INTERACTIVE CALCULUS Q4



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Licensing

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CHAPTER OVERVIEW

1: Vector-Valued Functions

A vector-valued function, also referred to as a vector function, is a mathematical function of one or more variables whose range is a set of multidimensional vectors or infinite-dimensional vectors. The input of a vector-valued function could be a scalar or a vector. Vector-valued functions provide a useful method for studying various curves both in the plane and in three-dimensional space. We can apply this concept to calculate the velocity, acceleration, arc length, and curvature of an object's trajectory. In this chapter, we examine these methods and show how they are used.

1.1: Prelude to Vector-Valued Functions

1.2: Vector-Valued Functions and Space Curves

1.2E: Exercises

1.3: Calculus of Vector-Valued Functions

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1.1: Prelude to Vector-Valued Functions

In 1705, using Sir Isaac Newton's new laws of motion, the astronomer Edmond Halley made a prediction. He stated that comets that had appeared in 1531, 1607, and 1682 were actually the same comet and that it would reappear in 1758. Halley was proved to be correct, although he did not live to see it. However, the comet was later named in his honor. Halley's Comet follows an elliptical path through the solar system, with the Sun appearing at one focus of the ellipse. This motion is predicted by Johannes Kepler's first law of planetary motion, which we mentioned briefly previously. Kepler's third law of planetary motion can be used with the calculus of vector-valued functions to find the average distance of Halley's Comet from the Sun.



Figure **1.1.1**: Halley's Comet appeared in view of Earth in 1986 and will appear again in 2061.

Vector-valued functions provide a useful method for studying various curves both in the plane and in three-dimensional space. We can apply this concept to calculate the velocity, acceleration, arc length, and curvature of an object's trajectory. In this chapter, we examine these methods and show how they are used.

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1.2: Vector-Valued Functions and Space Curves

Learning Objectives

- Write the general equation of a vector-valued function in component form and unit-vector form.
- Recognize parametric equations for a space curve.
- Describe the shape of a helix and write its equation.
- Define the limit of a vector-valued function.

Our study of vector-valued functions combines ideas from our earlier examination of single-variable calculus with our description of vectors in three dimensions from the preceding chapter. In this section, we extend concepts from earlier chapters and also examine new ideas concerning curves in three-dimensional space. These definitions and theorems support the presentation of material in the rest of this chapter and also in the remaining chapters of the text.

Definition of a Vector-Valued Function

Our first step in studying the calculus of vector-valued functions is to define what exactly a vector-valued function is. We can then look at graphs of vector-valued functions and see how they define curves in both two and three dimensions.

Definition: Vector-valued Functions

A vector-valued function is a function of the form

$$\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} \quad \text{or} \quad \vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} + h(t)\,\hat{\mathbf{k}}, \tag{1.2.1}$$

where the component functions f, g, and h, are real-valued functions of the parameter t. Vector-valued functions are also written in the form

$$\vec{\mathbf{r}}(t) = \langle f(t),\, g(t)\rangle \quad \text{or} \quad \vec{\mathbf{r}}(t) = \langle f(t),\, g(t),\, h(t)\rangle. \tag{1.2.2}$$

In both cases, the first form of the function defines a two-dimensional vector-valued function; the second form describes a three-dimensional vector-valued function.

The parameter t can lie between two real numbers: $a \le t \le b$. Another possibility is that the value of t might take on all real numbers. Last, the component functions themselves may have domain restrictions that enforce restrictions on the value of t. We often use t as a parameter because t can represent time.

Example 1.2.2: Finding the domain of a vector-valued function

State the domain of the vector-valued function $\vec{\mathbf{r}}(t) = \sqrt{2-t}\,\hat{\mathbf{i}} + \ln(t+3)\,\hat{\mathbf{j}} + e^t\,\hat{\mathbf{k}}$.

Solution

We first consider the natural domain of each component function. Note that we list the domains in both **set-builder notation** and **interval notation**.

Function:	**Domain:**
$\sqrt{2-t}$	$\{t \mid t \le 2\}$ or $(-\infty, 2]$
$\ln(t+3)$	$\{t \mid t > -3\}$ or $(-3, \infty)$
e^t	$(-\infty, \infty)$

The domain of $\vec{\mathbf{r}}$ is the intersection of these domains, so it must contain all values of t that work in all three, but no value of t that does not work in any one of these functions.

Hence, the domain of $\vec{\mathbf{r}}$ is: $D_{\vec{\mathbf{r}}} : \{t \mid -3 < t \le 2\}$ or $(-3, 2]$.

Note that only one form of the domain of $\vec{\mathbf{r}}$ need be given. The first, $\{t \mid -3 < t \le 2\}$, is in ***set-builder notation***, while the second, $(-3, 2]$, is in ***interval notation***.

Example 1.2.2: Evaluating Vector-Valued Functions and Determining Domains

For each of the following vector-valued functions, evaluate $\vec{\mathbf{r}}(0)$, $\vec{\mathbf{r}}\left(\frac{\pi}{2}\right)$, and $\vec{\mathbf{r}}\left(\frac{2\pi}{3}\right)$. Do any of these functions have domain restrictions?

1. $\vec{\mathbf{r}}(t) = 4\cos t\,\hat{\mathbf{i}} + 3\sin t\,\hat{\mathbf{j}}$
2. $\vec{\mathbf{r}}(t) = 3\tan t\,\hat{\mathbf{i}} + 4\sec t\,\hat{\mathbf{j}} + 5t\,\hat{\mathbf{k}}$

Solution

1. To calculate each of the function values, substitute the appropriate value of t into the function:

$$\begin{aligned}
\vec{\mathbf{r}}(0) &= 4\cos(0)\hat{\mathbf{i}} + 3\sin(0)\hat{\mathbf{j}} \\
&= 4\hat{\mathbf{i}} + 0\hat{\mathbf{j}} = 4\hat{\mathbf{i}} \\
\vec{\mathbf{r}}\left(\frac{\pi}{2}\right) &= 4\cos\left(\frac{\pi}{2}\right)\hat{\mathbf{i}} + 3\sin\left(\frac{\pi}{2}\right)\hat{\mathbf{j}} \\
&= 0\hat{\mathbf{i}} + 3\hat{\mathbf{j}} = 3\hat{\mathbf{j}} \\
\vec{\mathbf{r}}\left(\frac{2\pi}{3}\right) &= 4\cos\left(\frac{2\pi}{3}\right)\hat{\mathbf{i}} + 3\sin\left(\frac{2\pi}{3}\right)\hat{\mathbf{j}} \\
&= 4\left(-\tfrac{1}{2}\right)\hat{\mathbf{i}} + 3\left(\tfrac{\sqrt{3}}{2}\right)\hat{\mathbf{j}} = -2\hat{\mathbf{i}} + \tfrac{3\sqrt{3}}{2}\hat{\mathbf{j}}
\end{aligned}$$

   To determine whether this function has any domain restrictions, consider the component functions separately. The first component function is $f(t) = 4\cos t$ and the second component function is $g(t) = 3\sin t$. Neither of these functions has a domain restriction, so the domain of $\vec{\mathbf{r}}(t) = 4\cos t\,\hat{\mathbf{i}} + 3\sin t\,\hat{\mathbf{j}}$ is all real numbers.

2. To calculate each of the function values, substitute the appropriate value of t into the function:

$$\begin{aligned}
\vec{\mathbf{r}}(0) &= 3\tan(0)\hat{\mathbf{i}} + 4\sec(0)\hat{\mathbf{j}} + 5(0)\hat{\mathbf{k}} \\
&= 0\hat{\mathbf{i}} + 4j + 0\hat{\mathbf{k}} = 4\hat{\mathbf{j}} \\
\vec{\mathbf{r}}\left(\frac{\pi}{2}\right) &= 3\tan\left(\frac{\pi}{2}\right)\hat{\mathbf{i}} + 4\sec\left(\frac{\pi}{2}\right)\hat{\mathbf{j}} + 5\left(\frac{\pi}{2}\right)\hat{\mathbf{k}}, \text{ which does not exist} \\
\vec{\mathbf{r}}\left(\frac{2\pi}{3}\right) &= 3\tan\left(\frac{2\pi}{3}\right)\hat{\mathbf{i}} + 4\sec\left(\frac{2\pi}{3}\right)\hat{\mathbf{j}} + 5\left(\frac{2\pi}{3}\right)\hat{\mathbf{k}} \\
&= 3(-\sqrt{3})\hat{\mathbf{i}} + 4(-2)\hat{\mathbf{j}} + \frac{10\pi}{3}\hat{\mathbf{k}} \\
&= (-3\sqrt{3})\hat{\mathbf{i}} - 8\hat{\mathbf{j}} + \frac{10\pi}{3}\hat{\mathbf{k}}
\end{aligned}$$

   To determine whether this function has any domain restrictions, consider the component functions separately. The first component function is $f(t) = 3\tan t$, the second component function is $g(t) = 4\sec t$, and the third component function is $h(t) = 5t$. The first two functions are not defined for odd multiples of $\frac{\pi}{2}$, so the function is not defined for odd multiples of $\frac{\pi}{2}$. Therefore,

$$\mathrm{D}_{\vec{\mathbf{r}}} = \left\{t \,|\, t \neq \frac{(2n+1)\pi}{2}\right\},$$

   where n is any integer.



Exercise 1.2.1

For the vector-valued function $\vec{\mathbf{r}}(t) = (t^2 - 3t)\,\hat{\mathbf{i}} + (4t + 1)\,\hat{\mathbf{j}}$, evaluate $\vec{\mathbf{r}}(0)$, $\vec{\mathbf{r}}(1)$, and $\vec{\mathbf{r}}(-4)$. Does this function have any domain restrictions?

Hint

Substitute the appropriate values of t into the function.

Answer

$\vec{\mathbf{r}}(0) = \hat{\mathbf{j}}$, $\vec{\mathbf{r}}(1) = -2\hat{\mathbf{i}} + 5\hat{\mathbf{j}}$, $\vec{\mathbf{r}}(-4) = 28\hat{\mathbf{i}} - 15\hat{\mathbf{j}}$

The domain of $\vec{\mathbf{r}}(t) = (t^2 - 3t)\hat{\mathbf{i}} + (4t + 1)\hat{\mathbf{j}}$ is all real numbers.

Example **1.2.1** illustrates an important concept. The domain of a vector-valued function consists of real numbers. The domain can be all real numbers or a subset of the real numbers. The range of a vector-valued function consists of vectors. Each real number in the domain of a vector-valued function is mapped to either a two- or a three-dimensional vector.

Graphing Vector-Valued Functions

Recall that a plane vector consists of two quantities: direction and magnitude. Given any point in the plane (the *initial point*), if we move in a specific direction for a specific distance, we arrive at a second point. This represents the *terminal point* of the vector. We calculate the components of the vector by subtracting the coordinates of the initial point from the coordinates of the terminal point.

A vector is considered to be in *standard position* if the initial point is located at the origin. When graphing a vector-valued function, we typically graph the vectors in the domain of the function in standard position, because doing so guarantees the uniqueness of the graph. This convention applies to the graphs of three-dimensional vector-valued functions as well. The graph of a vector-valued function of the form

$$\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}}$$

consists of the set of all points $(f(t),\, g(t))$, and the path it traces is called a **plane curve**. The graph of a vector-valued function of the form

$$\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} + h(t)\,\hat{\mathbf{k}}$$

consists of the set of all points $(f(t),\, g(t),\, h(t))$, and the path it traces is called a **space curve**. Any representation of a plane curve or space curve using a vector-valued function is called a **vector parameterization** of the curve.

Each plane curve and space curve has an **orientation**, indicated by arrows drawn in on the curve, that shows the direction of motion along the curve as the value of the parameter t increases.

Example 1.2.3 : Graphing a Vector-Valued Function

Create a graph of each of the following vector-valued functions:

1. The plane curve represented by $\vec{\mathbf{r}}(t) = 4\cos t\,\hat{\mathbf{i}} + 3\sin t\,\hat{\mathbf{j}},\ 0 \le t \le 2\pi$
2. The plane curve represented by $\vec{\mathbf{r}}(t) = 4\cos(t^3)\,\hat{\mathbf{i}} + 3\sin(t^3)\,\hat{\mathbf{j}},\ 0 \le t \le \sqrt[3]{2\pi}$
3. The space curve represented by $\vec{\mathbf{r}}(t) = 4\cos t\,\hat{\mathbf{i}} + 4\sin t\,\hat{\mathbf{j}} + t\,\hat{\mathbf{k}},\ 0 \le t \le 4\pi$

Solution

1. As with any graph, we start with a table of values. We then graph each of the vectors in the second column of the table in standard position and connect the terminal points of each vector to form a curve (Figure 1.2.1). This curve turns out to be an ellipse centered at the origin.

Table 1.2.1: Table of Values for $\vec{\mathbf{r}}(t) = 4\cos t\,\hat{\mathbf{i}} + 3\sin t\,\hat{\mathbf{j}},\ 0 \le t \le 2\pi$

t	$\vec{\mathbf{r}}(t)$	t	$\vec{\mathbf{r}}(t)$
0	$4\hat{\mathbf{i}}$	π	$-4\hat{\mathbf{i}}$
$\frac{\pi}{4}$	$2\sqrt{2}\hat{\mathbf{i}} + \frac{3\sqrt{2}}{2}\hat{\mathbf{j}}$	$\frac{5\pi}{4}$	$-2\sqrt{2}\hat{\mathbf{i}} - \frac{3\sqrt{2}}{2}\hat{\mathbf{j}}$
$\frac{\pi}{2}$	$3\hat{\mathbf{j}}$	$\frac{3\pi}{2}$	$-3\hat{\mathbf{j}}$
$\frac{3\pi}{4}$	$-2\sqrt{2}\hat{\mathbf{i}} + \frac{3\sqrt{2}}{2}\hat{\mathbf{j}}$	$\frac{7\pi}{4}$	$2\sqrt{2}\hat{\mathbf{i}} - \frac{3\sqrt{2}}{2}\hat{\mathbf{j}}$
2π	$4\hat{\mathbf{i}}$		

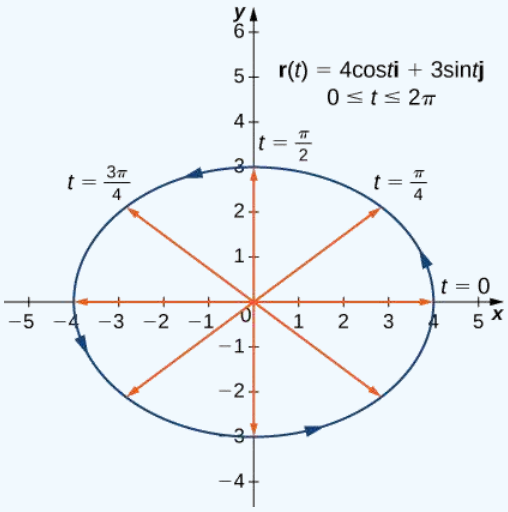


Figure 1.2.1: The graph of the first vector-valued function is an ellipse.

2. The table of values for $\vec{\mathbf{r}}(t) = 4\cos(t^3)\,\hat{\mathbf{i}} + 3\sin(t^3)\,\hat{\mathbf{j}},\ 0 \le t \le \sqrt[3]{2\pi}$ is as follows:

Table of Values for $\vec{\mathbf{r}}(t) = 4\cos(t^3)\,\hat{\mathbf{i}} + 3\sin(t^3)\,\hat{\mathbf{j}},\ 0 \le t \le \sqrt[3]{2\pi}$

t	$\vec{\mathbf{r}}(t)$	t	$\vec{\mathbf{r}}(t)$
0	$4\hat{\mathbf{i}}$	$\sqrt[3]{\pi}$	$-4\hat{\mathbf{i}}$
$\sqrt[3]{\frac{\pi}{4}}$	$2\sqrt{2}\hat{\mathbf{i}}+\frac{3\sqrt{2}}{2}\hat{\mathbf{j}}$	$\sqrt[3]{\frac{5\pi}{4}}$	$-2\sqrt{2}\hat{\mathbf{i}}-\frac{3\sqrt{2}}{2}\hat{\mathbf{j}}$
$\sqrt[3]{\frac{\pi}{2}}$	$3\hat{\mathbf{j}}$	$\sqrt[3]{\frac{3\pi}{2}}$	$-3\hat{\mathbf{j}}$
$\sqrt[3]{\frac{3\pi}{4}}$	$-2\sqrt{2}\hat{\mathbf{i}}+\frac{3\sqrt{2}}{2}\hat{\mathbf{j}}$	$\sqrt[3]{\frac{7\pi}{4}}$	$2\sqrt{2}\hat{\mathbf{i}}-\frac{3\sqrt{2}}{2}\hat{\mathbf{j}}$
$\sqrt[3]{2\pi}$	$4\hat{\mathbf{i}}$		

The graph of this curve is also an ellipse centered at the origin.

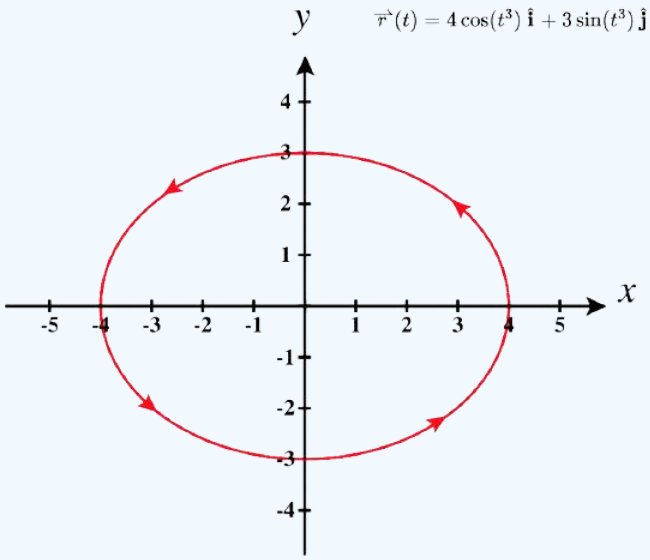


Figure 1.2.2: The graph of the second vector-valued function is also an ellipse.

3. We go through the same procedure for a three-dimensional vector function.

Table of Values for $\mathbf{r(t)} = 4\cos t\hat{\mathbf{i}} + 4\sin t\hat{\mathbf{j}} + t\hat{\mathbf{k}},\ 0 \le t \le 4\pi$

t	$\vec{\mathbf{r}}(t)$	t	$\vec{\mathbf{r}}(t)$
0	$4\hat{\mathbf{i}}$	π	$-4\hat{\mathbf{i}}+\pi\hat{\mathbf{k}}$
$\frac{\pi}{4}$	$2\sqrt{2}\hat{\mathbf{i}}+2\sqrt{2}\hat{\mathbf{j}}+\frac{\pi}{4}\hat{\mathbf{k}}$	$\frac{5\pi}{4}$	$-2\sqrt{2}\hat{\mathbf{i}}-2\sqrt{2}\hat{\mathbf{j}}+\frac{5\pi}{4}\hat{\mathbf{k}}$
$\frac{\pi}{2}$	$4\hat{\mathbf{j}}+\frac{\pi}{2}\hat{\mathbf{k}}$	$\frac{3\pi}{2}$	$-4\hat{\mathbf{j}}+\frac{3\pi}{2}\hat{\mathbf{k}}$
$\frac{3\pi}{4}$	$-2\sqrt{2}\hat{\mathbf{i}}+2\sqrt{2}\hat{\mathbf{j}}+\frac{3\pi}{4}\hat{\mathbf{k}}$	$\frac{7\pi}{4}$	$2\sqrt{2}\hat{\mathbf{i}}-2\sqrt{2}\hat{\mathbf{j}}+\frac{7\pi}{4}\hat{\mathbf{k}}$
2π	$4\hat{\mathbf{j}}+2\pi\hat{\mathbf{k}}$		

The values then repeat themselves, except for the fact that the coefficient of $\hat{\mathbf{k}}$ is always increasing (1.2.3). This curve is called a helix. Notice that if the $\hat{\mathbf{k}}$ component is eliminated, then the function becomes $\vec{\mathbf{r}}(t) = 4\cos t\hat{\mathbf{i}} + 4\sin t\hat{\mathbf{j}}$, which is a circle of radius 4 centered at the origin.

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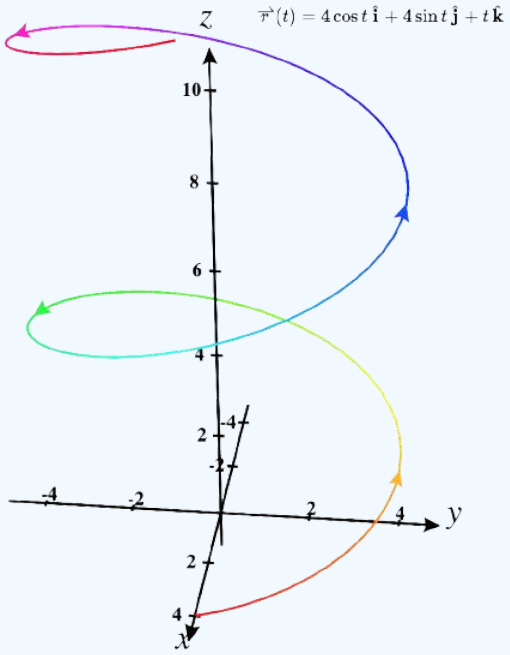


Figure 1.2.3: The graph of the third vector-valued function is a helix.

You may notice that the graphs in parts a. and b. are identical. This happens because the function describing curve b is a so-called reparameterization of the function describing curve a. In fact, any curve has an infinite number of reparameterizations; for example, we can replace t with $2t$ in any of the three previous curves without changing the shape of the curve. The interval over which t is defined may change, but that is all. We return to this idea later in this chapter when we study arc-length parameterization. As mentioned, the name of the shape of the curve of the graph in 1.2.3 is a *helix*. The curve resembles a spring, with a circular cross-section looking down along the z-axis. It is possible for a helix to be elliptical in cross-section as well. For example, the vector-valued function $\vec{\mathbf{r}}(t) = 4\cos t\,\hat{\mathbf{i}} + 3\sin t\,\hat{\mathbf{j}} + t\,\hat{\mathbf{k}}$ describes an elliptical helix. The projection of this helix into the xy-plane is an ellipse. Last, the arrows in the graph of this helix indicate the orientation of the curve as t progresses from 0 to 4π.



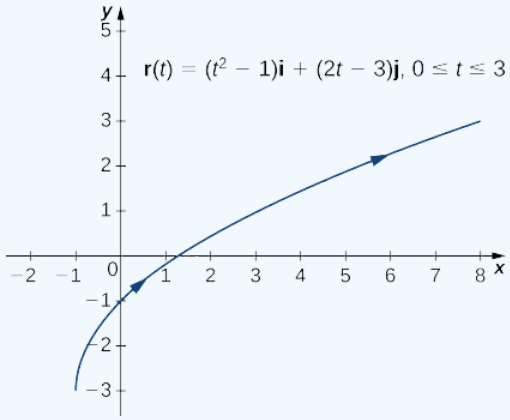
? Exercise 1.2.2

Create a graph of the vector-valued function $\vec{\mathbf{r}}(t) = (t^2 - 1)\hat{\mathbf{i}} + (2t - 3)\hat{\mathbf{j}}$, $0 \leq t \leq 3$.

Hint

Start by making a table of values, then graph the vectors for each value of t.

Answer



At this point, you may notice a similarity between vector-valued functions and parameterized curves. Indeed, given a vector-valued function $\vec{\mathbf{r}}(t) = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}}$ we can define $x = f(t)$ and $y = g(t)$. If a restriction exists on the values of t (for example, t is restricted to the interval $[a, b]$ for some constants $a < b$, then this restriction is enforced on the parameter. The graph of the parameterized function would then agree with the graph of the vector-valued function, except that the vector-valued graph would represent vectors rather than points. Since we can parameterize a curve defined by a function $y = f(x)$, it is also possible to represent an arbitrary plane curve by a vector-valued function.



Finding a Vector-Valued Function to Trace out the Graph of a Function $y = f(x)$

As you can see in the examples above, a vector-valued function traces out a curve in the plane or in space. What if we wish to write a vector-valued function that traces out the graph of a particular curve in the xy-plane?

What function's graph is traced out by the vector-valued function in Exercise 1.2.2 above: $\vec{\mathbf{r}}(t) = t\hat{\mathbf{i}} + t^3\hat{\mathbf{j}}$? It looks like the graph of $y = x^3$, doesn't it?

Remembering what was just said about the components of the vector-valued function corresponding to the parametic equations of a parameterized curve, we see that here we have:

$$x = t$$
$$y = t^3$$

Since $x = t$, we can replace t in the equation $y = t^3$ with x, giving us the function: $y = x^3$.

So we were correct in our guess.

How could we write a vector-valued function to trace out the graph of a function, $y = f(x)$?

Well, there are two orientations to consider: **left-to-right** and **right-to-left**.

Tracing a Function from Left-To-Right:

To trace out the graph of $y = f(x)$ from left-to-right, use: $\vec{\mathbf{r}}(t) = t\,\hat{\mathbf{i}} + f(t)\,\hat{\mathbf{j}}$

Note that what's important here is to have the x component be an increasing function. *Any increasing function will work.* We could use $x = t^3$, for example. But then we would need to remember to replace x in the function $f(x)$ with this expression t^3, giving us $y = f(t^3)$. This means the function $y = f(x)$ could also be parameterized from left-to-right by the vector-valued function: $\vec{\mathbf{r}}(t) = t^3\,\hat{\mathbf{i}} + f(t^3)\,\hat{\mathbf{j}}$

Tracing a Function from Right-To-Left:

To trace out the graph of $y = f(x)$ from right-to-left, use: $\vec{\mathbf{r}}(t) = -t\,\hat{\mathbf{i}} + f(-t)\,\hat{\mathbf{j}}$

Again note that we could use any decreasing function of t for the x component and obtain a vector-valued function that traces out the graph of $y = f(x)$ from right-to-left. Using $x = -t$ is just the simplest decreasing function we can choose.

Example 1.2.4: Finding a vector-valued function to trace the graph of $y = f(x)$

Determine a vector-valued function that will trace out the graph of $y = \cos x$ from left-to-right, and another one to trace it out from right-to-left.

Solution

Left-to-right: $\vec{\mathbf{r}}(t) = t\,\hat{\mathbf{i}} + \cos t\,\hat{\mathbf{j}}$

Right-to-left: $\vec{\mathbf{r}}(t) = -t\,\hat{\mathbf{i}} + \cos(-t)\,\hat{\mathbf{j}}$

Finding a Vector-Valued Function to Trace out the Graph of an Equation in x and y and Vice Versa

What if we wish to find a vector-valued function to trace out the graph of a circle, an ellipse, or a hyperbola, given its implicit equation?

Well, note that in Example 1.2.3, the vector-valued function $\vec{\mathbf{r}}(t) = 4\cos t\,\hat{\mathbf{i}} + 3\sin t\,\hat{\mathbf{j}}$ traced out the graph of the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$.

In this vector-valued function we see that:

$$x = 4\cos t \quad \text{and} \quad y = 3\sin t \tag{1.2.3}$$

What we need now is a way to convert this to an implicit equation involving x and y. To accomplish this, remember the Pythagorean identity,

$$\cos^2 t + \sin^2 t = 1 \tag{1.2.4}$$

.

Now all we need to do is solve the above equations for $\cos t$ and $\sin t$ and we can substitute into this identity to obtain an equation in x and y.

So:

$$\cos t = \frac{x}{4} \quad \text{and} \quad \sin t = \frac{y}{3} \tag{1.2.5}$$

Substituting into the identity gives us:

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2 = 1 \tag{1.2.6}$$

Simplifying this implicit equation gives us the implicit equation of the ellipse in Example 1.2.3 that we wrote above:

$$\frac{x^2}{16} + \frac{y^2}{9} = 1 \tag{1.2.7}$$

To go the other way and find a vector-valued function that traces out an ellipse requires us to simply take these steps in the opposite direction!

Example 1.2.5: Writing a vector-valued function for a given circle, ellipse, or Hyperbola

Write a vector-valued function that traces out each of the following implicit curves:

a. The ellipse: $\frac{x^2}{16} + \frac{y^2}{9} = 1$

b. The circle: $x^2 + y^2 = 4$

c. The hyperbola: $\frac{x^2}{25} - \frac{y^2}{16} = 1$

Solution

a. Let's just use the process shown above in reverse. First, let's rewrite the implicit equation so it shows a sum of quantities squared equals one.

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$

Now we need the identity we used above, $\cos^2 t + \sin^2 t = 1$.

Equating the parts being squared (note we actually have a choice here about which to make $\cos t$ and which to make $\sin t$), we get:

$$\frac{x}{4} = \cos t \quad \text{and} \quad \frac{y}{3} = \sin t$$

Now we just need to solve for x and y.

$$x = 4\cos t \quad \text{and} \quad y = 3\sin t$$

We can now write a vector-valued function that traces out this ellipse: $\vec{\mathbf{r}}(t) = 4\cos t\,\hat{\mathbf{i}} + 3\sin t\,\hat{\mathbf{j}}$

Note that we could also have written $\vec{\mathbf{r}}(t) = 4\sin t\,\hat{\mathbf{i}} + 3\cos t\,\hat{\mathbf{j}}$, since we could have chosen to switch the $\sin t$ and $\cos t$ above. It will trace out the same ellipse, but with the opposite orientation.

b. Tracing out a circle is fairly straightforward, not really needing the process we showed above, although it still may be helpful at first. Remember that all the vectors on the unit circle can be represented in the form: $\vec{\mathbf{v}} = \cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}$.

So the vector-valued function, $\vec{\mathbf{r}}(t) = \cos t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}}$, will trace out the unit circle with equation, $x^2 + y^2 = 1$.

To obtain a circle of radius 2 centered at the origin (which is the graph of $x^2 + y^2 = 4$), we just need to multiply through this vector-valued function by a scalar factor of 2.

Thus, a vector-valued function that will trace out this circle is: $\vec{\mathbf{r}}(t) = 2\cos t\,\hat{\mathbf{i}} + 2\sin t\,\hat{\mathbf{j}}$.

Note again that another possibility is: $\vec{\mathbf{r}}(t) = 2\sin t\,\hat{\mathbf{i}} + 2\cos t\,\hat{\mathbf{j}}$. It will trace out the same circle, but with the opposite orientation.

To use the technique above, you start by dividing each term in the equation by the square of the radius, here 4, thus putting the circle equation in "ellipse form". The rest of the steps follow the pattern shown in part a.

c. To trace out a hyperbola of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ or $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, we need to locate a trigonometric identity showing the difference of two squares equals 1. If you don't already have such an identity memorized, we can obtain one from the Pythagorean identity used above. That is,

$$\cos^2 t + \sin^2 t = 1$$

Dividing through each term by $\cos^2 t$

$$\frac{\cos^2 t}{\cos^2 t} + \frac{\sin^2 t}{\cos^2 t} = \frac{1}{\cos^2 t}$$

yields,

$$1 + \tan^2 t = \sec^2 t$$

Rewriting this equation gives us the identity we need:

$$\sec^2 t - \tan^2 t = 1$$

Now, the equation of this hyperbola is:

$$\frac{x^2}{25} - \frac{y^2}{16} = 1$$

Rewriting the left side to show the quantities that are squared:

$$\left(\frac{x}{5}\right)^2 - \left(\frac{y}{4}\right)^2 = 1$$

We can then equate the squared expressions corresponding terms:

$$\frac{x}{5} = \sec t \quad \text{and} \quad \frac{y}{4} = \tan t$$

Solving for x and y, we have:

$$x = 5\sec t \quad \text{and} \quad y = 4\tan t$$

So a vector-valued function that will trace out the hyperbola

$$\frac{x^2}{25} - \frac{y^2}{16} = 1 \tag{1.2.8}$$

is

$$\vec{\mathbf{r}}(t) = 5\sec t\,\hat{\mathbf{i}} + 4\tan t\,\hat{\mathbf{j}}.$$

Parameterizing a Piecewise Path

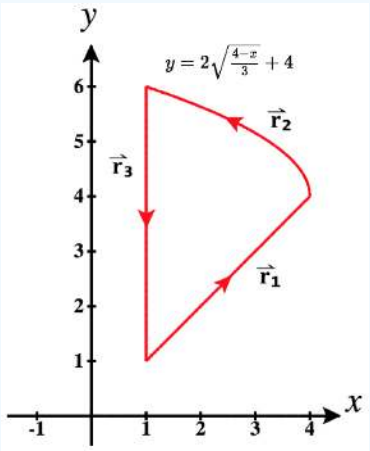
There are times when it is necessary to parameterize a path made up of pieces of different curves. This piecewise path may be open or form the boundary of a closed region as does the example shown in Figure 1.2.4. In addition to determining a vector-valued function to trace out each piece separately, with the indicated orientation, we also need to determine a suitable range of values for the parameter t.

Note that there are many ways to parameterize any one piece, so there are many correct ways to parameterize a path in this way.

✔ Example 1.2.6: Parameterizing a piecewise path

Determine a piecewise parameterization of the path shown in Figure 1.2.4, starting with $t = 0$ and continuing on through each piece.

Solution



Our first task is to identify the three pieces in this piecewise path.

Note how we labeled these sequentially as $\vec{\mathbf{r}}_1$, $\vec{\mathbf{r}}_2$, and $\vec{\mathbf{r}}_3$. Now we need to identify the function for each and write the corresponding vector-valued function with the correct orientation (left-to-right or right-to-left).

Determining $\vec{\mathbf{r}}_1$: The equation of the linear function in this piece is $y = x$.

Since it is oriented from left-to-right between $t = 1$ and $t = 4$, we can write:

$$\vec{\mathbf{r}}_{1a}(t) = t\,\hat{\mathbf{i}} + t\,\hat{\mathbf{j}} \quad \text{for} \quad 1 \le t \le 4$$

If we wish to begin this piece at $t = 0$, we just need to shift the value of t one unit to the left. One way to do this is to write $\vec{\mathbf{r}}_{1a}$ in terms of t_1 instead of t to make the translation easier to see.

Thus, we have $\vec{\mathbf{r}}_{1a}(t_1) = t_1\,\hat{\mathbf{i}} + t_1\,\hat{\mathbf{j}}$ for $1 \le t_1 \le 4$.

***Figure* 1.2.4:** *A closed piecewise path*

Subtracting 1 from each part of this range of parameter values, we have: $0 \le t_1 - 1 \le 3$.

Now we let $t = t_1 - 1$. Solving for t_1, we obtain: $t_1 = t + 1$.

Replacing t_1 with the expression $t + 1$ will effectively shift the range of parameter values one unit to the left.

So, starting with $t = 0$, we have:

$$\vec{\mathbf{r}}_1(t) = (t+1)\,\hat{\mathbf{i}} + (t+1)\,\hat{\mathbf{j}} \quad \text{for} \quad 0 \le t \le 3$$

Double-check that this vector-valued function will trace out this segment in the correct direction before going on to r_2.

Determining $\vec{\mathbf{r}}_2$: This piece has a label showing the function whose graph it traces along. If it were oriented from left-to-right, we would have:

$$\text{Left-to-right:} \quad \vec{\mathbf{r}}_{2a}(t) = t\,\hat{\mathbf{i}} + \left(2\sqrt{\frac{4-t}{3}} + 4\right)\hat{\mathbf{j}} \quad \text{for} \quad 1 \le t \le 4$$

But since we need it to be oriented from right-to-left, we need to replace t with $-t$ in the function and we need to divide through the range inequality by -1 to obtain the corresponding range. Thus we obtain:

$$\vec{\mathbf{r}}_{2b}(t) = -t\,\hat{\mathbf{i}} + \left(2\sqrt{\frac{4-(-t)}{3}} + 4\right)\hat{\mathbf{j}} \quad \text{for} \quad -4 \le t \le -1$$

Check that it works!

Now we wish to have this piece start at $t = 3$ just after the first one finishes. Again let's make this easier to see by writing r_{2b} in terms on t_2.

$$\vec{\mathbf{r}}_{2b}(t_2) = -t_2\,\hat{\mathbf{i}} + \left(2\sqrt{\frac{4-(-t_2)}{3}} + 4\right)\hat{\mathbf{j}} \quad \text{for} \quad -4 \le t_2 \le -1$$

To force r_2 to start with $t = 3$ instead of $t = -4$, we need to add 7 to each part of the inequality. This yields: $3 \le t_2 + 7 \le 6$.

Let $t = t_2 + 7$. Then solving for $-t_2$ (since this is what we need to replace in r_{2b}), we have: $-t_2 = 7 - t$.

Replacing $-t_2$ with $(7 - t)$ in $\vec{\mathbf{r}}_{2b}$, we obtain:

$$\vec{\mathbf{r}}_2(t) = (7-t)\,\hat{\mathbf{i}} + \left(2\sqrt{\frac{4-(7-t)}{3}} + 4\right)\hat{\mathbf{j}} \quad \text{for} \quad 3 \le t \le 6$$

This can be combined with our earlier result for r_1 to write a piecewise-defined vector-valued function that traces out the first two pieces, starting at $t = 0$:

$$\vec{\mathbf{r}}(t) = \begin{cases} (t+1)\,\hat{\mathbf{i}} + (t+1)\,\hat{\mathbf{j}}, & 0 \le t \le 3 \\ (7-t)\,\hat{\mathbf{i}} + \left(2\sqrt{\frac{t-3}{3}} + 4\right)\hat{\mathbf{j}}, & 3 < t \le 6 \end{cases}$$

Note that one small modification was made to the second range so that when $t = 3$, there is no confusion about which piece to evaluate.

Determining $\vec{\mathbf{r}}_3$: To determine this last piece we need to think a little differently. This is because it is a vertical segment, which cannot be represented with a function of the form, $y = f(x)$. Note that it could be represented by a function of the form $x = f(y)$. Letting $y = t$, we can write $x = f(t)$ and writing a parameterization in increasing y values (bottom-to-top), we'd get: $\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + t\,\hat{\mathbf{j}}$.

The equation of this line is $x = 1$. Thus, if we wished to parameterize this segment with upward orientation (increasing values of y), we have:

$$\vec{\mathbf{r}}_{3a}(t) = 1\,\hat{\mathbf{i}} + t\,\hat{\mathbf{j}} \quad \text{for} \quad 1 \le t \le 6$$

But since we wish to use a downward orientation (decreasing values of y), we need to use a decreasing function of t for y. As before, the simplest case is to use $y = -t$. Then, in the general case, we'd trace a function $x = f(y)$ in a downwards orientation with $\vec{\mathbf{r}}(t) = f(-t)\,\hat{\mathbf{i}} - t\,\hat{\mathbf{j}}$.

In the case of r_3, this gives us:

$$\vec{\mathbf{r}}_{3b}(t) = 1\,\hat{\mathbf{i}} - t\,\hat{\mathbf{j}} \quad \text{for} \quad -6 \le t \le -1$$

Note that since $x = 1,\ f(-t) = 1$, that is, it did not change the first component since it was constant and not a variable function of the parameter t.

Also note that since we negated t, we also had to negate the range, dividing it through by -1.

As above, to facilitate the translation, we'll replace t with t_3, giving us:

$$\vec{\mathbf{r}}_{3b}(t) = 1\,\hat{\mathbf{i}} - t_3\,\hat{\mathbf{j}} \quad \text{for} \quad -6 \le t_3 \le -1$$

Now, we wish this final piece to start at $t = 6$ where the second piece we formed above leaves off. We see that we need to add 12 to the range of paramater t to accomplish this, giving us a new range of $6 \le t_3 + 12 \le 11$.

Let $t = t_3 + 12$. Then solving for $-t_3$ (since this is what we need to replace in r_{3b}), we have: $-t_3 = 12 - t$.

Replacing $-t_3$ with $(12 - t)$ in $\vec{\mathbf{r}}_{3b}$, we obtain:

$$\vec{\mathbf{r}}_3(t) = 1\,\hat{\mathbf{i}} + (12-t)\,\hat{\mathbf{j}} \quad \text{for} \quad 6 \le t \le 11$$

Check that this still traces out this vertical segment from top-to-bottom.

We can now state the final answer as a single piecewise-defined vector-valued function that traces out this entire path, starting when $t = 0$.

$$\vec{\mathbf{r}}(t) = \begin{cases} (t+1)\,\hat{\mathbf{i}} + (t+1)\,\hat{\mathbf{j}}, & 0 \le t \le 3 \\ (7-t)\,\hat{\mathbf{i}} + \left(2\sqrt{\frac{t-3}{3}} + 4\right)\hat{\mathbf{j}}, & 3 < t \le 6 \\ 1\,\hat{\mathbf{i}} + (12-t)\,\hat{\mathbf{j}} & 6 < t \le 11 \end{cases}$$

Be sure to verify that this single vector-valued function does indeed trace out the entire path!

Limits and Continuity of a Vector-Valued Function

We now take a look at the limit of a vector-valued function. This is important to understand to study the calculus of vector-valued functions.

Definition: limit of a vector-valued function

A vector-valued function $\vec{\mathbf{r}}$ approaches the limit $\vec{\mathbf{L}}$ as t approaches a, written

$$\lim_{t\to a}\vec{\mathbf{r}}(t)=\vec{\mathbf{L}}, \tag{1.2.9}$$

provided

$$\lim_{t\to a}\|\vec{\mathbf{r}}(t)-\vec{\mathbf{L}}\|=0. \tag{1.2.10}$$

This is a rigorous definition of the limit of a vector-valued function. In practice, we use the following theorem:

Theorem: Limit of a vector-valued function

Let f, g, and h be functions of t. Then the limit of the vector-valued function $\vec{\mathbf{r}}(t)=f(t)\hat{\mathbf{i}}+g(t)\hat{\mathbf{j}}$ as t approaches a is given by

$$\lim_{t\to a}\vec{\mathbf{r}}(t)=[\lim_{t\to a}f(t)]\hat{\mathbf{i}}+[\lim_{t\to a}g(t)]\hat{\mathbf{j}}, \tag{1.2.11}$$

provided the limits $\lim_{t\to a}f(t)$ and $\lim_{t\to a}g(t)$ exist.

Similarly, the limit of the vector-valued function $\vec{\mathbf{r}}(t)=f(t)\hat{\mathbf{i}}+g(t)\hat{\mathbf{j}}+h(t)\hat{\mathbf{k}}$ as t approaches a is given by

$$\lim_{t\to a}\vec{\mathbf{r}}(t)=[\lim_{t\to a}f(t)]\hat{\mathbf{i}}+[\lim_{t\to a}g(t)]\hat{\mathbf{j}}+[\lim_{t\to a}h(t)]\hat{\mathbf{k}}, \tag{1.2.12}$$

provided the limits $\lim_{t\to a}f(t)$, $\lim_{t\to a}g(t)$ and $\lim_{t\to a}h(t)$ exist.

In the following example, we show how to calculate the limit of a vector-valued function.

Example 1.2.7: Evaluating the Limit of a Vector-Valued Function

For each of the following vector-valued functions, calculate $\lim_{t\to 3}\vec{\mathbf{r}}(t)$ for

a. $\vec{\mathbf{r}}(t)=(t^2-3t+4)\hat{\mathbf{i}}+(4t+3)\hat{\mathbf{j}}$
b. $\vec{\mathbf{r}}(t)=\frac{2t-4}{t+1}\hat{\mathbf{i}}+\frac{t}{t^2+1}\hat{\mathbf{j}}+(4t-3)\hat{\mathbf{k}}$

Solution

a. Use Equation 1.2.11 and substitute the value $t=3$ into the two component expressions:

$$\begin{aligned}\lim_{t\to 3}\vec{\mathbf{r}}(t) &= \lim_{t\to 3}\left[(t^2-3t+4)\hat{\mathbf{i}}+(4t+3)\hat{\mathbf{j}}\right]\\ &= \left[\lim_{t\to 3}(t^2-3t+4)\right]\hat{\mathbf{i}}+\left[\lim_{t\to 3}(4t+3)\right]\hat{\mathbf{j}}\\ &= 4\hat{\mathbf{i}}+15\hat{\mathbf{j}}\end{aligned}$$

b. Use Equation 1.2.12 and substitute the value $t=3$ into the three component expressions:

$$\begin{aligned}\lim_{t\to 3}\vec{\mathbf{r}}(t) &= \lim_{t\to 3}\left(\frac{2t-4}{t+1}\hat{\mathbf{i}}+\frac{t}{t^2+1}\hat{\mathbf{j}}+(4t-3)\hat{\mathbf{k}}\right)\\ &= \left[\lim_{t\to 3}\left(\frac{2t-4}{t+1}\right)\right]\hat{\mathbf{i}}+\left[\lim_{t\to 3}\left(\frac{t}{t^2+1}\right)\right]\hat{\mathbf{j}}+\left[\lim_{t\to 3}(4t-3)\right]\hat{\mathbf{k}}\\ &= \tfrac{1}{2}\hat{\mathbf{i}}+\tfrac{3}{10}\hat{\mathbf{j}}+9\hat{\mathbf{k}}\end{aligned}$$

Exercise 1.2.3

Calculate $\lim_{t\to 2}\vec{\mathbf{r}}(t)$ for the function $\vec{\mathbf{r}}(t)=\sqrt{t^2+3t-1}\,\hat{\mathbf{i}}-(4t-3)\hat{\mathbf{j}}-\sin\frac{(t+1)\pi}{2}\hat{\mathbf{k}}$

Hint

Use Equation 1.2.12 from the preceding theorem.

Answer

$$\lim_{t\to 2}\vec{\mathbf{r}}(t)=3\hat{\mathbf{i}}-5\hat{\mathbf{j}}+\hat{\mathbf{k}} \tag{1.2.13}$$

Now that we know how to calculate the limit of a vector-valued function, we can define **continuity at a point** for such a function.

Definitions

Let f, g, and h be functions of t. Then, the vector-valued function $\vec{\mathbf{r}}(t)=f(t)\hat{\mathbf{i}}+g(t)\hat{\mathbf{j}}$ is **continuous at point** $t=a$ if the following three conditions hold:

1. $\vec{\mathbf{r}}(a)$ exists
2. $\lim_{t\to a}\vec{\mathbf{r}}(t)$ exists
3. $\lim_{t\to a}\vec{\mathbf{r}}(t)=\vec{\mathbf{r}}(a)$

Similarly, the vector-valued function $\vec{\mathbf{r}}(t)=f(t)\hat{\mathbf{i}}+g(t)\hat{\mathbf{j}}+h(t)\hat{\mathbf{k}}$ is **continuous at point** $t=a$ if the following three conditions hold:

1. $\vec{\mathbf{r}}(a)$ exists
2. $\lim_{t\to a}\vec{\mathbf{r}}(t)$ exists
3. $\lim_{t\to a}\vec{\mathbf{r}}(t)=\vec{\mathbf{r}}(a)$

Summary

- A vector-valued function is a function of the form $\vec{\mathbf{r}}(t)=f(t)\hat{\mathbf{i}}+g(t)\hat{\mathbf{j}}$ or $\vec{\mathbf{r}}(t)=f(t)\hat{\mathbf{i}}+g(t)\hat{\mathbf{j}}+h(t)\hat{\mathbf{k}}$, where the component functions f, g, and h are real-valued functions of the parameter t.
- The graph of a vector-valued function of the form $\vec{\mathbf{r}}(t)=f(t)\hat{\mathbf{i}}+g(t)\hat{\mathbf{j}}$ is called a *plane curve*. The graph of a vector-valued function of the form $\vec{\mathbf{r}}(t)=f(t)\hat{\mathbf{i}}+g(t)\hat{\mathbf{j}}+h(t)\hat{\mathbf{k}}$ is called a *space curve*.
- It is possible to represent an arbitrary plane curve by a vector-valued function.
- To calculate the limit of a vector-valued function, calculate the limits of the component functions separately.

Key Equations

- **Vector-valued function**
  $\vec{\mathbf{r}}(t)=f(t)\hat{\mathbf{i}}+g(t)\hat{\mathbf{j}}$ or $\vec{\mathbf{r}}(t)=f(t)\hat{\mathbf{i}}+g(t)\hat{\mathbf{j}}+h(t)\hat{\mathbf{k}}$,or $\vec{\mathbf{r}}(t)=\langle f(t),g(t)\rangle$ or $\vec{\mathbf{r}}(t)=\langle f(t),g(t),h(t)\rangle$
- **Limit of a vector-valued function**
  $\lim_{t\to a}\vec{\mathbf{r}}(t)=[\lim_{t\to a}f(t)]\hat{\mathbf{i}}+[\lim_{t\to a}g(t)]\hat{\mathbf{j}}$ or $\lim_{t\to a}\vec{\mathbf{r}}(t)=[\lim_{t\to a}f(t)]\hat{\mathbf{i}}+[\lim_{t\to a}g(t)]\hat{\mathbf{j}}+[\lim_{t\to a}h(t)]\hat{\mathbf{k}}$

Glossary

component functions

the component functions of the vector-valued function $\vec{\mathbf{r}}(t)=f(t)\hat{\mathbf{i}}+g(t)\hat{\mathbf{j}}$ are $f(t)$ and $g(t)$, and the component functions of the vector-valued function $\vec{\mathbf{r}}(t)=f(t)\hat{\mathbf{i}}+g(t)\hat{\mathbf{j}}+h(t)\hat{\mathbf{k}}$ are $f(t)$, $g(t)$ and $h(t)$

helix

a three-dimensional curve in the shape of a spiral

limit of a vector-valued function

a vector-valued function $\vec{\mathbf{r}}(t)$ has a limit $\vec{\mathbf{L}}$ as t approaches a if $\lim t\to a\left|\vec{\mathbf{r}}(t)-\vec{\mathbf{L}}\right|=0$

plane curve

the set of ordered pairs $(f(t),g(t))$ together with their defining parametric equations $x=f(t)$ and $y=g(t)$

reparameterization

an alternative parameterization of a given vector-valued function

space curve

the set of ordered triples $(f(t), g(t), h(t))$ together with their defining parametric equations $x = f(t)$, $y = g(t)$ and $z = h(t)$

vector parameterization

any representation of a plane or space curve using a vector-valued function

vector-valued function

a function of the form $\vec{\mathbf{r}}(t) = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}}$ or $\vec{\mathbf{r}}(t) = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}} + h(t)\hat{\mathbf{k}}$,where the component functions f, g, and h are real-valued functions of the parameter t.

Contributors and Attributions

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1.2E: Exercises

Introduction to Vector-Valued Functions

1) Give the component functions $x = f(t)$ and $y = g(t)$ for the vector-valued function $\vec{\mathbf{r}}(t) = 3\sec t\,\hat{\mathbf{i}} + 2\tan t\,\hat{\mathbf{j}}$.

2) Given $\vec{\mathbf{r}}(t) = 3\sec t\hat{\mathbf{i}} + 2\tan t\hat{\mathbf{j}}$, find the following values (if possible).

a. $\vec{\mathbf{r}}\left(\frac{\pi}{4}\right)$
b. $\vec{\mathbf{r}}(\pi)$
c. $\vec{\mathbf{r}}\left(\frac{\pi}{2}\right)$

3) Sketch the curve of the vector-valued function $\vec{\mathbf{r}}(t) = 3\sec t\hat{\mathbf{i}} + 2\tan t\hat{\mathbf{j}}$ and give the orientation of the curve. Sketch asymptotes as a guide to the graph.

Limits of Vector-Valued Functions

4) Evaluate $\lim\limits_{t\to 0}\left(e^t\hat{\mathbf{i}} + \frac{\sin t}{t}\hat{\mathbf{j}} + e^{-t}\hat{\mathbf{k}}\right)$

5) Given the vector-valued function $\vec{\mathbf{r}}(t) = \langle \cos t, \sin t\rangle$ find the following values:

a. $\lim\limits_{t\to\frac{\pi}{4}} \vec{\mathbf{r}}(t)$
b. $\vec{\mathbf{r}}\left(\frac{\pi}{3}\right)$
c. Is $\vec{\mathbf{r}}(t)$ continuous at $t = \frac{\pi}{3}$?
d. Graph $\vec{\mathbf{r}}(t)$.

6) Given the vector-valued function $\vec{\mathbf{r}}(t) = \langle t, t^2 + 1\rangle$, find the following values:

a. $\lim\limits_{t\to -3} \vec{\mathbf{r}}(t)$
b. $\vec{\mathbf{r}}(-3)$
c. Is $\vec{\mathbf{r}}(t)$ continuous at $x = -3$?
d. $\vec{\mathbf{r}}(t+2) - \vec{\mathbf{r}}(t)$

7) Let $\vec{\mathbf{r}}(t) = e^t\hat{\mathbf{i}} + \sin t\hat{\mathbf{j}} + \ln t\hat{\mathbf{k}}$. Find the following values:

a. $\vec{\mathbf{r}}\left(\frac{\pi}{4}\right)$
b. $\lim\limits_{t\to\frac{\pi}{4}} \vec{\mathbf{r}}(t)$
c. Is $\vec{\mathbf{r}}(t)$ continuous at $t = \frac{\pi}{4}$?

For exercises 8 - 13, find the limit of the following vector-valued functions at the indicated value of t.

8) $\lim\limits_{t\to 4}\langle\sqrt{t-3}, \frac{\sqrt{t}-2}{t-4}, \tan(\frac{\pi}{t})\rangle$

9) $\lim\limits_{t\to\frac{\pi}{2}} \vec{\mathbf{r}}(t)$ for $\vec{\mathbf{r}}(t) = e^t\hat{\mathbf{i}} + \sin t\hat{\mathbf{j}} + \ln t\hat{\mathbf{k}}$

10) $\lim\limits_{t\to\infty}\langle e^{-2t}, \frac{2t+3}{3t-1}, \arctan(2t)\rangle$

11) $\lim\limits_{t\to e^2}\langle t\ln(t), \frac{\ln t}{t^2}, \sqrt{\ln(t^2)}\rangle$

12) $\lim\limits_{t\to\frac{\pi}{6}}\langle\cos 2t, \sin 2t, 1\rangle$

13) $\lim\limits_{t\to\infty} \vec{\mathbf{r}}(t)$ for $\vec{\mathbf{r}}(t) = 2e^{-t}\mathbf{i} + e^{-t}\hat{\mathbf{j}} + \ln(t-1)\hat{\mathbf{k}}$

Domain of a Vector-Valued Function

For problems 14 - 17, find the domain of the vector-valued functions.

14) Domain: $\vec{\mathbf{r}}(t) = \langle t^2, t, \sin t\rangle$

15) Domain: $\vec{\mathbf{r}}(t) = \langle t^2, \tan t, \ln t\rangle$

16) Domain: $\vec{\mathbf{r}}(t) = \langle t^2, \sqrt{t-3}, \frac{3}{2t+1}\rangle$

17) Domain: $\vec{\mathbf{r}}(t) = \langle \csc(t), \frac{1}{\sqrt{t-3}}, \ln(t-2) \rangle$

18) a. Find the domain of $\vec{\mathbf{r}}(t) = 2e^{-t}\hat{\mathbf{i}} + e^{-t}\hat{\mathbf{j}} + \ln(t-1)\hat{\mathbf{k}}$.

b. For what values of t is $\vec{\mathbf{r}}(t) = 2e^{-t}\hat{\mathbf{i}} + e^{-t}\hat{\mathbf{j}} + \ln(t-1)\hat{\mathbf{k}}$ continuous?

19) Domain: $\vec{\mathbf{r}}(t) = (\arccos t)\,\hat{\mathbf{i}} + \sqrt{2t-1}\,\hat{\mathbf{j}} + \ln(t)\,\hat{\mathbf{k}}$

Visualizing Vector-Valued Functions

20) Describe the curve defined by the vector-valued function $\vec{\mathbf{r}}(t) = (1+t)\hat{\mathbf{i}} + (2+5t)\hat{\mathbf{j}} + (-1+6t)\hat{\mathbf{k}}$.

21) Let $\vec{\mathbf{r}}(t) = \langle \cos t, t, \sin t \rangle$ and use it to answer the following questions.

a. For what values of t is $\vec{\mathbf{r}}(t)$ continuous?
b. Sketch the graph of $\vec{\mathbf{r}}(t)$.

22) Produce a careful sketch of the graph of $\vec{\mathbf{r}}(t) = t^2\,\hat{\mathbf{i}} + t\,\hat{\mathbf{j}}$.

In questions 23 - 25, use a graphing utility to sketch each of the vector-valued functions:

23) **[T]** $\vec{\mathbf{r}}(t) = 2\cos^2 t\hat{\mathbf{i}} + (2-\sqrt{t})\hat{\mathbf{j}}$

24) **[T]** $\vec{\mathbf{r}}(t) = \langle e^{\cos(3t)}, e^{-\sin(t)} \rangle$

25) **[T]** $\vec{\mathbf{r}}(t) = \langle 2 - \sin(2t), 3 + 2\cos t \rangle$

Finding Equations in x and y for the Path Traced out by Vector-Valued Functions

For questions 26-33, eliminate the parameter t, write the equation in Cartesian coordinates, then sketch the graph of the vector-valued functions.

26) $\vec{\mathbf{r}}(t) = 2t\hat{\mathbf{i}} + t^2\hat{\mathbf{j}}$
(*Hint:* Let $x = 2t$ and $y = t^2$. Solve the first equation for t in terms of x and substitute this result into the second equation.)

27) $\vec{\mathbf{r}}(t) = t^3\hat{\mathbf{i}} + 2t\hat{\mathbf{j}}$

28) $\vec{\mathbf{r}}(t) = \sin t\,\hat{\mathbf{i}} + \cos t\,\hat{\mathbf{j}}$

29) $\vec{\mathbf{r}}(t) = 3\cos t\,\hat{\mathbf{i}} + 3\sin t\,\hat{\mathbf{j}}$

30) $\vec{\mathbf{r}}(t) = \langle \sin t, 4\cos t \rangle$

31) $\vec{\mathbf{r}}(t) = 2\sin t\,\hat{\mathbf{i}} - 3\cos t\,\hat{\mathbf{j}}$

32) $\vec{\mathbf{r}}(t) = \tan t\,\hat{\mathbf{i}} - 2\sec t\,\hat{\mathbf{j}}$

33) $\vec{\mathbf{r}}(t) = 3\sec t\,\hat{\mathbf{i}} + 4\tan t\,\hat{\mathbf{j}}$

Finding a Vector-Valued Function to Trace out the Graph of an Equation in x and y

For questions 34 - 40, find a vector-valued function that traces out the given curve in the indicated direction.

34) $4x^2 + 9y^2 = 36$; clockwise and counterclockwise

35) $y = x^2$; from left to right

36) The line through P and Q where P is $(1, 4, -2)$ and Q is $(3, 9, 6)$

37) The circle, $x^2 + y^2 = 36$, oriented clockwise, with position $(-6, 0)$ at time $t = 0$.

38) The ellipse, $x^2 + \frac{y^2}{36} = 1$, oriented counterclockwise

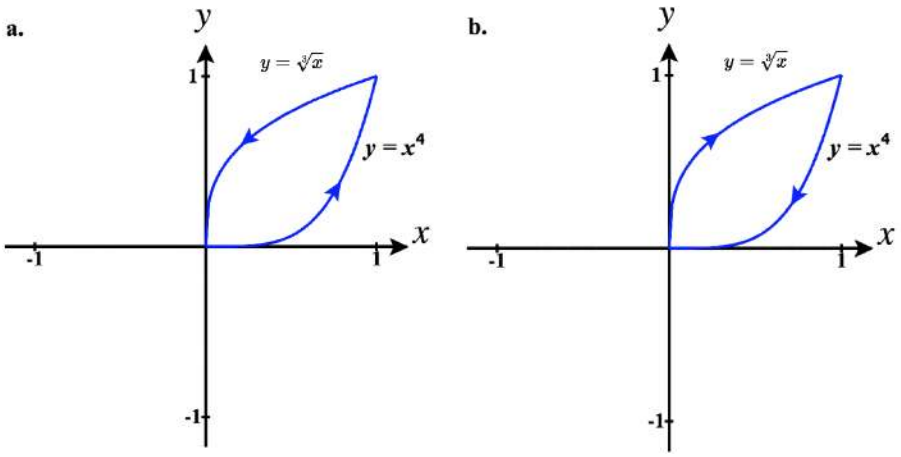
39) The hyperbola, $\frac{y^2}{36} - x^2 = 1$, top piece is oriented from left-to-right

40) The hyperbola, $\frac{x^2}{49} - \frac{y^2}{64} = 1$, right piece is oriented from bottom-to-top

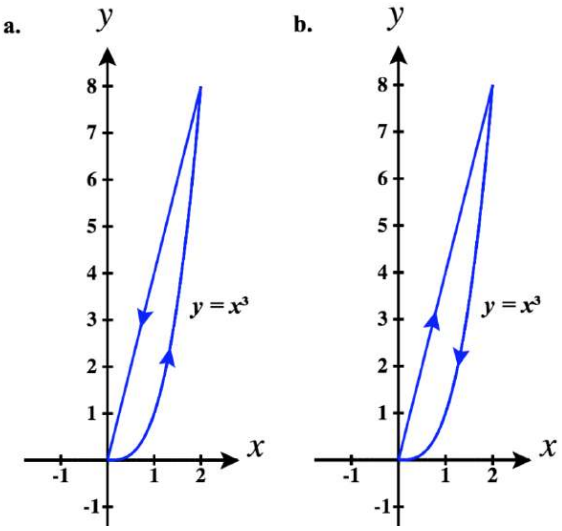
Parameterizing a Piecewise Path

For questions 41 - 44, provide a parameterization for each piecewise path. Try to write a parameterization that starts with $t = 0$ and progresses on through values of t as you move from one piece to another.

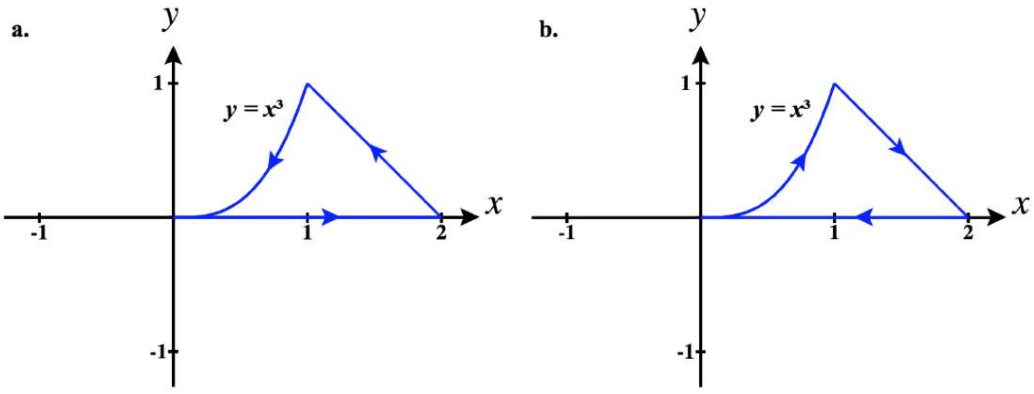
41)



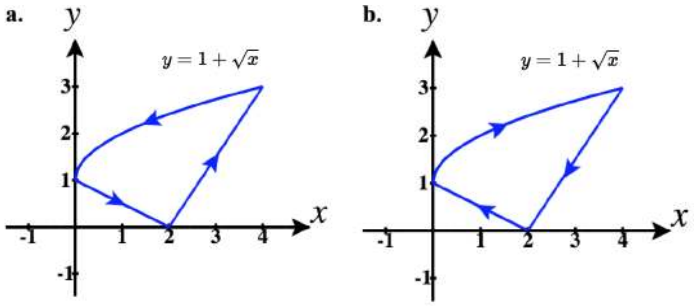
42)



43)



44)



Additional Vector-Valued Function Questions

For questions 45 - 48, consider the curve described by the vector-valued function $\vec{\mathbf{r}}(t) = (50e^{-t}\cos t)\hat{\mathbf{i}} + (50e^{-t}\sin t)\hat{\mathbf{j}} + (5 - 5e^{-t})\hat{\mathbf{k}}$.

45) What is the initial point of the path corresponding to $\vec{\mathbf{r}}(0)$?

46) What is $\lim_{t\to\infty} \vec{\mathbf{r}}(t)$?

47) **[T]** Use technology to sketch the curve.

48) Eliminate the parameter t to show that $z = 5 - \frac{r}{10}$ where $r^2 = x^2 + y^2$.

49) **[T]** Let $\vec{\mathbf{r}}(t) = \cos t\hat{\mathbf{i}} + \sin t\hat{\mathbf{j}} + 0.3\sin(2t)\hat{\mathbf{k}}$. Use technology to graph the curve (called the *roller-coaster curve*) over the interval $[0, 2\pi)$. Choose at least two views to determine the peaks and valleys.

50) **[T]** Use the result of the preceding problem to construct an equation of a roller coaster with a steep drop from the peak and steep incline from the "valley." Then, use technology to graph the equation.

51) Use the results of the preceding two problems to construct an equation of a path of a roller coaster with more than two turning points (peaks and valleys).

52) Complete the following investigation.

a. Graph the curve $\vec{\mathbf{r}}(t) = (4 + \cos(18t))\cos(t)\hat{\mathbf{i}} + (4 + \cos(18t)\sin(t))\hat{\mathbf{j}} + 0.3\sin(18t)\hat{\mathbf{k}}$ using two viewing angles of your choice to see the overall shape of the curve.
b. Does the curve resemble a "slinky"?
c. What changes to the equation should be made to increase the number of coils of the slinky?

Contributors

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Paul Seeburger (Monroe Community College) created problems 12, 14, 19, 22, 30-33, 37- 44.

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1.3: Calculus of Vector-Valued Functions

Learning Objectives

- Write an expression for the derivative of a vector-valued function.
- Find the tangent vector at a point for a given position vector.
- Find the unit tangent vector at a point for a given position vector and explain its significance.
- Calculate the definite integral of a vector-valued function.

To study the calculus of vector-valued functions, we follow a similar path to the one we took in studying real-valued functions. First, we define the derivative, then we examine applications of the derivative, then we move on to defining integrals. However, we will find some interesting new ideas along the way as a result of the vector nature of these functions and the properties of space curves.

Derivatives of Vector-Valued Functions

Now that we have seen what a vector-valued function is and how to take its limit, the next step is to learn how to differentiate a vector-valued function. The definition of the derivative of a vector-valued function is nearly identical to the definition of a real-valued function of one variable. However, because the range of a vector-valued function consists of vectors, the same is true for the range of the derivative of a vector-valued function.

Definition: Derivative of Vector-Valued Functions

The derivative of a vector-valued function $\vec{\mathbf{r}}(t)$ is

$$\vec{\mathbf{r}}'(t) = \lim_{\Delta t \to 0} \frac{\vec{\mathbf{r}}(t+\Delta t) - \vec{\mathbf{r}}(t)}{\Delta t} \tag{1.3.1}$$

provided the limit exists. If $\vec{\mathbf{r}}'(t)$ exists, then $\vec{\mathbf{r}}(t)$ is differentiable at t. If $\vec{\mathbf{r}}'(t)$ exists for all t in an open interval (a, b) then $\vec{\mathbf{r}}(t)$ is differentiable over the interval (a, b). For the function to be differentiable over the closed interval $[a, b]$, the following two limits must exist as well:

$$\vec{\mathbf{r}}'(a) = \lim_{\Delta t \to 0^+} \frac{\vec{\mathbf{r}}(a+\Delta t) - \vec{\mathbf{r}}(a)}{\Delta t} \tag{1.3.2}$$

and

$$\vec{\mathbf{r}}'(b) = \lim_{\Delta t \to 0^-} \frac{\vec{\mathbf{r}}(b+\Delta t) - \vec{\mathbf{r}}(b)}{\Delta t} \tag{1.3.3}$$

Graphical Interpretation of the Derivative: Recall that the derivative of a real-valued function can be interpreted as the slope of a tangent line or the instantaneous rate of change of the function. The derivative of a vector-valued function can be understood to be an instantaneous rate of change as well; for example, when the function represents the position of an object at a given point in time, the derivative represents its velocity at that same point in time.

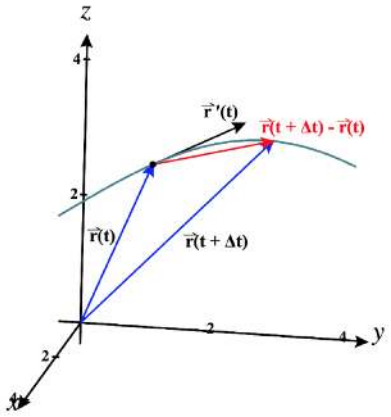


Figure 1.3.1

To understand what this derivative represents geometrically, it helps to consider the diagram shown in Figure 1.3.1. For the purposes of this illustration, consider $\vec{\mathbf{r}}(t)$ to the represent position of an object.

We see vectors $\vec{\mathbf{r}}(t)$ and $\vec{\mathbf{r}}(t+\Delta t)$, where Δt is a small change in time producing a second vector that is farther along in the direction of motion.

The difference of these two vectors gives us the **displacement vector,** $\vec{\mathbf{r}}(t+\Delta t)-\vec{\mathbf{r}}(t)$, shown here in red.

As $\Delta t \to 0$, $\vec{\mathbf{r}}(t+\Delta t)$ will approach $\vec{\mathbf{r}}(t)$, and the displacement vector $\vec{\mathbf{r}}(t+\Delta t)-\vec{\mathbf{r}}(t)$ will become closer and closer to being tangent to the curve at the terminal point of $\vec{\mathbf{r}}(t)$.

