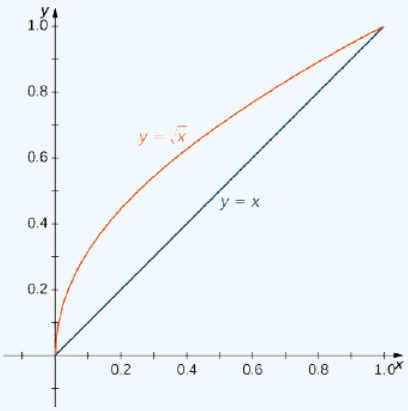

Figure 14.2b.15: The function f is continuous at all points of the region D except $(0,0)$.

The other way to express the same region D is

$$D = \{(x,y) : 0 \le y \le 1,\ y^2 \le x \le y\}.$$

Thus we can use Fubini's theorem for improper integrals and evaluate the integral as

$$\int_{y=0}^{y=1} \int_{x=y^2}^{x=y} \frac{e^y}{y}\, dx\, dy.$$

Therefore, we have

$$\int_{y=0}^{y=1} \int_{x=y^2}^{x=y} \frac{e^y}{y}\, dx\, dy = \int_{y=0}^{y=1} \frac{e^y}{y} x \Big|_{x=y^2}^{x=y} dy = \int_{y=0}^{y=1} \frac{e^y}{y}(y - y^2)\, dy$$
$$= \int_0^1 (e^y - ye^y)\, dy = e - 2.$$

As mentioned before, we also have an improper integral if the region of integration is unbounded. Suppose now that the function f is continuous in an unbounded rectangle R.

Theorem: Improper Integrals on an Unbounded Region

If R is an unbounded rectangle such as $R = \{(x,y) : a \le x \le \infty,\ c \le y \le \infty\}$, then when the limit exists, we have

$$\iint_R f(x,y)\, dA = \lim_{(b,d)\to(\infty,\infty)} \int_a^b \left(\int_c^d f(x,y)\, dy \right) dx$$
$$= \lim_{(b,d)\to(\infty,\infty)} \int_c^d \left(\int_a^b f(x,y)\, dx \right) dy.$$

The following example shows how this theorem can be used in certain cases of improper integrals.

Example 14.2b.11

Evaluate the integral $\iint_R xye^{-x^2-y^2}\, dA$ where R is the first quadrant of the plane.

Solution

The region R is the first quadrant of the plane, which is unbounded. So

$$\begin{aligned}
\iint_R xye^{-x^2-y^2}\,dA &= \lim_{(b,d)\to(\infty,\infty)} \int_{x=0}^{x=b} \left(\int_{y=0}^{y=d} xye^{-x^2-y^2}\,dy \right) dx \\
&= \lim_{(b,d)\to(\infty,\infty)} \int_{x=0}^{x=b} \left(-\frac{1}{2}x \int_{y=0}^{y=d} -2ye^{-x^2-y^2}\,dy \right) dx \\
&= \lim_{(b,d)\to(\infty,\infty)} \int_{x=0}^{x=b} -\frac{1}{2}x \left(e^{-x^2-y^2} \right) \Big|_{y=0}^{y=d} dx \\
&= \lim_{(b,d)\to(\infty,\infty)} \int_{x=0}^{x=b} -\frac{1}{2}x \left(e^{-x^2-d^2} - e^{-x^2} \right) dx \\
&= \lim_{(b,d)\to(\infty,\infty)} \int_{x=0}^{x=b} \frac{1}{2}xe^{-x^2} \left(1 - e^{-d^2} \right) dx \\
&= \lim_{(b,d)\to(\infty,\infty)} \left(-\frac{1}{4} \right) \left(1 - e^{-d^2} \right) \int_{x=0}^{x=b} -2xe^{-x^2}\,dx \\
&= \lim_{(b,d)\to(\infty,\infty)} \left(-\frac{1}{4} \right) \left(1 - e^{-d^2} \right) e^{-x^2} \Big|_{x=0}^{x=b} dx \\
&= \lim_{(b,d)\to(\infty,\infty)} \frac{1}{4} \left(1 - e^{-b^2} \right) \left(1 - e^{-d^2} \right) \\
&= \lim_{(b,d)\to(\infty,\infty)} \frac{1}{4} \left(1 - \cancel{e^{-b^2}}^{\,0} \right) \left(1 - \cancel{e^{-d^2}}^{\,0} \right) \\
&= \frac{1}{4}
\end{aligned}$$

Thus,

$$\iint_R xye^{-x^2-y^2}\,dA$$

is convergent and the value is $\frac{1}{4}$.

? Exercise 14.2b.9

$$\iint_D \frac{y}{\sqrt{1-x^2-y^2}} dA$$

where $D = \{(x,y) : x \geq 0,\ y \geq 0,\ x^2+y^2 \leq 1\}$.

In some situations in probability theory, we can gain insight into a problem when we are able to use double integrals over general regions. Before we go over an example with a double integral, we need to set a few definitions and become familiar with some important properties.

Definition: Joint Density Function

Consider a pair of continuous random variables X and Y such as the birthdays of two people or the number of sunny and rainy days in a month. The **joint density function** f of X and Y satisfies the probability that (X,Y) lies in a certain region D:

$$P((X,Y) \in D) = \iint_D f(x,y)\,dA.$$

Since the probabilities can never be negative and must lie between 0 and 1 the joint density function satisfies the following inequality and equation:

$$f(x,y) \geq 0 \text{ and } \iint_R f(x,y)\,dA = 1.$$

Definition: Independent Random Variables

The variables X and Y are said to be **independent random variables** if their joint density function is the product of their individual density functions:

$$f(x,y) = f_1(x)f_2(y).$$

Example 14.2b.12: Application to Probability

At Sydney's Restaurant, customers must wait an average of 15 minutes for a table. From the time they are seated until they have finished their meal requires an additional 40 minutes, on average. What is the probability that a customer spends less than an hour and a half at the diner, assuming that waiting for a table and completing the meal are independent events?

Solution

Waiting times are mathematically modeled by exponential density functions, with m being the average waiting time, as

$$f(t) = \begin{cases} 0, & \text{if } t < 0 \\ \frac{1}{m}e^{-t/m}, & \text{if } t \geq 0. \end{cases}$$

if X and Y are random variables for 'waiting for a table' and 'completing the meal,' then the probability density functions are, respectively,

$$f_1(x) = \begin{cases} 0, & \text{if } x < 0. \\ \frac{1}{15}e^{-x/15}, & \text{if } x \geq 0. \end{cases} \quad \text{and} \quad f_2(y) = \begin{cases} 0, & \text{if } y < 0 \\ \frac{1}{40}e^{-y/40}, & \text{if } y \geq 0. \end{cases}$$

Clearly, the events are independent and hence the joint density function is the product of the individual functions

$$f(x,y) = f_1(x)f_2(y) = \begin{cases} 0, & \text{if } x < 0 \text{ or } y < 0, \\ \frac{1}{600}e^{-x/15}e^{-y/40}, & \text{if } x, y \geq 0 \end{cases}$$

We want to find the probability that the combined time $X + Y$ is less than 90 minutes. In terms of geometry, it means that the region D is in the first quadrant bounded by the line $x + y = 90$ (Figure 14.2b.16).

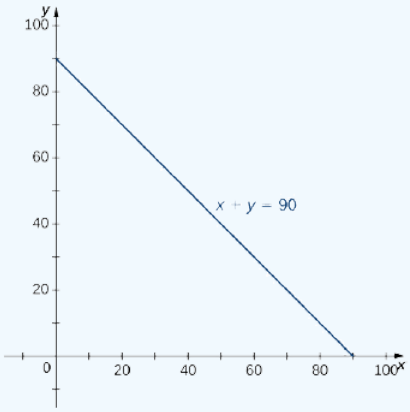

Figure 14.2b.16: The region of integration for a joint probability density function.

Hence, the probability that (X,Y) is in the region D is

$$P(X+Y \leq 90) = P((X,Y) \in D) = \iint_D f(x,y)\,dA = \iint_D \frac{1}{600}e^{-x/15}e^{-y/40}\,dA.$$

Since $x + y = 90$ is the same as $y = 90 - x$, we have a region of Type I, so

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$$D = \{(x,y)\,|\,0 \le x \le 90,\ 0 \le y \le 90 - x\},$$

$$\begin{aligned} P(X+Y \le 90) &= \frac{1}{600}\int_{x=0}^{x=90}\int_{y=0}^{y=90-x} e^{-x/15}e^{-y/40}dy\,dx \\ &= -\frac{1}{15}\int_{x=0}^{x=90} e^{-x/15}\int_{y=0}^{y=90-x} -\frac{1}{40}e^{-y/40}dy\,dx \\ &= -\frac{1}{15}\int_{x=0}^{x=90} e^{-x/15}e^{-y/40}\Big|_{y=0}^{y=90-x}\,dx \\ &= \ \ldots \\ &\approx 0.8328 \end{aligned}$$

Thus, there is an **83.3%** chance that a customer spends less than an hour and a half at the restaurant.

Another important application in probability that can involve improper double integrals is the calculation of expected values. First we define this concept and then show an example of a calculation.

Definition: Expected Values

In probability theory, we denote the **expected values** $E(X)$ and $E(Y)$ respectively, as the most likely outcomes of the events. The expected values $E(X)$ and $E(Y)$ are given by

$$E(X) = \iint_S x\,f(x,y)\,dA \text{ and } E(Y) = \iint_S y\,f(x,y)\,dA,$$

where S is the sample space of the random variables X and Y.

Example 14.2b.13: Finding Expected Value

Find the expected time for the events 'waiting for a table' and 'completing the meal' in Example **14.2b.12**.

Solution

Using the first quadrant of the rectangular coordinate plane as the sample space, we have improper integrals for $E(X)$ and $E(Y)$. The expected time for a table is

$$\begin{aligned} E(X) &= \iint_S x\frac{1}{600}e^{-x/15}e^{-y/40}\,dA \\ &= \frac{1}{600}\int_{x=0}^{x=\infty}\int_{y=0}^{y=\infty} xe^{-x/15}e^{-y/40}dA \\ &= \frac{1}{600}\lim_{(a,b)\to(\infty,\infty)}\int_{x=0}^{x=a}\int_{y=0}^{y=b} xe^{-x/15}e^{-y/40}dx\,dy \\ &= \frac{1}{600}\left(\lim_{a\to\infty}\int_{x=0}^{x=a} xe^{-x/15}dx\right)\left(\lim_{b\to\infty}\int_{y=0}^{y=b} e^{-y/40}dy\right) \\ &= \frac{1}{600}\left(\left(\lim_{a\to\infty}(-15e^{-x/15}(x+15))\right)\Big|_{x=0}^{x=a}\right)\left(\left(\lim_{b\to\infty}(-40e^{-y/40})\right)\Big|_{y=0}^{y=b}\right) \\ &= \frac{1}{600}\left(\lim_{a\to\infty}(-15e^{-a/15}(a+15)+225)\right)\left(\lim_{b\to\infty}(-40e^{-b/40}+40)\right) \\ &= \frac{1}{600}(225)(40) = 15. \end{aligned}$$

A similar calculation shows that $E(Y) = 40$. This means that the expected values of the two random events are the average waiting time and the average dining time, respectively.

? Exercise 14.2b.10

The joint density function for two random variables X and Y is given by

$$f(x,y) = \begin{cases} \frac{1}{16250}(x^2+y^2), & \text{if } 0 \le x \le 15,\ 0 \le y \le 10 \\ 0, & \text{otherwise} \end{cases}$$

Find the probability that X is at most 10 and Y is at least 5.

Key Concepts

- A general bounded region D on the plane is a region that can be enclosed inside a rectangular region. We can use this idea to define a double integral over a general bounded region.
- To evaluate an iterated integral of a function over a general nonrectangular region, we sketch the region and express it as a Type I or as a Type II region or as a union of several Type I or Type II regions that overlap only on their boundaries.
- We can use double integrals to find volumes, areas, and average values of a function over general regions, similarly to calculations over rectangular regions.
- We can use Fubini's theorem for improper integrals to evaluate some types of improper integrals.

Key Equations

- **Iterated integral over a Type I region**

$$\iint_D f(x,y)\,dA = \iint_D f(x,y)\,dy\,dx = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\right] dx$$

- **Iterated integral over a Type II region**

$$\iint_D f(x,y)\,dA = \iint_D (x,y)\,dx\,dy = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\right] dy$$

Glossary

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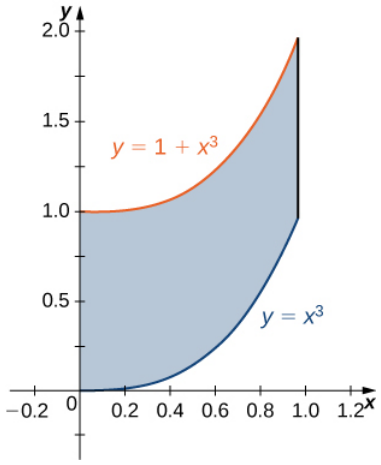
14.2bE: Double Integrals Part 2 (Exercises)

1) The region D bounded by $y = x^3,\ y = x^3 + 1,\ x = 0$, and $x = 1$ as given in the following figure.

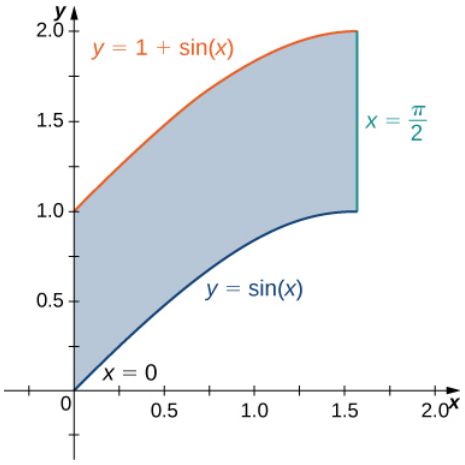


a. Classify this region as vertically simple (Type I) or horizontally simple (Type II).

b. Find the area of the region D.

c. Find the average value of the function $f(x, y) = 3xy$ on the region graphed in the previous exercise.

2) The region D bounded by $y = \sin x,\ y = 1 + \sin x,\ x = 0$, and $x = \frac{\pi}{2}$ as given in the following figure.



a. Classify this region as vertically simple (Type I) or horizontally simple (Type II).

b. Find the area of the region D.

c. Find the average value of the function $f(x, y) = \cos x$ on the region D.

3) The region D bounded by $x = y^2 - 1$ and $x = \sqrt{1 - y^2}$ as given in the following figure.

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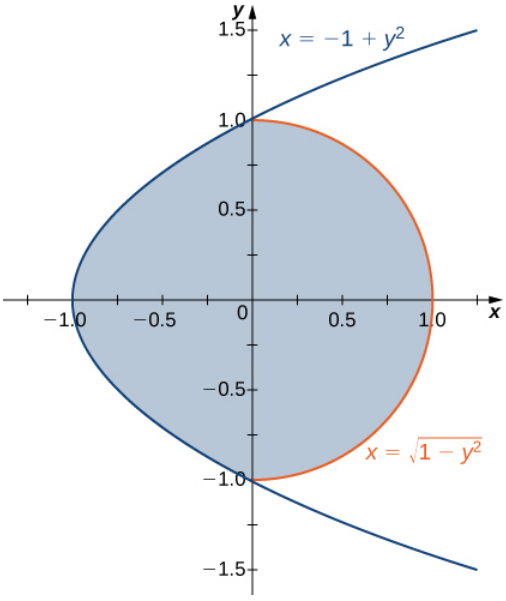
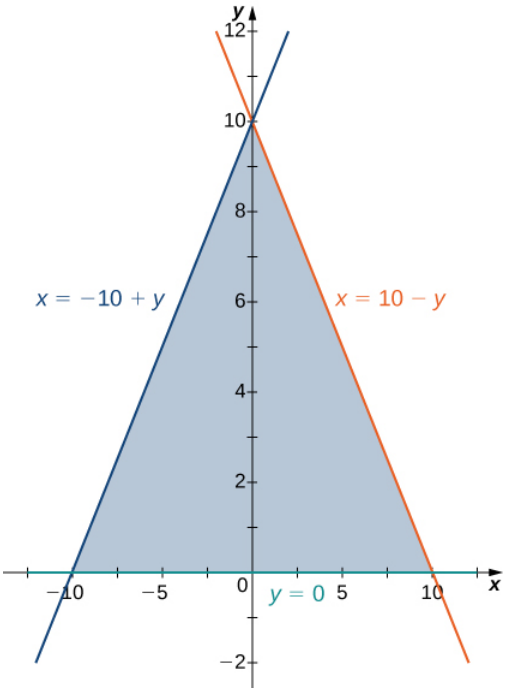


a. Classify this region as vertically simple (Type I) or horizontally simple (Type II).

b. Find the volume of the solid under the graph of the function $f(x,y) = xy + 1$ and above the region D.

4) The region D bounded by $y = 0,\ x = -10 + y,$ and $x = 10 - y$ as given in the following figure.



a. Classify this region as vertically simple (Type I) or horizontally simple (Type II).

b. Find the volume of the solid under the graph of the function $f(x,y) = x + y$ and above the region in the figure from the previous exercise.

5) The region D bounded by $y = 0,\ x = y - 1,\ x = \frac{\pi}{2}$ as given in the following figure.

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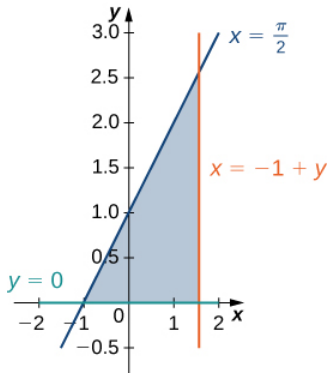
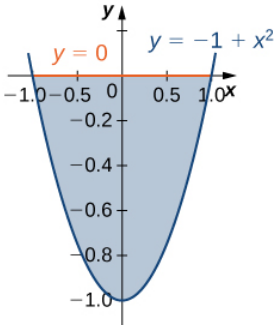


Classify this region as vertically simple (Type I) or horizontally simple (Type II).

6) The region D bounded by $y = 0$ and $y = x^2 - 1$ as given in the following figure.



Classify this region as vertically simple (Type I) or horizontally simple (Type II).

7) Let D be the region bounded by the curves of equations $y = \cos x$ and $y = 4 - x^2$ and the x-axis. Explain why D is neither of Type I nor II.

8) Let D be the region bounded by the curves of equations $y = x,\ y = -x$ and $y = 2 - x^2$. Explain why D is neither of Type I nor II.

In exercises 9 - 14, evaluate the double integral $\iint_D f(x,y)\,dA$ over the region D.

9) $f(x,y) = 1$ and

$D = \{(x,y) | \ 0 \le x \le \frac{\pi}{2},\ \sin x \le y \le 1 + \sin x\}$

10) $f(x,y) = 2$ and

$D = \{(x,y) | \ 0 \le y \le 1,\ y - 1 \le x \le \arccos y\}$

11) $f(x,y) = xy$ and

$D = \{(x,y) | \ -1 \le y \le 1,\ y^2 - 1 \le x \le \sqrt{1 - y^2}\}$

12) $f(x,y) = \sin y$ and D is the triangular region with vertices $(0,0),\ (0,3)$, and $(3,0)$

13) $f(x,y) = -x + 1$ and D is the triangular region with vertices $(0,0),\ (0,2)$, and $(2,2)$

14) $f(x,y) = 2x + 4y$ and

$D = \{(x,y) | \ 0 \le x \le 1,\ x^3 \le y \le x^3 + 1\}$

In exercises 15 - 20, evaluate the iterated integrals.

15) $\int_0^1 \int_{2\sqrt{x}}^{2\sqrt{x}+1} (xy + 1)\,dy\,dx$

16) $\int_0^3 \int_{2x}^{3x} (x + y^2)\,dy\,dx$

17) $\int_1^2 \int_{-u^2-1}^{-u} (8uv)\, dv\, du$

18) $\int_e^{e^2} \int_{\ln u}^{2} (v + \ln u)\, dv\, du$

19) $\int_0^{1/2} \int_{-\sqrt{1-4y^2}}^{\sqrt{1-4y^2}} 4\, dx\, dy$

20) $\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (2x + 4y^3)\, dx\, dy$

21) Let D be the region bounded by $y = 1 - x^2,\ y = 4 - x^2$, and the x- and y-axes.

a. Show that $\iint_D x\, dA = \int_0^1 \int_{1-x^2}^{4-x^2} x\, dy\, dx + \int_1^2 \int_0^{4-x^2} x\, dy\, dx$ by dividing the region D into two regions of Type I.

b. Evaluate the integral $\iint_D x\, dA$.

22) Let D be the region bounded by $y = 1,\ y = x,\ y = \ln x$, and the x-axis.

a. Show that $\iint_D y^2\, dA = \int_{-1}^0 \int_{-x}^{2-x^2} y^2 dy\, dx + \int_0^1 \int_x^{2-x^2} y^2 dy\, dx$ by dividing the region D into two regions of Type I, where $D = \{(x, y) \,|\, y \geq x, y \geq -x,\ y \leq 2 - x^2\}$.

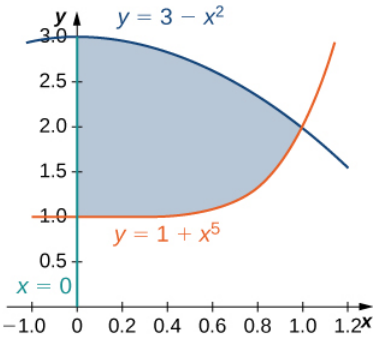
b. Evaluate the integral $\iint_D y^2\, dA$.

23) Let D be the region bounded by $y = x^2$, $y = x + 2$, and $y = -x$.

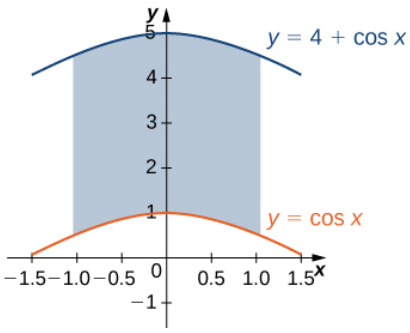
a. Show that $\iint_D x\, dA = \int_0^1 \int_{-y}^{\sqrt{y}} x\, dx\, dy + \int_1^4 \int_{y-2}^{\sqrt{y}} x\, dx\, dy$ by dividing the region D into two regions of Type II, where $D = \{(x, y) \,|\, y \geq x^2,\ y \geq -x,\ y \leq x + 2\}$.

b. Evaluate the integral $\iint_D x\, dA$.

24) The region D bounded by $x = 0, y = x^5 + 1$, and $y = 3 - x^2$ is shown in the following figure. Find the area $A(D)$ of the region D.



25) The region D bounded by $y = \cos x,\ y = 4 + \cos x$, and $x = \pm\frac{\pi}{3}$ is shown in the following figure. Find the area $A(D)$ of the region D.



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26) Find the area $A(D)$ of the region $D = \{(x,y) | \, y \geq 1 - x^2, y \leq 4 - x^2, \; y \geq 0, \; x \geq 0\}$.

27) Let D be the region bounded by $y = 1, \; y = x, \; y = \ln x$, and the x-axis. Find the area $A(D)$ of the region D.

28) Find the average value of the function $f(x,y) = \sin y$ on the triangular region with vertices $(0,0), \; (0,3)$, and $(3,0)$.

29) Find the average value of the function $f(x,y) = -x + 1$ on the triangular region with vertices $(0,0), \; (0,2)$, and $(2,2)$.

In exercises 30 - 33, change the order of integration and evaluate the integral.

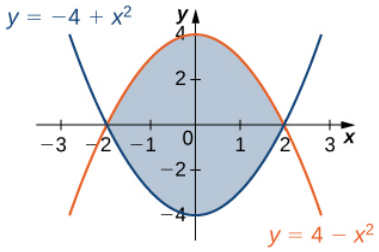
30) $\int_{-1}^{\pi/2} \int_{0}^{x+1} \sin x \, dy \, dx$

31) $\int_{0}^{1} \int_{x-1}^{1-x} x \, dy \, dx$

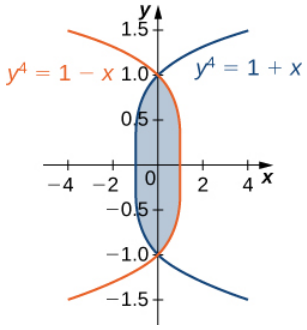
32) $\int_{-1}^{0} \int_{-\sqrt{y+1}}^{\sqrt{y+1}} y^2 dx \; dy$

33) $\int_{-1/2}^{1/2} \int_{-\sqrt{y^2+1}}^{\sqrt{y^2+1}} y \; dx \; dy$

34) The region D is shown in the following figure. Evaluate the double integral $\iint_D (x^2 + y) \; dA$ by using the easier order of integration.



35) The region D is shown in the following figure. Evaluate the double integral $\iint_D (x^2 - y^2) \; dA$ by using the easier order of integration.



36) Find the volume of the solid under the surface $z = 2x + y^2$ and above the region bounded by $y = x^5$ and $y = x$.

37) Find the volume of the solid under the plane $z = 3x + y$ and above the region determined by $y = x^7$ and $y = x$.

38) Find the volume of the solid under the plane $z = 3x + y$ and above the region bounded by $x = \tan y, \; x = -\tan y$, and $x = 1$.

39) Find the volume of the solid under the surface $z = x^3$ and above the plane region bounded by $x = \sin y, \; x = -\sin y$, and $x = 1$.

40) Let g be a positive, increasing, and differentiable function on the interval $[a,b]$. Show that the volume of the solid under the surface $z = g'(x)$ and above the region bounded by $y = 0, \; y = g(x), \; x = a$, and $x = b$ is given by $\frac{1}{2}(g^2(b) - g^2(a))$.

41) Let g be a positive, increasing, and differentiable function on the interval $[a,b]$ and let k be a positive real number. Show that the volume of the solid under the surface $z = g'(x)$ and above the region bounded by $y = g(x), \; y = g(x) + k, \; x = a$, and $x = b$ is given by $k(g(b) - g(a))$.

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42) Find the volume of the solid situated in the first octant and determined by the planes $z = 2$, $z = 0$, $x + y = 1$, $x = 0$, and $y = 0$.

43) Find the volume of the solid situated in the first octant and bounded by the planes $x + 2y = 1$, $x = 0$, $z = 4$, and $z = 0$.

44) Find the volume of the solid bounded by the planes $x + y = 1$, $x - y = 1$, $x = 0$, $z = 0$, and $z = 10$.

45) Find the volume of the solid bounded by the planes $x + y = 1$, $x - y = 1$, $x + y = -1$ $x - y = -1$, $z = 1$, and $z = 0$

46) Let S_1 and S_2 be the solids situated in the first octant under the planes $x + y + z = 1$ and $x + y + 2z = 1$ respectively, and let S be the solid situated between S_1, S_2, $x = 0$, and $y = 0$.

a. Find the volume of the solid S_1.
b. Find the volume of the solid S_2.
c. Find the volume of the solid S by subtracting the volumes of the solids S_1 and S_2.

47) Let S_1 and S_2 be the solids situated in the first octant under the planes $2x + 2y + z = 2$ and $x + y + z = 1$ respectively, and let S be the solid situated between S_1, S_2, $x = 0$, and $y = 0$.

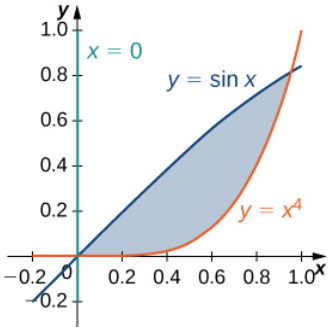
a. Find the volume of the solid S_1.
b. Find the volume of the solid S_2.
c. Find the volume of the solid S by subtracting the volumes of the solids S_1 and S_2.

48) Let S_1 and S_2 be the solids situated in the first octant under the plane $x + y + z = 2$ and under the sphere $x^2 + y^2 + z^2 = 4$, respectively. If the volume of the solid S_2 is $\frac{4\pi}{3}$ determine the volume of the solid S situated between S_1 and S_2 by subtracting the volumes of these solids.

49) Let S_1 and S_2 be the solids situated in the first octant under the plane $x + y + z = 2$ and bounded by the cylinder $x^2 + y^2 = 4$, respectively.

a. Find the volume of the solid S_1.
b. Find the volume of the solid S_2.
c. Find the volume of the solid S situated between S_1 and S_2 by subtracting the volumes of the solids S_1 and S_2.

50) [T] The following figure shows the region D bounded by the curves $y = \sin x$, $x = 0$, and $y = x^4$. Use a graphing calculator or CAS to find the x-coordinates of the intersection points of the curves and to determine the area of the region D. Round your answers to six decimal places.



51) [T] The region D bounded by the curves $y = \cos x$, $x = 0$, and $y = x^3$ is shown in the following figure. Use a graphing calculator or CAS to find the x-coordinates of the intersection points of the curves and to determine the area of the region D. Round your answers to six decimal places.

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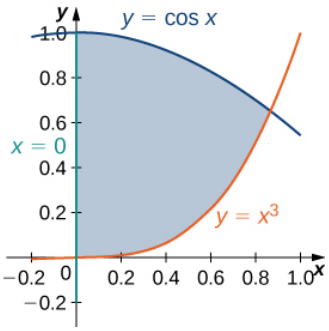


52) Suppose that (X, Y) is the outcome of an experiment that must occur in a particular region S in the xy-plane. In this context, the region S is called the sample space of the experiment and X and Y are random variables. If D is a region included in S, then the probability of (X, Y) being in D is defined as $P[(X, Y) \in D] = \iint_D p(x, y)dx\ dy$, where $p(x, y)$ is the joint probability density of the experiment. Here, $p(x, y)$ is a nonnegative function for which $\iint_S p(x, y)dx\ dy = 1$. Assume that a point (X, Y) is chosen arbitrarily in the square $[0, 3] \times [0, 3]$ with the probability density

$$p(x, y) = \frac{1}{9}(x, y) \in [0, 3] \times [0, 3],$$

$$p(x, y) = 0 \text{ otherwise}$$

Find the probability that the point (X, Y) is inside the unit square and interpret the result.

53) Consider X and Y two random variables of probability densities $p_1(x)$ and $p_2(x)$, respectively. The random variables X and Y are said to be independent if their joint density function is given by $p_(x, y) = p_1(x)p_2(y)$. At a drive-thru restaurant, customers spend, on average, 3 minutes placing their orders and an additional 5 minutes paying for and picking up their meals. Assume that placing the order and paying for/picking up the meal are two independent events X and Y. If the waiting times are modeled by the exponential probability densities

$$p_1(x) = \frac{1}{3}e^{-x/3}\ x \geq 0,$$

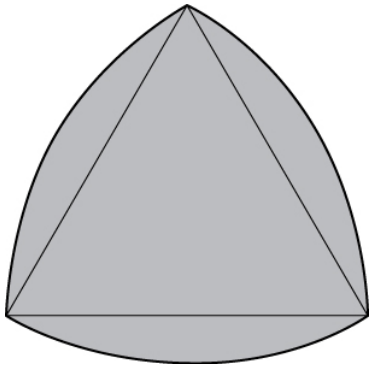
$$p_1(x) = 0 \text{ otherwise}$$

$$p_2(y) = \frac{1}{5}e^{-y/5}\ y \geq 0$$

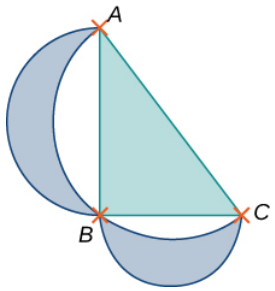
$$p_2(y) = 0 \text{ otherwise}$$

respectively, the probability that a customer will spend less than 6 minutes in the drive-thru line is given by $P[X + Y \leq 6] = \iint_D p(x, y)dx\ dy$, where $D = \{(x, y)|x \geq 0,\ y \geq 0,\ x + y \leq 6\}$. Find $P[X + Y \leq 6]$ and interpret the result.

54) [T] The Reuleaux triangle consists of an equilateral triangle and three regions, each of them bounded by a side of the triangle and an arc of a circle of radius s centered at the opposite vertex of the triangle. Show that the area of the Reuleaux triangle in the following figure of side length s is $\frac{s^2}{2}(\pi - \sqrt{3})$.



55) [T] Show that the area of the lunes of Alhazen, the two blue lunes in the following figure, is the same as the area of the right triangle ABC. The outer boundaries of the lunes are semicircles of diameters AB and AC respectively, and the inner boundaries are formed by the circumcircle of the triangle ABC.



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14.3: Double Integration with Polar Coordinates

We have used iterated integrals to evaluate double integrals, which give the signed volume under a surface, $z = f(x, y)$, over a region R of the xy-plane. The integrand is simply $f(x, y)$, and the bounds of the integrals are determined by the region R.

Some regions R are easy to describe using rectangular coordinates -- that is, with equations of the form $y = f(x)$, $x = a$, etc. However, some regions are easier to handle if we represent their boundaries with polar equations of the form $r = f(\theta)$, $\theta = \alpha$, etc.

The basic form of the double integral is $\iint_R f(x, y)\, dA$. We interpret this integral as follows: over the region R, sum up lots of products of heights (given by $f(x_i, y_i)$) and areas (given by ΔA_i). That is, dA represents "a little bit of area." In rectangular coordinates, we can describe a small rectangle as having area $dx\, dy$ or $dy\, dx$ -- the area of a rectangle is simply length×width -- a small change in x times a small change in y. Thus we replace dA in the double integral with $dx\, dy$ or $dy\, dx$.

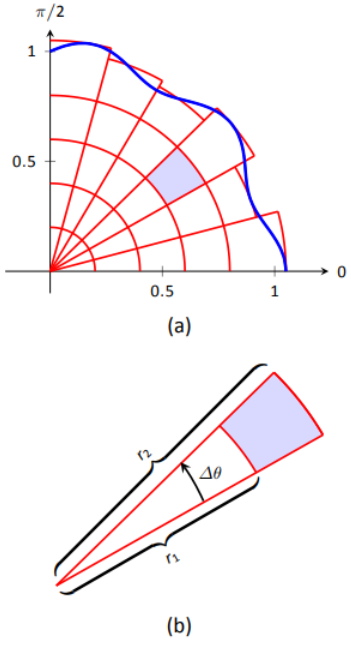

FIGURE 14.3.1

Now consider representing a region R with polar coordinates. Consider Figure 14.3.1*a*. Let R be the region in the first quadrant bounded by the curve. We can approximate this region using the natural shape of polar coordinates: portions of sectors of circles. In the figure, one such region is shaded, shown again in part (b) of the figure.

As the area of a sector of a circle with radius r, subtended by an angle θ, is $A = \frac{1}{2}r^2\theta$, we can find the area of the shaded region. The whole sector has area $\frac{1}{2}r_2^2\Delta\theta$, whereas the smaller, unshaded sector has area $\frac{1}{2}r_1^2\Delta\theta$. The area of the shaded region is the difference of these areas:

$$\Delta A_i = \frac{1}{2}r_2^2\Delta\theta - \frac{1}{2}r_1^2\Delta\theta = \frac{1}{2}\left(r_2^2 - r_1^2\right)\left(\Delta\theta\right) = \frac{r_2 + r_1}{2}\left(r_2 - r_1\right)\Delta\theta. \tag{14.3.1}$$

Note that $(r_2 + r_1)/2$ is just the average of the two radii.

To approximate the region R, we use many such subregions; doing so shrinks the difference $r_2 - r_1$ between radii to 0 and shrinks the change in angle $\Delta\theta$ also to 0. We represent these infinitesimal changes in radius and angle as dr and $d\theta$, respectively. Finally, as dr is small, $r_2 \approx r_1$, and so $(r_2 + r_1)/2 \approx r_1$. Thus, when dr and $d\theta$ are small,

$$\Delta A_i \approx r_i\, dr\, d\theta. \tag{14.3.2}$$

Taking a limit, where the number of subregions goes to infinity and both $r_2 - r_1$ and $\Delta\theta$ go to 0, we get

$$dA = r\, dr\, d\theta. \tag{14.3.3}$$

So to evaluate $\iint_R f(x,y)\ dA$, replace dA with $r\ dr\ d\theta$. Convert the function $z = f(x,y)$ to a function with polar coordinates with the substitutions $x = r\cos\theta$, $y = r\sin\theta$. Finally, find bounds $g_1(\theta) \le r \le g_2(\theta)$ and $\alpha \le \theta \le \beta$ that describe R. This is the key principle of this section, so we restate it here as a Key Idea.

Key Idea: Evaluating Double Integrals with Polar Coordinates

Let R be a plane region bounded by the polar equations $\alpha \le \theta \le \beta$ and $g_1(\theta) \le r \le g_2(\theta)$. Then

$$\iint_R f(x,y)\ dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f\big(r\cos\theta, r\sin\theta\big)\ r\ dr\ d\theta. \tag{14.3.4}$$

Examples will help us understand this Key Idea.

Example 14.3.1: Evaluating a double integral with polar coordinates

Find the signed volume under the plane $z = 4 - x - 2y$ over the circle with equation $x^2 + y^2 = 1$.

Solution

The bounds of the integral are determined solely by the region R over which we are integrating. In this case, it is a circle with equation $x^2 + y^2 = 1$. We need to find polar bounds for this region. It may help to review polar coordinates earlier in this text; bounds for this circle are $0 \le r \le 1$ and $0 \le \theta \le 2\pi$.

We replace $f(x,y)$ with $f(r\cos\theta, r\sin\theta)$. That means we make the following substitutions:

$$4 - x - 2y \quad \Rightarrow \quad 4 - r\cos\theta - 2r\sin\theta.$$

Finally, we replace dA in the double integral with $r\ dr\ d\theta$. This gives the final iterated integral, which we evaluate:

$$\begin{aligned}
\iint_R f(x,y)\ dA &= \int_0^{2\pi}\int_0^1 \big(4 - r\cos\theta - 2r\sin\theta\big) r\ dr\ d\theta \\
&= \int_0^{2\pi}\int_0^1 \big(4r - r^2(\cos\theta - 2\sin\theta)\big)\ dr\ d\theta \\
&= \int_0^{2\pi} \left(2r^2 - \frac{1}{3}r^3(\cos\theta - 2\sin\theta)\right)\Big|_0^1 d\theta \\
&= \int_0^{2\pi} \left(2 - \frac{1}{3}\big(\cos\theta - 2\sin\theta\big)\right) d\theta \\
&= \left(2\theta - \frac{1}{3}\big(\sin\theta + 2\cos\theta\big)\right)\Big|_0^{2\pi} \\
&= 4\pi \approx 12.566.
\end{aligned}$$

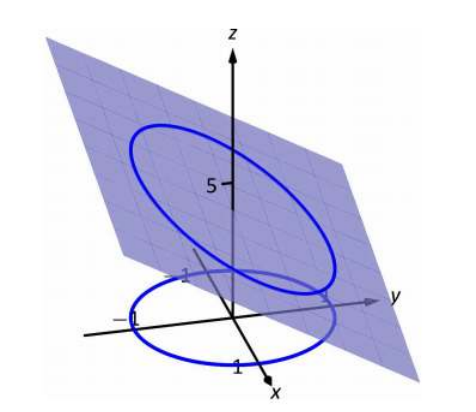

FIGURE 14.3.2

The surface and region R are shown in Figure 14.3.2.

Example 14.3.2: Evaluating a double integral with polar coordinates

Find the volume under the paraboloid $z = 4 - (x-2)^2 - y^2$ over the region bounded by the circles $(x-1)^2 + y^2 = 1$ and $(x-2)^2 + y^2 = 4$.

Solution

At first glance, this seems like a very hard volume to compute as the region R (shown in Figure 14.3.3*a*) has a hole in it, cutting out a strange portion of the surface, as shown in part (b) of the figure. However, by describing R in terms of polar equations, the volume is not very difficult to compute.

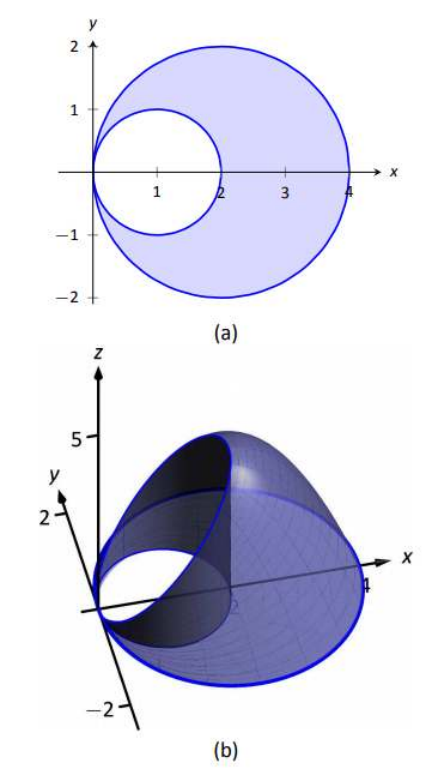

FIGURE 14.3.3

It is straightforward to show that the circle $(x-1)^2 + y^2 = 1$ has polar equation $r = 2\cos\theta$, and that the circle $(x-2)^2 + y^2 = 4$ has polar equation $r = 4\cos\theta$. Each of these circles is traced out on the interval $0 \le \theta \le \pi$. The bounds on r are $2\cos\theta \le r \le 4\cos\theta$.

Replacing x with $r\cos\theta$ in the integrand, along with replacing y with $r\sin\theta$, prepares us to evaluate the double integral $\iint_R f(x,y)\,dA$:

$$\begin{aligned}
\iint_R f(x,y)\,dA &= \int_0^{\pi}\int_{2\cos\theta}^{4\cos\theta}\Big(4-\big(r\cos\theta-2\big)^2-\big(r\sin\theta\big)^2\Big)r\,dr\,d\theta \\
&= \int_0^{\pi}\int_{2\cos\theta}^{4\cos\theta}\big(-r^3+4r^2\cos\theta\big)dr\,d\theta \\
&= \int_0^{\pi}\Big(-\frac{1}{4}r^4+\frac{4}{3}r^3\cos\theta\Big)\Big|_{2\cos\theta}^{4\cos\theta}d\theta \\
&= \int_0^{\pi}\Big(\Big[-\frac{1}{4}(256\cos^4\theta)+\frac{4}{3}(64\cos^4\theta)\Big]- \\
&\quad \Big[-\frac{1}{4}(16\cos^4\theta)+\frac{4}{3}(8\cos^4\theta)\Big]\Big)\,d\theta \\
&= \int_0^{\pi}\frac{44}{3}\cos^4\theta\,d\theta.
\end{aligned}$$

To integrate $\cos^4\theta$, rewrite it as $\cos^2\theta\cos^2\theta$ and employ the power-reducing formula twice:

$$\begin{aligned}\cos^4\theta &= \cos^2\theta\cos^2\theta \\ &= \frac{1}{2}\big(1+\cos(2\theta)\big)\frac{1}{2}\big(1+\cos(2\theta)\big) \\ &= \frac{1}{4}\big(1+2\cos(2\theta)+\cos^2(2\theta)\big) \\ &= \frac{1}{4}\Big(1+2\cos(2\theta)+\frac{1}{2}\big(1+\cos(4\theta)\big)\Big) \\ &= \frac{3}{8}+\frac{1}{2}\cos(2\theta)+\frac{1}{8}\cos(4\theta).\end{aligned}$$

Picking up from where we left off above, we have

$$\begin{aligned}&= \int_0^\pi \frac{44}{3}\cos^4\theta\,d\theta \\ &= \int_0^\pi \frac{44}{3}\left(\frac{3}{8}+\frac{1}{2}\cos(2\theta)+\frac{1}{8}\cos(4\theta)\right)d\theta \\ &= \frac{44}{3}\left(\frac{3}{8}\theta+\frac{1}{4}\sin(2\theta)+\frac{1}{32}\sin(4\theta)\right)\Big|_0^\pi \\ &= \frac{11}{2}\pi \approx 17.279.\end{aligned}$$

While this example was not trivial, the double integral would have been *much* harder to evaluate had we used rectangular coordinates.

Example 14.3.3: Evaluating a double integral with polar coordinates

Find the volume under the surface $f(x,y)=\dfrac{1}{x^2+y^2+1}$ over the sector of the circle with radius a centered at the origin in the first quadrant, as shown in Figure 14.3.4.

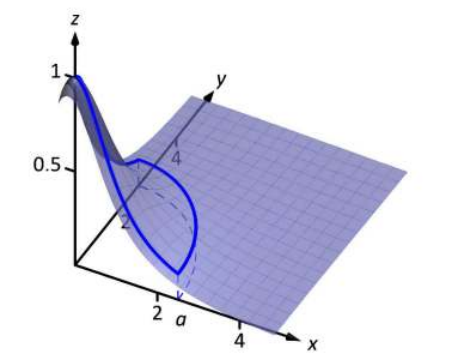

FIGURE 14.3.4

Solution

The region R we are integrating over is a circle with radius a, restricted to the first quadrant. Thus, in polar, the bounds on R are $0\le r\le a$, $0\le\theta\le\pi/2$. The integrand is rewritten in polar as

$$\frac{1}{x^2+y^2+1}\Rightarrow\frac{1}{r^2\cos^2\theta+r^2\sin^2\theta+1}=\frac{1}{r^2+1}. \tag{14.3.5}$$

We find the volume as follows:

$$\begin{aligned}\iint_R f(x,y)\,dA &= \int_0^{\pi/2}\int_0^a \frac{r}{r^2+1}\,dr\,d\theta \\ &= \int_0^{\pi/2} \frac{1}{2}\Big(\ln|r^2+1|\Big)\Big|_0^a\,d\theta \\ &= \int_0^{\pi/2} \frac{1}{2}\ln(a^2+1)\,d\theta \\ &= \left(\frac{1}{2}\ln(a^2+1)\theta\right)\Big|_0^{\pi/2} \\ &= \frac{\pi}{4}\ln(a^2+1).\end{aligned}$$

Figure **14.3.4** shows that f shrinks to near 0 very quickly. Regardless, as a grows, so does the volume, without bound.

Note: Previous work has shown that there is finite *area* under $\frac{1}{x^2+1}$ over the entire x-axis. However, Example **14.3.3** shows that there is infinite *volume* under $\dfrac{1}{x^2+y^2+1}$ over the entire xy-plane.

Example 14.3.4: Finding the volume of a sphere

Find the volume of a sphere with radius a.

Solution

The sphere of radius a, centered at the origin, has equation $x^2+y^2+z^2=a^2$; solving for z, we have $z=\sqrt{a^2-x^2-y^2}$. This gives the upper half of a sphere. We wish to find the volume under this top half, then double it to find the total volume.

The region we need to integrate over is the circle of radius a, centered at the origin. Polar bounds for this equation are $0 \le r \le a$, $0 \le \theta \le 2\pi$.

All together, the volume of a sphere with radius a is:

$$\begin{aligned}&2\iint_R \sqrt{a^2-x^2-y^2}\,dA \qquad (14.3.6)\\ &= 2\int_0^{2\pi}\int_0^a \sqrt{a^2-(r\cos\theta)^2-(r\sin\theta)^2}\,r\,dr\,d\theta \\ &= 2\int_0^{2\pi}\int_0^a r\sqrt{a^2-r^2}\,dr\,d\theta.\end{aligned}$$

We can evaluate this inner integral with substitution. With $u=a^2-r^2$, $du=-2r\,dr$. The new bounds of integration are $u(0)=a^2$ to $u(a)=0$. Thus we have:

$$\begin{aligned}&= \int_0^{2\pi}\int_{a^2}^0 \left(-u^{1/2}\right)du\,d\theta \\ &= \int_0^{2\pi}\left(-\frac{2}{3}u^{3/2}\right)\Big|_{a^2}^0\,d\theta \\ &= \int_0^{2\pi}\left(\frac{2}{3}a^3\right)d\theta \\ &= \left(\frac{2}{3}a^3\theta\right)\Big|_0^{2\pi} \\ &= \frac{4}{3}\pi a^3.\end{aligned}$$

Generally, the formula for the volume of a sphere with radius r is given as $4/3\pi r^3$; we have justified this formula with our calculation.

Example 14.3.5: Finding the volume of a solid

A sculptor wants to make a solid bronze cast of the solid shown in Figure **14.3.5**, where the base of the solid has boundary, in polar coordinates, $r=\cos(3\theta)$, and the top is defined by the plane $z=1-x+0.1y$. Find the volume of the solid.

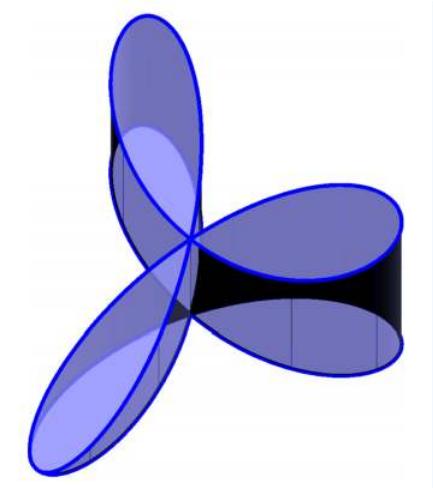

FIGURE 14.3.5

Solution

From the outset, we should recognize that knowing *how to set up* this problem is probably more important than knowing *how to compute the integrals*. The iterated integral to come is not "hard" to evaluate, though it is long, requiring lots of algebra. Once the proper iterated integral is determined, one can use readily--available technology to help compute the final answer.

The region R that we are integrating over is bound by $0 \le r \le \cos(3\theta)$, for $0 \le \theta \le \pi$ (note that this rose curve is traced out on the interval $[0, \pi]$, not $[0, 2\pi]$). This gives us our bounds of integration. The integrand is $z = 1 - x + 0.1y$; converting to polar, we have that the volume V is:

$$V = \iint_R f(x,y)\, dA = \int_0^{\pi} \int_0^{\cos(3\theta)} \big(1 - r\cos\theta + 0.1r\sin\theta\big) r\, dr\, d\theta.$$

Distributing the r, the inner integral is easy to evaluate, leading to

$$\int_0^{\pi} \left(\frac{1}{2}\cos^2(3\theta) - \frac{1}{3}\cos^3(3\theta)\cos\theta + \frac{0.1}{3}\cos^3(3\theta)\sin\theta\right) d\theta.$$

This integral takes time to compute by hand; it is rather long and cumbersome. The powers of cosine need to be reduced, and products like $\cos(3\theta)\cos\theta$ need to be turned to sums using the Product To Sum formulas in the back cover of this text.

We rewrite $\frac{1}{2}\cos^2(3\theta)$ as $\frac{1}{4}(1 + \cos(6\theta))$. We can also rewrite $\frac{1}{3}\cos^3(3\theta)\cos\theta$ as:

$$\frac{1}{3}\cos^3(3\theta)\cos\theta = \frac{1}{3}\cos^2(3\theta)\cos(3\theta)\cos\theta = \frac{1}{3}\frac{1+\cos(6\theta)}{2}\big(\cos(4\theta) + \cos(2\theta)\big).$$

This last expression still needs simplification, but eventually all terms can be reduced to the form $a\cos(m\theta)$ or $a\sin(m\theta)$ for various values of a and m.

We forgo the algebra and recommend the reader employ technology, such as WolframAlpha, to compute the numeric answer. Such technology gives:

$$\int_0^{\pi} \int_0^{\cos(3\theta)} \big(1 - r\cos\theta + 0.1r\sin\theta\big) r\, dr\, d\theta = \frac{\pi}{4} \approx 0.785u^3.$$

Since the units were not specified, we leave the result as almost 0.8 cubic units (meters, feet, etc.) Should the artist want to scale the piece uniformly, so that each rose petal had a length other than 1, she should keep in mind that scaling by a factor of k scales the volume by a factor of k^3.

We have used iterated integrals to find areas of plane regions and volumes under surfaces. Just as a single integral can be used to compute much more than "area under the curve," iterated integrals can be used to compute much more than we have thus far seen. The next two sections show two, among many, applications of iterated integrals.

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14.3b: Double Integrals in Polar Coordinates

Learning Objectives

- Recognize the format of a double integral over a polar rectangular region.
- Evaluate a double integral in polar coordinates by using an iterated integral.
- Recognize the format of a double integral over a general polar region.
- Use double integrals in polar coordinates to calculate areas and volumes.

Double integrals are sometimes much easier to evaluate if we change rectangular coordinates to polar coordinates. However, before we describe how to make this change, we need to establish the concept of a double integral in a polar rectangular region.

Polar Rectangular Regions of Integration

When we defined the double integral for a continuous function in rectangular coordinates—say, g over a region R in the xy-plane—we divided R into subrectangles with sides parallel to the coordinate axes. These sides have either constant x-values and/or constant y-values. In polar coordinates, the shape we work with is a polar rectangle, whose sides have constant r-values and/or constant θ-values. This means we can describe a polar rectangle as in Figure 14.3b.1a, with $R = \{(r, \theta) \mid a \le r \le b,\ \alpha \le \theta \le \beta\}$.

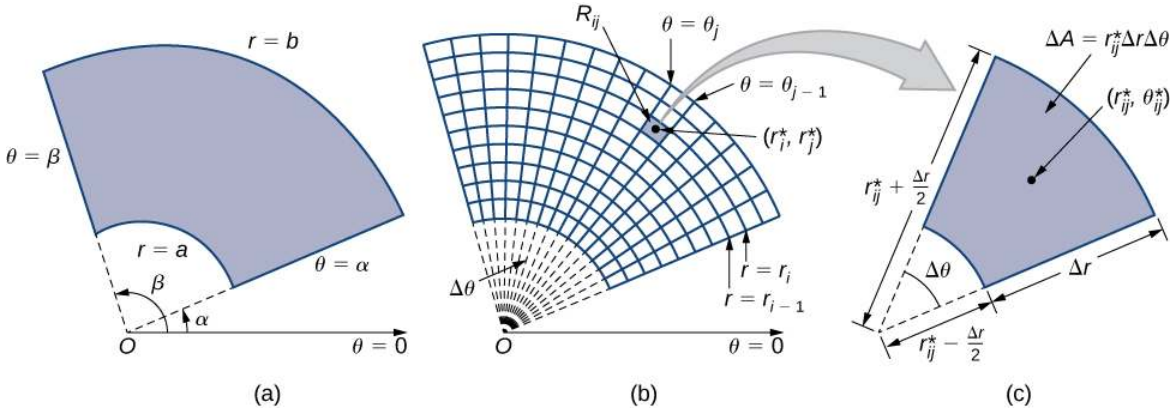

Figure 14.3b.1: (a) A polar rectangle R (b) divided into subrectangles R_{ij} (c) Close-up of a subrectangle.

In this section, we are looking to integrate over polar rectangles. Consider a function $f(r, \theta)$ over a polar rectangle R. We divide the interval $[a, b]$ into m subintervals $[r_{i-1}, r_i]$ of length $\Delta r = (b-a)/m$ and divide the interval $[\alpha, \beta]$ into n subintervals $[\theta_{i-1}, \theta_i]$ of width $\Delta\theta = (\beta - \alpha)/n$. This means that the circles $r = r_i$ and rays $\theta = \theta_i$ for $1 \le i \le m$ and $1 \le j \le n$ divide the polar rectangle R into smaller polar subrectangles R_{ij} (Figure 14.3b.1b).

As before, we need to find the area ΔA of the polar subrectangle R_{ij} and the "polar" volume of the thin box above R_{ij}. Recall that, in a circle of radius r the length s of an arc subtended by a central angle of θ radians is $s = r\theta$. Notice that the polar rectangle R_{ij} looks a lot like a trapezoid with parallel sides $r_{i-1}\Delta\theta$ and $r_i\Delta\theta$ and with a width Δr. Hence the area of the polar subrectangle R_{ij} is

$$\Delta A = \frac{1}{2}\Delta r(r_{i-1}\Delta\theta + r_i\Delta\theta).$$

Simplifying and letting

$$r_{ij}^* = \frac{1}{2}(r_{i-1} + r_i)$$

we have $\Delta A = r_{ij}^* \Delta r \Delta\theta$.

Therefore, the polar volume of the thin box above R_{ij} (Figure 14.3b.2) is

$$f(r_{ij}^*, \theta_{ij}^*)\Delta A = f(r_{ij}^*, \theta_{ij}^*) r_{ij}^* \Delta r \Delta\theta.$$

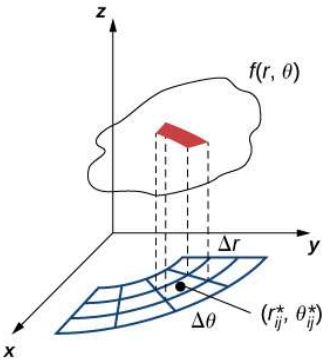

Figure 14.3b.2: Finding the volume of the thin box above polar rectangle R_{ij}.

Using the same idea for all the subrectangles and summing the volumes of the rectangular boxes, we obtain a double Riemann sum as

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_{ij}^*, \theta_{ij}^*) r_{ij}^* \Delta r \Delta\theta.$$

As we have seen before, we obtain a better approximation to the polar volume of the solid above the region R when we let m and n become larger. Hence, we define the polar volume as the limit of the double Riemann sum,

$$V = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_{ij}^*, \theta_{ij}^*) r_{ij}^* \Delta r \Delta\theta.$$

This becomes the expression for the double integral.

Definition: The double integral in polar coordinates

The double integral of the function $f(r, \theta)$ over the polar rectangular region R in the $r\theta$-plane is defined as

$$\iint_R f(r,\theta) dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_{ij}^*, \theta_{ij}^*) \Delta A \tag{14.3b.1}$$

$$= \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_{ij}^*, \theta_{ij}^*) r_{ij}^* \Delta r \Delta\theta. \tag{14.3b.2}$$

Again, just as in section on Double Integrals over Rectangular Regions, the double integral over a polar rectangular region can be expressed as an iterated integral in polar coordinates. Hence,

$$\iint_R f(r,\theta)\, dA = \iint_R f(r,\theta)\, r\, dr\, d\theta = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=a}^{r=b} f(r,\theta)\, r\, dr\, d\theta.$$

Notice that the expression for dA is replaced by $r\, dr\, d\theta$ when working in polar coordinates. Another way to look at the polar double integral is to change the double integral in rectangular coordinates by substitution. When the function f is given in terms of x and y using $x = r\cos\theta$, $y = r\sin\theta$, and $dA = r\, dr\, d\theta$ changes it to

$$\iint_R f(x,y)\, dA = \iint_R f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta.$$

Note that all the properties listed in section on Double Integrals over Rectangular Regions for the double integral in rectangular coordinates hold true for the double integral in polar coordinates as well, so we can use them without hesitation.

Example 14.3b.1A: Sketching a Polar Rectangular Region

Sketch the polar rectangular region

$$R = \{(r,\theta) \mid 1 \le r \le 3, 0 \le \theta \le \pi\}.$$

Solution

As we can see from Figure 14.3b.3, $r = 1$ and $r = 3$ are circles of radius 1 and 3 and $0 \le \theta \le \pi$ covers the entire top half of the plane. Hence the region R looks like a semicircular band.

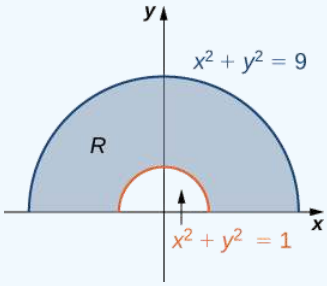

Figure 14.3b.3: The polar region R lies between two semicircles.

Now that we have sketched a polar rectangular region, let us demonstrate how to evaluate a double integral over this region by using polar coordinates.

Example 14.3b.1B: Evaluating a Double Integral over a Polar Rectangular Region

Evaluate the integral $\iint_R 3x\,dA$ over the region $R = \{(r,\theta)\,|\,1 \le r \le 2,\ 0 \le \theta \le \pi\}$.

Solution

First we sketch a figure similar to Figure 14.3b.3, but with outer radius $r = 2$. From the figure we can see that we have

$$\begin{aligned}\iint_R 3x\,dA &= \int_{\theta=0}^{\theta=\pi}\int_{r=1}^{r=2} 3r\cos\theta\, r\,dr\,d\theta \quad \text{Use an integral with correct limits of integration.}\\ &= \int_{\theta=0}^{\theta=\pi} \cos\theta\left[r^3\Big|_{r=1}^{r=2}\right]d\theta \quad \text{Integrate first with respect to } r.\\ &= \int_{\theta=0}^{\theta=\pi} 7\cos\theta\,d\theta\\ &= 7\sin\theta\Big|_{\theta=0}^{\theta=\pi} = 0.\end{aligned}$$

Exercise 14.3b.1

Sketch the region $D = \{(r,\theta)|1 \le r \le 2, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\}$, and evaluate $\iint_R x\,dA$.

Example 14.3b.2A: Evaluating a Double Integral by Converting from Rectangular Coordinates

Evaluate the integral

$$\iint_R (1 - x^2 - y^2)\,dA$$

where R is the unit circle on the xy-plane.

Solution

The region R is a unit circle, so we can describe it as $R = \{(r,\theta)\,|\,0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$.

Using the conversion $x = r\cos\theta$, $y = r\sin\theta$, and $dA = r\,dr\,d\theta$, we have

$$\begin{aligned}\iint_R (1-x^2-y^2)\,dA &= \int_0^{2\pi}\int_0^1 (1-r^2)\,r\,dr\,d\theta \\ &= \int_0^{2\pi}\int_0^1 (r-r^3)\,dr\,d\theta \\ &= \int_0^{2\pi}\left[\frac{r^2}{2}-\frac{r^4}{4}\right]_0^1 d\theta \\ &= \int_0^{2\pi}\frac{1}{4}\,d\theta = \frac{\pi}{2}.\end{aligned}$$

Example 14.3b.2B: Evaluating a Double Integral by Converting from Rectangular Coordinates

Evaluate the integral

$$\iint_R (x+y)\,dA$$

where $R = \{(x,y)\,|\,1 \le x^2+y^2 \le 4,\ x \le 0\}$.

Solution

We can see that R is an annular region that can be converted to polar coordinates and described as $R = \{(r,\theta)\,|\,1 \le r \le 2,\ \frac{\pi}{2} \le \theta \le \frac{3\pi}{2}\}$ (see the following graph).

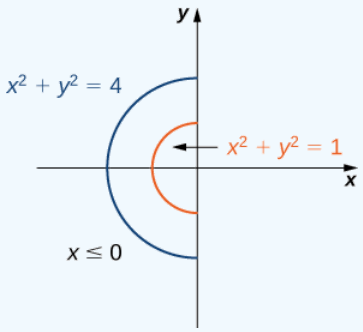

Figure 14.3b.4: The annular region of integration R.

Hence, using the conversion $x = r\cos\theta$, $y = r\sin\theta$, and $dA = r\,dr\,d\theta$, we have

$$\begin{aligned}\iint_R (x+y)\,dA &= \int_{\theta=\pi/2}^{\theta=3\pi/2}\int_{r=1}^{r=2} (r\cos\theta + r\sin\theta)r\,dr\,d\theta \\ &= \left(\int_{r=1}^{r=2} r^2\,dr\right)\left(\int_{\pi/2}^{3\pi/2}(\cos\theta+\sin\theta)\,d\theta\right) \\ &= \left[\frac{r^3}{3}\right]_1^2 [\sin\theta-\cos\theta]\Big|_{\pi/2}^{3\pi/2} \\ &= -\frac{14}{3}.\end{aligned}$$

Exercise 14.3b.2

Evaluate the integral

$$\iint_R (4-x^2-y^2)\,dA$$

where R is the circle of radius 2 on the xy-plane.

General Polar Regions of Integration

To evaluate the double integral of a continuous function by iterated integrals over general polar regions, we consider two types of regions, analogous to Type I and Type II as discussed for rectangular coordinates in section on Double Integrals over General Regions. It is more common to write polar equations as $r = f(\theta)$ than $\theta = f(r)$, so we describe a general polar region as $R = \{(r, \theta) \mid \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$ (Figure 14.3b.5).

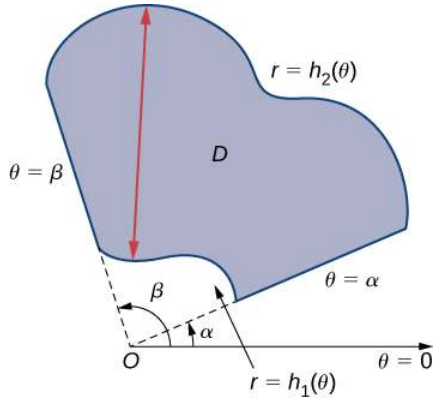

Figure 14.3b.5: A general polar region between $\alpha \le \theta \le \beta$ and $h_1(\theta) \le r \le h_2(\theta)$.

Theorem: Double Integrals over General Polar Regions

If $f(r, \theta)$ is continuous on a general polar region D as described above, then

$$\iint_D f(r, \theta)\, r\, dr\, d\theta = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=h_1(\theta)}^{r=h_2(\theta)} f(r, \theta)\, r\, dr\, d\theta.$$

Example 14.3b.3: Evaluating a Double Integral over a General Polar Region

Evaluate the integral

$$\iint_D r^2 \sin\theta\, r\, dr\, d\theta$$

where D is the region bounded by the polar axis and the upper half of the cardioid $r = 1 + \cos\theta$.

Solution

We can describe the region D as $\{(r, \theta) \mid 0 \le \theta \le \pi,\ 0 \le r \le 1 + \cos\theta\}$ as shown in Figure 14.3b.6.

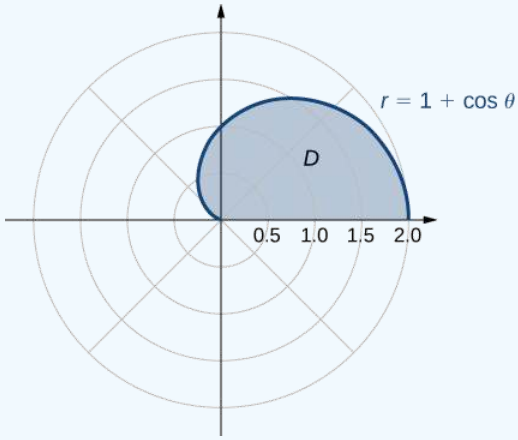

Figure 14.3b.6: The region D is the top half of a cardioid.

Hence, we have

$$\iint_D r^2 \sin\theta\, r\, dr\, d\theta = \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=1+\cos\theta} (r^2 \sin\theta)\, r\, dr\, d\theta$$
$$= \frac{1}{4} \int_{\theta=0}^{\theta=\pi} [r^4]\Big|_{r=0}^{r=1+\cos\theta} \sin\theta\, d\theta$$
$$= \frac{1}{4} \int_{\theta=0}^{\theta=\pi} (1+\cos\theta)^4 \sin\theta\, d\theta$$
$$= -\frac{1}{4}\left[\frac{(1+\cos\theta)^5}{5}\right]_0^{\pi} = \frac{8}{5}.$$

? Exercise 14.3b.3

Evaluate the integral

$$\iint_D r^2 \sin^2 2\theta\, r\, dr\, d\theta$$

where $D = \left\{(r,\theta) \,|\, 0 \le \theta \le \pi,\ 0 \le r \le 2\sqrt{\cos 2\theta}\right\}$.

Polar Areas and Volumes

As in rectangular coordinates, if a solid S is bounded by the surface $z = f(r,\theta)$, as well as by the surfaces $r = a$, $r = b$, $\theta = \alpha$, and $\theta = \beta$, we can find the volume V of S by double integration, as

$$V = \iint_R f(r,\theta)\, r\, dr\, d\theta = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=a}^{r=b} f(r,\theta)\, r\, dr\, d\theta.$$

If the base of the solid can be described as $D = \{(r,\theta) | \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$, then the double integral for the volume becomes

$$V = \iint_D f(r,\theta)\, r\, dr\, d\theta = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=h_1(\theta)}^{r=h_2(\theta)} f(r,\theta)\, r\, dr\, d\theta.$$

We illustrate this idea with some examples.

✔ Example 14.3b.4A: Finding a Volume Using a Double Integral

Find the volume of the solid that lies under the paraboloid $z = 1 - x^2 - y^2$ and above the unit circle on the xy-plane (Figure 14.3b.7).

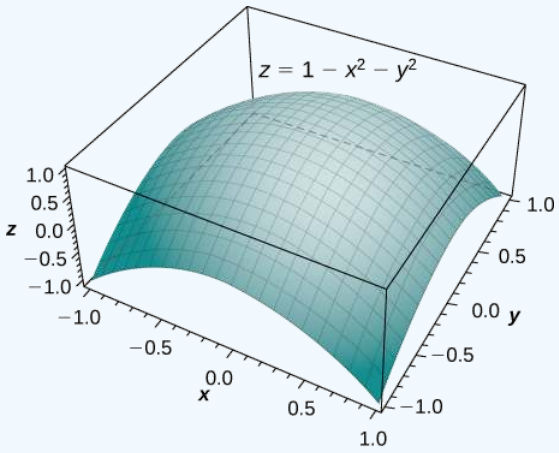

Figure 14.3b.7: Finding the volume of a solid under a paraboloid and above the unit circle.

Solution

By the method of double integration, we can see that the volume is the iterated integral of the form

$$\iint_R (1 - x^2 - y^2)\, dA$$

where $R = \{(r, \theta) \mid 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$.

This integration was shown before in Example **14.3b.2A**, so the volume is $\frac{\pi}{2}$ cubic units.

Example 14.3b.4B: Finding a Volume Using Double Integration

Find the volume of the solid that lies under the paraboloid $z = 4 - x^2 - y^2$ and above the disk $(x-1)^2 + y^2 = 1$ on the xy-plane. See the paraboloid in Figure **14.3b.8** intersecting the cylinder $(x-1)^2 + y^2 = 1$ above the xy-plane.

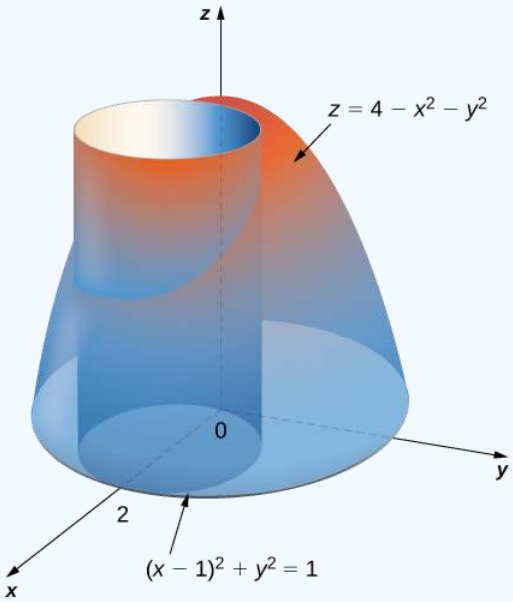

Figure **14.3b.8**: Finding the volume of a solid with a paraboloid cap and a circular base.

Solution

First change the disk $(x-1)^2 + y^2 = 1$ to polar coordinates. Expanding the square term, we have $x^2 - 2x + 1 + y^2 = 1$. Then simplify to get $x^2 + y^2 = 2x$, which in polar coordinates becomes $r^2 = 2r\cos\theta$ and then either $r = 0$ or $r = 2\cos\theta$. Similarly, the equation of the paraboloid changes to $z = 4 - r^2$. Therefore we can describe the disk $(x-1)^2 + y^2 = 1$ on the xy -plane as the region

$$D = \{(r, \theta) \mid 0 \le \theta \le \pi,\ 0 \le r \le 2\cos\theta\}.$$

Hence the volume of the solid bounded above by the paraboloid $z = 4 - x^2 - y^2$ and below by $r = 2\cos\theta$ is

$$\begin{aligned} V &= \iint_D f(r, \theta)\, r\, dr\, d\theta \\ &= \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=2\cos\theta} (4 - r^2)\, r\, dr\, d\theta \\ &= \int_{\theta=0}^{\theta=\pi} \left[4\frac{r^2}{2} - \frac{r^4}{4}\Big|_0^{2\cos\theta} \right] d\theta \\ &= \int_0^{\pi} [8\cos^2\theta - 4\cos^4\theta]\, d\theta \\ &= \left[\frac{5}{2}\theta + \frac{5}{2}\sin\theta\cos\theta - \sin\theta\cos^3\theta \right]_0^{\pi} = \frac{5}{2}\pi \text{ units}^3. \end{aligned}$$

Notice in the next example that integration is not always easy with polar coordinates. Complexity of integration depends on the function and also on the region over which we need to perform the integration. If the region has a more natural expression in polar coordinates or if f has a simpler antiderivative in polar coordinates, then the change in polar coordinates is appropriate; otherwise, use rectangular coordinates.

Example 14.3b.5A: Finding a Volume Using a Double Integral

Find the volume of the region that lies under the paraboloid $z = x^2 + y^2$ and above the triangle enclosed by the lines $y = x$, $x = 0$, and $x + y = 2$ in the xy-plane.

Solution

First examine the region over which we need to set up the double integral and the accompanying paraboloid.

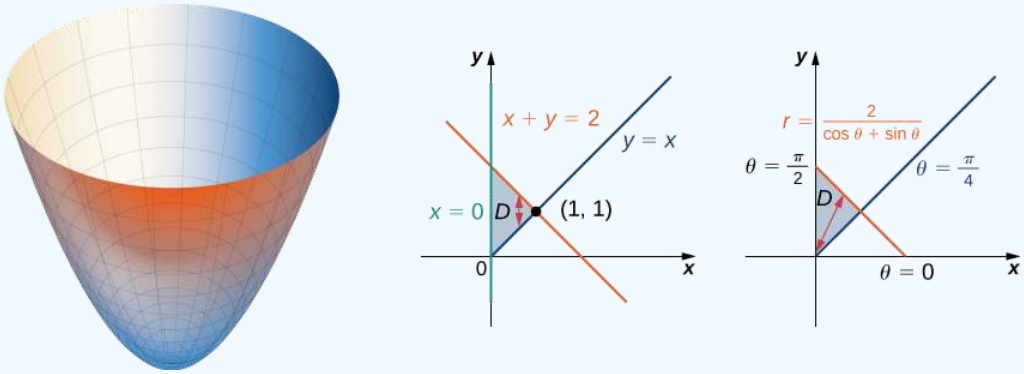

Figure 14.3b.9: Finding the volume of a solid under a paraboloid and above a given triangle.

The region D is $\{(x, y) \mid 0 \le x \le 1,\ x \le y \le 2 - x\}$. Converting the lines $y = x$, $x = 0$, and $x + y = 2$ in the xy-plane to functions of r and θ we have $\theta = \pi/4$, $\theta = \pi/2$, and $r = 2/(\cos\theta + \sin\theta)$, respectively. Graphing the region on the xy- plane, we see that it looks like $D = \{(r, \theta) \mid \pi/4 \le \theta \le \pi/2,\ 0 \le r \le 2/(\cos\theta + \sin\theta)\}$.

Now converting the equation of the surface gives $z = x^2 + y^2 = r^2$. Therefore, the volume of the solid is given by the double integral

$$\begin{aligned} V &= \iint_D f(r, \theta)\, r\, dr\, d\theta \\ &= \int_{\theta=\pi/4}^{\theta=\pi/2} \int_{r=0}^{r=2/(\cos\theta+\sin\theta)} r^2 r\, dr d\theta \\ &= \int_{\pi/4}^{\pi/2} \left[\frac{r^4}{4}\right]_0^{2/(\cos\theta+\sin\theta)} d\theta \\ &= \frac{1}{4}\int_{\pi/4}^{\pi/2} \left(\frac{2}{\cos\theta + \sin\theta}\right)^4 d\theta \\ &= \frac{16}{4}\int_{\pi/4}^{\pi/2} \left(\frac{1}{\cos\theta + \sin\theta}\right)^4 d\theta \\ &= 4\int_{\pi/4}^{\pi/2} \left(\frac{1}{\cos\theta + \sin\theta}\right)^4 d\theta. \end{aligned}$$

As you can see, this integral is very complicated. So, we can instead evaluate this double integral in rectangular coordinates as

$$V = \int_0^1 \int_x^{2-x} (x^2 + y^2)\, dy\, dx.$$

Evaluating gives

$$
\begin{aligned}
V &= \int_0^1 \int_x^{2-x} (x^2 + y^2)\, dy\, dx \\
&= \int_0^1 \left[x^2 y + \frac{y^3}{3} \right] \Bigg|_x^{2-x} dx \\
&= \int_0^1 \frac{8}{3} - 4x + 4x^2 - \frac{8x^3}{3}\, dx \\
&= \left[\frac{8x}{3} - 2x^2 + \frac{4x^3}{3} - \frac{2x^4}{3} \right] \Bigg|_0^1 \\
&= \frac{4}{3} \text{ units}^3.
\end{aligned}
$$

To answer the question of how the formulas for the volumes of different standard solids such as a sphere, a cone, or a cylinder are found, we want to demonstrate an example and find the volume of an arbitrary cone.

Example 14.3b.5B: Finding a Volume Using a Double Integral

Use polar coordinates to find the volume inside the cone $z = 2 - \sqrt{x^2 + y^2}$ and above the xy-plane.

Solution

The region D for the integration is the base of the cone, which appears to be a circle on the xy-plane (Figure 14.3b.10).

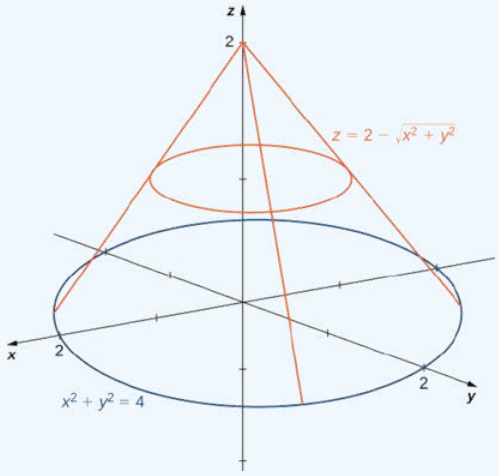

Figure 14.3b.10: Finding the volume of a solid inside the cone and above the xy-plane.

We find the equation of the circle by setting $z = 0$:

$$
\begin{aligned}
0 &= 2 - \sqrt{x^2 + y^2} \\
2 &= \sqrt{x^2 + y^2} \\
x^2 + y^2 &= 4.
\end{aligned}
$$

This means the radius of the circle is 2 so for the integration we have $0 \le \theta \le 2\pi$ and $0 \le r \le 2$. Substituting $x = r\cos\theta$ and $y = r\sin\theta$ in the equation $z = 2 - \sqrt{x^2 + y^2}$ we have $z = 2 - r$. Therefore, the volume of the cone is

$$
\int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} (2 - r)\, r\, dr\, d\theta = 2\pi \frac{4}{3} = \frac{8\pi}{3} \text{ cubic units.}
$$

Analysis

Note that if we were to find the volume of an arbitrary cone with radius a units and height h units, then the equation of the cone would be $z = h - \frac{h}{a}\sqrt{x^2 + y^2}$.

We can still use Figure 14.3b.10 and set up the integral as

$$
\int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=a} \left(h - \frac{h}{a} r \right) r\, dr\, d\theta.
$$

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Evaluating the integral, we get $\frac{1}{3}\pi a^2 h$.

Exercise 14.3b.5

Use polar coordinates to find an iterated integral for finding the volume of the solid enclosed by the paraboloids $z = x^2 + y^2$ and $z = 16 - x^2 - y^2$.

As with rectangular coordinates, we can also use polar coordinates to find areas of certain regions using a double integral. As before, we need to understand the region whose area we want to compute. Sketching a graph and identifying the region can be helpful to realize the limits of integration. Generally, the area formula in double integration will look like

$$\text{Area of } A = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} 1\, r\, dr\, d\theta.$$

Example 14.3b.6A: Finding an Area Using a Double Integral in Polar Coordinates

Evaluate the area bounded by the curve $r = \cos 4\theta$.

Solution

Sketching the graph of the function $r = \cos 4\theta$ reveals that it is a polar rose with eight petals (see the following figure).

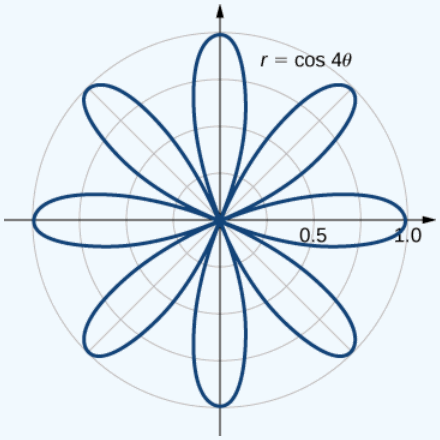

Figure 14.3b.11: Finding the area of a polar rose with eight petals.

Using symmetry, we can see that we need to find the area of one petal and then multiply it by 8. Notice that the values of θ for which the graph passes through the origin are the zeros of the function $\cos 4\theta$, and these are odd multiples of $\pi/8$. Thus, one of the petals corresponds to the values of θ in the interval $[-\pi/8, \pi/8]$. Therefore, the area bounded by the curve $r = \cos 4\theta$ is

$$\begin{aligned} A &= 8 \int_{\theta=-\pi/8}^{\theta=\pi/8} \int_{r=0}^{r=\cos 4\theta} 1\, r\, dr\, d\theta \\ &= 8 \int_{\theta=-\pi/8}^{\theta=\pi/8} \left[\frac{1}{2} r^2 \Big|_0^{\cos 4\theta} \right] d\theta \\ &= 8 \int_{-\pi/8}^{\pi/8} \frac{1}{2} \cos^2 4\theta\, d\theta \\ &= 8 \left[\frac{1}{4}\theta + \frac{1}{16} \sin 4\theta \cos 4\theta \Big|_{-\pi/8}^{\pi/8} \right] \\ &= 8 \left[\frac{\pi}{16} \right] = \frac{\pi}{2} \text{ units}^2. \end{aligned}$$

Example 14.3b.6B: Finding Area Between Two Polar Curves

Find the area enclosed by the circle $r = 3\cos\theta$ and the cardioid $r = 1 + \cos\theta$.

Solution

First and foremost, sketch the graphs of the region (Figure 14.3b.12).

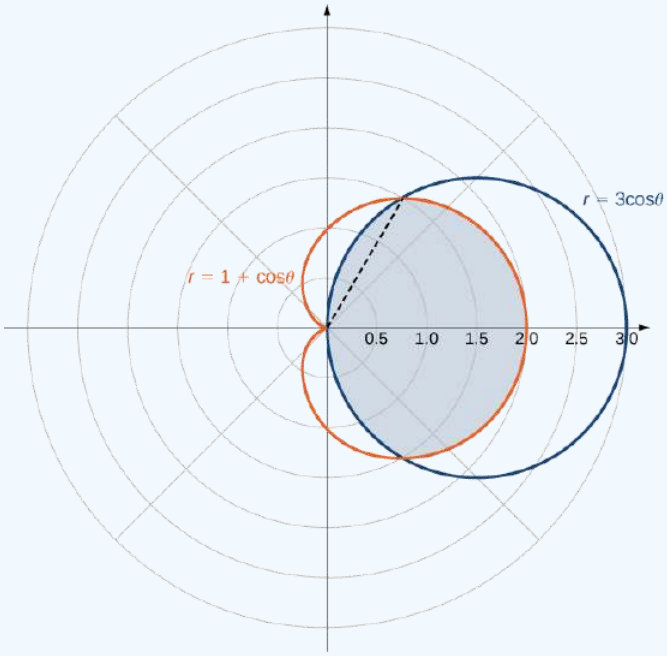

Figure 14.3b.12: Finding the area enclosed by both a circle and a cardioid.

We can from see the symmetry of the graph that we need to find the points of intersection. Setting the two equations equal to each other gives

$$3\cos\theta = 1 + \cos\theta.$$

One of the points of intersection is $\theta = \pi/3$. The area above the polar axis consists of two parts, with one part defined by the cardioid from $\theta = 0$ to $\theta = \pi/3$ and the other part defined by the circle from $\theta = \pi/3$ to $\theta = \pi/2$. By symmetry, the total area is twice the area above the polar axis. Thus, we have

$$A = 2\left[\int_{\theta=0}^{\theta=\pi/3}\int_{r=0}^{r=1+\cos\theta} 1\,r\,dr\,d\theta + \int_{\theta=\pi/3}^{\theta=\pi/2}\int_{r=0}^{r=3\cos\theta} 1\,r\,dr\,d\theta\right].$$

Evaluating each piece separately, we find that the area is

$$A = 2\left(\frac{1}{4}\pi + \frac{9}{16}\sqrt{3} + \frac{3}{8}\pi - \frac{9}{16}\sqrt{3}\right) = 2\left(\frac{5}{8}\pi\right) = \frac{5}{4}\pi \text{ square units.}$$

Exercise 14.3b.6

Find the area enclosed inside the cardioid $r = 3 - 3\sin\theta$ and outside the cardioid $r = 1 + \sin\theta$.

Example 14.3b.7: Evaluating an Improper Double Integral in Polar Coordinates

Evaluate the integral

$$\iint_{R^2} e^{-10(x^2+y^2)}\,dx\,dy.$$

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Solution

This is an improper integral because we are integrating over an unbounded region R^2. In polar coordinates, the entire plane R^2 can be seen as $0 \le \theta \le 2\pi$, $0 \le r \le \infty$.

Using the changes of variables from rectangular coordinates to polar coordinates, we have

$$\begin{aligned}\iint_{R^2} e^{-10(x^2+y^2)}\,dx\,dy &= \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=\infty} e^{-10r^2}\,r\,dr\,d\theta = \int_{\theta=0}^{\theta=2\pi}\left(\lim_{a\to\infty}\int_{r=0}^{r=a} e^{-10r^2}\,r\,dr\right)d\theta \\ &= \left(\int_{\theta=0}^{\theta=2\pi}\right)d\theta\left(\lim_{a\to\infty}\int_{r=0}^{r=a} e^{-10r^2}\,r\,dr\right) \\ &= 2\pi\left(\lim_{a\to\infty}\int_{r=0}^{r=a} e^{-10r^2}\,r\,dr\right) \\ &= 2\pi\lim_{a\to\infty}\left(-\frac{1}{20}\right)\left(e^{-10r^2}\Big|_0^a\right) \\ &= 2\pi\left(-\frac{1}{20}\right)\lim_{a\to\infty}\left(e^{-10a^2}-1\right) \\ &= \frac{\pi}{10}.\end{aligned}$$

? Exercise 14.3b.7

Evaluate the integral

$$\iint_{R^2} e^{-4(x^2+y^2)}dx\,dy.$$

Key Concepts

- To apply a double integral to a situation with circular symmetry, it is often convenient to use a double integral in polar coordinates. We can apply these double integrals over a polar rectangular region or a general polar region, using an iterated integral similar to those used with rectangular double integrals.
- The area dA in polar coordinates becomes $r\,dr\,d\theta$.
- Use $x = r\cos\theta$, $y = r\sin\theta$, and $dA = r\,dr\,d\theta$ to convert an integral in rectangular coordinates to an integral in polar coordinates.
- Use $r^2 = x^2 + y^2$ and $\theta = tan^{-1}\left(\frac{y}{x}\right)$ to convert an integral in polar coordinates to an integral in rectangular coordinates, if needed.
- To find the volume in polar coordinates bounded above by a surface $z = f(r,\theta)$ over a region on the xy-plane, use a double integral in polar coordinates.

Key Equations

- **Double integral over a polar rectangular region R**

$$\iint_R f(r,\theta)dA = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_{ij}^*,\theta_{ij}^*)\Delta A = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_{ij}^*,\theta_{ij}^*)r_{ij}^*\Delta r\Delta\theta$$

- **Double integral over a general polar region**

$$\iint_D f(r,\theta)\,r\,dr\,d\theta = \int_{\theta=\alpha}^{\theta=\beta}\int_{r=h_1(\theta)}^{r_2(\theta)} f(r,\theta)\,r\,dr\,d\theta$$

Glossary

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14.3E: Double Integrals in Polar Coordinates (Exercises)

Terms and Concepts

1. When evaluating $\int\int_R f(x,y)\,dA$ using polar coordinates, $f(x,y)$ is replaced with _____ and dA is replaced with _____.

2. Why would one be interested in evaluating a double integral with polar coordinates?

Defining Polar Regions

In exercises 3 - 6, express the region R in polar coordinates.

3) R is the region of the disk of radius 2 centered at the origin that lies in the first quadrant.

4) R is the region of the disk of radius 3 centered at the origin.

5) R is the region between the circles of radius 4 and radius 5 centered at the origin that lies in the second quadrant.

6) R is the region bounded by the y-axis and $x = \sqrt{1-y^2}$.

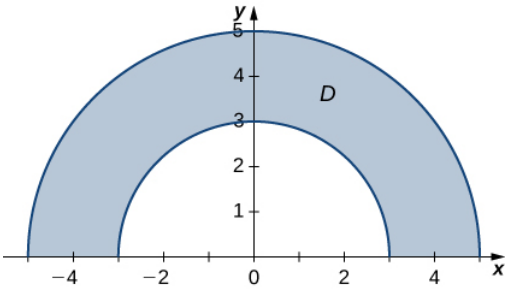
7) R is the region bounded by the x-axis and $y = \sqrt{2-x^2}$.

8) $R = \{(x,y) \mid x^2 + y^2 \le 4x\}$

9) $R = \{(x,y) \mid x^2 + y^2 \le 4y\}$

In exercises 10 - 15, the graph of the polar rectangular region D is given. Express D in polar coordinates. In exercises 10 - 13 the regions are bounded by circles centered at the origin, the coordinate axes, and/or lines of the form $y = \pm x$ (each of which should be able to be identified by inspection).

10)



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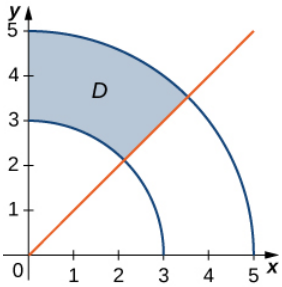

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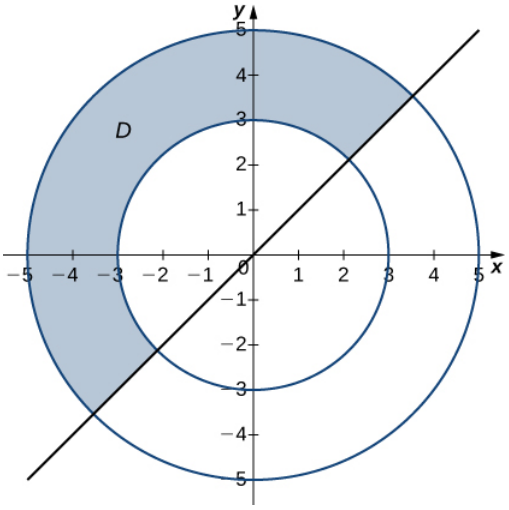

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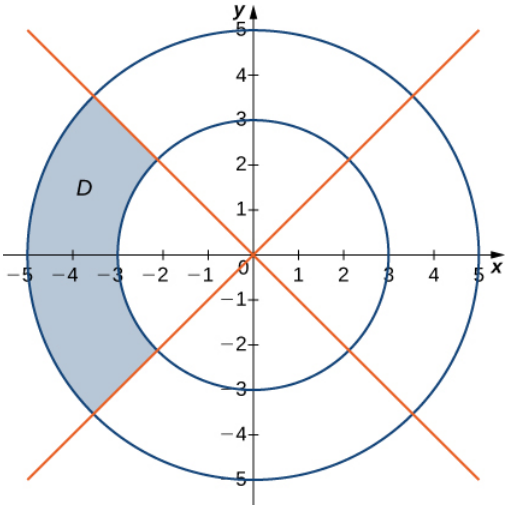
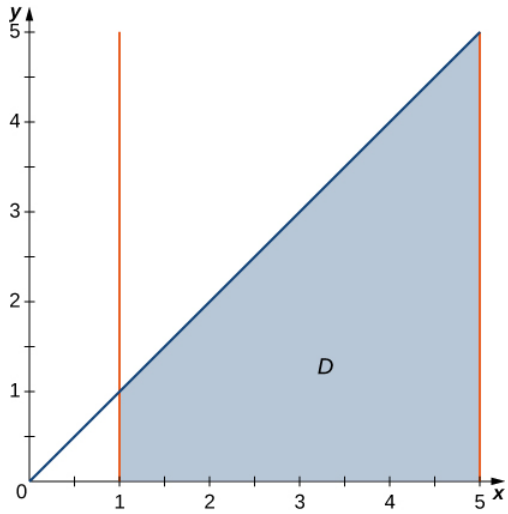


14) In the following graph, the region D is situated below $y = x$ and is bounded by $x = 1,\ x = 5$, and $y = 0$.



15) In the following graph, the region D is bounded by $y = x$ and $y = x^2$.

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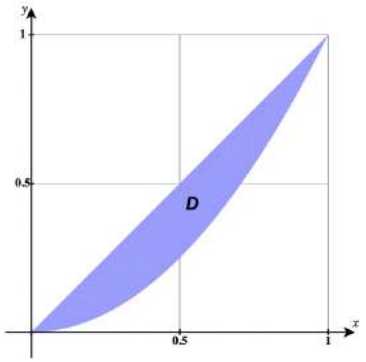


Evaluating Polar Double Integrals

In exercises 16 - 25, evaluate the double integral $\iint_R f(x,y)\,dA$ over the polar rectangular region R.

16) $f(x,y) = x^2 + y^2,\ R = \{(r,\theta) \mid 3 \le r \le 5,\ 0 \le \theta \le 2\pi\}$

17) $f(x,y) = x + y,\ R = \{(r,\theta) \mid 3 \le r \le 5,\ 0 \le \theta \le 2\pi\}$

18) $f(x,y) = x^2 + xy,\ R = \{(r,\theta) \mid 1 \le r \le 2,\ \pi \le \theta \le 2\pi\}$

19) $f(x,y) = x^4 + y^4,\ R = \{(r,\theta) \mid 1 \le r \le 2,\ \frac{3\pi}{2} \le \theta \le 2\pi\}$

20) $f(x,y) = \sqrt[3]{x^2 + y^2}$, where $R = \{(r,\theta) \mid 0 \le r \le 1,\ \frac{\pi}{2} \le \theta \le \pi\}$.

21) $f(x,y) = x^4 + 2x^2y^2 + y^4$, where $R = \{(r,\theta) \mid 3 \le r \le 4,\ \frac{\pi}{3} \le \theta \le \frac{2\pi}{3}\}$.

22) $f(x,y) = \sin(\arctan\frac{y}{x})$, where $R = \{(r,\theta) \mid 1 \le r \le 2,\ \frac{\pi}{6} \le \theta \le \frac{\pi}{3}\}$

23) $f(x,y) = \arctan\left(\frac{y}{x}\right)$, where $R = \{(r,\theta) \mid 2 \le r \le 3,\ \frac{\pi}{4} \le \theta \le \frac{\pi}{3}\}$

24) $\iint_R e^{x^2+y^2}\left[1 + 2\ \arctan\left(\frac{y}{x}\right)\right] dA,\ R = \{(r,\theta) \mid 1 \le r \le 2,\ \frac{\pi}{6} \le \theta \frac{\pi}{3}\}$

25) $\iint_R \left(e^{x^2+y^2} + x^4 + 2x^2y^2 + y^4\right) \arctan\left(\frac{y}{x}\right)\ dA,\ R = \{(r,\theta) \mid 1 \le r \le 2,\ \frac{\pi}{4} \le \theta \le \frac{\pi}{3}\}$

Converting Double Integrals to Polar Form

In exercises 26 - 29, the integrals have been converted to polar coordinates. Verify that the identities are true and choose the easiest way to evaluate the integrals, in rectangular or polar coordinates.

26) $\int_1^2 \int_0^x (x^2 + y^2)\,dy\,dx = \int_0^{\frac{\pi}{4}} \int_{\sec\theta}^{2\sec\theta} r^3\,dr\,d\theta$

27) $\int_2^3 \int_0^x \frac{x}{\sqrt{x^2+y^2}}\,dy\,dx = \int_0^{\pi/4} \int_{2\sec\theta}^{3\sec\theta} r\ \cos\theta\,dr\,d\theta$

28) $\int_0^1 \int_{x^2}^x \frac{1}{\sqrt{x^2+y^2}}\,dy\,dx = \int_0^{\pi/4} \int_0^{\tan\theta\ \sec\theta} dr\,d\theta$

29) $\int_0^1 \int_{x^2}^x \frac{y}{\sqrt{x^2+y^2}}\,dy\,dx = \int_0^{\pi/4} \int_0^{\tan\theta\ \sec\theta} r\ \sin\theta\,dr\,d\theta$

In exercises 30 - 37, draw the region of integration, R, labeling all limits of integration, convert the integrals to polar coordinates and evaluate them.

30) $\int_0^3 \int_0^{\sqrt{9-y^2}} (x^2 + y^2)\,dx\,dy$

31) $\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (x^2 + y^2)^2\,dx\,dy$

32) $\int_0^1 \int_0^{\sqrt{1-x^2}} (x + y)\,dy\,dx$

33) $\int_0^4 \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \sin(x^2+y^2)\, dy\, dx$

34) $\int_0^5 \int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} \sqrt{x^2+y^2}\, dy\, dx$

35) $\int_{-4}^4 \int_{-\sqrt{16-y^2}}^0 (2y-x)\, dx\, dy$

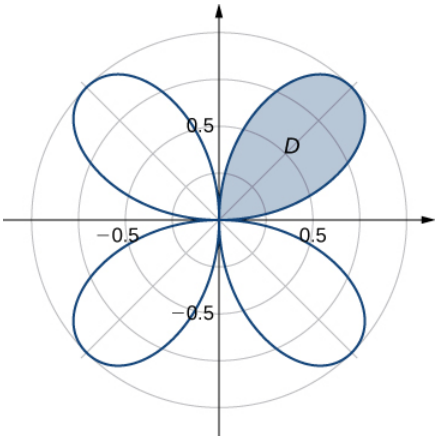
36) $\int_0^2 \int_y^{\sqrt{8-y^2}} (x+y)\, dx\, dy$

37) $\int_{-2}^{-1} \int_0^{\sqrt{4-x^2}} (x+5)\, dy\, dx + \int_{-1}^1 \int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} (x+5)\, dy\, dx + \int_1^2 \int_0^{\sqrt{4-x^2}} (x+5)\, dy\, dx$

38) Evaluate the integral $\iint_D r\, dA$ where D is the region bounded by the polar axis and the upper half of the cardioid $r = 1+\cos\theta$.

39) Find the area of the region D bounded by the polar axis and the upper half of the cardioid $r = 1+\cos\theta$.

40) Evaluate the integral $\iint_D r\, dA$, where D is the region bounded by the part of the four-leaved rose $r = \sin 2\theta$ situated in the first quadrant (see the following figure).

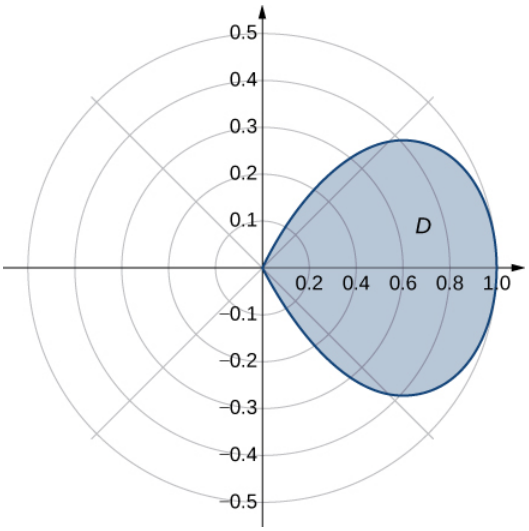


41) Find the total area of the region enclosed by the four-leaved rose $r = \sin 2\theta$ (see the figure in the previous exercise).

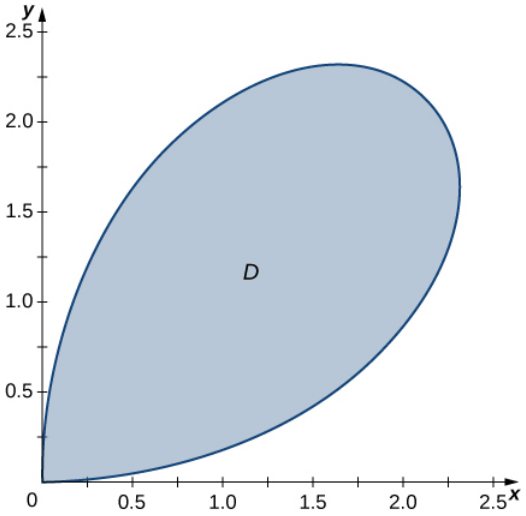
42) Find the area of the region D which is the region bounded by $y = \sqrt{4-x^2}$, $x = \sqrt{3}$, $x = 2$, and $y = 0$.

43) Find the area of the region D, which is the region inside the disk $x^2+y^2 \le 4$ and to the right of the line $x = 1$.

44) Determine the average value of the function $f(x,y) = x^2+y^2$ over the region D bounded by the polar curve $r = \cos 2\theta$, where $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$ (see the following graph).



45) Determine the average value of the function $f(x,y) = \sqrt{x^2+y^2}$ over the region D bounded by the polar curve $r = 3\sin 2\theta$, where $0 \le \theta \le \frac{\pi}{2}$ (see the following graph).



46) Find the volume of the solid situated in the first octant and bounded by the paraboloid $z = 1 - 4x^2 - 4y^2$ and the planes $x = 0$, $y = 0$, and $z = 0$.

47) Find the volume of the solid bounded by the paraboloid $z = 2 - 9x^2 - 9y^2$ and the plane $z = 1$.

48)

1. Find the volume of the solid S_1 bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = 0$ and $z = 1$.
2. Find the volume of the solid S_2 outside the double cone $z^2 = x^2 + y^2$ inside the cylinder $x^2 + y^2 = 1$, and above the plane $z = 0$.
3. Find the volume of the solid inside the cone $z^2 = x^2 + y^2$ and below the plane $z = 1$ by subtracting the volumes of the solids S_1 and S_2.

49)

1. Find the volume of the solid S_1 inside the unit sphere $x^2 + y^2 + z^2 = 1$ and above the plane $z = 0$.
2. Find the volume of the solid S_2 inside the double cone $(z-1)^2 = x^2 + y^2$ and above the plane $z = 0$.
3. Find the volume of the solid outside the double cone $(z-1)^2 = x^2 + y^2$ and inside the sphere $x^2 + y^2 + z^2 = 1$.

In Exercises 50-51, special double integrals are presented that are especially well suited for evaluation in polar coordinates.

50) The surface of a right circular cone with height h and base radius a can be described by the equation $f(x,y) = h - h\sqrt{\frac{x^2}{a^2} + \frac{y^2}{a^2}}$, where the tip of the cone lies at $(0,0,h)$ and the circular base lies in the xy-plane, centered at the origin.

Confirm that the volume of a right circular cone with height h and base radius a is $V = \frac{1}{3}\pi a^2 h$ by evaluating $\int\int_R f(x,y)\,dA$ in polar coordinates.

51) Consider $\int\int_R e^{-(x^2+y^2)}\,dA$.

(a) Why is this integral difficult to evaluate in rectangular coordinates, regardless of the region R?

(b) Let R be the region bounded by the circle of radius a centered at the origin. Evaluate the double integral using polar coordinates.

(c) Take the limit of your answer from (b), as $a \to \infty$. What does this imply about the volume under the surface of $e^{-(x^2+y^2)}$ over the entire xy-plane?

For the following two exercises, consider a spherical ring, which is a sphere with a cylindrical hole cut so that the axis of the cylinder passes through the center of the sphere (see the following figure).

52) If the sphere has radius 4 and the cylinder has radius 2 find the volume of the spherical ring.

53) A cylindrical hole of diameter 6 cm is bored through a sphere of radius 5 cm such that the axis of the cylinder passes through the center of the sphere. Find the volume of the resulting spherical ring.

54) Find the volume of the solid that lies under the double cone $z^2 = 4x^2 + 4y^2$, inside the cylinder $x^2 + y^2 = x$, and above the plane $z = 0$.

55) Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, inside the cylinder $x^2 + y^2 = 1$ and above the plane $z = 0$.

56) Find the volume of the solid that lies under the plane $x + y + z = 10$ and above the disk $x^2 + y^2 = 4x$.

57) Find the volume of the solid that lies under the plane $2x + y + 2z = 8$ and above the unit disk $x^2 + y^2 = 1$.

58) A radial function f is a function whose value at each point depends only on the distance between that point and the origin of the system of coordinates; that is, $f(x,y) = g(r)$, where $r = \sqrt{x^2 + y^2}$. Show that if f is a continuous radial function, then

$$\iint_D f(x,y)dA = (\theta_2 - \theta_1)[G(R_2) - G(R_1)],\ \textit{where}\ G'(r) = rg(r)$$

and $(x,y) \in D = \{(r,\theta)\,|\,R_1 \le r \le R_2,\ 0 \le \theta \le 2\pi\}$, with $0 \le R_1 < R_2$ and $0 \le \theta_1 < \theta_2 \le 2\pi$.

59) Use the information from the preceding exercise to calculate the integral $\iint_D (x^2 + y^2)^3 dA$, where D is the unit disk.

60) Let $f(x,y) = \dfrac{F'(r)}{r}$ be a continuous radial function defined on the annular region $D = \{(r,\theta)\,|\,R_1 \le r \le R_2,\ 0 \le \theta \le 2\pi\}$, where $r = \sqrt{x^2 + y^2}$, $0 < R_1 < R_2$, and F is a differentiable function.

Show that $\iint_D f(x,y)\,dA = 2\pi[F(R_2) - F(R_1)]$.

61) Apply the preceding exercise to calculate the integral $\iint_D \dfrac{e^{\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}}\,dx\,dy$ where D is the annular region between the circles of radii 1 and 2 situated in the third quadrant.

62) Let f be a continuous function that can be expressed in polar coordinates as a function of θ only; that is, $f(x,y) = h(\theta)$, where $(x,y) \in D = \{(r,\theta) \mid R_1 \le r \le R_2,\ \theta_1 \le \theta \le \theta_2\}$, with $0 \le R_1 < R_2$ and $0 \le \theta_1 < \theta_2 \le 2\pi$.

Show that $\iint_D f(x,y)\,dA = \frac{1}{2}(R_2^2 - R_1^2)[H(\theta_2) - H(\theta_1)]$, where H is an antiderivative of h.

63) Apply the preceding exercise to calculate the integral $\iint_D \frac{y^2}{x^2}\,dA$, where $D = \{(r,\theta) \mid 1 \le r \le 2,\ \frac{\pi}{6} \le \theta \le \frac{\pi}{3}\}$.

64) Let f be a continuous function that can be expressed in polar coordinates as a function of θ only; that is $f(x,y) = g(r)h(\theta)$, where $(x,y) \in \{(r,\theta) \mid R_1 \le r \le R_2,\ \theta_1 \le \theta \le \theta_2\}$ with $0 \le R_1 < R_2$ and $0 \le \theta_1 < \theta_2 \le 2\pi$. Show that

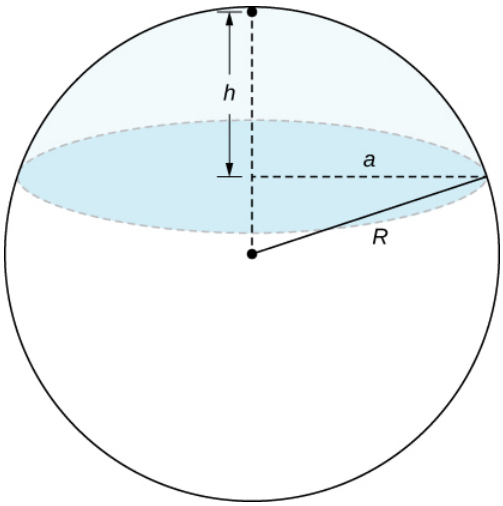
$$\iint_D f(x,y)\,dA = [G(R_2) - G(R_1)]\ [H(\theta_2) - H(\theta_1)],$$

where G and H are antiderivatives of g and h, respectively.

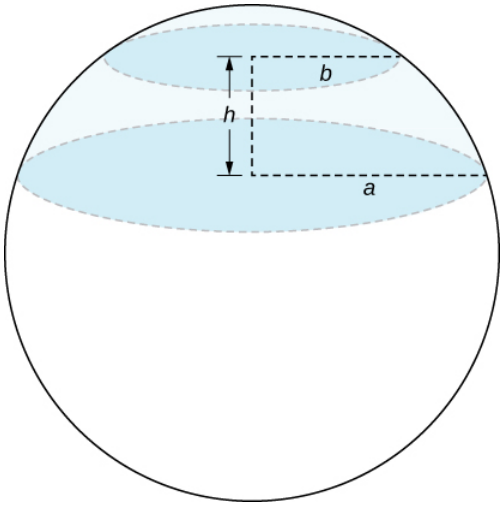
65) Evaluate $\iint_D \arctan\left(\frac{y}{x}\right)\sqrt{x^2+y^2}\,dA$, where $D = \{(r,\theta) \mid 2 \le r \le 3,\ \frac{\pi}{4} \le \theta \le \frac{\pi}{3}\}$.

66) A spherical cap is the region of a sphere that lies above or below a given plane.

a. Show that the volume of the spherical cap in the figure below is $\frac{1}{6}\pi h(3a^2 + h^2)$.



b. A spherical segment is the solid defined by intersecting a sphere with two parallel planes. If the distance between the planes is h show that the volume of the spherical segment in the figure below is $\frac{1}{6}\pi h(3a^2 + 3b^2 + h^2)$.



67) In statistics, the joint density for two independent, normally distributed events with a mean $\mu = 0$ and a standard distribution σ is defined by $p(x,y) = \frac{1}{2\pi\sigma^2}e^{-\frac{x^2+y^2}{2\sigma^2}}$. Consider (X,Y), the Cartesian coordinates of a ball in the resting position after it was released

from a position on the z-axis toward the xy-plane. Assume that the coordinates of the ball are independently normally distributed with a mean $\mu = 0$ and a standard deviation of σ (in feet). The probability that the ball will stop no more than a feet from the origin is given by

$$P[X^2 + Y^2 \leq a^2] = \iint_D p(x,y)dy\ dx,$$

where D is the disk of radius a centered at the origin. Show that

$$P[X^2 + Y^2 \leq a^2] = 1 - e^{-a^2/2\sigma^2}.$$

68) The double improper integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2+y^2/2}\,dy\,dx$$

may be defined as the limit value of the double integrals $\iint_D e^{-x^2+y^2/2}\,dA$ over disks D_a of radii a centered at the origin, as a increases without bound; that is,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2+y^2/2}dy\ dx = \lim_{a\to\infty}\iint_{D_a} e^{-x^2+y^2/2}\,dA.$$

Use polar coordinates to show that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2+y^2/2}\,dy\,dx = 2\pi.$

69) Show that $\int_{-\infty}^{\infty} e^{-x^2/2}\,dx = \sqrt{2\pi}$ by using the relation

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2+y^2/2}\,dy\,dx = \left(\int_{-\infty}^{\infty} e^{-x^2/2}dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2/2}dy\right).$$

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- Problems 1, 2, 34 - 37 and 50 - 51 are from Apex Calculus, Chapter 13.3
- Edited by Paul Seeburger (Monroe Community College)

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14.4: Triple Integrals

Learning Objectives

- Recognize when a function of three variables is integrable over a rectangular box.
- Evaluate a triple integral by expressing it as an iterated integral.
- Recognize when a function of three variables is integrable over a closed and bounded region.
- Simplify a calculation by changing the order of integration of a triple integral.
- Calculate the average value of a function of three variables.

Previously, we discussed the double integral of a function $f(x,y)$ of two variables over a rectangular region in the plane. In this section we define the triple integral of a function $f(x,y,z)$ of three variables over a rectangular solid box in space, $\mathbb{R}^3$. Later in this section we extend the definition to more general regions in $\mathbb{R}^3$.

Integrable Functions of Three Variables

We can define a rectangular box B in $\mathbb{R}^3$ as

$$B = \{(x,y,z) \mid a \le x \le b,\ c \le y \le d,\ e \le z \le f\}.$$

We follow a similar procedure to what we did in previously. We divide the interval $[a,b]$ into l subintervals $[x_{i-1}, x_i]$ of equal length Δx with

$$\Delta x = \frac{x_i - x_{i-1}}{l},$$

divide the interval $[c,d]$ into m subintervals $[y_{i-1}, y_i]$ of equal length Δy with

$$\Delta y = \frac{y_j - y_{j-1}}{m},$$

and divide the interval $[e,f]$ into n subintervals $[z_{i-1}, z_i]$ of equal length Δz with

$$\Delta z = \frac{z_k - z_{k-1}}{n}$$

Then the rectangular box B is subdivided into lmn subboxes:

$$B_{ijk} = [x_{i-1}, x_i] \times [y_{i-1}, y_i] \times [z_{i-1}, z_i],$$

as shown in Figure 14.4.1.

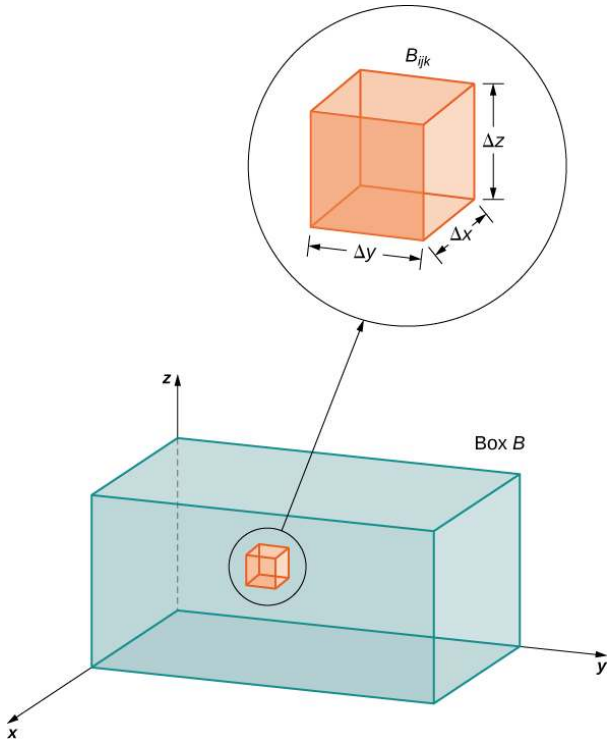

Figure 14.4.1: A rectangular box in $\mathbb{R}^3$ divided into subboxes by planes parallel to the coordinate planes.

For each i, j, and k, consider a sample point $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$ in each sub-box B_{ijk}. We see that its volume is $\Delta V = \Delta x \Delta y \Delta z$. Form the triple Riemann sum

$$\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\,\Delta x \Delta y \Delta z.$$

We define the triple integral in terms of the limit of a triple Riemann sum, as we did for the double integral in terms of a double Riemann sum.

Definition: The triple integral

The triple integral of a function $f(x, y, z)$ over a rectangular box B is defined as

$$\lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\,\Delta x \Delta y \Delta z = \iiint_B f(x, y, z)\,dV$$

if this limit exists.

When the triple integral exists on B the function $f(x, y, z)$ is said to be integrable on B. Also, the triple integral exists if $f(x, y, z)$ is continuous on B. Therefore, we will use continuous functions for our examples. However, continuity is sufficient but not necessary; in other words, f is bounded on B and continuous except possibly on the boundary of B. The sample point $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$ can be any point in the rectangular sub-box B_{ijk} and all the properties of a double integral apply to a triple integral. Just as the double integral has many practical applications, the triple integral also has many applications, which we discuss in later sections.

Now that we have developed the concept of the triple integral, we need to know how to compute it. Just as in the case of the double integral, we can have an iterated triple integral, and consequently, a version of **Fubini's theorem** for triple integrals exists.

Fubini's Theorem for Triple Integrals

If $f(x, y, z)$ is continuous on a rectangular box $B = [a, b] \times [c, d] \times [e, f]$, then

$$\iiint_B f(x,y,z)\,dV = \int_e^f \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz.$$

This integral is also equal to any of the other five possible orderings for the iterated triple integral.

For a, b, c, d, e and f real numbers, the iterated triple integral can be expressed in six different orderings:

$$\int_e^f \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz = \int_e^f \left(\int_c^d \left(\int_a^b f(x,y,z)\,dx \right) dy \right) dz \tag{14.4.1}$$

$$= \int_c^d \left(\int_e^f \left(\int_a^b f(x,y,z)\,dx \right) dz \right) dy \tag{14.4.2}$$

$$= \int_a^b \left(\int_e^f \left(\int_c^d f(x,y,z)\,dy \right) dz \right) dx \tag{14.4.3}$$

$$= \int_e^f \left(\int_a^b \left(\int_c^d f(x,y,z)\,dy \right) dx \right) dz \tag{14.4.4}$$

$$= \int_c^d \left(\int_a^b \left(\int_c^d f(x,y,z)\,dz \right) dx \right) dy \tag{14.4.5}$$

$$= \int_a^b \left(\int_c^d \left(\int_e^f f(x,y,z)\,dz \right) dy \right) dx \tag{14.4.6}$$

For a rectangular box, the order of integration does not make any significant difference in the level of difficulty in computation. We compute triple integrals using Fubini's Theorem rather than using the Riemann sum definition. We follow the order of integration in the same way as we did for double integrals (that is, from inside to outside).

Example 14.4.1: Evaluating a Triple Integral

Evaluate the triple integral

$$\int_{z=0}^{z=1} \int_{y=2}^{y=4} \int_{x=-1}^{x=5} (x + yz^2)\,dx\,dy\,dz.$$

Solution

The order of integration is specified in the problem, so integrate with respect to x first, then y, and then z.

$$\int_{z=0}^{z=1} \int_{y=2}^{y=4} \int_{x=-1}^{x=5} (x + yz^2)\,dx\,dy\,dz$$

$$= \int_{z=0}^{z=1} \int_{y=2}^{y=4} \left[\frac{x^2}{2} + xyz^2 \Big|_{x=-1}^{x=5} \right] dy\,dz \qquad \text{Integrate with respect to } x.$$

$$= \int_{z=0}^{z=1} \int_{y=2}^{y=4} \left[12 + 6yz^2 \right] dy\,dz \qquad \text{Evaluate.}$$

$$= \int_{z=0}^{z=1} \left[12y + 6\frac{y^2}{2} z^2 \Big|_{y=2}^{y=4} \right] dz \qquad \text{Integrate with respect to } y.$$

$$= \int_{z=0}^{z=1} \left[24 + 36z^2 \right] dz \qquad \text{Evaluate.}$$

$$= \left[24z + 36\frac{z^3}{3} \right]_{z=0}^{z=1} \qquad \text{Integrate with respect to } z.$$

$$= 36. \qquad \text{Evaluate.}$$

Example 14.4.2: Evaluating a Triple Integral

Evaluate the triple integral

$$\iiint_B x^2 yz\,dV$$

where $B = \{(x,y,z) \mid -2 \le x \le 1,\ 0 \le y \le 3,\ 1 \le z \le 5\}$ as shown in Figure 14.4.2.

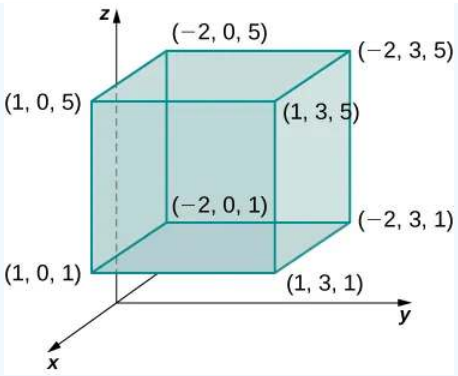

Figure 14.4.2: Evaluating a triple integral over a given rectangular box.

Solution

The order is not specified, but we can use the iterated integral in any order without changing the level of difficulty. Choose, say, to integrate y first, then x, and then z.

$$\begin{aligned}\iiint_B x^2yz\,dV &= \int_1^5\int_{-2}^1\int_0^3 [x^2yz]\,dy\,dx\,dz\\ &= \int_1^5\int_{-2}^1 \left[x^2\frac{y^2}{2}z\Big|_0^3\right]dx\,dz\\ &= \int_1^5\int_{-2}^1 \frac{9}{2}x^2z\,dx\,dz\\ &= \int_1^5 \left[\frac{9}{2}\frac{x^3}{3}z\Big|_{-2}^1\right]dz\\ &= \int_1^5 \frac{27}{2}z\,dz\\ &= \frac{27}{2}\frac{z^2}{2}\Big|_1^5 = \frac{27}{4}(25-1) = 162.\end{aligned}$$

Now try to integrate in a different order just to see that we get the same answer. Choose to integrate with respect to x first, then z, then y

$$\begin{aligned}\iiint_B x^2yz\,dV &= \int_0^3\int_1^5\int_{-2}^1 [x^2yz]\,dx\,dz\,dy\\ &= \int_0^3\int_1^5 \left[\frac{x^3}{3}yz\Big|_{-2}^1\right]dz\,dy\\ &= \int_0^3\int_1^5 3yz\,dz\,dy\\ &= \int_0^3 \left[3y\frac{z^2}{2}\Big|_1^5\right]dy\\ &= \int_0^3 36y\,dy\\ &= 36\frac{y^2}{2}\Big|_0^3 = 18(9-0) = 162.\end{aligned}$$

? Exercise 14.4.1

Evaluate the triple integral

$$\iiint_B z\sin x\cos y\,dV$$

where $B = \{(x,y,z)\,|\,0\le x\le\pi,\ \frac{3\pi}{2}\le y\le 2\pi,\ 1\le z\le 3\}$.

Triple Integrals over a General Bounded Region

We now expand the definition of the triple integral to compute a triple integral over a more **general bounded region** E in $\mathbb{R}^3$. The general bounded regions we will consider are of three types. First, let D be the bounded region that is a projection of E onto the xy-plane. Suppose the region E in $\mathbb{R}^3$ has the form

$$E = \{(x, y, z) \,|\, (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\}.$$

For two functions $z = u_1(x, y)$ and $u_2(x, y)$, such that $u_1(x, y) \leq u_2(x, y)$ for all (x, y) in D as shown in the following figure.

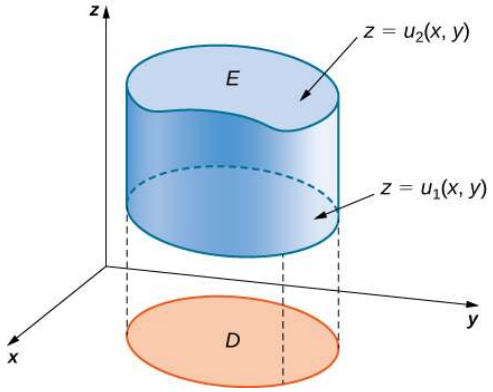

Figure **14.4.3**: We can describe region E as the space between $u_1(x, y)$ and $u_2(x, y)$ above the projection D of E onto the xy-plane.

Triple Integral over a General Region

The triple integral of a continuous function $f(x, y, z)$ over a general three-dimensional region

$$E = \{(x, y, z) \,|\, (x, y) \in D, \, u_1(x, y) \leq z \leq u_2(x, y)\}$$

in $\mathbb{R}^3$, where D is the projection of E onto the xy-plane, is

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz \right] dA.$$

Similarly, we can consider a general bounded region D in the xy-plane and two functions $y = u_1(x, z)$ and $y = u_2(x, z)$ such that $u_1(x, z) \leq u_2(x, z)$ for all (x, z) in D. Then we can describe the solid region E in $\mathbb{R}^3$ as

$$E = \{(x, y, z) \,|\, (x, z) \in D, \, u_1(x, z) \leq z \leq u_2(x, z)\}$$

where D is the projection of E onto the xy-plane and the triple integral is

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x, y, z)\, dy \right] dA.$$

Finally, if D is a general bounded region in the xy-plane and we have two functions $x = u_1(y, z)$ and $x = u_2(y, z)$ such that $u_1(y, z) \leq u_2(y, z)$ for all (y, z) in D, then the solid region E in $\mathbb{R}^3$ can be described as

$$E = \{(x, y, z) \,|\, (y, z) \in D, \, u_1(y, z) \leq z \leq u_2(y, z)\}$$

where D is the projection of E onto the xy-plane and the triple integral is

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x, y, z)\, dx \right] dA.$$

Note that the region D in any of the planes may be of Type I or Type II as described in previously. If D in the xy-plane is of Type I (Figure **14.4.4**), then

$$E = \{(x, y, z) \,|\, a \leq x \leq b, \, g_1(x) \leq y \leq g_2(x), \, u_1(x, y) \leq z \leq u_2(x, y)\}.$$

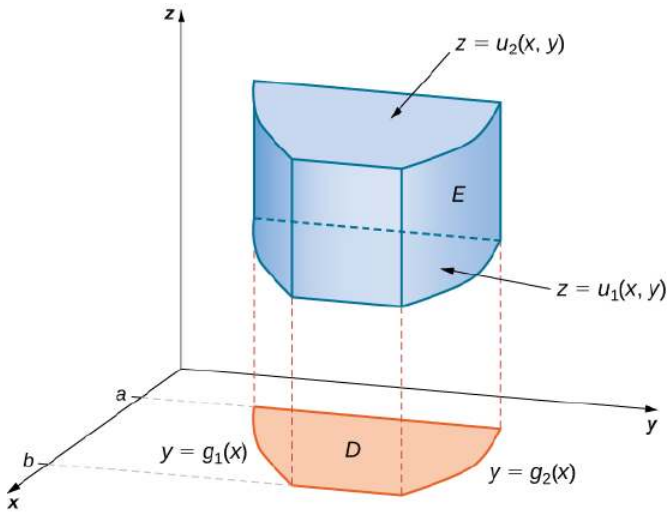

Figure 14.4.4: A box E where the projection D in the xy-plane is of Type I.

Then the triple integral becomes

$$\iiint_E f(x,y,z)\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx.$$

If D in the xy-plane is of Type II (Figure 14.4.5), then

$$E = \{(x,y,z)\,|\,c \le x \le d, h_1(x) \le y \le h_2(x),\, u_1(x,y) \le z \le u_2(x,y)\}.$$

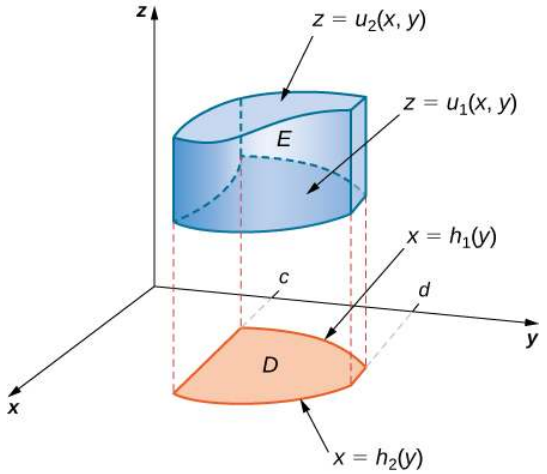

Figure 14.4.5: A box E where the projection D in the xy-plane is of Type II.

Then the triple integral becomes

$$\iiint_E f(x,y,z)\,dV = \int_{y=c}^{y=d} \int_{x=h_1(y)}^{x=h_2(y)} \int_{z=u_1(x,y)}^{z=u_2(x,y)} f(x,y,z)\,dz\,dx\,dy.$$

✔ Example 14.4.3.A: Evaluating a Triple Integral over a General Bounded Region

Evaluate the triple integral of the function $f(x,y,z) = 5x - 3y$ over the solid tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$.

Solution

Figure 14.4.6 shows the solid tetrahedron E and its projection D on the xy-plane.

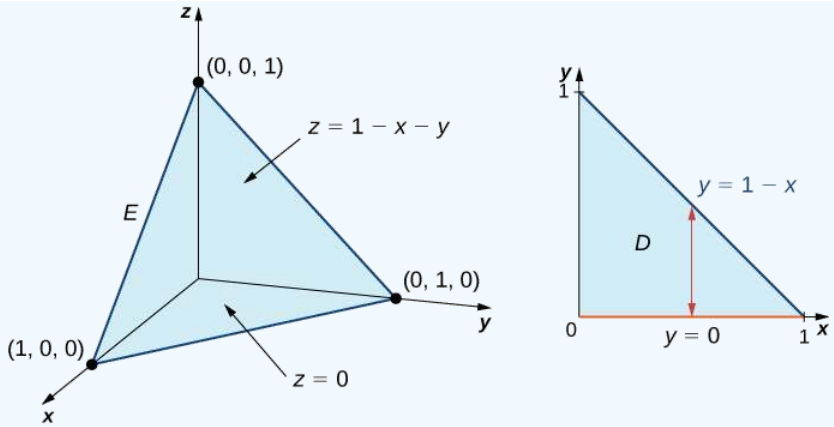

Figure **14.4.6**: The solid E has a projection D on the xy-plane of Type I.

