

OPEN ALGEBRA AND KNOTS



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Open Algebra and Knots

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TABLE OF CONTENTS

[Licensing](#)

[1: Rational Tangles](#)

- [1.1: Tangle Basics](#)
- [1.2: Tangle Arithmetic](#)

[2: Knots and Numbers](#)

- [2.1: Notations for Knots](#)
- [2.2: Numerical Knot Invariants](#)
- [2.3: Colorings of Knots](#)
- [2.4: Colorings of Tangles](#)

[3: Knots and Algebra](#)

- [3.1: Algebraic Structures](#)
- [3.2: Colorings and Quandles](#)
- [3.3: Knot Groups](#)
- [3.4: Invariant Polynomials](#)
- [3.5: Portfolio Problems](#)

[4: Workshop Project](#)

- [4.1: Project Overview](#)

[5: Knot Your Average Algebra](#)

- [5.1: Seminar Videos](#)

[Index](#)

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[Student Blogs](#)

[Detailed Licensing](#)

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CHAPTER OVERVIEW

1: Rational Tangles

Learning Objectives

- Discover how different types of twists on a tangle determine its tangle number.
- Argue for why the arithmetic of the rational numbers makes certain relationships among tangle numbers necessary.
- Calculate the fraction of a rational tangle in two different ways, and argue for why the fraction is an invariant of rational tangles.

"What I like doing is taking something that other people thought was complicated and difficult to understand, and finding a simple idea. So that any fool - and, in this case, you - can understand the complicated thing. ---John Conway"

As we discovered in our first class, *crossings* are one of the first ways for us to understand the connections between knots and algebra: somehow, if we can say "enough" about how a strand crosses itself, we can characterize the essential nature of a knot. So we'll begin by focusing as much as possible *only* on crossings, by studying objects known as tangles, in which crossings are created between two strands by twisting up their endpoints.

[1.1: Tangle Basics](#)

[1.2: Tangle Arithmetic](#)

References

1. Davis, T. (2017). *Conway's Rational Tangles*. Accessed at <http://www.geometer.org/mathcircles/tangle.pdf>.
2. Kauffman, L. H., & Lambropoulou, S. (2004). On the classification of rational tangles. *Advances in Applied Mathematics*, 33(2), 199-237. Available on arXiv at <http://arxiv.org/pdf/math/0311499.pdf>.
3. Tanton, J. (2012). *Understanding Rational Tangles*. Accessed at <http://mathteacherscircle.org/assets/session-materials/JTantonRationalTangles.pdf>.

Thumbnail: A knot diagram of the trefoil knot, the simplest non-trivial knot. (Public Domain; [Marnanel](#) via [Wikipedia](#))

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1.1: Tangle Basics

One of the aspects of knot theory that makes knots challenging is that knots must be understood in their wholeness: not on the basis of looking just at some of their crossings but looking at how all those crossings fit together to paint a *global* picture of the knot. Ultimately, the knot invariants we study later in the semester will help us to take this perspective. For now, though, we'll take the global questions out of the picture by slicing apart our knots and pinning down the cut ends, much like an entomologist might study a mounted butterfly. The objects we obtain for study in this process are called **tangles**, and the class of them that are easiest to study are the so-called rational tangles.

1.2.1 Constructing Rational Tangles

Rational Tangles

596.2 Rational Tangle Basics

Conway's Rational Tangles Matthew Salomone



Ex:



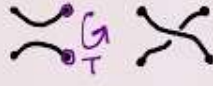
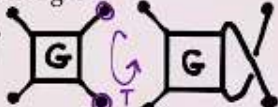
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Definition: Rational Tangle (Conway, 1967)

An **empty tangle** describes two horizontal, parallel strands: 

A **rational tangle** is any tangle which can be obtained by performing the following operations on an empty tangle:

- "Twisting"** top-right over bottom-right

From empty:  Generally:  T
- "Rotating"** the whole tangle 90° clockwise

From empty:  Generally:  R

Q: Numerical invariant for these tangles?

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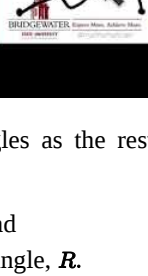


Rational Tangles

Conway's Rational Tangles



Ex:



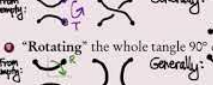
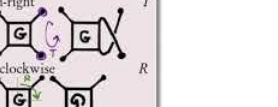
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From empty:  Generally:  T
- "Rotating"** the whole tangle 90° clockwise

From empty:  Generally:  R

Q: Numerical invariant for these tangles?

We begin by thinking of rational tangles as the result of applying combinations of two geometric operations to an "empty" horizontal tangle:

1. Twists of the rightmost strands, T , and
2. Quarter-turn rotations of the entire tangle, R .

So for instance, we might describe a tangle as a "word" built from these letters such as

$$\text{TTRTTTTRTTTTTRTRT.} \quad (1.1.1)$$

Ideally, what we'd like to find is a way to represent each of these operations as an operation happening to rational *numbers* and not just to rational tangles. In this process what we're hoping to obtain is:

- A **group** Γ of operations on tangles, and
- A (right-) **group action** of this group on the rational numbers, $\rho : \Gamma \rightarrow \text{Func}(\mathbb{Q} \rightarrow \mathbb{Q})$.

The latter definition might be new to you, but we'll hopefully see by example that it's pretty natural. To make it precise:

Definition 1.2.1 Group action.

Let G be a group and S be a set. A **(right) action** of G on S is a function

$$\rho : G \rightarrow \{ \text{bijections } S \rightarrow S \} \quad (1.1.2)$$

that respects the group operation. That is, for all $g, h \in G$ we have

1. $(\rho(g) \circ \rho(h))(x) = (\rho(h \cdot g))(x)$ and
2. $(\rho(g^{-1}))(x) = (\rho(g))^{-1}(x)$.

In effect, what a group action does is assign to each **element** $g \in G$ an invertible **function** $\rho(g) : S \rightarrow S$ in such a way that the product of two elements $h \cdot g \in G$ is assigned to the composition of the functions $\rho(g) \circ \rho(h)$. The reversal of this order is what makes this a right action, and permits us to read a product of group elements as acting from left to right, for example, (twist, then twist, then rotate, then twist, then rotate

$$\begin{array}{c} TTRTR \\ \downarrow \\ r(t(r(t(t(x)))))) \end{array} \quad (1.1.3)$$

1.2.2 How the Tangle Group Operates

Rational Tangles

596.2b A Presentation of the Tangle Group

Conway's Rational Tangles Matthew Salomone



Operations on rational tangles

We can say about the twist T and rotation R :

- ① "Pure" twists can continue forever: $I \neq T \neq T^2 \neq T^3 \neq \dots$
- ② "Pure" rotation undoes itself: $R^2 = I$

A twist can be undone by more twists & rotations.