Although the length of the displacement vector is approaching 0, we are actually taking the limit not of the displacement vector itself, but of the average velocity over the time Δt, found by dividing the displacement vector by the scalar Δt.

$$\textbf{Average Velocity} \quad = \quad \frac{\vec{\mathbf{r}}(t+\Delta t)-\vec{\mathbf{r}}(t)}{\Delta t} \tag{1.3.4}$$

Since Δt is also approaching 0, dividing by it will tend to lengthen the average velocity vector (as long as $\Delta t < 1$).

The limit gives us the vector $\vec{\mathbf{r}}'(t)$, the velocity of the motion along the curve, which is tangent to the curve at this point.

Many of the rules for calculating derivatives of real-valued functions can be applied to calculating the derivatives of vector-valued functions as well. Recall that the derivative of a real-valued function can be interpreted as the slope of a tangent line or the instantaneous rate of change of the function. The derivative of a vector-valued function can be understood to be an instantaneous rate of change as well; for example, when the function represents the position of an object at a given point in time, the derivative represents its velocity at that same point in time.

We now demonstrate taking the derivative of a vector-valued function.

Example 1.3.1: Finding the Derivative of a Vector-Valued Function

Use the definition to calculate the derivative of the function

$$\vec{\mathbf{r}}(t) = (3t+4)\,\hat{\mathbf{i}} + (t^2 - 4t + 3)\,\hat{\mathbf{j}}.$$

Solution

Let's use Equation 1.3.1:

$$
\begin{aligned}
\vec{\mathbf{r}}'(t) &= \lim_{\Delta t \to 0} \frac{\vec{\mathbf{r}}(t+\Delta t) - \vec{\mathbf{r}}(t)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{[(3(t+\Delta t)+4)\,\hat{\mathbf{i}} + ((t+\Delta t)^2 - 4(t+\Delta t)+3)\,\hat{\mathbf{j}}] - [(3t+4)\,\hat{\mathbf{i}} + (t^2-4t+3)\,\hat{\mathbf{j}}]}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{(3t+3\Delta t+4)\,\hat{\mathbf{i}} - (3t+4)\,\hat{\mathbf{i}} + (t^2+2t\Delta t+(\Delta t)^2 - 4t - 4\Delta t + 3)\,\hat{\mathbf{j}} - (t^2-4t+3)\,\hat{\mathbf{j}}}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{(3\Delta t)\,\hat{\mathbf{i}} + (2t\Delta t + (\Delta t)^2 - 4\Delta t)\,\hat{\mathbf{j}}}{\Delta t} \\
&= \lim_{\Delta t \to 0} (3\,\hat{\mathbf{i}} + (2t + \Delta t - 4)\,\hat{\mathbf{j}}) \\
&= 3\,\hat{\mathbf{i}} + (2t-4)\,\hat{\mathbf{j}}
\end{aligned}
$$

? Exercise 1.3.1

Use the definition to calculate the derivative of the function $\vec{\mathbf{r}}(t) = (2t^2+3)\,\hat{\mathbf{i}} + (5t-6)\,\hat{\mathbf{j}}$.

Hint

Use Equation 1.3.1.

Answer

$$\vec{\mathbf{r}}'(t) = 4t\,\hat{\mathbf{i}} + 5\,\hat{\mathbf{j}}$$

Notice that in the calculations in Example **1.3.1**, we could also obtain the answer by first calculating the derivative of each component function, then putting these derivatives back into the vector-valued function. This is always true for calculating the derivative of a vector-valued function, whether it is in two or three dimensions. We state this in the following theorem. The proof of this theorem follows directly from the definitions of the limit of a vector-valued function and the derivative of a vector-valued function.

Theorem 1.3.1: Differentiation of Vector-Valued Functions

Let f, g, and h be differentiable functions of t.

1. If $\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}}$ then

$$\vec{\mathbf{r}}'(t) = f'(t)\,\hat{\mathbf{i}} + g'(t)\,\hat{\mathbf{j}}. \tag{1.3.5}$$

2. If $\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} + h(t)\,\hat{\mathbf{k}}$ then

$$\vec{\mathbf{r}}'(t) = f'(t)\,\hat{\mathbf{i}} + g'(t)\,\hat{\mathbf{j}} + h'(t)\,\hat{\mathbf{k}}. \tag{1.3.6}$$

✓ Example 1.3.2: Calculating the Derivative of Vector-Valued Functions

Use Theorem **1.3.1** to calculate the derivative of each of the following functions.

a. $\vec{\mathbf{r}}(t) = (6t+8)\,\hat{\mathbf{i}} + (4t^2+2t-3)\,\hat{\mathbf{j}}$
b. $\vec{\mathbf{r}}(t) = 3\cos t\,\hat{\mathbf{i}} + 4\sin t\,\hat{\mathbf{j}}$
c. $\vec{\mathbf{r}}(t) = e^t \sin t\,\hat{\mathbf{i}} + e^t \cos t\,\hat{\mathbf{j}} - e^{2t}\,\hat{\mathbf{k}}$

Solution

We use Theorem **1.3.1** and what we know about differentiating functions of one variable.

a. The first component of

$$\vec{\mathbf{r}}(t) = (6t+8)\,\hat{\mathbf{i}} + (4t^2+2t-3)\,\hat{\mathbf{j}}$$

is $f(t) = 6t+8$. The second component is $g(t) = 4t^2+2t-3$. We have $f'(t) = 6$ and $g'(t) = 8t+2$, so the Theorem 1.3.1 gives $\vec{\mathbf{r}}'(t) = 6\,\hat{\mathbf{i}} + (8t+2)\,\hat{\mathbf{j}}$.

b. The first component is $f(t) = 3\cos t$ and the second component is $g(t) = 4\sin t$. We have $f'(t) = -3\sin t$ and $g'(t) = 4\cos t$, so we obtain $\vec{\mathbf{r}}'(t) = -3\sin t\,\hat{\mathbf{i}} + 4\cos t\,\hat{\mathbf{j}}$.

c. The first component of $\vec{\mathbf{r}}(t) = e^t \sin t\,\hat{\mathbf{i}} + e^t \cos t\,\hat{\mathbf{j}} - e^{2t}\hat{\mathbf{k}}$ is $f(t) = e^t \sin t$, the second component is $g(t) = e^t \cos t$, and the third component is $h(t) = -e^{2t}$. We have $f'(t) = e^t(\sin t + \cos t)$, $g'(t) = e^t(\cos t - \sin t)$, and $h'(t) = -2e^{2t}$, so the theorem gives $\vec{\mathbf{r}}'(t) = e^t(\sin t + \cos t)\,\hat{\mathbf{i}} + e^t(\cos t - \sin t)\,\hat{\mathbf{j}} - 2e^{2t}\hat{\mathbf{k}}$.

Exercise 1.3.2

Calculate the derivative of the function

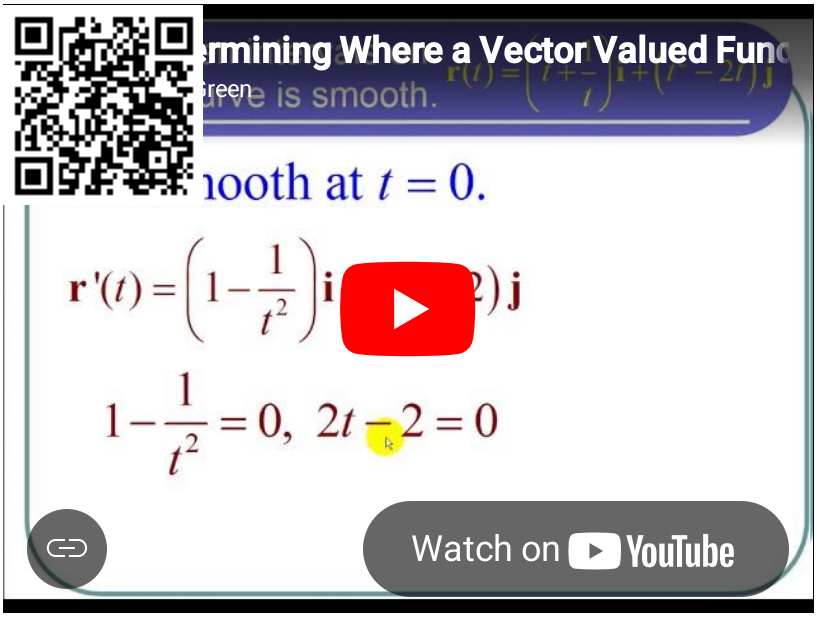
$$\vec{\mathbf{r}}(t) = (t\ln t)\,\hat{\mathbf{i}} + (5e^t)\,\hat{\mathbf{j}} + (\cos t - \sin t)\,\hat{\mathbf{k}}.$$

Hint

Identify the component functions and use Theorem 1.3.1.

Answer

$$\vec{\mathbf{r}}'(t) = (1+\ln t)\,\hat{\mathbf{i}} + 5e^t\,\hat{\mathbf{j}} - (\sin t + \cos t)\,\hat{\mathbf{k}}$$





Describing Motion with Vector-Valued Functions

As has been mentioned already in this section, vector-valued functions are often used to describe motion in the plane or in space. Now that we understand the derivative of a vector-valued function, let's make the relationships clear.

First, since the components of a vector-valued function represent the coordinates of a point on the corresponding curve, it may help to write

$$\vec{\mathbf{r}}(t) = x(t)\,\hat{\mathbf{i}} + y(t)\,\hat{\mathbf{j}} + z(t)\,\hat{\mathbf{k}} \tag{1.3.7}$$

This approach has already been used in Section 12.1.

Now we can define how to calculate the velocity, speed, acceleration, and jerk of the object's motion.

Definition: Velocity, speed, Acceleration, and JERK

Let $\vec{\mathbf{r}}(t)$ be a twice-differentiable vector-valued function of the parameter t that represents the position of an object as a function of time.

The **velocity vector** $\vec{\mathbf{v}}(t)$ of the object is given by

$$\text{Velocity } = \vec{\mathbf{v}}(t) = \vec{\mathbf{r}}'(t). \tag{1.3.8}$$

The ***speed*** is defined to be

$$\text{Speed } = \|\vec{\mathbf{v}}(t)\| = \|\vec{\mathbf{r}}'(t)\| = \frac{ds}{dt}. \tag{1.3.9}$$

The **acceleration vector** $\vec{\mathbf{a}}(t)$ is defined to be

$$\text{Acceleration } = \vec{\mathbf{a}}(t) = \vec{\mathbf{v}}'(t) = \vec{\mathbf{r}}''(t). \tag{1.3.10}$$

If $\vec{\mathbf{r}}(t)$ has a third derivative, we can also define the **jerk vector** $\vec{jerk}(t)$ to be

$$\text{Jerk } = \vec{\text{jerk}}(t) = \vec{\mathbf{a}}'(t) = \vec{\mathbf{r}}'''(t). \tag{1.3.11}$$

Now let's see how this looks in the context of a particular position function.

Example 1.3.3: Finding velocity, speed, acceleration, and jerk

Consider the position function,

$$\vec{\mathbf{r}}(t) = (1-t^2)\,\hat{\mathbf{i}} + t\,\hat{\mathbf{j}}, \text{ where time is in seconds} \tag{1.3.12}$$

a. Determine the velocity, speed, acceleration, and jerk as functions of time for this position function.

b. Evaluate each of these functions, including the position function, at times $t = 0$ sec and $t = 2$ sec.

c. Sketch a graph of this position function and draw in each of the non-zero vectors from part b on the graph of $\vec{\mathbf{r}}$, starting at the location of the object on the curve at the corresponding time.

Solution

a. Since $\vec{\mathbf{r}}(t) = (1-t^2)\,\hat{\mathbf{i}} + t\,\hat{\mathbf{j}}$, we have

$$\begin{aligned}
\vec{\mathbf{v}}(t) &= \vec{\mathbf{r}}'(t) = -2t\,\hat{\mathbf{i}} + 1\,\hat{\mathbf{j}},\\
\text{speed}(t) &= \|v(t)\| = \sqrt{(-2t)^2 + (1)^2} = \sqrt{4t^2+1},\\
\vec{\mathbf{a}}(t) &= \vec{\mathbf{v}}'(t) = -2\,\hat{\mathbf{i}},\\
\overrightarrow{\text{jerk}}(t) &= \vec{\mathbf{0}}
\end{aligned}$$

b. Evaluating these functions at time $t = 0$, we get:

$$\begin{aligned}
\vec{\mathbf{r}}(0) &= (1-0^2)\,\hat{\mathbf{i}} + 0\,\hat{\mathbf{j}} = \hat{\mathbf{i}}\\
\vec{\mathbf{v}}(0) &= -2(0)\,\hat{\mathbf{i}} + 1\,\hat{\mathbf{j}} = \hat{\mathbf{j}}\\
\text{speed}(0) &= \sqrt{4(0)^2+1} = \sqrt{1} = 1 \text{ unit/sec}\\
\vec{\mathbf{a}}(0) &= -2\,\hat{\mathbf{i}},\\
\overrightarrow{\text{jerk}}(0) &= \vec{\mathbf{0}}
\end{aligned}$$

And for time $t = 2$, we have

$$\begin{aligned}
\vec{\mathbf{r}}(2) &= (1-2^2)\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} = -3\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}}\\
\vec{\mathbf{v}}(2) &= -2(2)\,\hat{\mathbf{i}} + 1\,\hat{\mathbf{j}} = -4\,\hat{\mathbf{i}} + \hat{\mathbf{j}}\\
\text{speed}(2) &= \sqrt{4(2)^2+1} = \sqrt{17} \text{ units/sec}\\
\vec{\mathbf{a}}(0) &= -2\,\hat{\mathbf{i}},\\
\overrightarrow{\text{jerk}}(0) &= \vec{\mathbf{0}}
\end{aligned}$$

c. The graph of this curve is shown below in Figure 1.3.2. We then drew in and labeled the vectors $vecsv(0)$, $vecsa(0)$, $vecsv(2)$, and $vecsa(2)$.

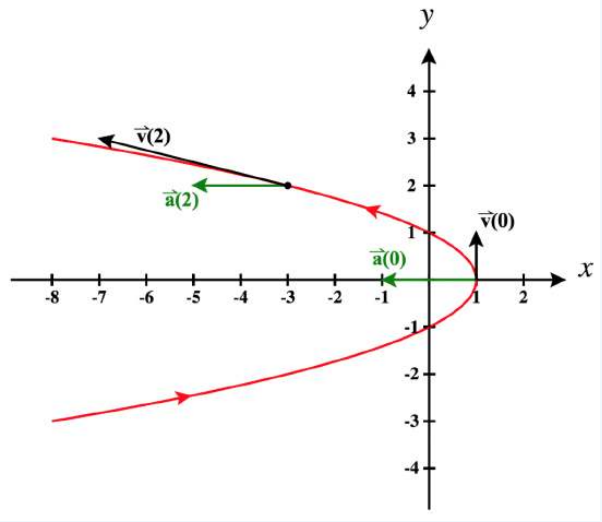


Figure **1.3.2**: Graphing velocity and acceleration vectors

Smooth Vector-Valued Functions

Definition: smoothness

Let $\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} + h(t)\,\hat{\mathbf{k}}$ be the parameterization of a curve that is differentiable on an open interval I.

Then $\vec{\mathbf{r}}(t)$ is **smooth** on the open interval I, if

$$\vec{\mathbf{r}}'(t) \neq \vec{\mathbf{0}}, \text{ for any value of } t \text{ in the interval } I.$$

To put this another way, $\vec{\mathbf{r}}(t)$ is **smooth** on the open interval I if:

1. $\vec{\mathbf{r}}(t)$ is defined and continuous on I, and
2. f', g' and h' are all continuous on I, that is, $\vec{\mathbf{r}}'(t)$ is defined and continuous on I, and
3. $\vec{\mathbf{r}}'(t) \neq \vec{\mathbf{0}}$ for all values of t in the interval I.

The concept of smoothness is very closely connected to the application of vector-valued functions to motion. If you have ever ridden rides at an amusement park, you can probably identify some where the ride felt smooth, and others where there are moments during the ride where it was not smooth. These moments often correspond to points in the motion where the relevant parameterization is not smooth by the definition above.

To identify open intervals in the domain of $\vec{\mathbf{r}}$ on which $\vec{\mathbf{r}}$ is smooth, we usually identify values of t for which $\vec{\mathbf{r}}$ is **not smooth**.

In addition to using the definition above to identify these points, it is often apparent from the graph of the vector-valued function $\vec{\mathbf{r}}(t)$. A **cusp**, i.e., a sharp turning point, is a common occurrence at points where a curve is not smooth. In Figure **1.3.3** below, you can quickly identify points at which the parameterization fails to be smooth by locating cusps in the graph.

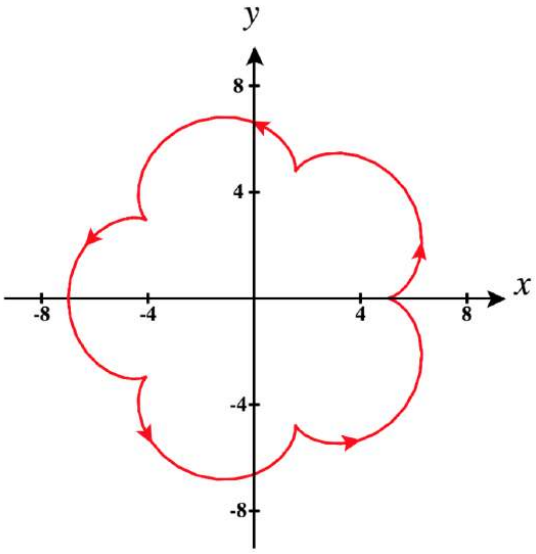


Figure **1.3.3**: This epicycloid is not smooth at the points between it's petals.

When we consider the motion described by $\vec{\mathbf{r}}(t)$, there are at least two additional events that can occur at points where the parameterization fails to be smooth: a reversal of motion or a momentary pause in the motion. To summarize these events:

If a vector-valued function $\vec{\mathbf{r}}(t)$ is **not smooth at time** t, we will observe that:

1. There is a cusp at the associated point on the graph of $\vec{\mathbf{r}}(t)$, or
2. The motion reverses itself at the associated point, causing the motion to travel back along the same path in the opposite direction, or
3. The motion actually stops and starts up again, with no visual cue, that is, where the curve appears smooth.

Let's consider a couple examples to see how this can look.

Example 1.3.4: Determining open intervals on which a vector-valued function is smooth

Determine the open intervals on which each of the following vector-valued functions is smooth:

a. $\vec{\mathbf{r}}(t) = (2t^3 - 3t^2)\ \hat{\mathbf{i}} + (t^2 - 2t)\ \hat{\mathbf{j}} + 2\ \hat{\mathbf{k}}$
b. $\vec{\mathbf{r}}(t) = t^3\ \hat{\mathbf{i}} + \sqrt{t^2 - 1}\,\hat{\mathbf{j}}$
c. $\vec{\mathbf{r}}(t) = t^4\ \hat{\mathbf{i}} + \ln(t^2 + 1)\,\hat{\mathbf{j}}$
d. $\vec{\mathbf{r}}(t) = (6\cos t - \cos(6t))\ \hat{\mathbf{i}} + (6\sin t - \sin(6t))\ \hat{\mathbf{j}}$, for $0 \le t \le 2\pi$

Solution

a. First we need to consider the domain of $\vec{\mathbf{r}}$. In this case, $\mathbf{D}_{\vec{\mathbf{r}}} : (-\infty, \infty)$.

Then we need to find $\vec{\mathbf{r}}'(t)$. Since $\vec{\mathbf{r}}(t) = (2t^3 - 3t^2)\ \hat{\mathbf{i}} + (t^2 - 2t)\ \hat{\mathbf{j}} + 2\ \hat{\mathbf{k}}$,

$$\vec{\mathbf{r}}'(t) = (6t^2 - 6t)\ \hat{\mathbf{i}} + (2t - 2)\ \hat{\mathbf{j}} \tag{1.3.13}$$

Next we need to consider the domain of $\vec{\mathbf{r}}'$. Here, $\mathbf{D}_{\vec{\mathbf{r}}'} : (-\infty, \infty)$.

Finally we need to determine if there are any values of t in $\mathbf{D}_{\vec{\mathbf{r}}}$ where $\vec{\mathbf{r}}'(t) = \vec{\mathbf{0}}$.

To do this, we set each component of $\vec{\mathbf{r}}'(t)$ equal to $\mathbf{0}$ and solve for t.

$$t^2 - 6t = 0 \quad \text{and} \quad 2t - 2 = 0 \tag{1.3.14}$$
$$6t(t - 1) = 0 \qquad 2(t - 1) = 0 \tag{1.3.15}$$
$$t = 0 \quad \text{or} \quad t = 1 \qquad t = 1 \tag{1.3.16}$$

Since $t = 1$ makes both components of $\vec{\mathbf{r}}'$ equal 0, we have $\vec{\mathbf{r}}'(1) = \vec{\mathbf{0}}$, so we know that $\vec{\mathbf{r}}$ is **not smooth** at $t = 1$.

Note however that $\vec{\mathbf{r}}$ is smooth at $t = 0$, because although one of the components of $\vec{\mathbf{r}}'$ is 0, the other one is not.

Therefore we can conclude that $\vec{\mathbf{r}}$ is smooth on the open intervals:

$$(-\infty, 1) \quad \text{and} \quad (1, \infty) \tag{1.3.17}$$

Note in the figure that there is a **cusp** in the curve when $t = 1$.

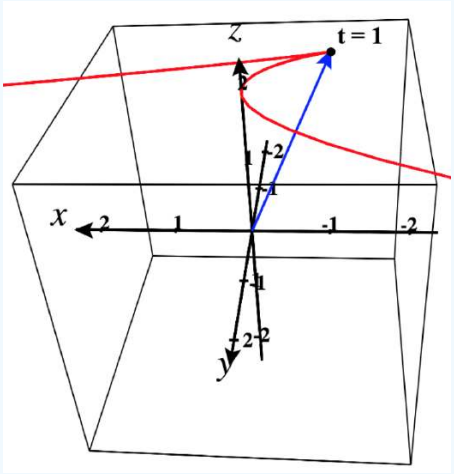


Figure 1.3.4: Plot of $\vec{\mathbf{r}}(t) = (2t^3 - 3t^2)\ \hat{\mathbf{i}} + (t^2 - 2t)\ \hat{\mathbf{j}} + 2\ \hat{\mathbf{k}}$

b. The domain of $\vec{\mathbf{r}}$ is $\mathbf{D}_{\vec{\mathbf{r}}} : (-\infty, -1] \cup [1, \infty)$, since $t^2 - 1$ is less than 0 between -1 and 1.

Now we calculate $\vec{\mathbf{r}}'(t)$. Since $\vec{\mathbf{r}}(t) = t^3\ \hat{\mathbf{i}} + \sqrt{t^2 - 1}\,\hat{\mathbf{j}}$, we have

$$\vec{\mathbf{r}}'(t) = 3t^2\ \hat{\mathbf{i}} + \frac{t}{\sqrt{t^2 - 1}}\ \hat{\mathbf{j}} \tag{1.3.18}$$

Then, since we cannot have zero in the denominator, $t = -1$ and $t = 1$ are not in the domain of $\vec{\mathbf{r}}'$. Thus, $\mathbf{D}_{\vec{\mathbf{r}}'} : (-\infty, -1) \cup (1, \infty)$.

Lastly we determine if there are any values of t in $\mathbf{D}_{\vec{\mathbf{r}}}$ where $\vec{\mathbf{r}}'(t) = \vec{\mathbf{0}}$.

Setting each component of $\vec{\mathbf{r}}'(t)$ equal to 0 and solve for t we get:

$$t^2 = 0 \quad \text{and} \quad \frac{t}{\sqrt{t^2 - 1}} = 0 \tag{1.3.19}$$

$$t = 0 \qquad\qquad t = 0 \tag{1.3.20}$$

If $t = 0$ were in the domain of $\vec{\mathbf{r}}'$, then $\vec{\mathbf{r}}$ would not be smooth there, but since it is not in the domain, we have nothing to remove, and we can conclude that $\vec{\mathbf{r}}$ is smooth on the entire domain of its derivative $\vec{\mathbf{r}}'$, that is,

$$\vec{\mathbf{r}} \text{ is smooth on the open intervals:} \quad (-\infty, -1) \quad \text{and} \quad (1, \infty). \tag{1.3.21}$$

Observe the two separate smooth pieces of this function in the graph below.

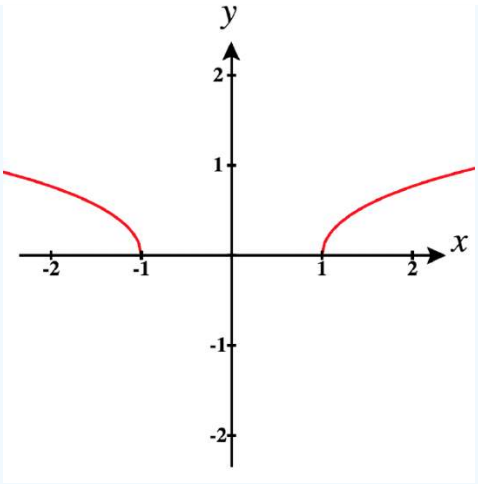


Figure 1.3.5: Plot of $\vec{\mathbf{r}}(t) = t^3\,\hat{\mathbf{i}} + \sqrt{t^2-1}\,\hat{\mathbf{j}}$

c.

We can extend to vector-valued functions the properties of the derivative that we presented previously. In particular, the constant multiple rule, the sum and difference rules, the product rule, and the chain rule all extend to vector-valued functions. However, in the case of the product rule, there are actually three extensions:

1. for a real-valued function multiplied by a vector-valued function,
2. for the dot product of two vector-valued functions, and
3. for the cross product of two vector-valued functions.

Theorem: Properties of the Derivative of Vector-Valued Functions

Let $\vec{\mathbf{r}}$ and $\vec{\mathbf{u}}$ be differentiable vector-valued functions of t, let f be a differentiable real-valued function of t, and let c be a scalar.

i.	$\frac{d}{dt}[c\vec{\mathbf{r}}(t)]$	$=$	$c\vec{\mathbf{r}}'(t)$	Scalar multiple
ii.	$\frac{d}{dt}[\vec{\mathbf{r}}(t) \pm \vec{\mathbf{u}}(t)]$	$=$	$\vec{\mathbf{r}}'(t) \pm \vec{\mathbf{u}}'(t)$	Sum and difference
iii.	$\frac{d}{dt}[f(t)\vec{\mathbf{u}}(t)]$	$=$	$f'(t)\vec{\mathbf{u}}(t) + f(t)\vec{\mathbf{u}}'(t)$	Scalar product
iv.	$\frac{d}{dt}[\vec{\mathbf{r}}(t) \cdot \vec{\mathbf{u}}(t)]$	$=$	$\vec{\mathbf{r}}'(t) \cdot \vec{\mathbf{u}}(t) + \vec{\mathbf{r}}(t) \cdot \vec{\mathbf{u}}'(t)$	Dot product
v.	$\frac{d}{dt}[\vec{\mathbf{r}}(t) \times \vec{\mathbf{u}}(t)]$	$=$	$\vec{\mathbf{r}}'(t) \times \vec{\mathbf{u}}(t) + \vec{\mathbf{r}}(t) \times \vec{\mathbf{u}}'(t)$	Cross product
vi.	$\frac{d}{dt}[\vec{\mathbf{r}}(f(t))]$	$=$	$\vec{\mathbf{r}}'(f(t)) \cdot f'(t)$	Chain rule
vii.	If $\vec{\mathbf{r}}(t) \cdot \vec{\mathbf{r}}(t)$	$=$	c, then $\vec{\mathbf{r}}(t) \cdot \vec{\mathbf{r}}'(t) = 0$.	

Proof

The proofs of the first two properties follow directly from the definition of the derivative of a vector-valued function. The third property can be derived from the first two properties, along with the product rule. Let $\vec{\mathbf{u}}(t) = g(t)\,\hat{\mathbf{i}} + h(t)\,\hat{\mathbf{j}}$. Then

$$\begin{aligned}\frac{d}{dt}[f(t)\vec{\mathbf{u}}(t)] &= \frac{d}{dt}[f(t)(g(t)\,\hat{\mathbf{i}} + h(t)\,\hat{\mathbf{j}})] \\ &= \frac{d}{dt}[f(t)g(t)\,\hat{\mathbf{i}} + f(t)h(t)\,\hat{\mathbf{j}}] \\ &= \frac{d}{dt}[f(t)g(t)]\,\hat{\mathbf{i}} + \frac{d}{dt}[f(t)h(t)]\,\hat{\mathbf{j}} \\ &= (f'(t)g(t) + f(t)g'(t))\,\hat{\mathbf{i}} + (f'(t)h(t) + f(t)h'(t))\,\hat{\mathbf{j}} \\ &= f'(t)\vec{\mathbf{u}}(t) + f(t)\vec{\mathbf{u}}'(t).\end{aligned}$$

To prove property iv. let $\vec{\mathbf{r}}(t) = f_1(t)\,\hat{\mathbf{i}} + g_1(t)\,\hat{\mathbf{j}}$ and $\vec{\mathbf{u}}(t) = f_2(t)\,\hat{\mathbf{i}} + g_2(t)\,\hat{\mathbf{j}}$. Then

$$\begin{aligned}\frac{d}{dt}[\vec{\mathbf{r}}(t)\cdot\vec{\mathbf{u}}(t)] &= \frac{d}{dt}[f_1(t)f_2(t) + g_1(t)g_2(t)] \\ &= f_1{}'(t)f_2(t) + f_1(t)f_2{}'(t) + g_1{}'(t)g_2(t) + g_1(t)g_2{}'(t) = f_1{}'(t)f_2(t) + g_1{}'(t)g_2(t) + f_1(t)f_2{}'(t) + g_1(t)g_2{}'(t) \\ &= (f_1{}'\,\hat{\mathbf{i}} + g_1{}'\,\hat{\mathbf{j}})\cdot(f_2\,\hat{\mathbf{i}} + g_2\,\hat{\mathbf{j}}) + (f_1\,\hat{\mathbf{i}} + g_1\,\hat{\mathbf{j}})\cdot(f_2{}'\,\hat{\mathbf{i}} + g_2{}'\,\hat{\mathbf{j}}) \\ &= \vec{\mathbf{r}}'(t)\cdot\vec{\mathbf{u}}(t) + \vec{\mathbf{r}}(t)\cdot\vec{\mathbf{u}}'(t).\end{aligned}$$

The proof of property v. is similar to that of property iv. Property vi. can be proved using the chain rule. Last, property vii. follows from property iv:

$$\begin{aligned}\frac{d}{dt}[\vec{\mathbf{r}}(t)\cdot\vec{\mathbf{r}}(t)] &= \frac{d}{dt}[c] \\ \vec{\mathbf{r}}'(t)\cdot\vec{\mathbf{r}}(t) + \vec{\mathbf{r}}(t)\cdot\vec{\mathbf{r}}'(t) &= 0 \\ 2\vec{\mathbf{r}}(t)\cdot\vec{\mathbf{r}}'(t) &= 0 \\ \vec{\mathbf{r}}(t)\cdot\vec{\mathbf{r}}'(t) &= 0\end{aligned}$$



Now for some examples using these properties.

Example 1.3.5: Using the Properties of Derivatives of Vector-Valued Functions

Given the vector-valued functions

$$\vec{\mathbf{r}}(t) = (6t+8)\,\hat{\mathbf{i}} + (4t^2 + 2t - 3)\,\hat{\mathbf{j}} + 5t\,\hat{\mathbf{k}}$$

and

$$\vec{\mathbf{u}}(t) = (t^2 - 3)\,\hat{\mathbf{i}} + (2t + 4)\,\hat{\mathbf{j}} + (t^3 - 3t)\,\hat{\mathbf{k}},$$

calculate each of the following derivatives using the properties of the derivative of vector-valued functions.

a. $\dfrac{d}{dt}[\vec{\mathbf{r}}(t) \cdot \vec{\mathbf{u}}(t)]$

b. $\dfrac{d}{dt}[\vec{\mathbf{u}}(t) \times \vec{\mathbf{u}}'(t)]$

Solution

We have $\vec{\mathbf{r}}'(t) = 6\,\hat{\mathbf{i}} + (8t + 2)\,\hat{\mathbf{j}} + 5\,\hat{\mathbf{k}}$ and $\vec{\mathbf{u}}'(t) = 2t\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + (3t^2 - 3)\,\hat{\mathbf{k}}$. Therefore, according to **property iv**:

1.
$$\begin{aligned}\frac{d}{dt}[\vec{\mathbf{r}}(t) \cdot \vec{\mathbf{u}}(t)] &= \vec{\mathbf{r}}'(t) \cdot \vec{\mathbf{u}}(t) + \vec{\mathbf{r}}(t) \cdot \vec{\mathbf{u}}'(t) \\ &= (6\,\hat{\mathbf{i}} + (8t + 2)\,\hat{\mathbf{j}} + 5\,\hat{\mathbf{k}}) \cdot ((t^2 - 3)\,\hat{\mathbf{i}} + (2t + 4)\,\hat{\mathbf{j}} + (t^3 - 3t)\,\hat{\mathbf{k}}) \\ &+((6t + 8)\,\hat{\mathbf{i}} + (4t^2 + 2t - 3)\,\hat{\mathbf{j}} + 5t\,\hat{\mathbf{k}}) \cdot (2t\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + (3t^2 - 3)\,\hat{\mathbf{k}}) \\ &= 6(t^2 - 3) + (8t + 2)(2t + 4) + 5(t^3 - 3t) \\ &+2t(6t + 8) + 2(4t^2 + 2t - 3) + 5t(3t^2 - 3) \\ &= 20t^3 + 42t^2 + 26t - 16.\end{aligned}$$

2. First, we need to adapt **property v** for this problem:

$$\frac{d}{dt}[\vec{\mathbf{u}}(t) \times \vec{\mathbf{u}}'(t)] = \vec{\mathbf{u}}'(t) \times \vec{\mathbf{u}}'(t) + \vec{\mathbf{u}}(t) \times \vec{\mathbf{u}}''(t).$$

Recall that the cross product of any vector with itself is zero. Furthermore, $\vec{\mathbf{u}}''(t)$ represents the second derivative of $\vec{\mathbf{u}}(t)$:

$$\vec{\mathbf{u}}''(t) = \frac{d}{dt}[\vec{\mathbf{u}}'(t)] = \frac{d}{dt}[2t\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + (3t^2 - 3)\,\hat{\mathbf{k}}] = 2\,\hat{\mathbf{i}} + 6t\,\hat{\mathbf{k}}.$$

Therefore,

$$\begin{aligned}\frac{d}{dt}[\vec{\mathbf{u}}(t) \times \vec{\mathbf{u}}'(t)] &= 0 + ((t^2 - 3)\,\hat{\mathbf{i}} + (2t + 4)\,\hat{\mathbf{j}} + (t^3 - 3t)\,\hat{\mathbf{k}}) \times (2\,\hat{\mathbf{i}} + 6t\,\hat{\mathbf{k}}) \\ &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ t^2 - 3 & 2t + 4 & t^3 - 3t \\ 2 & 0 & 6t \end{vmatrix} \\ &= 6t(2t + 4)\,\hat{\mathbf{i}} - (6t(t^2 - 3) - 2(t^3 - 3t))\,\hat{\mathbf{j}} - 2(2t + 4)\,\hat{\mathbf{k}} \\ &= (12t^2 + 24t)\,\hat{\mathbf{i}} + (12t - 4t^3)\,\hat{\mathbf{j}} - (4t + 8)\,\hat{\mathbf{k}}.\end{aligned}$$



Exercise 1.3.3

Calculate $\frac{d}{dt}[\vec{\mathbf{r}}(t) \cdot \vec{\mathbf{r}}'(t)]$ and $\frac{d}{dt}[\vec{\mathbf{u}}(t) \times \vec{\mathbf{r}}(t)]$ for the vector-valued functions:

- $\vec{\mathbf{r}}(t) = \cos t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}} - e^{2t}\hat{\mathbf{k}}$
- $\vec{\mathbf{u}}(t) = t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}} + \cos t\,\hat{\mathbf{k}}$,

Hint

Follow the same steps as in Example **1.3.3**.

Answer

$$\frac{d}{dt}[\vec{\mathbf{r}}(t) \cdot \vec{\mathbf{r}}'(t)] = 8e^{4t}$$

$$\frac{d}{dt}[\vec{\mathbf{u}}(t) \times \vec{\mathbf{r}}(t)] = -(e^{2t}(\cos t + 2\sin t) + \cos 2t)\,\hat{\mathbf{i}} + (e^{2t}(2t+1) - \sin 2t)\,\hat{\mathbf{j}} + (t\cos t + \sin t - \cos 2t)\,\hat{\mathbf{k}}$$

Tangent Vectors and Unit Tangent Vectors

Recall that the derivative at a point can be interpreted as the slope of the tangent line to the graph at that point. In the case of a vector-valued function, the derivative provides a tangent vector to the curve represented by the function. Consider the vector-valued function

$$\vec{\mathbf{r}}(t) = \cos t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}} \tag{1.3.22}$$

The derivative of this function is

$$\vec{\mathbf{r}}'(t) = -\sin t\,\hat{\mathbf{i}} + \cos t\,\hat{\mathbf{j}}$$

If we substitute the value $t = \pi/6$ into both functions we get

$$\vec{\mathbf{r}}\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}\,\hat{\mathbf{i}} + \frac{1}{2}\,\hat{\mathbf{j}}$$

and

$$\vec{\mathbf{r}}'\left(\frac{\pi}{6}\right) = -\frac{1}{2}\,\hat{\mathbf{i}} + \frac{\sqrt{3}}{2}\,\hat{\mathbf{j}}.$$

The graph of this function appears in Figure 1.3.1, along with the vectors $\vec{\mathbf{r}}\left(\frac{\pi}{6}\right)$ and $\vec{\mathbf{r}}'\left(\frac{\pi}{6}\right)$.

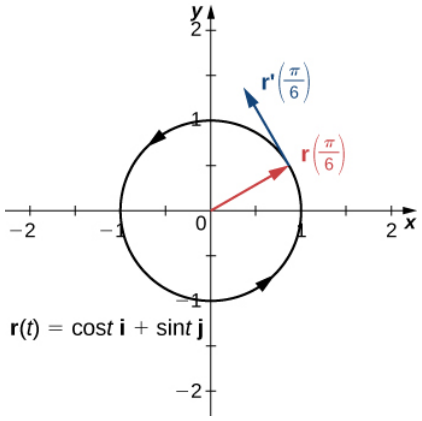


Figure 1.3.1: The tangent line at a point is calculated from the derivative of the vector-valued function $\vec{\mathbf{r}}(t)$.

Notice that the vector $\vec{\mathbf{r}}'\left(\frac{\pi}{6}\right)$ is tangent to the circle at the point corresponding to $t = \frac{\pi}{6}$. This is an example of a tangent vector to the plane curve defined by Equation 1.3.22.

Definition: principal unit tangent vector

Let C be a curve defined by a vector-valued function $\vec{\mathbf{r}}$, and assume that $\vec{\mathbf{r}}'(t)$ exists when $\mathbf{t} = \mathbf{t}_0$ A tangent vector $\vec{\mathbf{r}}$ at $t = t_0$ is any vector such that, when the tail of the vector is placed at point $\vec{\mathbf{r}}(t_0)$ on the graph, vector $\vec{\mathbf{r}}$ is tangent to curve C. Vector $\vec{\mathbf{r}}\,'(t_0)$ is an example of a tangent vector at point $t = t_0$. Furthermore, assume that $\vec{\mathbf{r}}'(t) \neq \mathbf{0}$. The principal unit tangent vector at t is defined to be

$$\vec{\mathbf{T}}(t) = \frac{\vec{\mathbf{r}}'(t)}{\|\vec{\mathbf{r}}'(t)\|}, \tag{1.3.23}$$

provided $\|\vec{\mathbf{r}}'(t)\| \neq 0$.

The unit tangent vector is exactly what it sounds like: a unit vector that is tangent to the curve. To calculate a unit tangent vector, first find the derivative $\vec{\mathbf{r}}\,'(t)$. Second, calculate the magnitude of the derivative. The third step is to divide the derivative by its magnitude.

Example 1.3.6: Finding a Unit Tangent Vector

Find the unit tangent vector for each of the following vector-valued functions:

a. $\vec{\mathbf{r}}(t) = \cos t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}}$
b. $\vec{\mathbf{u}}(t) = (3t^2 + 2t)\,\hat{\mathbf{i}} + (2 - 4t^3)\,\hat{\mathbf{j}} + (6t + 5)\,\hat{\mathbf{k}}$

Solution

a.
$$\begin{aligned} \textbf{First step:} \quad \vec{\mathbf{r}}'(t) &= -\sin t\,\hat{\mathbf{i}} + \cos t\,\hat{\mathbf{j}} \\ \textbf{Second step:} \quad \|\vec{\mathbf{r}}'(t)\| &= \sqrt{(-\sin t)^2 + (\cos t)^2} = 1 \\ \textbf{Third step:} \quad \vec{\mathbf{T}}(t) &= \frac{\vec{\mathbf{r}}'(t)}{\|\vec{\mathbf{r}}'(t)\|} = \frac{-\sin t\,\hat{\mathbf{i}} + \cos t\,\hat{\mathbf{j}}}{1} = -\sin t\,\hat{\mathbf{i}} + \cos t\,\hat{\mathbf{j}} \end{aligned}$$

https://math.libretexts.org/@go/page/84721

b.

$$\begin{aligned}
\text{First step:} \quad \vec{\mathbf{r}}'(t) &= (6t+2)\,\hat{\mathbf{i}} - 12t^2\,\hat{\mathbf{j}} + 6\,\hat{\mathbf{k}} \\
\text{Second step:} \quad \|\vec{\mathbf{r}}'(t)\| &= \sqrt{(6t+2)^2 + (-12t^2)^2 + 6^2} \\
&= \sqrt{144t^4 + 36t^2 + 24t + 40} \\
&= 2\sqrt{36t^4 + 9t^2 + 6t + 10} \\
\text{Third step:} \quad \vec{\mathbf{T}}(t) &= \frac{\vec{\mathbf{r}}'(t)}{\|\vec{\mathbf{r}}'(t)\|} = \frac{(6t+2)\,\hat{\mathbf{i}} - 12t^2\,\hat{\mathbf{j}} + 6\,\hat{\mathbf{k}}}{2\sqrt{36t^4 + 9t^2 + 6t + 10}} \\
&= \frac{3t+1}{\sqrt{36t^4 + 9t^2 + 6t + 10}}\,\hat{\mathbf{i}} - \frac{6t^2}{\sqrt{36t^4 + 9t^2 + 6t + 10}}\,\hat{\mathbf{j}} + \frac{3}{\sqrt{36t^4 + 9t^2 + 6t + 10}}\,\hat{\mathbf{k}}
\end{aligned}$$

Exercise 1.3.4

Find the unit tangent vector for the vector-valued function

$$\vec{\mathbf{r}}(t) = (t^2 - 3)\,\hat{\mathbf{i}} + (2t+1)\,\hat{\mathbf{j}} + (t-2)\,\hat{\mathbf{k}}.$$

Hint

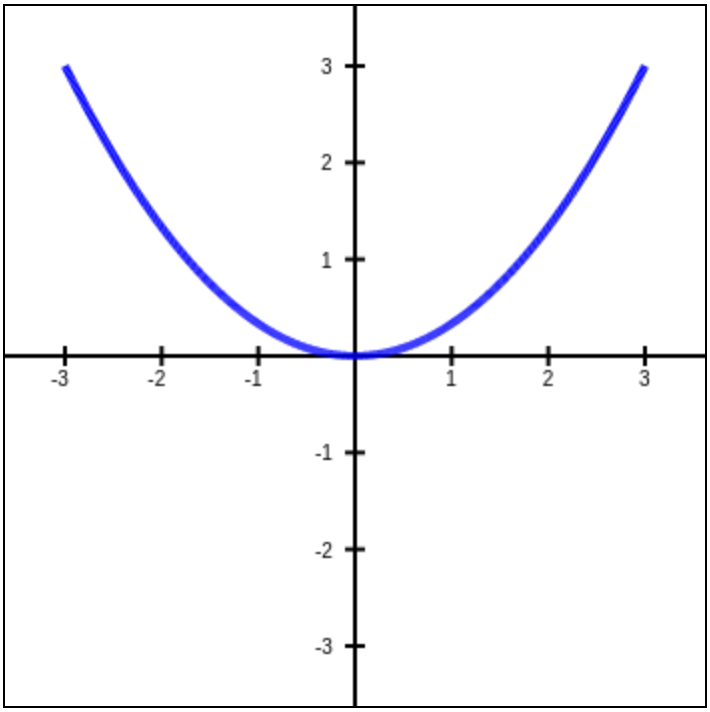
Follow the same steps as in Example **1.3.4.**

Answer

$$\vec{\mathbf{T}}(t) = \frac{2t}{\sqrt{4t^2+5}}\,\hat{\mathbf{i}} + \frac{2}{\sqrt{4t^2+5}}\,\hat{\mathbf{j}} + \frac{1}{\sqrt{4t^2+5}}\,\hat{\mathbf{k}}$$

Observing the Tangent Vectors of a Vectors Valued Function

Below shows the graph of a vector valued function, but a vector values function is more than just a static graph. If you hit the Animate button, you will see the tangent vector move and change as time, t progresses. You can also choose to observe the unit tangent vector.



$\vec{\mathbf{r}}(t) = < t, \frac{1}{3}t^2 >, -3 < t < 3$

◉ Tangent Vector ○ Unit Tangent Vector

Integrals of Vector-Valued Functions

We introduced antiderivatives of real-valued functions in Antiderivatives and definite integrals of real-valued functions in The Definite Integral. Each of these concepts can be extended to vector-valued functions. Also, just as we can calculate the derivative of a vector-valued function by differentiating the component functions separately, we can calculate the antiderivative in the same manner. Furthermore, the Fundamental Theorem of Calculus applies to vector-valued functions as well.

The antiderivative of a vector-valued function appears in applications. For example, if a vector-valued function represents the velocity of an object at time t, then its antiderivative represents position. Or, if the function represents the acceleration of the object at a given time, then the antiderivative represents its velocity.

Definition: Definite and Indefinite Integrals of Vector-Valued Functions

Let f, g, and h be integrable real-valued functions over the closed interval $[a, b]$.

1. The indefinite integral of a vector-valued function $\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}}$ is

$$\int [f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}}]\,dt = \left[\int f(t)\,dt\right]\hat{\mathbf{i}} + \left[\int g(t)\,dt\right]\hat{\mathbf{j}}. \tag{1.3.24}$$

The definite integral of a vector-valued function is

$$\int_a^b [f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}}]\,dt = \left[\int_a^b f(t)\,dt\right]\hat{\mathbf{i}} + \left[\int_a^b g(t)\,dt\right]\hat{\mathbf{j}}. \tag{1.3.25}$$

2. The indefinite integral of a vector-valued function $\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} + h(t)\,\hat{\mathbf{k}}$ is

$$\int [f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} + h(t)\,\hat{\mathbf{k}}]\,dt = \left[\int f(t)\,dt\right]\hat{\mathbf{i}} + \left[\int g(t)\,dt\right]\hat{\mathbf{j}} + \left[\int h(t)\,dt\right]\hat{\mathbf{k}}. \tag{1.3.26}$$

The definite integral of the vector-valued function is

$$\begin{aligned}&\int_a^b [f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} + h(t)\,\hat{\mathbf{k}}]\,dt \\ &= \left[\int_a^b f(t)\,dt\right]\hat{\mathbf{i}} + \left[\int_a^b g(t)\,dt\right]\hat{\mathbf{j}} + \left[\int_a^b h(t)\,dt\right]\hat{\mathbf{k}}.\end{aligned} \tag{1.3.27}$$

Since the indefinite integral of a vector-valued function involves indefinite integrals of the component functions, each of these component integrals contains an integration constant. They can all be different. For example, in the two-dimensional case, we can have

$$\int f(t)\,dt = F(t) + C_1 \text{ and } \int g(t)\,dt = G(t) + C_2,$$

where F and G are antiderivatives of f and g, respectively. Then

$$\begin{aligned}\int [f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}}]\,dt &= \left[\int f(t)\,dt\right]\hat{\mathbf{i}} + \left[\int g(t)\,dt\right]\hat{\mathbf{j}} \\ &= (F(t) + C_1)\,\hat{\mathbf{i}} + (G(t) + C_2)\,\hat{\mathbf{j}} \\ &= F(t)\,\hat{\mathbf{i}} + G(t)\,\hat{\mathbf{j}} + C_1\,\hat{\mathbf{i}} + C_2\,\hat{\mathbf{j}} \\ &= F(t)\,\hat{\mathbf{i}} + G(t)\,\hat{\mathbf{j}} + \vec{\mathbf{C}}\end{aligned}$$

where $\vec{\mathbf{C}} = C_1\,\hat{\mathbf{i}} + C_2\,\hat{\mathbf{j}}$. Therefore, the *integration constants* becomes a *constant vector*.

Example 1.3.7: Integrating Vector-Valued Functions

Calculate each of the following integrals:

a. $\int [(3t^2+2t)\,\hat{\mathbf{i}} + (3t-6)\,\hat{\mathbf{j}} + (6t^3+5t^2-4)\,\hat{\mathbf{k}}]\,dt$

b. $\int [\langle t, t^2, t^3\rangle \times \langle t^3, t^2, t\rangle]\,dt$

c. $\int_0^{\frac{\pi}{3}} [\sin 2t\,\hat{\mathbf{i}} + \tan t\,\hat{\mathbf{j}} + e^{-2t}\,\hat{\mathbf{k}}]\,dt$

Solution

a. We use the first part of the definition of the integral of a space curve:

$$\begin{aligned}\int [(3t^2+2t)\,\hat{\mathbf{i}} + (3t-6)\,\hat{\mathbf{j}} + (6t^3+5t^2-4)\,\hat{\mathbf{k}}]\,dt &= \left[\int 3t^2+2t\,dt\right]\hat{\mathbf{i}} + \left[\int 3t-6\,dt\right]\hat{\mathbf{j}} \\ &\quad + \left[\int 6t^3+5t^2-4\,dt\right]\hat{\mathbf{k}} \\ &= (t^3+t^2)\,\hat{\mathbf{i}} + \left(\frac{3}{2}t^2-6t\right)\hat{\mathbf{j}} \\ &\quad + \left(\frac{3}{2}t^4+\frac{5}{3}t^3-4t\right)\hat{\mathbf{k}} + \vec{\mathbf{C}}.\end{aligned}$$

b.

c. First calculate $\langle t, t^2, t^3\rangle \times \langle t^3, t^2, t\rangle$:

$$\begin{aligned}\langle t, t^2, t^3\rangle \times \langle t^3, t^2, t\rangle &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ t & t^2 & t^3 \\ t^3 & t^2 & t \end{vmatrix} \\ &= (t^2(t) - t^3(t^2))\,\hat{\mathbf{i}} - (t^2 - t^3(t^3))\,\hat{\mathbf{j}} + (t(t^2) - t^2(t^3))\,\hat{\mathbf{k}} \\ &= (t^3 - t^5)\,\hat{\mathbf{i}} + (t^6 - t^2)\,\hat{\mathbf{j}} + (t^3 - t^5)\,\hat{\mathbf{k}}.\end{aligned}$$

Next, substitute this back into the integral and integrate:

$$\begin{aligned}\int [\langle t, t^2, t^3\rangle \times \langle t^3, t^2, t\rangle]\,dt &= \int (t^3 - t^5)\,\hat{\mathbf{i}} + (t^6 - t^2)\,\hat{\mathbf{j}} + (t^3 - t^5)\,\hat{\mathbf{k}}\,dt \\ &= \left(\frac{t^4}{4} - \frac{t^6}{6}\right)\hat{\mathbf{i}} + \left(\frac{t^7}{7} - \frac{t^3}{3}\right)\hat{\mathbf{j}} + \left(\frac{t^4}{4} - \frac{t^6}{6}\right)\hat{\mathbf{k}} + \vec{\mathbf{C}}.\end{aligned}$$

d. Use the second part of the definition of the integral of a space curve:

$$\begin{aligned}\int_0^{\frac{\pi}{3}} [\sin 2t\,\hat{\mathbf{i}} + \tan t\,\hat{\mathbf{j}} + e^{-2t}\,\hat{\mathbf{k}}]\,dt &= \left[\int_0^{\frac{\pi}{3}} \sin 2t\,dt\right]\hat{\mathbf{i}} + \left[\int_0^{\frac{\pi}{3}} \tan t\,dt\right]\hat{\mathbf{j}} + \left[\int_0^{\frac{\pi}{3}} e^{-2t}\,dt\right]\hat{\mathbf{k}} \\ &= \left(-\tfrac{1}{2}\cos 2t\right)\Big|_0^{\pi/3}\hat{\mathbf{i}} - \left(\ln|\cos t|\right)\Big|_0^{\pi/3}\hat{\mathbf{j}} - \left(\tfrac{1}{2}e^{-2t}\right)\Big|_0^{\pi/3}\hat{\mathbf{k}} \\ &= \left(-\tfrac{1}{2}\cos\tfrac{2\pi}{3} + \tfrac{1}{2}\cos 0\right)\hat{\mathbf{i}} - \left(\ln\left(\cos\tfrac{\pi}{3}\right) - \ln(\cos 0)\right)\hat{\mathbf{j}} \\ &\quad - \left(\tfrac{1}{2}e^{-2\pi/3} - \tfrac{1}{2}e^{-2(0)}\right)\hat{\mathbf{k}} \\ &= \left(\tfrac{1}{4} + \tfrac{1}{2}\right)\hat{\mathbf{i}} - (-\ln 2)\,\hat{\mathbf{j}} - \left(\tfrac{1}{2}e^{-2\pi/3} - \tfrac{1}{2}\right)\hat{\mathbf{k}} \\ &= \tfrac{3}{4}\,\hat{\mathbf{i}} + (\ln 2)\,\hat{\mathbf{j}} + \left(\tfrac{1}{2} - \tfrac{1}{2}e^{-2\pi/3}\right)\hat{\mathbf{k}}.\end{aligned}$$



Exercise 1.3.5

Calculate the following integral:

$$\int_1^3 [(2t+4)\,\hat{\mathbf{i}} + (3t^2-4t)\,\hat{\mathbf{j}}]\,dt$$

Hint

Use the definition of the definite integral of a plane curve.

Answer

$$\int_1^3 [(2t+4)\,\hat{\mathbf{i}} + (3t^2-4t)\,\hat{\mathbf{j}}]\,dt = 16\,\hat{\mathbf{i}} + 10\,\hat{\mathbf{j}}$$

Integration in the Context of Motion

Above, we defined velocity as the derivative of position and acceleration as the derivative of velocity. Integration allows us to go the other way! As you learned in first semester calculus, integration allows us to generate an object's velocity as a function of time given its acceleration and an initial velocity. We can also find the object's position function by integrating the velocity and using an initial position.

Let's consider an example.

Example 1.3.8: Finding an object's velocity from its acceleration

An object has an acceleration over time given by $\vec{\mathbf{a}}(t) = \sin t\,\hat{\mathbf{i}} - 2t\,\hat{\mathbf{j}}$, and it's initial velocity was $\vec{\mathbf{v}}(0) = 2\,\hat{\mathbf{i}} + \hat{\mathbf{j}}$.

a. Find the object's velocity as a function of time.

b. Assuming the object was located at the point (2, 3, -1) when time $t = 0$, determine the object's position function and find its location at time $t = 3$ sec.

Solution

a. First, we find the velocity as the antiderivative of the acceleration.

$$\vec{\mathbf{v}}(t) = \int \vec{\mathbf{a}}(t)\, dt = \int \left(\sin t\, \hat{\mathbf{i}} - 2t\, \hat{\mathbf{j}}\right) dt$$
$$= -\cos t\, \hat{\mathbf{i}} - t^2\, \hat{\mathbf{j}} + \vec{\mathbf{C}}_1$$

Now we use the initial velocity to determine $\vec{\mathbf{C}}_1$.

$$\vec{\mathbf{v}}(0) = -\cos 0\, \hat{\mathbf{i}} - (0)^2\, \hat{\mathbf{j}} + \vec{\mathbf{C}}_1 = 2\hat{\mathbf{i}} + \hat{\mathbf{j}}$$
$$-\hat{\mathbf{i}} + \vec{\mathbf{C}}_1 = 2\hat{\mathbf{i}} + \hat{\mathbf{j}}$$

$$\text{And so} \quad \vec{\mathbf{C}}_1 = 3\hat{\mathbf{i}} + \hat{\mathbf{j}}$$

Incorporating this constant vector into our velocity function from above, we obtain the velocity describing this object's motion over time:

$$\vec{\mathbf{v}}(t) = (3 - \cos t)\, \hat{\mathbf{i}} + \left(1 - t^2\right) \hat{\mathbf{j}} \tag{1.3.28}$$

b. Since the object's position is at the point (2, 3, -1) when time $t = 0$, we know $\vec{\mathbf{r}}(0) = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}}$.

Now, to determine the object's position function, we integrate it's velocity.

$$\vec{\mathbf{r}}(t) = \int \vec{\mathbf{v}}(t)\, dt = \int \left[(3 - \cos t)\, \hat{\mathbf{i}} + \left(1 - t^2\right) \hat{\mathbf{j}}\right] dt$$
$$= (3t - \sin t)\, \hat{\mathbf{i}} + \left(t - \frac{t^3}{3}\right) \hat{\mathbf{j}} + \vec{\mathbf{C}}_2$$

Now we use the initial position to determine $\vec{\mathbf{C}}_2$.

$$\vec{\mathbf{r}}(0) = (3(0) - \sin 0)\, \hat{\mathbf{i}} + \left(0 - \frac{(0)^3}{3}\right) \hat{\mathbf{j}} + \vec{\mathbf{C}}_2 = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}}$$

$$\text{And so} \quad \vec{\mathbf{C}}_2 = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}}$$

Incorporating this constant vector into our position function from above, we obtain:

$$\vec{\mathbf{r}}(t) = (2 + 3t - \sin t)\, \hat{\mathbf{i}} + \left(3 + t - \frac{t^3}{3}\right) \hat{\mathbf{j}} - \hat{\mathbf{k}} \tag{1.3.29}$$

To find the object's position at time $t = 3$ seconds, we just evaluate this position function at $t = 3$.

$$\vec{\mathbf{r}}(3) = (2 + 3(3) - \sin 3)\, \hat{\mathbf{i}} + \left(3 + 3 - \frac{(3)^3}{3}\right) \hat{\mathbf{j}} - \hat{\mathbf{k}}$$
$$= (11 - \sin 3)\, \hat{\mathbf{i}} - 3\hat{\mathbf{j}} - \hat{\mathbf{k}}$$

This position vector indicates that the object will be located at the point $(11 - \sin 3, -3, -1)$ at time $t = 3$ seconds.

The Meaning of Definite Integrals of Vector-Valued Functions

Now let's turn our attention to the meaning of a definite integral of a vector-valued function. The context in which this will make the most sense is where the function we integrate is a velocity function. That is,

$$\int_a^b \vec{\mathbf{v}}(t)\, dt \tag{1.3.30}$$

We know that the antiderivative of velocity is position and that this definite integral gives us the change in position over the time interval, $a \le t \le b$. In other words,

$$\int_a^b \vec{\mathbf{v}}(t)\, dt = \vec{\mathbf{r}}(b) - \vec{\mathbf{r}}(a) \tag{1.3.31}$$

Thus, the definite integral of velocity over a time interval $a \le t \le b$ gives us the **displacement vector** that indicates the change in position over this time interval.

In general, the definite integral

$$\int_a^b \vec{\mathbf{r}}(t)\ dt = \vec{\mathbf{q}}(b) - \vec{\mathbf{q}}(a), \text{ where } \vec{\mathbf{q}}(t) \text{ is the antiderivative of } \vec{\mathbf{r}}(t),$$

gives us a change in the antiderivative of our vector-valued function over the given inteval $[a, b]$. This will always be a constant vector that would fit from tip to tip of the vectors given by the antiderivative function at $t = a$ and $t = b$, respectively (assuming the vectors were placed in standard position).

Summary

- To calculate the derivative of a vector-valued function, calculate the derivatives of the component functions, then put them back into a new vector-valued function.
- Many of the properties of differentiation of scalar functions also apply to vector-valued functions.
- The derivative of a vector-valued function $\vec{\mathbf{r}}(t)$ is also a tangent vector to the curve. The unit tangent vector $\vec{\mathbf{T}}(t)$ is calculated by dividing the derivative of a vector-valued function by its magnitude.
- The antiderivative of a vector-valued function is found by finding the antiderivatives of the component functions, then putting them back together in a vector-valued function.
- The definite integral of a vector-valued function is found by finding the definite integrals of the component functions, then putting them back together in a vector-valued function.

Key Equations

- **Derivative of a vector-valued function**

$$\vec{\mathbf{r}}'(t) = \lim_{\Delta t \to 0} \frac{\vec{\mathbf{r}}(t + \Delta t) - \vec{\mathbf{r}}(t)}{\Delta t}$$

- **Principal unit tangent vector**

$$\vec{\mathbf{T}}(t) = \frac{\vec{\mathbf{r}}'(t)}{\|\vec{\mathbf{r}}'(t)\|}$$

- **Indefinite integral of a vector-valued function**

$$\int [f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} + h(t)\,\hat{\mathbf{k}}]\,dt = \left[\int f(t)\,dt\right]\hat{\mathbf{i}} + \left[\int g(t)\,dt\right]\hat{\mathbf{j}} + \left[\int h(t)\,dt\right]\hat{\mathbf{k}}$$

- **Definite integral of a vector-valued function**

$$\int_a^b [f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} + h(t)\,\hat{\mathbf{k}}]\,dt = \left[\int_a^b f(t)\,dt\right]\hat{\mathbf{i}} + \left[\int_a^b g(t)\,dt\right]\hat{\mathbf{j}} + \left[\int_a^b h(t)\,dt\right]\hat{\mathbf{k}}$$

Glossary

definite integral of a vector-valued function

the vector obtained by calculating the definite integral of each of the component functions of a given vector-valued function, then using the results as the components of the resulting function

derivative of a vector-valued function

the derivative of a vector-valued function $\vec{\mathbf{r}}(t)$ is $\vec{\mathbf{r}}'(t) = \lim_{\Delta t \to 0} \frac{\vec{\mathbf{r}}(t+\Delta t) - \vec{\mathbf{r}}(t)}{\Delta t}$, provided the limit exists

indefinite integral of a vector-valued function

a vector-valued function with a derivative that is equal to a given vector-valued function

principal unit tangent vector

a unit vector tangent to a curve C

tangent vector

to $\vec{\mathbf{r}}(t)$ at $t = t_0$ any vector $\vec{\mathbf{v}}$ such that, when the tail of the vector is placed at point $\vec{\mathbf{r}}(t_0)$ on the graph, vector $\vec{\mathbf{v}}$ is tangent to curve C

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1.3E: Exercises

Taking Derivatives of Vector-Valued Functions

In questions 1 - 10, compute the derivative of each vector-valued function.

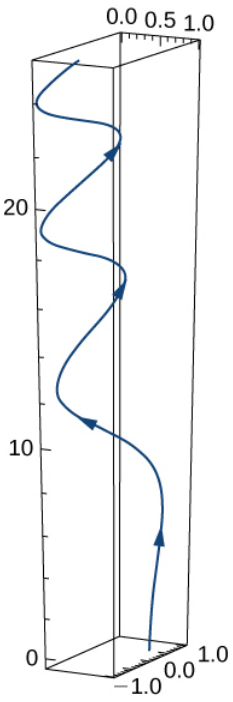
1) $\vec{\mathbf{r}}(t) = t^3\,\hat{\mathbf{i}} + 3t^2\,\hat{\mathbf{j}} + \frac{t^3}{6}\,\hat{\mathbf{k}}$

Answer

$\vec{\mathbf{r}}'(t) = 3t^2\,\hat{\mathbf{i}} + 6t\,\hat{\mathbf{j}} + \frac{1}{2}t^2\,\hat{\mathbf{k}}$

2) $\vec{\mathbf{r}}(t) = \sin(t)\,\hat{\mathbf{i}} + \cos(t)\,\hat{\mathbf{j}} + e^t\,\hat{\mathbf{k}}$

3) $\vec{\mathbf{r}}(t) = e^{-t}\,\hat{\mathbf{i}} + \sin(3t)\,\hat{\mathbf{j}} + 10\sqrt{t}\,\hat{\mathbf{k}}$. A sketch of the graph is shown here. Notice the varying periodic nature of the graph.



Answer

$\vec{\mathbf{r}}'(t) = -e^{-t}\,\hat{\mathbf{i}} + 3\cos(3t)\,\hat{\mathbf{j}} + \frac{5}{\sqrt{t}}\,\hat{\mathbf{k}}$

4) $\vec{\mathbf{r}}(t) = e^t\,\hat{\mathbf{i}} + 2e^t\,\hat{\mathbf{j}} + \hat{\mathbf{k}}$

5) $\vec{\mathbf{r}}(t) = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$

Answer

$\vec{\mathbf{r}}'(t) = \langle 0, 0, 0\rangle = \vec{\mathbf{0}}$

6) $\vec{\mathbf{r}}(t) = te^t\,\hat{\mathbf{i}} + t\ln(t)\,\hat{\mathbf{j}} + \sin(3t)\,\hat{\mathbf{k}}$

7) $\vec{\mathbf{r}}(t) = \left\langle \frac{1}{t+1},\ \arctan(t),\ \ln t^3 \right\rangle$

Answer

$\vec{\mathbf{r}}'(t) = \langle \frac{-1}{(t+1)^2}, \frac{1}{1+t^2}, \frac{3}{t}\rangle$

8) $\vec{\mathbf{r}}(t) = \langle \tan 2t,\ \sec 2t,\ \sin^2 t\rangle$

9) $\vec{\mathbf{r}}(t) = \langle 3,\ 4\sin(3t),\ t\cos(t)\rangle$

Answer

$\vec{\mathbf{r}}'(t) = \langle 0, 12\cos(3t), \cos t - t\sin t\rangle$

10) $\vec{\mathbf{r}}(t) = t^2\,\hat{\mathbf{i}} + te^{-2t}\,\hat{\mathbf{j}} - 5e^{-4t}\,\hat{\mathbf{k}}$

11) a. Describe and sketch the curve represented by the vector-valued function $\vec{\mathbf{r}}(t) = \langle 6t, 6t - t^2\rangle$.

b. Locate the highest point on the curve $\vec{\mathbf{r}}(t) = \langle 6t, 6t - t^2\rangle$ and give the value of the function at this point.

Answer

b. $\vec{\mathbf{r}}(t) = \langle 18, 9\rangle$ at $t = 3$

12) Find the parametric equations of the tangent line to the curve $\vec{\mathbf{r}}(t) = \langle t, t^2, t\rangle$ at $t = 2$.

13) Find the parametric equations of the tangent line to the curve $\vec{\mathbf{r}}(t) = \langle e^t, e^{-t}, 0\rangle$ at $t = 0$.

Answer

$x = 1 + t, \quad y = 1 - t, \quad z = 0$

14) Compute the first, second, and third derivatives of $\vec{\mathbf{r}}(t) = 3t\,\hat{\mathbf{i}} + 6\ln(t)\,\hat{\mathbf{j}} + 5e^{-3t}\,\hat{\mathbf{k}}$.

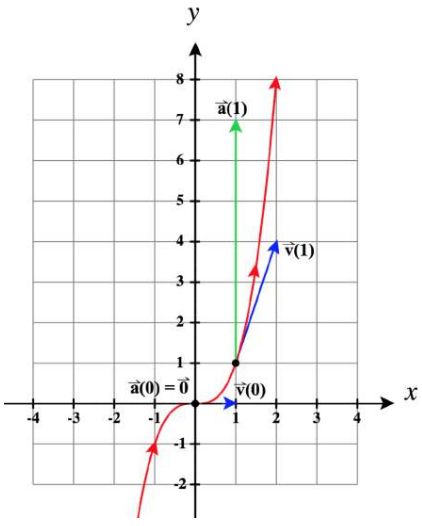
Describing Motion with Vector-Valued Functions

In questions 15 - 17, find the velocity and acceleration at the given times, plot the graph of the position function, and draw in the velocity and acceleration vectors at the corresponding locations on the curve.