We can describe the solid region tetrahedron as

$$E = \{(x, y, z) \,|\, 0 \le x \le 1,\ 0 \le y \le 1 - x,\ 0 \le z \le 1 - x - y\}.$$

Hence, the triple integral is

$$\iiint_E f(x, y, z)\, dV = \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} \int_{z=0}^{z=1-x-y} (5x - 3y)\, dz\, dy\, dx.$$

To simplify the calculation, first evaluate the integral $\int_{z=0}^{z=1-x-y} (5x - 3y)\, dz$. We have

$$\int_{z=0}^{z=1-x-y} (5x - 3y)\, dz = (5x - 3y)z \Big|_{z=0}^{z=1-x-y} = (5x - 3y)(1 - x - y).$$

Now evaluate the integral

$$\int_{y=0}^{y=1-x} (5x - 3y)(1 - x - y)\, dy,$$

obtaining

$$\int_{y=0}^{y=1-x} (5x - 3y)(1 - x - y)\, dy = \frac{1}{2}(x - 1)^2(6x - 1).$$

Finally evaluate

$$\int_{x=0}^{x=1} \frac{1}{2}(x - 1)^2(6x - 1)\, dx = \frac{1}{12}.$$

Putting it all together, we have

$$\iiint_E f(x, y, z)\, dV = \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} \int_{z=0}^{z=1-x-y} (5x - 3y)\, dz\, dy\, dx = \frac{1}{12}.$$

Just as we used the double integral

$$\iint_D 1\, dA$$

to find the area of a general bounded region D we can use

$$\iiint_E 1\, dV$$

to find the volume of a general solid bounded region E. The next example illustrates the method.

Example 14.4.3B: Finding a Volume by Evaluating a Triple Integral

Find the volume of a right pyramid that has the square base in the xy-plane $[-1,1] \times [-1,1]$ and vertex at the point $(0,0,1)$ as shown in the following figure.

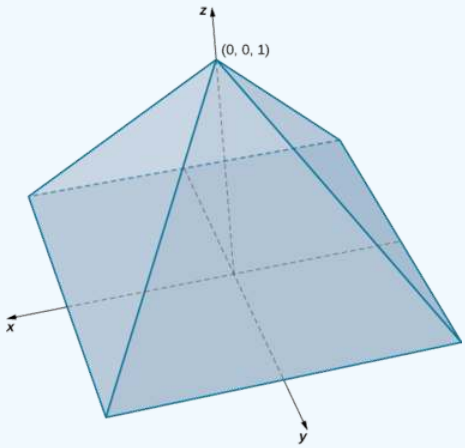

Figure 14.4.7: Finding the volume of a pyramid with a square base.

Solution

In this pyramid the value of z changes from 0 to 1 and at each height z the cross section of the pyramid for any value of z is the square

$$[-1+z,\, 1-z] \times [-1+z,\, 1-z].$$

Hence, the volume of the pyramid is

$$\iiint_E 1\, dV$$

where

$$E = \{(x,y,z) \,|\, 0 \le z \le 1,\, -1+z \le y \le 1-z,\, -1+z \le x \le 1-z\}.$$

Thus, we have

$$\begin{aligned} \iiint_E 1\, dV &= \int_{z=0}^{z=1} \int_{y=-1+z}^{y=1-z} \int_{x=-1+z}^{x=1-z} 1\, dx\, dy\, dz \\ &= \int_{z=0}^{z=1} \int_{y=-1+z}^{y=1-z} (2-2z)\, dy\, dz \\ &= \int_{z=0}^{z=1} (2-2z)^2\, dz = \frac{4}{3}. \end{aligned}$$

Hence, the volume of the pyramid is $\frac{4}{3}$ cubic units.

Exercise 14.4.3

Consider the solid sphere $E = \{(x,y,z) \,|\, x^2+y^2+z^2 = 9\}$. Write the triple integral

$$\iiint_E f(x,y,z)\, dV$$

for an arbitrary function f as an iterated integral. Then evaluate this triple integral with $f(x,y,z) = 1$. Notice that this gives the volume of a sphere using a triple integral.

Changing the Order of Integration

As we have already seen in double integrals over general bounded regions, changing the order of the integration is done quite often to simplify the computation. With a triple integral over a rectangular box, the order of integration does not change the level of difficulty of the calculation. However, with a triple integral over a general bounded region, choosing an appropriate order of integration can simplify the computation quite a bit. Sometimes making the change to polar coordinates can also be very helpful. We demonstrate two examples here.

Example 14.4.4: Changing the Order of Integration

Consider the iterated integral

$$\int_{x=0}^{x=1}\int_{y=0}^{y=x^2}\int_{z=0}^{z=y} f(x,y,z)\,dz\,dy\,dx.$$

The order of integration here is first with respect to z, then y, and then x. Express this integral by changing the order of integration to be first with respect to x, then z, and then y. Verify that the value of the integral is the same if we let $f(x,y,z) = xyz$.

Solution

The best way to do this is to sketch the region E and its projections onto each of the three coordinate planes. Thus, let

$$E = \{(x,y,z)\,|\,0 \le x \le 1,\ 0 \le y \le x^2,\ 0 \le z \le y\}.$$

and

$$\int_{x=0}^{x=1}\int_{y=0}^{y=x^2}\int_{z=0}^{z=x^2} f(x,y,z)\,dz\,dy\,dx = \iiint_E f(x,y,z)\,dV.$$

We need to express this triple integral as

$$\int_{y=c}^{y=d}\int_{z=v_1(y)}^{z=v_2(y)}\int_{x=u_1(y,z)}^{x=u_2(y,z)} f(x,y,z)\,dx\,dz\,dy.$$

Knowing the region E we can draw the following projections (Figure 14.4.8):

on the xy-plane is $D_1 = \{(x,y)\,|\,0 \le x \le 1,\ 0 \le y \le x^2\} = \{(x,y)\,|\,0 \le y \le 1,\ \sqrt{y} \le x \le 1\}$,

on the yz-plane is $D_2 = \{(y,z)\,|\,0 \le y \le 1,\ 0 \le z \le y^2\}$, and

on the xz-plane is $D_3 = \{(x,z)\,|\,0 \le x \le 1,\ 0 \le z \le x^2\}$.

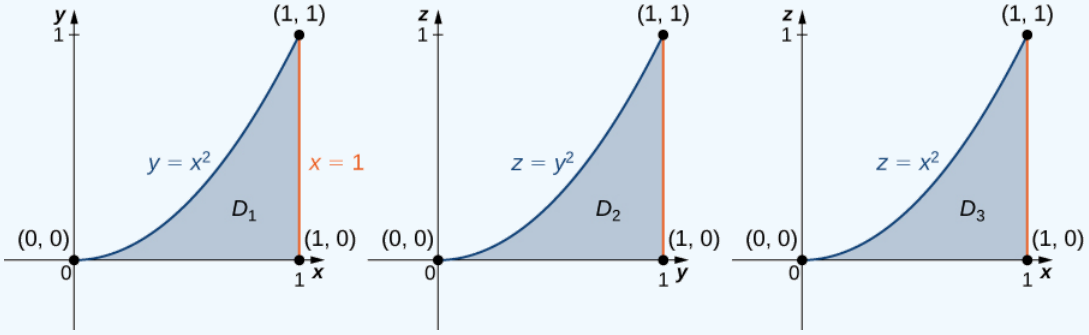

Figure 14.4.8. The three cross sections of E on the three coordinate planes.

Now we can describe the same region E as $\{(x,y,z)\,|\,0 \le y \le 1,\ 0 \le z \le y^2,\ \sqrt{y} \le x \le 1\}$, and consequently, the triple integral becomes

$$\int_{y=c}^{y=d}\int_{z=v_1(y)}^{z=v_2(y)}\int_{x=u_1(y,z)}^{x=u_2(y,z)} f(x,y,z)\,dx\,dz\,dy = \int_{y=0}^{y=1}\int_{z=0}^{z=x^2}\int_{x=\sqrt{y}}^{x=1} f(x,y,z)\,dx\,dz\,dy$$

Now assume that $f(x,y,z) = xyz$ in each of the integrals. Then we have

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$$\begin{aligned}\int_{x=0}^{x=1}\int_{y=0}^{y=x^2}\int_{z=0}^{z=y^2} xyz\,dz\,dy\,dx &= \int_{x=0}^{x=1}\int_{y=0}^{y=x^2}\left[xy\frac{z^2}{2}\Big|_{z=0}^{z=y^2}\right]dy\,dx\\ &= \int_{x=0}^{x=1}\int_{y=0}^{y=x^2}\left(x\frac{y^5}{2}\right)dy\,dx\\ &= \int_{x=0}^{x=1}\left[x\frac{y^6}{12}\Big|_{y=0}^{y=x^2}\right]dx\\ &= \int_{x=0}^{x=1}\frac{x^{13}}{12}dx = \frac{x^{14}}{168}\Big|_{x=0}^{x=1}\\ &= \frac{1}{168},\end{aligned}$$

$$\begin{aligned}\int_{y=0}^{y=1}\int_{z=0}^{z=y^2}\int_{x=\sqrt{y}}^{x=1} xyz\,dx\,dz\,dy &= \int_{y=0}^{y=1}\int_{z=0}^{z=y^2}\left[yz\frac{x^2}{2}\Big|_{\sqrt{y}}^{1}\right]dz\,dy\\ &= \int_{y=0}^{y=1}\int_{z=0}^{z=y^2}\left(\frac{yz}{2}-\frac{y^2z}{2}\right)dz\,dy\\ &= \int_{y=0}^{y=1}\left[\frac{yz^2}{4}-\frac{y^2z^2}{4}\Big|_{z=0}^{z=y^2}\right]dy\\ &= \int_{y=0}^{y=1}\left(\frac{y^5}{4}-\frac{y^6}{4}\right)dy\\ &= \left(\frac{y^6}{24}-\frac{y^7}{28}\right)\Big|_{y=0}^{y=1}\\ &= \frac{1}{168}.\end{aligned}$$

The answers match.

? Exercise 14.4.4

Write five different iterated integrals equal to the given integral

$$\int_{z=0}^{z=4}\int_{y=0}^{y=4-z}\int_{x=0}^{x=\sqrt{y}} f(x,y,z)\,dx\,dy\,dz.$$

✓ Example 14.4.5: Changing Integration Order and Coordinate Systems

Evaluate the triple integral

$$\iiint_E \sqrt{x^2+z^2}\,dV,$$

where E is the region bounded by the paraboloid $y=x^2+z^2$ (Figure 14.4.9) and the plane $y=4$.

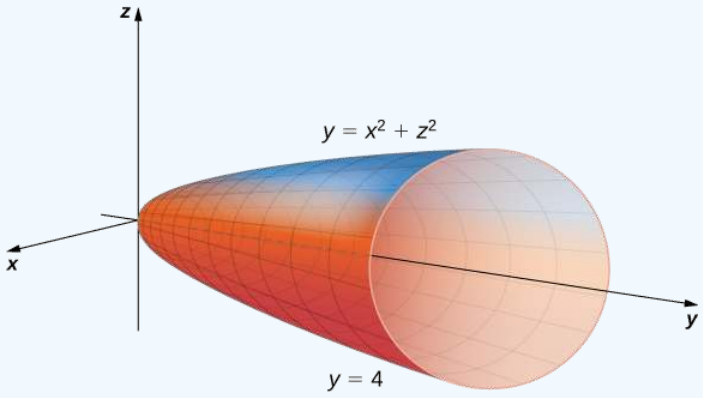

Figure 14.4.9. Integrating a triple integral over a paraboloid.

Solution

The projection of the solid region E onto the xy-plane is the region bounded above by $y = 4$ and below by the parabola $y = x^2$ as shown.

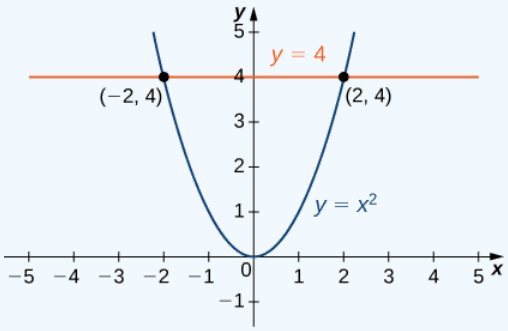

Figure 14.4.10. Cross section in the xy-plane of the paraboloid in Figure 14.4.9.

Thus, we have

$$E = \{(x, y, z) \mid -2 \le x \le 2,\ x^2 \le y \le 4,\ -\sqrt{y - x^2} \le z \le \sqrt{y - x^2}\}.$$

The triple integral becomes

$$\iiint_E \sqrt{x^2 + z^2}\, dV = \int_{x=-2}^{x=2} \int_{y=x^2}^{y=4} \int_{z=-\sqrt{y-x^2}}^{z=\sqrt{y-x^2}} \sqrt{x^2 + z^2}\, dz\, dy\, dx.$$

This expression is difficult to compute, so consider the projection of E onto the xz-plane. This is a circular disc $x^2 + z^2 \le 4$. So we obtain

$$\begin{aligned}\iiint_E \sqrt{x^2 + z^2}\, dV &= \int_{x=-2}^{x=2} \int_{y=x^2}^{y=4} \int_{z=-\sqrt{y-x^2}}^{z=\sqrt{y-x^2}} \sqrt{x^2 + z^2}\, dz\, dy\, dx \\ &= \int_{x=-2}^{x=2} \int_{z=-\sqrt{4-x^2}}^{z=\sqrt{4-x^2}} \int_{y=x^2+z^2}^{y=4} \sqrt{x^2 + z^2}\, dy\, dz\, dx.\end{aligned}$$

Here the order of integration changes from being first with respect to z then y and then x to being first with respect to y then to z and then to x. It will soon be clear how this change can be beneficial for computation. We have

$$\begin{aligned}&\int_{x=-2}^{x=2} \int_{z=\sqrt{4-x^2}}^{z=\sqrt{4-x^2}} \int_{y=x^2+z^2}^{y=4} \sqrt{x^2 + z^2}\, dy\, dz\, dx \\ &= \int_{x=-2}^{x=2} \int_{z=-\sqrt{4-x^2}}^{z=\sqrt{4-x^2}} (4 - x^2 - z^2)\sqrt{x^2 + z^2}\, dz\, dx.\end{aligned}$$

Now use the polar substitution $x = r\cos\theta$, $z = r\sin\theta$, and $dz\, dx = r\, dr\, d\theta$ in the xz-plane. This is essentially the same thing as when we used polar coordinates in the xy-plane, except we are replacing y by z. Consequently the limits of integration change

and we have, by using $r^2 = x^2 + z^2$,

$$\int_{x=-2}^{x=2} \int_{z=-\sqrt{4-x^2}}^{z=\sqrt{4-x^2}} (4 - x^2 - z^2)\sqrt{x^2 + z^2}\, dz\, dx = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} (4 - r^2) r r\, dr\, d\theta$$
$$= \int_0^{2\pi} \left[\frac{4r^3}{3} - \frac{r^5}{5} \Big|_0^2 \right] d\theta = \int_0^{2\pi} \frac{64}{15}\, d\theta = \frac{128\pi}{15}$$

Average Value of a Function of Three Variables

Recall that we found the average value of a function of two variables by evaluating the double integral over a region on the plane and then dividing by the area of the region. Similarly, we can find the average value of a function in three variables by evaluating the triple integral over a solid region and then dividing by the volume of the solid.

Average Value of a Function of Three Variables

If $f(x, y, z)$ is integrable over a solid bounded region E with positive volume $V(E)$, then the average value of the function is

$$f_{ave} = \frac{1}{V(E)} \iiint_E f(x, y, z)\, dV.$$

Note that the volume is

$$V(E) = \iiint_E 1\, dV.$$

Example 14.4.6: Finding an Average Temperature

The temperature at a point (x, y, z) of a solid E bounded by the coordinate planes and the plane $x + y + z = 1$ is $T(x, y, z) = (xy + 8z + 20)\,°\text{C}$. Find the average temperature over the solid.

Solution

Use the theorem given above and the triple integral to find the numerator and the denominator. Then do the division. Notice that the plane $x + y + z = 1$ has intercepts $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. The region E looks like

$$E = \{(x, y, z) \mid 0 \le x \le 1,\ 0 \le y \le 1 - x,\ 0 \le z \le 1 - x - y\}.$$

Hence the triple integral of the temperature is

$$\iiint_E f(x, y, z)\, dV = \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} \int_{z=0}^{z=1-x-y} (xy + 8z + 20)\, dz\, dy\, dx = \frac{147}{40}.$$

The volume evaluation is

$$V(E) = \iiint_E 1\, dV = \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} \int_{z=0}^{z=1-x-y} 1\, dz\, dy\, dx = \frac{1}{6}.$$

Hence the average value is

$$T_{ave} = \frac{147/40}{1/6} = \frac{6(147)}{40} = \frac{441}{20}\,°\text{C}$$

.

Exercise 14.4.6

Find the average value of the function $f(x, y, z) = xyz$ over the cube with sides of length 4 units in the first octant with one vertex at the origin and edges parallel to the coordinate axes.

Key Concepts

- To compute a triple integral we use Fubini's theorem, which states that if $f(x, y, z)$ is continuous on a rectangular box $B = [a, b] \times [c, d] \times [e, f]$, then

$$\iiint_B f(x, y, z)\, dV = \int_e^f \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz$$

  and is also equal to any of the other five possible orderings for the iterated triple integral.
- To compute the volume of a general solid bounded region E we use the triple integral

$$V(E) = \iiint_E 1\, dV.$$

- Interchanging the order of the iterated integrals does not change the answer. As a matter of fact, interchanging the order of integration can help simplify the computation.
- To compute the average value of a function over a general three-dimensional region, we use

$$f_{ave} = \frac{1}{V(E)} \iiint_E f(x, y, z)\, dV.$$

Key Equations

- **Triple integral**

$$\lim_{l,m,n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\, \Delta x \Delta y \Delta z = \iiint_B f(x, y, z)\, dV$$

Glossary

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14.4E: Triple Integrals (Exercises)

Terms and Concepts

1. The strategy for establishing bounds for triple integrals is "from _____ to _____, then from _____ to _____ and then from _____ to _____."

2. Give an informal interpretation of what $\iiint_Q dV$ means.

3. Give two uses of triple integration.

4. If an object has a constant density δ and a volume V, what is its mass?

Volume of Solid Regions

In Exercises 5-8, two surfaces $f_1(x,y)$ and $f_2(x,y)$ and a region R in the xy-plane are given. Set up and evaluate the triple integral that represents the volume between these surfaces over R.

5. $f_1(x,y) = 8 - x^2 - y^2$, $f_2(x,y) = 2x + y$;
R is the square with corners $(-1,-1)$ and $(1,1)$.

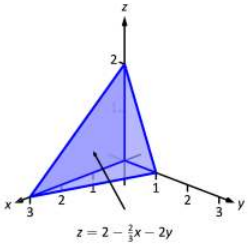
6. $f_1(x,y) = x^2 + y^2$, $f_2(x,y) = -x^2 - y^2$;
R is the square with corners $(0,0)$ and $(2,3)$.

7. $f_1(x,y) = \sin x \cos y$, $f_2(x,y) = \cos x \sin y + 2$;
R is the triangle with corners $(0,0)$, $(\pi,0)$ and (π,π).

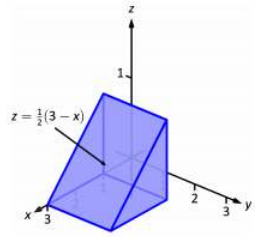
8. $f_1(x,y) = 2x^2 + 2y^2 + 3$, $f_2(x,y) = 6 - x^2 - y^2$;
R is the circle $x^2 + y^2 = 1$.

In Exercises 9-16, a domain D is described by its bounding surfaces, along with a graph. Set up the triple integral that gives the volume of D in the indicated order(s) of integration, and evaluate the triple integral to find this volume.

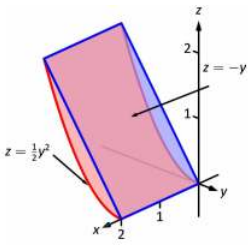
9. D is bounded by the coordinate planes and $z = 2 - \frac{2}{3}x - 2y$.
Evaluate the triple integral with order $dz\,dy\,dx$.



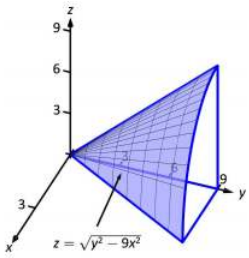
10. D is bounded by the planes $y = 0, y = 2, x = 1, z = 0$ and $z = (2 - x)/2$.
Evaluate the triple integral with order $dx\,dy\,dz$.



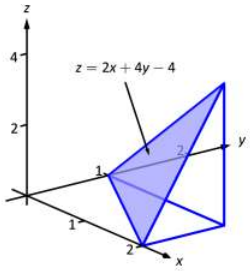
11. D is bounded by the planes $x = 0, x = 2, z = -y$ and by $z = y^2/2$.
Evaluate the triple integral with orders $dy\,dz\,dx$ and $dz\,dy\,dx$ to verify that you obtain the same volume either way.



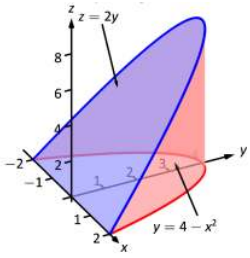
12. D is bounded by the planes $z = 0, y = 9, x = 0$ and by $z = \sqrt{y^2 - 9x^2}$.
Do not evaluate any triple integral. Just set this one up in the orders: $dz\,dy\,dx$, $dx\,dy\,dz$, and $dy\,dz\,dx$.



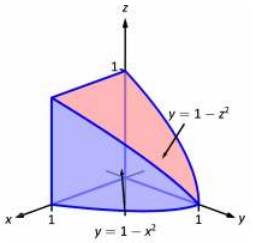
13. D is bounded by the planes $x = 2, y = 1, z = 0$ and $z = 2x + 4y - 4$.
Evaluate the triple integral with orders $dz\,dy\,dx$ and $dx\,dy\,dz$ to verify that you obtain the same volume either way.



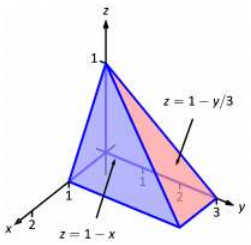
14. D is bounded by the plane $z = 2y$ and by $y = 4 - x^2$.
Evaluate the triple integral with order $dz\,dy\,dx$.



15. D is bounded by the coordinate planes and $y = 1 - x^2$ and $y = 1 - z^2$.
Do not evaluate any triple integral. Which order would be easier to evaluate: $dz\,dy\,dx$ or $dy\,dz\,dx$? Explain why.



16. D is bounded by the coordinate planes and by $z = 1 - y/3$ and $z = 1 - x$.
Evaluate the triple integral with order $dx\,dy\,dz$.



Evaluating General Triple Integrals

In exercises 17 - 20, evaluate the triple integrals over the rectangular solid box B.

17. $\iiint_B (2x + 3y^2 + 4z^3)\, dV$, where $B = \{(x, y, z) \mid 0 \le x \le 1,\ 0 \le y \le 2,\ 0 \le z \le 3\}$

18. $\iiint_B (xy + yz + xz)\, dV$, where $B = \{(x, y, z) \mid 1 \le x \le 2,\ 0 \le y \le 2,\ 1 \le z \le 3\}$

19. $\iiint_B (x\ cos\ y + z)\, dV$, where $B = \{(x, y, z) \mid 0 \le x \le 1,\ 0 \le y \le \pi,\ -1 \le z \le 1\}$

20. $\iiint_B (z\ sin\ x + y^2)\, dV$, where $B = \{(x, y, z) \mid 0 \le x \le \pi,\ 0 \le y \le 1,\ -1 \le z \le 2\}$

In Exercises 21 - 24, evaluate the triple integral.

21. $\int_{-\pi/2}^{\pi/2} \int_0^{\pi} \int_0^{\pi} (\cos x \sin y \sin z)\, dz\, dy\, dx$

22. $\int_0^1 \int_0^x \int_0^{x+y} (x + y + z)\, dz\, dy\, dx$

23. $\int_0^{\pi} \int_0^1 \int_0^z (\sin(yz))\, dx\, dy\, dz$

24. $\int_{\pi}^{\pi^2} \int_x^{x^3} \int_{-y^2}^{y^2} (\cos x \sin y \sin z)\, dz\, dy\, dx$

Average Value of a Function

25. Find the average value of the function $f(x, y, z) = x + y + z$ over the parallelepiped determined by $x = 0,\ x = 1,\ y = 0,\ y = 3,\ z = 0$, and $z = 5$.

26. Find the average value of the function $f(x, y, z) = xyz$ over the solid $E = [0, 1] \times [0, 1] \times [0, 1]$ situated in the first octant.

Approximating Triple Integrals

27. The midpoint rule for the triple integral $\iiint_B f(x, y, z)\, dV$ over the rectangular solid box B is a generalization of the midpoint rule for double integrals. The region B is divided into subboxes of equal sizes and the integral is approximated by the triple Riemann sum

$$\sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(\bar{x}_i, \bar{y}_j, \bar{z}_k) \Delta V,$$

where $(\bar{x}_i, \bar{y}_j, \bar{z}_k)$ is the center of the box B_{ijk} and ΔV is the volume of each subbox. Apply the midpoint rule to approximate

$$\iiint_B x^2\, dV$$

over the solid $B = \{(x, y, z) \mid 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\}$ by using a partition of eight cubes of equal size. Round your answer to three decimal places.

28. [T] a. Apply the midpoint rule to approximate

$$\iiint_B e^{-x^2}\,dV$$

over the solid $B = \{(x, y, z) | 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\}$ by using a partition of eight cubes of equal size. Round your answer to three decimal places.

b. Use a CAS to improve the above integral approximation in the case of a partition of n^3 cubes of equal size, where $n = 3, 4, \ldots, 10$.

Applications

29. Suppose that the temperature in degrees Celsius at a point (x, y, z) of a solid E bounded by the coordinate planes and the plane $x + y + z = 5$ is given by:

$$T(x, y, z) = xz + 5z + 10$$

Find the average temperature over the solid.

30. Suppose that the temperature in degrees Fahrenheit at a point (x, y, z) of a solid E bounded by the coordinate planes and the plane $x + y + z = 5$ is given by:

$$T(x, y, z) = x + y + xy$$

Find the average temperature over the solid.

31. If the charge density at an arbitrary point (x, y, z) of a solid E is given by the function $\rho(x, y, z)$, then the total charge inside the solid is defined as the triple integral $\iiint_E \rho(x, y, z)\,dV$. Assume that the charge density of the solid E enclosed by the paraboloids $x = 5 - y^2 - z^2$ and $x = y^2 + z^2 - 5$ is equal to the distance from an arbitrary point of E to the origin. Set up the integral that gives the total charge inside the solid E.

32. Show that the volume of a regular right hexagonal pyramid of edge length a is $\frac{a^3\sqrt{3}}{2}$ by using triple integrals.

Contributors

- Problems 17 - 20 and 25 - 32 are from Section 15.4, OpenStax Calculus 3 by

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- Problems 1 - 16 and 21 - 24 are from Apex Calculus, Section 13.6.
- Edited by Paul Seeburger (Monroe Community College)

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14.5: Triple Integrals in Cylindrical and Spherical Coordinates

Learning Objectives

- Evaluate a triple integral by changing to cylindrical coordinates.
- Evaluate a triple integral by changing to spherical coordinates.

Earlier in this chapter we showed how to convert a double integral in rectangular coordinates into a double integral in polar coordinates in order to deal more conveniently with problems involving circular symmetry. A similar situation occurs with triple integrals, but here we need to distinguish between cylindrical symmetry and spherical symmetry. In this section we convert triple integrals in rectangular coordinates into a triple integral in either cylindrical or spherical coordinates.

Also recall the chapter prelude, which showed the opera house l'Hemisphèric in Valencia, Spain. It has four sections with one of the sections being a theater in a five-story-high sphere (ball) under an oval roof as long as a football field. Inside is an IMAX screen that changes the sphere into a planetarium with a sky full of 9000 twinkling stars. Using triple integrals in spherical coordinates, we can find the volumes of different geometric shapes like these.

Review of Cylindrical Coordinates

As we have seen earlier, in two-dimensional space $\mathbb{R}^2$ a point with rectangular coordinates (x, y) can be identified with (r, θ) in polar coordinates and vice versa, where $x = r\cos\theta$, $y = r\sin\theta$, $r^2 = x^2 + y^2$ and $\tan\theta = \frac{y}{x}$ are the relationships between the variables.

In three-dimensional space $\mathbb{R}^3$ a point with rectangular coordinates (x, y, z) can be identified with cylindrical coordinates (r, θ, z) and vice versa. We can use these same conversion relationships, adding z as the vertical distance to the point from the $(xy$-plane as shown in 14.5.1.

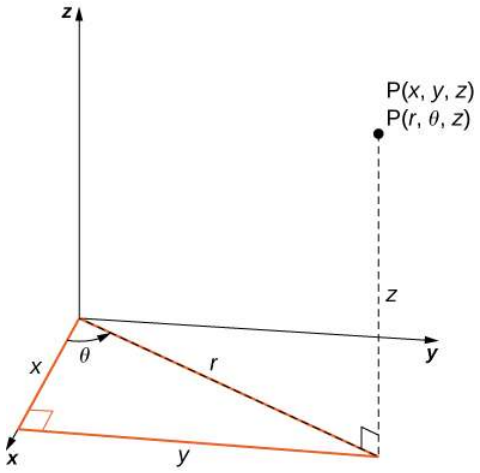


Figure 14.5.1: Cylindrical coordinates are similar to polar coordinates with a vertical z coordinate added.

To convert from rectangular to cylindrical coordinates, we use the conversion

- $x = r\cos\theta$
- $y = r\sin\theta$
- $z = z$

To convert from cylindrical to rectangular coordinates, we use

- $r^2 = x^2 + y^2$ and
- $\theta = \tan^{-1}\left(\frac{y}{x}\right)$
- $z = z$

Note that that z-coordinate remains the same in both cases.

In the two-dimensional plane with a rectangular coordinate system, when we say $x = k$ (constant) we mean an unbounded vertical line parallel to the y-axis and when $y = l$ (constant) we mean an unbounded horizontal line parallel to the x-axis. With the polar coordinate system, when we say $r = c$ (constant), we mean a circle of radius c units and when $\theta = \alpha$ (constant) we mean an infinite ray making an angle α with the positive x-axis.

Similarly, in three-dimensional space with rectangular coordinates (x, y, z) the equations $x = k$, $y = l$ and $z = m$ where k, l and m are constants, represent unbounded planes parallel to the yz-plane, xz-plane and xy-plane, respectively. With cylindrical coordinates (r, θ, z), by $r = c$, $\theta = \alpha$, and $z = m$, where c, α, and m are constants, we mean an unbounded vertical cylinder with the z-axis as its radial axis; a plane making a constant angle α with the xy-plane; and an unbounded horizontal plane parallel to the xy-plane, respectively. This means that the circular cylinder $x^2 + y^2 = c^2$ in rectangular coordinates can be represented simply as $r = c$ in cylindrical coordinates. (Refer to Cylindrical and Spherical Coordinates for more review.)

Integration in Cylindrical Coordinates

Triple integrals can often be more readily evaluated by using cylindrical coordinates instead of rectangular coordinates. Some common equations of surfaces in rectangular coordinates along with corresponding equations in cylindrical coordinates are listed in Table **14.5.1**. These equations will become handy as we proceed with solving problems using triple integrals.

Table **14.5.1**: Equations of Some Common Shapes

	Circular cylinder	**Circular cone**	**Sphere**	**Paraboloid**
Rectangular	$x^2 + y^2 = c^2$	$z^2 = c^2(x^2 + y^2)$	$x^2 + y^2 + z^2 = c^2$	$z = c(x^2 + y^2)$
Cylindrical	$r = c$	$z = cr$	$r^2 + z^2 = c^2$	$z = cr^2$

As before, we start with the simplest bounded region B in $\mathbb{R}^3$ to describe in cylindrical coordinates, in the form of a cylindrical box, $B = \{(r, \theta, z) | a \le r \le b, \alpha \le \theta \le \beta, c \le z \le d\}$ (Figure **14.5.2**). Suppose we divide each interval into l, m, and n subdivisions such that $\Delta r = \frac{b-a}{l}$, $\Delta\theta = \frac{\beta-\alpha}{m}$, and $\Delta z = \frac{d-c}{n}$. Then we can state the following definition for a triple integral in cylindrical coordinates.

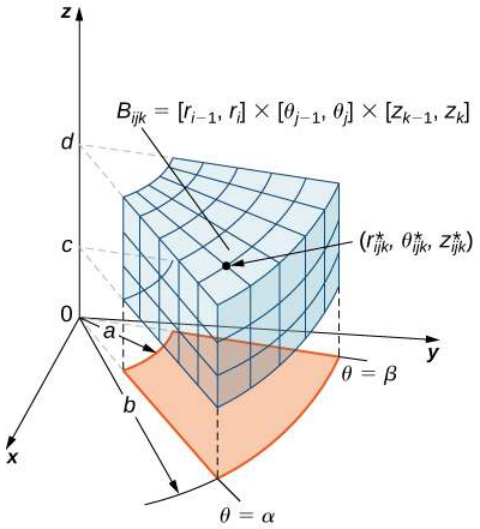


Figure **14.5.2**: A cylindrical box B described by cylindrical coordinates.

DEFINITION: triple integral in cylindrical coordinates

Consider the cylindrical box (expressed in cylindrical coordinates)

$$B = \{(r, \theta, z) | a \le r \le b, \alpha \le \theta \le \beta, c \le z \le d\}.$$

If the function $f(r, \theta, z)$ is continuous on B and if $(r^*_{ijk}, \theta^*_{ijk}, z^*_{ijk})$ is any sample point in the cylindrical subbox $B_{ijk} = |r_{i-1}, r_i| \times |\theta_{j-1}, \theta_j| \times |z_{k-1}, k_i|$ (Figure **14.5.2**), then we can define the *triple integral in cylindrical coordinates* as the limit of a triple Riemann sum, provided the following limit exists:

$$\lim_{l,m,n\to\infty}\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(r^*_{ijk},\theta^*_{ijk},z^*_{ijk})\,\Delta r\Delta\theta\Delta z.$$

Note that if $g(x,y,z)$ is the function in rectangular coordinates and the box B is expressed in rectangular coordinates, then the triple integral

$$\iiint_B g(x,y,z)\,dV$$

is equal to the triple integral

$$\iiint_B g(r\,\cos\theta,\, r\,\sin\,\theta,\, z)r\,dr\,d\theta\,dz$$

and we have

$$\iiint_B g(x,y,z)\,dV = \iiint_B g(r\,\cos\theta,\, r\,\sin\,\theta,\, z)\,r\,dr\,d\theta\,dz = \iiint_B f(r,\theta\,z)\,r\,dr\,d\theta\,dz.$$

As mentioned in the preceding section, all the properties of a double integral work well in triple integrals, whether in rectangular coordinates or cylindrical coordinates. They also hold for iterated integrals. To reiterate, in cylindrical coordinates, Fubini's theorem takes the following form:

Theorem: Fubini's Theorem in Cylindrical Coordinates

Suppose that $g(x,y,z)$ is continuous on a rectangular box B which when described in cylindrical coordinates looks like $B = \{(r,\theta,z)|a\le r\le b,\ \alpha\le\theta\le\beta,\ c\le z\le d\}$.

Then $g(x,y,z) = g(r\,\cos\theta, r\,\sin\,\theta, z) = f(r,\theta,z)$ and

$$\iiint_B g(x,y,z)\,dV = \int_c^d\int_\beta^\alpha\int_a^b f(r,\theta,z)\,r\,dr\,d\theta\,dz.$$

The iterated integral may be replaced equivalently by any one of the other five iterated integrals obtained by integrating with respect to the three variables in other orders.

Cylindrical coordinate systems work well for solids that are symmetric around an axis, such as cylinders and cones. Let us look at some examples before we define the triple integral in cylindrical coordinates on general cylindrical regions.

Example 14.5.1: Evaluating a Triple Integral over a Cylindrical Box

Evaluate the triple integral

$$\iiint_B (zr\,\sin\,\theta)r\,dr\,d\theta\,dz$$

where the cylindrical box B is $B = \{(r,\theta,z)|0\le r\le 2,\ 0\le\theta\le\pi/2,\ 0,\le z\le 4\}$.

Solution

As stated in Fubini's theorem, we can write the triple integral as the iterated integral

$$\iiint_B (zr\,\sin\,\theta)r\,dr\,d\theta\,dz = \int_{\theta=0}^{\theta=\pi/2}\int_{r=0}^{r=2}\int_{z=0}^{z=4}(zr\,\sin\,\theta)\,r\,dz\,dr\,d\theta.$$

The evaluation of the iterated integral is straightforward. Each variable in the integral is independent of the others, so we can integrate each variable separately and multiply the results together. This makes the computation much easier:

$$\int_{\theta=0}^{\theta=\pi/2}\int_{r=0}^{r=2}\int_{z=0}^{z=4}(zr\sin\theta)\,r\,dz\,dr\,d\theta=\left(\int_0^{\pi/2}\sin\theta\,d\theta\right)\left(\int_0^2 r^2dr\right)\left(\int_0^4 z\,dz\right)$$
$$=\left(-\cos\theta\Big|_0^{\pi/2}\right)\left(\frac{r^3}{3}\Big|_0^2\right)\left(\frac{z^2}{2}\Big|_0^4\right)=\frac{64}{3}.$$

? Exercise 14.5.1:

Evaluate the triple integral

$$\int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=1}\int_{z=0}^{z=4}(rz\sin\theta)\,r\,dz\,dr\,d\theta.$$

If the cylindrical region over which we have to integrate is a general solid, we look at the projections onto the coordinate planes. Hence the triple integral of a continuous function $f(r,\theta,z)$ over a general solid region $E=\{(r,\theta,z)|(r,\theta)\in D, u_1(r,\theta)\le z\le u_2(r,\theta)\}$ in $\mathbb{R}^3$ where D is the projection of E onto the $r\theta$-plane, is

$$\iiint_E f(r,\theta,z)\,r\,dr\,d\theta\,dz=\iint_D\left[\int_{u_1(r,\theta)}^{u_2(r,\theta)}f(r,\theta,z)dz\right]r\,dr\,d\theta.$$

In particular, if $D=\{(r,\theta)|g_1(\theta)\le r\le g_2(\theta),\alpha\le\theta\le\beta\}$, then we have

$$\iiint_E f(r,\theta,z)\,r\,dr\,d\theta=\int_{\theta=\alpha}^{\theta=\beta}\int_{r=g_1(\theta)}^{r=g_2(\theta)}\int_{z=u_1(r,\theta)}^{z=u_2(r,\theta)}f(r,\theta,z)\,r\,dz\,dr\,d\theta.$$

Similar formulas exist for projections onto the other coordinate planes. We can use polar coordinates in those planes if necessary.

✓ Example 14.5.2: Setting up a Triple Integral in Cylindrical Coordinates over a General Region

Consider the region E inside the right circular cylinder with equation $r=2\sin\theta$, bounded below by the $r\theta$-plane and bounded above by the sphere with radius 4 centered at the origin (Figure 15.5.3). Set up a triple integral over this region with a function $f(r,\theta,z)$ in cylindrical coordinates.

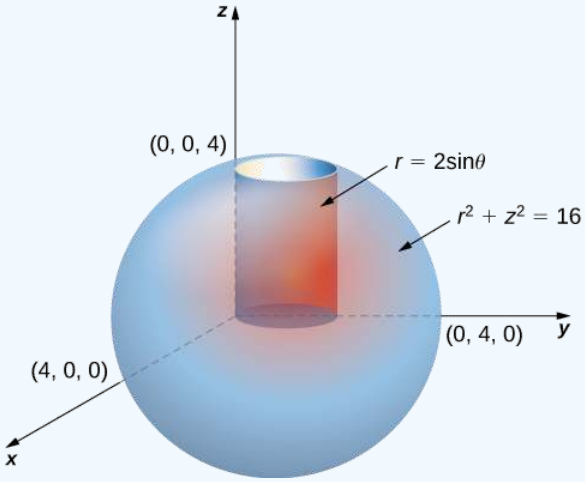


Figure 14.5.3: Setting up a triple integral in cylindrical coordinates over a cylindrical region.

Solution

First, identify that the equation for the sphere is $r^2+z^2=16$. We can see that the limits for z are from 0 to $z=\sqrt{16-r^2}$. Then the limits for r are from 0 to $r=2\sin\theta$. Finally, the limits for θ are from 0 to π. Hence the region is $E=\{(r,\theta,z)|0\le\theta\le\pi,\ 0\le r\le 2\sin\theta,\ 0\le z\le\sqrt{16-r^2}\}$. Therefore, the triple integral is

$$\iiint_E f(r,\theta,z)\,r\,dz\,dr\,d\theta=\int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=2\sin\theta}\int_{z=0}^{z=\sqrt{16-r^2}}f(r,\theta,z)\,r\,dz\,dr\,d\theta.$$

Exercise 14.5.2:

Consider the region inside the right circular cylinder with equation $r = 2 \sin \theta$ bounded below by the $r\theta$-plane and bounded above by $z = 4 - y$. Set up a triple integral with a function $f(r, \theta, z)$ in cylindrical coordinates.

Example 14.5.3: Setting up a Triple Integral in Two Ways

Let E be the region bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the paraboloid $z = 2 - x^2 - y^2$. (Figure 15.5.4). Set up a triple integral in cylindrical coordinates to find the volume of the region, using the following orders of integration:

a. $dz\,dr\,d\theta$

b. $dr\,dz\,d\theta$

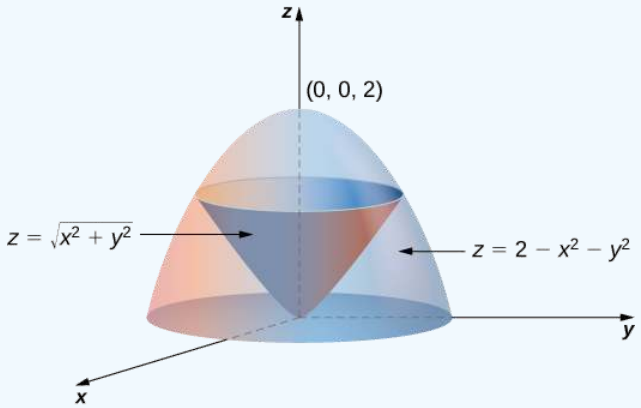


Figure 14.5.4: Setting up a triple integral in cylindrical coordinates over a conical region.

Solution

a. The cone is of radius 1 where it meets the paraboloid. Since $z = 2 - x^2 - y^2 = 2 - r^2$ and $z = \sqrt{x^2 + y^2} = r^2$ (assuming r is nonnegative), we have $2 - r^2 = r$. Solving, we have $r^2 + r - 2 = (r + 2)(r - 1) = 0$. Since $r \geq 0$, we have $r = 1$. Therefore $z = 1$. So the intersection of these two surfaces is a circle of radius 1 in the plane $z = 1$. The cone is the lower bound for z and the paraboloid is the upper bound. The projection of the region onto the xy-plane is the circle of radius 1 centered at the origin.

Thus, we can describe the region as $E = \{(r, \theta, z) | 0 \leq \theta \leq 2\pi,\ 0 \leq r \leq 1,\ r \leq z \leq 2 - r^2\}$.

Hence the integral for the volume is

$$V = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \int_{z=r}^{z=2-r^2} r\,dz\,dr\,d\theta.$$

b. We can also write the cone surface as $r = z$ and the paraboloid as $r^2 = 2 - z$. The lower bound for r is zero, but the upper bound is sometimes the cone and the other times it is the paraboloid. The plane $z = 1$ divides the region into two regions. Then the region can be described as

$$E = \{(r, \theta, z) | 0 \leq \theta \leq 2\pi,\ 0 \leq z \leq 1,\ 0 \leq r \leq z\}$$
$$\cup \{(r, \theta, z) | 0 \leq \theta \leq 2\pi,\ 1 \leq z \leq 2,\ 0 \leq r \leq \sqrt{2 - z}\}.$$

Now the integral for the volume becomes

$$V = \int_{\theta=0}^{\theta=2\pi} \int_{z=0}^{z=1} \int_{r=0}^{r=z} r\,dr\,dz\,d\theta + \int_{\theta=0}^{\theta=2\pi} \int_{z=1}^{z=2} \int_{r=0}^{r=\sqrt{2-z}} r\,dr\,dz\,d\theta.$$

Exercise 14.5.3:

Redo the previous example with the order of integration $d\theta\,dz\,dr$.

Example 14.5.4: Finding a Volume with Triple Integrals in Two Ways

Let E be the region bounded below by the $r\theta$-plane, above by the sphere $x^2 + y^2 + z^2 = 4$, and on the sides by the cylinder $x^2 + y^2 = 1$ (Figure 15.5.5). Set up a triple integral in cylindrical coordinates to find the volume of the region using the following orders of integration, and in each case find the volume and check that the answers are the same:

a. $dz\,dr\,d\theta$
b. $dr\,dz\,d\theta$.

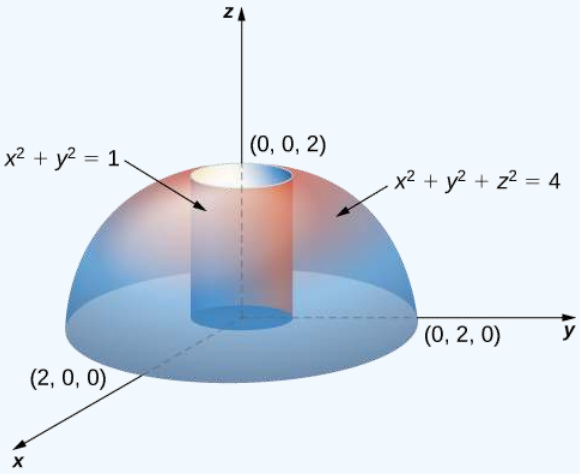


Figure 14.5.5: Finding a cylindrical volume with a triple integral in cylindrical coordinates.

Solution

a. Note that the equation for the sphere is

$$x^2 + y^2 + z^2 = 4 \text{ or } r^2 + z^2 = 4$$

and the equation for the cylinder is

$$x^2 + y^2 = 1 \text{ or } r^2 = 1.$$

Thus, we have for the region E

$$E = \{(r, \theta, z) | 0 \le z \le \sqrt{4 - r^2},\ 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$$

Hence the integral for the volume is

$$\begin{aligned} V(E) &= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \int_{z=0}^{z=\sqrt{4-r^2}} r\,dz\,dr\,d\theta \\ &= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \left[rz \Big|_{z=0}^{z=\sqrt{4-r^2}} \right] dr\,d\theta \\ &= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \left(r\sqrt{4 - r^2} \right) dr\,d\theta \\ &= \int_0^{2\pi} \left(\frac{8}{3} - \sqrt{3} \right) d\theta = 2\pi \left(\frac{8}{3} - \sqrt{3} \right) \text{ cubic units.} \end{aligned}$$

b. Since the sphere is $x^2 + y^2 + z^2 = 4$, which is $r^2 + z^2 = 4$, and the cylinder is $x^2 + y^2 = 1$, which is $r^2 = 1$, we have $1 + z^2 = 4$, that is, $z^2 = 3$. Thus we have two regions, since the sphere and the cylinder intersect at $(1, \sqrt{3})$ in the rz-plane

$$E_1 = \{(r, \theta, z) | 0 \le r \le \sqrt{4 - r^2},\ \sqrt{3} \le z \le 2,\ 0 \le \theta \le 2\pi\}$$

and

$$E_2 = \{(r, \theta, z) | 0 \le r \le 1,\ 0 \le z \le \sqrt{3},\ 0 \le \theta \le 2\pi\}.$$

Hence the integral for the volume is

$$V(E) = \int_{\theta=0}^{\theta=2\pi} \int_{z=\sqrt{3}}^{z=2} \int_{r=0}^{r=\sqrt{4-r^2}} r\,dr\,dz\,d\theta + \int_{\theta=0}^{\theta=2\pi} \int_{z=0}^{z=\sqrt{3}} \int_{r=0}^{r=1} r\,dr\,dz\,d\theta$$
$$= \sqrt{3}\pi + \left(\frac{16}{3} - 3\sqrt{3}\right)\pi$$
$$= 2\pi\left(\frac{8}{3} - \sqrt{3}\right) \text{ cubic units.}$$

? Exercise 14.5.4

Redo the previous example with the order of integration $d\theta\,dz\,dr$.

Review of Spherical Coordinates

In three-dimensional space $\mathbb{R}^3$ in the spherical coordinate system, we specify a point P by its distance ρ from the origin, the polar angle θ from the positive x-axis (same as in the cylindrical coordinate system), and the angle φ from the positive z-axis and the line OP (Figure 14.5.6). Note that $\rho > 0$ and $0 \le \varphi \le \pi$. (Refer to Cylindrical and Spherical Coordinates for a review.) Spherical coordinates are useful for triple integrals over regions that are symmetric with respect to the origin.

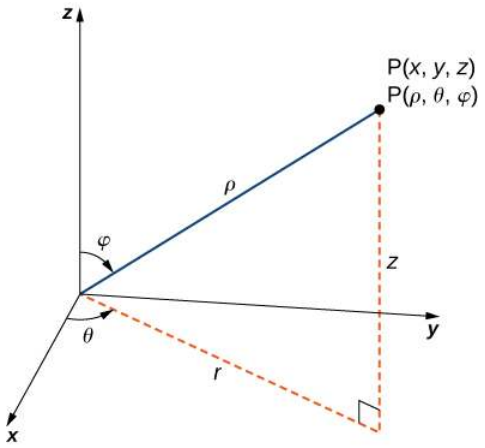


Figure 14.5.6: The spherical coordinate system locates points with two angles and a distance from the origin.

Recall the relationships that connect rectangular coordinates with spherical coordinates.

From spherical coordinates to rectangular coordinates:

$$x = \rho \sin\varphi \cos\theta,\ y = \rho \sin\varphi \sin\theta,\ and\ z = \rho \cos\varphi.$$

From rectangular coordinates to spherical coordinates:

$$\rho^2 = x^2 + y^2 + z^2,\ \tan\theta = \frac{y}{x},\ \varphi = \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right).$$

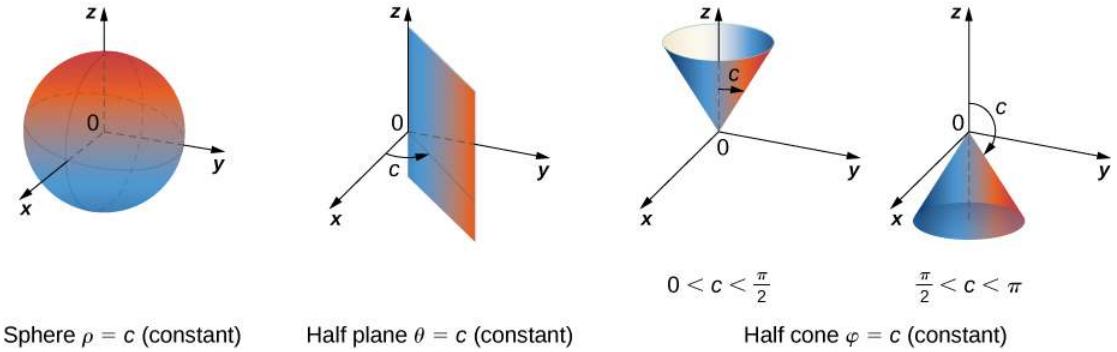
Other relationships that are important to know for conversions are

- $r = \rho \sin\varphi$
- $\theta = \theta$ These equations are used to convert from spherical coordinates to cylindrical coordinates.
- $z = \rho \cos\varphi$

and

- $\rho = \sqrt{r^2 + z^2}$
- $\theta = \theta$ These equations are used to convert from cylindrical coordinates to spherical coordinates.
- $\varphi = \arccos\left(\frac{z}{\sqrt{r^2+z^2}}\right)$

14.5.7 shows a few solid regions that are convenient to express in spherical coordinates.



Sphere $\rho = c$ (constant) Half plane $\theta = c$ (constant) Half cone $\varphi = c$ (constant)

Figure 14.5.7: Spherical coordinates are especially convenient for working with solids bounded by these types of surfaces. (The letter c indicates a constant.)

Integration in Spherical Coordinates

We now establish a triple integral in the spherical coordinate system, as we did before in the cylindrical coordinate system. Let the function $f(\rho, \theta, \varphi)$ be continuous in a bounded spherical box, $B = \{(\rho, \theta, \varphi) | a \le \rho \le b,\ \alpha \le \theta \le \beta,\ \gamma \le \varphi \le \psi\}$. We then divide each interval into l, m, n and n subdivisions such that $\Delta\rho = \frac{b-a}{l}$, $\Delta\theta = \frac{\beta-\alpha}{m}$. $\Delta\varphi = \frac{\psi-\gamma}{n}$. Now we can illustrate the following theorem for triple integrals in spherical coordinates with $(\rho^*_{ijk}, \theta^*_{ijk}, \varphi^*_{ijk})$ being any sample point in the spherical subbox B_{ijk}. For the volume element of the subbox ΔV in spherical coordinates, we have $\Delta V = (\Delta\rho)\,(\rho\Delta\varphi)\,(\rho \sin\varphi\,\Delta\theta)$, as shown in the following figure.

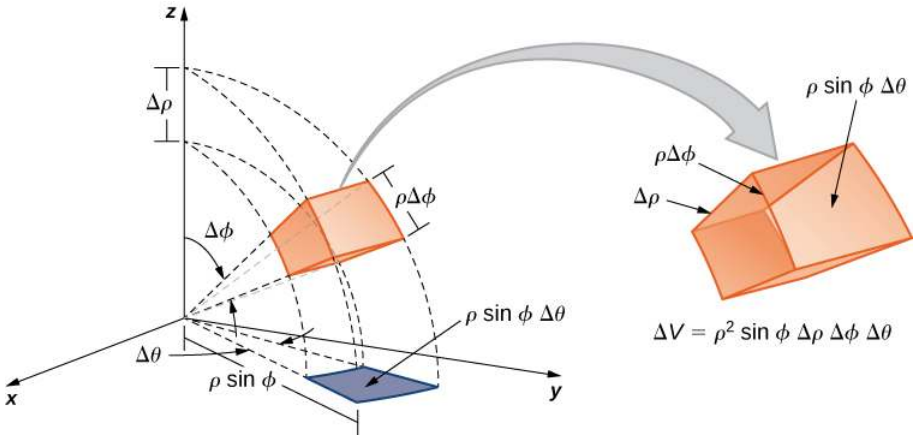


Figure 14.5.8: The volume element of a box in spherical coordinates.

Definition: triple integral in spherical coordinates

The **triple integral in spherical coordinates** is the limit of a triple Riemann sum,

$$\lim_{l,m,n\to\infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(\rho^*_{ijk}, \theta^*_{ijk}, \varphi^*_{ijk})(\rho^*_{ijk})^2 \sin\varphi \Delta\rho\,\Delta\theta\Delta\varphi$$

provided the limit exists.

As with the other multiple integrals we have examined, all the properties work similarly for a triple integral in the spherical coordinate system, and so do the iterated integrals. Fubini's theorem takes the following form.

Theorem: Fubini's Theorem for Spherical Coordinates

If $f(\rho, \theta, \varphi)$ is continuous on a spherical solid box $B = [a, b] \times [\alpha, \beta] \times [\gamma, \psi]$, then

$$\iiint_B f(\rho, \theta, \varphi)\,\rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta = \int_{\varphi=\gamma}^{\varphi=\psi} \int_{\theta=\alpha}^{\theta=\beta} \int_{\rho=a}^{\rho=b} f(\rho, \theta, \varphi)\,\rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta.$$

This iterated integral may be replaced by other iterated integrals by integrating with respect to the three variables in other orders.

As stated before, spherical coordinate systems work well for solids that are symmetric around a point, such as spheres and cones. Let us look at some examples before we consider triple integrals in spherical coordinates on general spherical regions.

Example 14.5.5: Evaluating a Triple Integral in Spherical Coordinates

Evaluate the iterated triple integral

$$\int_{\theta=0}^{\theta=2\pi}\int_{\varphi=0}^{\varphi=\pi/2}\int_{\rho=0}^{\rho=1}\rho^2\sin\,\varphi\,d\rho\,d\varphi\,d\theta.$$

Solution

As before, in this case the variables in the iterated integral are actually independent of each other and hence we can integrate each piece and multiply:

$$\int_0^{2\pi}\int_0^{\pi/2}\int_0^1\rho^2\sin\,\varphi\,d\rho\,d\varphi\,d\theta$$
$$=\int_0^{2\pi}d\theta\int_0^{\pi/2}\sin\,\varphi\,d\varphi\int_0^1\rho^2d\rho=(2\pi)\,(1)\,\left(\frac{1}{3}\right)=\frac{2\pi}{3}$$

The concept of triple integration in spherical coordinates can be extended to integration over a general solid, using the projections onto the coordinate planes. Note that dV and dA mean the increments in volume and area, respectively. The variables V and A are used as the variables for integration to express the integrals.

The triple integral of a continuous function $f(\rho,\theta,\varphi)$ over a general solid region

$$E=\{(\rho,\theta,\varphi)|(\rho,\theta)\in D,u_1(\rho,\theta)\le\varphi\le u_2(\rho,\theta)\}$$

in $\mathbb{R}^3$, where D is the projection of E onto the $\rho\theta$-plane, is

$$\iiint_E f(\rho,\theta,\varphi)\,dV=\iint_D\left[\int_{u_1(\rho,\theta)}^{u_2(\rho,\theta)}f(\rho,\theta,\varphi)\,d\varphi\right]\,dA.$$

In particular, if $D=\{(\rho,\theta)|g_1(\theta)\le\rho\le g_2(\theta),\,\alpha\le\theta\le\beta\}$, the we have

$$\iiint_E f(\rho,\theta,\varphi)\,dV=\int_\alpha^\beta\int_{g_1(\theta)}^{g_2(\theta)}\int_{u_1(\rho,\theta)}^{u_2(\rho,\theta)}f(\rho,\theta,\varphi)\rho^2\sin\,\varphi\,d\varphi\,d\rho\,d\theta.$$

Similar formulas occur for projections onto the other coordinate planes.

Example 14.5.6: Setting up a Triple Integral in Spherical Coordinates

Set up an integral for the volume of the region bounded by the cone $z=\sqrt{3(x^2+y^2)}$ and the hemisphere $z=\sqrt{4-x^2-y^2}$ (see the figure below).

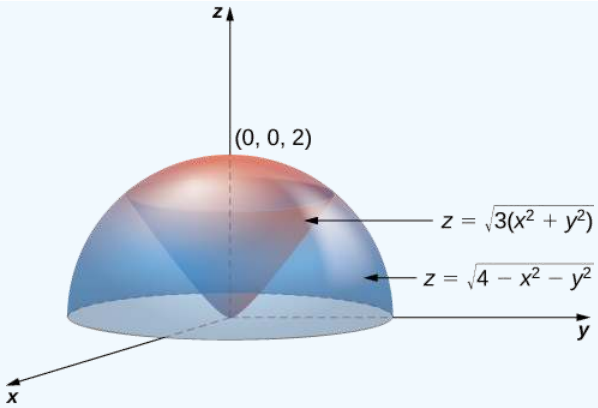


Figure 14.5.9: A region bounded below by a cone and above by a hemisphere.

Solution

Using the conversion formulas from rectangular coordinates to spherical coordinates, we have:

For the cone: $z = \sqrt{3(x^2 + y^2)}$ or $\rho \cos \varphi = \sqrt{3}\rho \sin \varphi$ or $\tan \varphi = \frac{1}{\sqrt{3}}$ or $\varphi = \frac{\pi}{6}$.

For the sphere: $z = \sqrt{4 - x^2 - y^2}$ or $z^2 + x^2 + y^2 = 4$ or $\rho^2 = 4$ or $\rho = 2$.

Thus, the triple integral for the volume is

$$V(E) = \int_{\theta=0}^{\theta=2\pi} \int_{\varphi=0}^{\varphi=\pi/6} \int_{\rho=0}^{\rho=2} \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta.$$

? Exercise 14.5.5

Set up a triple integral for the volume of the solid region bounded above by the sphere $\rho = 2$ and bounded below by the cone $\varphi = \pi/3$.

✓ Example 14.5.7: Interchanging Order of Integration in Spherical Coordinates

Let E be the region bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the sphere $z = x^2 + y^2 + z^2$ (Figure 15.5.10). Set up a triple integral in spherical coordinates and find the volume of the region using the following orders of integration:

a. $d\rho \, d\phi \, d\theta$
b. $d\varphi \, d\rho \, d\theta$

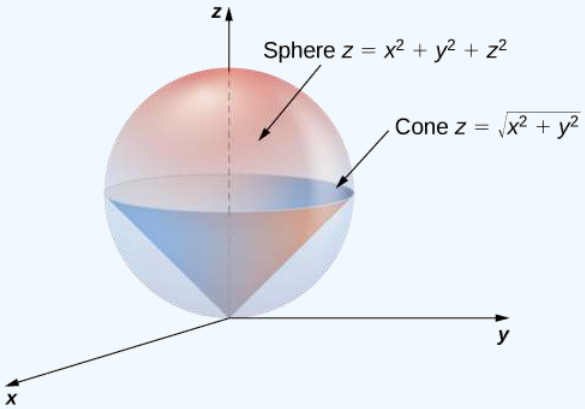


Figure 14.5.10:. A region bounded below by a cone and above by a sphere.

Solution

a. Use the conversion formulas to write the equations of the sphere and cone in spherical coordinates.

For the sphere:

$$x^2+y^2+z^2=z$$
$$\rho^2=\rho\cos\varphi$$
$$\rho=\cos\varphi.$$

For the cone:

$$z=\sqrt{x^2+y^2}$$
$$\rho\cos\varphi=\sqrt{\rho^2\sin^2\varphi\cos^2\theta+\rho^2\sin^2\varphi\sin^2\theta}$$
$$\rho\cos\varphi=\sqrt{\rho^2\sin^2\varphi(\cos^2\theta+\sin^2\theta)}$$
$$\rho\cos\varphi=\rho\sin\varphi$$
$$\cos\varphi=\sin\varphi$$
$$\varphi=\pi/4.$$

Hence the integral for the volume of the solid region E becomes

$$V(E)=\int_{\theta=0}^{\theta=2\pi}\int_{\varphi=0}^{\varphi=\pi/4}\int_{\rho=0}^{\rho=\cos\varphi}\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta.$$

b. Consider the $\varphi\rho$-plane. Note that the ranges for φ and ρ (from part a.) are

$$0\le\varphi\le\pi/4$$
$$0\le\rho\le\cos\varphi$$

The curve $\rho=\cos\varphi$ meets the line $\varphi=\pi/4$ at the point $(\pi/4,\sqrt{2}/2)$. Thus, to change the order of integration, we need to use two pieces:

$$0\le\rho\le\sqrt{2}/2,\ 0\le\varphi\le\pi/4$$

and

$$\sqrt{2}/2\le\rho\le1,\ 0\le\varphi\le\cos^{-1}\rho.$$

Hence the integral for the volume of the solid region E becomes

$$V(E)=\int_{\theta=0}^{\theta=2\pi}\int_{\rho=0}^{\rho=\sqrt{2}/2}\int_{\varphi=0}^{\varphi=\pi/4}\rho^2\sin\varphi\,d\varphi\,d\rho\,d\theta$$
$$+\int_{\theta=0}^{\theta=2\pi}\int_{\rho=\sqrt{2}/2}^{\rho=1}\int_{\varphi=0}^{\varphi=\cos^{-1}\rho}\rho^2\sin\varphi\,d\varphi\,d\rho\,d\theta$$

In each case, the integration results in $V(E)=\frac{\pi}{8}$.

Before we end this section, we present a couple of examples that can illustrate the conversion from rectangular coordinates to cylindrical coordinates and from rectangular coordinates to spherical coordinates.

✔ Example 14.5.8: Converting from Rectangular Coordinates to Cylindrical Coordinates

Convert the following integral into cylindrical coordinates:

$$\int_{y=-1}^{y=1}\int_{x=0}^{x=\sqrt{1-y^2}}\int_{z=x^2+y^2}^{z=\sqrt{x^2+y^2}}xyz\,dz\,dx\,dy.$$

Solution

The ranges of the variables are

$$-1\le y\le y$$
$$0\le x\le\sqrt{1-y^2}$$
$$x^2+y^2\le z\le\sqrt{x^2+y^2}.$$

The first two inequalities describe the right half of a circle of radius 1. Therefore, the ranges for θ and r are

$$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \text{ and } 0 \le r \le 1.$$

The limits of z are $r^2 \le z \le r$, hence

$$\begin{aligned}&\int_{y=-1}^{y=1}\int_{x=0}^{x=\sqrt{1-y^2}}\int_{z=x^2+y^2}^{z=\sqrt{x^2+y^2}} xyz\,dz\,dx\,dy \\ &= \int_{\theta=-\pi/2}^{\theta=\pi/2}\int_{r=0}^{r=1}\int_{z=r^2}^{z=r} r(r\cos\theta)(r\sin\theta)\,z\,dz\,dr\,d\theta.\end{aligned}$$

Example 14.5.9: Converting from Rectangular Coordinates to Spherical Coordinates

Convert the following integral into spherical coordinates:

$$\int_{y=0}^{y=3}\int_{x=0}^{x=\sqrt{9-y^2}}\int_{z=\sqrt{x^2+y^2}}^{z=\sqrt{18-x^2-y^2}} (x^2+y^2+z^2)\,dz\,dx\,dy.$$

Solution

The ranges of the variables are

$$\begin{aligned}0 \le y &\le 3 \\ 0 \le x &\le \sqrt{9-y^2} \\ \sqrt{x^2+y^2} \le z &\le \sqrt{18-x^2-y^2}.\end{aligned}$$

The first two ranges of variables describe a quarter disk in the first quadrant of the xy-plane. Hence the range for θ is $0 \le \theta \le \frac{\pi}{2}$.

The lower bound $z = \sqrt{x^2+y^2}$ is the upper half of a cone and the upper bound $z = \sqrt{18-x^2-y^2}$ is the upper half of a sphere. Therefore, we have $0 \le \rho \le \sqrt{18}$, which is $0 \le \rho \le 3\sqrt{2}$.

For the ranges of φ we need to find where the cone and the sphere intersect, so solve the equation

$$\begin{aligned}r^2+z^2 &= 18 \\ \left(\sqrt{x^2+y^2}\right)^2+z^2 &= 18 \\ z^2+z^2 &= 18 \\ 2z^2 &= 18 \\ z^2 &= 9 \\ z &= 3.\end{aligned}$$

This gives

$$\begin{aligned}3\sqrt{2}\cos\varphi &= 3 \\ \cos\varphi &= \frac{1}{\sqrt{2}} \\ \varphi &= \frac{\pi}{4}.\end{aligned}$$

Putting this together, we obtain

$$\begin{aligned}&\int_{y=0}^{y=3}\int_{x=0}^{x=\sqrt{9-y^2}}\int_{z=\sqrt{x^2+y^2}}^{z=\sqrt{18-x^2-y^2}} (x^2+y^2+z^2)\,dz\,dx\,dy \\ &= \int_{\varphi=0}^{\varphi=\pi/4}\int_{\theta=0}^{\theta=\pi/2}\int_{\rho=0}^{\rho=3\sqrt{2}} \rho^4\sin\varphi\,d\rho\,d\theta\,d\varphi.\end{aligned}$$

Exercise 14.5.6:

Use rectangular, cylindrical, and spherical coordinates to set up triple integrals for finding the volume of the region inside the sphere $x^2+y^2+z^2=4$ but outside the cylinder $x^2+y^2=1$.

Now that we are familiar with the spherical coordinate system, let's find the volume of some known geometric figures, such as spheres and ellipsoids.

Example 14.5.10: Chapter Opener: Finding the Volume of l'Hemisphèric

Find the volume of the spherical planetarium in l'Hemisphèric in Valencia, Spain, which is five stories tall and has a radius of approximately 50 ft, using the equation $x^2 + y^2 + z^2 = r^2$.

Figure 14.5.11: (credit: modification of work by Javier Yaya Tur, Wikimedia Commons)

Solution

We calculate the volume of the ball in the first octant, where $x \le 0$, $y \le 0$, and $z \le 0$, using spherical coordinates, and then multiply the result by 8 for symmetry. Since we consider the region D as the first octant in the integral, the ranges of the variables are

$$0 \le \varphi \le \frac{\pi}{2}, 0 \le \rho \le r, 0 \le \theta \le \frac{\pi}{2}.$$

Therefore,

$$\begin{aligned} V = \iiint_D dx\,dy\,dz &= 8\int_{\theta=0}^{\theta=\pi/2}\int_{\rho=0}^{\rho=\pi}\int_{\varphi=0}^{\varphi=\pi/2} \rho^2 \sin\theta\, d\varphi\, d\rho\, d\varphi \\ &= 8\int_{\varphi=0}^{\varphi=\pi/2} d\varphi \int_{\rho=0}^{\rho=r} \rho^2 d\rho \int_{\theta=0}^{\theta=\pi/2} \sin\theta\, d\theta \\ &= 8\left(\frac{\pi}{2}\right)\left(\frac{r^3}{3}\right)(1) \\ &= \frac{4}{3}\pi r^3. \end{aligned}$$

This exactly matches with what we knew. So for a sphere with a radius of approximately 50 ft, the volume is $\frac{4}{3}\pi(50)^3 \approx 523,600\ ft^3$.

For the next example we find the volume of an ellipsoid.

Example 14.5.11: Finding the Volume of an Ellipsoid

Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Solution

We again use symmetry and evaluate the volume of the ellipsoid using spherical coordinates. As before, we use the first octant $x \le 0$, $y \le 0$, and $z \le 0$ and then multiply the result by 8.

In this case the ranges of the variables are

$$0 \le \varphi \le \frac{\pi}{2}\ 0 \le \rho \le 1, \text{ and } 0 \le \theta \le \frac{\pi}{2}.$$

Also, we need to change the rectangular to spherical coordinates in this way:

$$x = a\rho\cos\varphi\sin\theta,\ y = b\rho\sin\varphi\sin\theta, \text{ and } z = cp\cos\theta.$$

Then the volume of the ellipsoid becomes

$$\begin{aligned} V &= \iiint_D dx\,dy\,dz \\ &= 8\int_{\theta=0}^{\theta=\pi/2}\int_{\rho=0}^{\rho=1}\int_{\varphi=0}^{\varphi=\pi/2} abc\,\rho^2\sin\theta\,d\varphi\,d\rho\,d\theta \\ &= 8abc\int_{\varphi=0}^{\varphi=\pi/2} d\varphi\int_{\rho=0}^{\rho=1}\rho^2 d\rho\int_{\theta=0}^{\theta=\pi/2}\sin\theta\,d\theta \\ &= 8abc\left(\frac{\pi}{2}\right)\left(\frac{1}{3}\right)(1) \\ &= \frac{4}{3}\pi abc. \end{aligned}$$

Example 14.5.12: Finding the Volume of the Space Inside an Ellipsoid and Outside a Sphere

Find the volume of the space inside the ellipsoid $\frac{x^2}{75^2}+\frac{y^2}{80^2}+\frac{z^2}{90^2}=1$ and outside the sphere $x^2+y^2+z^2=50^2$.

Solution

This problem is directly related to the l'Hemisphèric structure. The volume of space inside the ellipsoid and outside the sphere might be useful to find the expense of heating or cooling that space. We can use the preceding two examples for the volume of the sphere and ellipsoid and then substract.

First we find the volume of the ellipsoid using $a = 75$ ft, $b = 80$ ft, and $c = 90$ ft in the result from Example. Hence the volume of the ellipsoid is

$$V_{ellipsoid} = \frac{4}{3}\pi(75)(80)(90) \approx 2,262,000\ ft^3.$$

From Example, the volume of the sphere is

$$V_{sphere} \approx 523,600\ ft^3.$$

Therefore, the volume of the space inside the ellipsoid $\frac{x^2}{75^2}+\frac{y^2}{80^2}+\frac{z^2}{90^2}=1$ and outside the sphere $x^2+y^2+z^2=50^2$ is approximately

$$V_{Hemispheric} = V_{ellipsoid} - V_{sphere} = 1,738,400\ ft^3.$$

Student Project: Hot air balloons

Hot air ballooning is a relaxing, peaceful pastime that many people enjoy. Many balloonist gatherings take place around the world, such as the Albuquerque International Balloon Fiesta. The Albuquerque event is the largest hot air balloon festival in the world, with over 500 balloons participating each year.

Figure 14.5.12: Balloons lift off at the 2001 Albuquerque International Balloon Fiesta. (credit: David Herrera, Flickr)

As the name implies, hot air balloons use hot air to generate lift. (Hot air is less dense than cooler air, so the balloon floats as long as the hot air stays hot.) The heat is generated by a propane burner suspended below the opening of the basket. Once the balloon takes off, the pilot controls the altitude of the balloon, either by using the burner to heat the air and ascend or by using

a vent near the top of the balloon to release heated air and descend. The pilot has very little control over where the balloon goes, however—balloons are at the mercy of the winds. The uncertainty over where we will end up is one of the reasons balloonists are attracted to the sport.

In this project we use triple integrals to learn more about hot air balloons. We model the balloon in two pieces. The top of the balloon is modeled by a half sphere of radius 28

feet. The bottom of the balloon is modeled by a frustum of a cone (think of an ice cream cone with the pointy end cut off). The radius of the large end of the frustum is 28 feet and the radius of the small end of the frustum is 28 feet. A graph of our balloon model and a cross-sectional diagram showing the dimensions are shown in the following figure.

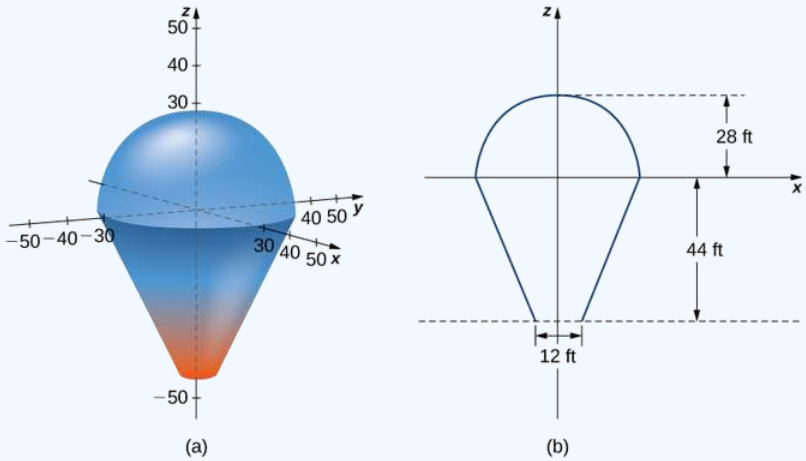


Figure 14.5.13: (a) Use a half sphere to model the top part of the balloon and a frustum of a cone to model the bottom part of the balloon. (b) A cross section of the balloon showing its dimensions.

We first want to find the volume of the balloon. If we look at the top part and the bottom part of the balloon separately, we see that they are geometric solids with known volume formulas. However, it is still worthwhile to set up and evaluate the integrals we would need to find the volume. If we calculate the volume using integration, we can use the known volume formulas to check our answers. This will help ensure that we have the integrals set up correctly for the later, more complicated stages of the project.

1. Find the volume of the balloon in two ways.

a. Use triple integrals to calculate the volume. Consider each part of the balloon separately. (Consider using spherical coordinates for the top part and cylindrical coordinates for the bottom part.)

b. Verify the answer using the formulas for the volume of a sphere, $V = \frac{4}{3}\pi r^3$, and for the volume of a cone, $V = \frac{1}{3}\pi r^2 h$.

In reality, calculating the temperature at a point inside the balloon is a tremendously complicated endeavor. In fact, an entire branch of physics (thermodynamics) is devoted to studying heat and temperature. For the purposes of this project, however, we are going to make some simplifying assumptions about how temperature varies from point to point within the balloon. Assume that just prior to liftoff, the temperature (in degrees Fahrenheit) of the air inside the balloon varies according to the function

$$T_0(r, \theta, z) = \frac{z - r}{10} + 210.$$

2. What is the average temperature of the air in the balloon just prior to liftoff? (Again, look at each part of the balloon separately, and do not forget to convert the function into spherical coordinates when looking at the top part of the balloon.)

Now the pilot activates the burner for 10 seconds. This action affects the temperature in a 12-foot-wide column 20 feet high, directly above the burner. A cross section of the balloon depicting this column in shown in the following figure

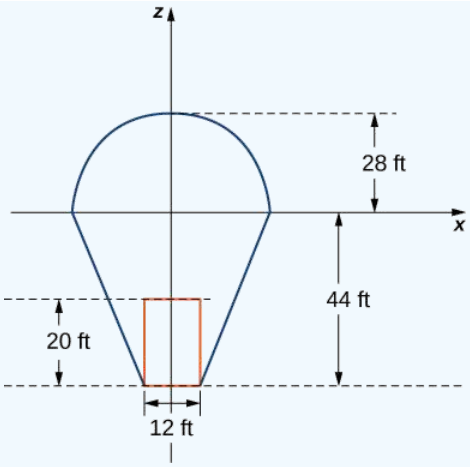


Figure 14.5.14: Activating the burner heats the air in a 20-foot-high, 12-foot-wide column directly above the burner.

Assume that after the pilot activates the burner for 10 seconds, the temperature of the air in the column described above *increases* according to the formula

$$H(r,\theta,z) = -2z - 48.$$

Then the temperature of the air in the column is given by

$$T_1(r,\theta,z) = \frac{z-r}{10} + 210 + (-2z - 48),$$

while the temperature in the remainder of the balloon is still given by

$$T_0(r,\theta,z) = \frac{z-r}{10} + 210.$$

3. Find the average temperature of the air in the balloon after the pilot has activated the burner for 10 seconds.

Key Concepts

- To evaluate a triple integral in cylindrical coordinates, use the iterated integral

$$\int_{\theta=\alpha}^{\theta=\beta} \int_{r=g_1(\theta)}^{r=g_2(\theta)} \int_{z=u_1(r,\theta)}^{u_2(r,\theta)} f(r,\theta,z) r \, dz \, dr \, d\theta.$$

- To evaluate a triple integral in spherical coordinates, use the iterated integral

$$\int_{\theta=\alpha}^{\theta=\beta} \int_{\rho=g_1(\theta)}^{\rho=g_2(\theta)} \int_{\varphi=u_1(r,\theta)}^{u_2(r,\theta)} f(\rho,\theta,\varphi)\, \rho^2 \sin\varphi \, d\varphi \, d\rho \, d\theta.$$

Key Equations

- **Triple integral in cylindrical coordinates**

$$\iiint_B g(s,y,z)dV = \iiint_B g(r\cos\theta,\, r\sin\theta,\, z) r \, dr \, d\theta \, dz = \iiint_B f(r,\theta,z) r \, dr \, d\theta \, dz$$

- **Triple integral in spherical coordinates**

$$\iiint_B f(\rho,\theta,\varphi)\rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta = \int_{\varphi=\gamma}^{\varphi=\psi} \int_{\theta=\alpha}^{\theta=\beta} \int_{\rho=a}^{\rho=b} f(\rho,\theta,\varphi)\rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

Glossary

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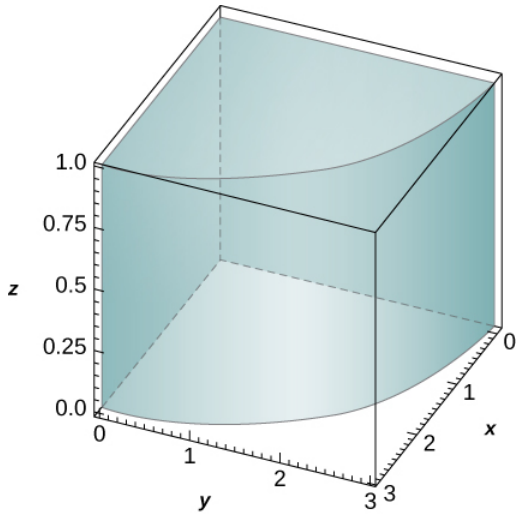
14.5E: Exercises for Section 14.5

In exercises 1 - 8, evaluate the triple integrals $\iiint_E f(x,y,z)\,dV$ over the solid E.

1. $f(x,y,z)=z, \quad B=\{(x,y,z)\,|\,x^2+y^2\le 9, \quad x\le 0, \quad y\le 0, \quad 0\le z\le 1\}$

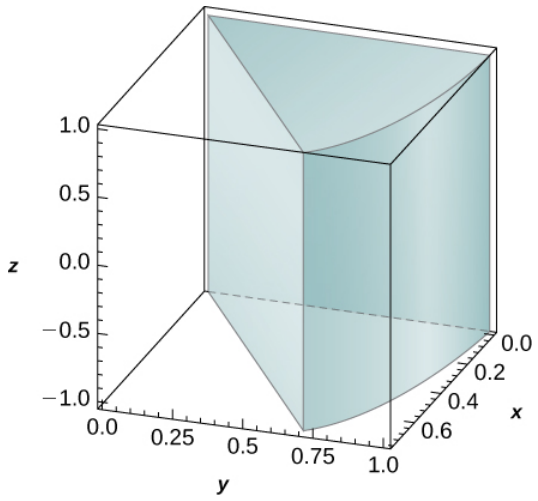


Answer

$\frac{9\pi}{8}$

2. $f(x,y,z)=xz^2,\ B=\{(x,y,z)\,|\,x^2+y^2\le 16,\ x\ge 0,\ y\le 0,\ -1\le z\le 1\}$

3. $f(x,y,z)=xy,\ B=\{(x,y,z)\,|\,x^2+y^2\le 1,\ x\ge 0,\ x\ge y,\ -1\le z\le 1\}$



Answer

$\frac{1}{8}$

4. $f(x,y,z)=x^2+y^2,\ B=\{(x,y,z)\,|\,x^2+y^2\le 4,\ x\ge 0,\ x\le y,\ 0\le z\le 3\}$

5. $f(x,y,z)=e^{\sqrt{x^2+y^2}},\ B=\{(x,y,z)\,|\,1\le x^2+y^2\le 4,\ y\le 0,\ x\le y\sqrt{3},\ 2\le z\le 3\}$

Answer

$\frac{\pi e^2}{6}$

6. $f(x,y,z)=\sqrt{x^2+y^2},\ B=\{(x,y,z)\,|\,1\le x^2+y^2\le 9,\ y\le 0,\ 0\le z\le 1\}$

7. a. Let B be a cylindrical shell with inner radius a outer radius b, and height c where $0<a<b$ and $c>0$. Assume that a function F defined on B can be expressed in cylindrical coordinates as $F(x,y,z)=f(r)+h(z)$, where f and h are differentiable functions. If $\int_a^b \bar{f}(r)\,dr=0$ and $\bar{h}(0)=0$, where $\bar{f}$ and $\bar{h}$ are antiderivatives of f and h, respectively, show that $\iiint_B F(x,y,z)\,dV=2\pi c(b\bar{f}(b)-a\bar{f}(a))+\pi(b^2-a^2)\bar{h}(c)$.

b. Use the previous result to show that $\iiint_B \left(z+\sin\sqrt{x^2+y^2}\right)\,dx\,dy\,dz=6\pi^2(\pi-2)$, where B is a cylindrical shell with inner radius π outer radius 2π, and height 2.

8. a. Let B be a cylindrical shell with inner radius a outer radius b and height c where $0<a<b$ and $c>0$. Assume that a function F defined on B can be expressed in cylindrical coordinates as $F(x,y,z)=f(r)g(\theta)f(z)$, where f, g, and h are differentiable functions. If $\int_a^b \tilde{f}(r)\,dr=0$, where $\tilde{f}$ is an antiderivative of f, show that $\iiint_B F(x,y,z)\,dV=[b\tilde{f}(b)-a\tilde{f}(a)][\tilde{g}(2\pi)-\tilde{g}(0)][\tilde{h}(c)-\tilde{h}(0)]$, where $\tilde{g}$ and $\tilde{h}$ are antiderivatives of g and h, respectively.

b. Use the previous result to show that $\iiint_B z\sin\sqrt{x^2+y^2}\,dx\,dy\,dz=-12\pi^2$, where B is a cylindrical shell with inner radius π outer radius 2π, and height 2.

In exercises 9 - 12, the boundaries of the solid E are given in cylindrical coordinates.

a. Express the region E in cylindrical coordinates.

b. Convert the integral $\iiint_E f(x,y,z)\,dV$ to cylindrical coordinates.

9. E is bounded by the right circular cylinder $r=4\sin\theta$, the $r\theta$-plane, and the sphere $r^2+z^2=16$.

Answer

a. $E=\{(r,\theta,z)\,|\,0\le\theta\le\pi,\ 0\le r\le 4\sin\theta,\ 0\le z\le\sqrt{16-r^2}\}$

b. $\int_0^{\pi}\int_0^{4\sin\theta}\int_0^{\sqrt{16-r^2}} f(r,\theta,z)r\,dz\,dr\,d\theta$

10. E is bounded by the right circular cylinder $r=\cos\theta$, the $r\theta$-plane, and the sphere $r^2+z^2=9$.

11. E is located in the first octant and is bounded by the circular paraboloid $z=9-3r^2$, the cylinder $r=\sqrt{3}$, and the plane $r(\cos\theta+\sin\theta)=20-z$.

Answer

a. $E=\{(r,\theta,z)\,|\,0\le\theta\le\frac{\pi}{2},\ 0\le r\le\sqrt{3},\ 9-r^2\le z\le 10-r(\cos\theta+\sin\theta)\}$

b. $\int_0^{\pi/2}\int_0^{\sqrt{3}}\int_{9-r^2}^{10-r(\cos\theta+\sin\theta)} f(r,\theta,z)r\,dz\,dr\,d\theta$

12. E is located in the first octant outside the circular paraboloid $z=10-2r^2$ and inside the cylinder $r=\sqrt{5}$ and is bounded also by the planes $z=20$ and $\theta=\frac{\pi}{4}$.

In exercises 13 - 16, the function f and region E are given.

a. Express the region E and the function f in cylindrical coordinates.

b. Convert the integral $\iiint_B f(x,y,z)\,dV$ into cylindrical coordinates and evaluate it.

13. $f(x,y,z)=x^2+y^2$, $E=\{(x,y,z)\,|\,0\le x^2+y^2\le 9,\ x\ge 0,\ y\ge 0,\ 0\le z\le x+3\}$

Answer

a. $E=\{(r,\theta,z)\,|\,0\le r\le 3,\ 0\le\theta\le\frac{\pi}{2},\ 0\le z\le r\ \cos\theta+3\}$,
$f(r,\theta,z)=\frac{1}{r\ \cos\theta+3}$

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b. $\int_0^3 \int_0^{\pi/2} \int_0^{r\ \cos\theta+3} \frac{r}{r\ \cos\theta+3}\, dz\ d\theta\ dr = \frac{9\pi}{4}$

14. $f(x,y,z) = x^2+y^2,\ E = \{(x,y,z) | 0 \le x^2+y^2 \le 4,\ y \ge 0,\ 0 \le z \le 3-x\}$

15. $f(x,y,z) = x,\ E = \{(x,y,z)\,|\,1 \le y^2+z^2 \le 9,\ 0 \le x \le 1-y^2-z^2\}$

Answer

a. $y = r\ \cos\theta,\ z = r\ \sin\theta,\ x = z,\ E = \{(r,\theta,z)\,|\,1 \le r \le 3,\ 0 \le \theta \le 2\pi,\ 0 \le z \le 1-r^2\},\ f(r,\theta,z) = z;$

b. $\int_1^3 \int_0^{2\pi} \int_0^{1-r^2} zr\ dz\ d\theta\ dr = \frac{356\pi}{3}$

16. $f(x,y,z) = y,\ E = \{(x,y,z)\,|\,1 \le x^2+z^2 \le 9,\ 0 \le y \le 1-x^2-z^2\}$

In exercises 17 - 24, find the volume of the solid E whose boundaries are given in rectangular coordinates.

17. E is above the xy-plane, inside the cylinder $x^2+y^2=1$, and below the plane $z=1$.

Answer

π

18. E is below the plane $z=1$ and inside the paraboloid $z=x^2+y^2$.

19. E is bounded by the circular cone $z=\sqrt{x^2+y^2}$ and $z=1$.

Answer

$\frac{\pi}{3}$

20. E is located above the xy-plane, below $z=1$, outside the one-sheeted hyperboloid $x^2+y^2-z^2=1$, and inside the cylinder $x^2+y^2=2$.

21. E is located inside the cylinder $x^2+y^2=1$ and between the circular paraboloids $z=1-x^2-y^2$ and $z=x^2+y^2$.

Answer

π

22. E is located inside the sphere $x^2+y^2+z^2=1$, above the xy-plane, and inside the circular cone $z=\sqrt{x^2+y^2}$.

23. E is located outside the circular cone $x^2+y^2=(z-1)^2$ and between the planes $z=0$ and $z=2$.

Answer

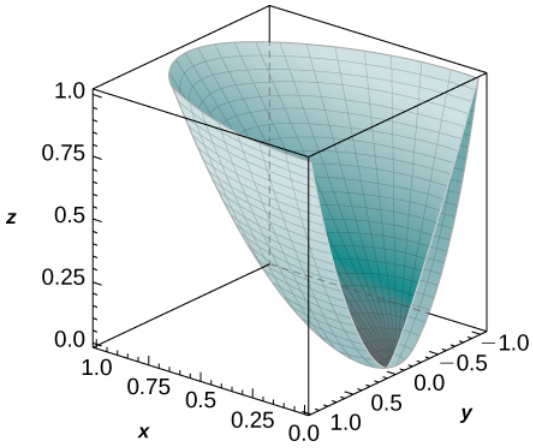
$\frac{4\pi}{3}$

24. E is located outside the circular cone $z=1-\sqrt{x^2+y^2}$, above the xy-plane, below the circular paraboloid, and between the planes $z=0$ and $z=2$.

25. [T] Use a computer algebra system (CAS) to graph the solid whose volume is given by the iterated integral in cylindrical coordinates $\int_{-\pi/2}^{\pi/2} \int_0^1 \int_{r^2}^r r\,dz\,dr\,d\theta$. Find the volume V of the solid. Round your answer to four decimal places.

Answer

$V = \frac{pi}{12} \approx 0.2618$



26. [T] Use a CAS to graph the solid whose volume is given by the iterated integral in cylindrical coordinates $\int_0^{\pi/2}\int_0^1\int_{r^4}^{r} r\,dz\,dr\,d\theta$. Find the volume E of the solid. Round your answer to four decimal places.

27. Convert the integral $\int_0^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{x^2+y^2}^{\sqrt{x^2+y^2}} xz\,dz\,dx\,dy$ into an integral in cylindrical coordinates.

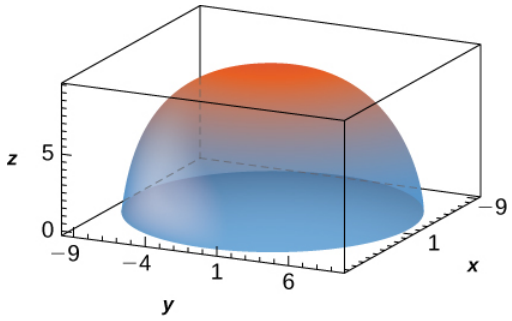
Answer

$$\int_0^1\int_0^{\pi}\int_{r^2}^{r} zr^2\cos\theta\,dz\,d\theta\,dr$$

28. Convert the integral $\int_0^2\int_0^y\int_0^1 (xy+z)\,dz\,dx\,dy$ into an integral in cylindrical coordinates.

In exercises 29 - 32, evaluate the triple integral $\iiint_B f(x,y,z)\,dV$ over the solid B.

29. $f(x,y,z)=1,\ B=\{(x,y,z)\,|\,x^2+y^2+z^2\le 90,\ z\ge 0\}$



[Hide Solution]

Answer

$180\pi\sqrt{10}$

30. $f(x,y,z)=1-\sqrt{x^2+y^2+z^2},\ B=\{(x,y,z)\,|\,x^2+y^2+z^2\le 9,\ y\ge 0,\ z\ge 0\}$

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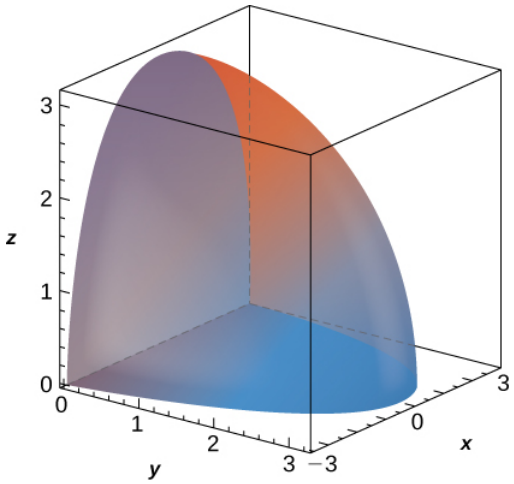


31. $f(x,y,z)=\sqrt{x^2+y^2},\ B$ is bounded above by the half-sphere $x^2+y^2+z^2=9$ with $z\geq 0$ and below by the cone $2z^2=x^2+y^2$.

Answer

$\frac{81\pi(\pi-2)}{16}$

32. $f(x,y,z)=\sqrt{x^2+y^2},\ B$ is bounded above by the half-sphere $x^2+y^2+z^2=16$ with $z\geq 0$ and below by the cone $2z^2=x^2+y^2$.

33. Show that if $F(\rho,\theta,\varphi)=f(\rho)g(\theta)h(\varphi)$ is a continuous function on the spherical box $B=\{(\rho,\theta,\varphi)\,|\,a\leq\rho\leq b,\ \alpha\leq\theta\leq\beta,\ \gamma\leq\varphi\leq\psi\}$, then $\iiint_B F\,dV=\left(\int_a^b \rho^2 f(\rho)\,dr\right)\left(\int_\alpha^\beta g(\theta)\,d\theta\right)\left(\int_\gamma^\psi h(\varphi)\,\sin\varphi\,d\varphi\right)$.

34. A function F is said to have spherical symmetry if it depends on the distance to the origin only, that is, it can be expressed in spherical coordinates as $F(x,y,z)=f(\rho)$, where $\rho=\sqrt{x^2+y^2+z^2}$. Show that $\iiint_B F(x,y,z)\,dV=2\pi\int_a^b \rho^2 f(\rho)\,d\rho$, where B is the region between the upper concentric hemispheres of radii a and b centered at the origin, with $0<a<b$ and F a spherical function defined on B.

Use the previous result to show that $\iiint_B (x^2+y^2+z^2)\sqrt{x^2+y^2+z^2}dV=21\pi$, where $B=\{(x,y,z)\,|\,1\leq x^2+y^2+z^2\leq 2,\ z\geq 0\}$.

35. Let B be the region between the upper concentric hemispheres of radii a and b centered at the origin and situated in the first octant, where $0<a<b$. Consider F a function defined on B whose form in spherical coordinates (ρ,θ,φ) is $F(x,y,z)=f(\rho)\cos\varphi$. Show that if $g(a)=g(b)=0$ and $\int_a^b h(\rho)\,d\rho=0$, then $\iiint_B F(x,y,z)\,dV=\frac{\pi^2}{4}[ah(a)-bh(b)]$, where g is an antiderivative of f and h is an antiderivative of g.

Use the previous result to show that $\iiint_B=\frac{z\cos\sqrt{x^2+y^2+z^2}}{\sqrt{x^2+y^2+z^2}}\,dV=\frac{3\pi^2}{2}$, where B is the region between the upper concentric hemispheres of radii π and 2π centered at the origin and situated in the first octant.

In exercises 36 - 39, the function f and region E are given.

a. Express the region E and function f in cylindrical coordinates.

b. Convert the integral $\iiint_B f(x,y,z)\,dV$ into cylindrical coordinates and evaluate it.

36. $f(x,y,z)=z;\ E=\{(x,y,z)\,|\,0\leq x^2+y^2+z^2\leq 1,\ z\geq 0\}$

37. $f(x,y,z)=x+y;\ E=\{(x,y,z)\,|\,1\leq x^2+y^2+z^2\leq 2,\ z\geq 0,\ y\geq 0\}$

Answer

a. $f(\rho,\theta,\varphi)=\rho\,\sin\varphi\,(\cos\theta+\sin\theta),\ E=\{(\rho,\theta,\varphi)\,|\,1\leq\rho\leq 2,\ 0\leq\theta\leq\pi,\ 0\leq\varphi\leq\frac{\pi}{2}\}$;

b. $\int_0^\pi\int_0^{\pi/2}\int_1^2 \rho^3\cos\varphi\,\sin\varphi\,d\rho\,d\varphi\,d\theta=\frac{15\pi}{8}$

38. $f(x,y,z)=2xy;\ E=\{(x,y,z)\,|\,\sqrt{x^2+y^2}\le z\le\sqrt{1-x^2-y^2},\ x\ge 0,\ y\ge 0\}$

39. $f(x,y,z)=z;\ E=\{(x,y,z)\,|\,x^2+y^2+z^2-2x\le 0,\ \sqrt{x^2+y^2}\le z\}$

Answer

a. $f(\rho,\theta,\varphi)=\rho\ \cos\varphi;\ E=\{(\rho,\theta,\varphi)\,|\,0\le\rho\le 2\ \cos\varphi,\ 0\le\theta\le\frac{\pi}{2},\ 0\le\varphi\le\frac{\pi}{4}\}$;

b. $\int_0^{\pi/2}\int_0^{\pi/4}\int_0^{2\ \cos\varphi}\rho^3\sin\varphi\ \cos\varphi\ d\rho\ d\varphi\ d\theta=\frac{7\pi}{24}$

In exercises 40 - 41, find the volume of the solid E whose boundaries are given in rectangular coordinates.

40. $E=\{(x,y,z)\,|\,\sqrt{x^2+y^2}\le z\le\sqrt{16-x^2-y^2},\ x\ge 0,\ y\ge 0\}$

41. $E=\{(x,y,z)\,|\,x^2+y^2+z^2-2z\le 0,\ \sqrt{x^2+y^2}\le z\}$

Answer

$\frac{\pi}{4}$

42. Use spherical coordinates to find the volume of the solid situated outside the sphere $\rho=1$ and inside the sphere $\rho=\cos\varphi$, with $\varphi\in[0,\frac{\pi}{2}]$.

43. Use spherical coordinates to find the volume of the ball $\rho\le 3$ that is situated between the cones $\varphi=\frac{\pi}{4}$ and $\varphi=\frac{\pi}{3}$.

Answer

$9\pi(\sqrt{2}-1)$

44. Convert the integral $\int_{-4}^{4}\int_{-\sqrt{16-y^2}}^{\sqrt{16-y^2}}\int_{-\sqrt{16-x^2-y^2}}^{\sqrt{16-x^2-y^2}}(x^2+y^2+z^2)\,dz\,dx\,dy$ into an integral in spherical coordinates.

45. Convert the integral $\int_0^4\int_0^{\sqrt{16-x^2}}\int_{-\sqrt{16-x^2-y^2}}^{\sqrt{16-x^2-y^2}}(x^2+y^2+z^2)^2\,dz\,dy\,dx$ into an integral in spherical coordinates.

Answer

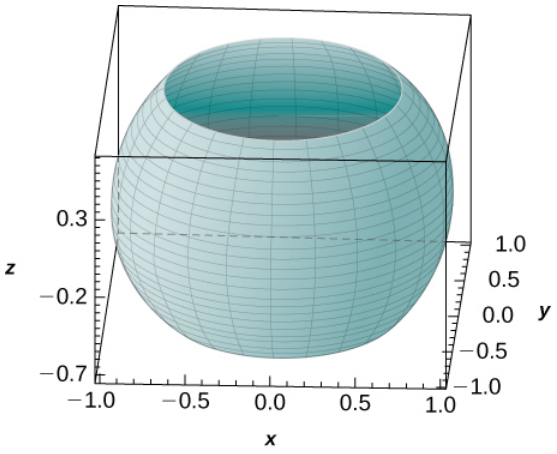
$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^4\rho^6\sin\varphi\,d\rho\,d\phi\,d\theta$

46. Convert the integral $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{16-x^2-y^2}}dz\,dy\,dx$ into an integral in spherical coordinates and evaluate it.

47. [T] Use a CAS to graph the solid whose volume is given by the iterated integral in spherical coordinates $\int_{\pi/2}^{\pi}\int_{5\pi}^{\pi/6}\int_0^2\rho^2\sin\varphi\ d\rho\ d\varphi\ d\theta$. Find the volume V of the solid. Round your answer to three decimal places.

Answer

$V=\frac{4\pi\sqrt{3}}{3}\approx 7.255$



48. [T] Use a CAS to graph the solid whose volume is given by the iterated integral in spherical coordinates as $\int_0^{2\pi}\int_{3\pi/4}^{\pi/4}\int_0^1 \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$. Find the volume V of the solid. Round your answer to three decimal places.

49. [T] Use a CAS to evaluate the integral $\iiint_E (x^2+y^2)\, dV$ where E lies above the paraboloid $z = x^2 + y^2$ and below the plane $z = 3y$.

Answer

$\frac{343\pi}{32}$

50. [T]

a. Evaluate the integral $\iiint_E e^{\sqrt{x^2+y^2+z^2}}\, dV$, where E is bounded by spheres $4x^2+4y^2+4z^2=1$ and $x^2+y^2+z^2=1$.

b. Use a CAS to find an approximation of the previous integral. Round your answer to two decimal places.

51. Express the volume of the solid inside the sphere $x^2+y^2+z^2=16$ and outside the cylinder $x^2+y^2=4$ as triple integrals in cylindrical coordinates and spherical coordinates, respectively.

Answer

$\int_0^{2\pi}\int_2^4\int_{-\sqrt{16-r^2}}^{\sqrt{16-r^2}} r\, dz\, dr\, d\theta$ and $\int_{\pi/6}^{5\pi/6}\int_0^{2\pi}\int_{2\csc\phi}^4 \rho^2 \sin\rho \, d\rho \, d\theta \, d\phi$

52. Express the volume of the solid inside the sphere $x^2+y^2+z^2=16$ and outside the cylinder $x^2+y^2=4$ that is located in the first octant as triple integrals in cylindrical coordinates and spherical coordinates, respectively.

53. The power emitted by an antenna has a power density per unit volume given in spherical coordinates by $p(\rho,\theta,\varphi) = \frac{P_0}{\rho^2}\cos^2\theta \, \sin^4\varphi$, where P_0 is a constant with units in watts. The total power within a sphere B of radius r meters is defined as $P = \iiint_B p(\rho,\theta,\varphi)\, dV$. Find the total power P.

Answer

$P = \frac{32P_0\pi}{3}$ watts

54. Use the preceding exercise to find the total power within a sphere B of radius 5 meters when the power density per unit volume is given by $p(\rho,\theta,\varphi) = \frac{30}{\rho^2}\cos^2\theta\sin^4\varphi$.

55. A charge cloud contained in a sphere B of radius r centimeters centered at the origin has its charge density given by $q(x,y,z) = k\sqrt{x^2+y^2+z^2}\,\frac{\mu C}{cm^3}$, where $k>0$. The total charge contained in B is given by $Q = \iiint_B q(x,y,z)\, dV$. Find the total charge Q.

Answer

$Q = kr^4\pi\mu C$

56. Use the preceding exercise to find the total charge cloud contained in the unit sphere if the charge density is $q(x, y, z) = 20\sqrt{x^2 + y^2 + z^2}\,\frac{\mu C}{cm^3}$.

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14.6: Calculating Centers of Mass and Moments of Inertia

Learning Objectives

- Use double integrals to locate the center of mass of a two-dimensional object.
- Use double integrals to find the moment of inertia of a two-dimensional object.
- Use triple integrals to locate the center of mass of a three-dimensional object.

We have already discussed a few applications of multiple integrals, such as finding areas, volumes, and the average value of a function over a bounded region. In this section we develop computational techniques for finding the center of mass and moments of inertia of several types of physical objects, using double integrals for a lamina (flat plate) and triple integrals for a three-dimensional object with variable density. The density is usually considered to be a constant number when the lamina or the object is homogeneous; that is, the object has uniform density.

Center of Mass in Two Dimensions

The center of mass is also known as the center of gravity if the object is in a uniform gravitational field. If the object has uniform density, the center of mass is the geometric center of the object, which is called the centroid. Figure **14.6.1** shows a point P as the center of mass of a lamina. The lamina is perfectly balanced about its center of mass.

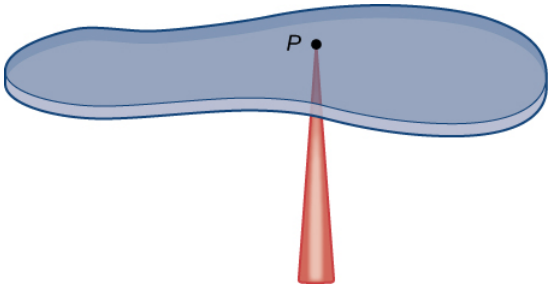

Figure **14.6.1**: A lamina is perfectly balanced on a spindle if the lamina's center of mass sits on the spindle.

To find the coordinates of the center of mass $P(\bar{x},\bar{y})$ of a lamina, we need to find the moment M_x of the lamina about the x-axis and the moment M_y about the y-axis. We also need to find the mass m of the lamina. Then

$$\bar{x}=\frac{M_y}{m}$$

and

$$\bar{y}=\frac{M_x}{m}.$$

Refer to Moments and Centers of Mass for the definitions and the methods of single integration to find the center of mass of a one-dimensional object (for example, a thin rod). We are going to use a similar idea here except that the object is a two-dimensional lamina and we use a double integral.

If we allow a constant density function, then $\bar{x}=\frac{M_y}{m}$ and $\bar{y}=\frac{M_x}{m}$ give the *centroid* of the lamina.

Suppose that the lamina occupies a region R in the xy-plane and let $\rho(x,y)$ be its density (in units of mass per unit area) at any point (x,y). Hence,

$$\rho(x,y)=\lim_{\Delta A\to 0}\frac{\Delta m}{\Delta A}$$

where Δm and ΔA are the mass and area of a small rectangle containing the point (x,y) and the limit is taken as the dimensions of the rectangle go to 0 (see the following figure).

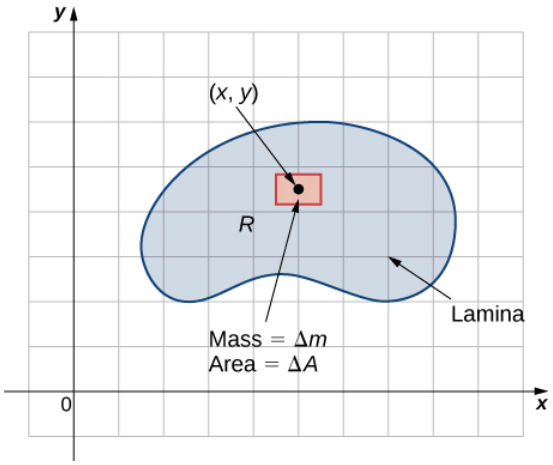

Figure **14.6.2**: The density of a lamina at a point is the limit of its mass per area in a small rectangle about the point as the area goes to zero.

Just as before, we divide the region R into tiny rectangles R_{ij} with area ΔA and choose (x_{ij}^*, y_{ij}^*) as sample points. Then the mass m_{ij} of each R_{ij} is equal to $\rho(x_{ij}^*, y_{ij}^*)\Delta A$ (Figure **14.6.2**). Let k and l be the number of subintervals in x and y respectively. Also, note that the shape might not always be rectangular but the limit works anyway, as seen in previous sections.

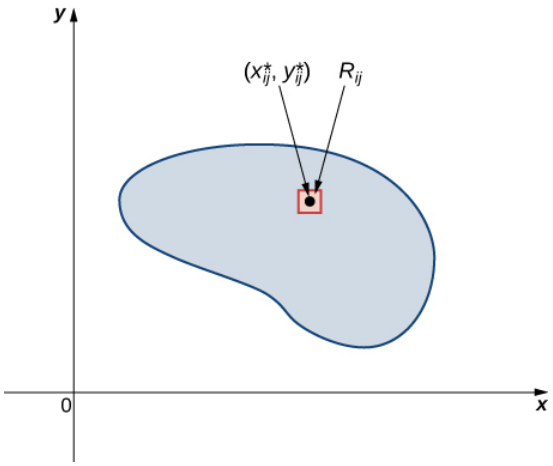

Figure **14.6.3**: Subdividing the lamina into tiny rectangles R_{ij} each containing a sample point (x_{ij}^*, y_{ij}^*).

Hence, the mass of the lamina is

$$m = \lim_{k,l\to\infty} \sum_{i=1}^{k} \sum_{j=1}^{l} m_{ij} = \lim_{k,l\to\infty} \sum_{i=1}^{k} \sum_{j=1}^{l} \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint_R \rho(x,y)dA.$$

Let's see an example now of finding the total mass of a triangular lamina.

Example 14.6.1: Finding the Total Mass of a Lamina

Consider a triangular lamina R with vertices $(0,0)$, $(0,3)$, $(3,0)$ and with density $\rho(x,y) = xy\, kg/m^2$. Find the total mass.

Solution

A sketch of the region R is always helpful, as shown in the following figure.

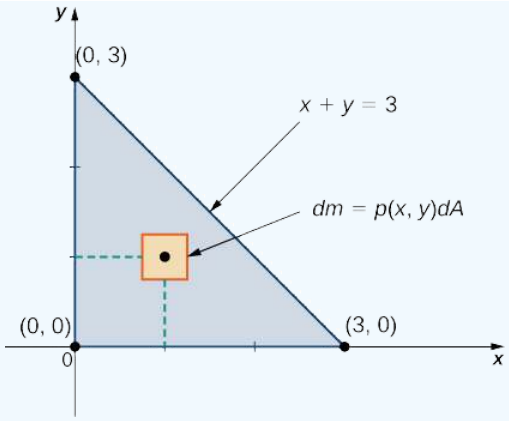

Figure **14.6.4**: A lamina in the xy-plane with density $\rho(x,y) = xy$.

Using the expression developed for mass, we see that

$$m = \iint_R dm = \iint_R \rho(x,y)dA = \int_{x=0}^{x=3}\int_{y=0}^{y=3-x} xy\,dy\,dx = \int_{x=0}^{x=3}\left[x\frac{y^2}{2}\Big|_{y=0}^{y=3}\right]dx$$
$$= \int_{x=0}^{x=3}\frac{1}{2}x(3-x)^2dx = \left[\frac{9x^2}{4} - x^3 + \frac{x^4}{8}\right]\Big|_{x=0}^{x=3} = \frac{27}{8}.$$

The computation is straightforward, giving the answer $m = \frac{27}{8}\ kg$.

? Exercise 14.6.1

Consider the same region R as in the previous example, and use the density function $\rho(x,y) = \sqrt{xy}$. Find the total mass.

Now that we have established the expression for mass, we have the tools we need for calculating moments and centers of mass. The moment M_z about the x-axis for R is the limit of the sums of moments of the regions R_{ij} about the x-axis. Hence

$$M_x = \lim_{k,l\to\infty}\sum_{i=1}^{k}\sum_{j=1}^{l}(y_{ij}^*)m_{ij} = \lim_{k,l\to\infty}\sum_{i=1}^{k}\sum_{j=1}^{l}(y_{ij}^*)\rho(x_{ij}^*, y_{ij}^*)\,\Delta A = \iint_R y\rho(x,y)\,dA$$

Similarly, the moment M_y about the y-axis for R is the limit of the sums of moments of the regions R_{ij} about the y-axis. Hence

$$M_y = \lim_{k,l\to\infty}\sum_{i=1}^{k}\sum_{j=1}^{l}(x_{ij}^*)m_{ij} = \lim_{k,l\to\infty}\sum_{i=1}^{k}\sum_{j=1}^{l}(x_{ij}^*)\rho(x_{ij}^*, y_{ij}^*)\,\Delta A = \iint_R x\rho(x,y)\,dA$$

✓ Example 14.6.2: Finding Moments

Consider the same triangular lamina R with vertices $(0,0)$, $(0,3)$, $(3,0)$ and with density $\rho(x,y) = xy$. Find the moments M_x and M_y.

Solution

Use double integrals for each moment and compute their values:

$$M_x = \iint_R y\rho(x,y)\,dA = \int_{x=0}^{x=3}\int_{y=0}^{y=3-x} xy^2\,dy\,dx = \frac{81}{20},$$
$$M_y = \iint_R x\rho(x,y)\,dA = \int_{x=0}^{x=3}\int_{y=0}^{y=3-x} x^2y\,dy\,dx = \frac{81}{20},$$

The computation is quite straightforward.

Exercise 14.6.2

Consider the same lamina R as above and use the density function $\rho(x,y)=\sqrt{xy}$. Find the moments M_x and M_y.

Finally we are ready to restate the expressions for the center of mass in terms of integrals. We denote the x-coordinate of the center of mass by $\bar{x}$ and the y-coordinate by $\bar{y}$. Specifically,

$$\bar{x}=\frac{M_y}{m}=\frac{\iint_R x\rho(x,y)\,dA}{\iint_R \rho(x,y)\,dA}$$

and

$$\bar{y}=\frac{M_x}{m}=\frac{\iint_R y\rho(x,y)\,dA}{\iint_R \rho(x,y)\,dA}$$

Example 14.6.3: center of mass

Again consider the same triangular region R with vertices $(0,0)$, $(0,3)$, $(3,0)$ and with density function $\rho(x,y)=xy$. Find the center of mass.

Solution

Using the formulas we developed, we have

$$\bar{x}=\frac{M_y}{m}=\frac{\iint_R x\rho(x,y)\,dA}{\iint_R \rho(x,y)\,dA}=\frac{81/20}{27/8}=\frac{6}{5},$$

$$\bar{y}=\frac{M_x}{m}=\frac{\iint_R y\rho(x,y)\,dA}{\iint_R \rho(x,y)\,dA}=\frac{81/20}{27/8}=\frac{6}{5}.$$

Therefore, the center of mass is the point $\left(\frac{6}{5},\frac{6}{5}\right)$.

Analysis

If we choose the density $\rho(x,y)$ instead to be uniform throughout the region (i.e., constant), such as the value 1 (any constant will do), then we can compute the centroid,

$$x_c=\frac{M_y}{m}=\frac{\iint_R x\,dA}{\iint_R dA}=\frac{9/2}{9/2}=1,$$

$$y_c=\frac{M_x}{m}=\frac{\iint_R y\,dA}{\iint_R dA}=\frac{9/2}{9/2}=1.$$

Notice that the center of mass $\left(\frac{6}{5},\frac{6}{5}\right)$ is not exactly the same as the centroid $(1,1)$ of the triangular region. This is due to the variable density of R If the density is constant, then we just use $\rho(x,y)=c$ (constant). This value cancels out from the formulas, so for a constant density, the center of mass coincides with the centroid of the lamina.

Exercise 14.6.3

Again use the same region R as above and use the density function $\rho(x,y)=\sqrt{xy}$. Find the center of mass.

Once again, based on the comments at the end of Example 14.6.3, we have expressions for the centroid of a region on the plane:

$$x_c=\frac{M_y}{m}=\frac{\iint_R x\,dA}{\iint_R dA}\text{ and }y_c=\frac{M_x}{m}=\frac{\iint_R y\,dA}{\iint_R dA}.$$

We should use these formulas and verify the centroid of the triangular region referred to in the last three examples.

Example 14.6.4: Finding Mass, Moments, and Center of Mass

Find the mass, moments, and the center of mass of the lamina of density $\rho(x,y) = x + y$ occupying the region R under the curve $y = x^2$ in the interval $0 \le x \le 2$ (see the following figure).

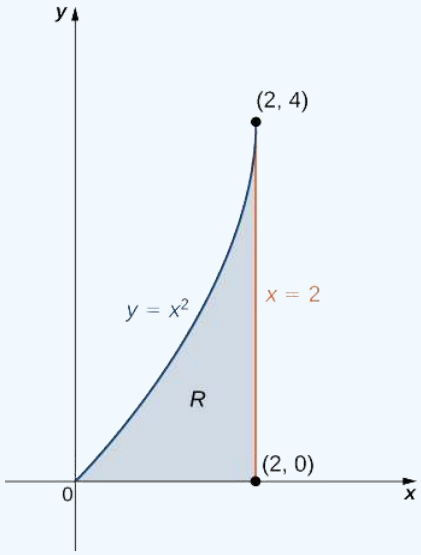

Figure 14.6.5: Locating the center of mass of a lamina R with density $\rho(x,y) = x + y$.

Solution

First we compute the mass m. We need to describe the region between the graph of $y = x^2$ and the vertical lines $x = 0$ and $x = 2$:

$$m = \iint_R dm = \iint_R \rho(x,y)\,dA = \int_{x=0} x = 2 \int_{y=0}^{y=x^2} (x+y)dy\,dx$$
$$= \int_{x=0}^{x=2} \left[xy + \frac{y^2}{2}\Big|_{y=0}^{y=x^2} \right] dx$$
$$= \int_{x=0}^{x=2} \left[x^3 + \frac{x^4}{2} \right] dx = \left[\frac{x^4}{4} + \frac{x^5}{10} \right]\Big|_{x=0}^{x=2} = \frac{36}{5}.$$

Now compute the moments M_x and M_y:

$$M_x = \iint_R y\rho(x,y)\,dA = \int_{x=0}^{x=2} \int_{y=0}^{y=x^2} y(x+y)\,dy\,dx = \frac{80}{7},$$
$$M_y = \iint_R x\rho(x,y)\,dA = \int_{x=0}^{x=2} \int_{y=0}^{y=x^2} x(x+y)\,dy\,dx = \frac{176}{15}.$$

Finally, evaluate the center of mass,

$$\bar{x} = \frac{M_y}{m} = \frac{\iint_R x\rho(x,y)\,dA}{\iint_R \rho(x,y)\,dA} = \frac{176/15}{36/5} = \frac{44}{27},$$
$$\bar{y} = \frac{M_x}{m} = \frac{\iint_R y\rho(x,y)\,dA}{\iint_R \rho(x,y)\,dA} = \frac{80/7}{36/5} = \frac{100}{63}.$$

Hence the center of mass is $(\bar{x}, \bar{y}) = \left(\frac{44}{27}, \frac{100}{63} \right)$.

Exercise 14.6.4

Calculate the mass, moments, and the center of mass of the region between the curves $y = x$ and $y = x^2$ with the density function $\rho(x, y) = x$ in the interval $0 \le x \le 1$.

Example 14.6.5: Finding a Centroid

Find the centroid of the region under the curve $y = e^x$ over the interval $1 \le x \le 3$ (Figure 14.6.6).

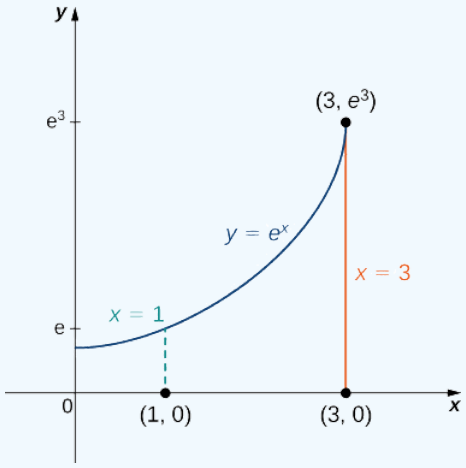

Figure 14.6.6: Finding a centroid of a region below the curve $y = e^x$.

Solution

To compute the centroid, we assume that the density function is constant and hence it cancels out:

$$x_c = \frac{M_y}{m} = \frac{\iint_R x\,dA}{\iint_R dA} \text{ and } y_c = \frac{M_x}{m} = \frac{\iint_R y\,dA}{\iint_R dA},$$

$$x_c = \frac{M_y}{m} = \frac{\iint_R x\,dA}{\iint_R dA} = \frac{\int_{x=1}^{x=3}\int_{y=0}^{y=e^x} x\,dy\,dx}{\int_{x=1}^{x=3}\int_{y=0}^{y=e^x} dy\,dx} = \frac{\int_{x=1}^{x=3} xe^x dx}{\int_{x=1}^{x=3} e^x dx} = \frac{2e^3}{e^3 - e} = \frac{2e^2}{e^2 - 1},$$

$$y_c = \frac{M_x}{m} = \frac{\iint_R y\,dA}{\iint_R dA} = \frac{\int_{x=1}^{x=3}\int_{y=0}^{y=e^x} y\,dy\,dx}{\int_{x=1}^{x=3}\int_{y=0}^{y=e^x} dy\,dx} = \frac{\int_{x=1}^{x=3} \frac{e^{2x}}{2} dx}{\int_{x=1}^{x=3} e^x dx}$$

$$= \frac{\frac{1}{4}e^2(e^4 - 1)}{e(e^2 - 1)} = \frac{1}{4}e(e^2 + 1).$$

Thus the centroid of the region is

$$(x_c, y_c) = \left(\frac{2e^2}{e^2 - 1}, \frac{1}{4}e(e^2 + 1)\right).$$

Exercise 14.6.5

Calculate the centroid of the region between the curves $y = x$ and $y = \sqrt{x}$ with uniform density in the interval $0 \le x \le 1$.

Moments of Inertia

For a clear understanding of how to calculate moments of inertia using double integrals, we need to go back to the general definition in Section 6.6. The moment of inertia of a particle of mass m about an axis is mr^2 where r is the distance of the particle from the axis. We can see from Figure 14.6.3 that the moment of inertia of the subrectangle R_{ij} about the x-axis is $(y_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*)\Delta A$. Similarly, the moment of inertia of the subrectangle R_{ij} about the y-axis is $(x_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*)\Delta A$. The moment of

inertia is related to the rotation of the mass; specifically, it measures the tendency of the mass to resist a change in rotational motion about an axis.

The moment of inertia I_x about the x-axis for the region R is the limit of the sum of moments of inertia of the regions R_{ij} about the x-axis. Hence

$$I_x = \lim_{k,l\to\infty} \sum_{i=1}^{k} \sum_{j=1}^{l} (y_{ij}^*)^2 m_{ij} = \lim_{k,l\to\infty} \sum_{i=1}^{k} \sum_{j=1}^{l} (y_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*)\, \Delta A = \iint_R y^2 \rho(x,y)\, dA.$$

Similarly, the moment of inertia I_y about the y-axis for R is the limit of the sum of moments of inertia of the regions R_{ij} about the y -axis. Hence

$$I_y = \lim_{k,l\to\infty} \sum_{i=1}^{k} \sum_{j=1}^{l} (x_{ij}^*)^2 m_{ij} = \lim_{k,l\to\infty} \sum_{i=1}^{k} \sum_{j=1}^{l} (x_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*)\, \Delta A = \iint_R x^2 \rho(x,y)\, dA.$$

Sometimes, we need to find the moment of inertia of an object about the origin, which is known as the polar moment of inertia. We denote this by I_0 and obtain it by adding the moments of inertia I_x and I_y. Hence

$$I_0 = I_x + I_y = \iint_R (x^2 + y^2) \rho(x,y)\, dA.$$

All these expressions can be written in polar coordinates by substituting $x = r\cos\theta$, $y = r\sin\theta$, and $dA = r\, dr\, d\theta$. For example, $I_0 = \iint_R r^2 \rho(r\cos\theta, r\sin\theta)\, dA$.

Example 14.6.6: Finding Moments of Inertia for a Triangular Lamina

Use the triangular region R with vertices $(0,0)$, $(2,2)$, and $(2,0)$ and with density $\rho(x,y) = xy$ as in previous examples. Find the moments of inertia.

Solution

Using the expressions established above for the moments of inertia, we have

$$I_x = \iint_R y^2 \rho(x,y)\, dA = \int_{x=0}^{x=2} \int_{y=0}^{y=x} xy^3\, dy\, dx = \frac{8}{3},$$

$$I_y = \iint_R x^2 \rho(x,y)\, dA = \int_{x=0}^{x=2} \int_{y=0}^{y=x} x^3 y\, dy\, dx = \frac{16}{3},$$

$$I_0 = \iint_R (x^2 + y^2) \rho(x,y)\, dA = \int_0^2 \int_0^x (x^2 + y^2) xy\, dy\, dx = I_x + I_y = 8$$

Exercise 14.6.6

Again use the same region R as above and the density function $\rho(x,y) = \sqrt{xy}$. Find the moments of inertia.

As mentioned earlier, the moment of inertia of a particle of mass m about an axis is mr^2 where r is the distance of the particle from the axis, also known as the **radius of gyration**.

Hence the radii of gyration with respect to the x-axis, the y-axis and the origin are

$$R_x = \sqrt{\frac{I_x}{m}},\ R_y = \sqrt{\frac{I_y}{m}},\ \text{and}\ R_0 = \sqrt{\frac{I_0}{m}},$$

respectively. In each case, the radius of gyration tells us how far (perpendicular distance) from the axis of rotation the entire mass of an object might be concentrated. The moments of an object are useful for finding information on the balance and torque of the object about an axis, but radii of gyration are used to describe the distribution of mass around its centroidal axis. There are many applications in engineering and physics. Sometimes it is necessary to find the radius of gyration, as in the next example.

Example 14.6.7: Finding the Radius of Gyration for a Triangular Lamina

Consider the same triangular lamina R with vertices $(0,0)$, $(2,2)$, and $(2,0)$ and with density $\rho(x,y) = xy$ as in previous examples. Find the radii of gyration with respect to the x-axis the y-axis and the origin.

Solution

If we compute the mass of this region we find that $m = 2$. We found the moments of inertia of this lamina in Example 14.6.4. From these data, the radii of gyration with respect to the x-axis, y-axis and the origin are, respectively,

$$R_x = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{8/3}{2}} = \sqrt{\frac{8}{6}} = \frac{2\sqrt{3}}{3}, \tag{14.6.1}$$

$$R_y = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{16/3}{2}} = \sqrt{\frac{8}{3}} = \frac{2\sqrt{6}}{3}, \tag{14.6.2}$$

$$R_0 = \sqrt{\frac{I_0}{m}} = \sqrt{\frac{8}{2}} = \sqrt{4} = 2. \tag{14.6.3}$$

Exercise 14.6.7

Use the same region R from Example 14.6.7 and the density function $\rho(x,y) = \sqrt{xy}$. Find the radii of gyration with respect to the x-axis, the y-axis, and the origin.

Center of Mass and Moments of Inertia in Three Dimensions

All the expressions of double integrals discussed so far can be modified to become triple integrals.

Definition

If we have a solid object Q with a density function $\rho(x,y,z)$ at any point (x,y,z) in space, then its mass is

$$m = \iiint_Q \rho(x,y,z)\,dV.$$

Its moments about the xy-plane the xz-plane and the yz-plane are

$$M_{xy} = \iiint_Q z\rho(x,y,z)\,dV,\ M_{xz} = \iiint_Q y\rho(x,y,z)\,dV,\ M_{yz} = \iiint_Q x\rho(x,y,z)\,dV.$$

If the center of mass of the object is the point $(\bar{x},\bar{y},\bar{z})$, then

$$\bar{x} = \frac{M_{yz}}{m},\ \bar{y} = \frac{M_{xz}}{m},\ \bar{z} = \frac{M_{xy}}{m}.$$

Also, if the solid object is homogeneous (with constant density), then the center of mass becomes the centroid of the solid. Finally, the moments of inertia about the yz-plane, xz-plane, and the xy-plane are

$$I_x = \iiint_Q (y^2+z^2)\,\rho(x,y,z)\,dV,$$

$$I_y = \iiint_Q (x^2+z^2)\,\rho(x,y,z)\,dV,$$

$$I_z = \iiint_Q (x^2+y^2)\,\rho(x,y,z)\,dV.$$

Example 14.6.8: Finding the Mass of a Solid

Suppose that Q is a solid region bounded by $x + 2y + 3z = 6$ and the coordinate planes and has density $\rho(x,y,z) = x^2yz$. Find the total mass.