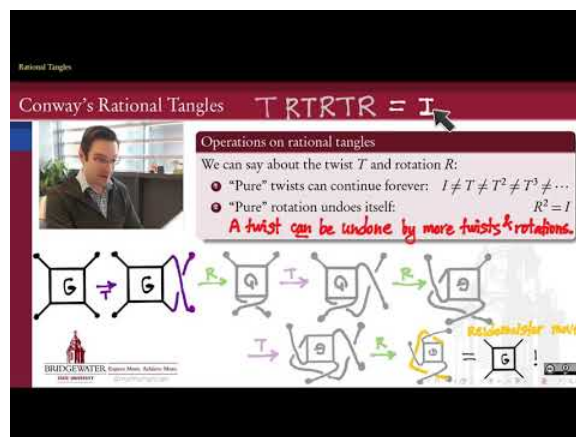


Redundant moves

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Determining the structure of the tangle group will help us to understand what rational tangles can look like after all the twisting and rotating is done. We can observe how each operation interacts with itself:

1. Iterated twists $T, T^2, T^3, T^4, \dots$ continue to add new crossings and change the tangle, so T is an element of *infinite order* in the tangle group.
2. Iterated rotations $R, R^2, R^3, R^4, \dots$ undo themselves. It's easy to see that since R is a quarter-turn, $R^4 = I$ will be the identity. What's more surprising, and will take until Section 1.3 to understand, is that for *rational* tangles, only two rotations are needed to return to the identity ($R^2 = I$). So R is an element of *order 2* in the tangle group.

The final step is to understand the *inverses* of each of these basic operations (**generators**) in the tangle group. What's clear from point (2) above is that R is its own inverse. What's less clear is how to "un-twist a twist," i.e. how to construct an inverse twist using only *more* of the same twist (combined with possibly some rotations). The video above does the best justice to the observation that

$$\underbrace{T R T R T R}_{T^{-1}} = I \quad (1.1.4)$$

and the discovery of this inverse completes our presentation of the tangle group.

Definition 1.2.2 Tangle group.

The **tangle group** Γ , defined as the set of all operations that carry one rational tangle into another, has the presentation

$$\Gamma = \langle T, R \mid R^2 = I = (TR)^3 \rangle. \quad (1.1.5)$$

When we attempt to understand this group more concretely, we might try, as in the video above, to examine its action on the set of four vertices of the tangle (the four people holding the ropes). This serves to represent each tangle group operation by the permutation that it induces on the four people holding the ropes. But it doesn't tell the whole story because, for example, the permutation induced by the twist T has infinite order in the tangle group (where successive twists always compound on each other) but order two in the permutation group (which only looks at the fact that the two people holding the twisting ropes have returned back to their starting places after two twists).

Rational Tangles

596.2c Discovering the Tangle Number Invariant

Conway's Rational Tangles

Matthew Salomone

A presentation of the tangle group

$$\Gamma = \langle T, R \mid R^2 = I = (TR)^3 \rangle$$

By its definition, Γ acts on the set of tangles.
What else could it act upon? What else might T, R transform?

How might these operations transform rational numbers?

Observation 1: T has infinite order just like 1 in $(\mathbb{Q}, +)$ does. $|1 \neq 1+1 \neq 1+1+1 \neq 1+1+1+1 \dots|$
So try $T \mapsto t(x) = x+1$.

Observation 2: R has order 2 so can't be simple addition.
What functions are equal to their own inverse function?

Watch on YouTube



Rational Tangles

Conway's Rational Tangles

Matthew Salomone

A presentation of the tangle group

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What functions are equal to their own inverse function?

What we ultimately want when we determine an action of the tangle group on the rational numbers is that the above situation *not* happen: we need the functions operating on the rational numbers to reflect *all* the structure revealed in the presentation of Definition 1.2.2. So what we need are functions $t: \mathbb{Q} \rightarrow \mathbb{Q}$ and $r: \mathbb{Q} \rightarrow \mathbb{Q}$ which satisfy all of the following.

1. t has infinite order, i.e. $t^n(x) = t(t(\dots t(x) \dots))$ must never equal the identity function $i(x) = x$ unless $n = 0$.
2. r has order two, i.e. $r^2(x) = r(r(x)) = i(x) = x$ for all $x \in \mathbb{Q}$.
3. $(r \circ t)$ has order three, i.e. $(r \circ t)^3(x) = r(t(r(t(r(t(x))))) = i(x) = x$ for all $x \in \mathbb{Q}$.

Beginning by positing that the simple "add one" function $t(x) = x + 1$ is a suitable choice to represent the infinite-order operation T , we then go in search of another function r which is an "involution" (i.e., it is equal to its own inverse function) and which also interacts with t in the manner specified by (3) above. We find that the opposite-reciprocal function miraculously fits the bill, and arrive at a right action of Γ on $\mathbb{Q}$ defined by

$$t(x) = x + 1 \quad r(x) = -\frac{1}{x}. \quad (1.1.6)$$

The video above concludes with an explanation of how this permits us to compute a rational number, called the **tangle number**, for a rational tangle once we assign to the empty tangle the number 0.

1.2.4 Exercises

Exercise 1.2.4.1.

Find an example of a knot diagram with an equal number of "over" crossings as "under" crossings, but the knot which it represents is *not* amphicheiral, i.e., $K \neq -K$.

Exercise 1.2.4.2.

(Adams, 2.12) Draw a sequence of pictures that show that the following two rational tangles are equivalent.

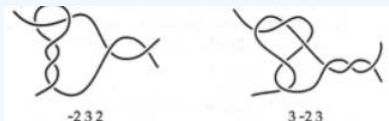


Figure 2.22 Two equivalent rational tangles.

Exercise 1.2.4.3.

Find an example of a 2-tangle which is *not* a rational tangle, and explain how you can tell that it is not rational. (Adams' text and the Kauffman/Lambropoulou paper are good places to start looking.)