15) $\vec{\mathbf{r}}(t) = t\,\hat{\mathbf{i}} + t^3\,\hat{\mathbf{j}}$, at $t = 0$ and at $t = 1$

Answer

	$t = 0$:	$t = 1$:
$\vec{\mathbf{r}}(t) = t\,\hat{\mathbf{i}} + t^3\,\hat{\mathbf{j}}$,	$\vec{\mathbf{r}}(0) = \vec{\mathbf{0}}$,	$\vec{\mathbf{r}}(1) = \hat{\mathbf{i}} + \hat{\mathbf{j}}$
$\vec{\mathbf{v}}(t) = \hat{\mathbf{i}} + 3t^2\,\hat{\mathbf{j}}$,	$\vec{\mathbf{v}}(0) = \hat{\mathbf{i}}$,	$\vec{\mathbf{v}}(1) = \hat{\mathbf{i}} + 3\,\hat{\mathbf{j}}$
$\vec{\mathbf{a}}(t) = 6t\,\hat{\mathbf{j}}$,	$\vec{\mathbf{a}}(0) = \vec{\mathbf{0}}$,	$\vec{\mathbf{a}}(1) = 6\,\hat{\mathbf{j}}$

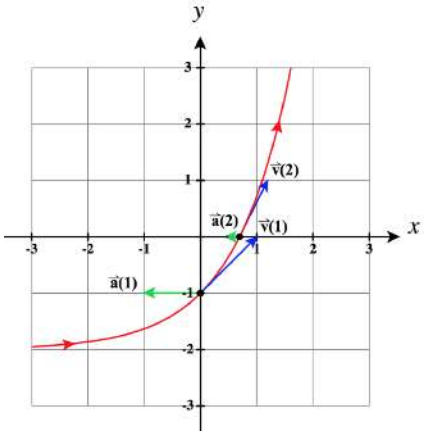


16) $\vec{\mathbf{r}}(t) = \cos t\,\hat{\mathbf{i}} + \sin 3t\,\hat{\mathbf{j}}$, at $t = 0$, at $t = \frac{\pi}{4}$, and at $t = \frac{\pi}{2}$

17) $\vec{\mathbf{r}}(t) = \ln t\,\hat{\mathbf{i}} + (t-2)\,\hat{\mathbf{j}}$, at $t = 1$ and at $t = 2$

Answer

	$t=1:$	$t=2:$
$\vec{\mathbf{r}}(t)=(\ln t)\,\hat{\mathbf{i}}+(t-2)\,\hat{\mathbf{j}},$	$\vec{\mathbf{r}}(1)=-\hat{\mathbf{j}},$	$\vec{\mathbf{r}}(2)=(\ln 2)\,\hat{\mathbf{i}}$
$\vec{\mathbf{v}}(t)=\frac{1}{t}\,\hat{\mathbf{i}}+\hat{\mathbf{j}},$	$\vec{\mathbf{v}}(1)=\hat{\mathbf{i}}+\hat{\mathbf{j}},$	$\vec{\mathbf{v}}(2)=\frac{1}{2}\hat{\mathbf{i}}+\hat{\mathbf{j}}$
$\vec{\mathbf{a}}(t)=-\frac{1}{t^2}\,\hat{\mathbf{i}},$	$\vec{\mathbf{a}}(1)=-\hat{\mathbf{i}},$	$\vec{\mathbf{a}}(2)=-\frac{1}{4}\,\hat{\mathbf{i}}$



In questions 18 - 24, find the velocity, speed, and acceleration of a particle with the given position function. Remember that the speed is the magnitude of the velocity represented by $\|\vec{\mathbf{v}}(t)\|$ or $\|\vec{\mathbf{r}}'(t)\|$.

18) $\vec{\mathbf{r}}(t)=e^{2t}\,\hat{\mathbf{i}}+\sin t\,\hat{\mathbf{j}}$

19) $\vec{\mathbf{r}}(t)=\cos t^3\,\hat{\mathbf{i}}+\sin t^3\,\hat{\mathbf{j}}$

Answer

$\vec{\mathbf{v}}(t)=-3t^2\sin t^3\,\hat{\mathbf{i}}+3t^2\cos t^3\,\hat{\mathbf{j}},\quad \text{speed}(t)=\|\vec{\mathbf{v}}(t)\|=3t^2,\quad \vec{\mathbf{a}}(t)=\left(-6t\sin t^3-9t^4\cos t^3\right)\hat{\mathbf{i}}+\left(6t\cos t^3-9t^4\sin t^3\right)\hat{\mathbf{j}}$

20) $\vec{\mathbf{r}}(t)=\langle e^t,e^{-t},0\rangle$

21) $\vec{\mathbf{r}}(t)=\langle t+\cos t,t-\sin t\rangle$

Answer

$\vec{\mathbf{v}}(t)=\langle 1-\sin t,1-\cos t\rangle,\quad \text{speed}(t)=\|\vec{\mathbf{v}}(t)\|=\sqrt{3-2(\sin t+\cos t)},\quad \vec{\mathbf{a}}(t)=\langle-\cos t,\sin t\rangle$

22) $\vec{\mathbf{r}}(t)=\dfrac{2t-1}{2t+1}\,\hat{\mathbf{i}}+\ln(1-4t^2)\,\hat{\mathbf{j}}$

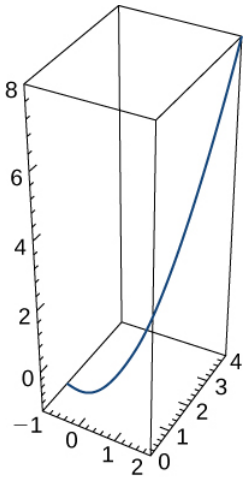
23) $\vec{\mathbf{r}}(t)=\cos 3t\,\hat{\mathbf{i}}+\sin 3t\,\hat{\mathbf{j}}+0.5t\,\hat{\mathbf{k}}$

Answer

$\vec{\mathbf{v}}(t)=-3\sin 3t\,\hat{\mathbf{i}}+3\cos 3t\,\hat{\mathbf{j}}+0.5\,\hat{\mathbf{k}},\quad \text{speed}(t)=\|\vec{\mathbf{v}}(t)\|=\sqrt{9.25}\ \text{units/sec},\quad \vec{\mathbf{a}}(t)=-9\cos 3t\,\hat{\mathbf{i}}-9\sin 3t\,\hat{\mathbf{j}}$

24) $\vec{\mathbf{r}}(t)=e^{-t}\,\hat{\mathbf{i}}+(\ln t)\,\hat{\mathbf{j}}+(\sin 7t)\,\hat{\mathbf{k}}$

25) Consider the position vector for a particle to be $\vec{\mathbf{r}}(t)=t\,\hat{\mathbf{i}}+t^2\,\hat{\mathbf{j}}+t^3\,\hat{\mathbf{k}}$. The graph is shown here:



a. Find the velocity vector at any time.

b. Find the speed of the particle at time $t = 2$ sec.

c. Find the acceleration at time $t = 2$ sec.

Answer

a. $\vec{\mathbf{v}}(t) = \hat{\mathbf{i}} + 2t\,\hat{\mathbf{j}} + 3t^2\,\hat{\mathbf{k}}$

b. $\sqrt{161}$ units/sec

Note that $\text{speed}(t) = \|\vec{\mathbf{v}}(t)\| = \sqrt{1^2 + (2t)^2 + (3t^2)^2} = \sqrt{1 + 4t^2 + 9t^4}$.

Hence, $\text{speed}(2) = \sqrt{1 + 16 + 9(16)} = \sqrt{161}$ units/sec.

c. Since $\vec{\mathbf{a}}(t) = 2\,\hat{\mathbf{j}} + 6t\,\hat{\mathbf{k}}$, $\vec{\mathbf{a}}(2) = 2\,\hat{\mathbf{j}} + 12\,\hat{\mathbf{k}}$

26) A particle travels along the path of an ellipse with the equation $\vec{\mathbf{r}}(t) = \cos t\,\hat{\mathbf{i}} + 2\sin t\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}$. Find the following:

a. Velocity of the particle

Answer

$\vec{\mathbf{v}}(t) = \langle -\sin t, 2\cos t, 0\rangle$

b. Speed of the particle at $t = \frac{\pi}{4}$

c. Acceleration of the particle at $t = \frac{\pi}{4}$

Answer

$\vec{\mathbf{a}}(t) = \langle -\frac{\sqrt{2}}{2}, -\sqrt{2}, 0\rangle$

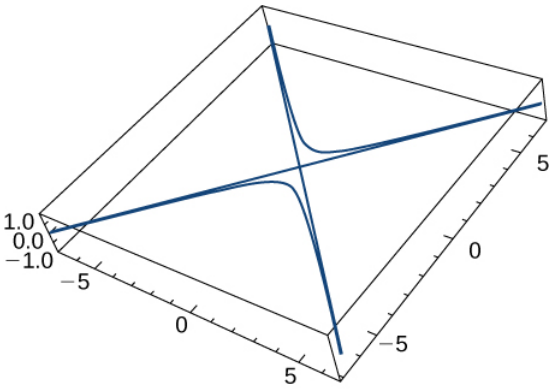
27) Show that if the speed of a particle traveling along a curve represented by a vector-valued function is constant, then the velocity function is always perpendicular to the acceleration function.

Answer

$$\begin{aligned} \|\vec{\mathbf{v}}(t)\| &= k \\ \vec{\mathbf{v}}(t)\cdot\vec{\mathbf{v}}(t) &= k^2 \\ \frac{d}{dt}\left(\vec{\mathbf{v}}(t)\cdot\vec{\mathbf{v}}(t)\right) &= \frac{d}{dt}\left(k^2\right) = 0 \\ \vec{\mathbf{v}}(t)\cdot\vec{\mathbf{v}}'(t) + \vec{\mathbf{v}}'(t)\cdot\vec{\mathbf{v}}(t) &= 0 \\ 2\vec{\mathbf{v}}(t)\cdot\vec{\mathbf{v}}'(t) &= 0 \\ \vec{\mathbf{v}}(t)\cdot\vec{\mathbf{v}}'(t) &= 0 \end{aligned}$$

The last statement implies that the velocity and acceleration are perpendicular or orthogonal.

28) Given the vector-valued function $\vec{\mathbf{r}}(t) = \langle \tan t, \sec t, 0\rangle$ (graph is shown here), find the following:



a. Velocity

Answer

$\vec{\mathbf{v}}(t) = \langle \sec^2 t,\ \sec t \tan t \rangle$

b. Speed

Answer

$\|\vec{\mathbf{v}}(t)\| = \sqrt{\sec^4 t + \sec^2 t \tan^2 t} = \sqrt{(\sec^2 t)(\sec^2 t + \tan^2 t)}$

c. Acceleration

Answer

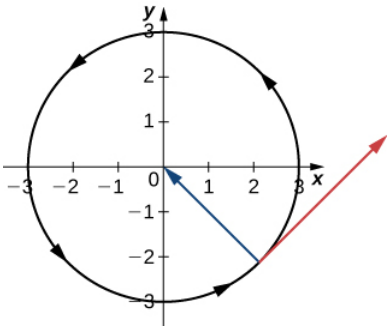
$\vec{\mathbf{a}}(t) = \langle 2\sec^2 t \tan t,\ \sec t \tan^2 t + \sec^3 t \rangle$

29) Find the minimum speed of a particle traveling along the curve $\vec{\mathbf{r}}(t) = \langle t + \cos t, t - \sin t \rangle$, where $t \in [0, 2\pi)$. Then also find its maximum speed on this interval.

Answer

Min. speed is $\sqrt{3 - 2\sqrt{2}} \approx 0.41421$ when $t = \frac{\pi}{4}$.
Max. speed is $\sqrt{3 + 2\sqrt{2}} \approx 2.41421$ when $t = \frac{5\pi}{4}$.

For questions 30 - 31, consider a particle that moves on a circular path of radius b according to the function $\vec{\mathbf{r}}(t) = b\cos(\omega t)\,\hat{\mathbf{i}} + b\sin(\omega t)\,\hat{\mathbf{j}}$, where ω is the angular velocity, $\frac{d\theta}{dt}$.



30) Show that the speed of the particle is proportional to the angular velocity.

31) Find the velocity function and show that $\vec{\mathbf{v}}(t)$ is always orthogonal to $\vec{\mathbf{r}}(t)$.

Answer

$\vec{\mathbf{r}}'(t) = -b\omega\sin(\omega t)\,\hat{\mathbf{i}} + b\omega\cos(\omega t)\,\hat{\mathbf{j}}$. To show orthogonality, note that $\vec{\mathbf{r}}'(t) \cdot \vec{\mathbf{r}}(t) = 0$.

Smoothness of Vector-Valued Functions

For questions 32 - 40,

a. Determine any values of t at which $\vec{\mathbf{r}}$ is not smooth.

b. Determine the open intervals on which $\vec{\mathbf{r}}$ is smooth.

c. Graph the vector-valued function and describe its behavior at the points where it is not smooth.

32) $\vec{\mathbf{r}}(t) = \langle 3t, 5t^2 - 1\rangle$

33) $\vec{\mathbf{r}}(t) = t^3\,\hat{\mathbf{i}} + 5t^2\,\hat{\mathbf{j}}$

Answer

a. $\vec{\mathbf{r}}$ is not smooth at $t = 0$, since $\vec{\mathbf{r}}'(0) = \vec{\mathbf{0}}$.
b. $\vec{\mathbf{r}}$ is smooth on the open intervals $(-\infty, 0)$ and $(0, \infty)$.
c. There is a cusp when $t = 0$.

34) $\vec{\mathbf{r}}(t) = \langle 5,\ 2\sin(t),\ \cos(t)\rangle$

35) $\vec{\mathbf{r}}(t) = \langle t^3 - 3t^2, 7\rangle$

Answer

a. $\vec{\mathbf{r}}$ is not smooth at $t = 0$ and $t = 2$, since $\vec{\mathbf{r}}'(0) = \vec{\mathbf{0}}$ and $\vec{\mathbf{r}}'(2) = \vec{\mathbf{0}}$.
b. $\vec{\mathbf{r}}$ is smooth on the open intervals $(-\infty, 0)$, $(0, 2)$, and $(2, \infty)$.
c. The motion on the curve reverses along the same path at both $t = 0$ and $t = 2$.

36) $\vec{\mathbf{r}}(t) = t^2\,\hat{\mathbf{i}} + t^3\,\hat{\mathbf{j}} - 5e^{-4t}\,\hat{\mathbf{k}}$

37) $\vec{\mathbf{r}}(t) = \left\langle \ln(t^2 + 4t + 5),\ \dfrac{t^3}{3} - 4t,\ 5 \right\rangle$

Answer

a. $\vec{\mathbf{r}}$ is not smooth at $t = -2$, since $\vec{\mathbf{r}}'(-2) = \vec{\mathbf{0}}$.
Since the domain of $\vec{\mathbf{r}}$ is $(-\infty, \infty)$, this is all we have to remove.
b. $\vec{\mathbf{r}}$ is smooth on the open intervals $(-\infty, -2)$ and $(-2, \infty)$.
c. There is a cusp when $t = -2$.

38) $\vec{\mathbf{r}}(t) = (5\cos t - \cos 5t)\,\hat{\mathbf{i}} + (5\sin t - \sin 5t)\,\hat{\mathbf{j}}$, for $0 \le t \le 2\pi$

39) $\vec{\mathbf{r}}(t) = \sqrt{t^3 + 9t^2}\,\hat{\mathbf{i}} + (t^2 + 12t)\,\hat{\mathbf{j}} + 7\hat{\mathbf{k}}$

Answer

a. The domain of $\vec{\mathbf{r}}$ is $[-9, \infty)$.
And $\vec{\mathbf{r}}$ is not smooth at $t = -6$, since $\vec{\mathbf{r}}'(-6) = \vec{\mathbf{0}}$.
The domain of $\vec{\mathbf{r}}'$ is $(-9, \infty)$, since $\vec{\mathbf{r}}'$ is undefined at $t = -9$.
b. $\vec{\mathbf{r}}$ is smooth on the open intervals $(-9, -6)$ and $(-6, \infty)$.
c. There is a cusp when $t = -6$.

40) $\vec{\mathbf{r}}(t) = \cos^3 t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}}$, for $0 \le t \le 2\pi$

Answer

a. The domain of $\vec{\mathbf{r}}$ is $(-\infty, \infty)$.
$\vec{\mathbf{r}}'(t) = -3(\cos^2 t)(\sin t)\,\hat{\mathbf{i}} + \cos t\,\hat{\mathbf{j}}$. It's domain is also $(-\infty, \infty)$.
But note that both components have a factor of $\cos t$, so both components will be 0 when $\cos t = 0$.
Therefore, $\vec{\mathbf{r}}$ is not smooth at $t = \frac{\pi}{2}$ and at $t = \frac{3\pi}{2}$, since $\vec{\mathbf{r}}'\left(\frac{\pi}{2}\right) = \vec{\mathbf{0}}$ and $\vec{\mathbf{r}}'\left(\frac{3\pi}{2}\right) = \vec{\mathbf{0}}$. Note then that $\vec{\mathbf{r}}$ is not smooth for any odd multiple of $\frac{\pi}{2}$, that is for $t = \frac{(2n+1)\pi}{2}$, for any integer value n.

b. $\vec{\mathbf{r}}$ is smooth on the open intervals $\left(\frac{(2n-1)\pi}{2}, \frac{(2n+1)\pi}{2}\right)$, for any integer value n.

c. There is a cusp when $t = \frac{(2n+1)\pi}{2}$, for any integer value n.

Properties of the Derivative

For questions, 41 - 43, evaluate each expression given that $\vec{\mathbf{r}}(t) = t\,\hat{\mathbf{i}} + t^2\,\hat{\mathbf{j}} - t^4\,\hat{\mathbf{k}}$ and $\vec{\mathbf{s}}(t) = \sin(t)\,\hat{\mathbf{i}} + e^t\,\hat{\mathbf{j}} + \cos(t)\,\hat{\mathbf{k}}$

41) $\frac{d}{dt}\left[\vec{\mathbf{r}}(t^2)\right]$

Answer

$\frac{d}{dt}\left[\vec{\mathbf{r}}(t^2)\right] = \langle 2t, 4t^3, -8t^7\rangle$

42) $\frac{d}{dt}\left[t^2 \cdot \vec{\mathbf{s}}(t)\right]$

43) $\frac{d}{dt}\left[\vec{\mathbf{r}}(t) \cdot \vec{\mathbf{s}}(t)\right]$

Answer

$\frac{d}{dt}\left[\vec{\mathbf{r}}(t) \cdot \vec{\mathbf{s}}(t)\right] = \sin t + 2te^t - 4t^3\cos t + t\cos t + t^2e^t + t^4\sin t$

44) Find $\vec{\mathbf{r}}'(t) \cdot \vec{\mathbf{r}}''(t)$ *for* $\vec{\mathbf{r}}(t) = -3t^5\,\hat{\mathbf{i}} + 5t\,\hat{\mathbf{j}} + 2t^2\,\hat{\mathbf{k}}$.

Answer

$\vec{\mathbf{r}}'(t) \cdot \vec{\mathbf{r}}''(t) = 900t^7 + 16t$

45) Given $\vec{\mathbf{r}}(t) = t\,\hat{\mathbf{i}} + 3t\,\hat{\mathbf{j}} + t^2\,\hat{\mathbf{k}}$ and $\vec{\mathbf{u}}(t) = 4t\,\hat{\mathbf{i}} + t^2\,\hat{\mathbf{j}} + t^3\,\hat{\mathbf{k}}$, find $\frac{d}{dt}\left(\vec{\mathbf{r}}(t) \times \vec{\mathbf{u}}(t)\right)$.

46) Evaluate $\frac{d}{dt}\left[\vec{\mathbf{u}}(t) \times \vec{\mathbf{u}}'(t)\right]$ given $\vec{\mathbf{u}}(t) = t^2\,\hat{\mathbf{i}} - 2t\,\hat{\mathbf{j}} + \hat{\mathbf{k}}$.

Answer

$\frac{d}{dt}\left[\vec{\mathbf{u}}(t) \times \vec{\mathbf{u}}'(t)\right] = 0\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + 4t\,\hat{\mathbf{k}}$

47) Given $\vec{\mathbf{r}}(t) = t\,\hat{\mathbf{i}} + 2\sin t\,\hat{\mathbf{j}} + 2\cos t\,\hat{\mathbf{k}}$ and $\vec{\mathbf{u}}(t) = \frac{1}{t}\,\hat{\mathbf{i}} + 2\sin t\,\hat{\mathbf{j}} + 2\cos t\,\hat{\mathbf{k}}$, find the following:

a. $\vec{\mathbf{r}}(t) \times \vec{\mathbf{u}}(t)$

Answer

$\vec{\mathbf{r}}(t) \times \vec{\mathbf{u}}(t) = \left\langle 0,\ 2(\cos t)\left(\frac{1}{t} - t\right),\ 2(\sin t)\left(t - \frac{1}{t}\right)\right\rangle$

b. $\frac{d}{dt}\left(\vec{\mathbf{r}}(t) \times \vec{\mathbf{u}}(t)\right)$

Answer

$\frac{d}{dt}\left(\vec{\mathbf{r}}(t) \times \vec{\mathbf{u}}(t)\right) = \left\langle 0,\ 2(\sin t)\left(t - \frac{1}{t}\right) - 2(\cos t)\left(1 + \frac{1}{t^2}\right),\ 2(\sin t)\left(1 + \frac{1}{t^2}\right) + 2(\cos t)\left(t - \frac{1}{t}\right)\right\rangle$

c. Now, use the product rule for the derivative of the cross product of two vectors and show this result is the same as the answer for the preceding problem.

Unit Tangent Vectors

For questions 48 - 51, find a *unit tangent vector* at the indicated value of t.

48) $\vec{\mathbf{r}}(t) = 3t^3\,\hat{\mathbf{i}} + 2t^2\,\hat{\mathbf{j}} + \frac{1}{t}\,\hat{\mathbf{k}}; \quad t = 1$

49) $\vec{\mathbf{r}}(t) = t\,\hat{\mathbf{i}} + \sin(2t)\,\hat{\mathbf{j}} + \cos(3t)\,\hat{\mathbf{k}}; \quad t = \frac{\pi}{3}$

Answer

$\vec{\mathbf{r}}'\left(\frac{\pi}{3}\right) = \langle 1, -1, 0\rangle$ is a tangent vector, so a unit tangent vector would be:
$\frac{1}{\sqrt{2}}\langle 1, -1, 0\rangle \quad = \quad \langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0\rangle$

50) $\vec{\mathbf{r}}(t) = \cos(2t)\,\hat{\mathbf{i}} + 2\sin t\,\hat{\mathbf{j}} + t^2\,\hat{\mathbf{k}}; \quad t = \frac{\pi}{2}$

51) $\vec{\mathbf{r}}(t) = 3e^t\,\hat{\mathbf{i}} + 2e^{-3t}\,\hat{\mathbf{j}} + 4e^{2t}\,\hat{\mathbf{k}}; \quad t = \ln(2)$

Answer

$\vec{\mathbf{r}}'(\ln(2)) = \langle 6, -\frac{3}{4}, 32\rangle$ is a tangent vector, so a unit tangent vector would be:
$$\frac{1}{\sqrt{1060.5625}}\left\langle 6, -\frac{3}{4}, 32\right\rangle \quad = \quad \left\langle \frac{24\sqrt{16969}}{16969}, -\frac{12\sqrt{16969}}{67876}, \frac{128\sqrt{16969}}{16969}\right\rangle$$

For questions 52 - 58, find the unit tangent vector $\vec{\mathbf{T}}(t)$ for the following parameterized curves.

52) $\vec{\mathbf{r}}(t) = t\,\hat{\mathbf{i}} + 3t\,\hat{\mathbf{j}} + t^2\,\hat{\mathbf{k}}$

53) $\vec{\mathbf{r}}(t) = 6\,\hat{\mathbf{i}} + \cos(3t)\,\hat{\mathbf{j}} + 3\sin(4t)\,\hat{\mathbf{k}}, \quad 0 \le t < 2\pi$

Answer

$$\vec{\mathbf{T}}(t) = \frac{1}{\sqrt{9\sin^2(3t) + 144\cos^2(4t)}}\langle 0, -3\sin(3t), 12\cos(4t)\rangle$$

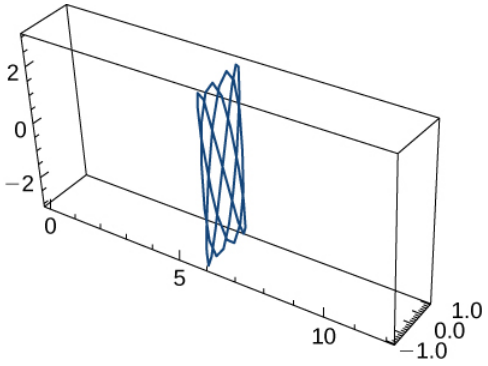
54) $\vec{\mathbf{r}}(t) = \langle t\cos t, t\sin t\rangle$

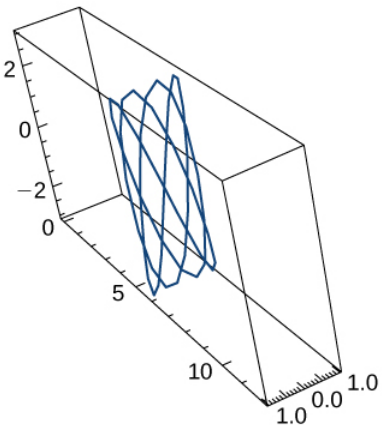
55) $\vec{\mathbf{r}}(t) = \langle t+1, 2t+1, 2t+2\rangle$

Answer

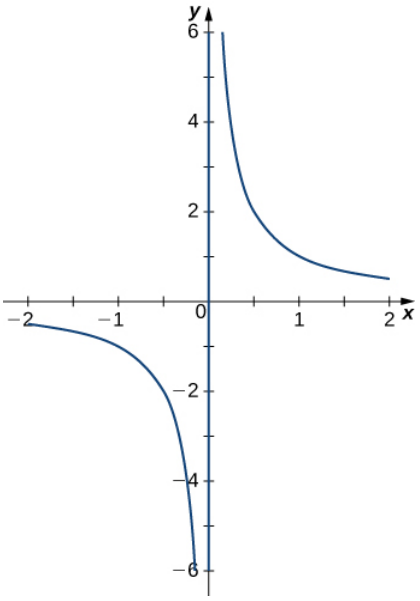
$\vec{\mathbf{T}}(t) = \frac{1}{3}\langle 1, 2, 2\rangle$

56) $\vec{\mathbf{r}}(t) = \cos t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}} + \sin t\,\hat{\mathbf{k}}, \quad 0 \le t < 2\pi$. Two views of this curve are presented here:





57) $\vec{\mathbf{r}}(t) = \left\langle t, \frac{1}{t} \right\rangle$. The graph is shown here:



Answer

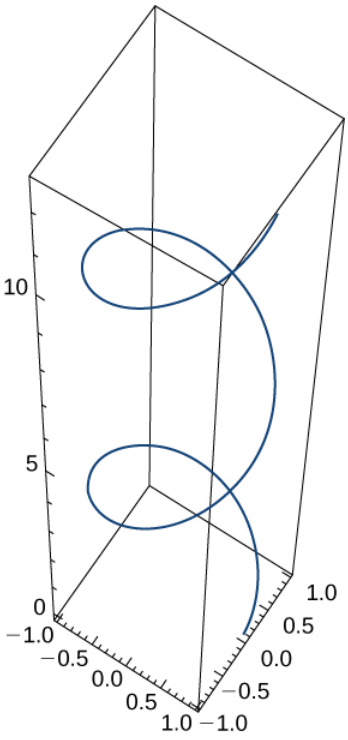
$$\vec{\mathbf{T}}(t) = \left\langle \frac{t^2}{\sqrt{t^4+1}}, \frac{-1}{\sqrt{t^4+1}} \right\rangle$$

58) $\vec{\mathbf{r}}(t) = 3\cos(4t)\,\hat{\mathbf{i}} + 3\sin(4t)\,\hat{\mathbf{j}} + 5t\,\hat{\mathbf{k}}, \quad 1 \le t \le 2$

Answer

$$\vec{\mathbf{T}}(t) = -\tfrac{12}{13}\sin(4t)\,\hat{\mathbf{i}} + \tfrac{12}{13}\cos(4t)\,\hat{\mathbf{j}} + \tfrac{5}{13}\,\hat{\mathbf{k}}$$

59) A particle travels along the path of a helix with the equation $\vec{\mathbf{r}}(t) = \cos(t)\,\hat{\mathbf{i}} + \sin(t)\,\hat{\mathbf{j}} + t\,\hat{\mathbf{k}}$. See the graph presented here:



Find the following:

a. Velocity of the particle at any time

Answer

$\vec{\mathbf{v}}(t) = \langle -\sin t, \cos t, 1 \rangle$

b. Speed of the particle at any time

c. Acceleration of the particle at any time

Answer

$\vec{\mathbf{a}}(t) = -\cos t\,\hat{\mathbf{i}} - \sin t\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}$

d. Find the unit tangent vector for the helix.

Integration of Vector-Valued Functions

Evaluate the following integrals:

60) $\int \left(e^t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}} + \frac{1}{2t-1}\,\hat{\mathbf{k}} \right) dt$

61) $\int_0^1 \vec{\mathbf{r}}(t)\,dt$, where $\vec{\mathbf{r}}(t) = \left\langle \sqrt[3]{t}, \frac{1}{t+1}, e^{-t} \right\rangle$

Answer

$\frac{3}{4}\,\hat{\mathbf{i}} + \ln(2)\,\hat{\mathbf{j}} + (1 - \frac{1}{e})\,\hat{\mathbf{k}}$

62) Evaluate $\int_0^3 \|t\,\hat{\mathbf{i}} + t^2\,\hat{\mathbf{j}}\| dt$.

Answer

$\frac{1}{3}(10^{\frac{3}{2}} - 1)$

63) The acceleration function, initial velocity, and initial position of a particle are

$$\vec{\mathbf{a}}(t) = -5\cos t\,\hat{\mathbf{i}} - 5\sin t\,\hat{\mathbf{j}},$$
$$\vec{\mathbf{v}}(0) = 9\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}}, \quad \text{and}$$
$$\vec{\mathbf{r}}(0) = 5\,\hat{\mathbf{i}}$$

Find $\vec{\mathbf{v}}(t)$ and $\vec{\mathbf{r}}(t)$.

Answer

$\vec{\mathbf{v}}(t) = (9 - 5\sin t)\,\hat{\mathbf{i}} + (-3 + 5\cos t)\,\hat{\mathbf{j}}$
$\vec{\mathbf{r}}(t) = (9t + 5\cos t)\,\hat{\mathbf{i}} + (-3t + 5\sin t)\,\hat{\mathbf{j}}$

64) Find the antiderivative of $\vec{\mathbf{r}}'(t) = \cos(2t)\,\hat{\mathbf{i}} - 2\sin t\,\hat{\mathbf{j}} + \frac{1}{1+t^2}\,\hat{\mathbf{k}}$ that satisfies the initial condition $\vec{\mathbf{r}}(0) = 3\,\hat{\mathbf{i}} - 2\,\hat{\mathbf{j}} + \hat{\mathbf{k}}$.

65) An object starts from rest at point $P(1, 2, 0)$ and moves with an acceleration of $\vec{\mathbf{a}}(t) = \hat{\mathbf{j}} + 2\,\hat{\mathbf{k}}$, where $\|\vec{\mathbf{a}}(t)\|$ is measured in feet per second per second. Find the location of the object after $t = 2$ sec.

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1.4: Arc Length and Curvature

Learning Objectives

- Determine the length of a particle's path in space by using the arc-length function.
- Explain the meaning of the curvature of a curve in space and state its formula.
- Describe the meaning of the normal and binormal vectors of a curve in space.

In this section, we study formulas related to curves in both two and three dimensions, and see how they are related to various properties of the same curve. For example, suppose a vector-valued function describes the motion of a particle in space. We would like to determine how far the particle has traveled over a given time interval, which can be described by the arc length of the path it follows. Or, suppose that the vector-valued function describes a road we are building and we want to determine how sharply the road curves at a given point. This is described by the curvature of the function at that point. We explore each of these concepts in this section.

Arc Length for Vector Functions

We have seen how a vector-valued function describes a curve in either two or three dimensions. Recall that the formula for the arc length of a curve defined by the parametric functions $x = x(t), y = y(t), t_1 \leq t \leq t_2$ is given by

$$s = \int_{t_1}^{t_2} \sqrt{(x'(t))^2 + (y'(t))^2}dt.$$

In a similar fashion, if we define a smooth curve using a vector-valued function $\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}}$, where $a \leq t \leq b$, the arc length is given by the formula

$$s = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}dt.$$

In three dimensions, if the vector-valued function is described by $\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} + h(t)\,\hat{\mathbf{k}}$ over the same interval $a \leq t \leq b$, the arc length is given by

$$s = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}dt.$$

Theorem: Arc-Length Formulas for Plane and Space curves

Plane curve: Given a smooth curve C defined by the function $\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}}$, where t lies within the interval $[a, b]$, the arc length of C over the interval is

$$s = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}dt \tag{1.4.1}$$

$$= \int_a^b \|\vec{\mathbf{r}}'(t)\|dt. \tag{1.4.2}$$

Space curve: Given a smooth curve C defined by the function $\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} + h(t)\,\hat{\mathbf{k}}$, where t lies within the interval $[a, b]$, the arc length of C over the interval is

$$s = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}dt \tag{1.4.3}$$

$$= \int_a^b \|\vec{\mathbf{r}}'(t)\|dt. \tag{1.4.4}$$

The two formulas are very similar; they differ only in the fact that a space curve has three component functions instead of two. Note that the formulas are defined for smooth curves: curves where the vector-valued function $\vec{\mathbf{r}}(t)$ is differentiable with a non-zero derivative. The smoothness condition guarantees that the curve has no cusps (or corners) that could make the formula problematic.

Example 1.4.1: Finding the Arc Length

Calculate the arc length for each of the following vector-valued functions:

a. $\vec{\mathbf{r}}(t) = (3t-2)\,\hat{\mathbf{i}} + (4t+5)\,\hat{\mathbf{j}}, \quad 1 \le t \le 5$

b. $\vec{\mathbf{r}}(t) = \langle t\cos t, t\sin t, 2t\rangle, 0 \le t \le 2\pi$

Solution

a. Using Equation 1.4.2, $\vec{\mathbf{r}}'(t) = 3\,\hat{\mathbf{i}} + 4\,\hat{\mathbf{j}}$, so

$$\begin{aligned} s &= \int_a^b \|\vec{\mathbf{r}}'(t)\|dt \\ &= \int_1^5 \sqrt{3^2+4^2}dt \\ &= \int_1^5 5dt = 5t\big|_1^5 = 20. \end{aligned}$$

b. Using Equation 1.4.4, $\vec{\mathbf{r}}'(t) = \langle \cos t - t\sin t, \sin t + t\cos t, 2\rangle$, so

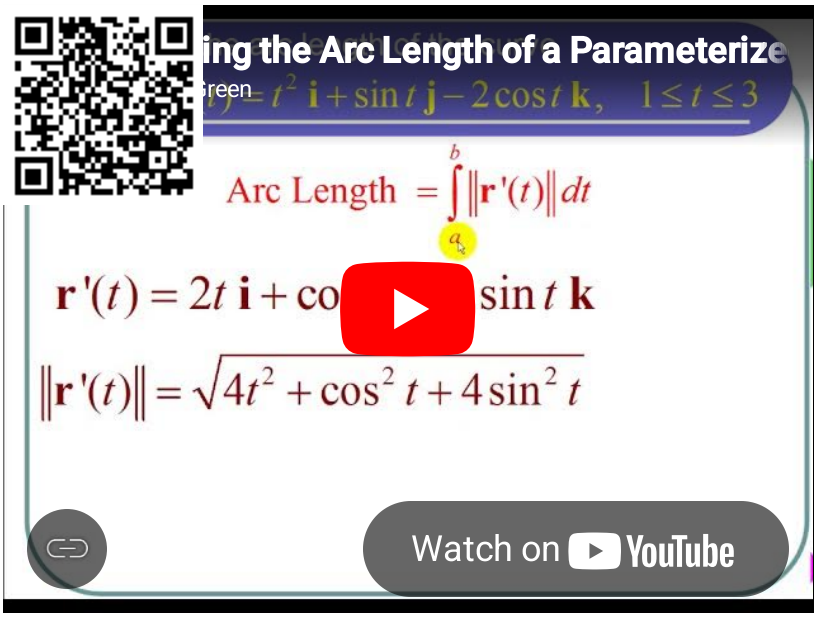
$$\begin{aligned} s &= \int_a^b \| \vec{\mathbf{r}}'(t) \| dt \\ &= \int_0^{2\pi} \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + 2^2}dt \\ &= \int_0^{2\pi} \sqrt{(\cos^2 t - 2t\sin t\cos t + t^2\sin^2 t) + (\sin^2 t + 2t\sin t\cos t + t^2\cos^2 t) + 4}dt \\ &= \int_0^{2\pi} \sqrt{\cos^2 t + \sin^2 t + t^2(\cos^2 t + \sin^2 t) + 4}dt \\ &= \int_0^{2\pi} \sqrt{t^2+5}dt \end{aligned}$$

Here we can use a table integration formula

$$\int \sqrt{u^2+a^2}du = \frac{u}{2}\sqrt{u^2+a^2} + \frac{a^2}{2}\ln\left|u+\sqrt{u^2+a^2}\right| + C,$$

so we obtain

$$\begin{aligned} \int_0^{2\pi} \sqrt{t^2+5}dt &= \frac{1}{2}\left(t\sqrt{t^2+5} + 5\ln\left|t+\sqrt{t^2+5}\right|\right)_0^{2\pi} \\ &= \frac{1}{2}\left(2\pi\sqrt{4\pi^2+5} + 5\ln\left(2\pi+\sqrt{4\pi^2+5}\right)\right) - \frac{5}{2}\ln\sqrt{5} \\ &\approx 25.343 \text{ units.} \end{aligned}$$



? Exercise 1.4.1

Calculate the arc length of the parameterized curve

$$\vec{\mathbf{r}}(t) = \langle 2t^2 + 1, 2t^2 - 1, t^3 \rangle, \quad 0 \le t \le 3.$$

Hint

Use Equation 1.4.4.

Answer

$\vec{\mathbf{r}}'(t) = \langle 4t, 4t, 3t^2 \rangle$, so $s = \frac{1}{27}(113^{3/2} - 32^{3/2}) \approx 37.785$ units

We now return to the helix introduced earlier in this chapter. A vector-valued function that describes a helix can be written in the form

$$\vec{\mathbf{r}}(t) = R\cos\left(\frac{2\pi Nt}{h}\right)\hat{\mathbf{i}} + R\sin\left(\frac{2\pi Nt}{h}\right)\hat{\mathbf{j}} + t\hat{\mathbf{k}}, 0 \le t \le h,$$

where R represents the radius of the helix, h represents the height (distance between two consecutive turns), and the helix completes N turns. Let's derive a formula for the arc length of this helix using Equation 1.4.4. First of all,

$$\vec{\mathbf{r}}'(t) = -\frac{2\pi NR}{h}\sin\left(\frac{2\pi Nt}{h}\right)\hat{\mathbf{i}} + \frac{2\pi NR}{h}\cos\left(\frac{2\pi Nt}{h}\right)\hat{\mathbf{j}} + \hat{\mathbf{k}}.$$

Therefore,

$$
\begin{aligned}
s &= \int_a^b \|\vec{\mathbf{r}}'(t)\| dt \\
&= \int_0^h \sqrt{\left(-\frac{2\pi NR}{h}\sin\left(\frac{2\pi Nt}{h}\right)\right)^2 + \left(\frac{2\pi NR}{h}\cos\left(\frac{2\pi Nt}{h}\right)\right)^2 + 1^2} dt \\
&= \int_0^h \sqrt{\frac{4\pi^2N^2R^2}{h^2}\left(\sin^2\left(\frac{2\pi Nt}{h}\right) + \cos^2\left(\frac{2\pi Nt}{h}\right)\right) + 1} dt \\
&= \int_0^h \sqrt{\frac{4\pi^2N^2R^2}{h^2} + 1} dt \\
&= \left[t\sqrt{\frac{4\pi^2N^2R^2}{h^2} + 1}\right]_0^h \\
&= h\sqrt{\frac{4\pi^2N^2R^2 + h^2}{h^2}} \\
&= \sqrt{4\pi^2N^2R^2 + h^2}.
\end{aligned}
$$

This gives a formula for the length of a wire needed to form a helix with N turns that has radius R and height h.

Arc-Length Parameterization

We now have a formula for the arc length of a curve defined by a vector-valued function. Let's take this one step further and examine what an **arc-length function** is.

If a vector-valued function represents the position of a particle in space as a function of time, then the arc-length function measures how far that particle travels as a function of time. The formula for the arc-length function follows directly from the formula for arc length:

$$
s = \int_a^t \sqrt{(f'(u))^2 + (g'(u))^2 + (h'(u))^2} du. \tag{1.4.5}
$$

If the curve is in two dimensions, then only two terms appear under the square root inside the integral. The reason for using the independent variable u is to distinguish between time and the variable of integration. Since $s(t)$ measures distance traveled as a function of time, $s'(t)$ measures the speed of the particle at any given time. Since we have a formula for $s(t)$ in Equation 1.4.5, we can differentiate both sides of the equation:

$$
\begin{aligned}
s'(t) &= \frac{d}{dt}\left[\int_a^t \sqrt{(f'(u))^2 + (g'(u))^2 + (h'(u))^2} du\right] \\
&= \frac{d}{dt}\left[\int_a^t \|\vec{\mathbf{r}}'(u)\| du\right] \\
&= \|\vec{\mathbf{r}}'(t)\|.
\end{aligned}
$$

If we assume that $\vec{\mathbf{r}}(t)$ defines a smooth curve, then the arc length is always increasing, so $s'(t) > 0$ for $t > a$. Last, if $\vec{\mathbf{r}}(t)$ is a curve on which $\|\vec{\mathbf{r}}'(t)\| = 1$ for all t, then

$$
s(t) = \int_a^t \|\vec{\mathbf{r}}'(u)\| du = \int_a^t 1 du = t - a,
$$

which means that t represents the arc length as long as $a = 0$.

Theorem: Arc-Length Function

Let $\vec{\mathbf{r}}(t)$ describe a smooth curve for $t \geq a$. Then the arc-length function is given by

$$
s(t) = \int_a^t \|\vec{\mathbf{r}}'(u)\| du \tag{1.4.6}
$$

Furthermore,

$$\frac{ds}{dt} = \|\vec{\mathbf{r}}'(t)\| > 0.$$

If $\|\vec{\mathbf{r}}'(t)\| = 1$ for all $t \geq a$, then the parameter t represents the arc length from the starting point at $t = a$.

A useful application of this theorem is to find an alternative parameterization of a given curve, called an **arc-length parameterization**. Recall that any vector-valued function can be reparameterized via a change of variables. For example, if we have a function $\vec{\mathbf{r}}(t) = \langle 3\cos t, 3\sin t\rangle, 0 \leq t \leq 2\pi$ that parameterizes a circle of radius 3, we can change the parameter from t to $4t$, obtaining a new parameterization $\vec{\mathbf{r}}(t) = \langle 3\cos 4t, 3\sin 4t\rangle$. The new parameterization still defines a circle of radius 3, but now we need only use the values $0 \leq t \leq \pi/2$ to traverse the circle once.

Suppose that we find the arc-length function $s(t)$ and are able to solve this function for t as a function of s. We can then reparameterize the original function $\vec{\mathbf{r}}(t)$ by substituting the expression for t back into $\vec{\mathbf{r}}(t)$. The vector-valued function is now written in terms of the parameter s. Since the variable s represents the arc length, we call this an *arc-length parameterization* of the original function $\vec{\mathbf{r}}(t)$. One advantage of finding the arc-length parameterization is that the distance traveled along the curve starting from $s = 0$ is now equal to the parameter s. The arc-length parameterization also appears in the context of curvature (which we examine later in this section) and line integrals.

Example 1.4.2: Finding an Arc-Length Parameterization

Find the arc-length parameterization for each of the following curves:

a. $\vec{\mathbf{r}}(t) = 4\cos t\,\hat{\mathbf{i}} + 4\sin t\,\hat{\mathbf{j}}, \quad t \geq 0$
b. $\vec{\mathbf{r}}(t) = \langle t+3, 2t-4, 2t\rangle, \quad t \geq 3$

Solution

a. First we find the arc-length function using Equation 1.4.5:

$$\begin{aligned} s(t) &= \int_a^t \|\vec{\mathbf{r}}'(u)\|\,du \\ &= \int_0^t \|\langle -4\sin u, 4\cos u\rangle\|\,du \\ &= \int_0^t \sqrt{(-4\sin u)^2 + (4\cos u)^2}\,du \\ &= \int_0^t \sqrt{16\sin^2 u + 16\cos^2 u}\,du \\ &= \int_0^t 4\,du = 4t, \end{aligned}$$

which gives the relationship between the arc length s and the parameter t as $s = 4t$; so, $t = s/4$. Next we replace the variable t in the original function $\vec{\mathbf{r}}(t) = 4\cos t\,\hat{\mathbf{i}} + 4\sin t\,\hat{\mathbf{j}}$ with the expression $s/4$ to obtain

$$\vec{\mathbf{r}}(s) = 4\cos\left(\frac{s}{4}\right)\hat{\mathbf{i}} + 4\sin\left(\frac{s}{4}\right)\hat{\mathbf{j}}.$$

This is the arc-length parameterization of $\vec{\mathbf{r}}(t)$. Since the original restriction on t was given by $t \geq 0$, the restriction on s becomes $s/4 \geq 0$, or $s \geq 0$.

b. The arc-length function is given by Equation 1.4.5:

$$\begin{aligned} s(t) &= \int_a^t \|\vec{\mathbf{r}}'(u)\|\, du \\ &= \int_3^t \|\langle 1,2,2\rangle\|\, du \\ &= \int_3^t \sqrt{1^2+2^2+2^2}\, du \\ &= \int_3^t 3\, du \\ &= 3t-9. \end{aligned}$$

Therefore, the relationship between the arc length s and the parameter t is $s = 3t-9$, so $t = \frac{s}{3}+3$. Substituting this into the original function $\vec{\mathbf{r}}(t) = \langle t+3, 2t-4, 2t\rangle$ yields

$$\vec{\mathbf{r}}(s) = \langle\left(\frac{s}{3}+3\right)+3,\, 2\left(\frac{s}{3}+3\right)-4,\, 2\left(\frac{s}{3}+3\right)\rangle = \langle\frac{s}{3}+6, \frac{2s}{3}+2, \frac{2s}{3}+6\rangle.$$

This is an arc-length parameterization of $\vec{\mathbf{r}}(t)$. The original restriction on the parameter t was $t \geq 3$, so the restriction on s is $(s/3)+3 \geq 3$, or $s \geq 0$.



Exercise 1.4.2

Find the arc-length function for the helix

$$\vec{\mathbf{r}}(t) = \langle 3\cos t, 3\sin t, 4t\rangle, \quad t \geq 0.$$

Then, use the relationship between the arc length and the parameter t to find an arc-length parameterization of $\vec{\mathbf{r}}(t)$.

Hint

Start by finding the arc-length function.

Answer

$s = 5t$, or $t = s/5$. Substituting this into $\vec{\mathbf{r}}(t) = \langle 3\cos t, 3\sin t, 4t\rangle$ gives

$$\vec{\mathbf{r}}(s) = \langle 3\cos\left(\frac{s}{5}\right), 3\sin\left(\frac{s}{5}\right), \frac{4s}{5}\rangle, \quad s \geq 0$$

Curvature

An important topic related to arc length is curvature. The concept of curvature provides a way to measure how sharply a smooth curve turns. A circle has constant curvature. The smaller the radius of the circle, the greater the curvature.

Think of driving down a road. Suppose the road lies on an arc of a large circle. In this case you would barely have to turn the wheel to stay on the road. Now suppose the radius is smaller. In this case you would need to turn more sharply to stay on the road. In the case of a curve other than a circle, it is often useful first to inscribe a circle to the curve at a given point so that it is tangent to the curve at that point and "hugs" the curve as closely as possible in a neighborhood of the point (Figure **1.4.1**). The curvature of the graph at that point is then defined to be the same as the curvature of the inscribed circle.

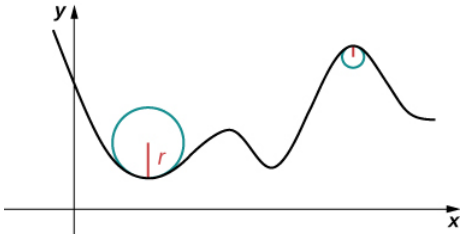


Figure **1.4.1**: The graph represents the curvature of a function $y = f(x)$. The sharper the turn in the graph, the greater the curvature, and the smaller the radius of the inscribed circle.

Definition: curvature

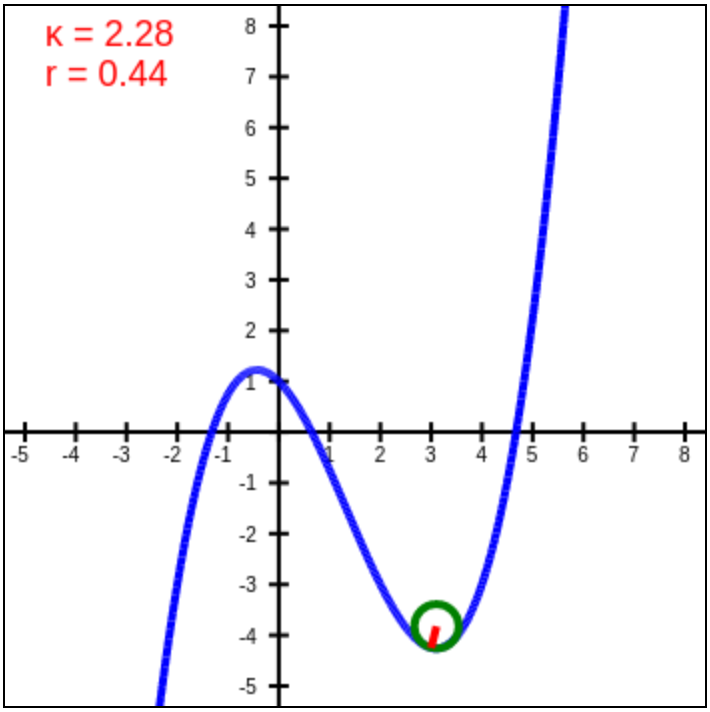
Let C be a smooth curve in the plane or in space given by $\vec{\mathbf{r}}(s)$, where s is the arc-length parameter. The curvature κ at s is

$$\kappa = \left\| \frac{d\vec{\mathbf{T}}}{ds} \right\| = \|\vec{\mathbf{T}}'(s)\|. \tag{1.4.7}$$

Visit this video for more information about the curvature of a space curve.

View the Circle of Curvature

Below you can see the graph of a function, its curvature (κ), its radius of curvature (r), and its circle of curvature. You can move the slider to see how the circle of curvature changes as you move along the curve. Click "Next" to see a new curve.



$f(x) = \frac{x^3}{4} - x^2 - x + 1$

x = 3

Next

The formula in the definition of curvature is not very useful in terms of calculation. In particular, recall that $\vec{\mathbf{T}}(t)$ represents the unit tangent vector to a given vector-valued function $\vec{\mathbf{r}}(t)$, and the formula for $\vec{\mathbf{T}}(t)$ is

$$\vec{\mathbf{T}}(t) = \frac{\vec{\mathbf{r}}'(t)}{\| \vec{\mathbf{r}}'(t) \|}. \tag{1.4.8}$$

To use the formula for curvature, it is first necessary to express $\vec{\mathbf{r}}(t)$ in terms of the arc-length parameter s, then find the unit tangent vector $\vec{\mathbf{T}}(s)$ for the function $\vec{\mathbf{r}}(s)$, then take the derivative of $\vec{\mathbf{T}}(s)$ with respect to s. This is a tedious process. Fortunately, there are equivalent formulas for curvature.

Theorem: Alternate Formulas of Curvature

If C is a smooth curve given by $\vec{\mathbf{r}}(t)$, then the curvature κ of C at t is given by

$$\kappa = \frac{\|\vec{\mathbf{T}}'(t)\|}{\|\vec{\mathbf{r}}'(t)\|}. \tag{1.4.9}$$

If C is a three-dimensional curve, then the curvature can be given by the formula

$$\kappa = \frac{\|\vec{\mathbf{r}}'(t) \times \vec{\mathbf{r}}''(t)\|}{\|\vec{\mathbf{r}}'(t)\|^3}. \tag{1.4.10}$$

If C is the graph of a function $y = f(x)$ and both y' and y'' exist, then the curvature κ at point (x, y) is given by

$$\kappa = \frac{|y''|}{[1 + (y')^2]^{3/2}}. \tag{1.4.11}$$

Proof

The first formula follows directly from the chain rule:

$$\frac{d\vec{\mathbf{T}}}{dt} = \frac{d\vec{\mathbf{T}}}{ds}\frac{ds}{dt},$$

where s is the arc length along the curve C. Dividing both sides by ds/dt, and taking the magnitude of both sides gives

$$\left\| \frac{d\vec{\mathbf{T}}}{ds} \right\| = \left\| \frac{\vec{\mathbf{T}}'(t)}{\frac{ds}{dt}} \right\|.$$

Since $ds/dt = \|\vec{\mathbf{r}}'(t)\|$, this gives the formula for the curvature κ of a curve C in terms of any parameterization of C:

$$\kappa = \frac{\|\vec{\mathbf{T}}'(t)\|}{\|\vec{\mathbf{r}}'(t)\|}.$$

In the case of a three-dimensional curve, we start with the formulas $\vec{\mathbf{T}}(t) = (\vec{\mathbf{r}}'(t))/\|\vec{\mathbf{r}}'(t)\|$ and $ds/dt = \|\vec{\mathbf{r}}'(t)\|$. Therefore, $\vec{\mathbf{r}}'(t) = (ds/dt)\vec{\mathbf{T}}(t)$. We can take the derivative of this function using the scalar product formula:

$$\vec{\mathbf{r}}''(t) = \frac{d^2s}{dt^2}\vec{\mathbf{T}}(t) + \frac{ds}{dt}\vec{\mathbf{T}}'(t).$$

Using these last two equations we get

$$\vec{\mathbf{r}}'(t) \times \vec{\mathbf{r}}''(t) = \frac{ds}{dt}\vec{\mathbf{T}}(t) \times \left(\frac{d^2s}{dt^2}\vec{\mathbf{T}}(t) + \frac{ds}{dt}\vec{\mathbf{T}}'(t)\right)$$
$$= \frac{ds}{dt}\frac{d^2s}{dt^2}\vec{\mathbf{T}}(t) \times \vec{\mathbf{T}}(t) + (\frac{ds}{dt})^2\vec{\mathbf{T}}(t) \times \vec{\mathbf{T}}'(t).$$

Since $\vec{\mathbf{T}}(t) \times \vec{\mathbf{T}}(t) = \mathbf{0}$, this reduces to

$$\vec{\mathbf{r}}'(t) \times \vec{\mathbf{r}}''(t) = \left(\frac{ds}{dt}\right)^2 \vec{\mathbf{T}}(t) \times \vec{\mathbf{T}}'(t).$$

Since $\vec{\mathbf{T}}'$ is parallel to $\vec{\mathbf{N}}$, and $\vec{\mathbf{T}}$ is orthogonal to $\vec{\mathbf{N}}$, it follows that $\vec{\mathbf{T}}$ and $\vec{\mathbf{T}}'$ are orthogonal. This means that $\|\vec{\mathbf{T}} \times \vec{\mathbf{T}}'\| = \|\vec{\mathbf{T}}\|\|\vec{\mathbf{T}}'\| \sin(\pi/2) = \|\vec{\mathbf{T}}'\|$, so

$$\|\vec{\mathbf{r}}'(t) \times \vec{\mathbf{r}}''(t)\| = \left(\frac{ds}{dt}\right)^2 \|\vec{\mathbf{T}}'(t)\|.$$

Now we solve this equation for $\|\vec{\mathbf{T}}'(t)\|$ and use the fact that $ds/dt = \|\vec{\mathbf{r}}'(t)\|$:

$$\|\vec{\mathbf{T}}'(t)\| = \frac{\|\vec{\mathbf{r}}'(t) \times \vec{\mathbf{r}}''(t)\|}{\|\vec{\mathbf{r}}'(t)\|^2}.$$

Then, we divide both sides by $\|\vec{\mathbf{r}}'(t)\|$. This gives

$$\kappa = \frac{\|\vec{\mathbf{T}}'(t)\|}{\|\vec{\mathbf{r}}'(t)\|} = \frac{\|\vec{\mathbf{r}}'(t) \times \vec{\mathbf{r}}''(t)\|}{\|\vec{\mathbf{r}}'(t)\|^3}.$$

This proves 1.4.10. To prove 1.4.11, we start with the assumption that curve C is defined by the function $y = f(x)$. Then, we can define $\vec{\mathbf{r}}(t) = x\,\hat{\mathbf{i}} + f(x)\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}$. Using the previous formula for curvature:

$$\vec{\mathbf{r}}'(t) = \hat{\mathbf{i}} + f'(x)\hat{\mathbf{j}}$$
$$\vec{\mathbf{r}}''(t) = f''(x)\hat{\mathbf{j}}$$
$$\vec{\mathbf{r}}'(t) \times \vec{\mathbf{r}}''(t) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & f'(x) & 0 \\ 0 & f''(x) & 0 \end{vmatrix} = f''(x)\hat{\mathbf{k}}.$$

Therefore,

$$\kappa = \frac{\|\vec{\mathbf{r}}'(t) \times \vec{\mathbf{r}}''(t)\|}{\|\vec{\mathbf{r}}'(t)\|^3} = \frac{|f''(x)|}{(1 + [f'(x)]^2)^{3/2}}$$

Example 1.4.3: Finding Curvature

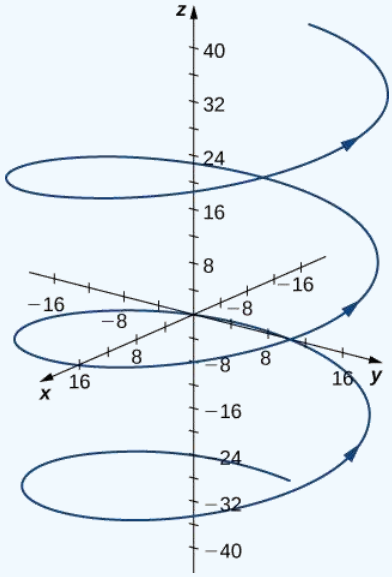
Find the curvature for each of the following curves at the given point:

a. $\vec{\mathbf{r}}(t) = 4\cos t\,\hat{\mathbf{i}} + 4\sin t\,\hat{\mathbf{j}} + 3t\,\hat{\mathbf{k}}, \quad t = \dfrac{4\pi}{3}$

b. $\mathbf{f(x) = \sqrt{4x - x^2}, x = 2}$

Solution

a. This function describes a helix.



The curvature of the helix at $t = (4\pi)/3$ can be found by using 1.4.9. First, calculate $\vec{\mathbf{T}}(t)$:

$$\begin{aligned}\vec{\mathbf{T}}(t) &= \frac{\vec{\mathbf{r}}'(t)}{\|\vec{\mathbf{r}}'(t)\|} \\ &= \frac{\langle -4\sin t, 4\cos t, 3\rangle}{\sqrt{(-4\sin t)^2 + (4\cos t)^2 + 3^2}} \\ &= \langle -\frac{4}{5}\sin t, \frac{4}{5}\cos t, \frac{3}{5}\rangle.\end{aligned}$$

Next, calculate $\vec{\mathbf{T}}'(t)$:

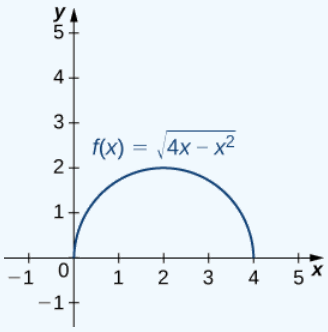
$$\vec{\mathbf{T}}'(t) = \langle -\frac{4}{5}\cos t, -\frac{4}{5}\sin t, 0\rangle.$$

Last, apply 1.4.9 :

$$\begin{aligned}\kappa &= \frac{\|\vec{\mathbf{T}}'(t)\|}{\|\vec{\mathbf{r}}'(t)\|} = \frac{\|\langle -\frac{4}{5}\cos t, -\frac{4}{5}\sin t, 0\rangle\|}{\|\langle -4\sin t, 4\cos t, 3\rangle\|} \\ &= \frac{\sqrt{(-\frac{4}{5}\cos t)^2 + (-\frac{4}{5}\sin t)^2 + 0^2}}{\sqrt{(-4\sin t)^2 + (4\cos t)^2 + 3^2}} \\ &= \frac{4/5}{5} = \frac{4}{25}.\end{aligned}$$

The curvature of this helix is constant at all points on the helix.

2. This function describes a semicircle.



To find the curvature of this graph, we must use 1.4.11. First, we calculate y' and y'':

$$\begin{aligned} y &= \sqrt{4x-x^2} = (4x-x^2)^{1/2} \\ y' &= \frac{1}{2}(4x-x^2)^{-1/2}(4-2x) = (2-x)(4x-x^2)^{-1/2} \\ y'' &= -(4x-x^2)^{-1/2} + (2-x)(-\frac{1}{2})(4x-x^2)^{-3/2}(4-2x) \\ &= -\frac{4x-x^2}{(4x-x^2)^{3/2}} - \frac{(2-x)^2}{(4x-x^2)^{3/2}} \\ &= \frac{x^2-4x-(4-4x+x^2)}{(4x-x^2)^{3/2}} \\ &= -\frac{4}{(4x-x^2)^{3/2}}. \end{aligned}$$

Then, we apply 1.4.11:

$$\begin{aligned} \kappa &= \frac{|y''|}{[1+(y')^2]^{3/2}} \\ &= \frac{\left|-\frac{4}{(4x-x^2)^{3/2}}\right|}{\left[1+((2-x)(4x-x^2)^{-1/2})^2\right]^{3/2}} = \frac{\left|\frac{4}{(4x-x^2)^{3/2}}\right|}{\left[1+\frac{(2-x)^2}{4x-x^2}\right]^{3/2}} \\ &= \frac{\left|\frac{4}{(4x-x^2)^{3/2}}\right|}{\left[\frac{4x-x^2+x^2-4x+4}{4x-x^2}\right]^{3/2}} = \left|\frac{4}{(4x-x^2)^{3/2}}\right| \cdot \frac{(4x-x^2)^{3/2}}{8} \\ &= \frac{1}{2}. \end{aligned}$$

The curvature of this circle is equal to the reciprocal of its radius. There is a minor issue with the absolute value in 1.4.11 ; however, a closer look at the calculation reveals that the denominator is positive for any value of x.





Exercise 1.4.3

Find the curvature of the curve defined by the function

$$y = 3x^2 - 2x + 4$$

at the point $x = 2$.

Hint

Use 1.4.11.

Answer

$\kappa = \frac{6}{101^{3/2}} \approx 0.0059$



The Normal and Binormal Vectors

We have seen that the derivative $\vec{\mathbf{r}}'(t)$ of a vector-valued function is a tangent vector to the curve defined by $\vec{\mathbf{r}}(t)$, and the unit tangent vector $\vec{\mathbf{T}}(t)$ can be calculated by dividing $\vec{\mathbf{r}}'(t)$ by its magnitude. When studying motion in three dimensions, two other vectors are useful in describing the motion of a particle along a path in space: the principal unit normal vector and the **binormal vector**.

Definition: binormal vectors

Let C be a three-dimensional **smooth** curve represented by $\vec{\mathbf{r}}$ over an open interval I. If $\vec{\mathbf{T}}'(t) \neq \vec{\mathbf{0}}$, then the principal unit normal vector at t is defined to be

$$\vec{\mathbf{N}}(t) = \frac{\vec{\mathbf{T}}'(t)}{\|\vec{\mathbf{T}}'(t)\|}. \tag{1.4.12}$$

The binormal vector at t is defined as

$$\vec{\mathbf{B}}(t) = \vec{\mathbf{T}}(t) \times \vec{\mathbf{N}}(t), \tag{1.4.13}$$

where $\vec{\mathbf{T}}(t)$ is the unit tangent vector.

Note that, by definition, the binormal vector is orthogonal to both the unit tangent vector and the normal vector. Furthermore, $\vec{\mathbf{B}}(t)$ is always a unit vector. This can be shown using the formula for the magnitude of a cross product.

$$\|\vec{\mathbf{B}}(t)\| = \|\vec{\mathbf{T}}(t) \times \vec{\mathbf{N}}(t)\| = \|\vec{\mathbf{T}}(t)\|\|\vec{\mathbf{N}}(t)\| \sin\theta, \tag{1.4.14}$$

where θ is the angle between $\vec{\mathbf{T}}(t)$ and $\vec{\mathbf{N}}(t)$. Since $\vec{\mathbf{N}}(t)$ is the derivative of a unit vector, property (vii) of the derivative of a vector-valued function tells us that $\vec{\mathbf{T}}(t)$ and $\vec{\mathbf{N}}(t)$ are orthogonal to each other, so $\theta = \pi/2$. Furthermore, they are both unit vectors, so their magnitude is 1. Therefore, $\|\vec{\mathbf{T}}(t)\|\|\vec{\mathbf{N}}(t)\| \sin\theta = (1)(1)\sin(\pi/2) = 1$ and $\vec{\mathbf{B}}(t)$ is a unit vector.

The principal unit normal vector can be challenging to calculate because the unit tangent vector involves a quotient, and this quotient often has a square root in the denominator. In the three-dimensional case, finding the cross product of the unit tangent vector and the unit normal vector can be even more cumbersome. Fortunately, we have alternative formulas for finding these two vectors, and they are presented in Motion in Space.

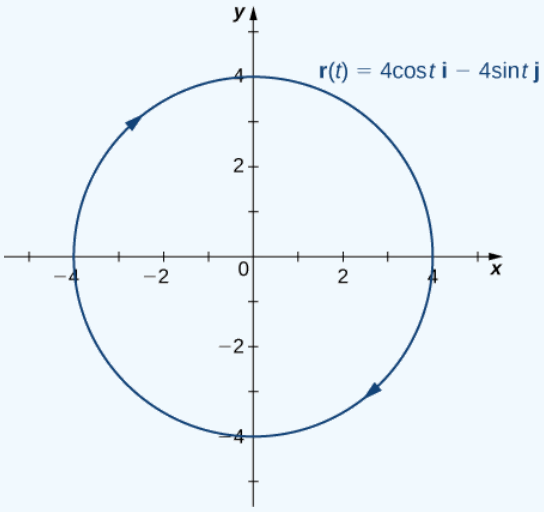
Example 1.4.4: Finding the Principal Unit Normal Vector and Binormal Vector

For each of the following vector-valued functions, find the principal unit normal vector. Then, if possible, find the binormal vector.

1. $\vec{\mathbf{r}}(t) = 4\cos t\,\hat{\mathbf{i}} - 4\sin t\,\hat{\mathbf{j}}$
2. $\vec{\mathbf{r}}(t) = (6t+2)\,\hat{\mathbf{i}} + 5t^2\,\hat{\mathbf{j}} - 8t\,\hat{\mathbf{k}}$

Solution

1. This function describes a circle.



To find the principal unit normal vector, we first must find the unit tangent vector $\vec{\mathbf{T}}(t)$:

$$\begin{aligned}\vec{\mathbf{T}}(t) &= \frac{\vec{\mathbf{r}}'(t)}{\|\vec{\mathbf{r}}'(t)\|} \\ &= \frac{-4\sin t\,\hat{\mathbf{i}} - 4\cos t\,\hat{\mathbf{j}}}{\sqrt{(-4\sin t)^2 + (-4\cos t)^2}} \\ &= \frac{-4\sin t\,\hat{\mathbf{i}} - 4\cos t\,\hat{\mathbf{j}}}{\sqrt{16\sin^2 t + 16\cos^2 t}} \\ &= \frac{-4\sin t\,\hat{\mathbf{i}} - 4\cos t\,\hat{\mathbf{j}}}{\sqrt{16(\sin^2 t + \cos^2 t)}} \\ &= \frac{-4\sin t\,\hat{\mathbf{i}} - 4\cos t\,\hat{\mathbf{j}}}{4} \\ &= -\sin t\,\hat{\mathbf{i}} - \cos t\,\hat{\mathbf{j}}.\end{aligned}$$

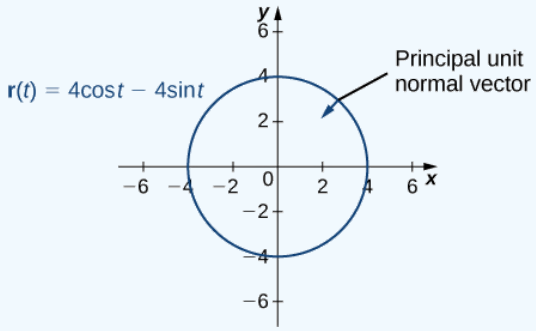
Next, we use 1.4.12 :

$$\begin{aligned}\vec{\mathbf{N}}(t) &= \frac{\vec{\mathbf{T}}'(t)}{\|\vec{\mathbf{T}}'(t)\|} \\ &= \frac{-\cos t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}}}{\sqrt{(-\cos t)^2 + (\sin t)^2}} \\ &= \frac{-\cos t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}}}{\sqrt{\cos^2 t + \sin^2 t}} \\ &= -\cos t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}}.\end{aligned}$$

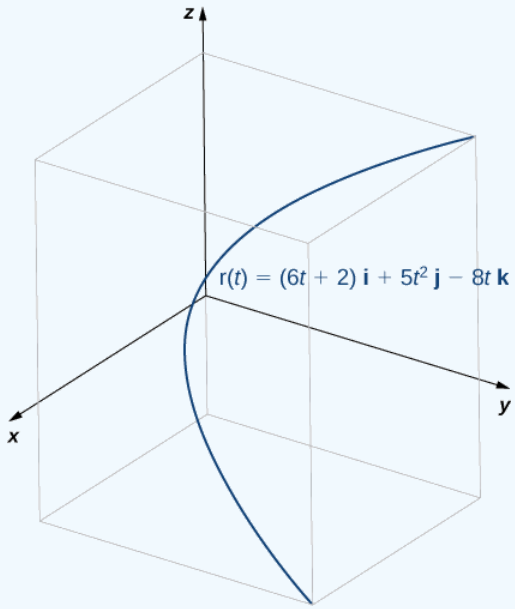
Notice that the unit tangent vector and the principal unit normal vector are orthogonal to each other for all values of t:

$$\begin{aligned}\vec{\mathbf{T}}(t)\cdot\vec{\mathbf{N}}(t) &= \langle -\sin t, -\cos t\rangle \cdot \langle -\cos t, \sin t\rangle \\ &= \sin t\cos t - \cos t\sin t \\ &= 0.\end{aligned}$$

Furthermore, the principal unit normal vector points toward the center of the circle from every point on the circle. Since $\vec{\mathbf{r}}(t)$ defines a curve in two dimensions, we cannot calculate the binormal vector.