Solution

The region Q is a tetrahedron (Figure 14.6.7) meeting the axes at the points $(6,0,0)$, $(0,3,0)$, and $(0,0,2)$. To find the limits of integration, let $z=0$ in the slanted plane $z=\frac{1}{3}(6-x-2y)$. Then for x and y find the projection of Q onto the xy-plane, which is bounded by the axes and the line $x+2y=6$. Hence the mass is

$$m=\iiint_Q \rho(x,y,z)\,dV=\int_{x=0}^{x=6}\int_{y=0}^{y=1/2(6-x)}\int_{z=0}^{z=1/3(6-x-2y)} x^2yz\,dz\,dy\,dx=\frac{108}{35}$$

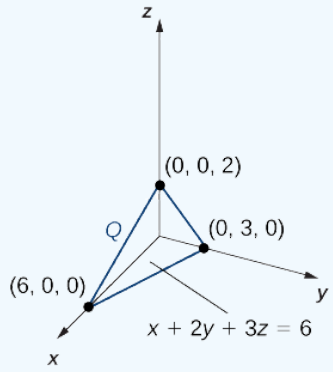

Figure 14.6.7: Finding the mass of a three-dimensional solid Q.

? Exercise 14.6.8

Consider the same region Q (Figure 14.6.7), and use the density function $\rho(x,y,z)=xy^2z$. Find the mass.

✓ Example 14.6.9: Finding the Center of Mass of a Solid

Suppose Q is a solid region bounded by the plane $x+2y+3z=6$ and the coordinate planes with density $\rho(x,y,z)=x^2yz$ (see Figure 14.6.7). Find the center of mass using decimal approximation.

Solution

We have used this tetrahedron before and know the limits of integration, so we can proceed to the computations right away. First, we need to find the moments about the xy-plane, the xz-plane, and the yz-plane:

$$M_{xy}=\iiint_Q z\rho(x,y,z)\,dV$$
$$=\int_{x=0}^{x=6}\int_{y=0}^{y=\frac{1}{2}(6-x)}\int_{z=0}^{z=\frac{1}{3}(6-x-2y)} x^2yz^2\,dz\,dy\,dx=\frac{54}{35}\approx 1.543,$$

$$M_{xz}=\iiint_Q y\rho(x,y,z)\,dV$$
$$=\int_{x=0}^{x=6}\int_{y=0}^{y=\frac{1}{2}(6-x)}\int_{z=0}^{z=\frac{1}{3}(6-x-2y)} x^2y^2z\,dz\,dy\,dx=\frac{81}{35}\approx 2.314,$$

$$M_{yz}=\iiint_Q x\rho(x,y,z)\,dV$$
$$=\int_{x=0}^{x=6}\int_{y=0}^{y=\frac{1}{2}(6-x)}\int_{z=0}^{z=\frac{1}{3}(6-x-2y)} x^3yz\,dz\,dy\,dx=\frac{243}{35}\approx 6.943.$$

Hence the center of mass is

$$\bar{x}=\frac{M_{yz}}{m},\ \bar{y}=\frac{M_{xz}}{m},\ \bar{z}=\frac{M_{xy}}{m},$$

$$\bar{x}=\frac{M_{yz}}{m}=\frac{243/35}{108/35}=\frac{243}{108}=2.25,$$

$$\bar{y} = \frac{M_{xz}}{m} = \frac{81/35}{108/35} = \frac{81}{108} = 0.75,$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{54/35}{108/35} = \frac{54}{108} = 0.5$$

The center of mass for the tetrahedron Q is the point $(2.25, 0.75, 0.5)$.

? Exercise 14.6.9

Consider the same region Q (Figure 14.6.7) and use the density function $\rho(x, y, z) = xy^2z$. Find the center of mass.

We conclude this section with an example of finding moments of inertia I_x, I_y, and I_z.

✓ Example 14.6.10: Finding the Moments of Inertia of a Solid

Suppose that Q is a solid region and is bounded by $x + 2y + 3z = 6$ and the coordinate planes with density $\rho(x, y, z) = x^2yz$ (see Figure 14.6.7). Find the moments of inertia of the tetrahedron Q about the yz-plane, the xz-plane, and the xy-plane.

Solution

Once again, we can almost immediately write the limits of integration and hence we can quickly proceed to evaluating the moments of inertia. Using the formula stated before, the moments of inertia of the tetrahedron Q about the yz-plane, the xz-plane, and the xy-plane are

$$I_x = \iiint_Q (y^2 + z^2)\rho(x, y, z)\, dV,$$

$$I_y = \iiint_Q (x^2 + z^2)\rho(x, y, z)\, dV,$$

and

$$I_z = \iiint_Q (x^2 + y^2)\rho(x, y, z)\, dV \text{ with } \rho(x, y, z) = x^2yz.$$

Proceeding with the computations, we have

$$I_x = \iiint_Q (y^2 + z^2)x^2\rho(x, y, z)\, dV$$

$$= \int_{x=0}^{x=6} \int_{y=0}^{y=\frac{1}{2}(6-x)} \int_{z=0}^{z=\frac{1}{3}(6-x-2y)} (y^2 + z^2)x^2yz\, dz\, dy\, dx = \frac{117}{35} \approx 3.343,$$

$$I_y = \iiint_Q (x^2 + z^2)x^2\rho(x, y, z)\, dV$$

$$= \int_{x=0}^{x=6} \int_{y=0}^{y=\frac{1}{2}(6-x)} \int_{z=0}^{z=\frac{1}{3}(6-x-2y)} (x^2 + z^2)x^2yz\, dz\, dy\, dx = \frac{684}{35} \approx 19.543,$$

$$I_z = \iiint_Q (x^2 + y^2)x^2\rho(x, y, z)\, dV$$

$$= \int_{x=0}^{x=6} \int_{y=0}^{y=\frac{1}{2}(6-x)} \int_{z=0}^{z=\frac{1}{3}(6-x-2y)} (x^2 + y^2)x^2yz\, dz\, dy\, dx = \frac{729}{35} \approx 20.829.$$

Thus, the moments of inertia of the tetrahedron Q about the yz-plane, the xz-plane, and the xy-plane are $117/35$, $684/35$, and $729/35$, respectively.

? Exercise 14.6.10

Consider the same region Q (Figure 14.6.7), and use the density function $\rho(x,y,z) = xy^2z$. Find the moments of inertia about the three coordinate planes.

Key Concepts

Finding the mass, center of mass, moments, and moments of inertia in double integrals:

- For a lamina R with a density function $\rho(x,y)$ at any point (x,y) in the plane, the mass is

$$m = \iint_R \rho(x,y)\,dA.$$

- The moments about the x-axis and y-axis are

$$M_x = \iint_R y\rho(x,y)\,dA \text{ and } M_y = \iint_R x\rho(x,y)\,dA.$$

- The center of mass is given by $\bar{x} = \dfrac{M_y}{m}$, $\bar{y} = \dfrac{M_x}{m}$.
- The center of mass becomes the centroid of the plane when the density is constant.
- The moments of inertia about the x-axis, y-axis, and the origin are

$$I_x = \iint_R y^2\rho(x,y)\,dA,\ I_y = \iint_R x^2\rho(x,y)\,dA,$$
$$\text{and } I_0 = I_x + I_y = \iint_R (x^2+y^2)\rho(x,y)\,dA.$$

Finding the mass, center of mass, moments, and moments of inertia in triple integrals:

- For a solid object Q with a density function $\rho(x,y,z)$ at any point (x,y,z) in space, the mass is

$$m = \iiint_Q \rho(x,y,z)\,dV.$$

- The moments about the xy-plane, the xz-plane, and the yz-plane are

$$M_{xy} = \iiint_Q z\rho(x,y,z)\,dV,\ M_{xz} = \iiint_Q y\rho(x,y,z)\,dV,\ M_{yz} = \iiint_Q x\rho(x,y,z)\,dV$$

- The center of mass is given by $\bar{x} = \dfrac{M_{yz}}{m}$, $\bar{y} = \dfrac{M_{xz}}{m}$, $\bar{z} = \dfrac{M_{xy}}{m}$.
- The center of mass becomes the centroid of the solid when the density is constant.
- The moments of inertia about the yz-plane, the xz-plane, and the xy-plane are

$$I_x = \iiint_Q (y^2+z^2)\,\rho(x,y,z)\,dV,\ I_y = \iiint_Q (x^2+z^2)\,\rho(x,y,z)\,dV,$$
$$I_z = \iiint_Q (x^2+y^2)\,\rho(x,y,z)\,dV.$$

Key Equations

- **Mass of a lamina**

$$m = \lim_{k,l\to\infty} \sum_{i=1}^{k}\sum_{j=1}^{l} m_{ij} = \lim_{k,l\to\infty} \sum_{i=1}^{k}\sum_{j=1}^{l} \rho(x_{ij}^*, y_{ij}^*)\,\Delta A = \iint_R \rho(x,y)\,dA$$

- **Moment about the x-axis**

$$M_x = \lim_{k,l\to\infty} \sum_{i=1}^{k}\sum_{j=1}^{l} (y_{ij}^*) m_{ij} = \lim_{k,l\to\infty} \sum_{i=1}^{k}\sum_{j=1}^{l} (y_{ij}^*)\rho(x_{ij}^*, y_{ij}^*)\,\Delta A = \iint_R y\rho(x,y)\,dA$$

- **Moment about the y-axis**

$$M_y = \lim_{k,l\to\infty} \sum_{i=1}^{k} \sum_{j=1}^{l} (x_{ij}^*) m_{ij} = \lim_{k,l\to\infty} \sum_{i=1}^{k} \sum_{j=1}^{l} (x_{ij}^*) \rho(x_{ij}^*, y_{ij}^*)\, \Delta A = \iint_R x\rho(x,y)\, dA$$

- **Center of mass of a lamina**

$$\bar{x} = \frac{M_y}{m} = \frac{\iint_R x\rho(x,y)\, dA}{\iint_R \rho(x,y)\, dA} \; and \; \bar{y} = \frac{M_x}{m} = \frac{\iint_R y\rho(x,y)\, dA}{\iint_R \rho(x,y)\, dA}$$

Glossary

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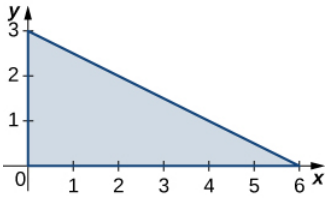
14.6E: Exercises for Section 14.6

In exercises 1 - 12, the region R occupied by a lamina is shown in a graph. Find the mass of R with the density function ρ.

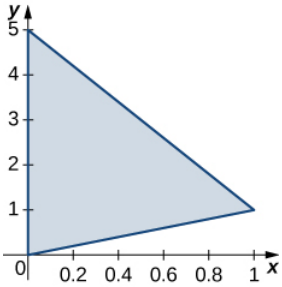
1. R is the triangular region with vertices $(0,0)$, $(0,3)$, and $(6,0)$; $\rho(x,y)=xy$.



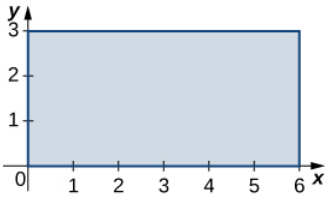
Answer

$\frac{27}{2}$

2. R is the triangular region with vertices $(0,0)$, $(1,1)$, and $(0,5)$; $\rho(x,y)=x+y$.



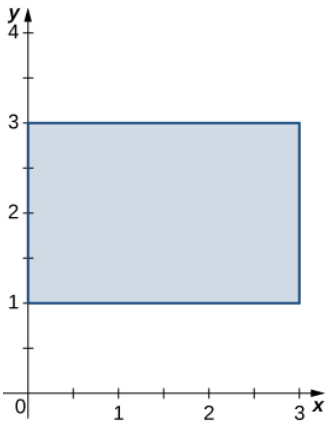
3. R is the rectangular region with vertices $(0,0)$, $(0,3)$, $(6,3)$ and $(6,0)$; $\rho(x,y)=\sqrt{xy}$.



Answer

$24\sqrt{2}$

4. R is the rectangular region with vertices $(0,1)$, $(0,3)$, $(3,3)$ and $(3,1)$; $\rho(x,y)=x^2y$.



5. R is the trapezoidal region determined by the lines $y=-\frac{1}{4}x+\frac{5}{2}$, $y=0$, $y=2$, and $x=0$; $\rho(x,y)=3xy$.

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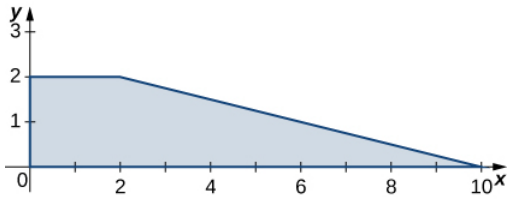
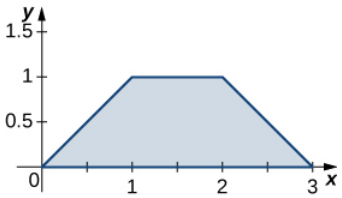


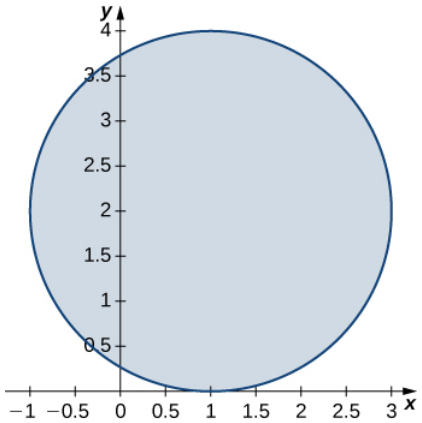
Answer

76

6. R is the trapezoidal region determined by the lines $y = 0,\ y = 1,\ y = x$ and $y = -x + 3;\ \rho(x, y) = 2x + y$.



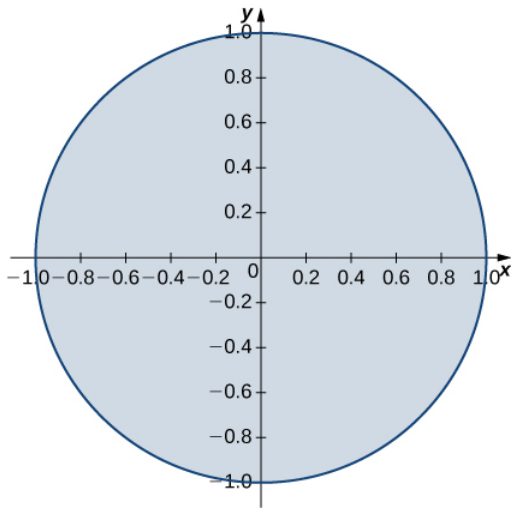
7. R is the disk of radius 2 centered at $(1, 2);\ \rho(x, y) = x^2 + y^2 - 2x - 4y + 5$.



Answer

8π

8. R is the unit disk; $\rho(x, y) = 3x^4 + 6x^2y^2 + 3y^4$.



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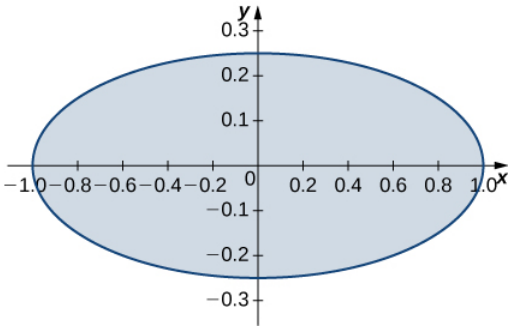
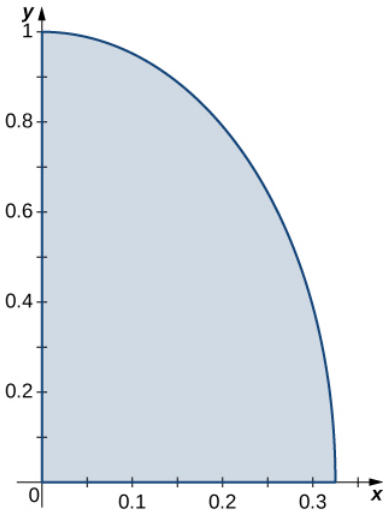
9. R is the region enclosed by the ellipse $x^2 + 4y^2 = 1;\ \rho(x, y) = 1$.



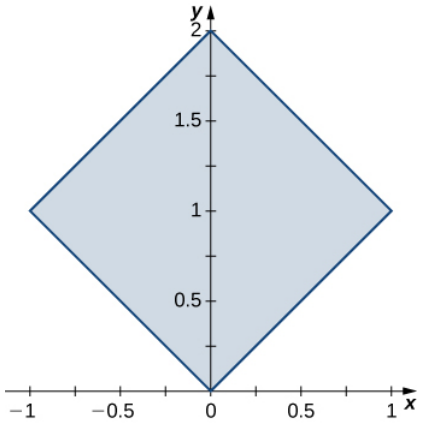
Answer

$\frac{\pi}{2}$

10. $R = \{(x, y) \mid 9x^2 + y^2 \leq 1,\ x \geq 0,\ y \geq 0\};\ \rho(x, y) = \sqrt{9x^2 + y^2}$.



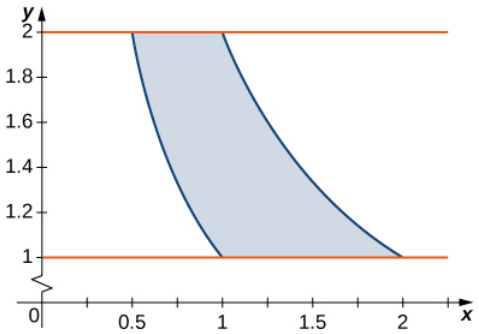
11. R is the region bounded by $y = x,\ y = -x,\ y = x + 2,\ y = -x + 2;\ \rho(x, y) = 1$.



Answer

2

12. R is the region bounded by $y = \frac{1}{x},\ y = \frac{2}{x},\ y = 1$, and $y = 2;\ \rho(x, y) = 4(x + y)$.



In exercises 13 - 24, consider a lamina occupying the region R and having the density function ρ given in the preceding group of exercises. Use a computer algebra system (CAS) to answer the following questions.

a. Find the moments M_x and M_y about the x-axis and y-axis, respectively.

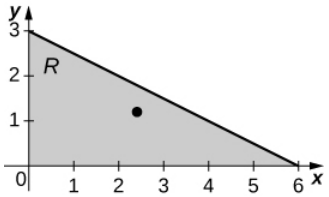
b. Calculate and plot the center of mass of the lamina.

c. [T] Use a CAS to locate the center of mass on the graph of R.

13. [T] R is the triangular region with vertices $(0,0)$, $(0,3)$, and $(6,0)$; $\rho(x,y)=xy$.

Answer

a. $M_x=\frac{81}{5}$, $M_y=\frac{162}{5}$;
b. $\bar{x}=\frac{12}{5}$, $\bar{y}=\frac{6}{5}$;
c.

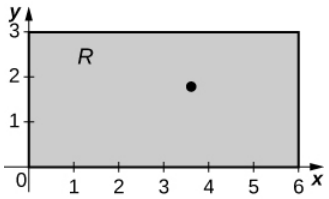


14. [T] R is the triangular region with vertices $(0,0)$, $(1,1)$, and $(0,5)$; $\rho(x,y)=x+y$.

15. [T] R is the rectangular region with vertices $(0,0)$, $(0,3)$, $(6,3)$, and $(6,0)$; $\rho(x,y)=\sqrt{xy}$.

Answer

a. $M_x=\frac{216\sqrt{2}}{5}$, $M_y=\frac{432\sqrt{2}}{5}$;
b. $\bar{x}=\frac{18}{5}$, $\bar{y}=\frac{9}{5}$;
c.



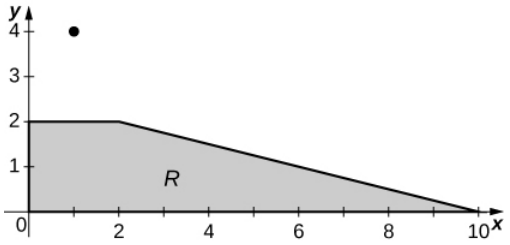
16. [T] R is the rectangular region with vertices $(0,1)$, $(0,3)$, $(3,3)$, and $(3,1)$; $\rho(x,y)=x^2y$.

17. [T] R is the trapezoidal region determined by the lines $y=-\frac{1}{4}x+\frac{5}{2}$, $y=0$, $y=2$, and $x=0$; $\rho(x,y)=3xy$.

Answer

a. $M_x=\frac{368}{5}$, $M_y=\frac{1552}{5}$;
b. $\bar{x}=\frac{92}{95}$, $\bar{y}=\frac{388}{95}$;

c.



18. [T] R is the trapezoidal region determined by the lines $y = 0,\ y = 1,\ y = x$, and $y = -x + 3;\ \rho(x, y) = 2x + y$.

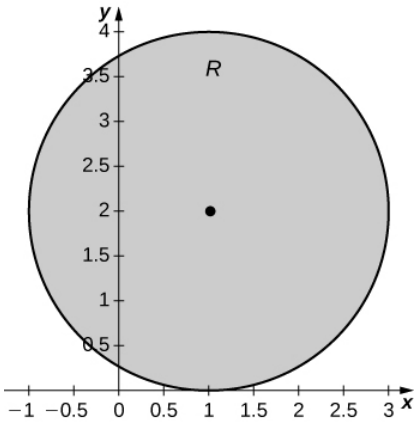
19. [T] R is the disk of radius 2 centered at $(1, 2);\ \rho(x, y) = x^2 + y^2 - 2x - 4y + 5$.

Answer

a. $M_x = 16\pi,\ M_y = 8\pi$;

b. $\bar{x} = 1,\ \bar{y} = 2$;

c.



20. [T] R is the unit disk; $\rho(x, y) = 3x^4 + 6x^2y^2 + 3y^4$.

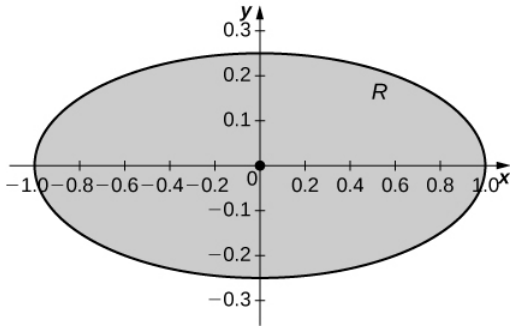
21. [T] R is the region enclosed by the ellipse $x^2 + 4y^2 = 1;\ \rho(x, y) = 1$.

Answer

a. $M_x = 0,\ M_y = 0)$;

b. $\bar{x} = 0,\ \bar{y} = 0$;

c.

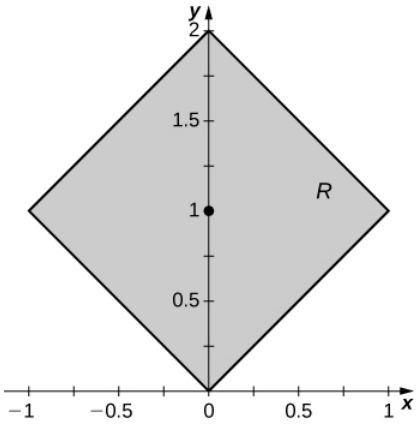


22. [T] $R = \{(x, y) \mid 9x^2 + y^2 \le 1,\ x \ge 0,\ y \ge 0\};\ \rho(x, y) = \sqrt{9x^2 + y^2}$.

23. [T] R is the region bounded by $y = x,\ y = -x,\ y = x + 2$, and $y = -x + 2;\ \rho(x, y) = 1$.

Answer

a. $M_x = 2,\ M_y = 0$);
b. $\bar{x} = 0,\ \bar{y} = 1$;
c.



24. [T] R is the region bounded by $y = \frac{1}{x},\ y = \frac{2}{x},\ y = 1$, and $y = 2;\ \rho(x, y) = 4(x + y)$.

In exercises 25 - 36, consider a lamina occupying the region R and having the density function ρ given in the first two groups of Exercises.

a. Find the moments of inertia I_x, I_y and I_0 about the x-axis, y-axis, and origin, respectively.

b. Find the radii of gyration with respect to the x-axis, y-axis, and origin, respectively.

25. R is the triangular region with vertices $(0, 0),\ (0, 3)$, and $(6, 0);\ \rho(x, y) = xy$.

Answer

a. $I_x = \frac{243}{10},\ I_y = \frac{486}{5}$, and $I_0 = \frac{243}{2}$;
b. $R_x = \frac{3\sqrt{5}}{5},\ R_y = \frac{6\sqrt{5}}{5}$, and $R_0 = 3$

26. R is the triangular region with vertices $(0, 0),\ (1, 1)$, and $(0, 5);\ \rho(x, y) = x + y$.

27. R is the rectangular region with vertices $(0, 0),\ (0, 3),\ (6, 3)$, and $(6, 0);\ \rho(x, y) = \sqrt{xy}$.

Answer

a. $I_x = \frac{2592\sqrt{2}}{7},\ I_y = \frac{648\sqrt{2}}{7}$, and $I_0 = \frac{3240\sqrt{2}}{7}$;
b. $R_x = \frac{6\sqrt{21}}{7},\ R_y = \frac{3\sqrt{21}}{7}$, and $R_0 = \frac{3\sqrt{106}}{7}$

28. R is the rectangular region with vertices $(0, 1),\ (0, 3),\ (3, 3)$, and $(3, 1);\ \rho(x, y) = x^2 y$.

29. R is the trapezoidal region determined by the lines $y = -\frac{1}{4}x + \frac{5}{2},\ y = 0,\ y = 2$, and x = 0; \space \rho (x,y) = 3xy\).

Answer

a. $I_x = 88,\ I_y = 1560$, and $I_0 = 1648$;
b. $R_x = \frac{\sqrt{418}}{19},\ R_y = \frac{\sqrt{7410}}{10}$, and $R_0 = \frac{2\sqrt{1957}}{19}$

30. R is the trapezoidal region determined by the lines $y = 0,\ y = 1,\ y = x$, and y = -x + 3; \space \rho (x,y) = 2x + y\).

31. R is the disk of radius 2 centered at $(1, 2);\ \rho(x, y) = x^2 + y^2 - 2x - 4y + 5$.

Answer

a. $I_x = \frac{128\pi}{3},\ I_y = \frac{56\pi}{3}$, and $I_0 = \frac{184\pi}{3}$;
b. $R_x = \frac{4\sqrt{3}}{3},\ R_y = \frac{\sqrt{21}}{2}$, and $R_0 = \frac{\sqrt{69}}{3}$

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32. R is the unit disk; $\rho(x,y) = 3x^4 + 6x^2y^2 + 3y^4$.

33. R is the region enclosed by the ellipse $x^2 + 4y^2 = 1;\ \rho(x,y) = 1$.

Answer

a. $I_x = \frac{\pi}{32},\ I_y = \frac{\pi}{8}$, and $I_0 = \frac{5\pi}{32}$;
b. $R_x = \frac{1}{4},\ R_y = \frac{1}{2}$, and $R_0 = \frac{\sqrt{5}}{4}$

34. $R = \{(x,y) \mid 9x^2 + y^2 \le 1,\ x \ge 0,\ y \ge 0\};\ \rho(x,y) = \sqrt{9x^2 + y^2}$.

35. R is the region bounded by $y = x,\ y = -x,\ y = x + 2$, and $y = -x + 2;\ \rho(x,y) = 1$.

Answer

a. $I_x = \frac{7}{3},\ I_y = \frac{1}{3}$, and $I_0 = \frac{8}{3}$;
b. $R_x = \frac{\sqrt{42}}{6},\ R_y = \frac{\sqrt{6}}{6}$, and $R_0 = \frac{2\sqrt{3}}{3}$

36. R is the region bounded by $y = \frac{1}{x},\ y = \frac{2}{x},\ y = 1$, and $y = 2;\ \rho(x,y) = 4(x+y)$.

37. Let Q be the solid unit cube. Find the mass of the solid if its density ρ is equal to the square of the distance of an arbitrary point of Q to the xy-plane.

Answer

$m = \frac{1}{3}$

38. Let Q be the solid unit hemisphere. Find the mass of the solid if its density ρ is proportional to the distance of an arbitrary point of Q to the origin.

39. The solid Q of constant density 1 is situated inside the sphere $x^2 + y^2 + z^2 = 16$ and outside the sphere $x^2 + y^2 + z^2 = 1$. Show that the center of mass of the solid is not located within the solid.

40. Find the mass of the solid $Q = \{(x,y,z) \mid 1 \le x^2 + z^2 \le 25,\ y \le 1 - x^2 - z^2\}$ whose density is $\rho(x,y,z) = k$, where $k > 0$.

41. [T] The solid $Q = \{(x,y,z) \mid x^2 + y^2 \le 9,\ 0 \le z \le 1,\ x \ge 0,\ y \ge 0\}$ has density equal to the distance to the xy-plane. Use a CAS to answer the following questions.

a. Find the mass of Q.

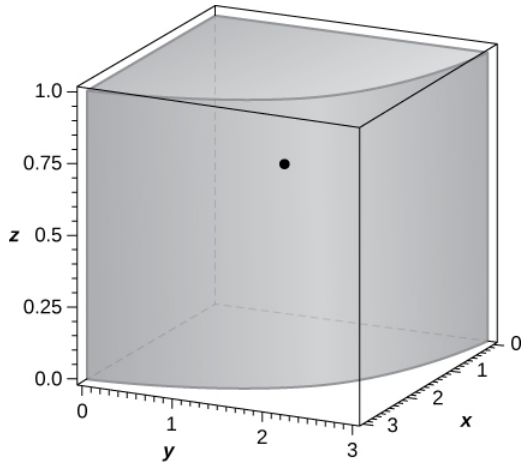
b. Find the moments $M_{xy},\ M_{xz}$ and M_{yz} about the xy-plane, xz-plane, and yz-plane, respectively.

c. Find the center of mass of Q.

d. Graph Q and locate its center of mass.

Answer

a. $m = \frac{9\pi}{4}$;
b. $M_{xy} = \frac{3\pi}{2},\ M_{xz} = \frac{81}{8},\ M_{yz} = \frac{81}{8}$;
c. $\bar{x} = \frac{9}{2\pi},\ \bar{y} = \frac{9}{2\pi},\ \bar{z} = \frac{2}{3}$;
d.



42. Consider the solid $Q = \{(x, y, z) \,|\, 0 \le x \le 1,\ 0 \le y \le 2,\ 0 \le z \le 3\}$ with the density function $\rho(x, y, z) = x + y + 1$.

a. Find the mass of Q.

b. Find the moments $M_{xy},\ M_{xz}$ and M_{yz} about the xy-plane, xz-plane, and yz-plane, respectively.

c. Find the center of mass of Q.

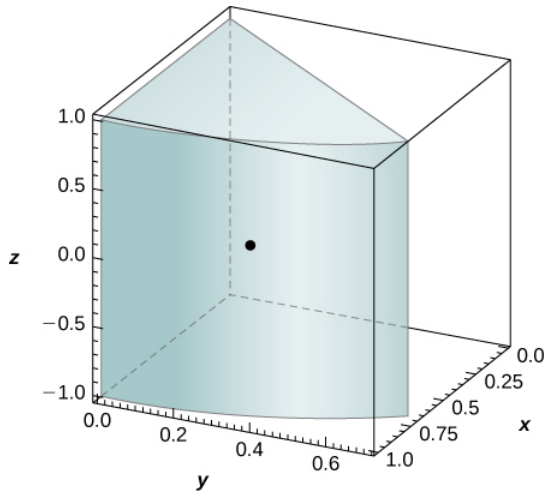
43. [T] The solid Q has the mass given by the triple integral $\int_{-1}^{1} \int_{0}^{\pi/4} \int_{0}^{1} r^2 \, dr \, d\theta \, dz$.

Use a CAS to answer the following questions.

- Show that the center of mass of Q is located in the xy-plane.
- Graph Q and locate its center of mass.

Answer

$\bar{x} = \frac{3\sqrt{2}}{2\pi}$, $\bar{y} = \frac{3(2-\sqrt{2})}{2\pi}$, $\bar{z} = 0$; 2. the solid Q and its center of mass are shown in the following figure.



44. The solid Q is bounded by the planes $x + 4y + z = 8,\ x = 0,\ y = 0$, and $z = 0$. Its density at any point is equal to the distance to the xz-plane. Find the moments of inertia of the solid about the xz-plane.

45. The solid Q is bounded by the planes $x + y + z = 3,\ x = 0,\ y = 0$, and $z = 0$. Its density is $\rho(x, y, z) = x + ay$, where $a > 0$. Show that the center of mass of the solid is located in the plane $z = \frac{3}{5}$ for any value of a.

46. Let Q be the solid situated outside the sphere $x^2 + y^2 + z^2 = z$ and inside the upper hemisphere $x^2 + y^2 + z^2 = R^2$, where $R > 1$. If the density of the solid is $\rho(x, y, z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$, find R such that the mass of the solid is $\frac{7\pi}{2}$.

47. The mass of a solid Q is given by $\int_0^2 \int_0^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{16-x^2-y^2}} (x^2+y^2+z^2)^n \, dz \, dy \, dx$, where n is an integer. Determine n such the mass of the solid is $(2-\sqrt{2})\pi$.

Answer

$n=-2$

48. Let Q be the solid bounded above the cone $x^2+y^2=z^2$ and below the sphere $x^2+y^2+z^2-4z=0$. Its density is a constant $k>0$. Find k such that the center of mass of the solid is situated 7 units from the origin.

49. The solid $Q=\{(x,y,z) \mid 0 \le x^2+y^2 \le 16,\ x \ge 0,\ y \ge 0,\ 0 \le z \le x\}$ has the density $\rho(x,y,z)=k$. Show that the moment M_{xy} about the xy-plane is half of the moment M_{yz} about the yz-plane.

50. The solid Q is bounded by the cylinder $x^2+y^2=a^2$, the paraboloid $b^2-z=x^2+y^2$, and the xy-plane, where $0<a<b$. Find the mass of the solid if its density is given by $\rho(x,y,z)=\sqrt{x^2+y^2}$.

51. Let Q be a solid of constant density k, where $k>0$, that is located in the first octant, inside the circular cone $x^2+y^2=9(z-1)^2$, and above the plane $z=0$. Show that the moment M_{xy} about the xy-plane is the same as the moment M_{yz} about the xz-plane.

52. The solid Q has the mass given by the triple integral $\int_0^1 \int_0^{\pi/2} \int_0^{r^3} (r^4+r) \, dz \, d\theta \, dr$.

a. Find the density of the solid in rectangular coordinates.

b. Find the moment M_{xy} about the xy-plane.

53. The solid Q has the moment of inertia I_x about the yz-plane given by the triple integral $\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_{\frac{1}{2}(x^2+y^2)}^{\sqrt{x^2+y^2}} (y^2+z^2)(x^2+y^2) \, dz \, dx \, dy$.

a. Find the density of Q.

b. Find the moment of inertia I_z about the xy-plane.

Answer

a. $\rho(x,y,z)=x^2+y^2$;
b. $\frac{16\pi}{7}$

54. The solid Q has the mass given by the triple integral $\int_0^{\pi/4} \int_0^{2 \sec\theta} \int_0^1 (r^3 \cos\theta \sin\theta + 2r) \, dz \, dr \, d\theta$.

a. Find the density of the solid in rectangular coordinates.

b. Find the moment M_{xz} about the xz-plane.

55. Let Q be the solid bounded by the xy-plane, the cylinder $x^2+y^2=a^2$, and the plane $z=1$, where $a>1$ is a real number. Find the moment M_{xy} of the solid about the xy-plane if its density given in cylindrical coordinates is $\rho(x,y,z)=\frac{d^2f}{dr^2}(r)$, where f is a differentiable function with the first and second derivatives continuous and differentiable on $(0,a)$.

Answer

$M_{xy}=\pi(f(0)-f(a)+af'(a))$

56. A solid Q has a volume given by $\iint_D \int_a^b dA \, dz$, where D is the projection of the solid onto the xy-plane and $a<b$ are real numbers, and its density does not depend on the variable z. Show that its center of mass lies in the plane $z=\frac{a+b}{2}$.

57. Consider the solid enclosed by the cylinder $x^2+z^2=a^2$ and the planes $y=b$ and $y=c$, where $a>0$ and $b<c$ are real numbers. The density of Q is given by $\rho(x,y,z)=f'(y)$, where f is a differential function whose derivative is continuous on (b,c). Show that if $f(b)=f(c)$, then the moment of inertia about the xz-plane of Q is null.

58. [T] The average density of a solid Q is defined as $\rho_{ave} = \frac{1}{V(Q)} \iiint_Q \rho(x,y,z)\, dV = \frac{m}{V(Q)}$, where $V(Q)$ and m are the volume and the mass of Q, respectively. If the density of the unit ball centered at the origin is $\rho(x,y,z) = e^{-x^2-y^2-z^2}$, use a CAS to find its average density. Round your answer to three decimal places.

59. Show that the moments of inertia I_x, I_y, and I_z about the yz-plane, xz-plane, and xy-plane, respectively, of the unit ball centered at the origin whose density is $\rho(x,y,z) = e^{-x^2-y^2-z^2}$ are the same. Round your answer to two decimal places.

Answer

$I_x = I_y = I_z \approx 0.84$

Contributors

Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.

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14.7: Change of Variables in Multiple Integrals (Jacobians)

Learning Objectives

- Determine the image of a region under a given transformation of variables.
- Compute the Jacobian of a given transformation.
- Evaluate a double integral using a change of variables.
- Evaluate a triple integral using a change of variables.

Recall from Substitution Rule the method of integration by substitution. When evaluating an integral such as

$$\int_2^3 x(x^2-4)^5 dx,$$

we substitute $u = g(x) = x^2 - 4$. Then $du = 2x\,dx$ or $x\,dx = \frac{1}{2}du$ and the limits change to $u = g(2) = 2^2 - 4 = 0$ and $u = g(3) = 9 - 4 = 5$. Thus the integral becomes

$$\int_0^5 \frac{1}{2}u^5 du$$

and this integral is much simpler to evaluate. In other words, when solving integration problems, we make appropriate substitutions to obtain an integral that becomes much simpler than the original integral.

We also used this idea when we transformed double integrals in rectangular coordinates to polar coordinates and transformed triple integrals in rectangular coordinates to cylindrical or spherical coordinates to make the computations simpler. More generally,

$$\int_a^b f(x)dx = \int_c^d f(g(u))g'(u)du,$$

Where $x = g(u)$, $dx = g'(u)du$, and $u = c$ and $u = d$ satisfy $c = g(a)$ and $d = g(b)$.

A similar result occurs in double integrals when we substitute

- $x = f(r,\theta) = r\,\cos\,\theta$
- $y = g(r,\theta) = r\,\sin\,\theta$, and
- $dA = dx\,dy = r\,dr\,d\theta$.

Then we get

$$\iint_R f(x,y)dA = \iint_S (r\,\cos\,\theta,\, r\,\sin\,\theta)r\,dr\,d\theta$$

where the domain R is replaced by the domain S in polar coordinates. Generally, the function that we use to change the variables to make the integration simpler is called a transformation or mapping.

Planar Transformations

A planar transformation T is a function that transforms a region G in one plane into a region R in another plane by a change of variables. Both G and R are subsets of R^2. For example, Figure 14.7.1 shows a region G in the uv-plane transformed into a region R in the xy-plane by the change of variables $x = g(u,v)$ and $y = h(u,v)$, or sometimes we write $x = x(u,v)$ and $y = y(u,v)$. We shall typically assume that each of these functions has continuous first partial derivatives, which means g_u, g_v, h_u, and h_v exist and are also continuous. The need for this requirement will become clear soon.

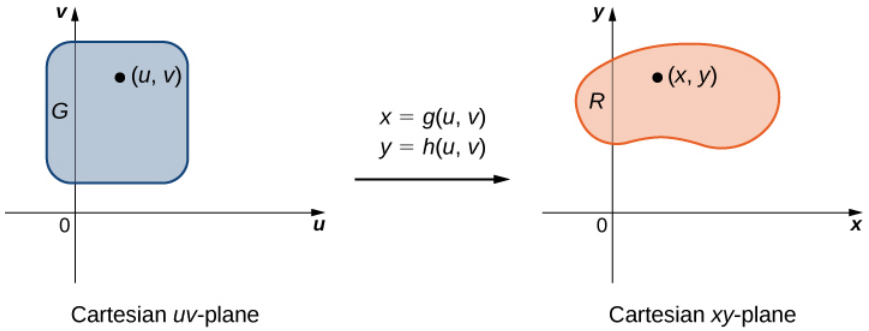


Figure 14.7.1: The transformation of a region G in the uv-plane into a region R in the xy-plane.

Definition: one-to-one transformation

A transformation $T: G \to R$, defined as $T(u,v) = (x,y)$, is said to be a one-to-one transformation if no two points map to the same image point.

To show that T s a one-to-one transformation, we assume $T(u_1, v_1) = T(u_2, v_2)$ and show that as a consequence we obtain$(u_1, v_1) = (u_2, v_2)$. If the transformation T is one-to-one in the domain G, then the inverse T^{-1} exists with the domain R such that $T^{-1} \circ T$ and $T \circ T^{-1}$ are identity functions.

Figure 14.7.2 shows the mapping $T(u,v) = (x,y)$ where x and y are related to u and v by the equations $x = g(u,v)$ and $y = h(u,v)$. The region G is the domain of T and the region R is the range of T, also known as the *image* of G under the transformation T.

Example 14.7.1A: Determining How the Transformation Works

Suppose a transformation T is defined as $T(r,\theta) = (x,y)$ where $x = r\cos\theta$, $y = r\sin\theta$. Find the image of the polar rectangle $G = \{(r,\theta) | 0 \le r \le 1,\ 0 \le \theta \le \pi/2\}$ in the $r\theta$-plane to a region R in the xy-plane. Show that T is a one-to-one transformation in G and find $T^{-1}(x,y)$.

Solution

Since r varies from 0 to 1 in the $r\theta$-plane, we have a circular disc of radius 0 to 1 in the xy-plane. Because θ varies from 0 to $\pi/2$ in the $r\theta$-plane, we end up getting a quarter circle of radius 1 in the first quadrant of the xy-plane (Figure 14.7.2). Hence R is a quarter circle bounded by $x^2 + y^2 = 1$ in the first quadrant.

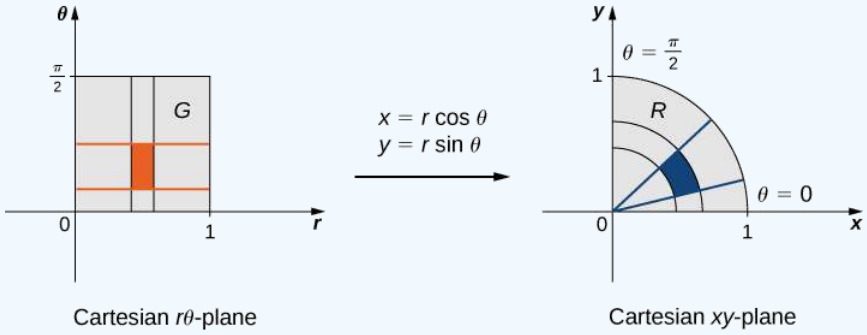


Figure 14.7.2: A rectangle in the $r\theta$-plane is mapped into a quarter circle in the xy-plane.

In order to show that T is a one-to-one transformation, assume $T(r_1,\theta_1) = T(r_2,\theta_2)$ and show as a consequence that $(r_1,\theta_1) = (r_2,\theta_2)$. In this case, we have

$$T(r_1,\theta_1) = T(r_2,\theta_2),$$

$$(x_1,y_1) = (x_1,y_1),$$

$$(r_1\cos\theta_1, r_1\sin\theta_1) = (r_2\cos\theta_2, r_2\sin\theta_2),$$

$$r_1\cos\theta_1 = r_2\cos\theta_2,\ r_1\sin\theta_1 = r_2\sin\theta_2.$$

Dividing, we obtain

$$\frac{r_1 \cos\theta_1}{r_1 \sin\theta_1} = \frac{r_2 \cos\theta_2}{r_2 \sin\theta_2}$$

$$\frac{\cos\theta_1}{\sin\theta_1} = \frac{\cos\theta_2}{\sin\theta_2}$$

$$\tan\theta_1 = \tan\theta_2$$

$$\theta_1 = \theta_2$$

since the tangent function is one-one function in the interval $0 \le \theta \le \pi/2$. Also, since $0 \le r \le 1$, we have $r_1 = r_2$, $\theta_1 = \theta_2$. Therefore, $(r_1, \theta_1) = (r_2, \theta_2)$ and T is a one-to-one transformation from G to R.

To find $T^{-1}(x, y)$ solve for r, θ in terms of x, y. We already know that $r^2 = x^2 + y^2$ and $\tan\theta = \frac{y}{x}$. Thus $T^{-1}(x, y) = (r, \theta)$ is defined as $r = \sqrt{x^2 + y^2}$ and $\tan^{-1}\left(\frac{y}{x}\right)$.

Example 14.7.1B: Finding the Image under T

Let the transformation T be defined by $T(u, v) = (x, y)$ where $x = u^2 - v^2$ and $y = uv$. Find the image of the triangle in the uv-plane with vertices $(0, 0)$, $(0, 1)$, and $(1, 1)$.

Solution

The triangle and its image are shown in Figure 14.7.3. To understand how the sides of the triangle transform, call the side that joins $(0, 0)$ and $(0, 1)$ side A, the side that joins $(0, 0)$ and $(1, 1)$ side B, and the side that joins $(1, 1)$ and $(0, 1)$ side C.

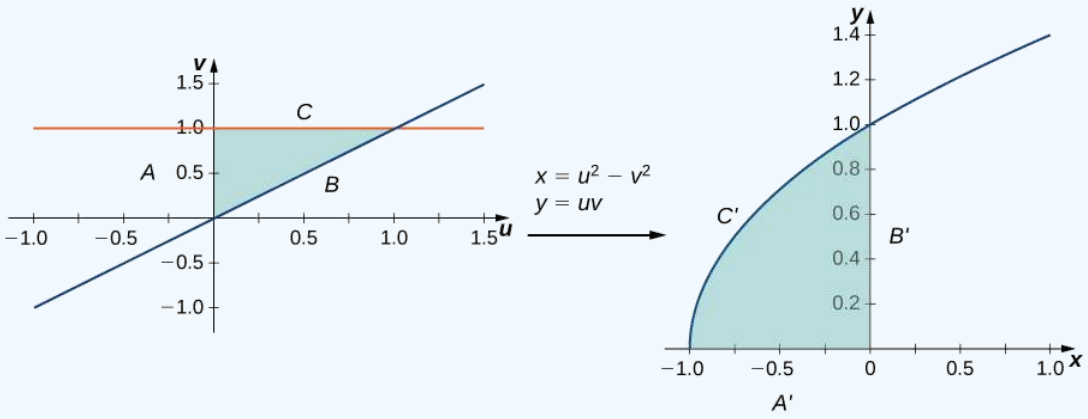


Figure 14.7.3: A triangular region in the uv-plane is transformed into an image in the xy-plane.

- For the side $A: u = 0,\ 0 \le v \le 1$ transforms to $x = -v^2,\ y = 0$ so this is the side A' that joins $(-1, 0)$ and $(0, 0)$.
- For the side $B: u = v,\ 0 \le u \le 1$ transforms to $x = 0,\ y = u^2$ so this is the side B' that joins $(0, 0)$ and $(0, 1)$.
- For the side $C: 0 \le u \le 1,\ v = 1$ transforms to $x = u^2 - 1,\ y = u$ (hence $x = y^2 - 1$ so this is the side C' that makes the upper half of the parabolic arc joining $(-1, 0)$ and $(0, 1)$.

All the points in the entire region of the triangle in the uv-plane are mapped inside the parabolic region in the xy-plane.

Exercise 14.7.1

Let a transformation T be defined as $T(u, v) = (x, y)$ where $x = u + v,\ y = 3v$. Find the image of the rectangle $G = \{(u, v) : 0 \le u \le 1,\ 0 \le v \le 2\}$ from the uv-plane after the transformation into a region R in the xy-plane. Show that T is a one-to-one transformation and find $T^{-1}(x, y)$.

Jacobians

Recall that we mentioned near the beginning of this section that each of the component functions must have continuous first partial derivatives, which means that g_u, g_v, h_u and h_v exist and are also continuous. A transformation that has this property is called a C^1

transformation (here C denotes continuous). Let $T(u,v) = (g(u,v),\ h(u,v))$, where $x = g(u,v)$ and $y = h(u,v)$ be a one-to-one C^1 transformation. We want to see how it transforms a small rectangular region S, Δu units by Δv units, in the uv-plane (Figure 14.7.4).

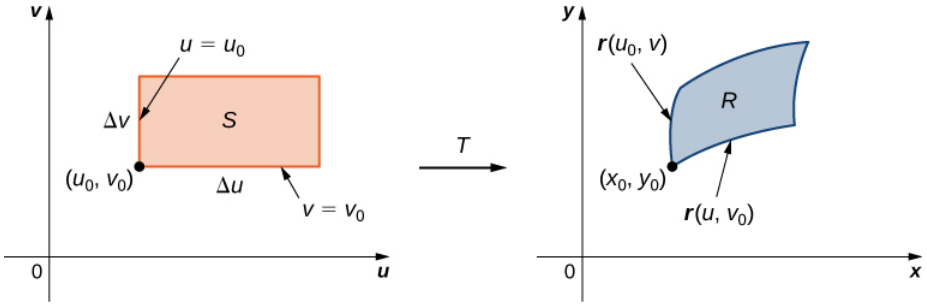


Figure 14.7.4: A small rectangle S in the uv-plane is transformed into a region R in the xy-plane.

Since $x = g(u,v)$ and $y = h(u,v)$, we have the position vector $r(u,v) = g(u,v)i + h(u,v)j$ of the image of the point (u,v). Suppose that $(u_0, v_0)$is the coordinate of the point at the lower left corner that mapped to $(x_0, y_0) = T(u_0, v_0)$ The line $v = v_0$ maps to the image curve with vector function $r(u, v_0)$, and the tangent vector at (x_0, y_0) to the image curve is

$$r_u = g_u(u_0, v_0)i + h_v(u_0, v_0)j = \frac{\partial x}{\partial u}i + \frac{\partial y}{\partial u}j.$$

Similarly, the line $u = u_0$ maps to the image curve with vector function $r(u_0, v)$, and the tangent vector at (x_0, y_0) to the image curve is

$$r_v = g_v(u_0, v_0)i + h_u(u_0, v_0)j = \frac{\partial x}{\partial v}i + \frac{\partial y}{\partial v}j.$$

Now, note that

$$r_u = \lim_{\Delta u \to 0} \frac{r(u_0 + \Delta u, v_0) - r(u_0, v_0)}{\Delta u} \ so \ r(u_0 + \Delta u, v_0) - r(u_0, v_0) \approx \Delta u r_u.$$

Similarly,

$$r_v = \lim_{\Delta v \to 0} \frac{r(u_0, v_0 + \Delta v) - r(u_0, v_0)}{\Delta v} \ so \ r(u_0, v_0 + \Delta v) - r(u_0, v_0) \approx \Delta v r_v.$$

This allows us to estimate the area ΔA of the image R by finding the area of the parallelogram formed by the sides $\Delta v r_v$ and $\Delta u r_u$. By using the cross product of these two vectors by adding the $\mathbf{k}$th component as 0, the area ΔA of the image R (refer to The Cross Product) is approximately $|\Delta u r_u \times \Delta v r_v| = |r_u \times r_v| \Delta u \Delta v$. In determinant form, the cross product is

$$r_u \times r_v = \begin{vmatrix} i & j & k \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} k = \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right) k$$

Since $|k| = 1$, we have

$\Delta A \approx |r_u \times r_v| \Delta u \Delta v = \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right) \Delta u \Delta v.$

Definition: Jacobian

The *Jacobian* of the C^1 transformation $T(u,v) = (g(u,v),\ h(u,v))$ is denoted by $J(u,v)$ and is defined by the 2×2 determinant

$$J(u,v) = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right).$$

Using the definition, we have

$$\Delta A \approx J(u,v)\Delta u\Delta v = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \Delta u\Delta v.$$

Note that the Jacobian is frequently denoted simply by

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)}.$$

Note also that

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}.$$

Hence the notation $J(u,v) = \frac{\partial(x,y)}{\partial(u,v)}$ suggests that we can write the Jacobian determinant with partials of x in the first row and partials of y in the second row.

Example 14.7.2A: Finding the Jacobian

Find the Jacobian of the transformation given in Example 14.7.1A.

Solution

The transformation in the example is $T(r,\theta) = (r\cos\theta,\ r\sin\theta)$ where $x = r\cos\theta$ and $y = r\sin\theta$. Thus the Jacobian is

$$J(r,\theta) = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$
$$= r\cos^2\theta + r\sin^2\theta = r(\cos^2\theta + \sin^2\theta) = r.$$

Example 14.7.2B: Finding the Jacobian

Find the Jacobian of the transformation given in Example 14.7.1B.

Solution

The transformation in the example is $T(u,v) = (u^2 - v^2, uv)$ where $x = u^2 - v^2$ and $y = uv$. Thus the Jacobian is

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ v & u \end{vmatrix} = 2u^2 + 2v^2.$$

Exercise 14.7.2

Find the Jacobian of the transformation given in the previous checkpoint: $T(u,v) = (u+v, 2v)$.

Change of Variables for Double Integrals

We have already seen that, under the change of variables $T(u,v) = (x,y)$ where $x = g(u,v)$ and $y = h(u,v)$, a small region ΔA in the xy-plane is related to the area formed by the product $\Delta u\Delta v$ in the uv-plane by the approximation

$$\Delta A \approx J(u,v)\Delta u,\ \Delta v.$$

Now let's go back to the definition of double integral for a minute:

$$\iint_R f(x,y)fA = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij},y_{ij})\Delta A.$$

Referring to Figure 14.7.5, observe that we divided the region S in the uv-plane into small subrectangles S_{ij} and we let the subrectangles R_{ij} in the xy-plane be the images of S_{ij} under the transformation $T(u,v) = (x,y)$.

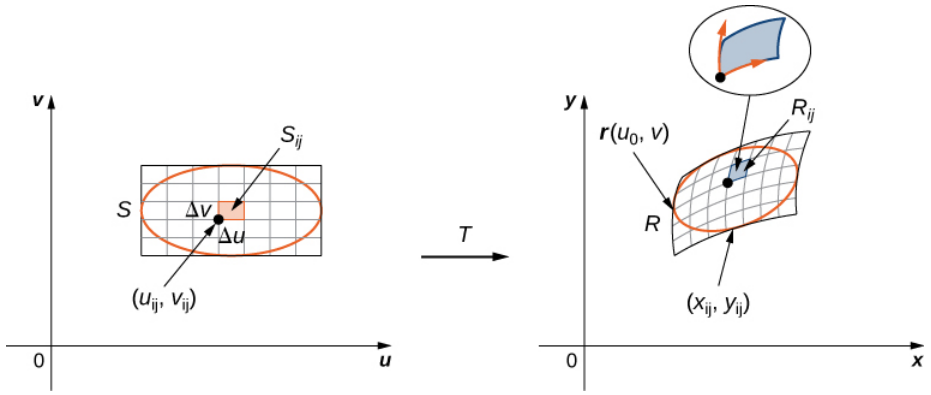


Figure **14.7.5**: The subrectangles S_{ij} in the uv-plane transform into subrectangles R_{ij} in the xy-plane.

Then the double integral becomes

$$\iint_R = f(x,y)dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}, y_{ij})\Delta A$$
$$= \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(g(u_{ij}, v_{ij}),\ h(u_{ij}, v_{ij}))|J(u_{ij}, v_{ij})|\Delta u \Delta v.$$

Notice this is exactly the double Riemann sum for the integral

$$\iint_S f(g(u,v),\ h(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv.$$

Change of Variables for Double Integrals

Let $T(u,v) = (x,y)$ where $x = g(u,v)$ and $y = h(u,v)$ be a one-to-one C^1 transformation, with a nonzero Jacobian on the interior of the region S in the uv-plane it maps S into the region R in the xy-plane. If f is continuous on R, then

$$\iint_R f(x,y)dA = \iint_S f(g(u,v),\ h(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv.$$

With this theorem for double integrals, we can change the variables from (x,y) to (u,v) in a double integral simply by replacing

$$dA = dx\, dy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv$$

when we use the substitutions $x = g(u,v)$ and $y = h(u,v)$ and then change the limits of integration accordingly. This change of variables often makes any computations much simpler.

Example 14.7.3: Changing Variables from Rectangular to Polar Coordinates

Consider the integral

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\, dy\, dx.$$

Use the change of variables $x = r\cos\theta$ and $y = r\sin\theta$, and find the resulting integral.

Solution

First we need to find the region of integration. This region is bounded below by $y = 0$ and above by $y = \sqrt{2x - x^2}$ (Figure **14.7.6**).

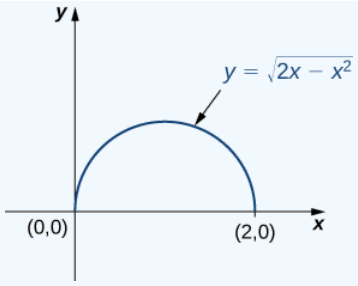


Figure 14.7.6: Changing a region from rectangular to polar coordinates.

Squaring and collecting terms, we find that the region is the upper half of the circle $x^2 + y^2 - 2x = 0$, that is $y^2 + (x-1)^2 = 1$. In polar coordinates, the circle is $r = 2\cos\theta$ so the region of integration in polar coordinates is bounded by $0 \le r \le \cos\theta$ and $0 \le \theta \le \frac{\pi}{2}$.

The Jacobian is $J(r,\theta) = r$, as shown in Example 14.7.2A. Since $r \ge 0$, we have $|J(r,\theta)| = r$.

The integrand $\sqrt{x^2+y^2}$ changes to r in polar coordinates, so the double iterated integral is

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\,dy\,dx = \int_0^{\pi/2} \int_0^{2\cos\theta} r|j(r,\theta)|\,dr\,d\theta = \int_0^{\pi/2} \int_0^{2\cos\theta} r^2\,dr\,d\theta.$$

Exercise 14.7.3

Considering the integral $\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2+y^2)\,dy\,dx$, use the change of variables $x = r\cos\theta$ and $y = r\sin\theta$ and find the resulting integral.

Notice in the next example that the region over which we are to integrate may suggest a suitable transformation for the integration. This is a common and important situation.

Example 14.7.4: Changing Variables

Consider the integral

$$\iint_R (x-y)\,dy\,dx,$$

where R is the parallelogram joining the points $(1,2)$, $(3,4)$, $(4,3)$, and $(6,5)$ (Figure 14.7.7). Make appropriate changes of variables, and write the resulting integral.

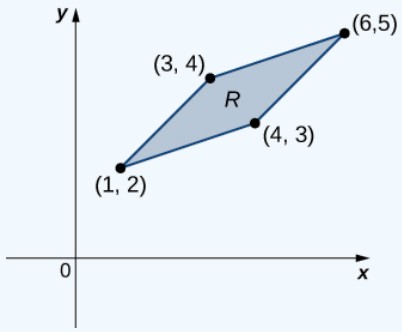


Figure 14.7.7: The region of integration for the given integral.

Solution

First, we need to understand the region over which we are to integrate. The sides of the parallelogram are $x - y + 1$, $x - y - 1 = 0$, $x - 3y + 5 = 0$ and $x - 3y + 9 = 0$ (Figure 14.7.8). Another way to look at them is $x - y = -1$, $x - y = 1$, $x - 3y = -5$, and $x - 3y = 9$.

Clearly the parallelogram is bounded by the lines $y = x + 1$, $y = x - 1$, $y = \frac{1}{3}(x+5)$, and $y = \frac{1}{3}(x+9)$.

Notice that if we were to make $u = x - y$ and $v = x - 3y$, then the limits on the integral would be $-1 \le u \le 1$ and $-9 \le v \le -5$.

To solve for x and y, we multiply the first equation by 3 and subtract the second equation, $3u - v = (3x - 3y) - (x - 3y) = 2x$. Then we have $x = \frac{3u-v}{2}$. Moreover, if we simply subtract the second equation from the first, we get $u - v = (x - y) - (x - 3y) = 2y$ and $y = \frac{u-v}{2}$.

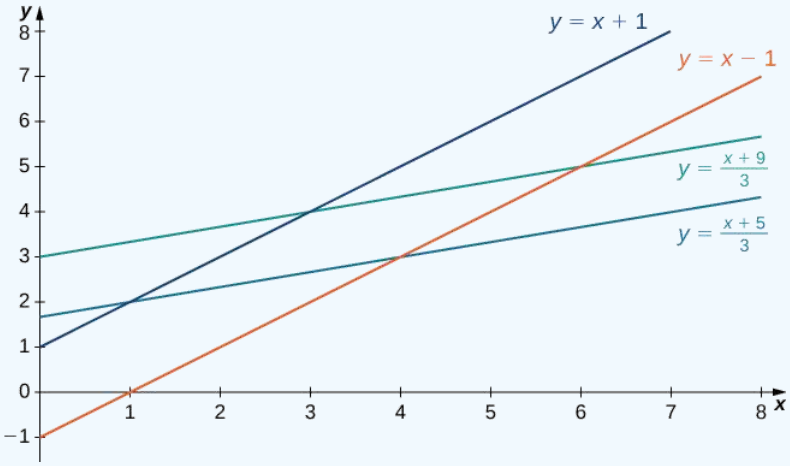


Figure 14.7.8: A parallelogram in the xy-plane that we want to transform by a change in variables.

Thus, we can choose the transformation

$$T(u,v) = \left(\frac{3u-v}{2}, \frac{u-v}{2}\right)$$

and compute the Jacobian $J(u,v)$. We have

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 3/2 & -1/2 \\ 1/2 & -1/2 \end{vmatrix} = -\frac{3}{4} + \frac{1}{4} = -\frac{1}{2}$$

Therefore, $|J(u,v)| = \frac{1}{2}$. Also, the original integrand becomes

$$x - y = \frac{1}{2}[3u - v - u + v] = \frac{1}{2}[3u - u] = \frac{1}{2}[2u] = u.$$

Therefore, by the use of the transformation T, the integral changes to

$$\iint_R (x-y)\,dy\,dx = \int_{-9}^{-5}\int_{-1}^{1} J(u,v)u\,du\,dv = \int_{-9}^{-5}\int_{-1}^{1} \left(\frac{1}{2}\right) u\,du\,dv,$$

which is much simpler to compute.

? Exercise 14.7.4

Make appropriate changes of variables in the integral

$$\iint_R \frac{4}{(x-y)^2}\,dy\,dx,$$

where R is the trapezoid bounded by the lines $x - y = 2$, $x - y = 4$, $x = 0$, and $y = 0$. Write the resulting integral.

We are ready to give a problem-solving strategy for change of variables.

Problem-Solving Strategy: Change of Variables

1. Sketch the region given by the problem in the xy-plane and then write the equations of the curves that form the boundary.
2. Depending on the region or the integrand, choose the transformations $x = g(u,v)$ and $y = h(u,v)$.

3. Determine the new limits of integration in the uv-plane.
4. Find the Jacobian $J(u,v)$.
5. In the integrand, replace the variables to obtain the new integrand.
6. Replace $dy\,dx$ or $dx\,dy$, whichever occurs, by $J(u,v)du\,dv$.

In the next example, we find a substitution that makes the integrand much simpler to compute.

Example 14.7.5: Evaluating an Integral

Using the change of variables $u = x - y$ and $v = x + y$, evaluate the integral

$$\iint_R (x-y)e^{x^2-y^2}\,dA,$$

where R is the region bounded by the lines $x + y = 1$ and $x + y = 3$ and the curves $x^2 - y^2 = -1$ and $x^2 - y^2 = 1$ (see the first region in Figure 14.7.9).

Solution

As before, first find the region R and picture the transformation so it becomes easier to obtain the limits of integration after the transformations are made (Figure 14.7.9).

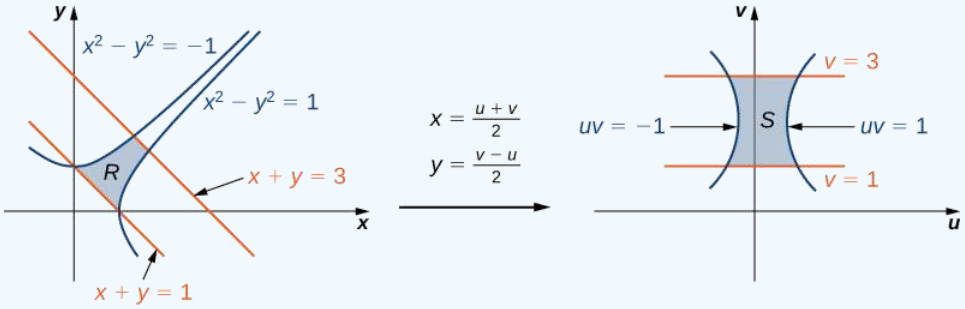


Figure 14.7.9: Transforming the region R into the region S to simplify the computation of an integral.

Given $u = x - y$ and $v = x + y$, we have $x = \frac{u+v}{2}$ and $y = \frac{v-u}{2}$ and hence the transformation to use is $T(u,v) = \left(\frac{u+v}{2}, \frac{v-u}{2}\right)$. The lines $x + y = 1$ and $x + y = 3$ become $v = 1$ and $v = 3$, respectively. The curves $x^2 - y^2 = 1$ and $x^2 - y^2 = -1$ become $uv = 1$ and $uv = -1$, respectively.

Thus we can describe the region S (see the second region Figure 14.7.9) as

$$S = \left\{(u,v) | 1 \le v \le 3, \frac{-1}{v} \le u \le \frac{1}{v}\right\}.$$

The Jacobian for this transformation is

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{vmatrix} = \frac{1}{2}.$$

Therefore, by using the transformation T, the integral changes to

$$\iint_R (x-y)e^{x^2-y^2}\,dA = \frac{1}{2}\int_1^3 \int_{-1/v}^{1/v} ue^{uv}\,du\,dv.$$

Doing the evaluation, we have

$$\frac{1}{2}\int_1^3 \int_{-1/v}^{1/v} ue^{uv}\,du\,dv = \frac{2}{3e} \approx 0.245.$$

Exercise 14.7.5

Using the substitutions $x = v$ and $y = \sqrt{u+v}$, evaluate the integral $\iint_R y \sin(y^2 - x)\, dA$, where R is the region bounded by the lines $y = \sqrt{x}$, $x = 2$ and $y = 0$.

Change of Variables for Triple Integrals

Changing variables in triple integrals works in exactly the same way. Cylindrical and spherical coordinate substitutions are special cases of this method, which we demonstrate here.

Suppose that G is a region in uvw-space and is mapped to D in xyz-space (Figure 14.7.10) by a one-to-one C^1 transformation $T(u, v, w) = (x, y, z)$ where $x = g(u, v, w)$, $y = h(u, v, w)$, and $z = k(u, v, w)$.

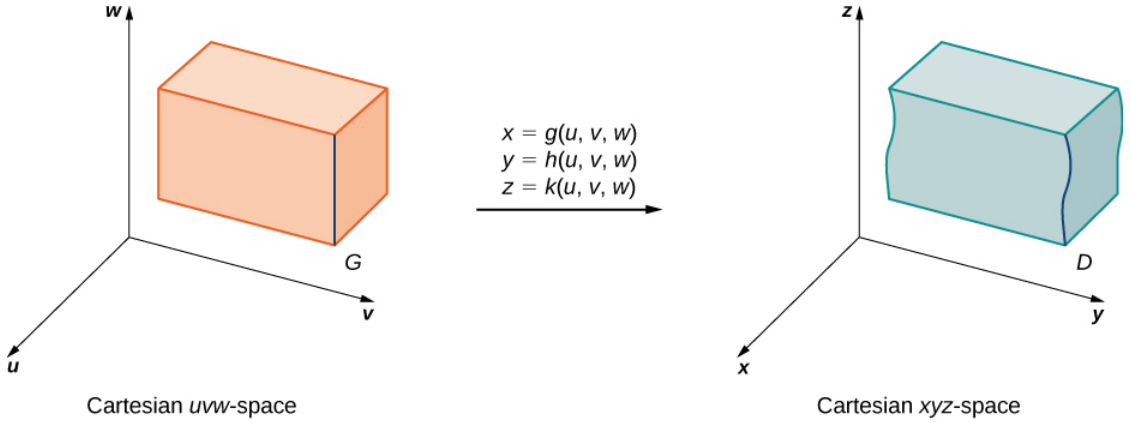


Figure 14.7.10: A region G in uvw-space mapped to a region D in xyz-space.

Then any function $F(x, y, z)$ defined on D can be thought of as another function $H(u, v, w)$ that is defined on G:

$$F(x, y, z) = F(g(u, v, w),\ h(u, v, w),\ k(u, v, w)) = H(u, v, w).$$

Now we need to define the Jacobian for three variables.

Definition: Jacobian determinant

The *Jacobian determinant* $J(u, v, w)$ in three variables is defined as follows:

$$J(u, v, w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

This is also the same as

$$J(u, v, w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

The Jacobian can also be simply denoted as $\frac{\partial(x,y,z)}{\partial(u,v,w)}$.

With the transformations and the Jacobian for three variables, we are ready to establish the theorem that describes change of variables for triple integrals.

Change of Variables for Triple Integrals

Let $T(u,v,w) = (x,y,z)$ where $x = g(u,v,w)$, $y = h(u,v,w)$, and $z = k(u,v,w)$, be a one-to-one C^1 transformation, with a nonzero Jacobian, that maps the region G in the uvw-space into the region D in the xyz-space. As in the two-dimensional case, if F is continuous on D, then

$$\iiint_D F(x,y,z)dV$$

$$= \iiint_G f(g(u,v,w)\,h(u,v,w),\,k(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw \tag{14.7.1}$$

$$= \iiint_G H(u,v,w)|J(u,v,w)|du\,dv\,dw. \tag{14.7.2}$$

Let us now see how changes in triple integrals for cylindrical and spherical coordinates are affected by this theorem. We expect to obtain the same formulas as in Triple Integrals in Cylindrical and Spherical Coordinates.

Example 14.7.6A: Obtaining Formulas in Triple Integrals for Cylindrical and Spherical Coordinates

Derive the formula in triple integrals for

a. cylindrical and
b. spherical coordinates.

Solution

A.

For cylindrical coordinates, the transformation is $T(r,\theta,z) = (x,y,z)$ from the Cartesian $r\theta z$-space to the Cartesian xyz-space (Figure 14.7.11). Here $x = r\cos\theta$, $y = r\sin\theta$ and $z = z$. The Jacobian for the transformation is

$$J(r,\theta,z) = \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix}$$

$$\begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

We know that $r \geq 0$, so $|J(r,\theta,z)| = r$. Then the triple integral is

$$\iiint_D f(x,y,z)dV = \iiint_G f(r\cos\theta,\, r\sin\theta,\, z)r\,dr\,d\theta\,dz.$$

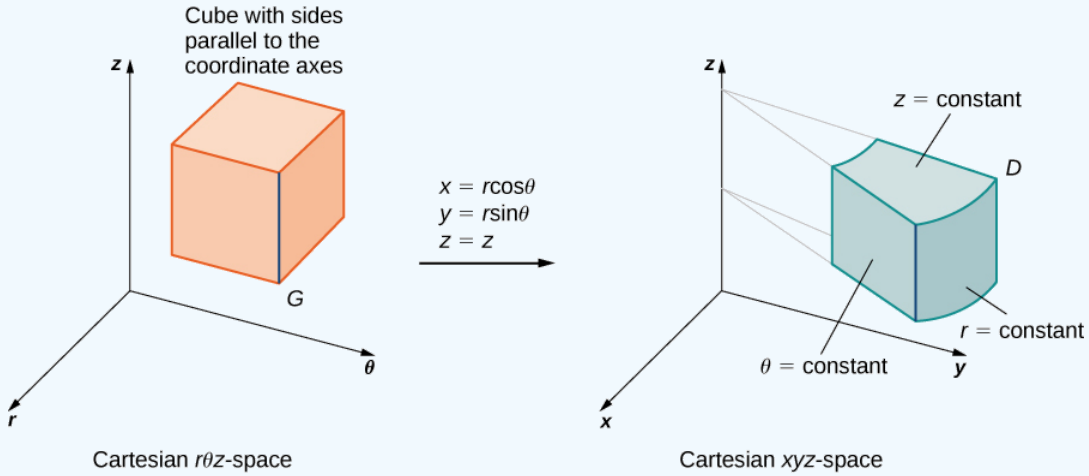


Figure 14.7.11: The transformation from rectangular coordinates to cylindrical coordinates can be treated as a change of variables from region G in $r\theta z$-space to region D in xyz-space.

B.

For spherical coordinates, the transformation is $T(\rho, \theta, \varphi)$ from the Cartesian $\rho\theta\varphi$-space to the Cartesian xyz-space (Figure 14.7.12). Here $x = \rho \sin\varphi \cos\theta$, $y = \rho \sin\varphi \sin\theta$, and $z = \rho \cos\varphi$. The Jacobian for the transformation is

$$J(\rho, \theta, \varphi) = \frac{\partial(x, y, z)}{\partial(\rho, \theta, \varphi)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \sin\varphi\cos\theta & -\rho\sin\varphi\sin\theta & \rho\cos\varphi\cos\theta \\ \sin\varphi\sin\theta & \rho\sin\varphi\cos\theta & \rho\cos\varphi\sin\theta \\ \cos\varphi & 0 & -\rho\sin\varphi \end{vmatrix}.$$

Expanding the determinant with respect to the third row:

$$\begin{aligned} &= \cos\varphi \begin{vmatrix} -\rho\sin\varphi\sin\theta & \rho\cos\varphi\cos\theta \\ \rho\sin\varphi\cos\theta & \rho\cos\varphi\sin\theta \end{vmatrix} - \rho\sin\varphi \begin{vmatrix} \sin\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\sin\varphi\cos\theta \end{vmatrix} \\ &= \cos\varphi(-\rho^2 \sin\varphi\cos\varphi\sin^2\theta - \rho^2\sin\varphi\cos\varphi\cos^2\theta) \\ &\quad -\rho\sin\varphi(\rho\sin^2\varphi\cos^2\theta + \rho\sin^2\varphi\sin^2\theta) \\ &= -\rho^2\sin\varphi\cos^2\varphi(\sin^2\theta + \cos^2\theta) - \rho^2\sin\varphi\sin^2\varphi(\sin^2\theta + \cos^2\theta) \\ &= -\rho^2\sin\varphi\cos^2\varphi - \rho^2\sin\varphi\sin^2\varphi \\ &= -\rho\sin\varphi(\cos^2\varphi + \sin^2\varphi) = -\rho^2\sin\varphi. \end{aligned}$$

Since $0 \le \varphi \le \pi$, we must have $\sin\varphi \ge 0$. Thus $|J(\rho, \theta, \varphi)| = |-\rho^2\sin\varphi| = \rho^2\sin\varphi$.

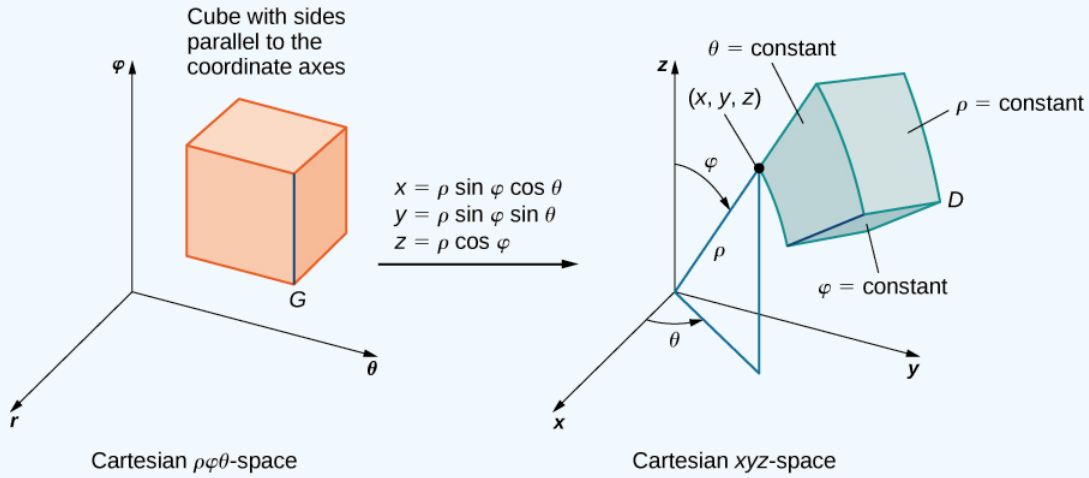


Figure 14.7.12: The transformation from rectangular coordinates to spherical coordinates can be treated as a change of variables from region G in $\rho\theta\varphi$-space to region D in xyz-space.

Then the triple integral becomes

$$\iiint_D f(x, y, z)dV = \iiint_G f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta.$$

Let's try another example with a different substitution.

Example 14.7.6B: Evaluating a Triple Integral with a Change of Variables

Evaluate the triple integral

$$\int_0^3 \int_0^4 \int_{y/2}^{(y/2)+1} \left(x + \frac{z}{3}\right) dx\, dy\, dz$$

In xyz-space by using the transformation

$u = (2x - y)/2$, $v = y/2$, and $w = z/3$.

Then integrate over an appropriate region in uvw-space.

Solution

As before, some kind of sketch of the region G in xyz-space over which we have to perform the integration can help identify the region D in uvw-space (Figure 14.7.13). Clearly G in xyz-space is bounded by the planes $x = y/2$, $x = (y/2) + 1$, $y = 0$, $y = 4$, $z = 0$, and $z = 4$. We also know that we have to use $u = (2x - y)/2$, $v = y/2$, and $w = z/3$ for the transformations. We need to solve for x, y and z. Here we find that $x = u + v$, $y = 2v$, and $z = 3w$.

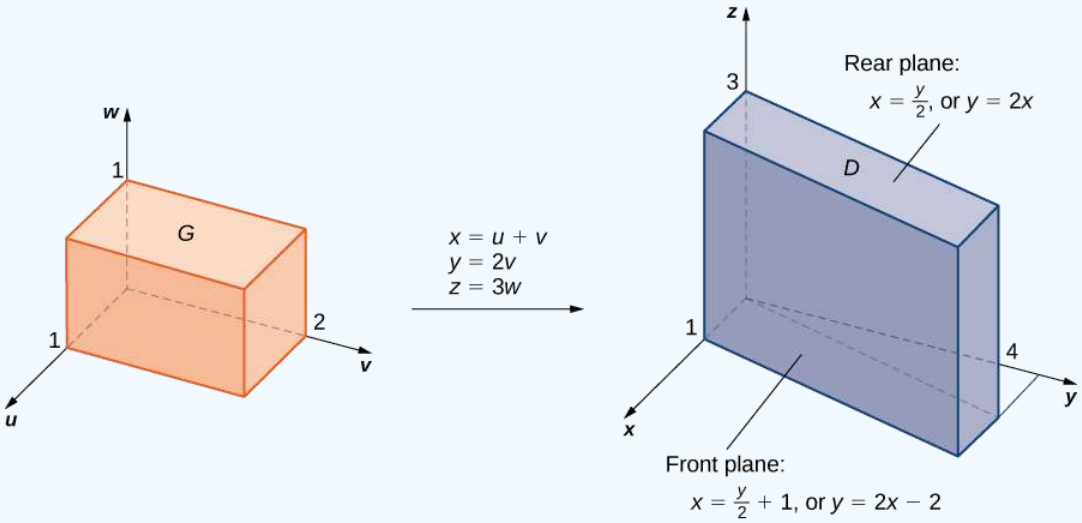


Figure 14.7.13: The region G in uvw-space is transformed to region D in xyz-space.

Using elementary algebra, we can find the corresponding surfaces for the region G and the limits of integration in uvw-space. It is convenient to list these equations in a table.

Equations in xyz for the region D	Corresponding equations in uvw for the region G	Limits for the integration in uvw
$x = y/2$	$u + v = 2v/2 = v$	$u = 0$
$x = y/2$	$u + v = (2v/2) + 1 = v + 1$	$u = 1$
$y = 0$	$2v = 0$	$v = 0$
$y = 4$	$2v = 4$	$v = 2$
$z = 0$	$3w = 0$	$w = 0$
$z = 3$	$3w = 3$	$w = 1$

Now we can calculate the Jacobian for the transformation:

$$J(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 6.$$

The function to be integrated becomes

$$f(x,y,z) = x + \frac{z}{3} = u + v + \frac{3w}{3} = u + v + w.$$

We are now ready to put everything together and complete the problem.

$$\int_0^3 \int_0^4 \int_{y/2}^{(y/2)+1} \left(x + \frac{z}{3}\right) dx\, dy\, dz = \int_0^1 \int_0^2 \int_0^1 (u+v+w)|J(u,v,w)|\, du\, dv\, dw$$
$$= \int_0^1 \int_0^2 \int_0^1 (u+v+w)|6|\, du\, dv\, dw$$
$$= 6\int_0^1 \int_0^2 \int_0^1 (u+v+w)\, du\, dv\, dw$$
$$= 6\int_0^1 \int_0^2 \left[\frac{u^2}{2} + vu + wu\right]_0^1 dv\, dw$$
$$= 6\int_0^1 \int_0^2 \left(\frac{1}{2} + v + u\right) dv\, dw$$
$$= 6\int_0^1 \left[\frac{1}{2}v + \frac{v^2}{2} + wv\right]_0^2 dw$$
$$= 6\int_0^1 (3 + 2w)\, dw = 6\left[3w + w^2\right]_0^1 = 24.$$

? Exercise 14.7.6

Let D be the region in xyz-space defined by $1 \le x \le 2$, $0 \le xy \le 2$, and $0 \le z \le 1$.

Evaluate $\iiint_D (x^2y + 3xyz)\, dx\, dy\, dz$ by using the transformation $u = x$, $v = xy$, and $w = 3z$.

Key Concepts

- A transformation T is a function that transforms a region G in one plane (space) into a region R. in another plane (space) by a change of variables.
- A transformation $T : G \to R$ defined as $T(u,v) = (x,y)$ (or $T(u,v,w) = (x,y,z)$)is said to be a one-to-one transformation if no two points map to the same image point.
- If f is continuous on R, then

$$\iint_R f(x,y)dA = \iint_S f(g(u,v),\, h(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv.$$

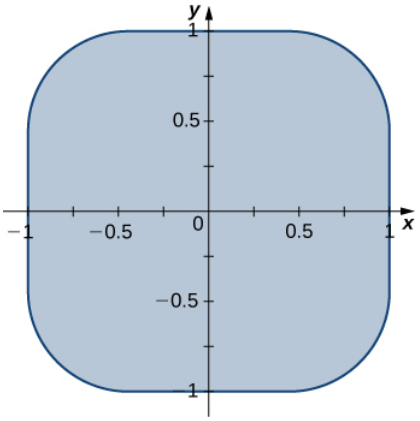
- If F is continuous on R, then

$$\iiint_R F(x,y,z)\, dV = \iiint_G F(g(u,v,w),\, h(u,v,w),\, k(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\, dv\, dw$$
$$= \iiint_G H(u,v,w)|J(u,v,w)|\, du\, dv\, dw.$$

[T] Lamé ovals (or superellipses) are plane curves of equations $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 1$, where a, b, and n are positive real numbers.

a. Use a CAS to graph the regions R bounded by Lamé ovals for $a = 1$, $b = 2$, $n = 4$ and $n = 6$ respectively.

b. Find the transformations that map the region R bounded by the Lamé oval $x^4 + y^4 = 1$ also called a squircle and graphed in the following figure, into the unit disk.



c. Use a CAS to find an approximation of the area $A(R)$ of the region R bounded by $x^4 + y^4 = 1$. Round your answer to two decimal places.

[T] Lamé ovals have been consistently used by designers and architects. For instance, Gerald Robinson, a Canadian architect, has designed a parking garage in a shopping center in Peterborough, Ontario, in the shape of a superellipse of the equation $(\frac{x}{a})^n + (\frac{y}{b})^n = 1$ with $\frac{a}{b} = \frac{9}{7}$ and $n = e$. Use a CAS to find an approximation of the area of the parking garage in the case $a = 900$ yards, $b = 700$ yards, and $n = 2.72$ yards.

[Hide Solution]

$A(R) \simeq 83,999.2$

Chapter Review Exercises

True or False? Justify your answer with a proof or a counterexample.

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dy\,dx$$

Fubini's theorem can be extended to three dimensions, as long as f is continuous in all variables.

[Hide solution]

True.

The integral

$$\int_0^{2\pi} \int_0^1 \int_0^1 dz\,dr\,d\theta$$

represents the volume of a right cone.

The Jacobian of the transformation for $x = u^2 - 2v$, $y = 3v - 2uv$ is given by $-4u^2 + 6u + 4v$.

[Hide Solution]

False.

Evaluate the following integrals.

$$\iint_R (5x^3y^2 - y^2)\,dA,\ R = \{(x,y) | 0 \le x \le 2,\ 1 \le y \le 4\}$$

$$\iint_D \frac{y}{3x^2 + 1} dA,\ D = \{(x,y) | 0 \le x \le 1,\ -x \le y \le x\}$$

[Hide Solution]

0

$$\iint_D \sin(x^2+y^2)dA$$

where D is a disk of radius 2 centered at the origin

$$\int_0^1 \int_0^1 xye^{x^2}\,dx\,dy$$

[Hide Solution]

$\frac{1}{4}$

$$\int_{-1}^1 \int_0^z \int_0^{x-z} 6dy\,dx\,dz$$

$$\iiint_R 3y\,dV,$$

where $R = \{(x,y,z)|0 \le x \le 1,\ 0 \le y \le x,\ 0 \le z \le \sqrt{9-y^2}\}$

[Hide Solution]

1.475

$$\int_0^2 \int_0^{2\pi} \int_r^1 r\,dz\,d\theta\,dr$$

$$\int_0^{2\pi} \int_0^{\pi/2} \int_1^3 \rho^2 \sin(\varphi) d\rho\,d\varphi,\ d\theta$$

[Hide Solution]

$\frac{52}{3}\pi$

$$\int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} dz\,dy\,sx$$

For the following problems, find the specified area or volume.

The area of region enclosed by one petal of $r = \cos(4\theta)$.

[Hide Solution]

$\frac{\pi}{16}$

The volume of the solid that lies between the paraboloid $z = 2x^2 + 2y^2$ and the plane $z = 8$.

The volume of the solid bounded by the cylinder $x^2 + y^2 = 16$ and from $z = 1$ to $z + x = 2$.

[Hide Solution]

93.291

The volume of the intersection between two spheres of radius 1, the top whose center is $(0, 0, 0.25)$ and the bottom, which is centered at $(0, 0, 0)$.

For the following problems, find the center of mass of the region.

$\rho(x, y) = xy$ on the circle with radius 1 in the first quadrant only.

[Hide Solution]

$\left(\frac{8}{15}, \frac{8}{15}\right)$

$\rho(x, y) = (y + 1)\sqrt{x}$ in the region bounded by $y = e^x$, $y = 0$, and $x = 1$.

$\rho(x, y, z) = z$ on the inverted cone with radius 2 and height 2.

$(0, 0, \frac{8}{5})$

The volume an ice cream cone that is given by the solid above $z = \sqrt{(x^2 + y^2)}$ and below $z^2 + x^2 + y^2 = z$.

The following problems examine Mount Holly in the state of Michigan. Mount Holly is a landfill that was converted into a ski resort. The shape of Mount Holly can be approximated by a right circular cone of height 1100 ft and radius 6000 ft.

If the compacted trash used to build Mount Holly on average has a density $400\ lb/ft^3$, find the amount of work required to build the mountain.

[Hide Solution]

$1.452\pi \times 10^{15}$ ft-lb

In reality, it is very likely that the trash at the bottom of Mount Holly has become more compacted with all the weight of the above trash. Consider a density function with respect to height: the density at the top of the mountain is still density $400\ lb/ft^3$ and the density increases. Every 100 feet deeper, the density doubles. What is the total weight of Mount Holly?

The following problems consider the temperature and density of Earth's layers.

[T] The temperature of Earth's layers is exhibited in the table below. Use your calculator to fit a polynomial of degree 3 to the temperature along the radius of the Earth. Then find the average temperature of Earth. (*Hint*: begin at 0 in the inner core and increase outward toward the surface)

Layer	Depth from center (km)	Temperature $°C$
Rocky Crust	0 to 40	0
Upper Mantle	40 to 150	870
Mantle	400 to 650	870
Inner Mantel	650 to 2700	870
Molten Outer Core	2890 to 5150	4300
Inner Core	5150 to 6378	7200

Source: http://www.enchantedlearning.com/sub...h/Inside.shtml

[Hide Solution]

$y = -1.238 \times 10^{-7}x^3 + 0.001196x^2 - 3.666x + 7208$; average temperature approximately $2800°C$

[T] The density of Earth's layers is displayed in the table below. Using your calculator or a computer program, find the best-fit quadratic equation to the density. Using this equation, find the total mass of Earth.

Layer	Depth from center (km)	Density (g/cm^3)
Inner Core	0	12.95
Outer Core	1228	11.05
Mantle	3488	5.00
Upper Mantle	6338	3.90
Crust	6378	2.55

Source: http://hyperphysics.phy-astr.gsu.edu...rthstruct.html

The following problems concern the Theorem of Pappus (see Moments and Centers of Mass for a refresher), a method for calculating volume using centroids. Assuming a region R, when you revolve around the x-axis the volume is given by V_x

$= 2\pi A\bar{y}$, and when you revolve around the y-axis the volume is given by $V_y = 2\pi A\bar{x}$, where A is the area of R. Consider the region bounded by $x^2 + y^2 = 1$ and above $y = x + 1$.

Find the volume when you revolve the region around the x-axis.

[Hide Solution]

$\frac{\pi}{3}$

Find the volume when you revolve the region around the y-axis.

Glossary

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14.7E: Exercises for Section 14.7

In exercises 1 - 6, the function $T : S \to R,\ T(u,v) = (x,y)$ on the region $S = \{(u,v) \,|\, 0 \le u \le 1,\ 0 \le v \le 1\}$ bounded by the unit square is given, where $R \in R^2$ is the image of S under T.

a. Justify that the function T is a C^1 transformation.

b. Find the images of the vertices of the unit square S through the function T.

c. Determine the image R of the unit square S and graph it.

1. $x = 2u,\ y = 3v$

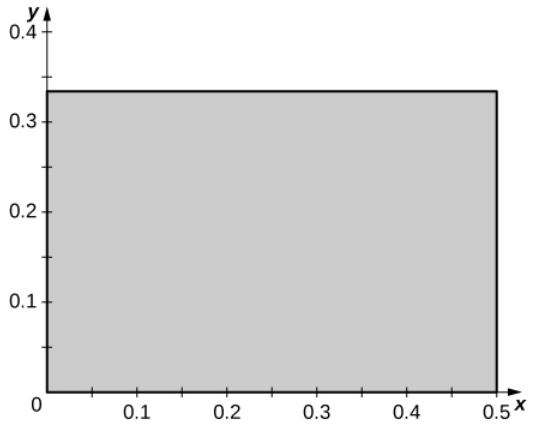
2. $x = \frac{u}{2},\ y = \frac{v}{3}$

Answer

a. $T(u,v) = (g(u,v),\ h(u,v),\ x = g(u,v) = \frac{u}{2}$ and $y = h(u,v) = \frac{v}{3}$. The functions g and h are continuous and differentiable, and the partial derivatives $g_u(u,v) = \frac{1}{2},\ g_v(u,v) = 0,\ h_u(u,v) = 0$ and $h_v(u,v) = \frac{1}{3}$ are continuous on S;

b. $T(0,0) = (0,0),\ T(1,0) = \left(\frac{1}{2},0\right),\ T(0,1) = \left(0,\frac{1}{3}\right)$, and $T(1,1) = \left(\frac{1}{2},\frac{1}{3}\right)$;

c. R is the rectangle of vertices $(0,0),\ \left(0,\frac{1}{3}\right),\ \left(\frac{1}{2},\frac{1}{3}\right)$, and $\left(0,\frac{1}{3}\right)$ in the xy-plane; the following figure.



3. $x = u - v,\ y = u + v$

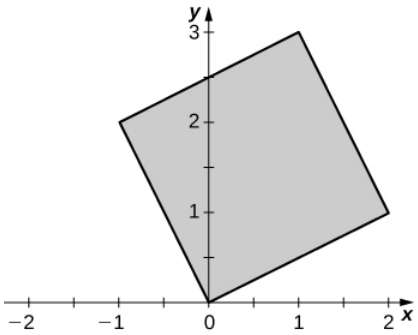
4. $x = 2u - v,\ y = u + 2v$

Answer

a. $T(u,v) = (g(u,v),\ h(u,v),\ x = g(u,v) = 2u - v$ and $y = h(u,v) = u + 2v$. The functions g and h are continuous and differentiable, and the partial derivatives $g_u(u,v) = 2,\ g_v(u,v) = -1,\ h_u(u,v) = 1$ and $h_v(u,v) = 2$ are continuous on S;

b. $T(0,0) = (0,0),\ T(1,0) = (2,1),\ T(0,1) = (-1,2)$, and $T(1,1) = (1,3)$;

c. R is the parallelogram of vertices $(0,0),\ (2,1)\ (1,3)$, and $(-1,2)$ in the xy-plane; the following figure.



https://math.libretexts.org/@go/page/66959

5. $x = u^2,\ y = v^2$

6. $x = u^3,\ y = v^3$

Answer

a. $T(u,v) = (g(u,v),\ h(u,v),\ x = g(u,v) = u^3$ and $y = h(u,v) = v^3$. The functions g and h are continuous and differentiable, and the partial derivatives $g_u(u,v) = 3u^2,\ g_v(u,v) = 0,\ h_u(u,v) = 0$ and $h_v(u,v) = 3v^2$ are continuous on S;

b. $T(0,0) = (0,0),\ T(1,0) = (1,0),\ T(0,1) = (0,1)$, and $T(1,1) = (1,1)$;

c. R is the unit square in the xy-plane see the figure in the answer to the previous exercise.

In exercises 7 - 12, determine whether the transformations $T : S \to R$ are one-to-one or not.

7. $x = u^2,\ y = v^2$, where S is the rectangle of vertices $(-1,0),\ (1,0),\ (1,1)$, and $(-1,1)$.

8. $x = u^4,\ y = u^2 + v$, where S is the triangle of vertices $(-2,0),\ (2,0)$, and $(0,2)$.

Answer

T is not one-to-one: two points of S have the same image. Indeed, $T(-2,0) = T(2,0) = (16,4)$.

9. $x = 2u,\ y = 3v$, where S is the square of vertices $(-1,1),\ (-1,-1),\ (1,-1)$, and $(1,1)$.

10. $T(u,v) = (2u - v, u)$, where S is the triangle with vertices $(-1,1),\ (-1,-1)$, and $(1,-1)$.

Answer

T is one-to-one: We argue by contradiction. $T(u_1,v_1) = T(u_2,v_2)$ implies $2u_1 - v_1 = 2u_2 - v_2$ and $u_1 = u_2$. Thus, $u_1 = u + 2$ and $v_1 = v_2$.

11. $x = u + v + w,\ y = u + v,\ z = w$, where $S = R = R^3$.

12. $x = u^2 + v + w,\ y = u^2 + v,\ z = w$, where $S = R = R^3$.

Answer

T is not one-to-one: $T(1,v,w) = (-1,v,w)$

In exercises 13 - 18, the transformations $T : R \to S$ are one-to-one. Find their related inverse transformations $T^{-1} : R \to S$.

13. $x = 4u,\ y = 5v$, where $S = R = R^2$.

14. $x = u + 2v,\ y = -u + v$, where $S = R = R^2$.

Answer

$u = \frac{x-2y}{3},\ v = \frac{x+y}{3}$

15. $x = e^{2u+v},\ y = e^{u-v}$, where $S = R^2$ and $R = \{(x,y) \mid x > 0,\ y > 0\}$

16. $x = \ln u,\ y = \ln(uv)$, where $S = \{(u,v) \mid u > 0,\ v > 0\}$ and $R = R^2$.

Answer

$u = e^x,\ v = e^{-x+y}$

17. $x = u + v + w,\ y = 3v,\ z = 2w$, where $S = R = R^3$.

18. $x = u + v,\ y = v + w,\ z = u + w$, where $S = R = R^3$.

Answer

$u = \frac{x-y+z}{2},\ v = \frac{x+y-z}{2},\ w = \frac{-x+y+z}{2}$

In exercises 19 - 22, the transformation $T : S \to R,\ T(u,v) = (x,y)$ and the region $R \subset R^2$ are given. Find the region $S \subset R^2$.

19. $x = au,\ y = bv,\ R = \{(x,y) \mid x^2 + y^2 \le a^2b^2\}$ where $a, b > 0$

20. $x = au,\ y = bc,\ R = \{(x,y) \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1\}$, where $a, b > 0$

Answer

$S = \{(u,v) \mid u^2 + v^2 \le 1\}$

21. $x = \frac{u}{a},\ y = \frac{v}{b},\ z = \frac{w}{c},\ R = \{(x,y) \mid x^2 + y^2 + z^2 \le 1\}$, where $a, b, c > 0$

22. $x = au,\ y = bv,\ z = cw,\ R = \{(x,y) \mid \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} \le 1,\ z > 0\}$, where $a, b, c > 0$

Answer

$R = \{(u,v,w) \mid u^2 - v^2 - w^2 \le 1,\ w > 0\}$

In exercises 23 - 32, find the Jacobian J of the transformation.

23. $x = u + 2v,\ y = -u + v$

24. $x = \frac{u^3}{2},\ y = \frac{v}{u^2}$

Answer

$\frac{3}{2}$

25. $x = e^{2u-v},\ y = e^{u+v}$

26. $x = ue^v,\ y = e^{-v}$

Answer

-1

27. $x = u\ \cos(e^v),\ y = u\ \sin(e^v)$

28. $x = v\ \sin(u^2),\ y = v\ \cos(u^2)$

Answer

$2uv$

29. $x = u\ \cosh v,\ y = u\ \sinh v,\ z = w$

30. $x = v\ \cosh\left(\frac{1}{u}\right),\ y = v\ \sinh\left(\frac{1}{u}\right),\ z = u + w^2$

Answer

$\frac{v}{u^2}$

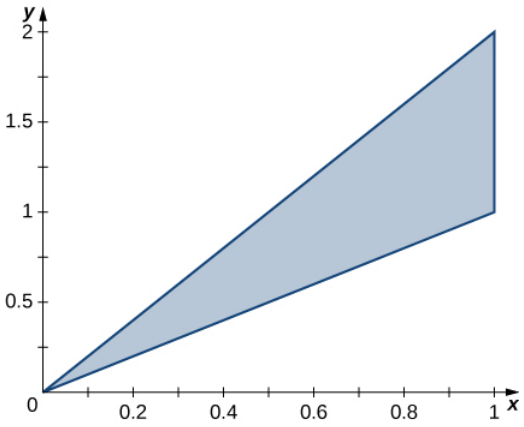
31. $x = u + v,\ y = v + w,\ z = u$

32. $x = u - v,\ y = u + v,\ z = u + v + w$

Answer

2

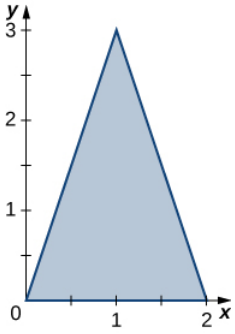
33. The triangular region R with the vertices $(0,0),\ (1,1)$, and $(1,2)$ is shown in the following figure.



a. Find a transformation $T : S \to R,\ T(u,v) = (x,y) = (au + bv + dv)$, where a, b, c, and d are real numbers with $ad - bc \neq 0$ such that $T^{-1}(0,0) = (0,0),\ T^{-1}(1,1) = (1,0)$, and $T^{-1}(1,2) = (0,1)$.

b. Use the transformation T to find the area $A(R)$ of the region R.

34. The triangular region R with the vertices $(0,0),\ (2,0)$, and $(1,3)$ is shown in the following figure.



a. Find a transformation $T : S \to R,\ T(u,v) = (x,y) = (au + bv + dv)$, where a, b, c, and d are real numbers with $ad - bc \neq 0$ such that $T^{-1}(0,0) = (0,0),\ T^{-1}(2,0) = (1,0)$, and $T^{-1}(1,3) = (0,1)$.

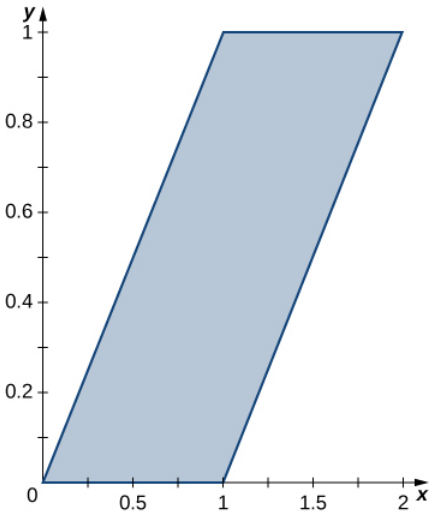
b. Use the transformation T to find the area $A(R)$ of the region R.

Answer

a. $T(u,v) = (2u + v,\ 3v)$

b. The area of R is $A(R) = \int_0^3 \int_{y/3}^{(6-y)/3} dx\,dy = \int_0^1 \int_0^{1-u} \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dv\,du = \int_0^1 \int_0^{1-u} 6\,dv\,du = 3.$

In exercises 35 - 36, use the transformation $u = y - x$, $v = y$, to evaluate the integrals on the parallelogram R of vertices $(0,0)$, $(1,0)$, $(2,1)$, and $(1,1)$ shown in the following figure.



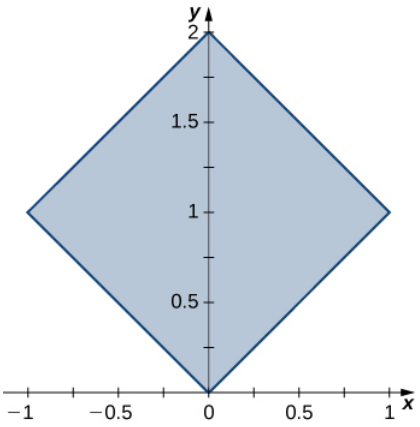
35. $\iint_R (y-x)\,dA$

36. $\iint_R (y^2-xy)\,dA$

Answer

$-\frac{1}{4}$

In exercises 37 - 38, use the transformation $y=x=u$, $x+y=v$ to evaluate the integrals on the square R determined by the lines $y=x$, $y=-x+2$, $y=x+2$, and $y=-x$ shown in the following figure.



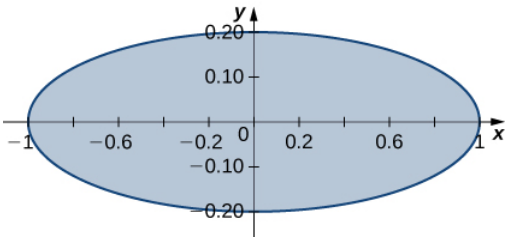
37. $\iint_R e^{x+y}\,dA$

38. $\iint_R \sin(x-y)\,dA$

Answer

$-1+\cos 2$

In exercises 39 - 40, use the transformation $x=u$, $5y=v$ to evaluate the integrals on the region R bounded by the ellipse $x^2+25y^2=1$ shown in the following figure.



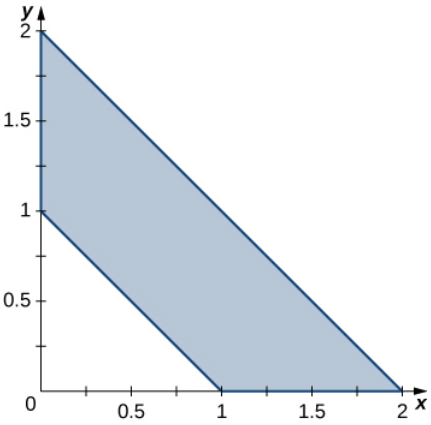
39. $\iint_R \sqrt{x^2+25y^2}\,dA$

40. $\iint_R (x^2+25y^2)^2\,dA$

Answer

$\frac{\pi}{15}$

In exercises 41 - 42, use the transformation $u=x+y$, $v=x-y$ to evaluate the integrals on the trapezoidal region R determined by the points $(1,0)$, $(2,0)$, $(0,2)$, and $(0,1)$ shown in the following figure.



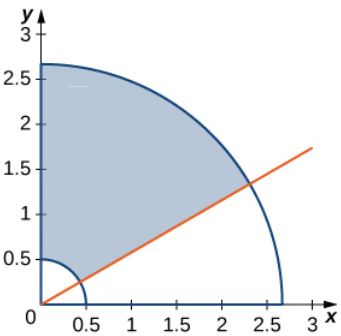
41. $\iint_R (x^2-2xy+y^2)\,e^{x+y}\,dA$

42. $\iint_R (x^3+3x^2y+3xy^2+y^3)\,dA$

Answer

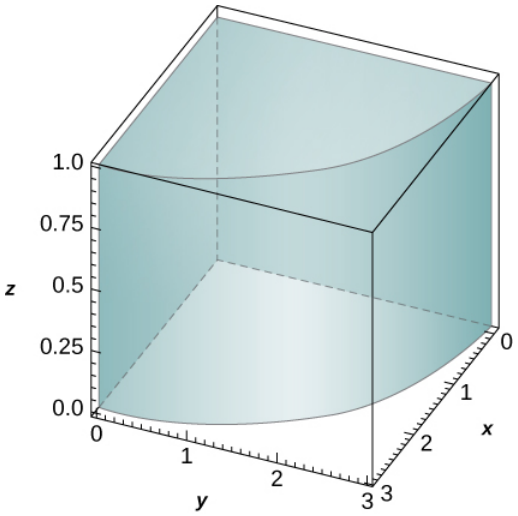
$\frac{31}{5}$

43. The circular annulus sector R bounded by the circles $4x^2+4y^2=1$ and $9x^2+9y^2=64$, the line $x=y\sqrt{3}$, and the y-axis is shown in the following figure. Find a transformation T from a rectangular region S in the $r\theta$-plane to the region R in the xy-plane. Graph S.



https://math.libretexts.org/@go/page/66959

44. The solid R bounded by the circular cylinder $x^2 + y^2 = 9$ and the planes $z = 0,\ z = 1,\ x = 0$, and $y = 0$ is shown in the following figure. Find a transformation T from a cylindrical box S in $r\theta z$-space to the solid R in xyz-space.



Answer

$T(r, \theta, z) = (r\ \cos\theta,\ r\ \sin\theta,\ z);\ S = [0, 3] \times [0, \frac{\pi}{2}] \times [0, 1]$ in the $r\theta z$-space

45. Show that

$$\iint_R f\left(\sqrt{\frac{x^2}{3} + \frac{y^2}{3}}\right) dA = 2\pi\sqrt{15} \int_0^1 f(\rho)\rho\ d\rho,$$

where f is a continuous function on $[0, 1]$ and R is the region bounded by the ellipse $5x^2 + 3y^2 = 15$.

46. Show that

$$\iiint_R f\left(\sqrt{16x^2 + 4y^2 + z^2}\right) dV = \frac{\pi}{2} \int_0^1 f(\rho)\rho^2 d\rho,$$

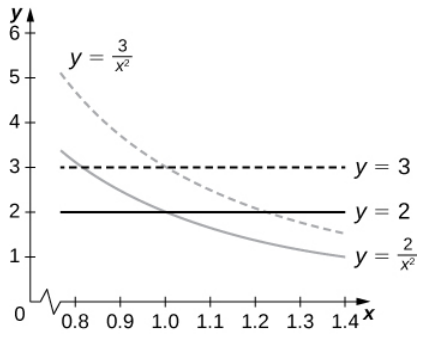
where f is a continuous function on $[0, 1]$ and R is the region bounded by the ellipsoid $16x^2 + 4y^2 + z^2 = 1$.

47. [T] Find the area of the region bounded by the curves $xy = 1,\ xy = 3,\ y = 2x$, and $y = 3x$ by using the transformation $u = xy$ and $v = \frac{y}{x}$. Use a computer algebra system (CAS) to graph the boundary curves of the region R.

48. [T] Find the area of the region bounded by the curves $x^2y = 2,\ x^2y = 3,\ y = x$, and $y = 2x$ by using the transformation $u = x^2y$ and $v = \frac{y}{x}$. Use a CAS to graph the boundary curves of the region R.

Answer

The area of R is $10 - 4\sqrt{6}$; the boundary curves of R are graphed in the following figure.



49. Evaluate the triple integral

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$$\int_0^1 \int_1^2 \int_z^{z+1} (y+1)\, dx\, dy\, dz$$

by using the transformation $u = x - z,\ v = 3y$, and $w = \frac{z}{2}$.

50. Evaluate the triple integral

$$\int_0^2 \int_4^6 \int_{3z}^{3z+2} (5 - 4y)\, dx\, dy\, dz$$

by using the transformation $u = x - 3z,\ v = 4y$, and $w = z$.

Answer

8

51. A transformation $T : R^2 \to R^2,\ T(u, v) = (x, y)$ of the form $x = au + bv,\ y = cu + dv$, where a, b, c, and d are real numbers, is called linear. Show that a linear transformation for which $ad - bc \neq 0$ maps parallelograms to parallelograms.

52. A transformation $T_\theta : R^2 \to R^2,\ T_\theta(u, v) = (x, y)$ of the form $x = u\ \cos\theta - v\ \sin\theta,\ y = u\ \sin\theta + v\ \cos\theta$, is called a rotation angle θ. Show that the inverse transformation of T_θ satisfies $T_\theta^{-1} = T_{-\theta}$ where $T_{-\theta}$ is the rotation of angle $-\theta$.

53. [T] Find the region S in the uv-plane whose image through a rotation of angle $\frac{\pi}{4}$ is the region R enclosed by the ellipse $x^2 + 4y^2 = 1$. Use a CAS to answer the following questions.

a. Graph the region S.

b. Evaluate the integral $\iint_S e^{-2uv}\, du\, dv$. Round your answer to two decimal places.

54. [T] The transformations $T_i : \mathbb{R}^2 \to \mathbb{R}^2,\ i = 1, \ldots, 4$, defined by $T_1(u, v) = (u, -v)$, $T_2(u, v) = (-u, v)$, $T_3(u, v) = (-u, -v)$, and $T_4(u, v) = (v, u)$ are called reflections about the x-axis, y-axis origin, and the line $y = x$, respectively.

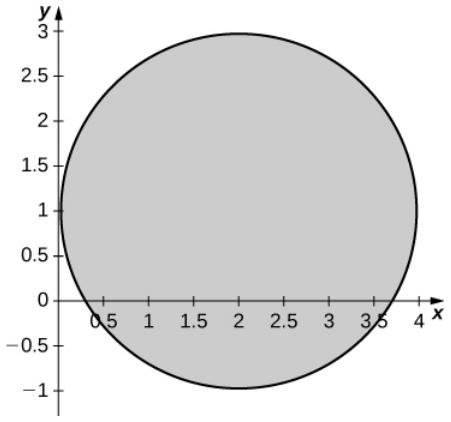
a. Find the image of the region $S = \{(u, v) \,|\, u^2 + v^2 - 2u - 4v + 1 \leq 0\}$ in the xy-plane through the transformation $T_1 \circ T_2 \circ T_3 \circ T_4$.

b. Use a CAS to graph R.

c. Evaluate the integral $\iint_S \sin(u^2)\, du\, dv$ by using a CAS. Round your answer to two decimal places.

Answer

a. $R = \{(x, y) \,|\, y^2 + x^2 - 2y - 4x + 1 \leq 0\}$;
b. R is graphed in the following figure;



c. **3.16**

55. [T] The transformations $T_{k,1,1} : \mathbb{R}^3 \to \mathbb{R}^3,\ T_{k,1,1}(u, v, w) = (x, y, z)$ of the form $x = ku,\ y = v,\ z = w$, where $k \neq 1$ is a positive real number, is called a stretch if $k > 1$ and a compression if $0 < k < 1$ in the x-direction. Use a CAS to evaluate the integral

$\iiint_S e^{-(4x^2+9y^2+25z^2)}\,dx\,dy\,dz$ on the solid $S = \{(x,y,z) \mid 4x^2 + 9y^2 + 25z^2 \leq 1\}$ by considering the compression $T_{2,3,5}(u,v,w) = (x,y,z)$ defined by $x = \frac{u}{2},\ y = \frac{v}{3}$, and $z = \frac{w}{5}$. Round your answer to four decimal places.

56. [T] The transformation $T_{a,0} : \mathbb{R}^2 \to \mathbb{R}^2,\ T_{a,0}(u,v) = (u + av, v)$, where $a \neq 0$ is a real number, is called a shear in the x-direction. The transformation, $T_{b,0} : R^2 \to R^2,\ T_{o,b}(u,v) = (u, bu + v)$, where $b \neq 0$ is a real number, is called a shear in the y-direction.

a. Find transformations $T_{0,2} \circ T_{3,0}$.

b. Find the image R of the trapezoidal region S bounded by $u = 0,\ v = 0,\ v = 1$, and $v = 2 - u$ through the transformation $T_{0,2} \circ T_{3,0}$.

c. Use a CAS to graph the image R in the xy-plane.

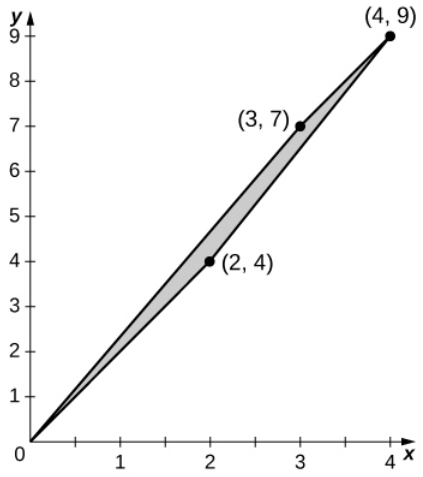
d. Find the area of the region R by using the area of region S.

Answer

a. $T_{0,2} \circ T_{3,0}(u,v) = (u + 3v, 2u + 7v)$;

b. The image S is the quadrilateral of vertices $(0,0),\ (3,7),\ (2,4)$, and $(4,9)$;

c. S is graphed in the following figure;



d. $\frac{3}{2}$

57. Use the transformation, $x = au,\ y = av,\ z = cw$ and spherical coordinates to show that the volume of a region bounded by the spheroid $\frac{x^2+y^2}{a^2} + \frac{z^2}{c^2} = 1$ is $\frac{4\pi a^2 c}{3}$.

58. Find the volume of a football whose shape is a spheroid $\frac{x^2+y^2}{a^2} + \frac{z^2}{c^2} = 1$ whose length from tip to tip is 11 inches and circumference at the center is 22 inches. Round your answer to two decimal places.

Answer

$\frac{2662}{3\pi} \approx 282.45\ in^3$

Contributors

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14R: Review Exercises for Chapter 14

True or False? **Justify your answer with a proof or a counterexample.**

1. $\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dy\,dx$

2. Fubini's theorem can be extended to three dimensions, as long as f is continuous in all variables.

Answer

True

3. The integral $\int_0^{2\pi} \int_0^1 \int_r^1 dz\,dr\,d\theta$ represents the volume of a right cone.

4. The Jacobian of the transformation for $x = u^2 - 2v,\ y = 3v - 2uv$ is given by $-4u^2 + 6u + 4v$.

Answer

False

Evaluate the following integrals.

5. $\iint_R (5x^3y^2 - y^2)\,dA$, where $R = \{(x,y) \mid 0 \le x \le 2,\ 1 \le y \le 4\}$

6. $\iint_D \frac{y}{3x^2+1}\,dA$, where $D = \{(x,y) \mid 0 \le x \le 1,\ -x \le y \le x\}$

Answer

0

7. $\iint_D \sin(x^2+y^2)\,dA$ where D is a disk of radius 2 centered at the origin.

8. $\int_0^1 \int_y^1 xye^{x^2}\,dx\,dy$

Answer

$\frac{1}{4}$

9. $\int_{-1}^1 \int_0^z \int_0^{x-z} 6\,dy\,dx\,dz$

10. $\iiint_R 3y\,dV$, where $R = \{(x,y,z) \mid 0 \le x \le 1,\ 0 \le y \le x,\ 0 \le z \le 9 - y^2\}$

Answer

1.475

11. $\int_0^2 \int_0^{2\pi} \int_r^1 r\,dz\,d\theta\,dr$

12. $\int_0^{2\pi} \int_0^{\pi/2} \int_1^3 \rho^2 \sin(\varphi)\,d\rho\,d\varphi\,d\theta$

Answer

$\frac{52\pi}{3}$

13. $\int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} dz\,dy\,dx$

For the following problems, find the specified area or volume.

14. The area of region enclosed by one petal of $r = \cos(4\theta)$.

Answer

$\frac{\pi}{16}$ units^3

15. The volume of the solid that lies between the paraboloid $z = 2x^2 + 2y^2$ and the plane $z = 8$.

16. The volume of the solid bounded by the cylinder $x^2 + y^2 = 16$ and from $z = 1$ to $z + x = 2$.

Answer

93.291 units^3

17. The volume of the intersection between two spheres of radius 1, the top whose center is $(0, 0, 0.25)$ and the bottom, which is centered at $(0, 0, 0)$.

For the following problems, find the center of mass of the region.

18. $\rho(x, y) = xy$ on the circle with radius 1 in the first quadrant only.

Answer

$(\frac{8}{15}, \frac{8}{15})$

19. $\rho(x, y) = (y + 1)\sqrt{x}$ in the region bounded by $y = e^x$, $y = 0$, and $x = 1$.

20. $\rho(x, y, z) = z$ on the inverted cone with radius 2 and height 2.

Answer

$(0, 0, \frac{8}{5})$

21. The volume an ice cream cone that is given by the solid above $z = \sqrt{x^2 + y^2}$ and below $z^2 + x^2 + y^2 = z$.

The following problems examine Mount Holly in the state of Michigan. Mount Holly is a landfill that was converted into a ski resort. The shape of Mount Holly can be approximated by a right circular cone of height 1100 ft and radius 6000 ft.

22. If the compacted trash used to build Mount Holly on average has a density 400 lb/ft^3, find the amount of work required to build the mountain.

Answer

$1.452\pi \times 10^{15}$ ft-lb

23. In reality, it is very likely that the trash at the bottom of Mount Holly has become more compacted with all the weight of the above trash. Consider a density function with respect to height: the density at the top of the mountain is still density 400 lb/ft^3, and the density increases. Every 100 feet deeper, the density doubles. What is the total weight of Mount Holly?

The following problems consider the temperature and density of Earth's layers.

24. **[T]** The temperature of Earth's layers is exhibited in the table below. Use your calculator to fit a polynomial of degree 3 to the temperature along the radius of the Earth. Then find the average temperature of Earth. (*Hint*: begin at 0 in the inner core and increase outward toward the surface)

Source: http://www.enchantedlearning.com/sub...h/Inside.shtml

Layer	Depth from center (km)	Temperature °C
Rocky Crust	0 to 40	0
Upper Mantle	40 to 150	870
Mantle	400 to 650	870
Inner Mantel	650 to 2700	870
Molten Outer Core	2890 to 5150	4300
Inner Core	5150 to 6378	7200

Answer

$y = -1.238 \times 10^{-7}x^3 + 0.001196x^2 - 3.666x + 7208$;
The average temperature is approximately 2800 °C.

25. **[T]** The density of Earth's layers is displayed in the table below. Using your calculator or a computer program, find the best-fit quadratic equation to the density. Using this equation, find the total mass of Earth.

Source: http://hyperphysics.phy-astr.gsu.edu...rthstruct.html

Layer	Depth from center (km)	Density (g/cm^3)
Inner Core	0	12.95
Outer Core	1228	11.05
Mantle	3488	5.00
Upper Mantle	6338	3.90
Crust	6378	2.55

The following problems concern the Theorem of Pappus (see Moments and Centers of Mass for a refresher), a method for calculating volume using centroids. Assuming a region R, when you revolve around the x-axis the volume is given by $V_x = 2\pi A\bar{y}$, and when you revolve around the y-axis the volume is given by $V_y = 2\pi A\bar{x}$, where A is the area of R. Consider the region bounded by $x^2 + y^2 = 1$ and above $y = x + 1$.

26. Find the volume when you revolve the region around the x-axis.

Answer

$\frac{\pi}{3}$ units^3

27. Find the volume when you revolve the region around the y-axis.

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More Chapter 14 Review Exercises

13.1: Iterated Integrals and Area

Terms and Concepts

1. When integrating $f_x(x,y)$ with respect to x, the constant of integration C is really which: $C(x)$ or $C(y)$? What does this mean?

2. Integrating an integral is called _____ _____.

3. When evaluating an iterated integral, we integrate from _____ to _____, then from _____ to _____.

4. One understanding of an iterated integral is that $\int_a^b \int_{g_1(x)}^{g_2(x)} dy\,dx$ gives the _____ of a plane region.

Problems

In Exercises 5-10, evaluate the integral and subsequent iterated integral.

5.

(a) $\int_2^5 (6x^2+4xy-3y^2)\,dy$

(b) $\int_{-3}^2 \int_2^5 (6x^2+4xy-3y^2)\,dy\,dx$

6.

(a) $\int_0^{\pi} (2x\cos y+\sin x)\,dx$

(b) $\int_0^{\pi/2} \int_0^{\pi} (2x\cos y+\sin x)\,dx\,dy$

7.

(a) $\int_1^x (x^2y-y+2)\,dy$

(b) $\int_0^2 \int_1^x (x^2y-y+2)\,dy\,dx$

8.

(a) $\int_y^{y^2} (x-y)\,dx$

(b) $\int_{-1}^1 \int_y^{y^2} (x-y)\,dx\,dy$

9.

(a) $\int_0^y (\cos x\sin y)\,dx$

(b) $\int_0^{\pi} \int_0^y (\cos x\sin y)\,dx\,dy$

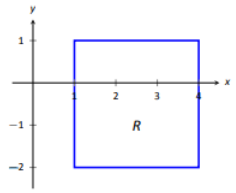
10.

(a) $\int_0^x \left(\frac{1}{1+x^2}\right) dy$

(b) $\int_1^2 \int_0^x \left(\frac{1}{1+x^2}\right) dy\,dx$

In Exercises 11-16, a graph of a planar region R is given. Give the iterated integrals, with both orders of integration $dy\,dx$ and $dx\,dy$, that give the area of R. Evaluate one of the iterated integrals to find the area.

11.



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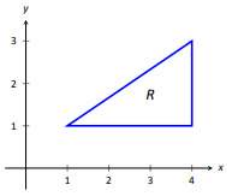

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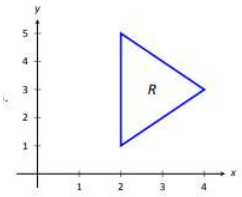

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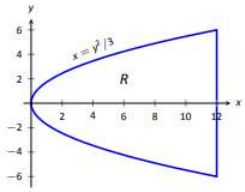

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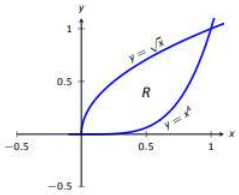

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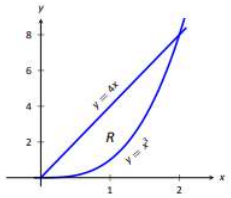

In Exercises 17-22, iterated integrals are given that compute the area of a region R in the xy-plane. Sketch the region R, and give the iterated integral(s) that give the area of R with the opposite order of integration.

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17. $\int_{-2}^{2}\int_{0}^{4-x^2} dy\,dx$

18. $\int_{0}^{1}\int_{5-5x}^{5-5x^2} dy\,dx$

19. $\int_{-2}^{2}\int_{0}^{2\sqrt{4-y^2}} dx\,dy$

20. $\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dy\,dx$

21. $\int_{0}^{1}\int_{-\sqrt{y}}^{\sqrt{y}} dx\,dy + \int_{1}^{4}\int_{y-2}^{\sqrt{y}} dx\,dy$

22. $\int_{-1}^{1}\int_{(x-1)/2}^{(1-x)/2} dy\,dx$

13.2: Double Integration and Volume

Terms and Concepts

1. An integral can be interpreted as giving the signed area over an interval; a double integral can be interpreted as giving the signed _____ over a region.

2. Explain why the following statement is false: "Fubini's Theorem states that $\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx = \int_a^b \int_{g_1(y)}^{g_2(y)} f(x,y)\,dx\,dy$."

3. Explain why if $f(x,y) > 0$ over a region R, then $\iint_R f(x,y)\,dA > 0$.

4. If $\iint_R f(x,y)\,dA = \iint_R g(x,y)\,dA$, does this imply $f(x,y) = g(x,y)$?

Problems

In Exercises 5-10,
(a) Evaluate the given iterated integral, and
(b) rewrite the integral using the other order of integration.

5. $\int_{1}^{2}\int_{-1}^{1}\left(\frac{x}{y}+3\right) dx\,dy$

6. $\int_{-\pi/2}^{\pi/2}\int_{0}^{\pi}(\sin x \cos y\,\mathrm{dy}\,\mathrm{dx})$

7. $\int_{0}^{4}\int_{0}^{-x/2+2}\left(3x^2 - y + 2\right) dy\,dx$

8. $\int_{1}^{3}\int_{y}^{3}\left(x^2y - xy^2\right) dx\,dy$

9. $\int_{0}^{2} 1 \int_{-\sqrt{1-y}}^{\sqrt{1-y}}(x+y+2)\,dx\,dy$

10. $\int_{0}^{9}\int_{y/3}^{\sqrt{3}}\left(xy^2\right) dx\,dy$

In Exercises 11-18:
(a) Sketch the region R given by the problem.
(b) Set up the iterated integrals, in both orders, that evaluate the given double integral for the described region R.
(c) Evaluate one of the iterated integrals to find the signed volume under the surface $z = f(x,y)$ over the region R.

11. $\iint_R x^2y\,dA$, where R is bounded by $y = \sqrt{x}$ and $y = x^2$.

12. $\iint_R x^2y\,dA$, where R is bounded by $y = \sqrt[3]{x}$ and $y = x^3$.

13. $\iint_R x^2 - y^2\,dA$, where R is the rectangle with corners $(-1,-1), (1,-1), (1,1)$ and $(-1,1)$.

14. $\iint_R ye^x\,dA$, where R is bounded by $x=0,\ x=y^2$ and $y=1$.

15. $\iint_R (6-3x-2y)\,dA$, where R is bounded by $x=0, y=0$ and $3x+2y=6$.

16. $\iint_R e^y\,dA$, where R is bounded by $y=\ln x$ and $y=\frac{1}{e-1}(x-1)$.

17. $\iint_R (x^3y-x)\,dA$, where R is the half of the circle $x^2+y^2=9$ in the first and second quadrants.

18. $\iint_R (4-sy)\,dA$, where R is bounded by $y=0, y=x/e$ and $y=\ln x$.

In Exercises 19-22, state why it is difficult/impossible to integrate the iterated integral in the given order of integration. Change the order of integration and evaluate the new iterated integral.

19. $\int_0^4 \int_{y/2}^2 e^{x^2}\,dx\,dy$

20. $\int_0^{\sqrt{\pi/2}} \int_x^{\sqrt{\pi/2}} \cos(y^2)\,dy\,dx$

21. $\int_0^1 \int_y^1 \frac{2y}{x^2+y^2}\,dx\,dy$

22. $\int_{-1}^1 \int_1^2 \frac{x\tan^2 y}{1+\ln y}\,dy\,dx$

In Exercises 23-26, find the average value of f over the region R. Notice how these functions and regions are related to the iterated integrals given in Exercises 5-8.

23. $f(x,y)=\frac{x}{y}+3$; R is the rectangle with opposite corners $(-1,1)$ and $(1,2)$.

24. $f(x,y)=\sin x\cos y$; R is bounded by $x=0, x=\pi, y=-\pi/2$ and $y=\pi/2$.

25. $f(x,y)=3x^2-y+2$; R is bounded by the lines $y=0, y=2-x/2$ and $x=0$.

26. $f(x,y)=x^2y-xy^2$; R is bounded by $y=x, y=1$ and $x=3$.

13.3: Double Integration with Polar Coordinates

Terms and Concepts

1. When evaluating $\iint_R f(x,y)\,dA$ using polar coordinates, $f(x,y)$ is replaced with _____ and dA is replaced with _____.