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1.2: Tangle Arithmetic

So far, our strategy for understanding the structure of rational tangles has been to see them as constructed through a series of geometric operations, twists and rotations, applied to an empty tangle. But that doesn't give us a very friendly way of determining the tangle number for a rational tangle, nor (especially!) for going the opposite direction. Our next goal is to find a more systematic way of determining the tangle number of a rational tangle, which is also capable of -- just as systematically, we hope -- determining the rational tangle associated with a given rational number. This system is built on a new, less geometric, more arithmetic vision of how rational tangles may be defined.

1.3.1 Arithmetic, not Geometry



Rather than thinking about rational tangles as the result of applying the *geometric* operations of twisting and rotating (**T** and **R**) to an empty tangle, what we want instead is a way of thinking about these as operations of *arithmetic*. Doing that will require us to define what it means to:

1. Add two tangles, $G + H$, and
2. Multiply two tangles, $G * H$,

both in a way that aligns with our original geometric notions and which have the effects that we expect on tangle number.

Rational Tangles

596.3b Rational Tangles, Flips, and Flypes

Operations that Don't Matter: Matthew Salomone

Flips and Flypes

Flips don't (up to isotopy) change a rational tangle! [KL] Lem. 2

$G^v \simeq G$ *really?* $G^h \simeq G$ *are we sure?*

...because neither do flypes: We can all agree that (in 3 dimensions) these are isotopies:

vertical flype $[-1] + G^v + [1]$ horizontal flype $[-1] + G^h + [1]$

Corollaries: $[n] + G \simeq G + [n]$ $\frac{1}{[n]} * G \simeq G * \frac{1}{[n]}$

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Rational Tangles

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Operations that Don't Matter: Matthew Salomone

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Corollaries: $[n] + G \simeq G + [n]$ $\frac{1}{[n]} * G \simeq G * \frac{1}{[n]}$

Addition and multiplication are the tangle operations that "matter" in arithmetic. Indeed, we now define rational tangles as those tangles which can be built through addition and multiplication of the simple twists $[1]$ and $[-1]$ beginning from an empty tangle.

But since the addition and multiplication are not in general commutative on tangles (that is, it's not always true that $G + H \simeq H + G$ or that $G * H \simeq H * G$), we now introduce a new set of tangle operations that *don't* make a difference to arithmetic, but they *do* help us get around the lack of commutativity by permitting us to **flype** left- and top-twists to turn them all into right- and bottom-twists only.

Rational Tangles

596.3c Rational Tangles are Continued Fractions

Right it Down! Matthew Salomone The standard form of a rational tangle

∴ All rational tangles can be constructed in this standard form:

- 1 Begin with the horizontal empty tangle $[0]$
- 2 Add $[t_1]$ to the right $[0] + [t_1]$
- 3 Multiply by $1/[s_1]$ on the bottom $([0] + [t_1]) * \frac{1}{[s_1]}$
- 4 Then add $[t_2]$ on the right, multiply $1/[s_2]$ on bottom, ...

AND: Each of these steps has a predictable effect on tangle number!

Addition is addition... Adding a twist adds to # and [KL] Lemma 4 covers multiplication

$G + [n] = G$ $G * \frac{1}{[n]} \approx \frac{1}{[n] + \frac{1}{G}}$

bottom twists conjugate addition w/ reciprocal

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Rational Tangles

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Addition is addition... Adding a twist adds to # and [KL] Lemma 4 covers multiplication

$G + [n] = G$ $G * \frac{1}{[n]} \approx \frac{1}{[n] + \frac{1}{G}}$

bottom twists conjugate addition w/ reciprocal

Since we now know that all rational tangles can be built solely using right- and bottom-twists, we can determine the arithmetic effects of each of those basic operations. If the tangle number of G is $x \in \mathbb{Q}$, then

1. Making a twist on the right adds to the tangle number:

$$G + [n] \rightsquigarrow x + n \quad (1.2.1)$$

and

2. Making a twist on the bottom *conjugates* that addition with rotation, i.e. the reciprocal. In other words, making a bottom twist is the same as taking a reciprocal, then making a right twist (adding), then taking the reciprocal again:

$$G * \frac{1}{[n]} \approx \frac{1}{[n] + \frac{1}{G}} \rightsquigarrow \frac{1}{n + \frac{1}{x}} \quad (1.2.2)$$

This gives us the key result of this section: that rational tangles, as well as their tangle numbers, each have (identical) representations as continued fractions.

Theorem 1.3.1 Continued Fraction Theorem.

Let G be a rational tangle. Then G is equivalent (through flypes, isotopy moves, and/or Reidemeister moves) to a tangle that has a continued fraction representation

$$G = [[a_1], [a_2], [a_3], \dots, [a_n]] = [a_1] + \frac{1}{[a_2] + \frac{1}{[a_3] + \dots + \frac{1}{[a_n]}}} \quad (1.2.3)$$

and the tangle number (or **fraction**) of G has the same expression as a rational number:

$$F(G) = [a_1, a_2, a_3, \dots, a_n] = a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots + \frac{1}{a_n}}}. \quad (1.2.4)$$

Moreover, both continued fraction representations have a **canonical form** in which all the integers a_i have the same sign and the length n is odd.

(1.3.4) Using Continued Fractions to Convert Tangles and Numbers

596.3d Computing tangle numbers with continued fractions
Matthew Salomone

What's the tangle number?

- > Count twists from outermost to innermost alternating horizontal & vertical.
- ★ Combine left with right top with bottom because FLYPES!

$[-2, 3, 1, -3, 5]$

Figure 4 - A rational tangle in twist form

Watch on YouTube



What's the tangle number?

- > Count twists from outermost to innermost alternating horizontal & vertical.
- ★ Combine left with right top with bottom because FLYPES!

$[-2, 3, 1, -3, 5]$

Figure 4 - A rational tangle in twist form

Theorem 1.3.1 shows that the (canonical) continued fraction representation is our bridge between rational tangles and rational numbers. Combined with an algorithm that we can use to compute the canonical continued fraction representation for any $x \in \mathbb{Q}$, this gives us all we need to travel between worlds.

1.3.5 Exercises

Exercise 1.3.5.1.

Sketch the rational tangles T_1, T_2, T_3 whose tangle numbers (fractions) are $F(T_1) = 4/5$, $F(T_2) = -5/8$, and $F(T_3) = 9/2$.

Exercise 1.3.5.2.

For each of the following knots K , find and sketch a rational tangle T for which $K = N(T)$ is the numerator closure of T . Adams' sketch in the diagram below can be an inspiration:

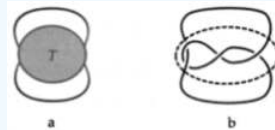


Figure 2.2.4 a) A rational link. b) The figure-eight knot.