2. This function looks like this:



To find the principal unit normal vector, we first find the unit tangent vector $\vec{\mathbf{T}}(t)$:

$$\begin{aligned}\vec{\mathbf{T}}(t) &= \frac{\vec{\mathbf{r}}'(t)}{\|\vec{\mathbf{r}}'(t)\|} \\ &= \frac{6\hat{\mathbf{i}} + 10t\hat{\mathbf{j}} - 8\hat{\mathbf{k}}}{\sqrt{6^2 + (10t)^2 + (-8)^2}} \\ &= \frac{6\hat{\mathbf{i}} + 10t\hat{\mathbf{j}} - 8\hat{\mathbf{k}}}{\sqrt{36 + 100t^2 + 64}} \\ &= \frac{6\hat{\mathbf{i}} + 10t\hat{\mathbf{j}} - 8\hat{\mathbf{k}}}{\sqrt{100(t^2 + 1)}} \\ &= \frac{3\hat{\mathbf{i}} - 5t\hat{\mathbf{j}} - 4\hat{\mathbf{k}}}{5\sqrt{t^2 + 1}} \\ &= \frac{3}{5}(t^2 + 1)^{-1/2}\hat{\mathbf{i}} - t(t^2 + 1)^{-1/2}\hat{\mathbf{j}} - \frac{4}{5}(t^2 + 1)^{-1/2}\hat{\mathbf{k}}.\end{aligned}$$

Next, we calculate $\vec{\mathbf{T}}'(t)$ and $\|\vec{\mathbf{T}}'(t)\|$:

https://math.libretexts.org/@go/page/84723

$$\begin{aligned}\vec{\mathbf{T}}'(t) &= \frac{3}{5}\left(-\frac{1}{2}\right)(t^2+1)^{-3/2}(2t)\,\hat{\mathbf{i}} - \left((t^2+1)^{-1/2} - t\left(\frac{1}{2}\right)(t^2+1)^{-3/2}(2t)\right)\hat{\mathbf{j}} \\ &\quad - \frac{4}{5}\left(-\frac{1}{2}\right)(t^2+1)^{-3/2}(2t)\,\hat{\mathbf{k}} \\ &= -\frac{3t}{5(t^2+1)^{3/2}}\,\hat{\mathbf{i}} - \frac{1}{(t^2+1)^{3/2}}\,\hat{\mathbf{j}} + \frac{4t}{5(t^2+1)^{3/2}}\,\hat{\mathbf{k}} \\ \|\vec{\mathbf{T}}'(t)\| &= \sqrt{\left(-\frac{3t}{5(t^2+1)^{3/2}}\right)^2 + \left(-\frac{1}{(t^2+1)^{3/2}}\right)^2 + \left(\frac{4t}{5(t^2+1)^{3/2}}\right)^2} \\ &= \sqrt{\frac{9t^2}{25(t^2+1)^3} + \frac{1}{(t^2+1)^3} + \frac{16t^2}{25(t^2+1)^3}} \\ &= \sqrt{\frac{25t^2+25}{25(t^2+1)^3}} \\ &= \sqrt{\frac{1}{(t^2+1)^2}} \\ &= \frac{1}{t^2+1}.\end{aligned}$$

Therefore, according to 1.4.12 :

$$\begin{aligned}\vec{\mathbf{N}}(t) &= \frac{\vec{\mathbf{T}}'(t)}{\|\vec{\mathbf{T}}'(t)\|} \\ &= \left(-\frac{3t}{5(t^2+1)^{3/2}}\,\hat{\mathbf{i}} - \frac{1}{(t^2+1)^{3/2}}\,\hat{\mathbf{j}} + \frac{4t}{5(t^2+1)^{3/2}}\,\hat{\mathbf{k}}\right)(t^2+1) \\ &= -\frac{3t}{5(t^2+1)^{1/2}}\,\hat{\mathbf{i}} - \frac{5}{5(t^2+1)^{1/2}}\,\hat{\mathbf{j}} + \frac{4t}{5(t^2+1)^{1/2}}\,\hat{\mathbf{k}} \\ &= -\frac{3t\,\hat{\mathbf{i}} + 5\,\hat{\mathbf{j}} - 4t\,\hat{\mathbf{k}}}{5\sqrt{t^2+1}}.\end{aligned}$$

Once again, the unit tangent vector and the principal unit normal vector are orthogonal to each other for all values of t:

$$\begin{aligned}\vec{\mathbf{T}}(t)\cdot\vec{\mathbf{N}}(t) &= \left(\frac{3\,\hat{\mathbf{i}} - 5t\,\hat{\mathbf{j}} - 4\,\hat{\mathbf{k}}}{5\sqrt{t^2+1}}\right)\cdot\left(-\frac{3t\,\hat{\mathbf{i}} + 5\,\hat{\mathbf{j}} - 4t\,\hat{\mathbf{k}}}{5\sqrt{t^2+1}}\right) \\ &= \frac{3(-3t) - 5t(-5) - 4(4t)}{25(t^2+1)} \\ &= \frac{-9t + 25t - 16t}{25(t^2+1)} \\ &= 0.\end{aligned}$$

Last, since $\vec{\mathbf{r}}(t)$ represents a three-dimensional curve, we can calculate the binormal vector using 1.4.13 :

$$
\begin{aligned}
\vec{\mathbf{B}}(t) &= \vec{\mathbf{T}}(t) \times \vec{\mathbf{N}}(t) \\
&= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{3}{5\sqrt{t^2+1}} & -\frac{5t}{5\sqrt{t^2+1}} & -\frac{4}{5\sqrt{t^2+1}} \\ -\frac{3t}{5\sqrt{t^2+1}} & -\frac{5}{5\sqrt{t^2+1}} & \frac{4t}{5\sqrt{t^2+1}} \end{vmatrix} \\
&= \left(\left(-\frac{5t}{5\sqrt{t^2+1}}\right)\left(\frac{4t}{5\sqrt{t^2+1}}\right) - \left(-\frac{4}{5\sqrt{t^2+1}}\right)\left(-\frac{5}{5\sqrt{t^2+1}}\right)\right)\hat{\mathbf{i}} \\
&- \left(\left(\frac{3}{5\sqrt{t^2+1}}\right)\left(\frac{4t}{5\sqrt{t^2+1}}\right) - \left(-\frac{4}{5\sqrt{t^2+1}}\right)\left(-\frac{3t}{5\sqrt{t^2+1}}\right)\right)\hat{\mathbf{j}} \\
&+ \left(\left(\frac{3}{5\sqrt{t^2+1}}\right)\left(-\frac{5}{5\sqrt{t^2+1}}\right) - \left(-\frac{5t}{5\sqrt{t^2+1}}\right)\left(-\frac{3t}{5\sqrt{t^2+1}}\right)\right)\hat{\mathbf{k}} \\
&= \left(\frac{-20t^2-20}{25(t^2+1)}\right)\hat{\mathbf{i}} + \left(\frac{-15-15t^2}{25(t^2+1)}\right)\hat{\mathbf{k}} \\
&= -20\left(\frac{t^2+1}{25(t^2+1)}\right)\hat{\mathbf{i}} - 15\left(\frac{t^2+1}{25(t^2+1)}\right)\hat{\mathbf{k}} \\
&= -\frac{4}{5}\hat{\mathbf{i}} - \frac{3}{5}\hat{\mathbf{k}}.
\end{aligned}
$$





Exercise 1.4.4

Find the unit normal vector for the vector-valued function $\vec{\mathbf{r}}(t) = (t^2 - 3t)\,\hat{\mathbf{i}} + (4t + 1)\,\hat{\mathbf{j}}$ and evaluate it at $t = 2$.

Hint

First, find $\vec{\mathbf{T}}(t)$, then use 1.4.12.

Answer

$$\vec{\mathbf{N}}(2) = \frac{\sqrt{2}}{2}(\hat{\mathbf{i}} - \hat{\mathbf{j}})$$

For any smooth curve in three dimensions that is defined by a vector-valued function, we now have formulas for the unit tangent vector $\vec{\mathbf{T}}$, the unit normal vector $\vec{\mathbf{N}}$, and the binormal vector $\vec{\mathbf{B}}$. The unit normal vector and the binormal vector form a plane that is perpendicular to the curve at any point on the curve, called the normal plane. In addition, these three vectors form a frame of reference in three-dimensional space called the *Frenet frame of reference* (also called the **TNB** frame) (Figure **1.4.2**). Last, the plane determined by the vectors $\vec{\mathbf{T}}$ and $\vec{\mathbf{N}}$ forms the osculating plane of C at any point P on the curve.

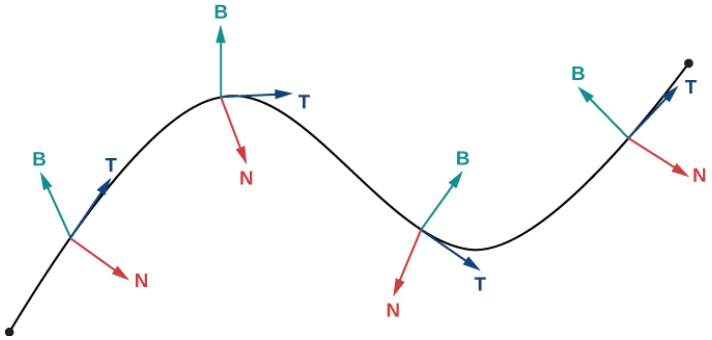


Figure **1.4.2**: This figure depicts a Frenet frame of reference. At every point P *on a three-dimensional curve, the unit tangent, unit normal, and binormal vectors form a three-dimensional frame of reference.*

Suppose we form a circle in the osculating plane of C at point P on the curve. Assume that the circle has the same curvature as the curve does at point P and let the circle have radius r. Then, the curvature of the circle is given by $\frac{1}{r}$. We call r the radius of curvature of the curve, and it is equal to the reciprocal of the curvature. If this circle lies on the concave side of the curve and is tangent to the curve at point P, then this circle is called the *osculating circle* of C at P, as shown in Figure **1.4.3**.

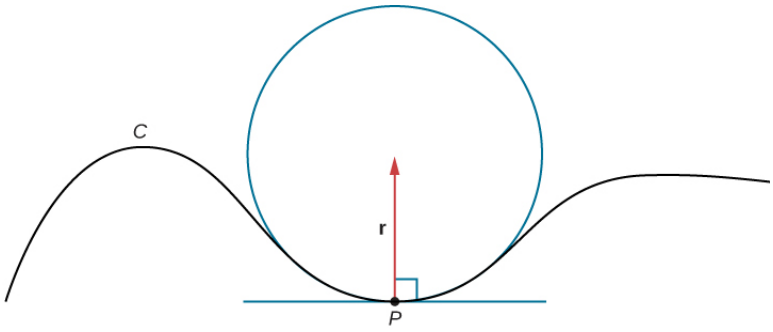


Figure 1.4.3: In this osculating circle, the circle is tangent to curve C at point P *and shares the same curvature.*

For more information on osculating circles, see this demonstration on curvature and torsion, this article on osculating circles, and this discussion of Serret formulas.

To find the equation of an osculating circle in two dimensions, we need find only the center and radius of the circle.

Example 1.4.5: Finding the Equation of an Osculating Circle

Find the equation of the osculating circle of the curve defined by the function $y = x^3 - 3x + 1$ at $x = 1$.

Solution

Figure 1.4.4 shows the graph of $y = x^3 - 3x + 1$.

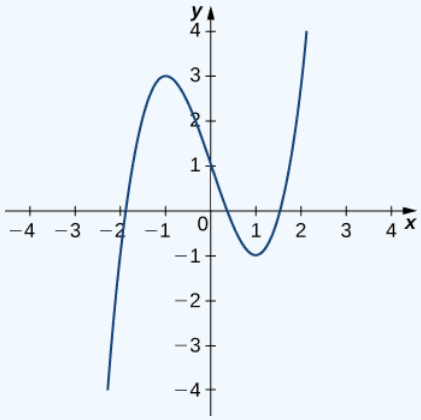


Figure 1.4.4: We want to find the osculating circle of this graph at the point where $x = 1$.

First, let's calculate the curvature at $x = 1$:

$$\kappa = \frac{|f''(x)|}{\left(1 + [f'(x)]^2\right)^{3/2}} = \frac{|6x|}{(1 + [3x^2 - 3]^2)^{3/2}}. \tag{1.4.15}$$

This gives $\kappa = 6$. Therefore, the radius of the osculating circle is given by $R = \frac{1}{\kappa} = \frac{1}{6}$. Next, we then calculate the coordinates of the center of the circle. When $x = 1$, the slope of the tangent line is zero. Therefore, the center of the osculating circle is directly above the point on the graph with coordinates $(1, -1)$. The center is located at $(1, -\frac{5}{6})$. The formula for a circle with radius r and center (h, k) is given by $(x - h)^2 + (y - k)^2 = r^2$. Therefore, the equation of the osculating circle is $(x - 1)^2 + (y + \frac{5}{6})^2 = \frac{1}{36}$. The graph and its osculating circle appears in the following graph.

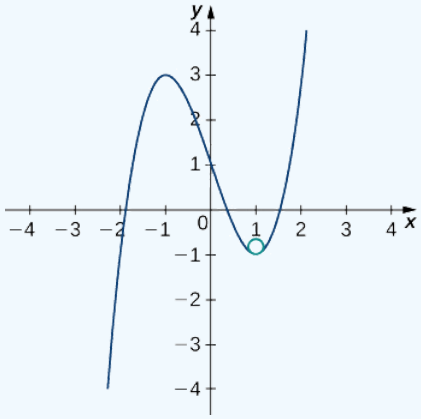


Figure 1.4.5: The osculating circle has radius $R = \frac{1}{6}$.

Exercise 1.4.5

Find the equation of the osculating circle of the curve defined by the vector-valued function $y = 2x^2 - 4x + 5$ at $x = 1$.

Hint

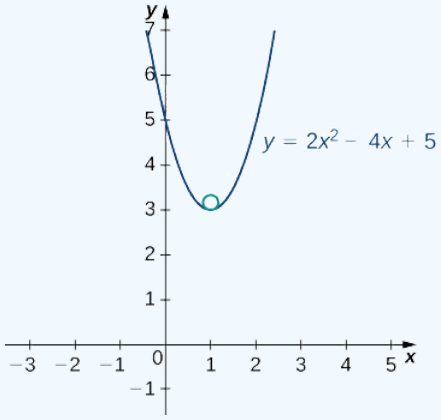
Use 1.4.11 to find the curvature of the graph, then draw a graph of the function around $x = 1$ to help visualize the circle in relation to the graph.

Answer

$\kappa = \frac{4}{[1+(4x-4)^2]^{3/2}}$

At the point $x = 1$, the curvature is equal to 4. Therefore, the radius of the osculating circle is $\frac{1}{4}$.

A graph of this function appears next:



The vertex of this parabola is located at the point $(1, 3)$. Furthermore, the center of the osculating circle is directly above the vertex. Therefore, the coordinates of the center are $(1, \frac{13}{4})$. The equation of the osculating circle is

$(x-1)^2 + (y-\frac{13}{4})^2 = \frac{1}{16}$.

Key Concepts

- The arc-length function for a vector-valued function is calculated using the integral formula $s(t) = \int_a^b \|\vec{\mathbf{r}}'(t)\|\, dt$. This formula is valid in both two and three dimensions.
- The curvature of a curve at a point in either two or three dimensions is defined to be the curvature of the inscribed circle at that point. The arc-length parameterization is used in the definition of curvature.
- There are several different formulas for curvature. The curvature of a circle is equal to the reciprocal of its radius.

- The principal unit normal vector at t is defined to be

$$\vec{\mathbf{N}}(t) = \frac{\vec{\mathbf{T}}'(t)}{\|\vec{\mathbf{T}}'(t)\|}.$$

- The binormal vector at t is defined as $\vec{\mathbf{B}}(t) = \vec{\mathbf{T}}(t) \times \vec{\mathbf{N}}(t)$, where $\vec{\mathbf{T}}(t)$ is the unit tangent vector.
- The Frenet frame of reference is formed by the unit tangent vector, the principal unit normal vector, and the binormal vector.
- The osculating circle is tangent to a curve at a point and has the same curvature as the tangent curve at that point.

Key Equations

- **Arc length of space curve**
  $s = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt = \int_a^b \|\vec{\mathbf{r}}'(t)\|\, dt$
- **Arc-length function**
  $s(t) = \int_a^t \sqrt{f'(u))^2 + (g'(u))^2 + (h'(u))^2}\, du \text{ or } s(t) = \int_a^t \|\vec{\mathbf{r}}'(u)\|\, du$
- $\kappa = \frac{\|\vec{\mathbf{T}}'(t)\|}{\|\vec{\mathbf{r}}'(t)\|}$ or $\kappa = \frac{\|\vec{\mathbf{r}}'(t) \times \vec{\mathbf{r}}''(t)\|}{\|\vec{\mathbf{r}}'(t)\|^3}$ or $\kappa = \frac{|y''|}{[1+(y')^2]^{3/2}}$
- **Principal unit normal vector**
  $\vec{\mathbf{N}}(t) = \frac{\vec{\mathbf{T}}'(t)}{\|\vec{\mathbf{T}}'(t)\|}$
- **Binormal vector**
  $\vec{\mathbf{B}}(t) = \vec{\mathbf{T}}(t) \times \vec{\mathbf{N}}(t)$

Glossary

arc-length function

a function $s(t)$ that describes the arc length of curve C as a function of t

arc-length parameterization

a reparameterization of a vector-valued function in which the parameter is equal to the arc length

binormal vector

a unit vector orthogonal to the unit tangent vector and the unit normal vector

curvature

the derivative of the unit tangent vector with respect to the arc-length parameter

Frenet frame of reference

(TNB frame) a frame of reference in three-dimensional space formed by the unit tangent vector, the unit normal vector, and the binormal vector

normal plane

a plane that is perpendicular to a curve at any point on the curve

osculating circle

a circle that is tangent to a curve C at a point P and that shares the same curvature

osculating plane

the plane determined by the unit tangent and the unit normal vector

principal unit normal vector

a vector orthogonal to the unit tangent vector, given by the formula $\frac{\vec{\mathbf{T}}'(t)}{\|\vec{\mathbf{T}}'(t)\|}$

radius of curvature

the reciprocal of the curvature

smooth

curves where the vector-valued function $\vec{\mathbf{r}}(t)$ is differentiable with a non-zero derivative

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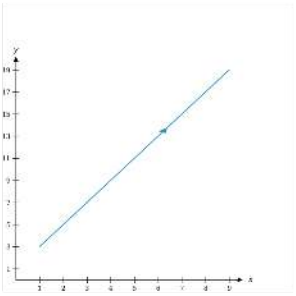
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1.4E: Exercises

Determining Arc Length

In questions 1 - 5, find the arc length of the curve on the given interval.

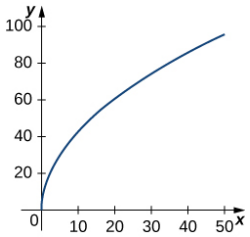
1) $\vec{\mathbf{r}}(t) = t^2\,\hat{\mathbf{i}} + (2t^2+1)\,\hat{\mathbf{j}}, \quad 1 \le t \le 3$. This portion of the graph is shown here:



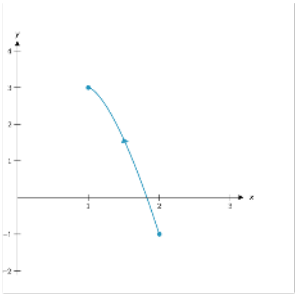
Answer

$8\sqrt{5}$ units ≈ 17.9 units

2) $\vec{\mathbf{r}}(t) = t^2\,\hat{\mathbf{i}} + 14t\,\hat{\mathbf{j}}, \quad 0 \le t \le 7$. This portion of the graph is shown here:



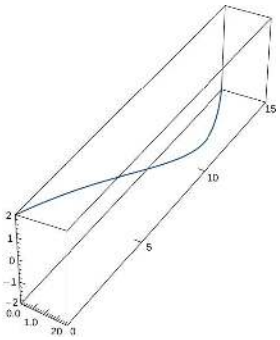
3) $\vec{\mathbf{r}}(t) = \langle t^2+1, 4t^3+3\rangle, \quad -1 \le t \le 0$. This portion of the graph is shown here:



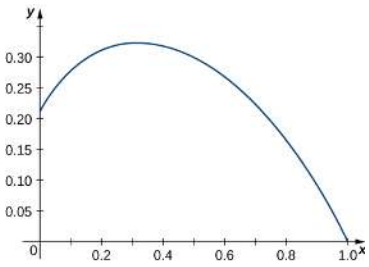
Answer

$\frac{1}{54}(37^{3/2}-1)$ units ≈ 4.15 units

4) $\vec{\mathbf{r}}(t) = \langle 2\sin t, 5t, 2\cos t\rangle, \quad 0 \le t \le \pi$. This portion of the graph is shown here:



5) $\vec{\mathbf{r}}(t) = \langle e^{-t\cos t}, e^{-t\sin t} \rangle$ over the interval $[0, \frac{\pi}{2}]$. Here is the portion of the graph on the indicated interval:



6) Set up an integral to represent the arc length from $t = 0$ to $t = 2$ along the curve traced out by $\vec{\mathbf{r}}(t) = \langle t, t^4 \rangle$. Then use technology to approximate this length to the nearest thousandth of a unit.

7) Find the length of one turn of the helix given by $\vec{\mathbf{r}}(t) = \frac{1}{2}\cos t\,\hat{\mathbf{i}} + \frac{1}{2}\sin t\,\hat{\mathbf{j}} + \frac{\sqrt{3}}{2}\,t\,\hat{\mathbf{k}}$.

Answer

Length $= 2\pi$ units

8) Find the arc length of the vector-valued function $\vec{\mathbf{r}}(t) = -t\,\hat{\mathbf{i}} + 4t\,\hat{\mathbf{j}} + 3t\,\hat{\mathbf{k}}$ over $[0, 1]$.

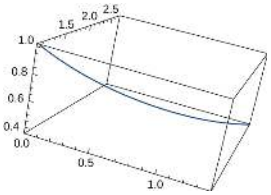
9) A particle travels in a circle with the equation of motion $\vec{\mathbf{r}}(t) = 3\cos t\,\hat{\mathbf{i}} + 3\sin t\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}$. Find the distance traveled around the circle by the particle.

Answer

6π units

10) Set up an integral to find the circumference of the ellipse with the equation $\vec{\mathbf{r}}(t) = \cos t\,\hat{\mathbf{i}} + 2\sin t\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}$.

11) Find the length of the curve $\vec{\mathbf{r}}(t) = \langle \sqrt{2}t, e^t, e^{-t} \rangle$ over the interval $0 \le t \le 1$. The graph is shown here:



Answer

$\left(e - \frac{1}{e}\right)$ units ≈ 2.35 units

12) Find the length of the curve $\vec{\mathbf{r}}(t) = \langle 2\sin t,\ 5t,\ 2\cos t \rangle$ for $t \in [-10, 10]$.

Unit Tangent Vectors and Unit Normal Vectors

13) The position function for a particle is $\vec{\mathbf{r}}(t) = a\cos(\omega t)\,\hat{\mathbf{i}} + b\sin(\omega t)\,\hat{\mathbf{j}}$. Find the unit tangent vector and the unit normal vector at $t = 0$.

Solution:

$$\vec{\mathbf{r}}'(t) = -a\omega \sin(\omega t)\,\hat{\mathbf{i}} + b\omega \cos(\omega t)\,\hat{\mathbf{j}}$$

$$\|\vec{\mathbf{r}}'(t)\| = \sqrt{a^2\omega^2 \sin^2(\omega t) + b^2\omega^2 \cos^2(\omega t)}$$

$$\vec{\mathbf{T}}(t) = \frac{\vec{\mathbf{r}}'(t)}{\|\vec{\mathbf{r}}'(t)\|} = \frac{-a\omega \sin(\omega t)\,\hat{\mathbf{i}} + b\omega \cos(\omega t)\,\hat{\mathbf{j}}}{\sqrt{a^2\omega^2 \sin^2(\omega t) + b^2\omega^2 \cos^2(\omega t)}}$$

$$\vec{\mathbf{T}}(0) = \frac{b\omega\,\hat{\mathbf{j}}}{\sqrt{(b\omega)^2}} = \frac{b\omega\,\hat{\mathbf{j}}}{|b\omega|}$$

If $b\omega > 0,\ \vec{\mathbf{T}}(0) = \hat{\mathbf{j}}$, and if $b\omega < 0,\ T(0) = -\hat{\mathbf{j}}$

Answer

If $b\omega > 0,\ \vec{\mathbf{T}}(0) = \hat{\mathbf{j}}$, and if $b\omega < 0,\ \vec{\mathbf{T}}(0) = -\hat{\mathbf{j}}$

If $a > 0,\ \vec{\mathbf{N}}(0) = -\hat{\mathbf{i}}$, and if $a < 0,\ \vec{\mathbf{N}}(0) = \hat{\mathbf{i}}$

14) Given $\vec{\mathbf{r}}(t) = a\cos(\omega t)\,\hat{\mathbf{i}} + b\sin(\omega t)\,\hat{\mathbf{j}}$, find the binormal vector $\vec{\mathbf{B}}(0)$.

15) Given $\vec{\mathbf{r}}(t) = \langle 2e^t, e^t \cos t, e^t \sin t\rangle$, determine the unit tangent vector $\vec{\mathbf{T}}(t)$.

Answer

$$\vec{\mathbf{T}}(t) = \left\langle \frac{2}{\sqrt{6}}, \frac{\cos t - \sin t}{\sqrt{6}}, \frac{\cos t + \sin t}{\sqrt{6}} \right\rangle$$

$$= \left\langle \frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{6}(\cos t - \sin t), \frac{\sqrt{6}}{6}(\cos t + \sin t) \right\rangle$$

16) Given $\vec{\mathbf{r}}(t) = \langle 2e^t,\ e^t \cos t,\ e^t \sin t\rangle$, find the unit tangent vector $\vec{\mathbf{T}}(t)$ evaluated at $t = 0$, $\vec{\mathbf{T}}(0)$.

17) Given $\vec{\mathbf{r}}(t) = \langle 2e^t,\ e^t \cos t,\ e^t \sin t\rangle$, determine the unit normal vector $\vec{\mathbf{N}}(t)$.

Answer

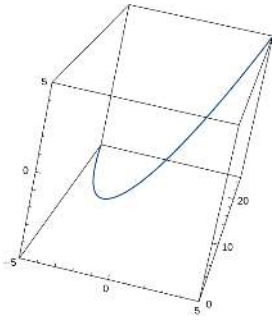
$$\vec{\mathbf{N}}(t) = \langle 0,\ -\tfrac{\sqrt{2}}{2}(\sin t + \cos t),\ \tfrac{\sqrt{2}}{2}(\cos t - \sin t)\rangle$$

18) Given $\vec{\mathbf{r}}(t) = \langle 2e^t,\ e^t \cos t,\ e^t \sin t\rangle$, find the unit normal vector $\vec{\mathbf{N}}(t)$ evaluated at $t = 0$, $\vec{\mathbf{N}}(0)$.

Answer

$$\vec{\mathbf{N}}(0) = \langle 0,\ -\tfrac{\sqrt{2}}{2},\ \tfrac{\sqrt{2}}{2}\rangle$$

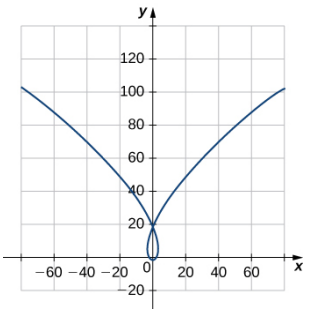
19) Given $\vec{\mathbf{r}}(t) = t\,\hat{\mathbf{i}} + t^2\,\hat{\mathbf{j}} + t\,\hat{\mathbf{k}}$, find the unit tangent vector $\vec{\mathbf{T}}(t)$. The graph is shown here:



Answer

$$\vec{\mathbf{T}}(t) = \frac{1}{\sqrt{4t^2 + 2}} < 1, 2t, 1 >$$

20) Find the unit tangent vector $\vec{\mathbf{T}}(t)$ and unit normal vector $\vec{\mathbf{N}}(t)$ at $t = 0$ for the plane curve $\vec{\mathbf{r}}(t) = \langle t^3 - 4t, 5t^2 - 2\rangle$. The graph is shown here:



21) Find the unit tangent vector $\vec{\mathbf{T}}(t)$ for $\vec{\mathbf{r}}(t) = 3t\,\hat{\mathbf{i}} + 5t^2\,\hat{\mathbf{j}} + 2t\,\hat{\mathbf{k}}$.

Answer

$$\vec{\mathbf{T}}(t) = \frac{1}{\sqrt{100t^2 + 13}}(3\,\hat{\mathbf{i}} + 10t\,\hat{\mathbf{j}} + 2\,\hat{\mathbf{k}})$$

22) Find the principal normal vector to the curve $\vec{\mathbf{r}}(t) = \langle 6\cos t, 6\sin t\rangle$ at the point determined by $t = \frac{\pi}{3}$.

23) Find $\vec{\mathbf{T}}(t)$ for the curve $\vec{\mathbf{r}}(t) = (t^3 - 4t)\,\hat{\mathbf{i}} + (5t^2 - 2)\,\hat{\mathbf{j}}$.

Answer

$$\vec{\mathbf{T}}(t) = \frac{1}{\sqrt{9t^4 + 76t^2 + 16}}\left((3t^2 - 4)\,\hat{\mathbf{i}} + 10t\,\hat{\mathbf{j}}\right)$$

24) Find $\vec{\mathbf{N}}(t)$ for the curve $\vec{\mathbf{r}}(t) = (t^3 - 4t)\,\hat{\mathbf{i}} + (5t^2 - 2)\,\hat{\mathbf{j}}$.

25) Find the unit tangent vector $\vec{\mathbf{T}}(t)$ for $\vec{\mathbf{r}}(t) = \langle 2\sin t,\ 5t,\ 2\cos t\rangle$.

Answer

$$\vec{\mathbf{T}}(t) = \langle \tfrac{2\sqrt{29}}{29}\cos t,\ \tfrac{5\sqrt{29}}{29},\ -\tfrac{2\sqrt{29}}{29}\sin t\rangle$$

26) Find the unit normal vector $\vec{\mathbf{N}}(t)$ for $\vec{\mathbf{r}}(t) = \langle 2\sin t,\ 5t,\ 2\cos t\rangle$.

Answer

$$\vec{\mathbf{N}}(t) = \langle -\sin t,\ 0,\ -\cos t\rangle$$

Arc Length Parameterizations

27) Find the arc-length function $\vec{\mathbf{s}}(t)$ for the line segment given by $\vec{\mathbf{r}}(t) = \langle 3 - 3t,\ 4t\rangle$. Then write the arc-length parameterization of r with s as the parameter.

Answer

Arc-length function: $s(t) = 5t$; The arc-length parameterization of $\vec{\mathbf{r}}(t)$: $\vec{\mathbf{r}}(s) = \left(3 - \frac{3s}{5}\right)\hat{\mathbf{i}} + \frac{4s}{5}\,\hat{\mathbf{j}}$

28) Parameterize the helix $\vec{\mathbf{r}}(t) = \cos t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}} + t\,\hat{\mathbf{k}}$ using the arc-length parameter s, from $t = 0$.

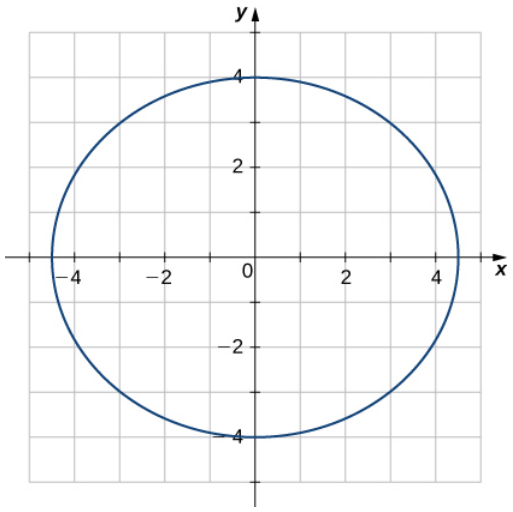
29) Parameterize the curve using the arc-length parameter s, at the point at which $t = 0$ for $\vec{\mathbf{r}}(t) = e^t \sin t\,\hat{\mathbf{i}} + e^t \cos t\,\hat{\mathbf{j}}$

Answer

$$\vec{\mathbf{r}}(s) = \left(1 + \frac{s}{\sqrt{2}}\right)\sin\left(\ln\left(1 + \frac{s}{\sqrt{2}}\right)\right)\hat{\mathbf{i}} + \left(1 + \frac{s}{\sqrt{2}}\right)\cos\left(\ln\left(1 + \frac{s}{\sqrt{2}}\right)\right)\hat{\mathbf{j}}$$

Curvature and the Osculating Circle

30) Find the curvature of the curve $\vec{\mathbf{r}}(t) = 5\cos t\,\hat{\mathbf{i}} + 4\sin t\,\hat{\mathbf{j}}$ at $t = \pi/3$. (*Note:* The graph is an ellipse.)



31) Find the x-coordinate at which the curvature of the curve $y = 1/x$ is a maximum value.

Answer

The maximum value of the curvature occurs at $x = 1$.

32) Find the curvature of the curve $\vec{\mathbf{r}}(t) = 5\cos t\,\hat{\mathbf{i}} + 5\sin t\,\hat{\mathbf{j}}$. Does the curvature depend upon the parameter t?

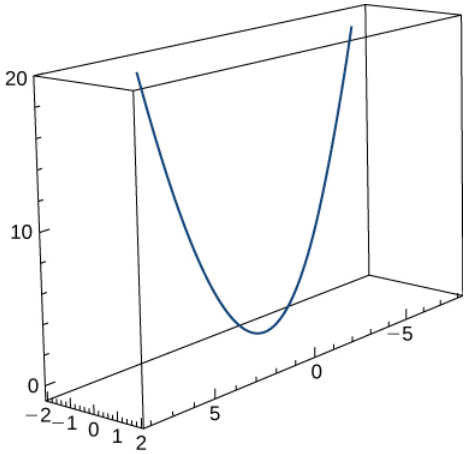
33) Find the curvature κ for the curve $y = x - \frac{1}{4}x^2$ at the point $x = 2$.

Answer

$\frac{1}{2}$

34) Find the curvature κ for the curve $y = \frac{1}{3}x^3$ at the point $x = 1$.

35) Find the curvature κ of the curve $\vec{\mathbf{r}}(t) = t\,\hat{\mathbf{i}} + 6t^2\,\hat{\mathbf{j}} + 4t\,\hat{\mathbf{k}}$. The graph is shown here:



Answer

$$\kappa \approx \frac{49.477}{(17 + 144t^2)^{3/2}}$$

36) Find the curvature of $\vec{\mathbf{r}}(t) = \langle 2\sin t, 5t, 2\cos t\rangle$.

37) Find the curvature of $\vec{\mathbf{r}}(t) = \sqrt{2}t\,\hat{\mathbf{i}} + e^t\,\hat{\mathbf{j}} + e^{-t}\,\hat{\mathbf{k}}$ at point $P(0, 1, 1)$.

Answer

$\frac{1}{2\sqrt{2}}$

38) At what point does the curve $y = e^x$ have maximum curvature?

39) What happens to the curvature as $x \to \infty$ for the curve $y = e^x$?

Answer

The curvature approaches zero.

40) Find the point of maximum curvature on the curve $y = \ln x$.

41) Find the equations of the normal plane and the osculating plane of the curve $\vec{\mathbf{r}}(t) = \langle 2\sin(3t), t, 2\cos(3t)\rangle$ at point $(0, \pi, -2)$.

Answer

$y = 6x + \pi$ and $x + 6y = 6\pi$

42) Find equations of the osculating circles of the ellipse $4y^2 + 9x^2 = 36$ at the points $(2, 0)$ and $(0, 3)$.

43) Find the equation for the osculating plane at point $t = \pi/4$ on the curve $\vec{\mathbf{r}}(t) = \cos(2t)\,\hat{\mathbf{i}} + \sin(2t)\,\hat{\mathbf{j}} + t\,\hat{\mathbf{k}}$.

Answer

$x + 2z = \frac{\pi}{2}$

44) Find the radius of curvature of $6y = x^3$ at the point $(2, \frac{4}{3})$.

45) Find the curvature at each point (x, y) on the hyperbola $\vec{\mathbf{r}}(t) = \langle a\cosh(t), b\sinh(t)\rangle$.

Answer

$$\frac{a^4b^4}{(b^4x^2 + a^4y^2)^{3/2}}$$

46) Calculate the curvature of the circular helix $\vec{\mathbf{r}}(t) = r\sin(t)\,\hat{\mathbf{i}} + r\cos(t)\,\hat{\mathbf{j}} + t\,\hat{\mathbf{k}}$.

47) Find the radius of curvature of $y = \ln(x + 1)$ at point $(2, \ln 3)$.

Answer

$\frac{10\sqrt{10}}{3}$

48) Find the radius of curvature of the hyperbola $xy = 1$ at point $(1, 1)$.

A particle moves along the plane curve C described by $\vec{\mathbf{r}}(t) = t\,\hat{\mathbf{i}} + t^2\,\hat{\mathbf{j}}$. Use this parameterization to answer questions 49 - 51.

49) Find the length of the curve over the interval $[0, 2]$.

Answer

$\frac{1}{4}\left[4\sqrt{17} + \ln\left(4 + \sqrt{17}\right)\right]$ units ≈ 4.64678 units

50) Find the curvature of the plane curve at $t = 0, 1, 2$.

51) Describe the curvature as t increases from $t = 0$ to $t = 2$.

Answer

The curvature is decreasing over this interval.

The surface of a large cup is formed by revolving the graph of the function $y = 0.25x^{1.6}$ from $x = 0$ to $x = 5$ about the y-axis (measured in centimeters).

52) **[T]** Use technology to graph the surface.

53) Find the curvature κ of the generating curve as a function of x.

Answer

$$\kappa = \frac{30}{x^{2/5}\left(25+4x^{6/5}\right)^{3/2}}$$

Note that initially your answer may be:

$$\frac{6}{25x^{2/5}\left(1+\frac{4}{25}x^{6/5}\right)^{3/2}}$$

We can simplify it as follows:

$$\begin{aligned}\frac{6}{25x^{2/5}\left(1+\frac{4}{25}x^{6/5}\right)^{3/2}} &= \frac{6}{25x^{2/5}\left[\frac{1}{25}\left(25+4x^{6/5}\right)\right]^{3/2}} \\ &= \frac{6}{25x^{2/5}\left(\frac{1}{25}\right)^{3/2}\left[25+4x^{6/5}\right]^{3/2}} \\ &= \frac{6}{\frac{25}{125}x^{2/5}\left[25+4x^{6/5}\right]^{3/2}} \\ &= \frac{30}{x^{2/5}\left(25+4x^{6/5}\right)^{3/2}}\end{aligned}$$

54) **[T]** Use technology to graph the curvature function.

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1.5: Motion in Space

Learning Objectives

- Describe the velocity and acceleration vectors of a particle moving in space.
- Explain the tangential and normal components of acceleration.
- State Kepler's laws of planetary motion.

We have now seen how to describe curves in the plane and in space, and how to determine their properties, such as arc length and curvature. All of this leads to the main goal of this chapter, which is the description of motion along plane curves and space curves. We now have all the tools we need; in this section, we put these ideas together and look at how to use them.

Motion Vectors in the Plane and in Space

Our starting point is using vector-valued functions to represent the position of an object as a function of time. All of the following material can be applied either to curves in the plane or to space curves. For example, when we look at the orbit of the planets, the curves defining these orbits all lie in a plane because they are elliptical. However, a particle traveling along a helix moves on a curve in three dimensions.

Definition: Speed, Velocity, and Acceleration

Let $\vec{\mathbf{r}}(t)$ be a twice-differentiable vector-valued function of the parameter t that represents the position of an object as a function of time.

The velocity vector $\vec{\mathbf{v}}(t)$ of the object is given by

$$\text{Velocity } = \vec{\mathbf{v}}(t) = \vec{\mathbf{r}}'(t). \tag{1.5.1}$$

The acceleration vector $\vec{\mathbf{a}}(t)$ is defined to be

$$\text{Acceleration } = \vec{\mathbf{a}}(t) = \vec{\mathbf{v}}'(t) = \vec{\mathbf{r}}''(t). \tag{1.5.2}$$

The *speed* is defined to be

$$\text{Speed } = v(t) = \|\vec{\mathbf{v}}(t)\| = \|\vec{\mathbf{r}}'(t)\| = \frac{ds}{dt}. \tag{1.5.3}$$

Since $\vec{\mathbf{r}}(t)$ can be in either two or three dimensions, these vector-valued functions can have either two or three components. In two dimensions, we define $\vec{\mathbf{r}}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}}$ and in three dimensions $\vec{\mathbf{r}}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}$. Then the velocity, acceleration, and speed can be written as shown in the following table.

Table **1.5.1**: Formulas for Position, Velocity, Acceleration, and Speed

Quantity	Two Dimensions	Three Dimensions
Position	$\vec{\mathbf{r}}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}}$	$\vec{\mathbf{r}}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}$
Velocity	$\vec{\mathbf{v}}(t) = x'(t)\hat{\mathbf{i}} + y'(t)\hat{\mathbf{j}}$	$\vec{\mathbf{v}}(t) = x'(t)\hat{\mathbf{i}} + y'(t)\hat{\mathbf{j}} + z'(t)\hat{\mathbf{k}}$
Acceleration	$\vec{\mathbf{a}}(t) = x''(t)\hat{\mathbf{i}} + y''(t)\hat{\mathbf{j}}$	$\vec{\mathbf{a}}(t) = x''(t)\hat{\mathbf{i}} + y''(t)\hat{\mathbf{j}} + z''(t)\hat{\mathbf{k}}$
Speed	$\|\vec{\mathbf{v}}(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2}$	$\|\vec{\mathbf{v}}(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$

Example 1.5.1: Studying Motion Along a Parabola

A particle moves in a parabolic path defined by the vector-valued function $\vec{\mathbf{r}}(t) = t^2\hat{\mathbf{i}} + \sqrt{5 - t^2}\hat{\mathbf{j}}$, where t measures time in seconds.

1. Find the velocity, acceleration, and speed as functions of time.

2. Sketch the curve along with the velocity vector at time $t = 1$.

Solution

1. We use Equations 1.5.1, 1.5.2, and 1.5.3:

$$\begin{aligned}
\vec{\mathbf{v}}(t) = \vec{\mathbf{r}}'(t) &= 2t\hat{\mathbf{i}} - \frac{t}{\sqrt{5-t^2}}\hat{\mathbf{j}} \\
\vec{\mathbf{a}}(t) = \vec{\mathbf{v}}'(t) &= 2\hat{\mathbf{i}} - 5(5-t^2)^{-\frac{3}{2}}\hat{\mathbf{j}} \\
||\vec{\mathbf{v}}(t)|| &= ||\,\vec{\mathbf{r}}'(t)|| \\
&= \sqrt{(2t)^2 + \left(-\frac{t}{\sqrt{5-t^2}}\right)^2} \\
&= \sqrt{4t^2 + \frac{t^2}{5-t^2}} \\
&= \sqrt{\frac{21t^2 - 4t^4}{5-t^2}}.
\end{aligned}$$

2. The graph of $\vec{\mathbf{r}}(t) = t^2\hat{\mathbf{i}} + \sqrt{5-t^2}\hat{\mathbf{j}}$ is a portion of a parabola (Figure 1.5.1).

When $t = 1$, $\vec{\mathbf{r}}(1) = (1)^2\hat{\mathbf{i}} + \sqrt{5-(1)^2}\hat{\mathbf{j}} \quad = \quad \hat{\mathbf{i}} + \sqrt{4}\hat{\mathbf{j}} \quad = \quad \hat{\mathbf{i}} + 2\hat{\mathbf{j}}$.

Thus the particle would be located at the point $(1, 2)$ when $t = 1$.

The velocity vector at $t = 1$ is

$$\begin{aligned}
\vec{\mathbf{v}}(1) = \vec{\mathbf{r}}'(1) &= 2(1)\hat{\mathbf{i}} - \frac{1}{\sqrt{5-1^2}}\hat{\mathbf{j}} \\
&= \quad 2\hat{\mathbf{i}} - \frac{1}{2}\hat{\mathbf{j}}
\end{aligned}$$

and the acceleration vector at $t = 1$ is

$$\vec{\mathbf{a}}(1) = \vec{\mathbf{v}}'(1) = 2\hat{\mathbf{i}} - 5(5-1^2)^{-3/2}\hat{\mathbf{j}} \quad = \quad 2\hat{\mathbf{i}} - \frac{5}{8}\hat{\mathbf{j}}. \tag{1.5.4}$$

Notice that the velocity vector is tangent to the path, as is always the case.

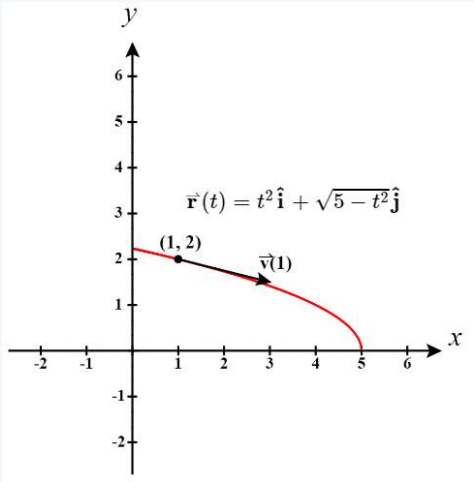


Figure 1.5.1: This graph depicts the velocity vector at time $t = 1$ for a particle moving in a parabolic path.



Exercise 1.5.1

A particle moves in a path defined by the vector-valued function $\vec{\mathbf{r}}(t) = (t^2 - 3t)\,\hat{\mathbf{i}} + (2t - 4)\,\hat{\mathbf{j}} + (t + 2)\,\hat{\mathbf{k}}$, where t measures time in seconds and where distance is measured in feet. Find the velocity, acceleration, and speed as functions of time.

Hint

Use Equations 1.5.1, 1.5.2, and 1.5.3.

Answer

$$\vec{\mathbf{v}}(t) = \vec{\mathbf{r}}\,'(t) = (2t - 3)\,\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$$
$$\vec{\mathbf{a}}(t) = \vec{\mathbf{v}}\,'(t) = 2\hat{\mathbf{i}}$$
$$||\vec{\mathbf{r}}\,'(t)|| = \sqrt{(2t - 3)^2 + 2^2 + 1^2} = \sqrt{4t^2 - 12t + 14} \tag{1.5.5}$$

The units for velocity and speed are feet per second, and the units for acceleration are feet per second squared.

To gain a better understanding of the velocity and acceleration vectors, imagine you are driving along a curvy road. If you do not turn the steering wheel, you would continue in a straight line and run off the road. The speed at which you are traveling when you run off the road, coupled with the direction, gives a vector representing your velocity, as illustrated in Figure 1.5.2.

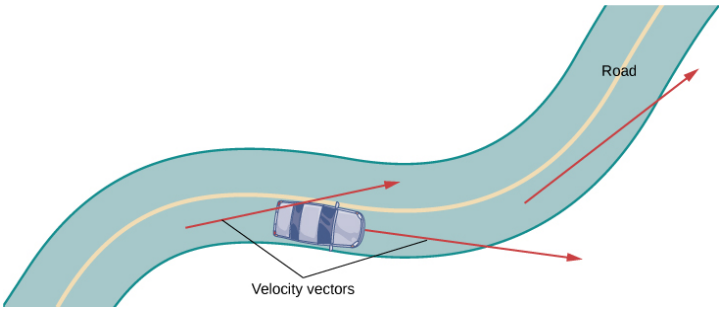


Figure 1.5.2: At each point along a road traveled by a car, the velocity vector of the car is tangent to the path traveled by the car.

However, the fact that you must turn the steering wheel to stay on the road indicates that your velocity is always changing (even if your speed is not) because your *direction* is constantly changing to keep you on the road. As you turn to the right, your acceleration

vector also points to the right. As you turn to the left, your acceleration vector points to the left. This indicates that your velocity and acceleration vectors are constantly changing, regardless of whether your actual speed varies (Figure 1.5.3).

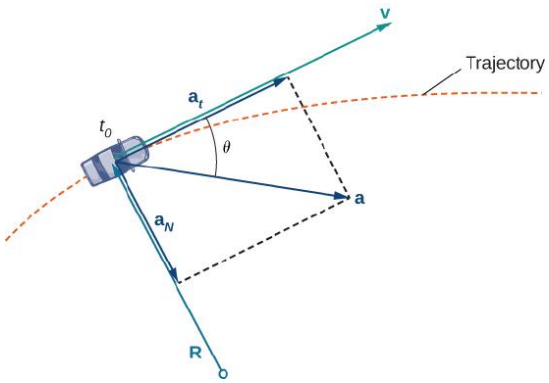


Figure 1.5.3: The dashed line represents the trajectory of an object (a car, for example). The acceleration vector points toward the inside of the turn at all times.

Components of the Acceleration Vector

We can combine some of the concepts discussed in Arc Length and Curvature with the acceleration vector to gain a deeper understanding of how this vector relates to motion in the plane and in space. Recall that the unit tangent vector $\vec{\mathbf{T}}$ and the unit normal vector $\vec{\mathbf{N}}$ form an osculating plane at any point P on the curve defined by a vector-valued function $\vec{\mathbf{r}}(t)$. The following theorem shows that the acceleration vector $\vec{\mathbf{a}}(t)$ lies in the osculating plane and can be written as a linear combination of the unit tangent and the unit normal vectors.

Theorem 1.5.1: The Plane of the Acceleration Vector

The acceleration vector $\vec{\mathbf{a}}(t)$ of an object moving along a curve traced out by a twice-differentiable function $\vec{\mathbf{r}}(t)$ lies in the plane formed by the unit tangent vector $\vec{\mathbf{T}}(t)$ and the principal unit normal vector $\vec{\mathbf{N}}(t)$ to C. Furthermore,

$$\vec{\mathbf{a}}(t) = v'(t)\vec{\mathbf{T}}(t) + [v(t)]^2\kappa\vec{\mathbf{N}}(t) \tag{1.5.6}$$

Here, $v(t) = \|\vec{\mathbf{v}}(t)\|$ is the speed of the object and κ is the curvature of C traced out by $\vec{\mathbf{r}}(t)$.

Proof

Because $\vec{\mathbf{v}}(t) = \vec{\mathbf{r}}'(t)$ and $\vec{\mathbf{T}}(t) = \dfrac{\vec{\mathbf{r}}'(t)}{||\vec{\mathbf{r}}'(t)||}$, we have $\vec{\mathbf{v}}(t) = ||\,\vec{\mathbf{r}}'(t)\,||\,\vec{\mathbf{T}}(t) = v(t)\vec{\mathbf{T}}(t)$.

Now we differentiate this equation:

$$\vec{\mathbf{a}}(t) = \vec{\mathbf{v}}'(t) = \frac{d}{dt}\left(v(t)\vec{\mathbf{T}}(t)\right) = v'(t)\vec{\mathbf{T}}(t) + v(t)\vec{\mathbf{T}}'(t)$$

Since $\vec{\mathbf{N}}(t) = \dfrac{\vec{\mathbf{T}}'(t)}{||\vec{\mathbf{T}}'(t)||}$, we know $\vec{\mathbf{T}}'(t) = ||\,\vec{\mathbf{T}}'(t)\,||\,\vec{\mathbf{N}}(t)$, so

$$\vec{\mathbf{a}}(t) = v'(t)\vec{\mathbf{T}}(t) + v(t)\,||\,\vec{\mathbf{T}}'(t)\,||\,\vec{\mathbf{N}}(t).$$

A formula for curvature is $\kappa = \dfrac{||\vec{\mathbf{T}}'(t)||}{||\vec{\mathbf{r}}'(t)||}$, so $\vec{\mathbf{T}}'(t) = \kappa\,||\,\vec{\mathbf{r}}'(t)\,|| = \kappa v(t)$.

This gives $\vec{\mathbf{a}}(t) = v'(t)\vec{\mathbf{T}}(t) + \kappa(v(t))^2\vec{\mathbf{N}}(t)$.

□

The coefficients of $\vec{\mathbf{T}}(t)$ and $\vec{\mathbf{N}}(t)$ are referred to as the **tangential component of acceleration** and the **normal component of acceleration**, respectively. We write $a_{\vec{\mathbf{T}}}$ to denote the tangential component and $a_{\vec{\mathbf{N}}}$ to denote the normal component.

Theorem 1.5.2: Tangential and Normal Components of Acceleration

Let $\vec{\mathbf{r}}(t)$ be a vector-valued function that denotes the position of an object as a function of time. Then $\vec{\mathbf{a}}(t) = \vec{\mathbf{r}}\,''(t)$ is the acceleration vector. The tangential and normal components of acceleration $a_{\vec{\mathbf{T}}}$ and $a_{\vec{\mathbf{N}}}$ are given by the formulas

$$a_{\vec{\mathbf{T}}} = \vec{\mathbf{a}} \cdot \vec{\mathbf{T}} = \frac{\vec{\mathbf{v}} \cdot \vec{\mathbf{a}}}{||\vec{\mathbf{v}}||} \tag{1.5.7}$$

and

$$a_{\vec{\mathbf{N}}} = \vec{\mathbf{a}} \cdot \vec{\mathbf{N}} = \frac{||\vec{\mathbf{v}} \times \vec{\mathbf{a}}||}{||\vec{\mathbf{v}}||} = \sqrt{||\vec{\mathbf{a}}||^2 - \left(a_{\vec{\mathbf{T}}}\right)^2}. \tag{1.5.8}$$

These components are related by the formula

$$\vec{\mathbf{a}}(t) = a_{\vec{\mathbf{T}}}\vec{\mathbf{T}}(t) + a_{\vec{\mathbf{N}}}\vec{\mathbf{N}}(t). \tag{1.5.9}$$

Here $\vec{\mathbf{T}}(t)$ is the unit tangent vector to the curve defined by $\vec{\mathbf{r}}(t)$, and $\vec{\mathbf{N}}(t)$ is the unit normal vector to the curve defined by $\vec{\mathbf{r}}(t)$.

The normal component of acceleration is also called the *centripetal component of acceleration* or sometimes the *radial component of acceleration*. To understand centripetal acceleration, suppose you are traveling in a car on a circular track at a constant speed. Then, as we saw earlier, the acceleration vector points toward the center of the track at all times. As a rider in the car, you feel a pull toward the *outside* of the track because you are constantly turning. This sensation acts in the opposite direction of centripetal acceleration. The same holds true for non-circular paths. The reason is that your body tends to travel in a straight line and resists the force resulting from acceleration that push it toward the side. Note that at point B in Figure 1.5.4 the acceleration vector is pointing backward. This is because the car is decelerating as it goes into the curve.

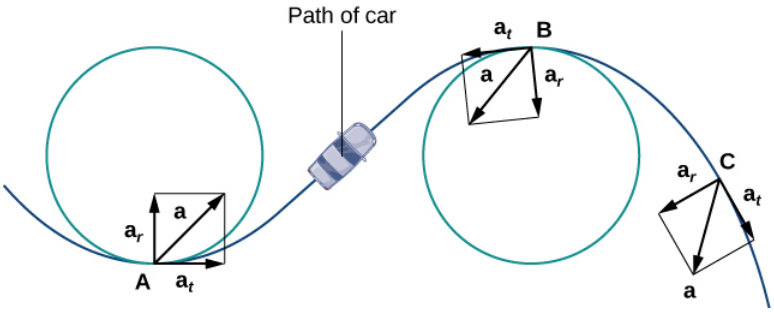


Figure 1.5.4: The tangential and normal components of acceleration can be used to describe the acceleration vector.

The tangential and normal unit vectors at any given point on the curve provide a frame of reference at that point. The tangential and normal components of acceleration are the projections of the acceleration vector onto $\vec{\mathbf{T}}$ and $\vec{\mathbf{N}}$, respectively.

Example 1.5.2: Finding Components of Acceleration

A particle moves in a path defined by the vector-valued function $\vec{\mathbf{r}}(t) = t^2\,\hat{\mathbf{i}} + (2t - 3)\,\hat{\mathbf{j}} + (3t^2 - 3t)\,\hat{\mathbf{k}}$, where t measures time in seconds and distance is measured in feet.

a. Find $a_{\vec{\mathbf{T}}}$ and $a_{\vec{\mathbf{N}}}$ as functions of t.
b. Find $a_{\vec{\mathbf{T}}}$ and $a_{\vec{\mathbf{N}}}$ at time $t = 2$.

Solution

a. Let's start deriving the velocityand acceleration functions:

$$\begin{aligned}\vec{\mathbf{v}}(t) &= \vec{\mathbf{r}}\,'(t) \\ &= 2t\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + (6t - 3)\,\hat{\mathbf{k}} \\ \vec{\mathbf{a}}(t) &= \vec{\mathbf{v}}\,'(t) \\ &= 2\,\hat{\mathbf{i}} + 6\,\hat{\mathbf{k}}\end{aligned}$$

Now we apply Equation 1.5.7:

$$\begin{aligned} a_{\vec{\mathbf{T}}} &= \frac{\vec{\mathbf{v}} \cdot \vec{\mathbf{a}}}{||\vec{\mathbf{v}}||} \\ &= \frac{(2t\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + (6t-3)\,\hat{\mathbf{k}}) \cdot (2\,\hat{\mathbf{i}} + 6\,\hat{\mathbf{k}})}{||2t\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + (6t-3)\,\hat{\mathbf{k}}||} \\ &= \frac{4t + 6(6t-3)}{\sqrt{(2t)^2 + 2^2 + (6t-3)^2}} \\ &= \frac{40t - 18}{40t^2 - 36t + 13} \end{aligned}$$

Now we can apply Equation 1.5.8:

$$\begin{aligned} a_{\vec{\mathbf{N}}} &= \sqrt{||\vec{\mathbf{a}}||^2 - a_{\vec{\mathbf{T}}}} \\ &= \sqrt{||2\,\hat{\mathbf{i}} + 6\,\hat{\mathbf{k}}||^2 - \left(\frac{40t-18}{\sqrt{40t^2-36+13}}\right)^2} \\ &= \sqrt{4 + 36 - \frac{(40t-18)^2}{40t^2 - 36t + 13}} \\ &= \sqrt{\frac{40(40t^2 - 36t + 13) - (1600t^2 - 1440t + 324)}{40t^2 - 36t + 13}} \\ &= \sqrt{\frac{196}{40t^2 - 36t + 13}} \\ &= \frac{14}{\sqrt{40t^2 - 36t + 13}} \end{aligned}$$

b. We must evaluate each of the answers from part a at $t = 2$:

$$\begin{aligned} a_{\vec{\mathbf{T}}}(2) &= \frac{40(2) - 18}{\sqrt{40(2)^2 - 36(2) + 13}} \\ &= \frac{80 - 18}{\sqrt{160 - 72 + 13}} \\ &= \frac{62}{\sqrt{101}} \\ a_{\vec{\mathbf{N}}}(2) &= \frac{14}{\sqrt{40(2)^2 - 36(2) + 13}} \\ &= \frac{14}{\sqrt{160 - 72 + 13}} = \frac{14}{\sqrt{101}}. \end{aligned}$$

The units of acceleration are feet per second squared, as are the units of the normal and tangential components of acceleration.

? Exercise 1.5.2

An object moves in a path defined by the vector-valued function $\vec{\mathbf{r}}(t) = 4t\,\hat{\mathbf{i}} + t^2\,\hat{\mathbf{j}}$, where t measures time in seconds.

a. Find $a_{\vec{\mathbf{T}}}$ and $a_{\vec{\mathbf{N}}}$ as functions of t.
b. Find $a_{\vec{\mathbf{T}}}$ and $a_{\vec{\mathbf{N}}}$ at time $t = -3$.

Hint

Use Equations 1.5.7 and 1.5.8

Answer

a.

$$a_{\vec{\mathbf{T}}} = \frac{\vec{\mathbf{v}}(t) \cdot \vec{\mathbf{a}}(t)}{||\vec{\mathbf{v}}(t)||} = \frac{\vec{\mathbf{r}}'(t) \cdot \vec{\mathbf{r}}''(t)}{||\vec{\mathbf{r}}'(t)||}$$
$$= \frac{(4\hat{\mathbf{i}} + 2t\hat{\mathbf{j}}) \cdot (2\hat{\mathbf{j}})}{||4\hat{\mathbf{i}} + 2t\hat{\mathbf{j}}||}$$
$$= \frac{4t}{\sqrt{4^2 + (2t)^2}}$$
$$= \frac{2t}{\sqrt{2 + t^2}}$$

$$a_{\vec{\mathbf{N}}} = \sqrt{||\vec{\mathbf{a}}||^2 - a_{\vec{\mathbf{T}}}^2}$$
$$= \sqrt{||2\hat{\mathbf{j}}||^2 - \left(\frac{2t}{\sqrt{2+t^2}}\right)^2}$$
$$= \sqrt{4 - \frac{4t^2}{2 + t^2}}$$

b.

$$a_{\vec{\mathbf{T}}}(-3) = \frac{2(-3)}{\sqrt{2 + (-3)^2}}$$
$$= \frac{-6}{\sqrt{11}}$$

$$a_{\vec{\mathbf{N}}}(-3) = \sqrt{4 - \frac{4(-3)^2}{2 + (-3)^2}}$$
$$= \sqrt{4 - \frac{36}{11}}$$
$$= \sqrt{\frac{8}{11}}$$
$$= \frac{2\sqrt{2}}{\sqrt{11}}$$

Finding the Components of Acceleration of a Truck on a Road

A self driving truck will be driving on a remote dirt road and it's possible position functions have already been determined and validated in several tests. Your job will be to validate the code that will determine the components of acceleration once it is written. You will be given the position function below and you must determine $a_{\vec{\mathbf{T}}}$ and $a_{\vec{\mathbf{N}}}$ at the given value of t rounded to four decimal places.



$a_{\vec{\mathbf{T}}}(t)$ = $a_{\vec{\mathbf{N}}}(t)$ =

Start

Click on the Start button to begin.

Projectile Motion

Now let's look at an application of vector functions. In particular, let's consider the effect of gravity on the motion of an object as it travels through the air, and how it determines the resulting trajectory of that object. In the following, we ignore the effect of air resistance. This situation, with an object moving with an initial velocity but with no forces acting on it other than gravity, is known as projectile motion. It describes the motion of objects from golf balls to baseballs, and from arrows to cannonballs.

First we need to choose a coordinate system. If we are standing at the origin of this coordinate system, then we choose the positive y-axis to be up, the negative y-axis to be down, and the positive x-axis to be forward (i.e., away from the thrower of the object). The effect of gravity is in a downward direction, so Newton's second law tells us that the force on the object resulting from gravity is equal to the mass of the object times the acceleration resulting from gravity, or $\vec{\mathbf{F}}_g = m\vec{\mathbf{a}}$, where $\vec{\mathbf{F}}_g$ represents the force from gravity and $\vec{\mathbf{a}} = -g\,\hat{\mathbf{j}}$ represents the acceleration resulting from gravity at Earth's surface. The value of g in the English system of measurement is approximately 32 ft/sec^2 and it is approximately 9.8 m/sec^2 in the metric system. This is the only force acting on the object. Since gravity acts in a downward direction, we can write the force resulting from gravity in the form $\vec{\mathbf{F}}_g = -mg\,\hat{\mathbf{j}}$, as shown in Figure 1.5.5.

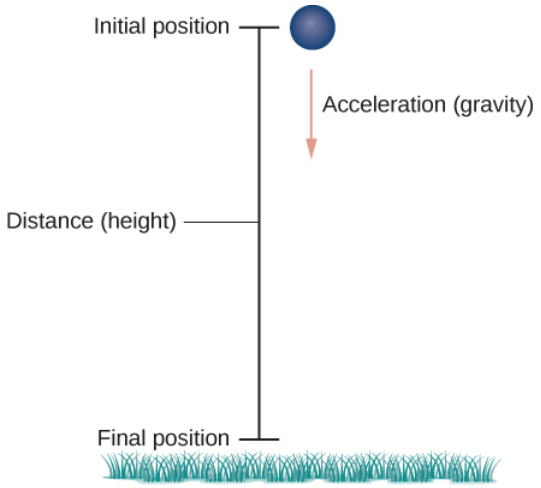


Figure 1.5.5: An object is falling under the influence of gravity.

Newton's second law also tells us that $F = m\vec{\mathbf{a}}$, where $\vec{\mathbf{a}}$ represents the acceleration vector of the object. This force must be equal to the force of gravity at all times, so we therefore know that

$$\begin{aligned} \vec{\mathbf{F}} &= \vec{\mathbf{F}}_g \\ m\vec{\mathbf{a}} &= -mg\,\hat{\mathbf{j}} \\ \vec{\mathbf{a}} &= -g\,\hat{\mathbf{j}}. \end{aligned}$$

Now we use the fact that the acceleration vector is the first derivative of the velocity vector. Therefore, we can rewrite the last equation in the form

$$\vec{\mathbf{v}}'(t) = -g\,\hat{\mathbf{j}} \tag{1.5.10}$$

By taking the antiderivative of each side of this equation we obtain

$$\vec{\mathbf{v}}(t) = \int -g\,\hat{\mathbf{j}}\,dt = -gt\,\hat{\mathbf{j}} + \vec{\mathbf{C}}_1 \tag{1.5.11}$$

for some constant vector $\vec{\mathbf{C}}_1$. To determine the value of this vector, we can use the velocity of the object at a fixed time, say at time $t = 0$. We call this velocity the *initial velocity*: $\vec{\mathbf{v}}(0) = \vec{\mathbf{v}}_0$. Therefore, $\vec{\mathbf{v}}(0) = -g(0)\,\hat{\mathbf{j}} + \vec{\mathbf{C}}_1 = \vec{\mathbf{v}}_0$ and $\vec{\mathbf{C}}_1 = \vec{\mathbf{v}}_0$. This gives the velocity vector as $\vec{\mathbf{v}}(t) = -gt\,\hat{\mathbf{j}} + \vec{\mathbf{v}}_0$.

Next we use the fact that velocity $\vec{\mathbf{v}}(t)$ is the derivative of position $\vec{\mathbf{s}}(t)$. This gives the equation

$$\vec{\mathbf{s}}'(t) = -gt\hat{\mathbf{j}} + \vec{\mathbf{v}}_0. \tag{1.5.12}$$

Taking the antiderivative of both sides of this equation leads to

$$\begin{aligned}\vec{\mathbf{s}}(t) &= \int -gt\hat{\mathbf{j}} + \vec{\mathbf{v}}_0 \, dt \\ &= -\frac{1}{2}gt^2\hat{\mathbf{j}} + \vec{\mathbf{v}}_0 t + \vec{\mathbf{C}}_2\end{aligned}$$

with another unknown constant vector $\vec{\mathbf{C}}_2$. To determine the value of $\vec{\mathbf{C}}_2$, we can use the position of the object at a given time, say at time $t = 0$. We call this position the *initial position*: $\vec{\mathbf{s}}(0) = \vec{\mathbf{s}}_0$. Therefore, $\vec{\mathbf{s}}(0) = -(1/2)g(0)^2\,\hat{\mathbf{j}} + \vec{\mathbf{v}}_0(0) + \vec{\mathbf{C}}_2 = \vec{\mathbf{s}}_0$. This gives the position of the object at any time as

$$\vec{\mathbf{s}}(t) = -12gt^2\hat{\mathbf{j}} + \vec{\mathbf{v}}_0 t + \vec{\mathbf{s}}_0. \tag{1.5.13}$$

Let's take a closer look at the initial velocity and initial position. In particular, suppose the object is thrown upward from the origin at an angle θ to the horizontal, with initial speed $\vec{\mathbf{v}}_0$. How can we modify the previous result to reflect this scenario? First, we can assume it is thrown from the origin. If not, then we can move the origin to the point from where it is thrown. Therefore, $\vec{\mathbf{s}}_0 = \vec{\mathbf{0}}$, as shown in Figure 1.5.6.