2. Why would one be interested in evaluating a double integral with polar coordinates?

Problems

In Exercises 3-10, a function $f(x,y)$ is given and a region R of the xy-plane is described. Set up and evaluate $\iint_R f(x,y)\,dA$.

3. $f(x,y)=3x-y+4$; R is the region enclosed by the circle $x^2+y^2=1$.

4. $f(x,y)=4x+4y$; R is the region enclosed by the circle $x^2+y^2=4$.

5. $f(x,y)=8-y$; R is the region enclosed by the circles with polar equations $r=\cos\theta$ and $r=3\cos\theta$.

6. $f(x,y)=4$; R is the region enclosed by the petal of the rose curve $r=\sin(2\theta)$ in the first quadrant.

7. $f(x,y)=\ln(x^2+y^2)$; R is the annulus enclosed by the circles \(x^2+y^2=1\text{ and }x^2+y^2=4.

8. $f(x,y)=1-x^2-y^2$; R is the region enclosed by the circle $x^2+y^2=1$.

9. $f(x,y)=x^2-y^2$; R is the region enclosed by the circle $x^2+y^2=36$ in the first and fourth quadrants.

10. $f(x,y)=(x-y)/(x+y)$; R is the region enclosed by the lines $y=x, y=0$ and the circle $x^2+y^2=1$ in the first quadrant.

In Exercises 11-14, an iterated integral in rectangular coordinates is given. Rewrite the integral using polar coordinates and evaluate the new double integral.

11. $\int_0^5 \int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} \sqrt{x^2+y^2}\,dy\,dx$

12. $\int_{-4}^{4} \int_{-\sqrt{16-y^2}}^{0} (2y-x)\,dx\,dy$

13. $\int_0^2 \int_y^{\sqrt{8-y^2}} (x+y)\,dx\,dy$

14. $\int_{-2}^{-1} \int_0^{\sqrt{4-x^2}} (x+5)\,dy\,dx + \int_{-1}^{1} \int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} (x+5)\,dy\,dx + \int_1^2 \int_0^{\sqrt{4-x^2}} (x+5)\,dy\,dx$

In Exercises 15-16, special double integrals are presented that are especially well suited for evaluation in polar coordinates.

15. Consider $\iint_R e^{-(x^2+y^2)}\,dA$.

(a) Why is this integral difficult to evaluate in rectangular coordinates, regardless of the region R?

(b) Let R be the region bounded by the circle of radius a centered at the origin. Evaluate the double integral using polar coordinates.

(c) Take the limit of your answer from (b), as $a \to \infty$. What does this imply about the volume under the surface of $e^{-(x^2+y^2)}$ over the entire xy-plane?

16. The surface of a right circular cone with height h and base radius a can be described by the equation $f(x,y) = h - h\sqrt{\frac{x^2}{a^2}+\frac{y^2}{a^2}}$, where the tip of the cone lies at $(0,0,h)$ and the circular base lies in the xy-plane, centered at the origin.

Confirm that the volume of a right circular cone with height h and base radius a is $V = \frac{1}{3}\pi a^2 h$ by evaluating $\iint_R f(x,y)\,dA$ in polar coordinates.

13.4: Center of Mass

Terms and Concepts

1. Why is it easy to use "mass" and "weight" interchangeably, even though they are different measures?

2. Given a point (x,y), the value of x is a measure of distance from the _____-axis.

3. We can think of $\iint_R dm$ as meaning "sum up lots of _____."

4. What is a "discrete planar system?"

5. Why does M_x use $\iint_R y\delta(x,y)\,dA$ instead of $\iint_R x\delta(x,y)\,dA$; that is, why do we use "y" and not "x"?

6. Describe a situation where the center of mass of a lamina does not lie within the region of the lamina itself.

Problems

In Exercises 7-10, point masses are given along a line or in the plane. Find the center of mass $\bar{x}$ or $(\bar{x},\bar{y})$, as appropriate. (All masses are in grams and distances are in cm.)

7. $m_1 = 4$ at $x = 1$; $m_2 = 3$ at $x = 3$; $m_3 = 5$ at $x = 10$

8. $m_1 = 2$ at $x = -3$; $m_2 = 2$ at $x = -1$; $m_3 = 3$ at $x = 0$; $m_4 = 3$ at $x = 7$

9. $m_1 = 2$ at $(-2,2)$; $m_2 = 2$ at $(2,-2)$; $m_3 = 20$ at $(0,4)$

10. $m_1 = 1$ at $(-1,1)$; $m_2 = 2$ at $(-1,1)$; $m_3 = 2$ at $(1,1)$; $m_4 = 1$ at $(1,-1)$

In Exercises 11-18, find the mass/weight of the lamina described by the region R in the plane and its density function $\delta(x,y)$.

11. R is the rectangle with corners $(1,-3),(1,2),(7,2)$ and $(7,-3)$; $\delta(x,y) = 5\text{gm/cm}^2$

12. R is the rectangle with corners $(1,-3),(1,2),(7,2)$ and $(7,-3)$; $\delta(x,y) = (x+y^2)\text{gm/cm}^2$

13. R is the triangle with corners $(-1,0),(1,0)$, and $(0,1)$; $\delta(x,y) = 2\text{lb/in}^2$

14. R is the triangle with corners $(0,0),(1,0)$, and $(0,1)$; $\delta(x,y) = (x^2+y^2+1)\text{lb/in}^2$

15. R is the circle centered at the origin with radius 2; $\delta(x,y) = (x+y+4)\text{kg/m}^2$

16. R is the circle sector bounded by $x^2+y^2=25$ in the first quadrant; $\delta(x,y) = (\sqrt{x^2+y^2}+1)\text{kg/m}^2$

17. R is the annulus in the first and second quadrants bounded by $x^2+y^2=9$ and $x^2+y^2=36$; $\delta(x,y) = 4\text{lb/ft}^2$

18. R is the annulus in the first and second quadrants bounded by $x^2+y^2=9$ and $x^+y^2=36$; $\delta(x,y) = \sqrt{x^2+y^2}\text{lb/ft}^2$

In Exercises 19-26, find the center of mass of the lamina described by the region R in the plane and its density function $\delta(x,y)$.

Note: these are the same lamina as in Exercises 11-18.

19. R is the rectangle with corners $(1,-3),(1,2),(7,2)$ and $(7,-3)$; $\delta(x,y) = 5\text{gm/cm}^2$

20. R is the rectangle with corners $(1,-3),(1,2),(7,2)$ and $(7,-3)$; $\delta(x,y) = (x+y^2)\text{gm/cm}^2$

21. R is the triangle with corners $(-1,0),(1,0),$ and $(0,1)$; $\delta(x,y) = 2\text{lb/in}^2$

22. R is the triangle with corners $(0,0),(1,0),$ and $(0,1)$; $\delta(x,y) = (x^2+y^2+1)\text{lb/in}^2$

23. R is the circle centered at the origin with radius 2; $\delta(x,y) = (x+y+4)\text{kg/m}^2$

24. R is the circle sector bounded by $x^2+y^2=25$ in the first quadrant; $\delta(x,y) = (\sqrt{x^2+y^2}+1)\text{kg/m}^2$

25. R is the annulus in the first and second quadrants bounded by $x^2+y^2=9$ and $x^2+y^2=36$; $\delta(x,y) = 4\text{lb/ft}^2$

26. R is the annulus in the first and second quadrants bounded by $x^2+y^2=9$ and $x^+y^2=36$; $\delta(x,y) = \sqrt{x^2+y^2}\text{lb/ft}^2$

The moment of inertia i is a measure of the tendency of lamina to resist rotating about an axis or continue to rotate about an axis. i_x is the moment of inertia about the x-axis, i_x is the moment of inertia about the x-axis, and i_o is the moment of inertia about the origin. These are computed as follows:

- $i_x = \iint_R y^2\,dm$
- $i_y = \iint_R x^2\,dm$
- $i_o = \iint_R (x^2+y^2)\,dm$

In Exercises 27-30, a lamina corresponding to a planar region R is given with a mass of 16 units. For each, compute i_x, i_y and i_o.

27. R is the 4 x 4 square with corners $(-2,-2)$ and $(2,2)$ with density $\delta(x,y) = 1$.

28. R is the 8 x 2 rectangle with corners $(-4,-1)$ and $(4,1)$ with density $\delta(x,y) = 1$.

29. R is the 4 x 2 rectangle with corners $(-2,-1)$ and $(2,1)$ with density $\delta(x,y) = 2$.

30. R is the circle with radius 2 centered at the origin with density $\delta(x,y) = 4/\pi$.

13.5: Surface Area

Terms and Concepts

1. "Surface area" is analogous to what previously studied concept?

2. To approximate the area of a small portion of a surface, we computed the area of its _____ plane.

3. We interpret $\iint_R dS$ as "sum up lots of little _____ _____."

4. Why is it important to know how to set up a double integral to compute surface area, even if the resulting integral is hard to evaluate?

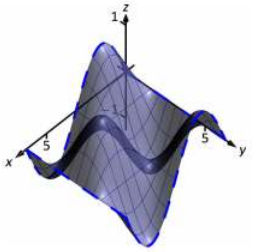
5. Why do $z = f(x,y)$ and $z = g(x,y) = f(x,y) + h$, for some real number h, have the same surface area over a region R?

6. Let $z = f(x,y)$ and $z = g(x,y) = 2f(x,y)$. Why is the surface area of g over a region R not twice the surface area of f over R?

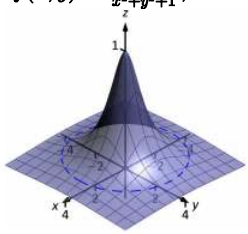
Problems

In Exercises 7-10, set up the iterated integral that computes the surfaces area of the given surface over the region R.

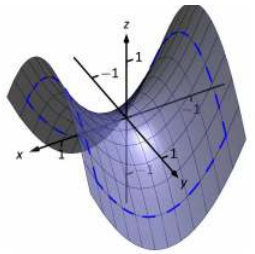
7. $f(x,y) = \sin x \cos y$; R is the rectangle with bounds $0 \le x \le 2\pi$, $0 \le y \le 2\pi$.



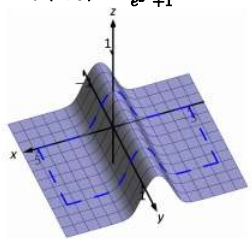
8. $f(x,y) = \frac{1}{x^2+y^2+1}$; R is the circle $x^2 + y^2 = 9$.



9. $f(x,y) = x^2 - y^2$; R is the rectangle with opposite corners $(-1,-1)$ and $1,1)$.



10. $f(x,y) = \frac{1}{e^{x^2}+1}$; R is the rectangle bounded by $-5 \le x \le 5$ and $0 \le y \le 1$.



In Exercises 11-19, find the area of the given surface over the region R.

11. $f(x,y) = 3x - 7y + 2$; R is the rectangle with opposite corners $(-1,0)$ and $(1,3)$.

12. $f(x,y) = 2x + 2y + 2$; R is the triangle with corners $(0,0), (1,0)$ and $(0,1)$.

13. $f(x,y) = x^2 + y^2 + 10$; R is the circle $x^2 + y^2 = 16$.

14. $f(x,y) = -2x + 4y^2 + 7$ over R, the triangle bounded by $y = -x, y = x, 0 \le y \le 1$.

15. $f(x,y) = x^2 + y$ over R, the triangle bounded by $y = 2x, y = 0$ and $x = 2$.

16. $f(x,y) = \frac{2}{3}x^{3/2}$ over R, the rectangle with opposite corners $(0,0)$ and $(1,1)$.

17. $f(x,y) = 10 - 2\sqrt{x^2+y^2}$ over R, the circle $x^2 + y^2 = 25$. (This is the cone with height 10 and base radius 5; be sure to compare your result with the known formula.)

18. Find the surface area of the sphere with radius 5 by doubling the surface area of $f(x,y) = \sqrt{25 - x^2 - y^2}$ over R, the circle $x^2 + y^2 = 25$. (Be sure to compare your result with the known formula.)

19. Find the surface area of the ellipse formed by restricting the plane $f(x,y) = cx + dy + h$ to the region R, the circle $x^2 + y^2 = 1$, where c, d and h are some constants. Your answer should be given in terms of c and d; why does the value of h not matter?

13.6: Volume Between Surfaces and Triple Integration

Terms and Concepts

1. The strategy for establishing bounds for triple integrals is "_____ to _____, _____ and _____ to _____."

2. Give an informal interpretation of what "$\int\int\int_D dV$" means.

3. Give two uses of triple integration.

4. If an object has a constant density δ and a volume V, what is its mass?

Problems

In Exercises 5-8, two surfaces $f_1(x,y)$ and $f_2(x,y)$ and a region R in the xy-plane are given. Set up and evaluate the double integral that finds the volume between these surfaces over R.

5. $f_1(x,y) = 8 - x^2 - y^2,\ f_2(x,y) = 2x + y$;
R is the square with corners $(-1,-1)$ and $(1,1)$.

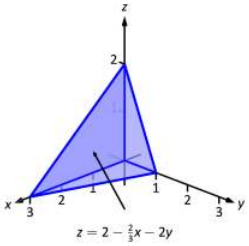
6. $f_1(x,y) = x^2 + y^2,\ f_2(x,y) = -x^2 - y^2$;
R is the square with corners $(0,0)$ and $(2,3)$.

7. $f_1(x,y) = \sin x \cos y,\ f_2(x,y) = \cos x \sin y + 2$;
R is the triangle with corners $(0,0), (\pi,0)$ and (π,π).

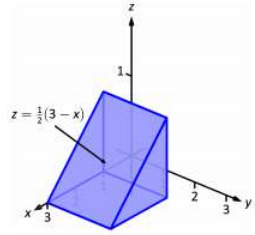
8. $f_1(x,y) = 2x^2 + 2y^2 + 3,\ f_2(x,y) = 6 - x^2 - y^2$;
R is the circle $x^2 + y^2 = 1$.

In Exercises 9-16, a domain D is described by its bounding surfaces, along with a graph. Set up the triple integrals that give the volume of D in all 6 orders of integration, and find the volume of D by evaluating the indicated triple integral.

9. D is bounded by the coordinate planes and $z = 2 - 2x/3 - 2y$.
Evaluate the triple integral with order *dz dy dz*.



10. D is bounded by the planes $y = 0, y = 2, x = 1, z = 0$ and $z = (2 - x)/2$.
Evaluate the triple integral with order *dx dy dz*.



11. D is bounded by the planes $x = 0, x = 2, z = -y$ and by $z = y^2/2$.
Evaluate the triple integral with order *dy dz dx*.

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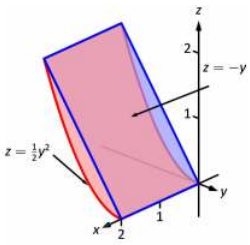
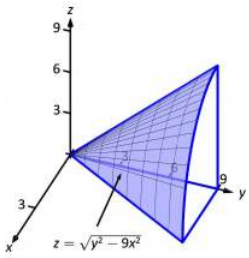
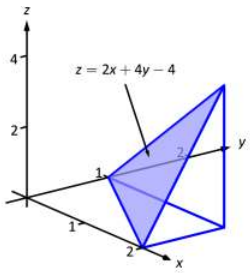


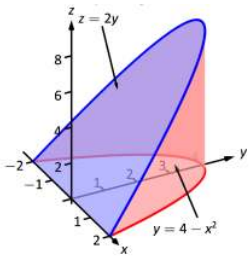
12. D is bounded by the planes \(z=0,y=9, x=0\text{ and by }\)z=\sqrt{y^2-9x^2}\).
Do not evaluate any triple integral.



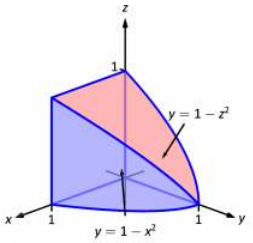
13. D is bounded by the planes $x = 2, y = 1, z = 0$ and $z = 2x + 4y - 4$.
Evaluate the triple integral with order *dx dy dz*.



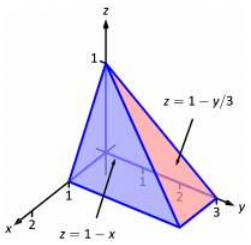
14. D is bounded by the plane $z = 2y$ and by $y = 4 - x^2$.
Evaluate the triple integral with order *dz dy dz*.



15. D is bounded by the coordinate planes and $y = 1 - x^2$ and $y = 1 - z^2$.
Do not evaluate any triple integral. Which order is easier to evaluate: *dz dy dx* or *dy dz dx*? Explain why.



16. D is bounded by the coordinate planes and by $z = 1 - y/3$ and $z = 1 - x$.
Evaluate the triple integral with order *dx dy dz*.



In Exercises 17-20, evaluate the triple integral.

17. $\int_{-\pi/2}^{\pi/2}\int_{0}^{\pi}\int_{0}^{\pi}(\cos x \sin y \sin z)dz\,dy\,dx$

18. $\int_{0}^{1}\int_{0}^{x}\int_{0}^{x+y}(x+y+z)dz\,dy\,dx$

19. $\int_{0}^{\pi}\int_{0}^{1}\int_{0}^{z}(\sin(yz))dx\,dy\,dz$

20. $\int_{\pi}^{\pi^2}\int_{x}^{x^3}\int_{-y^2}^{y^2}(\cos x \sin y \sin z)dz\,dy\,dx$

In Exercises 21-24, find the center of mass of the solid represented by the indicated space region D with density function $\delta(x,y,z)$.

21. D is bounded by the coordinate planes and $z = 2 - 2x/3 - 2y$; $\delta(x,y,z) = 10\text{g/cm}^3$.
(Note: this is the same region as used in Exercise 9.)

22. D is bounded by the planes $y = 0, y = 2, x = 1, z = 0$ and $z = (3-x)/2$; $\delta(x,y,z) = 2\text{g/cm}^3$.
(Note: this is the same region as used in Exercise 10.)

23. D is bounded by the planes $x = 2, y = 1, z = 0$ and $z = 2x + 4y - 4$; $\delta(x,y,z) = x^2\text{lb/in}^3$.
(Note: this is the same region as used in Exercise 13.)

24. D is bounded by the planes $z = 2y$ and by $y = 4 - x^2$. $\delta(x,y,z) = y^2\text{lb/in}^3$.
(Note: this is the same region as used in Exercise 14.)

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CHAPTER OVERVIEW

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15.0: Introduction to Vector Field Chapter

Hurricanes are huge storms that can produce tremendous amounts of damage to life and property, especially when they reach land. Predicting where and when they will strike and how strong the winds will be is of great importance for preparing for protection or evacuation. Scientists rely on studies of rotational vector fields for their forecasts.

Figure 15.0.1: Hurricanes form from rotating winds driven by warm temperatures over the ocean. Meteorologists forecast the motion of hurricanes by studying the rotating vector fields of their wind velocity. Shown is Cyclone Catarina in the South Atlantic Ocean in 2004, as seen from the International Space Station. (credit: modification of work by NASA)

In this chapter, we learn to model new kinds of integrals over fields such as magnetic fields, gravitational fields, or velocity fields. We also learn how to calculate the work done on a charged particle traveling through a magnetic field, the work done on a particle with mass traveling through a gravitational field, and the volume per unit time of water flowing through a net dropped in a river.

All these applications are based on the concept of a vector field, which we explore in this chapter. Vector fields have many applications because they can be used to model real fields such as electromagnetic or gravitational fields. A deep understanding of physics or engineering is impossible without an understanding of vector fields. Furthermore, vector fields have mathematical properties that are worthy of study in their own right. In particular, vector fields can be used to develop several higher-dimensional versions of the Fundamental Theorem of Calculus.

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15.1: Vector Fields

Learning Objectives

- Recognize a vector field in a plane or in space.
- Sketch a vector field from a given equation.
- Identify a conservative field and its associated potential function.

Vector fields are an important tool for describing many physical concepts, such as gravitation and electromagnetism, which affect the behavior of objects over a large region of a plane or of space. They are also useful for dealing with large-scale behavior such as atmospheric storms or deep-sea ocean currents. In this section, we examine the basic definitions and graphs of vector fields so we can study them in more detail in the rest of this chapter.

Examples of Vector Fields

How can we model the gravitational force exerted by multiple astronomical objects? How can we model the velocity of water particles on the surface of a river? Figure 15.1.1 gives visual representations of such phenomena.

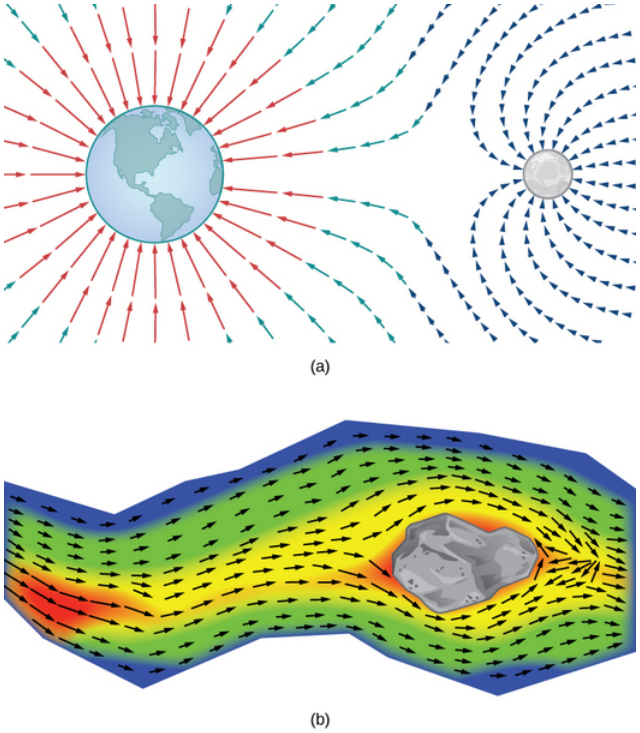

Figure 15.1.1 (a) The gravitational field exerted by two astronomical bodies on a small object. (b) The vector velocity field of water on the surface of a river shows the varied speeds of water. Red indicates that the magnitude of the vector is greater, so the water flows more quickly; blue indicates a lesser magnitude and a slower speed of water flow.

Figure 15.1.1*a* shows a gravitational field exerted by two astronomical objects, such as a star and a planet or a planet and a moon. At any point in the figure, the vector associated with a point gives the net gravitational force exerted by the two objects on an object of unit mass. The vectors of largest magnitude in the figure are the vectors closest to the larger object. The larger object has greater mass, so it exerts a gravitational force of greater magnitude than the smaller object.

Figure 15.1.1*b* shows the velocity of a river at points on its surface. The vector associated with a given point on the river's surface gives the velocity of the water at that point. Since the vectors to the left of the figure are small in magnitude, the water is flowing slowly on that part of the surface. As the water moves from left to right, it encounters some rapids around a rock. The speed of the water increases, and a whirlpool occurs in part of the rapids.

Each figure illustrates an example of a vector field. Intuitively, a vector field is a map of vectors. In this section, we study vector fields in $\mathbb{R}^2$ and $\mathbb{R}^3$.

DEFINITION: Vector Field

- A vector field $\vec{\mathbf{F}}$ in $\mathbb{R}^2$ is an assignment of a two-dimensional vector $\vec{\mathbf{F}}(x,y)$ to each point (x,y) of a subset D of $\mathbb{R}^2$. The subset D is the domain of the vector field.
- A vector field $\vec{\mathbf{F}}$ in $\mathbb{R}^3$ is an assignment of a three-dimensional vector $\vec{\mathbf{F}}(x,y,z)$ to each point (x,y,z) of a subset D of $\mathbb{R}^3$. The subset D is the domain of the vector field.

Vector Fields in $\mathbb{R}^2$

A vector field in $\mathbb{R}^2$ can be represented in either of two equivalent ways. The first way is to use a vector with components that are two-variable functions:

$$\vec{\mathbf{F}}(x,y) = \langle P(x,y), Q(x,y) \rangle$$

The second way is to use the standard unit vectors:

$$\vec{\mathbf{F}}(x,y) = P(x,y)\,\hat{\mathbf{i}} + Q(x,y)\,\hat{\mathbf{j}}.$$

A vector field is said to be *continuous* if its component functions are continuous.

Example 15.1.1: Finding a Vector Associated with a Given Point

Let $\vec{\mathbf{F}}(x,y) = (2y^2 + x - 4)\,\hat{\mathbf{i}} + \cos(x)\,\hat{\mathbf{j}}$ be a vector field in $\mathbb{R}^2$. Note that this is an example of a continuous vector field since both component functions are continuous. What vector is associated with point $(0,-1)$?

Solution

Substitute the point values for x and y:

$$\begin{aligned}\vec{\mathbf{F}}(0,-1) &= (2(-1)^2 + 0 - 4)\,\hat{\mathbf{i}} + \cos(0)\,\hat{\mathbf{j}} \\ &= -2\,\hat{\mathbf{i}} + \hat{\mathbf{j}}.\end{aligned}$$

Exercise 15.1.1

Let $\vec{\mathbf{G}}(x,y) = x^2y\,\hat{\mathbf{i}} - (x+y)\,\hat{\mathbf{j}}$ be a vector field in $\mathbb{R}^2$. What vector is associated with the point $(-2,3)$?

Drawing a Vector Field

We can now represent a vector field in terms of its components of functions or unit vectors, but representing it visually by sketching it is more complex because the domain of a vector field is in $\mathbb{R}^2$, as is the range. Therefore the "graph" of a vector field in $\mathbb{R}^2$ lives in four-dimensional space. Since we cannot represent four-dimensional space visually, we instead draw vector fields in $\mathbb{R}^2$ in a plane itself. To do this, draw the vector associated with a given point at the point in a plane. For example, suppose the vector associated with point $(4,-1)$ is $\langle 3,1 \rangle$. Then, we would draw vector $\langle 3,1 \rangle$ at point $(4,-1)$.

We should plot enough vectors to see the general shape, but not so many that the sketch becomes a jumbled mess. If we were to plot the image vector at each point in the region, it would fill the region completely and is useless. Instead, we can choose points at the intersections of grid lines and plot a sample of several vectors from each quadrant of a rectangular coordinate system in $\mathbb{R}^2$.

There are two types of vector fields in $\mathbb{R}^2$ on which this chapter focuses: radial fields and rotational fields. Radial fields model certain gravitational fields and energy source fields, and rotational fields model the movement of a fluid in a vortex. In a radial field, all vectors either point directly toward or directly away from the origin. Furthermore, the magnitude of any vector depends only on its distance from the origin. In a radial field, the vector located at point (x,y) is perpendicular to the circle centered at the origin that contains point (x,y), and all other vectors on this circle have the same magnitude.

Example 15.1.2: Drawing a Radial Vector Field

Sketch the vector field $\vec{\mathbf{F}}(x,y) = \frac{x}{2}\hat{\mathbf{i}} + \frac{y}{2}\hat{\mathbf{j}}$.

Solution

To sketch this vector field, choose a sample of points from each quadrant and compute the corresponding vector. The following table gives a representative sample of points in a plane and the corresponding vectors.

Table 15.1.1

(x,y)	$\vec{\mathbf{F}}(x,y)$	(x,y)	$\vec{\mathbf{F}}(x,y)$	(x,y)	$\vec{\mathbf{F}}(x,y)$
$(1,0)$	$\langle \frac{1}{2}, 0\rangle$	$(2,0)$	$\langle 1, 0\rangle$	$(1,1)$	$\langle \frac{1}{2}, \frac{1}{2}\rangle$
$(0,1)$	$\langle 0, \frac{1}{2}\rangle$	$(0,2)$	$\langle 0, 1\rangle$	$(-1,1)$	$\langle -\frac{1}{2}, \frac{1}{2}\rangle$
$(-1,0)$	$\langle -\frac{1}{2}, 0\rangle$	$(-2,0)$	$\langle -1, 0\rangle$	$(-1,-1)$	$\langle -\frac{1}{2}, -\frac{1}{2}\rangle$
$(0,-1)$	$\langle 0, -\frac{1}{2}\rangle$	$(0,-2)$	$\langle 0, -1\rangle$	$(1,-1)$	$\langle \frac{1}{2}, -\frac{1}{2}\rangle$

Figure 15.1.2a shows the vector field. To see that each vector is perpendicular to the corresponding circle, Figure 15.1.2b shows circles overlain on the vector field.

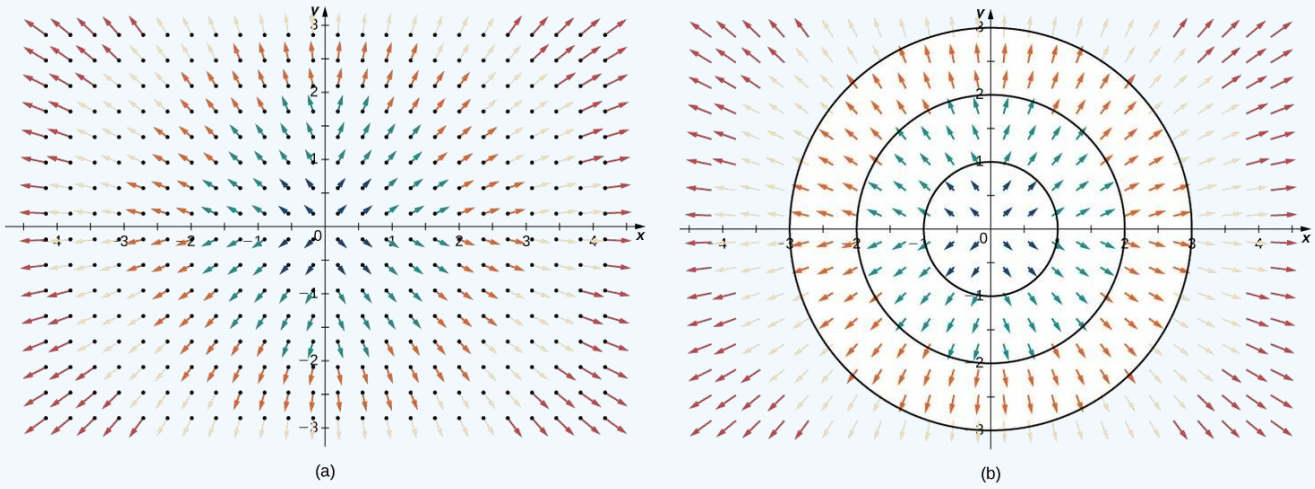

Figure 15.1.2: (a) A visual representation of the radial vector field $\vec{\mathbf{F}}(x,y) = \frac{x}{2}\hat{\mathbf{i}} + \frac{y}{2}\hat{\mathbf{j}}$. (b) The radial vector field $\vec{F}(x,y) = \frac{x}{2}\hat{\mathbf{i}} + \frac{y}{2}\hat{\mathbf{j}}$ with overlaid circles. Notice that each vector is perpendicular to the circle on which it is located.

Exercise 15.1.2

Draw the radial field $\vec{\mathbf{F}}(x,y) = -\frac{x}{3}\hat{\mathbf{i}} - \frac{y}{3}\hat{\mathbf{j}}$.

In contrast to radial fields, in a *rotational field*, the vector at point (x,y) is tangent (not perpendicular) to a circle with radius $r = \sqrt{x^2+y^2}$. In a standard rotational field, all vectors point either in a clockwise direction or in a counterclockwise direction, and the magnitude of a vector depends only on its distance from the origin. Both of the following examples are clockwise rotational fields, and we see from their visual representations that the vectors appear to rotate around the origin.

Example 15.1.3: Drawing a Rotational Vector Field

Sketch the vector field $\vec{\mathbf{F}}(x,y) = \langle y, -x\rangle$.

Solution

Create a table (see the one that follows) using a representative sample of points in a plane and their corresponding vectors. Figure **15.1.3** shows the resulting vector field.

Table **15.1.2**

(x,y)	$\vec{\mathbf{F}}(x,y)$	(x,y)	$\vec{\mathbf{F}}(x,y)$	(x,y)	$\vec{\mathbf{F}}(x,y)$
$(1,0)$	$\langle 0,-1\rangle$	$(2,0)$	$\langle 0,-2\rangle$	$(1,1)$	$\langle 1,-1\rangle$
$(0,1)$	$\langle 1,0\rangle$	$(0,2)$	$\langle 2,0\rangle$	$(-1,1)$	$\langle 1,1\rangle$
$(-1,0)$	$\langle 0,1\rangle$	$(-2,0)$	$\langle 0,2\rangle$	$(-1,-1)$	$\langle -1,1\rangle$
$(0,-1)$	$\langle -1,0\rangle$	$(0,-2)$	$\langle -2,0\rangle$	$(1,-1)$	$\langle -1,-1\rangle$

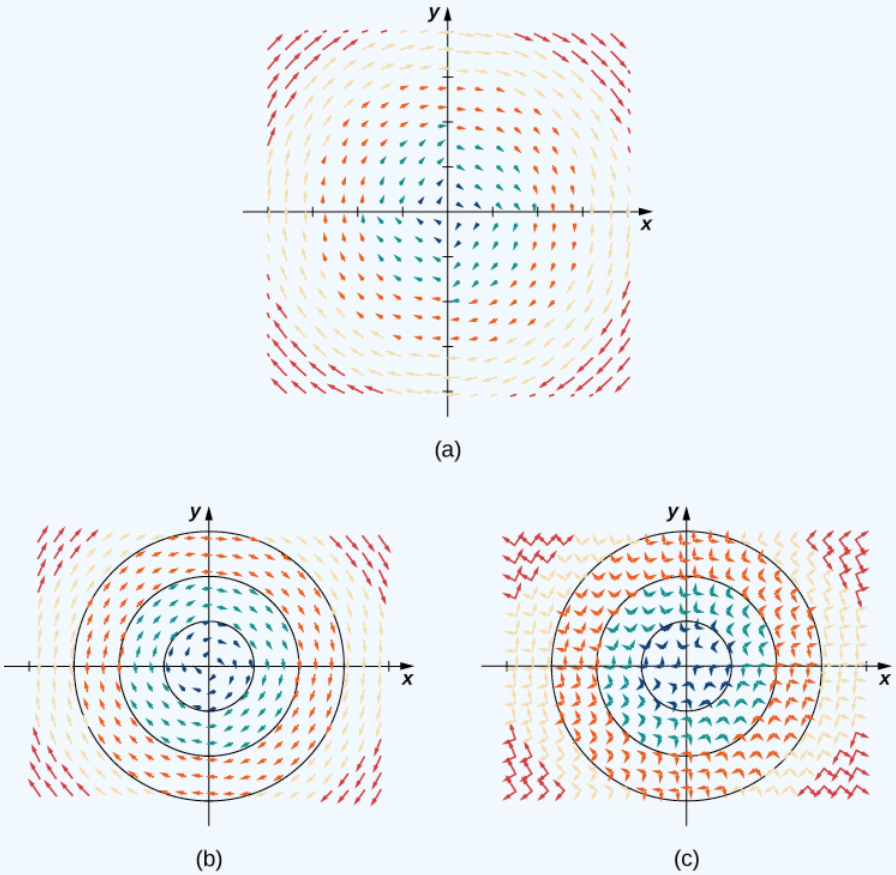

Figure **15.1.3**: (a) A visual representation of vector field $\vec{\mathbf{F}}(x,y)=\langle y,-x\rangle$. (b) Vector field $\vec{\mathbf{F}}(x,y)=\langle y,-x\rangle$ with circles centered at the origin. (c) Vector $\vec{\mathbf{F}}(a,b)$ is perpendicular to radial vector $\langle a,b\rangle$ at point (a,b).

Analysis

Note that vector $\vec{\mathbf{F}}(a,b)=\langle b,-a\rangle$ points clockwise and is perpendicular to radial vector $\langle a,b\rangle$. (We can verify this assertion by computing the dot product of the two vectors: $\langle a,b\rangle\cdot\langle -b,a\rangle=-ab+ab=0$.) Furthermore, vector $\langle b,-a\rangle$ has length $r=\sqrt{a^2+b^2}$. Thus, we have a complete description of this rotational vector field: the vector associated with point (a,b) is the vector with length r tangent to the circle with radius r, and it points in the clockwise direction.

Sketches such as that in Figure **15.1.3** are often used to analyze major storm systems, including hurricanes and cyclones. In the northern hemisphere, storms rotate counterclockwise; in the southern hemisphere, storms rotate clockwise. (This is an effect caused by Earth's rotation about its axis and is called the Coriolis Effect.)

Figure 15.1.4: (credit: modification of work by NASA)

Example 15.1.4: Sketching a Vector Field

Sketch vector field $\vec{\mathbf{F}}(x,y) = \frac{y}{x^2+y^2}\hat{\mathbf{i}}, -\frac{x}{x^2+y^2}\hat{\mathbf{j}}$.

Solution

To visualize this vector field, first note that the dot product $\vec{\mathbf{F}}(a,b) \cdot (a\,\hat{\mathbf{i}} + b\,\hat{\mathbf{j}})$ is zero for any point (a,b). Therefore, each vector is tangent to the circle on which it is located. Also, as $(a,b) \to (0,0)$, the magnitude of $\vec{\mathbf{F}}(a,b)$ goes to infinity. To see this, note that

$$||\vec{\mathbf{F}}(a,b)|| = \sqrt{\frac{a^2+b^2}{(a^2+b^2)^2}} = \sqrt{\frac{1}{a^2+b^2}}.$$

Since $\frac{1}{a^2+b^2} \to \infty$ as $(a,b) \to (0,0)$, then $||\vec{\mathbf{F}}(a,b)|| \to \infty$ as $(a,b) \to (0,0)$. This vector field looks similar to the vector field in Example 15.1.3, but in this case the magnitudes of the vectors close to the origin are large. Table 15.1.3 shows a sample of points and the corresponding vectors, and Figure 15.1.5 shows the vector field. Note that this vector field models the whirlpool motion of the river in Figure 15.1.5(b). The domain of this vector field is all of $\mathbb{R}^2$ except for point $(0,0)$.

Table 15.1.3

(x,y)	$\vec{\mathbf{F}}(x,y)$	(x,y)	$\vec{\mathbf{F}}(x,y)$	(x,y)	$\vec{\mathbf{F}}(x,y)$
$(1,0)$	$\langle 0,-1\rangle$	$(2,0)$	$\langle 0,-\frac{1}{2}\rangle$	$(1,1)$	$\langle \frac{1}{2},-\frac{1}{2}\rangle$
$(0,1)$	$\langle 1,0\rangle$	$(0,2)$	$\langle \frac{1}{2},0\rangle$	$(-1,1)$	$\langle \frac{1}{2},\frac{1}{2}\rangle$
$(-1,0)$	$\langle 0,1\rangle$	$(-2,0)$	$\langle 0,\frac{1}{2}\rangle$	$(-1,-1)$	$\langle -\frac{1}{2},\frac{1}{2}\rangle$
$(0,-1)$	$\langle -1,0\rangle$	$(0,-2)$	$\langle -\frac{1}{2},0\rangle$	$(1,-1)$	$\langle -\frac{1}{2},-\frac{1}{2}\rangle$

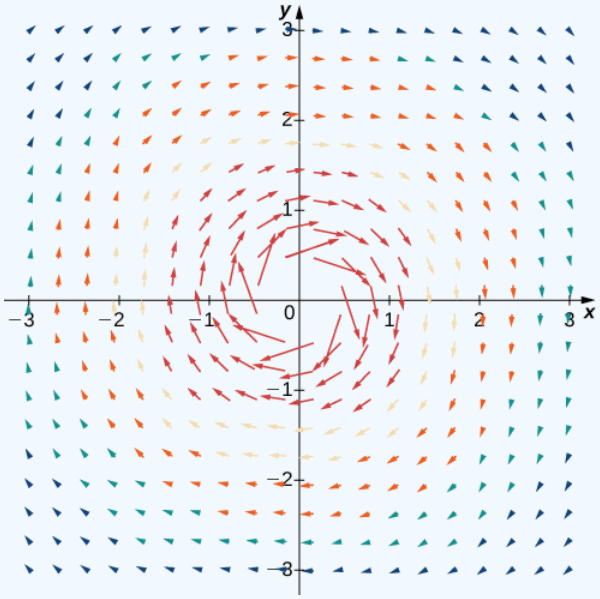

Figure 15.1.5: A visual representation of vector field $\vec{\mathbf{F}}(x,y) = \frac{y}{x^2+y^2}\hat{\mathbf{i}} - \frac{x}{x^2+y^2}\hat{\mathbf{j}}$. *This vector field could be used to model whirlpool motion of a fluid.*

? Exercise 15.1.4

Sketch vector field $\vec{\mathbf{F}}(x,y) = \langle -2y, 2x\rangle$. Is the vector field radial, rotational, or neither?

✓ Example 15.1.5: Velocity Field of a Fluid

Suppose that $\vec{\mathbf{v}}(x,y) = -\frac{2y}{x^2+y^2}\hat{\mathbf{i}} + \frac{2x}{x^2+y^2}\hat{\mathbf{j}}$ is the velocity field of a fluid. How fast is the fluid moving at point $(1,-1)$? (Assume the units of speed are meters per second.)

Solution

To find the velocity of the fluid at point $(1,-1)$, substitute the point into $\vec{\mathbf{v}}$:

$$\vec{\mathbf{v}}(1,-1) = \frac{-2(-1)}{1+1}\hat{\mathbf{i}} + \frac{2(1)}{1+1}\hat{\mathbf{j}} = \hat{\mathbf{i}} + \hat{\mathbf{j}}.$$

The speed of the fluid at $(1,-1)$ is the magnitude of this vector. Therefore, the speed is $||\hat{\mathbf{i}} + \hat{\mathbf{j}}|| = \sqrt{2}$ m/sec.

? Exercise 15.1.5

Vector field $\vec{\mathbf{v}}(x,y) = \langle 4|x|, 1\rangle$ models the velocity of water on the surface of a river. What is the speed of the water at point $(2,3)$? Use meters per second as the units.

We have examined vector fields that contain vectors of various magnitudes, but just as we have unit vectors, we can also have a unit vector field. A vector field $\vec{\mathbf{F}}$ is a **unit vector field** if the magnitude of each vector in the field is 1. In a unit vector field, the only relevant information is the direction of each vector.

✓ Example 15.1.6: A Unit Vector Field

Show that vector field $\vec{\mathbf{F}}(x,y) = \left\langle \frac{y}{\sqrt{x^2+y^2}}, -\frac{x}{\sqrt{x^2+y^2}} \right\rangle$ is a unit vector field.

Solution

To show that $\vec{\mathbf{F}}$ is a unit field, we must show that the magnitude of each vector is 1. Note that

$$\sqrt{\left(\frac{y}{\sqrt{x^2+y^2}}\right)^2+\left(-\frac{x}{\sqrt{x^2+y^2}}\right)^2}=\sqrt{\frac{y^2}{x^2+y^2}+\frac{x^2}{x^2+y^2}}$$
$$=\sqrt{\frac{x^2+y^2}{x^2+y^2}}$$
$$=1$$

Therefore, $\vec{\mathbf{F}}$ is a unit vector field.

? Exercise 15.1.6

Is vector field $\vec{\mathbf{F}}(x,y)=\langle -y,\, x\rangle$ a unit vector field?

Why are unit vector fields important? Suppose we are studying the flow of a fluid, and we care only about the direction in which the fluid is flowing at a given point. In this case, the speed of the fluid (which is the magnitude of the corresponding velocity vector) is irrelevant, because all we care about is the direction of each vector. Therefore, the unit vector field associated with velocity is the field we would study.

If $\vec{\mathbf{F}}=\langle P,Q,R\rangle$ is a vector field, then the corresponding unit vector field is $\left\langle \frac{P}{||\vec{\mathbf{F}}||}, \frac{Q}{||\vec{\mathbf{F}}||}, \frac{R}{||\vec{\mathbf{F}}||}\right\rangle$. Notice that if $\vec{\mathbf{F}}(x,y)=\langle y,\, -x\rangle$ is the vector field from Example **15.1.6**, then the magnitude of $\vec{\mathbf{F}}$ is $\sqrt{x^2+y^2}$, and therefore the corresponding unit vector field is the field $\vec{\mathbf{G}}$ from the previous example.

If $\vec{\mathbf{F}}$ is a vector field, then the process of dividing $\vec{\mathbf{F}}$ by its magnitude to form unit vector field $\vec{\mathbf{F}}/||\vec{\mathbf{F}}||$ is called normalizing the field $\vec{\mathbf{F}}$.

Vector Fields in $\mathbb{R}^3$

We have seen several examples of vector fields in $\mathbb{R}^2$; let's now turn our attention to vector fields in $\mathbb{R}^3$. These vector fields can be used to model gravitational or electromagnetic fields, and they can also be used to model fluid flow or heat flow in three dimensions. A two-dimensional vector field can really only model the movement of water on a two-dimensional slice of a river (such as the river's surface). Since a river flows through three spatial dimensions, to model the flow of the entire depth of the river, we need a vector field in three dimensions.

The extra dimension of a three-dimensional field can make vector fields in $\mathbb{R}^3$ more difficult to visualize, but the idea is the same. To visualize a vector field in $\mathbb{R}^3$, plot enough vectors to show the overall shape. We can use a similar method to visualizing a vector field in $\mathbb{R}^2$ by choosing points in each octant.

Just as with vector fields in $\mathbb{R}^2$, we can represent vector fields in $\mathbb{R}^3$ with component functions. We simply need an extra component function for the extra dimension. We write either

$$\vec{\mathbf{F}}(x,y,z)=\langle P(x,y,z),Q(x,y,z),R(x,y,z)\rangle$$

or

$$\vec{\mathbf{F}}(x,y,z)=P(x,y,z)\hat{\mathbf{i}}+Q(x,y,z)\hat{\mathbf{j}}+R(x,y,z)\hat{\mathbf{k}}.$$

✓ Example 15.1.7: Sketching a Vector Field in Three Dimensions

Describe vector field $\vec{\mathbf{F}}(x,y,z)=\langle 1,\, 1,\, z\rangle$.

Solution

For this vector field, the x- and y-components are constant, so every point in $\mathbb{R}^3$ has an associated vector with x- and y-components equal to one. To visualize $\vec{\mathbf{F}}$, we first consider what the field looks like in the xy-plane. In the xy-plane, $z=0$. Hence, each point of the form $(a,b,0)$ has vector $\langle 1,1,0\rangle$ associated with it. For points not in the xy-plane but slightly above it, the associated vector has a small but positive z-component, and therefore the associated vector points slightly upward. For

points that are far above the xy-plane, the z-component is large, so the vector is almost vertical. Figure 15.1.6 shows this vector field.

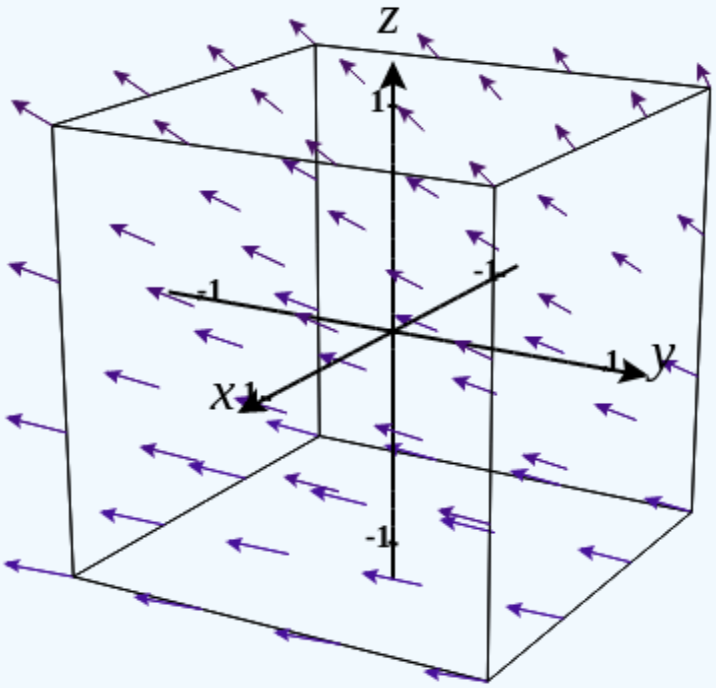

Figure 15.1.6: A visual representation of vector field $\vec{\mathbf{F}}(x, y, z) = \langle 1, 1, z \rangle$.

? Exercise 15.1.7

Sketch vector field $\vec{\mathbf{G}}(x, y, z) = \langle 2, \frac{z}{2}, 1 \rangle$.

In the next example, we explore one of the classic cases of a three-dimensional vector field: a gravitational field.

✓ Example 15.1.8: Describing a Gravitational Vector Field

Newton's law of gravitation states that $\vec{\mathbf{F}} = -G\frac{m_1 m_2}{r^2}\hat{\mathbf{r}}$, where G is the universal gravitational constant. It describes the gravitational field exerted by an object (object 1) of mass m_1 located at the origin on another object (object 2) of mass m_2 located at point (x, y, z). Field $\vec{\mathbf{F}}$ denotes the gravitational force that object 1 exerts on object 2, r is the distance between the two objects, and $\hat{\mathbf{r}}$ indicates the unit vector from the first object to the second. The minus sign shows that the gravitational force attracts toward the origin; that is, the force of object 1 is attractive. Sketch the vector field associated with this equation.

Solution

Since object 1 is located at the origin, the distance between the objects is given by $r = \sqrt{x^2 + y^2 + z^2}$. The unit vector from object 1 to object 2 is $\hat{\mathbf{r}} = \frac{\langle x, y, z \rangle}{||\langle x, y, z \rangle||}$, and hence $\hat{\mathbf{r}} = \langle \frac{x}{r}, \frac{y}{r}, \frac{z}{r} \rangle$. Therefore, gravitational vector field $\vec{\mathbf{F}}$ exerted by object 1 on object 2 is

$$\vec{\mathbf{F}} = -Gm_1m_2\langle \frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3} \rangle.$$

This is an example of a radial vector field in $\mathbb{R}^3$.

Figure 15.1.7 shows what this gravitational field looks like for a large mass at the origin. Note that the magnitudes of the vectors increase as the vectors get closer to the origin.

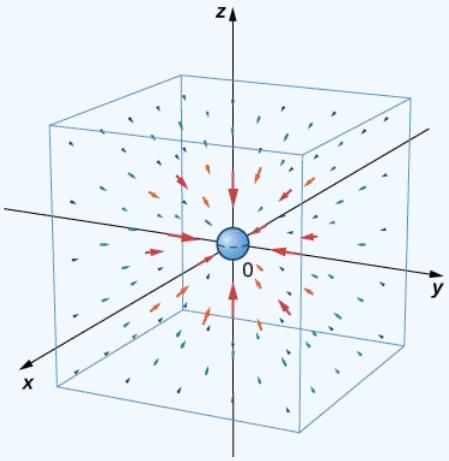

Figure 15.1.7: A visual representation of gravitational vector field $\vec{\mathbf{F}} = -Gm_1m_2\langle \frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3} \rangle$ for a large mass at the origin.

? Exercise 15.1.8

The mass of asteroid 1 is 750,000 kg and the mass of asteroid 2 is 130,000 kg. Assume asteroid 1 is located at the origin, and asteroid 2 is located at $(15, -5, 10)$, measured in units of 10 to the eighth power kilometers. Given that the universal gravitational constant is $G = 6.67384 \times 10^{-11} m^3 kg^{-1} s^{-2}$, find the gravitational force vector that asteroid 1 exerts on asteroid 2.

Gradient Fields (Conservative Fields)

In this section, we study a special kind of vector field called a gradient field or a **conservative field**. These vector fields are extremely important in physics because they can be used to model physical systems in which energy is conserved. Gravitational fields and electric fields associated with a static charge are examples of gradient fields.

Recall that if f is a (scalar) function of x and y, then the gradient of f is

$$\text{grad } f = \vec{\nabla} f(x, y) = f_x(x, y)\hat{\mathbf{i}} + f_y(x, y)\hat{\mathbf{j}}.$$

We can see from the form in which the gradient is written that $\vec{\nabla} f$ is a vector field in $\mathbb{R}^2$. Similarly, if f is a function of x, y, and z, then the gradient of f is

$$\text{grad } f = \vec{\nabla} f(x, y, z) = f_x(x, y, z)\hat{\mathbf{i}} + f_y(x, y, z)\hat{\mathbf{j}} + f_z(x, y, z)\hat{\mathbf{k}}.$$

The gradient of a three-variable function is a vector field in $\mathbb{R}^3$. A gradient field is a vector field that can be written as the gradient of a function, and we have the following definition.

DEFINITION: Gradient Field

A vector field $\vec{\mathbf{F}}$ in $\mathbb{R}^2$ or in $\mathbb{R}^3$ is a gradient field if there exists a scalar function f such that $\vec{\nabla} f = \vec{\mathbf{F}}$.

✓ Example 15.1.9: Sketching a Gradient Vector Field

Use technology to plot the gradient vector field of $f(x, y) = x^2y^2$.

Solution

The gradient of f is $\vec{\nabla} f(x,y) = \langle 2xy^2,\ 2x^2y \rangle$. To sketch the vector field, use a computer algebra system such as Mathematica. Figure 15.1.8 shows $\vec{\nabla} f$.

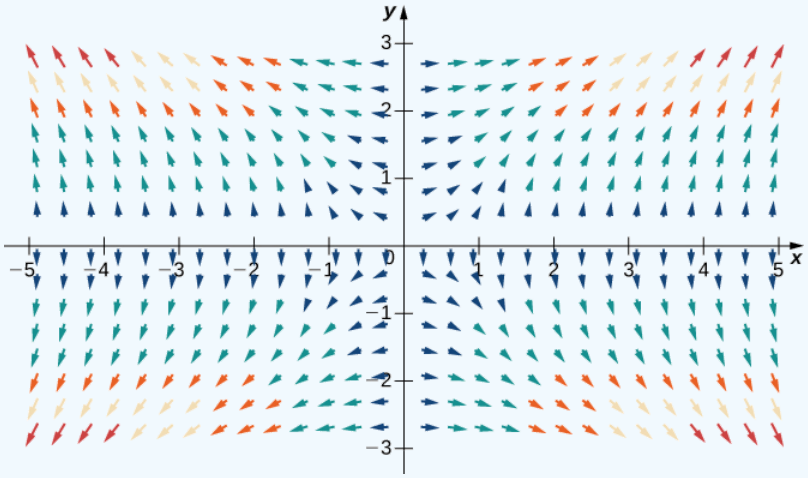

Figure 15.1.8: The gradient vector field is $\vec{\nabla} f$, where $f(x,y) = x^2y^2$.

? Exercise 15.1.9

Use technology to plot the gradient vector field of $f(x,y) = \sin x \cos y$.

Consider the function $f(x,y) = x^2y^2$ from Example 15.1.9. Figure 15.1.9 shows the level curves of this function overlaid on the function's gradient vector field. The gradient vectors are perpendicular to the level curves, and the magnitudes of the vectors get larger as the level curves get closer together, because closely grouped level curves indicate the graph is steep, and the magnitude of the gradient vector is the largest value of the directional derivative. Therefore, you can see the local steepness of a graph by investigating the corresponding function's gradient field.

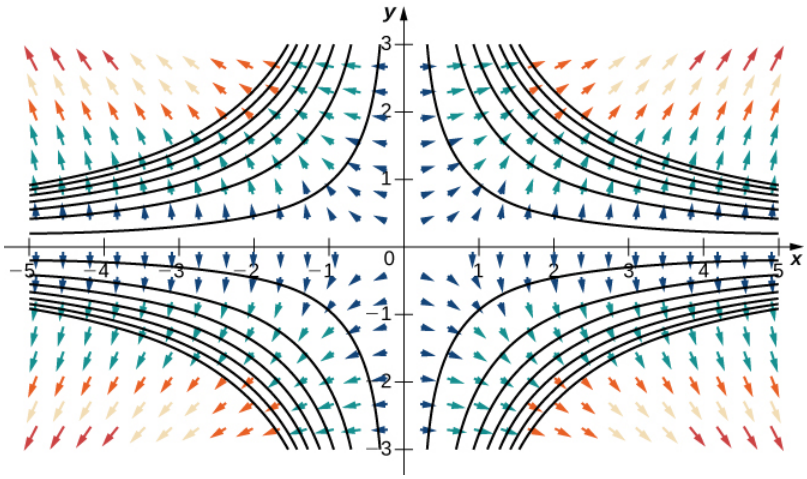

Figure 15.1.9: The gradient field of $f(x,y) = x^2y^2$ and several level curves of f. Notice that as the level curves get closer together, the magnitude of the gradient vectors increases.

As we learned earlier, a vector field $\vec{\mathbf{F}}$ is a conservative vector field, or a gradient field if there exists a scalar function f such that $\vec{\nabla} f = \vec{\mathbf{F}}$. In this situation, f is called a **potential function** for $\vec{\mathbf{F}}$. Conservative vector fields arise in many applications, particularly in physics. The reason such fields are called *conservative* is that they model forces of physical systems in which energy is conserved. We study conservative vector fields in more detail later in this chapter.

You might notice that, in some applications, a potential function f for $\vec{\mathbf{F}}$ is defined instead as a function such that $-\vec{\nabla} f = \vec{\mathbf{F}}$. This is the case for certain contexts in physics, for example.

Example 15.1.10: Verifying a Potential Function

Is $f(x,y,z) = x^2yz - \sin(xy)$ a potential function for vector field

$$\vec{\mathbf{F}}(x,y,z) = \langle 2xyz - y\cos(xy), x^2z - x\cos(xy), x^2y\rangle?$$

Solution

We need to confirm whether $\vec{\nabla} f = \vec{\mathbf{F}}$. We have

$$f_x(x,y) = 2xyz - y\cos(xy)$$
$$f_y(x,y) = x^2z - x\cos(xy).$$
$$f_z(x,y) = x^2y$$

Therefore, $\vec{\nabla} f = \vec{\mathbf{F}}$ and f is a potential function for $\vec{\mathbf{F}}$.

Exercise 15.1.10

Is $f(x,y,z) = x^2\cos(yz) + y^2z^2$ a potential function for $\vec{\mathbf{F}}(x,y,z) = \langle 2x\cos(yz), -x^2z\sin(yz) + 2yz^2, y^2\rangle$?

Example 15.1.11: Verifying a Potential Function

The velocity of a fluid is modeled by field $\vec{\mathbf{v}}(x,y) = \langle xy, \frac{x^2}{2} - y\rangle$. Verify that $f(x,y) = \dfrac{x^2y}{2} - \dfrac{y^2}{2}$ is a potential function for $\vec{\mathbf{v}}$.

Solution

To show that f is a potential function, we must show that $\vec{\nabla} f = \vec{\mathbf{v}}$. Note that $f_x(x,y) = xy$ and $f_y(x,y) = \dfrac{x^2}{2} - y$. Therefore, $\vec{\nabla} f(x,y) = \langle xy, \frac{x^2}{2} - y\rangle$ and f is a potential function for $\vec{\mathbf{v}}$ (Figure 15.1.10).

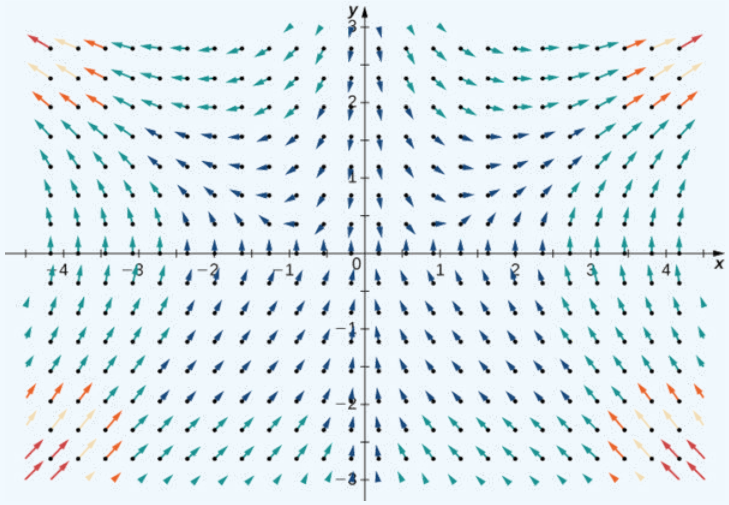

Figure 15.1.10: Velocity field $\vec{\mathbf{v}}(x,y)$ has a potential function and is a conservative field.

Exercise 15.1.11

Verify that $f(x,y) = x^3y^2 + x$ is a potential function for velocity field $\vec{\mathbf{v}}(x,y) = \langle 3x^2y^2 + 1, 2x^3y\rangle$.

If $\vec{\mathbf{F}}$ is a conservative vector field, then there is at least one potential function f such that $\vec{\nabla} f = \vec{\mathbf{F}}$. But, could there be more than one potential function? If so, is there any relationship between two potential functions for the same vector field? Before answering these questions, let's recall some facts from single-variable calculus to guide our intuition. Recall that if $k(x)$ is an integrable function, then k has infinitely many antiderivatives. Furthermore, if $\vec{\mathbf{F}}$ and $\vec{\mathbf{G}}$ are both antiderivatives of k, then $\vec{\mathbf{F}}$ and $\vec{\mathbf{G}}$ **differ only by a constant**. That is, there is some number C such that $\vec{\mathbf{F}}(x) = \vec{\mathbf{G}}(x) + C$.

Now let $\vec{\mathbf{F}}$ be a conservative vector field and let f and g be potential functions for $\vec{\mathbf{F}}$. Since the gradient is like a derivative, $\vec{\mathbf{F}}$ being conservative means that $\vec{\mathbf{F}}$ is "integrable" with "antiderivatives" f and g. Therefore, if the analogy with single-variable calculus is valid, we expect there is some constant C such that $f(x) = g(x) + C$. The next theorem says that this is indeed the case.

To state the next theorem with precision, we need to assume the domain of the vector field is connected and open. To be connected means if P_1 and P_2 are any two points in the domain, then you can walk from P_1 to P_2 along a path that stays entirely inside the domain.

UNIQUENESS OF POTENTIAL FUNCTIONS

Let $\vec{\mathbf{F}}$ be a conservative vector field on an open and connected domain and let f and g be functions such that $\vec{\nabla} f = \vec{\mathbf{F}}$ and $\vec{\nabla} g = \vec{\mathbf{F}}$. Then, there is a constant C such that $f = g + C$.

Proof

Since f and g are both potential functions for $\vec{\mathbf{F}}$, then $\vec{\nabla}(f - g) = \vec{\nabla} f - \vec{\nabla} g = \vec{\mathbf{F}} - \vec{\mathbf{F}} = \vec{\mathbf{0}}$. Let $h = f - g$, then we have $\vec{\nabla} h = \vec{\mathbf{0}}$.We would like to show that h is a constant function.

Assume h is a function of x and y (the logic of this proof extends to any number of independent variables). Since $\vec{\nabla} h = \vec{\mathbf{0}}$, we have $h_x(x, y) = 0$ and $h_y(x, y) = 0$. The expression $h_x(x, y) = 0$ implies that h is a constant function with respect to x—that is, $h(x, y) = k_1(y)$ for some function k_1. Similarly, $h_y(x, y) = 0$ implies $h(x, y) = k_2(x)$ for some function k_2. Therefore, function h depends only on y and also depends only on x. Thus, $h(x, y) = C$ for some constant C on the connected domain of $\vec{\mathbf{F}}$. Note that we really do need connectedness at this point; if the domain of $\vec{\mathbf{F}}$ came in two separate pieces, then k could be a constant C_1 on one piece but could be a different constant C_2 on the other piece. Since $f - g = h = C$, we have that $f = g + C$, as desired.

□

Conservative vector fields also have a special property called the ***cross-partial property***. This property helps test whether a given vector field is conservative.

THE CROSS-PARTIAL PROPERTY OF CONSERVATIVE VECTOR FIELDS

Let $\vec{\mathbf{F}}$ be a vector field in two or three dimensions such that the component functions of $\vec{\mathbf{F}}$ have continuous second-order mixed-partial derivatives on the domain of $\vec{\mathbf{F}}$.

If $\vec{\mathbf{F}}(x, y) = \langle P(x, y), Q(x, y) \rangle$ is a conservative vector field in $\mathbb{R}^2$, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

If $\vec{\mathbf{F}}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle$ is a conservative vector field in $\mathbb{R}^3$, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$
$$\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$$
$$\frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}.$$

Proof

Since $\vec{\mathbf{F}}$ is conservative, there is a function $f(x, y)$ such that $\vec{\nabla} f = \vec{\mathbf{F}}$. Therefore, by the definition of the gradient, $f_x = P$ and $f_y = Q$. By Clairaut's theorem, $f_{xy} = f_{yx}$, But, $f_{xy} = P_y$ and $f_{yx} = Q_x$, and thus $P_y = Q_x$.

□

Clairaut's theorem gives a fast proof of the cross-partial property of conservative vector fields in $\mathbb{R}^3$, just as it did for vector fields in $\mathbb{R}^2$.

The Cross-Partial Property of Conservative Vector Fields shows that most vector fields are not conservative. The cross-partial property is difficult to satisfy in general, so most vector fields won't have equal cross-partials.

Show that rotational vector field $\vec{\mathbf{F}}(x,y) = \langle y, -x \rangle$ is not conservative.

Solution

Let $P(x,y) = y$ and $Q(x,y) = -x$. If $\vec{\mathbf{F}}$ is conservative, then the cross-partials would be equal—that is, P_y would equal Q_x.Therefore, to show that $\vec{\mathbf{F}}$ is not conservative, check that $P_y \neq Q_x$. Since $P_y = 1$ and $Q_x = -1$, the vector field is not conservative.

? Exercise 15.1.12

Show that vector field $\vec{\mathbf{F}}(x,y) = xy\,\hat{\mathbf{i}} - x^2y\,\hat{\mathbf{j}}$ is not conservative.

✓ Example 15.1.13: Showing a Vector Field Is Not Conservative

Is vector field $\vec{\mathbf{F}}(x,y,z) = \langle 7, -2, x^3 \rangle$ conservative?

Solution

Let $P(x,y,z) = 7$, $Q(x,y,z) = -2$, and $R(x,y,z) = x^3$. If $\vec{\mathbf{F}}$ is conservative, then all three cross-partial equations will be satisfied —that is, if $\vec{\mathbf{F}}$ is conservative, then P_y would equal Q_x, Q_z would equal R_y, and R_x would equal P_z. Note that

$$P_y = Q_x = R_y = Q_z = 0$$

so the first two necessary equalities hold. However, $R_x(x,y,z) = x^3$ and $P_z(x,y,z) = 0$ so $R_x \neq P_z$. Therefore, $\vec{\mathbf{F}}$ is not conservative.

? Exercise 15.1.13

Is vector field $\vec{\mathbf{G}}(x,y,z) = \langle y, x, xyz \rangle$ conservative?

We conclude this section with a word of warning: The Cross-Partial Property of Conservative Vector Fields says that if $\vec{\mathbf{F}}$ is conservative, then $\vec{\mathbf{F}}$ has the cross-partial property. The theorem does *not* say that, if $\vec{\mathbf{F}}$ has the cross-partial property, then $\vec{\mathbf{F}}$ is conservative (the converse of an implication is not logically equivalent to the original implication). In other words, The Cross-Partial Property of Conservative Vector Fields can only help determine that a field is not conservative; it does not let you conclude that a vector field is conservative.

For example, consider vector field $\vec{\mathbf{F}}(x,y) = \langle x^2y, \frac{x^3}{3} \rangle$. This field has the cross-partial property, so it is natural to try to use The Cross-Partial Property of Conservative Vector Fields to conclude this vector field is conservative. However, this is a misapplication of the theorem. We learn later how to conclude that $\vec{\mathbf{F}}$ is conservative.

Key Concepts

- A vector field assigns a vector $\vec{\mathbf{F}}(x,y)$ to each point (x,y) in a subset D of $\mathbb{R}^2$ or $\mathbb{R}^3$. $\vec{\mathbf{F}}(x,y,z)$ to each point (x,y,z) in a subset D of $\mathbb{R}^3$.
- Vector fields can describe the distribution of vector quantities such as forces or velocities over a region of the plane or of space. They are in common use in such areas as physics, engineering, meteorology, oceanography.
- We can sketch a vector field by examining its defining equation to determine relative magnitudes in various locations and then drawing enough vectors to determine a pattern.
- A vector field $\vec{\mathbf{F}}$ is called conservative if there exists a scalar function f such that $\vec{\nabla} f = \vec{\mathbf{F}}$.

Key Equations

- **Vector field in $\mathbb{R}^2$**
  $\vec{\mathbf{F}}(x, y) = \langle P(x, y), Q(x, y) \rangle$
  or
  $\vec{\mathbf{F}}(x, y) = P(x, y)\,\hat{\mathbf{i}} + Q(x, y)\,\hat{\mathbf{j}}$
- **Vector field in $\mathbb{R}^3$**
  $\vec{\mathbf{F}}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle$
  or
  $\vec{\mathbf{F}}(x, y, z) = P(x, y, z)\,\hat{\mathbf{i}} + Q(x, y, z)\,\hat{\mathbf{j}} + R(x, y, z)\,\hat{\mathbf{k}}$

Glossary

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15.1E: Exercises for Section 15.1

1. The domain of vector field $\vec{\mathbf{F}} = \vec{\mathbf{F}}(x, y)$ is a set of points (x, y) in a plane, and the range of $\vec{\mathbf{F}}$ is a set of *what* in the plane?

Answer

Vectors

For exercises 2 - 4, determine whether the statement is *true or false*.

2. Vector field $\vec{\mathbf{F}} = \langle 3x^2, 1 \rangle$ is a gradient field for both $\phi_1(x, y) = x^3 + y$ and $\phi_2(x, y) = y + x^3 + 100$.

3. Vector field $\vec{\mathbf{F}} = \dfrac{\langle y, x \rangle}{\sqrt{x^2 + y^2}}$ is constant in direction and magnitude on a unit circle.

Answer

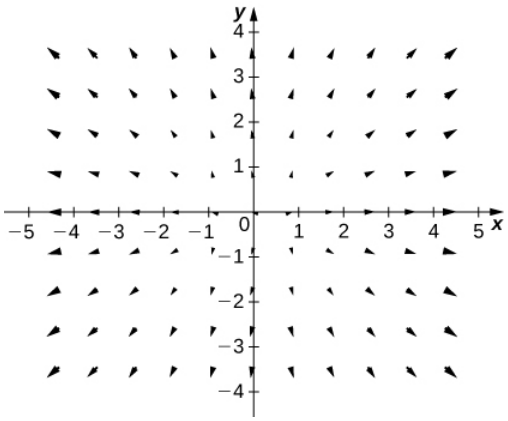
False

4. Vector field $\vec{\mathbf{F}} = \dfrac{\langle y, x \rangle}{\sqrt{x^2 + y^2}}$ is neither a radial field nor a rotation field.

For exercises 5 - 13, describe each vector field by drawing some of its vectors.

5. [T] $\vec{\mathbf{F}}(x, y) = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}$

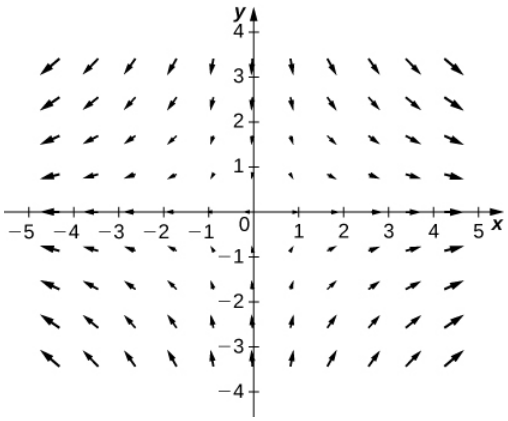
Answer



6. [T] $\vec{\mathbf{F}}(x, y) = -y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}}$

7. [T] $\vec{\mathbf{F}}(x, y) = x\,\hat{\mathbf{i}} - y\,\hat{\mathbf{j}}$

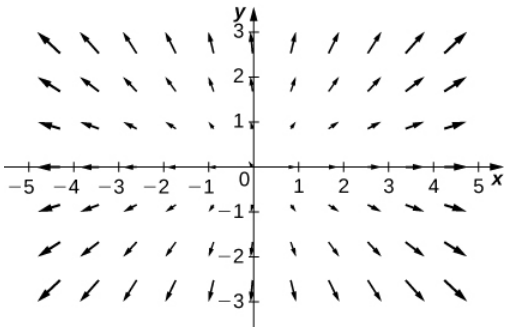
Answer



8. [T] $\vec{\mathbf{F}}(x,y) = \hat{\mathbf{i}} + \hat{\mathbf{j}}$

9. [T] $\vec{\mathbf{F}}(x,y) = 2x\,\hat{\mathbf{i}} + 3y\,\hat{\mathbf{j}}$

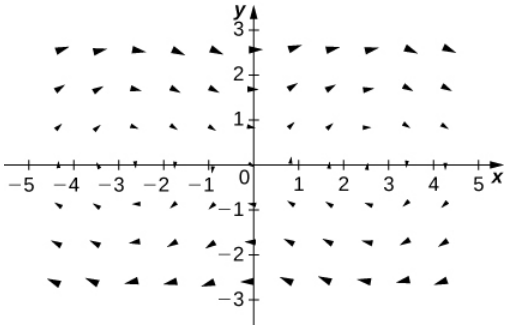
Answer



10. [T] $\vec{\mathbf{F}}(x,y) = 3\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}}$

11. [T] $\vec{\mathbf{F}}(x,y) = y\,\hat{\mathbf{i}} + \sin x\,\hat{\mathbf{j}}$

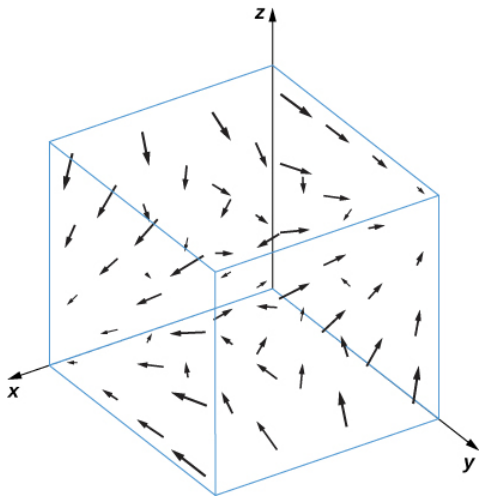
Answer



12. [T] $\vec{\mathbf{F}}(x,y,z) = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$

13. [T] $\vec{\mathbf{F}}(x,y,z) = 2x\,\hat{\mathbf{i}} - 2y\,\hat{\mathbf{j}} - 2z\,\hat{\mathbf{k}}$

Answer



14. [T] $\vec{\mathbf{F}}(x,y,z) = yz\,\hat{\mathbf{i}} - xz\,\hat{\mathbf{j}}$

For exercises 15 - 20, find the gradient vector field of each function f.

15. $f(x,y) = x\sin y + \cos y$

Answer

$\vec{\mathbf{F}}(x,y) = \sin(y)\,\hat{\mathbf{i}} + (x\cos y - \sin y)\,\hat{\mathbf{j}}$

16. $f(x,y,z) = ze^{-xy}$

17. $f(x,y,z) = x^2y + xy + y^2z$

Answer

$\vec{\mathbf{F}}(x,y,z) = (2xy+y)\,\hat{\mathbf{i}} + (x^2 + x + 2yz)\,\hat{\mathbf{j}} + y^2\,\hat{\mathbf{k}}$

18. $f(x,y) = x^2\sin(5y)$

19. $f(x,y) = \ln(1 + x^2 + 2y^2)$

Answer

$\vec{\mathbf{F}}(x,y) = \dfrac{2x}{1+x^2+2y^2}\,\hat{\mathbf{i}} + \dfrac{4y}{1+x^2+2y^2}\,\hat{\mathbf{j}}$

20. $f(x,y,z) = x\cos\left(\frac{y}{z}\right)$

21. What is vector field $\vec{\mathbf{F}}(x,y)$ with a value at (x,y) that is of unit length and points toward $(1,0)$?

Answer

$\vec{\mathbf{F}}(x,y) = \dfrac{(1-x)\,\hat{\mathbf{i}} - y\,\hat{\mathbf{j}}}{\sqrt{(1-x)^2 + y^2}}$

For exercises 22 - 24, write formulas for the vector fields with the given properties.

22. All vectors are parallel to the x-axis and all vectors on a vertical line have the same magnitude.

23. All vectors point toward the origin and have constant length.

Answer

$\vec{\mathbf{F}}(x,y) = \dfrac{(y\,\hat{\mathbf{i}} - x\,\hat{\mathbf{j}})}{\sqrt{x^2+y^2}}$

24. All vectors are of unit length and are perpendicular to the position vector at that point.

25. Give a formula $\vec{\mathbf{F}}(x,y) = M(x,y)\,\hat{\mathbf{i}} + N(x,y)\,\hat{\mathbf{j}}$ for the vector field in a plane that has the properties that $\vec{\mathbf{F}} = \vec{\mathbf{0}}$ at $(0,0)$ and that at any other point (a,b), $\vec{\mathbf{F}}$ is tangent to circle $x^2 + y^2 = a^2 + b^2$ and points in the clockwise direction with magnitude $\|\vec{\mathbf{F}}\| = \sqrt{a^2+b^2}$.

Answer

$\vec{\mathbf{F}}(x,y) = y\,\hat{\mathbf{i}} - x\,\hat{\mathbf{j}}$

26. Is vector field $\vec{\mathbf{F}}(x,y) = (P(x,y), Q(x,y)) = (\sin x + y)\,\hat{\mathbf{i}} + (\cos y + x)\,\hat{\mathbf{j}}$ a gradient field?

27. Find a formula for vector field $\vec{\mathbf{F}}(x,y) = M(x,y)\,\hat{\mathbf{i}} + N(x,y)\,\hat{\mathbf{j}}$ given the fact that for all points (x,y), $\vec{\mathbf{F}}$ points toward the origin and $\|\vec{\mathbf{F}}\| = \dfrac{10}{x^2+y^2}$.

Answer

$\vec{\mathbf{F}}(x,y) = \dfrac{-10}{(x^2+y^2)^{3/2}}(x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}})$

For exercises 28 - 29, assume that an electric field in the xy-plane caused by an infinite line of charge along the x-axis is a gradient field with potential function $V(x,y) = c\ln\left(\frac{r_0}{\sqrt{x^2+y^2}}\right)$, where $c > 0$ is a constant and r_0 is a reference distance at which

the potential is assumed to be zero.

28. Find the components of the electric field in the x- and y-directions, where $\vec{\mathbf{E}}(x,y) = -\vec{\nabla}V(x,y)$.

29. Show that the electric field at a point in the xy-plane is directed outward from the origin and has magnitude $\|\vec{\mathbf{E}}\| = \frac{c}{r}$, where $r = \sqrt{x^2+y^2}$.

Answer

$$\|\vec{\mathbf{E}}\| = \frac{c}{|r|^2}r = \frac{c}{|r|}\frac{r}{|r|}$$

A ***flow line*** (or *streamline*) of a vector field $\vec{\mathbf{F}}$ is a curve $\vec{\mathbf{r}}(t)$ such that $d\vec{\mathbf{r}}/dt = \vec{\mathbf{F}}(\vec{\mathbf{r}}(t))$. If $\vec{\mathbf{F}}$ represents the velocity field of a moving particle, then the flow lines are paths taken by the particle. Therefore, flow lines are tangent to the vector field.

For exercises 30 and 31, show that the given curve $\vec{\mathbf{c}}(t)$ is a flow line of the given velocity vector field $\vec{\mathbf{F}}(x,y,z)$.

30. $\vec{\mathbf{c}}(t) = \langle e^{2t}, \ln|t|, \frac{1}{t}\rangle$, $t \neq 0$; $\quad \vec{\mathbf{F}}(x,y,z) = \langle 2x, z, -z^2\rangle$

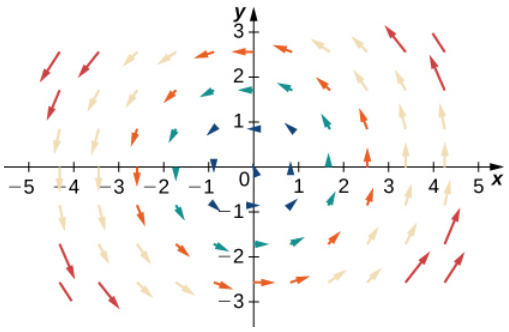
31. $\vec{\mathbf{c}}(t) = \langle \sin t, \cos t, e^t\rangle$; $\quad \vec{\mathbf{F}}(x,y,z) = \langle y, -x, z\rangle$

Answer

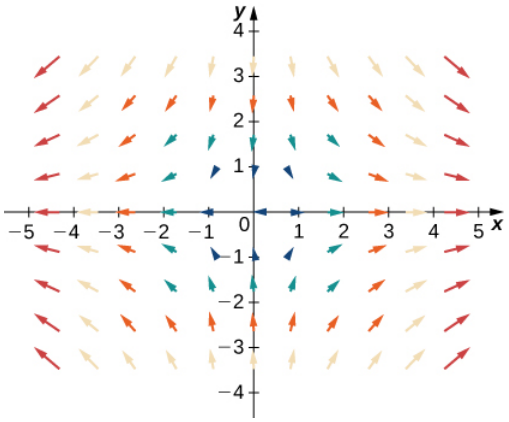
$$\vec{\mathbf{c}}'(t) = \langle \cos t, -\sin t, e^{-t}\rangle = \vec{\mathbf{F}}(\vec{\mathbf{c}}(t))$$

For exercises 32 - 34, let $\vec{\mathbf{F}} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}$, $\vec{\mathbf{G}} = -y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}}$, and $\vec{\mathbf{H}} = x\,\hat{\mathbf{i}} - y\,\hat{\mathbf{j}}$. Match $\vec{\mathbf{F}}$, $\vec{\mathbf{G}}$, and $\vec{\mathbf{H}}$ with their graphs.

32.



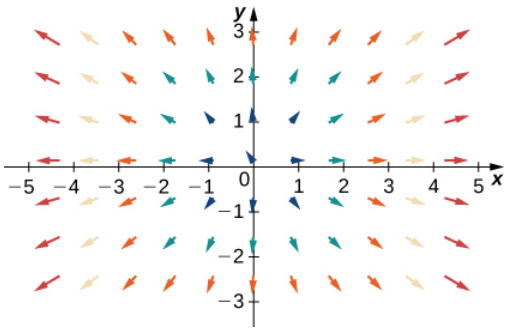
33.



Answer

$\vec{\mathbf{H}}$

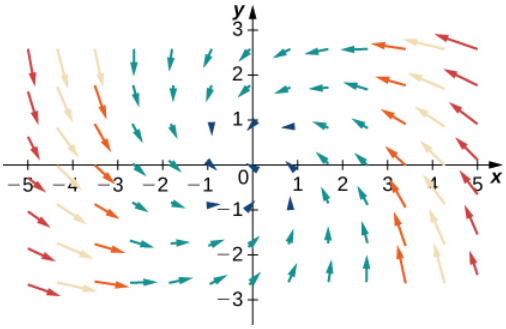
34.



For exercises 35 - 38, let $\vec{\mathbf{F}} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}$, $\vec{\mathbf{G}} = -y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}}$, and $\vec{\mathbf{H}} = -x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}$. Match the vector fields with their graphs in (I)−(IV).

a. $\vec{\mathbf{F}} + \vec{\mathbf{G}}$
b. $\vec{\mathbf{F}} + \vec{\mathbf{H}}$
c. $\vec{\mathbf{G}} + \vec{\mathbf{H}}$
d. $-\vec{\mathbf{F}} + \vec{\mathbf{G}}$

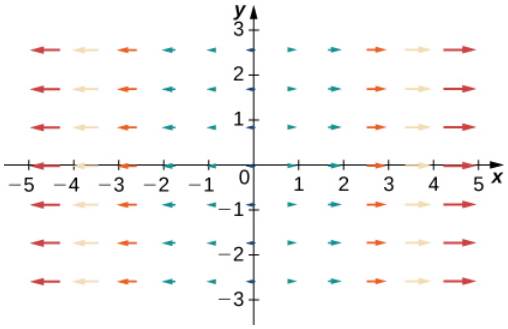
35.



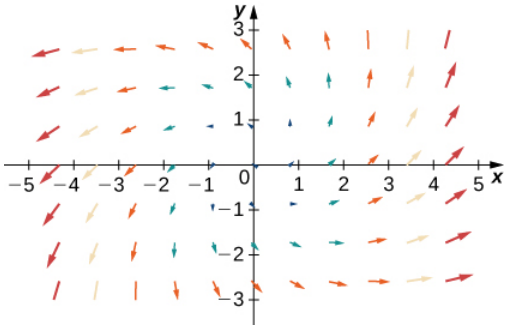
Answer

d. $-\vec{\mathbf{F}} + \vec{\mathbf{G}}$

36.



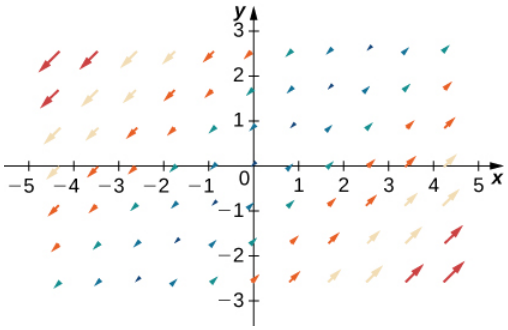
37.



Answer

a. $\vec{\mathbf{F}} + \vec{\mathbf{G}}$

38.



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15.2: Line Integrals

Learning Objectives

- Calculate a scalar line integral along a curve.
- Calculate a vector line integral along an oriented curve in space.
- Use a line integral to compute the work done in moving an object along a curve in a vector field.
- Describe the flux and circulation of a vector field.

We are familiar with single-variable integrals of the form $\int_a^b f(x)\,dx$, where the domain of integration is an interval $[a,b]$. Such an interval can be thought of as a curve in the xy-plane, since the interval defines a line segment with endpoints $(a,0)$ and $(b,0)$—in other words, a line segment located on the x-axis. Suppose we want to integrate over *any* curve in the plane, not just over a line segment on the x-axis. Such a task requires a new kind of integral, called a *line integral.*

Line integrals have many applications to engineering and physics. They also allow us to make several useful generalizations of the Fundamental Theorem of Calculus. And, they are closely connected to the properties of vector fields, as we shall see.

Scalar Line Integrals

A line integral gives us the ability to integrate multivariable functions and vector fields over arbitrary curves in a plane or in space. There are two types of line integrals: scalar line integrals and vector line integrals. Scalar line integrals are integrals of a scalar function over a curve in a plane or in space. Vector line integrals are integrals of a vector field over a curve in a plane or in space. Let's look at scalar line integrals first.

A scalar line integral is defined just as a single-variable integral is defined, except that for a scalar line integral, the integrand is a function of more than one variable and the domain of integration is a curve in a plane or in space, as opposed to a curve on the x-axis.

For a scalar line integral, we let C be a smooth curve in a plane or in space and let f be a function with a domain that includes C. We chop the curve into small pieces. For each piece, we choose point P in that piece and evaluate f at P. (We can do this because all the points in the curve are in the domain of f.) We multiply $f(P)$ by the arc length of the piece Δs, add the product $f(P)\Delta s$ over all the pieces, and then let the arc length of the pieces shrink to zero by taking a limit. The result is the scalar line integral of the function over the curve.

For a formal description of a scalar line integral, let C be a smooth curve in space given by the parameterization $\vec{\mathbf{r}}(t) = \langle x(t), y(t), z(t)\rangle$, $a \le t \le b$. Let $f(x,y,z)$ be a function with a domain that includes curve C. To define the line integral of the function f over C, we begin as most definitions of an integral begin: we chop the curve into small pieces. Partition the parameter interval $[a,b]$ into n subintervals $[t_{i-1}, t_i]$ of equal width for $1 \le i \le n$, where $t_0 = a$ and $t_n = b$ (Figure 15.2.1). Let t_i^* be a value in the i^{th} interval $[t_{i-1}, t_i]$. Denote the endpoints of $\vec{\mathbf{r}}(t_0)$, $\vec{\mathbf{r}}(t_1)$,…, $\vec{\mathbf{r}}(t_n)$ by P_0,…, P_n. Points P_i divide curve C into n pieces C_1, C_2,…, C_n,with lengths Δs_1, Δs_2,…, Δs_n, respectively. Let P_i^* denote the endpoint of $\vec{\mathbf{r}}(t_i^*)$ for $1 \le i \le n$. Now, we evaluate the function f at point P_i^* for $1 \le i \le n$. Note that P_i^* is in piece C_1, and therefore P_i^* is in the domain of f. Multiply $f(P_i^*)$ by the length Δs_1 of C_1, which gives the area of the "sheet" with base C_1, and height $f(P_i^*)$. This is analogous to using rectangles to approximate area in a single-variable integral. Now, we form the sum $\sum_{i=1}^{n} f(P_i^*)\,\Delta s_i$.

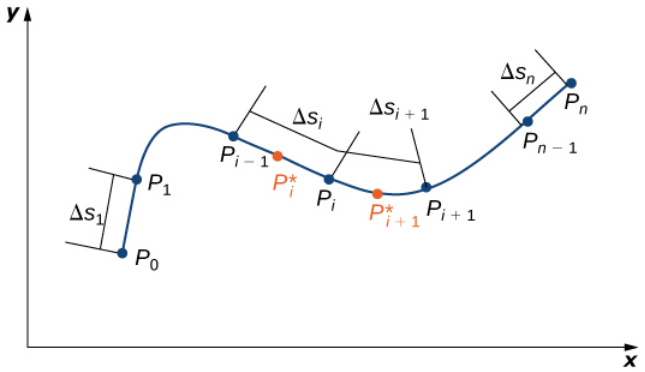

Figure 15.2.1: Curve C has been divided into n pieces, and a point inside each piece has been chosen.

Note the similarity of this sum versus a Riemann sum; in fact, this definition is a generalization of a Riemann sum to arbitrary curves in space. Just as with Riemann sums and integrals of form $\int_a^b g(x)\,dx$, we define an integral by letting the width of the pieces of the curve shrink to zero by taking a limit. The result is the scalar line integral of f along C.

You may have noticed a difference between this definition of a scalar line integral and a single-variable integral. In this definition, the arc lengths Δs_1, Δs_2,…, Δs_n aren't necessarily the same; in the definition of a single-variable integral, the curve in the x-axis is partitioned into pieces of equal length. This difference does not have any effect in the limit. As we shrink the arc lengths to zero, their values become close enough that any small difference becomes irrelevant.

DEFINITION: Scalar Line Integral

Let f be a function with a domain that includes the smooth curve C that is parameterized by $\vec{\mathbf{r}}(t) = \langle x(t), y(t), z(t)\rangle$, $a \le t \le b$. The *scalar line integral* of f along C is

$$\int_C f(x,y,z)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(P_i^*)\,\Delta s_i \tag{15.2.1}$$

if this limit exists (t_i^* and Δs_i are defined as in the previous paragraphs). If C is a planar curve, then C can be represented by the parametric equations $x = x(t)$, $y = y(t)$, and $a \le t \le b$. If C is smooth and $f(x,y)$ is a function of two variables, then the scalar line integral of f along C is defined similarly as

$$\int_C f(x,y)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(P_i^*)\,\Delta s_i, \tag{15.2.2}$$

if this limit exists.

If f is a continuous function on a smooth curve C, then $\int_C f\,ds$ always exists. Since $\int_C f\,ds$ is defined as a limit of Riemann sums, the continuity of f is enough to guarantee the existence of the limit, just as the integral $\int_a^b g(x)\,dx$ exists if g is continuous over $[a,b]$.

Before looking at how to compute a line integral, we need to examine the geometry captured by these integrals. Suppose that $f(x,y) \ge 0$ for all points (x,y) on a smooth planar curve C. Imagine taking curve C and projecting it "up" to the surface defined by $f(x,y)$, thereby creating a new curve C' that lies in the graph of $f(x,y)$ (Figure 15.2.2). Now we drop a "sheet" from C' down to the xy-plane. The area of this sheet is $\int_C f(x,y)ds$. If $f(x,y) \le 0$ for some points in C, then the value of $\int_C f(x,y)\,ds$ is the area above the xy-plane less the area below the xy-plane. (Note the similarity with integrals of the form $\int_a^b g(x)\,dx$.)

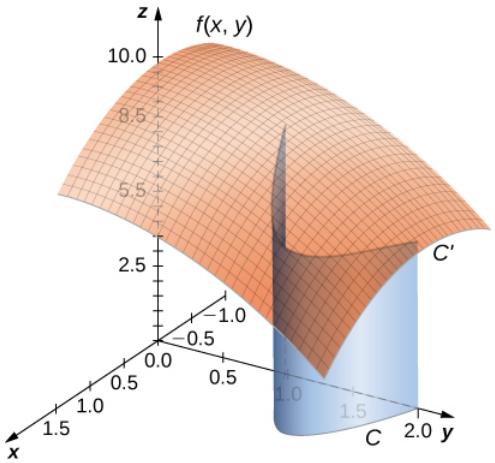

Figure 15.2.2: The area of the blue sheet is $\int_C f(x,y)\,ds$.

From this geometry, we can see that line integral $\int_C f(x,y)\,ds$ does not depend on the parameterization $\vec{\mathbf{r}}(t)$ of C. As long as the curve is traversed exactly once by the parameterization, the area of the sheet formed by the function and the curve is the same. This same kind of geometric argument can be extended to show that the line integral of a three-variable function over a curve in space does not depend on the parameterization of the curve.

Example 15.2.1: Finding the Value of a Line Integral

Find the value of integral $\int_C 2\,ds$, where C is the upper half of the unit circle.

Solution

The integrand is $f(x,y) = 2$. Figure 15.2.3 shows the graph of $f(x,y) = 2$, curve C, and the sheet formed by them. Notice that this sheet has the same area as a rectangle with width π and length 2. Therefore, $\int_C 2\,ds = 2\pi$ units^2.

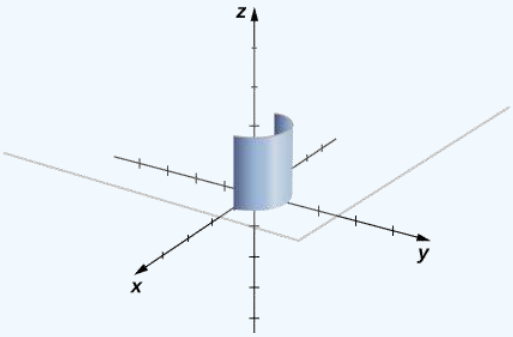

Figure 15.2.3: The sheet that is formed by the upper half of the unit circle in a plane and the graph of $f(x,y) = 2$.

To see that $\int_C 2\,ds = 2\pi$ using the definition of line integral, we let $\vec{\mathbf{r}}(t)$ be a parameterization of C. Then, $f(\vec{\mathbf{r}}(t_i)) = 2$ for any number t_i in the domain of $\vec{\mathbf{r}}$. Therefore,

$$\begin{aligned}\int_C f\,ds &= \lim_{n\to\infty}\sum_{i=1}^{n} f(\vec{\mathbf{r}}(t_i^*))\,\Delta s_i \\ &= \lim_{n\to\infty}\sum_{i=1}^{n} 2\,\Delta s_i \\ &= 2\lim_{n\to\infty}\sum_{i=1}^{n} \Delta s_i \\ &= 2(\text{length of } C) \\ &= 2\pi \text{ units}^2.\end{aligned}$$

Exercise 15.2.1

Find the value of $\int_C (x+y)\,ds$, where C is the curve parameterized by $x=t$, $y=t$, $0\le t\le 1$.

Note that in a scalar line integral, the integration is done with respect to arc length s, which can make a scalar line integral difficult to calculate. To make the calculations easier, we can translate $\int_C f\,ds$ to an integral with a variable of integration that is t.

Let $\vec{\mathbf{r}}(t)=\langle x(t),y(t),z(t)\rangle$ for $a\le t\le b$ be a parameterization of C. Since we are assuming that C is smooth, $\vec{\mathbf{r}}\,'(t)=\langle x'(t),y'(t),z'(t)\rangle$ is continuous for all t in $[a,b]$. In particular, $x'(t)$, $y'(t)$, and $z'(t)$ exist for all t in $[a,b]$. According to the arc length formula, we have

$$\text{length}(C_i)=\Delta s_i=\int_{t_{i-1}}^{t_i}\|\vec{\mathbf{r}}\,'(t)\|\,dt.$$

If width $\Delta t_i=t_i-t_{i-1}$ is small, then function $\int_{t_{i-1}}^{t_i}\|\vec{\mathbf{r}}\,'(t)\|\,dt\approx\|\vec{\mathbf{r}}\,'(t_i^*)\|\,\Delta t_i$, $\|\vec{\mathbf{r}}\,'(t)\|$ is almost constant over the interval $[t_{i-1},t_i]$.Therefore,

$$\int_{t_{i-1}}^{t_i}\|\vec{\mathbf{r}}\,'(t)\|\,dt\approx\|\vec{\mathbf{r}}\,'(t_i^*)\|\,\Delta t_i, \tag{15.2.3}$$

and we have

$$\sum_{i=1}^{n} f(\vec{\mathbf{r}}(t_i^*))\,\Delta s_i\approx\sum_{i=1}^{n} f(\vec{\mathbf{r}}(t_i^*))\|\vec{\mathbf{r}}\,'(t_i^*)\|\,\Delta t_i.$$

See Figure 15.2.4.

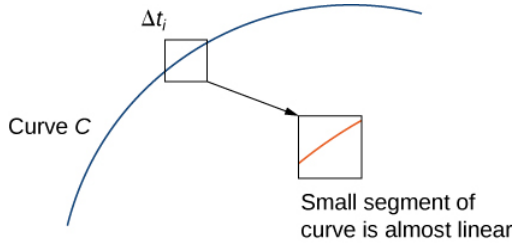

Figure 15.2.4: If we zoom in on the curve enough by making Δt_i very small, then the corresponding piece of the curve is approximately linear.

Note that

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(\vec{\mathbf{r}}(t_i^*))\|\vec{\mathbf{r}}\,'(t_i^*)\|\,\Delta t_i=\int_a^b f(\vec{\mathbf{r}}(t))\|\vec{\mathbf{r}}\,'(t)\|\,dt.$$

In other words, as the widths of intervals $[t_{i-1},t_i]$ shrink to zero, the sum $\sum_{i=1}^{n} f(\vec{\mathbf{r}}(t_i^*))\|\vec{\mathbf{r}}\,'(t_i^*)\|\,\Delta t_i$ converges to the integral $\int_a^b f(\vec{\mathbf{r}}(t))\|\vec{\mathbf{r}}\,'(t)\|\,dt$. Therefore, we have the following theorem.

Theorem: Evaluating a Scalar Line Integral

Let f be a continuous function with a domain that includes the smooth curve C with parameterization $\vec{\mathbf{r}}(t)$, $a\le t\le b$. Then

$$\int_C f\,ds=\int_a^b f(\vec{\mathbf{r}}(t))\|\vec{\mathbf{r}}\,'(t)\|\,dt. \tag{15.2.4}$$

Although we have labeled Equation 15.2.3 as an equation, it is more accurately considered an approximation because we can show that the left-hand side of Equation 15.2.3 approaches the right-hand side as $n\to\infty$. In other words, letting the widths of the pieces shrink to zero makes the right-hand sum arbitrarily close to the left-hand sum. Since

$$\|\vec{\mathbf{r}}'(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2},$$

we obtain the following theorem, which we use to compute scalar line integrals.

Theorem: Scalar Line Integral Calculation

Let f be a continuous function with a domain that includes the smooth curve C with parameterization $\vec{\mathbf{r}}(t) = \langle x(t), y(t), z(t)\rangle$, $a \le t \le b$. Then

$$\int_C f(x,y,z)\,ds = \int_a^b f(\vec{\mathbf{r}}(t))\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\,dt.$$

Similarly,

$$\int_C f(x,y)\,ds = \int_a^b f(\vec{\mathbf{r}}(t))\sqrt{(x'(t))^2 + (y'(t))^2}\,dt$$

if C is a planar curve and f is a function of two variables.

Note that a consequence of this theorem is the equation $ds = \|\vec{\mathbf{r}}'(t)\|\,dt$. In other words, the change in arc length can be viewed as a change in the t-domain, scaled by the magnitude of vector $\vec{\mathbf{r}}'(t)$.

Example 15.2.2: Evaluating a Line Integral

Find the value of integral $\int_C (x^2 + y^2 + z)\,ds$, where C is part of the helix parameterized by $\vec{\mathbf{r}}(t) = \langle \cos t, \sin t, t\rangle$, $0 \le t \le 2\pi$.

Solution

To compute a scalar line integral, we start by converting the variable of integration from arc length s to t. Then, we can use Equation 15.2.1 to compute the integral with respect to t. Note that

$$f(\vec{\mathbf{r}}(t)) = \cos^2 t + \sin^2 t + t = 1 + t$$

and

$$\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} = \sqrt{(-\sin(t))^2 + \cos^2(t) + 1} = \sqrt{2}.$$

Therefore,

$$\int_C (x^2 + y^2 + z)\,ds = \int_0^{2\pi} (1 + t)\sqrt{2}\,dt.$$

Notice that Equation 15.2.1 translated the original difficult line integral into a manageable single-variable integral. Since

$$\begin{aligned}\int_0^{2\pi} (1 + t)\sqrt{2}\,dt &= \left[\sqrt{2}t + \frac{\sqrt{2}t^2}{2}\right]_0^{2\pi} \\ &= 2\sqrt{2}\pi + 2\sqrt{2}\pi^2,\end{aligned}$$

we have

$$\int_C (x^2 + y^2 + z)\,ds = 2\sqrt{2}\pi + 2\sqrt{2}\pi^2.$$

Exercise 15.2.2

Evaluate $\int_C (x^2 + y^2 + z)ds$, where C is the curve with parameterization $\vec{\mathbf{r}}(t) = \langle \sin(3t), \cos(3t)\rangle$, $0 \le t \le \frac{\pi}{4}$.

Example 15.2.3: Independence of Parameterization

Find the value of integral $\int_C (x^2+y^2+z)\,ds$, where C is part of the helix parameterized by $\vec{\mathbf{r}}(t)=\langle\cos(2t),\sin(2t),2t\rangle$, $0\le t\le\pi$. Notice that this function and curve are the same as in the previous example; the only difference is that the curve has been reparameterized so that time runs twice as fast.

Solution

As with the previous example, we use Equation 15.2.1 to compute the integral with respect to t. Note that $f(\vec{\mathbf{r}}(t))=\cos^2(2t)+\sin^2(2t)+2t=2t+1$ and

$$\begin{aligned}\sqrt{(x'(t))^2+(y'(t))^2+(z'(t))^2}&=\sqrt{(-\sin t+\cos t+4)}\\&=22\end{aligned}$$

so we have

$$\begin{aligned}\int_C (x^2+y^2+z)ds&=2\sqrt{2}\int_0^{\pi}(1+2t)dt\\&=2\sqrt{2}\Big[t+t^2\Big]_0^{\pi}\\&=2\sqrt{2}(\pi+\pi^2).\end{aligned}$$

Notice that this agrees with the answer in the previous example. Changing the parameterization did not change the value of the line integral. Scalar line integrals are independent of parameterization, as long as the curve is traversed exactly once by the parameterization.

Exercise 15.2.3

Evaluate line integral $\int_C (x^2+yz)\,ds$, where C is the line with parameterization $\vec{\mathbf{r}}(t)=\langle 2t,5t,-t\rangle$, $0\le t\le 10$. Reparameterize C with parameterization $s(t)=\langle 4t,10t,-2t\rangle$, $0\le t\le 5$, recalculate line integral $\int_C (x^2+yz)\,ds$, and notice that the change of parameterization had no effect on the value of the integral.

Now that we can evaluate line integrals, we can use them to calculate arc length. If $f(x,y,z)=1$, then

$$\begin{aligned}\int_C f(x,y,z)\,ds&=\lim_{n\to\infty}\sum_{i=1}^{n} f(t_i^*)\,\Delta s_i\\&=\lim_{n\to\infty}\sum_{i=1}^{n}\Delta s_i\\&=\lim_{n\to\infty}\text{length}(C)\\&=\text{length}(C).\end{aligned}$$

Therefore, $\int_C 1\,ds$ is the arc length of C.

Example 15.2.4: Calculating Arc Length

A wire has a shape that can be modeled with the parameterization $\vec{\mathbf{r}}(t)=\langle\cos t,\sin t,\frac{2}{3}t^{3/2}\rangle$, $0\le t\le 4\pi$. Find the length of the wire.

Solution

The length of the wire is given by $\int_C 1\,ds$, where C is the curve with parameterization $\vec{\mathbf{r}}$. Therefore,

$$
\begin{aligned}
\text{The length of the wire} &= \int_C 1\,ds \\
&= \int_0^{4\pi} ||\vec{\mathbf{r}}'(t)||\,dt \\
&= \int_0^{4\pi} \sqrt{(-\sin t)^2 + \cos^2 t + t}dt \\
&= \int_0^{4\pi} \sqrt{1+t}dt \\
&= \frac{2(1+t)^{\frac{3}{2}}}{3}\Bigg|_0^{4\pi} \\
&= \frac{2}{3}\left((1+4\pi)^{3/2} - 1\right).
\end{aligned}
$$

? Exercise 15.2.4

Find the length of a wire with parameterization $\vec{\mathbf{r}}(t) = \langle 3t+1, 4-2t, 5+2t\rangle$, $0 \le t \le 4$.

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15.2: Line Integrals - Part 2

In this part of the section, we are going to develop a second type of line integral called a vector line integral. Although it's result is also a scalar, it is defined a little differently than the scalar line integral defined earlier in this section. The previous definition is repeated below for easy reference.

DEFINITION: Scalar Line Integral

Let f be a function with a domain that includes the smooth curve C that is parameterized by $\vec{\mathbf{r}}(t) = \langle x(t), y(t), z(t) \rangle$, $a \leq t \leq b$. The *scalar line integral* of f along C is

$$\int_C f(x,y,z)\, ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(P_i^*)\, \Delta s_i \tag{15.2.1}$$

if this limit exists t_i^* and Δs_i are defined as in the previous paragraphs). If C is a planar curve, then C can be represented by the parametric equations $x = x(t)$, $y = y(t)$, and $a \leq t \leq b$. If C is smooth and $f(x,y)$ is a function of two variables, then the scalar line integral of f along C is defined similarly as

$$\int_C f(x,y)\, ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(P_i^*)\, \Delta s_i, \tag{15.2.2}$$

if this limit exists.

Vector Line Integrals

The second type of line integrals are vector line integrals, in which we integrate along a curve through a vector field. For example, let

$$\vec{\mathbf{F}}(x,y,z) = P(x,y,z)\,\hat{\mathbf{i}} + Q(x,y,z)\,\hat{\mathbf{j}} + R(x,y,z)\,\hat{\mathbf{k}}$$

be a continuous vector field in $\mathbb{R}^3$ that represents a force on a particle, and let C be a smooth curve in $\mathbb{R}^3$ contained in the domain of $\vec{\mathbf{F}}$. How would we compute the work done by $\vec{\mathbf{F}}$ in moving a particle along C?

To answer this question, first note that a particle could travel in two directions along a curve: a forward direction and a backward direction. The work done by the vector field depends on the direction in which the particle is moving. Therefore, we must specify a direction along curve C; such a specified direction is called an orientation of a curve. The specified direction is the positive direction along C; the opposite direction is the negative direction along C. When C has been given an orientation, C is called an oriented curve (Figure 15.2.5). The work done on the particle depends on the direction along the curve in which the particle is moving.

A closed curve is one for which there exists a parameterization $\vec{\mathbf{r}}(t)$, $a \leq t \leq b$, such that $\vec{\mathbf{r}}(a) = \vec{\mathbf{r}}(b)$, and the curve is traversed exactly once. In other words, the parameterization is one-to-one on the domain (a, b).

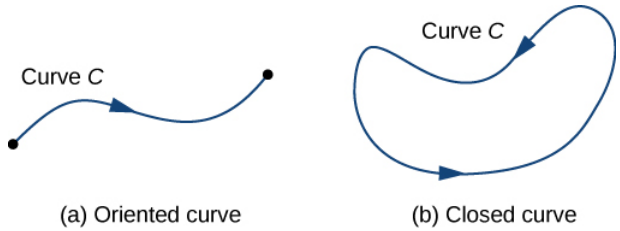

Figure 15.2.5: (a) An oriented curve between two points. (b) A closed oriented curve.

Let $\vec{\mathbf{r}}(t)$ be a parameterization of C for $a \leq t \leq b$ such that the curve is traversed exactly once by the particle and the particle moves in the positive direction along C. Divide the parameter interval $[a, b]$ into n subintervals $[t_{i-1}, t_i]$, $0 \leq i \leq n$, of equal width. Denote the endpoints of $r(t_0)$, $r(t_1)$,…, $r(t_n)$ by P_0,…,P_n. Points P_i divide C into n pieces. Denote the length of the piece from P_{i-1} to P_i by Δs_i. For each i, choose a value t_i^* in the subinterval $[t_{i-1}, t_i]$. Then, the endpoint of $\vec{\mathbf{r}}(t_i^*)$ is a point in the piece of C between P_{i-1}

and P_i (Figure 15.2.6). If Δs_i is small, then as the particle moves from P_{i-1} to P_i along C, it moves approximately in the direction of $\vec{\mathbf{T}}(P_i)$, the unit tangent vector at the endpoint of $\vec{\mathbf{r}}(t_i^*)$. Let P_i^* denote the endpoint of $\vec{\mathbf{r}}(t_i^*)$. Then, the work done by the force vector field in moving the particle from P_{i-1} to P_i is $\vec{\mathbf{F}}(P_i^*) \cdot (\Delta s_i \vec{\mathbf{T}}(P_i^*))$, so the total work done along C is

$$\sum_{i=1}^{n} \vec{\mathbf{F}}(P_i^*) \cdot (\Delta s_i \vec{\mathbf{T}}(P_i^*)) = \sum_{i=1}^{n} \vec{\mathbf{F}}(P_i^*) \cdot \vec{\mathbf{T}}(P_i^*)\, \Delta s_i.$$

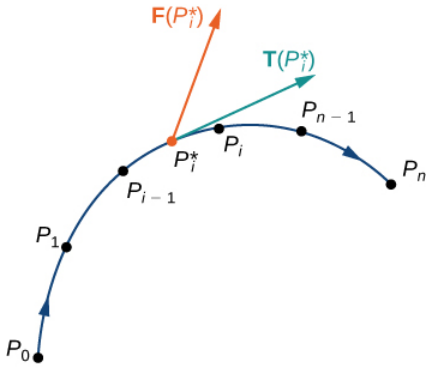

Figure 15.2.6: Curve C is divided into n pieces, and a point inside each piece is chosen. The dot product of any tangent vector in the ith piece with the corresponding vector $\vec{\mathbf{F}}$ is approximated by $\vec{\mathbf{F}}(P_i^*) \cdot \vec{\mathbf{T}}(P_i^*)$.

Letting the arc length of the pieces of C get arbitrarily small by taking a limit as $n \to \infty$ gives us the work done by the field in moving the particle along C. Therefore, the work done by $\vec{\mathbf{F}}$ in moving the particle in the positive direction along C is defined as

$$W = \int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds,$$

which gives us the concept of a vector line integral.

DEFINITION: Line Integral of a Vector Field

The vector line integral of vector field $\vec{\mathbf{F}}$ along oriented smooth curve C is

$$\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds = \lim_{n\to\infty} \sum_{i=1}^{n} \vec{\mathbf{F}}(P_i^*) \cdot \vec{\mathbf{T}}(P_i^*) \Delta s_i$$

if that limit exists.

With scalar line integrals, neither the orientation nor the parameterization of the curve matters. As long as the curve is traversed exactly once by the parameterization, the value of the line integral is unchanged. With vector line integrals, the orientation of the curve does matter. If we think of the line integral as computing work, then this makes sense: if you hike up a mountain, then the gravitational force of Earth does negative work on you. If you walk down the mountain by the exact same path, then Earth's gravitational force does positive work on you. In other words, reversing the path changes the work value from negative to positive in this case. Note that if C is an oriented curve, then we let $-C$ represent the same curve but with opposite orientation.

As with scalar line integrals, it is easier to compute a vector line integral if we express it in terms of the parameterization function $\vec{\mathbf{r}}$ and the variable t. To translate the integral $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}} ds$ in terms of t, note that unit tangent vector $\vec{\mathbf{T}}$ along C is given by $\vec{\mathbf{T}} = \dfrac{\vec{\mathbf{r}}'(t)}{\|\vec{\mathbf{r}}'(t)\|}$ (assuming $\|\vec{\mathbf{r}}'(t)\| \neq 0$). Since $ds = \|\vec{\mathbf{r}}'(t)\|\, dt$, as we saw when discussing scalar line integrals, we have

$$\vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds = \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \frac{\vec{\mathbf{r}}'(t)}{\|\vec{\mathbf{r}}'(t)\|} \|\vec{\mathbf{r}}'(t)\| dt = \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}'(t)\, dt.$$

Thus, we have the following formula for computing vector line integrals:

$$\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds = \int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}'(t)\, dt. \tag{15.2.3}$$

Because of Equation 15.2.3, we often use the notation $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ for the line integral $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds$.

If $\vec{\mathbf{r}}(t) = \langle x(t), y(t), z(t)\rangle$, then $\dfrac{d\vec{\mathbf{r}}}{dt}$ denotes vector $\langle x'(t), y'(t), z'(t)\rangle$, and $d\vec{\mathbf{r}} = \vec{\mathbf{r}}'(t)\, dt$.

Example 15.2.5: Evaluating a Vector Line Integral

Find the value of integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where C is the semicircle parameterized by $\vec{\mathbf{r}}(t) = \langle \cos t, \sin t\rangle$, $0 \le t \le \pi$ and $\vec{\mathbf{F}} = \langle -y, x\rangle$.

Solution

We can use Equation 15.2.3 to convert the variable of integration from s to t. We then have

$$\vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) = \langle -\sin t, \cos t\rangle \text{ and } \vec{\mathbf{r}}'(t) = \langle -\sin t, \cos t\rangle.$$

Therefore,

$$\begin{aligned}\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_0^\pi \langle -\sin t, \cos t\rangle \cdot \langle -\sin t, \cos t\rangle\, dt \\ &= \int_0^\pi \sin^2 t + \cos^2 t\, dt \\ &= \int_0^\pi 1\, dt = \pi.\end{aligned}$$

See Figure 15.2.7.

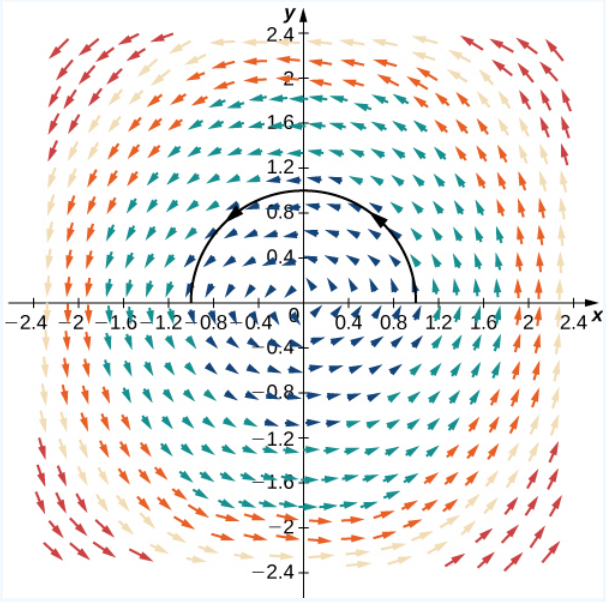

Figure 15.2.7: This figure shows curve $\vec{\mathbf{r}}(t) = \langle \cos t, \sin t\rangle$, $0 \le t \le \pi$ in vector field $\vec{\mathbf{F}} = \langle -y, x\rangle$.

Example 15.2.6: Reversing Orientation

Find the value of integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where C is the semicircle parameterized by $\vec{\mathbf{r}}(t) = \langle \cos(t+\pi), \sin t\rangle$, $0 \le t \le \pi$ and $\vec{\mathbf{F}} = \langle -y, x\rangle$.

Solution

Notice that this is the same problem as Example 15.2.5, except the orientation of the curve has been traversed. In this example, the parameterization starts at $\vec{\mathbf{r}}(0) = \langle -1, 0\rangle$ and ends at $\vec{\mathbf{r}}(\pi) = \langle 1, 0\rangle$. By Equation 15.2.3,

$$\begin{aligned}\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_0^{\pi} \langle -\sin t, \cos(t+\pi)\rangle \cdot \langle -\sin(t+\pi), \cos t\rangle dt \\ &= \int_0^{\pi} \langle -\sin t, -\cos t\rangle \cdot \langle \sin t, \cos t\rangle dt \\ &= \int_0^{\pi} (-\sin^2 t - \cos^2 t) dt \\ &= \int_0^{\pi} -1 dt \\ &= -\pi.\end{aligned}$$

Notice that this is the negative of the answer in Example 15.2.5. It makes sense that this answer is negative because the orientation of the curve goes against the "flow" of the vector field.

Let C be an oriented curve and let $-C$ denote the same curve but with the orientation reversed. Then, the previous two examples illustrate the following fact:

$$\int_{-C} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = -\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}.$$

That is, reversing the orientation of a curve changes the sign of a line integral.

? Exercise 15.2.6

Let $\vec{\mathbf{F}} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}$ be a vector field and let C be the curve with parameterization $\langle t, t^2\rangle$ for $0 \le t \le 2$. Which is greater: $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds$ or $\int_{-C} \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds$?

Hint

Imagine moving along the path and computing the dot product $\vec{\mathbf{F}} \cdot \vec{\mathbf{T}}$ as you go.

Answer

$$\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds$$

Another standard notation for integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ is $\int_C P\,dx + Q\,dy + R\,dz$. In this notation, P, Q, and R are functions, and we think of $d\vec{\mathbf{r}}$ as vector $\langle dx, dy, dz\rangle$. To justify this convention, recall that $d\vec{\mathbf{r}} = \vec{\mathbf{T}}\, ds = \vec{\mathbf{r}}'(t)\, dt = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle dt$. Therefore,

$$\vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \langle P, Q, R\rangle \cdot \langle dx, dy, dz\rangle = P\,dx + Q\,dy + R\,dz.$$

If $d\vec{\mathbf{r}} = \langle dx, dy, dz\rangle$, then $\frac{d\vec{r}}{dt} = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$, which implies that $d\vec{\mathbf{r}} = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle dt$. Therefore

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C P\,dx + Q\,dy + R\,dz \tag{15.2.4}$$

$$= \int_a^b \left(P\big(\vec{\mathbf{r}}(t)\big)\frac{dx}{dt} + Q\big(\vec{\mathbf{r}}(t)\big)\frac{dy}{dt} + R\big(\vec{\mathbf{r}}(t)\big)\frac{dz}{dt} \right) dt. \tag{15.2.5}$$

✓ Example 15.2.7: Finding the Value of an Integral of the Form $\int_C P\,dx + Q\,dy + R\,dz$

Find the value of integral $\int_C z\,dx + x\,dy + y\,dz$, where C is the curve parameterized by $\vec{\mathbf{r}}(t) = \langle t^2, \sqrt{t}, t\rangle$, $1 \le t \le 4$.

Solution

As with our previous examples, to compute this line integral we should perform a change of variables to write everything in terms of t. In this case, Equation 15.2.5 allows us to make this change:

$$\begin{aligned}\int_C z\,dx + x\,dy + y\,dz &= \int_1^4 \left(t(2t) + t^2\left(\frac{1}{2\sqrt{t}}\right) + \sqrt{t}\right)\,dt \\ &= \int_1^4 \left(2t^2 + \frac{t^{3/2}}{2} + \sqrt{t}\right)\,dt \\ &= \left[\frac{2t^3}{3} + \frac{t^{5/2}}{5} + \frac{2t^{3/2}}{3}\right]_{t=1}^{t=4} \\ &= \frac{793}{15}.\end{aligned}$$

? Exercise 15.2.7

Find the value of $\int_C 4x\,dx + z\,dy + 4y^2\,dz$, where C is the curve parameterized by $\vec{\mathbf{r}}(t) = \langle 4\cos(2t), 2\sin(2t), 3\rangle$, $0 \le t \le \frac{\pi}{4}$.

Hint

Write the integral in terms of t using Equation 15.2.5.

Answer

-26

We have learned how to integrate smooth oriented curves. Now, suppose that C is an oriented curve that is not smooth, but can be written as the union of finitely many smooth curves. In this case, we say that C is a piecewise smooth curve. To be precise, curve C is piecewise smooth if C can be written as a union of n smooth curves C_1, C_2,…, C_n such that the endpoint of C_i is the starting point of C_{i+1} (Figure 15.2.8). When curves C_i satisfy the condition that the endpoint of C_i is the starting point of C_{i+1}, we write their union as $C_1 + C_2 + \cdots + C_n$.

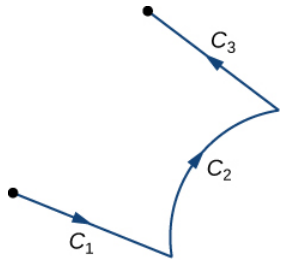

Figure 15.2.8: The union of C_1, C_2, C_3 is a piecewise smooth curve.

The next theorem summarizes several key properties of vector line integrals.

Theorem: Properties of Vector Line Integrals

Let $\vec{\mathbf{F}}$ and $\vec{\mathbf{G}}$ be continuous vector fields with domains that include the oriented smooth curve C. Then

1. $\int_C (\vec{\mathbf{F}} + \vec{\mathbf{G}}) \cdot d\vec{\mathbf{r}} = \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_C \vec{\mathbf{G}} \cdot d\vec{\mathbf{r}}$
2. $\int_C k\vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = k\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where k is a constant
3. $\int_{-C} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = -\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$
4. Suppose instead that C is a piecewise smooth curve in the domains of $\vec{\mathbf{F}}$ and $\vec{\mathbf{G}}$, where $C = C_1 + C_2 + \cdots + C_n$ and $C_1, C_2, \ldots, C_n$ are smooth curves such that the endpoint of C_i is the starting point of C_{i+1}. Then

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \cdots + \int_{C_n} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}.$$

Notice the similarities between these items and the properties of single-variable integrals. Properties i. and ii. say that line integrals are linear, which is true of single-variable integrals as well. Property iii. says that reversing the orientation of a curve changes the sign of the integral. If we think of the integral as computing the work done on a particle traveling along C, then this makes sense. If

the particle moves backward rather than forward, then the value of the work done has the opposite sign. This is analogous to the equation $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$. Finally, if $[a_1, a_2]$, $[a_2, a_3]$,…, $[a_{n-1}, a_n]$ are intervals, then

$$\int_{a_1}^{a_n} f(x)\,dx = \int_{a_1}^{a_2} f(x)\,dx + \int_{a_1}^{a_3} f(x)\,dx + \cdots + \int_{a_{n-1}}^{a_n} f(x)\,dx,$$

which is analogous to property iv.

Example 15.2.8: Using Properties to Compute a Vector Line Integral

Find the value of integral $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\,ds$, where C is the rectangle (oriented counterclockwise) in a plane with vertices $(0,0)$, $(2,0)$, $(2,1)$, and $(0,1)$, and where $\vec{\mathbf{F}} = \langle x - 2y, y - x \rangle$ (Figure 15.2.9).

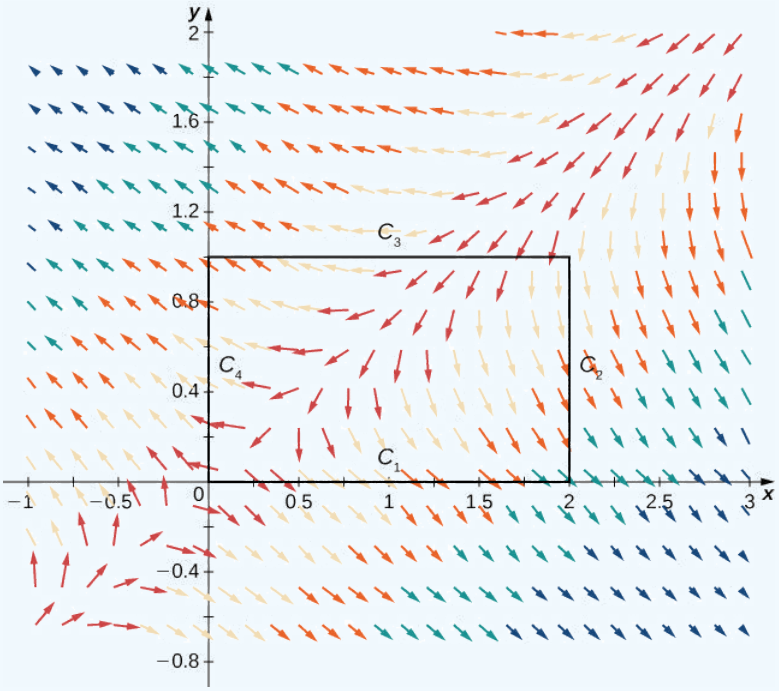

Figure 15.2.9: Rectangle and vector field for Example 15.2.8.

Solution

Note that curve C is the union of its four sides, and each side is smooth. Therefore C is piecewise smooth. Let C_1 represent the side from $(0,0)$ to $(2,0)$, let C_2 represent the side from $(2,0)$ to $(2,1)$, let C_3 represent the side from $(2,1)$ to $(0,1)$, and let C_4 represent the side from $(0,1)$ to $(0,0)$ (Figure 15.2.9). Then,

$$\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\,dr = \int_{C_1} \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\,dr + \int_{C_2} \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\,dr + \int_{C_3} \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\,dr + \int_{C_4} \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\,dr.$$

We want to compute each of the four integrals on the right-hand side using Equation 15.2.1. Before doing this, we need a parameterization of each side of the rectangle. Here are four parameterizations (note that they traverse C counterclockwise):

$$C_1 : \langle t, 0 \rangle, 0 \le t \le 2$$
$$C_2 : \langle 2, t \rangle, 0 \le t \le 1$$
$$C_3 : \langle 2 - t, 1 \rangle, 0 \le t \le 2$$
$$C_4 : \langle 0, 1 - t \rangle, 0 \le t \le 1.$$

Therefore,

$$\int_{C_1} \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, dr = \int_0^2 \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}'(t)\, dt$$
$$= \int_0^2 \langle t - 2(0), 0 - t \rangle \cdot \langle 1, 0 \rangle\, dt = \int_0^2 t\, dt$$
$$= \left[\tfrac{t^2}{2}\right]_0^2 = 2.$$

Notice that the value of this integral is positive, which should not be surprising. As we move along curve C_1 from left to right, our movement flows in the general direction of the vector field itself. At any point along C_1, the tangent vector to the curve and the corresponding vector in the field form an angle that is less than 90°. Therefore, the tangent vector and the force vector have a positive dot product all along C_1, and the line integral will have positive value.

The calculations for the three other line integrals are done similarly:

$$\int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_0^1 \langle 2 - 2t, t - 2 \rangle \cdot \langle 0, 1 \rangle\, dt$$
$$= \int_0^1 (t - 2)\, dt$$
$$= \left[\tfrac{t^2}{2} - 2t\right]_0^1 = -\frac{3}{2},$$

$$\int_{C_3} \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds = \int_0^2 \langle (2 - t) - 2, 1 - (2 - t) \rangle \cdot \langle -1, 0 \rangle\, dt$$
$$= \int_0^2 t\, dt = 2,$$

and

$$\int_{C_4} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_0^1 \langle -2(1 - t), 1 - t \rangle \cdot \langle 0, -1 \rangle\, dt$$
$$= \int_0^1 (t - 1)\, dt$$
$$= \left[\tfrac{t^2}{2} - t\right]_0^1 = -\frac{1}{2}.$$

Thus, we have $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 2$.

? Exercise 15.2.8

Calculate line integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where $\vec{\mathbf{F}}$ is vector field $\langle y^2, 2xy + 1 \rangle$ and C is a triangle with vertices $(0, 0)$, $(4, 0)$, and $(0, 5)$, oriented counterclockwise.

Hint

Write the triangle as a union of its three sides, then calculate three separate line integrals.

Answer

0

Applications of Line Integrals

Scalar line integrals have many applications. They can be used to calculate the length or mass of a wire, the surface area of a sheet of a given height, or the electric potential of a charged wire given a linear charge density. Vector line integrals are extremely useful in physics. They can be used to calculate the work done on a particle as it moves through a force field, or the flow rate of a fluid

across a curve. Here, we calculate the mass of a wire using a scalar line integral and the work done by a force using a vector line integral.