- a. $K = 3_1$, the trefoil.
- b. $K = 4_1$, the figure-eight.
- c. $K = 6_3$.

Exercise 1.3.5.3.

Determine the tangle number for each of the tangles you found in Exercise 1.3.5.2.

Exercise 1.3.5.4.

Refer to the two equivalent tangles from Exercise 1.2.4.2, which Adams notates as $T_1 = -2/3 = [[2], [3], [-2]]$ and $T_2 = 3/2 = [[3], [-2], [3]]$

Verify that the tangle numbers (fractions) agree, i.e., that $F(T_1) = F(T_2)$.

Exercise 1.3.5.5.

([KL] Lem. 5.3 and Lem. 6.3) Let $\frac{p}{q} = [a_1, a_2, a_3, \dots, a_n]$ be a rational number written as a continued fraction.

- a. Prove that $[0, a_1, a_2, a_3, \dots, a_n] = \frac{q}{p}$ represents its reciprocal.
- b. What relationship exists between the rational tangles

$$T_1 = [[a_1], [a_2], [a_3], \dots, [a_n]] \quad \text{and} \quad T_2 = [[0], [a_1], [a_2], [a_3], \dots, [a_n]]? \quad (1.2.5)$$

Explain the relationship using pictures, and in words explain why this relationship is not surprising.

CHAPTER OVERVIEW

2: Knots and Numbers

Learning Objectives

- Determine and contrast several ways to notate (tabulate) a knot using its crossings.
- Use numerical invariants for knots including the unknotting, bridge, and crossing numbers, to investigate relationships among classes of knots.

Studying rational tangles was a way to focus in a limited fashion on how crossings interact with one another to build intricate local structures that define a knot. But as an invariant for knots, the tangle number isn't perfect: it's most useful for rational knots, and even then, it can be challenging to rearrange a knot diagram into a twist-form rational tangle.

What we'd like instead are more *global* invariants that work for knots, invariants that capture the whole structure of the topology without relying upon making a specific set of choices along the way. This will come at the cost of needing invariants capable of conveying more algebraic information than a single rational number does: polynomials on one hand, and algebraic groups on the other.

[2.1: Notations for Knots](#)

[2.2: Numerical Knot Invariants](#)

[2.3: Colorings of Knots](#)

[2.4: Colorings of Tangles](#)

References

4. Adams, C. C. (2004). The Knot Book: an elementary introduction to the mathematical theory of knots. American Mathematical Society, ISBN 0-8218-3678-1. Chapters 2 and 3.
5. Austin, D. (2016). Knot quandaries quelled by quandles. American Mathematical Society Feature Column, accessed at <http://www.ams.org/publicoutreach/feature-column/fc-2016-03>.
6. Portnoy, N. and Mattman, T. (Undated). Knot Theory for Preservice and Practicing Secondary Mathematics Teachers. Accessed at <http://www.csuchico.edu/math/mattman/NSF.html>.
7. Rolfsen, D. (1990). Knots and Links. Corrected reprint of the 1976 original. Mathematics Lecture Series (7). American Mathematical Society. Chapter 3

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2.1: Notations for Knots

Rational Tangles
Knots and Knot Invariants

596.4a Numerical Knot Invariants: Intro

Matthew Salomone Numerical Knot Invariants





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Dr. Matt Salomone

[@mattsalomone](#)

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(2.2.1) Conway's notation

Rational Tangles
Knots and Knot Invariants

596.4b Conway's Notation for Rational Knots

Making the Tangle Number a Knot Invariant: Conway's Notation



Definition of Rational Knot

A knot K is called a **rational knot** if there exists a rational tangle T such that $K = N(T)$ is the “numerator closure” of T :



Theorem: For a given rational knot K , this tangle is well defined: $N(T_1) \simeq N(T_2) \Rightarrow T_1 \simeq T_2$.



T_1



T_2




596.4c Dowker's Notation for Knots

Matthew Salomone

Conway's notation tells us how twists are "adjacent" in a knot.
Ex: $7_3 = 43$

Dowker's Notation
Traverse any knot, (twice) counting each crossing. List pairs.

Reconstruct an alternating projection

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2.2.2 Dowker's notation

596.4c Dowker's Notation for Knots

Matthew Salomone

Conway's notation tells us how twists are "adjacent" in a knot.
Ex: $7_3 = 43$

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2.2: Numerical Knot Invariants

(2.3.1) What is a Prime Knot?

Rational Tangles

Knots and Knot Invariants

596.4d Prime and Composite Knots

Matthew Salomone

Connected Sums and Prime Knots

Definition: Connected sum of knots

If K_1 and K_2 are knots, their connected sum $K_1 \# K_2$ is formed by

Compare: Adding tangles.

Remove small arc from each

Join the cut ends

This knot is clearly composite.

Definition: Prime and Composite Knots

A knot K is **composite** if there exist nontrivial K_1, K_2 such that $K = K_1 \# K_2$.

A knot K is **prime** if it is not composite.

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Rational Tangles

Knots and Knot Invariants

Connected Sums and Prime Knots

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2.3.2 The Unknotting Number

Rational Tangles

Knots and Knot Invariants

596.4e Three Numerical Knot Invariants

Matthew Salomone

Other Numerical Knot Invariants: Bridge Number

Definition: Bridge number $b(K)$

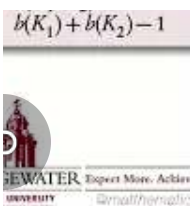
The bridge number $b(K)$ of a knot K is the minimal number of unknotted arcs on either side of a plane bisecting K .

Arcs above

Some results

- K is rational $\Rightarrow b(K) = 2$
- $b(K_1 \# K_2) = \max\{b(K_1), b(K_2)\}$

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Rational Tangles

Knots and Knot Invariants

Other Numerical Knot Invariants: Bridge Number

Definition: Bridge number $b(K)$

The bridge number $b(K)$ of a knot K is the minimal number of unknotted arcs on either side of a plane bisecting K .

Some results

- 1. K is rational $\Rightarrow b(K) = 2$
- 2. $b(K_1 \# K_2) = b(K_1) + b(K_2) - 1$

Diagram: A knot 3_1 is shown bisected by a horizontal plane. Two arcs, labeled 1 and 2, are shown on either side of the plane. The text "Arcs above" and "Arcs below" is written. Below the diagram, it says $b(3_1) = 2.$ and B .