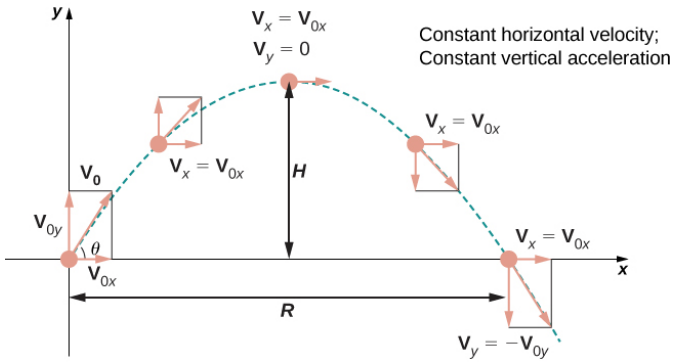


Figure 1.5.6: Projectile motion when the object is thrown upward at an angle θ. The horizontal motion is at constant velocity and the vertical motion is at constant acceleration.

We can rewrite the initial velocity vector in the form $\vec{\mathbf{v}}_0 = v_0\cos\theta\,\hat{\mathbf{i}} + v_0\sin\theta\,\hat{\mathbf{j}}$. Then the equation for the position function $\vec{\mathbf{s}}(t)$ becomes

$$\begin{aligned}\vec{\mathbf{s}}(t) &= -\frac{1}{2}gt^2\hat{\mathbf{j}} + v_0 t\cos\theta\,\hat{\mathbf{i}} + v_0 t\sin\theta\,\hat{\mathbf{j}} \\ &= v_0 t\cos\theta\,\hat{\mathbf{i}} + v_0 t\sin\theta\,\hat{\mathbf{j}} - \frac{1}{2}gt^2\hat{\mathbf{j}} \\ &= v_0 t\cos\theta\,\hat{\mathbf{i}} + \left(v_0 t\sin\theta - \frac{1}{2}gt^2\right)\hat{\mathbf{j}}.\end{aligned}$$

The coefficient of $\hat{\mathbf{i}}$ represents the horizontal component of $\vec{\mathbf{s}}(t)$ and is the horizontal distance of the object from the origin at time t. The maximum value of the horizontal distance (measured at the same initial and final altitude) is called the range R. The coefficient of $\hat{\mathbf{j}}$ represents the vertical component of $\vec{\mathbf{s}}(t)$ and is the altitude of the object at time t. The maximum value of the vertical distance is the height H.

✔ Example 1.5.3: Motion of a Cannonball

During an Independence Day celebration, a cannonball is fired from a cannon on a cliff toward the water. The cannon is aimed at an angle of 30° above horizontal and the initial speed of the cannonball is 600 ft/sec. The cliff is 100 ft above the water (Figure 1.5.7).

a. Find the maximum height of the cannonball.
b. How long will it take for the cannonball to splash into the sea?
c. How far out to sea will the cannonball hit the water?

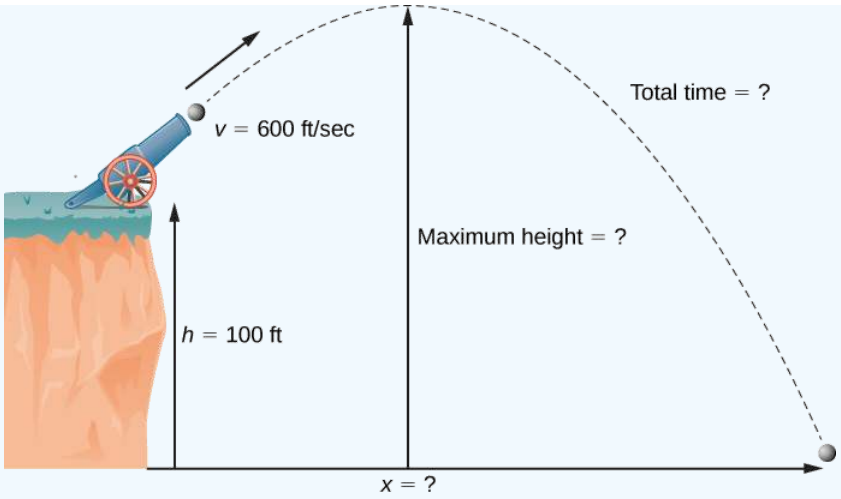


Figure 1.5.7: The flight of a cannonball (ignoring air resistance) is projectile motion.

Solution

We use the equation

$$\vec{\mathbf{s}}(t) = v_0 t\cos\theta\,\hat{\mathbf{i}} + \left(v_0 t\sin\theta - \frac{1}{2}gt^2\right)\hat{\mathbf{j}}$$

with $\theta = 30°$, $g = 32\frac{\text{ft}}{\text{sec}^2}$, and $v_0 = 600\frac{\text{ft}}{\text{sec}^2}$. Then the position equation becomes

$$\begin{aligned}\vec{\mathbf{s}}(t) &= 600t(\cos 30°)\,\hat{\mathbf{i}} + \left(600t\sin 30° - \frac{1}{2}(32)t^2\right)\hat{\mathbf{j}}\\ &= 300t\sqrt{3}\,\hat{\mathbf{i}} + \left(300t - 16t^2\right)\hat{\mathbf{j}}\end{aligned}$$

a. The cannonball reaches its maximum height when the vertical component of its velocity is zero, because the cannonball is neither rising nor falling at that point. The velocity vector is

$$\begin{aligned}\vec{\mathbf{v}}(t) &= \vec{\mathbf{s}}'(t)\\ &= 300\sqrt{3}\,\hat{\mathbf{i}} + (300 - 32t)\,\hat{\mathbf{j}}\end{aligned}$$

Therefore, the vertical component of velocity is given by the expression $300 - 32t$. Setting this expression equal to zero and solving for t gives $t = 9.375$ sec. The height of the cannonball at this time is given by the vertical component of the position vector, evaluated at $t = 9.375$.

$$\begin{aligned}\vec{\mathbf{s}}(9.375) &= 300(9.375)\sqrt{3}\,\hat{\mathbf{i}} + (300(9.375) - 16(9.375)^2)\,\hat{\mathbf{j}}\\ &= 4871.39\,\hat{\mathbf{i}} + 1406.25\,\hat{\mathbf{j}}\end{aligned}$$

Therefore, the maximum height of the cannonball is 1406.39 ft above the cannon, or 1506.39 ft above sea level.

b. When the cannonball lands in the water, it is 100 ft below the cannon. Therefore, the vertical component of the position vector is equal to −100. Setting the vertical component of $\vec{\mathbf{s}}(t)$ equal to −100 and solving, we obtain

$$\begin{aligned}300t - 16t^2 &= -100\\ 16t^2 - 300t - 100 &= 0\\ 4t^2 - 75 - 25 &= 0\end{aligned}$$

$$\begin{aligned}t &= \frac{75 \pm \sqrt{(-75)^2 - 4(4)(-25)}}{2(4)}\\ &= \frac{75 \pm \sqrt{6025}}{8}\\ &= \frac{75 \pm 5\sqrt{241}}{8}\end{aligned}$$

The positive value of t that solves this equation is approximately 19.08. Therefore, the cannonball hits the water after approximately 19.08 sec.

c. To find the distance out to sea, we simply substitute the answer from part (b) into $\vec{\mathbf{s}}(t)$:

$$\begin{aligned}\vec{\mathbf{s}}(19.08) &= 300(19.08)\sqrt{3}\,\hat{\mathbf{i}} + \left(300(19.08) - 16(19.08)^2\right)\hat{\mathbf{j}} \\ &= 9914.26\,\hat{\mathbf{i}} - 100.7424\,\hat{\mathbf{j}}\end{aligned}$$

Therefore, the ball hits the water about 9914.26 ft away from the base of the cliff. Notice that the vertical component of the position vector is very close to −100, which tells us that the ball just hit the water. Note that 9914.26 feet is not the true range of the cannon since the cannonball lands in the ocean at a location below the cannon. The range of the cannon would be determined by finding how far out the cannonball is when its height is 100 ft above the water (the same as the altitude of the cannon).



? Exercise 1.5.3

An archer fires an arrow at an angle of 40° above the horizontal with an initial speed of 98 m/sec. The height of the archer is 171.5 cm. Find the horizontal distance the arrow travels before it hits the ground.

Hint

The equation for the position vector needs to account for the height of the archer in meters.

Answer

967.15 m

One final question remains: In general, what is the maximum distance a projectile can travel, given its initial speed? To determine this distance, we assume the projectile is fired from ground level and we wish it to return to ground level. In other words, we want to determine an equation for the range. In this case, the equation of projectile motion is

$$\vec{\mathbf{s}} = v_0 t\cos\theta\,\hat{\mathbf{i}} + \left(v_0 t\sin\theta - \frac{1}{2}gt^2\right)\hat{\mathbf{j}}. \tag{1.5.14}$$

Setting the second component equal to zero and solving for t yields

$$v_0 t \sin\theta - \frac{1}{2} g t^2 = 0$$
$$t\left(v_0 \sin\theta - \frac{1}{2} g t\right) = 0$$

Therefore, either $t = 0$ or $t = \dfrac{2v_0 \sin\theta}{g}$. We are interested in the second value of t, so we substitute this into $\vec{\mathbf{s}}(t)$, which gives

$$\begin{aligned}\vec{\mathbf{s}}\left(\frac{2v_0 \sin\theta}{g}\right) &= v_0\left(\frac{2v_0 \sin\theta}{g}\right)\cos\theta\,\hat{\mathbf{i}} + \left(v_0\left(\frac{2v_0 \sin\theta}{g}\right)\sin\theta - \frac{1}{2}g\left(\frac{2v_0 \sin\theta}{g}\right)^2\right)\hat{\mathbf{j}} \\ &= \left(\frac{2v_0^2 \sin\theta\cos\theta}{g}\right)\hat{\mathbf{i}} \\ &= \frac{v_0^2 \sin 2\theta}{g}\,\hat{\mathbf{i}}.\end{aligned}$$

Thus, the expression for the range of a projectile fired at an angle θ is

$$R = \frac{v_0^2 \sin 2\theta}{g}\,\hat{\mathbf{i}}. \tag{1.5.15}$$

The only variable in this expression is θ. To maximize the distance traveled, take the derivative of the coefficient of $\mathbf{i}$ with respect to θ and set it equal to zero:

$$\begin{aligned}\frac{d}{d\theta}\left(\frac{v_0^2 \sin 2\theta}{g}\right) &= 0 \\ \frac{2v_0^2 \cos 2\theta}{g} &= 0 \\ \theta &= 45^\circ\end{aligned}$$

This value of θ) is the smallest positive value that makes the derivative equal to zero. Therefore, in the absence of air resistance, the best angle to fire a projectile (to maximize the range) is at a 45° angle. The distance it travels is given by

$$\vec{\mathbf{s}}\left(\frac{2v_0 \sin 45^\circ}{g}\right) = \frac{v_0^2 \sin 90^\circ}{g}\,\hat{\mathbf{i}} = \frac{v_0^2}{g}\,\hat{\mathbf{i}} \tag{1.5.16}$$

Therefore, the range for an angle of 45° is $\frac{v_0^2}{g}$ units.



https://math.libretexts.org/@go/page/84725

Kepler's Laws

During the early 1600s, Johannes Kepler was able to use the amazingly accurate data from his mentor Tycho Brahe to formulate his three laws of planetary motion, now known as Kepler's laws of planetary motion. These laws also apply to other objects in the solar system in orbit around the Sun, such as comets (e.g., Halley's comet) and asteroids. Variations of these laws apply to satellites in orbit around Earth.

Theorem 1.5.2: Kepler's Laws of Planetary Motion

1. The path of any planet about the Sun is elliptical in shape, with the center of the Sun located at one focus of the ellipse (the law of ellipses).
2. A line drawn from the center of the Sun to the center of a planet sweeps out equal areas in equal time intervals (the law of equal areas) (Figure 1.5.8).
3. The ratio of the squares of the periods of any two planets is equal to the ratio of the cubes of the lengths of their semimajor orbital axes (the Law of Harmonies).

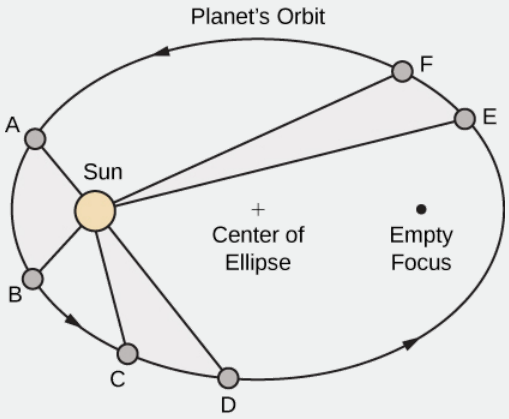


Figure 1.5.8: Kepler's first and second laws are pictured here. The Sun is located at a focus of the elliptical orbit of any planet. Furthermore, the shaded areas are all equal, assuming that the amount of time measured as the planet moves is the same for each region.

Kepler's third law is especially useful when using appropriate units. In particular, *1 astronomical unit* is defined to be the average distance from Earth to the Sun, and is now recognized to be 149,597,870,700 m or, approximately 93,000,000 mi. We therefore write 1 A.U. = 93,000,000 mi. Since the time it takes for Earth to orbit the Sun is 1 year, we use Earth years for units of time. Then, substituting 1 year for the period of Earth and 1 A.U. for the average distance to the Sun, Kepler's third law can be written as

$$T_p^2 = D_p^3$$

for any planet in the solar system, where T_P is the period of that planet measured in Earth years and D_P is the average distance from that planet to the Sun measured in astronomical units. Therefore, if we know the average distance from a planet to the Sun (in astronomical units), we can then calculate the length of its year (in Earth years), and vice versa.

Kepler's laws were formulated based on observations from Brahe; however, they were not proved formally until Sir Isaac Newton was able to apply calculus. Furthermore, Newton was able to generalize Kepler's third law to other orbital systems, such as a moon orbiting around a planet. Kepler's original third law only applies to objects orbiting the Sun.

Proof

Let's now prove Kepler's first law using the calculus of vector-valued functions. First we need a coordinate system. Let's place the Sun at the origin of the coordinate system and let the vector-valued function $\vec{\mathbf{r}}(t)$ represent the location of a planet as a function of time. Newton proved Kepler's law using his second law of motion and his law of universal gravitation. Newton's second law of motion can be written as $\vec{\mathbf{F}} = m\vec{\mathbf{a}}$, where $\vec{\mathbf{F}}$ represents the net force acting on the planet. His law of universal

gravitation can be written in the form $\vec{\mathbf{F}} = -\frac{GmM}{||\vec{\mathbf{r}}||^2} \cdot \frac{\vec{\mathbf{r}}}{||\vec{\mathbf{r}}||}$, which indicates that the force resulting from the gravitational attraction of the Sun points back toward the Sun, and has magnitude $\frac{GmM}{||\vec{\mathbf{r}}||^2}$ (Figure 1.5.9).

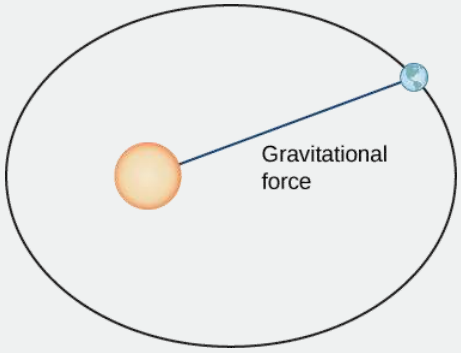


Figure 1.5.9: The gravitational force between Earth and the Sun is equal to the mass of the earth times its acceleration.

Setting these two forces equal to each other, and using the fact that $\vec{\mathbf{a}}(t) = \vec{\mathbf{v}}'(t)$, we obtain

$$m\vec{\mathbf{v}}'(t) = -\frac{GmM}{\|\vec{\mathbf{r}}\|^2} \cdot \frac{\vec{\mathbf{r}}}{\|\vec{\mathbf{r}}\|},$$

which can be rewritten as

$$\frac{d\vec{\mathbf{v}}}{dt} = -\frac{GM}{||\vec{\mathbf{r}}||^3}\vec{\mathbf{r}}.$$

This equation shows that the vectors $d\vec{\mathbf{v}}/dt$ and $\vec{\mathbf{r}}$ are parallel to each other, so $d\vec{\mathbf{v}}/dt \times \vec{\mathbf{r}} = \vec{\mathbf{0}}$. Next, let's differentiate $\vec{\mathbf{r}} \times \vec{\mathbf{v}}$ with respect to time:

$$\frac{d}{dt}(\vec{\mathbf{r}} \times \vec{\mathbf{v}}) = \frac{d\vec{\mathbf{r}}}{dt} \times \vec{\mathbf{v}} + \vec{\mathbf{r}} \times \frac{d\vec{\mathbf{v}}}{dt} = \vec{\mathbf{v}} \times \vec{\mathbf{v}} + \vec{\mathbf{0}} = \vec{\mathbf{0}}. \tag{1.5.17}$$

This proves that $\vec{\mathbf{r}} \times \vec{\mathbf{v}}$ is a constant vector, which we call $\vec{\mathbf{C}}$. Since $\vec{\mathbf{r}}$ and $\vec{\mathbf{v}}$ are both perpendicular to $\vec{\mathbf{C}}$ for all values of t, they must lie in a plane perpendicular to $\vec{\mathbf{C}}$. Therefore, the motion of the planet lies in a plane.

Next we calculate the expression $d\vec{\mathbf{v}}/dt \times \vec{\mathbf{C}}$:

$$\frac{d\vec{\mathbf{v}}}{dt} \times \vec{\mathbf{C}} = -\frac{GM}{||\vec{\mathbf{r}}||^3}\vec{\mathbf{r}} \times (\vec{\mathbf{r}} \times \vec{\mathbf{v}}) = -\frac{GM}{||\vec{\mathbf{r}}||^3}[(\vec{\mathbf{r}} \cdot \vec{\mathbf{v}})\vec{\mathbf{r}} - (\vec{\mathbf{r}} \cdot \vec{\mathbf{r}})\vec{\mathbf{v}}]. \tag{1.5.18}$$

The last equality in Equation 1.5.17 is from the triple cross product formula (Introduction to Vectors in Space). We need an expression for $\vec{\mathbf{r}} \cdot \vec{\mathbf{v}}$. To calculate this, we differentiate $\vec{\mathbf{r}} \cdot \vec{\mathbf{r}}$ with respect to time:

$$\frac{d}{dt}(\vec{\mathbf{r}} \cdot \vec{\mathbf{r}}) = \frac{d\vec{\mathbf{r}}}{dt} \cdot \vec{\mathbf{r}} + \vec{\mathbf{r}} \cdot \frac{d\vec{\mathbf{r}}}{dt} = 2\vec{\mathbf{r}} \cdot \frac{d\vec{\mathbf{r}}}{dt} = 2\vec{\mathbf{r}} \cdot \vec{\mathbf{v}}. \tag{1.5.19}$$

Since $\vec{\mathbf{r}} \cdot \vec{\mathbf{r}} = ||\vec{\mathbf{r}}||^2$, we also have

$$\frac{d}{dt}(\vec{\mathbf{r}} \cdot \vec{\mathbf{r}}) = \frac{d}{dt}||\vec{\mathbf{r}}||^2 = 2||\vec{\mathbf{r}}|| \frac{d}{dt}||\vec{\mathbf{r}}||. \tag{1.5.20}$$

Combining Equation 1.5.19 and Equation 1.5.20, we get

$$2\vec{\mathbf{r}} \cdot \vec{\mathbf{v}} = 2||\vec{\mathbf{r}}|| \frac{d}{dt}||\vec{\mathbf{r}}||$$
$$\vec{\mathbf{r}} \cdot \vec{\mathbf{v}} = ||\vec{\mathbf{r}}|| \frac{d}{dt}||\vec{\mathbf{r}}||.$$

Substituting this into Equation 1.5.18 gives us

$$\begin{aligned}\frac{d\vec{\mathbf{v}}}{dt}\times\vec{\mathbf{C}} &= -\frac{GM}{||\vec{\mathbf{r}}||^3}[(\vec{\mathbf{r}}\cdot\vec{\mathbf{v}})\vec{\mathbf{r}}-(\vec{\mathbf{r}}\cdot\vec{\mathbf{r}})\vec{\mathbf{v}}] \\ &= -\frac{GM}{||\vec{\mathbf{r}}||^3}\left[||\vec{\mathbf{r}}\left(\frac{d}{dt}||\vec{\mathbf{r}}||\right)\vec{\mathbf{r}}-||\vec{\mathbf{r}}||^2\vec{\mathbf{v}}\right] \\ &= -GM\left[\frac{1}{||\vec{\mathbf{r}}||^2}\left(\frac{d}{dt}||\vec{\mathbf{r}}||\right)\vec{\mathbf{r}}-\frac{1}{||\vec{\mathbf{r}}||}\vec{\mathbf{v}}\right] \\ &= GM\left[\frac{\vec{\mathbf{v}}}{||\vec{\mathbf{r}}||}-\frac{\vec{\mathbf{r}}}{||\vec{\mathbf{r}}||^2}\left(\frac{d}{dt}||\vec{\mathbf{r}}||\right)\right]. \end{aligned} \tag{1.5.21}$$

However,

$$\begin{aligned}\frac{d}{dt}\frac{\vec{\mathbf{r}}}{||\vec{\mathbf{r}}||} &= \frac{\frac{d}{dt}(\vec{\mathbf{r}})||\vec{\mathbf{r}}||-\vec{\mathbf{r}}\frac{d}{dt}||\vec{\mathbf{r}}||}{||\vec{\mathbf{r}}||^2} \\ &= \frac{\frac{d\vec{\mathbf{r}}}{dt}}{||\vec{\mathbf{r}}||}-\frac{\vec{\mathbf{r}}}{||\vec{\mathbf{r}}||^2}\frac{d}{dt}||\vec{\mathbf{r}}|| \\ &= \frac{\vec{\mathbf{v}}}{||\vec{\mathbf{r}}||}-\frac{\vec{\mathbf{r}}}{||\vec{\mathbf{r}}||^2}\frac{d}{dt}||\vec{\mathbf{r}}||. \end{aligned}$$

Therefore, Equation 1.5.21 becomes

$$\frac{d\vec{\mathbf{v}}}{dt}\times\vec{\mathbf{C}} = GM\left(\frac{d}{dt}\frac{\vec{\mathbf{r}}}{||\vec{\mathbf{r}}||}\right).$$

Since $\vec{\mathbf{C}}$ is a constant vector, we can integrate both sides and obtain

$$\vec{\mathbf{v}}\times\vec{\mathbf{C}} = GM\frac{\vec{\mathbf{r}}}{||\vec{\mathbf{r}}||}+\vec{\mathbf{D}},$$

where $\vec{\mathbf{D}}$ is a constant vector. Our goal is to solve for $||\vec{\mathbf{r}}||$. Let's start by calculating $\vec{\mathbf{r}}\cdot(\vec{\mathbf{v}}\times\vec{\mathbf{C}}$:

$$\vec{\mathbf{r}}\cdot(\vec{\mathbf{v}}\times\vec{\mathbf{C}} = GM\frac{||\vec{\mathbf{r}}||^2}{||\vec{\mathbf{r}}||}+\vec{\mathbf{r}}\cdot\vec{\mathbf{D}} = GM||\vec{\mathbf{r}}||+\vec{\mathbf{r}}\cdot\vec{\mathbf{D}}.$$

However, $\vec{\mathbf{r}}\cdot(\vec{\mathbf{v}}\times\vec{\mathbf{C}}) = (\vec{\mathbf{r}}\times\vec{\mathbf{v}})\cdot\vec{\mathbf{C}}$, so

$$(\vec{\mathbf{r}}\times\vec{\mathbf{v}})\cdot\vec{\mathbf{C}} = GM||\vec{\mathbf{r}}||+\vec{\mathbf{r}}\cdot\vec{\mathbf{D}}.$$

Since $\vec{\mathbf{r}}\times\vec{\mathbf{v}} = \vec{\mathbf{C}}$, we have

$$||\vec{\mathbf{C}}||^2 = GM||\vec{\mathbf{r}}||+\vec{\mathbf{r}}\cdot\vec{\mathbf{D}}.$$

Note that $\vec{\mathbf{r}}\cdot\vec{\mathbf{D}} = ||\vec{\mathbf{r}}||\,||\vec{\mathbf{D}}||\cos\theta$, where θ is the angle between $\vec{\mathbf{r}}$ and $\vec{\mathbf{D}}$. Therefore,

$$||\vec{\mathbf{C}}||^2 = GM||\vec{\mathbf{r}}||+||\vec{\mathbf{r}}||\,||\vec{\mathbf{D}}||\cos\theta$$

Solving for $||\vec{\mathbf{r}}||$,

$$||\vec{\mathbf{r}}|| = \frac{||\vec{\mathbf{C}}||^2}{GM+||\vec{\mathbf{D}}||\cos\theta} = \frac{||\vec{\mathbf{C}}||^2}{GM}\left(\frac{1}{1+e\cos\theta}\right).$$

where $e = ||\vec{\mathbf{D}}||/GM$. This is the polar equation of a conic with a focus at the origin, which we set up to be the Sun. It is a hyperbola if $e>1$, a parabola if $e=1$, or an ellipse if $e<1$. Since planets have closed orbits, the only possibility is an ellipse. However, at this point it should be mentioned that hyperbolic comets do exist. These are objects that are merely passing through the solar system at speeds too great to be trapped into orbit around the Sun. As they pass close enough to the Sun, the gravitational field of the Sun deflects the trajectory enough so the path becomes hyperbolic.

□

Kepler's third law of planetary motion can be modified to the case of one object in orbit around an object other than the Sun, such as the Moon around the Earth. In this case, Kepler's third law becomes

$$P^2 = \frac{4\pi^2 a^3}{G(m+M)}, \tag{1.5.22}$$

where m is the mass of the Moon and M is the mass of Earth, a represents the length of the major axis of the elliptical orbit, and P represents the period.

Example 1.5.4: Using Kepler's Third Law for Nonheliocentric Orbits

Given that the mass of the Moon is 7.35×10^{22} kg, the mass of Earth is 5.97×10^{24} kg, $G = 6.67 \times 10^{-11} \text{m/kg} \cdot \text{sec}^2$, and the period of the moon is 27.3 days, let's find the length of the major axis of the orbit of the Moon around Earth.

Solution

It is important to be consistent with units. Since the universal gravitational constant contains seconds in the units, we need to use seconds for the period of the Moon as well:

$$27.3 \text{ days} \times \frac{24 \text{ hr}}{1 \text{ day}} \times \frac{3600 \text{ esc}}{1 \text{ hour}} = 2,358,720 \text{ sec}$$

Substitute all the data into Equation 1.5.22 and solve for a:

$$\begin{aligned}
(2,358,720 sec)^2 &= \frac{4\pi^2 a^3}{\left(6.67 \times 10^{-11} \frac{m}{\text{kg} \times \text{sec}^2}\right)(7.35 \times 10^{22}\text{kg} + 5.97 \times 10^{24}\text{kg})} \\
5.563 \times 10^{12} &= \frac{4\pi^2 a^3}{(6.67 \times 10^{-11}\text{m}^3)(6.04 \times 10^{24})} \\
(5.563 \times 10^{12})(6.67 \times 10^{-11}\text{m}^3)(6.04 \times 10^{24}) &= 4\pi^2 a^3 \\
a^3 &= \frac{2.241 \times 10^{27}}{4\pi^2}\text{m}^3 \\
a &= 3.84 \times 10^8 \text{m} \\
&\approx 384,000 \text{ km}.
\end{aligned}$$

Analysis

According to solarsystem.nasa.gov, the actual average distance from the Moon to Earth is 384,400 km. This is calculated using reflectors left on the Moon by Apollo astronauts back in the 1960s.

Exercise 1.5.4

Titan is the largest moon of Saturn. The mass of Titan is approximately $1.35 \times 10^{23} kg$. The mass of Saturn is approximately 5.68×10^{26} kg. Titan takes approximately 16 days to orbit Saturn. Use this information, along with the universal gravitation constant $G = 6.67 \times 10^{-11} \text{m/kg} \cdot \text{sec}^2$ to estimate the distance from Titan to Saturn.

Hint

Make sure your units agree, then use Equation 1.5.22.

Answer

$$a \approx 1.224 \times 10^9 \text{m} = 1,224,000 \text{km} \tag{1.5.23}$$

Example 1.5.5: Halley's Comet

We now return to the chapter opener, which discusses the motion of Halley's comet around the Sun. Kepler's first law states that Halley's comet follows an elliptical path around the Sun, with the Sun as one focus of the ellipse. The period of Halley's comet is approximately 76.1 years, depending on how closely it passes by Jupiter and Saturn as it passes through the outer solar system. Let's use $T = 76.1$ years. What is the average distance of Halley's comet from the Sun?



Solution

Using the equation $T^2 = D^3$ with $T = 76.1$, we obtain $D^3 = 5791.21$, so $D \approx 17.96$ A.U. This comes out to approximately 1.67×10^9 mi.

A natural question to ask is: What are the maximum (aphelion) and minimum (perihelion) distances from Halley's Comet to the Sun? The eccentricity of the orbit of Halley's Comet is 0.967 (Source: http://nssdc.gsfc.nasa.gov/planetary...cometfact.html). Recall that the formula for the eccentricity of an ellipse is $e = c/a$, where a is the length of the semimajor axis and c is the distance from the center to either focus. Therefore, $0.967 = c/17.96$ and $c \approx 17.37$ A.U. Subtracting this from a gives the perihelion distance $p = a - c = 17.96 - 17.37 = 0.59$ A.U. According to the National Space Science Data Center (Source: http://nssdc.gsfc.nasa.gov/planetary...cometfact.html), the perihelion distance for Halley's comet is 0.587 A.U. To calculate the aphelion distance, we add

$$P = a + c = 17.96 + 17.37 = 35.33 \text{ A.U.}$$

This is approximately 3.3×10^9 mi. The average distance from Pluto to the Sun is 39.5 A.U. (Source: http://www.oarval.org/furthest.htm), so it would appear that Halley's Comet stays just within the orbit of Pluto.

NAVIGATING A BANKED TURN

How fast can a racecar travel through a circular turn without skidding and hitting the wall? The answer could depend on several factors:

- The weight of the car;
- The friction between the tires and the road;
- The radius of the circle;
- The "steepness" of the turn.

In this project we investigate this question for NASCAR racecars at the Bristol Motor Speedway in Tennessee. Before considering this track in particular, we use vector functions to develop the mathematics and physics necessary for answering questions such as this.

A car of mass m moves with constant angular speed ω around a circular curve of radius R (Figure 1.5.9). The curve is banked at an angle θ. If the height of the car off the ground is h, then the position of the car at time t is given by the function $\vec{\mathbf{r}}(t) = < R\cos(\omega t), R\sin(\omega t), h >$.

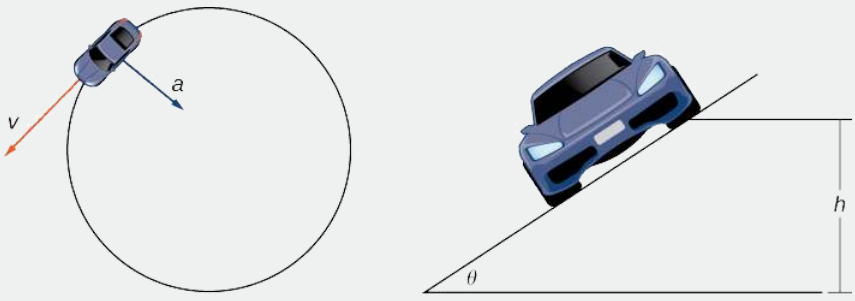


Figure 1.5.9: Views of a race car moving around a track.

1. Find the velocity function $\vec{\mathbf{v}}(t)$ of the car. Show that $\vec{\mathbf{v}}$ is tangent to the circular curve. This means that, without a force to keep the car on the curve, the car will shoot off of it.
2. Show that the speed of the car is ωR. Use this to show that $(2\pi 4)/\|\vec{\mathbf{v}}\| = (2\pi)/\omega$.
3. Find the acceleration $\vec{\mathbf{a}}$. Show that this vector points toward the center of the circle and that $\|\vec{\mathbf{a}}\| = R\omega^2$.
4. The force required to produce this circular motion is called the *centripetal force*, and it is denoted $\vec{\mathbf{F}}_{cent}$. This force points toward the center of the circle (not toward the ground). Show that $\|\vec{\mathbf{F}}_{cent}\| = \left(m|\vec{\mathbf{v}}|^2\right)/R$.

As the car moves around the curve, three forces act on it: gravity, the force exerted by the road (this force is perpendicular to the ground), and the friction force (Figure 1.5.10). Because describing the frictional force generated by the tires and the road is complex, we use a standard approximation for the frictional force. Assume that $\vec{\mathbf{f}} = \mu\vec{\mathbf{N}}$ for some positive constant μ. The constant μ is called the *coefficient of friction*.

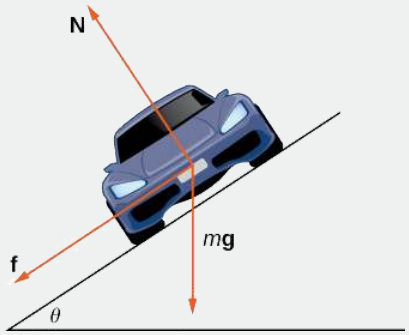


Figure 1.5.10: The car has three forces acting on it: gravity (denoted by $m\vec{\mathbf{g}}$), *the friction force* $\vec{\mathbf{f}}$, *and the force exerted by the road* $\vec{\mathbf{N}}$.

Let v_{max} denote the maximum speed the car can attain through the curve without skidding. In other words, v_{max} is the fastest speed at which the car can navigate the turn. When the car is traveling at this speed, the magnitude of the centripetal force is

$$\|\vec{\mathbf{F}}_{cent}\| = \frac{m(v_{max})^2}{R}. \tag{1.5.24}$$

The next three questions deal with developing a formula that relates the speed v_{max} to the banking angle θ.

5. Show that $\vec{\mathbf{N}}\cos\theta = m\vec{\mathbf{g}} + \vec{\mathbf{f}}\sin\theta$. Conclude that $\vec{\mathbf{N}} = (m\vec{\mathbf{g}})/(\cos\theta - \mu\sin\theta)$.
6. The centripetal force is the sum of the forces in the horizontal direction, since the centripetal force points toward the center of the circular curve. Show that

$$\vec{\mathbf{F}}_{cent} = \vec{\mathbf{N}}\sin\theta + \vec{\mathbf{f}}\cos\theta. \tag{1.5.25}$$

   Conclude that

$$\vec{\mathbf{F}}_{cent} = \frac{\sin\theta + \mu\cos\theta}{cos\theta - \mu\sin\theta} m\vec{\mathbf{g}}. \tag{1.5.26}$$

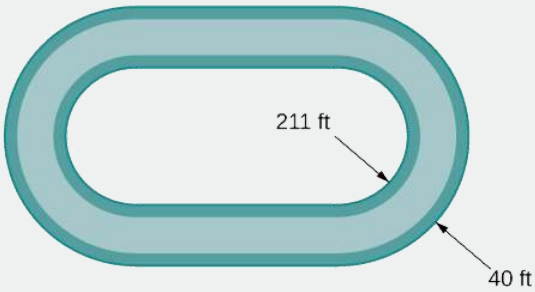
7. Show that $(v_{max})^2 = ((\sin\theta + \mu\ cos\theta)/(\cos\theta - \mu\sin\theta))gR$. Conclude that the maximum speed does not actually depend on the mass of the car.

   Now that we have a formula relating the maximum speed of the car and the banking angle, we are in a position to answer the questions like the one posed at the beginning of the project.

   The Bristol Motor Speedway is a NASCAR short track in Bristol, Tennessee. The track has the approximate shape shown in Figure 1.5.11. Each end of the track is approximately semicircular, so when cars make turns they are traveling along an approximately circular curve. If a car takes the inside track and speeds along the bottom of turn 1, the car travels along a semicircle of radius approximately 211 ft with a banking angle of 24°. If the car decides to take the outside track and speeds along the top of turn 1, then the car travels along a semicircle with a banking angle of 28°. (The track has variable angle banking.)



(a)



(b)

Figure **1.5.11**: At the Bristol Motor Speedway, Bristol, Tennessee (a), the turns have an inner radius of about 211 ft and a width of 40 ft (b). (credit: part (a) photo by Raniel Diaz, Flickr)

The coefficient of friction for a normal tire in dry conditions is approximately 0.7. Therefore, we assume the coefficient for a NASCAR tire in dry conditions is approximately 0.98.

Before answering the following questions, note that it is easier to do computations in terms of feet and seconds, and then convert the answers to miles per hour as a final step.

8. In dry conditions, how fast can the car travel through the bottom of the turn without skidding?
9. In dry conditions, how fast can the car travel through the top of the turn without skidding?
10. In wet conditions, the coefficient of friction can become as low as 0.1. If this is the case, how fast can the car travel through the bottom of the turn without skidding?
11. Suppose the measured speed of a car going along the outside edge of the turn is 105 mph. Estimate the coefficient of friction for the car's tires.

Key Concepts

- If $\vec{\mathbf{r}}(t)$ represents the position of an object at time t, then $\vec{\mathbf{r}}'(t)$ represents the velocity and $\vec{\mathbf{r}}''(t)$ represents the acceleration of the object at time t. The magnitude of the velocity vector is speed.
- The acceleration vector always points toward the concave side of the curve defined by $\vec{\mathbf{r}}(t)$. The tangential and normal components of acceleration $a_{\vec{\mathbf{T}}}$ and $a_{\vec{\mathbf{N}}}$ are the projections of the acceleration vector onto the unit tangent and unit normal vectors to the curve.
- Kepler's three laws of planetary motion describe the motion of objects in orbit around the Sun. His third law can be modified to describe motion of objects in orbit around other celestial objects as well.
- Newton was able to use his law of universal gravitation in conjunction with his second law of motion and calculus to prove Kepler's three laws.

Key Equations

- **Velocity**

$$\vec{\mathbf{v}}(t) = \vec{\mathbf{r}}'(t)$$

- **Acceleration**

$$\vec{\mathbf{a}}(t) = \vec{\mathbf{v}}'(t) = \vec{\mathbf{r}}''(t)$$

- **Speed**

$$v(t) = ||\vec{\mathbf{v}}(t)|| = ||\vec{\mathbf{r}}'(t)|| = \frac{ds}{dt}$$

- **Tangential component of acceleration**

$$a_{\vec{\mathbf{T}}} = \vec{\mathbf{a}} \cdot \vec{\mathbf{T}} = \frac{\vec{\mathbf{v}} \cdot \vec{\mathbf{a}}}{||\vec{\mathbf{v}}||}$$

- **Normal component of acceleration**

$$a_{\vec{\mathbf{N}}} = \vec{\mathbf{a}} \cdot \vec{\mathbf{N}} = \frac{||\vec{\mathbf{v}} \times \vec{\mathbf{a}}||}{||\vec{\mathbf{v}}||} = \sqrt{||\vec{\mathbf{a}}||^2 - a_{\vec{\mathbf{T}}}}$$

Glossary

acceleration vector
the second derivative of the position vector

Kepler's laws of planetary motion
three laws governing the motion of planets, asteroids, and comets in orbit around the Sun

normal component of acceleration
the coefficient of the unit normal vector $\vec{\mathbf{N}}$ when the acceleration vector is written as a linear combination of $\vec{\mathbf{T}}$ and $\vec{\mathbf{N}}$

projectile motion
motion of an object with an initial velocity but no force acting on it other than gravity

tangential component of acceleration
the coefficient of the unit tangent vector $\vec{\mathbf{T}}$ when the acceleration vector is written as a linear combination of $\vec{\mathbf{T}}$ and $\vec{\mathbf{N}}$

velocity vector
the derivative of the position vector

Contributors and Attributions

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- Edited by Paul Seeburger
  Paul Seeburger added finding point $(1, 2)$ when $t = 1$ and added part 3 (finding the equations of the tangent line) in Example 1.5.1.
  He also created Figure 1.5.1.

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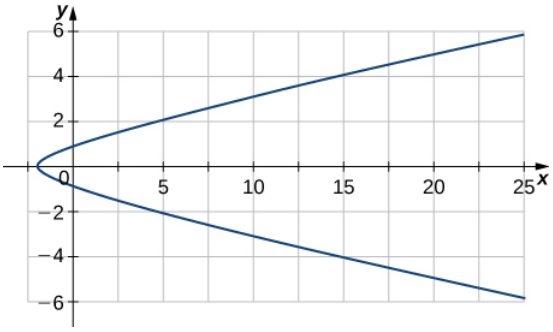
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1.5E: Exercises

1) Given $\vec{\mathbf{r}}(t) = (3t^2 - 2)\,\hat{\mathbf{i}} + (2t - \sin t)\,\hat{\mathbf{j}}$,

a. find the velocity of a particle moving along this curve.

b. find the acceleration of a particle moving along this curve.



Answer

a. $\vec{\mathbf{v}}(t) = 6t\,\hat{\mathbf{i}} + (2 - \cos t)\,\hat{\mathbf{i}}$

b. $\vec{\mathbf{a}}(t) = 6\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{i}}$

In questions 2 - 5, given the position function, find the velocity, acceleration, and speed in terms of the parameter t.

2) $\vec{\mathbf{r}}(t) = e^{-t}\,\hat{\mathbf{i}} + t^2\,\hat{\mathbf{j}} + \tan t\,\hat{\mathbf{k}}$

3) $\vec{\mathbf{r}}(t) = \langle 3\cos t,\ 3\sin t,\ t^2 \rangle$

Answer

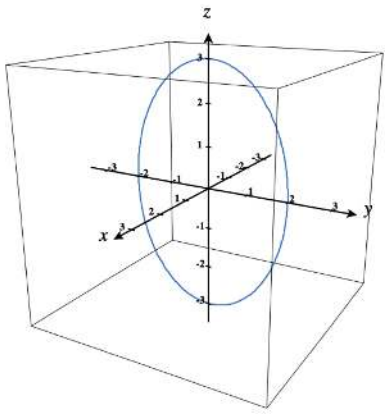
$\vec{\mathbf{v}}(t) = -3\sin t\,\hat{\mathbf{i}} + 3\cos t\,\hat{\mathbf{j}} + 2t\,\hat{\mathbf{k}}$

$\vec{\mathbf{a}}(t) = -3\cos t\,\hat{\mathbf{i}} - 3\sin t\,\hat{\mathbf{j}} + 2\,\hat{\mathbf{k}}$

$\text{Speed}(t) = \|\vec{\mathbf{v}}(t)\| = \sqrt{9 + 4t^2}$

4) $\vec{\mathbf{r}}(t) = t^5\,\hat{\mathbf{i}} + (3t^2 + 2t - 5)\,\hat{\mathbf{j}} + (3t - 1)\,\hat{\mathbf{k}}$

5) $\vec{\mathbf{r}}(t) = 2\cos t\,\hat{\mathbf{j}} + 3\sin t\,\hat{\mathbf{k}}$. The graph is shown here:



Answer

$\vec{\mathbf{v}}(t) = -2\sin t\,\hat{\mathbf{j}} + 3\cos t\,\hat{\mathbf{k}}$

$\vec{\mathbf{a}}(t) = -2\cos t\,\hat{\mathbf{j}} - 3\sin t\,\hat{\mathbf{k}}$

$\text{Speed}(t) = \|\vec{\mathbf{v}}(t)\| = \sqrt{4\sin^2 t + 9\cos^2 t} = \sqrt{4 + 5\cos^2 t}$

In questions 6 - 8, find the velocity, acceleration, and speed of a particle with the given position function.

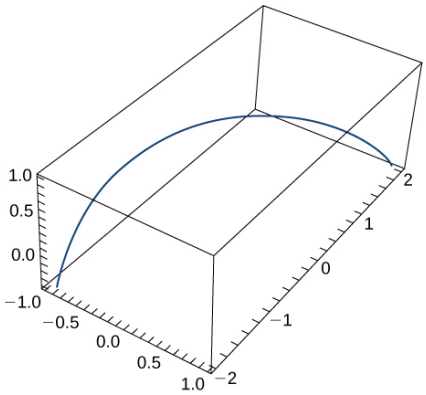
6) $\vec{\mathbf{r}}(t) = \langle t^2 - 1, t \rangle$

7) $\vec{\mathbf{r}}(t) = \langle e^t, e^{-t} \rangle$

Answer

$\vec{\mathbf{v}}(t) = \langle e^t, -e^{-t} \rangle,$
$\vec{\mathbf{a}}(t) = \langle e^t, e^{-t} \rangle,$
$\|\vec{\mathbf{v}}(t)\| = \sqrt{e^{2t} + e^{-2t}}$

8) $\vec{\mathbf{r}}(t) = \langle \sin t, t, \cos t \rangle$. The graph is shown here:



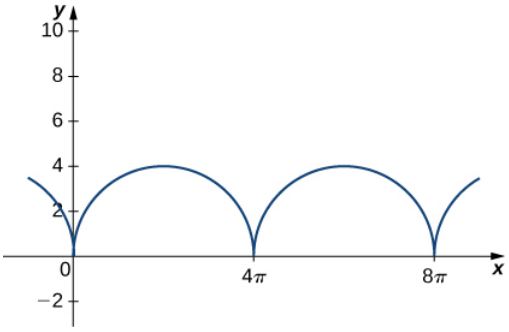
9) The position function of an object is given by $\vec{\mathbf{r}}(t) = \langle t^2, 5t, t^2 - 16t \rangle$. At what time is the speed a minimum?

Answer

$t = 4$

10) Let $\vec{\mathbf{r}}(t) = r\cosh(\omega t)\,\hat{\mathbf{i}} + r\sinh(\omega t)\,\hat{\mathbf{j}}$. Find the velocity and acceleration vectors and show that the acceleration is proportional to $\vec{\mathbf{r}}(t)$.

11) Consider the motion of a point on the circumference of a rolling circle. As the circle rolls, it generates the cycloid $\vec{\mathbf{r}}(t) = (\omega t - \sin(\omega t))\,\hat{\mathbf{i}} + (1 - \cos(\omega t))\,\hat{\mathbf{j}}$, where ω is the angular velocity of the circle and b is the radius of the circle:

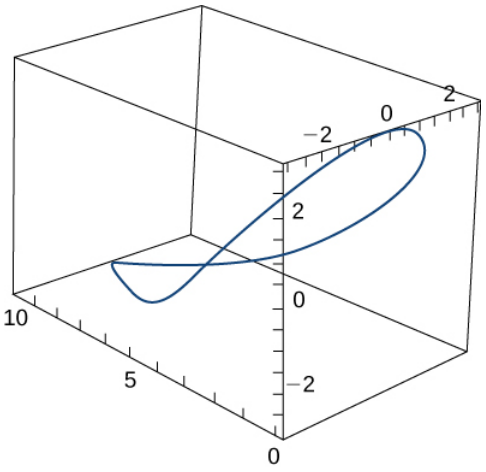


Find the equations for the velocity, acceleration, and speed of the particle at any time.

Answer

$\vec{\mathbf{v}}(t) = (\omega - \omega\cos(\omega t))\,\hat{\mathbf{i}} + (\omega\sin(\omega t))\,\hat{\mathbf{j}}$
$\vec{\mathbf{a}}(t) = (\omega^2\sin(\omega t))\,\hat{\mathbf{i}} + (\omega^2\cos(\omega t))\,\hat{\mathbf{j}}$
$\text{speed}(t) = \sqrt{(\omega - \omega\cos(\omega t))^2 + (\omega\sin(\omega t))^2}$
$= \sqrt{\omega^2 - 2\omega^2\cos(\omega t) + \omega^2\cos^2(\omega t) + \omega^2\sin^2(\omega t)}$
$= \sqrt{2\omega^2(1 - \cos(\omega t))}$

12) A person on a hang glider is spiraling upward as a result of the rapidly rising air on a path having position vector $\vec{\mathbf{r}}(t) = (3\cos t)\,\hat{\mathbf{i}} + (3\sin t)\,\hat{\mathbf{j}} + t^2\,\hat{\mathbf{k}}$. The path is similar to that of a helix, although it is not a helix. The graph is shown here:



Find the following quantities:

a. The velocity and acceleration vectors

b. The glider's speed at any time

Answer

$\|\vec{\mathbf{v}}(t)\| = \sqrt{9 + 4t^2}$

c. The times, if any, at which the glider's acceleration is orthogonal to its velocity

13) Given that $\vec{\mathbf{r}}(t) = \langle e^{-5t}\sin t,\ e^{-5t}\cos t,\ 4e^{-5t}\rangle$ is the position vector of a moving particle, find the following quantities:

a. The velocity of the particle

Answer

$\vec{\mathbf{v}}(t) = \langle e^{-5t}(\cos t - 5\sin t),\ -e^{-5t}(\sin t + 5\cos t),\ -20e^{-5t}\rangle$

b. The speed of the particle

c. The acceleration of the particle

Answer

$\vec{\mathbf{a}}(t) = \langle e^{-5t}(-\sin t - 5\cos t) - 5e^{-5t}(\cos t - 5\sin t),\ -e^{-5t}(\cos t - 5\sin t) + 5e^{-5t}(\sin t + 5\cos t),\ 100e^{-5t}\rangle$

14) Find the maximum speed of a point on the circumference of an automobile tire of radius 1 ft when the automobile is traveling at 55 mph.

15) Find the position vector-valued function $\vec{\mathbf{r}}(t)$, given that $\vec{\mathbf{a}}(t) = \hat{\mathbf{i}} + e^t\,\hat{\mathbf{j}}$, $\vec{\mathbf{v}}(0) = 2\,\hat{\mathbf{j}}$, and $\vec{\mathbf{r}}(0) = 2\,\hat{\mathbf{i}}$.

16) Find $\vec{\mathbf{r}}(t)$ given that $\vec{\mathbf{a}}(t) = -32\,\hat{\mathbf{j}}$, $\vec{\mathbf{v}}(0) = 600\sqrt{3}\,\hat{\mathbf{i}} + 600\,\hat{\mathbf{j}}$, and $\vec{\mathbf{r}}(0) = \vec{\mathbf{0}}$.

17) The acceleration of an object is given by $\vec{\mathbf{a}}(t) = t\,\hat{\mathbf{j}} + t\,\hat{\mathbf{k}}$. The velocity at $t = 1$ sec is $\vec{\mathbf{v}}(1) = 5\,\hat{\mathbf{j}}$ and the position of the object at $t = 1$ sec is $\vec{\mathbf{r}}(1) = 0\,\hat{\mathbf{i}} + 0\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}$. Find the object's position at any time.

Answer

$\vec{\mathbf{r}}(t) = 0\,\hat{\mathbf{i}} + \left(\frac{1}{6}t^3 + 4.5t - \frac{14}{3}\right)\hat{\mathbf{j}} + \left(\frac{1}{6}t^3 - \frac{1}{2}t + \frac{1}{3}\right)\hat{\mathbf{k}}$

Projectile Motion

18) A projectile is shot in the air from ground level with an initial velocity of 500 m/sec at an angle of 60° with the horizontal.

a. At what time does the projectile reach maximum height?

Answer

44.185 sec

b. What is the approximate maximum height of the projectile?

c. At what time is the maximum range of the projectile attained?

Answer

$t = 88.37$ sec

d. What is the maximum range?

e. What is the total flight time of the projectile?

Answer

$t = 88.37$ sec

19) A projectile is fired at a height of 1.5 m above the ground with an initial velocity of 100 m/sec and at an angle of 30° above the horizontal. Use this information to answer the following questions:

a. Determine the maximum height of the projectile.

b. Determine the range of the projectile.

Answer

The range is approximately 886.29 m.

20) A golf ball is hit in a horizontal direction off the top edge of a building that is 100 ft tall. How fast must the ball be launched to land 450 ft away?

21) A projectile is fired from ground level at an angle of 8° with the horizontal. The projectile is to have a range of 50 m. Find the minimum velocity (speed) necessary to achieve this range.

Answer

$v = 42.16$ m/sec

22) Prove that an object moving in a straight line at a constant speed has an acceleration of zero.

Finding Components of Acceleration & Kepler's Laws

23) Find the tangential and normal components of acceleration for $\vec{\mathbf{r}}(t) = t^2\,\hat{\mathbf{i}} + 2t\,\hat{\mathbf{j}}$ when $t = 1$.

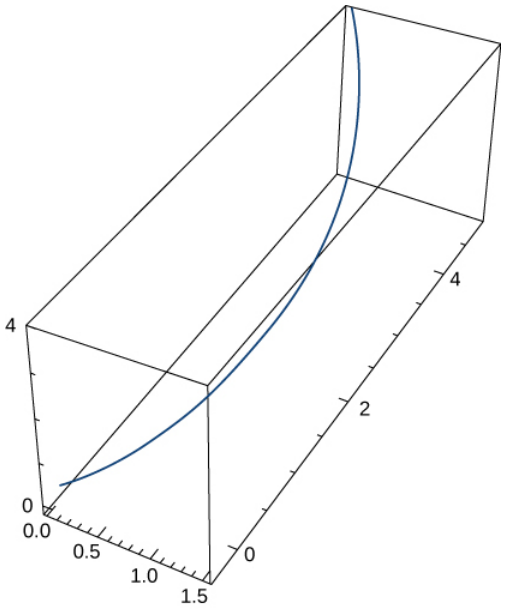
Answer

$a_{\vec{\mathbf{T}}} = \sqrt{2}, \quad a_{\vec{\mathbf{N}}} = \sqrt{2}$

In questions 24 - 30, find the tangential and normal components of acceleration.

24) $\vec{\mathbf{r}}(t) = \langle \cos(2t),\ \sin(2t),\ 1 \rangle$

25) $\vec{\mathbf{r}}(t) = \langle e^t \cos t,\ e^t \sin t,\ e^t \rangle$. The graph is shown here:



Answer

$a_{\vec{\mathbf{T}}} = \sqrt{3}e^t, \quad a_{\vec{\mathbf{N}}} = \sqrt{2}e^t$

26) $\vec{\mathbf{r}}(t) = \langle \frac{2}{3}(1+t)^{3/2}, \frac{2}{3}(1-t)^{3/2}, \sqrt{2}t \rangle$

27) $\vec{\mathbf{r}}(t) = \left\langle 2t, t^2, \frac{t^3}{3} \right\rangle$

Answer

$a_{\vec{\mathbf{T}}} = 2t, \quad a_{\vec{\mathbf{N}}} = 2$

28) $\vec{\mathbf{r}}(t) = t^2\,\hat{\mathbf{i}} + t^2\,\hat{\mathbf{j}} + t^3\,\hat{\mathbf{k}}$

29) $\vec{\mathbf{r}}(t) = \langle 6t, 3t^2, 2t^3 \rangle$

Answer

$a_{\vec{\mathbf{T}}} = \dfrac{6t + 12t^3}{\sqrt{1 + t^2 + t^4}}, \quad a_{\vec{\mathbf{N}}} = 6\sqrt{\dfrac{1 + 4t^2 + t^4}{1 + t^2 + t^4}}$

30) $\vec{\mathbf{r}}(t) = 3\cos(2\pi t)\,\hat{\mathbf{i}} + 3\sin(2\pi t)\,\hat{\mathbf{j}}$

Answer

$a_{\vec{\mathbf{T}}} = 0, \quad a_{\vec{\mathbf{N}}} = 12\pi^2$

31) Find the tangential and normal components of acceleration for $\vec{\mathbf{r}}(t) = a\cos(\omega t)\,\hat{\mathbf{i}} + b\sin(\omega t)\,\hat{\mathbf{j}}$ at $t = 0$.

Answer

$a_{\vec{\mathbf{T}}} = 0, \quad a_{\vec{\mathbf{N}}} = a\omega^2$

32) Suppose that the position function for an object in three dimensions is given by the equation $\vec{\mathbf{r}}(t) = t\cos(t)\,\hat{\mathbf{i}} + t\sin(t)\,\hat{\mathbf{j}} + 3t\,\hat{\mathbf{k}}$.

a. Show that the particle moves on a circular cone.

b. Find the angle between the velocity and acceleration vectors when $t = 1.5$.

c. Find the tangential and normal components of acceleration when $t = 1.5$.

Answer

c. $a_{\vec{\mathbf{T}}} = 0.43 \text{ m/sec}^2, \quad a_{\vec{\mathbf{N}}} = 2.46 \text{ m/sec}^2$

33) The force on a particle is given by $\vec{\mathbf{f}}(t) = (\cos t)\,\hat{\mathbf{i}} + (\sin t)\,\hat{\mathbf{j}}$. The particle is located at point $(c, 0)$ at $t = 0$. The initial velocity of the particle is given by $\vec{\mathbf{v}}(0) = v_0\,\hat{\mathbf{j}}$. Find the path of the particle of mass m. (Recall, $\vec{\mathbf{F}} = m\vec{\mathbf{a}}$.)

Answer

$$\vec{\mathbf{r}}(t) = \left(\frac{-\cos t}{m} + c + \frac{1}{m}\right)\hat{\mathbf{i}} + \left(\frac{-\sin t}{m} + \left(v_0 + \frac{1}{m}\right)t\right)\hat{\mathbf{j}}$$

34) An automobile that weighs 2700 lb makes a turn on a flat road while traveling at 56 ft/sec. If the radius of the turn is 70 ft, what is the required frictional force to keep the car from skidding?

35) Using Kepler's laws, it can be shown that $v_0 = \sqrt{\frac{2GM}{r_0}}$ is the minimum speed needed when $\theta = 0$ so that an object will escape from the pull of a central force resulting from mass M. Use this result to find the minimum speed when $\theta = 0$ for a space capsule to escape from the gravitational pull of Earth if the probe is at an altitude of 300 km above Earth's surface.

Answer

10.94 km/sec

36) Find the time in years it takes the dwarf planet Pluto to make one orbit about the Sun given that $a = 39.5$ A.U.

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1.6: Chapter 6 Review Exercises

***True or False*? Justify your answer with a proof or a counterexample.**

1. A parametric equation that passes through points P and Q can be given by $\vec{\mathbf{r}}(t) = \langle t^2,\ 3t+1,\ t-2\rangle$, where $P(1,4,-1)$ and $Q(16,11,2)$.

2. $\dfrac{d}{dt}\left[\vec{\mathbf{u}}(t)\times\vec{\mathbf{u}}(t)\right] = 2\vec{\mathbf{u}}'(t)\times\vec{\mathbf{u}}(t)$

Answer

False, $\dfrac{d}{dt}\left[\vec{\mathbf{u}}(t)\times\vec{\mathbf{u}}(t)\right] = \vec{\mathbf{0}}$.

3. The curvature of a circle of radius r is constant everywhere. Furthermore, the curvature is equal to $1/r$.

4. The speed of a particle with a position function $\vec{\mathbf{r}}(t)$ is $\dfrac{\vec{\mathbf{r}}'(t)}{\|\vec{\mathbf{r}}'(t)\|}$.

Answer

False, it is $\|\vec{\mathbf{r}}'(t)\|$

Find the domains of the vector-valued functions.

5. $\vec{\mathbf{r}}(t) = \langle \sin(t),\ \ln(t),\ \sqrt{t}\rangle$

6. $\vec{\mathbf{r}}(t) = \left\langle e^t,\ \dfrac{1}{\sqrt{4-t}},\ \sec t\right\rangle$

Answer

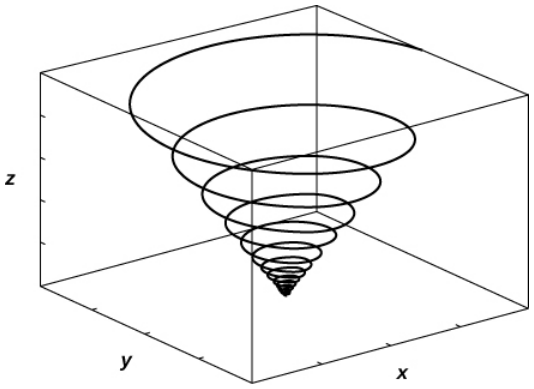
$t<4,\ t\neq\dfrac{n\pi}{2}$

Sketch the curves for the following vector equations. Use a calculator if needed.

7. **[T]** $\vec{\mathbf{r}}(t) = \langle t^2, t^3\rangle$

8. **[T]** $\vec{\mathbf{r}}(t) = \langle \sin(20t)e^{-t},\ \cos(20t)e^{-t},\ e^{-t}\rangle$

Answer



Find a vector function that describes the following curves.

9. Intersection of the cylinder $x^2+y^2=4$ with the plane $x+z=6$

10. Intersection of the cone $z=\sqrt{x^2+y^2}$ and plane $z=y-4$

Answer

$\vec{\mathbf{r}}(t) = \left\langle t,\ 2-\frac{t^2}{8},\ -2-\frac{t^2}{8}\right\rangle$

Find the derivatives of $\vec{\mathbf{u}}(t)$, $\vec{\mathbf{u}}'(t)$, $\vec{\mathbf{u}}'(t) \times \vec{\mathbf{u}}(t)$, $\vec{\mathbf{u}}(t) \times \vec{\mathbf{u}}'(t)$, and $\vec{\mathbf{u}}(t) \cdot \vec{\mathbf{u}}'(t)$. Find the unit tangent vector.

11. $\vec{\mathbf{u}}(t) = \langle e^t, e^{-t} \rangle$

12. $\vec{\mathbf{u}}(t) = \langle t^2, 2t+6, 4t^5 - 12 \rangle$

Answer

$\vec{\mathbf{u}}'(t) = \langle 2t, 2, 20t^4 \rangle$,
$\vec{\mathbf{u}}''(t) = \langle 2, 0, 80t^3 \rangle$,
$\frac{d}{dt}\left[\vec{\mathbf{u}}'(t) \times \vec{\mathbf{u}}(t)\right] = \langle -480t^3 - 160t^4, 24 + 75t^2, 12 + 4t \rangle$,
$\frac{d}{dt}\left[\vec{\mathbf{u}}(t) \times \vec{\mathbf{u}}'(t)\right] = \langle 480t^3 + 160t^4, -24 - 75t^2, -12 - 4t \rangle$,
$\frac{d}{dt}\left[\vec{\mathbf{u}}(t) \cdot \vec{\mathbf{u}}'(t)\right] = 720t^8 - 9600t^3 + 6t^2 + 4$,
unit tangent vector: $\vec{\mathbf{T}}(t) = \frac{2t}{\sqrt{400t^8 + 4t^2 + 4}}\,\hat{\mathbf{i}} + \frac{2}{\sqrt{400t^8 + 4t^2 + 4}}\,\hat{\mathbf{j}} + \frac{20t^4}{\sqrt{400t^8 + 4t^2 + 4}}\,\hat{\mathbf{k}}$

Evaluate the following integrals.

13. $\int \left(\tan(t)\sec(t)\,\hat{\mathbf{i}} - te^{3t}\,\hat{\mathbf{j}}\right)\,dt$

14. $\int_1^4 \vec{\mathbf{u}}(t)\,dt$, with $\vec{\mathbf{u}}(t) = \left\langle \frac{\ln t}{t}, \frac{1}{\sqrt{t}}, \sin\left(\frac{t\pi}{4}\right) \right\rangle$

Answer

$\frac{\ln(4^2)}{2}\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + \frac{2(2+\sqrt{2})}{\pi}\,\hat{\mathbf{k}}$

Find the length for the following curves.

15. $\vec{\mathbf{r}}(t) = \langle 3t, 4\cos t, 4\sin t \rangle$ for $1 \le t \le 4$

16. $\vec{\mathbf{r}}(t) = 2\,\hat{\mathbf{i}} + t\,\hat{\mathbf{j}} + 3t^2\,\hat{\mathbf{k}}$ for $0 \le t \le 1$

Answer

$\frac{\sqrt{37}}{2} + \frac{1}{12}\sinh^{-1} 6$

Reparameterize the following functions with respect to their arc length measured from $t = 0$ in direction of increasing t.

17. $\vec{\mathbf{r}}(t) = 2t\,\hat{\mathbf{i}} + (4t-5)\,\hat{\mathbf{j}} + (1-3t)\,\hat{\mathbf{k}}$

18. $\vec{\mathbf{r}}(t) = \cos(2t)\,\hat{\mathbf{i}} + 8t\,\hat{\mathbf{j}} - \sin(2t)\,\hat{\mathbf{k}}$

Answer

$\vec{\mathbf{r}}(t(s)) = \cos\left(\frac{2s}{\sqrt{65}}\right)\,\hat{\mathbf{i}} + \frac{8s}{\sqrt{65}}\,\hat{\mathbf{j}} - \sin\left(\frac{2s}{\sqrt{65}}\right)\,\hat{\mathbf{k}}$

Find the curvature for the following vector functions.

19. $\vec{\mathbf{r}}(t) = (2\sin t)\,\hat{\mathbf{i}} - 4t\,\hat{\mathbf{j}} + (2\cos t)\,\hat{\mathbf{k}}$

20. $\vec{\mathbf{r}}(t) = \sqrt{2}e^t\,\hat{\mathbf{i}} + \sqrt{2}e^{-t}\,\hat{\mathbf{j}} + 2t\,\hat{\mathbf{k}}$

Answer

$\frac{e^{2t}}{\left(e^{2t}+1\right)^2}$

21. Find the unit tangent vector, the unit normal vector, and the binormal vector for $\vec{\mathbf{r}}(t) = 2\cos t\,\hat{\mathbf{i}} + 3t\,\hat{\mathbf{j}} + 2sint\,\hat{\mathbf{k}}$.

22. Find the tangential and normal acceleration components with the position vector $\vec{\mathbf{r}}(t) = \langle \cos t, \sin t, e^t \rangle$.

Answer

$$a_T = \frac{e^{2t}}{1+e^{2t}},$$

$$a_N = \frac{\sqrt{2e^{2t}+4e^{2t}\sin t\cos t+1}}{1+e^{2t}}$$

23. A Ferris wheel car is moving at a constant speed v and has a constant radius r. Find the tangential and normal acceleration of the Ferris wheel car.

24. The position of a particle is given by $\vec{\mathbf{r}}(t) = \langle t^2, \ln t, \sin(\pi t)\rangle$, where t is measured in seconds and r is measured in meters. Find the velocity, acceleration, and speed functions. What are the position, velocity, speed, and acceleration of the particle at 1 sec?

Answer

$\vec{\mathbf{v}}(t) = \left\langle 2t, \frac{1}{t}, \pi\cos(\pi t)\right\rangle$ m/sec,
$\vec{\mathbf{a}}(t) = \left\langle 2, -\frac{1}{t^2}, -\pi^2\sin(\pi t)\right\rangle$ m/sec^2,
$\text{speed}(t) = \sqrt{4t^2+\frac{1}{t^2}+\pi^2\cos^2(\pi t)}$ m/sec;
At $t=1$, $\vec{\mathbf{r}}(1) = \langle 1,0,0\rangle$ m, $\vec{\mathbf{v}}(1) = \langle 2,-1,\pi\rangle$ m/sec, $\vec{\mathbf{a}}(1) = \langle 2,-1,0\rangle$ m/sec^2, and $\text{speed}(1) = \sqrt{5+\pi^2}$ m/sec

The following problems consider launching a cannonball out of a cannon. The cannonball is shot out of the cannon with an angle θ and initial velocity $\vec{\mathbf{v}}_0$. The only force acting on the cannonball is gravity, so we begin with a constant acceleration $\vec{\mathbf{a}}(t) = -g\hat{\mathbf{j}}$.

25. Find the velocity vector function $\vec{\mathbf{v}}(t)$.

26. Find the position vector $\vec{\mathbf{r}}(t)$ and the parametric representation for the position.

Answer

$\vec{\mathbf{r}}(t) = \vec{\mathbf{v}}_0 t - \frac{gt^2}{2}\hat{\mathbf{j}}$,
$\vec{\mathbf{r}}(t) = \langle v_0(\cos\theta)t,\ v_0(\sin\theta)t, -\frac{gt^2}{2}\rangle$ where $v_0 = \|\vec{\mathbf{v}}_0\|$.

27. At what angle do you need to fire the cannonball for the horizontal distance to be greatest? What is the total distance it would travel?

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CHAPTER OVERVIEW

2: Multiple Integration

In this chapter we extend the concept of a definite integral of a single variable to double and triple integrals of functions of two and three variables, respectively. We examine applications involving integration to compute volumes, masses, and centroids of more general regions. We will also see how the use of other coordinate systems (such as polar, cylindrical, and spherical coordinates) makes it simpler to compute multiple integrals over some types of regions and functions. In the preceding chapter, we discussed differential calculus with multiple independent variables. Now we examine integral calculus in multiple dimensions. Just as a partial derivative allows us to differentiate a function with respect to one variable while holding the other variables constant, we will see that an iterated integral allows us to integrate a function with respect to one variable while holding the other variables constant.