Suppose that a piece of wire is modeled by curve C in space. The mass per unit length (the linear density) of the wire is a continuous function $\rho(x,y,z)$. We can calculate the total mass of the wire using the scalar line integral $\int_C \rho(x,y,z)\,ds$. The reason is that mass is density multiplied by length, and therefore the density of a small piece of the wire can be approximated by $\rho(x^*,y^*,z^*)\,\Delta s$ for some point (x^*,y^*,z^*) in the piece. Letting the length of the pieces shrink to zero with a limit yields the line integral $\int_C \rho(x,y,z)\,ds$.

Example 15.2.9: Calculating the Mass of a Wire

Calculate the mass of a spring in the shape of a curve parameterized by $\langle t, 2\cos t, 2\sin t\rangle$, $0 \le t \le \dfrac{\pi}{2}$, with a density function given by $\rho(x,y,z) = e^x + yz$ kg/m (Figure 15.2.10).

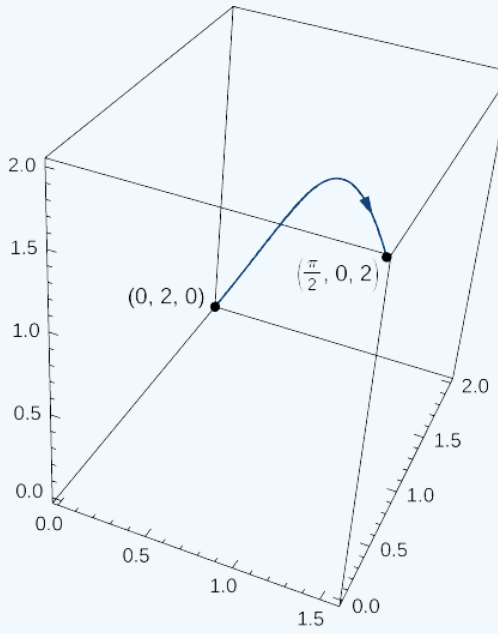

Figure 15.2.10: The wire from Example 15.2.9.

Solution

To calculate the mass of the spring, we must find the value of the scalar line integral $\int_C (e^x + yz)\,ds$, where C is the given helix. To calculate this integral, we write it in terms of t using Equation 15.2.1:

$$\begin{aligned}\int_C (e^x + yz)\,ds &= \int_0^{\frac{\pi}{2}} \left((e^t + 4\cos t\sin t)\sqrt{1 + (-2\cos t)^2 + (2\sin t)^2}\right) dt \\ &= \int_0^{\frac{\pi}{2}} \left((e^t + 4\cos t\sin t)\sqrt{5}\right) dt \\ &= \sqrt{5}\left[e^t + 2\sin^2 t\right]_{t=0}^{t=\pi/2} \\ &= \sqrt{5}(e^{\pi/2} + 1).\end{aligned}$$

Therefore, the mass is $\sqrt{5}(e^{\pi/2}+1)$ kg.

Exercise 15.2.9

Calculate the mass of a spring in the shape of a helix parameterized by $\vec{\mathbf{r}}(t) = \langle \cos t, \sin t, t\rangle$, $0 \le t \le 6\pi$, with a density function given by $\rho(x,y,z) = x + y + z$ kg/m.

Hint

Calculate the line integral of ρ over the curve with parameterization $\vec{\mathbf{r}}$.

Answer

$18\sqrt{2}\pi^2$ kg

When we first defined vector line integrals, we used the concept of work to motivate the definition. Therefore, it is not surprising that calculating the **work done by a vector field** representing a force is a standard use of vector line integrals. Recall that if an object moves along curve C in force field $\vec{\mathbf{F}}$, then the work required to move the object is given by $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$.

Example 15.2.10: Calculating Work

How much work is required to move an object in vector force field $\vec{\mathbf{F}} = \langle yz, xy, xz \rangle$ along path $\vec{\mathbf{r}}(t) = \langle t^2, t, t^4 \rangle$, $0 \leq t \leq 1$? See Figure 15.2.11.

Solution

Let C denote the given path. We need to find the value of $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$. To do this, we use Equation 15.2.3 :

$$\begin{aligned} \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_0^1 (\langle t^5, t^3, t^6 \rangle \cdot \langle 2t, 1, 4t^3 \rangle)\, dt \\ &= \int_0^1 (2t^6 + t^3 + 4t^9)\, dt \\ &= \left[\frac{2t^7}{7} + \frac{t^4}{4} + \frac{2t^{10}}{5} \right]_{t=0}^{t=1} = \frac{131}{140} \text{ units of work.} \end{aligned}$$

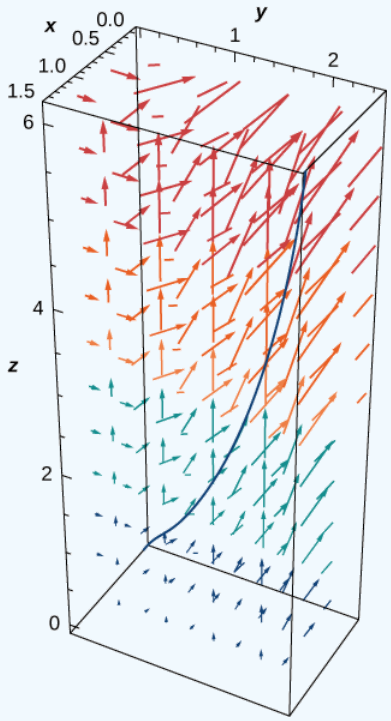

Figure 15.2.11: The curve and vector field for Example 15.2.10.

Flux

We close this section by discussing two key concepts related to line integrals: flux across a plane curve and circulation along a plane curve. Flux is used in applications to calculate fluid flow across a curve, and the concept of circulation is important for

characterizing conservative gradient fields in terms of line integrals. Both these concepts are used heavily throughout the rest of this chapter. The idea of flux is especially important for Green's theorem, and in higher dimensions for Stokes' theorem and the divergence theorem.

Let C be a plane curve and let $\vec{\mathbf{F}}$ be a vector field in the plane. Imagine C is a membrane across which fluid flows, but C does not impede the flow of the fluid. In other words, C is an idealized membrane invisible to the fluid. Suppose $\vec{\mathbf{F}}$ represents the velocity field of the fluid. How could we quantify the rate at which the fluid is crossing C?

Recall that the line integral of $\vec{\mathbf{F}}$ along C is $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds$—in other words, the line integral is the dot product of the vector field with the unit tangential vector with respect to arc length. If we replace the unit tangential vector with unit normal vector $\vec{\mathbf{N}}(t)$ and instead compute integral $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds$, we determine the flux across C. To be precise, the definition of integral $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds$ is the same as integral $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds$, except the $\vec{\mathbf{T}}$ in the Riemann sum is replaced with $\vec{\mathbf{N}}$. Therefore, the flux across C is defined as

$$\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds = \lim_{n\to\infty} \sum_{i=1}^{n} \vec{\mathbf{F}}(P_i^*) \cdot \vec{\mathbf{N}}(P_i^*)\, \Delta s_i,$$

where P_i^* and Δs_i are defined as they were for integral $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds$. Therefore, a flux integral is an integral that is *perpendicular* to a vector line integral, because $\vec{\mathbf{N}}$ and $\vec{\mathbf{T}}$ are perpendicular vectors.

If $\vec{\mathbf{F}}$ is a velocity field of a fluid and C is a curve that represents a membrane, then the flux of $\vec{\mathbf{F}}$ across C is the quantity of fluid flowing across C per unit time, or the rate of flow.

More formally, let C be a plane curve parameterized by $\vec{\mathbf{r}}(t) = \langle x(t), y(t)\rangle$, $a \le t \le b$. Let $\vec{\mathbf{n}}(t) = \langle y'(t), -x'(t)\rangle$ be the vector that is normal to C at the endpoint of $\vec{\mathbf{r}}(t)$ and points to the right as we traverse C in the positive direction (Figure 15.2.12). Then, $\vec{\mathbf{N}}(t) = \dfrac{\vec{\mathbf{n}}(t)}{\|\vec{\mathbf{n}}(t)\|}$ is the unit normal vector to C at the endpoint of $\vec{\mathbf{r}}(t)$ that points to the right as we traverse C.

DEFINITION: Flux

The *flux* of $\vec{\mathbf{F}}$ across C is line integral

$$\int_C \vec{\mathbf{F}} \cdot \frac{\vec{\mathbf{n}}(t)}{\|\vec{\mathbf{n}}(t)\|}\, ds.$$

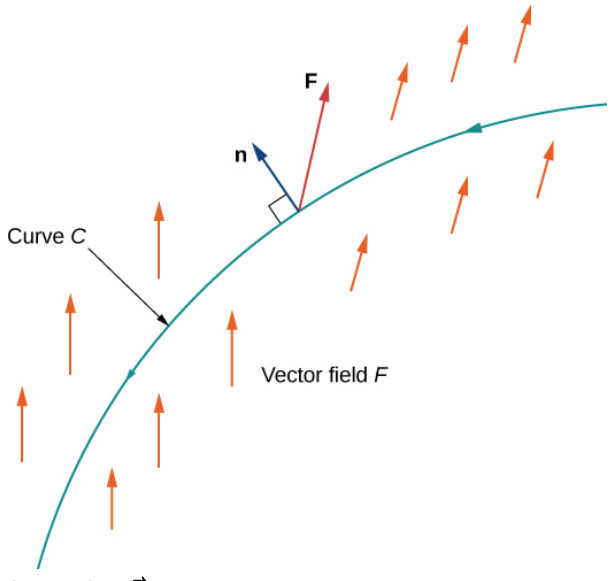

Figure 15.2.12: The flux of vector field $\vec{\mathbf{F}}$ *across curve* C *is computed by an integral similar to a vector line integral.*

We now give a formula for calculating the flux across a curve. This formula is analogous to the formula used to calculate a vector line integral (see Equation 15.2.3).

Theorem: Calculating Flux Across a Curve

Let $\vec{\mathbf{F}}$ be a vector field and let C be a smooth curve with parameterization $r(t) = \langle x(t), y(t) \rangle$, $a \leq t \leq b$.Let $\vec{\mathbf{n}}(t) = \langle y'(t), -x'(t) \rangle$. The flux of $\vec{\mathbf{F}}$ across C is

$$\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds = \int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{n}}(t)\, dt. \tag{15.2.6}$$

Proof

Before deriving the formula, note that

$$\|\vec{\mathbf{n}}(t)\| = \|\langle y'(t), -x'(t) \rangle\| = \sqrt{(y'(t))^2 + (x'(t))^2} = \|\vec{\mathbf{r}}'(t)\|.$$

Therefore,

$$\begin{aligned} \int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds &= \int_C \vec{\mathbf{F}} \cdot \frac{\vec{\mathbf{n}}(t)}{\|\vec{\mathbf{n}}(t)\|}\, ds \\ &= \int_a^b \vec{\mathbf{F}} \cdot \frac{\vec{\mathbf{n}}(t)}{\|\vec{\mathbf{n}}(t)\|} \|\vec{\mathbf{r}}'(t)\|\, dt \\ &= \int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{n}}(t)\, dt. \end{aligned}$$

□

Example 15.2.11: Flux across a Curve

Calculate the flux of $\vec{\mathbf{F}} = \langle 2x, 2y \rangle$ across a unit circle oriented counterclockwise (Figure 15.2.13).

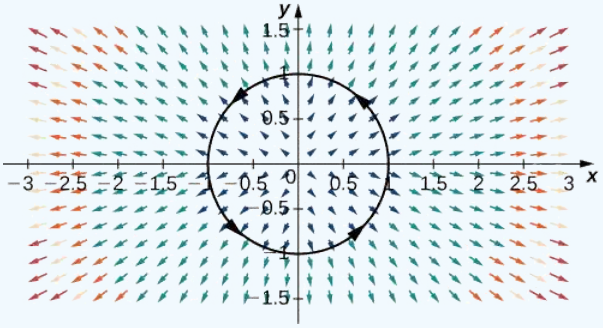

Figure 15.2.13: A unit circle in vector field $\vec{\mathbf{F}} = \langle 2x, 2y \rangle$.

Solution

To compute the flux, we first need a parameterization of the unit circle. We can use the standard parameterization $\vec{\mathbf{r}}(t) = \langle \cos t, \sin t \rangle$, $0 \leq t \leq 2\pi$. The normal vector to a unit circle is $\langle \cos t, \sin t \rangle$. Therefore, the flux is

$$\begin{aligned} \int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds &= \int_0^{2\pi} \langle 2\cos t, 2\sin t \rangle \cdot \langle \cos t, \sin t \rangle\, dt \\ &= \int_0^{2\pi} (2\cos^2 t + 2\sin^2 t)\, dt \\ &= 2\int_0^{2\pi} (\cos^2 t + \sin^2 t)\, dt \\ &= 2\int_0^{2\pi} dt = 4\pi. \end{aligned}$$

? Exercise 15.2.11

Calculate the flux of $\vec{\mathbf{F}} = \langle x + y, 2y \rangle$ across the line segment from $(0, 0)$ to $(2, 3)$, where the curve is oriented from left to right.

Hint

Use Equation 15.2.6.

Answer

$3/2$

Let $\vec{\mathbf{F}}(x, y) = \langle P(x, y), Q(x, y) \rangle$ be a two-dimensional vector field. Recall that integral $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds$ is sometimes written as $\int_C P\, dx + Q\, dy$. Analogously, flux $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds$ is sometimes written in the notation $\int_C -Q\, dx + P\, dy$, because the unit normal vector $\vec{\mathbf{N}}$ is perpendicular to the unit tangent $\vec{\mathbf{T}}$. Rotating the vector $d\vec{\mathbf{r}} = \langle dx, dy \rangle$ by 90° results in vector $\langle dy, -dx \rangle$. Therefore, the line integral in Example 15.2.8 can be written as $\int_C -2y\, dx + 2x\, dy$.

Circulation

Now that we have defined flux, we can turn our attention to circulation. The line integral of vector field $\vec{\mathbf{F}}$ along an oriented closed curve is called the circulation of $\vec{\mathbf{F}}$ along C. Circulation line integrals have their own notation: $\oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds$. The circle on the integral symbol denotes that C is "circular" in that it has no endpoints. Example 15.2.5 shows a calculation of circulation.

To see where the term *circulation* comes from and what it measures, let $\vec{\mathbf{v}}$ represent the velocity field of a fluid and let C be an oriented closed curve. At a particular point P, the closer the direction of $\vec{\mathbf{v}}(P)$ is to the direction of $\vec{\mathbf{T}}(P)$, the larger the value of the dot product $\vec{\mathbf{v}}(P) \cdot \vec{\mathbf{T}}(P)$. The maximum value of $\vec{\mathbf{v}}(P) \cdot \vec{\mathbf{T}}(P)$ occurs when the two vectors are pointing in the exact same direction; the minimum value of $\vec{\mathbf{v}}(P) \cdot \vec{\mathbf{T}}(P)$ occurs when the two vectors are pointing in opposite directions. Thus, the value of the circulation $\oint_C \vec{\mathbf{v}} \cdot \vec{\mathbf{T}}\, ds$ measures the tendency of the fluid to move in the direction of C.

✓ Example 15.2.12: Calculating Circulation

Let $\vec{\mathbf{F}} = \langle -y, x \rangle$ be the vector field from Example 15.2.3 and let C represent the unit circle oriented counterclockwise. Calculate the circulation of $\vec{\mathbf{F}}$ along C.

Solution

We use the standard parameterization of the unit circle: $\vec{\mathbf{r}}(t) = \langle \cos t, \sin t \rangle$, $0 \le t \le 2\pi$. Then, $\vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) = \langle -\sin t, \cos t \rangle$ and $\vec{\mathbf{r}}\,'(t) = \langle -\sin t, \cos t \rangle$. Therefore, the circulation of $\vec{\mathbf{F}}$ along C is

$$\begin{aligned} \oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds &= \int_0^{2\pi} \langle -\sin t, \cos t \rangle \cdot \langle -\sin t, \cos t \rangle\, dt \\ &= \int_0^{2\pi} (\sin^2 t + \cos^2 t)\, dt \\ &= \int_0^{2\pi} dt = 2\pi \text{ units of work.} \end{aligned}$$

Notice that the circulation is positive. The reason for this is that the orientation of C "flows" with the direction of $\vec{\mathbf{F}}$. At any point along the circle, the tangent vector and the vector from $\vec{\mathbf{F}}$ form an angle of less than 90°, and therefore the corresponding dot product is positive.

In Example 15.2.12, what if we had oriented the unit circle clockwise? We denote the unit circle oriented clockwise by $-C$. Then

$$\oint_{-C} \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds = -\oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds = -2\pi \text{ units of work.}$$

Notice that the circulation is negative in this case. The reason for this is that the orientation of the curve flows against the direction of $\vec{\mathbf{F}}$.

? Exercise 15.2.12

Calculate the circulation of $\vec{\mathbf{F}}(x,y) = \langle -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \rangle$ along a unit circle oriented counterclockwise.

Hint

Use Equation 15.2.6.

Answer

2π units of work

✓ Example 15.2.13: Calculating Work

Calculate the work done on a particle that traverses circle C of radius 2 centered at the origin, oriented counterclockwise, by field $\vec{\mathbf{F}}(x,y) = \langle -2, y \rangle$. Assume the particle starts its movement at $(1, 0)$.

Solution

The work done by $\vec{\mathbf{F}}$ on the particle is the circulation of $\vec{\mathbf{F}}$ along C: $\oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds$. We use the parameterization $\vec{\mathbf{r}}(t) = \langle 2\cos t, 2\sin t \rangle$, $0 \le t \le 2\pi$ for C. Then, $\vec{\mathbf{r}}'(t) = \langle -2\sin t, 2\cos t \rangle$ and $\vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) = \langle -2, 2\sin t \rangle$. Therefore, the circulation of $\vec{\mathbf{F}}$ along C is

$$\begin{aligned}
\oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds &= \int_0^{2\pi} \langle -2, 2\sin t \rangle \cdot \langle -2\sin t, 2\cos t \rangle\, dt \\
&= \int_0^{2\pi} (4\sin t + 4\sin t \cos t)\, dt \\
&= \Big[-4\cos t + 4\sin^2 t \Big]_0^{2\pi} \\
&= \big(-4\cos(2\pi) + 2\sin^2(2\pi)\big) - \big(-4\cos(0) + 4\sin^2(0)\big) \\
&= -4 + 4 = 0 \text{ units of work.}
\end{aligned}$$

The force field does zero work on the particle.

Notice that the circulation of $\vec{\mathbf{F}}$ along C is zero. Furthermore, notice that since $\vec{\mathbf{F}}$ is the gradient of $f(x,y) = -2x + \frac{y^2}{2}$, $\vec{\mathbf{F}}$ is conservative. We prove in a later section that under certain broad conditions, the circulation of a conservative vector field along a closed curve is zero.

? Exercise 15.2.14

Calculate the work done by field $\vec{\mathbf{F}}(x,y) = \langle 2x, 3y \rangle$ on a particle that traverses the unit circle. Assume the particle begins its movement at $(-1, 0)$.

Hint

Use Equation 15.2.6.

Answer

0 units of work

Key Concepts

- Line integrals generalize the notion of a single-variable integral to higher dimensions. The domain of integration in a single-variable integral is a line segment along the x-axis, but the domain of integration in a line integral is a curve in a plane or in

space.

- If C is a curve, then the length of C is $\int_C ds$.
- There are two kinds of line integral: scalar line integrals and vector line integrals. Scalar line integrals can be used to calculate the mass of a wire; vector line integrals can be used to calculate the work done on a particle traveling through a field.
- Scalar line integrals can be calculated using Equation 15.2.1; vector line integrals can be calculated using Equation 15.2.3.
- Two key concepts expressed in terms of line integrals are flux and circulation. Flux measures the rate that a field crosses a given line; circulation measures the tendency of a field to move in the same direction as a given closed curve.

Key Equations

- **Calculating a scalar line integral**
  $$\int_C f(x,y,z)\,ds = \int_a^b f(\vec{\mathbf{r}}(t))\sqrt{(x'(t))^2+(y'(t))^2+(z'(t))^2}\,dt$$
- **Calculating a vector line integral**
  $$\int_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} = \int_C \vec{\mathbf{F}}\cdot\vec{\mathbf{T}}\,ds = \int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}(t))\cdot\vec{\mathbf{r}}'(t)\,dt$$
  or
  $$\int_C P\,dx+Q\,dy+R\,dz = \int_a^b \left(P\big(\vec{\mathbf{r}}(t)\big)\frac{dx}{dt}+Q\big(\vec{\mathbf{r}}(t)\big)\frac{dy}{dt}+R\big(\vec{\mathbf{r}}(t)\big)\frac{dz}{dt}\right)dt$$
- **Calculating flux**
  $$\int_C \vec{\mathbf{F}}\cdot\frac{\vec{\mathbf{n}}(t)}{\|\vec{\mathbf{n}}(t)\|}\,ds = \int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}(t))\cdot\vec{\mathbf{n}}(t)\,dt$$

Glossary

circulation

the tendency of a fluid to move in the direction of curve C. If C is a closed curve, then the circulation of $\vec{\mathbf{F}}$ along C is line integral $\int_C \vec{\mathbf{F}}\cdot\vec{\mathbf{T}}\,ds$, which we also denote $\oint_C \vec{\mathbf{F}}\cdot\vec{\mathbf{T}}\,ds$.

closed curve

a curve for which there exists a parameterization $\vec{\mathbf{r}}(t), a\le t\le b$, such that $\vec{\mathbf{r}}(a)=\vec{\mathbf{r}}(b)$, and the curve is traversed exactly once

flux

the rate of a fluid flowing across a curve in a vector field; the flux of vector field $\vec{\mathbf{F}}$ across plane curve C is line integral $\int_C \vec{\mathbf{F}}\cdot\frac{\vec{\mathbf{n}}(t)}{\|\vec{\mathbf{n}}(t)\|}\,ds$

line integral

the integral of a function along a curve in a plane or in space

orientation of a curve

the orientation of a curve C is a specified direction of C

piecewise smooth curve

an oriented curve that is not smooth, but can be written as the union of finitely many smooth curves

scalar line integral

the scalar line integral of a function f along a curve C with respect to arc length is the integral $\int_C f\,ds$, it is the integral of a scalar function f along a curve in a plane or in space; such an integral is defined in terms of a Riemann sum, as is a single-variable integral

vector line integral

the vector line integral of vector field $\vec{\mathbf{F}}$ along curve C is the integral of the dot product of $\vec{\mathbf{F}}$ with unit tangent vector $\vec{\mathbf{T}}$ of C with respect to arc length, $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds$; such an integral is defined in terms of a Riemann sum, similar to a single-variable integral

16.2: Line Integrals

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15.2E: Exercises for Section 15.2

1. *True or False?* Line integral $\int_C f(x,y)\,ds$ is equal to a definite integral if C is a smooth curve defined on $[a,b]$ and if function f is continuous on some region that contains curve C.

Answer

True

2. *True or False?* Vector functions $\vec{\mathbf{r}}_1 = t\,\hat{\mathbf{i}} + t^2\,\hat{\mathbf{j}}, \quad 0 \le t \le 1$, and $\vec{\mathbf{r}}_2 = (1-t)\,\hat{\mathbf{i}} + (1-t)^2\,\hat{\mathbf{j}}, \quad 0 \le t \le 1$, define the same oriented curve.

3. *True or False?* $\int_{-C}(P\,dx + Q\,dy) = \int_C (P\,dx - Q\,dy)$

Answer

False

4. *True or False?* A piecewise smooth curve C consists of a finite number of smooth curves that are joined together end to end.

5. *True or False?* If C is given by $x(t) = t, \quad y(t) = t, \quad 0 \le t \le 1$, then $\int_C xy\,ds = \int_0^1 t^2\,dt$.

Answer

False

For the following exercises, use a computer algebra system (CAS) to evaluate the line integrals over the indicated path.

6. [T] $\int_C (x+y)\,ds$

$C: x = t, y = (1-t), z = 0$ from $(0,1,0)$ to $(1,0,0)$

7. [T] $\int_C (x-y)ds$

$C: \vec{\mathbf{r}}(t) = 4t\,\hat{\mathbf{i}} + 3t\,\hat{\mathbf{j}}$ when $0 \le t \le 2$

Answer

$\int_C (x-y)\,ds = 10$

8. [T] $\int_C (x^2 + y^2 + z^2)\,ds$

$C: \vec{\mathbf{r}}(t) = \sin t\,\hat{\mathbf{i}} + \cos t\,\hat{\mathbf{j}} + 8t\,\hat{\mathbf{k}}$ when $0 \le t \le \dfrac{\pi}{2}$

9. [T] Evaluate $\int_C xy^4\,ds$, where C is the right half of circle $x^2 + y^2 = 16$ and is traversed in the clockwise direction.

Answer

$\int_C xy^4\,ds = \dfrac{8192}{5}$

10. [T] Evaluate $\int_C 4x^3 ds$, where C is the line segment from $(-2,-1)$ to $(1,2)$.

For the following exercises, find the work done.

11. Find the work done by vector field $\vec{\mathbf{F}}(x,y,z) = x\,\hat{\mathbf{i}} + 3xy\,\hat{\mathbf{j}} - (x+z)\,\hat{\mathbf{k}}$ on a particle moving along a line segment that goes from $(1,4,2)$ to $(0,5,1)$.

Answer

$W = 8$ units of work

12. Find the work done by a person weighing 150 lb walking exactly one revolution up a circular, spiral staircase of radius 3 ft if the person rises 10 ft.

13. Find the work done by force field $\vec{\mathbf{F}}(x, y, z) = -\frac{1}{2}x\,\hat{\mathbf{i}} - \frac{1}{2}y\hat{\mathbf{j}} + \frac{1}{4}\,\hat{\mathbf{k}}$ on a particle as it moves along the helix $\vec{\mathbf{r}}(t) = \cos t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}} + t\,\hat{\mathbf{k}}$ from point $(1, 0, 0)$ to point $(-1, 0, 3\pi)$.

Answer

$W = \frac{3\pi}{4}$ units of work

14. Find the work done by vector field $\vec{\mathbf{F}}(x, y) = y\,\hat{\mathbf{i}} + 2x\,\hat{\mathbf{j}}$ in moving an object along path C, which joins points $(1, 0)$ and $(0, 1)$.

15. Find the work done by force $\vec{\mathbf{F}}(x, y) = 2y\,\hat{\mathbf{i}} + 3x\hat{\mathbf{j}} + (x + y)\,\hat{\mathbf{k}}$ in moving an object along curve $\vec{\mathbf{r}}(t) = \cos(t)\,\hat{\mathbf{i}} + \sin(t)\,\hat{\mathbf{j}} + 16\,\hat{\mathbf{k}}$, where $0 \le t \le 2\pi$.

Answer

$W = \pi$ units of work

16. Find the mass of a wire in the shape of a circle of radius 2 centered at (3, 4) with linear mass density $\rho(x, y) = y^2$.

For the following exercises, evaluate the line integrals.

17. Evaluate $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where $\vec{\mathbf{F}}(x, y) = -1\,\hat{\mathbf{j}}$, and C is the part of the graph of $y = \frac{1}{2}x^3 - x$ from $(2, 2)$ to $(-2, -2)$.

Answer

$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 4$ units of work

18. Evaluate $\int_\gamma (x^2 + y^2 + z^2)^{-1} ds$, where γ is the helix $x = \cos t, y = \sin t, z = t$, with $0 \le t \le T$.

19. Evaluate $\int_C yz\,dx + xz\,dy + xy\,dz$ over the line segment from $(1, 1, 1)$ to $(3, 2, 0)$.

Answer

$\int_C yz\,dx + xz\,dy + xy\,dz = -1$

20. Let C be the line segment from point (0, 1, 1) to point (2, 2, 3). Evaluate line integral $\int_C y\,ds$.

21. [T] Use a computer algebra system to evaluate the line integral $\int_C y^2\,dx + x\,dy$, where C is the arc of the parabola $x = 4 - y^2$ from $(-5, -3)$ to $(0, 2)$.

Answer

$\int_C (y^2)\,dx + (x)\,dy = \frac{245}{6}$

22. [T] Use a computer algebra system to evaluate the line integral $\int_C (x + 3y^2)\,dy$ over the path C given by $x = 2t, y = 10t$, where $0 \le t \le 1$.

23. [T] Use a CAS to evaluate line integral $\int_C xy\,dx + y\,dy$ over path C given by $x = 2t, y = 10t$, where $0 \le t \le 1$.

Answer

$\int_C xy\,dx + y\,dy = \frac{190}{3}$

24. Evaluate line integral $\int_C (2x - y)\,dx + (x + 3y)\,dy$, where C lies along the x-axis from $x = 0$ to $x = 5$.

26. [T] Use a CAS to evaluate $\int_C \frac{y}{2x^2 - y^2}\,ds$, where C is defined by the parametric equations $x = t, y = t$, for $1 \le t \le 5$.

Answer

$$\int_C \frac{y}{2x^2 - y^2}\,ds = \sqrt{2}\ln 5$$

27. [T] Use a CAS to evaluate $\int_C xy\,ds$, where C is defined by the parametric equations $x = t^2, y = 4t$, for $0 \le t \le 1$.

In the following exercises, find the work done by force field $\vec{\mathbf{F}}$ on an object moving along the indicated path.

28. $\vec{\mathbf{F}}(x, y) = -x\,\hat{\mathbf{i}} - 2y\,\hat{\mathbf{j}}$

$C : y = x^3$ from $(0, 0)$ to $(2, 8)$

Answer

$W = -66$ units of work

29. $\vec{\mathbf{F}}(x, y) = 2x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}$

$<C$: counterclockwise around the triangle with vertices $(0, 0), (1, 0)$, and $(1, 1)$

30. $\vec{\mathbf{F}}(x, y, z) = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} - 5z\,\hat{\mathbf{k}}$

$C : \vec{\mathbf{r}}(t) = 2\cos t\,\hat{\mathbf{i}} + 2\sin t\,\hat{\mathbf{j}} + t\,\hat{\mathbf{k}},\ 0 \le t \le 2\pi$

Answer

$W = -10\pi^2$ units of work

31. Let $\vec{\mathbf{F}}$ be vector field $\vec{\mathbf{F}}(x, y) = (y^2 + 2xe^y + 1)\,\hat{\mathbf{i}} + (2xy + x^2e^y + 2y)\,\hat{\mathbf{j}}$. Compute the work of integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where C is the path $\vec{\mathbf{r}}(t) = \sin t\,\hat{\mathbf{i}} + \cos t\,\hat{\mathbf{j}}, \quad 0 \le t \le \frac{\pi}{2}$.

32. Compute the work done by force $\vec{\mathbf{F}}(x, y, z) = 2x\,\hat{\mathbf{i}} + 3y\,\hat{\mathbf{j}} - z\,\hat{\mathbf{k}}$ along path $\vec{\mathbf{r}}(t) = t\,\hat{\mathbf{i}} + t^2\,\hat{\mathbf{j}} + t^3\,\hat{\mathbf{k}}$, where $0 \le t \le 1$.

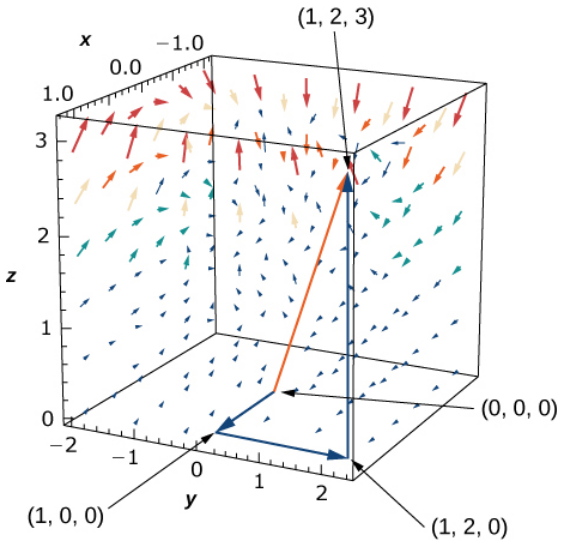
Answer

$W = 2$ units of work

33. Evaluate $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where $\vec{\mathbf{F}}(x, y) = \frac{1}{x + y}\,\hat{\mathbf{i}} + \frac{1}{x + y}\,\hat{\mathbf{j}}$ and C is the segment of the unit circle going counterclockwise from $(1, 0)$ to $(0, 1)$.

34. Force $\vec{\mathbf{F}}(x, y, z) = zy\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}} + z^2x\,\hat{\mathbf{k}}$ acts on a particle that travels from the origin to point $(1, 2, 3)$. Calculate the work done if the particle travels:

a. along the path $(0, 0, 0) \to (1, 0, 0) \to (1, 2, 0) \to (1, 2, 3)$ along straight-line segments joining each pair of endpoints;
b. along the straight line joining the initial and final points.
c. Is the work the same along the two paths?



Answer

a. $W = 11$ units of work;

b. $W = \frac{39}{4} = 9\frac{3}{4}$ units of work;

c. No

35. Find the work done by vector field $\vec{\mathbf{F}}(x, y, z) = x\,\hat{\mathbf{i}} + 3xy\,\hat{\mathbf{j}} - (x + z)\,\hat{\mathbf{k}}$ on a particle moving along a line segment that goes from $(1, 4, 2)$ to $(0, 5, 1)$.

36. How much work is required to move an object in vector field $\vec{\mathbf{F}}(x, y) = y\,\hat{\mathbf{i}} + 3x\,\hat{\mathbf{j}}$ along the upper part of ellipse $\frac{x^2}{4} + y^2 = 1$ from $(2, 0)$ to $(-2, 0)$?

Answer

$W = 2\pi$ units of work

37. A vector field is given by $\vec{\mathbf{F}}(x, y) = (2x + 3y)\,\hat{\mathbf{i}} + (3x + 2y)\,\hat{\mathbf{j}}$. Evaluate the line integral of the field around a circle of unit radius traversed in a clockwise fashion.

38. Evaluate the line integral of scalar function xy along parabolic path $y = x^2$ connecting the origin to point $(1, 1)$.

Answer

$$\int_C f\,ds = \frac{25\sqrt{5} + 1}{120}$$

39. Find $\int_C y^2\,dx + (xy - x^2)\,dy$ along $C : y = 3x$ from $(0, 0)$ to $(1, 3)$.

40. Find $\int_C y^2\,dx + (xy - x^2)\,dy$ along $C : y^2 = 9x$ from $(0, 0)$ to $(1, 3)$.

Answer

$$\int_C y^2\,dx + (xy - x^2)\,dy = 6.15$$

For the following exercises, use a CAS to evaluate the given line integrals.

41. [T] Evaluate $\vec{\mathbf{F}}(x, y, z) = x^2 z\,\hat{\mathbf{i}} + 6y\,\hat{\mathbf{j}} + yz^2\,\hat{\mathbf{k}}$, where C is represented by $\vec{\mathbf{r}}(t) = t\,\hat{\mathbf{i}} + t^2\,\hat{\mathbf{j}} + \ln t\,\hat{\mathbf{k}}, 1 \le t \le 3$.

42. [T] Evaluate line integral $\int_\gamma xe^y\,ds$ where, γ is the arc of curve $x = e^y$ from $(1, 0)$ to $(e, 1)$.

Answer

$$\int_\gamma xe^y\,ds \approx 7.157$$

43. [T] Evaluate the integral $\int_\gamma xy^2\,ds$, where γ is a triangle with vertices $(0,1,2),(1,0,3)$, and $(0,-1,0)$.

44. [T] Evaluate line integral $\int_\gamma (y^2-xy)\,dx$, where γ is curve $y=\ln x$ from $(1,0)$ toward $(e,1)$.

Answer

$$\int_\gamma (y^2-xy)\,dx \approx -1.379$$

45. [T] Evaluate line integral $\int_\gamma xy^4\,ds$, where γ is the right half of circle $x^2+y^2=16$.

46. [T] Evaluate $\int C\vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}, \int C\vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}$, where $\vec{\mathbf{F}}(x,y,z)=x^2y\,\hat{\mathbf{i}}+(x-z)\,\hat{\mathbf{j}}+xyz\,\hat{\mathbf{k}}$ and

$C:\vec{\mathbf{r}}(t)=t\,\hat{\mathbf{i}}+t^2\,\hat{\mathbf{j}}+2\,\hat{\mathbf{k}}, 0\le t\le 1$.

Answer

$\int_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} \approx -1.133$ units of work

47. Evaluate $\int_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}$, where $\vec{\mathbf{F}}(x,y)=2x\sin y\,\hat{\mathbf{i}}+(x^2\cos y-3y^2)\,\hat{\mathbf{j}}$ and

C is any path from $(-1,0)$ to $(5,1)$.

48. Find the line integral of $\vec{\mathbf{F}}(x,y,z)=12x^2\,\hat{\mathbf{i}}-5xy\hat{\mathbf{j}}+xz\,\hat{\mathbf{k}}$ over path C defined by $y=x^2, z=x^3$ from point $(0,0,0)$ to point $(2,4,8)$.

Answer

$\int_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} \approx 22.857$ units of work

49. Find the line integral of $\int_C (1+x^2y)\,ds$, where C is ellipse $\vec{\mathbf{r}}(t)=2\cos t\,\hat{\mathbf{i}}+3\sin t\,\hat{\mathbf{j}}$ from $0\le t\le \pi$.

For the following exercises, find the flux.

50. Compute the flux of $\vec{\mathbf{F}}=x^2\,\hat{\mathbf{i}}+y\,\hat{\mathbf{j}}$ across a line segment from $(0,0)$ to $(1,2)$.

Answer

$\text{flux}=-\frac{1}{3}$

51. Let $\vec{\mathbf{F}}=5\,\hat{\mathbf{i}}$ and let C be curve $y=0$, with $0\le x\le 4$. Find the flux across C.

52. Let $\vec{\mathbf{F}}=5\,\hat{\mathbf{j}}$ and let C be curve $y=0$, with $0\le x\le 4$. Find the flux across C.

Answer

$\text{flux}=-20$

53. Let $\vec{\mathbf{F}}=-y\,\hat{\mathbf{i}}+x\,\hat{\mathbf{j}}$ and let $C:\vec{\mathbf{r}}(t)=\cos t\,\hat{\mathbf{i}}+\sin t\,\hat{\mathbf{j}}$ for $0\le t\le 2\pi$. Calculate the flux across C.

54. Let $\vec{\mathbf{F}}=(x^2+y^3)\,\hat{\mathbf{i}}+(2xy)\,\hat{\mathbf{j}}$. Calculate flux $\vec{\mathbf{F}}$ orientated counterclockwise across the curve $C:x^2+y^2=9$.

Answer

$\text{flux}=0$

Complete the rest of the exercises as stated.

55. Find the line integral of $\int_C z^2\,dx+y\,dy+2y\,dz$, where C consists of two parts: C_1 and C_2. C_1 is the intersection of cylinder $x^2+y^2=16$ and plane $z=3$ from $(0,4,3)$ to $(-4,0,3)$. C_2 is a line segment from $(-4,0,3)$ to $(0,1,5)$.

56. A spring is made of a thin wire twisted into the shape of a circular helix $x = 2\cos t,\ y = 2\sin t,\ z = t$. Find the mass of two turns of the spring if the wire has a constant mass density of ρ grams per cm.

Answer

$m = 4\pi\rho\sqrt{5}$ grams

57. A thin wire is bent into the shape of a semicircle of radius a. If the linear mass density at point P is directly proportional to its distance from the line through the endpoints, find the mass of the wire.

58. An object moves in force field $\vec{\mathbf{F}}(x, y, z) = y^2\,\hat{\mathbf{i}} + 2(x+1)y\,\hat{\mathbf{j}}$ counterclockwise from point $(2, 0)$ along elliptical path $x^2 + 4y^2 = 4$ to $(-2, 0)$, and back to point $(2, 0)$ along the x-axis. How much work is done by the force field on the object?

Answer

$W = 0$ units of work

59. Find the work done when an object moves in force field $\vec{\mathbf{F}}(x, y, z) = 2x\,\hat{\mathbf{i}} - (x+z)\,\hat{\mathbf{j}} + (y-x)\,\hat{\mathbf{k}}$ along the path given by $\vec{\mathbf{r}}(t) = t^2\,\hat{\mathbf{i}} + (t^2 - t)\,\hat{\mathbf{j}} + 3\,\hat{\mathbf{k}},\ 0 \le t \le 1$.

60. If an inverse force field $\vec{\mathbf{F}}$ is given by $\vec{\mathbf{F}}(x, y, z) = \dfrac{k}{\|r\|^3} r$, where k is a constant, find the work done by $\vec{\mathbf{F}}$ as its point of application moves along the x-axis from $A(1, 0, 0)$ to $B(2, 0, 0)$.

Answer

$W = \frac{k}{2}$ units of work

61. David and Sandra plan to evaluate line integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ along a path in the xy-plane from $(0, 0)$ to $(1, 1)$. The force field is $\vec{\mathbf{F}}(x, y) = (x + 2y)\,\hat{\mathbf{i}} + (-x + y^2)\,\hat{\mathbf{j}}$. David chooses the path that runs along the x-axis from $(0, 0)$ to $(1, 0)$ and then runs along the vertical line $x = 1$ from $(1, 0)$ to the final point $(1, 1)$. Sandra chooses the direct path along the diagonal line $y = x$ from $(0, 0)$ to $(1, 1)$. Whose line integral is larger and by how much?

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15.3: Conservative Vector Fields

Learning Objectives

- Describe simple and closed curves; define connected and simply connected regions.
- Explain how to find a potential function for a conservative vector field.
- Use the Fundamental Theorem for Line Integrals to evaluate a line integral in a vector field.
- Explain how to test a vector field to determine whether it is conservative.

In this section, we continue the study of conservative vector fields. We examine the Fundamental Theorem for Line Integrals, which is a useful generalization of the Fundamental Theorem of Calculus to line integrals of conservative vector fields. We also discover show how to test whether a given vector field is conservative, and determine how to build a potential function for a vector field known to be conservative.

Curves and Regions

Before continuing our study of conservative vector fields, we need some geometric definitions. The theorems in the subsequent sections all rely on integrating over certain kinds of curves and regions, so we develop the definitions of those curves and regions here. We first define two special kinds of curves: closed curves and simple curves. As we have learned, a closed curve is one that begins and ends at the same point. A simple curve is one that does not cross itself. A curve that is both closed and simple is a simple closed curve (Figure 15.3.1).

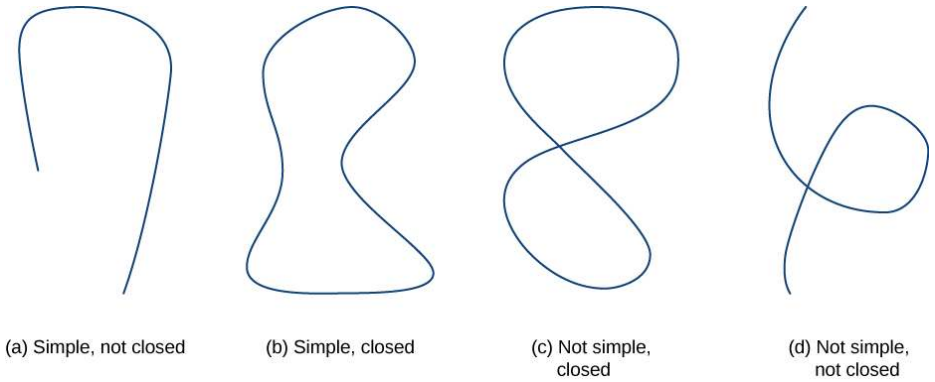


Figure 15.3.1. Types of curves that are simple or not simple and closed or not closed.

DEFINITION: Closed Curves

Curve C is a *closed curve* if there is a parameterization $\vec{\mathbf{r}}(t)$, $a \le t \le b$ of C such that the parameterization traverses the curve exactly once and $\vec{\mathbf{r}}(a) = \vec{\mathbf{r}}(b)$. Curve C is a simple curve if C does not cross itself. That is, C is simple if there exists a parameterization $\vec{\mathbf{r}}(t)$, $a \le t \le b$ of C such that $\vec{\mathbf{r}}$ is one-to-one over (a, b). It is possible for $\vec{\mathbf{r}}(a) = \vec{\mathbf{r}}(b)$, meaning that the simple curve is also closed.

Example 15.3.1: Determining Whether a Curve Is Simple and Closed

Is the curve with parameterization $\vec{\mathbf{r}}(t) = \left\langle \cos t, \frac{\sin(2t)}{2} \right\rangle$, $0 \le t \le 2\pi$ a simple closed curve?

Solution

Note that $\vec{\mathbf{r}}(0) = \langle 1, 0 \rangle = \vec{\mathbf{r}}(2\pi)$; therefore, the curve is closed. The curve is not simple, however. To see this, note that $\vec{\mathbf{r}}\left(\frac{\pi}{2}\right) = \langle 0, 0 \rangle = \vec{\mathbf{r}}\left(\frac{3\pi}{2}\right)$, and therefore the curve crosses itself at the origin (Figure 15.3.2).

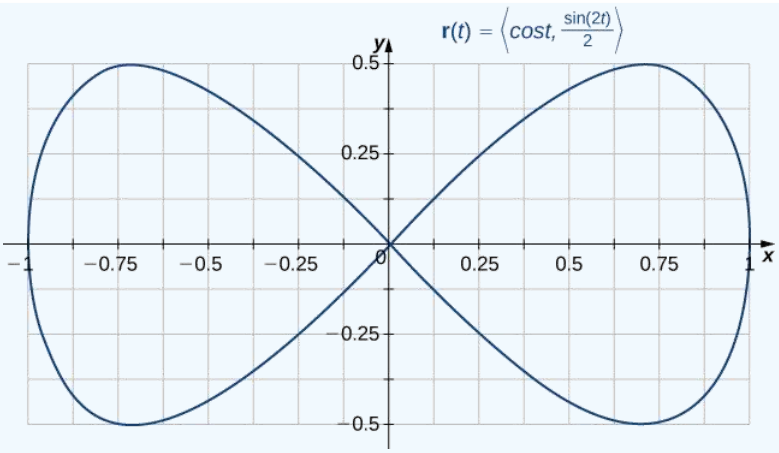


Figure **15.3.2.** *A curve that is closed but not simple.*

? Exercise 15.3.1

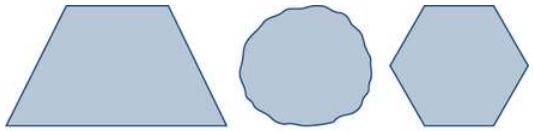
Is the curve given by parameterization $\vec{\mathbf{r}}(t) = \langle 2\cos t, 3\sin t\rangle$, $0 \le t \le 6\pi$, a simple closed curve?

Many of the theorems in this chapter relate an integral over a region to an integral over the boundary of the region, where the region's boundary is a simple closed curve or a union of simple closed curves. To develop these theorems, we need two geometric definitions for regions: that of a connected region and that of a simply connected region. A connected region is one in which there is a path in the region that connects any two points that lie within that region. A simply connected region is a connected region that does not have any holes in it. These two notions, along with the notion of a simple closed curve, allow us to state several generalizations of the Fundamental Theorem of Calculus later in the chapter. These two definitions are valid for regions in any number of dimensions, but we are only concerned with regions in two or three dimensions.

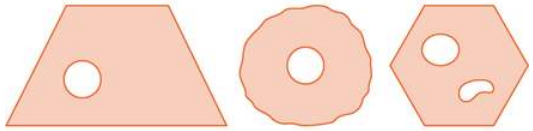
DEFINITION: connected regions

A region D is a *connected region* if, for any two points P_1 and P_2, there is a path from P_1 to P_2 with a trace contained entirely inside D. A region D is a simply connected region if D is connected for any simple closed curve C that lies inside D, and curve C can be shrunk continuously to a point while staying entirely inside D. In two dimensions, a region is simply connected if it is connected and has no holes.

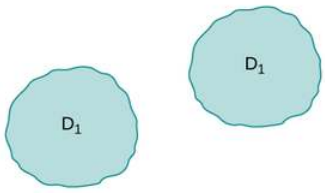
All simply connected regions are connected, but not all connected regions are simply connected (Figure **15.3.3**).



(a) Simply connected regions



(b) Connected regions that are not simply connected

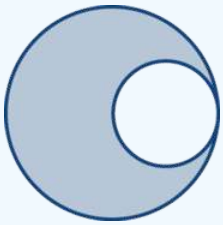


(c) A region that is not connected

Figure 15.3.3: Not all connected regions are simply connected. (a) Simply connected regions have no holes. (b) Connected regions that are not simply connected may have holes but you can still find a path in the region between any two points. (c) A region that is not connected has some points that cannot be connected by a path in the region.

? Exercise 15.3.2

Is the region in the below image connected? Is the region simply connected?



Fundamental Theorem for Line Integrals

Now that we understand some basic curves and regions, let's generalize the Fundamental Theorem of Calculus to line integrals. Recall that the Fundamental Theorem of Calculus says that if a function f has an antiderivative F, then the integral of f from a to b depends only on the values of F at a and at b—that is,

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

If we think of the gradient as a derivative, then the same theorem holds for vector line integrals. We show how this works using a motivational example.

✓ Example 15.3.2: Evaluating a Line Integral and the Antiderivatives of the Endpoints

Let $\vec{\mathbf{F}}(x,y) = \langle 2x, 4y\rangle$. Calculate $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where C is the line segment from $(0,0)$ to $(2,2)$ (Figure 15.3.4).

Solution

We use the method from the previous section to calculate $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$. Curve C can be parameterized by $\vec{\mathbf{r}}(t) = \langle 2t, 2t\rangle$, $0 \le t \le 1$. Then, $\vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) = \langle 4t, 8t\rangle$ and $\vec{\mathbf{r}}'(t) = \langle 2, 2\rangle$, which implies that

$$\begin{aligned}\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_0^1 \langle 4t, 8t\rangle \cdot \langle 2, 2\rangle dt \\ &= \int_0^1 (8t + 16t)dt = \int_0^1 24t dt \\ &= \left[12t^2\right]_0^1 = 12.\end{aligned}$$

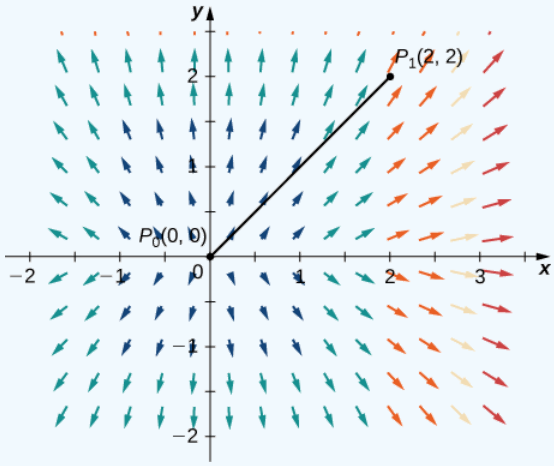


Figure 15.3.4: The value of line integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ *depends only on the value of the potential function of* $\vec{\mathbf{F}}$ *at the endpoints of the curve.*

Notice that $\vec{\mathbf{F}} = \vec{\nabla} f$, where $f(x, y) = x^2 + 2y^2$. If we think of the gradient as a derivative, then f is an "antiderivative" of $\vec{\mathbf{F}}$. In the case of single-variable integrals, the integral of derivative $g'(x)$ is $g(b) - g(a)$, where a is the start point of the interval of integration and b is the endpoint. If vector line integrals work like single-variable integrals, then we would expect integral $\vec{\mathbf{F}}$ to be $f(P_1) - f(P_0)$, where P_1 is the endpoint of the curve of integration and P_0 is the start point. Notice that this is the case for this example:

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C \vec{\nabla} f \cdot d\vec{\mathbf{r}} = 12$$

and

$$f(2, 2) - f(0, 0) = 4 + 8 - 0 = 12.$$

In other words, the integral of a "derivative" can be calculated by evaluating an "antiderivative" at the endpoints of the curve and subtracting, just as for single-variable integrals.

The following theorem says that, under certain conditions, what happened in the previous example holds for any gradient field. The same theorem holds for vector line integrals, which we call the **Fundamental Theorem for Line Integrals**.

Theorem: THE FUNDAMENTAL THEOREM FOR LINE INTEGRALS

Let C be a piecewise smooth curve with parameterization $\vec{\mathbf{r}}(t)$, $a \le t \le b$. Let f be a function of two or three variables with first-order partial derivatives that exist and are continuous on C. Then,

$$\int_C \vec{\nabla} f \cdot d\vec{\mathbf{r}} = f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(a)). \tag{15.3.1}$$

Proof

First,

$$\int_C \vec{\nabla} f \cdot d\vec{\mathbf{r}} = \int_a^b \vec{\nabla} f(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}'(t)\, dt.$$

By the chain rule,

$$\frac{d}{dt}(f(\vec{\mathbf{r}}(t)) = \vec{\nabla} f(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}'(t)$$

Therefore, by the Fundamental Theorem of Calculus,

$$\begin{aligned}\int_C \vec{\nabla} f \cdot d\vec{\mathbf{r}} &= \int_a^b \vec{\nabla} f(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}'(t) dt \\ &= \int_a^b \frac{d}{dt}(f(\vec{\mathbf{r}}(t)) dt \\ &= \left[f(\vec{\mathbf{r}}(t))\right]_{t=a}^{t=b} \\ &= f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(a)).\end{aligned}$$

□

We know that if $\vec{\mathbf{F}}$ is a conservative vector field, there is a potential function f such that $\vec{\nabla} f = \vec{\mathbf{F}}$. Therefore

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C \vec{\nabla} f \cdot d\vec{\mathbf{r}} = f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(a)).$$

In other words, just as with the Fundamental Theorem of Calculus, computing the line integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where $\vec{\mathbf{F}}$ is conservative, is a two-step process:

1. Find a potential function ("antiderivative") f for $\vec{\mathbf{F}}$ and
2. Compute the value of f at the endpoints of C and calculate their difference $f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(a))$.

Keep in mind, however, there is one major difference between the Fundamental Theorem of Calculus and the Fundamental Theorem for Line Integrals:
A function of one variable that is continuous must have an antiderivative. However, a vector field, even if it is continuous, does not need to have a potential function.

Example 15.3.3: Applying the Fundamental Theorem

Calculate integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where $\vec{\mathbf{F}}(x, y, z) = \langle 2x \ln y, \frac{x^2}{y} + z^2, 2yz \rangle$ and C is a curve with parameterization $\vec{\mathbf{r}}(t) = \langle t^2, t, t \rangle$, $1 \leq t \leq e$

a. without using the Fundamental Theorem of Line Integrals and
b. using the Fundamental Theorem of Line Integrals.

Solution

1. First, let's calculate the integral without the Fundamental Theorem for Line Integrals and instead use the method we learned in the previous section:

$$\begin{aligned}\int_C \vec{\mathbf{F}} \cdot dr &= \int_1^e \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}'(t)\, dt \\ &= \int_1^e \langle 2t^2 \ln t, \frac{t^4}{t} + t^2, 2t^2 \rangle \cdot \langle 2t, 1, 1 \rangle\, dt \\ &= \int_1^e (4t^3 \ln t + t^3 + 3t^2)\, dt \\ &= \int_1^e 4t^3 \ln t\, dt + \int_1^e (t^3 + 3t^2)\, dt \\ &= \int_1^e 4t^3 \ln t\, dt + \left[\frac{t^4}{4} + t^3\right]_1^e \\ &= \int_1^e 4t^3 \ln t\, dt + \frac{e^4}{4} + e^3 - \frac{1}{4} - 1 \\ &= \int_1^e 4t^3 \ln t\, dt + \frac{e^4}{4} + e^3 - \frac{5}{4}\end{aligned}$$

Integral $\int_1^e t^3 \ln t \, dt$ requires integration by parts. Let $u = \ln t$ and $dv = t^3$. Then $u = \ln t$, $dv = t^3$ and

$$du = \frac{1}{t}\,dt, \quad v = \frac{t^4}{4}.$$

Therefore,

$$\begin{aligned}\int_1^e t^3 \ln t \, dt &= \left[\frac{t^4}{4}\ln t\right]_1^e - \frac{1}{4}\int_1^e t^3\,dt \\ &= \frac{e^4}{4} - \frac{1}{4}\left(\frac{e^4}{4} - \frac{1}{4}\right).\end{aligned}$$

Thus,

$$\begin{aligned}\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 4\int_1^e t^3 \ln t \, dt \quad &+ \quad \frac{e^4}{4} + e^3 - \frac{5}{4} \\ &= 4\left(\frac{e^4}{4} - \frac{1}{4}\left(\frac{e^4}{4} - \frac{1}{4}\right)\right) + \frac{e^4}{4} + e^3 - \frac{5}{4} \\ &= e^4 - \frac{e^4}{4} + \frac{1}{4} + \frac{e^4}{4} + e^3 - \frac{5}{4} \\ &= e^4 + e^3 - 1.\end{aligned}$$

2. Given that $f(x,y,z) = x^2 \ln y + yz^2$ is a potential function for $\vec{\mathbf{F}}$, let's use the Fundamental Theorem for Line Integrals to calculate the integral. Note that

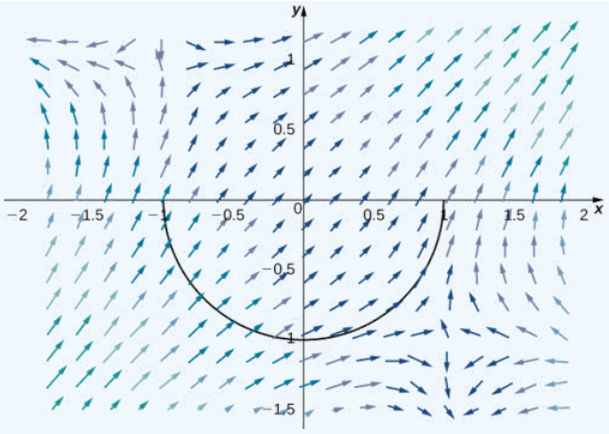
$$\begin{aligned}\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_C \vec{\nabla} f \cdot d\vec{\mathbf{r}} \\ &= f(\vec{\mathbf{r}}(e)) - f(\vec{\mathbf{r}}(1)) \\ &= f(e^2, e, e) - f(1,1,1) \\ &= e^4 + e^3 - 1.\end{aligned}$$

This calculation is much more straightforward than the calculation we did in (a). As long as we have a potential function, calculating a line integral using the Fundamental Theorem for Line Integrals is much easier than calculating without the theorem.

Example **15.3.3** illustrates a nice feature of the Fundamental Theorem of Line Integrals: it allows us to calculate more easily many vector line integrals. As long as we have a potential function, calculating the line integral is only a matter of evaluating the potential function at the endpoints and subtracting.

? Exercise 15.3.3

Given that $f(x,y) = (x-1)^2 y + (y+1)^2 x$ is a potential function for $\vec{\mathbf{F}}(x,y) = \langle 2xy - 2y + (y+1)^2, (x-1)^2 + 2yx + 2x\rangle$, calculate integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where C is the lower half of the unit circle oriented counterclockwise.



The Fundamental Theorem for Line Integrals has two important consequences. The first consequence is that if $\vec{\mathbf{F}}$ is conservative and C is a closed curve, then the circulation of $\vec{\mathbf{F}}$ along C is zero—that is, $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 0$. To see why this is true, let f be a potential function for $\vec{\mathbf{F}}$. Since C is a closed curve, the terminal point $\vec{\mathbf{r}}(b)$ of C is the same as the initial $\vec{\mathbf{r}}(a)$ of C—that is, $\vec{\mathbf{r}}(a) = \vec{\mathbf{r}}(b)$. Therefore, by the Fundamental Theorem for Line Integrals,

$$\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \oint_C \vec{\nabla} f \cdot d\vec{\mathbf{r}} \tag{15.3.2}$$

$$= f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(a)) \tag{15.3.3}$$

$$= f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(b)) \tag{15.3.4}$$

$$= 0. \tag{15.3.5}$$

Recall that the reason a conservative vector field $\vec{\mathbf{F}}$ is called "conservative" is because such vector fields model forces in which energy is conserved. We have shown gravity to be an example of such a force. If we think of vector field $\vec{\mathbf{F}}$ in integral $\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ as a gravitational field, then the equation $\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 0$ follows. If a particle travels along a path that starts and ends at the same place, then the work done by gravity on the particle is zero.

The second important consequence of the Fundamental Theorem for Line Integrals (Equation 15.3.1) is that line integrals of conservative vector fields are independent of path—meaning, they depend only on the endpoints of the given curve, and do not depend on the path between the endpoints.

DEFINITION: Path Independence

Let $\vec{\mathbf{F}}$ be a vector field with domain D; it is independent of path (or path independent) if

$$\int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$$

for any paths C_1 and C_2 in D with the same initial and terminal points.

The second consequence is stated formally in the following theorem.

Theorem: CONSERVATIVE FIELDS

If $\vec{\mathbf{F}}$ is a *conservative vector field*, then $\vec{\mathbf{F}}$ is independent of path.

Proof

Let D denote the domain of $\vec{\mathbf{F}}$ and let C_1 and C_2 be two paths in D with the same initial and terminal points (Figure 15.3.5). Call the initial point P_1 and the terminal point P_2. Since $\vec{\mathbf{F}}$ is conservative, there is a potential function f for $\vec{\mathbf{F}}$. By the Fundamental Theorem for Line Integrals,

$$\int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = f(P_2) - f(P_1) = \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}.$$

Therefore, $\int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ and $\vec{\mathbf{F}}$ is independent of path.

□

To visualize what independence of path means, imagine three hikers climbing from base camp to the top of a mountain. Hiker 1 takes a steep route directly from camp to the top. Hiker 2 takes a winding route that is not steep from camp to the top. Hiker 3 starts by taking the steep route but halfway to the top decides it is too difficult for him. Therefore he returns to camp and takes the non-steep path to the top. All three hikers are traveling along paths in a gravitational field. Since gravity is a force in which energy is conserved, the gravitational field is conservative. By independence of path, the total amount of work done by gravity on each of the hikers is the same because they all started in the same place and ended in the same place. The work done by the hikers includes other factors such as friction and muscle movement, so the total amount of energy each one expended is not the same, but the net energy expended against gravity is the same for all three hikers.

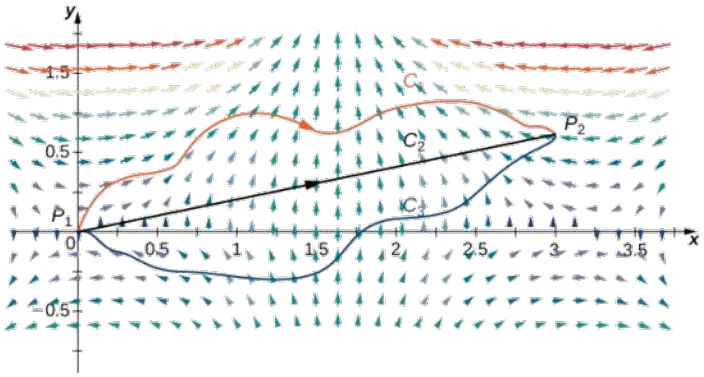


Figure **15.3.4**: The vector field is conservative, and therefore independent of path.

We have shown that if $\vec{\mathbf{F}}$ is conservative, then $\vec{\mathbf{F}}$ is independent of path. It turns out that if the domain of $\vec{\mathbf{F}}$ is open and connected, then the converse is also true. That is, if $\vec{\mathbf{F}}$ is independent of path and the domain of $\vec{\mathbf{F}}$ is open and connected, then $\vec{\mathbf{F}}$ is conservative. Therefore, the set of conservative vector fields on open and connected domains is precisely the set of vector fields independent of path.

Theorem: THE PATH INDEPENDENCE TEST FOR CONSERVATIVE FIELDS

If $\vec{\mathbf{F}}$ is a continuous vector field that is independent of path and the domain D of $\vec{\mathbf{F}}$ is open and connected, then $\vec{\mathbf{F}}$ is conservative.

Proof

We prove the theorem for vector fields in $\mathbb{R}^2$. The proof for vector fields in $\mathbb{R}^3$ is similar. To show that $\vec{\mathbf{F}} = \langle P, Q \rangle$ is conservative, we must find a potential function f for $\vec{\mathbf{F}}$. To that end, let X be a fixed point in D. For any point (x, y) in D, let C be a path from X to (x, y). Define $f(x, y)$ by $f(x, y) = \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$. (Note that this definition of f makes sense only because $\vec{\mathbf{F}}$ is independent of path. If $\vec{\mathbf{F}}$ was not independent of path, then it might be possible to find another path C' from X to (x, y) such that $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \neq \int_{C'} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, and in such a case $f(x, y)$ would not be a function.) We want to show that f has the property $\vec{\nabla} f = \vec{\mathbf{F}}$.

Since domain D is open, it is possible to find a disk centered at (x, y) such that the disk is contained entirely inside D. Let (a, y) with $a < x$ be a point in that disk. Let C be a path from X to (x, y) that consists of two pieces: C_1 and C_2. The first piece, C_1, is any path from C to (a, y) that stays inside D; C_2 is the horizontal line segment from (a, y) to (x, y) (Figure **15.3.6**). Then

$$f(x, y) = \int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}.$$

The first integral does not depend on x, so

$$f_x(x,y) = \frac{\partial}{\partial x}\int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}.$$

If we parameterize C_2 by $\vec{\mathbf{r}}(t) = \langle t, y\rangle$, $a \le t \le x$, then

$$\begin{aligned} f_x(x,y) &= \frac{\partial}{\partial x}\int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \\ &= \frac{\partial}{\partial x}\int_a^x \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}'(t)\, dt \\ &= \frac{\partial}{\partial x}\int_a^x \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \frac{d}{dt}(\langle t, y\rangle)\, dt \\ &= \frac{\partial}{\partial x}\int_a^x \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \langle 1, 0\rangle\, dt \\ &= \frac{\partial}{\partial x}\int_a^x P(t,y)\, dt. \end{aligned}$$

By the Fundamental Theorem of Calculus (part 1),

$$f_x(x,y) = \frac{\partial}{\partial x}\int_a^x P(t,y)\, dt = P(x,y).$$

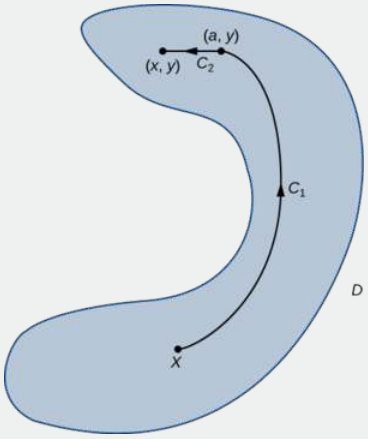


Figure 15.3.6. Here, C_1 is any path from C to (a,y) that stays inside D, and C_2 is the horizontal line segment from (a,y) to (x,y).

A similar argument using a vertical line segment rather than a horizontal line segment shows that $f_y(x,y) = Q(x,y)$.

Therefore $\vec{\nabla} f = \vec{\mathbf{F}}$ and $\vec{\mathbf{F}}$ is conservative.

□

We have spent a lot of time discussing and proving the theorems above, but we can summarize them simply: a vector field $\vec{\mathbf{F}}$ on an open and connected domain is conservative if and only if it is independent of path. This is important to know because conservative vector fields are extremely important in applications, and these theorems give us a different way of viewing what it means to be conservative using path independence.

✔ Example 15.3.4: Showing That a Vector Field Is Not Conservative

Use path independence to show that vector field $\vec{\mathbf{F}}(x,y) = \langle x^2 y, y + 5\rangle$ is not conservative.

Solution

We can indicate that $\vec{\mathbf{F}}$ is not conservative by showing that $\vec{\mathbf{F}}$ is not path independent. We do so by giving two different paths, C_1 and C_2, that both start at $(0,0)$ and end at $(1,1)$, and yet $\int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \ne \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$.

Let C_1 be the curve with parameterization $\vec{\mathbf{r}}_1(t) = \langle t,\ t\rangle$, $0 \le t \le 1$ and let C_2 be the curve with parameterization $\vec{\mathbf{r}}_2(t) = \langle t,\ t^2\rangle$, $0 \le t \le 1$ (Figure 15.3.7.). Then

$$\int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_0^1 \vec{\mathbf{F}}(\vec{\mathbf{r}}_1(t)) \cdot \vec{\mathbf{r}}_1{}'(t)\, dt$$
$$= \int_0^1 \langle t^3, t+5 \rangle \cdot \langle 1, 1 \rangle \, dt = \int_0^1 (t^3 + t + 5)\, dt$$
$$= \left[\frac{t^4}{4} + \frac{t^2}{2} + 5t \right]_0^1 = \frac{23}{4}$$

and

$$\int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_0^1 \vec{\mathbf{F}}(\vec{\mathbf{r}}_2(t)) \cdot \vec{\mathbf{r}}_2{}'(t)\, dt$$
$$= \int_0^1 \langle t^4, t^2+5 \rangle \cdot \langle 1, 2t \rangle \, dt = \int_0^1 (t^4 + 2t^3 + 10t)\, dt$$
$$= \left[\frac{t^5}{5} + \frac{t^4}{2} + 5t^2 \right]_0^1 = \frac{57}{10}.$$

Since $\int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \neq \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, the value of a line integral of $\vec{\mathbf{F}}$ depends on the path between two given points. Therefore, $\vec{\mathbf{F}}$ is not independent of path, and $\vec{\mathbf{F}}$ is not conservative.

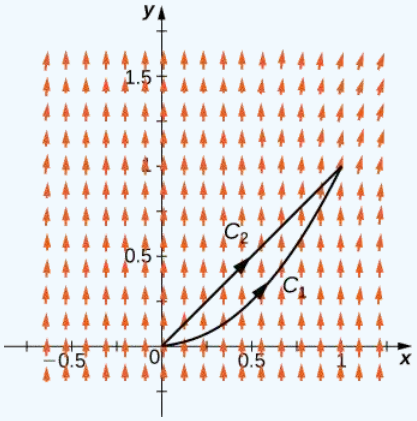


Figure 15.3.7: Curves C_1 and C_2 are both oriented from left to right.

? Exercise 15.3.4

Show that $\vec{\mathbf{F}}(x, y) = \langle xy,\ x^2y^2 \rangle$ is not path independent by considering the line segment from $(0, 0)$ to $(0, 2)$ and the piece of the graph of $y = \frac{x^2}{2}$ that goes from $(0, 0)$ to $(0, 2)$.

Conservative Vector Fields and Potential Functions

As we have learned, the Fundamental Theorem for Line Integrals says that if $\vec{\mathbf{F}}$ is conservative, then calculating $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ has two steps: first, find a potential function f for $\vec{\mathbf{F}}$ and, second, calculate $f(P_1) - f(P_0)$, where P_1 is the endpoint of C and P_0 is the starting point. To use this theorem for a conservative field $\vec{\mathbf{F}}$, we must be able to find a potential function f for $\vec{\mathbf{F}}$. Therefore, we must answer the following question: Given a conservative vector field $\vec{\mathbf{F}}$, how do we find a function f such that $\vec{\nabla} f = \vec{\mathbf{F}}$? Before giving a general method for finding a potential function, let's motivate the method with an example.

✓ Example 15.3.5: Finding a Potential Function

Find a potential function for $\vec{\mathbf{F}}(x, y) = \langle 2xy^3, 3x^2y^2 + \cos(y) \rangle$, thereby showing that $\vec{\mathbf{F}}$ is conservative.

Solution

Suppose that $f(x, y)$ is a potential function for $\vec{\mathbf{F}}$. Then, $\vec{\nabla} f = \vec{\mathbf{F}}$, and therefore

$$f_x(x, y) = 2xy^3 \quad \text{and} \quad f_y(x, y) = 3x^2y^2 + \cos y.$$

Integrating the equation $f_x(x,y) = 2xy^3$ with respect to x yields the equation

$$f(x,y) = x^2y^3 + h(y).$$

Notice that since we are integrating a two-variable function with respect to x, we must add a constant of integration that is a constant with respect to x, but may still be a function of y. The equation $f(x,y) = x^2y^3 + h(y)$ can be confirmed by taking the partial derivative with respect to x:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2y^3) + \frac{\partial}{\partial x}(h(y)) = 2xy^3 + 0 = 2xy^3.$$

Since f is a potential function for $\vec{\mathbf{F}}$,

$$f_y(x,y) = 3x^2y^2 + \cos(y),$$

and therefore

$$3x^2y^2 + g'(y) = 3x^2y^2 + \cos(y).$$

This implies that $h'(y) = \cos y$, so $h(y) = \sin y + C$. Therefore, *any* function of the form $f(x,y) = x^2y^3 + \sin(y) + C$ is a potential function. Taking, in particular, $C = 0$ gives the potential function $f(x,y) = x^2y^3 + \sin(y)$.

To verify that f is a potential function, note that $\vec{\nabla} f(x,y) = \langle 2xy^3, 3x^2y^2 + \cos y\rangle = \vec{\mathbf{F}}$.

? Exercise 15.3.5

Find a potential function for $\vec{\mathbf{F}}(x,y) = \langle e^x y^3 + y, 3e^x y^2 + x\rangle$.

The logic of the previous example extends to finding the potential function for any conservative vector field in $\mathbb{R}^2$. Thus, we have the following problem-solving strategy for finding potential functions:

PROBLEM-SOLVING STRATEGY: FINDING A POTENTIAL FUNCTION FOR A CONSERVATIVE VECTOR FIELD $\vec{\mathbf{F}}(x,y) = \langle P(x,y), Q(x,y)\rangle$

1. Integrate P with respect to x. This results in a function of the form $g(x,y) + h(y)$, where $h(y)$ is unknown.
2. Take the partial derivative of $g(x,y) + h(y)$ with respect to y, which results in the function $g_y(x,y) + h'(y)$.
3. Use the equation $g_y(x,y) + h'(y) = Q(x,y)$ to find $h'(y)$.
4. Integrate $h'(y)$ to find $h(y)$.
5. Any function of the form $f(x,y) = g(x,y) + h(y) + C$, where C is a constant, is a potential function for $\vec{\mathbf{F}}$.

We can adapt this strategy to find potential functions for vector fields in $\mathbb{R}^3$, as shown in the next example.

✓ Example 15.3.6: Finding a Potential Function in $\mathbb{R}^3$

Find a potential function for $F(x,y,z) = \langle 2xy, x^2 + 2yz^3, 3y^2z^2 + 2z\rangle$, thereby showing that $\vec{\mathbf{F}}$ is conservative.

Solution

Suppose that f is a potential function. Then, $\vec{\nabla} f = \vec{\mathbf{F}}$ and therefore $f_x(x,y,z) = 2xy$. Integrating this equation with respect to x yields the equation $f(x,y,z) = x^2y + g(y,z)$ for some function g. Notice that, in this case, the constant of integration with respect to x is a function of y and z.

Since f is a potential function,

$$x^2 + 2yz^3 = f_y(x,y,z) = x^2 + g_y(y,z).$$

Therefore,

$$g_y(y,z) = 2yz^3.$$

Integrating this function with respect to y yields

$$g(y,z) = y^2z^3 + h(z)$$

for some function $h(z)$ of z alone. (Notice that, because we know that g is a function of only y and z, we do not need to write $g(y,z) = y^2z^3 + h(x,z)$.) Therefore,

$$f(x,y,z) = x^2y + g(y,z) = x^2y + y^2z^3 + h(z).$$

To find f, we now must only find h. Since f is a potential function,

$$3y^2z^2 + 2z = g_z(y,z) = 3y^2z^2 + h'(z).$$

This implies that $h'(z) = 2z$, so $h(z) = z^2 + C$. Letting $C = 0$ gives the potential function

$$f(x,y,z) = x^2y + y^2z^3 + z^2.$$

To verify that f is a potential function, note that $\vec{\nabla} f(x,y,z) = \langle 2xy, x^2 + 2yz^3, 3y^2z^2 + 2z\rangle = \vec{\mathbf{F}}(x,y,z)$.

? Exercise 15.3.6

Find a potential function for $\vec{\mathbf{F}}(x,y,z) = \langle 12x^2, \cos y \cos z, 1 - \sin y \sin z\rangle$.

We can apply the process of finding a potential function to a gravitational force. Recall that, if an object has unit mass and is located at the origin, then the gravitational force in $\mathbb{R}^2$ that the object exerts on another object of unit mass at the point (x,y) is given by vector field

$$\vec{\mathbf{F}}(x,y) = -G\left\langle \frac{x}{(x^2+y^2)^{3/2}}, \frac{y}{(x^2+y^2)^{3/2}} \right\rangle,$$

where G is the universal gravitational constant. In the next example, we build a potential function for $\vec{\mathbf{F}}$, thus confirming what we already know: that gravity is conservative.

✓ Example 15.3.7: Finding a Potential Function

Find a potential function f for $\vec{\mathbf{F}}(x,y) = -G\left\langle \frac{x}{(x^2+y^2)^{3/2}}, \frac{y}{(x^2+y^2)^{3/2}} \right\rangle$.

Solution

Suppose that f is a potential function. Then, $\vec{\nabla} f = \vec{\mathbf{F}}$ and therefore

$$f_x(x,y) = \frac{-Gx}{(x^2+y^2)^{3/2}}.$$

To integrate this function with respect to x, we can use u-substitution. If $u = x^2 + y^2$, then $\frac{du}{2} = x\,dx$, so

$$\begin{aligned}\int \frac{-Gx}{(x^2+y^2)^{3/2}}\,dx &= \int \frac{-G}{2u^{3/2}}\,du \\ &= \frac{G}{\sqrt{u}} + h(y) \\ &= \frac{G}{\sqrt{x^2+y^2}} + h(y)\end{aligned}$$

for some function $h(y)$. Therefore,

$$f(x,y) = \frac{G}{\sqrt{x^2+y^2}} + h(y).$$

Since f is a potential function for $\vec{\mathbf{F}}$,

$$f_y(x,y) = \frac{-Gy}{(x^2+y^2)^{3/2}}$$

Since $f(x,y) = \dfrac{G}{\sqrt{x^2+y^2}} + h(y)$, $f_y(x,y)$ also equals $\dfrac{-Gy}{(x^2+y^2)^{3/2}} + h'(y)$.

Therefore,

$$\frac{-Gy}{(x^2+y^2)^{3/2}} + h'(y) = \frac{-Gy}{(x^2+y^2)^{3/2}},$$

which implies that $h'(y) = 0$. Thus, we can take $h(y)$ to be any constant; in particular, we can let $h(y) = 0$. The function

$$f(x,y) = \frac{G}{\sqrt{x^2+y^2}}$$

is a potential function for the gravitational field $\vec{\mathbf{F}}$. To confirm that f is a potential function, note that

$$\begin{aligned}\vec{\nabla} f(x,y) &= \left\langle -\frac{1}{2}\frac{G}{(x^2+y^2)^{3/2}}(2x), -\frac{1}{2}\frac{G}{(x^2+y^2)^{3/2}}(2y)\right\rangle \\ &= \left\langle \frac{-Gx}{(x^2+y^2)^{3/2}}, \frac{-Gy}{(x^2+y^2)^{3/2}}\right\rangle \\ &= \vec{\mathbf{F}}(x,y).\end{aligned}$$

? Exercise 15.3.7

Find a potential function f for the three-dimensional gravitational force $\vec{\mathbf{F}}(x,y,z) = \left\langle \dfrac{-Gx}{(x^2+y^2+z^2)^{3/2}}, \dfrac{-Gy}{(x^2+y^2+z^2)^{3/2}}, \dfrac{-Gz}{(x^2+y^2+z^2)^{3/2}} \right\rangle$.

Testing a Vector Field

Until now, we have worked with vector fields that we know are conservative, but if we are not told that a vector field is conservative, we need to be able to test whether it is conservative. Recall that, if $\vec{\mathbf{F}}$ is conservative, then $\vec{\mathbf{F}}$ has the cross-partial property (see The Cross-Partial Property of Conservative Vector Fields). That is, if $\vec{\mathbf{F}} = \langle P, Q, R\rangle$ is conservative, then $P_y = Q_x$, $P_z = R_x$, and $Q_z = R_y$. So, if $\vec{\mathbf{F}}$ has the cross-partial property, then is $\vec{\mathbf{F}}$ conservative? If the domain of $\vec{\mathbf{F}}$ is open and simply connected, then the answer is yes.

Theorem: THE CROSS-PARTIAL TEST FOR CONSERVATIVE FIELDS

If $\vec{\mathbf{F}} = \langle P, Q, R\rangle$ is a vector field on an open, simply connected region D and $P_y = Q_x$, $P_z = R_x$, and $Q_z = R_y$ throughout D, then $\vec{\mathbf{F}}$ is conservative.

Although a proof of this theorem is beyond the scope of the text, we can discover its power with some examples. Later, we see why it is necessary for the region to be simply connected.

Combining this theorem with the cross-partial property, we can determine whether a given vector field is conservative:

Theorem: CROSS-PARTIAL PROPERTY OF CONSERVATIVE FIELDS

Let $\vec{\mathbf{F}} = \langle P, Q, R\rangle$ be a vector field on an open, simply connected region D. Then $P_y = Q_x$, $P_z = R_x$, and $Q_z = R_y$ throughout D if and only if $\vec{\mathbf{F}}$ is conservative.

The version of this theorem in $\mathbb{R}^2$ is also true. If $\vec{\mathbf{F}}(x,y) = \langle P, Q\rangle$ is a vector field on an open, simply connected domain in $\mathbb{R}^2$, then $\vec{\mathbf{F}}$ is conservative if and only if $P_y = Q_x$.

Example 15.3.8: Determining Whether a Vector Field Is Conservative

Determine whether vector field $\vec{\mathbf{F}}(x,y,z) = \langle xy^2z, x^2yz, z^2\rangle$ is conservative.

Solution

Note that the domain of $\vec{\mathbf{F}}$ is all of $\mathbb{R}^2$ and $\mathbb{R}^3$ is simply connected. Therefore, we can use **The Cross-Partial Property of Conservative Vector Fields** to determine whether $\vec{\mathbf{F}}$ is conservative. Let

$$P(x,y,z) = xy^2z$$

$$Q(x,y,z) = x^2yz$$

and

$$R(x,y,z) = z^2.$$

Since $Q_z(x,y,z) = x^2y$ and $R_y(x,y,z) = 0$, the vector field is not conservative.

Example 15.3.9: Determining Whether a Vector Field Is Conservative

Determine vector field $\vec{\mathbf{F}}(x,y) = \langle x\ln(y), \dfrac{x^2}{2y}\rangle$ is conservative.

Solution

Note that the domain of $\vec{\mathbf{F}}$ is the part of $\mathbb{R}^2$ in which $y > 0$. Thus, the domain of $\vec{\mathbf{F}}$ is part of a plane above the x-axis, and this domain is simply connected (there are no holes in this region and this region is connected). Therefore, we can use **The Cross-Partial Property of Conservative Vector Fields** to determine whether $\vec{\mathbf{F}}$ is conservative. Let

$$P(x,y) = x\ln(y) \quad \text{and} \quad Q(x,y) = \frac{x^2}{2y}.$$

Then $P_y(x,y) = \dfrac{x}{y} = Q_x(x,y)$ and thus $\vec{\mathbf{F}}$ is conservative.

Exercise 15.3.8

Determine whether $\vec{\mathbf{F}}(x,y) = \langle \sin x\cos y,\ \cos x\sin y\rangle$ is conservative.

When using **The Cross-Partial Property of Conservative Vector Fields**, it is important to remember that a theorem is a tool, and like any tool, it can be applied only under the right conditions. In the case of **The Cross-Partial Property of Conservative Vector Fields**, the theorem can be applied only if the domain of the vector field is simply connected.

To see what can go wrong when misapplying the theorem, consider the vector field from Example **15.3.4**:

$$\vec{\mathbf{F}}(x,y) = \frac{y}{x^2+y^2}\,\hat{\mathbf{i}} + \frac{-x}{x^2+y^2}\,\hat{\mathbf{j}}.$$

This vector field satisfies the cross-partial property, since

$$\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) = \frac{(x^2+y^2)-y(2y)}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

and

$$\frac{\partial}{\partial x}\left(\frac{-x}{x^2+y^2}\right) = \frac{-(x^2+y^2)+x(2x)}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}.$$

Since $\vec{\mathbf{F}}$ satisfies the cross-partial property, we might be tempted to conclude that $\vec{\mathbf{F}}$ is conservative. However, $\vec{\mathbf{F}}$ is not conservative. To see this, let

$$\vec{\mathbf{r}}(t) = \langle \cos t, \sin t\rangle, \quad 0 \le t \le \pi$$

be a parameterization of the upper half of a unit circle oriented counterclockwise (denote this C_1) and let

$$\vec{\mathbf{s}}(t) = \langle \cos t, -\sin t \rangle, \quad 0 \le t \le \pi$$

be a parameterization of the lower half of a unit circle oriented clockwise (denote this C_2). Notice that C_1 and C_2 have the same starting point and endpoint. Since $\sin^2 t + \cos^2 t = 1$,

$$\vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}'(t) = \langle \sin(t), -\cos(t) \rangle \cdot \langle -\sin(t), \cos(t) \rangle = -1$$

and

$$\vec{\mathbf{F}}(\vec{\mathbf{s}}(t)) \cdot \vec{\mathbf{s}}'(t) = \langle -\sin t, -\cos t \rangle \cdot \langle -\sin t, -\cos t \rangle = \sin^2 t + \cos^2 t = 1.$$

Therefore,

$$\int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_0^{\pi} -1 \, dt = -\pi$$

and

$$\int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_0^{\pi} 1 \, dt = \pi.$$

Thus, C_1 and C_2 have the same starting point and endpoint, but $\int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \neq \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$. Therefore, $\vec{\mathbf{F}}$ is not independent of path and $\vec{\mathbf{F}}$ is not conservative.

To summarize: $\vec{\mathbf{F}}$ satisfies the cross-partial property and yet $\vec{\mathbf{F}}$ is not conservative. What went wrong? Does this contradict **The Cross-Partial Property of Conservative Vector Fields**? The issue is that the domain of $\vec{\mathbf{F}}$ is all of $\mathbb{R}^2$ except for the origin. In other words, the domain of $\vec{\mathbf{F}}$ has a hole at the origin, and therefore the domain is not simply connected. Since the domain is not simply connected, **The Cross-Partial Property of Conservative Vector Fields** does not apply to $\vec{\mathbf{F}}$.

We close this section by looking at an example of the usefulness of the Fundamental Theorem for Line Integrals. Now that we can test whether a vector field is conservative, we can always decide whether the Fundamental Theorem for Line Integrals can be used to calculate a vector line integral. If we are asked to calculate an integral of the form $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, then our first question should be: Is $\vec{\mathbf{F}}$ conservative? If the answer is yes, then we should find a potential function and use the Fundamental Theorem for Line Integrals to calculate the integral. If the answer is no, then the Fundamental Theorem for Line Integrals cannot help us and we have to use other methods, such as using the method from the previous section (using $\vec{\mathbf{F}}(\vec{\mathbf{r}}(t))$ and $\vec{\mathbf{r}}'(t)$).

Example 15.3.10: Using the Fundamental Theorem for Line Integrals

Calculate line integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where $\vec{\mathbf{F}}(x, y, z) = \langle 2xe^y z + e^x z, \; x^2 e^y z, \; x^2 e^y + e^x \rangle$ and C is any smooth curve that goes from the origin to $(1, 1, 1)$.

Solution

Before trying to compute the integral, we need to determine whether $\vec{\mathbf{F}}$ is conservative and whether the domain of $\vec{\mathbf{F}}$ is simply connected. The domain of $\vec{\mathbf{F}}$ is all of $\mathbb{R}^3$, which is connected and has no holes. Therefore, the domain of $\vec{\mathbf{F}}$ is simply connected. Let

$$P(x, y, z) = 2xe^y z + e^x z, \quad Q(x, y, z) = x^2 e^y z, \quad \text{and} \quad R(x, y, z) = x^2 e^y + e^x$$

so that $\vec{\mathbf{F}}(x, y, z) = \langle P, Q, R \rangle$. Since the domain of $\vec{\mathbf{F}}$ is simply connected, we can check the cross partials to determine whether $\vec{\mathbf{F}}$ is conservative. Note that

$$\begin{aligned} P_y(x, y, z) &= 2xe^y z = Q_x(x, y, z) \\ P_z(x, y, z) &= 2xe^y + e^x = R_x(x, y, z) \\ Q_z(x, y, z) &= x^2 e^y = R_y(x, y, z). \end{aligned}$$

Therefore, $\vec{\mathbf{F}}$ is conservative.

To evaluate $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ using the Fundamental Theorem for Line Integrals, we need to find a potential function f for $\vec{\mathbf{F}}$. Let f be a potential function for $\vec{\mathbf{F}}$. Then, $\vec{\nabla} f = \vec{\mathbf{F}}$, and therefore $f_x(x,y,z) = 2xe^y z + e^x z$. Integrating this equation with respect to x gives $f(x,y,z) = x^2 e^y z + e^x z + h(y,z)$ for some function h. Differentiating this equation with respect to y gives $x^2 e^y z + h_y(y,z) = Q(x,y,z) = x^2 e^y z$, which implies that $h_y(y,z) = 0$. Therefore, h is a function of z only, and $f(x,y,z) = x^2 e^y z + e^x z + h(z)$. To find h, note that $f_z = x^2 e^y + e^x + h'(z) = R = x^2 e^y + e^x$. Therefore, $h'(z) = 0$ and we can take $h(z) = 0$. A potential function for $\vec{\mathbf{F}}$ is $f(x,y,z) = x^2 e^y z + e^x z$.

Now that we have a potential function, we can use the Fundamental Theorem for Line Integrals to evaluate the integral. By the theorem,

$$\begin{aligned} \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_C \vec{\nabla} f \cdot d\vec{\mathbf{r}} \\ &= f(1,1,1) - f(0,0,0) \\ &= 2e. \end{aligned}$$

Analysis

Notice that if we hadn't recognized that $\vec{\mathbf{F}}$ is conservative, we would have had to parameterize C and use the method from the previous section. Since curve C is unknown, using the Fundamental Theorem for Line Integrals is much simpler.

? Exercise 15.3.9

Calculate integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where $\vec{\mathbf{F}}(x,y) = \langle \sin x \sin y, 5 - \cos x \cos y \rangle$ and C is a semicircle with starting point $(0,\pi)$ and endpoint $(0,-\pi)$.

✓ Example 15.3.11: Work Done on a Particle

Let $\vec{\mathbf{F}}(x,y) = \langle 2xy^2, 2x^2 y \rangle$ be a force field. Suppose that a particle begins its motion at the origin and ends its movement at any point in a plane that is not on the x-axis or the y-axis. Furthermore, the particle's motion can be modeled with a smooth parameterization. Show that $\vec{\mathbf{F}}$ does positive work on the particle.

Solution

We show that $\vec{\mathbf{F}}$ does positive work on the particle by showing that $\vec{\mathbf{F}}$ is conservative and then by using the Fundamental Theorem for Line Integrals.

To show that $\vec{\mathbf{F}}$ is conservative, suppose $f(x,y)$ were a potential function for $\vec{\mathbf{F}}$. Then, $\vec{\nabla} f(x,y) = \vec{\mathbf{F}}(x,y) = \langle 2xy^2, 2x^2 y \rangle$ and therefore $f_x(x,y) = 2xy^2$ and $f_y(x,y) = 2x^2 y$. The equation $fx(x,y) = 2xy^2$ implies that $f(x,y) = x^2 y^2 + h(y)$. Deriving both sides with respect to y yields $f_y(x,y) = 2x^2 y + h'(y)$. Therefore, $h'(y) = 0$ and we can take $h(y) = 0$.

If $f(x,y) = x^2 y^2$, then note that $\vec{\nabla} f(x,y) = \langle 2xy^2, 2x^2 y \rangle = \vec{\mathbf{F}}$, and therefore f is a potential function for $\vec{\mathbf{F}}$.

Let (a,b) be the point at which the particle stops is motion, and let C denote the curve that models the particle's motion. The work done by $\vec{\mathbf{F}}$ on the particle is $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$. By the Fundamental Theorem for Line Integrals,

$$\begin{aligned} \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_C \nabla f \cdot d\vec{\mathbf{r}} \\ &= f(a,b) - f(0,0) \\ &= a^2 b^2. \end{aligned}$$

Since $a \neq 0$ and $b \neq 0$, by assumption, $a^2 b^2 > 0$. Therefore, $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} > 0$, and $\vec{\mathbf{F}}$ does positive work on the particle.

Analysis

Notice that this problem would be much more difficult without using the Fundamental Theorem for Line Integrals. To apply the tools we have learned, we would need to give a curve parameterization and use the method from the previous section. Since the path of motion C can be as exotic as we wish (as long as it is smooth), it can be very difficult to parameterize the motion of the particle.

? Exercise 15.3.10

Let $\vec{\mathbf{F}}(x,y) = \langle 4x^3y^4, 4x^4y^3 \rangle$, and suppose that a particle moves from point $(4,4)$ to $(1,1)$ along any smooth curve. Is the work done by $\vec{\mathbf{F}}$ on the particle positive, negative, or zero?

Key Concepts

- The theorems in this section require curves that are closed, simple, or both, and regions that are connected or simply connected.
- The line integral of a conservative vector field can be calculated using the Fundamental Theorem for Line Integrals. This theorem is a generalization of the Fundamental Theorem of Calculus in higher dimensions. Using this theorem usually makes the calculation of the line integral easier.
- Conservative fields are independent of path. The line integral of a conservative field depends only on the value of the potential function at the endpoints of the domain curve.
- Given vector field $\vec{\mathbf{F}}$, we can test whether $\vec{\mathbf{F}}$ is conservative by using the cross-partial property. If $\vec{\mathbf{F}}$ has the cross-partial property and the domain is simply connected, then $\vec{\mathbf{F}}$ is conservative (and thus has a potential function). If $\vec{\mathbf{F}}$ is conservative, we can find a potential function by using the Problem-Solving Strategy.
- The circulation of a conservative vector field on a simply connected domain over a closed curve is zero.

Key Equations

- **Fundamental Theorem for Line Integrals**
  $\int_C \vec{\nabla} f \cdot d\vec{\mathbf{r}} = f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(a))$
- **Circulation of a conservative field over curve C that encloses a simply connected region**
  $\oint_C \vec{\nabla} f \cdot d\vec{\mathbf{r}} = 0$

Glossary

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15.3E: Exercises for Section 15.3

1. *True* or *False?* If vector field $\vec{\mathbf{F}}$ is conservative on the open and connected region D, then line integrals of $\vec{\mathbf{F}}$ are path independent on D, regardless of the shape of D.

Answer

True

2. *True* or *False?* Function $\vec{\mathbf{r}}(t) = \vec{\mathbf{a}} + t(\vec{\mathbf{b}} - \vec{\mathbf{a}})$, where $0 \le t \le 1$, parameterizes the straight-line segment from $\vec{\mathbf{a}}$ to $\vec{\mathbf{b}}$.

Answer

True

3. *True* or *False?* Vector field $\vec{\mathbf{F}}(x, y, z) = (y \sin z)\,\hat{\mathbf{i}} + (x \sin z)\,\hat{\mathbf{j}} + (xy \cos z)\,\hat{\mathbf{k}}$ is conservative.

Answer

True

4. *True* or *False?* Vector field $\vec{\mathbf{F}}(x, y, z) = y\,\hat{\mathbf{i}} + (x + z)\,\hat{\mathbf{j}} - y\,\hat{\mathbf{k}}$ is conservative.

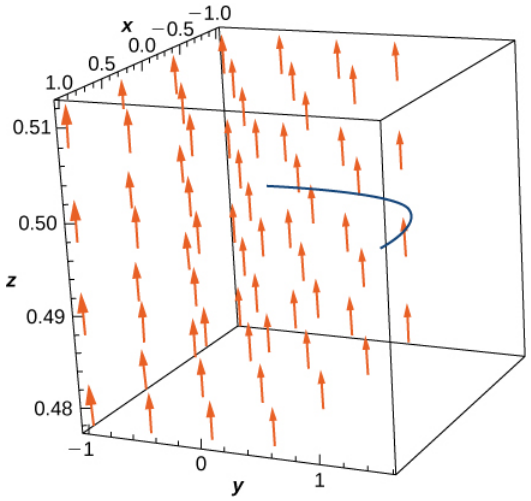
5. Justify the Fundamental Theorem of Line Integrals for $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ in the case when $\vec{\mathbf{F}}(x, y) = (2x + 2y)\,\hat{\mathbf{i}} + (2x + 2y)\,\hat{\mathbf{j}}$ and C is a portion of the positively oriented circle $x^2 + y^2 = 25$ from $(5, 0)$ to $(3, 4)$.

Answer

$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 24$ units of work

6. [T] Find $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where $\vec{\mathbf{F}}(x, y) = (ye^{xy} + \cos x)\,\hat{\mathbf{i}} + \left(xe^{xy} + \frac{1}{y^2+1}\right)\hat{\mathbf{j}}$ and C is a portion of curve $y = \sin x$ from $x = 0$ to $x = \frac{\pi}{2}$.

7. [T] Evaluate line integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where $\vec{\mathbf{F}}(x, y) = (e^x \sin y - y)\,\hat{\mathbf{i}} + (e^x \cos y - x - 2)\,\hat{\mathbf{j}}$, and C is the path given by $\vec{\mathbf{r}}(t) = (t^3 \sin \frac{\pi t}{2})\,\hat{\mathbf{i}} - (\frac{\pi}{2}\cos(\frac{\pi t}{2} + \frac{\pi}{2}))\,\hat{\mathbf{j}}$ for $0 \le t \le 1$.



Answer

$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \left(e - \frac{3\pi}{2}\right)$ units of work

For the following exercises, determine whether the vector field is conservative and, if it is, find the potential function.

8. $\vec{\mathbf{F}}(x,y) = 2xy^3\,\hat{\mathbf{i}} + 3y^2x^2\,\hat{\mathbf{j}}$

9. $\vec{\mathbf{F}}(x,y) = (-y + e^x \sin y)\,\hat{\mathbf{i}} + ((x+2)e^x \cos y)\,\hat{\mathbf{j}}$

Answer

Not conservative

10. $\vec{\mathbf{F}}(x,y) = (e^{2x} \sin y)\,\hat{\mathbf{i}} + (e^{2x} \cos y)\,\hat{\mathbf{j}}$

11. $\vec{\mathbf{F}}(x,y) = (6x + 5y)\,\hat{\mathbf{i}} + (5x + 4y)\,\hat{\mathbf{j}}$

Answer

Conservative, $f(x,y) = 3x^2 + 5xy + 2y^2 + k$

12. $\vec{\mathbf{F}}(x,y) = (2x\cos(y) - y\cos(x))\,\hat{\mathbf{i}} + (-x^2\sin(y) - \sin(x))\,\hat{\mathbf{j}}$

13. $\vec{\mathbf{F}}(x,y) = (ye^x + \sin(y))\,\hat{\mathbf{i}} + (e^x + x\cos(y))\,\hat{\mathbf{j}}$

Answer

Conservative, $f(x,y) = ye^x + x\sin(y) + k$

For the following exercises, evaluate the line integrals using the Fundamental Theorem of Line Integrals.

14. $\oint_C (y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}}) \cdot d\vec{\mathbf{r}}$, where C is any path from $(0,0)$ to $(2,4)$

15. $\oint_C (2y\,dx + 2x\,dy)$, where C is the line segment from $(0,0)$ to $(4,4)$

Answer

$\oint_C (2y\,dx + 2x\,dy) = 32$ units of work

16. [T] $\oint_C \left[\arctan\frac{y}{x} - \frac{xy}{x^2+y^2}\right] dx + \left[\frac{x^2}{x^2+y^2} + e^{-y}(1-y)\right] dy$, where C is any smooth curve from $(1,1)$ to $(-1,2)$.

17. Find the conservative vector field for the potential function $f(x,y) = 5x^2 + 3xy + 10y^2$.

Answer

$\vec{\mathbf{F}}(x,y) = (10x + 3y)\,\hat{\mathbf{i}} + (3x + 20y)\,\hat{\mathbf{j}}$

For the following exercises, determine whether the vector field is conservative and, if so, find a potential function.

18. $\vec{\mathbf{F}}(x,y) = (12xy)\,\hat{\mathbf{i}} + 6(x^2 + y^2)\,\hat{\mathbf{j}}$

19. $\vec{\mathbf{F}}(x,y) = (e^x \cos y)\,\hat{\mathbf{i}} + 6(e^x \sin y)\,\hat{\mathbf{j}}$

Answer

$\vec{\mathbf{F}}$ is not conservative.

20. $\vec{\mathbf{F}}(x,y) = (2xye^{x^2y})\,\hat{\mathbf{i}} + 6(x^2e^{x^2y})\,\hat{\mathbf{j}}$

21. $\vec{\mathbf{F}}(x,y,z) = (ye^z)\,\hat{\mathbf{i}} + (xe^z)\,\hat{\mathbf{j}} + (xye^z)\,\hat{\mathbf{k}}$

Answer

$\vec{\mathbf{F}}$ is conservative and a potential function is $f(x,y,z) = xye^z + k$.

22. $\vec{\mathbf{F}}(x,y,z) = (\sin y)\,\hat{\mathbf{i}} - (x\cos y)\,\hat{\mathbf{j}} + \hat{\mathbf{k}}$

23. $\vec{\mathbf{F}}(x,y,z) = \frac{1}{y}\,\hat{\mathbf{i}} - \frac{x}{y^2}\,\hat{\mathbf{j}} + (2z-1)\,\hat{\mathbf{k}}$

Answer

$\vec{\mathbf{F}}$ is conservative and a potential function is $f(x,y,z) = \dfrac{x}{y} + z^2 - z + k$.

24. $\vec{\mathbf{F}}(x,y,z) = 3z^2\,\hat{\mathbf{i}} - \cos y\,\hat{\mathbf{j}} + 2xz\,\hat{\mathbf{k}}$

25. $\vec{\mathbf{F}}(x,y,z) = (2xy)\,\hat{\mathbf{i}} + (x^2 + 2yz)\,\hat{\mathbf{j}} + y^2\,\hat{\mathbf{k}}$

Answer

$\vec{\mathbf{F}}$ is conservative and a potential function is $f(x,y,z) = x^2y + y^2z + k$.

For the following exercises, determine whether the given vector field is conservative and find a potential function.

26. $\vec{\mathbf{F}}(x,y) = (e^x \cos y)\,\hat{\mathbf{i}} + 6(e^x \sin y)\,\hat{\mathbf{j}}$

27. $\vec{\mathbf{F}}(x,y) = (2xye^{x^2y})\,\hat{\mathbf{i}} + (x^2e^{x^2y})\,\hat{\mathbf{j}}$

Answer

$\vec{\mathbf{F}}$ is conservative and a potential function is $f(x,y) = e^{x^2y} + k$

For the following exercises, evaluate the integral using the Fundamental Theorem of Line Integrals.

28. Evaluate $\int_C \vec{\nabla} f \cdot d\vec{\mathbf{r}}$, where $f(x,y,z) = \cos(\pi x) + \sin(\pi y) - xyz$ and C is any path that starts at $(1,12,2)$ and ends at $(2,1,-1)$.

29. [T] Evaluate $\int_C \vec{\nabla} f \cdot d\vec{\mathbf{r}}$, where $f(x,y) = xy + e^x$ and C is a straight line from $(0,0)$ to $(2,1)$.

Answer

$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = (e^2 + 1)$ units of work

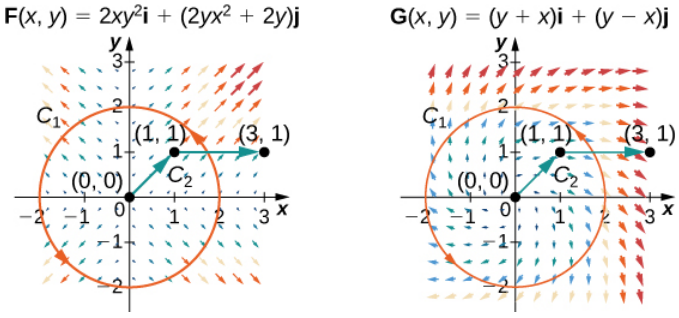
30. [T] Evaluate $\int_C \vec{\nabla} f \cdot d\vec{\mathbf{r}}$, where $f(x,y) = x^2y - x$ and C is any path in a plane from (1, 2) to (3, 2).

31. Evaluate $\int_C \vec{\nabla} f \cdot d\vec{\mathbf{r}}$, where $f(x,y,z) = xyz^2 - yz$ and C has initial point $(1,2,3)$ and terminal point $(3,5,2)$.

Answer

$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 38$ units of work

For the following exercises, let $\vec{\mathbf{F}}(x,y) = 2xy^2\,\hat{\mathbf{i}} + (2yx^2 + 2y)\,\hat{\mathbf{j}}$ and $G(x,y) = (y + x)\,\hat{\mathbf{i}} + (y - x)\,\hat{\mathbf{j}}$, and let C_1 be the curve consisting of the circle of radius 2, centered at the origin and oriented counterclockwise, and C_2 be the curve consisting of a line segment from $(0,0)$ to $(1,1)$ followed by a line segment from $(1,1)$ to $(3,1)$.



32. Calculate the line integral of $\vec{\mathbf{F}}$ over C_1.

33. Calculate the line integral of $\vec{\mathbf{G}}$ over C_1.

Answer

$\oint_{C_1} \vec{\mathbf{G}} \cdot d\vec{\mathbf{r}} = -8\pi$ units of work

34. Calculate the line integral of $\vec{\mathbf{F}}$ over C_2.

35. Calculate the line integral of $\vec{\mathbf{G}}$ over C_2.

Answer

$\oint_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 7$ units of work

36. [T] Let $\vec{\mathbf{F}}(x,y,z) = x^2\,\hat{\mathbf{i}} + z\sin(yz)\,\hat{\mathbf{j}} + y\sin(yz)\,\hat{\mathbf{k}}$. Calculate $\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where C is a path from $A = (0,0,1)$ to $B = (3,1,2)$.

37. [T] Find line integral $\oint_C \vec{\mathbf{F}} \cdot d\boldsymbol{r}$ of vector field $\vec{\mathbf{F}}(x,y,z) = 3x^2z\,\hat{\mathbf{i}} + z^2\,\hat{\mathbf{j}} + (x^3 + 2yz)\,\hat{\mathbf{k}}$ along curve C parameterized by $\vec{\mathbf{r}}(t) = \left(\frac{\ln t}{\ln 2}\right)\hat{\mathbf{i}} + t^{3/2}\,\hat{\mathbf{j}} + t\cos(\pi t), 1 \le t \le 4$.

Answer

$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 159$ units of work

For exercises 38 - 40, show that the following vector fields are conservative. Then calculate $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ for the given curve.

38. $\vec{\mathbf{F}}(x,y) = (xy^2 + 3x^2y)\,\hat{\mathbf{i}} + (x+y)x^2\,\hat{\mathbf{j}}$; C is the curve consisting of line segments from $(1,1)$ to $(0,2)$ to $(3,0)$.

39. $\vec{\mathbf{F}}(x,y) = \dfrac{2x}{y^2+1}\,\hat{\mathbf{i}} - \dfrac{2y(x^2+1)}{(y^2+1)^2}\,\hat{\mathbf{j}}$; C is parameterized by $x = t^3 - 1,\ y = t^6 - t$, for $0 \le t \le 1$.

Answer

$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = -1$ units of work

40. [T] $\vec{\mathbf{F}}(x,y) = \left[\cos(xy^2) - xy^2\sin(xy^2)\right]\hat{\mathbf{i}} - 2x^2y\sin(xy^2)\,\hat{\mathbf{j}}$; C is the curve $\langle e^t, e^{t+1}\rangle$, for $-1 \le t \le 0$.

41. The mass of Earth is approximately $6 \times 10^{27} g$ and that of the Sun is 330,000 times as much. The gravitational constant is $6.7 \times 10^{-8} cm^3/s^2 \cdot g$. The distance of Earth from the Sun is about $1.5 \times 10^{12} cm$. Compute, approximately, the work necessary to increase the distance of Earth from the Sun by $1\ cm$.

Answer

4×10^{31} erg

42. [T] Let $\vec{\mathbf{F}}(x,y,z) = (e^x \sin y)\,\hat{\mathbf{i}} + (e^x \cos y)\,\hat{\mathbf{j}} + z^2\,\hat{\mathbf{k}}$. Evaluate the integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where $\vec{\mathbf{r}}(t) = \langle \sqrt{t}, t^3, e^{\sqrt{t}}\rangle$, for $0 \le t \le 1$.

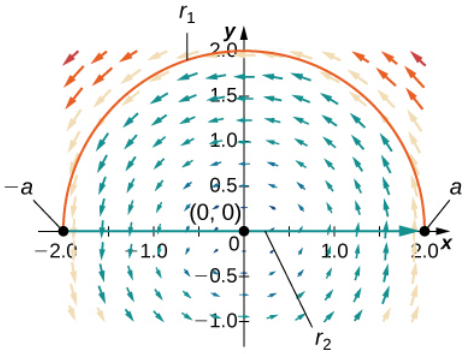
43. [T] Let $C : [1,2] \to \mathbb{R}^2$ be given by $x = e^{t-1}, y = \sin\left(\frac{\pi}{t}\right)$. Use a computer to compute the integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C 2x\cos y\,dx - x^2 \sin y\,dy$, where $\vec{\mathbf{F}}(x,y) = (2x\cos y)\,\hat{\mathbf{i}} - (x^2\sin y)\,\hat{\mathbf{j}}$.

Answer

$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}} = 0.4687$ units of work

44. [T] Use a computer algebra system to find the mass of a wire that lies along the curve $\vec{\mathbf{r}}(t) = (t^2 - 1)\,\hat{\mathbf{j}} + 2t\,\hat{\mathbf{k}}$, where $0 \le t \le 1$, if the density is given by $d(t) = \dfrac{3}{2}t$.

45. Find the circulation and flux of field $\vec{\mathbf{F}}(x,y) = -y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}}$ around and across the closed semicircular path that consists of semicircular arch $\vec{\mathbf{r}}_1(t) = (a\cos t)\,\hat{\mathbf{i}} + (a\sin t)\,\hat{\mathbf{j}}, \quad 0 \le t \le \pi$, followed by line segment $\vec{\mathbf{r}}_2(t) = t\,\hat{\mathbf{i}}, \quad -a \le t \le a$.



Answer

$\textbf{circulation} = \pi a^2$ and $\textbf{flux} = 0$

46. Compute $\int_C \cos x \cos y \, dx - \sin x \sin y \, dy$, where $\vec{\mathbf{r}}(t) = \langle t, t^2 \rangle, \quad 0 \le t \le 1.$

47. Complete the proof of the theorem titled THE PATH INDEPENDENCE TEST FOR CONSERVATIVE FIELDS by showing that $f_y = Q(x, y)$.

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15.4: Green's Theorem

Learning Objectives

- Apply the circulation form of Green's theorem.
- Apply the flux form of Green's theorem.
- Calculate circulation and flux on more general regions.

In this section, we examine Green's theorem, which is an extension of the Fundamental Theorem of Calculus to two dimensions. Green's theorem has two forms: a circulation form and a flux form, both of which require region D in the double integral to be simply connected. However, we will extend Green's theorem to regions that are not simply connected.

Put simply, Green's theorem relates a line integral around a simply closed plane curve C and a double integral over the region enclosed by C. The theorem is useful because it allows us to translate difficult line integrals into more simple double integrals, or difficult double integrals into more simple line integrals.

Extending the Fundamental Theorem of Calculus

Recall that the Fundamental Theorem of Calculus says that

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

As a geometric statement, this equation says that the integral over the region below the graph of $F'(x)$ and above the line segment $[a, b]$ depends only on the value of F at the endpoints a and b of that segment. Since the numbers a and b are the boundary of the line segment $[a, b]$, the theorem says we can calculate integral $\int_a^b F'(x)\,dx$ based on information about the boundary of line segment $[a, b]$ (Figure 15.4.1). The same idea is true of the Fundamental Theorem for Line Integrals:

$$\int_C \vec{\nabla} f \cdot d\vec{\mathbf{r}} = f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(a)).$$

When we have a potential function (an "antiderivative"), we can calculate the line integral based solely on information about the boundary of curve C.

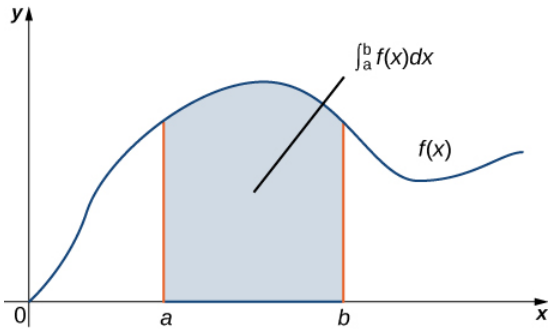


Figure 15.4.1: The Fundamental Theorem of Calculus says that the integral over line segment $[a, b]$ depends only on the values of the antiderivative at the endpoints of $[a, b]$.

Green's theorem takes this idea and extends it to calculating double integrals. Green's theorem says that we can calculate a double integral over region D based solely on information about the boundary of D. Green's theorem also says we can calculate a line integral over a simple closed curve C based solely on information about the region that C encloses. In particular, Green's theorem connects a double integral over region D to a line integral around the boundary of D.

Circulation Form of Green's Theorem

The first form of Green's theorem that we examine is the circulation form. This form of the theorem relates the vector line integral over a simple, closed plane curve C to a double integral over the region enclosed by C. Therefore, the circulation of a vector field along a simple closed curve can be transformed into a double integral and vice versa.

GREEN'S THEOREM (CIRCULATION FORM)

Let D be an open, simply connected region with a boundary curve C that is a piecewise smooth, simple closed curve oriented counterclockwise (Figure 15.4.2). Let $\vec{\mathbf{F}} = \langle P, Q \rangle$ be a vector field with component functions that have continuous partial derivatives on D. Then,

$$\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \oint_C P\,dx + Q\,dy \tag{15.4.1}$$

$$= \iint_D (Q_x - P_y)\,dA. \tag{15.4.2}$$

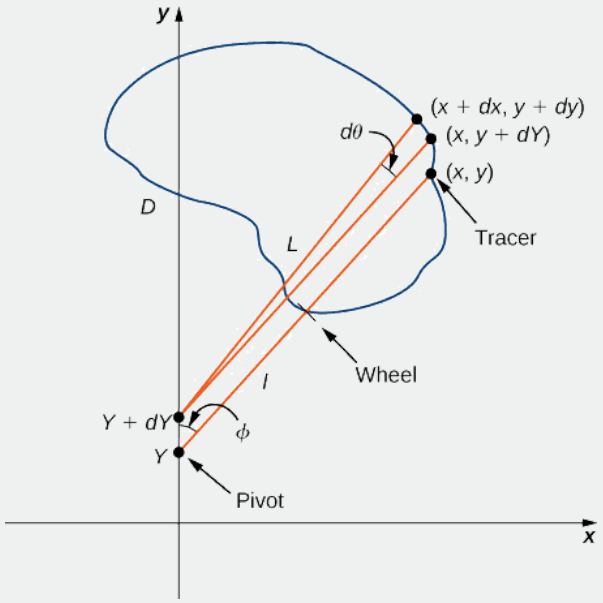


Figure 15.4.2: The circulation form of Green's theorem relates a line integral over curve C to a double integral over region D.

Notice that Green's theorem can be used only for a two-dimensional vector field $\vec{\mathbf{F}}$. If $\vec{\mathbf{F}}$ is a three-dimensional field, then Green's theorem does not apply. Since

$$\int_C P\,dx + Q\,dy = \int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\,ds$$

this version of Green's theorem is sometimes referred to as the ***tangential form* of Green's theorem.**

The proof of Green's theorem is rather technical, and beyond the scope of this text. Here we examine a proof of the theorem in the special case that D is a rectangle. For now, notice that we can quickly confirm that the theorem is true for the special case in which $\vec{\mathbf{F}} = \langle P, Q \rangle$ is conservative. In this case,

$$\oint_C P\,dx + Q\,dy = 0$$

because the circulation is zero in conservative vector fields. $\vec{\mathbf{F}}$ satisfies the cross-partial condition, so $P_y = Q_x$. Therefore,

$$\iint_D (Q_x - P_y)\,dA = \int_D 0\,dA = 0 = \oint_C P\,dx + Q\,dy$$

which confirms Green's theorem in the case of **conservative vector fields**.

Proof

Let's now prove that the circulation form of Green's theorem is true when the region D is a rectangle. Let D be the rectangle $[a, b] \times [c, d]$ oriented counterclockwise. Then, the boundary C of D consists of four piecewise smooth pieces C_1, C_2, C_3, and C_4 (Figure 15.4.3). We parameterize each side of D as follows:

$$C_1 : \vec{\mathbf{r}}_1(t) = \langle t, c \rangle,\ a \le t \le b$$

$$C_2 : \vec{\mathbf{r}}_2(t) = \langle b, t \rangle,\ c \le t \le d$$

$$-C_3 : \vec{\mathbf{r}}_3(t) = \langle t, d \rangle,\ a \le t \le b$$

$$-C_4 : \vec{\mathbf{r}}_4(t) = \langle a, t \rangle,\ c \le t \le d.$$

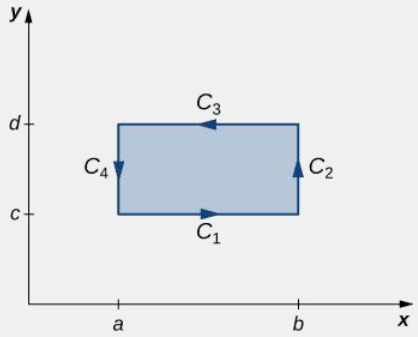


Figure 15.4.3: Rectangle D is oriented counterclockwise.

Then,

$$\begin{aligned}
\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{C_3} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{C_4} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \\
&= \int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} - \int_{-C_3} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} - \int_{-C_4} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \\
&= \int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}_1(t)) \cdot \vec{\mathbf{r}}_1'(t)\, dt + \int_c^d \vec{\mathbf{F}}(\vec{\mathbf{r}}_2(t)) \cdot \vec{\mathbf{r}}_2'(t)\, dt - \int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}_3(t)) \cdot \vec{\mathbf{r}}_3'(t)\, dt \\
&\quad - \int_c^d \vec{\mathbf{F}}(\vec{\mathbf{r}}_4(t)) \cdot \vec{\mathbf{r}}_4'(t)\, dt \\
&= \int_a^b P(t,c)\, dt + \int_c^d Q(b,t)\, dt - \int_a^b P(t,d)\, dt - \int_c^d Q(a,t)\, dt \\
&= \int_a^b (P(t,c) - P(t,d))\, dt + \int_c^d (Q(b,t) - Q(a,t))\, dt \\
&= -\int_a^b (P(t,d) - P(t,c))\, dt + \int_c^d (Q(b,t) - Q(a,t))\, dt.
\end{aligned}$$

By the Fundamental Theorem of Calculus,

$$P(t,d) - P(t,c) = \int_c^d \frac{\partial}{\partial y} P(t,y) dy$$

and

$$Q(b,t) - Q(a,t) = \int_a^b \frac{\partial}{\partial x} Q(x,t)\, dx.$$

Therefore,

$$\begin{aligned}
&-\int_a^b (P(t,d) - P(t,c))\, dt + \int_c^d (Q(b,t) - Q(a,t))\, dt \\
&= -\int_a^b \int_c^d \frac{\partial}{\partial y} P(t,y)\, dy\, dt + \int_c^d \int_a^b \frac{\partial}{\partial x} Q(x,t)\, dx\, dt.
\end{aligned}$$

But,

$$-\int_a^b \int_c^d \frac{\partial}{\partial y} P(t,y)\, dy\, dt$$

$$+\int_c^d \int_a^b \frac{\partial}{\partial x} Q(x,t)\,dx\,dt = -\int_a^b \int_c^d \frac{\partial}{\partial y} P(x,y)\,dy\,dx$$
$$+\int_c^d \int_a^b \frac{\partial}{\partial x} Q(x,y)\,dx\,dy$$
$$= \int_a^b \int_c^d (Q_x - P_y)\,dy\,dx$$
$$= \iint_D (Q_x - P_y)\,dA.$$

Therefore, $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_D (Q_x - P_y)\,dA$ and we have proved Green's theorem in the case of a rectangle.

□

To prove Green's theorem over a general region D, we can decompose D into many tiny rectangles and use the proof that the theorem works over rectangles. The details are technical, however, and beyond the scope of this text.

Example 15.4.1: Applying Green's Theorem over a Rectangle

Calculate the line integral

$$\oint_C x^2 y\,dx + (y-3)\,dy,$$

where C is a rectangle with vertices $(1,1)$, $(4,1)$, $(4,5)$, and $(1,5)$ oriented counterclockwise.

Solution

Let $\vec{\mathbf{F}}(x,y) = \langle P(x,y), Q(x,y)\rangle = \langle x^2 y, y-3\rangle$. Then, $Q_x(x,y) = 0$ and $P_y(x,y) = x^2$. Therefore, $Q_x - P_y = -x^2$.

Let D be the rectangular region enclosed by C (Figure 15.4.4). By Green's theorem,

$$\oint_C x^2 y\,dx + (y-3)\,dy = \iint_D (Q_x - P_y)\,dA$$
$$= \iint_D -x^2\,dA = \int_1^5 \int_1^4 -x^2\,dx\,dy$$
$$= \int_1^5 -\frac{x^3}{3}\Big|_1^4\,dy$$
$$= \int_1^5 \left(-\frac{64}{3} + \frac{1}{3}\right)\,dy$$
$$= \int_1^5 -21\,dy = -84.$$

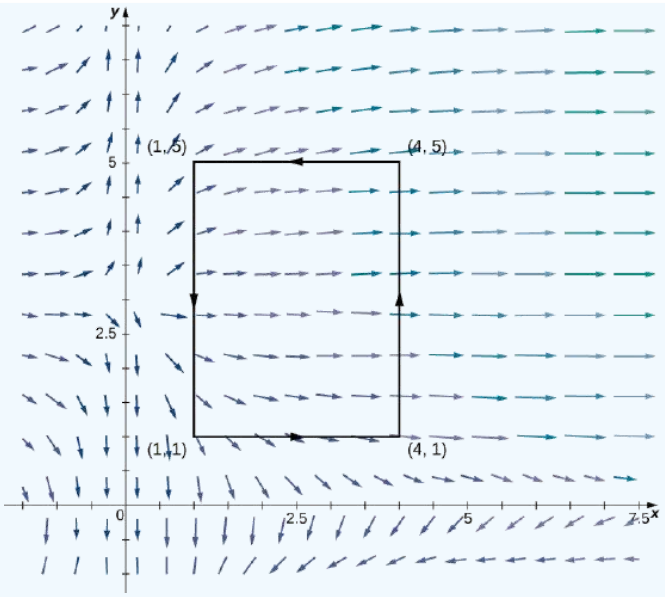


Figure 15.4.4: The line integral over the boundary of the rectangle can be transformed into a double integral over the rectangle.

Analysis

If we were to evaluate this line integral without using Green's theorem, we would need to parameterize each side of the rectangle, break the line integral into four separate line integrals, and use the methods from the section titled **Line Integrals** to evaluate each integral. Furthermore, since the vector field here is not conservative, we cannot apply the Fundamental Theorem for Line Integrals. Green's theorem makes the calculation much simpler.

Example 15.4.2: Applying Green's Theorem to Calculate Work

Calculate the work done on a particle by force field

$$\vec{\mathbf{F}}(x,y) = \langle y + \sin x, e^y - x \rangle$$

as the particle traverses circle $x^2 + y^2 = 4$ exactly once in the counterclockwise direction, starting and ending at point $(2, 0)$.

Solution

Let C denote the circle and let D be the disk enclosed by C. The work done on the particle is

$$W = \oint_C (y + \sin x)\, dx + (e^y - x)\, dy.$$

As with Example 15.4.1, this integral can be calculated using tools we have learned, but it is easier to use the double integral given by Green's theorem (Figure 15.4.5).

Let $\vec{\mathbf{F}}(x,y) = \langle P(x,y), Q(x,y) \rangle = \langle y + \sin x, e^y - x \rangle$. Then, $Q_x = -1$ and $P_y = 1$. Therefore, $Q_x - P_y = -2$.

By Green's theorem,

$$\begin{aligned} W &= \oint_C (y + \sin(x))dx + (e^y - x)\, dy \\ &= \iint_D (Q_x - P_y)\, dA \\ &= \iint_D -2\, dA \\ &= -2(area(D)) = -2\pi(2^2) = -8\pi. \end{aligned}$$

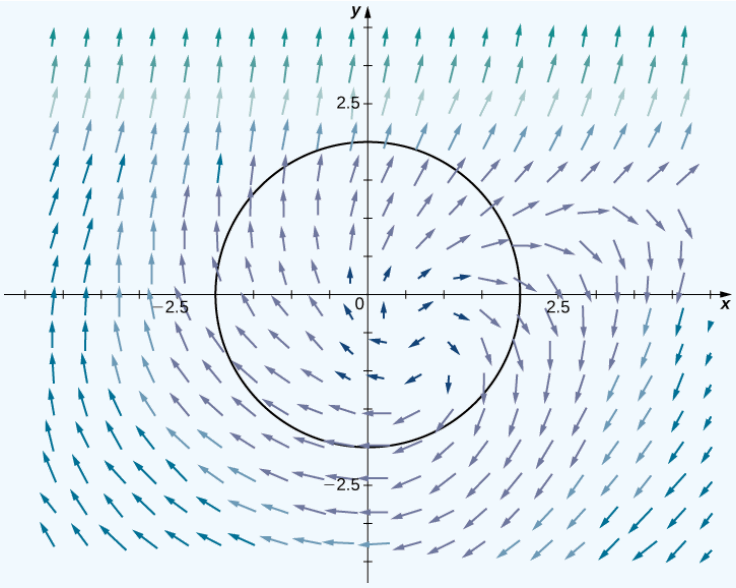


Figure 15.4.5: The line integral over the boundary circle can be transformed into a double integral over the disk enclosed by the circle.

Exercise 15.4.2

Use Green's theorem to calculate line integral

$$\oint_C \sin(x^2)\,dx + (3x - y)\,dy.$$

where C is a right triangle with vertices $(-1, 2)$, $(4, 2)$, and $(4, 5)$ oriented counterclockwise.

In the preceding two examples, the double integral in Green's theorem was easier to calculate than the line integral, so we used the theorem to calculate the line integral. In the next example, the double integral is more difficult to calculate than the line integral, so we use Green's theorem to translate a double integral into a line integral.

Example 15.4.3: Applying Green's Theorem over an Ellipse

Calculate the area enclosed by ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (Figure 15.4.6).

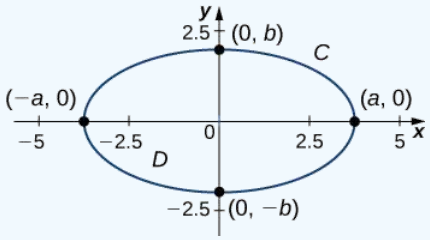


Figure 15.4.6: Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is denoted by C.

Solution

Let C denote the ellipse and let D be the region enclosed by C. Recall that ellipse C can be parameterized by

- $x = a\cos t$,
- $y = b\sin t$,
- $0 \le t \le 2\pi$.

Calculating the area of D is equivalent to computing double integral $\iint_D dA$. To calculate this integral without Green's theorem, we would need to divide D into two regions: the region above the x-axis and the region below. The area of the ellipse

is

$$\int_{-a}^{a}\int_{0}^{\sqrt{b^2-(bx/a)^2}} dy\,dx + \int_{-a}^{a}\int_{-\sqrt{b^2-(bx/a)^2}}^{0} dy\,dx.$$

These two integrals are not straightforward to calculate (although when we know the value of the first integral, we know the value of the second by symmetry). Instead of trying to calculate them, we use Green's theorem to transform $\iint_D dA$ into a line integral around the boundary C.

Consider vector field

$$\vec{\mathbf{F}}(x,y) = \langle P, Q\rangle = \langle -\frac{y}{2}, \frac{x}{2}\rangle.$$

Then, $Q_x = \frac{1}{2}$ and $P_y = -\frac{1}{2}$, and therefore $Q_x - P_y = 1$. Notice that $\vec{\mathbf{F}}$ was chosen to have the property that $Q_x - P_y = 1$. Since this is the case, Green's theorem transforms the line integral of $\vec{\mathbf{F}}$ over C into the double integral of 1 over D.