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2.3.3 The Bridge Number

Rational Tangles

Knots and Knot Invariants

596.4e Three Numerical Knot Invariants

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Other Numerical Knot Invariants: Bridge Number

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Rational Tangles

Knots and Knot Invariants

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Rational Tangles
Knots and Knot Invariants

596.4e Three Numerical Knot Invariants

Other Numerical Knot Invariants: Matthew Salomone
Bridge Number



Some results

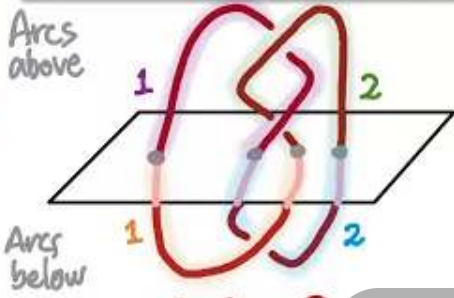
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Definition: Bridge number $b(K)$

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$b(3_1) = 2.$

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Rational Tangles
Knots and Knot Invariants

Other Numerical Knot Invariants: Bridge Number



Some results

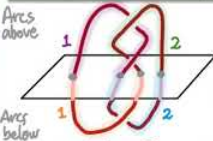
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$b(3_1) = 2.$

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2.3: Colorings of Knots

Knots and Knot Invariants

596.6 Introduction to Knot Coloring

Matthew Salomone





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2.4.1 Colorability as an Invariant

Knots and Knot Invariants

596.6b k-Colorability is a Knot Invariant

Matthew Salomone



Definition: A k -coloring of a knot diagram

Let K be a knot represented by a diagram with arcs $X = \{x_1, x_2, x_3, \dots, x_d\}$. Then a k -coloring of K is a function

$$\chi : X \rightarrow \{0, 1, 2, \dots, k-1\}$$

Colors $\leftrightarrow$ Integers

for which each crossing of three arcs x_i, x_j, x_k satisfies either:

① $\chi(x_i) = \chi(x_j) = \chi(x_k)$

or

② $\chi(x_i), \chi(x_j), \chi(x_k)$ are all distinct.

Can we 3-color?



What Does It Mean to "Color" a Knot?

Definition: A k -coloring of a knot diagram
 Let K be a knot represented by a diagram with arcs $X = \{x_1, x_2, x_3, \dots, x_d\}$. Then a k -coloring of K is a function $\chi: X \rightarrow \{0, 1, 2, \dots, k-1\}$ for which each crossing of three arcs x_i, x_j, x_k satisfies either:
 $\chi(x_i) = \chi(x_j) = \chi(x_k)$ or $\chi(x_i), \chi(x_j), \chi(x_k)$ are all distinct.

Colors $\leftrightarrow$ Integers

Can we 3-color this knot? $\{0, 1, 2\}$

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2.4.2 Colorability as an Algebra Problem

596.6c Tricolorability's "Magic Equation"

How To Detect Whether a Knot is 3-Colorable?

Matthew Salomone

The "magic equation" for three colors $\{0, 1, 2\}$

True for $\{x, y, z\} =$			False for $\{x, y, z\} =$		
0	0	0	0	0	1
1	1	1	1	1	0
2	2	2	2	2	1
0	1	2	0	1	2
1	2	0	1	2	0
2	0	1	2	0	1

$x + y + z \equiv 0 \pmod 3$

K is tricolorable if: at every crossing, $x + y + z \equiv 0 \pmod 3$

multiple of 3

not multiple of 3

Crossing

Equ

Watch on YouTube



How To Detect Whether a Knot is 3-Colorable?

The "magic equation" for three colors $\{0, 1, 2\}$

True for $\{x, y, z\} =$			False for $\{x, y, z\} =$		
0	0	0	0	0	1
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Rational Tangles
Knots and Knot Invariants

596.6d Tricoloration of an 8-Crossing Knot

Tricolorability by Brute Force:
An Algorithm



Tricolorability $\Leftrightarrow$ A system of $(\mathbb{Z}_3)$ -linear equations

Each solution of this system of equations in $\mathbb{Z}_3 = \{0, 1, 2\}$ gives a valid coloring... but, does it use all three colors?

Is 8_{18} tricolorable?

Check the 8×8 system:

1	1	0	1	1	1	0	0
1	1	0	0	0	1	1	1
1	1	0	0	0	1	1	1
1	1	0	0	0	1	1	1
1	1	0	0	0	1	1	1
1	1	0	0	0	1	1	1
1	1	0	0	0	1	1	1
1	1	0	0	0	1	1	1

Crossing Equation

A	x_1	$+x_6+x_7$	$\equiv 0$
B	$x_1+x_2+x_4$		$\equiv 0$
C	x_2	$+x_7+x_9$	$\equiv 0$
D	$x_2+x_3+x_5$		$\equiv 0$
E	$x_2+x_4+x_6$		$\equiv 0$
F	$x_4+x_5+x_7$		$\equiv 0$
G	$x_5+x_6+x_9$		$\equiv 0$
H	x_1+x_3	$+x_9$	$\equiv 0$



N free variables $\Leftrightarrow 3^N$ tricolorings

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Rational Tangles
Knots and Knot Invariants

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N free variables $\Leftrightarrow 3^N$ tricolorings

Rational Tangles

Knots and Knot Invariants

596.6e Using k -colorability to Distinguish Knots

Beyond Tricolorable: Matthew Salomone

How Does k -Colorability Distinguish Knots?

When $k > 3$ we lose the "magic equation" *But, gain a NEW one!*

$$x + y + z \equiv 0 \pmod{3} \xrightarrow{\text{Solve for } z} z \equiv -x - y \xrightarrow{-1 \equiv 2} z \equiv -x + 2y \pmod{3}$$

Tricolorable $\begin{cases} k=3 \\ t=2 \end{cases}$ $z \equiv t \cdot x + (1-t) \cdot y \pmod{k}$ *Require k, t CO-PRIME*

$k=3, t=2$ (Honest 3-coloring):
 $k=5, t=4$ (Like a 5-coloring):
 $z \equiv -x + 2y \pmod{5}$

1 free variable (trivial) 1 free variable (trivial)

One disagreement $\Rightarrow$ different knots!

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Rational Tangles

Knots and Knot Invariants

Beyond Tricolorable: How Does k -Colorability Distinguish Knots?