2.1: Prelude to Multiple Integration

2.2: Double Integrals over Rectangular Regions

2.2E: Exercises

2.3: Double Integrals over General Regions

2.3E: Exercises

2.4: Double Integrals in Polar Coordinates

2.4E: Exercises

2.5: Triple Integrals

2.5E: Exercises

2.6: Triple Integrals in Cylindrical and Spherical Coordinates

2.6E: Exercises

2.7: Calculating Centers of Mass and Moments of Inertia

2.7E: Exercises

2.8: Change of Variables in Multiple Integrals

2.8E: Exercises

2.9: Chapter 2 Review Exercises

Thumbnail: Double integral as volume under a surface $z = 10 - x^2 - y^2/8$. The rectangular region at the bottom of the body is the domain of integration, while the surface is the graph of the two-variable function to be integrated. (Public Domain; Oleg Alexandrov).

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2.1: Prelude to Multiple Integration

In this chapter we extend the concept of a definite integral of a single variable to double and triple integrals of functions of two and three variables, respectively. We examine applications involving integration to compute volumes, masses, and centroids of more general regions. We will also see how the use of other coordinate systems (such as polar, cylindrical, and spherical coordinates) makes it simpler to compute multiple integrals over some types of regions and functions. As an example, we will use polar coordinates to find the volume of structures such as l'Hemisfèric (Figure **2.1.1**).



Figure **2.1.1**: The City of Arts and Sciences in Valencia, Spain, has a unique structure along an axis of just two kilometers that was formerly the bed of the River Turia. The l'Hemisfèric has an IMAX cinema with three systems of modern digital projections onto a concave screen of 900 square meters. An oval roof over 100 meters long has been made to look like a huge human eye that comes alive and opens up to the world as the "Eye of Wisdom." (credit: modification of work by Javier Yaya Tur, Wikimedia Commons)

In the preceding chapter, we discussed differential calculus with multiple independent variables. Now we examine integral calculus in multiple dimensions. Just as a partial derivative allows us to differentiate a function with respect to one variable while holding the other variables constant, we will see that an iterated integral allows us to integrate a function with respect to one variable while holding the other variables constant.

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2.2: Double Integrals over Rectangular Regions

Learning Objectives

- Recognize when a function of two variables is integrable over a rectangular region.
- Recognize and use some of the properties of double integrals.
- Evaluate a double integral over a rectangular region by writing it as an iterated integral.
- Use a double integral to calculate the area of a region, volume under a surface, or average value of a function over a plane region.

In this section we investigate double integrals and show how we can use them to find the volume of a solid over a rectangular region in the xy-plane. Many of the properties of double integrals are similar to those we have already discussed for single integrals.

Volumes and Double Integrals

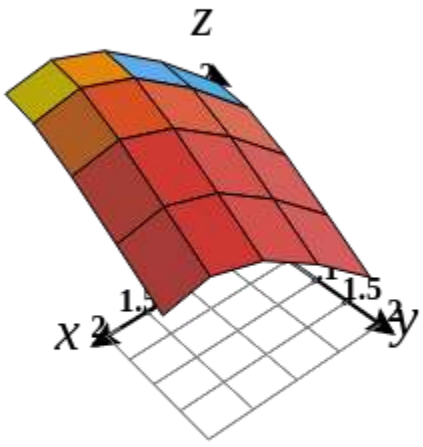
We begin by considering the space above a rectangular region R. Consider a continuous function $f(x,y) \geq 0$ of two variables defined on the closed rectangle R:

$$R = [a,b] \times [c,d] = \{(x,y) \in \mathbb{R}^2 | \, a \leq x \leq b, \, c \leq y \leq d\}$$

Here $[a,b] \times [c,d]$ denotes the Cartesian product of the two closed intervals $[a,b]$ and $[c,d]$. It consists of rectangular pairs (x,y) such that $a \leq x \leq b$ and $c \leq y \leq d$. The graph of f represents a surface above the xy-plane with equation $z = f(x,y)$ where z is the height of the surface at the point (x,y). Let S be the solid that lies above R and under the graph of f (Figure 2.2.1). The base of the solid is the rectangle R in the xy-plane. We want to find the volume V of the solid S.

Visualizing the Definition of the Double Integral

Below is the graph of $z = 1 + \sin(x) + \cos(y)$ above the grid on the xy-plane cut up into several pieces. Use the buttons to incrementally increase or decrease the number of pieces along the x and y axes.



4 Partitions

Decrease Partitions | Increase Partitions

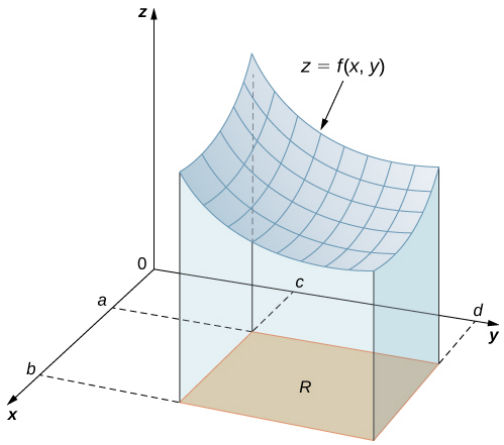


Figure 2.2.1: The graph of $f(x, y)$ *over the rectangle R in the* xy*-plane is a curved surface.*

Visualizing the Definition of the Double Integral

Below is the graph of $z = 1 + \sin(x) + \cos(y)$ above the grid on the xy-plane cut up into several pieces. Use the buttons to incrementally increase or decrease the number of pieces along the x and y axes.

4 Partitions

Decrease Partitions | Increase Partitions

We divide the region R into small rectangles R_{ij}, each with area ΔA and with sides Δx and Δy (Figure 2.2.2). We do this by dividing the interval $[a, b]$ into m subintervals and dividing the interval $[c, d]$ into n subintervals. Hence $\Delta x = \frac{b-a}{m}$, $\Delta y = \frac{d-c}{n}$, and $\Delta A = \Delta x \Delta y$.

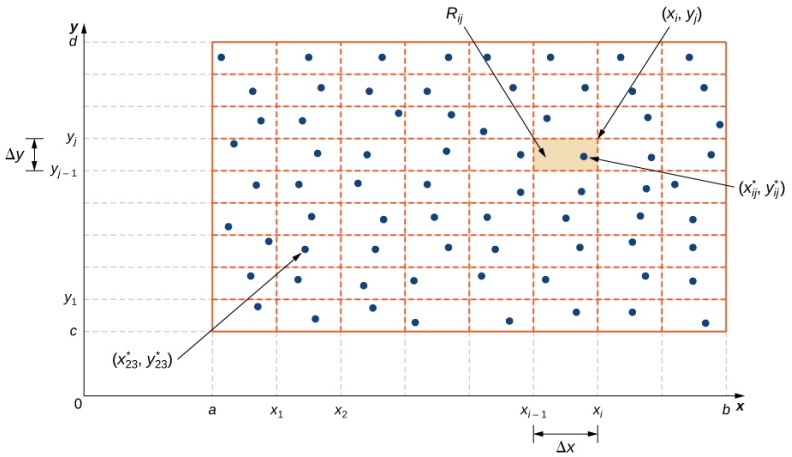


Figure 2.2.2: Rectangle R is divided into small rectangles R_{ij} each with area ΔA.

The volume of a thin rectangular box above R_{ij} is $f(x^*_{ij}, y^*_{ij})\,\Delta A$, where (x^*_{ij}, y^*_{ij}) is an arbitrary sample point in each R_{ij} as shown in the following figure, $f(x^*_{ij}, y^*_{ij})$ is the height of the corresponding thin rectangular box, and ΔA is the area of each rectangle R_{ij}.

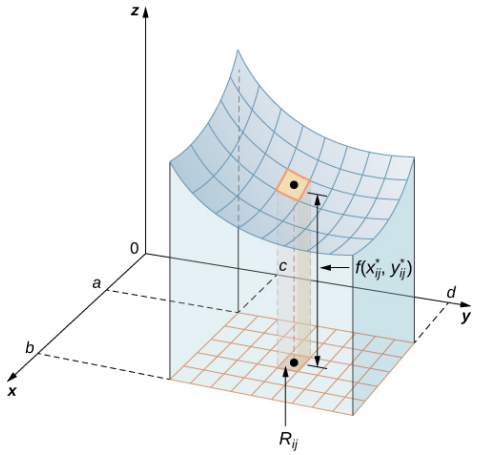


Figure 2.2.3: A thin rectangular box above R_{ij} with height $f(x^*_{ij}, y^*_{ij})$.

Using the same idea for all the subrectangles, we obtain an approximate volume of the solid S as

$$V \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x^*_{ij}, y^*_{ij})\Delta A.$$

This sum is known as a **double Riemann sum** and can be used to approximate the value of the volume of the solid. Here the double sum means that for each subrectangle we evaluate the function at the chosen point, multiply by the area of each rectangle, and then add all the results.

As we have seen in the single-variable case, we obtain a better approximation to the actual volume if m and n become larger.

$$V = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x^*_{ij}, y^*_{ij})\Delta A$$

or

$$V = \lim_{\Delta x,\, \Delta y\to 0} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x^*_{ij}, y^*_{ij})\Delta A.$$

Note that the sum approaches a limit in either case and the limit is the volume of the solid with the base R. Now we are ready to define the double integral.

Definition

The double integral of the function $f(x, y)$ over the rectangular region R in the xy-plane is defined as

$$\iint_R f(x, y)dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A. \tag{2.2.1}$$

If $f(x,y) \geq 0$, then the volume V of the solid S, which lies above R in the xy-plane and under the graph of f, is the double integral of the function $f(x,y)$ over the rectangle R. If the function is ever negative, then the double integral can be considered a "signed" volume in a manner similar to the way we defined net signed area in The Definite Integral.

Example 2.2.1: Setting up a Double Integral and Approximating It by Double Sums

Consider the function $z = f(x, y) = 3x^2 - y$ over the rectangular region $R = [0, 2] \times [0, 2]$ (Figure 2.2.4).

a. Set up a double integral for finding the value of the signed volume of the solid S that lies above R and "under" the graph of f.

b. Divide R into four squares with $m = n = 2$, and choose the sample point as the upper right corner point of each square (1,1), (2,1),(1,2), and (2,2) (Figure 2.2.4) to approximate the signed volume of the solid S that lies above R and "under" the graph of f.

c. Divide R into four squares with $m = n = 2$, and choose the sample point as the midpoint of each square: (1/2, 1/2), (3/2, 1/2), (1/2,3/2), and (3/2, 3/2) to approximate the signed volume.

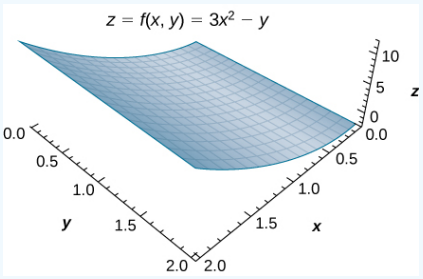


Figure 2.2.4: The function $z = f(x, y)$ graphed over the rectangular region $R = [0, 2] \times [0, 2]$.

Solution

a. As we can see, the function $z = f(x, y) = 3x^2 - y$ is above the plane. To find the signed volume of S, we need to divide the region R into small rectangles R_{ij}, each with area ΔA and with sides Δx and Δy, and choose (x_{ij}^*, y_{ij}^*) as sample points in each R_{ij}. Hence, a double integral is set up as

$$V = \iint_R (3x^2 - y)dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} [3(x_{ij}^*)^2 - y_{ij}^*]\Delta A.$$

b. Approximating the signed volume using a Riemann sum with $m = n = 2$ we have $\Delta A = \Delta x \Delta y = 1 \times 1 = 1$. Also, the sample points are (1, 1), (2, 1), (1, 2), and (2, 2) as shown in the following figure.

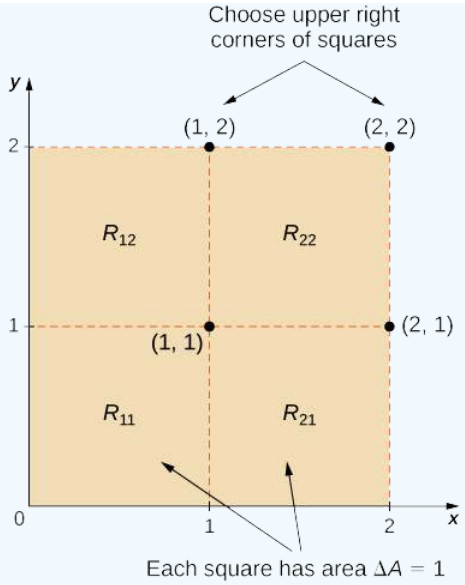


Figure 2.2.5: Subrectangles for the rectangular region $R = [0,2] \times [0,2]$.

Hence,

$$\begin{aligned} V &\approx \sum_{i=1}^{2}\sum_{j=1}^{2} f(x_{ij}^*, y_{ij}^*)\Delta A \\ &= \sum_{i=1}^{2}(f(x_{i1}^*, y_{i1}^*) + f(x_{i2}^*, y_{i2}^*))\Delta A \\ &= f(x_{11}^*, y_{11}^*)\Delta A + f(x_{21}^*, y_{21}^*)\Delta A + f(x_{12}^*, y_{12}^*)\Delta A + f(x_{22}^*, y_{22}^*)\Delta A \\ &= f(1,1)(1) + f(2,1)(1) + f(1,2)(1) + f(2,2)(1) \\ &= (3-1)(1) + (12-1)(1) + (3-2)(1) + (12-2)(1) \\ &= 2 + 11 + 1 + 10 = 24. \end{aligned}$$

c. Approximating the signed volume using a Riemann sum with $m = n = 2$ we have$\Delta A = \Delta x \Delta y = 1 \times 1 = 1$. In this case the sample points are (1/2, 1/2), (3/2, 1/2), (1/2, 3/2), and (3/2, 3/2).
Hence,

$$\begin{aligned} V &\approx \sum_{i=1}^{2}\sum_{j=1}^{2} f(x_{ij}^*, y_{ij}^*)\Delta A \\ &= f(x_{11}^*, y_{11}^*)\Delta A + f(x_{21}^*, y_{21}^*)\Delta A + f(x_{12}^*, y_{12}^*)\Delta A + f(x_{22}^*, y_{22}^*)\Delta A \\ &= f(1/2,1/2)(1) + f(3/2,1/2)(1) + f(1/2,3/2)(1) + f(3/2,3/2)(1) \\ &= \left(\frac{3}{4} - \frac{1}{4}\right)(1) + \left(\frac{27}{4} - \frac{1}{2}\right)(1) + \left(\frac{3}{4} - \frac{3}{2}\right)(1) + \left(\frac{27}{4} - \frac{3}{2}\right)(1) \\ &= \frac{2}{4} + \frac{25}{4} + \left(-\frac{3}{4}\right) + \frac{21}{4} = \frac{45}{4} = 11. \end{aligned}$$

Analysis

Notice that the approximate answers differ due to the choices of the sample points. In either case, we are introducing some error because we are using only a few sample points. Thus, we need to investigate how we can achieve an accurate answer.

? Exercise 2.2.1

Use the same function$z = f(x,y) = 3x^2 - y$ over the rectangular region $R = [0,2] \times [0,2]$.

Divide R into the same four squares with $m = n = 2$, and choose the sample points as the upper left corner point of each square (0,1), (1,1), (0,2), and (1,2) (Figure 2.2.5) to approximate the signed volume of the solid S that lies above R and "under" the

graph of f.

Hint

Follow the steps of the previous example.

Answer

$$V \approx \sum_{i=1}^{2}\sum_{j=1}^{2} f(x_{ij}^*, y_{ij}^*)\,\Delta A = 0$$

Note that we developed the concept of double integral using a rectangular region R. This concept can be extended to any general region. However, when a region is not rectangular, the subrectangles may not all fit perfectly into R, particularly if the base area is curved. We examine this situation in more detail in the next section, where we study regions that are not always rectangular and subrectangles may not fit perfectly in the region R. Also, the heights may not be exact if the surface $z = f(x, y)$ is curved. However, the errors on the sides and the height where the pieces may not fit perfectly within the solid S approach 0 as m and n approach infinity. Also, the double integral of the function $z = f(x, y)$ exists provided that the function f is not too discontinuous. If the function is bounded and continuous over R except on a finite number of smooth curves, then the double integral exists and we say that ff is integrable over R.

Since $\Delta A = \Delta x \Delta y = \Delta y \Delta x$, we can express dA as $dx\,dy$ or $dy\,dx$. This means that, when we are using rectangular coordinates, the double integral over a region R denoted by

$$\iint_R f(x, y)\,dA$$

can be written as

$$\iint_R f(x, y)\,dx\,dy$$

or

$$\iint_R f(x, y)\,dy\,dx.$$

Now let's list some of the properties that can be helpful to compute double integrals.

Properties of Double Integrals

The properties of double integrals are very helpful when computing them or otherwise working with them. We list here six properties of double integrals. Properties 1 and 2 are referred to as the linearity of the integral, property 3 is the additivity of the integral, property 4 is the monotonicity of the integral, and property 5 is used to find the bounds of the integral. Property 6 is used if $f(x, y)$ is a product of two functions $g(x)$ and $h(y)$.

Theorem: PROPERTIES OF DOUBLE INTEGRALS

Assume that the functions $f(x, y)$ and $g(x, y)$ are integrable over the rectangular region R; S and T are subregions of R; and assume that m and M are real numbers.

i. The sum $f(x, y) + g(x, y)$ is integrable and

$$\iint_R [f(x, y) + g(x, y)]\,dA = \iint_R f(x, y)\,dA + \iint_R g(x, y)\,dA.$$

ii. If c is a constant, then $cf(x, y)$ is integrable and

$$\iint_R cf(x, y)\,dA = c\iint_R f(x, y)\,dA.$$

iii. If $R = S \cup T$ and $S \cap T = \emptyset$ except an overlap on the boundaries, then

$$\iint_R f(x,y)\,dA = \iint_S f(x,y)\,dA + \iint_T f(x,y)\,dA.$$

iv. If $f(x,y) \geq g(x,y)$ for (x,y) in R, then

$$\iint_R f(x,y)\,dA \geq \iint_R g(x,y)\,dA.$$

v. If $m \leq f(x,y) \leq M$ and $A(R) = \textbf{the area of } R$, then

$$m \cdot A(R) \leq \iint_R f(x,y)\,dA \leq M \cdot A(R).$$

vi. In the case where $f(x,y)$ can be factored as a product of a function $g(x)$ of x only and a function $h(y)$ of y only, then over the region $R = \{(x,y) \mid a \leq x \leq b,\ c \leq y \leq d\}$, the double integral can be written as

$$\iint_R f(x,y)\,dA = \left(\int_a^b g(x)\,dx\right)\left(\int_c^d h(y)\,dy\right).$$

These properties are used in the evaluation of double integrals, as we will see later. We will become skilled in using these properties once we become familiar with the computational tools of double integrals. So let's get to that now.

Iterated Integrals

So far, we have seen how to set up a double integral and how to obtain an approximate value for it. We can also imagine that evaluating double integrals by using the definition can be a very lengthy process if we choose larger values for m and n.Therefore, we need a practical and convenient technique for computing double integrals. In other words, we need to learn how to compute double integrals without employing the definition that uses limits and double sums.

The basic idea is that the evaluation becomes easier if we can break a double integral into single integrals by integrating first with respect to one variable and then with respect to the other. The key tool we need is called an iterated integral.

Definitions: iterated integrals

Assume a, b, c, and d are real numbers. We define an *iterated integral* for a function $f(x,y)$ over the rectangular region $R = [a,b] \times [c,d]$ as

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx \tag{2.2.2}$$

or

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy. \tag{2.2.3}$$

The notation $\int_a^b \left[\int_c^d f(x,y)\,dy\right] dx$ means that we integrate $f(x,y)$ with respect to y while holding x constant. Similarly, the notation $\int_c^d \left[\int_a^b f(x,y)\,dx\right] dy$ means that we integrate $f(x,y)$ with respect to x while holding y constant. The fact that double integrals can be split into iterated integrals is expressed in Fubini's theorem. Think of this theorem as an essential tool for evaluating double integrals.

Theorem: FUBINI'S THEOREM

Suppose that $f(x,y)$ is a function of two variables that is continuous over a rectangular region $R = \{(x,y) \in \mathbb{R}^2 \mid a \leq x \leq b,\ c \leq y \leq d\}$. Then we see from Figure **2.2.6** that the double integral of f over the region equals an iterated integral,

$$\iint_R f(x,y)\,dA = \iint_R f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy. \tag{2.2.4}$$

More generally, Fubini's theorem is true if f is bounded on R and f is discontinuous only on a finite number of continuous curves. In other words, f has to be integrable over R.

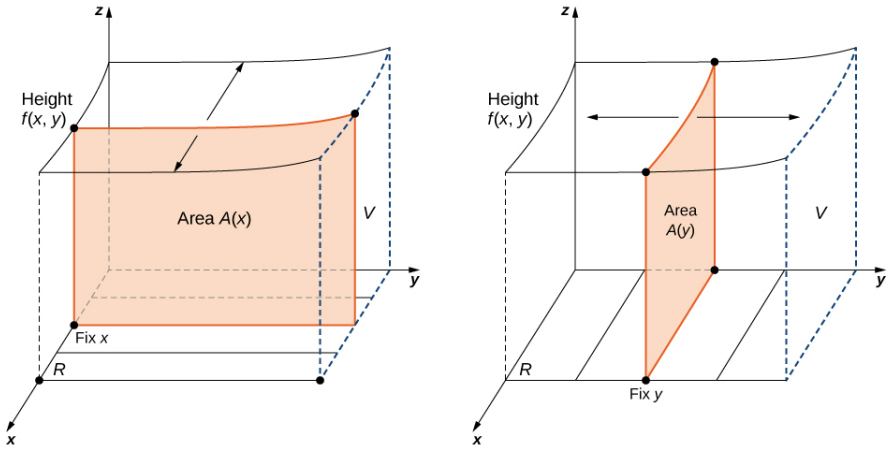


Figure 2.2.6: (a) Integrating first with respect to y and then with respect to x to find the area $A(x)$ and then the volume V; (b) integrating first with respect to x and then with respect to y to find the area $A(y)$ and then the volume V.

Example 2.2.2: Using Fubini's Theorem

Use Fubini's theorem to compute the double integral $\iint_R f(x,y)\,dA$ where $f(x,y) = x$ and $R = [0,2] \times [0,1]$.

Solution

Fubini's theorem offers an easier way to evaluate the double integral by the use of an iterated integral. Note how the boundary values of the region R become the upper and lower limits of integration.

$$\begin{aligned}\iint_R f(x,y)\,dA &= \iint_R f(x,y)\,dx\,dy \\ &= \int_{y=0}^{y=1}\int_{x=0}^{x=2} x\,dx\,dy \\ &= \int_{y=0}^{y=1}\left[\frac{x^2}{2}\Big|_{x=0}^{x=2}\right] dy \\ &= \int_{y=0}^{y=1} 2\,dy = 2y\Big|_{y=0}^{y=1} = 2\end{aligned}$$

The double integration in this example is simple enough to use Fubini's theorem directly, allowing us to convert a double integral into an iterated integral. Consequently, we are now ready to convert all double integrals to iterated integrals and demonstrate how the properties listed earlier can help us evaluate double integrals when the function $f(x,y)$ is more complex. Note that the order of integration can be changed (see Example 7).

Example 2.2.3: Illustrating Properties i and ii

Evaluate the double integral

$$\iint_R (xy - 3xy^2)\,dA, \text{ where } R = \{(x,y) \mid 0 \le x \le 2,\ 1 \le y \le 2\}.$$

Solution

This function has two pieces: one piece is xy and the other is $3xy^2$. Also, the second piece has a constant 3. Notice how we use properties i and ii to help evaluate the double integral.

$$\iint_R (xy - 3xy^2)\,dA = \iint_R xy\,dA + \iint_R (-3xy^2)\,dA$$ Property i: Integral of a sum is the sum of the integrals.

$$= \int_{y=1}^{y=2}\int_{x=0}^{x=2} xy\,dx\,dy - \int_{y=1}^{y=2}\int_{x=0}^{x=2} 3xy^2\,dx\,dy$$ Convert double integrals to iterated integrals.

$$= \int_{y=1}^{y=2}\left(\frac{x^2}{2}y\right)\Big|_{x=0}^{x=2} dy - 3\int_{y=1}^{y=2}\left(\frac{x^2}{2}y^2\right)\Big|_{x=0}^{x=2} dy$$ Integrate with respect to x, holding y constant.

$$= \int_{y=1}^{y=2} 2y\,dy - \int_{y=1}^{y=2} 6y^2 dy$$ Property ii: Placing the constant before the integral.

$$= 2\int_1^2 y\,dy - 6\int_1^2 y^2\,dy$$ Integrate with respect to y.

$$= 2\frac{y^2}{2}\Big|_1^2 - 6\frac{y^3}{3}\Big|_1^2$$

$$= y^2\Big|_1^2 - 2y^3\Big|_1^2$$

$$= (4-1) - 2(8-1) = 3 - 2(7) = 3 - 14 = -11.$$

Example 2.2.4: Illustrating Property v.

Over the region $R = \{(x,y) \mid 1 \le x \le 3,\ 1 \le y \le 2\}$, we have $2 \le x^2 + y^2 \le 13$. Find a lower and an upper bound for the integral $\iint_R (x^2 + y^2)\,dA$.

Solution

For a lower bound, integrate the constant function 2 over the region R. For an upper bound, integrate the constant function 13 over the region R.

$$\int_1^2\int_1^3 2\,dx\,dy = \int_1^2 [2x\Big|_1^3]\,dy = \int_1^2 2(2)dy = 4y\Big|_1^2 = 4(2-1) = 4$$

$$\int_1^2\int_1^3 13dx\,dy = \int_1^2 [13x\Big|_1^3]\,dy = \int_1^2 13(2)\,dy = 26y\Big|_1^2 = 26(2-1) = 26.$$

Hence, we obtain $4 \le \iint_R (x^2 + y^2)\,dA \le 26$.

Example 2.2.5: Illustrating Property vi

Evaluate the integral $\iint_R e^y \cos x\,dA$ over the region $R = \{(x,y) \mid 0 \le x \le \frac{\pi}{2},\ 0 \le y \le 1\}$.

Solution

This is a great example for property vi because the function $f(x,y)$ is clearly the product of two single-variable functions e^y and $\cos x$. Thus we can split the integral into two parts and then integrate each one as a single-variable integration problem.

$$\iint_R e^y \cos x \, dA = \int_0^1 \int_0^{\pi/2} e^y \cos x \, dx \, dy$$
$$= \left(\int_0^1 e^y dy \right) \left(\int_0^{\pi/2} \cos x \, dx \right)$$
$$= (e^y \Big|_0^1)(\sin x \Big|_0^{\pi/2})$$
$$= e - 1.$$

? Exercise 2.2.2

a. Use the properties of the double integral and Fubini's theorem to evaluate the integral

$$\int_0^1 \int_{-1}^3 (3 - x + 4y) \, dy \, dx.$$

b. Show that $0 \leq \iint_R \sin \pi x \, \cos \pi y \, dA \leq \frac{1}{32}$ where $R = \left(0, \frac{1}{4}\right) \left(\frac{1}{4}, \frac{1}{2}\right)$.

Hint

Use properties i. and ii. and evaluate the iterated integral, and then use property v.

Answer

a. **26**

b. Answers may vary.

As we mentioned before, when we are using rectangular coordinates, the double integral over a region R denoted by $\iint_R f(x,y) \, dA$ can be written as $\iint_R f(x,y) \, dx \, dy$ or $\iint_R f(x,y) \, dy \, dx$. The next example shows that the results are the same regardless of which order of integration we choose.

✓ Example 2.2.6: Evaluating an Iterated Integral in Two Ways

Let's return to the function $f(x,y) = 3x^2 - y$ from Example 1, this time over the rectangular region $R = [0,2] \times [0,3]$. Use Fubini's theorem to evaluate $\iint_R f(x,y) \, dA$ in two different ways:

a. First integrate with respect to y and then with respect to x;
b. First integrate with respect to x and then with respect to y.

Solution

Figure 2.2.6 shows how the calculation works in two different ways.

a. First integrate with respect to y and then integrate with respect to x:

$$\iint_R f(x,y) \, dA = \int_{x=0}^{x=2} \int_{y=0}^{y=3} (3x^2 - y) \, dy \, dx$$
$$= \int_{x=0}^{x=2} \left(\int_{y=0}^{y=3} (3x^2 - y) \, dy \right) dx = \int_{x=0}^{x=2} \left[3x^2 y - \frac{y^2}{2} \Big|_{y=0}^{y=3} \right] dx$$
$$= \int_{x=0}^{x=2} \left(9x^2 - \frac{9}{2} \right) dx = 3x^3 - \frac{9}{2} x \Big|_{x=0}^{x=2} = 15.$$

b. First integrate with respect to x and then integrate with respect to y:

$$\begin{aligned}\iint_R f(x,y)\,dA &= \int_{y=0}^{y=3}\int_{x=0}^{x=2}(3x^2-y)\,dx\,dy \\ &= \int_{y=0}^{y=3}\left(\int_{x=0}^{x=2}(3x^2-y)\,dx\right)dy \\ &= \int_{y=0}^{y=3}\left[x^3-xy\Big|_{x=0}^{x=2}\right]dy \\ &= \int_{y=0}^{y=3}(8-2y)\,dy = 8y-y^2\Big|_{y=0}^{y=3} = 15.\end{aligned}$$

Analysis

With either order of integration, the double integral gives us an answer of 15. We might wish to interpret this answer as a volume in cubic units of the solid S below the function $f(x,y)=3x^2-y$ over the region $R=[0,2]\times[0,3]$. However, remember that the interpretation of a double integral as a (non-signed) volume works only when the integrand f is a nonnegative function over the base region R.

? Exercise 2.2.3

Evaluate

$$\int_{y=-3}^{y=2}\int_{x=3}^{x=5}(2-3x^2+y^2)\,dx\,dy.$$

Hint

Use Fubini's theorem.

Answer

$-\frac{1340}{3}$

In the next example we see that it can actually be beneficial to switch the order of integration to make the computation easier. We will come back to this idea several times in this chapter.

✓ Example 2.2.7: Switching the Order of Integration

Consider the double integral $\iint_R x\sin(xy)\,dA$ over the region $R=\{(x,y)\,|\,0\le x\le\pi,\ 1\le y\le 2\}$ (Figure 2.2.7).

a. Express the double integral in two different ways.
b. Analyze whether evaluating the double integral in one way is easier than the other and why.
c. Evaluate the integral.

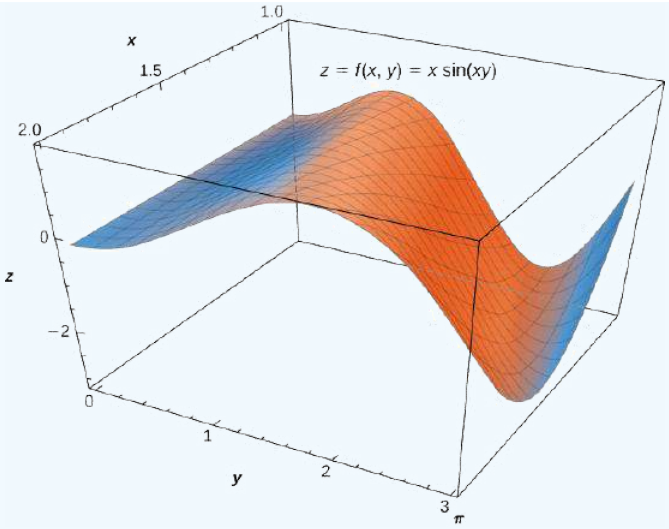


Figure **2.2.7**: The function $z = f(x, y) = x \sin(xy)$ over the rectangular region $R = [0, \pi] \times [1, 2]$.

a. We can express $\iint_R x \sin(xy)\, dA$ in the following two ways: first by integrating with respect to y and then with respect to x; second by integrating with respect to x and then with respect to y.

$$\iint_R x \sin(xy)\, dA = \int_{x=0}^{x=\pi} \int_{y=1}^{y=2} x \sin(xy)\, dy\, dx$$

Integrate first with respect to y.

$$= \int_{y=1}^{y=2} \int_{x=0}^{x=\pi} x \sin(xy)\, dx\, dy$$

Integrate first with respect to x.

b. If we want to integrate with respect to y first and then integrate with respect to x, we see that we can use the substitution $u = xy$, which gives $du = x\, dy$. Hence the inner integral is simply $\int \sin u\, du$ and we can change the limits to be functions of x,

$$\iint_R x \sin(xy)\, dA = \int_{x=0}^{x=\pi} \int_{y=1}^{y=2} x \sin(xy)\, dy\, dx = \int_{x=0}^{x=\pi} \left[\int_{u=x}^{u=2x} \sin(u)\, du \right] dx.$$

However, integrating with respect to x first and then integrating with respect to y requires integration by parts for the inner integral, with $u = x$ and $dv = \sin(xy)dx$

Then $du = dx$ and $v = -\frac{\cos(xy)}{y}$, so

$$\iint_R x \sin(xy)\, dA = \int_{y=1}^{y=2} \int_{x=0}^{x=\pi} x \sin(xy)\, dx\, dy$$
$$= \int_{y=1}^{y=2} \left[-\frac{x \cos(xy)}{y} \Big|_{x=0}^{x=\pi} + \frac{1}{y} \int_{x=0}^{x=\pi} \cos(xy)\, dx \right] dy.$$

Since the evaluation is getting complicated, we will only do the computation that is easier to do, which is clearly the first method.

c. Evaluate the double integral using the easier way.

$$\iint_R x\sin(xy)\,dA = \int_{x=0}^{x=\pi}\int_{y=1}^{y=2} x\sin(xy)\,dy\,dx$$
$$= \int_{x=0}^{x=\pi}\left[\int_{u=x}^{u=2x}\sin(u)\,du\right]dx = \int_{x=0}^{x=\pi}\left[-\cos u\Big|_{u=x}^{u=2x}\right]dx$$
$$= \int_{x=0}^{x=\pi}(-\cos 2x+\cos x)\,dx$$
$$= \left(-\frac{1}{2}\sin 2x+\sin x\right)\Big|_{x=0}^{x=\pi} = 0.$$

? Exercise 2.2.4

Evaluate the integral $\iint_R xe^{xy}\,dA$ where $R=[0,1]\times[0,\ln 5]$.

Hint

Integrate with respect to y first.

Answer

$\frac{4-\ln 5}{\ln 5}$

Applications of Double Integrals

Double integrals are very useful for finding the area of a region bounded by curves of functions. We describe this situation in more detail in the next section. However, if the region is a rectangular shape, we can find its area by integrating the constant function $f(x,y)=1$ over the region R.

Definition: area of the region

The area of the region R is given by

$$A(R)=\iint_R 1\,dA. \tag{2.2.5}$$

This definition makes sense because using $f(x,y)=1$ and evaluating the integral make it a product of length and width. Let's check this formula with an example and see how this works.

✓ Example 2.2.8: Finding Area Using a Double Integral

Find the area of the region $R=\{(x,y)\,|\,0\le x\le 3,\ 0\le y\le 2\}$ by using a double integral, that is, by integrating 1 over the region R.

Solution

The region is rectangular with length 3 and width 2, so we know that the area is 6. We get the same answer when we use a double integral:

$$A(R)=\int_0^2\int_0^3 1\,dx\,dy=\int_0^2\left[x\big|_0^3\right]dy=\int_0^2 3dy=3\int_0^2 dy=3y\Big|_0^2=3(2)=6\,\text{units}^2.$$

We have already seen how double integrals can be used to find the volume of a solid bounded above by a function $f(x,y)\ge 0$ over a region R provided $f(x,y)\ge 0$ for all (x,y) in R. Here is another example to illustrate this concept.

Example 2.2.9: Volume of an Elliptic Paraboloid

Find the volume V of the solid S that is bounded by the elliptic paraboloid $2x^2 + y^2 + z = 27$, the planes $x = 3$ and $y = 3$, and the three coordinate planes.

Solution

First notice the graph of the surface $z = 27 - 2x^2 - y^2$ in Figure 2.2.8(a) and above the square region $R_1 = [-3, 3] \times [-3, 3]$. However, we need the volume of the solid bounded by the elliptic paraboloid $2x^2 + y^2 + z = 27$, the planes $x = 3$ and $y = 3$, and the three coordinate planes.

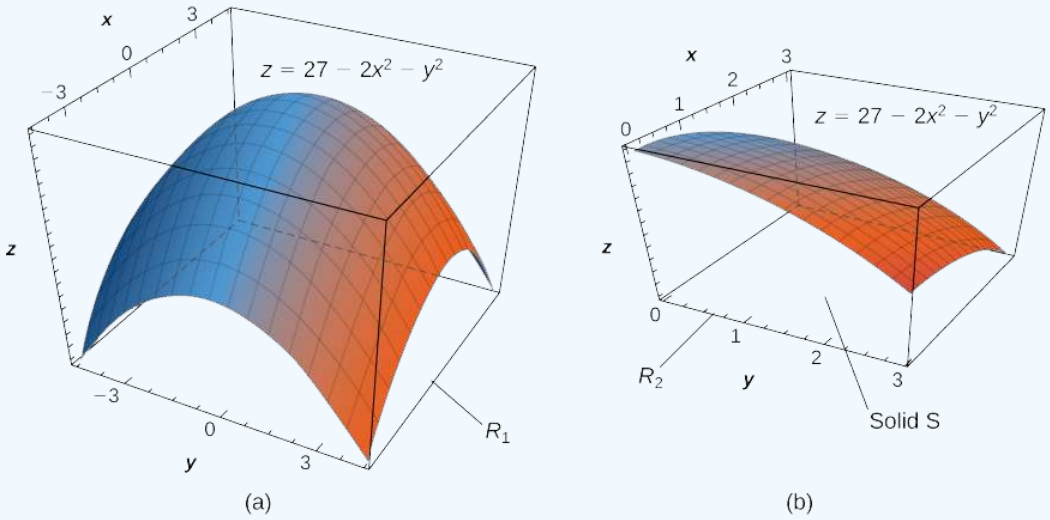


Figure 2.2.8: (a) The surface $z = 27 - 2x^2 - y^2$ above the square region $R1 = [-3, 3] \times [-3, 3]$. (b) The solid S lies under the surface $z = 27 - 2x^2 - y^2$ above the square region $R1 = [0, 3] \times [0, 3]$.

Now let's look at the graph of the surface in Figure 2.2.8(b). We determine the volume V by evaluating the double integral over R_2:

$$\begin{aligned} V &= \iint_R z\,dA = \iint_R (27 - 2x^2 - y^2)\,dA \\ &= \int_{y=0}^{y=3} \int_{x=0}^{x=3} (27 - 2x^2 - y^2)\,dx\,dy && \text{Convert to literal integral.} \\ &= \int_{y=0}^{y=3} \left[27x - \frac{2}{3}x^3 - y^2 x\right]\Big|_{x=0}^{x=3} dy && \text{Integrate with respect to } x. \\ &= \int_{y=0}^{y=3} (63 - 3y^2)dy = 63y - y^3\Big|_{y=0}^{y=3} = 162. \end{aligned}$$

Exercise 2.2.5

Find the volume of the solid bounded above by the graph of $f(x, y) = xy\sin(x^2 y)$ and below by the xy-plane on the rectangular region $R = [0, 1] \times [0, \pi]$.

Hint

Graph the function, set up the integral, and use an iterated integral.

Answer

$\frac{\pi}{2}$

Recall that we defined the average value of a function of one variable on an interval $[a, b]$ as

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx. \tag{2.2.6}$$

Similarly, we can define the average value of a function of two variables over a region R. The main difference is that we divide by an area instead of the width of an interval.

Definition: AVERAGE VALUE OF A FUNCTION

The **average value of a function** of two variables over a region R is

$$F_{ave} = \frac{1}{\text{Area of } R}\iint_R f(x,y)\,dx\,dy. \tag{2.2.7}$$

In the next example we find the average value of a function over a rectangular region. This is a good example of obtaining useful information for an integration by making individual measurements over a grid, instead of trying to find an algebraic expression for a function.

Example 2.2.10: Calculating Average Storm Rainfall

The weather map in Figure 2.2.9 shows an unusually moist storm system associated with the remnants of Hurricane Karl, which dumped 4–8 inches (100–200 mm) of rain in some parts of the Midwest on September 22–23, 2010. The area of rainfall measured 300 miles east to west and 250 miles north to south. Estimate the average rainfall over the entire area in those two days.

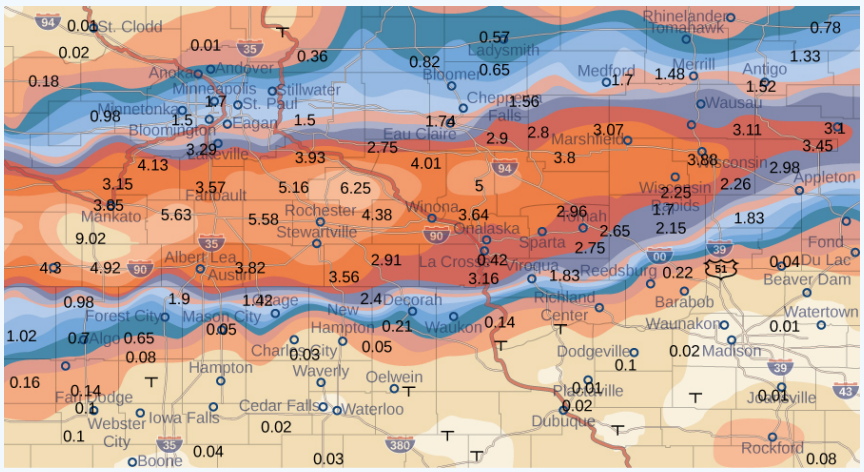


Figure 2.2.9: Effects of Hurricane Karl, which dumped 4–8 inches (100–200 mm) of rain in some parts of southwest Wisconsin, southern Minnesota, and southeast South Dakota over a span of 300 miles east to west and 250 miles north to south.

Solution

Place the origin at the southwest corner of the map so that all the values can be considered as being in the first quadrant and hence all are positive. Now divide the entire map into six rectangles ($m = 2$ and $n = 3$), as shown in Figure 2.2.9. Assume $f(x,y)$ denotes the storm rainfall in inches at a point approximately x miles to the east of the origin and y miles to the north of the origin. Let R represent the entire area of $250 \times 300 = 75000$ square miles. Then the area of each subrectangle is

$$\Delta A = \frac{1}{6}(75000) = 12500.$$

Assume $(x_{ij}*, y_{ij}*)$ are approximately the midpoints of each subrectangle R_{ij}. Note the color-coded region at each of these points, and estimate the rainfall. The rainfall at each of these points can be estimated as:

- At (x_{11}, y_{11}), the rainfall is 0.08.
- At (x_{12}, y_{12}), the rainfall is 0.08.
- At (x_{13}, y_{13}), the rainfall is 0.01.
- At (x_{21}, y_{21}), the rainfall is 1.70.

- At (x_{22}, y_{22}), the rainfall is 1.74.
- At (x_{23}, y_{23}), the rainfall is 3.00.

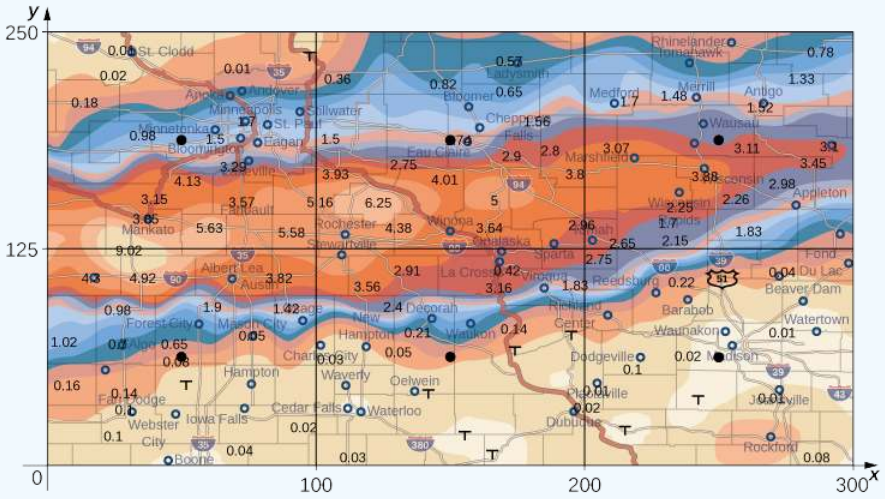


Figure **2.2.10**: Storm rainfall with rectangular axes and showing the midpoints of each subrectangle.

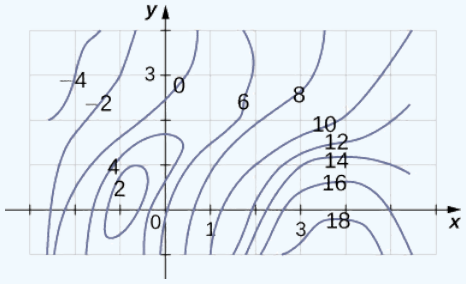
According to our definition, the average storm rainfall in the entire area during those two days was

$$
\begin{aligned}
f_{ave} = \frac{1}{Area\ R}\iint_R = f(x,y)\,dx\,dy &= \frac{1}{75000}\iint_R f(x,y)\,dx\,dy \\
&\approx \frac{1}{75000}\sum_{i=1}^{3}\sum_{j=1}^{2} f(x_{ij}^*, y_{ij}^*)\Delta A \\
&= \frac{1}{75000} \\
&\quad [f(x_{11}^*, y_{11}^*)\Delta A + f(x_{12}^*, y_{12}^*)\Delta A + f(x_{13}^*, y_{13}^*)\Delta A + f(x_{21}^*, y_{21}^*)\Delta A \\
&\quad + f(x_{22}^*, y_{22}^*)\Delta A + f(x_{23}^*, y_{23}^*)\Delta A] \\
&\approx \frac{1}{75000}[0.08 + 0.08 + 0.01 + 1.70 + 1.74 + 3.00]\Delta A \\
&= \frac{1}{75000}[0.08 + 0.08 + 0.01 + 1.70 + 1.74 + 3.00]12500 \\
&= \frac{1}{6}[0.08 + 0.08 + 0.01 + 1.70 + 1.74 + 3.00] \\
&\approx 1.10 \text{ in.}
\end{aligned}
$$

During September 22–23, 2010 this area had an average storm rainfall of approximately 1.10 inches.

? Exercise 2.2.6

A contour map is shown for a function $f(x,y)$ on the rectangle $R = [-3,6] \times [-1,4]$.



a. Use the midpoint rule with $m = 3$ and $n = 2$ to estimate the value of $\iint_R f(x,y)\,dA$.

b. Estimate the average value of the function $f(x,y)$.

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Hint

Divide the region into six rectangles, and use the contour lines to estimate the values for $f(x,y)$.

Answer

Answers to both parts a. and b. may vary.

Key Concepts

- We can use a double Riemann sum to approximate the volume of a solid bounded above by a function of two variables over a rectangular region. By taking the limit, this becomes a double integral representing the volume of the solid.
- Properties of double integral are useful to simplify computation and find bounds on their values.
- We can use Fubini's theorem to write and evaluate a double integral as an iterated integral.
- Double integrals are used to calculate the area of a region, the volume under a surface, and the average value of a function of two variables over a rectangular region.

Key Equations

- $$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i j*, y_i j*)\,\Delta A$$
- $$\int_a^b \int_c^d f(x,y)\,dx\,dy = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx$$

  or

  $$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy$$
- $$f_{ave} = \frac{1}{\text{Area of } R}\iint_R f(x,y)\,dx\,dy$$

Glossary

double integral

of the function $f(x,y)$ over the region R in the xy-plane is defined as the limit of a double Riemann sum,

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A.$$

double Riemann sum

of the function $f(x,y)$ over a rectangular region R is

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A,$$

where R is divided into smaller subrectangles R_{ij} and (x_{ij}^*, y_{ij}^*) is an arbitrary point in R_{ij}

Fubini's theorem

if $f(x,y)$ is a function of two variables that is continuous over a rectangular region $R = \{(x,y) \in \mathbb{R}^2 \,|\, a \le x \le b,\, c \le y \le d\}$, then the double integral of f over the region equals an iterated integral,

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dx\,dy = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

iterated integral

for a function $f(x,y)$ over the region R is

a. $\int_a^b \int_c^d f(x,y)\,dx\,dy = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx,$

b. $\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy,$

where a, b, c, and d are any real numbers and $R = [a,b] \times [c,d]$

Contributors and Attributions

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2.2E: Exercises

In exercises 1 and 2, use the midpoint rule with $m = 4$ and $n = 2$ to estimate the volume of the solid bounded by the surface $z = f(x, y)$, the vertical planes $x = 1$, $x = 2$, $y = 1$, and $y = 2$, and the horizontal plane $z = 0$.

1) $f(x, y) = 4x + 2y + 8xy$

Answer

27

2) $f(x, y) = 16x^2 + \frac{y}{2}$

In exercises 3 and 4, estimate the volume of the solid under the surface $z = f(x, y)$ and above the rectangular region R by using a Riemann sum with $m = n = 2$ and the sample points to be the lower left corners of the sub-rectangles of the partition.

3) $f(x, y) = \sin x - \cos y$, $R = [0, \pi] \times [0, \pi]$

Answer

0

4) $f(x, y) = \cos x + \cos y$, $R = [0, \pi] \times [0, \frac{\pi}{2}]$

5) Use the midpoint rule with $m = n = 2$ to estimate $\iint_R f(x, y)\, dA$, where the values of the function f on $R = [8, 10] \times [9, 11]$ are given in the following table.

	y				
x	9	9.5	10	10.5	11
8	9.8	5	6.7	5	5.6
8.5	9.4	4.5	8	5.4	3.4
9	8.7	4.6	6	5.5	3.4
9.5	6.7	6	4.5	5.4	6.7
10	6.8	6.4	5.5	5.7	6.8

Answer

21.3

6) The values of the function f on the rectangle $R = [0, 2] \times [7, 9]$ are given in the following table. Estimate the double integral $\iint_R f(x, y)\, dA$ by using a Riemann sum with $m = n = 2$. Select the sample points to be the upper right corners of the sub-squares of R.

	$y_0 = 7$	$y_1 = 8$	$y_2 = 9$
$x_0 = 0$	10.22	10.21	9.85
$x_1 = 1$	6.73	9.75	9.63
$x_2 = 2$	5.62	7.83	8.21

7) The depth of a children's 4-ft by 4-ft swimming pool, measured at 1-ft intervals, is given in the following table.

a. Estimate the volume of water in the swimming pool by using a Riemann sum with $m = n = 2$. Select the sample points using the midpoint rule on $R = [0, 4] \times [0, 4]$.
b. Find the average depth of the swimming pool.

	y				
x	0	1	2	3	4
0	1	1.5	2	2.5	3
1	1	1.5	2	2.5	3
2	1	1.5	1.5	2.5	3
3	1	1	1.5	2	2.5
4	1	1	1	1.5	2

Answer

a. 28 ft^3

b. 1.75 ft.

8) The depth of a 3-ft by 3-ft hole in the ground, measured at 1-ft intervals, is given in the following table.

a. Estimate the volume of the hole by using a Riemann sum with $m = n = 3$ and the sample points to be the upper left corners of the sub-squares of R.

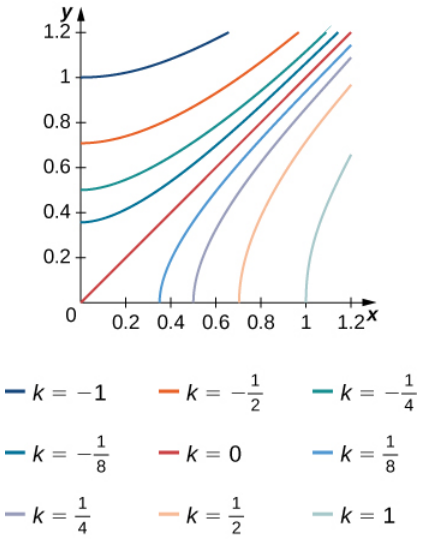
b. Find the average depth of the hole.

	y			
x	0	1	2	3
0	6	6.5	6.4	6
1	6.5	7	7.5	6.5
2	6.5	6.7	6.5	6
3	6	6.5	5	5.6

9) The level curves $f(x, y) = k$ of the function f are given in the following graph, where k is a constant.

a. Apply the midpoint rule with $m = n = 2$ to estimate the double integral $\iint_R f(x, y)\, dA$, where $R = [0.2, 1] \times [0, 0.8]$.

b. Estimate the average value of the function f on R.



Answer

a. Answers will vary somewhat, but should be close to these: $f(0.4, 0.2) \approx 0.125$, $f(0.4, 0.6) \approx -0.2$, $f(0.8, 0.2) \approx 0.6$, and $f(0.8, 0.6) \approx 0.3$.

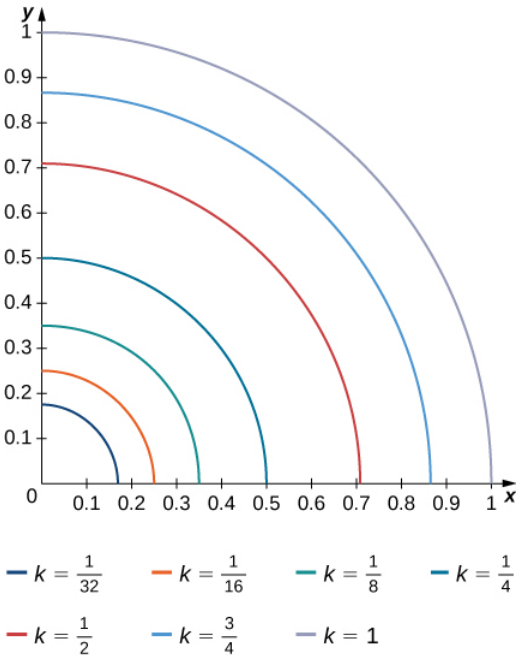
Since the area of each rectangle is $\Delta A_i = (0.4)(0.4) = 0.16 \text{ units}^2$, $\iint_R f(x, y)\, dA = \int_{0.2}^{1} \int_{0}^{0.8} f(x, y)\, dy\, dx \approx f(0.4, 0.2)(0.16) + f(0.4, 0.6)(0.16) + f(0.8, 0.2)(0.16) + f(0.8, 0.6)(0.16) = 0.132$

b. $f_{ave} = \dfrac{\int_{0.2}^{1} \int_{0}^{0.8} f(x, y)\, dy\, dx}{\text{area of } R} \approx \dfrac{\int_{0.2}^{1} \int_{0}^{0.8} f(x, y)\, dy\, dx}{0.64} \approx 0.206$

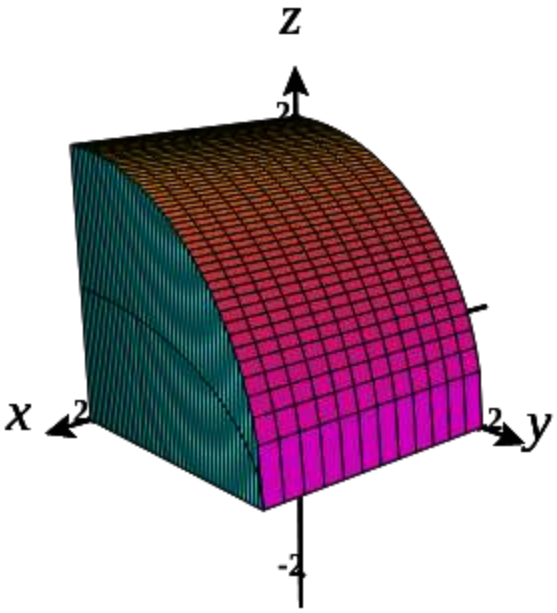
10) The level curves $f(x, y) = k$ of the function f are given in the following graph, where k is a constant.

a. Apply the midpoint rule with $m = n = 2$ to estimate the double integral $\iint_R f(x, y)\, dA$, where $R = [0.1, 0.5] \times [0.1, 0.5]$.

b. Estimate the average value of the function f on R.



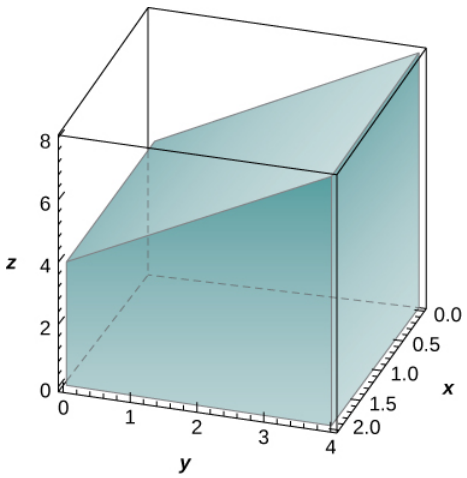
11) The solid lying under the surface $z = \sqrt{4 - y^2}$ and above the rectangular region$R = [0, 2] \times [0, 2]$ is illustrated in the following graph. Evaluate the double integral $\iint_R f(x, y)$, where $f(x, y) = \sqrt{4 - y^2}$ by finding the volume of the corresponding solid.



Answer

2π

12) The solid lying under the plane $z = y + 4$ and above the rectangular region $R = [0, 2] \times [0, 4]$ is illustrated in the following graph. Evaluate the double integral $\iint_R f(x, y)\, dA$, where $f(x, y) = y + 4$, by finding the volume of the corresponding solid.



In the exercises 13 - 20, calculate the integrals by reversing the order of integration.

13) $\int_{-1}^{1} \left(\int_{-2}^{2} (2x + 3y + 5)\, dx \right)\, dy$

Answer

40

14) $\int_0^2 \left(\int_0^1 (x + 2e^y + 3)\, dx \right)\, dy$

15) $\int_1^{27} \left(\int_1^2 (\sqrt[3]{x} + \sqrt[3]{y})\, dy \right)\, dx$

Answer

$\frac{81}{2} + 39\sqrt[3]{2}$

16) $\int_1^{16} \left(\int_1^8 (\sqrt[4]{x} + 2\sqrt[3]{y})\, dy \right)\, dx$

17) $\int_{\ln 2}^{\ln 3} \left(\int_0^1 e^{x+y}\, dy \right)\, dx$

Answer

$e - 1$

18) $\int_0^2 \left(\int_0^1 3^{x+y}\, dy \right)\, dx$

19) $\int_1^6 \left(\int_2^9 \frac{\sqrt{y}}{x^2}\, dy \right)\, dx$

Answer

$15 - \frac{10\sqrt{2}}{9}$

20) $\int_1^9 \left(\int_4^2 \frac{\sqrt{x}}{y^2}\, dy \right)\, dx$

In exercises 21 - 34, evaluate the iterated integrals by choosing the order of integration.

21) $\int_0^{\pi} \int_0^{\pi/2} \sin(2x)\cos(3y)\, dx\, dy$

Answer

0

22) $\int_{\pi/12}^{\pi/8} \int_{\pi/4}^{\pi/3} [\cot x + \tan(2y)]\, dx\, dy$

23) $\int_1^e \int_1^e \left[\frac{1}{x}\sin(\ln x) + \frac{1}{y}\cos(\ln y) \right]\, dx\, dy$

Answer

$(e-1)(1 + \sin 1 - \cos 1)$

24) $\int_1^e \int_1^e \frac{\sin(\ln x)\cos(\ln y)}{xy}\, dx\, dy$

25) $\int_1^2 \int_1^2 \left(\frac{\ln y}{x} + \frac{x}{2y+1} \right)\, dy\, dx$

Answer

$\frac{3}{4}\ln\left(\frac{5}{3}\right) + 2(\ln 2)^2 - \ln 2$

26) $\int_1^e \int_1^2 x^2 \ln(x)\, dy\, dx$

27) $\int_1^{\sqrt{3}} \int_1^2 y \arctan\left(\frac{1}{x}\right)\, dy\, dx$

Answer

$\frac{1}{8}[(2\sqrt{3}-3)\pi+6\ \ln 2]$

28) $\int_0^1 \int_0^{1/2} (\arcsin x + \arcsin y)\, dy\, dx$

29) $\int_0^1 \int_1^2 xe^{x+4y}\, dy\, dx$

Answer

$\frac{1}{4}e^4(e^4-1)$

30) $\int_1^2 \int_0^1 xe^{x-y}\, dy\, dx$

31) $\int_1^e \int_1^e \left(\frac{\ln y}{\sqrt{y}} + \frac{\ln x}{\sqrt{x}}\right) dy\, dx$

Answer

$4(e-1)(2-\sqrt{e})$

32) $\int_1^e \int_1^e \left(\frac{x\ \ln y}{\sqrt{y}} + \frac{y\ \ln x}{\sqrt{x}}\right) dy\, dx$

33) $\int_0^1 \int_1^2 \left(\frac{x}{x^2+y^2}\right) dy\, dx$

Answer

$-\frac{\pi}{4} + \ln\left(\frac{5}{4}\right) - \frac{1}{2}\ln 2 + \arctan 2$

34) $\int_0^1 \int_1^2 \frac{y}{x+y^2}\, dy\, dx$

In exercises 35 - 38, find the average value of the function over the given rectangles.

35) $f(x,y) = -x+2y$, $R = [0,1]\times[0,1]$

Answer

$\frac{1}{2}$

36) $f(x,y) = x^4+2y^3$, $R = [1,2]\times[2,3]$

37) $f(x,y) = \sinh x + \sinh y$, $R = [0,1]\times[0,2]$

Answer

$\frac{1}{2}(2\ \cosh 1 + \cosh 2 - 3)$.

38) $f(x,y) = \arctan(xy)$, $R = [0,1]\times[0,1]$

39) Let f and g be two continuous functions such that $0 \le m_1 \le f(x) \le M_1$ for any $x \in [a,b]$ and $0 \le m_2 \le g(y) \le M_2$ for any $y \in [c,d]$. Show that the following inequality is true:

$$m_1 m_2 (b-a)(c-d) \le \int_a^b \int_c^d f(x)g(y)\, dydx \le M_1 M_2 (b-a)(c-d).$$

In exercises 40 - 43, use property v. of double integrals and the answer from the preceding exercise to show that the following inequalities are true.

40) $\frac{1}{e^2} \le \iint_R e^{-x^2-y^2}\, dA \le 1$, where $R = [0,1]\times[0,1]$

41) $\frac{\pi^2}{144} \le \iint_R \sin x \cos y\, dA \le \frac{\pi^2}{48}$, where $R = [\frac{\pi}{6}, \frac{\pi}{3}]\times[\frac{\pi}{6}, \frac{\pi}{3}]$

42) $0 \le \iint_R e^{-y}\ \cos x\, dA \le \frac{\pi}{2}$, where $R = [0, \frac{\pi}{2}]\times[0, \frac{\pi}{2}]$

43) $0 \le \iint_R (\ln x)(\ln y)\, dA \le (e-1)^2$, where $R = [1, e] \times [1, e]$

44) Let f and g be two continuous functions such that $0 \le m_1 \le f(x) \le M_1$ for any $x \in [a, b]$ and $0 \le m_2 \le g(y) \le M_2$ for any $y \in [c, d]$. Show that the following inequality is true:

$$(m_1 + m_2)(b-a)(c-d) \le \int_a^b \int_c^d |f(x) + g(y)|\ dy\, dx \le (M_1 + M_2)(b-a)(c-d)$$

In exercises 45 - 48, use property v. of double integrals and the answer from the preceding exercise to show that the following inequalities are true.

45) $\frac{2}{e} \le \iint_R (e^{-x^2} + e^{-y^2})\, dA \le 2$, where $R = [0, 1] \times [0, 1]$

46) $\frac{\pi^2}{36} \iint_R (\sin x + \cos y)\, dA \le \frac{\pi^2\sqrt{3}}{36}$, where $R = [\frac{\pi}{6}, \frac{\pi}{3}] \times [\frac{\pi}{6}, \frac{\pi}{3}]$

47) $\frac{\pi}{2} e^{-\pi/2} \le \iint_R (\cos x + e^{-y})\, dA \le \pi$, where $R = [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]$

48) $\frac{1}{e} \le \iint_R (e^{-y} - \ln x)\, dA \le 2$, where $R = [0, 1] \times [0, 1]$

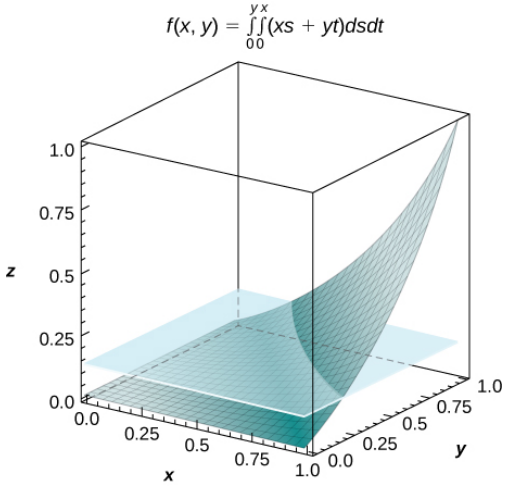
In exercises 49 - 50, the function f is given in terms of double integrals.

a. Determine the explicit form of the function f.
b. Find the volume of the solid under the surface $z = f(x, y)$ and above the region R.
c. Find the average value of the function f on R.
d. Use a computer algebra system (CAS) to plot $z = f(x, y)$ and $z = f_{ave}$ in the same system of coordinates.

49) [T] $f(x, y) = \int_0^y \int_0^x (xs + yt) ds\, dt$, where $(x, y) \in R = [0, 1] \times [0, 1]$

Answer

a. $f(x, y) = \frac{1}{2} xy(x^2 + y^2)$;
b. $V = \int_0^1 \int_0^1 f(x, y)\, dx\, dy = \frac{1}{8}$;
c. $f_{ave} = \frac{1}{8}$;
d.



50) [T] $f(x, y) = \int_0^x \int_0^y [\cos(s) + \cos(t)]\, dt\, ds$, where $(x, y) \in R = [0, 3] \times [0, 3]$

51) Show that if f and g are continuous on $[a, b]$ and $[c, d]$, respectively, then

$$\int_a^b \int_c^d |f(x) + g(y)|\, dy\, dx = (d-c) \int_a^b f(x)\, dx$$
$$+ \int_a^b \int_c^d g(y)\, dy\, dx = (b-a) \int_c^d g(y)\, dy + \int_c^d \int_a^b f(x)\, dx\, dy.$$

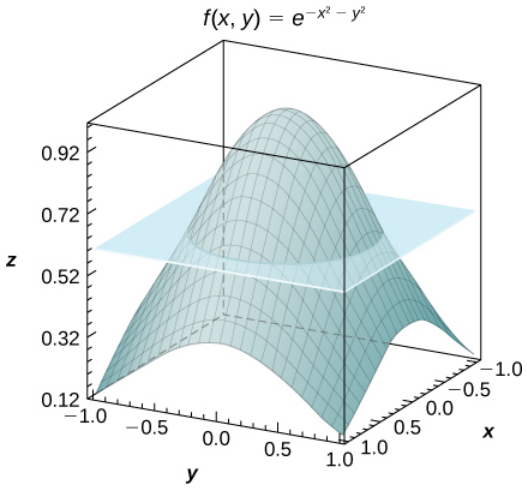
52) Show that $\int_a^b \int_c^d yf(x) + xg(y)\, dy\, dx = \frac{1}{2}(d^2 - c^2)\left(\int_a^b f(x)\, dx\right) + \frac{1}{2}(b^2 - a^2)\left(\int_c^d g(y)\, dy\right)$.

53) [T] Consider the function $f(x, y) = e^{-x^2-y^2}$, where $(x, y) \in R = [-1, 1] \times [-1, 1]$.

a. Use the midpoint rule with $m = n = 2, 4, ..., 10$ to estimate the double integral $I = \iint_R e^{-x^2-y^2} dA$. Round your answers to the nearest hundredths.
b. For $m = n = 2$, find the average value of f over the region R. Round your answer to the nearest hundredths.
c. Use a CAS to graph in the same coordinate system the solid whose volume is given by $\iint_R e^{-x^2-y^2} dA$ and the plane $z = f_{ave}$.

Answer

a. For $m = n = 2$, $I = 4e^{-0.5} \approx 2.43$
b. $f_{ave} = e^{-0.5} \simeq 0.61$;
c.



54) [T] Consider the function $f(x, y) = \sin(x^2)\ \cos(y^2)$, where $(x, y \in R = [-1, 1] \times [-1, 1]$.

a. Use the midpoint rule with $m = n = 2, 4, ..., 10$ to estimate the double integral $I = \iint_R \sin(x^2) \cos(y^2)\ dA$. Round your answers to the nearest hundredths.
b. For $m = n = 2$, find the average value of $f$over the region R. Round your answer to the nearest hundredths.
c. Use a CAS to graph in the same coordinate system the solid whose volume is given by $\iint_R \sin(x^2) \cos(y^2)\ dA$ and the plane $z = f_{ave}$.

In exercises 55 - 56, the functions f_n are given, where $n \geq 1$ is a natural number.

a. Find the volume of the solids S_n under the surfaces $z = f_n(x, y)$ and above the region R.
b. Determine the limit of the volumes of the solids S_n as n increases without bound.