By Green's theorem,

$$\begin{aligned}\iint_D dA &= \iint_D (Q_x - P_y)\,dA \\ &= \int_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} = \frac{1}{2}\int_C -y\,dx + x\,dy \\ &= \frac{1}{2}\int_0^{2\pi} -b\sin t(-a\sin t) + a(\cos t)b\cos t\,dt \\ &= \frac{1}{2}\int_0^{2\pi} ab\cos^2 t + ab\sin^2 t\,dt \\ &= \frac{1}{2}\int_0^{2\pi} ab\,dt = \pi ab.\end{aligned}$$

Therefore, the area of the ellipse is πab units^2.

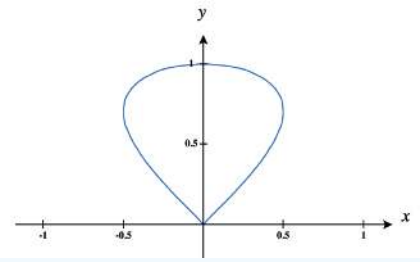
In Example 15.4.3, we used vector field $\vec{\mathbf{F}}(x,y) = \langle P, Q\rangle = \langle -\frac{y}{2}, \frac{x}{2}\rangle$ to find the area of any ellipse. The logic of the previous example can be extended to derive a formula for the area of any region D. Let D be any region with a boundary that is a simple closed curve C oriented counterclockwise. If $\vec{\mathbf{F}}(x,y) = \langle P, Q\rangle = \langle -\frac{y}{2}, \frac{x}{2}\rangle$, then $Q_x - P_y = 1$. Therefore, by the same logic as in Example 15.4.3,

$$\text{area of } D = \iint_D dA = \frac{1}{2}\oint_C -y\,dx + x\,dy. \tag{15.4.3}$$

It's worth noting that if $\vec{\mathbf{F}} = \langle P, Q\rangle$ is any vector field with $Q_x - P_y = 1$, then the logic of the previous paragraph works. So. Equation 15.4.3 is not the only equation that uses a vector field's mixed partials to get the area of a region.

? Exercise 15.4.3

Find the area of the region enclosed by the curve with parameterization $\vec{\mathbf{r}}(t) = \langle \sin t\cos t, \sin t\rangle$, $0 \le t \le \pi$.



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Flux Form of Green's Theorem

The circulation form of Green's theorem relates a double integral over region D to line integral $\oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}} ds$, where C is the boundary of D. The flux form of Green's theorem relates a double integral over region D to the flux across boundary C. The flux of a fluid across a curve can be difficult to calculate using the flux line integral. This form of Green's theorem allows us to translate a difficult flux integral into a double integral that is often easier to calculate.

GREEN'S THEOREM (FLUX FORM)

Let D be an open, simply connected region with a boundary curve C that is a piecewise smooth, simple closed curve that is oriented counterclockwise (Figure 15.4.7). Let $\vec{\mathbf{F}} = \langle P, Q \rangle$ be a vector field with component functions that have continuous partial derivatives on an open region containing D. Then,

$$\oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}} \, ds = \iint_D P_x + Q_y \, dA. \tag{15.4.4}$$

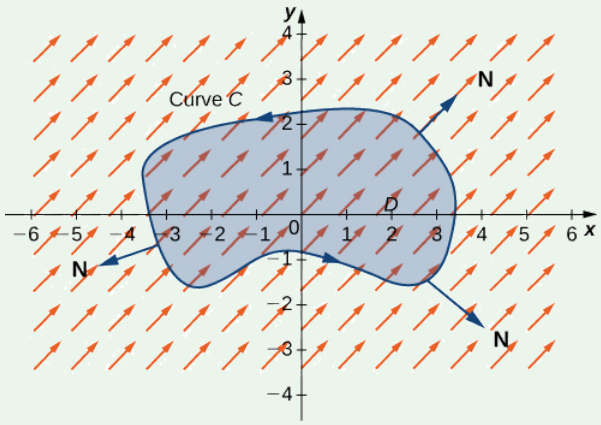


Figure 15.4.7: The flux form of Green's theorem relates a double integral over region D to the flux across curve C.

Because this form of Green's theorem contains unit normal vector $\vec{\mathbf{N}}$, it is sometimes referred to as the *normal form* of Green's theorem.

Proof

Recall that $\oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}} \, ds = \oint_C -Q \, dx + P \, dy$. Let $M = -Q$ and $N = P$. By the circulation form of Green's theorem,

$$\begin{aligned} \oint_C -Q \, dx + P \, dy &= \oint_C M \, dx + N \, dy \\ &= \iint_D N_x - M_y \, dA \\ &= \iint_D P_x - (-Q)_y \, dA \\ &= \iint_D P_x + Q_y \, dA. \end{aligned}$$

□

Example 15.4.4A: Applying Green's Theorem for Flux across a Circle

Let C be a circle of radius r centered at the origin (Figure 15.4.8) and let $\vec{\mathbf{F}}(x, y) = \langle x, y \rangle$. Calculate the flux across C.

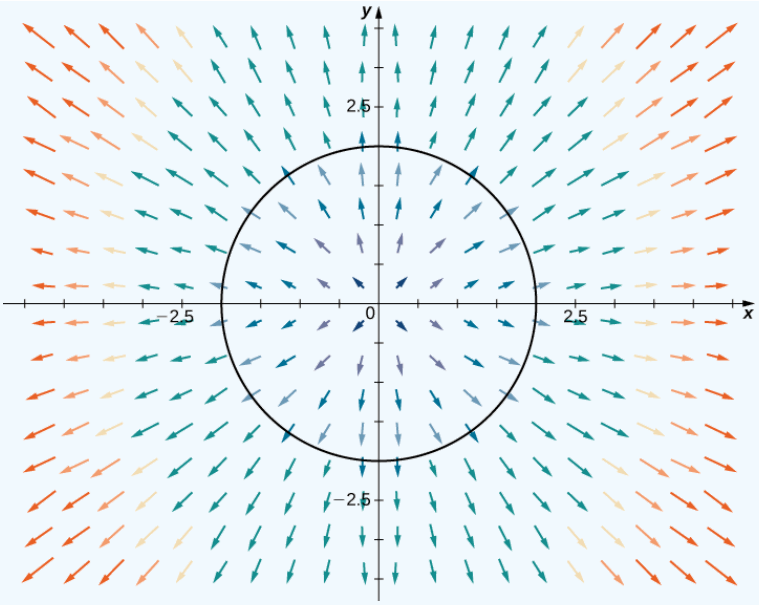


Figure 15.4.8: Curve C is a circle of radius r centered at the origin.

Solution

Let D be the disk enclosed by C. The flux across C is $\oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds$. We could evaluate this integral using tools we have learned, but Green's theorem makes the calculation much more simple. Let $P(x,y) = x$ and $Q(x,y) = y$ so that $\vec{\mathbf{F}} = \langle P, Q\rangle$. Note that $P_x = 1 = Q_y$, and therefore $P_x + Q_y = 2$. By Green's theorem,

$$\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds = \iint_D 2\, dA = 2 \iint_D dA.$$

Since $\iint_D dA$ is the area of the circle, $\iint_D dA = \pi r^2$. Therefore, the flux across C is $2\pi r^2$.

Example 15.4.4B: Applying Green's Theorem for Flux across a Triangle

Let S be the triangle with vertices $(0,0)$, $(1,0)$, and $(0,3)$ oriented clockwise (Figure 15.4.9). Calculate the flux of $\vec{\mathbf{F}}(x,y) = \langle P(x,y), Q(x,y)\rangle = \langle x^2 + e^y, x + y\rangle$ across S.

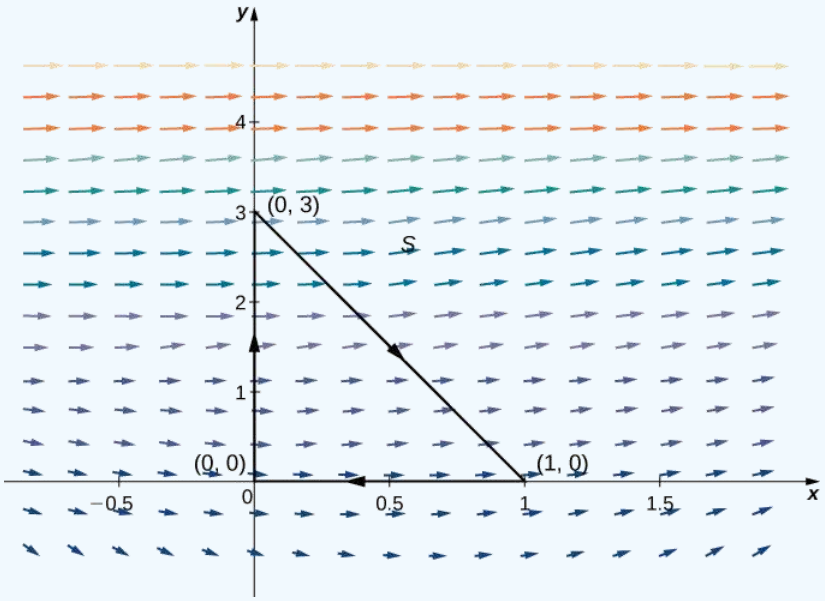


Figure 15.4.9: Curve S is a triangle with vertices $(0,0)$, $(1,0)$, and $(0,3)$ oriented clockwise.

Solution

To calculate the flux without Green's theorem, we would need to break the flux integral into three line integrals, one integral for each side of the triangle. Using Green's theorem to translate the flux line integral into a single double integral is much more simple.

Let D be the region enclosed by S. Note that $P_x = 2x$ and $Q_y = 1$; therefore, $P_x + Q_y = 2x + 1$. Green's theorem applies only to simple closed curves oriented counterclockwise, but we can still apply the theorem because $\oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds = -\oint_{-S} \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds$ and $-S$ is oriented counterclockwise. By Green's theorem, the flux is

$$\begin{aligned}\oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds &= \oint_{-S} \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds \\ &= -\iint_D (P_x + Q_y)\, dA \\ &= -\iint_D (2x+1)\, dA.\end{aligned}$$

Notice that the top edge of the triangle is the line $y = -3x + 3$. Therefore, in the iterated double integral, the y-values run from $y = 0$ to $y = -3x + 3$, and we have

$$\begin{aligned}-\iint_D (2x+1)\, dA &= -\int_0^1 \int_0^{-3x+3} (2x+1)\, dy\, dx \\ &= -\int_0^1 (2x+1)y\Big|_0^{-3x+3} dx \\ &= -\int_0^1 (2x+1)(-3x+3)\, dx \\ &= -\int_0^1 (-6x^2 + 3x + 3)\, dx \\ &= -\left[-2x^3 + \frac{3x^2}{2} + 3x\right]\Big|_0^1 \\ &= -\frac{5}{2}.\end{aligned}$$

? Exercise 15.4.4

Calculate the flux of $\vec{\mathbf{F}}(x, y) = \langle x^3, y^3 \rangle$ across a unit circle oriented counterclockwise.

✓ Example 15.4.5: Applying Green's Theorem for Water Flow across a Rectangle

Water flows from a spring located at the origin. The velocity of the water is modeled by vector field $\vec{\mathbf{v}}(x, y) = \langle 5x + y, x + 3y \rangle$ m/sec. Find the amount of water per second that flows across the rectangle with vertices $(-1, -2)$, $(1, -2)$, $(1, 3)$,and $(-1, 3)$, oriented counterclockwise (Figure 15.4.10).

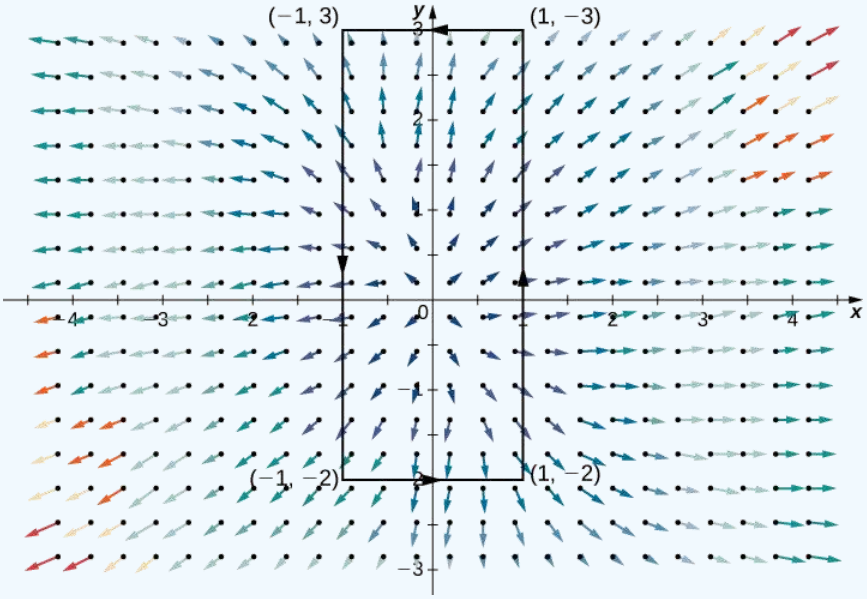


Figure 15.4.10: Water flows across the rectangle with vertices $(-1,-2)$, $(1,-2)$, $(1,3)$,and $(-1,3)$, oriented counterclockwise.

Solution

Let C represent the given rectangle and let D be the rectangular region enclosed by C. To find the amount of water flowing across C, we calculate flux $\int_C \vec{\mathbf{v}} \cdot d\vec{\mathbf{r}}$. Let $P(x,y) = 5x + y$ and $Q(x,y) = x + 3y$ so that $\vec{\mathbf{v}} = \langle P, Q \rangle$. Then, $P_x = 5$ and $Q_y = 3$. By Green's theorem,

$$\begin{aligned} \int_C \vec{\mathbf{v}} \cdot d\vec{\mathbf{r}} &= \iint_D (P_x + Q_y)\, dA \\ &= \iint_D 8\, dA \\ &= 8(area\ of\ D) = 80. \end{aligned}$$

Therefore, the water flux is 80 m^2/sec.

Recall that if vector field $\vec{\mathbf{F}}$ is conservative, then $\vec{\mathbf{F}}$ does no work around closed curves—that is, the circulation of $\vec{\mathbf{F}}$ around a closed curve is zero. In fact, if the domain of $\vec{\mathbf{F}}$ is simply connected, then $\vec{\mathbf{F}}$ is conservative if and only if the circulation of $\vec{\mathbf{F}}$ around any closed curve is zero. If we replace "circulation of $\vec{\mathbf{F}}$" with "flux of $\vec{\mathbf{F}}$," then we get a definition of a source-free vector field. The following statements are all equivalent ways of defining a source-free field $\vec{\mathbf{F}} = \langle P, Q \rangle$ on a simply connected domain (note the similarities with properties of conservative vector fields):

1. The flux $\oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds$ across any closed curve C is zero.
2. If C_1 and C_2 are curves in the domain of $\vec{\mathbf{F}}$ with the same starting points and endpoints, then $\int_{C_1} \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds = \int_{C_2} \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds$. In other words, flux is independent of path.
3. There is a stream function $g(x,y)$ for $\vec{\mathbf{F}}$. A stream function for $\vec{\mathbf{F}} = \langle P, Q \rangle$ is a function g such that $P = g_y$ and $Q = -g_x$.Geometrically, $\vec{\mathbf{F}} = \langle a, b \rangle$ is tangential to the level curve of g at (a,b). Since the gradient of g is perpendicular to the level curve of g at (a,b), stream function g has the property $\vec{\mathbf{F}}(a,b) \cdot \vec{\nabla} g(a,b) = 0$ for any point (a,b) in the domain of g. (Stream functions play the same role for source-free fields that potential functions play for conservative fields.)
4. $P_x + Q_y = 0$

Example 15.4.6: Finding a Stream Function

Verify that rotation vector field $\vec{\mathbf{F}}(x,y) = \langle y, -x \rangle$ is source free, and find a stream function for $\vec{\mathbf{F}}$.

Solution

Note that the domain of $\vec{\mathbf{F}}$ is all of $\mathbb{R}^2$, which is simply connected. Therefore, to show that $\vec{\mathbf{F}}$ is source free, we can show any of items 1 through 4 from the previous list to be true. In this example, we show that item 4 is true. Let $P(x,y) = y$ and $Q(x,y) = -x$. Then $P_x = 0 = Q_y$, and therefore $P_x + Q_y = 0$. Thus, $\vec{\mathbf{F}}$ is source free.

To find a stream function for $\vec{\mathbf{F}}$, proceed in the same manner as finding a potential function for a conservative field. Let g be a stream function for $\vec{\mathbf{F}}$. Then $g_y = y$, which implies that

$$g(x,y) = \frac{y^2}{2} + h(x).$$

Since $-g_x = Q = -x$, we have $h'(x) = x$. Therefore,

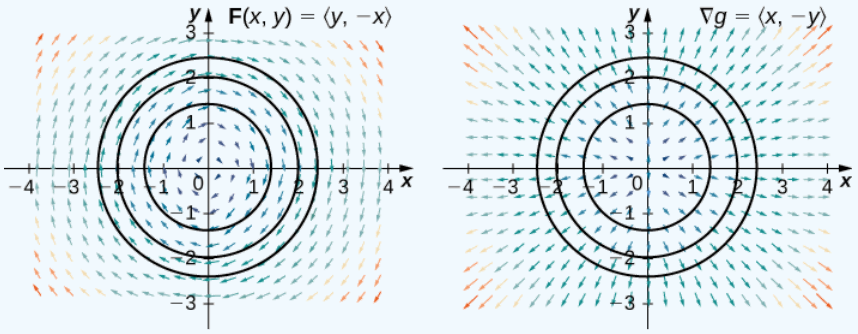
$$h(x) = \frac{x^2}{2} + C.$$

Letting $C = 0$ gives stream function

$$g(x,y) = \frac{x^2}{2} + \frac{y^2}{2}.$$

To confirm that g is a stream function for $\vec{\mathbf{F}}$, note that $g_y = y = P$ and $-g_x = -x = Q$.

Notice that source-free rotation vector field $\vec{\mathbf{F}}(x,y) = \langle y, -x\rangle$ is perpendicular to conservative radial vector field $\vec{\nabla} g = \langle x, y\rangle$ (Figure 15.4.11).



(a) (b)

Figure 15.4.11: (a) In this image, we see the three-level curves of g and vector field $\vec{\mathbf{F}}$. Note that the $\vec{\mathbf{F}}$ vectors on a given level curve are tangent to the level curve. (b) In this image, we see the three-level curves of g and vector field $\vec{\nabla} g$. The gradient vectors are perpendicular to the corresponding level curve. Therefore, $\vec{\mathbf{F}}(a,b) \cdot \vec{\nabla} g(a,b) = 0$ for any point in the domain of g.

? Exercise 15.4.6

Find a stream function for vector field $\vec{\mathbf{F}}(x,y) = \langle x \sin y, \cos y\rangle$.

Vector fields that are both conservative and source free are important vector fields. One important feature of conservative and source-free vector fields on a simply connected domain is that any potential function f of such a field satisfies Laplace's equation $f_{xx} + f_{yy} = 0$. Laplace's equation is foundational in the field of partial differential equations because it models such phenomena as gravitational and magnetic potentials in space, and the velocity potential of an ideal fluid. A function that satisfies Laplace's equation is called a *harmonic* function. Therefore any potential function of a conservative and source-free vector field is harmonic.

To see that any potential function of a conservative and source-free vector field on a simply connected domain is harmonic, let f be such a potential function of vector field $\vec{\mathbf{F}} = \langle P, Q\rangle$. Then, $f_x = P$ and $f_x = Q$ because $\vec{\nabla} f = \vec{\mathbf{F}}$. Therefore, $f_{xx} = P_x$ and $f_{yy} = Q_y$. Since $\vec{\mathbf{F}}$ is source free, $f_{xx} + f_{yy} = P_x + Q_y = 0$, and we have that f is harmonic.

✓ Example 15.4.7: Satisfying Laplace's Equation

For vector field $\vec{\mathbf{F}}(x,y) = \langle e^x \sin y, e^x \cos y\rangle$, verify that the field is both conservative and source free, find a potential function for $\vec{\mathbf{F}}$, and verify that the potential function is harmonic.

Solution

Let $P(x,y) = e^x \sin y$ and $Q(x,y) = e^x \cos y$. Notice that the domain of $\vec{\mathbf{F}}$ is all of two-space, which is simply connected. Therefore, we can check the cross-partials of $\vec{\mathbf{F}}$ to determine whether $\vec{\mathbf{F}}$ is conservative. Note that $P_y = e^x \cos y = Q_x$, so $\vec{\mathbf{F}}$ is conservative. Since $P_x = e^x \sin y$ and $Q_y = e^x \sin y$, $P_x + Q_y = 0$ and the field is source free.

To find a potential function for $\vec{\mathbf{F}}$, let f be a potential function. Then, $\vec{\nabla} f = \vec{\mathbf{F}}$, so $f_x(x,y) = e^x \sin y$. Integrating this equation with respect to x gives $f(x,y) = e^x \sin y + h(y)$. Since $f_y(x,y) = e^x \cos y$, differentiating f with respect to y gives $e^x \cos y = e^x \cos y + h'(y)$. Therefore, we can take $h(y) = 0$, and $f(x,y) = e^x \sin y$ is a potential function for f.

To verify that f is a harmonic function, note that $f_{xx}(x,y) = \dfrac{\partial}{\partial x}(e^x \sin y) = e^x \sin y$ and

$f_{yy}(x,y) = \dfrac{\partial}{\partial x}(e^x \cos y) = -e^x \sin y$. Therefore, $f_{xx} + f_{yy} = 0$, and f satisfies Laplace's equation.

? Exercise 15.4.7

Is the function $f(x,y) = e^{x+5y}$ harmonic?

Green's Theorem on General Regions

Green's theorem, as stated, applies only to regions that are simply connected—that is, Green's theorem as stated so far cannot handle regions with holes. Here, we extend Green's theorem so that it does work on regions with finitely many holes (Figure 15.4.12).

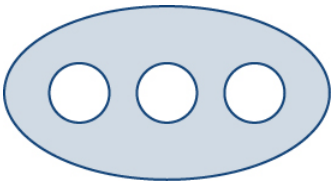


Figure 15.4.12: Green's theorem, as stated, does not apply to a nonsimply connected region with three holes like this one.

Before discussing extensions of Green's theorem, we need to go over some terminology regarding the boundary of a region. Let D be a region and let C be a component of the boundary of D. We say that C is *positively oriented* if, as we walk along C in the direction of orientation, region D is always on our left. Therefore, the counterclockwise orientation of the boundary of a disk is a positive orientation, for example. Curve C is *negatively oriented* if, as we walk along C in the direction of orientation, region D is always on our right. The clockwise orientation of the boundary of a disk is a negative orientation, for example.

Let D be a region with finitely many holes (so that D has finitely many boundary curves), and denote the boundary of D by ∂D (Figure 15.4.13). To extend Green's theorem so it can handle D, we divide region D into two regions, D_1 and D_2 (with respective boundaries ∂D_1 and ∂D_2), in such a way that $D = D_1 \cup D_2$ and neither D_1 nor D_2 has any holes (Figure 15.4.13).

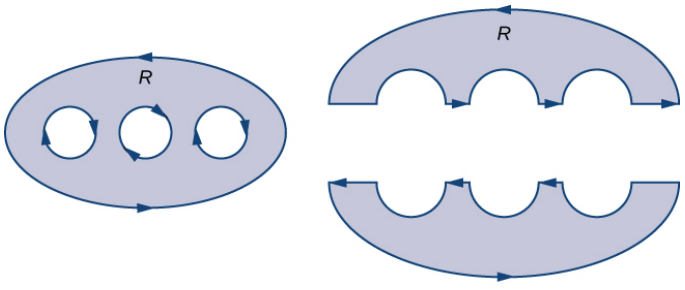


Figure 15.4.13: (a) Region D with an oriented boundary has three holes. (b) Region D split into two simply connected regions has no holes.

Assume the boundary of D is oriented as in the figure, with the inner holes given a negative orientation and the outer boundary given a positive orientation. The boundary of each simply connected region D_1 and D_2 is positively oriented. If $\vec{\mathbf{F}}$ is a vector field defined on D, then Green's theorem says that

$$\oint_{\partial D} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \oint_{\partial D_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \oint_{\partial D_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \tag{15.4.5}$$

$$= \iint_{D_1} Q_x - P_y \, dA + \iint_{D_2} Q_x - P_y \, dA \tag{15.4.6}$$

$$= \iint_{D} (Q_x - P_y) \, dA. \tag{15.4.7}$$

Therefore, Green's theorem still works on a region with holes.

To see how this works in practice, consider annulus D in Figure 15.4.14 and suppose that $F = \langle P, Q \rangle$ is a vector field defined on this annulus. Region D has a hole, so it is not simply connected. Orient the outer circle of the annulus counterclockwise and the inner circle clockwise (Figure 15.4.14) so that, when we divide the region into D_1 and D_2, we are able to keep the region on our left as we walk along a path that traverses the boundary. Let D_1 be the upper half of the annulus and D_2 be the lower half. Neither of these regions has holes, so we have divided D into two simply connected regions.

We label each piece of these new boundaries as P_i for some i, as in Figure 15.4.14. If we begin at P and travel along the oriented boundary, the first segment is P_1, then P_2, P_3, and P_4. Now we have traversed D_1 and returned to P. Next, we start at P again and traverse D_2. Since the first piece of the boundary is the same as P_4 in D_1, but oriented in the opposite direction, the first piece of D_2 is $-P_4$. Next, we have P_5, then $-P_2$, and finally P_6.

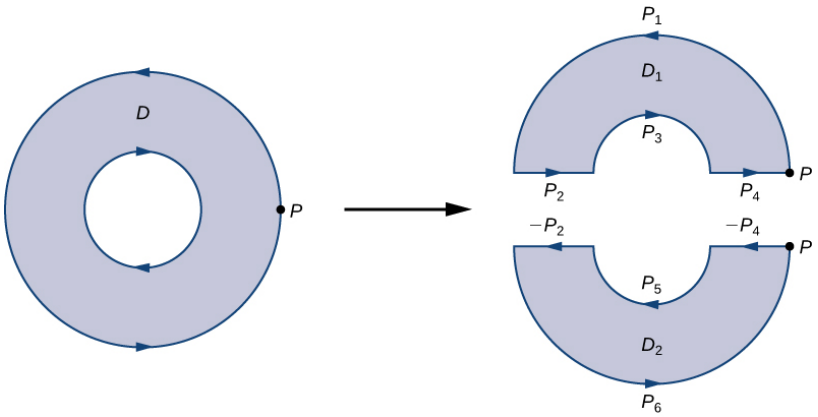


Figure 15.4.14: Breaking the annulus into two separate regions gives us two simply connected regions. The line integrals over the common boundaries cancel out.

Figure 15.4.14 shows a path that traverses the boundary of D. Notice that this path traverses the boundary of region D_1, returns to the starting point, and then traverses the boundary of region D_2. Furthermore, as we walk along the path, the region is always on our left. Notice that this traversal of the P_i paths covers the entire boundary of region D. If we had only traversed one portion of the boundary of D, then we cannot apply Green's theorem to D.

The boundary of the upper half of the annulus, therefore, is $P_1 \cup P_2 \cup P_3 \cup P_4$ and the boundary of the lower half of the annulus is $-P_4 \cup P_5 \cup -P_2 \cup P_6$. Then, Green's theorem implies

$$\int \vec{\mathbf{F}}$$

$$\int_{\partial D} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{P_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{P_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{P_3} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{P_4} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \tag{15.4.8}$$
$$+ \int_{-P_4} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{P_5} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{-P_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{P_6} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$$
$$= \int_{P_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{P_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{P_3} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{P_4} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \tag{15.4.9}$$
$$+ \int_{P_4} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{P_5} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{-P_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{P_6} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$$
$$= \int_{P_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{P_3} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{P_5} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{P_6} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \tag{15.4.10}$$
$$= \oint_{\partial D_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \oint_{\partial D_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \tag{15.4.11}$$
$$= \iint_{D_1} (Q_x - P_y)\, dA + \iint_{D_2} (Q_x - P_y)\, dA \tag{15.4.12}$$
$$= \iint_{D} (Q_x - P_y)\, dA. \tag{15.4.13}$$

Therefore, we arrive at the equation found in Green's theorem—namely,

$$\oint_{\partial D} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_{D} (Q_x - P_y)\, dA.$$

The same logic implies that the flux form of Green's theorem can also be extended to a region with finitely many holes:

$$\oint_{C} \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds = \iint_{D} (P_x + Q_y)\, dA.$$

Example 15.4.8A: Using Green's Theorem on a Region with Holes

Calculate the integral

$$\oint_{\partial D} (\sin x - \frac{y^3}{3})dx + (\frac{y^3}{3} + \sin y)dy,$$

where D is the annulus given by the polar inequalities $1 \leq r \leq 2$, $0 \leq \theta \leq 2\pi$.

Solution

Although D is not simply connected, we can use the extended form of Green's theorem to calculate the integral. Since the integration occurs over an annulus, we convert to polar coordinates:

$$\begin{aligned} \oint_{\partial D} (\sin x - \frac{y^3}{3})\, dx + (\frac{x^3}{3} + \sin y)\, dy &= \iint_{D} (Q_x - P_y)\, dA \\ &= \iint_{D} (x^2 + y^2)\, dA \\ &= \int_0^{2\pi} \int_1^2 r^3\, dr d\theta = \int_0^{2\pi} \frac{15}{4}\, d\theta \\ &= \frac{15\pi}{2}. \end{aligned}$$

Example 15.4.8B: Using the Extended Form of Green's Theorem

Let $\vec{\mathbf{F}} = \langle P, Q \rangle = \left\langle \frac{y}{x^2+y^2}, -\frac{x}{x^2+y^2} \right\rangle$ and let C be any simple closed curve in a plane oriented counterclockwise. What are the possible values of $\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$?

Solution

We use the extended form of Green's theorem to show that $\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ is either 0 or -2π—that is, no matter how crazy curve C is, the line integral of $\vec{\mathbf{F}}$ along C can have only one of two possible values. We consider two cases: the case when C encompasses the origin and the case when C does not encompass the origin.

Case 1: C Does Not Encompass the Origin

In this case, the region enclosed by C is simply connected because the only hole in the domain of $\vec{\mathbf{F}}$ is at the origin. We showed in our discussion of cross-partials that $\vec{\mathbf{F}}$ satisfies the cross-partial condition. If we restrict the domain of $\vec{\mathbf{F}}$ just to C and the region it encloses, then $\vec{\mathbf{F}}$ with this restricted domain is now defined on a simply connected domain. Since $\vec{\mathbf{F}}$ satisfies the cross-partial property on its restricted domain, the field $\vec{\mathbf{F}}$ is conservative on this simply connected region and hence the circulation $\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ is zero.

Case 2: C Does Encompass the Origin

In this case, the region enclosed by C is not simply connected because this region contains a hole at the origin. Let C_1 be a circle of radius a centered at the origin so that C_1 is entirely inside the region enclosed by C (Figure 15.4.15). Give C_1 a clockwise orientation.

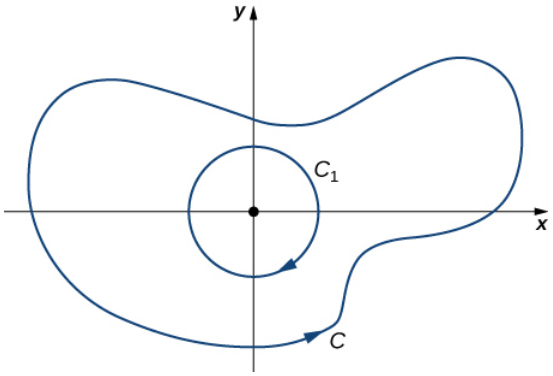


Figure 15.4.15: Choose circle C_1 centered at the origin that is contained entirely inside C.

Let D be the region between C_1 and C, and C is orientated counterclockwise. By the extended version of Green's theorem,

$$\begin{aligned}\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \iint_D Qx_P_y \, dA \\ &= \iint_D -\frac{y^2 - x^2}{(x^2+y^2)^2} + \frac{y^2 - x^2}{(x^2+y^2)^2} dA \\ &= 0,\end{aligned}$$

and therefore

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = -\int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}.$$

Since C_1 is a specific curve, we can evaluate $\int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$. Let

$$x = a\cos t, \quad y = a\sin t, \quad 0 \le t \le 2\pi$$

be a parameterization of C_1. Then,

$$\begin{aligned}\int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_0^{2\pi} \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}'(t) \, dt \\ &= \int_0^{2\pi} \langle -\frac{\sin(t)}{a}, -\frac{\cos(t)}{a} \rangle \cdot \langle -a\sin(t), -a\cos(t) \rangle \, dt \\ &= \int_0^{2\pi} \sin^2(t) + \cos^2(t) \, dt \\ &= \int_0^{2\pi} dt = 2\pi.\end{aligned}$$

Therefore, $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = -2\pi$.

? Exercise 15.4.8

Calculate integral $\oint_{\partial D} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where D is the annulus given by the polar inequalities $2 \leq r \leq 5$, $0 \leq \theta \leq 2\pi$, and $\vec{\mathbf{F}}(x, y) = \langle x^3, 5x + e^y \sin y \rangle$.

MEASURING AREA FROM A BOUNDARY: THE PLANIMETER

Imagine you are a doctor who has just received a magnetic resonance image of your patient's brain. The brain has a tumor (Figure **15.4.16**). How large is the tumor? To be precise, what is the area of the red region? The red cross-section of the tumor has an irregular shape, and therefore it is unlikely that you would be able to find a set of equations or inequalities for the region and then be able to calculate its area by conventional means. You could approximate the area by chopping the region into tiny squares (a Riemann sum approach), but this method always gives an answer with some error.

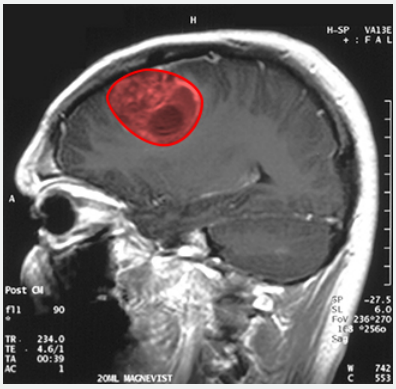


Figure **15.4.16**: This magnetic resonance image of a patient's brain shows a tumor, which is highlighted in red. (credit: modification of work by Christaras A, Wikimedia Commons)

Instead of trying to measure the area of the region directly, we can use a device called a *rolling planimeter* to calculate the area of the region exactly, simply by measuring its boundary. In this project you investigate how a planimeter works, and you use Green's theorem to show the device calculates area correctly.

A rolling planimeter is a device that measures the area of a planar region by tracing out the boundary of that region (Figure **15.4.17**). To measure the area of a region, we simply run the tracer of the planimeter around the boundary of the region. The planimeter measures the number of turns through which the wheel rotates as we trace the boundary; the area of the shape is proportional to this number of wheel turns. We can derive the precise proportionality equation using Green's theorem. As the tracer moves around the boundary of the region, the tracer arm rotates and the roller moves back and forth (but does not rotate).

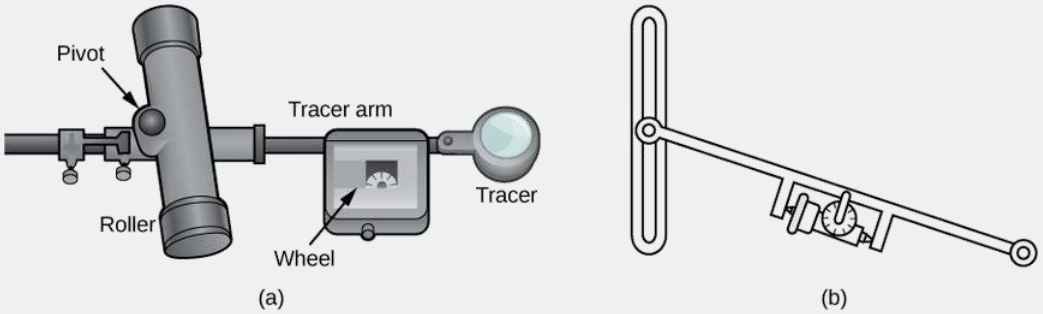


Figure **15.4.17**: (a) A rolling planimeter. The pivot allows the tracer arm to rotate. The roller itself does not rotate; it only moves back and forth. (b) An interior view of a rolling planimeter. Notice that the wheel cannot turn if the planimeter is moving back and forth with the tracer arm perpendicular to the roller.

Let C denote the boundary of region D, the area to be calculated. As the tracer traverses curve C, assume the roller moves along the y-axis (since the roller does not rotate, one can assume it moves along a straight line). Use the coordinates (x, y) to represent points on boundary C, and coordinates $(0, Y)$ to represent the position of the pivot. As the planimeter traces C, the pivot moves along the y-axis while the tracer arm rotates on the pivot.

Watch a short animation of a planimeter in action.

Begin the analysis by considering the motion of the tracer as it moves from point (x, y) counterclockwise to point $(x + dx, y + dy)$ that is close to (x, y) (Figure 15.4.18). The pivot also moves, from point $(0, Y)$ to nearby point $(0, Y + dY)$. How much does the wheel turn as a result of this motion? To answer this question, break the motion into two parts. First, roll the pivot along the *y*-axis from $(0, Y)$ to $(0, Y + dY)$ without rotating the tracer arm. The tracer arm then ends up at point $(x, y + dY)$ while maintaining a constant angle ϕ with the *x*-axis. Second, rotate the tracer arm by an angle $d\theta$ without moving the roller. Now the tracer is at point $(x + dx, y + dy)$. Let ll be the distance from the pivot to the wheel and let L be the distance from the pivot to the tracer (the length of the tracer arm).

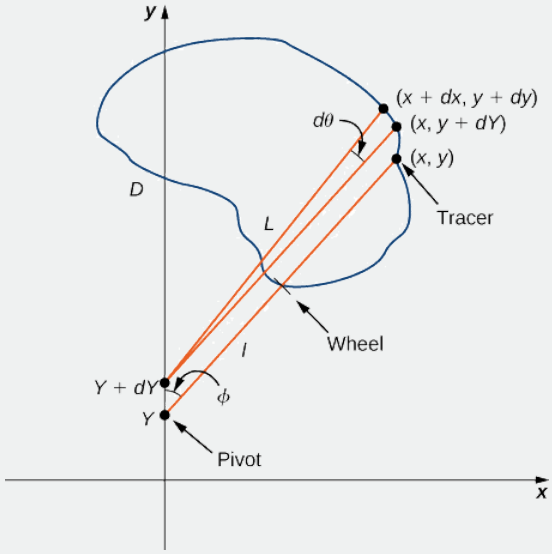


Figure 15.4.18: Mathematical analysis of the motion of the planimeter.

1. Explain why the total distance through which the wheel rolls the small motion just described is $\sin\phi dY + ld\theta = \frac{x}{L}dY + ld\theta$.
2. Show that $\oint_C d\theta = 0$.
3. Use step 2 to show that the total rolling distance of the wheel as the tracer traverses curve C is

   Total wheel roll $= \frac{1}{L}\oint_C x dY$.

   Now that you have an equation for the total rolling distance of the wheel, connect this equation to Green's theorem to calculate area D enclosed by C.
4. Show that $x^2 + (y - Y)^2 = L^2$.
5. Assume the orientation of the planimeter is as shown in Figure 15.4.18. Explain why $Y \le y$, and use this inequality to show there is a unique value of Y for each point (x, y): $Y = y = \sqrt{L^2 - x^2}$.
6. Use step 5 to show that $dY = dy + \frac{x}{L^2 - x^2}dx$.
7. Use Green's theorem to show that $\oint_C \frac{x}{L^2 - x^2}dx = 0$.
8. Use step 7 to show that the total wheel roll is

$$\text{Total wheel roll} \quad = \quad 1L\oint_C x\,dy.$$

   It took a bit of work, but this equation says that the variable of integration *Y* in step 3 can be replaced with *y*.
9. Use Green's theorem to show that the area of D is $\oint_C x\,dy$. The logic is similar to the logic used to show that the area of D $= 12\oint_C -y\,dx + x\,dy$.
10. Conclude that the area of D equals the length of the tracer arm multiplied by the total rolling distance of the wheel.

You now know how a planimeter works and you have used Green's theorem to justify that it works. To calculate the area of a planar region D, use a planimeter to trace the boundary of the region. The area of the region is the length of the tracer arm multiplied by the distance the wheel rolled.

Key Concepts

- Green's theorem relates the integral over a connected region to an integral over the boundary of the region. Green's theorem is a version of the Fundamental Theorem of Calculus in one higher dimension.
- Green's Theorem comes in two forms: a circulation form and a flux form. In the circulation form, the integrand is $\vec{\mathbf{F}} \cdot \vec{\mathbf{T}}$. In the flux form, the integrand is $\vec{\mathbf{F}} \cdot \vec{\mathbf{N}}$.
- Green's theorem can be used to transform a difficult line integral into an easier double integral, or to transform a difficult double integral into an easier line integral.
- A vector field is source free if it has a stream function. The flux of a source-free vector field across a closed curve is zero, just as the circulation of a conservative vector field across a closed curve is zero.

Key Equations

- **Green's theorem, circulation form**
  $\oint_C P\,dx + Q\,dy = \iint_D Q_x - P_y\,dA$, where C is the boundary of D
- **Green's theorem, flux form**
  $\oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\,ds = \iint_D P_x + Q_y\,dA$, where C is the boundary of D
- **Green's theorem, extended version**
  $\oint_{\partial D} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_D Q_x - P_y\,dA$

Glossary

Contributions

Paul Seeburger created a corrected image and the worked out solution for Exercise 15.4.3.

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15.4E: Exercises for Section 15.4

For the following exercises, evaluate the line integrals by applying Green's theorem.

1. $\int_C 2xy\,dx + (x+y)\,dy$, where C is the path from $(0,0)$ to $(1,1)$ along the graph of $y = x^3$ and from $(1,1)$ to $(0,0)$ along the graph of $y = x$ oriented in the counterclockwise direction

2. $\int_C 2xy\,dx + (x+y)\,dy$, where C is the boundary of the region lying between the graphs of $y = 0$ and $y = 4 - x^2$ oriented in the counterclockwise direction

Answer

$\int_C 2xy\,dx + (x+y)\,dy = \frac{32}{3}$ units of work

3. $\int_C 2\arctan\left(\frac{y}{x}\right)\,dx + \ln(x^2+y^2)\,dy$, where C is defined by $x = 4 + 2\cos\theta$, $y = 4\sin\theta$ oriented in the counterclockwise direction

4. $\int_C \sin x\cos y\,dx + (xy + \cos x\sin y)\,dy$, where C is the boundary of the region lying between the graphs of $y = x$ and $y = \sqrt{x}$ oriented in the counterclockwise direction

Answer

$\int_C \sin x\cos y\,dx + (xy + \cos x\sin y)\,dy = \frac{1}{12}$ units of work

5. $\int_C xy\,dx + (x+y)\,dy$, where C is the boundary of the region lying between the graphs of $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$ oriented in the counterclockwise direction

6. $\oint_C (-y\,dx + x\,dy)$, where C consists of line segment C_1 from $(-1,0)$ to $(1,0)$, followed by the semicircular arc C_2 from $(1,0)$ back to $(-1,0)$

Answer

$\oint_C (-y\,dx + x\,dy) = \pi$ units of work

For the following exercises, use Green's theorem.

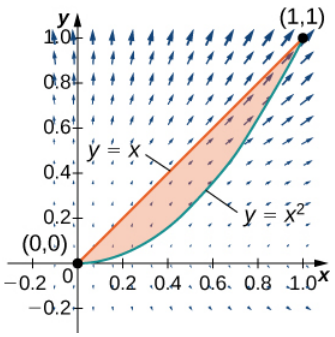
7. Let C be the curve consisting of line segments from $(0,0)$ to $(1,1)$ to $(0,1)$ and back to $(0,0)$. Find the value of $\int_C xy\,dx + \sqrt{y^2+1}\,dy$.

8. Evaluate line integral $\int_C xe^{-2x}\,dx + (x^4 + 2x^2y^2)\,dy$, where C is the boundary of the region between circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$, and is a positively oriented curve.

Answer

$\int_C xe^{-2x}\,dx + (x^4 + 2x^2y^2)\,dy = 0$ units of work

9. Find the counterclockwise circulation of field $\vec{\mathbf{F}}(x,y) = xy\,\hat{\mathbf{i}} + y^2\,\hat{\mathbf{j}}$ around and over the boundary of the region enclosed by curves $y = x^2$ and $y = x$ in the first quadrant and oriented in the counterclockwise direction.

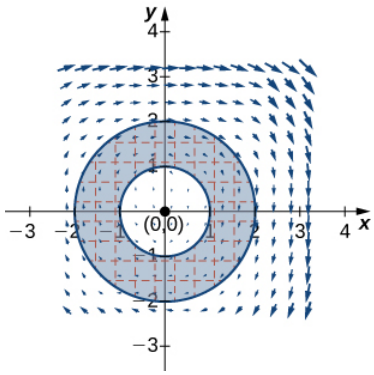


10. Evaluate $\oint_C y^3\,dx - x^3y^2\,dy$, where C is the positively oriented circle of radius 2 centered at the origin.

Answer

$\oint_C y^3\,dx - x^3y^2\,dy = -20\pi$ units of work

11. Evaluate $\oint_C y^3\,dx - x^3\,dy$, where C includes the two circles of radius 2 and radius 1 centered at the origin, both with positive orientation.



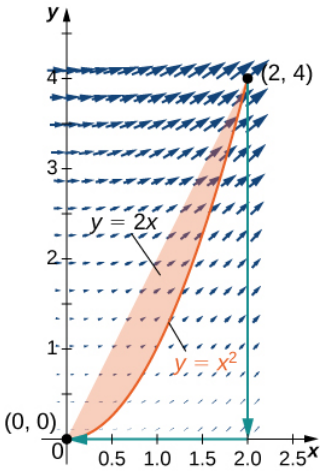
12. Calculate $\oint_C -x^2y\,dx + xy^2\,dy$, where C is a circle of radius 2 centered at the origin and oriented in the counterclockwise direction.

Answer

$\oint_C -x^2y\,dx + xy^2\,dy = 8\pi$ units of work

13. Calculate integral $\oint_C 2[y + x\sin(y)]\,dx + [x^2\cos(y) - 3y^2]\,dy$ along triangle C with vertices $(0,0)$, $(1,0)$ and $(1,1)$, oriented counterclockwise, using Green's theorem.

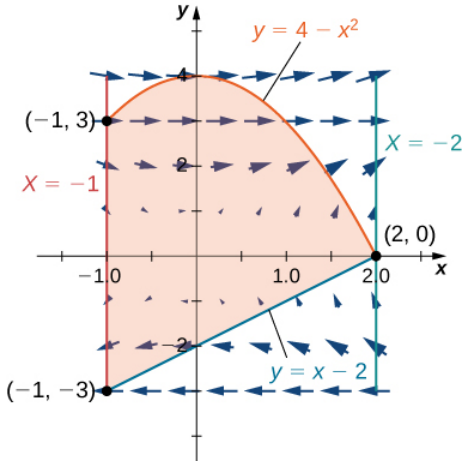
14. Evaluate integral $\oint_C (x^2 + y^2)\,dx + 2xy\,dy$, where C is the curve that follows parabola $y = x^2$ from $(0,0)$, $(2,4)$, then the line from $(2,4)$ to $(2,0)$, and finally the line from $(2,0)$ to $(0,0)$.



Answer

$\oint_C (x^2 + y^2)\,dx + 2xy\,dy = 0$ units of work

15. Evaluate line integral $\oint_C (y - \sin(y)\cos(y))\,dx + 2x\sin^2(y)\,dy$, where C is oriented in a counterclockwise path around the region bounded by $x = -1$, $x = 2$, $y = 4 - x^2$, and $y = x - 2$.



For the following exercises, use Green's theorem to find the area.

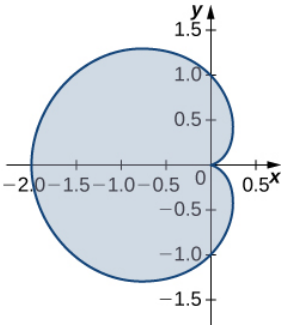
16. Find the area between ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and circle $x^2 + y^2 = 25$.

Answer

$A = 19\pi$ units^2

17. Find the area of the region enclosed by parametric equation

$$\vec{\mathbf{p}}(\theta) = (\cos(\theta) - \cos^2(\theta))\,\hat{\mathbf{i}} + (\sin(\theta) - \cos(\theta)\sin(\theta))\,\hat{\mathbf{j}} \text{ for } 0 \le \theta \le 2\pi.$$



18. Find the area of the region bounded by hypocycloid $\vec{\mathbf{r}}(t) = \cos^3(t)\,\hat{\mathbf{i}} + \sin^3(t)\,\hat{\mathbf{j}}$. The curve is parameterized by $t \in [0, 2\pi]$.

Answer

$A = \frac{3}{8\pi}$ units^2

19. Find the area of a pentagon with vertices $(0, 4)$, $(4, 1)$, $(3, 0)$, $(-1, -1)$, and $(-2, 2)$.

20. Use Green's theorem to evaluate $\int_{C^+} (y^2 + x^3)\,dx + x^4\,dy$, where C^+ is the perimeter of square $[0, 1] \times [0, 1]$ oriented counterclockwise.

Answer

$\int_{C^+} (y^2 + x^3)\,dx + x^4\,dy = 0$

21. Use Green's theorem to prove the area of a disk with radius a is $A = \pi a^2$ units^2.

22. Use Green's theorem to find the area of one loop of a four-leaf rose $r = 3\sin 2\theta$. (*Hint*: $x\,dy - y\,dx = r^2\,d\theta$).

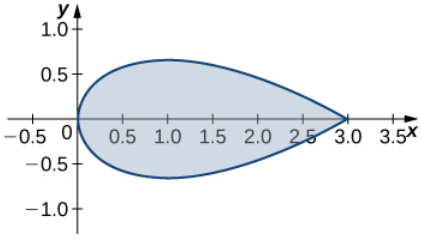
Answer

$A = \frac{9\pi}{8}$ units^2

23. Use Green's theorem to find the area under one arch of the cycloid given by the parametric equations: $x = t - \sin t,\ y = 1 - \cos t,\ t \geq 0$.

24. Use Green's theorem to find the area of the region enclosed by curve

$$\vec{\mathbf{r}}(t) = t^2\,\hat{\mathbf{i}} + \left(\tfrac{t^3}{3} - t\right)\hat{\mathbf{j}}, \text{ for } -\sqrt{3} \leq t \leq \sqrt{3}.$$



Answer

$A = \frac{8\sqrt{3}}{5}$ units^2

25. [T] Evaluate Green's theorem using a computer algebra system to evaluate the integral $\int_C xe^y\,dx + e^x\,dy$, where C is the circle given by $x^2 + y^2 = 4$ and is oriented in the counterclockwise direction.

26. Evaluate $\int_C (x^2y - 2xy + y^2)\,ds$, where C is the boundary of the unit square $0 \leq x \leq 1,\ 0 \leq y \leq 1$, traversed counterclockwise.

Answer

$$\int_C (x^2y - 2xy + y^2)\, ds = 3$$

27. Evaluate $\int_C \frac{-(y+2)\,dx + (x-1)\,dy}{(x-1)^2 + (y+2)^2}$, where C is any simple closed curve with an interior that does not contain point $(1, -2)$ traversed counterclockwise.

28. Evaluate $\int_C \frac{x\,dx + y\,dy}{x^2 + y^2}$, where C is any piecewise, smooth simple closed curve enclosing the origin, traversed counterclockwise.

Answer

$$\int_C \frac{x\,dx + y\,dy}{x^2 + y^2} = 2\pi$$

For the following exercises, use Green's theorem to calculate the work done by force $\vec{\mathbf{F}}$ on a particle that is moving counterclockwise around closed path C.

29. $\vec{\mathbf{F}}(x, y) = xy\,\hat{\mathbf{i}} + (x + y)\,\hat{\mathbf{j}}, \quad C : x^2 + y^2 = 4$

30. $\vec{\mathbf{F}}(x, y) = (x^{3/2} - 3y)\,\hat{\mathbf{i}} + (6x + 5\sqrt{y})\,\hat{\mathbf{j}}, \quad C$: boundary of a triangle with vertices $(0, 0)$, $(5, 0)$, and $(0, 5)$

Answer

$W = \frac{225}{2}$ units of work

31. Evaluate $\int_C (2x^3 - y^3)\,dx + (x^3 + y^3)\,dy$, where C is a unit circle oriented in the counterclockwise direction.

32. A particle starts at point $(-2, 0)$, moves along the x-axis to $(2, 0)$, and then travels along semicircle $y = \sqrt{4 - x^2}$ to the starting point. Use Green's theorem to find the work done on this particle by force field $\vec{\mathbf{F}}(x, y) = x\,\hat{\mathbf{i}} + (x^3 + 3xy^2)\,\hat{\mathbf{j}}$.

Answer

$W = 12\pi$ units of work

33. David and Sandra are skating on a frictionless pond in the wind. David skates on the inside, going along a circle of radius 2 in a counterclockwise direction. Sandra skates once around a circle of radius 3, also in the counterclockwise direction. Suppose the force of the wind at point (x, y) is $\vec{\mathbf{F}}(x, y) = (x^2y + 10y)\,\hat{\mathbf{i}} + (x^3 + 2xy^2)\,\hat{\mathbf{j}}$. Use Green's theorem to determine who does more work.

34. Use Green's theorem to find the work done by force field $\vec{\mathbf{F}}(x, y) = (3y - 4x)\,\hat{\mathbf{i}} + (4x - y)\,\hat{\mathbf{j}}$ when an object moves once counterclockwise around ellipse $4x^2 + y^2 = 4$.

Answer

$W = 2\pi$ units of work

35. Use Green's theorem to evaluate line integral $\oint_C e^{2x} \sin 2y\,dx + e^{2x} \cos 2y\,dy$, where C is ellipse $9(x - 1)^2 + 4(y - 3)^2 = 36$ oriented counterclockwise.

36. Evaluate line integral $\oint_C y^2\,dx + x^2\,dy$, where C is the boundary of a triangle with vertices $(0, 0)$, $(1, 1)$, and $(1, 0)$, with the counterclockwise orientation.

Answer

$\oint_C y^2\,dx + x^2\,dy = \frac{1}{3}$ units of work

37. Use Green's theorem to evaluate line integral $\int_C \vec{\mathbf{h}} \cdot d\vec{\mathbf{r}}$ if $\vec{\mathbf{h}}(x, y) = e^y\,\hat{\mathbf{i}} - \sin \pi x\,\hat{\mathbf{j}}$, where C is a triangle with vertices $(1, 0)$, $(0, 1)$, and $(-1, 0)$, traversed counterclockwise.

38. Use Green's theorem to evaluate line integral $\int_C \sqrt{1+x^3}\,dx + 2xy\,dy$ where C is a triangle with vertices $(0,0)$, $(1,0)$, and $(1,3)$ oriented clockwise.

Answer

$\int_C \sqrt{1+x^3}\,dx + 2xy\,dy = 3$ units of work

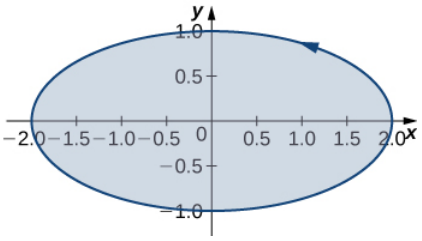
39. Use Green's theorem to evaluate line integral $\int_C x^2y\ dx\ -xy^2\ dy$ where C is a circle $x^2\ +y^2\ =4$ oriented counterclockwise.

40. Use Green's theorem to evaluate line integral $\int_C \left(3y - e^{\sin x}\right)\ dx + \left(7x + \sqrt{y^4+1}\right)\ dy$ where C is circle $x^2\ +y^2\ =9$ oriented in the counterclockwise direction.

Answer

$\int_C \left(3y - e^{\sin x}\right)\ dx + \left(7x + \sqrt{y^4+1}\right)\ dy = 36\pi$ units of work

41. Use Green's theorem to evaluate line integral $\int_C (3x-5y)\,dx + (x-6y)\,dy$, where C is ellipse $\frac{x^2}{4} + y^2 = 1$ and is oriented in the counterclockwise direction.



42. Let C be a triangular closed curve from $(0,0)$ to $(1,0)$ to $(1,1)$ and finally back to $(0,0)$. Let $\vec{\mathbf{F}}(x,y) = 4y\,\hat{\mathbf{i}} + 6x^2\,\hat{\mathbf{j}}$. Use Green's theorem to evaluate $\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$.

Answer

$\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 2$ units of work

43. Use Green's theorem to evaluate line integral $\oint_C y\ dx - x\ dy$, where C is circle $x^2\ +y^2\ =a^2$ oriented in the clockwise direction.

44. Use Green's theorem to evaluate line integral $\oint_C (y+x)\,dx + (x+\sin y)\,dy$, where C is any smooth simple closed curve joining the origin to itself oriented in the counterclockwise direction.

Answer

$\oint_C (y+x)\,dx + (x+\sin y)\,dy = 0$ units of work

45. Use Green's theorem to evaluate line integral $\oint_C \left(y - \ln(x^2+y^2)\right)\ dx + \left(2\arctan\frac{y}{x}\right)\ dy$, where C is the positively oriented circle $(x-2)^2 + (y-3)^2 = 1$.

46. Use Green's theorem to evaluate $\oint_C xy\ dx\ + x^3y^3\ dy$, where C is a triangle with vertices $(0,0)$, $(1,0)$, and $(1,2)$ with positive orientation.

Answer

$\oint_C xy\,dx + x^3y^3\,dy = 2221$ units of work

47. Use Green's theorem to evaluate line integral $\int_C \sin y\, dx + x\cos y\, dy$, where C is ellipse $x^2 + xy + y^2 = 1$ oriented in the counterclockwise direction.

48. Let $\vec{\mathbf{F}}(x,y) = (\cos(x^5) - 13y^3)\,\hat{\mathbf{i}} + 13x^3\,\hat{\mathbf{j}}$. Find the counterclockwise circulation $\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where C is a curve consisting of the line segment joining $(-2,0)$ and $(-1,0)$, half circle $y = \sqrt{1-x^2}$, the line segment joining $(1,0)$ and $(2,0)$, and half circle $y = \sqrt{4-x^2}$.

Answer

$\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 15\pi^4$ units of work

49. Use Green's theorem to evaluate line integral $\int_C \sin(x^3)\, dx + 2ye^{x^2}\, dy$, where C is a triangular closed curve that connects the points $(0,0)$, $(2,2)$, and $(0,2)$ counterclockwise.

50. Let C be the boundary of square $0 \le x \le \pi,\ 0 \le y \le \pi$, traversed counterclockwise. Use Green's theorem to find $\int_C \sin(x+y)\, dx + \cos(x+y)\, dy$.

Answer

$\int_C \sin(x+y)\, dx + \cos(x+y)\, dy = 4$ units of work

51. Use Green's theorem to evaluate line integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where $\vec{\mathbf{F}}(x,y) = (y^2 - x^2)\,\hat{\mathbf{i}} + (x^2 + y^2)\,\hat{\mathbf{j}}$, and C is a triangle bounded by $y = 0,\ x = 3$, and $y = x$, oriented counterclockwise.

52. Use Green's Theorem to evaluate integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where $\vec{\mathbf{F}}(x,y) = (xy^2)\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}}$, and C is a unit circle oriented in the counterclockwise direction.

Answer

$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \pi$ units of work

53. Use Green's theorem in a plane to evaluate line integral $\oint_C (xy + y^2)\, dx + x^2\, dy$, where C is a closed curve of a region bounded by $y = x$ and $y = x^2$ oriented in the counterclockwise direction.

54. Calculate the outward flux of $\vec{\mathbf{F}}(x,y) = -x\,\hat{\mathbf{i}} + 2y\,\hat{\mathbf{j}}$ over a square with corners $(\pm 1, \pm 1)$, where the unit normal is outward pointing and oriented in the counterclockwise direction.

Answer

$\oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds = 4$

55. **[T]** Let C be circle $x^2 + y^2 = 4$ oriented in the counterclockwise direction. Evaluate $\oint_C \left[(3y - e^{\arctan x})\, dx + \left(7x + \sqrt{y^4+1}\right)\, dy\right]$ using a computer algebra system.

56. Find the flux of field $\vec{\mathbf{F}}(x,y) = -x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}$ across $x^2 + y^2 = 16$ oriented in the counterclockwise direction.

Answer

$\oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds = 32\pi$

57. Let $\vec{\mathbf{F}} = (y^2 - x^2)\,\hat{\mathbf{i}} + (x^2 + y^2)\,\hat{\mathbf{j}}$, and let C be a triangle bounded by $y = 0$, $x = 3$, and $y = x$ oriented in the counterclockwise direction. Find the outward flux of $\vec{\mathbf{F}}$ through C.

58. **[T]** Let C be unit circle $x^2 + y^2 = 1$ traversed once counterclockwise. Evaluate $\int_C \left[-y^3 + \sin(xy) + xy\cos(xy)\right]\, dx + \left[x^3 + x^2\cos(xy)\right]\, dy$ by using a computer algebra system.

Answer

$\int_C \left[-y^3 + \sin(xy) + xy\cos(xy)\right]\, dx + \left[x^3 + x^2\cos(xy)\right]\, dy = 4.7124$ units of work

59. [T] Find the outward flux of vector field $\vec{\mathbf{F}}(x, y) = xy^2\,\hat{\mathbf{i}} + x^2y\,\hat{\mathbf{j}}$ across the boundary of annulus $R = \{(x, y) : 1 \leq x^2 + y^2 \leq 4\} = \{(r, \theta) : 1 \leq r \leq 2,\ 0 \leq \theta \leq 2\pi\}$ using a computer algebra system.

60. Consider region R bounded by parabolas $y = x^2$ and $x = y^2$. Let C be the boundary of R oriented counterclockwise. Use Green's theorem to evaluate $\oint_C \left(y + e^{\sqrt{x}}\right)\, dx + \left(2x + \cos(y^2)\right)\, dy$.

Answer

$\oint_C \left(y + e^{\sqrt{x}}\right)\, dx + \left(2x + \cos(y^2)\right)\, dy = 13$ units of work

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15.5: Divergence and Curl

Learning Objectives

- Determine divergence from the formula for a given vector field.
- Determine curl from the formula for a given vector field.
- Use the properties of curl and divergence to determine whether a vector field is conservative.

In this section, we examine two important operations on a vector field: divergence and curl. They are important to the field of calculus for several reasons, including the use of curl and divergence to develop some higher-dimensional versions of the Fundamental Theorem of Calculus. In addition, curl and divergence appear in mathematical descriptions of fluid mechanics, electromagnetism, and elasticity theory, which are important concepts in physics and engineering. We can also apply curl and divergence to other concepts we already explored. For example, under certain conditions, a vector field is conservative if and only if its curl is zero.

In addition to defining curl and divergence, we look at some physical interpretations of them, and show their relationship to conservative and source-free vector fields.

Divergence

Divergence is an operation on a vector field that tells us how the field behaves toward or away from a point. Locally, the divergence of a vector field $\vec{\mathbf{F}}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ at a particular point P is a measure of the "outflowing-ness" of the vector field at P. If $\vec{\mathbf{F}}$ represents the velocity of a fluid, then the divergence of $\vec{\mathbf{F}}$ at P measures the net rate of change with respect to time of the amount of fluid flowing away from P (the tendency of the fluid to flow "out of" P). In particular, if the amount of fluid flowing into P is the same as the amount flowing out, then the divergence at P is zero.

Definition: divergence in $\mathbb{R}^3$

If $\vec{\mathbf{F}} = \langle P, Q, R \rangle$ is a vector field in $\mathbb{R}^3$ and P_x, Q_y, and R_z all exist, then the divergence of $\vec{\mathbf{F}}$ is defined by

$$\operatorname{div} F = P_x + Q_y + R_z \tag{15.5.1}$$

$$= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}. \tag{15.5.2}$$

Note the divergence of a vector field is not a vector field, but a scalar function. In terms of the gradient operator

$$\vec{\nabla} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$$

divergence can be written symbolically as the dot product

$$\operatorname{div} \vec{\mathbf{F}} = \vec{\nabla} \cdot \vec{\mathbf{F}}.$$

Note this is merely helpful notation, because the dot product of a vector of operators and a vector of functions is not meaningfully defined given our current definition of dot product.

If $\vec{\mathbf{F}} = \langle P, Q \rangle$ is a vector field in $\mathbb{R}^2$, and P_x and Q_y both exist, then the divergence of $\vec{\mathbf{F}}$ is defined similarly as

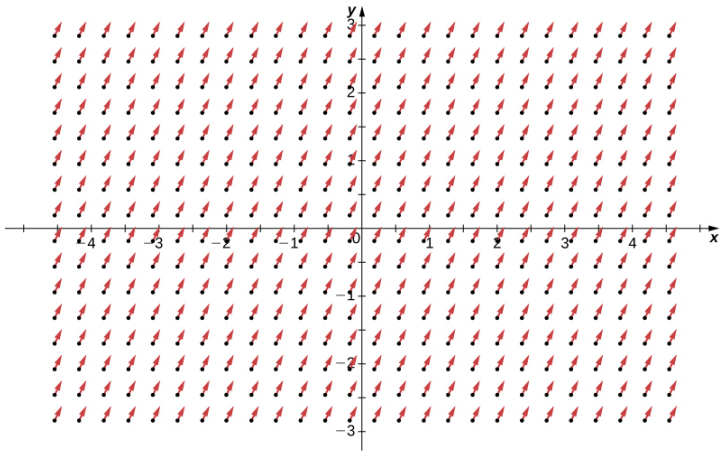
$$\begin{aligned} \operatorname{div} \vec{\mathbf{F}} &= P_x + Q_y \\ &= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \\ &= \vec{\nabla} \cdot \vec{\mathbf{F}}. \end{aligned}$$

To illustrate this point, consider the two vector fields in Figure 15.5.1. At any particular point, the amount flowing in is the same as the amount flowing out, so at every point the "outflowing-ness" of the field is zero. Therefore, we expect the divergence of both fields to be zero, and this is indeed the case, as

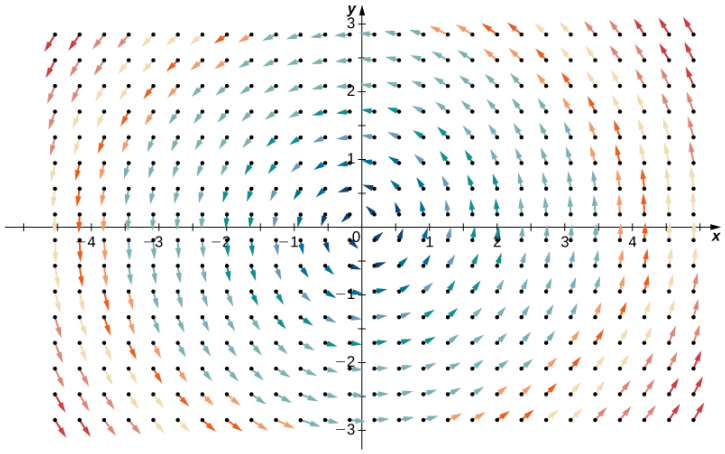
$$\text{div}(\langle 1,2\rangle) = \frac{\partial}{\partial x}(1) + \frac{\partial}{\partial y}(2) = 0$$

and

$$\text{div}(\langle -y,x\rangle) = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x) = 0.$$



(a)



(b)

Figure 15.5.1: (a) Vector field $\langle 1,2\rangle$ has zero divergence. (b) Vector field $\langle -y,x\rangle$ also has zero divergence.

By contrast, consider radial vector field $\vec{\mathbf{R}}(x,y) = \langle -x,-y\rangle$ in Figure 15.5.2. At any given point, more fluid is flowing in than is flowing out, and therefore the "outgoingness" of the field is negative. We expect the divergence of this field to be negative, and this is indeed the case, as

$$\text{div}(\vec{\mathbf{R}}) = \frac{\partial}{\partial x}(-x) + \frac{\partial}{\partial y}(-y) = -2.$$

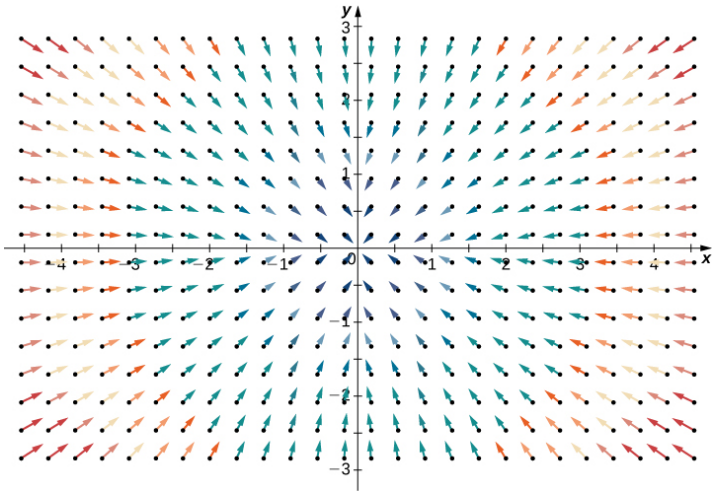


Figure 15.5.2: This vector field has negative divergence.

To get a global sense of what divergence is telling us, suppose that a vector field in $\mathbb{R}^2$ represents the velocity of a fluid. Imagine taking an elastic circle (a circle with a shape that can be changed by the vector field) and dropping it into a fluid. If the circle maintains its exact area as it flows through the fluid, then the divergence is zero. This would occur for both vector fields in Figure 15.5.1. On the other hand, if the circle's shape is distorted so that its area shrinks or expands, then the divergence is not zero. Imagine dropping such an elastic circle into the radial vector field in Figure 15.5.2 so that the center of the circle lands at point $(3,3)$. The circle would flow toward the origin, and as it did so the front of the circle would travel more slowly than the back, causing the circle to "scrunch" and lose area. This is how you can see a negative divergence.

✔ Example 15.5.1: Calculating Divergence at a Point

If $\vec{\mathbf{F}}(x,y,z) = e^x\hat{i} + yz\hat{j} - yz^2\hat{k}$, then find the divergence of $\vec{\mathbf{F}}$ at $(0,2,-1)$.

Solution

The divergence of $\vec{\mathbf{F}}$ is

$$\frac{\partial}{\partial x}(e^x) + \frac{\partial}{\partial y}(yz) - \frac{\partial}{\partial z}(yz^2) = e^x + z - 2yz.$$

Therefore, the divergence at $(0,2,-1)$ is $e^0 - 1 + 4 = 4$. If $\vec{\mathbf{F}}$ represents the velocity of a fluid, then more fluid is flowing out than flowing in at point $(0,2,-1)$.

? Exercise 15.5.1

Find $\operatorname{div} \vec{\mathbf{F}}$ for

$$\vec{\mathbf{F}}(x,y,z) = \langle xy,\ 5 - z^2,\ x^2 + y^2 \rangle.$$

One application for divergence occurs in physics, when working with magnetic fields. A magnetic field is a vector field that models the influence of electric currents and magnetic materials. Physicists use divergence in Gauss's law for magnetism, which states that if $\vec{\mathbf{B}}$ is a magnetic field, then $\vec{\nabla} \cdot \vec{\mathbf{B}} = 0$; in other words, the divergence of a magnetic field is zero.

✔ Example 15.5.2: Determining Whether a Field Is Magnetic

Is it possible for $\vec{\mathbf{F}}(x,y) = \langle x^2y,\ y - xy^2 \rangle$ to be a magnetic field?

Solution

If $\vec{\mathbf{F}}$ were magnetic, then its divergence would be zero. The divergence of $\vec{\mathbf{F}}$ is

$$\frac{\partial}{\partial x}(x^2y)+\frac{\partial}{\partial y}(y-xy^2)=2xy+1-2xy=1$$

and therefore $\vec{\mathbf{F}}$ cannot model a magnetic field (Figure 15.5.3).

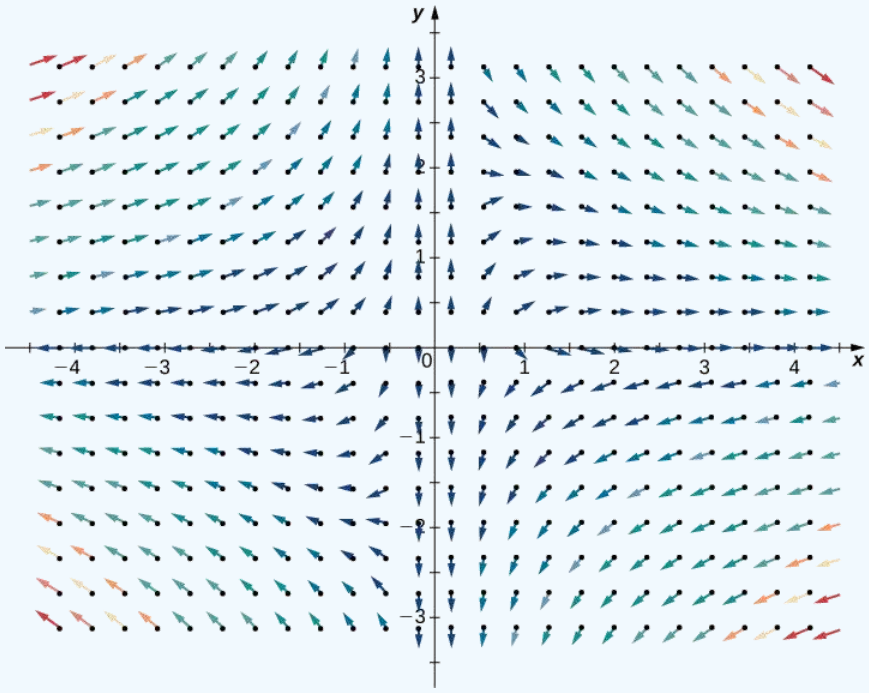


Figure 15.5.3: The divergence of vector field $\vec{\mathbf{F}}(x,y)=\langle x^2y,\ y-xy^2\rangle$ is one, so it cannot model a magnetic field.

Another application for divergence is detecting whether a field is source free. Recall that a source-free field is a vector field that has a stream function; equivalently, a source-free field is a field with a flux that is zero along any closed curve. The next two theorems say that, under certain conditions, source-free vector fields are precisely the vector fields with zero divergence.

Theorem: Divergence of a Source-Free Vector Field

If $\vec{\mathbf{F}}=\langle P,Q\rangle$ is a source-free continuous vector field with differentiable component functions, then $\operatorname{div}\vec{\mathbf{F}}=0$.

Proof

Since $\vec{\mathbf{F}}$ is source free, there is a function $g(x,y)$ with $g_y=P$ and $-g_x=Q$. Therefore, $\vec{\mathbf{F}}=\langle g_y,-g_x\rangle$ and $\operatorname{div}\vec{\mathbf{F}}=g_{yx}-g_{xy}=0$ by Clairaut's theorem.

□

The converse of Divergence of a Source-Free Vector Field is true on simply connected regions, but the proof is too technical to include here. Thus, we have the following theorem, which can test whether a vector field in $\mathbb{R}^2$ is source free.

Theorem: Divergence Test for Source-Free Vector Fields

Let $\vec{\mathbf{F}}=\langle P,Q\rangle$ be a continuous vector field with differentiable component functions with a domain that is simply connected. Then, $\operatorname{div}\vec{\mathbf{F}}=0$ if and only if $\vec{\mathbf{F}}$ is source free.

Example 15.5.3: Determining Whether a Field Is Source Free

Is field $\vec{\mathbf{F}}(x,y)=\langle x^2y,\ 5-xy^2\rangle$ source free?

Solution

Note the domain of $\vec{\mathbf{F}}$ is $\mathbb{R}^2$ which is simply connected. Furthermore, $\vec{\mathbf{F}}$ is continuous with differentiable component functions. Therefore, we can use the Divergence Test for Source-Free Vector Fields to analyze $\vec{\mathbf{F}}$. The divergence of $\vec{\mathbf{F}}$ is

$$\frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(5 - xy^2) = 2xy - 2xy = 0.$$

Therefore, $\vec{\mathbf{F}}$ is source free by the Divergence Test for Source-Free Vector Fields.

? Exercise 15.5.2

Let $\vec{\mathbf{F}}(x, y) = \langle -ay, bx \rangle$ be a rotational field where a and b are positive constants. Is $\vec{\mathbf{F}}$ source free?

Recall that the flux form of Green's theorem says that

$$\oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds = \iint_D P_x + Q_y\, dA,$$

where C is a simple closed curve and D is the region enclosed by C. Since $P_x + Q_y = \operatorname{div} \vec{\mathbf{F}}$, Green's theorem is sometimes written as

$$\oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds = \iint_D \operatorname{div} \vec{\mathbf{F}}\, dA.$$

Therefore, Green's theorem can be written in terms of divergence. If we think of divergence as a derivative of sorts, then Green's theorem says the "derivative" of $\vec{\mathbf{F}}$ on a region can be translated into a line integral of $\vec{\mathbf{F}}$ along the boundary of the region. This is analogous to the Fundamental Theorem of Calculus, in which the derivative of a function f on a line segment $[a, b]$ can be translated into a statement about f on the boundary of $[a, b]$. Using divergence, we can see that Green's theorem is a higher-dimensional analog of the Fundamental Theorem of Calculus.

We can use all of what we have learned in the application of divergence. Let $\vec{\mathbf{v}}$ be a vector field modeling the velocity of a fluid. Since the divergence of $\vec{\mathbf{v}}$ at point P measures the "outflowing-ness" of the fluid at P, $\operatorname{div} v(P) > 0$ implies that more fluid is flowing out of P than flowing in. Similarly, $\operatorname{div} v(P) < 0$ implies the more fluid is flowing in to P than is flowing out, and $\operatorname{div} \vec{\mathbf{v}}(P) = 0$ implies the same amount of fluid is flowing in as flowing out.

✓ Example 15.5.4: Determining Flow of a Fluid

Suppose $\vec{\mathbf{v}}(x, y) = \langle -xy, y \rangle$, $y > 0$ models the flow of a fluid. Is more fluid flowing into point $(1, 4)$ than flowing out?

Solution

To determine whether more fluid is flowing into $(1, 4)$ than is flowing out, we calculate the divergence of $\vec{\mathbf{v}}$ at $(1, 4)$:

$$div(\vec{\mathbf{v}}) = \frac{\partial}{\partial x}(-xy) + \frac{\partial}{\partial y}(y) = -y + 1.$$

To find the divergence at $(1, 4)$ substitute the point into the divergence: $-4 + 1 = -3$. Since the divergence of $\vec{\mathbf{v}}$ at $(1, 4)$ is negative, more fluid is flowing in than flowing out (Figure 15.5.4).

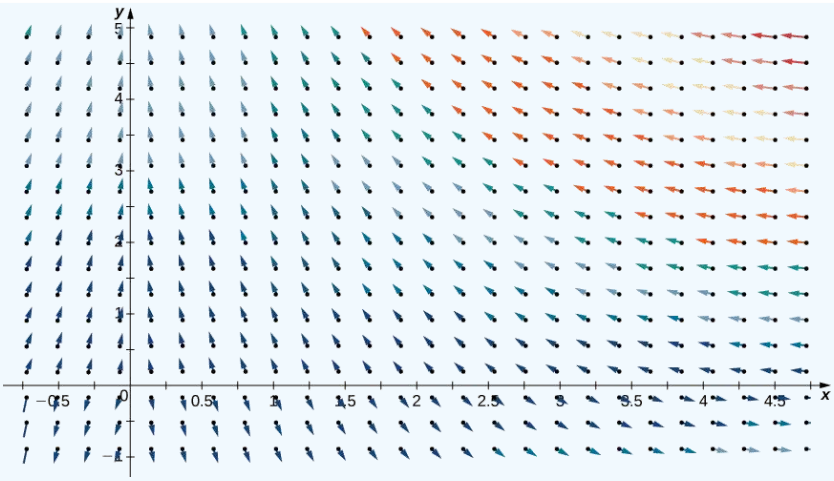


Figure 15.5.4: Vector field $\vec{\mathbf{v}}(x,y) = \langle -xy, y\rangle$ has negative divergence at $(1,4)$

? Exercise 15.5.3

For vector field $\vec{\mathbf{v}}(x,y) = \langle -xy, y\rangle$, $y > 0$, find all points P such that the amount of fluid flowing in to P equals the amount of fluid flowing out of P.

Curl

The second operation on a vector field that we examine is the curl, which measures the extent of rotation of the field about a point. Suppose that $\vec{\mathbf{F}}$ represents the velocity field of a fluid. Then, the curl of $\vec{\mathbf{F}}$ at point P is a vector that measures the tendency of particles near P to rotate about the axis that points in the direction of this vector. The magnitude of the curl vector at P measures how quickly the particles rotate around this axis. In other words, the curl at a point is a measure of the vector field's "spin" at that point. Visually, imagine placing a paddlewheel into a fluid at P, with the axis of the paddlewheel aligned with the curl vector (Figure 15.5.5). The curl measures the tendency of the paddlewheel to rotate.

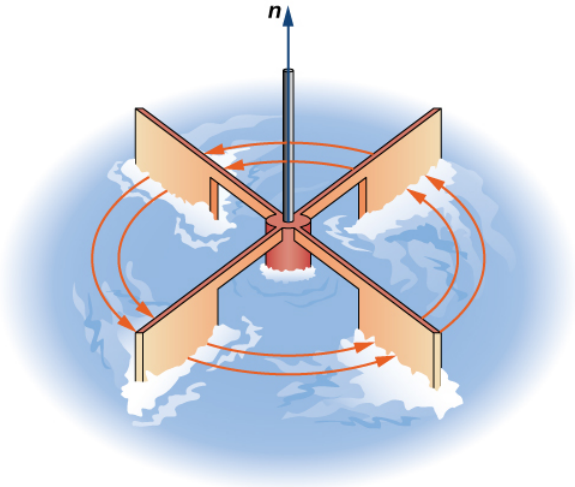


Figure 15.5.5: To visualize curl at a point, imagine placing a small paddlewheel into the vector field at a point.

Consider the vector fields in Figure 15.5.1. In part (a), the vector field is constant and there is no spin at any point. Therefore, we expect the curl of the field to be zero, and this is indeed the case. Part (b) shows a rotational field, so the field has spin. In particular, if you place a paddlewheel into a field at any point so that the axis of the wheel is perpendicular to a plane, the wheel rotates counterclockwise. Therefore, we expect the curl of the field to be nonzero, and this is indeed the case (the curl is $2\hat{\mathbf{k}}$).

To see what curl is measuring globally, imagine dropping a leaf into the fluid. As the leaf moves along with the fluid flow, the curl measures the tendency of the leaf to rotate. If the curl is zero, then the leaf doesn't rotate as it moves through the fluid.

Definition: Curl

If $\vec{\mathbf{F}} = \langle P, Q, R \rangle$ is a vector field in $\mathbb{R}^3$, and P_x, Q_y, and R_z all exist, then the curl of $\vec{\mathbf{F}}$ is defined by

$$\operatorname{curl} \vec{\mathbf{F}} = (R_y - Q_z)\,\hat{\mathbf{i}} + (P_z - R_x)\,\hat{\mathbf{j}} + (Q_x - P_y)\,\hat{\mathbf{k}} \tag{15.5.3}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{\mathbf{k}}. \tag{15.5.4}$$

Note that the curl of a vector field is a vector field, in contrast to divergence.

The definition of curl can be difficult to remember. To help with remembering, we use the notation $\vec{\nabla} \times \vec{\mathbf{F}}$ to stand for a "determinant" that gives the curl formula:

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}.$$

The determinant of this matrix is

$$(R_y - Q_z)\,\hat{\mathbf{i}} - (R_x - P_z)\,\hat{\mathbf{j}} + (Q_x - P_y)\,\hat{\mathbf{k}}$$
$$= (R_y - Q_z)\,\hat{\mathbf{i}} + (P_z - R_x)\,\hat{\mathbf{j}} + (Q_x - P_y)\,\hat{\mathbf{k}} = \operatorname{curl} \vec{\mathbf{F}}.$$

Thus, this matrix is a way to help remember the formula for curl. Keep in mind, though, that the word *determinant* is used very loosely. A determinant is not really defined on a matrix with entries that are three vectors, three operators, and three functions.

If $\vec{\mathbf{F}} = \langle P, Q \rangle$ is a vector field in $\mathbb{R}^2$, then the curl of $\vec{\mathbf{F}}$, by definition, is

$$\operatorname{curl} \vec{\mathbf{F}} = (Q_x - P_y)\,\hat{\mathbf{k}} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{\mathbf{k}}.$$

Example 15.5.5: Finding the Curl of a Three-Dimensional Vector Field

Find the curl of $\vec{\mathbf{F}}(P, Q, R) = \langle x^2 z, e^y + xz, xyz \rangle$.

Solution

The curl is

$$\begin{aligned} \operatorname{curl} f &= \vec{\nabla} \times \vec{\mathbf{F}} \\ &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} \\ &= (R_y - Q_z)\,\hat{\mathbf{i}} + (P_z - R_x)\,\hat{\mathbf{j}} + (Q_x - P_y)\,\hat{\mathbf{k}} \\ &= (xz - x)\,\hat{\mathbf{i}} + (x^2 - yz)\,\hat{\mathbf{j}} + z\hat{\mathbf{k}}. \end{aligned}$$

Exercise 15.5.4

Find the curl of $\vec{\mathbf{F}} = \langle \sin x \cos z,\ \sin y \sin z,\ \cos x \cos y \rangle$ at point $\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right)$.

Example 15.5.6: Finding the Curl of a Two-Dimensional Vector Field

Find the curl of $\vec{\mathbf{F}} = \langle P, Q \rangle = \langle y, 0 \rangle$.

Solution

Notice that this vector field consists of vectors that are all parallel. In fact, each vector in the field is parallel to the x-axis. This fact might lead us to the conclusion that the field has no spin and that the curl is zero. To test this theory, note that

$$\text{curl}\,\vec{\mathbf{F}} = (Q_x - P_y)\,\hat{\mathbf{k}} = -\hat{\mathbf{k}} \neq \vec{\mathbf{0}}.$$

Therefore, this vector field does have spin. To see why, imagine placing a paddlewheel at any point in the first quadrant (Figure 15.5.6). The larger magnitudes of the vectors at the top of the wheel cause the wheel to rotate. The wheel rotates in the clockwise (negative) direction, causing the coefficient of the curl to be negative.

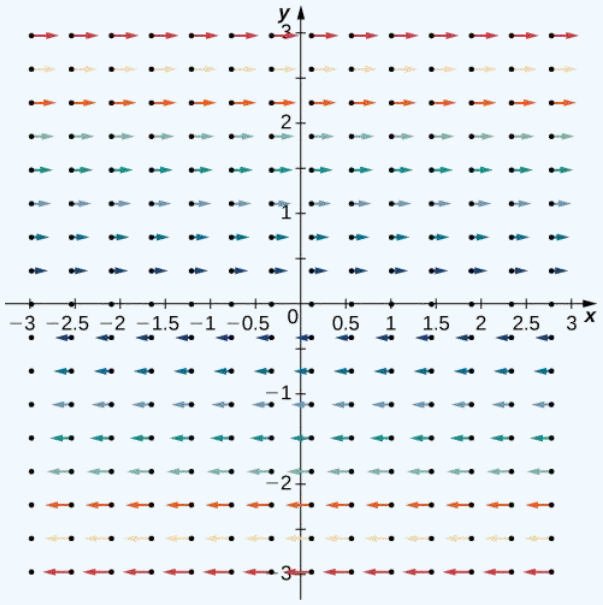


Figure 15.5.6: Vector field $\vec{\mathbf{F}}(x,y) = \langle y, 0\rangle$ consists of vectors that are all parallel.

Note that if $\vec{\mathbf{F}} = \langle P, Q\rangle$ is a vector field in a plane, then $\text{curl}\,\vec{\mathbf{F}} \cdot \hat{\mathbf{k}} = (Q_x - P_y)\,\hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = Q_x - P_y$. Therefore, the circulation form of Green's theorem is sometimes written as

$$\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_D \text{curl}\,\vec{\mathbf{F}} \cdot \hat{\mathbf{k}}\, dA,$$

where C is a simple closed curve and D is the region enclosed by C. Therefore, the circulation form of Green's theorem can be written in terms of the curl. If we think of curl as a derivative of sorts, then Green's theorem says that the "derivative" of $\vec{\mathbf{F}}$ on a region can be translated into a line integral of $\vec{\mathbf{F}}$ along the boundary of the region. This is analogous to the Fundamental Theorem of Calculus, in which the derivative of a function f on line segment $[a,b]$ can be translated into a statement about f on the boundary of $[a,b]$. Using curl, we can see the circulation form of Green's theorem is a higher-dimensional analog of the Fundamental Theorem of Calculus.

We can now use what we have learned about curl to show that gravitational fields have no "spin." Suppose there is an object at the origin with mass m_1 at the origin and an object with mass m_2. Recall that the gravitational force that object 1 exerts on object 2 is given by field

$$\vec{\mathbf{F}}(x,y,z) = -Gm_1m_2\left\langle \frac{x}{(x^2+y^2+z^2)^{3/2}}, \frac{y}{(x^2+y^2+z^2)^{3/2}}, \frac{z}{(x^2+y^2+z^2)^{3/2}}\right\rangle.$$

Example 15.5.7: Determining the Spin of a Gravitational Field

Show that a gravitational field has no spin.

Solution

To show that $\vec{\mathbf{F}}$ has no spin, we calculate its curl. Let

- $P(x,y,z) = \dfrac{x}{(x^2+y^2+z^2)^{3/2}}$,
- $Q(x,y,z) = \dfrac{y}{(x^2+y^2+z^2)^{3/2}}$, and
- $R(x,y,z) = \dfrac{z}{(x^2+y^2+z^2)^{3/2}}$.

Then,

$$\begin{aligned}\operatorname{curl}\vec{\mathbf{F}} &= -Gm_1m_2[(R_y-Q_z)\hat{\mathbf{i}}+(P_z-R_x)\hat{\mathbf{j}}+(Q_x-P_y)\hat{\mathbf{k}}] \\ &= -Gm_1m_2\begin{pmatrix} \left(\frac{-3yz}{(x^2+y^2+z^2)^{5/2}}-\left(\frac{-3yz}{(x^2+y^2+z^2)^{5/2}}\right)\right)\hat{\mathbf{i}} \\ +\left(\frac{-3xz}{(x^2+y^2+z^2)^{5/2}}-\left(\frac{-3xz}{(x^2+y^2+z^2)^{5/2}}\right)\right)\hat{\mathbf{j}} \\ +\left(\frac{-3xy}{(x^2+y^2+z^2)^{5/2}}-\left(\frac{-3xy}{(x^2+y^2+z^2)^{5/2}}\right)\right)\hat{\mathbf{k}}\end{pmatrix} \\ &= \vec{\mathbf{0}}.\end{aligned}$$

Since the curl of the gravitational field is zero, the field has no spin.

? Exercise 15.5.7

Field $\vec{\mathbf{v}}(x,y)=\langle -\frac{y}{x^2+y^2},\frac{x}{x^2+y^2}\rangle$ models the flow of a fluid. Show that if you drop a leaf into this fluid, as the leaf moves over time, the leaf does not rotate.

Using Divergence and Curl

Now that we understand the basic concepts of divergence and curl, we can discuss their properties and establish relationships between them and conservative vector fields.

If $\vec{\mathbf{F}}$ is a vector field in $\mathbb{R}^3$ then the curl of $\vec{\mathbf{F}}$ is also a vector field in $\mathbb{R}^3$. Therefore, we can take the divergence of a curl. The next theorem says that the result is always zero. This result is useful because it gives us a way to show that some vector fields are not the curl of any other field. To give this result a physical interpretation, recall that divergence of a velocity field $\vec{\mathbf{v}}$ at point P measures the tendency of the corresponding fluid to flow out of P. Since $\operatorname{div}(\operatorname{curl}\vec{\mathbf{v}})=0$, the net rate of flow in vector field $\operatorname{curl}\vec{\mathbf{v}}$ at any point is zero. Taking the curl of vector field $\vec{\mathbf{F}}$ eliminates whatever divergence was present in $\vec{\mathbf{F}}$.

Theorem: Divergence of the Curl

Let $\vec{\mathbf{F}}=\langle P,Q,R\rangle$ be a vector field in $\mathbb{R}^3$ such that the component functions all have continuous second-order partial derivatives. Then,

$$\operatorname{div}(\operatorname{curl}\vec{\mathbf{F}})=\vec{\nabla}\cdot(\vec{\nabla}\times\vec{\mathbf{F}})=0.$$

Proof

By the definitions of divergence and curl, and by Clairaut's theorem,

$$\begin{aligned}\operatorname{div}(\operatorname{curl}\vec{\mathbf{F}}) &= \operatorname{div}[(R_y-Q_z)\hat{\mathbf{i}}+(P_z-R_x)\hat{\mathbf{j}}+(Q_x-P_y)\hat{\mathbf{k}}] \\ &= R_{yx}-Q_{xz}+P_{yz}-R_{yx}+Q_{xz}-P_{zy} \\ &= 0.\end{aligned}$$

□

✓ Example 15.5.8: Showing That a Vector Field Is Not the Curl of Another

Show that $\vec{\mathbf{F}}(x,y,z)=e^x\hat{\mathbf{i}}+yz\hat{\mathbf{j}}+xz^2\hat{\mathbf{k}}$ is not the curl of another vector field. That is, show that there is no other vector $\vec{\mathbf{G}}$ with $\operatorname{curl}\vec{\mathbf{G}}=\vec{\mathbf{F}}$.

Solution

Notice that the domain of $\vec{\mathbf{F}}$ is all of $\mathbb{R}^3$ and the second-order partials of $\vec{\mathbf{F}}$ are all continuous. Therefore, we can apply the previous theorem to $\vec{\mathbf{F}}$.

The divergence of $\vec{\mathbf{F}}$ is $e^x + z + 2xz$. If $\vec{\mathbf{F}}$ were the curl of vector field $\vec{\mathbf{G}}$, then $\operatorname{div} \vec{\mathbf{F}} = \operatorname{div}(\operatorname{curl} \vec{\mathbf{G}}) = 0$. But, the divergence of $\vec{\mathbf{F}}$ is not zero, and therefore $\vec{\mathbf{F}}$ is not the curl of any other vector field.

Exercise 15.5.8

Is it possible for $\vec{\mathbf{G}}(x, y, z) = \langle \sin x, \cos y, \sin(xyz) \rangle$ to be the curl of a vector field?

With the next two theorems, we show that if $\vec{\mathbf{F}}$ is a conservative vector field then its curl is zero, and if the domain of $\vec{\mathbf{F}}$ is simply connected then the converse is also true. This gives us another way to test whether a vector field is conservative.

Theorem: Curl of a Conservative Vector Field

If $\vec{\mathbf{F}} = \langle P, Q, R \rangle$ is conservative, then $\operatorname{curl} \vec{\mathbf{F}} = \vec{\mathbf{0}}$.

Proof

Since conservative vector fields satisfy the cross-partials property, all the cross-partials of $\vec{\mathbf{F}}$ are equal. Therefore,

$$\begin{aligned} \operatorname{curl} \vec{\mathbf{F}} &= (R_y - Q_z)\hat{\mathbf{i}} + (P_z - R_x)\hat{\mathbf{j}} + (Q_x - P_y)\hat{\mathbf{k}} \\ &= \vec{\mathbf{0}}. \end{aligned}$$

□

The same theorem is true for vector fields in a plane.

Since a conservative vector field is the gradient of a scalar function, the previous theorem says that $\operatorname{curl}(\vec{\nabla} f) = \vec{\mathbf{0}}$ for any scalar function f. In terms of our curl notation, $\vec{\nabla} \times \vec{\nabla}(f) = \vec{\mathbf{0}}$. This equation makes sense because the cross product of a vector with itself is always the zero vector. Sometimes equation $\vec{\nabla} \times \vec{\nabla}(f) = \vec{\mathbf{0}}$ is simplified as $\vec{\nabla} \times \vec{\nabla} = \vec{\mathbf{0}}$.

Theorem: Curl Test for a Conservative Field

Let $\vec{\mathbf{F}} = \langle P, Q, R \rangle$ be a vector field in space on a simply connected domain. If $\operatorname{curl} \vec{\mathbf{F}} = \vec{\mathbf{0}}$, then $\vec{\mathbf{F}}$ is conservative.

Proof

Since $\operatorname{curl} \vec{\mathbf{F}} = \vec{\mathbf{0}}$, we have that $R_y = Q_z$, $P_z = R_x$, and $Q_x = P_y$. Therefore, $\vec{\mathbf{F}}$ satisfies the cross-partials property on a simply connected domain, and the Cross-Partial Property of Conservative Fields implies that $\vec{\mathbf{F}}$ is conservative.

□

The same theorem is also true in a plane. Therefore, if $\vec{\mathbf{F}}$ is a vector field in a plane or in space and the domain is simply connected, then $\vec{\mathbf{F}}$ is conservative if and only if $\operatorname{curl} \vec{\mathbf{F}} = \vec{\mathbf{0}}$.

Example 15.5.9: Testing Whether a Vector Field Is Conservative

Use the curl to determine whether $\vec{\mathbf{F}}(x, y, z) = \langle yz, xz, xy \rangle$ is conservative.

Solution

Note that the domain of $\vec{\mathbf{F}}$ is all of $\mathbb{R}^3$ which is simply connected (Figure 15.5.7). Therefore, we can test whether $\vec{\mathbf{F}}$ is conservative by calculating its curl.

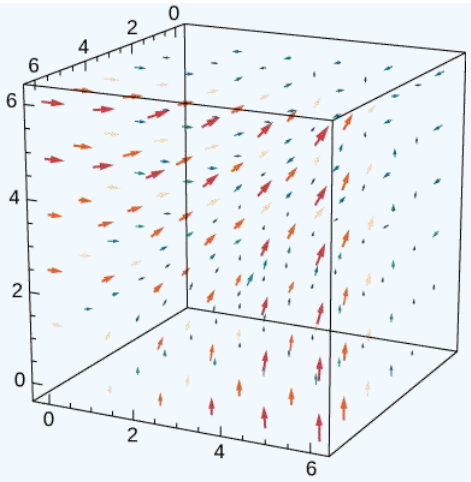


Figure 15.5.7: The curl of vector field $\vec{\mathbf{F}}(x,y,z) = \langle yz, xz, xy\rangle$ is zero.

The curl of $\vec{\mathbf{F}}$ is

$$\left(\frac{\partial}{\partial y}xy - \frac{\partial}{\partial z}xz\right)\hat{\mathbf{i}} + \left(\frac{\partial}{\partial y}yz - \frac{\partial}{\partial z}xy\right)\hat{\mathbf{j}} + \left(\frac{\partial}{\partial y}xz - \frac{\partial}{\partial z}yz\right)\hat{\mathbf{k}}$$
$$= (x-x)\hat{\mathbf{i}} + (y-y)\hat{\mathbf{j}} + (z-z)\hat{\mathbf{k}} = \vec{\mathbf{0}}.$$

Thus, $\vec{\mathbf{F}}$ is conservative.

We have seen that the curl of a gradient is zero. What is the divergence of a gradient? If f is a function of two variables, then $\text{div}(\vec{\nabla} f) = \vec{\nabla} \cdot (\vec{\nabla} f) = f_{xx} + f_{yy}$. We abbreviate this "double dot product" as $\vec{\nabla}^2$. This operator is called the *Laplace operator*, and in this notation Laplace's equation becomes $\vec{\nabla}^2 f = 0$. Therefore, a harmonic function is a function that becomes zero after taking the divergence of a gradient.

Similarly, if f is a function of three variables then

$$\text{div}(\vec{\nabla} f) = \vec{\nabla} \cdot (\vec{\nabla} f) = f_{xx} + f_{yy} + f_{zz}.$$

Using this notation we get Laplace's equation for harmonic functions of three variables:

$$\vec{\nabla}^2 f = 0.$$

Harmonic functions arise in many applications. For example, the potential function of an electrostatic field in a region of space that has no static charge is harmonic.

✔ Example 15.5.10: Finding a Potential Function

Is it possible for $f(x,y) = x^2 + x - y$ to be the potential function of an electrostatic field that is located in a region of $\mathbb{R}^2$ free of static charge?

Solution

If f were such a potential function, then f would be harmonic. Note that $f_{xx} = 2$ and $f_{yy} = 0$, and so $f_{xx} + f_{yy} \neq 0$. Therefore, f is not harmonic and f cannot represent an electrostatic potential.

? Exercise 15.5.10

Is it possible for function $f(x,y) = x^2 - y^2 + x$ to be the potential function of an electrostatic field located in a region of $\mathbb{R}^2$ free of static charge?

Key Concepts

- The divergence of a vector field is a scalar function. Divergence measures the "outflowing-ness" of a vector field. If $\vec{\mathbf{v}}$ is the velocity field of a fluid, then the divergence of $\vec{\mathbf{v}}$ at a point is the outflow of the fluid less the inflow at the point.
- The curl of a vector field is a vector field. The curl of a vector field at point P measures the tendency of particles at P to rotate about the axis that points in the direction of the curl at P.
- A vector field with a simply connected domain is conservative if and only if its curl is zero.

Key Equations

- **Curl**

$$\vec{\nabla} \times \vec{\mathbf{F}} = (R_y - Q_z)\hat{\mathbf{i}} + (P_z - R_x)\hat{\mathbf{j}} + (Q_x - P_y)\hat{\mathbf{k}}$$

- **Divergence**

$$\vec{\nabla} \cdot \vec{\mathbf{F}} = P_x + Q_y + R_z$$

- **Divergence of curl is zero**

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{\mathbf{F}}) = 0$$

- **Curl of a gradient is the zero vector**

$$\vec{\nabla} \times (\vec{\nabla} f) = 0$$

Glossary

Check Your Understanding

Let $\mathbf{F} = 10x\mathbf{i} + 8y\mathbf{j} + 7z\mathbf{k}$. Compute the divergence and the curl.

A. div $\mathbf{F} =$ [ ]

B. curl $\mathbf{F} =$ [ ] $\mathbf{i}+$ [ ] $\mathbf{j}+$ [ ] $\mathbf{k}$

Your overall score for this problem is 0%

Preview My Answers | Check Answers | Show correct answers

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Let $\mathbf{F} = (10yz)\,\mathbf{i} + (8xz)\,\mathbf{j} + (7xy)\,\mathbf{k}$. Compute the following:

A. div $\mathbf{F} =$ [ ]

B. curl $\mathbf{F} =$ [ ] $\mathbf{i}+$ [ ] $\mathbf{j}+$ [ ] $\mathbf{k}$

C. div curl $\mathbf{F} =$ [ ]

Note: Your answers should be expressions of x, y and/or z; e.g. "3xy" or "z" or "5"

Your overall score for this problem is 0%

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15.5E: Divergence and Curl (Exercises)

For the following exercises, determine whether the statement is *True* or *False*.

1. If the coordinate functions of $\vec{\mathbf{F}} : \mathbb{R}^3 \to \mathbb{R}^3$ have continuous second partial derivatives, then $\text{curl}\,(\text{div}\,\vec{\mathbf{F}})$ equals zero.

2. $\vec{\nabla} \cdot (x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}) = 1$.

3. All vector fields of the form $\vec{\mathbf{F}}(x, y, z) = f(x)\,\hat{\mathbf{i}} + g(y)\,\hat{\mathbf{j}} + h(z)\,\hat{\mathbf{k}}$ are conservative.

4. If $\text{curl}\,\vec{\mathbf{F}} = \vec{\mathbf{0}}$, then $\vec{\mathbf{F}}$ is conservative.

5. If $\vec{\mathbf{F}}$ is a constant vector field then $\text{div}\,\vec{\mathbf{F}} = 0$.

6. If $\vec{\mathbf{F}}$ is a constant vector field then $\text{curl}\,\vec{\mathbf{F}} = \vec{\mathbf{0}}$.

For the following exercises, find the curl of $\vec{\mathbf{F}}$.

7. $\vec{\mathbf{F}}(x, y, z) = xy^2z^4\,\hat{\mathbf{i}} + (2x^2y + z)\,\hat{\mathbf{j}} + y^3z^2\,\hat{\mathbf{k}}$

8. $\vec{\mathbf{F}}(x, y, z) = x^2z\,\hat{\mathbf{i}} + y^2x\,\hat{\mathbf{j}} + (y + 2z)\,\hat{\mathbf{k}}$

9. $\vec{\mathbf{F}}(x, y, z) = 3xyz^2\,\hat{\mathbf{i}} + y^2 \sin z\,\hat{\mathbf{j}} + xe^{2z}\,\hat{\mathbf{k}}$

10. $\vec{\mathbf{F}}(x, y, z) = x^2yz\,\hat{\mathbf{i}} + xy^2z\,\hat{\mathbf{j}} + xyz^2\,\hat{\mathbf{k}}$

11. $\vec{\mathbf{F}}(x, y, z) = (x \cos y)\,\hat{\mathbf{i}} + xy^2\,\hat{\mathbf{j}}$

12. $\vec{\mathbf{F}}(x, y, z) = (x - y)\,\hat{\mathbf{i}} + (y - z)\,\hat{\mathbf{j}} + (z - x)\,\hat{\mathbf{k}}$

13. $\vec{\mathbf{F}}(x, y, z) = xyz\,\hat{\mathbf{i}} + x^2y^2z^2\,\hat{\mathbf{j}} + y^2z^3\,\hat{\mathbf{k}}$

14. $\vec{\mathbf{F}}(x, y, z) = xy\,\hat{\mathbf{i}} + yz\,\hat{\mathbf{j}} + xz\,\hat{\mathbf{k}}$

15. $\vec{\mathbf{F}}(x, y, z) = x^2\,\hat{\mathbf{i}} + y^2\,\hat{\mathbf{j}} + z^2\,\hat{\mathbf{k}}$

16. $\vec{\mathbf{F}}(x, y, z) = ax\,\hat{\mathbf{i}} + by\,\hat{\mathbf{j}} + c\,\hat{\mathbf{k}}$ for constants a, b, c.

For the following exercises, find the divergence of $\vec{\mathbf{F}}$.

17. $\vec{\mathbf{F}}(x, y, z) = x^2z\,\hat{\mathbf{i}} + y^2x\,\hat{\mathbf{j}} + (y + 2z)\,\hat{\mathbf{k}}$

18. $\vec{\mathbf{F}}(x, y, z) = 3xyz^2\,\hat{\mathbf{i}} + y^2 \sin z\,\hat{\mathbf{j}} + xe^2z\,\hat{\mathbf{k}}$

19. $\vec{\mathbf{F}}(x, y) = (\sin x)\,\hat{\mathbf{i}} + (\cos y)\,\hat{\mathbf{j}}$

20. $\vec{\mathbf{F}}(x, y, z) = x^2\,\hat{\mathbf{i}} + y^2\,\hat{\mathbf{j}} + z^2\,\hat{\mathbf{k}}$

21. $\vec{\mathbf{F}}(x, y, z) = (x - y)\,\hat{\mathbf{i}} + (y - z)\,\hat{\mathbf{j}} + (z - x)\,\hat{\mathbf{k}}$

22. $\vec{\mathbf{F}}(x, y) = \dfrac{x}{\sqrt{x^2 + y^2}}\,\hat{\mathbf{i}} + \dfrac{y}{\sqrt{x^2 + y^2}}\,\hat{\mathbf{j}}$

23. $\vec{\mathbf{F}}(x, y) = x\,\hat{\mathbf{i}} - y\,\hat{\mathbf{j}}$

24. $\vec{\mathbf{F}}(x, y, z) = ax\,\hat{\mathbf{i}} + by\,\hat{\mathbf{j}} + c\,\hat{\mathbf{k}}$ for constants a, b, c.

25. $\vec{\mathbf{F}}(x, y, z) = xyz\,\hat{\mathbf{i}} + x^2y^2z^2\,\hat{\mathbf{j}} + y^2z^3\,\hat{\mathbf{k}}$

26. $\vec{\mathbf{F}}(x, y, z) = xy\,\hat{\mathbf{i}} + yz\,\hat{\mathbf{j}} + xz\,\hat{\mathbf{k}}$

For exercises 27 & 28, determine whether each of the given scalar functions is harmonic.

27. $u(x, y, z) = e^{-x}(\cos y - \sin y)$

28. $w(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$

29. If $\vec{\mathbf{F}}(x, y, z) = 2\,\hat{\mathbf{i}} + 2xj + 3yk$ and $\vec{\mathbf{G}}(x, y, z) = x\,\hat{\mathbf{i}} - y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$, find $\text{curl}\,(\vec{\mathbf{F}} \times \vec{\mathbf{G}})$.

30. If $\vec{\mathbf{F}}(x, y, z) = 2\,\hat{\mathbf{i}} + 2xj + 3yk$ and $\vec{\mathbf{G}}(x, y, z) = x\,\hat{\mathbf{i}} - y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$, find $\text{div}\,(\vec{\mathbf{F}} \times \vec{\mathbf{G}})$.

31. Find $\text{div}\,\vec{\mathbf{F}}$, given that $\vec{\mathbf{F}} = \vec{\nabla} f$, where $f(x,y,z) = xy^3z^2$.

32. Find the divergence of $\vec{\mathbf{F}}$ for vector field $\vec{\mathbf{F}}(x,y,z) = (y^2+z^2)(x+y)\,\hat{\mathbf{i}} + (z^2+x^2)(y+z)\,\hat{\mathbf{j}} + (x^2+y^2)(z+x)\,\hat{\mathbf{k}}$.

33. Find the divergence of $\vec{\mathbf{F}}$ for vector field $\vec{\mathbf{F}}(x,y,z) = f_1(y,z)\,\hat{\mathbf{i}} + f_2(x,z)\,\hat{\mathbf{j}} + f_3(x,y)\,\hat{\mathbf{k}}$.

For exercises 34 - 36, use $r = |\vec{\mathbf{r}}|$ and $\vec{\mathbf{r}}(x,y,z) = \langle x,y,z\rangle$.

34. Find the $\text{curl}\,\vec{\mathbf{r}}$

35. Find the $\text{curl}\,\dfrac{\vec{\mathbf{r}}}{r}$.

36. Find the $\text{curl}\,\dfrac{\vec{\mathbf{r}}}{r^3}$.

37. Let $\vec{\mathbf{F}}(x,y) = \dfrac{-y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}}}{x^2+y^2}$, where $\vec{\mathbf{F}}$ is defined on $\{(x,y) \in \mathbb{R} | (x,y) \neq (0,0)\}$. Find $\text{curl}\,\vec{\mathbf{F}}$.

For the following exercises, use a computer algebra system to find the curl of the given vector fields.

38. [T] $\vec{\mathbf{F}}(x,y,z) = \arctan\left(\dfrac{x}{y}\right)\hat{\mathbf{i}} + \ln\sqrt{x^2+y^2}\,\hat{\mathbf{j}} + \hat{\mathbf{k}}$

39. [T] $\vec{\mathbf{F}}(x,y,z) = \sin(x-y)\,\hat{\mathbf{i}} + \sin(y-z)\,\hat{\mathbf{j}} + \sin(z-x)\,\hat{\mathbf{k}}$

For the following exercises, find the divergence of $\vec{\mathbf{F}}$ at the given point.

40. $\vec{\mathbf{F}}(x,y,z) = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ at $(2,-1,3)$

41. $\vec{\mathbf{F}}(x,y,z) = xyz\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$ at $(1,2,3)$

42. $\vec{\mathbf{F}}(x,y,z) = e^{-xy}\,\hat{\mathbf{i}} + e^{xz}\,\hat{\mathbf{j}} + e^{yz}\,\hat{\mathbf{k}}$ at $(3,2,0)$

43. $\vec{\mathbf{F}}(x,y,z) = xyz\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$ at $(1,2,1)$

44. $\vec{\mathbf{F}}(x,y,z) = e^x \sin y\,\hat{\mathbf{i}} - e^x \cos y\,\hat{\mathbf{j}}$ at $(0,0,3)$

For exercises 45- 49, find the curl of $\vec{\mathbf{F}}$ at the given point.

45. $\vec{\mathbf{F}}(x,y,z) = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ at $(2,-1,3)$

46. $\vec{\mathbf{F}}(x,y,z) = xyz\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + x\,\hat{\mathbf{k}}$ at $(1,2,3)$

47. $\vec{\mathbf{F}}(x,y,z) = e^{-xy}\,\hat{\mathbf{i}} + e^{xz}\,\hat{\mathbf{j}} + e^{yz}\,\hat{\mathbf{k}}$ at $(3,2,0)$

48. $\vec{\mathbf{F}}(x,y,z) = xyz\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$ at $(1,2,1)$

49. $\vec{\mathbf{F}}(x,y,z) = e^x \sin y\,\hat{\mathbf{i}} - e^x \cos y\,\hat{\mathbf{j}}$ at $(0,0,3)$

50. Let $\vec{\mathbf{F}}(x,y,z) = (3x^2y + az)\,\hat{\mathbf{i}} + x^3\,\hat{\mathbf{j}} + (3x+3z^2)\,\hat{\mathbf{k}}$. For what value of a is $\vec{\mathbf{F}}$ conservative?

51. Given vector field $\vec{\mathbf{F}}(x,y) = \dfrac{1}{x^2+y^2}\langle -y,x\rangle$ on domain $D = \dfrac{\mathbb{R}^2}{\{(0,0)\}} = \{(x,y) \in \mathbb{R}^2 | (x,y) \neq (0,0)\}$, is $\vec{\mathbf{F}}$ conservative?

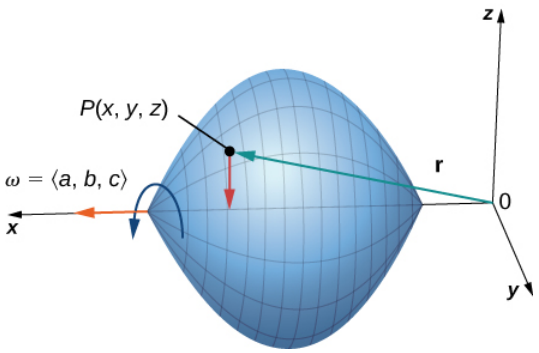
52. Given vector field $\vec{\mathbf{F}}(x,y) = \dfrac{1}{x^2+y^2}\langle x,y\rangle$ on domain $D = \dfrac{\mathbb{R}^2}{\{(0,0)\}}$, is $\vec{\mathbf{F}}$ conservative?

53. Find the work done by force field $\vec{\mathbf{F}}(x,y) = e^{-y}\,\hat{\mathbf{i}} - xe^{-y}\,\hat{\mathbf{j}}$ in moving an object from $P(0,1)$ to $Q(2,0)$. Is the force field conservative?

54. Compute divergence $\vec{\mathbf{F}}(x,y,z) = (\sinh x)\,\hat{\mathbf{i}} + (\cosh y)\,\hat{\mathbf{j}} - xyz\,\hat{\mathbf{k}}$.

55. Compute $\text{curl}\ \vec{\mathbf{F}} = (\sinh x)\,\hat{\mathbf{i}} + (\cosh y)\,\hat{\mathbf{j}} - xyz\,\hat{\mathbf{k}}$.

For the following exercises, consider a rigid body that is rotating about the x-axis counterclockwise with constant angular velocity $\vec{\omega} = \langle a,b,c\rangle$. If P is a point in the body located at $\vec{\mathbf{r}} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$, the velocity at P is given by vector field $\vec{\mathbf{F}} = \vec{\omega} \times \vec{\mathbf{r}}$.



56. Express $\vec{\mathbf{F}}$ in terms of $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ vectors.

57. Find $\operatorname{div} F$.

58. Find $\operatorname{curl} F$

In the following exercises, suppose that $\vec{\nabla} \cdot \vec{\mathbf{F}} = 0$ and $\vec{\nabla} \cdot \vec{\mathbf{G}} = 0$.

59. Does $\vec{\mathbf{F}} + \vec{\mathbf{G}}$ necessarily have zero divergence?

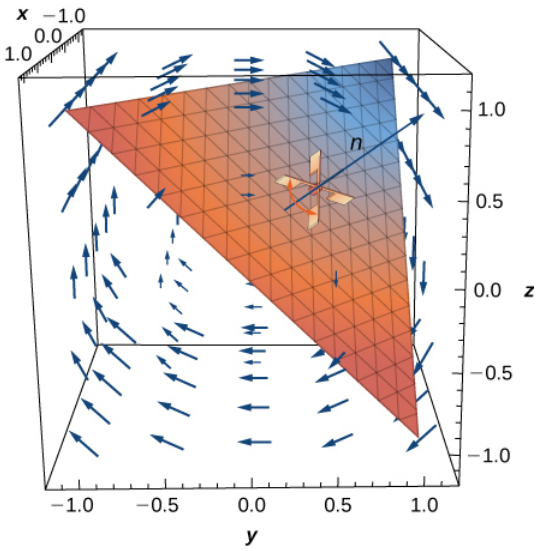
60. Does $\vec{\mathbf{F}} \times \vec{\mathbf{G}}$ necessarily have zero divergence?

In the following exercises, suppose a solid object in $\mathbb{R}^3$ has a temperature distribution given by $T(x, y, z)$. The heat flow vector field in the object is $\vec{\mathbf{F}} = -k\vec{\nabla}T$, where $k > 0$ is a property of the material. The heat flow vector points in the direction opposite to that of the gradient, which is the direction of greatest temperature decrease. The divergence of the heat flow vector is $\vec{\nabla} \cdot \vec{\mathbf{F}} = -k\vec{\nabla} \cdot \vec{\nabla}T = -k\vec{\nabla}^2 T$.

61. Compute the heat flow vector field.

62. Compute the divergence.

63. [T] Consider rotational velocity field $\vec{\mathbf{v}} = \langle 0, 10z, -10y \rangle$. If a paddlewheel is placed in plane $x + y + z = 1$ with its axis normal to this plane, using a computer algebra system, calculate how fast the paddlewheel spins in revolutions per unit time.



Contributors

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15.6: Surface Integrals

Learning Objectives

- Find the parametric representations of a cylinder, a cone, and a sphere.
- Describe the surface integral of a scalar-valued function over a parametric surface.
- Use a surface integral to calculate the area of a given surface.
- Explain the meaning of an oriented surface, giving an example.
- Describe the surface integral of a vector field.
- Use surface integrals to solve applied problems.

We have seen that a line integral is an integral over a path in a plane or in space. However, if we wish to integrate over a surface (a two-dimensional object) rather than a path (a one-dimensional object) in space, then we need a new kind of integral that can handle integration over objects in higher dimensions. We can extend the concept of a line integral to a surface integral to allow us to perform this integration.

Surface integrals are important for the same reasons that line integrals are important. They have many applications to physics and engineering, and they allow us to develop higher dimensional versions of the Fundamental Theorem of Calculus. In particular, surface integrals allow us to generalize Green's theorem to higher dimensions, and they appear in some important theorems we discuss in later sections.

Parametric Surfaces

A surface integral is similar to a line integral, except the integration is done over a surface rather than a path. In this sense, surface integrals expand on our study of line integrals. Just as with line integrals, there are two kinds of surface integrals: a surface integral of a scalar-valued function and a surface integral of a vector field.

However, before we can integrate over a surface, we need to consider the surface itself. Recall that to calculate a scalar or vector line integral over curve C, we first need to parameterize C. In a similar way, to calculate a surface integral over surface S, we need to parameterize S. That is, we need a working concept of a *parameterized surface* (or a *parametric surface*), in the same way that we already have a concept of a parameterized curve.

A parameterized surface is given by a description of the form

$$\vec{\mathbf{r}}(u,v) = \langle x(u,v),\, y(u,v),\, z(u,v)\rangle.$$

Notice that this parameterization involves two parameters, u and v, because a surface is two-dimensional, and therefore two variables are needed to trace out the surface. The parameters u and v vary over a region called the parameter domain, or parameter space—the set of points in the uv-plane that can be substituted into $\vec{\mathbf{r}}$. Each choice of u and v in the parameter domain gives a point on the surface, just as each choice of a parameter t gives a point on a parameterized curve. The entire surface is created by making all possible choices of u and v over the parameter domain.

Definition: Parameter Domain

Given a parameterization of surface

$$\vec{\mathbf{r}}(u,v) = \langle x(u,v),\, y(u,v),\, z(u,v)\rangle.$$

the *parameter domain* of the parameterization is the set of points in the uv-plane that can be substituted into $\vec{\mathbf{r}}$.

Example 15.6.1: Parameterizing a Cylinder

Describe surface S parameterized by

$$\vec{\mathbf{r}}(u,v) = \langle \cos u,\, \sin u,\, v\rangle,\ -\infty < u < \infty,\ -\infty < v < \infty.$$

Solution

To get an idea of the shape of the surface, we first plot some points. Since the parameter domain is all of $\mathbb{R}^2$, we can choose any value for u and v and plot the corresponding point. If $u = v = 0$, then $\vec{\mathbf{r}}(0,0) = \langle 1,0,0\rangle$, so point (1, 0, 0) is on S. Similarly, points $\vec{\mathbf{r}}(\pi,2) = (-1,0,2)$ and $\vec{\mathbf{r}}\left(\frac{\pi}{2},4\right) = (0,1,4)$ are on S.

Although plotting points may give us an idea of the shape of the surface, we usually need quite a few points to see the shape. Since it is time-consuming to plot dozens or hundreds of points, we use another strategy. To visualize S, we visualize two families of curves that lie on S. In the first family of curves we hold u constant; in the second family of curves we hold v constant. This allows us to build a "skeleton" of the surface, thereby getting an idea of its shape.

- Suppose that u is a constant K. Then the curve traced out by the parameterization is $\langle \cos K, \sin K, v\rangle$, which gives a vertical line that goes through point $(\cos K, \sin K, v\rangle$ in the xy-plane.
- Suppose that v is a constant K. Then the curve traced out by the parameterization is $\langle \cos u, \sin u, K\rangle$, which gives a circle in plane $z = K$ with radius 1 and center $(0,0,K)$.

If u is held constant, then we get vertical lines; if v is held constant, then we get circles of radius 1 centered around the vertical line that goes through the origin. Therefore the surface traced out by the parameterization is cylinder $x^2 + y^2 = 1$ (Figure 15.6.1).

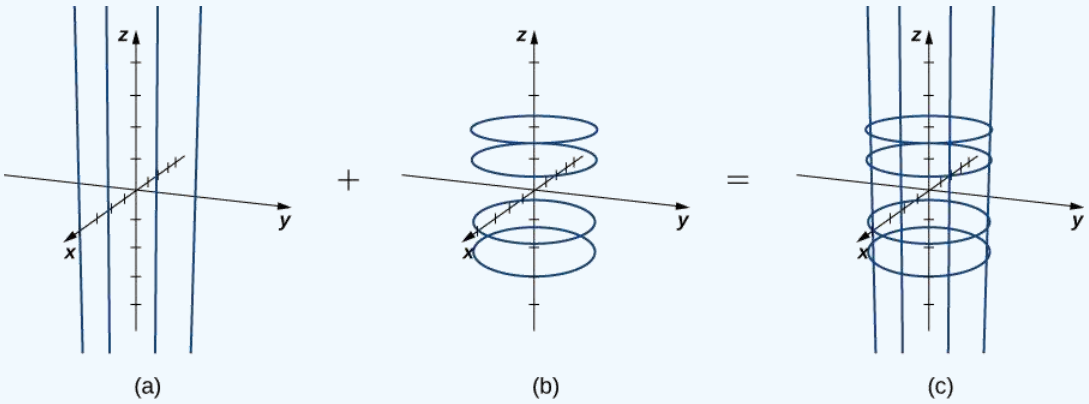


Figure 15.6.1: (a) Lines $\langle \cos K, \sin K, v\rangle$ for $K = 0$, $\frac{\pi}{2}$, π, and $\frac{3\pi}{2}$. (b) Circles $\langle \cos u, \sin u, K\rangle$ for $K = -2, -1, 1$, and 2 (c) The lines and circles together. As u and v vary, they describe a cylinder.

Notice that if $x = \cos u$ and $y = \sin u$, then $x^2 + y^2 = 1$, so points from S do indeed lie on the cylinder. Conversely, each point on the cylinder is contained in some circle $\langle \cos u, \sin u, k\rangle$ for some k, and therefore each point on the cylinder is contained in the parameterized surface (Figure 15.6.2).

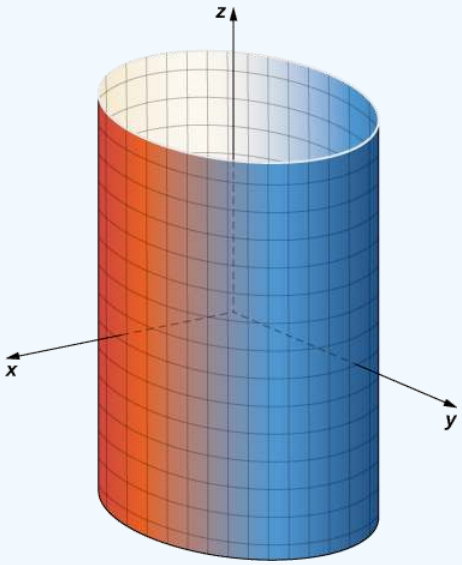


Figure 15.6.2: Cylinder $x^2 + y^2 = r^2$ has parameterization $\vec{\mathbf{r}}(u,v) = \langle r\cos u, r\sin u, v\rangle$, $0 \le u \le 2\pi$, $-\infty < v < \infty$.

Analysis

Notice that if we change the parameter domain, we could get a different surface. For example, if we restricted the domain to $0 \leq u \leq \pi$, $-\infty < v < 6$, then the surface would be a half-cylinder of height 6.

? Exercise 15.6.1

Describe the surface with parameterization

$$\vec{\mathbf{r}}(u,v) = \langle 2\cos u,\ 2\sin u,\ v\rangle,\ 0 \leq u \leq 2\pi,\ -\infty < v < \infty$$

It follows from Example 15.6.1 that we can parameterize all cylinders of the form $x^2 + y^2 = R^2$. If S is a cylinder given by equation $x^2 + y^2 = R^2$, then a parameterization of S is $\vec{\mathbf{r}}(u,v) = \langle R\cos u,\ R\sin u,\ v\rangle$, $0 \leq u \leq 2\pi$, $-\infty < v < \infty$.

We can also find different types of surfaces given their parameterization, or we can find a parameterization when we are given a surface.

✓ Example 15.6.2: Describing a Surface

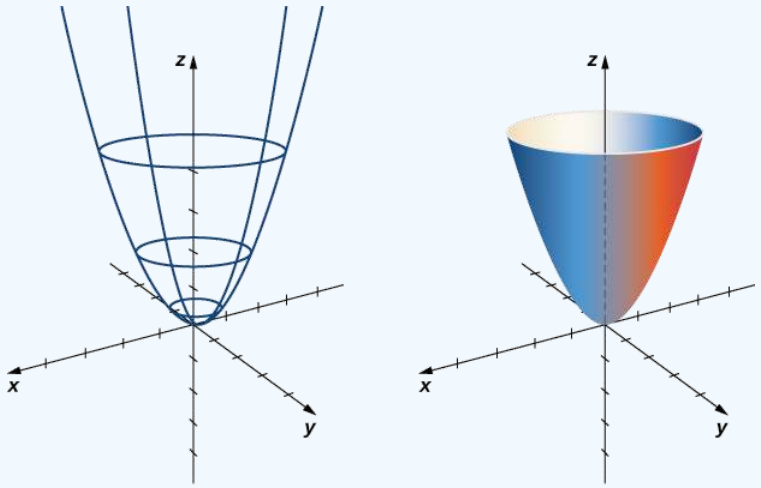
Describe surface S parameterized by $\vec{\mathbf{r}}(u,v) = \langle u\cos v,\ u\sin v,\ u^2\rangle$, $0 \leq u < \infty$, $0 \leq v < 2\pi$.