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2.4: Colorings of Tangles

2.5.1 An Integer Coloring Scheme for Rational Tangles

596.7a Integer Colorings for Rational Tangles
Matthew Salomone

Tangle coloring: an infinite box of crayons

Tricoloring: over $\mathbb{Z}_3$ $\uparrow$ $\text{mod } 3$ $\downarrow$ Tangle coloring: over $\mathbb{Z}$

$z \equiv -x + 2y \pmod{3}$ Limit 3 colors $z = -x + 2y$

one is different $\Leftrightarrow$ All three are different

Over = avg of unders
Choosing two strands defines the rest.
Check well defined!

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596.7a Integer Colorings for Rational Tangles
Matthew Salomone

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2.5.2 Coloring as a Rational Tangle Invariant

596.7b Integer Coloration as a Rational Tangle Invariant
Matthew Salomone

A Coloring Invariant for Rational Tangles : *Proof by induction.* [KL] §5

Properties of the color matrix of a rational tangle G

① *Diagonal sum rule:*
We observed $b - a = d - c \Leftrightarrow b = a + d - c \Leftrightarrow b + c = a + d$
 $c - a = d - b \Leftrightarrow b + c - a = d \Leftrightarrow b + c = a + d$

$a + d = b + c$
 $b + c = a + d$
 $b + c = a + d$

Proof: Base cases

$x + y = x + y$ $x + y = x + y$ $2x = 2x$ $2y = 2y$ $2x - y = 2x - y$

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CHAPTER OVERVIEW

3: Knots and Algebra

Learning Objectives

- Calculate algebraic structures that are invariants for knots, including the fundamental quandle and the knot group, and use these to distinguish among knots.
- Calculate polynomials that are invariants for knots, including the Alexander, HOMFLY, and Jones polynomials, and use these to distinguish among knots.

Numerical invariants such as we studied in Chapter 2 are powerful, especially insofar as they can help us to discover relationships among different classes of knots. But while the Conway notation comes closest, none of these numerical invariants can carry enough information about the type of a knot to permit us to *surely* distinguish one knot from another. While they are sensitive (different numbers imply different knots), they are far from specific (same number does *not* imply same knot).

In our search for a **complete invariant** for knots, then, we need a structure capable of holding more information than can a single number. In this culminating chapter we'll work our way through algebraic structures of increasing complexity, each of which adds much-needed specificity. The rogue's gallery of structures range from the familiar (polynomials) to the recognizable (groups) to the exotic (quandles), each with its own advantages and limitations. But reckoning with how each one encodes information about a knot is the key to understanding the role of algebraic thinking in knot theory in particular, and in mathematics more generally.

[3.1: Algebraic Structures](#)

[3.2: Colorings and Quandles](#)

[3.3: Knot Groups](#)

[3.4: Invariant Polynomials](#)

[3.5: Portfolio Problems](#)

References

8. Adams, C. C. (2004). *The Knot Book: an elementary introduction to the mathematical theory of knots*. American Mathematical Society, ISBN 0-8218-3678-1. Chapter 5.
9. Austin, D. (2016). Knot quandaries quelled by quandles. American Mathematical Society Feature Column, accessed at <http://www.ams.org/publicoutreach/feature-column/fc-2016-03>.
10. Rolfsen, D. (1990). *Knots and Links*. Corrected reprint of the 1976 original. Mathematics Lecture Series (7). American Mathematical Society. Chapter 3.

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3.1: Algebraic Structures

3.2.1 The Familiar: Groups, Rings, Fields

{

3.2.2 The Less-Familiar: Quandles

{

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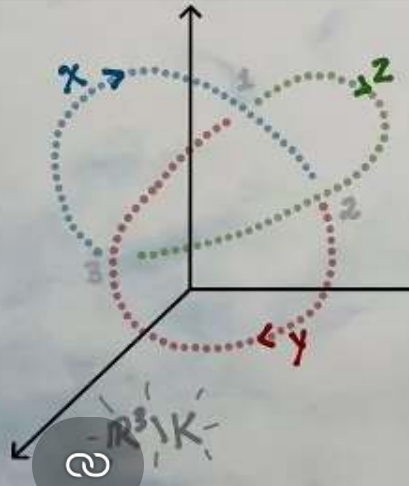
3.2: Colorings and Quandles

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3.3: Knot Groups

596.10 A Presentation for the Knot Group
Matthew Salomone

Knot Group $G(K) = \pi_1(\mathbb{R}^3 \setminus K)$ ← is the fundamental group of the knot's COMPLEMENT in space.



Fundamental Quandle

Arcs
x y z

Crossing relation ▸

- 1 $z = y \triangleright x$
- 2 $y = x \triangleright z$
- 3 $x = z \triangleright y$

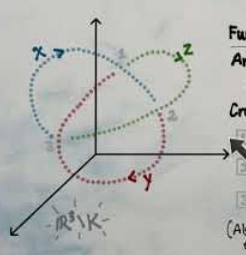
(Also includes other elements like $x \triangleright y$, etc.)

Knot Group

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Fundamental Quandle

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- 3 $x = z \triangleright y$

(Also includes other elements like $x \triangleright y$, etc.)

Knot Group

596.10 Simplifying the trefoil knot group

Matthew Salomone

$\pi_1(\mathbb{R}^3 \setminus K) = \langle a, b, c \mid c = \bar{a}ba, b = \bar{a}a\bar{c}, a = \bar{b}cb \rangle$

$= \langle a, b \mid bab = aba \rangle$

$(ababab = (ba)(baa))$

$= \langle p, q \mid p^3 = q^2 \rangle$

★ Substitute one relation into other(s)

Can we do better?

Play around with the relation.

If $p = ab$
 $q = ba^2$

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Simplifying Wirtinger presentations

$\pi_1(\mathbb{R}^3 \setminus K) = \langle a, b, c \mid c = \bar{a}ba, b = \bar{a}a\bar{c}, a = \bar{b}cb \rangle$

$= \langle a, b \mid bab = aba \rangle$

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If $p = ab$
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3.4: Invariant Polynomials

3.5.1 The Alexander Polynomial

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3.5: Portfolio Problems

Writeups of the following problems taken from our Class Outlines 2 will constitute the Algebra Portfolio for this semester.

\(3.6.1\) Exercises

Exercise 3.6.1.1.

Use a presentation matrix for the figure-eight knot 4_1 to determine its Alexander polynomial. Then, repeat the process using a *different* 3×3 minor of the matrix; how does this change your process? Your answer?

Exercise 3.6.1.2.

Following the trefoil example from class, determine the full set of elements in the fundamental kei $\mathcal{K}(4_1)$ of the figure-eight knot, and a table of operations for this kei. (Note that unlike the trefoil, not all elements of this kei will be arcs in the knot diagram.)

Exercise 3.6.1.3.