55) $f(x, y) = x^n + y^n + xy,\ (x, y) \in R = [0, 1] \times [0, 1]$

Answer

a. $\frac{2}{n+1} + \frac{1}{4}$
b. $\frac{1}{4}$

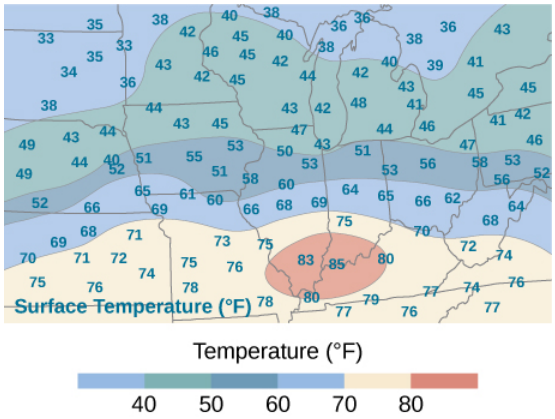
56) $f(x, y) = \frac{1}{x^n} + \frac{1}{y^n},\ (x, y) \in R = [1, 2] \times [1, 2]$

57) Show that the average value of a function f on a rectangular region $R = [a, b] \times [c, d]$ is $f_{ave} \approx \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)$,where (x_{ij}^*, y_{ij}^*) are the sample points of the partition of R, where $1 \leq i \leq m$ and $1 \leq j \leq n$.

58) Use the midpoint rule with $m = n$ to show that the average value of a function f on a rectangular region $R = [a, b] \times [c, d]$ is approximated by

$$f_{ave} \approx \frac{1}{n^2} \sum_{i,j=1}^{n} f\left(\frac{1}{2}(x_{i=1} + x_i),\ \frac{1}{2}(y_{j=1} + y_j)\right).$$

59) An isotherm map is a chart connecting points having the same temperature at a given time for a given period of time. Use the preceding exercise and apply the midpoint rule with $m = n = 2$ to find the average temperature over the region given in the following figure.



Answer

$56.5°$ F; here $f(x_1^*, y_1^*) = 71,\ f(x_2^*, y_1^*) = 72,\ f(x_2^*, y_1^*) = 40,\ f(x_2^*, y_2^*) = 43$, where x_i^* and y_j^* are the midpoints of the subintervals of the partitions of $[a, b]$ and $[c, d]$, respectively.

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2.3: Double Integrals over General Regions

Learning Objectives

- Recognize when a function of two variables is integrable over a general region.
- Evaluate a double integral by computing an iterated integral over a region bounded by two vertical lines and two functions of x, or two horizontal lines and two functions of y.
- Simplify the calculation of an iterated integral by changing the order of integration.
- Use double integrals to calculate the volume of a region between two surfaces or the area of a plane region.
- Solve problems involving double improper integrals.

Previously, we studied the concept of double integrals and examined the tools needed to compute them. We learned techniques and properties to integrate functions of two variables over rectangular regions. We also discussed several applications, such as finding the volume bounded above by a function over a rectangular region, finding area by integration, and calculating the average value of a function of two variables.

In this section we consider double integrals of functions defined over a general bounded region D on the plane. Most of the previous results hold in this situation as well, but some techniques need to be extended to cover this more general case.

General Regions of Integration

An example of a general bounded region D on a plane is shown in Figure **2.3.1**. Since D is bounded on the plane, there must exist a rectangular region R on the same plane that encloses the region D that is, a rectangular region R exists such that D is a subset of $R (D \subseteq R)$.

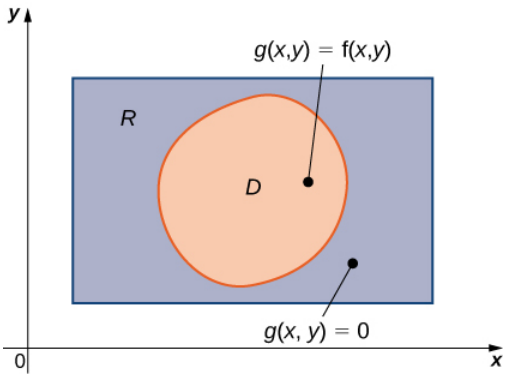


Figure **2.3.1**: For a region D that is a subset of R, we can define a function $g(x, y)$ *to equal* $f(x, y)$ *at every point in* D *and* 0 *at every point of* R *not in* D.

Suppose $z = f(x, y)$ is defined on a general planar bounded region D as in Figure **2.3.1**. In order to develop double integrals of f over D we extend the definition of the function to include all points on the rectangular region R and then use the concepts and tools from the preceding section. But how do we extend the definition of f to include all the points on R? We do this by defining a new function $g(x, y)$ on R as follows:

$$g(x,y) = \begin{cases} f(x,y) \text{if } (x,y) \text{ is in } D \\ 0 \text{if } (x,y) \text{ is in } R \text{ but not in } D \end{cases} \tag{2.3.1}$$

Note that we might have some technical difficulties if the boundary of D is complicated. So we assume the boundary to be a piecewise smooth and continuous simple closed curve. Also, since all the results developed in the section on Double Integrals over Rectangular Regions used an integrable function $f(x, y)$ we must be careful about $g(x, y)$ and verify that $g(x, y)$ is an integrable function over the rectangular region R. This happens as long as the region D is bounded by simple closed curves. For now we will concentrate on the descriptions of the regions rather than the function and extend our theory appropriately for integration.

We consider two types of planar bounded regions.

Definition: Type I and Type II regions

A region D in the (x, y)-plane is of Type I if it lies between two vertical lines and the graphs of two continuous functions $g_1(x)$ and $g_2(x)$. That is (Figure 2.3.2),

$$D = \{(x, y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}. \tag{2.3.2}$$

A region D in the xy-plane is of Type II if it lies between two horizontal lines and the graphs of two continuous functions $h_1(y)$ and $h_2(y)$. That is (Figure 2.3.3),

$$D = \{(x, y) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y)\}. \tag{2.3.3}$$

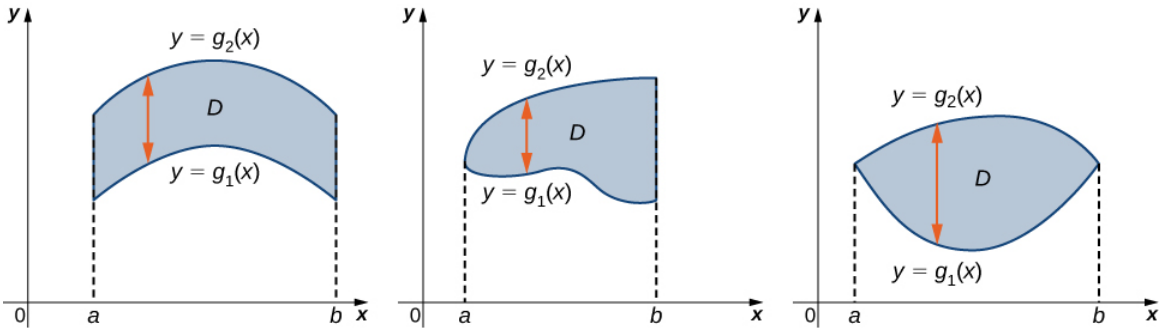


Figure 2.3.2. A Type I region lies between two vertical lines and the graphs of two functions of x.

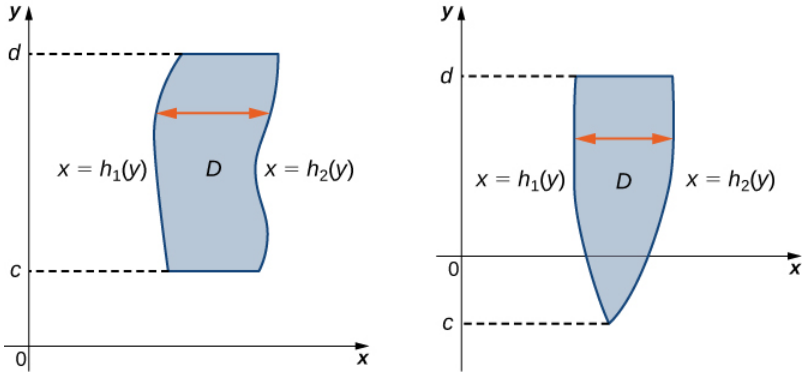


Figure 2.3.3: A Type II region lies between two horizontal lines and the graphs of two functions of y.

Example 2.3.1: Describing a Region as Type I and Also as Type II

Consider the region in the first quadrant between the functions $y = \sqrt{x}$ and $y = x^3$ (Figure 2.3.4). Describe the region first as Type I and then as Type II.

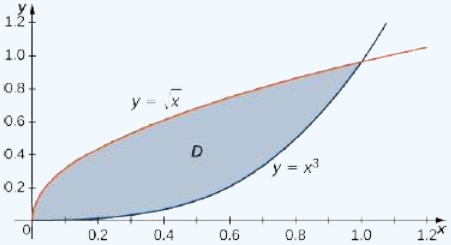


Figure 2.3.4: Region D can be described as Type I or as Type II.

When describing a region as Type I, we need to identify the function that lies above the region and the function that lies below the region. Here, region D is bounded above by $y = \sqrt{x}$ and below by $y = x^3$ in the interval for x in $[0, 1]$. Hence, as Type I, D is described as the set $\{(x, y) \mid 0 \le x \le 1,\ x^3 \le y \le \sqrt[3]{x}\}$.

However, when describing a region as Type II, we need to identify the function that lies on the left of the region and the function that lies on the right of the region. Here, the region D is bounded on the left by $x = y^2$ and on the right by $x = \sqrt[3]{y}$ in

the interval for y in $[0,1]$. Hence, as Type II, D is described as the set $\{(x,y)\,|\,0\le y\le 1,\ y^2\le x\le \sqrt[3]{y}\}$.

? Exercise 2.3.1

Consider the region in the first quadrant between the functions $y=2x$ and $y=x^2$. Describe the region first as Type I and then as Type II.

Hint

Graph the functions, and draw vertical and horizontal lines.

Answer

Type I and Type II are expressed as $\{(x,y)\,|\,0\le x\le 2,\ x^2\le y\le 2x\}$ and $\{(x,y)|\,0\le y\le 4,\ \frac{1}{2}y\le x\le \sqrt{y}\}$, respectively.

Double Integrals over Non-rectangular Regions

To develop the concept and tools for evaluation of a double integral over a general, nonrectangular region, we need to first understand the region and be able to express it as Type I or Type II or a combination of both. Without understanding the regions, we will not be able to decide the limits of integrations in double integrals. As a first step, let us look at the following theorem.

THeorem: Double Integrals over Nonrectangular Regions

Suppose $g(x,y)$ is the extension to the rectangle R of the function $f(x,y)$ defined on the regions D and R as shown in Figure 2.3.1 inside R. Then $g(x,y)$ is integrable and we define the double integral of $f(x,y)$ over D by

$$\iint_D f(x,y)\,dA=\iint_R g(x,y)\,dA. \tag{2.3.4}$$

The right-hand side of this equation is what we have seen before, so this theorem is reasonable because R is a rectangle and $\iint_R g(x,y)dA$ has been discussed in the preceding section. Also, the equality works because the values of $g(x,y)$ are 0 for any point (x,y) that lies outside D and hence these points do not add anything to the integral. However, it is important that the rectangle R contains the region D.

As a matter of fact, if the region D is bounded by smooth curves on a plane and we are able to describe it as Type I or Type II or a mix of both, then we can use the following theorem and not have to find a rectangle R containing the region.

Theorem: Fubini's Theorem (Strong Form)

For a function $f(x,y)$ that is continuous on a region D of Type I, we have

$$\iint_D f(x,y)\,dA=\iint_D f(x,y)\,dy\,dx=\int_a^b\left[\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\right]dx. \tag{2.3.5}$$

Similarly, for a function $f(x,y)$ that is continuous on a region D of Type II, we have

$$\iint_D f(x,y)\,dA=\iint_D f(x,y)\,dx\,dy=\int_c^d\left[\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\right]dy. \tag{2.3.6}$$

The integral in each of these expressions is an iterated integral, similar to those we have seen before. Notice that, in the inner integral in the first expression, we integrate $f(x,y)$ with x being held constant and the limits of integration being $g_1(x)$ and $g_2(x)$. In the inner integral in the second expression, we integrate $f(x,y)$ with y being held constant and the limits of integration are $h_1(x)$ and $h_2(x)$.

Example 2.3.2: Evaluating an Iterated Integral over a Type I Region

Evaluate the integral $\iint_D x^2 e^{xy}\, dA$ where D is shown in Figure 2.3.5.

Solution

First construct the region as a Type I region (Figure 2.3.5). Here $D = \{(x, y) \mid 0 \le x \le 2,\ \frac{1}{2}x \le y \le 1\}$. Then we have

$$\iint_D x^2 e^{xy}\, dA = \int_{x=0}^{x=2} \int_{y=1/2x}^{y=1} x^2 e^{xy}\, dy\, dx.$$

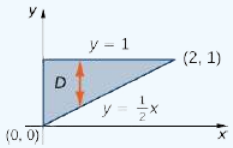


Figure 2.3.5: We can express region D as a Type I region and integrate from $y = \frac{1}{2}x$ to $y = 1$ between the lines $x = 0$ and $x = 2$.

Therefore, we have

$$\begin{aligned}
\int_{x=0}^{x=2} \int_{y=\frac{1}{2}x}^{y=1} x^2 e^{xy}\, dy\, dx &= \int_{x=0}^{x=2} \left[\int_{y=\frac{1}{2}x}^{y=1} x^2 e^{xy}\, dy\right] dx && \text{Iterated integral for a Type I region.} \\
&= \int_{x=0}^{x=2} \left[x^2 \frac{e^{xy}}{x}\right]\Big|_{y=1/2x}^{y=1} dx && \text{Integrate with respect to } y \\
&= \int_{x=0}^{x=2} \left[xe^x - xe^{x^2/2}\right] dx && \text{Integrate with respect to } x \\
&= \left[xe^x - e^x - e^{\frac{1}{2}x^2}\right]\Big|_{x=0}^{x=2} = 2.
\end{aligned}$$

In Example 2.3.2, we could have looked at the region in another way, such as $D = \{(x, y) \mid 0 \le y \le 1,\ 0 \le x \le 2y\}$ (Figure 2.3.6).

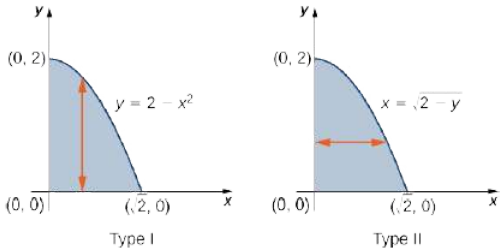


Figure 2.3.6.

This is a Type II region and the integral would then look like

$$\iint_D x^2 e^{xy}\, dA = \int_{y=0}^{y=1} \int_{x=0}^{x=2y} x^2 e^{xy}\, dx\, dy. \tag{2.3.7}$$

However, if we integrate first with respect to x this integral is lengthy to compute because we have to use integration by parts twice.



Example 2.3.3: Evaluating an Iterated Integral over a Type II Region

Evaluate the integral

$$\iint_D (3x^2 + y^2)\, dA$$

where $D = \{(x, y) \mid -2 \le y \le 3,\ y^2 - 3 \le x \le y + 3\}$.

Solution

Notice that D can be seen as either a Type I or a Type II region, as shown in Figure 2.3.7. However, in this case describing D as Type I is more complicated than describing it as Type II. Therefore, we use D as a Type II region for the integration.

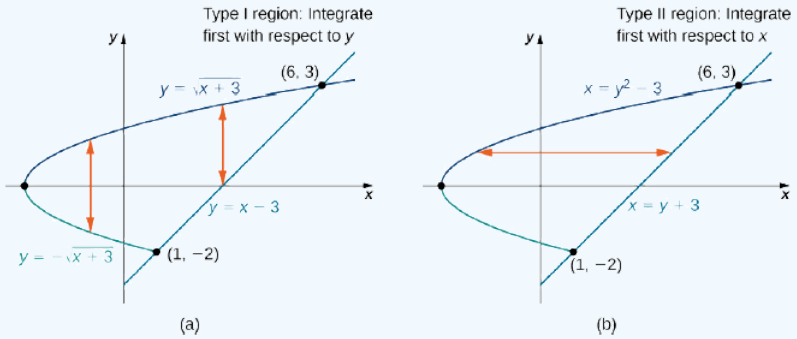


Figure 2.3.7: The region D in this example can be either (a) Type I or (b) Type II.

Choosing this order of integration, we have

$$\begin{aligned}\iint_D (3x^2+y^2)\,dA &= \int_{y=-2}^{y=3}\int_{x=y^2-3}^{x=y+3}(3x^2+y^2)\,dx\,dy \\ &= \int_{y=-2}^{y=3}(x^3+xy^2)\Big|_{y^2-3}^{y+3}\,dy && \text{Iterated integral, Type II region} \\ &= \int_{y=-2}^{y=3}\big((y+3)^3+(y+3)y^2 \\ &\quad -(y^2-3)y^2\big)\,dy \\ &= \int_{-2}^{3}(54+27y-12y^2+2y^3+8y^4 && \text{Integrate with respect to } x. \\ &\quad -y^6)\,dy \\ &= \left[54y+\frac{27y^2}{2}-4y^3+\frac{y^4}{2}+\frac{8y^5}{5}-\frac{y^7}{7}\right]_{-2}^{3} \\ &= \frac{2375}{7}.\end{aligned}$$

? Exercise 2.3.2

Sketch the region D and evaluate the iterated integral

$$\iint_D xy\,dy\,dx \tag{2.3.8}$$

where D is the region bounded by the curves $y=\cos\ x$ and $y=\sin\ x$ in the interval $[-3\pi/4,\ \pi/4]$.

Hint

Express D as a Type I region, and integrate with respect to y first.

Answer

$\frac{\pi}{4}$

Recall from Double Integrals over Rectangular Regions the properties of double integrals. As we have seen from the examples here, all these properties are also valid for a function defined on a non-rectangular bounded region on a plane. In particular, property 3 states:

If $R=S\cup T$ and $S\cap T=0$ except at their boundaries, then

$$\iint_R f(x,y)\,dA=\iint_S f(x,y)\,dA+\iint_T f(x,y)\,dA. \tag{2.3.9}$$

Similarly, we have the following property of double integrals over a non-rectangular bounded region on a plane.

Theorem: Decomposing Regions into Smaller Regions

Suppose the region D can be expressed as $D=D_1\cup D_2$ where D_1 and D_2 do not overlap except at their boundaries. Then

$$\iint_D f(x,y)\,dA=\iint_{D_1} f(x,y)\,dA+\iint_{D_2} f(x,y)\,dA. \tag{2.3.10}$$

This theorem is particularly useful for non-rectangular regions because it allows us to split a region into a union of regions of Type I and Type II. Then we can compute the double integral on each piece in a convenient way, as in the next example.

Example 2.3.4: Decomposing Regions

Express the region D shown in Figure 2.3.8 as a union of regions of Type I or Type II, and evaluate the integral

$$\iint_D (2x + 5y)\, dA.$$

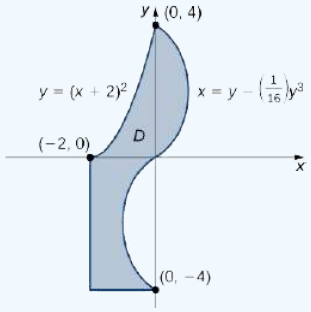


Figure 2.3.8: This region can be decomposed into a union of three regions of Type I or Type II.

Solution

The region D is not easy to decompose into any one type; it is actually a combination of different types. So we can write it as a union of three regions D_1, D_2, and D_3 where, $D_1 = \{(x, y) \mid -2 \le x \le 0,\ 0 \le y \le (x+2)^2\}$, $D_2 = \{(x, y) \mid 0 \le y \le 4,\ 0 \le x \le \left(y - \frac{1}{16}y^3\right)\}$, and $D_3 = \{(x, y) \mid -4 \le y \le 0,\ -2 \le x \le \left(y - \frac{1}{16}y^3\right)\}$. These regions are illustrated more clearly in Figure 2.3.9.

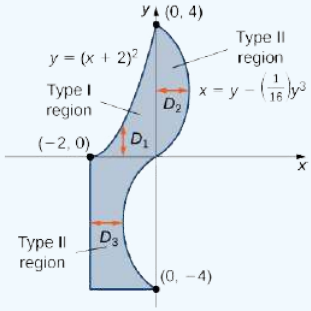


Figure 2.3.9: Breaking the region into three subregions makes it easier to set up the integration.

Here D_1 is Type I and D_2 and D_3 are both of Type II. Hence,

$$\begin{aligned}
\iint_D (2x + 5y)\, dA &= \iint_{D_1} (2x + 5y)\, dA + \iint_{D_2} (2x + 5y)\, dA + \iint_{D_3} (2x + 5y)\, dA \\
&= \int_{x=-2}^{x=0} \int_{y=0}^{y=(x+2)^2} (2x + 5y)\, dy\, dx + \int_{y=0}^{y=4} \int_{x=0}^{x=y-(1/16)y^3} (2 + 5y)\, dx\, dy \\
&\quad + \int_{y=-4}^{y=0} \int_{x=-2}^{x=y-(1/16)y^3} (2x + 5y)\, dx\, dy \\
&= \int_{x=-2}^{x=0} \left[\frac{1}{2}(2 + x)^2(20 + 24x + 5x^2)\right] dx + \int_{y=0}^{y=4} \left[\frac{1}{256}y^6 - \frac{7}{16}y^4 + 6y^2\right] dy \\
&\quad + \int_{y=-4}^{y=0} \left[\frac{1}{256}y^6 - \frac{7}{16}y^4 + 6y^2 + 10y - 4\right] dy \\
&= \frac{40}{3} + \frac{1664}{35} - \frac{1696}{35} = \frac{1304}{105}.
\end{aligned}$$

Now we could redo this example using a union of two Type II regions (see the Checkpoint).



? Exercise 2.3.3

Consider the region bounded by the curves $y = \ln x$ and $y = e^x$ in the interval $[1, 2]$. Decompose the region into smaller regions of Type II.

Hint

Sketch the region, and split it into three regions to set it up.

Answer

$$\{(x,y)\,|\,0 \le y \le 1,\ 1 \le x \le e^y\} \cup \{(x,y)\,|\,1 \le y \le e,\ 1 \le x \le 2\} \cup \{(x,y)\,|\,e \le y \le e^2,\ \ln y \le x \le 2\}$$

? Exercise 2.3.4

Redo Example **2.3.4** using a union of two Type II regions.

Hint

$$\{(x,y)\,|\,0 \le y \le 4,\ 2 + \sqrt{y} \le x \le \left(y - \frac{1}{16}y^3\right)\} \cup \{(x,y)\,|\,-4 \le y \le 0,\ -2 \le x \le \left(y - \frac{1}{16}y^{13}\right)\}$$

Answer

Same as in the example shown.

Changing the Order of Integration

As we have already seen when we evaluate an iterated integral, sometimes one order of integration leads to a computation that is significantly simpler than the other order of integration. Sometimes the order of integration does not matter, but it is important to learn to recognize when a change in order will simplify our work.

✓ Example 2.3.5: Changing the Order of Integration

Reverse the order of integration in the iterated integral

$$\int_{x=0}^{x=\sqrt{2}} \int_{y=0}^{y=2-x^2} xe^{x^2}\,dy\,dx.$$

Then evaluate the new iterated integral.

Solution

The region as presented is of Type I. To reverse the order of integration, we must first express the region as Type II. Refer to Figure **2.3.10**.

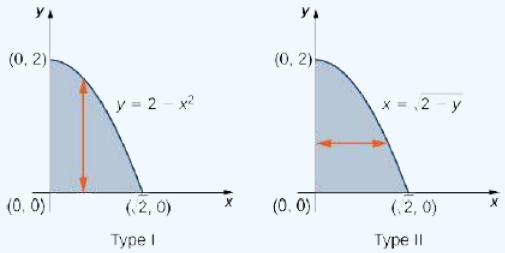


Figure **2.3.10**: Converting a region from Type I to Type II.

We can see from the limits of integration that the region is bounded above by $y = 2 - x^2$ and below by $y = 0$ where x is in the interval $[0, \sqrt{2}]$. By reversing the order, we have the region bounded on the left by $x = 0$ and on the right by $x = \sqrt{2-y}$ where y is in the interval $[0, 2]$. We solved $y = 2 - x^2$ in terms of x to obtain $x = \sqrt{2-y}$.

Hence

$$\int_0^{\sqrt{2}} \int_0^{2-x^2} xe^{x^2}\, dy\, dx = \int_0^2 \int_0^{\sqrt{2-y}} xe^{x^2}\, dx\, dy \qquad \text{Reverse the order of integration then use substitution.}$$

$$= \int_0^2 \left[\frac{1}{2} e^{x^2} \Big|_0^{\sqrt{2-y}} \right] dy$$

$$= \int_0^2 \frac{1}{2}(e^{2-y} - 1)\, dy$$

$$= -\frac{1}{2}(e^{2-y} + y)\Big|_0^2$$

$$= \frac{1}{2}(e^2 - 3).$$



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Example 2.3.6: Evaluating an Iterated Integral by Reversing the Order of Integration

Consider the iterated integral

$$\iint_R f(x,y)\,dx\,dy \tag{2.3.11}$$

where $z = f(x,y) = x - 2y$ over a triangular region R that has sides on $x = 0$, $y = 0$, and the line $x + y = 1$. Sketch the region, and then evaluate the iterated integral by

a. integrating first with respect to y and then
b. integrating first with respect to x.

Solution

A sketch of the region appears in Figure 2.3.11.

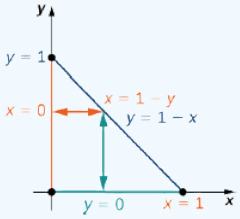


Figure 2.3.11: A triangular region R for integrating in two ways.

We can complete this integration in two different ways.

a. One way to look at it is by first integrating y from $y = 0$ to $y = 1 - x$ vertically and then integrating x from $x = 0$ to $x = 1$:

$$\begin{aligned}\iint_R f(x,y)\,dx\,dy &= \int_{x=0}^{x=1}\int_{y=0}^{y=1-x}(x-2y)\,dy\,dx \\ &= \int_{x=0}^{x=1}\left[xy - 2y^2\right]_{y=0}^{y=1-x}dx\end{aligned}$$

$$\begin{aligned}\int_{x=0}^{x=1}\left[x(1-x)-(1-x)^2\right]dx &= \int_{x=0}^{x=1}[-1+3x-2x^2]dx \\ &= \left[-x+\frac{3}{2}x^2-\frac{2}{3}x^3\right]_{x=0}^{x=1} = -\frac{1}{6}.\end{aligned}$$

b. The other way to do this problem is by first integrating x from $x = 0$ to $x = 1 - y$ horizontally and then integrating y from $y = 0$ to $y = 1$:

$$\begin{aligned}\iint_D (3x^2+y^2)\,dA &= \int_{y=-2}^{y=3}\int_{x=y^2-3}^{x=y+3}(3x^2+y^2)\,dx\,dy \\ &= \int_{y=-2}^{y=3}(x^3+xy^2)\Big|_{y^2-3}^{y+3}\,dy && \text{Iterated integral, Type II region} \\ &= \int_{y=-2}^{y=3}\big((y+3)^3+(y+3)y^2 \\ &\quad -(y^2-3)y^2\big)\,dy \\ &= \int_{-2}^{3}(54+27y-12y^2+2y^3+8y^4 && \text{Integrate with respect to } x. \\ &\quad -y^6)\,dy \\ &= \left[54y+\frac{27y^2}{2}-4y^3+\frac{y^4}{2}+\frac{8y^5}{5}-\frac{y^7}{7}\right]_{-2}^{3} \\ &= \frac{2375}{7}.\end{aligned}$$



Exercise 2.3.5

Evaluate the iterated integral $\iint_D (x^2 + y^2)\, dA$ over the region D in the first quadrant between the functions $y = 2x$ and $y = x^2$. Evaluate the iterated integral by integrating first with respect to y and then integrating first with resect to x.

Hint

Sketch the region and follow Example 2.3.6.

Answer

$\frac{216}{35}$

Calculating Volumes, Areas, and Average Values

We can use double integrals over general regions to compute volumes, areas, and average values. The methods are the same as those in Double Integrals over Rectangular Regions, but without the restriction to a rectangular region, we can now solve a wider variety of problems.

Example 2.3.7: Finding the Volume of a Tetrahedron

Find the volume of the solid bounded by the planes $x = 0,\ y = 0,\ z = 0$, and $2x + 3y + z = 6$.

Solution

The solid is a tetrahedron with the base on the xy-plane and a height $z = 6 - 2x - 3y$. The base is the region D bounded by the lines, $x = 0$, $y = 0$ and $2x + 3y = 6$ where $z = 0$ (Figure 2.3.12). Note that we can consider the region D as Type I or as Type II, and we can integrate in both ways.

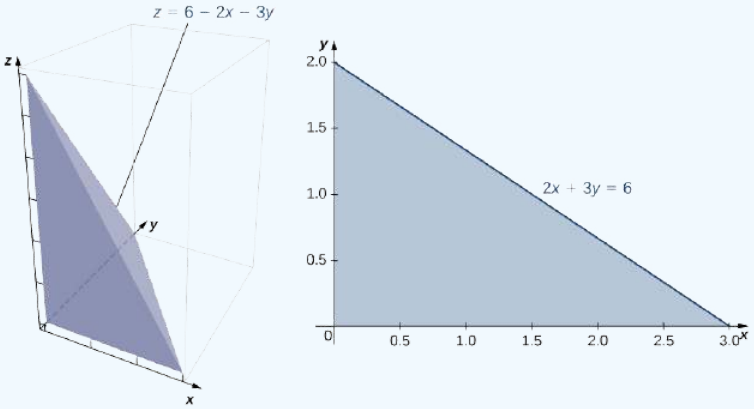


Figure **2.3.12**: A tetrahedron consisting of the three coordinate planes and the plane $z = 6 - 2x - 3y$, with the base bound by $x = 0,\ y = 0$, and $2x + 3y = 6$.

First, consider D as a Type I region, and hence $D = \{(x, y) \,|\, 0 \le x \le 3,\ 0 \le y \le 2 - \frac{2}{3}x\}$.

Therefore, the volume is

$$V = \int_{x=0}^{x=3} \int_{y=0}^{y=2-(2x/3)} (6 - 2x - 3y)\, dy\, dx = \int_{x=0}^{x=3} \left[\left(6y - 2xy - \frac{3}{2}y^2 \right) \Big|_{y=0}^{y=2-(2x/3)} \right] dx$$

$$= \int_{x=0}^{x=3} \left[\frac{2}{3}(x-3)^2 \right] dx = 6.$$

Now consider D as a Type II region, so $D = \{(x, y) \,|\, 0 \le y \le 2,\ 0 \le x \le 3 - \frac{3}{2}y\}$. In this calculation, the volume is

$$V = \int_{y=0}^{y=2} \int_{x=0}^{x=3-(3y/2)} (6 - 2x - 3y)\, dx\, dy = \int_{y=0}^{y=2} \left[(6x - x^2 - 3xy) \Big|_{x=0}^{x=3-(3y/2)} \right] dy$$

$$= \int_{y=0}^{y=2} \left[\frac{9}{4}(y-2)^2 \right] dy = 6.$$

Therefore, the volume is 6 cubic units.



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Exercise 2.3.6

Find the volume of the solid bounded above by $f(x,y) = 10 - 2x + y$ over the region enclosed by the curves $y = 0$ and $y = e^x$ where x is in the interval $[0,1]$.

Hint

Sketch the region, and describe it as Type I.

Answer

$\frac{e^2}{4} + 10e - \frac{49}{4}$ cubic units

Finding the area of a rectangular region is easy, but finding the area of a non-rectangular region is not so easy. As we have seen, we can use double integrals to find a rectangular area. As a matter of fact, this comes in very handy for finding the area of a general non-rectangular region, as stated in the next definition.

Definition: double integrals

The area of a plane-bounded region D is defined as the double integral

$$\iint_D 1\, dA. \tag{2.3.12}$$

We have already seen how to find areas in terms of single integration. Here we are seeing another way of finding areas by using double integrals, which can be very useful, as we will see in the later sections of this chapter.

Example 2.3.8: Finding the Area of a Region

Find the area of the region bounded below by the curve $y = x^2$ and above by the line $y = 2x$ in the first quadrant (Figure 2.3.13).

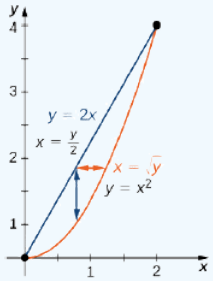


Figure 2.3.13: The region bounded by $y = x^2$ and $y = 2x$.

Solution

We just have to integrate the constant function $f(x,y) = 1$ over the region. Thus, the area A of the bounded region is $\int_{x=0}^{x=2} \int_{y=x^2}^{y=2x} dy\, dx$ *or* $\int_{y=0}^{y=4} \int_{x=y/2}^{x=\sqrt{y}} dx\, dy$:

$$\begin{aligned} A &= \iint_D 1\, dx\, dy \\ &= \int_{x=0}^{x=2} \int_{y=x^2}^{y=2x} 1\, dy\, dx \\ &= \int_{x=0}^{x=2} \left[y \Big|_{y=x^2}^{y=2x} \right] dx \\ &= \int_{x=0}^{x=2} (2x - x^2)\, dx \\ &= x^2 - \frac{x^3}{3} \Big|_0^2 = \frac{4}{3}. \end{aligned}$$

Exercise 2.3.7

Find the area of a region bounded above by the curve $y = x^3$ and below by $y = 0$ over the interval $[0, 3]$.

Hint

Sketch the region.

Answer

$\frac{81}{4}$ square units

We can also use a double integral to find the average value of a function over a general region. The definition is a direct extension of the earlier formula.

Definition

If $f(x, y)$ is integrable over a plane-bounded region D with positive area $A(D)$, then the average value of the function is

$$f_{ave} = \frac{1}{A(D)} \iint_D f(x, y)\, dA. \tag{2.3.13}$$

Note that the area is $A(D) = \iint_D 1\, dA$.

Example 2.3.9: Finding an Average Value

Find the average value of the function $f(x, y) = 7xy^2$ on the region bounded by the line $x = y$ and the curve $x = \sqrt{y}$ (Figure 2.3.14).

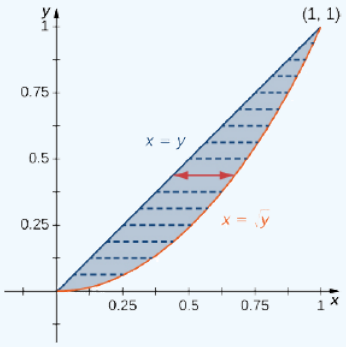


Figure 2.3.14: The region bounded by $x = y$ and $x = \sqrt{y}$.

Solution

First find the area $A(D)$ where the region D is given by the figure. We have

$$A(D) = \iint_D 1\, dA = \int_{y=0}^{y=1} \int_{x=y}^{x=\sqrt{y}} 1\, dx\, dy = \int_{y=0}^{y=1} \left[x|_{x=y}^{x=\sqrt{y}} \right] dy \tag{2.3.14}$$

$$= \int_{y=0}^{y=1} (\sqrt{y} - y)\, dy = \frac{2}{3} y^{2/3} - \frac{y^2}{2} \Big|_0^1 = \frac{1}{6}$$

Then the average value of the given function over this region is

$$f_{ave} = \frac{1}{A(D)} \iint_D f(x,y)\,dA = \frac{1}{A(D)} \int_{y=0}^{y=1} \int_{x=y}^{x=\sqrt{y}} 7xy^2\,dx\,dy = \frac{1}{1/6} \int_{y=0}^{y=1} \left[\frac{7}{2} x^2 y^2 \Big|_{x=y}^{x=\sqrt{y}} \right] dy$$
$$= 6 \int_{y=0}^{y=1} \left[\frac{7}{2} y^2 (y - y^2) \right] dy = 6 \int_{y=0}^{y=1} \left[\frac{7}{2} (y^3 - y^4) \right] dy = \frac{42}{2} \left(\frac{y^4}{4} - \frac{y^5}{5} \right) \Big|_0^1 = \frac{42}{40} = \frac{21}{20}.$$

? Exercise 2.3.8

Find the average value of the function $f(x,y) = xy$ over the triangle with vertices $(0,0)$, $(1,0)$ and $(1,3)$.

Hint

Express the line joining $(0,0)$ and $(1,3)$ as a function $y = g(x)$.

Answer

$\frac{3}{4}$



Improper Double Integrals

An **improper double integral** is an integral $\iint_D f\,dA$ where either D is an unbounded region or f is an unbounded function. For example, $D = \{(x,y) \mid |x-y| \geq 2\}$ is an unbounded region, and the function $f(x,y) = 1/(1-x^2-2y^2)$ over the ellipse $x^2 + 3y^2 \geq 1$ is an unbounded function. Hence, both of the following integrals are improper integrals:

i. $$\iint_D xy\,dA \text{ where } D = \{(x,y) \mid\mid x-y| \geq 2\}; \tag{2.3.15}$$

ii. $$\iint_D \frac{1}{1-x^2-2y^2}\,dA \text{ where } D = \{(x,y) | x^2 + 3y^2 \leq 1\}. \tag{2.3.16}$$

In this section we would like to deal with improper integrals of functions over rectangles or simple regions such that f has only finitely many discontinuities. Not all such improper integrals can be evaluated; however, a form of Fubini's theorem does apply for some types of improper integrals.

Theorem: Fubini's Theorem for Improper Integrals

If D is a bounded rectangle or simple region in the plane defined by

$\{(x,y) : a \le x \le b,\ g(x) \le y \le h(x)\}$ and also by

$\{(x,y) : c \le y \le d,\ j(y) \le x \le k(y)\}$ and f is a nonnegative function on D with finitely many discontinuities in the interior of D then

$$\iint_D f\, dA = \int_{x=a}^{x=b} \int_{y=g(x)}^{y=h(x)} f(x,y)\, dy\, dx = \int_{y=c}^{y=d} \int_{x=j(y)}^{x=k(y)} f(x,y)\, dx\, dy \tag{2.3.17}$$

It is very important to note that we required that the function be nonnegative on D for the theorem to work. We consider only the case where the function has finitely many discontinuities inside D.

Example 2.3.10: Evaluating a Double Improper Integral

Consider the function $f(x,y) = \frac{e^y}{y}$ over the region $D = \{(x,y) : 0 \le x \le 1,\ x \le y \le \sqrt{x}\}$.

Notice that the function is nonnegative and continuous at all points on D except $(0,0)$. Use Fubini's theorem to evaluate the improper integral.

Solution

First we plot the region D (Figure 2.3.15); then we express it in another way.

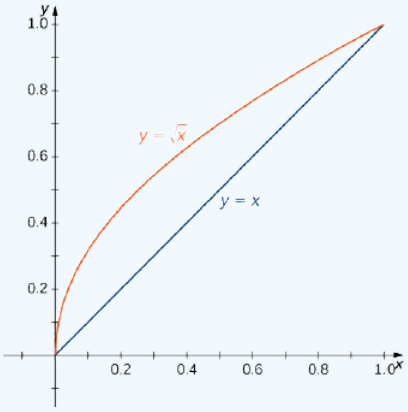


Figure 2.3.15: The function f is continuous at all points of the region D except $(0,0)$.

The other way to express the same region D is

$$D = \{(x,y) : 0 \le y \le 1,\ y^2 \le x \le y\}.$$

Thus we can use Fubini's theorem for improper integrals and evaluate the integral as

$$\int_{y=0}^{y=1} \int_{x=y^2}^{x=y} \frac{e^y}{y}\, dx\, dy.$$

Therefore, we have

$$\begin{aligned} \int_{y=0}^{y=1} \int_{x=y^2}^{x=y} \frac{e^y}{y}\, dx\, dy &= \int_{y=0}^{y=1} \frac{e^y}{y} x \Big|_{x=y^2}^{x=y} dy = \int_{y=0}^{y=1} \frac{e^y}{y}(y - y^2)\, dy \\ &= \int_0^1 (e^y - ye^y)\, dy = e - 2. \end{aligned} \tag{2.3.18}$$

As mentioned before, we also have an improper integral if the region of integration is unbounded. Suppose now that the function f is continuous in an unbounded rectangle R.

Theorem: Improper Integrals on an Unbounded Region

If R is an unbounded rectangle such as $R = \{(x,y) : a \le x \le \infty,\ c \le y \le \infty\}$, then when the limit exists, we have

$$\iint_R f(x,y)\,dA = \lim_{(b,d)\to(\infty,\infty)} \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx \tag{2.3.19}$$
$$= \lim_{(b,d)\to(\infty,\infty)} \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy.$$

The following example shows how this theorem can be used in certain cases of improper integrals.

Example 2.3.11

Evaluate the integral $\iint_R xye^{-x^2-y^2}\,dA$ where R is the first quadrant of the plane.

Solution

The region R is the first quadrant of the plane, which is unbounded. So

$$\begin{aligned}\iint_R xye^{-x^2-y^2}\,dA &= \lim_{(b,d)\to(\infty,\infty)} \int_{x=0}^{x=b} \left(\int_{y=0}^{y=d} xye^{-x^2-y^2}\,dy \right) dx \\ &= \lim_{(b,d)\to(\infty,\infty)} \int_{y=0}^{x=b} xye^{-x^2-y^2}\,dy \\ &= \lim_{(b,d)\to(\infty,\infty)} \frac{1}{4}\left(1-e^{-b^2}\right)\left(1-e^{-d^2}\right) = \frac{1}{4}\end{aligned}$$

Thus,

$$\iint_R xye^{-x^2-y^2}\,dA$$

is convergent and the value is $\frac{1}{4}$.

Exercise 2.3.9

$$\iint_D \frac{y}{\sqrt{1-x^2-y^2}}\,dA \tag{2.3.20}$$

where $D = \{(x,y) : x \ge 0,\ y \ge 0,\ x^2+y^2 \le 1\}$.

Hint

Notice that the integral is nonnegative and discontinuous on $x^2+y^2=1$. Express the region D as $D = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le \sqrt{1-x^2}\}$ and integrate using the method of substitution.

Answer

$\frac{\pi}{4}$

In some situations in probability theory, we can gain insight into a problem when we are able to use double integrals over general regions. Before we go over an example with a double integral, we need to set a few definitions and become familiar with some important properties.

Definition: Joint Density Function

Consider a pair of continuous random variables X and Y such as the birthdays of two people or the number of sunny and rainy days in a month. The **joint density function** f of X and Y satisfies the probability that (X,Y) lies in a certain region D:

$$P((X,Y) \in D) = \iint_D f(x,y)\,dA. \tag{2.3.21}$$

Since the probabilities can never be negative and must lie between 0 and 1 the joint density function satisfies the following inequality and equation:

$$f(x,y) \geq 0 \text{ and } \iint_R f(x,y)\,dA = 1. \tag{2.3.22}$$

Definition: Independent Random Variables

The variables X and Y are said to be **independent random variables** if their joint density function is the product of their individual density functions:

$$f(x,y) = f_1(x)f_2(y). \tag{2.3.23}$$

Example 2.3.12: Application to Probability

At Sydney's Restaurant, customers must wait an average of 15 minutes for a table. From the time they are seated until they have finished their meal requires an additional 40 minutes, on average. What is the probability that a customer spends less than an hour and a half at the diner, assuming that waiting for a table and completing the meal are independent events?

Solution

Waiting times are mathematically modeled by exponential density functions, with m being the average waiting time, as

$$f(t) = \begin{cases} 0 \text{if } t < 0 \\ \frac{1}{m} e^{-t/m} \text{if } t \geq 0. \end{cases}$$

if X and Y are random variables for 'waiting for a table' and 'completing the meal,' then the probability density functions are, respectively,

$$f_1(x) = \begin{cases} 0 \text{if } x < 0. \\ \frac{1}{15} e^{-x/15} \text{if } x \geq 0. \end{cases} \quad \text{and} \quad f_2(y) = \begin{cases} 0 \text{if } y < 0 \\ \frac{1}{40} e^{-y/40} \text{if } y \geq 0. \end{cases} \tag{2.3.24}$$

Clearly, the events are independent and hence the joint density function is the product of the individual functions

$$f(x,y) = f_1(x)f_2(y) = \begin{cases} 0 \text{if } x < 0 \text{ or } y < 0, \\ \frac{1}{600} e^{-x/15} \text{if } x, y \geq 0 \end{cases}$$

We want to find the probability that the combined time $X+Y$ is less than 90 minutes. In terms of geometry, it means that the region D is in the first quadrant bounded by the line $x+y=90$ (Figure 2.3.16).

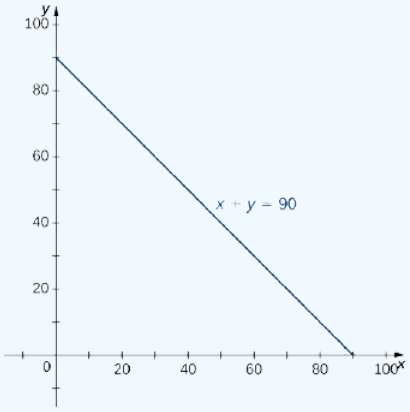


Figure **2.3.16**: The region of integration for a joint probability density function.

Hence, the probability that (X, Y) is in the region D is

$$P(X+Y \leq 90) = P((X,Y) \in D) = \iint_D f(x,y)\,dA = \iint_D \frac{1}{600} e^{-x/15} e^{-y/40}\,dA. \tag{2.3.25}$$

Since $x + y = 90$ is the same as $y = 90 - x$, we have a region of Type I, so

$$D = \{(x,y) \,|\, 0 \leq x \leq 90,\ 0 \leq y \leq 90 - x\}, \tag{2.3.26}$$

$$P(X+Y \leq 90) = \frac{1}{600} \int_{x=0}^{x=90} \int_{y=0}^{y=90-x} e^{-(/15} e^{-y/40} dx\ dy$$

$$= \frac{1}{600} \int_{x=0}^{x=90} \int_{y=0}^{y=90-x} e^{-x/15} e^{-y/40} dx\ dy \tag{2.3.27}$$

$$= \frac{1}{600} \int_{x=0}^{x=90} \int_{y=0}^{y=90-x} e^{-(x/15+y/40)} dx\ dy = 0.8328 \tag{2.3.28}$$

Thus, there is an 83.2% chance that a customer spends less than an hour and a half at the restaurant.



Another important application in probability that can involve improper double integrals is the calculation of expected values. First we define this concept and then show an example of a calculation.

Definition: Expected Values

In probability theory, we denote the **expected values** $E(X)$ and $E(Y)$ respectively, as the most likely outcomes of the events. The expected values $E(X)$ and $E(Y)$ are given by

$$E(X) = \iint_S x\, f(x,y)\, dA \text{ and } E(Y) = \iint_S y\, f(x,y)\, dA, \tag{2.3.29}$$

where S is the sample space of the random variables X and Y.

Example 2.3.13: Finding Expected Value

Find the expected time for the events 'waiting for a table' and 'completing the meal' in Example **2.3.12**.

Solution

Using the first quadrant of the rectangular coordinate plane as the sample space, we have improper integrals for $E(X)$ and $E(Y)$. The expected time for a table is

$$E(X) = \iint_S x \frac{1}{600} e^{-x/15} e^{-y/40} dA = \frac{1}{600} \int_{x=0}^{x=\infty} \int_{y=0}^{y=\infty} x e^{-x/15} e^{-y/40} dA \tag{2.3.30}$$

$$= \frac{1}{600} \lim_{(a,b)\to(\infty,\infty)} \int_{x=0}^{x=a} \int_{y=0}^{y=b} x e^{-x/15} e^{-y/40} dx\, dy \tag{2.3.31}$$

$$= \frac{1}{600} \left(\lim_{a\to\infty} \int_{x=0}^{x=a} x e^{-x/15} dx \right) \left(\lim_{b\to\infty} \int_{y=0}^{y=b} e^{-y/40} dy \right) \tag{2.3.32}$$

$$\frac{1}{600} \left(\left(\lim_{a\to\infty} (-15e^{-x/15}(x+15)) \right) \Big|_{x=0}^{x=a} \right) \left(\left(\lim_{b\to\infty} (-40e^{-y/40}) \right) \Big|_{y=0}^{y=b} \right) \tag{2.3.33}$$

$$= \frac{1}{600} \left(\lim_{a\to\infty} (-15e^{-a/15}(x+15) + 225) \right) \left(\lim_{b\to\infty} (-40e^{-b/40} + 40) \right) \tag{2.3.34}$$

$$= \frac{1}{600}(225)(40) = 15. \tag{2.3.35}$$

A similar calculation shows that $E(Y) = 40$. This means that the expected values of the two random events are the average waiting time and the average dining time, respectively.

Exercise 2.3.10

The joint density function for two random variables X and Y is given by

$$f(x,y) = \begin{cases} \frac{1}{600}(x^2+y^2), & \text{if } \leq x \leq 15,\ 0 \leq y \leq 10 \\ 0, & \text{otherwise} \end{cases}$$

Find the probability that X is at most 10 and Y is at least 5.

Hint

Compute the probability

$$P(X \leq 10,\ Y \geq 5) = \int_{x=-\infty}^{10} \int_{y=5}^{y=10} \frac{1}{6000}(x^2+y^2) dy\, dx.$$

Answer

$\frac{55}{72} \approx 0.7638$

Key Concepts

- A general bounded region D on the plane is a region that can be enclosed inside a rectangular region. We can use this idea to define a double integral over a general bounded region.

- To evaluate an iterated integral of a function over a general nonrectangular region, we sketch the region and express it as a Type I or as a Type II region or as a union of several Type I or Type II regions that overlap only on their boundaries.
- We can use double integrals to find volumes, areas, and average values of a function over general regions, similarly to calculations over rectangular regions.
- We can use Fubini's theorem for improper integrals to evaluate some types of improper integrals.

Key Equations

- **Iterated integral over a Type I region**

$$\iint_D f(x,y)\,dA = \iint_D f(x,y)\,dy\,dx = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\right] dx \tag{2.3.36}$$

- **Iterated integral over a Type II region**

$$\iint_D f(x,y)\,dA = \iint_D (x,y)\,dx\,dy = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\right] dy \tag{2.3.37}$$

Glossary

improper double integral

a double integral over an unbounded region or of an unbounded function

Type I

a region D in the xy- plane is Type I if it lies between two vertical lines and the graphs of two continuous functions $g_1(x)$ and $g_2(x)$

Type II

a region D in the xy-plane is Type II if it lies between two horizontal lines and the graphs of two continuous functions $h_1(y)$ and $h_2(h)$

Contributors and Attributions

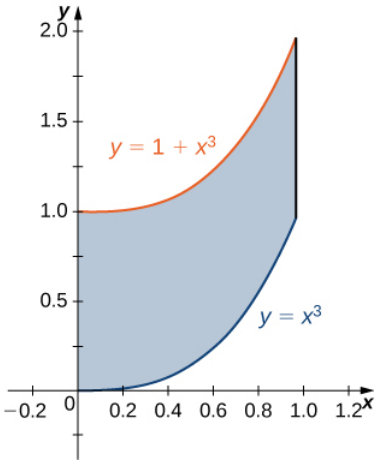
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2.3E: Exercises

1) The region D bounded by $y = x^3,\ y = x^3 + 1,\ x = 0$, and $x = 1$ as given in the following figure.



a. Classify this region as vertically simple (Type I) or horizontally simple (Type II).

Type:

Type I but not Type II

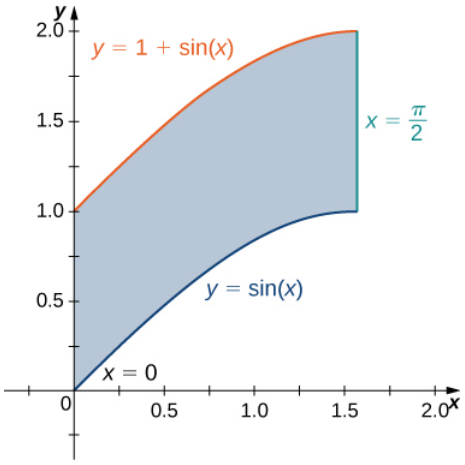
b. Find the area of the region D.

c. Find the average value of the function $f(x, y) = 3xy$ on the region graphed in the previous exercise.

Answer

$\frac{27}{20}$

2) The region D bounded by $y = \sin x,\ y = 1 + \sin x,\ x = 0$, and $x = \frac{\pi}{2}$ as given in the following figure.



a. Classify this region as vertically simple (Type I) or horizontally simple (Type II).

Type:

Type I but not Type II

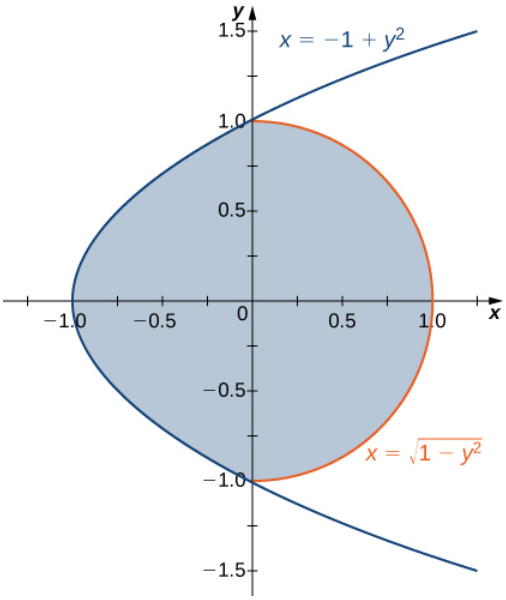
b. Find the area of the region D.

Answer

$\frac{\pi}{2}$ units^2

c. Find the average value of the function $f(x, y) = \cos x$ on the region D.

3) The region D bounded by $x = y^2 - 1$ and $x = \sqrt{1 - y^2}$ as given in the following figure.



a. Classify this region as vertically simple (Type I) or horizontally simple (Type II).

Type:

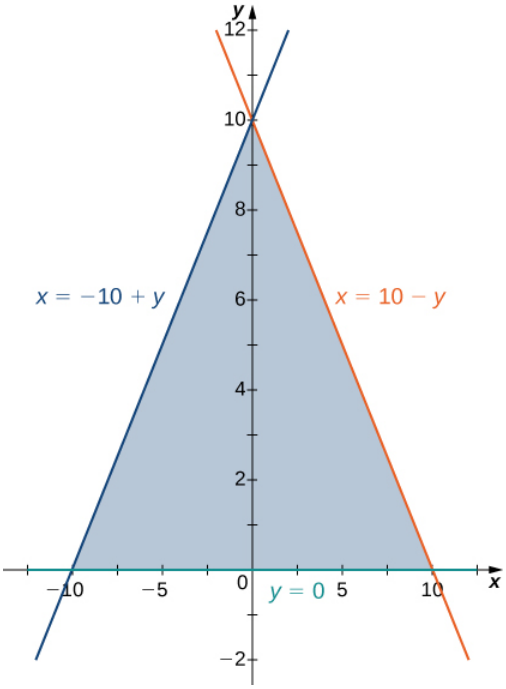
Type II but not Type I

b. Find the volume of the solid under the graph of the function $f(x, y) = xy + 1$ and above the region D.

Answer

$\frac{1}{6}(8 + 3\pi)$ units^3

4) The region D bounded by $y = 0,\ x = -10 + y$, and $x = 10 - y$ as given in the following figure.



a. Classify this region as vertically simple (Type I) or horizontally simple (Type II).

Type:

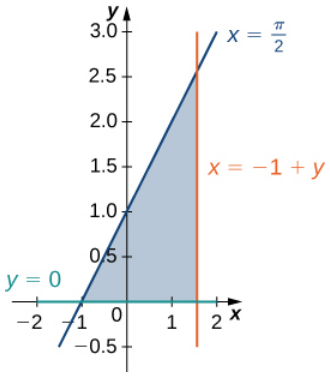
Type II but not Type I

b. Find the volume of the solid under the graph of the function $f(x,y) = x + y$ and above the region in the figure from the previous exercise.

Answer

$\frac{1000}{3}$ units^3

5) The region D bounded by $y = 0,\ x = y - 1,\ x = \frac{\pi}{2}$ as given in the following figure.

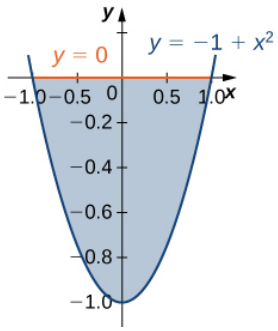


Classify this region as vertically simple (Type I) or horizontally simple (Type II).

Type:

Type I and Type II

6) The region D bounded by $y = 0$ and $y = x^2 - 1$ as given in the following figure.



Classify this region as vertically simple (Type I) or horizontally simple (Type II).

Type:

Type I and Type II

7) Let D be the region bounded by the curves of equations $y = \cos x$ and $y = 4 - x^2$ and the x-axis. Explain why D is neither of Type I nor II.

Answer

The region D is not of Type I: it does not lie between two vertical lines and the graphs of two continuous functions $g_1(x)$ and $g_2(x)$. The region is not of Type II: it does not lie between two horizontal lines and the graphs of two continuous functions $h_1(y)$ and $h_2(y)$.

8) Let D be the region bounded by the curves of equations $y = x,\ y = -x$ and $y = 2 - x^2$. Explain why D is neither of Type I nor II.

In exercises 9 - 14, evaluate the double integral $\iint_D f(x,y)\,dA$ over the region D.

9) $f(x,y) = 1$ and

$D = \{(x,y) | \ 0 \le x \le \frac{\pi}{2},\ \sin x \le y \le 1 + \sin x\}$

Answer

$\frac{\pi}{2}$

10) $f(x,y) = 2$ and

$D = \{(x,y) | \ 0 \le y \le 1,\ y - 1 \le x \le \arccos y\}$

11) $f(x,y) = xy$ and

$D = \{(x,y) | \ -1 \le y \le 1,\ y^2 - 1 \le x \le \sqrt{1 - y^2}\}$

Answer

0

12) $f(x,y) = \sin y$ and D is the triangular region with vertices $(0,0),\ (0,3)$, and $(3,0)$

13) $f(x,y) = -x + 1$ and D is the triangular region with vertices $(0,0),\ (0,2)$, and $(2,2)$

Answer

$\frac{2}{3}$

14) $f(x,y) = 2x + 4y$ and

$D = \{(x,y) | \ 0 \le x \le 1,\ x^3 \le y \le x^3 + 1\}$

In exercises 15 - 20, evaluate the iterated integrals.

15) $\int_0^1 \int_{2\sqrt{x}}^{2\sqrt{x}+1} (xy+1)\,dy\,dx$

Answer

$\frac{41}{20}$

16) $\int_0^3 \int_{2x}^{3x} (x+y^2)\,dy\,dx$

17) $\int_1^2 \int_{-u^2-1}^{-u} (8uv)\,dv\,du$

Answer

-63

18) $\int_e^{e^2} \int_{\ln u}^{2} (v+\ln u)\,dv\,du$

19) $\int_0^{1/2} \int_{-\sqrt{1-4y^2}}^{\sqrt{1-4y^2}} 4\,dx\,dy$

Answer

π

20) $\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (2x+4y^3)\,dx\,dy$

21) Let D be the region bounded by $y=1-x^2,\ y=4-x^2$, and the x- and y-axes.

a. Show that $\iint_D x\,dA = \int_0^1 \int_{1-x^2}^{4-x^2} x\,dy\,dx + \int_1^2 \int_0^{4-x^2} x\,dy\,dx$ by dividing the region D into two regions of Type I.

b. Evaluate the integral $\iint_D x\,dA$.

22) Let D be the region bounded by $y=1,\ y=x,\ y=\ln x$, and the x-axis.

a. Show that $\iint_D y^2\,dA = \int_{-1}^0 \int_{-x}^{2-x^2} y^2 dy\,dx + \int_0^1 \int_x^{2-x^2} y^2 dy\,dx$ by dividing the region D into two regions of Type I, where $D = \{(x,y)\,|\,y \geq x, y \geq -x,\ y \leq 2-x^2\}$.

b. Evaluate the integral $\iint_D y^2\,dA$.

23) Let D be the region bounded by $y=x^2$, $y=x+2$, and $y=-x$.

a. Show that $\iint_D x\,dA = \int_0^1 \int_{-y}^{\sqrt{y}} x\,dx\,dy + \int_1^4 \int_{y-2}^{\sqrt{y}} x\,dx\,dy$ by dividing the region D into two regions of Type II, where $D = \{(x,y)\,|\,y \geq x^2,\ y \geq -x,\ y \leq x+2\}$.

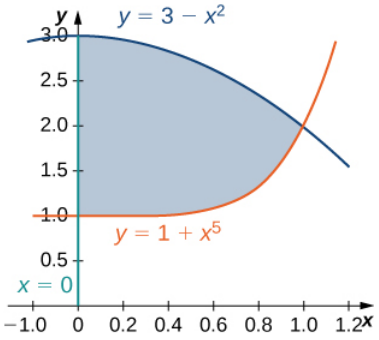
b. Evaluate the integral $\iint_D x\,dA$.

Answer

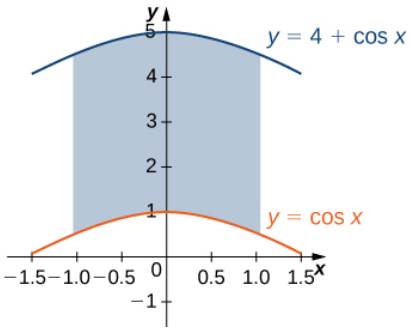
a. Answers may vary;

b. $\frac{7}{3}$

24) The region D bounded by $x=0, y=x^5+1$, and $y=3-x^2$ is shown in the following figure. Find the area $A(D)$ of the region D.



25) The region D bounded by $y = \cos x$, $y = 4 + \cos x$, and $x = \pm\frac{\pi}{3}$ is shown in the following figure. Find the area $A(D)$ of the region D.



Answer

$\frac{8\pi}{3}$

26) Find the area $A(D)$ of the region $D = \{(x,y) | y \geq 1 - x^2, y \leq 4 - x^2,\ y \geq 0,\ x \geq 0\}$.

27) Let D be the region bounded by $y = 1,\ y = x,\ y = \ln x$, and the x-axis. Find the area $A(D)$ of the region D.

Answer

$\left(e - \frac{3}{2}\right)$ units^2

28) Find the average value of the function $f(x,y) = \sin y$ on the triangular region with vertices $(0,0),\ (0,3)$, and $(3,0)$.

29) Find the average value of the function $f(x,y) = -x + 1$ on the triangular region with vertices $(0,0),\ (0,2)$, and $(2,2)$.

Answer

The average value of f on this triangular region is $\frac{1}{3}$.

In exercises 30 - 33, change the order of integration and evaluate the integral.

30) $\int_{-1}^{\pi/2} \int_{0}^{x+1} \sin x \, dy \, dx$

31) $\int_{0}^{1} \int_{x-1}^{1-x} x \, dy \, dx$

Answer

$$\int_0^1 \int_{x-1}^{1-x} x \, dy \, dx = \int_{-1}^0 \int_0^{y+1} x \, dx \, dy + \int_0^1 \int_0^{1-y} x \, dx \, dy = \frac{1}{3}$$

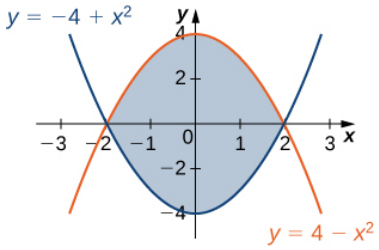
32) $\int_{-1}^{0} \int_{-\sqrt{y+1}}^{\sqrt{y+1}} y^2 dx \, dy$

33) $\int_{-1/2}^{1/2} \int_{-\sqrt{y^2+1}}^{\sqrt{y^2+1}} y \, dx \, dy$

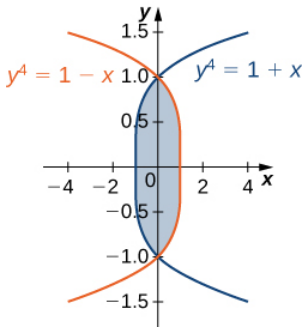
Answer

$$\int_{-1/2}^{1/2}\int_{-\sqrt{y^2+1}}^{\sqrt{y^2+1}} y\,dx\,dy = \int_{1}^{2}\int_{-\sqrt{x^2-1}}^{\sqrt{x^2-1}} y\,dy\,dx = 0$$

34) The region D is shown in the following figure. Evaluate the double integral $\iint_D (x^2+y)\,dA$ by using the easier order of integration.



35) The region D is shown in the following figure. Evaluate the double integral $\iint_D (x^2-y^2)\,dA$ by using the easier order of integration.



Answer

$$\iint_D (x^2-y^2)dA = \int_{-1}^{1}\int_{y^4-1}^{1-y^4} (x^2-y^2)dx\,dy = \frac{464}{4095}$$

36) Find the volume of the solid under the surface $z = 2x + y^2$ and above the region bounded by $y = x^5$ and $y = x$.

37) Find the volume of the solid under the plane $z = 3x + y$ and above the region determined by $y = x^7$ and $y = x$.

Answer

$\frac{4}{5}$ units^3

38) Find the volume of the solid under the plane $z = 3x + y$ and above the region bounded by $x = \tan y,\ x = -\tan y$, and $x = 1$.

39) Find the volume of the solid under the surface $z = x^3$ and above the plane region bounded by $x = \sin y,\ x = -\sin y$, and $x = 1$.

Answer

$\frac{5\pi}{32}$ units^3

40) Let g be a positive, increasing, and differentiable function on the interval $[a, b]$. Show that the volume of the solid under the surface $z = g'(x)$ and above the region bounded by $y = 0,\ y = g(x),\ x = a$, and $x = b$ is given by $\frac{1}{2}(g^2(b) - g^2(a))$.

41) Let g be a positive, increasing, and differentiable function on the interval $[a, b]$ and let k be a positive real number. Show that the volume of the solid under the surface $z = g'(x)$ and above the region bounded by $y = g(x),\ y = g(x) + k,\ x = a$, and $x = b$ is given by $k(g(b) - g(a))$.

42) Find the volume of the solid situated in the first octant and determined by the planes $z = 2,\ z = 0,\ x + y = 1,\ x = 0$, and $y = 0$.

43) Find the volume of the solid situated in the first octant and bounded by the planes $x+2y=1$, $x=0,\ z=4$, and $z=0$.

Answer

$1\ \text{units}^3$

44) Find the volume of the solid bounded by the planes $x+y=1,\ x-y=1,\ x=0,\ z=0$, and $z=10$.

45) Find the volume of the solid bounded by the planes $x+y=1,\ x-y=1,\ x+y=-1\ x-y=-1,\ z=1$, and $z=0$

Answer

$2\ \text{units}^3$

46) Let S_1 and S_2 be the solids situated in the first octant under the planes $x+y+z=1$ and $x+y+2z=1$ respectively, and let S be the solid situated between $S_1,\ S_2,\ x=0$, and $y=0$.

a. Find the volume of the solid S_1.
b. Find the volume of the solid S_2.
c. Find the volume of the solid S by subtracting the volumes of the solids S_1 and S_2.

47) Let S_1 and S_2 be the solids situated in the first octant under the planes $2x+2y+z=2$ and $x+y+z=1$ respectively, and let S be the solid situated between $S_1,\ S_2,\ x=0$, and $y=0$.

a. Find the volume of the solid S_1.
b. Find the volume of the solid S_2.
c. Find the volume of the solid S by subtracting the volumes of the solids S_1 and S_2.

Answer

a. $\frac{1}{3}\ \text{units}^3$
b. $\frac{1}{6}\ \text{units}^3$
c. $\frac{1}{6}\ \text{units}^3$

48) Let S_1 and S_2 be the solids situated in the first octant under the plane $x+y+z=2$ and under the sphere $x^2+y^2+z^2=4$, respectively. If the volume of the solid S_2 is $\frac{4\pi}{3}$ determine the volume of the solid S situated between S_1 and S_2 by subtracting the volumes of these solids.

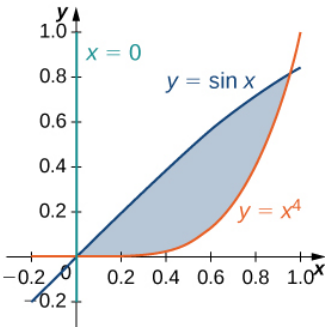
49) Let S_1 and S_2 be the solids situated in the first octant under the plane $x+y+z=2$ and bounded by the cylinder $x^2+y^2=4$, respectively.

a. Find the volume of the solid S_1.
b. Find the volume of the solid S_2.
c. Find the volume of the solid S situated between S_1 and S_2 by subtracting the volumes of the solids S_1 and S_2.

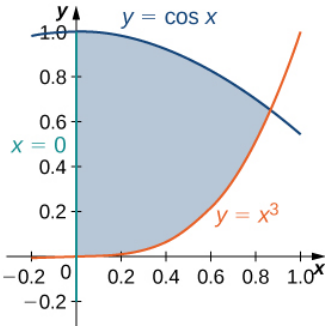
Answer

a. $\frac{4}{3}\ \text{units}^3$
b. $2\pi\ \text{units}^3$
c. $\frac{6\pi-4}{3}\ \text{units}^3$

50) [T] The following figure shows the region D bounded by the curves $y=\sin x,\ x=0$, and $y=x^4$. Use a graphing calculator or CAS to find the x-coordinates of the intersection points of the curves and to determine the area of the region D. Round your answers to six decimal places.