Solution

Notice that if u is held constant, then the resulting curve is a circle of radius u in plane $z = u$. Therefore, as u increases, the radius of the resulting circle increases. If v is held constant, then the resulting curve is a vertical parabola. Therefore, we expect the surface to be an elliptic paraboloid. To confirm this, notice that

$$\begin{aligned} x^2 + y^2 &= (u\cos v)^2 + (u\sin v)^2 \\ &= u^2\cos^2 v + u^2 sin^2 v \\ &= u^2 \\ &= z \end{aligned}$$

Therefore, the surface is the elliptic paraboloid $x^2 + y^2 = z$ (Figure 15.6.3).



(a) (b)

Figure 15.6.3: (a) Circles arise from holding u constant; the vertical parabolas arise from holding v constant. (b) An elliptic paraboloid results from all choices of u and v in the parameter domain.

? Exercise 15.6.2

Describe the surface parameterized by $\vec{\mathbf{r}}(u,v) = \langle u\cos v,\ u\sin v,\ u\rangle$, $-\infty < u < \infty$, $0 \leq v < 2\pi$.

Example 15.6.3: Finding a Parameterization

Give a parameterization of the cone $x^2 + y^2 = z^2$ lying on or above the plane $z = -2$.

Solution

The horizontal cross-section of the cone at height $z = u$ is circle $x^2 + y^2 = u^2$. Therefore, a point on the cone at height u has coordinates $(u\cos v, u\sin v, u)$ for angle v. Hence, a parameterization of the cone is $\vec{\mathbf{r}}(u, v) = \langle u\cos v, u\sin v, u\rangle$. Since we are not interested in the entire cone, only the portion on or above plane $z = -2$, the parameter domain is given by $-2 < u < \infty$, $0 \le v < 2\pi$ (Figure 15.6.4).

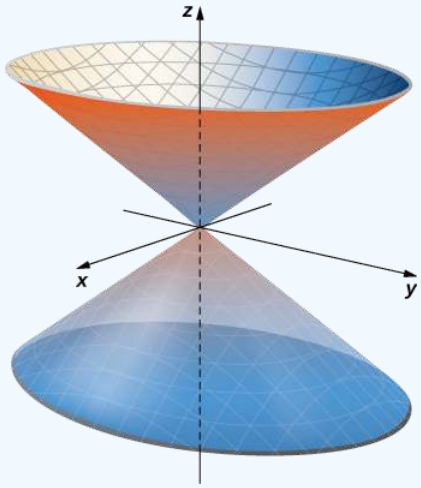


Figure 15.6.4: Cone $x^2 + y^2 = z^2$ has parametrization $\vec{\mathbf{r}}(u, v) = \langle u\cos v, u\sin v\rangle$, $-\infty < u < \infty$, $0 \le v \le 2\pi$.

Exercise 15.6.3

Give a parameterization for the portion of cone $x^2 + y^2 = z^2$ lying in the first octant.

We have discussed parameterizations of various surfaces, but two important types of surfaces need a separate discussion: spheres and graphs of two-variable functions. To parameterize a sphere, it is easiest to use spherical coordinates. The sphere of radius ρ centered at the origin is given by the parameterization

$$\vec{\mathbf{r}}(\phi, \theta) = \langle \rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi\rangle, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi.$$

The idea of this parameterization is that as ϕ sweeps downward from the positive z-axis, a circle of radius $\rho\sin\phi$ is traced out by letting θ run from 0 to 2π. To see this, let ϕ be fixed. Then

$$\begin{aligned} x^2 + y^2 &= (\rho\cos\theta\sin\phi)^2 + (\rho\sin\theta\sin\phi)^2 \\ &= \rho^2\sin^2\phi(\cos^2\theta + \sin^2\theta) \\ &= \rho^2\sin^2\phi \\ &= (\rho\sin\phi)^2. \end{aligned}$$

This results in the desired circle (Figure 15.6.5).

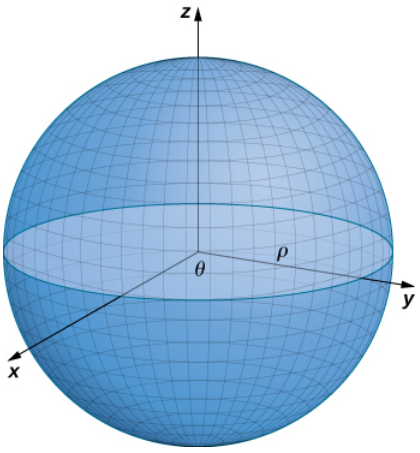


Figure 15.6.5: The sphere of radius ρ has parameterization $\vec{\mathbf{r}}(\phi,\theta) = \langle \rho\cos\theta\sin\phi,\ \rho\sin\theta\sin\phi,\ \rho\cos\phi\rangle$, $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$.

Finally, to parameterize the graph of a two-variable function, we first let $z = f(x,y)$ be a function of two variables. The simplest parameterization of the graph of f is $\vec{\mathbf{r}}(x,y) = \langle x, y, f(x,y)\rangle$, where x and y vary over the domain of f (Figure 15.6.6). For example, the graph of $f(x,y) = x^2y$ can be parameterized by $\vec{\mathbf{r}}(x,y) = \langle x, y, x^2y\rangle$, where the parameters x and y vary over the domain of f. If we only care about a piece of the graph of f - say, the piece of the graph over rectangle $[1,3] \times [2,5]$ - then we can restrict the parameter domain to give this piece of the surface:

$$\vec{\mathbf{r}}(x,y) = \langle x, y, x^2y\rangle,\ 1 \le x \le 3,\ 2 \le y \le 5.$$

Similarly, if S is a surface given by equation $x = g(y,z)$ or equation $y = h(x,z)$, then a parameterization of S is $\vec{\mathbf{r}}(y,z) = \langle g(y,z),\ y, z\rangle$ or $\vec{\mathbf{r}}(x,z) = \langle x, h(x,z), z\rangle$, respectively. For example, the graph of paraboloid $2y = x^2 + z^2$ can be parameterized by $\vec{\mathbf{r}}(x,y) = \left\langle x, \dfrac{x^2+z^2}{2}, z\right\rangle$, $0 \le x < \infty$, $0 \le z < \infty$. Notice that we do not need to vary over the entire domain of y because x and z are squared.

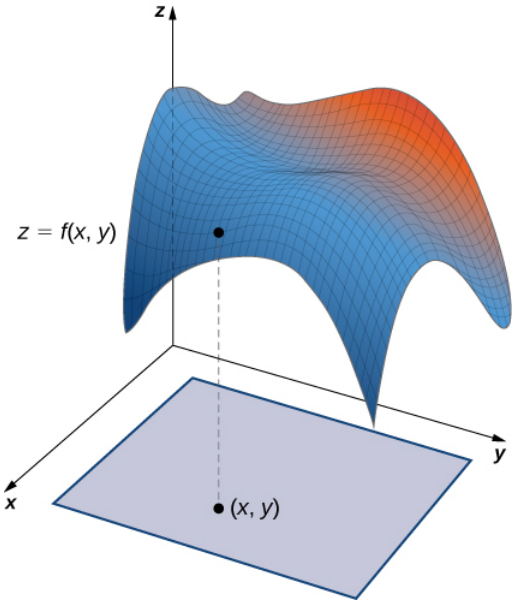


Figure 15.6.6: The simplest parameterization of the graph of a function is $\vec{\mathbf{r}}(x,y) = \langle x, y, f(x,y)\rangle$.

Let's now generalize the notions of smoothness and regularity to a parametric surface. Recall that curve parameterization $\vec{\mathbf{r}}(t)$, $a \le t \le b$ is regular (or **smooth**) if $\vec{\mathbf{r}}'(t) \neq \vec{\mathbf{0}}$ for all t in $[a,b]$. For a curve, this condition ensures that the image of $\vec{\mathbf{r}}$ really is a curve, and not just a point. For example, consider curve parameterization $\vec{\mathbf{r}}(t) = \langle 1, 2\rangle$, $0 \le t \le 5$. The image of this parameterization is simply point $(1,2)$, which is not a curve. Notice also that $\vec{\mathbf{r}}'(t) = \vec{\mathbf{0}}$. The fact that the derivative is the zero vector indicates we are not actually looking at a curve.

Analogously, we would like a notion of regularity (or smoothness) for surfaces so that a surface parameterization really does trace out a surface. To motivate the definition of regularity of a surface parameterization, consider the parameterization

$$\vec{\mathbf{r}}(u,v) = \langle 0,\ \cos v,\ 1\rangle,\ 0 \le u \le 1,\ 0 \le v \le \pi.$$

Although this parameterization appears to be the parameterization of a surface, notice that the image is actually a line (Figure 15.6.7). How could we avoid parameterizations such as this? Parameterizations that do not give an actual surface? Notice that $\vec{\mathbf{r}}_u = \langle 0,0,0\rangle$ and $\vec{\mathbf{r}}_v = \langle 0,-\sin v,0\rangle$, and the corresponding cross product is zero. The analog of the condition $\vec{\mathbf{r}}'(t) = \vec{\mathbf{0}}$ is that $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v$ is not zero for point (u,v) in the parameter domain, which is a regular parameterization.

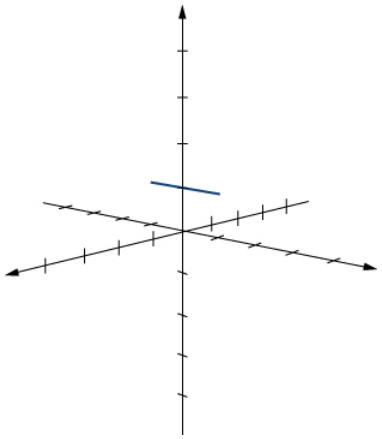


Figure 15.6.7: The image of parameterization $\vec{\mathbf{r}}(u,v) = \langle 0,\cos v,1\rangle,\ 0 \le u \le 1,\ 0 \le v \le \pi$ is a line.

Definition: Regular Parameterization

Parameterization $\vec{\mathbf{r}}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$ is a **regular parameterization** if $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v$ is not zero for point (u,v) in the parameter domain.

If parameterization $\vec{r}$ is regular, then the image of $\vec{r}$ is a two-dimensional object, as a surface should be. Throughout this chapter, parameterizations $\vec{\mathbf{r}}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$are assumed to be regular.

Recall that curve parameterization $\vec{\mathbf{r}}(t),\ a \le t \le b$ is smooth if $\vec{\mathbf{r}}'(t)$ is continuous and $\vec{\mathbf{r}}'(t) \ne \vec{\mathbf{0}}$ for all t in $[a,b]$. Informally, a curve parameterization is smooth if the resulting curve has no sharp corners. The definition of a smooth surface parameterization is similar. Informally, a surface parameterization is *smooth* if the resulting surface has no sharp corners.

Definition: Smooth Surfaces

A surface parameterization $\vec{\mathbf{r}}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$ is *smooth* if vector $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v$ is not zero for any choice of u and v in the parameter domain.

A surface may also be *piecewise smooth* if it has smooth faces but also has locations where the directional derivatives do not exist.

Example 15.6.4: Identifying Smooth and Nonsmooth Surfaces

Which of the figures in Figure 15.6.8 is smooth?

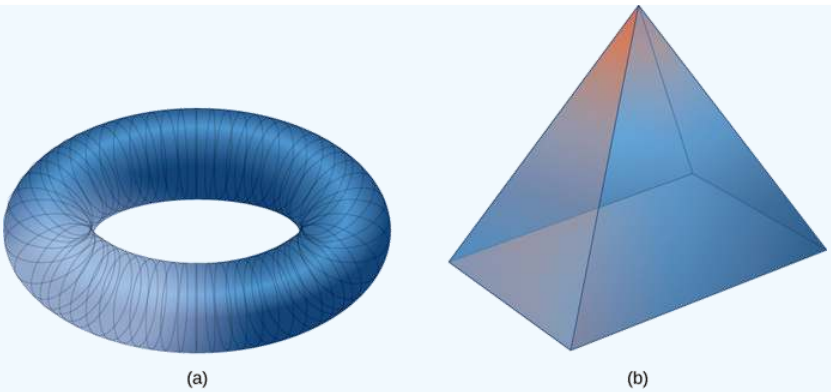


Figure 15.6.8: (a) This surface is smooth. (b) This surface is piecewise smooth.

Solution

The surface in Figure 15.6.8*a* can be parameterized by

$$\vec{\mathbf{r}}(u,v) = \langle (2+\cos v)\cos u,\ (2+\cos v)\sin u,\ \sin v\rangle,\ 0 \le u < 2\pi,\ 0 \le v < 2\pi$$

(we can use technology to verify). Notice that vectors

$$\vec{\mathbf{r}}_u = \langle -(2+\cos v)\sin u,\ (2+\cos v)\cos u, 0\rangle$$

and

$$\vec{\mathbf{r}}_v = \langle -\sin v\cos u,\ -\sin v\sin u,\ \cos v\rangle$$

exist for any choice of u and v in the parameter domain, and

$$\begin{aligned}\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -(2+\cos v)\sin u & (2+\cos v)\cos u & 0 \\ -\sin v\cos u & -\sin v\sin u & \cos v \end{vmatrix} \\ &= [(2+\cos v)\cos u\cos v]\hat{\mathbf{i}} + [2+\cos v)\sin u\cos v]\hat{\mathbf{j}} \\ &\quad + [(2+\cos v)\sin v\sin^2 u + (2+\cos v)\sin v\cos^2 u]\hat{\mathbf{k}} \\ &= [(2+\cos v)\cos u\cos v]\hat{\mathbf{i}} + [(2+\cos v)\sin u\cos v]\hat{\mathbf{j}} + [(2+\cos v)\sin v]\hat{\mathbf{k}}.\end{aligned}$$

The $\hat{\mathbf{k}}$ component of this vector is zero only if $v = 0$ or $v = \pi$. If $v = 0$ or $v = \pi$, then the only choices for u that make the $\hat{\mathbf{j}}$ component zero are $u = 0$ or $u = \pi$. But, these choices of u do not make the $\hat{\mathbf{i}}$ component zero. Therefore, $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v$ is not zero for any choice of u and v in the parameter domain, and the parameterization is smooth. Notice that the corresponding surface has no sharp corners.

In the pyramid in Figure 15.6.8*b*, the sharpness of the corners ensures that directional derivatives do not exist at those locations. Therefore, the pyramid has no smooth parameterization. However, the pyramid consists of four smooth faces, and thus this surface is piecewise smooth.

? Exercise 15.6.4

Is the surface parameterization $\vec{\mathbf{r}}(u,v) = \langle u^{2v}, v+1, \sin u\rangle,\ 0 \le u \le 2,\ 0 \le v \le 3$ smooth?

Surface Area of a Parametric Surface

Our goal is to define a surface integral, and as a first step we have examined how to parameterize a surface. The second step is to define the surface area of a parametric surface. The notation needed to develop this definition is used throughout the rest of this chapter.

Let S be a surface with parameterization $\vec{\mathbf{r}}(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v)\rangle$ over some parameter domain D. We assume here and throughout that the surface parameterization $\vec{\mathbf{r}}(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v)\rangle$ is continuously differentiable—meaning, each component function has continuous partial derivatives. Assume for the sake of simplicity that D is a rectangle (although the

following material can be extended to handle nonrectangular parameter domains). Divide rectangle D into subrectangles D_{ij} with horizontal width Δu and vertical length Δv. Suppose that i ranges from 1 to m and j ranges from 1 to n so that D is subdivided into mn rectangles. This division of D into subrectangles gives a corresponding division of surface S into pieces S_{ij}. Choose point P_{ij} in each piece S_{ij}. Point P_{ij} corresponds to point (u_i, v_j) in the parameter domain.

Note that we can form a grid with lines that are parallel to the u-axis and the v-axis in the uv-plane. These grid lines correspond to a set of grid curves on surface S that is parameterized by $\vec{\mathbf{r}}(u, v)$. Without loss of generality, we assume that P_{ij} is located at the corner of two grid curves, as in Figure 15.6.9. If we think of $\vec{\mathbf{r}}$ as a mapping from the uv-plane to $\mathbb{R}^3$, the grid curves are the image of the grid lines under $\vec{\mathbf{r}}$. To be precise, consider the grid lines that go through point (u_i, v_j). One line is given by $x = u_i,\ y = v$; the other is given by $x = u,\ y = v_j$. In the first grid line, the horizontal component is held constant, yielding a vertical line through (u_i, v_j). In the second grid line, the vertical component is held constant, yielding a horizontal line through (u_i, v_j). The corresponding grid curves are $\vec{\mathbf{r}}(u_i, v)$ and (u, v_j) and these curves intersect at point P_{ij}.

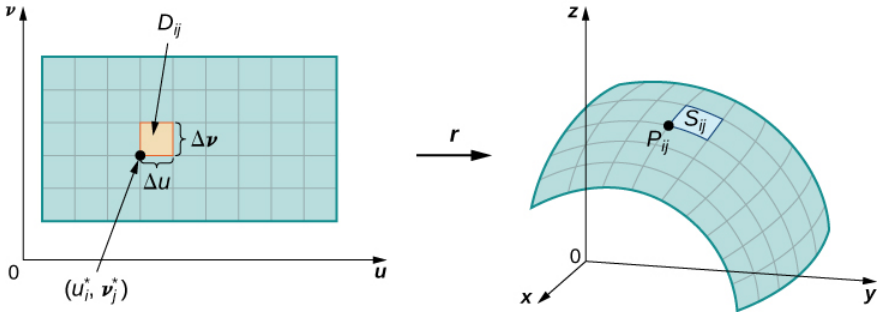


Figure 15.6.9: Grid lines on a parameter domain correspond to grid curves on a surface.

Now consider the vectors that are tangent to these grid curves. For grid curve $\vec{\mathbf{r}}(u_i, v)$, the tangent vector at P_{ij} is

$$\vec{\mathbf{t}}_v(P_{ij}) = \vec{\mathbf{r}}_v(u_i, v_j) = \langle x_v(u_i, v_j),\ y_v(u_i, v_j),\ z_v(u_i, v_j)\rangle.$$

For grid curve $\vec{\mathbf{r}}(u, v_j)$, the tangent vector at P_{ij} is

$$\vec{\mathbf{t}}_u(P_{ij}) = \vec{\mathbf{r}}_u(u_i, v_j) = \langle x_u(u_i, v_j),\ y_u(u_i, v_j),\ z_u(u_i, v_j)\rangle.$$

If vector $\vec{\mathbf{N}} = \vec{\mathbf{t}}_u(P_{ij}) \times \vec{\mathbf{t}}_v(P_{ij})$ exists and is not zero, then the tangent plane at P_{ij} exists (Figure 15.6.10). If piece S_{ij} is small enough, then the tangent plane at point P_{ij} is a good approximation of piece S_{ij}.

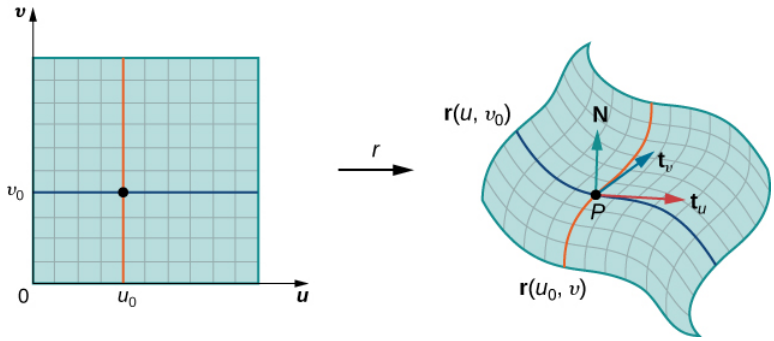


Figure 15.6.10: If the cross product of vectors $\vec{\mathbf{t}}_u$ and $\vec{\mathbf{t}}_v$ exists, then there is a tangent plane.

The tangent plane at P_{ij} contains vectors $\vec{\mathbf{t}}_u(P_{ij})$ and $\vec{\mathbf{t}}_v(P_{ij})$ and therefore the parallelogram spanned by $\vec{\mathbf{t}}_u(P_{ij})$ and $\vec{\mathbf{t}}_v(P_{ij})$ is in the tangent plane. Since the original rectangle in the uv-plane corresponding to S_{ij} has width Δu and length Δv, the parallelogram that we use to approximate S_{ij} is the parallelogram spanned by $\Delta u\vec{\mathbf{t}}_u(P_{ij})$ and $\Delta v\vec{\mathbf{t}}_v(P_{ij})$. In other words, we scale the tangent vectors by the constants Δu and Δv to match the scale of the original division of rectangles in the parameter domain. Therefore, the area of the parallelogram used to approximate the area of S_{ij} is

$$\Delta S_{ij} \approx ||(\Delta u\vec{\mathbf{t}}_u(P_{ij})) \times (\Delta v\vec{\mathbf{t}}_v(P_{ij}))|| = ||\vec{\mathbf{t}}_u(P_{ij}) \times \vec{\mathbf{t}}_v(P_{ij})||\Delta u\,\Delta v.$$

Varying point P_{ij} over all pieces S_{ij} and the previous approximation leads to the following definition of surface area of a parametric surface (Figure 15.6.11).

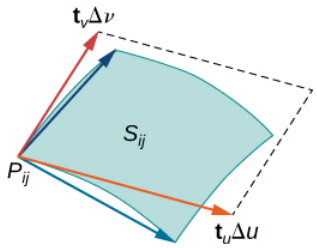


Figure 15.6.11: The parallelogram spanned by $\vec{\mathbf{t}}_u$ *and* $\vec{\mathbf{t}}_v$ *approximates the piece of surface* S_{ij}.

Definition: Smooth Parameterization of Surface

Let $\vec{\mathbf{r}}(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v)\rangle$ with parameter domain D be a **smooth parameterization of surface** S. Furthermore, assume that S is traced out only once as (u,v) varies over D. The surface area of S is

$$\iint_D ||\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v||\, dA, \tag{15.6.1}$$

where $\vec{\mathbf{t}}_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle$

and

$$\vec{\mathbf{t}}_v = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle.$$

Example 15.6.5: Calculating Surface Area

Calculate the lateral surface area (the area of the "side," not including the base) of the right circular cone with height h and radius r.

Solution

Before calculating the surface area of this cone using Equation 15.6.1, we need a parameterization. We assume this cone is in $\mathbb{R}^3$ with its vertex at the origin (Figure 15.6.12). To obtain a parameterization, let α be the angle that is swept out by starting at the positive z-axis and ending at the cone, and let $k = \tan\alpha$. For a height value v with $0 \le v \le h$, the radius of the circle formed by intersecting the cone with plane $z = v$ is kv. Therefore, a parameterization of this cone is

$$\vec{\mathbf{s}}(u,v) = \langle kv\cos u,\ kv\sin u,\ v\rangle,\ 0 \le u < 2\pi,\ 0 \le v \le h.$$

The idea behind this parameterization is that for a fixed v-value, the circle swept out by letting u vary is the circle at height v and radius kv. As v increases, the parameterization sweeps out a "stack" of circles, resulting in the desired cone.

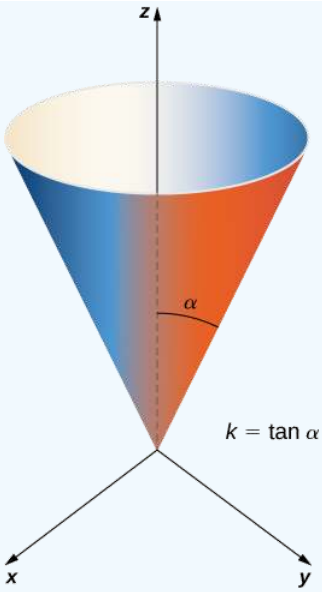


Figure 15.6.12: The right circular cone with radius $r = kh$ and height h has parameterization $\vec{\mathbf{s}}(u,v) = \langle kv\cos u,\ kv\sin u,\ v\rangle$, $0 \le u < 2\pi$, $0 \le v \le h$.

With a parameterization in hand, we can calculate the surface area of the cone using Equation 15.6.1. The tangent vectors are $\vec{\mathbf{t}}_u = \langle -kv\sin u,\ kv\cos u,\ 0\rangle$ and $\vec{\mathbf{t}}_v = \langle k\cos u,\ k\sin u,\ 1\rangle$. Therefore,

$$\begin{aligned}\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -kv\sin u & kv\cos u & 0 \\ k\cos u & k\sin u & 1 \end{vmatrix} \\ &= \langle kv\cos u,\ kv\sin u,\ -k^2v\sin^2 u - k^2v\cos^2 u\rangle \\ &= \langle kv\cos u,\ kv\sin u,\ -k^2v\rangle.\end{aligned}$$

The magnitude of this vector is

$$\begin{aligned}||\langle kv\cos u,\ kv\sin u,\ -k^2v\rangle|| &= \sqrt{k^2v^2\cos^2 u + k^2v^2\sin^2 u + k^4v^2} \\ &= \sqrt{k^2v^2 + k^4v^2} \\ &= kv\sqrt{1+k^2}.\end{aligned}$$

By Equation 15.6.1, the surface area of the cone is

$$\begin{aligned}\iint_D ||\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v||\,dA &= \int_0^h \int_0^{2\pi} kv\sqrt{1+k^2}\,du\,dv \\ &= 2\pi k\sqrt{1+k^2}\int_0^h v\,dv \\ &= 2\pi k\sqrt{1+k^2}\left[\frac{v^2}{2}\right]_0^h \\ &= \pi kh^2\sqrt{1+k^2}.\end{aligned}$$

Since $k = \tan\alpha = r/h$,

$$\begin{aligned}\pi k h^2\sqrt{1+k^2} &= \pi \frac{r}{h} h^2 \sqrt{1+\frac{r^2}{h^2}} \\ &= \pi r h \sqrt{1+\frac{r^2}{h^2}} \\ &= \pi r \sqrt{h^2 + h^2\left(\frac{r^2}{h^2}\right)} \\ &= \pi r \sqrt{h^2+r^2}.\end{aligned}$$

Therefore, the lateral surface area of the cone is $\pi r\sqrt{h^2+r^2}$.

Analysis

The surface area of a right circular cone with radius r and height h is usually given as $\pi r^2 + \pi r\sqrt{h^2+r^2}$. The reason for this is that the circular base is included as part of the cone, and therefore the area of the base πr^2 is added to the lateral surface area $\pi r\sqrt{h^2+r^2}$ that we found.

? Exercise 15.6.5

Find the surface area of the surface with parameterization $\vec{\mathbf{r}}(u,v) = \langle u+v, u^2, 2v\rangle$, $0 \le u \le 3$, $0 \le v \le 2$.

✓ Example 15.6.6: Calculating Surface Area

Show that the surface area of the sphere $x^2+y^2+z^2 = r^2$ is $4\pi r^2$.

Solution

The sphere has parameterization

$$r\cos\theta\sin\phi,\ r\sin\theta\sin\phi,\ r\cos\phi\rangle,\ 0 \le \theta < 2\pi,\ 0 \le \phi \le \pi.$$

The tangent vectors are

$$\vec{\mathbf{t}}_\theta = \langle -r\sin\theta\sin\phi,\ r\cos\theta\sin\phi,\ 0\rangle$$

and

$$\vec{\mathbf{t}}_\phi = \langle r\cos\theta\cos\phi,\ r\sin\theta\cos\phi,\ -r\sin\phi\rangle.$$

Therefore,

$$\begin{aligned}\vec{\mathbf{t}}_\phi \times \vec{\mathbf{t}}_\theta &= \langle r^2\cos\theta\sin^2\phi,\ r^2\sin\theta\sin^2\phi,\ r^2\sin^2\theta\sin\phi\cos\phi + r^2\cos^2\theta\sin\phi\cos\phi\rangle \\ &= \langle r^2\cos\theta\sin^2\phi,\ r^2\sin\theta\sin^2\phi,\ r^2\sin\phi\cos\phi\rangle.\end{aligned}$$

Now,

$$\begin{aligned}||\vec{\mathbf{t}}_\phi \times \vec{\mathbf{t}}_\theta|| &= \sqrt{r^4\sin^4\phi\cos^2\theta + r^4\sin^4\phi\sin^2\theta + r^4\sin^2\phi\cos^2\phi} \\ &= \sqrt{r^4\sin^4\phi + r^4\sin^2\phi\cos^2\phi} \\ &= r^2\sqrt{\sin^2\phi} \\ &= r\sin\phi.\end{aligned}$$

Notice that $\sin\phi \ge 0$ on the parameter domain because $0 \le \phi < \pi$, and this justifies equation $\sqrt{\sin^2\phi} = \sin\phi$. The surface area of the sphere is

$$\int_0^{2\pi}\int_0^{\pi} r^2\sin\phi\,d\phi\,d\theta = r^2\int_0^{2\pi} 2\,d\theta = 4\pi r^2.$$

We have derived the familiar formula for the surface area of a sphere using surface integrals.

Exercise 15.6.6

Show that the surface area of cylinder $x^2 + y^2 = r^2$, $0 \le z \le h$ is $2\pi rh$. Notice that this cylinder does not include the top and bottom circles.

In addition to parameterizing surfaces given by equations or standard geometric shapes such as cones and spheres, we can also parameterize surfaces of revolution. Therefore, we can calculate the surface area of a surface of revolution by using the same techniques. Let $y = f(x) \ge 0$ be a positive single-variable function on the domain $a \le x \le b$ and let S be the surface obtained by rotating f about the x-axis (Figure 15.6.13). Let θ be the angle of rotation. Then, S can be parameterized with parameters x and θ by

$$\vec{\mathbf{r}}(x, \theta) = \langle x, f(x)\cos\theta, f(x)\sin\theta \rangle, \; a \le x \le b, \; 0 \le x \le 2\pi.$$

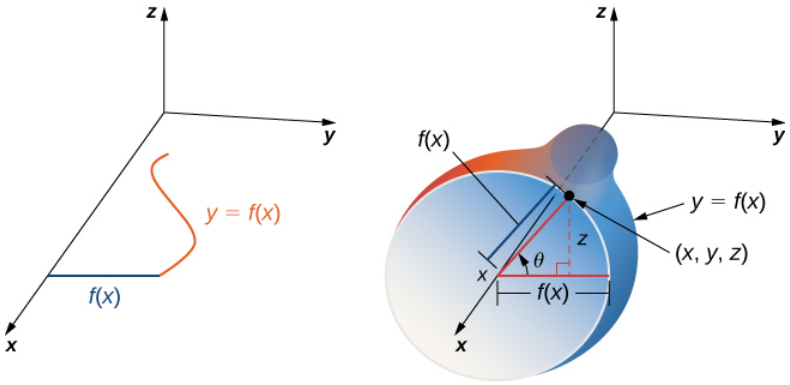


Figure 15.6.13: We can parameterize a surface of revolution by $\vec{\mathbf{r}}(x, \theta) = \langle x, f(x)\cos\theta, f(x)\sin\theta \rangle$, $a \le x \le b$, $0 \le x < 2\pi$.

Example 15.6.7: Calculating Surface Area

Find the area of the surface of revolution obtained by rotating $y = x^2$, $0 \le x \le b$ about the x-axis (Figure 15.6.14).

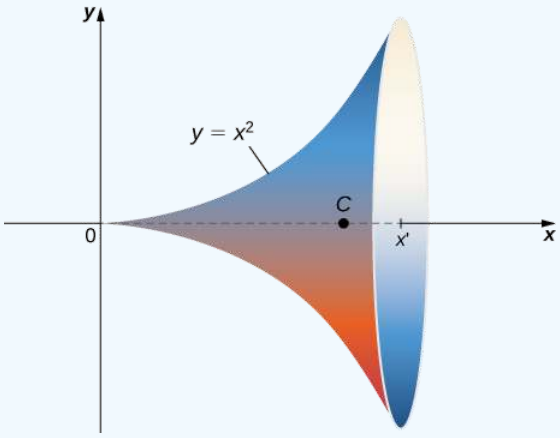


Figure 15.6.14: A surface integral can be used to calculate the surface area of this solid of revolution.

Solution

This surface has parameterization $\vec{\mathbf{r}}(x, \theta) = \langle x, x^2\cos\theta, x^2\sin\theta \rangle$, $0 \le x \le b$, $0 \le x < 2\pi$.

The tangent vectors are $\vec{\mathbf{t}}_x = \langle 1, 2x\cos\theta, 2x\sin\theta \rangle$ and $\vec{\mathbf{t}}_\theta = \langle 0, -x^2\sin\theta, -x^2\cos\theta \rangle$.

Therefore,

$$\begin{aligned} \vec{\mathbf{t}}_x \times \vec{\mathbf{t}}_\theta &= \langle 2x^3\cos^2\theta + 2x^3\sin^2\theta, -x^2\cos\theta, -x^2\sin\theta \rangle \\ &= \langle 2x^3, -x^2\cos\theta, -x^2\sin\theta \rangle \end{aligned}$$

and

$$\begin{aligned}\vec{\mathbf{t}}_x \times \vec{\mathbf{t}}_\theta &= \sqrt{4x^6 + x^4\cos^2\theta + x^4\sin^2\theta} \\ &= \sqrt{4x^6 + x^4} \\ &= x^2\sqrt{4x^2+1}\end{aligned}$$

The area of the surface of revolution is

$$\begin{aligned}\int_0^b \int_0^{2\pi} x^2\sqrt{4x^2+1}\,d\theta\,dx &= 2\pi\int_0^b x^2\sqrt{4x^2+1}\,dx \\ &= 2\pi\left[\frac{1}{64}\left(2\sqrt{4x^2+1}(8x^3+x)\,\sinh^{-1}(2x)\right)\right]_0^b \\ &= 2\pi\left[\frac{1}{64}\left(2\sqrt{4b^2+1}(8b^3+b)\,\sinh^{-1}(2b)\right)\right].\end{aligned}$$

? Exercise 15.6.7

Use Equation 15.6.1 to find the area of the surface of revolution obtained by rotating curve $y = \sin x$, $0 \le x \le \pi$ about the x-axis.

Surface Integral of a Scalar-Valued Function

Now that we can parameterize surfaces and we can calculate their surface areas, we are able to define surface integrals. First, let's look at the surface integral of a scalar-valued function. Informally, the surface integral of a scalar-valued function is an analog of a scalar line integral in one higher dimension. The domain of integration of a scalar line integral is a parameterized curve (a one-dimensional object); the domain of integration of a scalar surface integral is a parameterized surface (a two-dimensional object). Therefore, the definition of a surface integral follows the definition of a line integral quite closely. For scalar line integrals, we chopped the domain curve into tiny pieces, chose a point in each piece, computed the function at that point, and took a limit of the corresponding Riemann sum. For scalar surface integrals, we chop the domain *region* (no longer a curve) into tiny pieces and proceed in the same fashion.

Let S be a piecewise smooth surface with parameterization $\vec{\mathbf{r}}(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v)\rangle$ with parameter domain D and let $f(x,y,z)$ be a function with a domain that contains S. For now, assume the parameter domain D is a rectangle, but we can extend the basic logic of how we proceed to any parameter domain (the choice of a rectangle is simply to make the notation more manageable). Divide rectangle D into subrectangles D_{ij} with horizontal width Δu and vertical length Δv. Suppose that i ranges from 1 to m and j ranges from 1 to n so that D is subdivided into mn rectangles. This division of D into subrectangles gives a corresponding division of S into pieces S_{ij}. Choose point P_{ij} in each piece S_{ij} evaluate P_{ij} at f, and multiply by area S_{ij} to form the Riemann sum

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij})\,\Delta S_{ij}.$$

To define a surface integral of a scalar-valued function, we let the areas of the pieces of S shrink to zero by taking a limit.

Definition: Surface Integral of a Scalar-Valued Function

The **surface integral of a scalar-valued function** of f over a piecewise smooth surface S is

$$\iint_S f(x,y,z)\,dA = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij})\Delta S_{ij}.$$

Again, notice the similarities between this definition and the definition of a scalar line integral. In the definition of a line integral we chop a curve into pieces, evaluate a function at a point in each piece, and let the length of the pieces shrink to zero by taking the limit of the corresponding Riemann sum. In the definition of a surface integral, we chop a surface into pieces, evaluate a function at a point in each piece, and let the area of the pieces shrink to zero by taking the limit of the corresponding Riemann sum. Thus, a surface integral is similar to a line integral but in one higher dimension.

The definition of a scalar line integral can be extended to parameter domains that are not rectangles by using the same logic used earlier. The basic idea is to chop the parameter domain into small pieces, choose a sample point in each piece, and so on. The exact shape of each piece in the sample domain becomes irrelevant as the areas of the pieces shrink to zero.

Scalar surface integrals are difficult to compute from the definition, just as scalar line integrals are. To develop a method that makes surface integrals easier to compute, we approximate surface areas ΔS_{ij} with small pieces of a tangent plane, just as we did in the previous subsection. Recall the definition of vectors $\vec{\mathbf{t}}_u$ and $\vec{\mathbf{t}}_v$:

$$\vec{\mathbf{t}}_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle \text{ and } \vec{\mathbf{t}}_v = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle.$$

From the material we have already studied, we know that

$$\Delta S_{ij} \approx ||\vec{\mathbf{t}}_u(P_{ij}) \times \vec{\mathbf{t}}_v(P_{ij})|| \, \Delta u \, \Delta v.$$

Therefore,

$$\iint_S f(x,y,z)\, dS \approx \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij}) ||\vec{\mathbf{t}}_u(P_{ij}) \times \vec{\mathbf{t}}_v(P_{ij})|| \, \Delta u \, \Delta v.$$

This approximation becomes arbitrarily close to $\lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij}) \Delta S_{ij}$ as we increase the number of pieces S_{ij} by letting m and n go to infinity. Therefore, we have the following equation to calculate **scalar surface integrals**:

$$\iint_S f(x,y,z)\, dS = \iint_D f(\vec{\mathbf{r}}(u,v)) ||\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v|| \, dA. \tag{15.6.2}$$

Equation 15.6.2 allows us to calculate a surface integral by transforming it into a double integral. This equation for surface integrals is analogous to the equation for line integrals:

$$\iint_C f(x,y,z)\, ds = \int_a^b f(\vec{\mathbf{r}}(t)) ||\vec{\mathbf{r}}'(t)|| \, dt.$$

In this case, vector $\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v$ is perpendicular to the surface, whereas vector $\vec{\mathbf{r}}'(t)$ is tangent to the curve.

Example 15.6.8: Calculating a Surface Integral

Calculate surface integral

$$\iint_S 5\, dS,$$

where S is the surface with parameterization $\vec{\mathbf{r}}(u,v) = \langle u, u^2, v \rangle$ for $0 \le u \le 2$ and $0 \le v \le u$.

Solution

Notice that this parameter domain D is a triangle, and therefore the parameter domain is not rectangular. This is not an issue though, because Equation 15.6.2 does not place any restrictions on the shape of the parameter domain.

To use Equation 15.6.2 to calculate the surface integral, we first find vectors $\vec{\mathbf{t}}_u$ and $\vec{\mathbf{t}}_v$. Note that $\vec{\mathbf{t}}_u = \langle 1, 2u, 0 \rangle$ and $\vec{\mathbf{t}}_v = \langle 0, 0, 1 \rangle$. Therefore,

$$\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 2u & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle 2u, -1, 0 \rangle$$

and

$$||\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v|| = \sqrt{1 + 4u^2}.$$

By Equation 15.6.2,

$$\begin{aligned}\iint_S 5\,dS &= 5\iint_D \sqrt{1+4u^2}\,dA \\ &= 5\int_0^2\int_0^u \sqrt{1+4u^2}\,dv\,du = 5\int_0^2 u\sqrt{1+4u^2}\,du \\ &= 5\left[\frac{(1+4u^2)^{3/2}}{3}\right]_0^2 \\ &= \frac{5(17^{3/2}-1)}{3} \approx 115.15.\end{aligned}$$

Example 15.6.9: Calculating the Surface Integral of a Cylinder

Calculate surface integral

$$\iint_S (x+y^2)\,dS,$$

where S is cylinder $x^2+y^2=4$, $0 \le z \le 3$ (Figure 15.6.15).

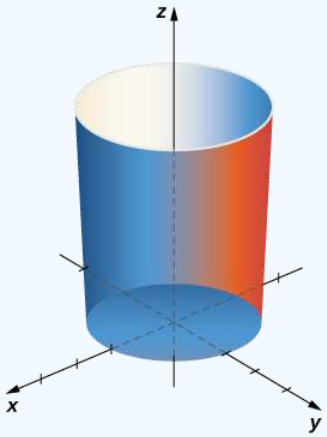


Figure 15.6.15: Integrating function $f(x,y,z)=x+y^2$ over a cylinder.

Solution

To calculate the surface integral, we first need a parameterization of the cylinder. A parameterization is $\vec{\mathbf{r}}(u,v) = \langle \cos u,\ \sin u,\ v\rangle, 0 \le u \le 2\pi,\ 0 \le v \le 3$.

The tangent vectors are $\vec{\mathbf{t}}_u = \langle \sin u,\ \cos u,\ 0\rangle$ and $\vec{\mathbf{t}}_v = \langle 0, 0, 1\rangle$. Then,

$$\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -\sin u & \cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle \cos u,\ \sin u,\ 0\rangle$$

and $||\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v|| = \sqrt{\cos^2 u + \sin^2 u} = 1$. By Equation 15.6.2,

$$\begin{aligned}\iint_S f(x,y,z)dS &= \iint_D f(\vec{\mathbf{r}}(u,v))||\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v||\,dA \\ &= \int_0^3\int_0^{2\pi} (\cos u + \sin^2 u)\,du\,dv \\ &= \int_0^3 \left[\sin u + \frac{u}{2} - \frac{\sin(2u)}{4}\right]_0^{2\pi} dv \\ &= \int_0^3 \pi\,dv = 3\pi.\end{aligned}$$

Exercise 15.6.8

Calculate

$$\iint_S (x^2 - z)\, dS,$$

where S is the surface with parameterization $\vec{\mathbf{r}}(u, v) = \langle v, u^2 + v^2, 1\rangle$, $0 \le u \le 2$, $0 \le v \le 3$.

Example 15.6.10: Calculating the Surface Integral of a Piece of a Sphere

Calculate surface integral

$$\iint_S f(x, y, z)\, dS,$$

where $f(x, y, z) = z^2$ and S is the surface that consists of the piece of sphere $x^2 + y^2 + z^2 = 4$ that lies on or above plane $z = 1$ and the disk that is enclosed by intersection plane $z = 1$ and the given sphere (Figure 15.6.16).

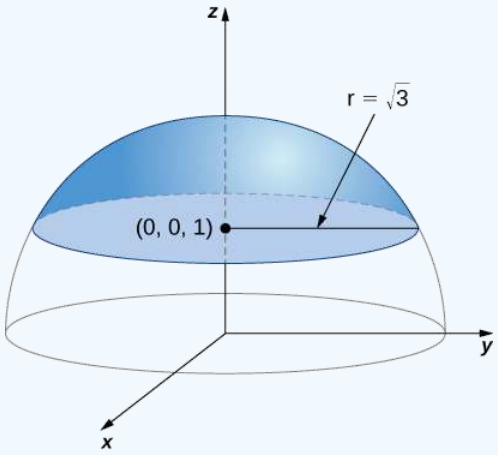


Figure 15.6.16: Calculating a surface integral over surface S.

Solution

Notice that S is not smooth but is piecewise smooth; S can be written as the union of its base S_1 and its spherical top S_2, and both S_1 and S_2 are smooth. Therefore, to calculate

$$\iint_S z^2 dS,$$

we write this integral as

$$\iint_{S_1} z^2\, dS + \iint_{S_2} z^2\, dS$$

and we calculate integrals

$$\iint_{S_1} z^2\, dS$$

and

$$\iint_{S_2} Z^2\, dS.$$

First, we calculate $\iint_{S_1} z^2\, dS$. To calculate this integral we need a parameterization of S_1. This surface is a disk in plane $z = 1$ centered at $(0, 0, 1)$. To parameterize this disk, we need to know its radius. Since the disk is formed where plane $z = 1$ intersects sphere $x^2 + y^2 + z^2 = 4$, we can substitute $z = 1$ into equation $x^2 + y^2 + z^2 = 4$:

$$x^2 + y^2 + 1 = 4 \Rightarrow x^2 + y^2 = 3.$$

Therefore, the radius of the disk is $\sqrt{3}$ and a parameterization of S_1 is $\vec{\mathbf{r}}(u,v) = \langle u\ \cos v,\ u\ \sin v,\ 1\rangle$, $0 \le u \le \sqrt{3}$, $0 \le v \le 2\pi$. The tangent vectors are $\vec{\mathbf{t}}_u = \langle \cos v,\ \sin v,\ 0\rangle$ and $\vec{\mathbf{t}}_v = \langle -u\ \sin v,\ u\ \cos v,\ 0\rangle$, and thus

$$\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 0 \end{vmatrix} = \langle 0,\ 0,\ u\cos^2 v + u\sin^2 v\rangle = \langle 0, 0, u\rangle.$$

The magnitude of this vector is u. Therefore,

$$\begin{aligned} \iint_{S_1} z^2\, dS &= \int_0^{\sqrt{3}} \int_0^{2\pi} f(r(u,v))||t_u \times t_v||\, dv\, du \\ &= \int_0^{\sqrt{3}} \int_0^{2\pi} u\, dv\, du \\ &= 2\pi \int_0^{\sqrt{3}} u\, du \\ &= 2\pi\sqrt{3}. \end{aligned}$$

Now we calculate

$$\iint_{S_2} dS.$$

To calculate this integral, we need a parameterization of S_2. The parameterization of full sphere $x^2 + y^2 + z^2 = 4$ is

$$\vec{\mathbf{r}}(\phi,\theta) = \langle 2\cos\theta\sin\phi,\ 2\sin\theta\sin\phi,\ 2\cos\phi\rangle,\ 0 \le \theta \le 2\pi, 0 \le \phi \le \pi.$$

Since we are only taking the piece of the sphere on or above plane $z = 1$, we have to restrict the domain of ϕ. To see how far this angle sweeps, notice that the angle can be located in a right triangle, as shown in Figure 15.6.17 (the $\sqrt{3}$ comes from the fact that the base of S is a disk with radius $\sqrt{3}$). Therefore, the tangent of ϕ is $\sqrt{3}$, which implies that ϕ is $\pi/6$. We now have a parameterization of S_2:

$\vec{\mathbf{r}}(\phi,\theta) = \langle 2\cos\theta\sin\phi,\ 2\sin\theta\sin\phi,\ 2\cos\phi\rangle,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi/3.$

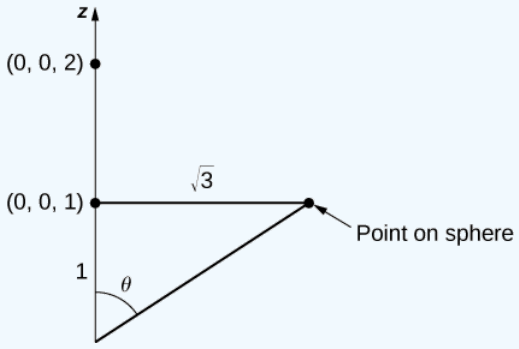


Figure 15.6.17: The maximum value of ϕ has a tangent value of $\sqrt{3}$.

The tangent vectors are $\vec{\mathbf{t}}_\phi = \langle 2\cos\theta\cos\phi,\ 2\sin\theta\cos\phi,\ -2\sin\phi\rangle$ and $\vec{\mathbf{t}}_\theta = \langle -2\sin\theta\sin\phi,\ u\cos\theta\sin\phi,\ 0\rangle$, and thus

$$\begin{aligned} \vec{\mathbf{t}}_\phi \times \vec{\mathbf{t}}_\theta &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2\cos\theta\cos\phi & 2\sin\theta\cos\phi & -2\sin\phi \\ -2\sin\theta\sin\phi & 2\cos\theta\sin\phi & 0 \end{vmatrix} \\ &= \langle 4\cos\theta\sin^2\phi,\ 4\sin\theta\sin^2\phi,\ 4\cos^2\theta\cos\phi\sin\phi + 4\sin^2\theta\cos\phi\sin\phi\rangle \\ &= \langle 4\cos\theta\sin^2\phi,\ 4\sin\theta\sin^2\phi,\ 4\cos\phi\sin\phi\rangle. \end{aligned}$$

The magnitude of this vector is

$$\vec{\mathbf{t}}_\phi \times \vec{\mathbf{t}}_\theta = \sqrt{16\cos^2\theta\sin^4\phi + 16\sin^2\theta\sin^4\phi + 16\cos^2\phi\sin^2\phi}$$
$$= 4\sqrt{\sin^4\phi + \cos^2\phi\sin^2\phi}.$$

Therefore,

$$\begin{aligned}\iint_{S_2} z\,dS &= \int_0^{\pi/6}\int_0^{2\pi} f(\vec{\mathbf{r}}(\phi,\theta))\|\vec{\mathbf{t}}_\phi \times \vec{\mathbf{t}}_\theta\|\,d\theta\,d\phi \\ &= \int_0^{\pi/6}\int_0^{2\pi} 16\cos^2\phi\sqrt{\sin^4\phi + \cos^2\phi\sin^2\phi}\,d\theta\,d\phi \\ &= 32\pi\int_0^{\pi/6}\cos^2\phi\sqrt{\sin^4\phi + \cos^2\phi\sin^2\phi}\,d\phi \\ &= 32\pi\int_0^{\pi/6}\cos^2\phi\sin\phi\sqrt{\sin^2\phi + \cos^2\phi}\,d\phi \\ &= 32\pi\int_0^{\pi/6}\cos^2\phi\sin\phi\,d\phi \\ &= 32\pi\left[-\frac{\cos^3\phi}{3}\right]_0^{\pi/6} \\ &= 32\pi\left[\frac{1}{3} - \frac{\sqrt{3}}{8}\right] = \frac{32\pi}{3} - 4\sqrt{3}.\end{aligned}$$

Since

$$\iint_S z^2\,dS = \iint_{S_1} z^2\,dS + \iint_{S_2} z^2\,dS,$$

we have

$$\iint_S z^2\,dS = (2\pi - 4)\sqrt{3} + \frac{32\pi}{3}.$$

Analysis

In this example we broke a surface integral over a piecewise surface into the addition of surface integrals over smooth subsurfaces. There were only two smooth subsurfaces in this example, but this technique extends to finitely many smooth subsurfaces.

? Exercise 15.6.9

Calculate line integral $\iint_S (x-y)\,dS$, where S is cylinder $x^2 + y^2 = 1$, $0 \le z \le 2$, including the circular top and bottom.

Scalar surface integrals have several real-world applications. Recall that scalar line integrals can be used to compute the mass of a wire given its density function. In a similar fashion, we can use scalar surface integrals to compute the mass of a sheet given its density function. If a thin sheet of metal has the shape of surface S and the density of the sheet at point (x,y,z) is $\rho(x,y,z)$ then mass m of the sheet is

$$m = \iint_S \rho(x,y,z)\,dS. \tag{15.6.3}$$

✓ Example 15.6.11: Calculating the Mass of a Sheet

A flat sheet of metal has the shape of surface $z = 1 + x + 2y$ that lies above rectangle $0 \le x \le 4$ and $0 \le y \le 2$. If the density of the sheet is given by $\rho(x,y,z) = x^2yz$, what is the mass of the sheet?

Solution

Let S be the surface that describes the sheet. Then, the mass of the sheet is given by $m = \iint_S x^2yx\,dS$. To compute this surface integral, we first need a parameterization of S. Since S is given by the function $f(x,y) = 1 + x + 2y$, a parameterization of S is

$\vec{\mathbf{r}}(x,y) = \langle x, y, 1+x+2y\rangle$, $0 \le x \le 4$, $0 \le y \le 2$.

The tangent vectors are $\vec{\mathbf{t}}_x = \langle 1,0,1\rangle$ and $\vec{\mathbf{t}}_y = \langle 1,0,2\rangle$. Therefore, $\vec{\mathbf{t}}_x + \vec{\mathbf{t}}_y = \langle -1,-2,1\rangle$ and $||\vec{\mathbf{t}}_x \times \vec{\mathbf{t}}_y|| = \sqrt{6}$.

By the definition of the line integral (Section 16.2),

$$\begin{aligned} m &= \iint_S x^2yz\,dS \\ &= \sqrt{6}\int_0^4 \int_0^2 x^2y(1+x+2y)\,dy\,dx \\ &= \sqrt{6}\int_0^4 \frac{22x^2}{3} + 2x^3\,dx \\ &= \frac{2560\sqrt{6}}{9} \approx 696.74. \end{aligned}$$

? Exercise 15.6.10

A piece of metal has a shape that is modeled by paraboloid $z = x^2 + y^2$, $0 \le z \le 4$, and the density of the metal is given by $\rho(x,y,z) = z+1$. Find the mass of the piece of metal.

Orientation of a Surface

Recall that when we defined a scalar line integral, we did not need to worry about an orientation of the curve of integration. The same was true for scalar surface integrals: we did not need to worry about an "orientation" of the surface of integration.

On the other hand, when we defined vector line integrals, the curve of integration needed an orientation. That is, we needed the notion of an oriented curve to define a vector line integral without ambiguity. Similarly, when we define a surface integral of a vector field, we need the notion of an oriented surface. An oriented surface is given an "upward" or "downward" orientation or, in the case of surfaces such as a sphere or cylinder, an "outward" or "inward" orientation.

Let S be a smooth surface. For any point (x,y,z) on S, we can identify two unit normal vectors $\vec{\mathbf{N}}$ and $-\vec{\mathbf{N}}$. If it is possible to choose a unit normal vector $\vec{\mathbf{N}}$ at every point (x,y,z) on S so that $\vec{\mathbf{N}}$ varies continuously over S, then S is "*orientable*." Such a choice of unit normal vector at each point gives the **orientation of a surface** S. If you think of the normal field as describing water flow, then the side of the surface that water flows toward is the "negative" side and the side of the surface at which the water flows away is the "positive" side. Informally, a choice of orientation gives S an "outer" side and an "inner" side (or an "upward" side and a "downward" side), just as a choice of orientation of a curve gives the curve "forward" and "backward" directions.

Closed surfaces such as spheres are orientable: if we choose the outward normal vector at each point on the surface of the sphere, then the unit normal vectors vary continuously. This is called the *positive orientation of the closed surface* (Figure 15.6.18). We also could choose the inward normal vector at each point to give an "inward" orientation, which is the negative orientation of the surface.

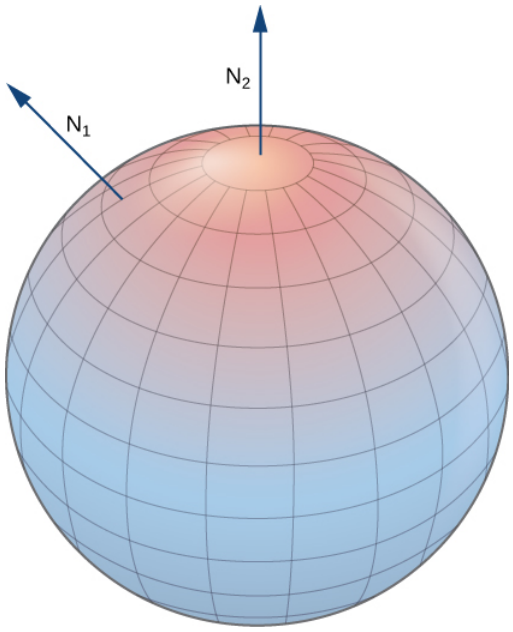


Figure 15.6.18: An oriented sphere with positive orientation.

A portion of the graph of any smooth function $z = f(x, y)$ is also orientable. If we choose the unit normal vector that points "above" the surface at each point, then the unit normal vectors vary continuously over the surface. We could also choose the unit normal vector that points "below" the surface at each point. To get such an orientation, we parameterize the graph of f in the standard way: $\vec{\mathbf{r}}(x, y) = \langle x,\ y,\ f(x, y)\rangle$, where x and y vary over the domain of f. Then, $\vec{\mathbf{t}}_x = \langle 1, 0, f_x\rangle$ and $\vec{\mathbf{t}}_y = \langle 0, 1, f_y\rangle$, and therefore the cross product $\vec{\mathbf{t}}_x \times \vec{\mathbf{t}}_y$ (which is normal to the surface at any point on the surface) is $\langle -f_x, -f_y, 1\rangle$Since the z-component of this vector is one, the corresponding unit normal vector points "upward," and the upward side of the surface is chosen to be the "positive" side.

Let S be a smooth orientable surface with parameterization $\vec{\mathbf{r}}(u, v)$. For each point $\vec{\mathbf{r}}(a, b)$ on the surface, vectors $\vec{\mathbf{t}}_u$ and $\vec{\mathbf{t}}_v$ lie in the tangent plane at that point. Vector $\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v$ is normal to the tangent plane at $\vec{\mathbf{r}}(a, b)$ and is therefore normal to S at that point. Therefore, the choice of unit normal vector

$$\vec{\mathbf{N}} = \frac{\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v}{||\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v||}$$

gives an orientation of surface S.

Example 15.6.12: Choosing an Orientation

Give an orientation of cylinder $x^2 + y^2 = r^2,\ 0 \le z \le h$.

Solution

This surface has parameterization $\vec{\mathbf{r}}(u, v) = \langle r\cos u,\ r\sin u,\ v\rangle,\ 0 \le u < 2\pi,\ 0 \le v \le h$.

The tangent vectors are $\vec{\mathbf{t}}_u = \langle -r\sin u,\ r\cos u,\ 0\rangle$ and $\vec{\mathbf{t}}_v = \langle 0, 0, 1\rangle$. To get an orientation of the surface, we compute the unit normal vector

$$\vec{\mathbf{N}} = \frac{\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v}{||\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v||}$$

In this case, $\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v = \langle r\cos u,\ r\sin u,\ 0\rangle$ and therefore

$$||\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v|| = \sqrt{r^2\cos^2 u + r^2\sin^2 u} = r.$$

An orientation of the cylinder is

$$\vec{\mathbf{N}}(u, v) = \frac{\langle r\cos u,\ r\sin u,\ 0\rangle}{r} = \langle \cos u,\ \sin u,\ 0\rangle.$$

Notice that all vectors are parallel to the xy-plane, which should be the case with vectors that are normal to the cylinder. Furthermore, all the vectors point outward, and therefore this is an outward orientation of the cylinder (Figure 15.6.19).

Figure 15.6.19: If all the vectors normal to a cylinder point outward, then this is an outward orientation of the cylinder.

? Exercise 15.6.11

Give the "upward" orientation of the graph of $f(x,y) = xy$.

Since every curve has a "forward" and "backward" direction (or, in the case of a closed curve, a clockwise and counterclockwise direction), it is possible to give an orientation to any curve. Hence, it is possible to think of every curve as an oriented curve. This is not the case with surfaces, however. Some surfaces cannot be oriented; such surfaces are called *nonorientable*. Essentially, a surface can be oriented if the surface has an "inner" side and an "outer" side, or an "upward" side and a "downward" side. Some surfaces are twisted in such a fashion that there is no well-defined notion of an "inner" or "outer" side.

The classic example of a nonorientable surface is the Möbius strip. To create a Möbius strip, take a rectangular strip of paper, give the piece of paper a half-twist, and the glue the ends together (Figure 15.6.20). Because of the half-twist in the strip, the surface has no "outer" side or "inner" side. If you imagine placing a normal vector at a point on the strip and having the vector travel all the way around the band, then (because of the half-twist) the vector points in the opposite direction when it gets back to its original position. Therefore, the strip really only has one side.

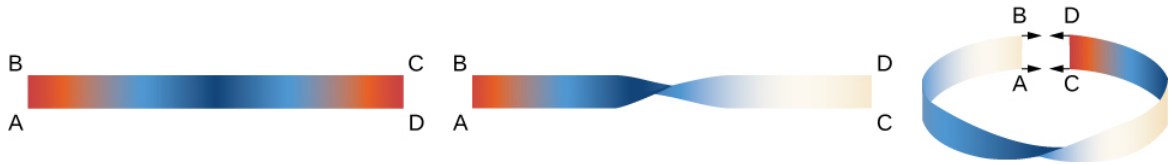


Figure 15.6.20: The construction of a Möbius strip.

Since some surfaces are *nonorientable*, it is not possible to define a vector surface integral on all piecewise smooth surfaces. This is in contrast to vector line integrals, which can be defined on any piecewise smooth curve.

Surface Integral of a Vector Field

With the idea of orientable surfaces in place, we are now ready to define a surface integral of a vector field. The definition is analogous to the definition of the flux of a vector field along a plane curve. Recall that if $\vec{\mathbf{F}}$ is a two-dimensional vector field and C is a plane curve, then the definition of the flux of $\vec{\mathbf{F}}$ along C involved chopping C into small pieces, choosing a point inside each piece, and calculating $\vec{\mathbf{F}} \cdot \vec{\mathbf{N}}$ at the point (where $\vec{\mathbf{N}}$ is the unit normal vector at the point). The definition of a surface integral of a vector field proceeds in the same fashion, except now we chop surface S into small pieces, choose a point in the small (two-dimensional) piece, and calculate $\vec{\mathbf{F}} \cdot \vec{\mathbf{N}}$ at the point.

To place this definition in a real-world setting, let S be an oriented surface with unit normal vector $\vec{\mathbf{N}}$. Let $\vec{\mathbf{v}}$ be a velocity field of a fluid flowing through S, and suppose the fluid has density $\rho(x,y,z)$ Imagine the fluid flows through S, but S is completely permeable so that it does not impede the fluid flow (Figure 15.6.21). The mass flux of the fluid is the rate of mass flow per unit area. The mass flux is measured in mass per unit time per unit area. How could we calculate the mass flux of the fluid across S?

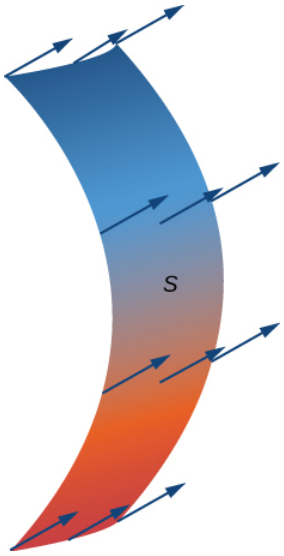


Figure 15.6.21: Fluid flows across a completely permeable surface S.

The rate of flow, measured in mass per unit time per unit area, is $\rho\vec{\mathbf{N}}$. To calculate the mass flux across S, chop S into small pieces S_{ij}. If S_{ij} is small enough, then it can be approximated by a tangent plane at some point P in S_{ij}. Therefore, the unit normal vector at P can be used to approximate $\vec{\mathbf{N}}(x,y,z)$ across the entire piece S_{ij} because the normal vector to a plane does not change as we move across the plane. The component of the vector ρv at P in the direction of $\vec{\mathbf{N}}$ is $\rho\vec{\mathbf{v}}\cdot\vec{\mathbf{N}}$ at P. Since S_{ij} is small, the dot product $\rho v\cdot N$ changes very little as we vary across S_{ij} and therefore $\rho\vec{\mathbf{v}}\cdot\vec{\mathbf{N}}$ can be taken as approximately constant across S_{ij}. To approximate the mass of fluid per unit time flowing across S_{ij} (and not just locally at point P), we need to multiply $(\rho\vec{\mathbf{v}}\cdot\vec{\mathbf{N}})(P)$ by the area of S_{ij}. Therefore, the mass of fluid per unit time flowing across S_{ij} in the direction of $\vec{\mathbf{N}}$ can be approximated by $(\rho\vec{\mathbf{v}}\cdot\vec{\mathbf{N}})\Delta S_{ij}$ where $\vec{\mathbf{N}}$, ρ and $\vec{\mathbf{v}}$ are all evaluated at P (Figure 15.6.22). This is analogous to the flux of two-dimensional vector field $\vec{\mathbf{F}}$ across plane curve C, in which we approximated flux across a small piece of C with the expression $(\vec{\mathbf{F}}\cdot\vec{\mathbf{N}})\,\Delta s$. To approximate the mass flux across S, form the sum

$$\sum_{i=1}^{m}\sum_{j=1}^{n}(\rho\vec{\mathbf{v}}\cdot\vec{\mathbf{N}})\Delta S_{ij}.$$

As pieces S_{ij} get smaller, the sum

$$\sum_{i=1}^{m}\sum_{j=1}^{n}(\rho\vec{\mathbf{v}}\cdot\vec{\mathbf{N}})\Delta S_{ij}$$

gets arbitrarily close to the mass flux. Therefore, the mass flux is

$$\iint_S \rho\vec{\mathbf{v}}\cdot\vec{\mathbf{N}}\,dS=\lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n}(\rho\vec{\mathbf{v}}\cdot\vec{\mathbf{N}})\Delta S_{ij}.$$

This is a surface integral of a vector field. Letting the vector field $\rho\vec{\mathbf{v}}$ be an arbitrary vector field $\vec{\mathbf{F}}$ leads to the following definition.

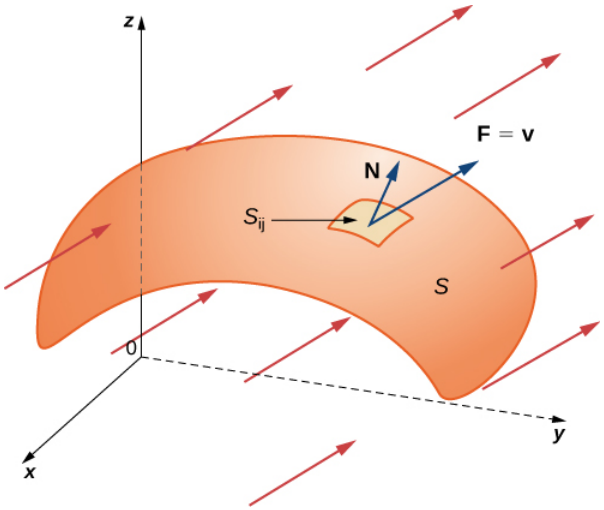


Figure 15.6.22: The mass of fluid per unit time flowing across S_{ij} in the direction of $\vec{\mathbf{N}}$ can be approximated by $(\rho\vec{\mathbf{v}}\cdot\vec{\mathbf{N}})\Delta S_{ij}$.

Definition: Surface Integrals

Let $\vec{\mathbf{F}}$ be a continuous vector field with a domain that contains oriented surface S with unit normal vector $\vec{\mathbf{N}}$. The surface integral of $\vec{\mathbf{F}}$ over S is

$$\iint_S \vec{\mathbf{F}}\cdot d\vec{\mathbf{S}} = \iint_S \vec{\mathbf{F}}\cdot\vec{\mathbf{N}}\,dS. \tag{15.6.4}$$

Notice the parallel between this definition and the definition of vector line integral $\int_C \vec{\mathbf{F}}\cdot\vec{\mathbf{N}}\,dS$. A surface integral of a vector field is defined in a similar way to a flux line integral across a curve, except the domain of integration is a surface (a two-dimensional object) rather than a curve (a one-dimensional object). Integral $\iint_S \vec{\mathbf{F}}\cdot\vec{\mathbf{N}}\,dS$ is called the *flux of* $\vec{\mathbf{F}}$ *across* S, just as integral $\int_C \vec{\mathbf{F}}\cdot\vec{\mathbf{N}}\,dS$ is the flux of $\vec{\mathbf{F}}$ across curve C. A surface integral over a vector field is also called a **flux integral**.

Just as with vector line integrals, surface integral $\iint_S \vec{\mathbf{F}}\cdot\vec{\mathbf{N}}\,dS$ is easier to compute after surface S has been parameterized. Let $\vec{\mathbf{r}}(u,v)$ be a parameterization of S with parameter domain D. Then, the unit normal vector is given by $\vec{\mathbf{N}} = \dfrac{\vec{\mathbf{t}}_u\times\vec{\mathbf{t}}_v}{||\vec{\mathbf{t}}_u\times\vec{\mathbf{t}}_v||}$ and, from Equation 15.6.4, we have

$$\begin{aligned}\int_C \vec{\mathbf{F}}\cdot\vec{\mathbf{N}}\,dS &= \iint_S \vec{\mathbf{F}}\cdot\frac{\vec{\mathbf{t}}_u\times\vec{\mathbf{t}}_v}{||\vec{\mathbf{t}}_u\times\vec{\mathbf{t}}_v||}\,dS\\ &= \iint_D \left(\vec{\mathbf{F}}(\vec{\mathbf{r}}(u,v))\cdot\frac{\vec{\mathbf{t}}_u\times\vec{\mathbf{t}}_v}{||\vec{\mathbf{t}}_u\times\vec{\mathbf{t}}_v||}\right)||\vec{\mathbf{t}}_u\times\vec{\mathbf{t}}_v||\,dA\\ &= \iint_D (\vec{\mathbf{F}}(\vec{\mathbf{r}}(u,v))\cdot(\vec{\mathbf{t}}_u\times\vec{\mathbf{t}}_v))\,dA.\end{aligned}$$

Therefore, to compute a surface integral over a vector field we can use the equation

$$\iint_S \vec{\mathbf{F}}\cdot\vec{\mathbf{N}}\,dS = \iint_D (\vec{\mathbf{F}}(\vec{\mathbf{r}}(u,v))\cdot(\vec{\mathbf{t}}_u\times\vec{\mathbf{t}}_v))\,dA. \tag{15.6.5}$$

Example 15.6.13: Calculating a Surface Integral

Calculate the surface integral

$$\iint_S \vec{\mathbf{F}}\cdot\vec{\mathbf{N}}\,dS,$$

where $\vec{\mathbf{F}} = \langle -y, x, 0\rangle$ and S is the surface with parameterization

$$\vec{\mathbf{r}}(u,v) = \langle u, v^2 - u, u + v\rangle,\ 0 \le u \le 3,\ 0 \le v \le 4.$$

Solution

The tangent vectors are $\vec{\mathbf{t}}_u = \langle 1, -1, 1\rangle$ and $\vec{\mathbf{t}}_v = \langle 0, 2v, 1\rangle$. Therefore,

$$\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v = \langle -1 - 2v, -1, 2v\rangle.$$

By Equation 15.6.5,

$$\begin{aligned}
\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} &= \int_0^4 \int_0^3 F(\vec{\mathbf{r}}(u,v)) \cdot (\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v)\, du\, dv \\
&= \int_0^4 \int_0^3 \langle u - v^2,\ u,\ 0\rangle \cdot \langle -1 - 2v,\ -1,\ 2v\rangle\, du\, dv \\
&= \int_0^4 \int_0^3 [(u - v^2)(-1 - 2v) - u]\, du\, dv \\
&= \int_0^4 \int_0^3 (2v^3 + v^2 - 2uv - 2u)\, du\, dv \\
&= \int_0^4 [2v^3 u + v^2 u - v u^2 - u^2]\Big|_0^3\, dv \\
&= \int_0^4 (6v^3 + 3v^2 - 9v - 9)\, dv \\
&= \left[\frac{3v^4}{2} + v^3 - \frac{9v^2}{2} - 9v\right]_0^4 \\
&= 340.
\end{aligned}$$

Therefore, the flux of $\vec{\mathbf{F}}$ across S is 340.

? Exercise 15.6.12

Calculate surface integral

$$\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, dS,$$

where $\vec{\mathbf{F}} = \langle 0, -z, y\rangle$ and S is the portion of the unit sphere in the first octant with outward orientation.

✓ Example 15.6.14: Calculating Mass Flow Rate

Let $\vec{\mathbf{v}}(x,y,z) = \langle 2x,\ 2y,\ z\rangle$ represent a velocity field (with units of meters per second) of a fluid with constant density 80 kg/m^3. Let S be hemisphere $x^2 + y^2 + z^2 = 9$ with $z \le 0$ such that S is oriented outward. Find the mass flow rate of the fluid across S.

Solution

A parameterization of the surface is

$$\vec{\mathbf{r}}(\phi,\theta) = \langle 3\cos\theta\sin\phi,\ 3\sin\theta\sin\phi,\ 3\cos\phi\rangle,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi/2.$$

As in Example 15.6.7, the tangent vectors are $\vec{\mathbf{t}}_\theta = \langle -3\sin\theta\sin\phi,\ 3\cos\theta\sin\phi,\ 0\rangle$ and $\vec{\mathbf{t}}_\phi = \langle 3\cos\theta\cos\phi,\ 3\sin\theta\cos\phi,\ -3\sin\phi\rangle$, and their cross product is

$$\vec{\mathbf{t}}_\phi \times \vec{\mathbf{t}}_\theta = \langle 9\cos\theta\sin^2\phi,\ 9\sin\theta\sin^2\phi,\ 9\sin\phi\cos\phi\rangle.$$

Notice that each component of the cross product is positive, and therefore this vector gives the outward orientation. Therefore we use the orientation

$$\vec{\mathbf{N}} = \langle 9\cos\theta\sin^2\phi,\ 9\sin\theta\sin^2\phi,\ 9\sin\phi\cos\phi\rangle$$

for the sphere.

By Equation 15.6.4,

$$\begin{aligned}\iint_S \rho v \cdot dS &= 80\int_0^{2\pi}\int_0^{\pi/2} v(r(\phi,\theta))\cdot(t_\phi \times t_\theta)\,d\phi\,d\theta \\ &= 80\int_0^{2\pi}\int_0^{\pi/2} \langle 6\cos\theta\sin\phi,\ 6\sin\theta\sin\phi,\ 3\cos\phi\rangle \\ &\quad \cdot \langle 9\cos\theta\sin^2\phi,\ 9\sin\theta\sin^2\phi,\ 9\sin\phi\cos\phi\rangle\,d\phi\,d\theta \\ &= 80\int_0^{2\pi}\int_0^{\pi/2} 54\sin^3\phi + 27\cos^2\phi\sin\phi\,d\phi\,d\theta \\ &= 80\int_0^{2\pi}\int_0^{\pi/2} 54(1-\cos^2\phi)\sin\phi + 27\cos^2\phi\sin\phi\,d\phi\,d\theta \\ &= 80\int_0^{2\pi}\int_0^{\pi/2} 54\sin\phi - 27\cos^2\phi\sin\phi\,d\phi\,d\theta \\ &= 80\int_0^{2\pi}\Big[-54\cos\phi + 9\cos^3\phi\Big]_{\phi=0}^{\phi=2\pi} d\theta \\ &= 80\int_0^{2\pi} 45\,d\theta \\ &= 7200\pi.\end{aligned}$$

Therefore, the mass flow rate is $7200\pi\,\text{kg/sec/m}^2$.

? Exercise 15.6.13

Let $\vec{\mathbf{v}}(x,y,z) = \langle x^2+y^2,\ z,\ 4y\rangle$ m/sec represent a velocity field of a fluid with constant density 100 kg/m^3. Let S be the half-cylinder $\vec{\mathbf{r}}(u,v) = \langle \cos u,\ \sin u,\ v\rangle$, $0 \le u \le \pi$, $0 \le v \le 2$ oriented outward. Calculate the mass flux of the fluid across S.

In Example 15.6.14, we computed the mass flux, which is the rate of mass flow per unit area. If we want to find the flow rate (measured in volume per time) instead, we can use flux integral

$$\iint_S \vec{\mathbf{v}}\cdot\vec{\mathbf{N}}\,dS,$$

which leaves out the density. Since the flow rate of a fluid is measured in volume per unit time, flow rate does not take mass into account. Therefore, we have the following characterization of the flow rate of a fluid with velocity $\vec{\mathbf{v}}$ across a surface S:

$$\text{Flow rate of fluid across S} = \iint_S \vec{\mathbf{v}}\cdot d\vec{\mathbf{S}}.$$

To compute the flow rate of the fluid in Example 15.6.14, we simply remove the density constant, which gives a flow rate of 90π m^3/sec.

Both mass flux and flow rate are important in physics and engineering. Mass flux measures how much mass is flowing across a surface; flow rate measures how much volume of fluid is flowing across a surface.

In addition to modeling fluid flow, surface integrals can be used to model heat flow. Suppose that the temperature at point (x,y,z) in an object is $T(x,y,z)$. Then the heat flow is a vector field proportional to the negative temperature gradient in the object. To be precise, the heat flow is defined as vector field $F = -k\nabla T$, where the constant k is the *thermal conductivity* of the substance from which the object is made (this constant is determined experimentally). The rate of heat flow across surface S in the object is given by the flux integral

$$\iint_S \vec{\mathbf{F}}\cdot d\vec{\mathbf{S}} = \iint_S -k\vec{\nabla}T\cdot d\vec{\mathbf{S}}.$$

✓ Example 15.6.15: Calculating Heat Flow

A cast-iron solid cylinder is given by inequalities $x^2+y^2 \le 1$, $1 \le z \le 4$. The temperature at point (x,y,z) in a region containing the cylinder is $T(x,y,z) = (x^2+y^2)z$. Given that the thermal conductivity of cast iron is 55, find the heat flow across the boundary of the solid if this boundary is oriented outward.

Solution

Let S denote the boundary of the object. To find the heat flow, we need to calculate flux integral

$$\iint_S -k\vec{\nabla}T \cdot d\vec{\mathbf{S}}.$$

Notice that S is not a smooth surface but is piecewise smooth, since S is the union of three smooth surfaces (the circular top and bottom, and the cylindrical side). Therefore, we calculate three separate integrals, one for each smooth piece of S. Before calculating any integrals, note that the gradient of the temperature is $\vec{\nabla}T = \langle 2xz,\ 2yz,\ x^2 + y^2 \rangle$.

First we consider the circular bottom of the object, which we denote S_1. We can see that S_1 is a circle of radius 1 centered at point $(0, 0, 1)$ sitting in plane $z = 1$. This surface has parameterization $\vec{\mathbf{r}}(u, v) = \langle v\cos u,\ v\sin u,\ 1 \rangle$, $0 \le u < 2\pi$, $0 \le v \le 1$.

Therefore,

$\vec{\mathbf{t}}_u = \langle -v\sin u,\ v\cos u,\ 0 \rangle$ and $\vec{\mathbf{t}}_v = \langle \cos u,\ v\sin u,\ 0 \rangle$, and $\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v = \langle 0,\ 0, -v\sin^2 u - v\cos^2 u \rangle = \langle 0, 0, -v \rangle$.

Since the surface is oriented outward and S_1 is the bottom of the object, it makes sense that this vector points downward. By Equation 15.6.5 the heat flow across S_1 is

$$\begin{aligned}
\iint_{S_1} -k\vec{\nabla}T \cdot d\vec{\mathbf{S}} &= -55\int_0^{2\pi}\int_0^1 \vec{\nabla}T(u, v) \cdot (\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v)\, dv\, du \\
&= -55 \int_0^{2\pi}\int_0^1 \langle 2v\cos u,\ 2v\sin u,\ v^2\cos^2 u + v^2\sin^2 u \rangle \cdot \langle 0, 0, -v \rangle\, dv\, du \\
&= -55\int_0^{2\pi}\int_0^1 \langle 2v\cos u,\ 2v\sin u,\ v^2 \rangle \cdot \langle 0, 0, -v \rangle\, dv\, du \\
&= -55\int_0^{2\pi}\int_0^1 -v^3\, dv\, du \\
&= -55\int_0^{2\pi} -\frac{1}{4}\, du \\
&= \frac{55\pi}{2}.
\end{aligned}$$

Now let's consider the circular top of the object, which we denote S_2. We see that S_2 is a circle of radius 1 centered at point $(0, 0, 4)$, sitting in plane $z = 4$. This surface has parameterization $\vec{\mathbf{r}}(u, v) = \langle v\cos u,\ v\sin u,\ 4 \rangle$, $0 \le u < 2\pi$, $0 \le v \le 1$.

Therefore, $\vec{\mathbf{t}}_u = \langle -v\sin u,\ v\cos u,\ 0 \rangle$ and $\vec{\mathbf{t}}_v = \langle \cos u,\ v\sin u,\ 0 \rangle$, and $\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v = \langle 0,\ 0, -v\sin^2 u - v\cos^2 u \rangle = \langle 0, 0, -v \rangle$.

Since the surface is oriented outward and S_1 is the top of the object, we instead take vector $\vec{\mathbf{t}}_v \times \vec{\mathbf{t}}_u = \langle 0, 0, v \rangle$. By Equation 15.6.5, the heat flow across S_1 is

$$\begin{aligned}
\iint_{S_2} -k\vec{\nabla}T \cdot d\vec{\mathbf{S}} &= -55\int_0^{2\pi}\int_0^1 \vec{\nabla}T(u, v) \cdot (\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v)\, dv\, du \\
&= -55 \int_0^{2\pi}\int_0^1 \langle 8v\cos u,\ 8v\sin u,\ v^2\cos^2 u + v^2\sin^2 u \rangle \cdot \langle 0, 0, -v \rangle\, dv\, du \\
&= -55\int_0^{2\pi}\int_0^1 \langle 8v\cos u,\ 8v\sin u,\ v^2 \rangle \cdot \langle 0, 0, -v \rangle\ dv\, du \\
&= -55\int_0^{2\pi}\int_0^1 -v^3\, dv\, du = -55\int_0^{2\pi} -\frac{1}{4}\, du = -\frac{55\pi}{2}.
\end{aligned}$$

Last, let's consider the cylindrical side of the object. This surface has parameterization $\vec{\mathbf{r}}(u, v) = \langle \cos u,\ \sin u,\ v \rangle$, $0 \le u < 2\pi$, $1 \le v \le 4$. By Example 15.6.9, we know that $\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v = \langle \cos u,\ \sin u,\ 0 \rangle$. By Equation 15.6.5,

$$\begin{aligned}\iint_{S_3} -k\vec{\nabla}T \cdot d\vec{\mathbf{S}} &= -55\int_0^{2\pi}\int_1^4 \vec{\nabla}T(u,v)\cdot(\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v)\,dv\,du \\ &= -55 \\ &\quad \int_0^{2\pi}\int_1^4 \langle 2v\cos u,\, 2v\sin u,\, \cos^2 u + \sin^2 u\rangle \cdot \langle \cos u,\, \sin u,\, 0\rangle\,dv\,du \\ &= -55\int_0^{2\pi}\int_0^1 \langle 2v\cos^2 u,\, 2v\sin u,\, 1\rangle \cdot \langle \cos u,\, \sin u,\, 0\rangle\,dv\,du \\ &= -55\int_0^{2\pi}\int_0^1 (2v\cos^2 u + 2v\sin^2 u)\,dv\,du \\ &= -55\int_0^{2\pi}\int_0^1 2v\,dv\,du \\ &= -55\int_0^{2\pi} du \\ &= -110\pi.\end{aligned}$$

Therefore, the rate of heat flow across S is

$$\frac{55\pi}{2} - \frac{55\pi}{2} - 110\pi = -110\pi.$$

? Exercise 15.6.14

A cast-iron solid ball is given by inequality $x^2 + y^2 + z^2 \le 1$. The temperature at a point in a region containing the ball is $T(x,y,z) = \frac{1}{3}(x^2+y^2+z^2)$. Find the heat flow across the boundary of the solid if this boundary is oriented outward.

Key Concepts

- Surfaces can be parameterized, just as curves can be parameterized. In general, surfaces must be parameterized with two parameters.
- Surfaces can sometimes be oriented, just as curves can be oriented. Some surfaces, such as a Möbius strip, cannot be oriented.
- A surface integral is like a line integral in one higher dimension. The domain of integration of a surface integral is a surface in a plane or space, rather than a curve in a plane or space.
  - The integrand of a surface integral can be a scalar function or a vector field. To calculate a surface integral with an integrand that is a function, use Equation 15.6.2. To calculate a surface integral with an integrand that is a vector field, use Equation 15.6.5.
- If S is a surface, then the area of S is

$$\iint_S dS.$$

Key Equations

- **Scalar surface integral**

$$\iint_S f(x,y,z)\,dS = \iint_D f(\vec{\mathbf{r}}(u,v))\|\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v\|\,dA$$

- **Flux integral**

$$\iint_S \vec{\mathbf{F}}\cdot\vec{\mathbf{N}}\,dS = \iint_S \vec{\mathbf{F}}\cdot d\vec{\mathbf{S}} = \iint_D \vec{\mathbf{F}}(\vec{\mathbf{r}}(u,v))\cdot(\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v)\,dA$$

Glossary

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15.6E: Exercises for Section 15.6

In exercises 1 - 4, determine whether the statements are *true or false*.

1. If surface S is given by $\{(x, y, z) : 0 \le x \le 1, 0 \le y \le 1, z = 10\}$, then $\iint_S f(x, y, z)\, dS = \int_0^1 \int_0^1 f(x, y, 10)\, dx\, dy$.

Answer

True

2. If surface S is given by $\{(x, y, z) : 0 \le x \le 1, 0 \le y \le 1, z = x\}$, then $\iint_S f(x, y, z)\, dS = \int_0^1 \int_0^1 f(x, y, x)\, dx\, dy$.

3. Surface $\vec{\mathbf{r}} = \langle v \cos u,\ v \sin u,\ v^2 \rangle$, for $0 \le u \le \pi$, $0 \le v \le 2$ is the same surface $\vec{\mathbf{r}} = \langle \sqrt{v} \cos 2u,\ \sqrt{v} \sin 2u,\ v \rangle$, for $0 \le u \le \frac{\pi}{2}$, $0 \le v \le 4$.

Answer

True

4. Given the standard parameterization of a sphere, normal vectors $\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v$ are outward normal vectors.

In exercises 5 - 10, find parametric descriptions for the following surfaces.

5. Plane $3x - 2y + z = 2$

Answer

$\vec{\mathbf{r}}(u, v) = \langle u,\ v,\ 2 - 3u + 2v \rangle$ for $-\infty \le u < \infty$ and $-\infty \le v < \infty$.

6. Paraboloid $z = x^2 + y^2$, for $0 \le z \le 9$.

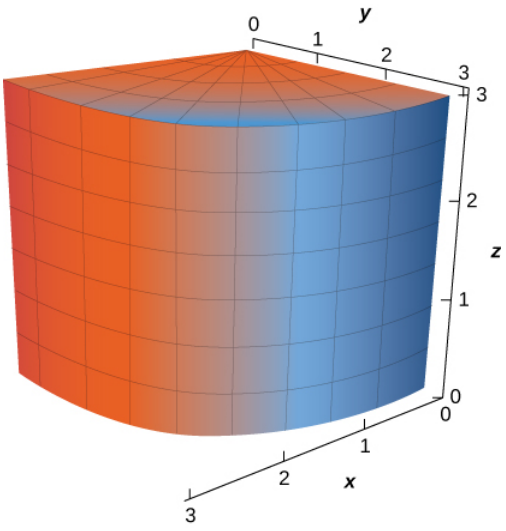
7. Plane $2x - 4y + 3z = 16$

Answer

$\vec{\mathbf{r}}(u, v) = \langle u,\ v,\ \frac{1}{3}(16 - 2u + 4v) \rangle$ for $|u| < \infty$ and $|v| < \infty$.

8. The frustum of cone $z^2 = x^2 + y^2$, for $2 \le z \le 8$

9. The portion of cylinder $x^2 + y^2 = 9$ in the first octant, for $0 \le z \le 3$



Answer

$\vec{\mathbf{r}}(u, v) = \langle 3 \cos u,\ 3 \sin u,\ v \rangle$ for $0 \le u \le \frac{\pi}{2}$, $0 \le v \le 3$

https://math.libretexts.org/@go/page/66962

10. A cone with base radius r and height h, where r and h are positive constants.

For exercises 11 - 12, use a computer algebra system to approximate the area of the following surfaces using a parametric description of the surface.

11. [T] Half cylinder $\{(r,\theta,z) : r = 4,\ 0 \le \theta \le \pi,\ 0 \le z \le 7\}$

Answer

$A = 87.9646$

12. [T] Plane $z = 10 - z - y$ above square $|x| \le 2,\ |y| \le 2$

In exercises 13 - 15, let S be the hemisphere $x^2 + y^2 + z^2 = 4$, with $z \ge 0$, and evaluate each surface integral, in the counterclockwise direction.

13. $\iint_S z\,dS$

Answer

$\iint_S z\,dS = 8\pi$

14. $\iint_S (x - 2y)\,dS$

15. $\iint_S (x^2 + y^2)\,dS$

Answer

$\iint_S (x^2 + y^2)\,dS = 16\pi$

In exercises 16 - 18, evaluate $\int\int_S \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\,dS$ for vector field $\vec{\mathbf{F}}$ where $\vec{\mathbf{N}}$ is an outward normal vector to surface S.

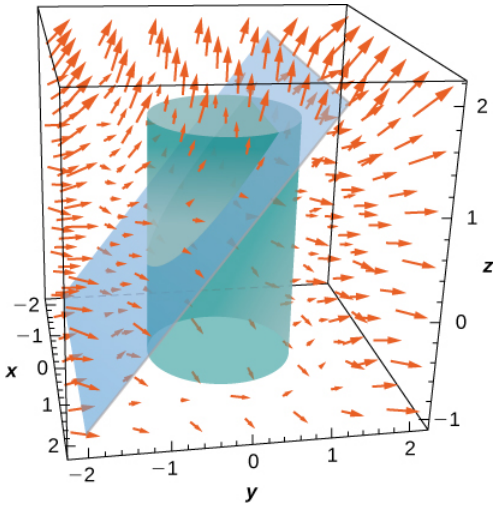
16. $\vec{\mathbf{F}}(x,y,z) = x\,\hat{\mathbf{i}} + 2y\,\hat{\mathbf{j}} = 3z\,\hat{\mathbf{k}}$, and S is that part of plane $15x - 12y + 3z = 6$ that lies above unit square $0 \le x \le 1,\ 0 \le y \le 1$.

17. $\vec{\mathbf{F}}(x,y) = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}$, and S is hemisphere $z = \sqrt{1 - x^2 - y^2}$.

Answer

$\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\,dS = \frac{4\pi}{3}$

18. $\vec{\mathbf{F}}(x,y,z) = x^2\,\hat{\mathbf{i}} + y^2\,\hat{\mathbf{j}} + z^2\,\hat{\mathbf{k}}$, and S is the portion of plane $z = y + 1$ that lies inside cylinder $x^2 + y^2 = 1$.



In exercises 19 - 20, approximate the mass of the homogeneous lamina that has the shape of given surface S. Round to four decimal places.

19. [T] S is surface $z = 4 - x - 2y$, with $z \geq 0$, $x \geq 0$, $y \geq 0$; $\xi = x$.

Answer

$m \approx 13.0639$

20. [T] S is surface $z = x^2 + y^2$, with $z \leq 1$; $\xi = z$.

21. [T] S is surface $x^2 + y^2 + x^2 = 5$, with $z \geq 1$; $\xi = \theta^2$.

Answer

$m \approx 228.5313$

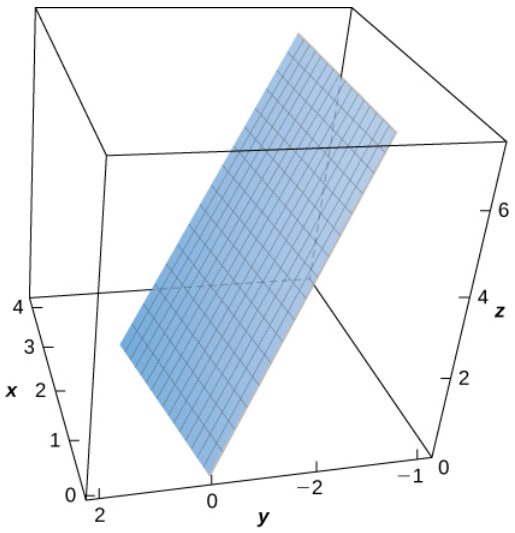
22. Evaluate $\iint_S (y^2 z\,\hat{\mathbf{i}} + y^3\,\hat{\mathbf{j}} + xz\,\hat{\mathbf{k}}) \cdot dS$, where S is the surface of cube $-1 \leq x \leq 1$, $-1 \leq y \leq 1$, and $0 \leq z \leq 2$ in a counterclockwise direction.

23. Evaluate surface integral $\iint_S g\,dS$, where $g(x, y, z) = xz + 2x^2 - 3xy$ and S is the portion of plane $2x - 3y + z = 6$ that lies over unit square $R : 0 \leq x \leq 1$, $0 \leq y \leq 1$.

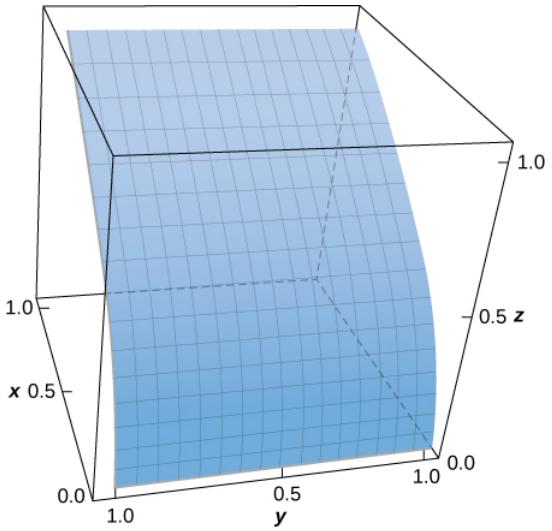
Answer

$\iint_S g\,dS = 3\sqrt{4}$

24. Evaluate $\iint_S (x + y + z)\,dS$, where S is the surface defined parametrically by $\vec{\mathbf{R}}(u, v) = (2u + v)\,\hat{\mathbf{i}} + (u - 2v)\,\hat{\mathbf{j}} + (u + 3v)\,\hat{\mathbf{k}}$ for $0 \leq u \leq 1$, and $0 \leq v \leq 2$.



25. [T] Evaluate $\iint_S (x - y^2 + z)\,dS$, where S is the surface defined parametrically by $\vec{\mathbf{R}}(u, v) = u^2\,\hat{\mathbf{i}} + v\,\hat{\mathbf{j}} + u\,\hat{\mathbf{k}}$ for $0 \leq u \leq 1$, $0 \leq v \leq 1$.



Answer

$$\iint_S (x^2 + y - z)\, dS \approx 0.9617$$

26. [T] Evaluate where S is the surface defined by $\vec{\mathbf{R}}(u, v) = u\,\hat{\mathbf{i}} - u^2\,\hat{\mathbf{j}} + v\,\hat{\mathbf{k}}$, $0 \le u \le 2$, $0 \le v \le 1$ for $0 \le u \le 1$, $0 \le v \le 2$.

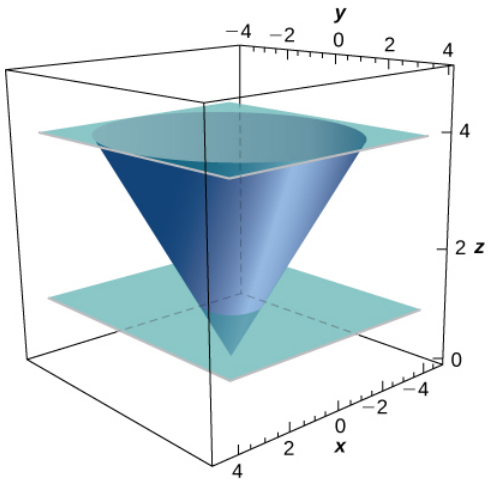
27. Evaluate $\iint_S (x^2 + y^2)\, dS$, where S is the surface bounded above hemisphere $z = \sqrt{1 - x^2 - y^2}$, and below by plane $z = 0$.

Answer

$$\iint_S (x^2 + y^2)\, dS = \frac{4\pi}{3}$$

28. Evaluate $\iint_S (x^2 + y^2 + z^2)\, dS$, where S is the portion of plane that lies inside cylinder $x^2 + y^2 = 1$.

29. [T] Evaluate $\iint_S x^2 z\, dS$, where S is the portion of cone $z^2 = x^2 + y^2$ that lies between planes $z = 1$ and $z = 4$.

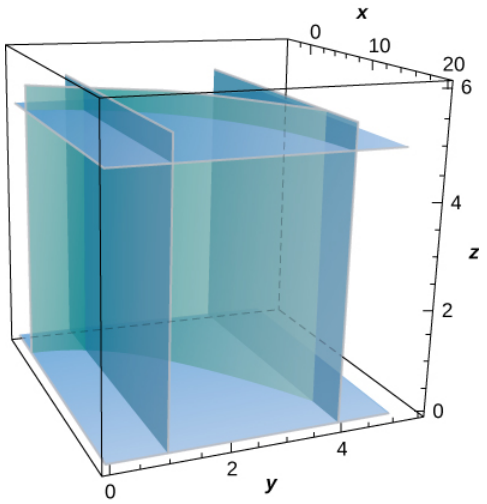


Answer

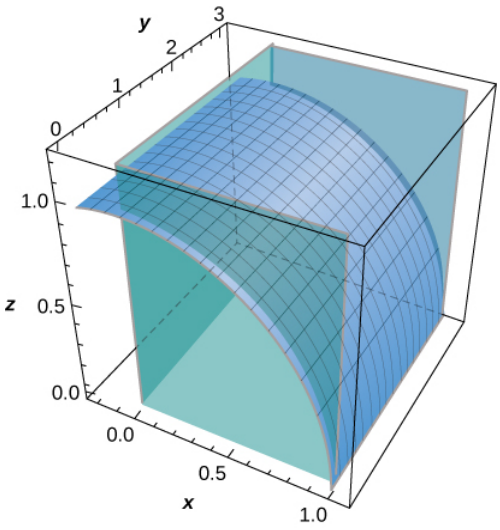
$$\text{div}\,\vec{\mathbf{F}} = a + b$$

$$\iint_S x^2 z\, dS = \frac{1023\sqrt{2\pi}}{5}$$

30. [T] Evaluate $\iint_S \frac{xz}{y}\,dS$, where S is the portion of cylinder $x = y^2$ that lies in the first octant between planes $z = 0$, $z = 5$, and $y = 4$.



31. [T] Evaluate $\iint_S (z + y)\,dS$, where S is the part of the graph of $z = \sqrt{1 - x^2}$ in the first octant between the xy-plane and plane $y = 3$.



Answer

$\iint_S (z + y)\,dS \approx 10.1$

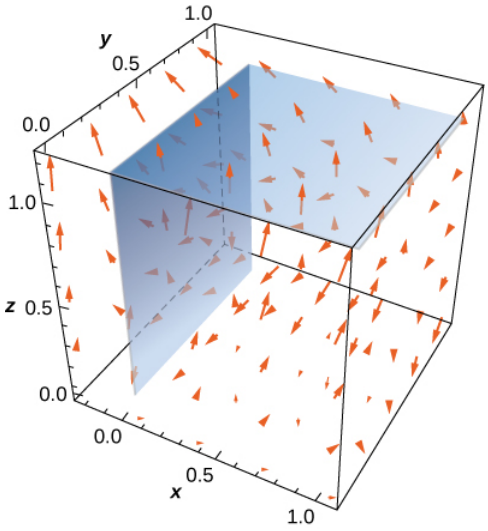
32. Evaluate $\iint_S xyz\,dS$ if S is the part of plane $z = x + y$ that lies over the triangular region in the xy-plane with vertices (0, 0, 0), (1, 0, 0), and (0, 2, 0).

33. Find the mass of a lamina of density $\xi(x, y, z) = z$ in the shape of hemisphere $z = (a^2 - x^2 - y^2)^{1/2}$.

Answer

$m = \pi a^3$

34. Compute $\int\int_S \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\,dS$, where $\vec{\mathbf{F}}(x, y, z) = x\,\hat{\mathbf{i}} - 5y\,\hat{\mathbf{j}} + 4z\,\hat{\mathbf{k}}$ and $\vec{\mathbf{N}}$ is an outward normal vector S, where S is the union of two squares $S_1 : x = 0,\ 0 \le y \le 1,\ 0 \le z \le 1$ and $S_2 : x = 0,\ 0 \le x \le 1,\ 0 \le y \le 1$.



35. Compute $\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, dS$, where $\vec{\mathbf{F}}(x,y,z) = xy\,\hat{\mathbf{i}} + z\,\hat{\mathbf{j}} + (x+y)\,\hat{\mathbf{k}}$ and $\vec{\mathbf{N}}$ is an outward normal vector S, where S is the triangular region cut off from plane $x+y+z=1$ by the positive coordinate axes.

Answer

$$\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, dS = \frac{13}{24}$$

36. Compute $\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, dS$, where $\vec{\mathbf{F}}(x,y,z) = 2yz\,\hat{\mathbf{i}} + (\tan^{-1} xz)\,\hat{\mathbf{j}} + e^{xy}\,\hat{\mathbf{k}}$ and $\vec{\mathbf{N}}$ is an outward normal vector S, where S is the surface of sphere $x^2+y^2+z^2=1$.

37. Compute $\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, dS$, where $\vec{\mathbf{F}}(x,y,z) = xyz\,\hat{\mathbf{i}} + xyz\,\hat{\mathbf{j}} + xyz\,\hat{\mathbf{k}}$ and $\vec{\mathbf{N}}$ is an outward normal vector S, where S is the surface of the five faces of the unit cube $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$ missing $z=0$.

Answer

$$\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, dS = \frac{3}{4}$$

For exercises 38 - 39, express the surface integral as an iterated double integral by using a projection on S on the yz-plane.

38. $\iint_S xy^2z^3\, dS$; S is the first-octant portion of plane $2x+3y+4z=12$.

39. $\iint_S (x^2-2y+z)\, dS$; S is the portion of the graph of $4x+y=8$ bounded by the coordinate planes and plane $z=6$.

Answer

$$\int_0^8 \int_0^6 \left(4-3y+\frac{1}{16}y^2+z\right)\left(\frac{1}{4}\sqrt{17}\right) dz\, dy$$

For exercises 40 - 41, express the surface integral as an iterated double integral by using a projection on S on the xz-plane.

40. $\iint_S xy^2z^3\, dS$; S is the first-octant portion of plane $2x+3y+4z=12$.

41. $\iint_S (x^2-2y+z)\, dS$; is the portion of the graph of $4x+y=8$ bounded by the coordinate planes and plane $z=6$.

Answer

$$\int_0^2 \int_0^6 \left[x^2-2(8-4x)+z\right]\sqrt{17}\, dz\, dx$$

42. Evaluate surface integral $\iint_S yz\, dS$, where S is the first-octant part of plane $x+y+z=\lambda$, where λ is a positive constant.

43. Evaluate surface integral $\iint_S (x^2z + y^2z)\, dS$, where S is hemisphere $x^2 + y^2 + z^2 = a^2$, $z \geq 0$.

Answer

$$\iint_S (x^2z + y^2z)\, dS = \frac{\pi a^5}{2}$$

44. Evaluate surface integral $\iint_S z\, dA$, where S is surface $z = \sqrt{x^2 + y^2}$, $0 \leq z \leq 2$.

45. Evaluate surface integral $\iint_S x^2yz\, dS$, where S is the part of plane $z = 1 + 2x + 3y$ that lies above rectangle $0 \leq x \leq 3$ and $0 \leq y \leq 2$.

Answer

$$\iint_S x^2yz\, dS = 171\sqrt{14}$$

46. Evaluate surface integral $\iint_S yz\, dS$, where S is plane $x + y + z = 1$ that lies in the first octant.

47. Evaluate surface integral $\iint_S yz\, dS$, where S is the part of plane $z = y + 3$ that lies inside cylinder $x^2 + y^2 = 1$.

Answer

$$\iint_S yz\, dS = \frac{\sqrt{2}\pi}{4}$$

For exercises 48 - 50, use geometric reasoning to evaluate the given surface integrals.

48. $\iint_S \sqrt{x^2 + y^2 + z^2}\, dS$, where S is surface $x^2 + y^2 + z^2 = 4$, $z \geq 0$

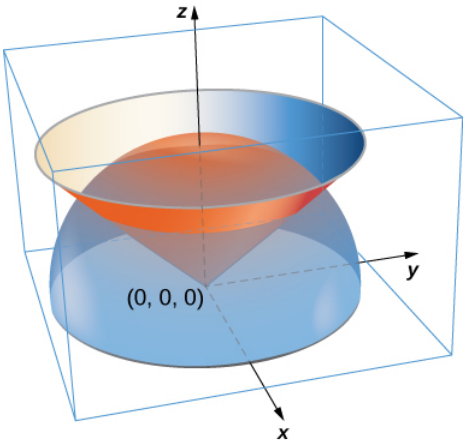
49. $\iint_S (x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}) \cdot dS$, where S is surface $x^2 + y^2 = 4$, $1 \leq z \leq 3$, oriented with unit normal vectors pointing outward

Answer

$$\iint_S (x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}) \cdot dS = 16\pi$$

50. $\iint_S (z\,\hat{\mathbf{k}}) \cdot dS$, where S is disc $x^2 + y^2 \leq 9$ on plane $z = 4$ oriented with unit normal vectors pointing upward

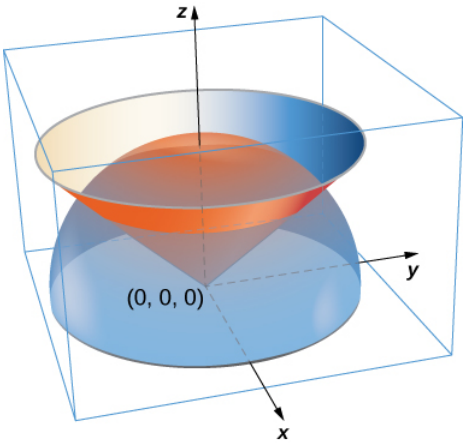
51. A lamina has the shape of a portion of sphere $x^2 + y^2 + z^2 = a^2$ that lies within cone $z = \sqrt{x^2 + y^2}$. Let S be the spherical shell centered at the origin with radius a, and let C be the right circular cone with a vertex at the origin and an axis of symmetry that coincides with the z-axis. Determine the mass of the lamina if $\delta(x, y, z) = x^2y^2z$.



Answer

$$m = \frac{\pi a^7}{192}$$

52. A lamina has the shape of a portion of sphere $x^2 + y^2 + z^2 = a^2$ that lies within cone $z = \sqrt{x^2 + y^2}$. Let S be the spherical shell centered at the origin with radius a, and let C be the right circular cone with a vertex at the origin and an axis of symmetry that coincides with the z-axis. Suppose the vertex angle of the cone is ϕ_0, with $0 \leq \phi_0 < \frac{\pi}{2}$. Determine the mass of that portion of the shape enclosed in the intersection of S and C. Assume $\delta(x, y, z) = x^2 y^2 z$.



53. A paper cup has the shape of an inverted right circular cone of height 6 in. and radius of top 3 in. If the cup is full of water weighing $62.5\, lb/ft^3$, find the magnitude of the total force exerted by the water on the inside surface of the cup.

Answer

$F \approx 4.57\, lb$

For exercises 54 - 55, the heat flow vector field for conducting objects is $\vec{\mathbf{F}} = -k\vec{\nabla}T$, where $T(x, y, z)$ is the temperature in the object and $k > 0$ is a constant that depends on the material. Find the outward flux of $\vec{\mathbf{F}}$ across the following surfaces S for the given temperature distributions and assume $k = 1$.

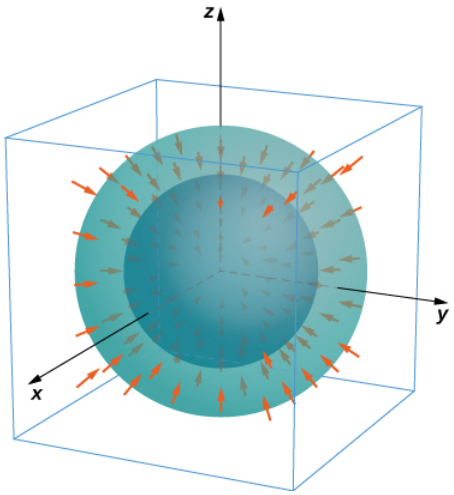
54. $T(x, y, z) = 100e^{-x-y}$; S consists of the faces of cube $|x| \leq 1$, $|y| \leq 1$, $|z| \leq 1$.

55. $T(x, y, z) = -\ln(x^2 + y^2 + z^2)$; S is sphere $x^2 + y^2 + z^2 = a^2$.

Answer

$8\pi a$

For exercises 56 - 57, consider the radial fields $\vec{\mathbf{F}} = \frac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{p/2}} = \frac{r}{|r|^p}$, where p is a real number. Let S consist of spheres A and B centered at the origin with radii $0 < a < b$. The total outward flux across S consists of the outward flux across the outer sphere B less the flux into S across inner sphere A.



56. Find the total flux across S with $p = 0$.

57. Show that for $p = 3$ the flux across S is independent of a and b.

Answer

The net flux is zero.

Contributors

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15.7: Stokes' Theorem

Learning Objectives

- Explain the meaning of Stokes' theorem.
- Use Stokes' theorem to evaluate a line integral.
- Use Stokes' theorem to calculate a surface integral.
- Use Stokes' theorem to calculate a curl.

In this section, we study Stokes' theorem, a higher-dimensional generalization of Green's theorem. This theorem, like the Fundamental Theorem for Line Integrals and Green's theorem, is a generalization of the Fundamental Theorem of Calculus to higher dimensions. Stokes' theorem relates a vector surface integral over surface S in space to a line integral around the boundary of S. Therefore, just as the theorems before it, Stokes' theorem can be used to reduce an integral over a geometric object S to an integral over the boundary of S. In addition to allowing us to translate between line integrals and surface integrals, Stokes' theorem connects the concepts of curl and circulation. Furthermore, the theorem has applications in fluid mechanics and electromagnetism. We use Stokes' theorem to derive Faraday's law, an important result involving electric fields.

Stokes' Theorem

Stokes' theorem says we can calculate the flux of $curl\,\vec{\mathbf{F}}$ across surface S by knowing information only about the values of $\vec{\mathbf{F}}$ along the boundary of S. Conversely, we can calculate the line integral of vector field $\vec{\mathbf{F}}$ along the boundary of surface S by translating to a double integral of the curl of $\vec{\mathbf{F}}$ over S.

Let S be an oriented smooth surface with unit normal vector $\vec{\mathbf{N}}$. Furthermore, suppose the boundary of S is a simple closed curve C. The orientation of S induces the positive orientation of C if, as you walk in the positive direction around C with your head pointing in the direction of $\vec{\mathbf{N}}$, the surface is always on your left. With this definition in place, we can state *Stokes' theorem*.

Theorem 15.7.1: Stokes' Theorem

Let S be a piecewise smooth oriented surface with a boundary that is a simple closed curve C with positive orientation (Figure 15.7.1). If $\vec{\mathbf{F}}$ is a vector field with component functions that have continuous partial derivatives on an open region containing S, then

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_S curl\,\vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}. \tag{15.7.1}$$

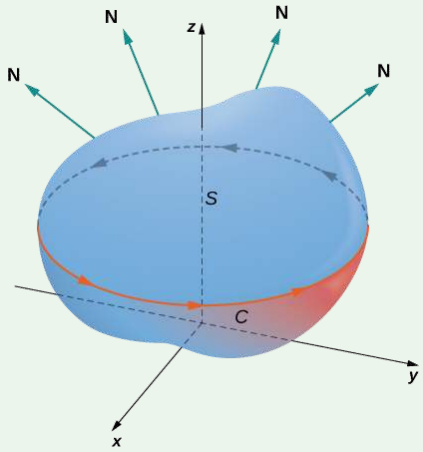


Figure 15.7.1: Stokes' theorem relates the flux integral over the surface to a line integral around the boundary of the surface. Note that the orientation of the curve is positive.

Suppose surface S is a flat region in the xy-plane with upward orientation. Then the unit normal vector is $\vec{\mathbf{k}}$ and surface integral

$$\iint_S curl\,\vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$$

is actually the double integral

$$\iint_S curl\,\vec{\mathbf{F}} \cdot \vec{\mathbf{k}}\, dA.$$

In this special case, Stokes' theorem gives

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_S curl\,\vec{\mathbf{F}} \cdot \vec{\mathbf{k}}\, dA.$$

However, this is the flux form of Green's theorem, which shows us that Green's theorem is a special case of Stokes' theorem. Green's theorem can only handle surfaces in a plane, but Stokes' theorem can handle surfaces in a plane or in space.

The complete proof of Stokes' theorem is beyond the scope of this text. We look at an intuitive explanation for the truth of the theorem and then see proof of the theorem in the special case that surface S is a portion of a graph of a function, and S, the boundary of S, and $\vec{\mathbf{F}}$ are all fairly tame.

Proof

First, we look at an informal proof of the theorem. This proof is not rigorous, but it is meant to give a general feeling for why the theorem is true. Let S be a surface and let D be a small piece of the surface so that D does not share any points with the boundary of S. We choose D to be small enough so that it can be approximated by an oriented square E. Let D inherit its orientation from S, and give E the same orientation. This square has four sides; denote them E_l, E_r, E_u, and E_d for the left, right, up, and down sides, respectively. On the square, we can use the flux form of Green's theorem:

$$\int_{E_l+E_d+E_r+E_u} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_E curl\,\vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, d\vec{\mathbf{S}} = \iint_E curl\,\vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}.$$

To approximate the flux over the entire surface, we add the values of the flux on the small squares approximating small pieces of the surface (Figure 15.7.2).

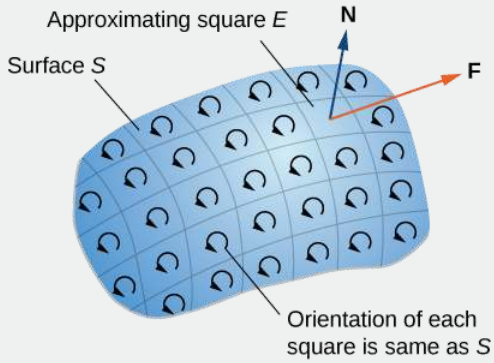


Figure 15.7.2: Chop the surface into small pieces. The pieces should be small enough that they can be approximated by a square.

By Green's theorem, the flux across each approximating square is a line integral over its boundary. Let F be an approximating square with an orientation inherited from S and with a right side E_l (so F is to the left of E). Let F_r denote the right side of F; then, $E_l = -F_r$. In other words, the right side of F is the same curve as the left side of E, just oriented in the opposite direction. Therefore,

$$\int_{E_l} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = -\int_{F_r} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}.$$

As we add up all the fluxes over all the squares approximating surface S, line integrals

$$\int_{E_l} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$$

and

$$\int_{F_r} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$$

cancel each other out. The same goes for the line integrals over the other three sides of E. These three line integrals cancel out with the line integral of the lower side of the square above E, the line integral over the left side of the square to the right of E, and the line integral over the upper side of the square below E (Figure 15.7.3). After all this cancelation occurs over all the approximating squares, the only line integrals that survive are the line integrals over sides approximating the boundary of S. Therefore, the sum of all the fluxes (which, by Green's theorem, is the sum of all the line integrals around the boundaries of approximating squares) can be approximated by a line integral over the boundary of S. In the limit, as the areas of the approximating squares go to zero, this approximation gets arbitrarily close to the flux.

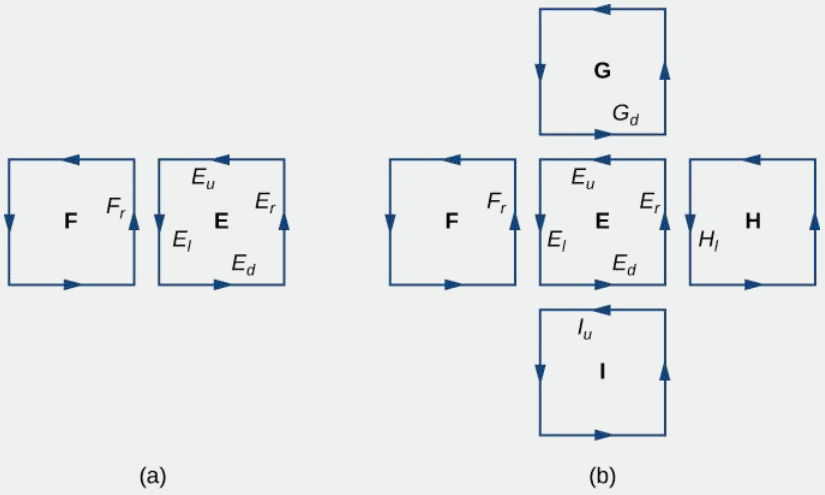


Figure 15.7.3: (a) The line integral along E_l cancels out the line integral along F_r because $E_l = -F_r$. (b) The line integral along any of the sides of E cancels out with the line integral along a side of an adjacent approximating square.

Let's now look at a rigorous proof of the theorem in the special case that S is the graph of function $z = f(x, y)$, where x and y vary over a bounded, simply connected region D of finite area (Figure 15.7.4). Furthermore, assume that f has continuous second-order partial derivatives. Let C denote the boundary of S and let C' denote the boundary of D. Then, D is the "shadow" of S in the plane and C' is the "shadow" of C. Suppose that S is oriented upward. The counterclockwise orientation of C is positive, as is the counterclockwise orientation of C'. Let $\vec{\mathbf{F}}(x, y, z) = \langle P, Q, R \rangle$ be a vector field with component functions that have continuous partial derivatives.

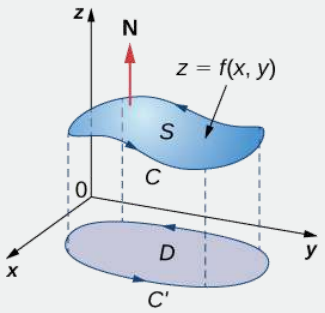


Figure 15.7.4: D is the "shadow," or projection, of S in the plane and C' is the projection of C.

We take the standard parameterization of $S: x = x, y = y, z = g(x, y)$. The tangent vectors are $\vec{\mathbf{t}}_x = \langle 1, 0, g_x \rangle$ and $\vec{\mathbf{t}}_y = \langle 0, 1, g_y \rangle$, and therefore $\vec{\mathbf{t}}_x \times \vec{\mathbf{t}}_y = \langle -g_x, -g_y, 1 \rangle$.

$$\iint_S curl\, \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_D [-(R_y - Q_z)z_x - (P_z - R_x)z_y + (Q_x - P_y)]\, dA,$$

where the partial derivatives are all evaluated at $(x, y, g(x, y))$, making the integrand depend on x and y only. Suppose $\langle x(t), y(t) \rangle$, $a \le t \le b$ is a parameterization of C'. Then, a parameterization of C is $\langle x(t), y(t), g(x(t), y(t)) \rangle$, $a \le t \le b$. Armed with

these parameterizations, the Chain rule, and Green's theorem, and keeping in mind that P, Q and R are all functions of x and y, we can evaluate line integral

$$\begin{aligned}\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_a^b (Px'(t) + Qy'(t) + Rz'(t))\, dt \\ &= \int_a^b \left[Px'(t) + Qy'(t) + R\left(\frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}\right)\right] dt \\ &= \int_a^b \left[\left(P + R\frac{\partial z}{\partial x}\right)x'(t) + \left(Q + R\frac{\partial z}{\partial y}\right)y'(t)\right] dt \\ &= \int_{C'} \left(P + R\frac{\partial z}{\partial x}\right) dx + \left(Q + R\frac{\partial z}{\partial y}\right) dy \\ &= \iint_D \left[\frac{\partial}{\partial x}\left(Q + R\frac{\partial z}{\partial y}\right) - \frac{\partial}{\partial y}\left(P + R\frac{\partial z}{\partial x}\right)\right] dA \\ &= \iint_D \left(\frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial R}{\partial x}\frac{\partial z}{\partial y} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} + R\frac{\partial^2 z}{\partial x \partial y}\right) \\ &\quad - \left(\frac{\partial P}{\partial y} + \frac{\partial P}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial y}\frac{\partial z}{\partial x} + R\frac{\partial^2 z}{\partial y \partial x}\right)\end{aligned}$$

By Clairaut's theorem,

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$$

Therefore, four of the terms disappear from this double integral, and we are left with

$$\iint_D [-(R_y - Q_z)Z_x - (P_z - R_x)z_y + (Q_x - P_y)]\, dA,$$

which equals

$$\iint_S curl\, \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}.$$

□

We have shown that Stokes' theorem is true in the case of a function with a domain that is a simply connected region of finite area. We can quickly confirm this theorem for another important case: when vector field $\vec{\mathbf{F}}$ is a conservative field. If $\vec{\mathbf{F}}$ is conservative, the curl of $\vec{\mathbf{F}}$ is zero, so

$$\iint_S curl\, \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = 0.$$

Since the boundary of S is a closed curve, the integral

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}.$$

is also zero.

✔ Example 15.7.1: Verifying Stokes' Theorem for a Specific Case

Verify that Stokes' theorem is true for vector field $\vec{\mathbf{F}}(x, y) = \langle -z, x, 0\rangle$ and surface S, where S is the hemisphere, oriented outward, with parameterization $\vec{\mathbf{r}}(\phi, \theta) = \langle \sin\phi\, \cos\theta,\, \sin\phi\, \sin\theta,\, \cos\phi\rangle$, $0 \le \theta \le \pi$, $0 \le \phi \le \pi$ as shown in Figure 15.7.5.

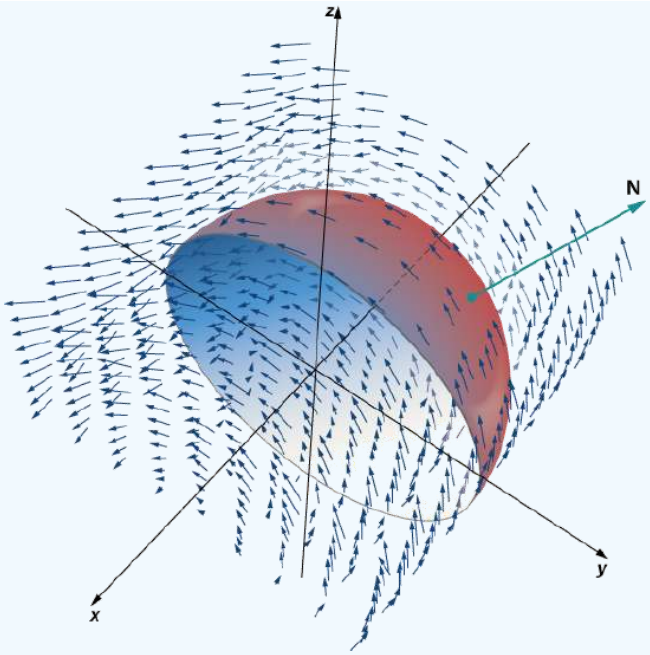


Figure 15.7.5: Verifying Stokes' theorem for a hemisphere in a vector field.

Solution

Let C be the boundary of S. Note that C is a circle of radius 1, centered at the origin, sitting in plane $y = 0$. This circle has parameterization $\langle \cos t, 0, \sin t \rangle$, $0 \le t \le 2\pi$. the equation for scalar surface integrals

$$\begin{aligned} \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_0^{2\pi} \langle -\sin t, \cos t, 0 \rangle \cdot \langle -\sin t, 0, \cos t \rangle \, dt \\ &= \int_0^{2\pi} \sin^2 t \, dt \\ &= \pi. \end{aligned}$$

By the equation for vector line integrals,

$$\begin{aligned} \iint_S curl\, \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} &= \iint_D curl\, \vec{\mathbf{F}}(\vec{\mathbf{r}}(\phi, \theta)) \cdot (\vec{\mathbf{t}}_\phi \times \vec{\mathbf{t}}_\theta) \, dA \\ &= \iint_D \langle 0, -1, 1 \rangle \cdot \langle \cos\theta \sin^2\phi, \sin\theta \sin^2\phi, \sin\phi \cos\phi \rangle \, dA \\ &= \int_0^{\pi} \int_0^{\pi} (\sin\phi \cos\phi - \sin\theta \sin^2\phi) \, d\phi d\theta \\ &= \frac{\pi}{2} \int_0^{\pi} \sin\theta \, d\theta \\ &= \pi. \end{aligned}$$

Therefore, we have verified Stokes' theorem for this example.

? Exercise 15.7.1

Verify that Stokes' theorem is true for vector field $\vec{\mathbf{F}}(x, y, z) = \langle y, x, -z \rangle$ and surface S, where S is the upwardly oriented portion of the graph of $f(x, y) = x^2 y$ over a triangle in the xy-plane with vertices $(0, 0)$, $(2, 0)$, and $(0, 2)$.

Applying Stokes' Theorem

Stokes' theorem translates between the flux integral of surface S to a line integral around the boundary of S. Therefore, the theorem allows us to compute surface integrals or line integrals that would ordinarily be quite difficult by translating the line integral into a surface integral or vice versa. We now study some examples of each kind of translation.

Example 15.7.2: Calculating a Surface Integral

Calculate surface integral

$$\iint_S curl\,\vec{\mathbf{F}} \cdot d\vec{\mathbf{S}},$$

where S is the surface, oriented outward, in Figure 15.7.6 and $\vec{\mathbf{F}} = \langle z,\ 2xy,\ x+y \rangle$.

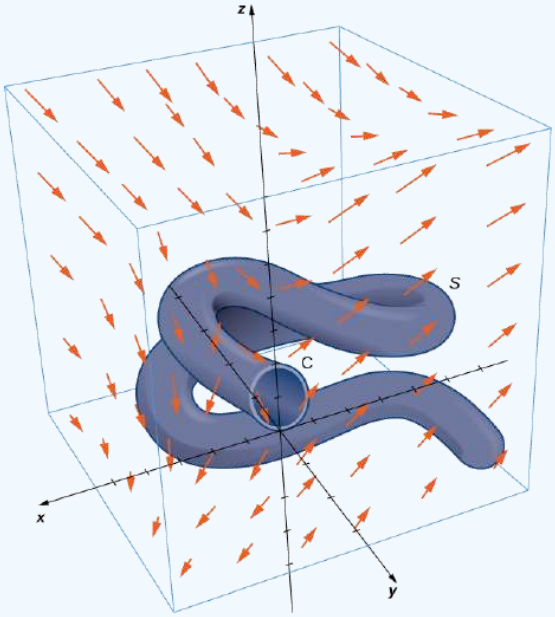


Figure 15.7.6: A complicated surface in a vector field.

Solution

Note that to calculate

$$\iint_S curl\,\vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$$

without using Stokes' theorem, we would need the equation for scalar surface integrals. Use of this equation requires a parameterization of S. Surface S is complicated enough that it would be extremely difficult to find a parameterization. Therefore, the methods we have learned in previous sections are not useful for this problem. Instead, we use Stokes' theorem, noting that the boundary C of the surface is merely a single circle with radius 1.

The curl of $\vec{\mathbf{F}}$ is $\langle 1, 1, 2y \rangle$. By Stokes' theorem,

$$\iint_S curl\,\vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}},$$

where C has parameterization $\langle \cos t,\ \sin t,\ 1 \rangle, 0 \le t \le 2\pi$. By the equation for vector line integrals,

$$\begin{aligned}\iint_S curl\, F \cdot d\vec{\mathbf{S}} &= \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \\ &= \int_0^2 \langle 1,\, \sin t \cos t,\, \cos t + \sin t \rangle \cdot \langle -\sin t,\, \cos t,\, 0 \rangle \, dt \\ &= \int_0^{2\pi} (-\sin t + 2 \sin t \cos^2 t)\, dt \\ &= \left[\cos t - \frac{2\cos^3 t}{3} \right]_0^{2\pi} \\ &= \cos(2\pi) - \frac{2\cos^3(2\pi)}{3} - \left(\cos(0) - \frac{2\cos^3(0)}{3} \right) \\ &= 0.\end{aligned}$$

An amazing consequence of Stokes' theorem is that if S' is any other smooth surface with boundary C and the same orientation as S, then

$$\iint_S curl\, \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 0$$

because Stokes' theorem says the surface integral depends on the line integral around the boundary only.

In Example **15.7.2**, we calculated a surface integral simply by using information about the boundary of the surface. In general, let S_1 and S_2 be smooth surfaces with the same boundary C and the same orientation. By Stokes' theorem,

$$\iint_{S_1} curl\, \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_{S_2} curl\, \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}. \tag{15.7.2}$$

Therefore, if

$$\iint_{S_1} curl\, \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$$

is difficult to calculate but

$$\iint_{S_2} curl\, \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$$

is easy to calculate, Stokes' theorem allows us to calculate the easier surface integral. In Example **15.7.2**, we could have calculated

$$\iint_S curl\, \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$$

by calculating

$$\iint_{S'} curl\, \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}},$$

where $\vec{\mathbf{S}}'$ is the disk enclosed by boundary curve C (a much more simple surface with which to work).

Equation 15.7.2 shows that flux integrals of curl vector fields are surface independent in the same way that line integrals of gradient fields are path independent. Recall that if $\vec{\mathbf{F}}$ is a two-dimensional conservative vector field defined on a simply connected domain, f is a *potential function* for $\vec{\mathbf{F}}$, and C is a curve in the domain of $\vec{\mathbf{F}}$, then

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$$

depends only on the endpoints of C. Therefore if C' is any other curve with the same starting point and endpoint as C (that is, C' has the same orientation as C), then

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{C'} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$$

In other words, the value of the integral depends on the boundary of the path only; it does not really depend on the path itself.