Thinking about the definition of the knot group, make a conjecture and explain your reasoning: *The knot group of the unknot is...* (Hint: using the simplest diagram of the unknot is probably enough to get a good idea.)

Exercise 3.6.1.4.

Determine the (Wirtinger presentation of the) knot group of the figure-eight knot. Then, try to simplify this presentation so that it contains only *two* generators.

Hint

In the Wirtinger presentation, one relation is *always* redundant with the others. So, begin by erasing any one relation; then, try to simplify those that remain.

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CHAPTER OVERVIEW

4: Workshop Project

4.1: Project Overview

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4.1: Project Overview

Learning Objectives

- Understand algebraic thinking by identifying it in a wide variety of contexts.
- Contrast several models of teacher professional development, including the Math Teachers Circle model.
- Design resources for, and facilitate, a workshop for in-service teachers on knot theory and algebraic thinking.

A major theme of this course is the discovery of “familiar” arithmetic and algebra in unfamiliar places. These explorations are meant to illuminate that algebra is far more than a set of procedures suitable for simplifying and solving equations involving numbers; algebra is a name for fundamental *structures* and *ways of reasoning* that can be found in places where numbers aren't even present.

This, principally, highlights both a connection and a distinction between quantitative reasoning, which emphasizes concrete, contextual, and critical uses of number skills, and mathematical reasoning, which exposes the abstract, fundamental, and logical principles of which the former is one special case. Quantitative reasoning makes these skills practical; mathematical reasoning makes them durable.

Our semester project will be to design and deliver a professional development workshop for elementary and secondary math educators, designed to help them (and by extension their students) discover what algebraic thinking is, through activities using tangles, knots, links, and/or braids. The materials developed for this project will be shared here, and after testing them, I hope we can share them more widely through professional development networks for math educators.

References

- [1] Ernst, D. and Hodge, A. (2014). *Math Teachers' Circles: What Makes a Good One?* In *Math Ed Matters*, Mathematical Association of America blog, retrieved at <https://maamathedmatters.blogspot.com/2014/11/math-teachers-circles-what-makes-good.html>.
- [2] Zucker, J. (2012). *Be Less Helpful*. In MTCircular, American Institute of Mathematics, Autumn 2012, pp. 4-7. Retrieved at http://www.mathteacherscircle.org/assets/toolkits/beginning/mathematics/BeLessHelpful_MTCircularAutumn2012.pdf.

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CHAPTER OVERVIEW

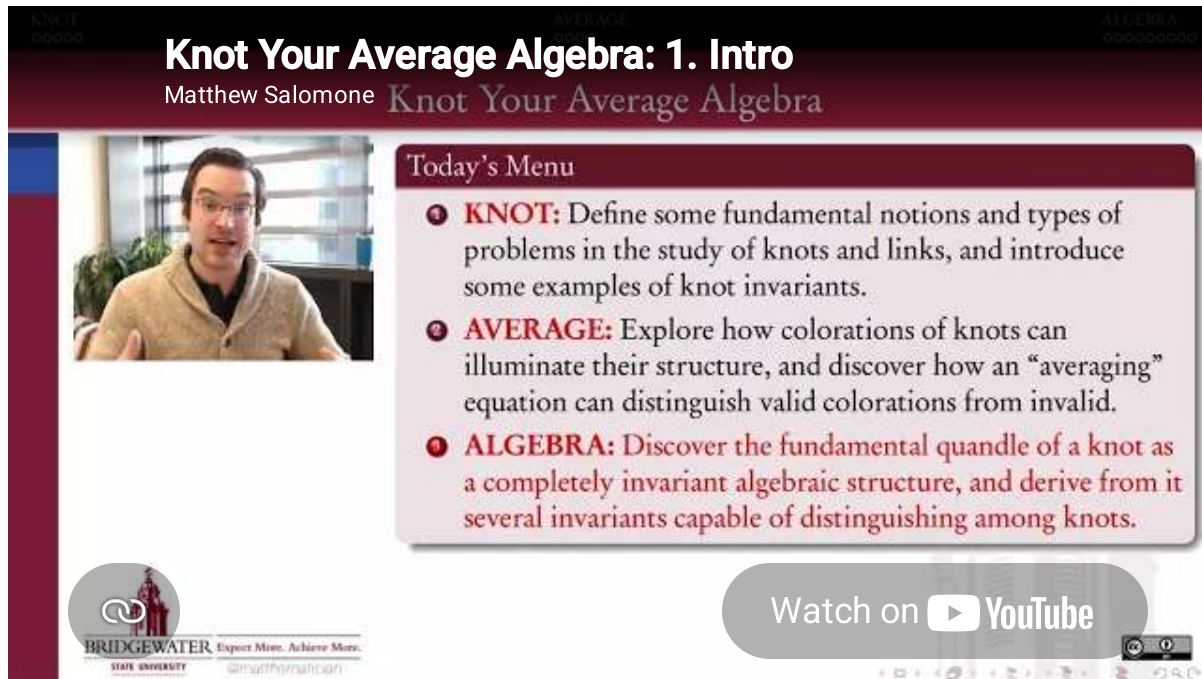
5: Knot Your Average Algebra

5.1: Seminar Videos

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5.1: Seminar Videos

The following are videos of "Knot Your Average Algebra," presented at the Bridgewater State University mathematics department seminar on March 22, 2018.

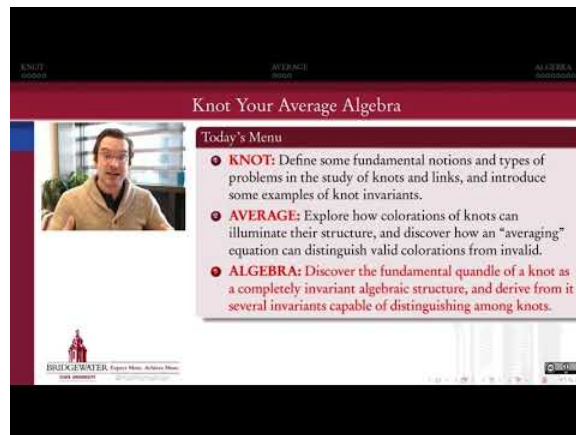


Knot Your Average Algebra: 1. Intro
Matthew Salomone

Today's Menu

- 1 **KNOT:** Define some fundamental notions and types of problems in the study of knots and links, and introduce some examples of knot invariants.
- 2 **AVERAGE:** Explore how colorations of knots can illuminate their structure, and discover how an "averaging" equation can distinguish valid colorations from invalid.
- 3 **ALGEBRA:** Discover the fundamental quandle of a knot as a completely invariant algebraic structure, and derive from it several invariants capable of distinguishing among knots.