Analogously, suppose that S and S' are surfaces with the same boundary and same orientation, and suppose that $\vec{\mathbf{G}}$ is a three-dimensional vector field that can be written as the curl of another vector field $\vec{\mathbf{F}}$ (so that $\vec{\mathbf{F}}$ is like a "potential field" of $\vec{\mathbf{G}}$). By Equation 15.7.2,

$$\iint_S \vec{\mathbf{G}} \cdot d\vec{\mathbf{S}} = \iint_S curl\, \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_{S'} curl\, \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_{S'} \vec{\mathbf{G}} \cdot d\vec{\mathbf{S}}.$$

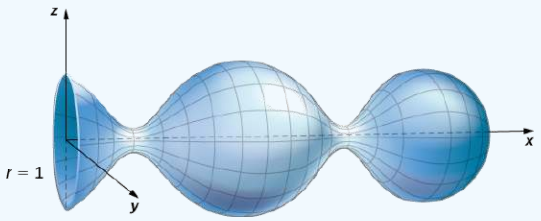
Therefore, the flux integral of $\vec{\mathbf{G}}$ does not depend on the surface, only on the boundary of the surface. Flux integrals of vector fields that can be written as the curl of a vector field are surface independent in the same way that line integrals of vector fields that can be written as the gradient of a scalar function are path independent.

? Exercise 15.7.2

Use Stokes' theorem to calculate surface integral

$$\iint_S curl\, \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}},$$

where $\vec{\mathbf{F}}(x, y, z) = \langle z, x, y \rangle$ and S is the surface as shown in the following figure. The boundary curve, C is oriented clockwise when looking along the positive y-axis.



✓ Example 15.7.3: Calculating a Line Integral

Calculate the line integral

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}},$$

where $\vec{\mathbf{F}} = \langle xy,\ x^2 + y^2 + z^2,\ yz \rangle$ and C is the boundary of the parallelogram with vertices $(0, 0, 1)$, $(0, 1, 0)$, $(2, 0, -1)$, and $(2, 1, -2)$.

Solution

To calculate the line integral directly, we need to parameterize each side of the parallelogram separately, calculate four separate line integrals, and add the result. This is not overly complicated, but it is time-consuming.

By contrast, let's calculate the line integral using Stokes' theorem. Let S denote the surface of the parallelogram. Note that S is the portion of the graph of $z = 1 - x - y$ for (x, y) varying over the rectangular region with vertices $(0, 0)$, $(0, 1)$, $(2, 0)$, and $(2, 1)$ in the xy-plane. Therefore, a parameterization of S is $\langle x, y,\ 1 - x - y \rangle$, $0 \le x \le 2$, $0 \le y \le 1$. The curl of $\vec{\mathbf{F}}$ is $\langle -z,\ 0,\ x \rangle$,and Stokes' theorem and the equation for scalar surface integrals

$$\begin{aligned}\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \iint_S curl\, \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} \\ &= \int_0^2 \int_0^1 curl\, \vec{\mathbf{F}}(x,y) \cdot (\vec{\mathbf{t}}_x \times \vec{\mathbf{t}}_y)\, dy\, dx \\ &= \int_0^2 \int_0^1 \langle -(1-x-y),\, 0,\, x\rangle \cdot (\langle 1,\, 0, -1\rangle \times \langle 0,\, 1,\, -1\rangle)\, dy\, dx \\ &= \int_0^2 \int_0^1 \langle x+y-1,\, 0,\, x\rangle \cdot \langle 1,1,1\rangle\, dy\, dx \\ &= \int_0^2 \int_0^1 2x+y-1\, dy\, dx \\ &= 3.\end{aligned}$$

? Exercise 15.7.3

Use Stokes' theorem to calculate line integral

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}},$$

where $\vec{\mathbf{F}} = \langle z, x, y\rangle$ and C is the boundary of a triangle with vertices $(0,0,1)$, $(3,0,-2)$, and $(0,1,2)$.

Interpretation of Curl

In addition to translating between line integrals and flux integrals, Stokes' theorem can be used to justify the physical interpretation of curl that we have learned. Here we investigate the relationship between curl and circulation, and we use Stokes' theorem to state Faraday's law—an important law in electricity and magnetism that relates the curl of an electric field to the rate of change of a magnetic field.

Recall that if C is a closed curve and $\vec{\mathbf{F}}$ is a vector field defined on C, then the circulation of $\vec{\mathbf{F}}$ around C is line integral

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}.$$

If $\vec{\mathbf{F}}$ represents the velocity field of a fluid in space, then the circulation measures the tendency of the fluid to move in the direction of C.

Let $\vec{\mathbf{F}}$ be a continuous vector field and let D_r be a small disk of radius r with center P_0 (Figure 15.7.7). If D_r is small enough, then $(curl\, \vec{\mathbf{F}})(P) \approx (curl\, \vec{\mathbf{F}})(P_0)$ for all points P in D_r because the curl is continuous. Let C_r be the boundary circle of D_r: By Stokes' theorem,

$$\int_{C_r} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_{D_r} curl\, \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, d\vec{\mathbf{S}} \approx \iint_{D_r} (curl\, \vec{\mathbf{F}})(P_0) \cdot \vec{\mathbf{N}}(P_0)\, d\vec{\mathbf{S}}.$$

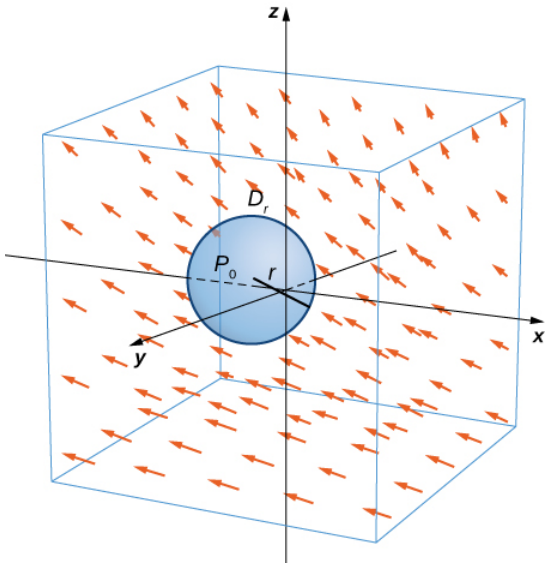


Figure 15.7.7: Disk D_r is a small disk in a continuous vector field.

The quantity $(curl\,\vec{\mathbf{F}})(P_0)\cdot\vec{\mathbf{N}}(P_0)$ is constant, and therefore

$$\iint_{D_r}(curl\,\vec{\mathbf{F}})(P_0)\cdot\vec{\mathbf{N}}(P_0)\,d\vec{\mathbf{S}}=\pi r^2[(curl\,\vec{\mathbf{F}})(P_0)\cdot\vec{\mathbf{N}}(P_0)].$$

Thus

$$\int_{C_r}\vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}\approx\pi r^2[(curl\,\vec{\mathbf{F}})(P_0)\cdot\vec{\mathbf{N}}(P_0)],$$

and the approximation gets arbitrarily close as the radius shrinks to zero. Therefore Stokes' theorem implies that

$$(curl\,\vec{\mathbf{F}})(P_0)\cdot\vec{\mathbf{N}}(P_0)=\lim_{r\to0^+}\frac{1}{\pi r^2}\int_{C_r}\vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}.$$

This equation relates the curl of a vector field to the circulation. Since the area of the disk is πr^2, this equation says we can view the curl (in the limit) as the circulation per unit area. Recall that if $\vec{\mathbf{F}}$ is the velocity field of a fluid, then circulation

$$\oint_{C_r}\vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}=\oint_{C_r}\vec{\mathbf{F}}\cdot\vec{\mathbf{T}}\,ds$$

is a measure of the tendency of the fluid to move around C_r: The reason for this is that $\vec{\mathbf{F}}\cdot\vec{\mathbf{T}}$ is a component of $\vec{\mathbf{F}}$ in the direction of $\vec{\mathbf{T}}$, and the closer the direction of $\vec{\mathbf{F}}$ is to $\vec{\mathbf{T}}$, the larger the value of $\vec{\mathbf{F}}\cdot\vec{\mathbf{T}}$ (remember that if $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are vectors and $\vec{\mathbf{b}}$ is fixed, then the dot product $\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}$ is maximal when $\vec{\mathbf{a}}$ points in the same direction as $\vec{\mathbf{b}}$). Therefore, if $\vec{\mathbf{F}}$ is the velocity field of a fluid, then $curl\,\vec{\mathbf{F}}\cdot\vec{\mathbf{N}}$ is a measure of how the fluid rotates about axis $\vec{\mathbf{N}}$. The effect of the curl is largest about the axis that points in the direction of $\vec{\mathbf{N}}$, because in this case $curl\,\vec{\mathbf{F}}\cdot\vec{\mathbf{N}}$ is as large as possible.

To see this effect in a more concrete fashion, imagine placing a tiny paddlewheel at point P_0 (Figure 15.7.8). The paddlewheel achieves its maximum speed when the axis of the wheel points in the direction of curl $\vec{\mathbf{F}}$. This justifies the interpretation of the curl we have learned: curl is a measure of the rotation in the vector field about the axis that points in the direction of the normal vector $\vec{\mathbf{N}}$, and Stokes' theorem justifies this interpretation.

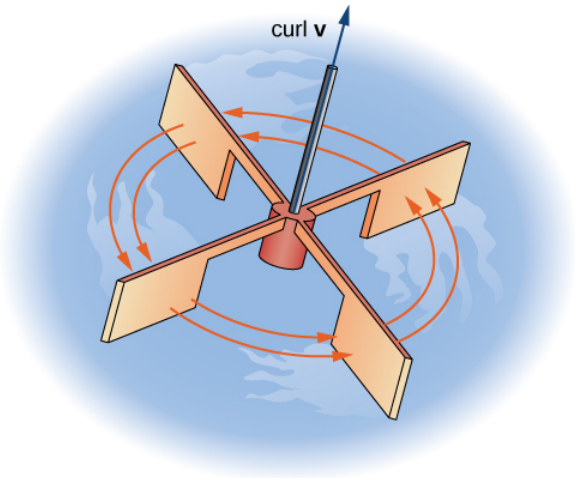


Figure **15.7.8**: To visualize curl at a point, imagine placing a tiny paddlewheel at that point in the vector field.

Now that we have learned about Stokes' theorem, we can discuss applications in the area of electromagnetism. In particular, we examine how we can use Stokes' theorem to translate between two equivalent forms of Faraday's law. Before stating the two forms of Faraday's law, we need some background terminology.

Let C be a closed curve that models a thin wire. In the context of electric fields, the wire may be moving over time, so we write $C(t)$ to represent the wire. At a given time t, curve $C(t)$ may be different from original curve C because of the movement of the wire, but we assume that $C(t)$ is a closed curve for all times t. Let $D(t)$ be a surface with $C(t)$ as its boundary, and orient $C(t)$ so that $D(t)$ has positive orientation. Suppose that $C(t)$is in a magnetic field $\vec{\mathbf{B}}(t)$ that can also change over time. In other words, $\vec{\mathbf{B}}$ has the form

$$\vec{\mathbf{B}}(x,y,z) = \langle P(x,y,z),\, Q(x,y,z),\, R(x,y,z)\rangle,$$

where P, Q, and R can all vary continuously over time. We can produce current along the wire by changing field $\vec{\mathbf{B}}(t)$ (this is a consequence of Ampere's law). Flux $\phi(t) = \iint_{D(t)} \vec{\mathbf{B}}(t)\cdot d\vec{\mathbf{S}}$ creates electric field $\vec{\mathbf{E}}(t)$ that does work. The integral form of Faraday's law states that

$$Work = \int_{C(t)} \vec{\mathbf{E}}(t)\cdot d\vec{\mathbf{r}} = -\frac{\partial \phi}{\partial t}.$$

In other words, the work done by $\vec{\mathbf{E}}$ is the line integral around the boundary, which is also equal to the rate of change of the flux with respect to time. The differential form of Faraday's law states that

$$curl\, \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}.$$

Using Stokes' theorem, we can show that the differential form of Faraday's law is a consequence of the integral form. By Stokes' theorem, we can convert the line integral in the integral form into surface integral

$$-\frac{\partial \phi}{\partial t} = \int_{C(t)} \vec{\mathbf{E}}(t)\cdot d\vec{\mathbf{r}} = \iint_{D(t)} curl\, \vec{\mathbf{E}}(t)\cdot d\vec{\mathbf{S}}.$$

Since

$$\phi(t) = \iint_{D(t)} B(t)\cdot d\vec{\mathbf{S}},$$

then as long as the integration of the surface does not vary with time we also have

$$-\frac{\partial \phi}{\partial t} = \iint_{D(t)} -\frac{\partial \vec{\mathbf{B}}}{\partial t}\cdot d\vec{\mathbf{S}}.$$

Therefore,

$$\iint_{D(t)} -\frac{\partial \vec{\mathbf{B}}}{\partial t} \cdot d\vec{\mathbf{S}} = \iint_{D(t)} curl\, \vec{\mathbf{E}} \cdot d\vec{\mathbf{S}}.$$

To derive the differential form of Faraday's law, we would like to conclude that $curl\, \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}$: In general, the equation

$$\iint_{D(t)} -\frac{\partial \vec{\mathbf{B}}}{\partial t} \cdot d\vec{\mathbf{S}} = \iint_{D(t)} curl\, \vec{\mathbf{E}} \cdot d\vec{\mathbf{S}}$$

is not enough to conclude that $curl\, \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}$: The integral symbols do not simply "cancel out," leaving equality of the integrands. To see why the integral symbol does not just cancel out in general, consider the two single-variable integrals $\int_0^1 x\, dx$ and $\int_0^1 f(x)\, dx$, where

$$f(x) = \begin{cases} 1, & \text{if } 0 \le x \le 1/2 \\ 0, & \text{if } 1/2 \le x \le 1. \end{cases}$$

Both of these integrals equal $\frac{1}{2}$, so $\int_0^1 x\, dx = \int_0^1 f(x)\, dx$.

However, $x \neq f(x)$. Analogously, with our equation

$$\iint_{D(t)} -\frac{\partial \vec{\mathbf{B}}}{\partial t} \cdot d\vec{\mathbf{S}} = \iint_{D(t)} curl\, \vec{\mathbf{E}} \cdot d\vec{\mathbf{S}},$$

we cannot simply conclude that $curl\, \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}$ just because their integrals are equal. However, in our context, equation

$$\iint_{D(t)} -\frac{\partial \vec{\mathbf{B}}}{\partial t} \cdot d\vec{\mathbf{S}} = \iint_{D(t)} curl\, \vec{\mathbf{E}} \cdot d\vec{\mathbf{S}}$$

is true for *any* region, however small (this is in contrast to the single-variable integrals just discussed). If $\vec{\mathbf{F}}$ and $\vec{\mathbf{G}}$ are three-dimensional vector fields such that

$$\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_S \vec{\mathbf{G}} \cdot d\vec{\mathbf{S}}$$

for any surface S, then it is possible to show that $\vec{\mathbf{F}} = \vec{\mathbf{G}}$ by shrinking the area of S to zero by taking a limit (the smaller the area of S, the closer the value of $\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$ to the value of $\vec{\mathbf{F}}$ at a point inside S). Therefore, we can let area $D(t)$ shrink to zero by taking a limit and obtain the differential form of Faraday's law:

$$curl\, \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}.$$

In the context of electric fields, the curl of the electric field can be interpreted as the negative of the rate of change of the corresponding magnetic field with respect to time.

Example 15.7.4: Using Faraday's Law

Calculate the curl of electric field $\vec{\mathbf{E}}$ if the corresponding magnetic field is constant field $\vec{\mathbf{B}}(t) = \langle 1, -4, 2 \rangle$.

Solution

Since the magnetic field does not change with respect to time, $-\frac{\partial \vec{\mathbf{B}}}{\partial t} = \vec{\mathbf{0}}$. By Faraday's law, the curl of the electric field is therefore also zero.

Analysis

A consequence of Faraday's law is that the curl of the electric field corresponding to a constant magnetic field is always zero.

? Exercise 15.7.4

Calculate the curl of electric field $\vec{\mathbf{E}}$ if the corresponding magnetic field is $\vec{\mathbf{B}}(t) = \langle tx, ty, -2tz \rangle$, $0 \leq t < \infty$.

Key Concepts

- Stokes' theorem relates a flux integral over a surface to a line integral around the boundary of the surface. Stokes' theorem is a higher dimensional version of Green's theorem, and therefore is another version of the Fundamental Theorem of Calculus in higher dimensions.
- Stokes' theorem can be used to transform a difficult surface integral into an easier line integral, or a difficult line integral into an easier surface integral.
- Through Stokes' theorem, line integrals can be evaluated using the simplest surface with boundary C.
- Faraday's law relates the curl of an electric field to the rate of change of the corresponding magnetic field. Stokes' theorem can be used to derive Faraday's law.

Key Equations

- **Stokes' theorem**

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_S curl\, \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$$

Glossary

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15.7E: Exercises for Section 15.7

In exercises 1 - 6, without using Stokes' theorem, calculate directly both the flux of $curl\,\vec{\mathbf{F}}\cdot\vec{\mathbf{N}}$ over the given surface and the circulation integral around its boundary, assuming all are oriented clockwise.

1. $\vec{\mathbf{F}}(x,y,z)=y^2\,\hat{\mathbf{i}}+z^2\,\hat{\mathbf{j}}+x^2\,\hat{\mathbf{k}}$; S is the first-octant portion of plane $x+y+z=1$.

2. $\vec{\mathbf{F}}(x,y,z)=z\,\hat{\mathbf{i}}+x\,\hat{\mathbf{j}}+y\,\hat{\mathbf{k}}$; S is hemisphere $z=(a^2-x^2-y^2)^{1/2}$.

Answer

$$\iint_S (curl\,\vec{\mathbf{F}}\cdot\vec{\mathbf{N}})\,dS=\pi a^2$$

3. $\vec{\mathbf{F}}(x,y,z)=y^2\,\hat{\mathbf{i}}+2x\,\hat{\mathbf{j}}+5\,\hat{\mathbf{k}}$; S is hemisphere $z=(4-x^2-y^2)^{1/2}$.

4. $\vec{\mathbf{F}}(x,y,z)=z\,\hat{\mathbf{i}}+2x\,\hat{\mathbf{j}}+3y\,\hat{\mathbf{k}}$; S is upper hemisphere $z=\sqrt{9-x^2-y^2}$.

Answer

$$\iint_S (curl\,\vec{\mathbf{F}}\cdot\vec{\mathbf{N}})\,dS=18\pi$$

5. $\vec{\mathbf{F}}(x,y,z)=(x+2z)\,\hat{\mathbf{i}}+(y-x)\,\hat{\mathbf{j}}+(z-y)\,\hat{\mathbf{k}}$; S is a triangular region with vertices $(3,0,0)$, $(0,3/2,0)$, and $(0,0,3)$.

6. $\vec{\mathbf{F}}(x,y,z)=2y\,\hat{\mathbf{i}}+6z\,\hat{\mathbf{j}}+3x\,\hat{\mathbf{k}}$; S is a portion of paraboloid $z=4-x^2-y^2$ and is above the xy-plane.

Answer

$$\iint_S (curl\,\vec{\mathbf{F}}\cdot\vec{\mathbf{N}})\,dS=-8\pi$$

In exercises 7 - 9, use Stokes' theorem to evaluate $\iint_S (curl\,\vec{\mathbf{F}}\cdot\vec{\mathbf{N}})\,dS$ for the vector fields and surface.

7. $\vec{\mathbf{F}}(x,y,z)=xy\,\hat{\mathbf{i}}-z\,\hat{\mathbf{j}}$ and S is the surface of the cube $0\le x\le 1$, $0\le y\le 1$, $0\le z\le 1$, except for the face where $z=0$ and using the outward unit normal vector.

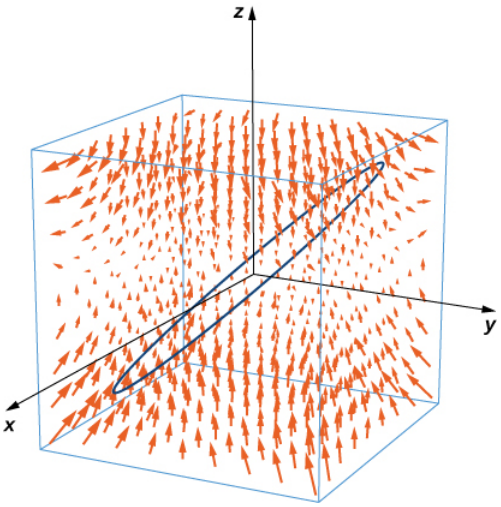
8. $\vec{\mathbf{F}}(x,y,z)=xy\,\hat{\mathbf{i}}+x^2\,\hat{\mathbf{j}}+z^2\,\hat{\mathbf{k}}$; and C is the intersection of paraboloid $z=x^2+y^2$ and plane $z=y$, and using the outward normal vector.

Answer

$$\iint_S (curl\,\vec{\mathbf{F}}\cdot\vec{\mathbf{N}})\,dS=0$$

9. $\vec{\mathbf{F}}(x,y,z)=4y\,\hat{\mathbf{i}}+z\,\hat{\mathbf{j}}+2y\,\hat{\mathbf{k}}$; and C is the intersection of sphere $x^2+y^2+z^2=4$ with plane $z=0$, and using the outward normal vector.

10. Use Stokes' theorem to evaluate $\int_C \left[2xy^2z\,dx+2x^2yz\,dy+(x^2y^2-2z)\,dz\right]$, where C is the curve given by $x=\cos t$, $y=\sin t$, $0\le t\le 2\pi$, traversed in the direction of increasing t.



Answer

$$\int_C \vec{\mathbf{F}} \cdot d\vec{S} = 0$$

11. [T] Use a computer algebraic system (CAS) and Stokes' theorem to approximate line integral $\int_C (y\,dx + z\,dy + x\,dz)$, where C is the intersection of plane $x + y = 2$ and surface $x^2 + y^2 + z^2 = 2(x + y)$, traversed counterclockwise viewed from the origin.

12. [T] Use a CAS and Stokes' theorem to approximate line integral $\int_C (3y\,dx + 2z\,dy - 5x\,dz)$, where C is the intersection of the xy-plane and hemisphere $z = \sqrt{1 - x^2 - y^2}$, traversed counterclockwise viewed from the top—that is, from the positive z-axis toward the xy-plane.

Answer

$$\int_C \vec{\mathbf{F}} \cdot d\vec{S} = -9.4248$$

13. [T] Use a CAS and Stokes' theorem to approximate line integral $\int_C [(1 + y)\,z\,dx + (1 + z)x\,dy + (1 + x)y\,dz]$, where C is a triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ oriented counterclockwise.

14. Use Stokes' theorem to evaluate $\iint_S curl\ \vec{\mathbf{F}} \cdot d\vec{S}$, where $\vec{\mathbf{F}}(x, y, z) = e^{xy} cos\ z\ \hat{\mathbf{i}} + x^2 z\ \hat{\mathbf{j}} + xy\ \hat{\mathbf{k}}$, and S is half of sphere $x = \sqrt{1 - y^2 - z^2}$, oriented out toward the positive x-axis.

Answer

$$\iint_S \vec{\mathbf{F}} \cdot d\vec{S} = 0$$

15. [T] Use a CAS and Stokes' theorem to evaluate $\iint_S (curl\ \vec{\mathbf{F}} \cdot \vec{\mathbf{N}})\,dS$, where $\vec{\mathbf{F}}(x, y, z) = x^2 y\,\hat{\mathbf{i}} + xy^2\,\hat{\mathbf{j}} + z^3\,\hat{\mathbf{k}}$ and C is the curve of the intersection of plane $3x + 2y + z = 6$ and cylinder $x^2 + y^2 = 4$, oriented clockwise when viewed from above.

16. [T] Use a CAS and Stokes' theorem to evaluate $\iint_S curl\ \vec{\mathbf{F}} \cdot d\vec{S}$, where $\vec{\mathbf{F}}(x, y, z) = \left(\sin(y + z) - yx^2 - \frac{y^3}{3}\right)\hat{\mathbf{i}} + x\ \cos(y + z)\,\hat{\mathbf{j}} + \cos(2y)\,\hat{\mathbf{k}}$ and S consists of the top and the four sides but not the bottom of the cube with vertices $(\pm 1, \pm 1, \pm 1)$, oriented outward.

Answer

$$\iint_S curl\ \vec{\mathbf{F}} \cdot d\vec{S} = 2.6667$$

17. [T] Use a CAS and Stokes' theorem to evaluate $\iint_S curl\ \vec{\mathbf{F}} \cdot d\vec{S}$, where $\vec{\mathbf{F}}(x, y, z) = z^2\,\hat{\mathbf{i}} + 3xy\,\hat{\mathbf{j}} + x^3 y^3\,\hat{\mathbf{k}}$ and S is the top part of $z = 5 - x^2 - y^2$ above plane $z = 1$ and S is oriented upward.

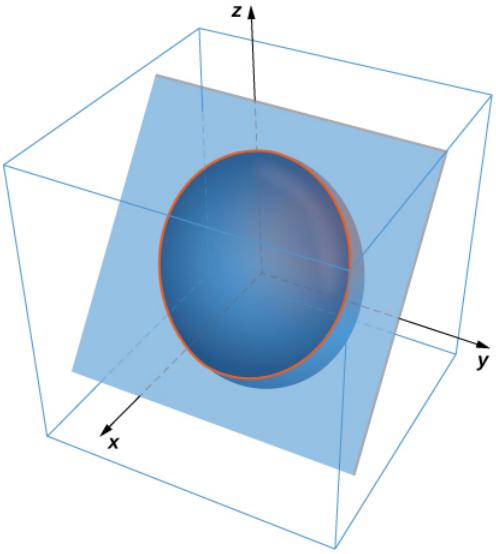
18. Use Stokes' theorem to evaluate $\iint_S (curl\, \vec{\mathbf{F}} \cdot \vec{\mathbf{N}})dS$, where $\vec{\mathbf{F}}(x,y,z) = z^2\,\hat{\mathbf{i}} + y^2\,\hat{\mathbf{j}} + x\,\hat{\mathbf{k}}$ and S is a triangle with vertices $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ with counterclockwise orientation.

Answer

$$\iint_S (curl\, \vec{\mathbf{F}} \cdot \vec{\mathbf{N}})dS = -\frac{1}{6}$$

19. Use Stokes' theorem to evaluate line integral $\int_C (z\,dx + x\,dy + y\,dz)$, where C is a triangle with vertices $(3,0,0)$, $(0,0,2)$, and $(0,6,0)$ traversed in the given order.

20. Use Stokes' theorem to evaluate $\int_C \left(\frac{1}{2}y^2\,dx + z\,dy + x\,dz\right)$, where C is the curve of intersection of plane $x + z = 1$ and ellipsoid $x^2 + 2y^2 + z^2 = 1$, oriented clockwise from the origin.



Answer

$$\int_C \left(\frac{1}{2}y^2\,dx + z\,dy + x\,dz\right) = -\frac{\pi}{4}$$

21. Use Stokes' theorem to evaluate $\iint_S (curl\, \vec{\mathbf{F}} \cdot \vec{\mathbf{N}})dS$, where $\vec{\mathbf{F}}(x,y,z) = x\,\hat{\mathbf{i}} + y^2\,\hat{\mathbf{j}} + ze^{xy}\,\hat{\mathbf{k}}$ and S is the part of surface $z = 1 - x^2 - 2y^2$ with $z \geq 0$, oriented counterclockwise.

22. Use Stokes' theorem for vector field $\vec{\mathbf{F}}(x,y,z) = z\,\hat{\mathbf{i}} + 3x\,\hat{\mathbf{j}} + 2z\hat{\mathbf{k}}$ where S is surface $z = 1 - x^2 - 2y^2$, $z \geq 0$, C is boundary circle $x^2 + y^2 = 1$, and S is oriented in the positive z-direction.

Answer

$$\iint_S (curl\, \vec{\mathbf{F}} \cdot \vec{\mathbf{N}})dS = -3\pi$$

23. Use Stokes' theorem for vector field $\vec{\mathbf{F}}(x,y,z) = -\frac{3}{2}y^2\,\hat{\mathbf{i}} - 2xy\,\hat{\mathbf{j}} + yz\hat{\mathbf{k}}$, where S is that part of the surface of plane $x + y + z = 1$ contained within triangle C with vertices $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, traversed counterclockwise as viewed from above.

24. A certain closed path C in plane $2x + 2y + z = 1$ is known to project onto unit circle $x^2 + y^2 = 1$ in the xy-plane. Let C be a constant and let $\vec{\mathbf{R}}(x,y,z) = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$. Use Stokes' theorem to evaluate $\int_C (c\,\hat{\mathbf{k}} \times \vec{\mathbf{R}}) \cdot dS$.

Answer

$$\int_C (c\,\hat{\mathbf{k}} \times \vec{\mathbf{R}}) \cdot dS = 2\pi c$$

25. Use Stokes' theorem and let C be the boundary of surface $z = x^2 + y^2$ with $0 \le x \le 2$ and $0 \le y \le 1$ oriented with upward facing normal. Define $\vec{\mathbf{F}}(x, y, z) = \left(\sin(x^3) + xz\right)\hat{\mathbf{i}} + (x - yz)\hat{\mathbf{j}} + \cos(z^4)\hat{\mathbf{k}}$ and evaluate $\int_C \vec{\mathbf{F}} \cdot d\vec{S}$.

26. Let S be hemisphere $x^2 + y^2 + z^2 = 4$ with $z \ge 0$, oriented upward. Let $\vec{\mathbf{F}}(x, y, z) = x^2 e^{yz}\hat{\mathbf{i}} + y^2 e^{xz}\hat{\mathbf{j}} + z^2 e^{xy}\hat{\mathbf{k}}$ be a vector field. Use Stokes' theorem to evaluate $\iint_S curl\, \vec{\mathbf{F}} \cdot d\vec{S}$.

Answer

$\iint_S curl\, \vec{\mathbf{F}} \cdot d\vec{S} = 0$

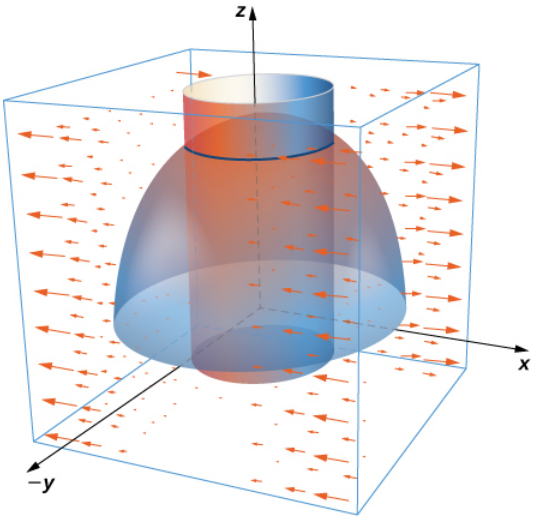
27. Let $\vec{\mathbf{F}}(x, y, z) = xy\hat{\mathbf{i}} + (e^{z^2} + y)\hat{\mathbf{j}} + (x + y)\hat{\mathbf{k}}$ and let S be the graph of function $y = \frac{x^2}{9} + \frac{z^2}{9} - 1$ with $z \le 0$ oriented so that the normal vector S has a positive y component. Use Stokes' theorem to compute integral $\iint_S curl\, \vec{\mathbf{F}} \cdot d\vec{S}$.

28. Use Stokes' theorem to evaluate $\oint \vec{\mathbf{F}} \cdot d\vec{S}$, where $\vec{\mathbf{F}}(x, y, z) = y\hat{\mathbf{i}} + z\hat{\mathbf{j}} + x\hat{\mathbf{k}}$ and C is a triangle with vertices $(0, 0, 0)$, $(2, 0, 0)$ and $0, -2, 2)$ oriented counterclockwise when viewed from above.

Answer

$\oint \vec{\mathbf{F}} \cdot d\vec{S} = -4$

29. Use the surface integral in Stokes' theorem to calculate the circulation of field $\vec{\mathbf{F}}$, $\vec{\mathbf{F}}(x, y, z) = x^2 y^3\hat{\mathbf{i}} + \hat{\mathbf{j}} + z\hat{\mathbf{k}}$ around C, which is the intersection of cylinder $x^2 + y^2 = 4$ and hemisphere $x^2 + y^2 + z^2 = 16$, $z \ge 0$, oriented counterclockwise when viewed from above.



30. Use Stokes' theorem to compute $\iint_S curl\, \vec{\mathbf{F}} \cdot d\vec{S}$. where $\vec{\mathbf{F}}(x, y, z) = \hat{\mathbf{i}} + xy^2\hat{\mathbf{j}} + xy^2\hat{\mathbf{k}}$ and S is a part of plane $y + z = 2$ inside cylinder $x^2 + y^2 = 1$ and oriented counterclockwise.

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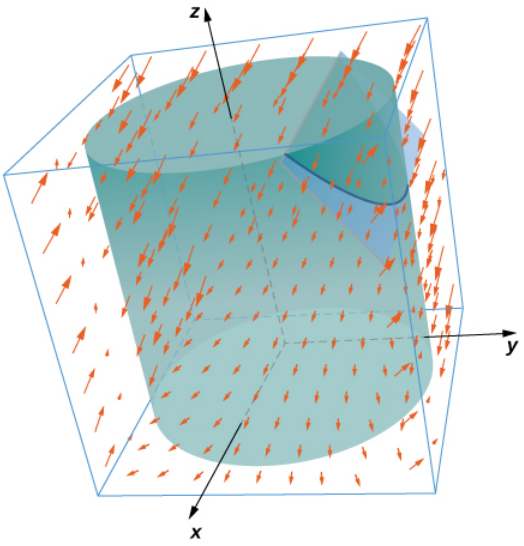


Answer

$$\iint_S curl\, \vec{\mathbf{F}} \cdot dS = 0$$

31. Use Stokes' theorem to evaluate $\iint_S curl\, \vec{\mathbf{F}} \cdot dS$, where $\vec{\mathbf{F}}(x, y, z) = -y^2\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}} + z^2\,\hat{\mathbf{k}}$ and S is the part of plane $x + y + z = 1$ in the positive octant and oriented counterclockwise $x \geq 0,\ y \geq 0,\ z \geq 0$.

32. Let $\vec{\mathbf{F}}(x, y, z) = xy\,\hat{\mathbf{i}} + 2z\,\hat{\mathbf{j}} - 2y\,\hat{\mathbf{k}}$ and let C be the intersection of plane $x + z = 5$ and cylinder $x^2 + y^2 = 9$, which is oriented counterclockwise when viewed from the top. Compute the line integral of $\vec{\mathbf{F}}$ over C using Stokes' theorem.

Answer

$$\iint_S curl\, \vec{\mathbf{F}} \cdot dS = -36\pi$$

33. [T] Use a CAS and let $\vec{\mathbf{F}}(x, y, z) = xy^2\,\hat{\mathbf{i}} + (yz - x)\,\hat{\mathbf{j}} + e^{yxz}\,\hat{\mathbf{k}}$. Use Stokes' theorem to compute the surface integral of curl $\vec{\mathbf{F}}$ over surface S with inward orientation consisting of cube $[0, 1] \times [0, 1] \times [0, 1]$ with the right side missing.

34. Let S be ellipsoid $\frac{x^2}{4} + \frac{y^2}{9} + z^2 = 1$ oriented counterclockwise and let $\vec{\mathbf{F}}$ be a vector field with component functions that have continuous partial derivatives.

Answer

$$\iint_S curl\, \vec{\mathbf{F}} \cdot \vec{\mathbf{N}} = 0$$

35. Let S be the part of paraboloid $z = 9 - x^2 - y^2$ with $z \geq 0$. Verify Stokes' theorem for vector field $\vec{\mathbf{F}}(x, y, z) = 3z\,\hat{\mathbf{i}} + 4x\,\hat{\mathbf{j}} + 2y\,\hat{\mathbf{k}}$.

36. [T] Use a CAS and Stokes' theorem to evaluate $\oint \vec{\mathbf{F}} \cdot dS$, if $\vec{\mathbf{F}}(x, y, z) = (3z - \sin x)\,\hat{\mathbf{i}} + (x^2 + e^y)\,\hat{\mathbf{j}} + (y^3 - \cos z)\,\hat{\mathbf{k}}$, where C is the curve given by $x = \cos t,\ y = \sin t,\ z = 1;\ 0 \leq t \leq 2\pi$.

Answer

$$\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 0$$

37. [T] Use a CAS and Stokes' theorem to evaluate $\vec{\mathbf{F}}(x, y, z) = 2y\,\hat{\mathbf{i}} + e^z\,\hat{\mathbf{j}} - \arctan x\,\hat{\mathbf{k}}$ with S as a portion of paraboloid $z = 4 - x^2 - y^2$ cut off by the xy-plane oriented counterclockwise.

38. [T] Use a CAS to evaluate $\iint_S curl(F) \cdot dS$, where $\vec{\mathbf{F}}(x, y, z) = 2z\,\hat{\mathbf{i}} + 3x\,\hat{\mathbf{j}} + 5y\,\hat{\mathbf{k}}$ and S is the surface parametrically by $\vec{\mathbf{r}}(r, \theta) = r\cos\theta\,\hat{\mathbf{i}} + r\sin\theta\,\hat{\mathbf{j}} + (4 - r^2)\,\hat{\mathbf{k}}\ (0 \leq \theta \leq 2\pi,\ 0 \leq r \leq 3)$.

Answer

$$\iint_S curl(F) \cdot dS = 84.8230$$

39. Let S be paraboloid $z = a(1 - x^2 - y^2)$, for $z \geq 0$, where $a > 0$ is a real number. Let $\vec{\mathbf{F}}(x, y, z) = \langle x - y,\ y + z,\ z - x \rangle$. For what value(s) of a (if any) does $\iint_S (\vec{\nabla} \times \vec{\mathbf{F}}) \cdot \vec{\mathbf{n}}\, dS$ have its maximum value?

For application exercises 40 - 41, the goal is to evaluate $A = \iint_S (\vec{\nabla} \times \vec{\mathbf{F}}) \cdot \vec{\mathbf{n}}\, dS$, where $\vec{\mathbf{F}} = \langle xz, -xz, xy \rangle$ and S is the upper half of ellipsoid $x^2 + y^2 + 8z^2 = 1$, where $z \geq 0$.

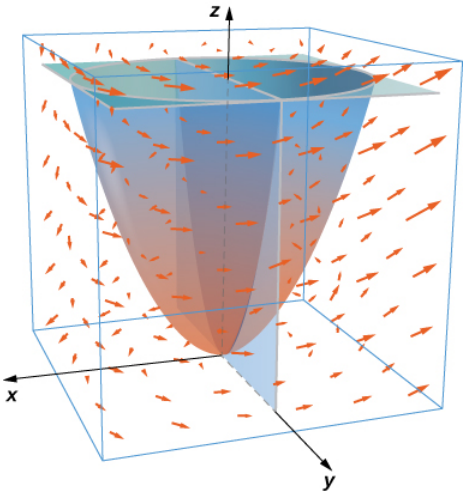
40. Evaluate a surface integral over a more convenient surface to find the value of A.

Answer

$$A = \iint_S (\vec{\nabla} \times \vec{\mathbf{F}}) \cdot \vec{\mathbf{n}}\, dS = 0$$

41. Evaluate A using a line integral.

42. Take paraboloid $z = x^2 + y^2$, for $0 \leq z \leq 4$, and slice it with plane $y = 0$. Let S be the surface that remains for $y \geq 0$, including the planar surface in the xz-plane. Let C be the semicircle and line segment that bounded the cap of S in plane $z = 4$ with counterclockwise orientation. Let $\vec{\mathbf{F}} = \langle 2z + y,\ 2x + z,\ 2y + x \rangle$. Evaluate $\iint_S (\vec{\nabla} \times \vec{\mathbf{F}}) \cdot \vec{\mathbf{n}}\, dS$.



Answer

$$\iint_S (\vec{\nabla} \times \vec{\mathbf{F}}) \cdot \vec{\mathbf{n}}\, dS = 2\pi$$

For exercises 43 - 45, let S be the disk enclosed by curve C : $\vec{\mathbf{r}}(t) = \langle \cos\varphi \cos t,\ \sin t,\ \sin\varphi \cos t \rangle$, for $0 \leq t \leq 2\pi$, where $0 \leq \varphi \leq \frac{\pi}{2}$ is a fixed angle.

43. What is the length of C in terms of φ?

44. What is the circulation of C of vector field $\vec{\mathbf{F}} = \langle -y, -z, x \rangle$ as a function of φ?

Answer

$$C = \pi(\cos\varphi - \sin\varphi)$$

45. For what value of φ is the circulation a maximum?

46. Circle C in plane $x + y + z = 8$ has radius 4 and center $(2, 3, 3)$. Evaluate $\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ for $\vec{\mathbf{F}} = \langle 0, -z, 2y \rangle$, where C has a counterclockwise orientation when viewed from above.

Answer

$$\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 48\pi$$

47. Velocity field $v = \langle 0, 1 - x^2, 0 \rangle$, for $|x| \leq 1$ and $|z| \leq 1$, represents a horizontal flow in the y-direction. Compute the curl of $\vec{\mathbf{v}}$ in a clockwise rotation.

48. Evaluate integral $\iint_S (\vec{\nabla} \times \vec{\mathbf{F}}) \cdot \vec{\mathbf{n}}\, dS$, where $\vec{\mathbf{F}} = -xz\hat{\mathbf{i}} + yz\hat{\mathbf{j}} + xye^z\hat{\mathbf{k}}$ and S is the cap of paraboloid $z = 5 - x^2 - y^2$ above plane $z = 3$, and $\vec{\mathbf{n}}$ points in the positive z-direction on S.

Answer

$$\iint_S (\vec{\nabla} \times \vec{\mathbf{F}}) \cdot \vec{\mathbf{n}} = 0$$

In exercises 49 - 50, use Stokes' theorem to find the circulation of the following vector fields around any smooth, simple closed curve C.

49. $\vec{\mathbf{F}} = \vec{\nabla}(x \sin ye^z)$

50. $\vec{\mathbf{F}} = \langle y^2z^3,\ z2xyz^3, 3xy^2z^2 \rangle$

Answer

0

Contributors

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15.8: The Divergence Theorem

Learning Objectives

- Explain the meaning of the divergence theorem.
- Use the divergence theorem to calculate the flux of a vector field.
- Apply the divergence theorem to an electrostatic field.

We have examined several versions of the Fundamental Theorem of Calculus in higher dimensions that relate the integral around an oriented boundary of a domain to a "derivative" of that entity on the oriented domain. In this section, we state the divergence theorem, which is the final theorem of this type that we will study. The divergence theorem has many uses in physics; in particular, the divergence theorem is used in the field of partial differential equations to derive equations modeling heat flow and conservation of mass. We use the theorem to calculate flux integrals and apply it to electrostatic fields.

Overview of Theorems

Before examining the divergence theorem, it is helpful to begin with an overview of the versions of the Fundamental Theorem of Calculus we have discussed:

1. The Fundamental Theorem of Calculus:

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

   This theorem relates the integral of derivative f' over line segment $[a, b]$ along the x-axis to a difference of f evaluated on the boundary.
2. The Fundamental Theorem for Line Integrals:

$$\int_C \vec{\nabla} f \cdot d\vec{\mathbf{r}} = f(P_1) - f(P_0),$$

   where P_0 is the initial point of C and P_1 is the terminal point of C. The Fundamental Theorem for Line Integrals allows path C to be a path in a plane or in space, not just a line segment on the x-axis. If we think of the gradient as a derivative, then this theorem relates an integral of derivative $\vec{\nabla} f$ over path C to a difference of f evaluated on the boundary of C.
3. Green's theorem, circulation form:

$$\iint_D (Q_x - P_y)\,dA = \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}.$$

   Since $Q_x - P_y = \text{curl } \vec{\mathbf{F}} \cdot \hat{\mathbf{k}}$ and curl is a derivative of sorts, Green's theorem relates the integral of derivative curl $\vec{\mathbf{F}}$ over planar region D to an integral of $\vec{\mathbf{F}}$ over the boundary of D.
4. Green's theorem, flux form:

$$\iint_D (P_x + Q_y)\,dA = \int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\,dS.$$

   Since $P_x + Q_y = \text{div } \vec{\mathbf{F}}$ and divergence is a derivative of sorts, the flux form of Green's theorem relates the integral of derivative div $\vec{\mathbf{F}}$ over planar region D to an integral of $\vec{\mathbf{F}}$ over the boundary of D.
5. Stokes' theorem:

$$\iint_S curl\, \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}.$$

   If we think of the curl as a derivative of sorts, then Stokes' theorem relates the integral of derivative curl $\vec{\mathbf{F}}$ over surface S (not necessarily planar) to an integral of $\vec{\mathbf{F}}$ over the boundary of S.

Stating the Divergence Theorem

The divergence theorem follows the general pattern of these other theorems. If we think of divergence as a derivative of sorts, then the divergence theorem relates a triple integral of derivative div $\vec{\mathbf{F}}$ over a solid to a flux integral of $\vec{\mathbf{F}}$ over the boundary of the solid. More specifically, the divergence theorem relates a flux integral of vector field $\vec{\mathbf{F}}$ over a closed surface S to a triple integral of the divergence of $\vec{\mathbf{F}}$ over the solid enclosed by S.

The Divergence Theorem

Let S be a piecewise, smooth closed surface that encloses solid E in space. Assume that S is oriented outward, and let $\vec{\mathbf{F}}$ be a vector field with continuous partial derivatives on an open region containing E (Figure 15.8.1). Then

$$\iiint_E \operatorname{div} \vec{\mathbf{F}}\, dV = \iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}. \tag{15.8.1}$$

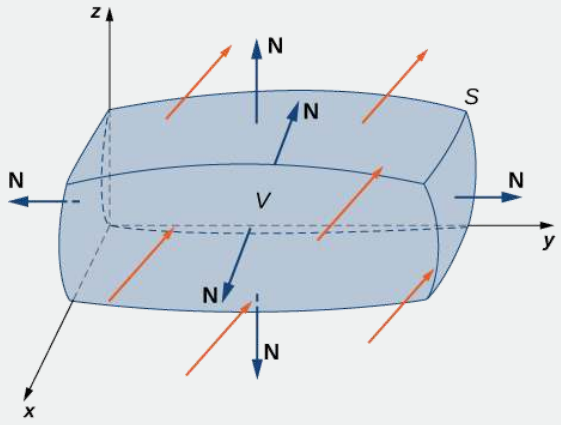


Figure 15.8.1: The divergence theorem relates a flux integral across a closed surface S to a triple integral over solid E enclosed by the surface.

Recall that the flux form of Green's theorem states that

$$\iint_D \operatorname{div} \vec{\mathbf{F}}\, dA = \int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, dS.$$

Therefore, the divergence theorem is a version of Green's theorem in one higher dimension.

The proof of the divergence theorem is beyond the scope of this text. However, we look at an informal proof that gives a general feel for why the theorem is true, but does not prove the theorem with full rigor. This explanation follows the informal explanation given for why Stokes' theorem is true.

Proof

Let B be a small box with sides parallel to the coordinate planes inside E (Figure 15.8.2a). Let the center of B have coordinates (x, y, z) and suppose the edge lengths are Δx, Δy, and Δz. (Figure 15.8.1b). The normal vector out of the top of the box is $\hat{\mathbf{k}}$ and the normal vector out of the bottom of the box is $-\hat{\mathbf{k}}$. The dot product of $\vec{\mathbf{F}} = \langle P, Q, R \rangle$ with $\hat{\mathbf{k}}$ is R and the dot product with $-\hat{\mathbf{k}}$ is $-R$. The area of the top of the box (and the bottom of the box) ΔS is $\Delta x \Delta y$.

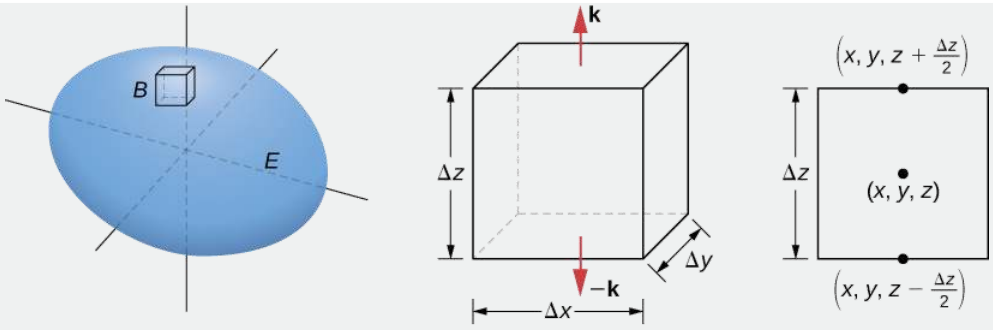


Figure 15.8.2: (a) A small box B inside surface E has sides parallel to the coordinate planes. (b) Box B has side lengths Δx, Δy, and Δz (c) If we look at the side view of B, we see that, since (x, y, z) is the center of the box, to get to the top of the box we must travel a vertical distance of $\frac{\Delta z}{2}$ up from (x, y, z). Similarly, to get to the bottom of the box we must travel a distance $\frac{\Delta z}{2}$ down from (x, y, z).

The flux out of the top of the box can be approximated by $R\left(x, y, z+\frac{\Delta z}{2}\right)\,\Delta x\,\Delta y$ (Figure 15.8.2c) and the flux out of the bottom of the box is $-R\left(x, y, z-\frac{\Delta z}{2}\right)\,\Delta x\,\Delta y$. If we denote the difference between these values as ΔR, then the net flux in the vertical direction can be approximated by $\Delta R\,\Delta x\,\Delta y$. However,

$$\Delta R\,\Delta x\,\Delta y = \left(\frac{\Delta R}{\Delta z}\right)\Delta x\,\Delta y\Delta z \approx \left(\frac{\partial R}{\partial z}\right)\Delta V.$$

Therefore, the net flux in the vertical direction can be approximated by $\left(\frac{\partial R}{\partial z}\right)\Delta V$. Similarly, the net flux in the x-direction can be approximated by $\left(\frac{\partial P}{\partial x}\right)\,\Delta V$ and the net flux in the y-direction can be approximated by $\left(\frac{\partial Q}{\partial y}\right)\,\Delta V$. Adding the fluxes in all three directions gives an approximation of the total flux out of the box:

$$\text{Total flux } \approx \left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)\Delta V = \text{div } \vec{\mathbf{F}}\,\Delta V.$$

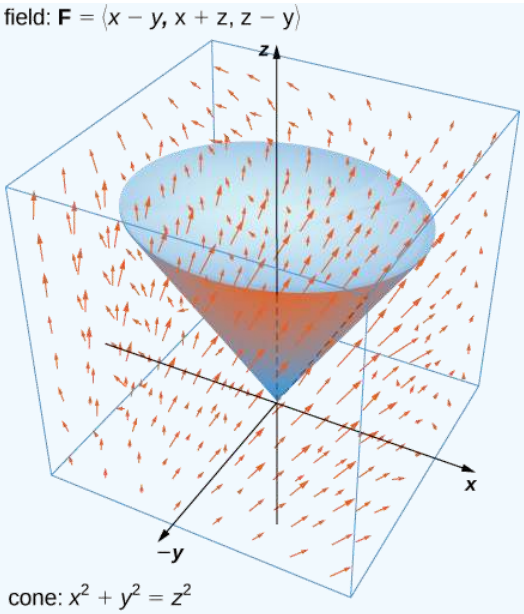
This approximation becomes arbitrarily close to the value of the total flux as the volume of the box shrinks to zero.

The sum of $\text{div } \vec{\mathbf{F}}\,\Delta V$ over all the small boxes approximating E is approximately $\iiint_E \text{div } \vec{\mathbf{F}}\,dV$. On the other hand, the sum of $\text{div } \vec{\mathbf{F}}\,\Delta V$ over all the small boxes approximating E is the sum of the fluxes over all these boxes. Just as in the informal proof of Stokes' theorem, adding these fluxes over all the boxes results in the cancelation of a lot of the terms. If an approximating box shares a face with another approximating box, then the flux over one face is the negative of the flux over the shared face of the adjacent box. These two integrals cancel out. When adding up all the fluxes, the only flux integrals that survive are the integrals over the faces approximating the boundary of E. As the volumes of the approximating boxes shrink to zero, this approximation becomes arbitrarily close to the flux over S.

□

Example 15.8.1: Verifying the Divergence Theorem

Verify the divergence theorem for vector field $\vec{\mathbf{F}} = \langle x-y,\, x+z,\, z-y\rangle$ and surface S that consists of cone $x^2+y^2=z^2$, $0 \le z \le 1$, and the circular top of the cone (see the following figure). Assume this surface is positively oriented.



Solution

Let E be the solid cone enclosed by S. To verify the theorem for this example, we show that

$$\iiint_E \operatorname{div} \vec{\mathbf{F}}\, dV = \iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$$

by calculating each integral separately.

To compute the triple integral, note that $\operatorname{div} \vec{\mathbf{F}} = P_x + Q_y + R_z = 2$, and therefore the triple integral is

$$\begin{aligned} \iiint_E \operatorname{div} \vec{\mathbf{F}}\, dV &= 2 \iiint_E dV \\ &= 2\,(volume\ of\ E). \end{aligned}$$

The volume of a right circular cone is given by $\pi r^2 \frac{h}{3}$. In this case, $h = r = 1$. Therefore,

$$\iiint_E \operatorname{div} \vec{\mathbf{F}}\, dV = 2\,(volume\ of\ E) = \frac{2\pi}{3}.$$

To compute the flux integral, first note that S is piecewise smooth; S can be written as a union of smooth surfaces. Therefore, we break the flux integral into two pieces: one flux integral across the circular top of the cone and one flux integral across the remaining portion of the cone. Call the circular top S_1 and the portion under the top S_2. We start by calculating the flux across the circular top of the cone. Notice that S_1 has parameterization

$$\vec{\mathbf{r}}(u, v) = \langle u \cos v,\ u \sin v,\ 1\rangle,\ 0 \le u \le 1,\ 0 \le v \le 2\pi.$$

Then, the tangent vectors are $\vec{\mathbf{t}}_u = \langle \cos v,\ \sin v,\ 0\rangle$ and $\vec{\mathbf{t}}_v = \langle -u \sin v,\ u \cos v, 0\rangle$. Therefore, the flux across S_1 is

$$\begin{aligned} \iint_{S_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} &= \int_0^1 \int_0^{2\pi} \vec{\mathbf{F}}(\vec{\mathbf{r}}(u, v)) \cdot (\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v)\, dA \\ &= \int_0^1 \int_0^{2\pi} \langle u \cos v - u \sin v,\ u \cos v + 1,\ 1 - u \sin v\rangle \cdot \langle 0, 0, u\rangle\, dv\, du \\ &= \int_0^1 \int_0^{2\pi} u - u^2 \sin v\, dv du \\ &= \pi. \end{aligned}$$

We now calculate the flux over S_2. A parameterization of this surface is

$$\vec{\mathbf{r}}(u, v) = \langle u \cos v,\ u \sin v,\ u\rangle,\ 0 \le u \le 1,\ 0 \le v \le 2\pi.$$

The tangent vectors are $\vec{\mathbf{t}}_u = \langle \cos v,\ \sin v,\ 1\rangle$ and $\vec{\mathbf{t}}_v = \langle -u\sin v,\ u\cos v, 0\rangle$, so the cross product is

$$\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v = \langle -u\cos v,\ -u\sin v,\ u\rangle.$$

Notice that the negative signs on the x and y components induce the negative (or inward) orientation of the cone. Since the surface is positively oriented, we use vector $\vec{\mathbf{t}}_v \times \vec{\mathbf{t}}_u = \langle u\cos v,\ u\sin v,\ -u\rangle$ in the flux integral. The flux across S_2 is then

$$\begin{aligned}\iint_{S_2} \vec{\mathbf{F}}\cdot d\vec{\mathbf{S}} &= \int_0^1\int_0^{2\pi} \vec{\mathbf{F}}(\vec{\mathbf{r}}(u,v))\cdot(\vec{\mathbf{t}}_u\times\vec{\mathbf{t}}_v)\,dA \\ &= \int_0^1\int_0^{2\pi} \langle u\cos v - u\sin v,\ u\cos v + u,\ u - u\sin v\rangle \\ &\quad\cdot\langle u\cos v,\ u\sin v,\ -u\rangle\,dv\,du \\ &= \int_0^1\int_0^{2\pi} u^2\cos^2 v + 2u^2\sin v - u^2\,dv\,du \\ &= -\frac{\pi}{3}\end{aligned}$$

The total flux across S is

$$\iint_S \vec{\mathbf{F}}\cdot d\vec{\mathbf{S}} = \iint_{S_1}\vec{\mathbf{F}}\cdot d\vec{\mathbf{S}} + \iint_{S_2}\vec{\mathbf{F}}\cdot d\vec{\mathbf{S}} = \frac{2\pi}{3} = \iiint_E \operatorname{div}\vec{\mathbf{F}}\,dV,$$

and we have verified the divergence theorem for this example.

? Exercise 15.8.1

Verify the divergence theorem for vector field $\vec{\mathbf{F}}(x,y,z) = \langle x+y+z,\ y,\ 2x-y\rangle$ and surface S given by the cylinder $x^2+y^2=1$, $0\le z\le 3$ plus the circular top and bottom of the cylinder. Assume that S is positively oriented.

Recall that the divergence of continuous field $\vec{\mathbf{F}}$ at point P is a measure of the "outflowing-ness" of the field at P. If $\vec{\mathbf{F}}$ represents the velocity field of a fluid, then the divergence can be thought of as the rate per unit volume of the fluid flowing out less the rate per unit volume flowing in. The divergence theorem confirms this interpretation. To see this, let P be a point and let B_r be a ball of small radius r centered at P (Figure 15.8.3). Let S_r be the boundary sphere of B_r. Since the radius is small and $\vec{\mathbf{F}}$ is continuous, $\operatorname{div}\vec{\mathbf{F}}(Q) \approx \operatorname{div}\vec{\mathbf{F}}(P)$ for all other points Q in the ball. Therefore, the flux across S_r can be approximated using the divergence theorem:

$$\iint_{S_r}\vec{\mathbf{F}}\cdot d\vec{\mathbf{S}} = \iiint_{B_r}\operatorname{div}\vec{\mathbf{F}}\,dV \approx \iiint_{B_r}\operatorname{div}\vec{\mathbf{F}}(P)\,dV.$$

Since $divF(P)$ is a constant,

$$\iiint_{B_r}\operatorname{div}\vec{\mathbf{F}}(P)\,dV = \operatorname{div}\vec{\mathbf{F}}(P)\,V(B_r).$$

Therefore, flux

$$\iint_{S_r}\vec{\mathbf{F}}\cdot d\vec{\mathbf{S}}$$

can be approximated by $\vec{\mathbf{F}}(P)\,V(B_r)$. This approximation gets better as the radius shrinks to zero, and therefore

$$\operatorname{div}\vec{\mathbf{F}}(P) = \lim_{r\to 0}\frac{1}{V(B_r)}\iint_{S_r}\vec{\mathbf{F}}\cdot d\vec{\mathbf{S}}.$$

This equation says that the divergence at P is the net rate of outward flux of the fluid per unit volume.

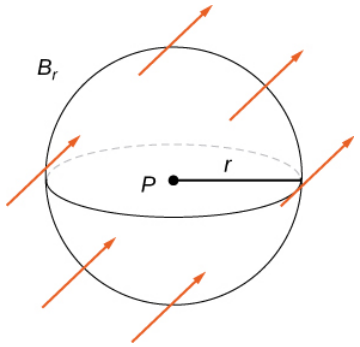


Figure 15.8.3: Ball B_r of small radius r centered at P.

Using the Divergence Theorem

The divergence theorem translates between the flux integral of closed surface S and a triple integral over the solid enclosed by S. Therefore, the theorem allows us to compute flux integrals or triple integrals that would ordinarily be difficult to compute by translating the flux integral into a triple integral and vice versa.

Example 15.8.2: Applying the Divergence Theorem

Calculate the surface integral

$$\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}},$$

where S is cylinder $x^2+y^2=1$, $0 \le z \le 2$, including the circular top and bottom, and $\vec{\mathbf{F}} = \left\langle \frac{x^3}{3}+yz,\ \frac{y^3}{3}-\sin(xz),\ z-x-y \right\rangle$.

Solution

We could calculate this integral without the divergence theorem, but the calculation is not straightforward because we would have to break the flux integral into three separate integrals: one for the top of the cylinder, one for the bottom, and one for the side. Furthermore, each integral would require parameterizing the corresponding surface, calculating tangent vectors and their cross product..

By contrast, the divergence theorem allows us to calculate the single triple integral

$$\iiint_E \operatorname{div} \vec{\mathbf{F}}\, dV,$$

where E is the solid enclosed by the cylinder. Using the divergence theorem (Equation 15.8.1) and converting to cylindrical coordinates, we have

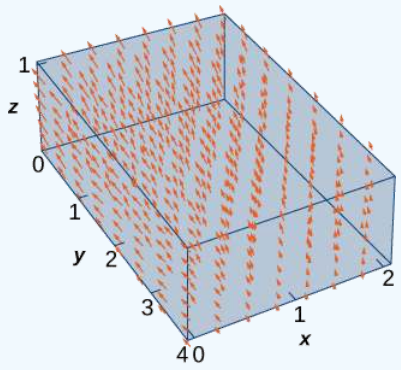
$$\begin{aligned}
\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} &= \iiint_E \operatorname{div} \vec{\mathbf{F}}\, dV, \\
&= \iiint_E (x^2+y^2+1)\, dV \\
&= \int_0^{2\pi}\int_0^1\int_0^2 (r^2+1)\, r\, dz\, dr\, d\theta \\
&= \frac{3}{2}\int_0^{2\pi} d\theta \\
&= 3\pi.
\end{aligned}$$

Exercise 15.8.2

Use the divergence theorem to calculate flux integral

$$\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}},$$

where S is the boundary of the box given by $0 \le x \le 2,\ 0 \le y \le 4,\ 0 \le z \le 1$ and $\vec{\mathbf{F}} = \langle x^2 + yz,\ y - z,\ 2x + 2y + 2z \rangle$ (see the following figure).



Example 15.8.3: Applying the Divergence Theorem

Let $\vec{\mathbf{v}} = \langle -\frac{y}{z},\ \frac{x}{z},\ 0 \rangle$ be the velocity field of a fluid. Let C be the solid cube given by $1 \le x \le 4,\ 2 \le y \le 5,\ 1 \le z \le 4$, and let S be the boundary of this cube (see the following figure). Find the flow rate of the fluid across S.

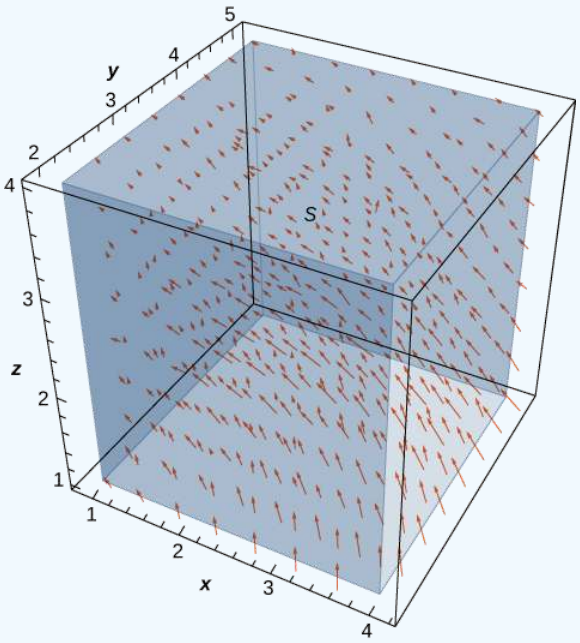


Figure 15.8.4: Vector field $\vec{\mathbf{v}} = \langle -\frac{y}{z},\ \frac{x}{z},\ 0 \rangle$.

Solution

The flow rate of the fluid across S is $\iint_S \vec{\mathbf{v}} \cdot d\vec{\mathbf{S}}$. Before calculating this flux integral, let's discuss what the value of the integral should be. Based on Figure 15.8.4, we see that if we place this cube in the fluid (as long as the cube doesn't encompass the origin), then the rate of fluid entering the cube is the same as the rate of fluid exiting the cube. The field is rotational in nature and, for a given circle parallel to the xy-plane that has a center on the z-axis, the vectors along that circle are all the same magnitude. That is how we can see that the flow rate is the same entering and exiting the cube. The flow into the cube cancels with the flow out of the cube, and therefore the flow rate of the fluid across the cube should be zero.

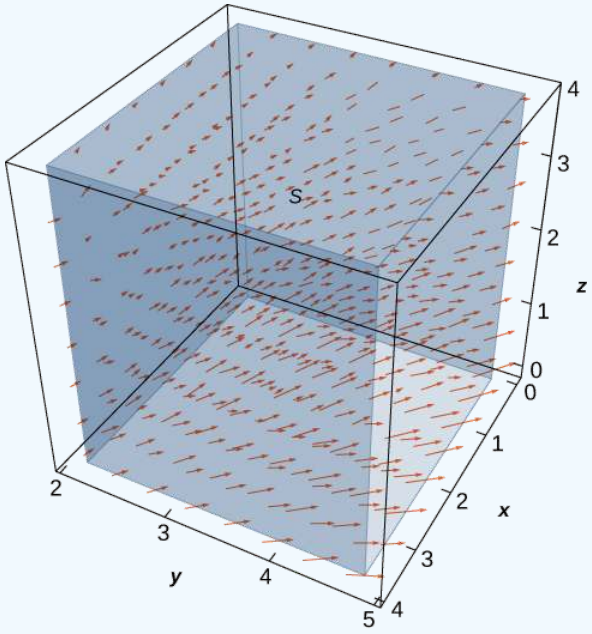
To verify this intuition, we need to calculate the flux integral. Calculating the flux integral directly requires breaking the flux integral into six separate flux integrals, one for each face of the cube. We also need to find tangent vectors, compute their cross product. However, using the divergence theorem makes this calculation go much more quickly:

$$\iint_S \vec{\mathbf{v}} \cdot d\vec{\mathbf{S}} = \iiint_C \operatorname{div} \vec{\mathbf{v}}\, dV$$
$$= \iiint_C 0\, dV = 0.$$

Therefore the flux is zero, as expected.

? Exercise 15.8.3

Let $\vec{\mathbf{v}} = \langle \frac{x}{z}, \frac{y}{z}, 0 \rangle$ be the velocity field of a fluid. Let C be the solid cube given by $1 \le x \le 4$, $2 \le y \le 5$, $1 \le z \le 4$, and let S be the boundary of this cube (see the following figure). Find the flow rate of the fluid across S.



Example illustrates a remarkable consequence of the divergence theorem. Let S be a piecewise, smooth closed surface and let $\vec{\mathbf{F}}$ be a vector field defined on an open region containing the surface enclosed by S. If $\vec{\mathbf{F}}$ has the form $F = \langle f(y,z), g(x,z), h(x,y) \rangle$, then the divergence of $\vec{\mathbf{F}}$ is zero. By the divergence theorem, the flux of $\vec{\mathbf{F}}$ across S is also zero. This makes certain flux integrals incredibly easy to calculate. For example, suppose we wanted to calculate the flux integral $\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$ where S is a cube and

$$\vec{\mathbf{F}} = \langle \sin(y)\, e^{yz}, x^2 z^2, \cos(xy)\, e^{\sin x} \rangle.$$

Calculating the flux integral directly would be difficult, if not impossible, using techniques we studied previously. At the very least, we would have to break the flux integral into six integrals, one for each face of the cube. But, because the divergence of this field is zero, the divergence theorem immediately shows that the flux integral is zero.

We can now use the divergence theorem to justify the physical interpretation of divergence that we discussed earlier. Recall that if $\vec{\mathbf{F}}$ is a continuous three-dimensional vector field and P is a point in the domain of $\vec{\mathbf{F}}$, then the divergence of $\vec{\mathbf{F}}$ at P is a measure of the "outflowing-ness" of $\vec{\mathbf{F}}$ at P. If $\vec{\mathbf{F}}$ represents the velocity field of a fluid, then the divergence of $\vec{\mathbf{F}}$ at P is a measure of the net flow rate out of point P (the flow of fluid out of P less the flow of fluid in to P). To see how the divergence theorem justifies this interpretation, let B_r be a ball of very small radius r with center P, and assume that B_r is in the domain of $\vec{\mathbf{F}}$. Furthermore, assume that B_r has a positive, outward orientation. Since the radius of B_r is small and $\vec{\mathbf{F}}$ is continuous, the divergence of $\vec{\mathbf{F}}$ is approximately constant on B_r. That is, ifv P' is any point in B_r, then $\operatorname{div} \vec{\mathbf{F}}(P) \approx \operatorname{div} \vec{\mathbf{F}}(P')$. Let S_r denote the boundary sphere of B_r. We can approximate the flux across S_r using the divergence theorem as follows:

$$\begin{aligned}\iint_{S_r} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} &= \iiint_{B_r} \operatorname{div} \vec{\mathbf{F}}\, dV \\ &\approx \iiint_{B_r} \operatorname{div} \vec{\mathbf{F}}(P)\, dV \\ &= \operatorname{div} \vec{\mathbf{F}}(P)\, V(B_r).\end{aligned}$$

As we shrink the radius r to zero via a limit, the quantity $\operatorname{div} \vec{\mathbf{F}}(P)\, V(B_r)$ gets arbitrarily close to the flux. Therefore,

$$\operatorname{div} \vec{\mathbf{F}}(P) = \lim_{r \to 0} \frac{1}{V(B_r)} \iint_{S_r} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$$

and we can consider the divergence at P as measuring the net rate of outward flux per unit volume at P. Since "outflowing-ness" is an informal term for the net rate of outward flux per unit volume, we have justified the physical interpretation of divergence we discussed earlier, and we have used the divergence theorem to give this justification.

Application to Electrostatic Fields

The divergence theorem has many applications in physics and engineering. It allows us to write many physical laws in both an integral form and a differential form (in much the same way that Stokes' theorem allowed us to translate between an integral and differential form of Faraday's law). Areas of study such as fluid dynamics, electromagnetism, and quantum mechanics have equations that describe the conservation of mass, momentum, or energy, and the divergence theorem allows us to give these equations in both integral and differential forms.

One of the most common applications of the divergence theorem is to **electrostatic fields**. An important result in this subject is **Gauss' law**. This law states that if S is a closed surface in electrostatic field $\vec{\mathbf{E}}$, then the flux of $\vec{\mathbf{E}}$ across S is the total charge enclosed by S (divided by an electric constant). We now use the divergence theorem to justify the special case of this law in which the electrostatic field is generated by a stationary point charge at the origin.

If (x, y, z) is a point in space, then the distance from the point to the origin is $r = \sqrt{x^2 + y^2 + z^2}$. Let $\vec{\mathbf{F}}_r$ denote radial vector field $\vec{\mathbf{F}}_r = \frac{1}{r^2} \left\langle \frac{x}{r}, \frac{y}{r}, \frac{z}{r} \right\rangle$.The vector at a given position in space points in the direction of unit radial vector $\left\langle \frac{x}{r}, \frac{y}{r}, \frac{z}{r} \right\rangle$ and is scaled by the quantity $1/r^2$. Therefore, the magnitude of a vector at a given point is inversely proportional to the square of the vector's distance from the origin. Suppose we have a stationary charge of q Coulombs at the origin, existing in a vacuum. The charge generates electrostatic field $\vec{\mathbf{E}}$ given by

$$\vec{\mathbf{E}} = \frac{q}{4\pi\epsilon_0} \vec{\mathbf{F}}_r,$$

where the approximation $\epsilon_0 = 8.854 \times 10^{-12}$ farad (F)/m is an electric constant. (The constant ϵ_0 is a measure of the resistance encountered when forming an electric field in a vacuum.) Notice that $\vec{\mathbf{E}}$ is a radial vector field similar to the gravitational field described in [link]. The difference is that this field points outward whereas the gravitational field points inward. Because

$$\vec{\mathbf{E}} = \frac{q}{4\pi\epsilon_0} \vec{\mathbf{F}}_r = \frac{q}{4\pi\epsilon_0} \left(\frac{1}{r^2} \left\langle \frac{x}{r}, \frac{y}{r}, \frac{z}{r} \right\rangle \right),$$

we say that electrostatic fields obey an inverse-square law. That is, the electrostatic force at a given point is inversely proportional to the square of the distance from the source of the charge (which in this case is at the origin). Given this vector field, we show that the flux across closed surface S is zero if the charge is outside of S, and that the flux is $q/epsilon_0$ if the charge is inside of S. In other words, the flux across S is the charge inside the surface divided by constant ϵ_0. This is a special case of Gauss' law, and here we use the divergence theorem to justify this special case.

To show that the flux across S is the charge inside the surface divided by constant ϵ_0, we need two intermediate steps. First we show that the divergence of $\vec{\mathbf{F}}_r$ is zero and then we show that the flux of $\vec{\mathbf{F}}_r$ across any smooth surface S is either zero or 4π. We can then justify this special case of Gauss' law.

Example 15.8.4: The Divergence of F_τ is Zero

Verify that the divergence of $\vec{\mathbf{F}}_\tau$ is zero where $\vec{\mathbf{F}}_\tau$ is defined (away from the origin).

Solution

Since $\tau = \sqrt{x^2+y^2+z^2}$, the quotient rule gives us

$$\begin{aligned}\frac{\partial}{\partial x}\left(\frac{x}{\tau^3}\right) &= \frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right)\\ &= \frac{(x^2+y^2+z^2)^{3/2} - x\left[\frac{3}{2}(x^2+y^2+z^2)^{1/2}2x\right]}{(x^2+y^2+z^2)^3}\\ &= \frac{\tau^3 - 3x^2\tau}{\tau^6} = \frac{\tau^2-3x^2}{\tau^5}.\end{aligned}$$

Similarly,

$$\frac{\partial}{\partial y}\left(\frac{y}{\tau^3}\right) = \frac{\tau^2-3y^2}{\tau^5} \text{ and } \frac{\partial}{\partial z}\left(\frac{z}{\tau^3}\right) = \frac{\tau^2-3z^2}{\tau^5}.$$

Therefore,

$$\begin{aligned}\text{div}\,\vec{\mathbf{F}}_\tau &= \frac{\tau^2-3x^2}{\tau^5} + \frac{\tau^2-3y^2}{\tau^5} + \frac{\tau^2-3z^2}{\tau^5}\\ &= \frac{3\tau^2 - 3(x^2+y^2+z^2)}{\tau^5}\\ &= \frac{3\tau^2-3\tau^2}{\tau^5} = 0.\end{aligned}$$

Notice that since the divergence of $\vec{\mathbf{F}}_\tau$ is zero and $\vec{\mathbf{E}}$ is $\vec{\mathbf{F}}_\tau$ scaled by a constant, the divergence of electrostatic field $\vec{\mathbf{E}}$ is also zero (except at the origin).

Flux across a Smooth Surface

Let S be a connected, piecewise smooth closed surface and let $\vec{\mathbf{F}}_\tau = \frac{1}{\tau^2}\left\langle \frac{x}{\tau}, \frac{y}{\tau}, \frac{z}{\tau}\right\rangle$. Then,

$$\iint_S \vec{\mathbf{F}}_\tau \cdot d\vec{\mathbf{S}} = \begin{cases} 0, & \text{if } S \text{ does not encompass the origin} \\ 4\pi, & \text{if } S \text{ encompasses the origin.}\end{cases}$$

In other words, this theorem says that the flux of $\vec{\mathbf{F}}_\tau$ across any piecewise smooth closed surface S depends only on whether the origin is inside of S.

Proof

The logic of this proof follows the logic of [link], only we use the divergence theorem rather than Green's theorem.

First, suppose that S does not encompass the origin. In this case, the solid enclosed by S is in the domain of $\vec{\mathbf{F}}_\tau$, and since the divergence of $\vec{\mathbf{F}}_\tau$ is zero, we can immediately apply the divergence theorem and find that

$$\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$$

is zero.

Now suppose that S does encompass the origin. We cannot just use the divergence theorem to calculate the flux, because the field is not defined at the origin. Let S_a be a sphere of radius a inside of S centered at the origin. The outward normal vector field on the sphere, in spherical coordinates, is

$$\vec{\mathbf{t}}_\phi \times \vec{\mathbf{t}}_\theta = \langle a^2\cos\theta\sin^2\phi,\ a^2\sin\theta\sin^2\phi,\ a^2\sin\phi\cos\phi\rangle$$

(see [link]). Therefore, on the surface of the sphere, the dot product $\vec{\mathbf{F}}_\tau \cdot \vec{\mathbf{N}}$ (in spherical coordinates) is

$$\begin{aligned}\vec{\mathbf{F}}_\tau \cdot \vec{\mathbf{N}} &= \left\langle \frac{\sin\phi\cos\theta}{a^2}, \frac{\sin\phi\sin\theta}{a^2}, \frac{\cos\phi}{a^2} \right\rangle \\ &\quad \cdot \langle a^2\cos\theta\sin^2\phi, a^2\sin\theta\sin^2\phi, a^2\sin\phi\cos\phi\rangle \\ &= \sin\phi(\langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi\rangle \cdot \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi\rangle) \\ &= \sin\phi.\end{aligned}$$

The flux of $\vec{\mathbf{F}}_\tau$ across S_a is

$$\iint_{S_a} \vec{\mathbf{F}}_\tau \cdot \vec{\mathbf{N}} dS = \int_0^{2\pi}\int_0^{\pi} \sin\phi\, d\phi\, d\theta = 4\pi.$$

Now, remember that we are interested in the flux across S, not necessarily the flux across S_a. To calculate the flux across S, let E be the solid between surfaces S_a and S. Then, the boundary of E consists of S_a and S. Denote this boundary by $S - S_a$ to indicate that S is oriented outward but now S_a is oriented inward. We would like to apply the divergence theorem to solid E. Notice that the divergence theorem, as stated, can't handle a solid such as E because E has a hole. However, the divergence theorem can be extended to handle solids with holes, just as Green's theorem can be extended to handle regions with holes. This allows us to use the divergence theorem in the following way. By the divergence theorem,

$$\begin{aligned}\iint_{S-S_a} \vec{\mathbf{F}}_\tau \cdot d\vec{\mathbf{S}} &= \iint_S \vec{\mathbf{F}}_\tau \cdot d\vec{\mathbf{S}} - \iint_{S_a} \vec{\mathbf{F}}_\tau \cdot d\vec{\mathbf{S}} \\ &= \iiint_E \operatorname{div} \vec{\mathbf{F}}_\tau\, dV \\ &= \iiint_E 0\, dV = 0.\end{aligned}$$

Therefore,

$$\iint_S \vec{\mathbf{F}}_\tau \cdot d\vec{\mathbf{S}} = \iint_{S_a} \vec{\mathbf{F}}_\tau \cdot d\vec{\mathbf{S}} = 4\pi,$$

and we have our desired result.

□

Now we return to calculating the flux across a smooth surface in the context of electrostatic field $\vec{\mathbf{E}} = \frac{q}{4\pi\epsilon_0}\vec{\mathbf{F}}_\tau$ of a point charge at the origin. Let S be a piecewise smooth closed surface that encompasses the origin. Then

$$\begin{aligned}\iint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{S}} &= \iint_S \frac{q}{4\pi\epsilon_0}\vec{\mathbf{F}}_\tau \cdot d\vec{\mathbf{S}} \\ &= \frac{q}{4\pi\epsilon_0}\iint_S \vec{\mathbf{F}}_\tau \cdot d\vec{\mathbf{S}} \\ &= \frac{q}{\epsilon_0}.\end{aligned}$$

If S does not encompass the origin, then

$$\iint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{S}} = \frac{q}{4\pi\epsilon_0}\iint_S \vec{\mathbf{F}}_\tau \cdot d\vec{\mathbf{S}} = 0.$$

Therefore, we have justified the claim that we set out to justify: the flux across closed surface S is zero if the charge is outside of S, and the flux is q/ϵ_0 if the charge is inside of S.

This analysis works only if there is a single point charge at the origin. In this case, Gauss' law says that the flux of $\vec{\mathbf{E}}$ across S is the total charge enclosed by S. Gauss' law can be extended to handle multiple charged solids in space, not just a single point charge at the origin. The logic is similar to the previous analysis, but beyond the scope of this text. In full generality, Gauss' law states that if S is a piecewise smooth closed surface and Q is the total amount of charge inside of S, then the flux of $\vec{\mathbf{E}}$ across S is Q/ϵ_0.

Example 15.8.5: Using Gauss' law

Suppose we have four stationary point charges in space, all with a charge of 0.002 Coulombs (C). The charges are located at $(0,0,1)$, $(1,1,4)$, $(-1,0,0)$, and $(-2,-2,2)$. Let $\vec{\mathbf{E}}$ denote the electrostatic field generated by these point charges. If S is the sphere of radius 2 oriented outward and centered at the origin, then find

$$\iint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{S}}.$$

Solution

According to Gauss' law, the flux of $\vec{\mathbf{E}}$ across S is the total charge inside of S divided by the electric constant. Since S has radius 2, notice that only two of the charges are inside of S: the charge at $(0,1,1)$ and the charge at $(-1,0,0)$. Therefore, the total charge encompassed by S is 0.004 and, by Gauss' law,

$$\iint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{S}} = \frac{0.004}{8.854 \times 10^{-12}} \approx 4.418 \times 10^9\, V - m.$$

Exercise 15.8.4

Work the previous example for surface S that is a sphere of radius 4 centered at the origin, oriented outward.

Key Concepts

- The divergence theorem relates a surface integral across closed surface S to a triple integral over the solid enclosed by S. The divergence theorem is a higher dimensional version of the flux form of Green's theorem, and is therefore a higher dimensional version of the Fundamental Theorem of Calculus.
- The divergence theorem can be used to transform a difficult flux integral into an easier triple integral and vice versa.
- The divergence theorem can be used to derive Gauss' law, a fundamental law in electrostatics.

Key Equations

- **Divergence theorem**

$$\iiint_E \operatorname{div} \vec{\mathbf{F}}\, dV = \iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$$

Glossary

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15.8E: Exercises for Section 15.8

For exercises 1 - 9, use a computer algebraic system (CAS) and the divergence theorem to evaluate surface integral $\int_S \vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\, ds$ for the given choice of $\vec{\mathbf{F}}$ and the boundary surface S. For each closed surface, assume $\vec{\mathbf{N}}$ is the outward unit normal vector.

1. [T] $\vec{\mathbf{F}}(x,y,z) = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$; S is the surface of cube $0 \le x \le 1$, $0 \le y \le 1$, $0 < z \le 1$.

2. [T] $\vec{\mathbf{F}}(x,y,z) = (\cos yz)\,\hat{\mathbf{i}} + e^{xz}\,\hat{\mathbf{j}} + 3z^2\,\hat{\mathbf{k}}$; S is the surface of hemisphere $z = \sqrt{4 - x^2 - y^2}$ together with disk $x^2 + y^2 \le 4$ in the xy-plane.

Answer

$$\int_S \vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\, ds = 75.3982$$

3. [T] $\vec{\mathbf{F}}(x,y,z) = (x^2 + y^2 - x^2)\,\hat{\mathbf{i}} + x^2 y\,\hat{\mathbf{j}} + 3z\,\hat{\mathbf{k}}$; S is the surface of the five faces of unit cube $0 \le x \le 1$, $0 \le y \le 1$, $0 < z \le 1$.

4. [T] $\vec{\mathbf{F}}(x,y,z) = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$; S is the surface of paraboloid $z = x^2 + y^2$ for $0 \le z \le 9$.

Answer

$$\int_S \vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\, ds = 127.2345$$

5. [T] $\vec{\mathbf{F}}(x,y,z) = x^2\,\hat{\mathbf{i}} + y^2\,\hat{\mathbf{j}} + z^2\,\hat{\mathbf{k}}$; S is the surface of sphere $x^2 + y^2 + z^2 = 4$.

6. [T] $\vec{\mathbf{F}}(x,y,z) = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + (z^2 - 1)\,\hat{\mathbf{k}}$; S is the surface of the solid bounded by cylinder $x^2 + y^2 = 4$ and planes $z = 0$ and $z = 1$.

Answer

$$\int_S \vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\, ds = 37.699$$

7. [T] $\vec{\mathbf{F}}(x,y,z) = x^2\,\hat{\mathbf{i}} + y^2\,\hat{\mathbf{j}} + z^2\,\hat{\mathbf{k}}$; S is the surface bounded above by sphere $\rho = 2$ and below by cone $\varphi = \frac{\pi}{4}$ in spherical coordinates. (Think of S as the surface of an "ice cream cone.")

8. [T] $\vec{\mathbf{F}}(x,y,z) = x^3\,\hat{\mathbf{i}} + y^3\,\hat{\mathbf{j}} + 3a^2 z\,\hat{\mathbf{k}}$ ($constant\ a > 0$); S is the surface bounded by cylinder $x^2 + y^2 = a^2$ and planes $z = 0$ and $z = 1$.

Answer

$$\int_S \vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\, ds = \frac{9\pi a^4}{2}$$

9. [T] Surface integral $\iint_S \vec{\mathbf{F}} \cdot dS$, where S is the solid bounded by paraboloid $z = x^2 + y^2$ and plane $z = 4$, and $\vec{\mathbf{F}}(x,y,z) = (x + y^2 z^2)\,\hat{\mathbf{i}} + (y + z^2 x^2)\,\hat{\mathbf{j}} + (z + x^2 y^2)\,\hat{\mathbf{k}}$

10. Use the divergence theorem to calculate surface integral $\iint_S \vec{\mathbf{F}} \cdot dS$, where $\vec{\mathbf{F}}(x,y,z) = (e^{y^2}\,\hat{\mathbf{i}} + (y + \sin(z^2))\,\hat{\mathbf{j}} + (z - 1)\,\hat{\mathbf{k}}$ and S is upper hemisphere $x^2 + y^2 + z^2 = 1$, $z \ge 0$, oriented upward.

Answer

$$\iint_S \vec{\mathbf{F}} \cdot dS = \frac{\pi}{3}$$

11. Use the divergence theorem to calculate surface integral $\iint_S \vec{\mathbf{F}} \cdot dS$, where $\vec{\mathbf{F}}(x,y,z) = x^4\,\hat{\mathbf{i}} - x^3 z^2\,\hat{\mathbf{j}} + 4xy^2 z\,\hat{\mathbf{k}}$ and S is the surface bounded by cylinder $x^2 + y^2 = 1$ and planes $z = x + 2$ and $z = 0$.

12. Use the divergence theorem to calculate surface integral $\iint_S \vec{\mathbf{F}} \cdot dS$, when $\vec{\mathbf{F}}(x,y,z) = x^2 z^3\,\hat{\mathbf{i}} + 2xyz^3\,\hat{\mathbf{j}} + xz^4\,\hat{\mathbf{k}}$ and S is the surface of the box with vertices $(\pm 1, \pm 2, \pm 3)$.

Answer

$$\iint_S \vec{\mathbf{F}} \cdot dS = 0$$

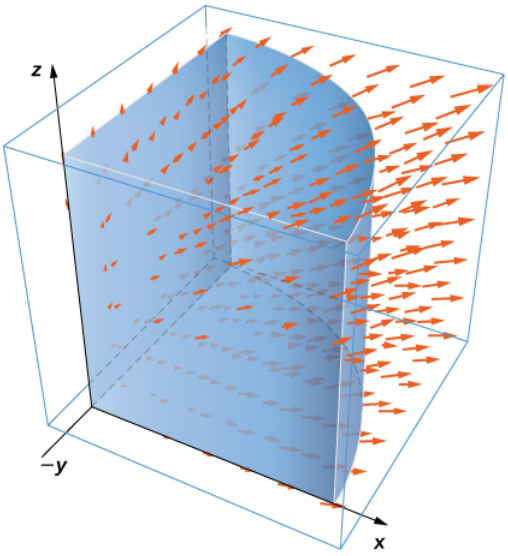
13. Use the divergence theorem to calculate surface integral $\iint_S \vec{\mathbf{F}} \cdot dS$, when $\vec{\mathbf{F}}(x, y, z) = z \tan^{-1}(y^2)\,\hat{\mathbf{i}} + z^3 \ln(x^2 + 1)\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$ and S is a part of paraboloid $x^2 + y^2 + z = 2$ that lies above plane $z = 1$ and is oriented upward.

14. [T] Use a CAS and the divergence theorem to calculate flux $\iint_S \vec{\mathbf{F}} \cdot dS$, where $\vec{\mathbf{F}}(x, y, z) = (x^3 + y^3)\,\hat{\mathbf{i}} + (y^3 + z^3)\,\hat{\mathbf{j}} + (z^3 + x^3)\,\hat{\mathbf{k}}$ and S is a sphere with center $(0, 0)$ and radius 2.

Answer

$$\iint_S \vec{\mathbf{F}} \cdot dS = 241.2743$$

15. Use the divergence theorem to compute the value of flux integral $\iint_S \vec{\mathbf{F}} \cdot dS$, where $\vec{\mathbf{F}}(x, y, z) = (y^3 + 3x)\,\hat{\mathbf{i}} + (xz + y)\,\hat{\mathbf{j}} + (z + x^4 \cos(x^2 y))\,\hat{\mathbf{k}}$ and S is the area of the region bounded by $x^2 + y^2 = 1$, $x \geq 0$, $y \geq 0$, and $0 \leq z \leq 1$.



16. Use the divergence theorem to compute flux integral $\iint_S \vec{\mathbf{F}} \cdot dS$, where $\vec{\mathbf{F}}(x, y, z) = y\,\hat{\mathbf{j}} - z\,\hat{\mathbf{k}}$ and S consists of the union of paraboloid $y = x^2 + z^2$, $0 \leq y \leq 1$, and disk $x^2 + z^2 \leq 1$, $y = 1$, oriented outward. What is the flux through just the paraboloid?

Answer

$$\iint_S \vec{\mathbf{F}} \cdot dS = -\pi$$

17. Use the divergence theorem to compute flux integral $\iint_S \vec{\mathbf{F}} \cdot dS$, where $\vec{\mathbf{F}}(x, y, z) = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z^4\,\hat{\mathbf{k}}$ and S is a part of cone $z = \sqrt{x^2 + y^2}$ beneath top plane $z = 1$ oriented downward.

18. Use the divergence theorem to calculate surface integral $\iint_S \vec{\mathbf{F}} \cdot dS$ for $\vec{\mathbf{F}}(x, y, z) = x^4\,\hat{\mathbf{i}} - x^3 z^2\,\hat{\mathbf{j}} + 4xy^2 z\,\hat{\mathbf{k}}$, where S is the surface bounded by cylinder $x^2 + y^2 = 1$ and planes $z = x + 2$ and $z = 0$.

Answer

$$\iint_S \vec{\mathbf{F}} \cdot dS = \frac{2\pi}{3}$$

19. Consider $\vec{\mathbf{F}}(x, y, z) = x^2\,\hat{\mathbf{i}} + xy\,\hat{\mathbf{j}} + (z + 1)\,\hat{\mathbf{k}}$. Let E be the solid enclosed by paraboloid $z = 4 - x^2 - y^2$ and plane $z = 0$ with normal vectors pointing outside E. Compute flux $\vec{\mathbf{F}}$ across the boundary of E using the divergence theorem.

In exercises 20 - 23, use a CAS along with the divergence theorem to compute the net outward flux for the fields across the given surfaces S.

20. [T] $\vec{\mathbf{F}} = \langle x, -2y, 3z \rangle$; S is sphere $\{(x, y, z) : x^2 + y^2 + z^2 = 6\}$.

Answer

$15\sqrt{6}\pi$

21. [T] $\vec{\mathbf{F}} = \langle x, 2y, z \rangle$; S is the boundary of the tetrahedron in the first octant formed by plane $x + y + z = 1$.

22. [T] $\vec{\mathbf{F}} = \langle y - 2x, x^3 - y, y^2 - z \rangle$; S is sphere $\{(x, y, z) : x^2 + y^2 + z^2 = 4\}$.

Answer

$-\frac{128}{3}\pi$

23. [T] $\vec{\mathbf{F}} = \langle x, y, z \rangle$; S is the surface of paraboloid $z = 4 - x^2 - y^2$, for $z \geq 0$, plus its base in the xy-plane.

For exercises 24 - 26, use a CAS and the divergence theorem to compute the net outward flux for the vector fields across the boundary of the given regions D.

24. [T] $\vec{\mathbf{F}} = \langle z - x, x - y, 2y - z \rangle$; D is the region between spheres of radius 2 and 4 centered at the origin.

Answer

-703.7168

25. [T] $\vec{\mathbf{F}} = \dfrac{\vec{\mathbf{r}}}{\|\vec{\mathbf{r}}\|} = \dfrac{\langle x, y, z \rangle}{\sqrt{x^2 + y^2 + z^2}}$; D is the region between spheres of radius 1 and 2 centered at the origin.

26. [T] $\vec{\mathbf{F}} = \langle x^2, -y^2, z^2 \rangle$; D is the region in the first octant between planes $z = 4 - x - y$ and $z = 2 - x - y$.

Answer

20

27. Let $\vec{\mathbf{F}}(x, y, z) = 2x\,\hat{\mathbf{i}} - 3xy\,\hat{\mathbf{j}} + xz^2\,\hat{\mathbf{k}}$. Use the divergence theorem to calculate $\iint_S \vec{\mathbf{F}} \cdot d\vec{S}$, where S is the surface of the cube with corners at $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(1, 1, 0)$, $(0, 0, 1)$, $(1, 0, 1)$, $(0, 1, 1)$, and $(1, 1, 1)$, oriented outward.

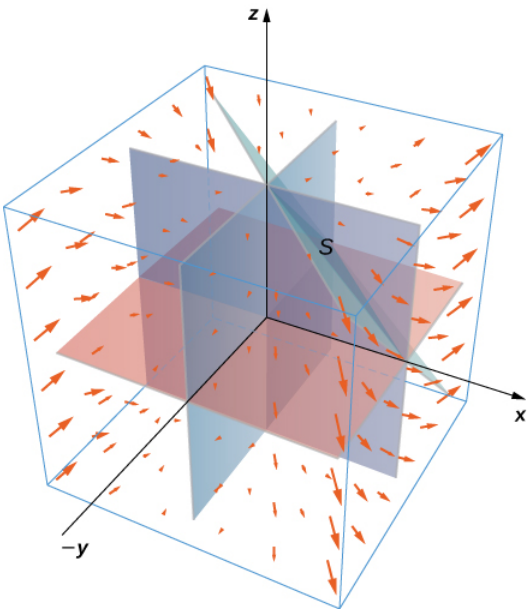
28. Use the divergence theorem to find the outward flux of field $\vec{\mathbf{F}}(x, y, z) = (x^3 - 3y)\,\hat{\mathbf{i}} + (2yz + 1)\,\hat{\mathbf{j}} + xyz\,\hat{\mathbf{k}}$ through the cube bounded by planes $x = \pm 1$, $y = \pm 1$, and $z = \pm 1$.

Answer

$\iint_S \vec{\mathbf{F}} \cdot d\vec{S} = 8$

29. Let $\vec{\mathbf{F}}(x, y, z) = 2x\,\hat{\mathbf{i}} - 3y\,\hat{\mathbf{j}} + 5z\,\hat{\mathbf{k}}$ and let S be hemisphere $z = \sqrt{9 - x^2 - y^2}$ together with disk $x^2 + y^2 \leq 9$ in the xy-plane. Use the divergence theorem.

30. Evaluate $\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\, dS$, where $\vec{\mathbf{F}}(x, y, z) = x^2\,\hat{\mathbf{i}} + xy\,\hat{\mathbf{j}} + x^3y^3\,\hat{\mathbf{k}}$ and S is the surface consisting of all faces except the tetrahedron bounded by plane $x + y + z = 1$ and the coordinate planes, with outward unit normal vector $\vec{\mathbf{N}}$.



Answer

$$\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\, dS = \frac{1}{8}$$

31. Find the net outward flux of field $\vec{\mathbf{F}} = \langle bz - cy,\ cx - az,\ ay - bx \rangle$ across any smooth closed surface in R^3 where a, b, and c are constants.

32. Use the divergence theorem to evaluate $\iint_S ||\vec{\mathbf{R}}||\vec{\mathbf{R}} \cdot \vec{\mathbf{n}}\, ds$, where $\vec{\mathbf{R}}(x, y, z) = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$ and S is sphere $x^2 + y^2 + z^2 = a^2$, with constant $a > 0$.

Answer

$$\iint_S ||\vec{\mathbf{R}}||\vec{\mathbf{R}} \cdot \vec{\mathbf{n}}\, ds = 4\pi a^4$$

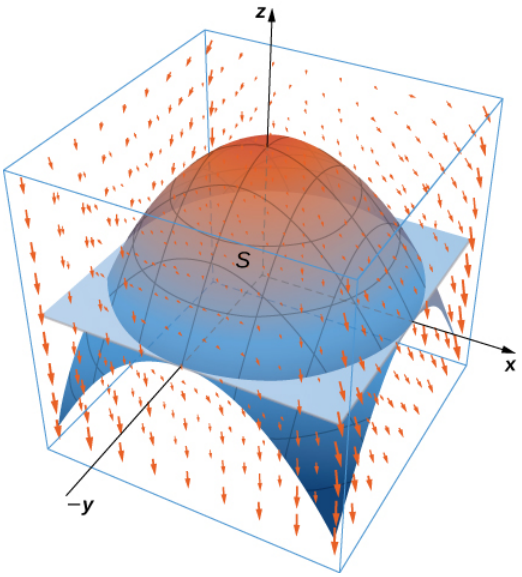
33. Use the divergence theorem to evaluate $\iint_S \vec{\mathbf{F}} \cdot dS$, where $\vec{\mathbf{F}}(x, y, z) = y^2 z\,\hat{\mathbf{i}} + y^3\,\hat{\mathbf{j}} + xz\,\hat{\mathbf{k}}$ and S is the boundary of the cube defined by $-1 \le x \le 1$, $-1 \le y \le 1$, and $0 \le z \le 2$.

34. Let R be the region defined by $x^2 + y^2 + z^2 \le 1$. Use the divergence theorem to find $\iiint_R z^2\, dV$.

Answer

$$\iiint_R z^2 dV = \frac{4\pi}{15}$$

35. Let E be the solid bounded by the xy-plane and paraboloid $z = 4 - x^2 - y^2$ so that S is the surface of the paraboloid piece together with the disk in the xy-plane that forms its bottom. If $\vec{\mathbf{F}}(x, y, z) = (xz \sin(yz) + x^3)\,\hat{\mathbf{i}} + \cos(yz)\,\hat{\mathbf{j}} + (3zy^2 - e^{x^2+y^2})\,\hat{\mathbf{k}}$, find $\iint_S \vec{\mathbf{F}} \cdot dS$ using the divergence theorem.



36. Let E be the solid unit cube with diagonally opposite corners at the origin and $(1,1,1)$, and faces parallel to the coordinate planes. Let S be the surface of E, oriented with the outward-pointing normal. Use a CAS to find $\iint_S \vec{\mathbf{F}} \cdot dS$ using the divergence theorem if $\vec{\mathbf{F}}(x,y,z) = 2xy\,\hat{\mathbf{i}} + 3ye^z\,\hat{\mathbf{j}} + x\sin z\,\hat{\mathbf{k}}$.

Answer

$$\iint_S \vec{\mathbf{F}} \cdot dS = 6.5759$$

37. Use the divergence theorem to calculate the flux of $\vec{\mathbf{F}}(x,y,z) = x^3\,\hat{\mathbf{i}} + y^3\,\hat{\mathbf{j}} + z^3\,\hat{\mathbf{k}}$ through sphere $x^2 + y^2 + z^2 = 1$.

38. Find $\iint_S \vec{\mathbf{F}} \cdot dS$, where $\vec{\mathbf{F}}(x,y,z) = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$ and S is the outwardly oriented surface obtained by removing cube $[1,2] \times [1,2] \times [1,2]$ from cube $[0,2] \times [0,2] \times [0,2]$.

Answer

$$\iint_S \vec{\mathbf{F}} \cdot dS = 21$$

39. Consider radial vector field $\vec{\mathbf{F}} = \dfrac{\vec{\mathbf{r}}}{\|\vec{\mathbf{r}}\|} = \dfrac{\langle x,y,z\rangle}{(x^2+y^2+z^2)^{1/2}}$. Compute the surface integral, where S is the surface of a sphere of radius a centered at the origin.

40. Compute the flux of water through parabolic cylinder $S : y = x^2$, from $0 \le x \le 2$, $0 \le z \le 3$, if the velocity vector is $\vec{\mathbf{F}}(x,y,z) = 3z^2\,\hat{\mathbf{i}} + 6\,\hat{\mathbf{j}} + 6xz\,\hat{\mathbf{k}}$.

Answer

$$\iint_S \vec{\mathbf{F}} \cdot dS = 72$$

41. [T] Use a CAS to find the flux of vector field $\vec{\mathbf{F}}(x,y,z) = z\,\hat{\mathbf{i}} + z\,\hat{\mathbf{j}} + \sqrt{x^2+y^2}\,\hat{\mathbf{k}}$ across the portion of hyperboloid $x^2 + y^2 = z^2 + 1$ between planes $z = 0$ and $z = \dfrac{\sqrt{3}}{3}$, oriented so the unit normal vector points away from the z-axis.

42. Use a CAS to find the flux of vector field $\vec{\mathbf{F}}(x,y,z) = (e^y + x)\,\hat{\mathbf{i}} + (3\cos(xz) - y)\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$ through surface S, where S is given by $z^2 = 4x^2 + 4y^2$ from $0 \le z \le 4$, oriented so the unit normal vector points downward.

Answer

$$\iint_S \vec{\mathbf{F}} \cdot dS = -33.5103$$

43. [T] Use a CAS to compute $\iint_S \vec{\mathbf{F}} \cdot dS$, where $\vec{\mathbf{F}}(x,y,z) = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + 2z\hat{\mathbf{k}}$ and S is a part of sphere $x^2 + y^2 + z^2 = 2$ with $0 \le z \le 1$.

44. Evaluate $\iint_S \vec{\mathbf{F}} \cdot dS$, where $\vec{\mathbf{F}}(x,y,z) = bxy^2\,\hat{\mathbf{i}} + bx^2y\,\hat{\mathbf{j}} + (x^2+y^2)z^2\,\hat{\mathbf{k}}$ and S is a closed surface bounding the region and consisting of solid cylinder $x^2 + y^2 \le a^2$ and $0 \le z \le b$.

Answer

$$\iint_S \vec{\mathbf{F}} \cdot dS = \pi a^4 b^2$$

45. [T] Use a CAS to calculate the flux of $\vec{\mathbf{F}}(x,y,z) = (x^3 + y\,\sin z)\,\hat{\mathbf{i}} + (y^3 + z\,\sin x)\,\hat{\mathbf{j}} + 3z\,\hat{\mathbf{k}}$ across surface S, where S is the boundary of the solid bounded by hemispheres $z = \sqrt{4 - x^2 - y^2}$ and $z = \sqrt{1 - x^2 - y^2}$, and plane $z = 0$.

46. Use the divergence theorem to evaluate $\iint_S \vec{\mathbf{F}} \cdot dS$, where $\vec{\mathbf{F}}(x,y,z) = xy\,\hat{\mathbf{i}} - \frac{1}{2}y^2\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$ and S is the surface consisting of three pieces: $z = 4 - 3x^2 - 3y^2$, $1 \le z \le 4$ on the top; $x^2 + y^2 = 1$, $0 \le z \le 1$ on the sides; and $z = 0$ on the bottom.

Answer

$$\iint_S \vec{\mathbf{F}} \cdot dS = \frac{5}{2}\pi$$

47. [T] Use a CAS and the divergence theorem to evaluate $\iint_S \vec{\mathbf{F}} \cdot dS$, where $\vec{\mathbf{F}}(x,y,z) = (2x + y\,\cos z)\,\hat{\mathbf{i}} + (x^2 - y)\,\hat{\mathbf{j}} + y^2 z\,\hat{\mathbf{k}}$ and S is sphere $x^2 + y^2 + z^2 = 4$ orientated outward.

48. Use the divergence theorem to evaluate $\iint_S \vec{\mathbf{F}} \cdot dS$, where $\vec{\mathbf{F}}(x,y,z) = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$ and S is the boundary of the solid enclosed by paraboloid $y = x^2 + z^2 - 2$, cylinder $x^2 + z^2 = 1$, and plane $x + y = 2$, and S is oriented outward.

Answer

$$\iint_S \vec{\mathbf{F}} \cdot dS = \frac{21\pi}{2}$$

For the following exercises, **Fourier's law of heat transfer** states that the heat flow vector $\vec{\mathbf{F}}$ at a point is proportional to the negative gradient of the temperature; that is, $\vec{\mathbf{F}} = -k\vec{\nabla}T$, which means that heat energy flows hot regions to cold regions. The constant $k > 0$ is called the *conductivity*, which has metric units of joules per meter per second-kelvin or watts per meter-kelvin. A temperature function for region D s given. Use the divergence theorem to find net outward heat flux $\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\, dS = -k \iint_S \vec{\nabla}T \cdot N\, dS$ across the boundary S of D, where $k = 1$.

49. $T(x,y,z) = 100 + x + 2y + z$;

$D = \{(x,y,z) : 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\}$

50. $T(x,y,z) = 100 + e^{-z}$;

$D = \{(x,y,z) : 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\}$

Answer

$-(1 - e^{-1})$

51. $T(x,y,z) = 100e^{-x^2-y^2-z^2}$; D is the sphere of radius a centered at the origin.

Contributors

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CHAPTER OVERVIEW

Appendices

Trigonometric Identities

Table of Derivatives

Table of Integrals

Table of Laplace Transforms

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Trigonometric Identities

Pythagorean Identities

$$\cos^2 x + \sin^2 x = 1$$

$$\sec^2 x - \tan^2 x = 1$$

Double-Angle Identities

$$\sin 2x = 2 \sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1$$

Half-Angle Identities

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

Angle Sum and Difference Identities

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$

$$\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$$

$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

$$\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$$

Angle Reflections and Shifts

$$\sin(-x) = -\sin x$$

$$\cos(-x) = \cos x$$

$$\tan(-x) = -\tan x$$

$$\sin\left(x \pm \tfrac{\pi}{2}\right) = \pm \cos x$$

$$\cos\left(x \pm \tfrac{\pi}{2}\right) = \mp \sin x$$

Angle Supplement Identities

$$\sin(\pi - x) = \sin x$$

$$\cos(\pi - x) = -\cos x$$

$$\tan(\pi - x) = -\tan x$$

Periodicity Identities

$$\sin(x + 2\pi) = \sin x$$

$$\cos(x + 2\pi) = \cos x$$

$$\tan(x + \pi) = \tan x$$

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Table of Derivatives

General Formulas

1. $\frac{d}{dx}(c) = 0$
2. $\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$
3. $\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$
4. $\frac{d}{dx}(x^n) = nx^{n-1}, \quad \text{for real numbers } n$
5. $\frac{d}{dx}(cf(x)) = cf'(x)$
6. $\frac{d}{dx}(f(x) - g(x)) = f'(x) - g'(x)$
7. $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$
8. $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$

Trigonometric Functions

9. $\frac{d}{dx}(\sin x) = \cos x$
10. $\frac{d}{dx}(\tan x) = \sec^2 x$
11. $\frac{d}{dx}(\sec x) = \sec x \tan x$
12. $\frac{d}{dx}(\cos x) = -\sin x$
13. $\frac{d}{dx}(\cot x) = -\csc^2 x$
14. $\frac{d}{dx}(\csc x) = -\csc x \cot x$

Inverse Trigonometric Functions

15. $\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$
16. $\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$
17. $\frac{d}{dx}(\operatorname{arcsec} x) = \frac{1}{|x|\sqrt{x^2-1}}$
18. $\frac{d}{dx}(\arccos x) = \frac{-1}{\sqrt{1-x^2}}$
19. $\frac{d}{dx}(\operatorname{arccot} x) = \frac{-1}{1+x^2}$
20. $\frac{d}{dx}(\operatorname{arccsc} x) = \frac{-1}{|x|\sqrt{x^2-1}}$

Exponential and Logarithmic Functions

21. $\frac{d}{dx}(e^x) = e^x$
22. $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$
23. $\frac{d}{dx}(b^x) = b^x \ln b$
24. $\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}$

Hyperbolic Functions

25. $\frac{d}{dx}(\sinh x) = \cosh x$
26. $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$
27. $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$
28. $\frac{d}{dx}(\cosh x) = \sinh x$
29. $\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$
30. $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$

Inverse Hyperbolic Functions

31. $\frac{d}{dx}(\operatorname{arcsinh} x) = \frac{1}{\sqrt{x^2+1}}$
32. $\frac{d}{dx}(\operatorname{arctanh} x) = \frac{1}{1-x^2} \quad (|x| < 1)$
33. $\frac{d}{dx}(\operatorname{arcsech} x) = \frac{-1}{x\sqrt{1-x^2}} \quad (0 < x < 1)$
34. $\frac{d}{dx}(\operatorname{arccosh} x) = \frac{1}{\sqrt{x^2-1}} \quad (x > 1)$
35. $\frac{d}{dx}(\operatorname{arccoth} x) = \frac{1}{1-x^2} \quad (|x| > 1)$
36. $\frac{d}{dx}(\operatorname{arccsch} x) = \frac{-1}{|x|\sqrt{1+x^2}} \quad (x \neq 0)$

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- Modified to change inverse trig notation by Paul Seeburger (Monroe Community College)

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Table of Integrals

For this course, all work must be shown to obtain most of these integral forms. **Of the integration formulas listed below, the only ones that can be applied without further work are #1 - 10, 15 - 17, and 49 and 50.** And even these will require work to be shown if a substitution is involved.

All others may be helpful for checking your final answers, but cannot be used to skip the necessary work to show you understand how to use the integration techniques taught in this course.

As you look through these formulas, you should be able to recognize which integration technique was needed to obtain the general formula. It may be very useful for you to try to obtain the general formula yourself using the techniques we learn in this course.

Basic Integrals

1. $\int u^n\,du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$
2. $\int \frac{du}{u} = \ln|u| + C$
3. $\int e^u\,du = e^u + C$
4. $\int a^u\,du = \frac{a^u}{\ln a} + C$
5. $\int \sin u\,du = -\cos u + C$
6. $\int \cos u\,du = \sin u + C$
7. $\int \sec^2 u\,du = \tan u + C$
8. $\int \csc^2 u\,du = -\cot u + C$
9. $\int \sec u \tan u\,du = \sec u + C$
10. $\int \csc u \cot u\,du = -\csc u + C$
11. $\int \tan u\,du = \ln|\sec u| + C$
12. $\int \cot u\,du = \ln|\sin u| + C$
13. $\int \sec u\,du = \ln|\sec u + \tan u| + C$
14. $\int \csc u\,du = \ln|\csc u - \cot u| + C$
15. $\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin\left(\frac{u}{a}\right) + C$
16. $\int \frac{du}{a^2 + u^2} = \frac{1}{a}\arctan\left(\frac{u}{a}\right) + C$
17. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\operatorname{arcsec}\left(\frac{|u|}{a}\right) + C$

Trigonometric Integrals

18. $\int \sin^2 u\,du = \frac{1}{2}u - \frac{1}{4}\sin 2u + C$
19. $\int \cos^2 u\,du = \frac{1}{2}u + \frac{1}{4}\sin 2u + C$
20. $\int \tan^2 u\,du = \tan u - u + C$

21. $\int \cot^2 u\,du = -\cot u - u + C$

22. $\int \sin^3 u\,du = -\frac{1}{3}(2+\sin^2 u)\cos u + C$

23. $\int \cos^3 u\,du = \frac{1}{3}(2+\cos^2 u)\sin u + C$

24. $\int \tan^3 u\,du = \frac{1}{2}\tan^2 u + \ln|\cos u| + C$

25. $\int \cot^3 u\,du = -\frac{1}{2}\cot^2 u - \ln|\sin u| + C$

26. $\int \sec^3 u\,du = \frac{1}{2}\sec u\tan u + \frac{1}{2}\ln|\sec u + \tan u| + C$

27. $\int \csc^3 u\,du = -\frac{1}{2}\csc u\cot u + \frac{1}{2}\ln|\csc u - \cot u| + C$

28. $\int \sin^n u\,du = \frac{-1}{n}\sin^{n-1} u\cos u + \frac{n-1}{n}\int \sin^{n-2} u\,du$

29. $\int \cos^n u\,du = \frac{1}{n}\cos^{n-1} u\sin u + \frac{n-1}{n}\int \cos^{n-2} u\,du$

30. $\int \tan^n u\,du = \frac{1}{n-1}\tan^{n-1} u - \int \tan^{n-2} u\,du$

31. $\int \cot^n u\,du = \frac{-1}{n-1}\cot^{n-1} u - \int \cot^{n-2} u\,du$

32. $\int \sec^n u\,du = \frac{1}{n-1}\tan u\sec^{n-2} u + \frac{n-2}{n-1}\int \sec^{n-2} u\,du$

33. $\int \csc^n u\,du = \frac{-1}{n-1}\cot u\csc^{n-2} u + \frac{n-2}{n-1}\int \csc^{n-2} u\,du$

34. $\int \sin au\sin bu\,du = \frac{\sin(a-b)u}{2(a-b)} - \frac{\sin(a+b)u}{2(a+b)} + C$

35. $\int \cos au\cos bu\,du = \frac{\sin(a-b)u}{2(a-b)} + \frac{\sin(a+b)u}{2(a+b)} + C$

36. $\int \sin au\cos bu\,du = -\frac{\cos(a-b)u}{2(a-b)} - \frac{\cos(a+b)u}{2(a+b)} + C$

37. $\int u\sin u\,du = \sin u - u\cos u + C$

38. $\int u\cos u\,du = \cos u + u\sin u + C$

39. $\int u^n \sin u\,du = -u^n\cos u + n\int u^{n-1}\cos u\,du$

40. $\int u^n \cos u\,du = u^n\sin u - n\int u^{n-1}\sin u\,du$

41. $\int \sin^n u\cos^m u\,du =$ Use the methods for powers of sine and cosine

Exponential and Logarithmic Integrals

42. $\int ue^{au}\,du = \frac{1}{a^2}(au-1)e^{au} + C$

43. $\int u^n e^{au}\,du = \frac{1}{a}u^n e^{au} - \frac{n}{a}\int u^{n-1}e^{au}\,du$

44. $\int e^{au}\sin bu\,du = \frac{e^{au}}{a^2+b^2}(a\sin bu - b\cos bu) + C$

45. $\int e^{au}\cos bu\,du = \frac{e^{au}}{a^2+b^2}(a\cos bu + b\sin bu) + C$

46. $\int \ln u\,du = u\ln u - u + C$

47. $\int u^n \ln u\,du = \frac{u^{n+1}}{(n+1)^2}[(n+1)\ln u - 1] + C$

48. $\int \frac{1}{u \ln u}\, du = \ln|\ln u| + C$

Hyperbolic Integrals

49. $\int \sinh u\, du = \cosh u + C$

50. $\int \cosh u\, du = \sinh u + C$

51. $\int \tanh u\, du = \ln \cosh u + C$

52. $\int \coth u\, du = \ln|\sinh u| + C$

53. $\int \operatorname{sech} u\, du = \arctan|\sinh u| + C$

54. $\int \operatorname{csch} u\, du = \ln|\tanh \frac{1}{2}u| + C$

55. $\int \operatorname{sech}^2 u\, du = \tanh u + C$

56. $\int \operatorname{csch}^2 u\, du = -\coth u + C$

57. $\int \operatorname{sech} u \tanh u\, du = -\operatorname{sech} u + C$

58. $\int \operatorname{csch} u \coth u\, du = -\operatorname{csch} u + C$

Inverse Trigonometric Integrals

59. $\int \arcsin u\, du = u \arcsin u + \sqrt{1-u^2} + C$

60. $\int \arccos u\, du = u \arccos u - \sqrt{1-u^2} + C$

61. $\int \arctan u\, du = u \arctan u - \frac{1}{2}\ln(1+u^2) + C$

62. $\int u \arcsin u\, du = \frac{2u^2-1}{4}\arcsin u + \frac{u\sqrt{1-u^2}}{4} + C$

63. $\int u \arccos u\, du = \frac{2u^2-1}{4}\arccos u - \frac{u\sqrt{1-u^2}}{4} + C$

64. $\int u \arctan u\, du = \frac{u^2+1}{2}\arctan u - \frac{u}{2} + C$

65. $\int u^n \arcsin u\, du = \frac{1}{n+1}\left[u^{n+1}\arcsin u - \int \frac{u^{n+1}\, du}{\sqrt{1-u^2}}\right], \quad n \neq -1$

66. $\int u^n \arccos u\, du = \frac{1}{n+1}\left[u^{n+1}\arccos u + \int \frac{u^{n+1}\, du}{\sqrt{1-u^2}}\right], \quad n \neq -1$

67. $\int u^n \arctan u\, du = \frac{1}{n+1}\left[u^{n+1}\arctan u - \int \frac{u^{n+1}\, du}{1+u^2}\right], \quad n \neq -1$

Integrals Involving $a^2 + u^2$, $a > 0$

68. $\int \sqrt{a^2+u^2}\, du = \frac{u}{2}\sqrt{a^2+u^2} + \frac{a^2}{2}\ln\left(u + \sqrt{a^2+u^2}\right) + C$

69. $\int u^2\sqrt{a^2+u^2}\, du = \frac{u}{8}(a^2+2u^2)\sqrt{a^2+u^2} - \frac{a^4}{8}\ln\left(u + \sqrt{a^2+u^2}\right) + C$

70. $\int \frac{\sqrt{a^2+u^2}}{u}\, du = \sqrt{a^2+u^2} - a\ln\left|\frac{a+\sqrt{a^2+u^2}}{u}\right| + C$

71. $\int \frac{\sqrt{a^2+u^2}}{u^2}\, du = -\frac{\sqrt{a^2+u^2}}{u} + \ln\left(u + \sqrt{a^2+u^2}\right) + C$

72. $\int \frac{du}{\sqrt{a^2+u^2}} = \ln\left(u+\sqrt{a^2+u^2}\right)+C$

73. $\int \frac{u^2}{\sqrt{a^2+u^2}}\,du = \frac{u}{2}\left(\sqrt{a^2+u^2}\right) - \frac{a^2}{2}\ln\left(u+\sqrt{a^2+u^2}\right)+C$

74. $\int \frac{du}{u\sqrt{a^2+u^2}} = \frac{-1}{a}\ln\left|\frac{\sqrt{a^2+u^2}+a}{u}\right|+C$

75. $\int \frac{du}{u^2\sqrt{a^2+u^2}} = -\frac{\sqrt{a^2+u^2}}{a^2u}+C$

76. $\int \frac{du}{(a^2+u^2)^{3/2}} = \frac{u}{a^2\sqrt{a^2+u^2}}+C$

Integrals Involving $u^2 - a^2$, $a > 0$

77. $\int \sqrt{u^2-a^2}\,du = \frac{u}{2}\sqrt{u^2-a^2} - \frac{a^2}{2}\ln\left|u+\sqrt{u^2-a^2}\right|+C$

78. $\int u^2\sqrt{u^2-a^2}\,du = \frac{u}{8}(2u^2-a^2)\sqrt{u^2-a^2} - \frac{a^4}{8}\ln\left|u+\sqrt{u^2-a^2}\right|+C$

79. $\int \frac{\sqrt{u^2-a^2}}{u}\,du = \sqrt{u^2-a^2} - a\arccos\frac{a}{|u|}+C$

80. $\int \frac{\sqrt{u^2-a^2}}{u^2}\,du = -\frac{\sqrt{u^2-a^2}}{u} + \ln\left|u+\sqrt{u^2-a^2}\right|+C$

81. $\int \frac{du}{\sqrt{u^2-a^2}} = \ln\left|u+\sqrt{u^2-a^2}\right|+C$

82. $\int \frac{u^2}{\sqrt{u^2-a^2}}\,du = \frac{u}{2}\sqrt{u^2-a^2} + \frac{a^2}{2}\ln\left|u+\sqrt{u^2-a^2}\right|+C$

83. $\int \frac{du}{u^2\sqrt{u^2-a^2}} = \frac{\sqrt{u^2-a^2}}{a^2u}+C$

84. $\int \frac{du}{(u^2-a^2)^{3/2}} = -\frac{u}{a^2\sqrt{u^2-a^2}}+C$

Integrals Involving $a^2 - u^2$, $a > 0$

85. $\int \sqrt{a^2-u^2}\,du = \frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\arcsin\frac{u}{a}+C$

86. $\int u^2\sqrt{a^2-u^2}\,du = \frac{u}{8}(2u^2-a^2)\sqrt{a^2-u^2} + \frac{a^4}{8}\arcsin\frac{u}{a}+C$

87. $\int \frac{\sqrt{a^2-u^2}}{u}\,du = \sqrt{a^2-u^2} - a\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right|+C$

88. $\int \frac{\sqrt{a^2-u^2}}{u^2}\,du = \frac{-1}{u}\sqrt{a^2-u^2} - \arcsin\frac{u}{a}+C$

89. $\int \frac{u^2}{\sqrt{a^2-u^2}}\,du = \frac{1}{2}\left(-u\sqrt{a^2-u^2} + a^2\arcsin\frac{u}{a}\right)+C$

90. $\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right|+C$

91. $\int \frac{du}{u^2\sqrt{a^2-u^2}} = -\frac{1}{a^2u}\sqrt{a^2-u^2}+C$

92. $\int \left(a^2-u^2\right)^{3/2}du = -\frac{u}{8}\left(2u^2-5a^2\right)\sqrt{a^2-u^2} + \frac{3a^4}{8}\arcsin\frac{u}{a}+C$

93. $\int \frac{du}{(a^2-u^2)^{3/2}} = -\frac{u}{a^2\sqrt{a^2-u^2}}+C$

Integrals Involving $2au - u^2$, $a > 0$

94. $\int \sqrt{2au-u^2}\,du = \frac{u-a}{2}\sqrt{2au-u^2} + \frac{a^2}{2}\arccos\left(\frac{a-u}{a}\right)+C$

95. $\int \frac{du}{\sqrt{2au-u^2}} = \arccos\left(\frac{a-u}{a}\right) + C$

96. $\int u\sqrt{2au-u^2}\,du = \frac{2u^2-au-3a^2}{6}\sqrt{2au-u^2} + \frac{a^3}{2}\arccos\left(\frac{a-u}{a}\right) + C$

97. $\int \frac{du}{u\sqrt{2au-u^2}} = -\frac{\sqrt{2au-u^2}}{au} + C$

Integrals Involving $a + bu$, $a \neq 0$

98. $\int \frac{u}{a+bu}\,du = \frac{1}{b^2}(a+bu-a\ln|a+bu|) + C$

99. $\int \frac{u^2}{a+bu}\,du = \frac{1}{2b^3}\left[(a+bu)^2 - 4a(a+bu) + 2a^2\ln|a+bu|\right] + C$

100. $\int \frac{du}{u(a+bu)} = \frac{1}{a}\ln\left|\frac{u}{a+bu}\right| + C$

101. $\int \frac{du}{u^2(a+bu)} = -\frac{1}{au} + \frac{b}{a^2}\ln\left|\frac{a+bu}{u}\right| + C$

102. $\int \frac{u}{(a+bu)^2}\,du = \frac{a}{b^2(a+bu)} + \frac{1}{b^2}\ln|a+bu| + C$

103. $\int \frac{u}{u(a+bu)^2}\,du = \frac{1}{a(a+bu)} - \frac{1}{a^2}\ln\left|\frac{a+bu}{u}\right| + C$

104. $\int \frac{u^2}{(a+bu)^2}\,du = \frac{1}{b^3}\left(a+bu-\frac{a^2}{a+bu}-2a\ln|a+bu|\right) + C$

105. $\int u\sqrt{a+bu}\,du = \frac{2}{15b^2}(3bu-2a)(a+bu)^{3/2} + C$

106. $\int \frac{u}{\sqrt{a+bu}}\,du = \frac{2}{3b^2}(bu-2a)\sqrt{a+bu} + C$

107. $\int \frac{u^2}{\sqrt{a+bu}}\,du = \frac{2}{15b^3}(8a^2+3b^2u^2-4abu)\sqrt{a+bu} + C$

108. $\int \frac{du}{u\sqrt{a+bu}} = \begin{cases} \frac{1}{\sqrt{a}}\ln\left|\frac{\sqrt{a+bu}-\sqrt{a}}{\sqrt{a+bu}+\sqrt{a}}\right| + C, & \text{if } a > 0 \\ \frac{\sqrt{2}}{\sqrt{-a}}\arctan\sqrt{\frac{a+bu}{-a}} + C, & \text{if } a < 0 \end{cases}$

109. $\int \frac{\sqrt{a+bu}}{u}\,du = 2\sqrt{a+bu} + a\int \frac{du}{u\sqrt{a+bu}}$

110. $\int \frac{\sqrt{a+bu}}{u^2}\,du = -\frac{\sqrt{a+bu}}{u} + \frac{b}{2}\int \frac{du}{u\sqrt{a+bu}}$

111. $\int u^n\sqrt{a+bu}\,du = \frac{2}{b(2n+3)}\left[u^n(a+bu)^{3/2} - na\int u^{n-1}\sqrt{a+bu}\,du\right]$

112. $\int \frac{u^n}{\sqrt{a+bu}}\,du = \frac{2u^n\sqrt{a+bu}}{b(2n+1)} - \frac{2na}{b(2n+1)}\int \frac{u^{n-1}}{\sqrt{a+bu}}\,du$

113. $\int \frac{du}{u^n\sqrt{a+bu}} = -\frac{\sqrt{a+bu}}{a(n-1)u^{n-1}} - \frac{b(2n-3)}{2a(n-1)}\int \frac{du}{u^{n-1}\sqrt{a+bu}}$

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Table of Laplace Transforms

Table of Laplace Transforms

$f(t)=\mathscr{L}^{-1}\{F(s)\}$	$F(s)=\mathscr{L}\{F(s)\}$	$f(t)=\mathscr{L}^{-1}\{F(s)\}$	$F(s)=\mathscr{L}\{F(s)\}$
1. 1	$\frac{1}{s}$	2. t	$\frac{1}{s^2}$
3. $t^n,\ n=1,2,3,\cdots$	$\frac{n!}{s^{n+1}}$	4. e^{at}	$\frac{1}{s-a}$
5. $\sin at$	$\frac{a}{s^2+a^2}$	6. $\cos at$	$\frac{s}{s^2+a^2}$
7. $\sinh at$	$\frac{a}{s^2-a^2}$	8. $\cosh at$	$\frac{s}{s^2-a^2}$
9. $e^{at}\cdot f(t)$	$F(s-a)$	**Unit Step or Heavyside Function** 10. $\mathscr{U}(t-a)$	$\frac{e^{-as}}{s}$
11. $f(t-a)\cdot\mathscr{U}(t-a)$	$e^{-as}\cdot F(s)$	12. $f(t)\cdot\mathscr{U}(t-a)$	$e^{-as}\cdot\mathscr{L}\{f(t+a)\}$
13. $f'(t)$	$sF(s)-f(0)$	14. $f''(t)$	$s^2F(s)-s\cdot f(0)-f'(0)$
15. $t\cdot f(t)$	$-F'(s)$	16. $f^{(n)}(t)$	$s^nF(s)-s^{(n-1)}f(0)-\cdots$ $-sf^{(n-2)}(0)-f^{(n-1)}(0)$
17. $t^n\cdot f(t)$	$(-1)^n\frac{d^n}{ds^n}\big(F(s)\big)$	18. $\frac{1}{t}\cdot f(t)$	$\int_s^\infty F(w)\,dw$
Dirac Delta Function 19. $\delta(t-a)$	e^{-as}	**Convolution** 20. $f(t)*g(t)$	$F(s)\cdot G(s)$
Dirac Delta Function 21. $\delta(t-a)f(t)$	$f(a)\cdot e^{-as}$	**Periodic function with period T** 22. $f(t+T)=f(t)$	$\frac{F_T(s)}{1-e^{-Ts}}=\frac{\int_0^T e^{-st}f(t)\,dt}{1-e^{-Ts}}$

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