Watch on  YouTube

KNOT
000000
AVERAGE
000000
ALGEBRA
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Knot Your Average Algebra: 2. Knots and Invariants

Matthew Salomone

The Mixed Blessing of Knot Diagrams



Opposite height function
 $\Rightarrow -K$
 reverse all crossings

Alternating diagram
 $\Rightarrow$

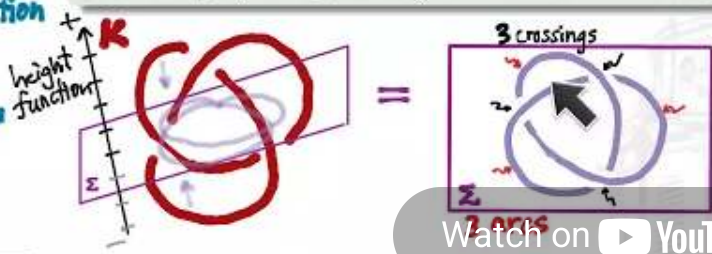


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Definition: Knot Diagrams, Crossings, and Arcs

Let K be a knot, and fix a hyperplane $\Sigma \subset \mathbb{R}^3$. The knot diagram of K on Σ is the projection of the image of K onto Σ .

A crossing is a point on a knot diagram with two pre-images under the projection, labeling over- and under-crossing arcs.



Watch on  YouTube



KNOT
000000
AVERAGE
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ALGEBRA
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The Mixed Blessing of Knot Diagrams



Opposite height function
 $\Rightarrow -K$
 reverse all crossings

Alternating diagram
 $\Rightarrow$

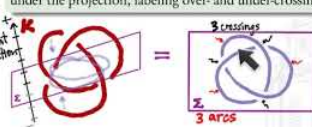


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KNOT THEORY
AVERAGE
ALGEBRA

Knot Your Average Algebra: 3. Average

Matthew Salomone

Colorability is a Knot Invariant



Hope:
Colorability is
THE (complete!)
knot invariant.

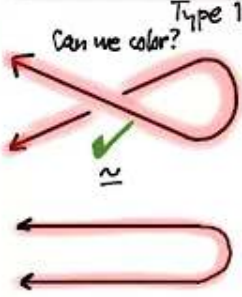


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Theorem: Colorability doesn't depend on diagram

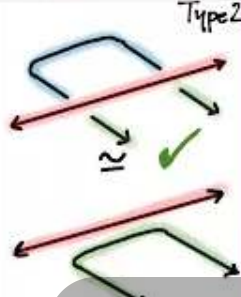
Let K be a knot and D_1, D_2 be two diagrams of K . Then D_1 is k -colorable if and only if D_2 is k -colorable.

Type 1

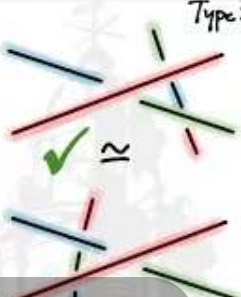


Can we color?

Type 2



Type 3



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Knot Your Average Algebra: 3. Average

Matthew Salomone

Colorability is a Knot Invariant



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Colorability is
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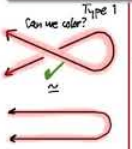


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Theorem: Colorability doesn't depend on diagram

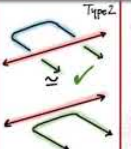
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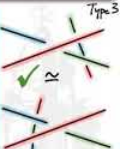


Can we color?

Type 2



Type 3



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Quandles and Related Structures

ALGEBRA

Knot Your Average Algebra: 4b. Algebra Examples

Matthew Salomone

...Determine the Knot Group $\pi_1(S^3 \setminus K)$

Deducing the knot group: Wirtinger presentation

There is a functor $\mathcal{W} : \{\text{f.p. quandles}\} \rightarrow \{\text{f.p. groups}\}$ that realizes the quandle operation as conjugation of group elements:

$$x \triangleright y \xrightarrow{\mathcal{W}} y^{-1}xy.$$

$xy = yz$ in $\pi_1(K^c)$.

$\frac{\ln \mathcal{Q}(K)}{Z = x \triangleright y} \mapsto \frac{\ln \pi_1(S^3 \setminus K)}{Z = y^{-1}xy}$

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Theorem: For a knot K , we have $\mathcal{W}(\mathcal{Q}(K)) = \pi_1(S^3 \setminus K)$.



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Quandles and Related Structures

ALGEBRA

Knot Your Average Algebra: 4b. Algebra Examples

Matthew Salomone

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Theorem: For a knot K , we have $\mathcal{W}(\mathcal{Q}(K)) = \pi_1(S^3 \setminus K)$.

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 - [1: Rational Tangles](#) - *GNU Free Documentation License*
 - [1.1: Tangle Basics](#) - *GNU Free Documentation License*
 - [1.2: Tangle Arithmetic](#) - *GNU Free Documentation License*
 - [2: Knots and Numbers](#) - *GNU Free Documentation License*
 - [2.1: Notations for Knots](#) - *GNU Free Documentation License*
 - [2.2: Numerical Knot Invariants](#) - *GNU Free Documentation License*
 - [2.3: Colorings of Knots](#) - *GNU Free Documentation License*
 - [2.4: Colorings of Tangles](#) - *GNU Free Documentation License*
 - [3: Knots and Algebra](#) - *GNU Free Documentation License*
 - [3.1: Algebraic Structures](#) - *GNU Free Documentation License*
 - [3.2: Colorings and Quandles](#) - *GNU Free Documentation License*
 - [3.3: Knot Groups](#) - *GNU Free Documentation License*
 - [3.4: Invariant Polynomials](#) - *GNU Free Documentation License*
 - [3.5: Portfolio Problems](#) - *GNU Free Documentation License*
 - [4: Workshop Project](#) - *Undeclared*
 - [4.1: Project Overview](#) - *Undeclared*
 - [5: Knot Your Average Algebra](#) - *Undeclared*
 - [5.1: Seminar Videos](#) - *Undeclared*
 - [Back Matter](#) - *Undeclared*
 - [Index](#) - *Undeclared*
 - [Glossary](#) - *Undeclared*
 - [GNU Free Documentation License](#) - *Undeclared*
 - [Student Blogs](#) - *Undeclared*
 - [Detailed Licensing](#) - *Undeclared*