51) [T] The region D bounded by the curves $y = \cos x,\ x = 0$, and $y = x^3$ is shown in the following figure. Use a graphing calculator or CAS to find the x-coordinates of the intersection points of the curves and to determine the area of the region D. Round your answers to six decimal places.



Answer

0 and 0.865474; $A(D) = 0.621135 \text{ units}^3$

52) Suppose that (X, Y) is the outcome of an experiment that must occur in a particular region S in the xy-plane. In this context, the region S is called the sample space of the experiment and X and Y are random variables. If D is a region included in S, then the probability of (X, Y) being in D is defined as $P[(X, Y) \in D] = \iint_D p(x, y)dx\ dy$, where $p(x, y)$ is the joint probability density of the experiment. Here, $p(x, y)$ is a nonnegative function for which $\iint_S p(x, y)dx\ dy = 1$. Assume that a point (X, Y) is chosen arbitrarily in the square $[0, 3] \times [0, 3]$ with the probability density

$$p(x, y) = \frac{1}{9}(x, y) \in [0, 3] \times [0, 3],$$

$$p(x, y) = 0 \text{ otherwise}$$

Find the probability that the point (X, Y) is inside the unit square and interpret the result.

53) Consider X and Y two random variables of probability densities $p_1(x)$ and $p_2(x)$, respectively. The random variables X and Y are said to be independent if their joint density function is given by $p(x, y) = p_1(x)p_2(y)$. At a drive-thru restaurant, customers spend, on average, 3 minutes placing their orders and an additional 5 minutes paying for and picking up their meals. Assume that placing the order and paying for/picking up the meal are two independent events X and Y. If the waiting times are modeled by the exponential probability densities

$$p_1(x) = \frac{1}{3}e^{-x/3}\ x \geq 0,$$

$$p_1(x) = 0 \text{ otherwise}$$

$$p_2(y) = \frac{1}{5}e^{-y/5}\ y \geq 0$$

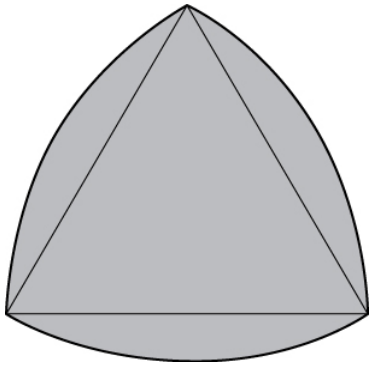
$$p_2(y) = 0 \text{ otherwise}$$

respectively, the probability that a customer will spend less than 6 minutes in the drive-thru line is given by $P[X+Y \leq 6] = \iint_D p(x,y)dx\,dy$, where $D = \{(x,y)|x \geq 0,\ y \geq 0,\ x+y \leq 6\}$. Find $P[X+Y \leq 6]$ and interpret the result.

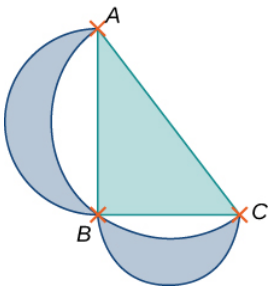
Answer

$P[X+Y \leq 6] = 1 + \frac{3}{2e^2} - \frac{5}{e^{6/5}} \approx 0.45$; there is a 45% chance that a customer will spend 6 minutes in the drive-thru line.

54) [T] The Reuleaux triangle consists of an equilateral triangle and three regions, each of them bounded by a side of the triangle and an arc of a circle of radius s centered at the opposite vertex of the triangle. Show that the area of the Reuleaux triangle in the following figure of side length s is $\frac{s^2}{2}(\pi - \sqrt{3})$.



55) [T] Show that the area of the lunes of Alhazen, the two blue lunes in the following figure, is the same as the area of the right triangle ABC. The outer boundaries of the lunes are semicircles of diameters AB and AC respectively, and the inner boundaries are formed by the circumcircle of the triangle ABC.



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2.4: Double Integrals in Polar Coordinates

Learning Objectives

- Recognize the format of a double integral over a polar rectangular region.
- Evaluate a double integral in polar coordinates by using an iterated integral.
- Recognize the format of a double integral over a general polar region.
- Use double integrals in polar coordinates to calculate areas and volumes.

Double integrals are sometimes much easier to evaluate if we change rectangular coordinates to polar coordinates. However, before we describe how to make this change, we need to establish the concept of a double integral in a polar rectangular region.

Polar Rectangular Regions of Integration

When we defined the double integral for a continuous function in rectangular coordinates—say, g over a region R in the xy-plane—we divided R into subrectangles with sides parallel to the coordinate axes. These sides have either constant x-values and/or constant y-values. In polar coordinates, the shape we work with is a polar rectangle, whose sides have constant r-values and/or constant θ-values. This means we can describe a polar rectangle as in Figure 2.4.1a, with $R = \{(r, \theta) \,|\, a \le r \le b,\ \alpha \le \theta \le \beta\}$.

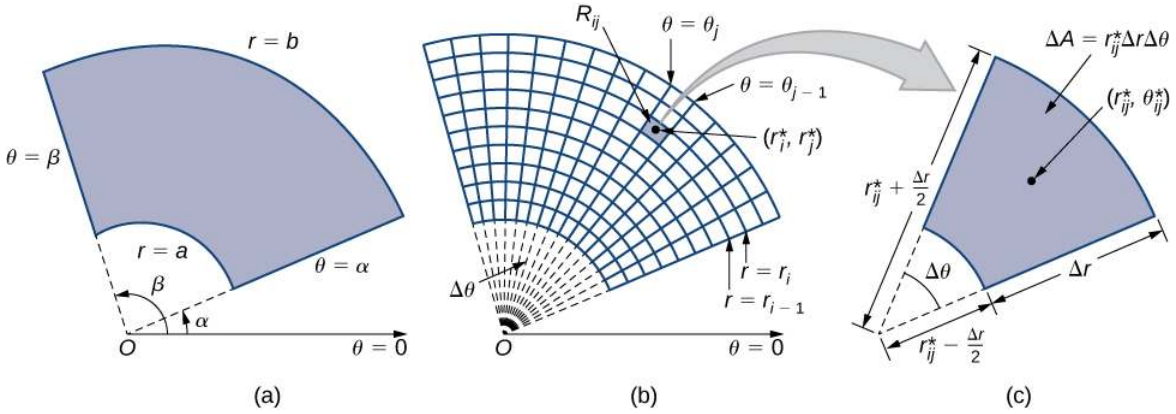


Figure 2.4.1: (a) A polar rectangle R (b) divided into subrectangles R_{ij} (c) Close-up of a subrectangle.

In this section, we are looking to integrate over polar rectangles. Consider a function $f(r, \theta)$ over a polar rectangle R. We divide the interval $[a, b]$ into m subintervals $[r_{i-1}, r_i]$ of length $\Delta r = (b - a)/m$ and divide the interval $[\alpha, \beta]$ into n subintervals $[\theta_{i-1}, \theta_i]$ of width $\Delta\theta = (\beta - \alpha)/n$. This means that the circles $r = r_i$ and rays $\theta = \theta_i$ for $1 \le i \le m$ and $1 \le j \le n$ divide the polar rectangle R into smaller polar subrectangles R_{ij} (Figure 2.4.1b).

As before, we need to find the area ΔA of the polar subrectangle R_{ij} and the "polar" volume of the thin box above R_{ij}. Recall that, in a circle of radius r the length s of an arc subtended by a central angle of θ radians is $s = r\theta$. Notice that the polar rectangle R_{ij} looks a lot like a trapezoid with parallel sides $r_{i-1}\Delta\theta$ and $r_i\Delta\theta$ and with a width Δr. Hence the area of the polar subrectangle R_{ij} is

$$\Delta A = \frac{1}{2}\Delta r(r_{i-1}\Delta\theta + r_i\Delta\theta).$$

Simplifying and letting

$$r_{ij}^* = \frac{1}{2}(r_{i-1} + r_i)$$

we have $\Delta A = r_{ij}^* \Delta r \Delta\theta$.

Therefore, the polar volume of the thin box above R_{ij} (Figure 2.4.2) is

$$f(r_{ij}^*, \theta_{ij}^*)\Delta A = f(r_{ij}^*, \theta_{ij}^*) r_{ij}^* \Delta r \Delta\theta.$$

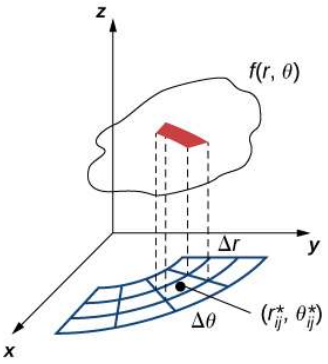


Figure 2.4.2: Finding the volume of the thin box above polar rectangle R_{ij}.

Using the same idea for all the subrectangles and summing the volumes of the rectangular boxes, we obtain a double Riemann sum as

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_{ij}^*, \theta_{ij}^*) r_{ij}^* \Delta r \Delta\theta.$$

As we have seen before, we obtain a better approximation to the polar volume of the solid above the region R when we let m and n become larger. Hence, we define the polar volume as the limit of the double Riemann sum,

$$V = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_{ij}^*, \theta_{ij}^*) r_{ij}^* \Delta r \Delta\theta.$$

This becomes the expression for the double integral.

Definition: The double integral in polar coordinates

The double integral of the function $f(r, \theta)$ over the polar rectangular region R in the $r\theta$-plane is defined as

$$\iint_R f(r,\theta) dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_{ij}^*, \theta_{ij}^*) \Delta A \tag{2.4.1}$$

$$= \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_{ij}^*, \theta_{ij}^*) r_{ij}^* \Delta r \Delta\theta. \tag{2.4.2}$$

Again, just as in section on Double Integrals over Rectangular Regions, the double integral over a polar rectangular region can be expressed as an iterated integral in polar coordinates. Hence,

$$\iint_R f(r,\theta)\, dA = \iint_R f(r,\theta)\, r\, dr\, d\theta = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=a}^{r=b} f(r,\theta)\, r\, dr\, d\theta. \tag{2.4.3}$$

Notice that the expression for dA is replaced by $r\, dr\, d\theta$ when working in polar coordinates. Another way to look at the polar double integral is to change the double integral in rectangular coordinates by substitution. When the function f is given in terms of x and y using $x = r\cos\theta$, $y = r\sin\theta$, and $dA = r\, dr\, d\theta$ changes it to

$$\iint_R f(x,y)\, dA = \iint_R f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta. \tag{2.4.4}$$

Note that all the properties listed in section on Double Integrals over Rectangular Regions for the double integral in rectangular coordinates hold true for the double integral in polar coordinates as well, so we can use them without hesitation.

Example 2.4.1A: Sketching a Polar Rectangular Region

Sketch the polar rectangular region

$$R = \{(r,\theta) \mid 1 \le r \le 3, 0 \le \theta \le \pi\}.$$

Solution

As we can see from Figure 2.4.3, $r = 1$ and $r = 3$ are circles of radius 1 and 3 and $0 \le \theta \le \pi$ covers the entire top half of the plane. Hence the region R looks like a semicircular band.

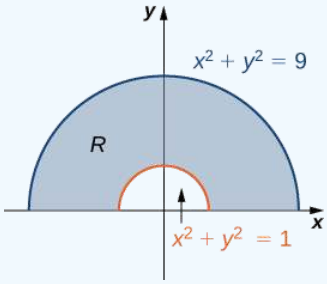


Figure 2.4.3: The polar region R lies between two semicircles.

Now that we have sketched a polar rectangular region, let us demonstrate how to evaluate a double integral over this region by using polar coordinates.

Example 2.4.1B: Evaluating a Double Integral over a Polar Rectangular Region

Evaluate the integral $\iint_R 3x\,dA$ over the region $R = \{(r, \theta) \mid 1 \le r \le 2,\ 0 \le \theta \le \pi\}$.

Solution

First we sketch a figure similar to Figure 2.4.3, but with outer radius $r = 2$. From the figure we can see that we have

$$\begin{aligned}
\iint_R 3x\,dA &= \int_{\theta=0}^{\theta=\pi} \int_{r=1}^{r=2} 3r\cos\theta\, r\,dr\,d\theta \quad \text{Use an integral with correct limits of integration.} \\
&= \int_{\theta=0}^{\theta=\pi} \cos\theta \left[r^3\Big|_{r=1}^{r=2}\right] d\theta \quad \text{Integrate first with respect to } r. \\
&= \int_{\theta=0}^{\theta=\pi} 7\cos\theta\,d\theta \\
&= 7\sin\theta\Big|_{\theta=0}^{\theta=\pi} = 0.
\end{aligned}$$



Exercise 2.4.1

Sketch the region $D = \{(r, \theta) | 1 \leq r \leq 2, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\}$, and evaluate $\iint_R x \, dA$.

Hint

Follow the steps in Example 2.4.1A.

Answer

$\frac{14}{3}$

Example 2.4.2A: Evaluating a Double Integral by Converting from Rectangular Coordinates

Evaluate the integral

$$\iint_R (1 - x^2 - y^2) \, dA$$

where R is the unit circle on the xy-plane.

Solution

The region R is a unit circle, so we can describe it as $R = \{(r, \theta) \,|\, 0 \leq r \leq 1, \, 0 \leq \theta \leq 2\pi\}$.

Using the conversion $x = r \cos \theta$, $y = r \sin \theta$, and $dA = r \, dr \, d\theta$, we have

$$\begin{aligned} \iint_R (1 - x^2 - y^2) \, dA &= \int_0^{2\pi} \int_0^1 (1 - r^2) \, r \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^1 (r - r^3) \, dr \, d\theta \\ &= \int_0^{2\pi} \left[\frac{r^2}{2} - \frac{r^4}{4} \right]_0^1 d\theta \\ &= \int_0^{2\pi} \frac{1}{4} \, d\theta = \frac{\pi}{2}. \end{aligned}$$



Example 2.4.2B: Evaluating a Double Integral by Converting from Rectangular Coordinates

Evaluate the integral

$$\iint_R (x+y)\,dA$$

where $R = \{(x,y) \mid 1 \le x^2 + y^2 \le 4,\ x \le 0\}$.

Solution

We can see that R is an annular region that can be converted to polar coordinates and described as $R = \{(r,\theta) \mid 1 \le r \le 2,\ \frac{\pi}{2} \le \theta \le \frac{3\pi}{2}\}$ (see the following graph).

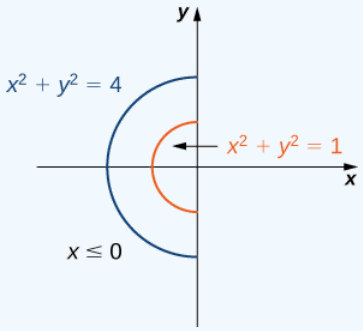


Figure 2.4.4: The annular region of integration R.

Hence, using the conversion $x = r\cos\theta$, $y = r\sin\theta$, and $dA = r\,dr\,d\theta$, we have

$$\begin{aligned}\iint_R (x+y)\,dA &= \int_{\theta=\pi/2}^{\theta=3\pi/2}\int_{r=1}^{r=2}(r\cos\theta + r\sin\theta)r\,dr\,d\theta \\ &= \left(\int_{r=1}^{r=2} r^2\,dr\right)\left(\int_{\pi/2}^{3\pi/2}(\cos\theta+\sin\theta)\,d\theta\right) \\ &= \left[\frac{r^3}{3}\right]_1^2 [\sin\theta - \cos\theta]\Big|_{\pi/2}^{3\pi/2} \\ &= -\frac{14}{3}.\end{aligned}$$

Exercise 2.4.2

Evaluate the integral

$$\iint_R (4 - x^2 - y^2)\,dA$$

where R is the circle of radius 2 on the xy-plane.

Hint

Follow the steps in the previous example.

Answer

8π

General Polar Regions of Integration

To evaluate the double integral of a continuous function by iterated integrals over general polar regions, we consider two types of regions, analogous to Type I and Type II as discussed for rectangular coordinates in section on Double Integrals over General Regions. It is more common to write polar equations as $r = f(\theta)$ than $\theta = f(r)$, so we describe a general polar region as $R = \{(r,\theta) \mid \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$ (Figure 2.4.5).

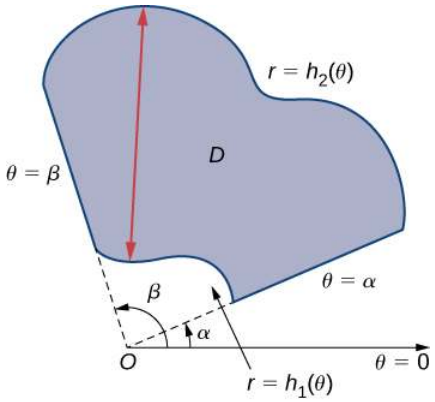


Figure 2.4.5: A general polar region between $\alpha \le \theta \le \beta$ and $h_1(\theta) \le r \le h_2(\theta)$.

Theorem: Double Integrals over General Polar Regions

If $f(r, \theta)$ is continuous on a general polar region D as described above, then

$$\iint_D f(r,\theta)\, r\, dr\, d\theta = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=h_1(\theta)}^{r=h_2(\theta)} f(r,\theta)\, r\, dr\, d\theta. \tag{2.4.5}$$

Example 2.4.3: Evaluating a Double Integral over a General Polar Region

Evaluate the integral

$$\iint_D r^2 \sin\theta\, r\, dr\, d\theta$$

where D is the region bounded by the polar axis and the upper half of the cardioid $r = 1 + \cos\theta$.

Solution

We can describe the region D as $\{(r, \theta) \mid 0 \le \theta \le \pi,\ 0 \le r \le 1 + \cos\theta\}$ as shown in Figure 2.4.6.

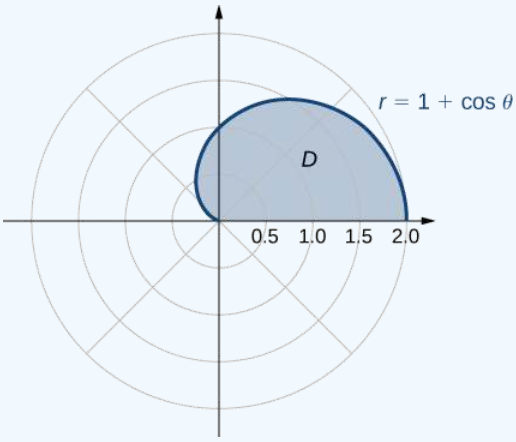


Figure 2.4.6: The region D is the top half of a cardioid.

Hence, we have

$$\iint_D r^2 \sin\theta\, r\, dr\, d\theta = \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=1+\cos\theta} (r^2 \sin\theta)\, r\, dr\, d\theta$$
$$= \frac{1}{4} \int_{\theta=0}^{\theta=\pi} [r^4]\Big|_{r=0}^{r=1+\cos\theta} \sin\theta\, d\theta$$
$$= \frac{1}{4} \int_{\theta=0}^{\theta=\pi} (1+\cos\theta)^4 \sin\theta\, d\theta$$
$$= -\frac{1}{4}\left[\frac{(1+\cos\theta)^5}{5}\right]_0^{\pi} = \frac{8}{5}.$$

? Exercise 2.4.3

Evaluate the integral

$$\iint_D r^2 \sin^2 2\theta\, r\, dr\, d\theta$$

where $D = \left\{(r,\theta) \,|\, 0 \le \theta \le \pi,\ 0 \le r \le 2\sqrt{\cos 2\theta}\right\}$.

Hint

Graph the region and follow the steps in the previous example.

Answer

$\frac{\pi}{8}$



Polar Areas and Volumes

As in rectangular coordinates, if a solid S is bounded by the surface $z = f(r,\theta)$, as well as by the surfaces $r = a$, $r = b$, $\theta = \alpha$, and $\theta = \beta$, we can find the volume V of S by double integration, as

$$V = \iint_R f(r,\theta)\, r\, dr\, d\theta = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=a}^{r=b} f(r,\theta)\, r\, dr\, d\theta. \tag{2.4.6}$$

If the base of the solid can be described as $D = \{(r,\theta) | \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$, then the double integral for the volume becomes

$$V = \iint_D f(r,\theta)\, r\, dr\, d\theta = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=h_1(\theta)}^{r=h_2(\theta)} f(r,\theta)\, r\, dr\, d\theta. \tag{2.4.7}$$

We illustrate this idea with some examples.

Example 2.4.4A: Finding a Volume Using a Double Integral

Find the volume of the solid that lies under the paraboloid $z = 1 - x^2 - y^2$ and above the unit circle on the xy-plane (Figure 2.4.7).

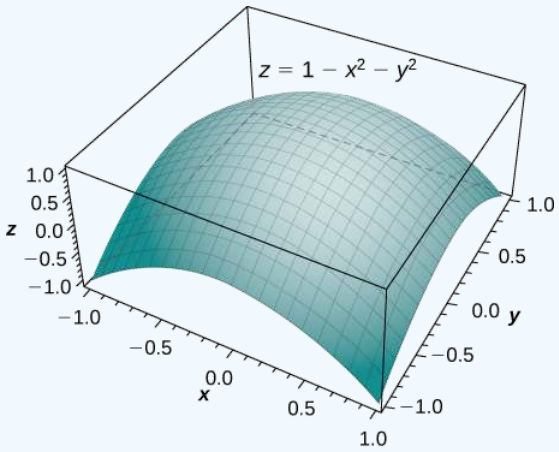


Figure 2.4.7: Finding the volume of a solid under a paraboloid and above the unit circle.

Solution

By the method of double integration, we can see that the volume is the iterated integral of the form

$$\iint_R (1 - x^2 - y^2)\, dA$$

where $R = \{(r, \theta) \mid 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$.

This integration was shown before in Example 2.4.2A, so the volume is $\frac{\pi}{2}$ cubic units.

Example 2.4.4B: Finding a Volume Using Double Integration

Find the volume of the solid that lies under the paraboloid $z = 4 - x^2 - y^2$ and above the disk $(x - 1)^2 + y^2 = 1$ on the xy-plane. See the paraboloid in Figure 2.4.8 intersecting the cylinder $(x - 1)^2 + y^2 = 1$ above the xy-plane.

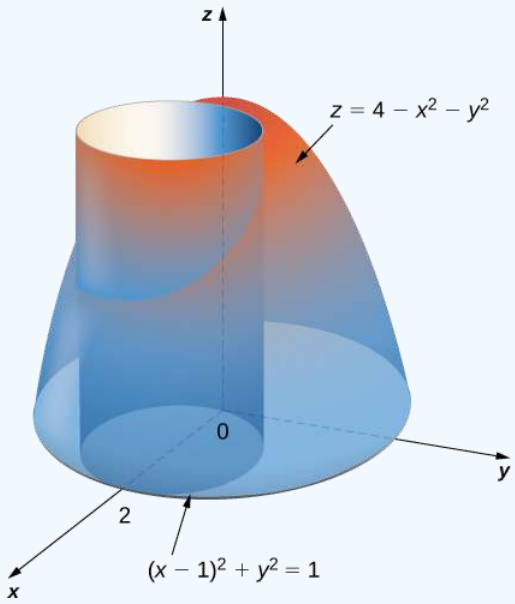


Figure 2.4.8: Finding the volume of a solid with a paraboloid cap and a circular base.

Solution

First change the disk $(x-1)^2+y^2=1$ to polar coordinates. Expanding the square term, we have $x^2-2x+1+y^2=1$. Then simplify to get $x^2+y^2=2x$, which in polar coordinates becomes $r^2=2r\cos\theta$ and then either $r=0$ or $r=2\cos\theta$. Similarly, the equation of the paraboloid changes to $z=4-r^2$. Therefore we can describe the disk $(x-1)^2+y^2=1$ on the xy -plane as the region

$$D=\{(r,\theta)\,|\,0\le\theta\le\pi,\ 0\le r\le 2\cos\theta\}.$$

Hence the volume of the solid bounded above by the paraboloid $z=4-x^2-y^2$ and below by $r=2\cos\theta$ is

$$\begin{aligned}V&=\iint_D f(r,\theta)\,r\,dr\,d\theta\\&=\int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=2\cos\theta}(4-r^2)\,r\,dr\,d\theta\\&=\int_{\theta=0}^{\theta=\pi}\left[4\frac{r^2}{2}-\frac{r^4}{4}\Big|_0^{2\cos\theta}\right]d\theta\\&=\int_0^{\pi}[8\cos^2\theta-4\cos^4\theta]\,d\theta\\&=\left[\frac{5}{2}\theta+\frac{5}{2}\sin\theta\cos\theta-\sin\theta\cos^3\theta\right]_0^{\pi}=\frac{5}{2}\pi\text{ units}^3.\end{aligned}$$

Notice in the next example that integration is not always easy with polar coordinates. Complexity of integration depends on the function and also on the region over which we need to perform the integration. If the region has a more natural expression in polar coordinates or if f has a simpler antiderivative in polar coordinates, then the change in polar coordinates is appropriate; otherwise, use rectangular coordinates.

Example 2.4.5A: Finding a Volume Using a Double Integral

Find the volume of the region that lies under the paraboloid $z=x^2+y^2$ and above the triangle enclosed by the lines $y=x$, $x=0$, and $x+y=2$ in the xy-plane.

Solution

First examine the region over which we need to set up the double integral and the accompanying paraboloid.

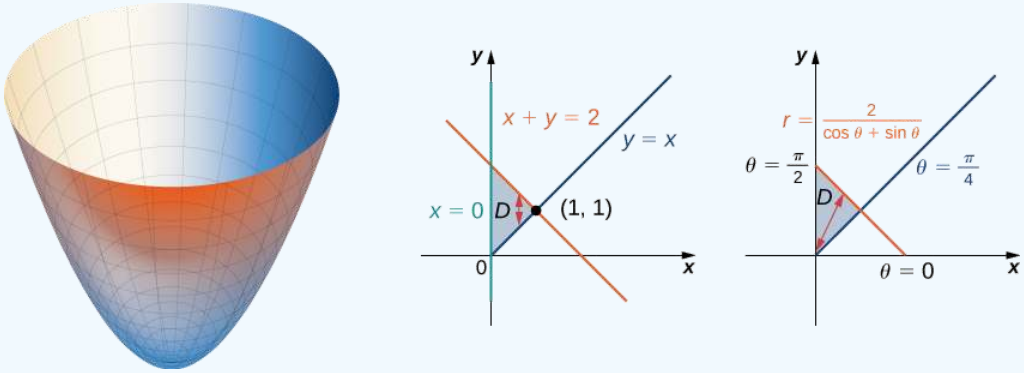


Figure 2.4.9: Finding the volume of a solid under a paraboloid and above a given triangle.

The region D is $\{(x,y)\,|\,0\le x\le 1,\ x\le y\le 2-x\}$. Converting the lines $y=x$, $x=0$, and $x+y=2$ in the xy-plane to functions of r and θ we have $\theta=\pi/4$, $\theta=\pi/2$, and $r=2/(\cos\theta+\sin\theta)$, respectively. Graphing the region on the xy- plane, we see that it looks like $D=\{(r,\theta)\,|\,\pi/4\le\theta\le\pi/2,\ 0\le r\le 2/(\cos\theta+\sin\theta)\}$.

Now converting the equation of the surface gives $z=x^2+y^2=r^2$. Therefore, the volume of the solid is given by the double integral

$$\begin{aligned} V &= \iint_D f(r,\theta)\, r\, dr\, d\theta \\ &= \int_{\theta=\pi/4}^{\theta=\pi/2} \int_{r=0}^{r=2/(\cos\theta+\sin\theta)} r^2 r\, dr d\theta \\ &= \int_{\pi/4}^{\pi/2} \left[\frac{r^4}{4}\right]_0^{2/(\cos\theta+\sin\theta)} d\theta \\ &= \frac{1}{4}\int_{\pi/4}^{\pi/2} \left(\frac{2}{\cos\theta+\sin\theta}\right)^4 d\theta \\ &= \frac{16}{4}\int_{\pi/4}^{\pi/2} \left(\frac{1}{\cos\theta+\sin\theta}\right)^4 d\theta \\ &= 4\int_{\pi/4}^{\pi/2} \left(\frac{1}{\cos\theta+\sin\theta}\right)^4 d\theta. \end{aligned}$$

As you can see, this integral is very complicated. So, we can instead evaluate this double integral in rectangular coordinates as

$$V = \int_0^1 \int_x^{2-x} (x^2+y^2)\, dy\, dx.$$

Evaluating gives

$$\begin{aligned} V &= \int_0^1 \int_x^{2-x} (x^2+y^2)\, dy\, dx \\ &= \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]\Big|_x^{2-x} dx \\ &= \int_0^1 \frac{8}{3} - 4x + 4x^2 - \frac{8x^3}{3}\, dx \\ &= \left[\frac{8x}{3} - 2x^2 + \frac{4x^3}{3} - \frac{2x^4}{3}\right]\Big|_0^1 \\ &= \frac{4}{3}\ \text{units}^3. \end{aligned}$$

To answer the question of how the formulas for the volumes of different standard solids such as a sphere, a cone, or a cylinder are found, we want to demonstrate an example and find the volume of an arbitrary cone.

Example 2.4.5B: Finding a Volume Using a Double Integral

Use polar coordinates to find the volume inside the cone $z = 2 - \sqrt{x^2+y^2}$ and above the xy-plane.

Solution

The region D for the integration is the base of the cone, which appears to be a circle on the xy-plane (Figure 2.4.10).

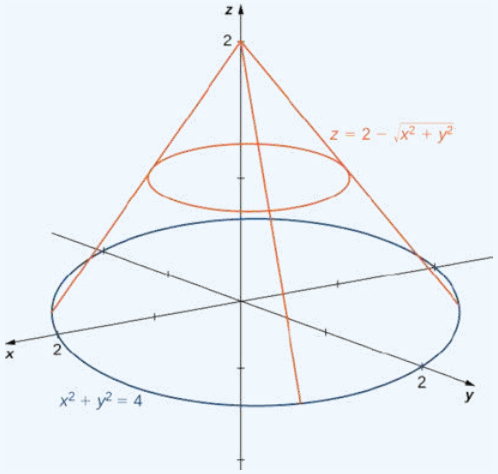


Figure 2.4.10: Finding the volume of a solid inside the cone and above the xy-plane.

We find the equation of the circle by setting $z = 0$:

$$\begin{aligned} 0 &= 2 - \sqrt{x^2 + y^2} \\ 2 &= \sqrt{x^2 + y^2} \\ x^2 + y^2 &= 4. \end{aligned}$$

This means the radius of the circle is 2 so for the integration we have $0 \le \theta \le 2\pi$ and $0 \le r \le 2$. Substituting $x = r\cos\theta$ and $y = r\sin\theta$ in the equation $z = 2 - \sqrt{x^2 + y^2}$ we have $z = 2 - r$. Therefore, the volume of the cone is

$$\int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} (2 - r)\, r\, dr\, d\theta = 2\pi \frac{4}{3} = \frac{8\pi}{3} \text{ cubic units.}$$

Analysis

Note that if we were to find the volume of an arbitrary cone with radius a units and height h units, then the equation of the cone would be $z = h - \frac{h}{a}\sqrt{x^2 + y^2}$.

We can still use Figure **2.4.10** and set up the integral as

$$\int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=a} \left(h - \frac{h}{a} r\right) r\, dr\, d\theta.$$

Evaluating the integral, we get $\frac{1}{3}\pi a^2 h$.

? Exercise 2.4.5

Use polar coordinates to find an iterated integral for finding the volume of the solid enclosed by the paraboloids $z = x^2 + y^2$ and $z = 16 - x^2 - y^2$.

Hint

Sketching the graphs can help.

Answer

$$V = \int_0^{2\pi} \int_0^{2\sqrt{2}} (16 - 2r^2)\, r\, dr\, d\theta = 64\pi \text{ cubic units.}$$

As with rectangular coordinates, we can also use polar coordinates to find areas of certain regions using a double integral. As before, we need to understand the region whose area we want to compute. Sketching a graph and identifying the region can be helpful to realize the limits of integration. Generally, the area formula in double integration will look like

$$\text{Area of } A = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} 1\, r\, dr\, d\theta. \tag{2.4.8}$$

✓ Example 2.4.6A: Finding an Area Using a Double Integral in Polar Coordinates

Evaluate the area bounded by the curve $r = \cos 4\theta$.

Solution

Sketching the graph of the function $r = \cos 4\theta$ reveals that it is a polar rose with eight petals (see the following figure).

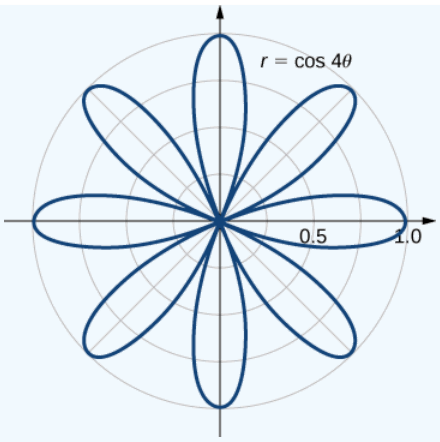


Figure 2.4.11: Finding the area of a polar rose with eight petals.

Using symmetry, we can see that we need to find the area of one petal and then multiply it by 8. Notice that the values of θ for which the graph passes through the origin are the zeros of the function $\cos\ 4\theta$, and these are odd multiples of $\pi/8$. Thus, one of the petals corresponds to the values of θ in the interval $[-\pi/8, \pi/8]$. Therefore, the area bounded by the curve $r = \cos\ 4\theta$ is

$$\begin{aligned} A &= 8\int_{\theta=-\pi/8}^{\theta=\pi/8}\int_{r=0}^{r=\cos\ 4\theta} 1\, r\, dr\, d\theta \\ &= 8\int_{\theta=-\pi/8}^{\theta=\pi/8}\left[\frac{1}{2}r^2\Big|_0^{\cos\ 4\theta}\right] d\theta \\ &= 8\int_{-\pi/8}^{\pi/8}\frac{1}{2}\cos^2 4\theta\, d\theta \\ &= 8\left[\frac{1}{4}\theta + \frac{1}{16}\sin 4\theta\cos 4\theta\Big|_{-\pi/8}^{\pi/8}\right] \\ &= 8\left[\frac{\pi}{16}\right] = \frac{\pi}{2}\ \text{units}^2. \end{aligned}$$

Example 2.4.6B: Finding Area Between Two Polar Curves

Find the area enclosed by the circle $r = 3\ \cos\ \theta$ and the cardioid $r = 1 + \cos\ \theta$.

Solution

First and foremost, sketch the graphs of the region (Figure 2.4.12).

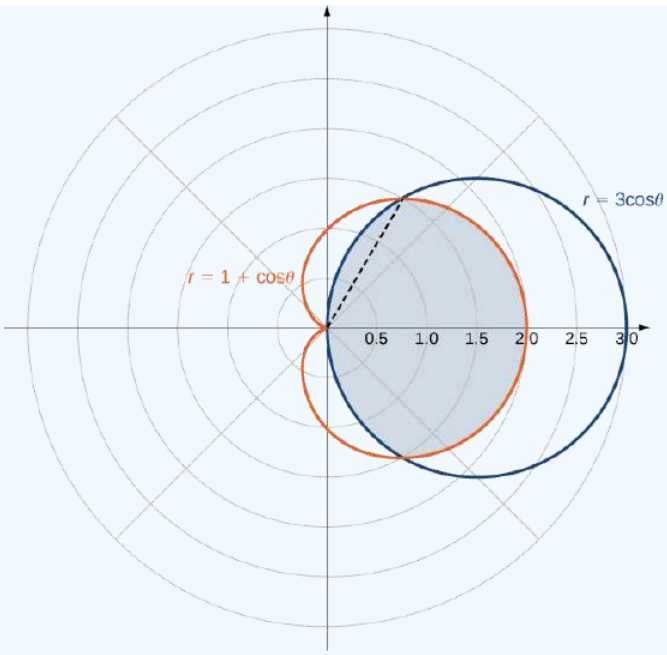


Figure 2.4.12: Finding the area enclosed by both a circle and a cardioid.

We can from see the symmetry of the graph that we need to find the points of intersection. Setting the two equations equal to each other gives

$$3\cos\theta = 1 + \cos\theta.$$

One of the points of intersection is $\theta = \pi/3$. The area above the polar axis consists of two parts, with one part defined by the cardioid from $\theta = 0$ to $\theta = \pi/3$ and the other part defined by the circle from $\theta = \pi/3$ to $\theta = \pi/2$. By symmetry, the total area is twice the area above the polar axis. Thus, we have

$$A = 2\left[\int_{\theta=0}^{\theta=\pi/3}\int_{r=0}^{r=1+\cos\theta} 1\,r\,dr\,d\theta + \int_{\theta=\pi/3}^{\theta=\pi/2}\int_{r=0}^{r=3\cos\theta} 1\,r\,dr\,d\theta\right].$$

Evaluating each piece separately, we find that the area is

$$A = 2\left(\frac{1}{4}\pi + \frac{9}{16}\sqrt{3} + \frac{3}{8}\pi - \frac{9}{16}\sqrt{3}\right) = 2\left(\frac{5}{8}\pi\right) = \frac{5}{4}\pi \text{ square units.}$$

? Exercise 2.4.6

Find the area enclosed inside the cardioid $r = 3 - 3\sin\theta$ and outside the cardioid $r = 1 + \sin\theta$.

Hint

Sketch the graph, and solve for the points of intersection.

Answer

$$A = 2\int_{-\pi/2}^{\pi/6}\int_{1+\sin\theta}^{3-3\sin\theta} r\,dr\,d\theta = \left(8\pi + 9\sqrt{3}\right) \text{ units}^2$$

✓ Example 2.4.7: Evaluating an Improper Double Integral in Polar Coordinates

Evaluate the integral

$$\iint_{R^2} e^{-10(x^2+y^2)}\,dx\,dy.$$

Solution

This is an improper integral because we are integrating over an unbounded region R^2. In polar coordinates, the entire plane R^2 can be seen as $0 \le \theta \le 2\pi$, $0 \le r \le \infty$.

Using the changes of variables from rectangular coordinates to polar coordinates, we have

$$\begin{aligned}\iint_{R^2} e^{-10(x^2+y^2)}\,dx\,dy &= \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=\infty} e^{-10r^2}\,r\,dr\,d\theta = \int_{\theta=0}^{\theta=2\pi}\left(\lim_{a\to\infty}\int_{r=0}^{r=a} e^{-10r^2}r\,dr\right)d\theta \\ &= \left(\int_{\theta=0}^{\theta=2\pi}\right)d\theta\left(\lim_{a\to\infty}\int_{r=0}^{r=a} e^{-10r^2}r\,dr\right) \\ &= 2\pi\left(\lim_{a\to\infty}\int_{r=0}^{r=a} e^{-10r^2}r\,dr\right) \\ &= 2\pi\lim_{a\to\infty}\left(-\frac{1}{20}\right)\left(e^{-10r^2}\Big|_0^a\right) \\ &= 2\pi\left(-\frac{1}{20}\right)\lim_{a\to\infty}\left(e^{-10a^2}-1\right) \\ &= \frac{\pi}{10}.\end{aligned}$$

? Exercise 2.4.7

Evaluate the integral

$$\iint_{R^2} e^{-4(x^2+y^2)}dx\,dy.$$

Hint

Convert to the polar coordinate system.

Answer

$\frac{\pi}{4}$

Key Concepts

- To apply a double integral to a situation with circular symmetry, it is often convenient to use a double integral in polar coordinates. We can apply these double integrals over a polar rectangular region or a general polar region, using an iterated integral similar to those used with rectangular double integrals.
- The area dA in polar coordinates becomes $r\,dr\,d\theta$.
- Use $x = r\cos\theta$, $y = r\sin\theta$, and $dA = r\,dr\,d\theta$ to convert an integral in rectangular coordinates to an integral in polar coordinates.
- Use $r^2 = x^2 + y^2$ and $\theta = tan^{-1}\left(\frac{y}{x}\right)$ to convert an integral in polar coordinates to an integral in rectangular coordinates, if needed.
- To find the volume in polar coordinates bounded above by a surface $z = f(r,\theta)$ over a region on the xy-plane, use a double integral in polar coordinates.

Key Equations

- **Double integral over a polar rectangular region R**

$$\iint_R f(r,\theta)dA = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_{ij}^*,\theta_{ij}^*)\Delta A = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_{ij}^*,\theta_{ij}^*)r_{ij}^*\Delta r\Delta\theta$$

- **Double integral over a general polar region**

$$\iint_D f(r,\theta)\,r\,dr\,d\theta = \int_{\theta=\alpha}^{\theta=\beta}\int_{r=h_1(\theta)}^{r_2(\theta)} f(r,\theta)\,r\,dr\,d\theta$$

Glossary

polar rectangle

the region enclosed between the circles $r = a$ and $r = b$ and the angles $\theta = \alpha$ and $\theta = \beta$; it is described as $R = \{(r, \theta) \mid a \leq r \leq b, \alpha \leq \theta \leq \beta\}$

Contributors and Attributions

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2.4E: Exercises

Terms and Concepts

1. When evaluating $\int\int_R f(x,y)\,dA$ using polar coordinates, $f(x,y)$ is replaced with _____ and dA is replaced with _____.

Answer

$f(x,y)$ is replaced with $f(r\cos\theta, r\sin\theta)$ and dA is replaced with $r\,dr\,d\theta$.

2. Why would one be interested in evaluating a double integral with polar coordinates?

Defining Polar Regions

In exercises 3 - 6, express the region R in polar coordinates.

3) R is the region of the disk of radius 2 centered at the origin that lies in the first quadrant.

Answer

$R = \{(r,\theta)\,|\,0 \le r \le 2,\ 0 \le \theta \le \frac{\pi}{2}\}$

4) R is the region of the disk of radius 3 centered at the origin.

5) R is the region between the circles of radius 4 and radius 5 centered at the origin that lies in the second quadrant.

Answer

$R = \{(r,\theta)\,|\,4 \le r \le 5,\ \frac{\pi}{2} \le \theta \le \pi\}$

6) R is the region bounded by the y-axis and $x = \sqrt{1-y^2}$.

7) R is the region bounded by the x-axis and $y = \sqrt{2-x^2}$.

Answer

$R = \{(r,\theta)\,|\,0 \le r \le \sqrt{2},\ 0 \le \theta \le \pi\}$

8) $R = \{(x,y)\,|\,x^2+y^2 \le 4x\}$

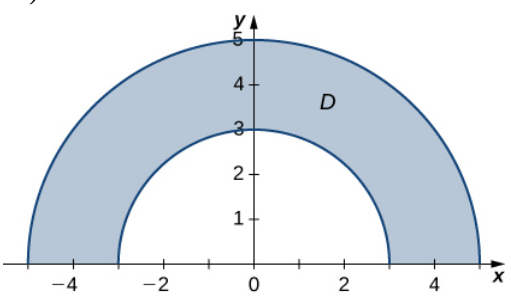
9) $R = \{(x,y)\,|\,x^2+y^2 \le 4y\}$

Answer

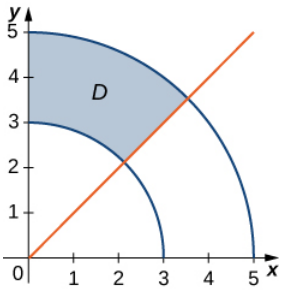
$R = \{(r,\theta)\,|\,0 \le r \le 4\,\sin\theta,\ 0 \le \theta \le \pi\}$

In exercises 10 - 15, the graph of the polar rectangular region D is given. Express D in polar coordinates. In exercises 10 - 13 the regions are bounded by circles centered at the origin, the coordinate axes, and/or lines of the form $y = \pm x$ (each of which should be able to be identified by inspection).

10)



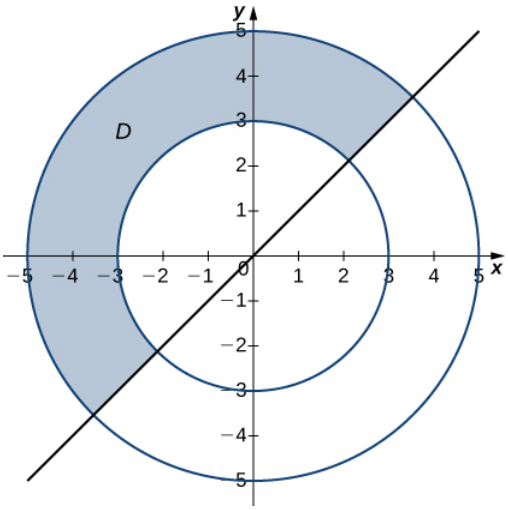
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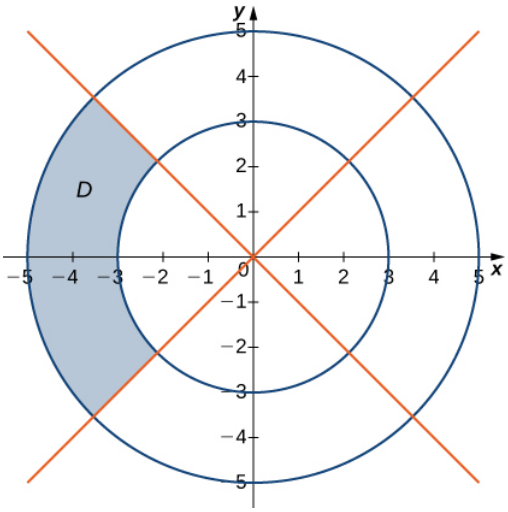
Answer

$D = \{(r, \theta) \mid 3 \leq r \leq 5,\ \frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}\}$

12)



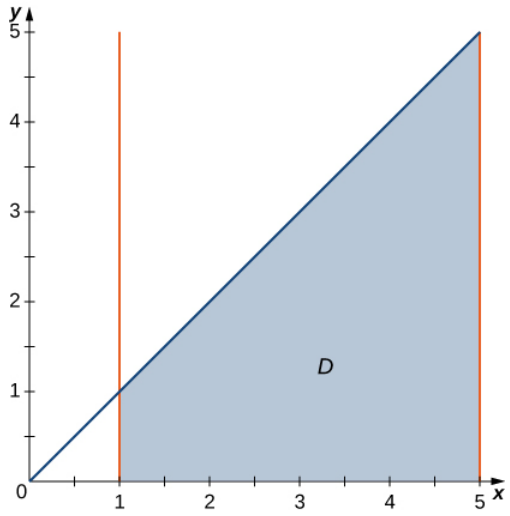
13)



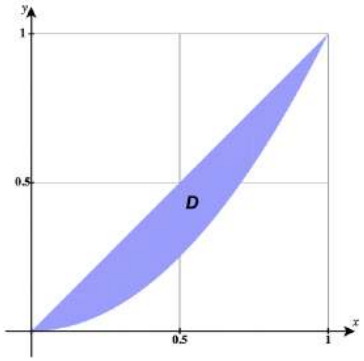
Answer

$D = \{(r, \theta) \mid 3 \leq r \leq 5,\ \frac{3\pi}{4} \leq \theta \leq \frac{5\pi}{4}\}$

14) In the following graph, the region D is situated below $y = x$ and is bounded by $x = 1,\ x = 5$, and $y = 0$.



15) In the following graph, the region D is bounded by $y = x$ and $y = x^2$.



Answer

$D = \{(r, \theta) \mid 0 \le r \le \tan\theta \sec\theta,\ 0 \le \theta \le \frac{\pi}{4}\}$

Evaluating Polar Double Integrals

In exercises 16 - 25, evaluate the double integral $\iint_R f(x,y)\,dA$ over the polar rectangular region R.

16) $f(x,y) = x^2 + y^2$, $R = \{(r, \theta) \mid 3 \le r \le 5,\ 0 \le \theta \le 2\pi\}$

17) $f(x,y) = x + y$, $R = \{(r, \theta) \mid 3 \le r \le 5,\ 0 \le \theta \le 2\pi\}$

Answer

0

18) $f(x,y) = x^2 + xy$, $R = \{(r, \theta) \mid 1 \le r \le 2,\ \pi \le \theta \le 2\pi\}$

19) $f(x,y) = x^4 + y^4$, $R = \{(r, \theta) \mid 1 \le r \le 2,\ \frac{3\pi}{2} \le \theta \le 2\pi\}$

Answer

$\frac{63\pi}{16}$

20) $f(x,y) = \sqrt[3]{x^2 + y^2}$, where $R = \{(r, \theta) \mid 0 \le r \le 1,\ \frac{\pi}{2} \le \theta \le \pi\}$.

21) $f(x,y) = x^4 + 2x^2y^2 + y^4$, where $R = \{(r, \theta) \mid 3 \le r \le 4,\ \frac{\pi}{3} \le \theta \le \frac{2\pi}{3}\}$.

Answer

$\frac{3367\pi}{18}$

22) $f(x,y) = \sin(\arctan \frac{y}{x})$, where $R = \{(r,\theta) \mid 1 \le r \le 2,\ \frac{\pi}{6} \le \theta \le \frac{\pi}{3}\}$

23) $f(x,y) = \arctan\left(\frac{y}{x}\right)$, where $R = \{(r,\theta) \mid 2 \le r \le 3,\ \frac{\pi}{4} \le \theta \le \frac{\pi}{3}\}$

Answer

$\frac{35\pi^2}{576}$

24) $\iint_R e^{x^2+y^2}\left[1 + 2\ \arctan\left(\frac{y}{x}\right)\right] dA,\ R = \{(r,\theta) \mid 1 \le r \le 2,\ \frac{\pi}{6} \le \theta \frac{\pi}{3}\}$

25) $\iint_R \left(e^{x^2+y^2} + x^4 + 2x^2y^2 + y^4\right) \arctan\left(\frac{y}{x}\right)\ dA,\ R = \{(r,\theta) \mid 1 \le r \le 2,\ \frac{\pi}{4} \le \theta \le \frac{\pi}{3}\}$

Answer

$\frac{7}{576}\pi^2(21 - e + e^4)$

Converting Double Integrals to Polar Form

In exercises 26 - 29, the integrals have been converted to polar coordinates. Verify that the identities are true and choose the easiest way to evaluate the integrals, in rectangular or polar coordinates.

26) $\int_1^2 \int_0^x (x^2 + y^2)\, dy\, dx = \int_0^{\frac{\pi}{4}} \int_{\sec\theta}^{2\ \sec\theta} r^3\, dr\, d\theta$

27) $\int_2^3 \int_0^x \frac{x}{\sqrt{x^2+y^2}}\, dy\, dx = \int_0^{\pi/4} \int_{2\sec\theta}^{3\sec\theta} r\ \cos\theta\, dr\, d\theta$

Answer

$\frac{5}{2}\ln(1+\sqrt{2})$

28) $\int_0^1 \int_{x^2}^x \frac{1}{\sqrt{x^2+y^2}}\, dy\, dx = \int_0^{\pi/4} \int_0^{\tan\theta\ \sec\theta} dr\, d\theta$

29) $\int_0^1 \int_{x^2}^x \frac{y}{\sqrt{x^2+y^2}}\, dy\, dx = \int_0^{\pi/4} \int_0^{\tan\theta\ \sec\theta} r\ \sin\theta\, dr\, d\theta$

Answer

$\frac{1}{6}(2 - \sqrt{2})$

In exercises 30 - 37, draw the region of integration, R, labeling all limits of integration, convert the integrals to polar coordinates and evaluate them.

30) $\int_0^3 \int_0^{\sqrt{9-y^2}} (x^2 + y^2)\ dx\ dy$

31) $\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \left(x^2+y^2\right)^2\ dx\ dy$

Answer

$\int_0^{\pi} \int_0^2 r^5\, dr\, d\theta \quad = \quad \frac{32\pi}{3}$

32) $\int_0^1 \int_0^{\sqrt{1-x^2}} (x + y)\ dy\ dx$

33) $\int_0^4 \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \sin(x^2 + y^2)\ dy\ dx$

Answer

$\int_{-\pi/2}^{\pi/2} \int_0^4 r\ \sin(r^2)\ dr\ d\theta \quad = \quad \pi\ \sin^2 8$

34) $\int_0^5 \int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} \sqrt{x^2+y^2}\, dy\, dx$

35) $\int_{-4}^{4} \int_{-\sqrt{16-y^2}}^{0} (2y-x)\, dx\, dy$

Answer

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_0^4 \left(2r\sin\theta - r\cos\theta\right) r\, dr\, d\theta \quad = \quad \frac{128}{3}$$

36) $\int_0^2 \int_y^{\sqrt{8-y^2}} (x+y)\, dx\, dy$

37) $\int_{-2}^{-1} \int_0^{\sqrt{4-x^2}} (x+5)\, dy\, dx + \int_{-1}^{1} \int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} (x+5)\, dy\, dx + \int_1^2 \int_0^{\sqrt{4-x^2}} (x+5)\, dy\, dx$

Answer

$$\int_0^{\pi} \int_1^2 \left(r\cos\theta + 5\right) r\, dr\, d\theta \quad = \quad \frac{15\pi}{2}$$

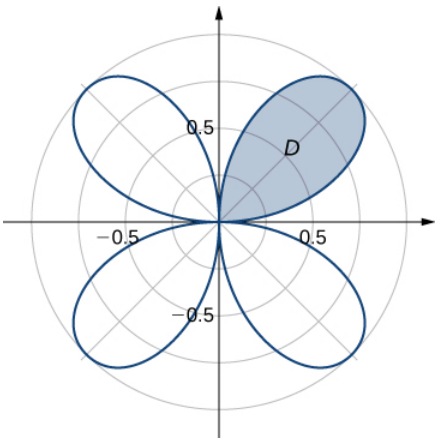
38) Evaluate the integral $\iint_D r\, dA$ where D is the region bounded by the polar axis and the upper half of the cardioid $r = 1+\cos\theta$.

39) Find the area of the region D bounded by the polar axis and the upper half of the cardioid $r = 1+\cos\theta$.

Answer

$\frac{3\pi}{4}$

40) Evaluate the integral $\iint_D r\, dA$, where D is the region bounded by the part of the four-leaved rose $r = \sin 2\theta$ situated in the first quadrant (see the following figure).



41) Find the total area of the region enclosed by the four-leaved rose $r = \sin 2\theta$ (see the figure in the previous exercise).

Answer

$\frac{\pi}{2}$

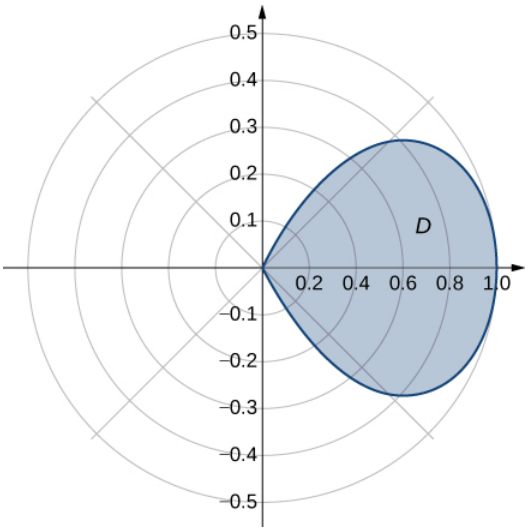
42) Find the area of the region D which is the region bounded by $y = \sqrt{4-x^2}$, $x = \sqrt{3}$, $x = 2$, and $y = 0$.

43) Find the area of the region D, which is the region inside the disk $x^2+y^2 \le 4$ and to the right of the line $x = 1$.

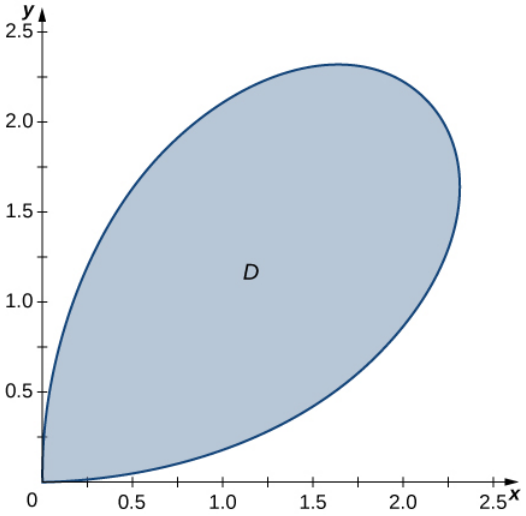
Answer

$\frac{1}{3}(4\pi - 3\sqrt{3})$

44) Determine the average value of the function $f(x,y) = x^2+y^2$ over the region D bounded by the polar curve $r = \cos 2\theta$, where $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$ (see the following graph).



45) Determine the average value of the function $f(x,y) = \sqrt{x^2+y^2}$ over the region D bounded by the polar curve $r = 3\sin 2\theta$, where $0 \le \theta \le \frac{\pi}{2}$ (see the following graph).



Answer

$\frac{16}{3\pi}$

46) Find the volume of the solid situated in the first octant and bounded by the paraboloid $z = 1 - 4x^2 - 4y^2$ and the planes $x = 0$, $y = 0$, and $z = 0$.

47) Find the volume of the solid bounded by the paraboloid $z = 2 - 9x^2 - 9y^2$ and the plane $z = 1$.

Answer

$\frac{\pi}{18}$

48)

1. Find the volume of the solid S_1 bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = 0$ and $z = 1$.
2. Find the volume of the solid S_2 outside the double cone $z^2 = x^2 + y^2$ inside the cylinder $x^2 + y^2 = 1$, and above the plane $z = 0$.
3. Find the volume of the solid inside the cone $z^2 = x^2 + y^2$ and below the plane $z = 1$ by subtracting the volumes of the solids S_1 and S_2.

49)

1. Find the volume of the solid S_1 inside the unit sphere $x^2+y^2+z^2=1$ and above the plane $z=0$.
2. Find the volume of the solid S_2 inside the double cone $(z-1)^2=x^2+y^2$ and above the plane $z=0$.
3. Find the volume of the solid outside the double cone $(z-1)^2=x^2+y^2$ and inside the sphere $x^2+y^2+z^2=1$.

Answer

a. $\frac{2\pi}{3}$; b. $\frac{\pi}{2}$; c. $\frac{\pi}{6}$

In Exercises 50-51, special double integrals are presented that are especially well suited for evaluation in polar coordinates.

50) The surface of a right circular cone with height h and base radius a can be described by the equation $f(x,y)=h-h\sqrt{\frac{x^2}{a^2}+\frac{y^2}{a^2}}$, where the tip of the cone lies at $(0,0,h)$ and the circular base lies in the xy-plane, centered at the origin.
Confirm that the volume of a right circular cone with height h and base radius a is $V=\frac{1}{3}\pi a^2 h$ by evaluating $\int\int_R f(x,y)\,dA$ in polar coordinates.

51) Consider $\int\int_R e^{-(x^2+y^2)}\,dA$.
(a) Why is this integral difficult to evaluate in rectangular coordinates, regardless of the region R?
(b) Let R be the region bounded by the circle of radius a centered at the origin. Evaluate the double integral using polar coordinates.
(c) Take the limit of your answer from (b), as $a\to\infty$. What does this imply about the volume under the surface of $e^{-(x^2+y^2)}$ over the entire xy-plane?

For the following two exercises, consider a spherical ring, which is a sphere with a cylindrical hole cut so that the axis of the cylinder passes through the center of the sphere (see the following figure).



52) If the sphere has radius 4 and the cylinder has radius 2 find the volume of the spherical ring.

53) A cylindrical hole of diameter 6 cm is bored through a sphere of radius 5 cm such that the axis of the cylinder passes through the center of the sphere. Find the volume of the resulting spherical ring.

Answer

$\frac{256\pi}{3}$ cm^3

54) Find the volume of the solid that lies under the double cone $z^2=4x^2+4y^2$, inside the cylinder $x^2+y^2=x$, and above the plane $z=0$.

55) Find the volume of the solid that lies under the paraboloid $z=x^2+y^2$, inside the cylinder $x^2+y^2=1$ and above the plane $z=0$.

Answer

$\frac{3\pi}{32}$

56) Find the volume of the solid that lies under the plane $x+y+z=10$ and above the disk $x^2+y^2=4x$.

57) Find the volume of the solid that lies under the plane $2x+y+2z=8$ and above the unit disk $x^2+y^2=1$.

Answer

4π

58) A radial function f is a function whose value at each point depends only on the distance between that point and the origin of the system of coordinates; that is, $f(x,y) = g(r)$, where $r = \sqrt{x^2+y^2}$. Show that if f is a continuous radial function, then

$$\iint_D f(x,y)dA = (\theta_2 - \theta_1)[G(R_2) - G(R_1)], \text{ where } G'(r) = rg(r)$$

and $(x,y) \in D = \{(r,\theta) \mid R_1 \le r \le R_2,\ 0 \le \theta \le 2\pi\}$, with $0 \le R_1 < R_2$ and $0 \le \theta_1 < \theta_2 \le 2\pi$.

59) Use the information from the preceding exercise to calculate the integral $\iint_D (x^2+y^2)^3 dA$, where D is the unit disk.

Answer

$\frac{\pi}{4}$

60) Let $f(x,y) = \dfrac{F'(r)}{r}$ be a continuous radial function defined on the annular region $D = \{(r,\theta) \mid R_1 \le r \le R_2,\ 0 \le \theta \le 2\pi\}$, where $r = \sqrt{x^2+y^2}$, $0 < R_1 < R_2$, and F is a differentiable function.

Show that $\iint_D f(x,y)\,dA = 2\pi[F(R_2) - F(R_1)]$.

61) Apply the preceding exercise to calculate the integral $\iint_D \dfrac{e^{\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}}\,dx\,dy$ where D is the annular region between the circles of radii 1 and 2 situated in the third quadrant.

Answer

$\frac{1}{2}\pi e(e-1)$

62) Let f be a continuous function that can be expressed in polar coordinates as a function of θ only; that is, $f(x,y) = h(\theta)$, where $(x,y) \in D = \{(r,\theta) \mid R_1 \le r \le R_2,\ \theta_1 \le \theta \le \theta_2\}$, with $0 \le R_1 < R_2$ and $0 \le \theta_1 < \theta_2 \le 2\pi$.

Show that $\iint_D f(x,y)\,dA = \dfrac{1}{2}(R_2^2 - R_1^2)[H(\theta_2) - H(\theta_1)]$, where H is an antiderivative of h.

63) Apply the preceding exercise to calculate the integral $\iint_D \dfrac{y^2}{x^2}\,dA$, where $D = \{(r,\theta) \mid 1 \le r \le 2,\ \frac{\pi}{6} \le \theta \le \frac{\pi}{3}\}$.

Answer

$\sqrt{3} - \frac{\pi}{4}$

64) Let f be a continuous function that can be expressed in polar coordinates as a function of θ only; that is $f(x,y) = g(r)h(\theta)$, where $(x,y) \in \{(r,\theta) \mid R_1 \le r \le R_2,\ \theta_1 \le \theta \le \theta_2\}$ with $0 \le R_1 < R_2$ and $0 \le \theta_1 < \theta_2 \le 2\pi$. Show that

$$\iint_D f(x,y)\,dA = [G(R_2) - G(R_1)]\ [H(\theta_2) - H(\theta_1)],$$

where G and H are antiderivatives of g and h, respectively.

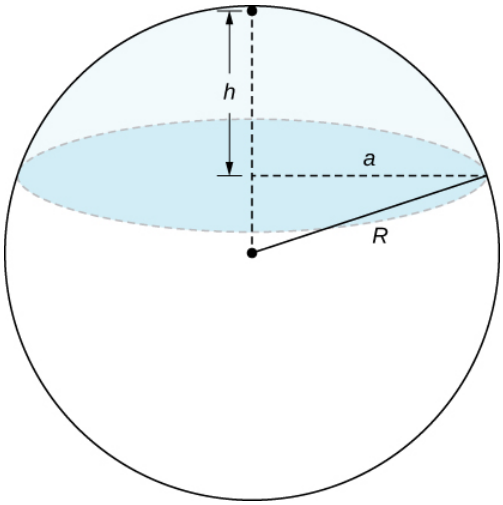
65) Evaluate $\iint_D \arctan\left(\dfrac{y}{x}\right)\sqrt{x^2+y^2}\,dA$, where $D = \{(r,\theta) \mid 2 \le r \le 3,\ \frac{\pi}{4} \le \theta \le \frac{\pi}{3}\}$.

Answer

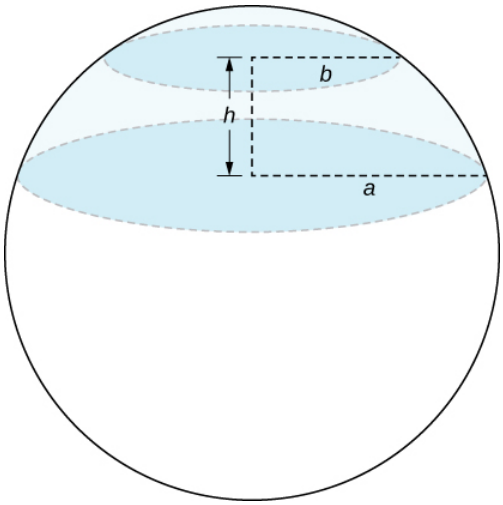
$\frac{133}{864}\pi^2$

66) A spherical cap is the region of a sphere that lies above or below a given plane.

a. Show that the volume of the spherical cap in the figure below is $\frac{1}{6}\pi h(3a^2+h^2)$.



b. A spherical segment is the solid defined by intersecting a sphere with two parallel planes. If the distance between the planes is h show that the volume of the spherical segment in the figure below is $\frac{1}{6}\pi h(3a^2+3b^2+h^2)$.



67) In statistics, the joint density for two independent, normally distributed events with a mean $\mu = 0$ and a standard distribution σ is defined by $p(x,y) = \frac{1}{2\pi\sigma^2}e^{-\frac{x^2+y^2}{2\sigma^2}}$. Consider (X,Y), the Cartesian coordinates of a ball in the resting position after it was released from a position on the z-axis toward the xy-plane. Assume that the coordinates of the ball are independently normally distributed with a mean $\mu = 0$ and a standard deviation of σ (in feet). The probability that the ball will stop no more than a feet from the origin is given by

$$P[X^2+Y^2 \le a^2] = \iint_D p(x,y)dy\,dx,$$

where D is the disk of radius a centered at the origin. Show that

$$P[X^2+Y^2 \le a^2] = 1 - e^{-a^2/2\sigma^2}.$$

68) The double improper integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2+y^2/2}\,dy\,dx$$

may be defined as the limit value of the double integrals $\iint_D e^{-x^2+y^2/2}\,dA$ over disks D_a of radii a centered at the origin, as a increases without bound; that is,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2+y^2/2}dy\,dx = \lim_{a\to\infty}\iint_{D_a} e^{-x^2+y^2/2}\,dA.$$

Use polar coordinates to show that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2+y^2/2}\,dy\,dx = 2\pi.$

69) Show that $\int_{-\infty}^{\infty} e^{-x^2/2}\,dx = \sqrt{2\pi}$ by using the relation

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2+y^2/2}\,dy\,dx = \left(\int_{-\infty}^{\infty} e^{-x^2/2}dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2/2}dy\right).$$

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- Problems 1, 2, 34 - 37 and 50 - 51 are from Apex Calculus, Chapter 13.3
- Edited by Paul Seeburger (Monroe Community College)

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2.5: Triple Integrals

Learning Objectives

- Recognize when a function of three variables is integrable over a rectangular box.
- Evaluate a triple integral by expressing it as an iterated integral.
- Recognize when a function of three variables is integrable over a closed and bounded region.
- Simplify a calculation by changing the order of integration of a triple integral.
- Calculate the average value of a function of three variables.

Previously, we discussed the double integral of a function $f(x,y)$ of two variables over a rectangular region in the plane. In this section we define the triple integral of a function $f(x,y,z)$ of three variables over a rectangular solid box in space, $\mathbb{R}^3$. Later in this section we extend the definition to more general regions in $\mathbb{R}^3$.

Integrable Functions of Three Variables

We can define a rectangular box B in $\mathbb{R}^3$ as

$$B = \{(x,y,z) \,|\, a \le x \le b,\, c \le y \le d,\, e \le z \le f\}. \tag{2.5.1}$$

We follow a similar procedure to what we did in previously. We divide the interval $[a,b]$ into l subintervals $[x_{i-1}, x_i]$ of equal length Δx with

$$\Delta x = \frac{x_i - x_{i-1}}{l}, \tag{2.5.2}$$

divide the interval $[c,d]$ into m subintervals $[y_{i-1}, y_i]$ of equal length Δy with

$$\Delta y = \frac{y_j - y_{j-1}}{m}, \tag{2.5.3}$$

and divide the interval $[e,f]$ into n subintervals $[z_{i-1}, z_i]$ of equal length Δz with

$$\Delta z = \frac{z_k - z_{k-1}}{n} \tag{2.5.4}$$

Then the rectangular box B is subdivided into lmn subboxes:

$$B_{ijk} = [x_{i-1}, x_i] \times [y_{i-1}, y_i] \times [z_{i-1}, z_i], \tag{2.5.5}$$

as shown in Figure 2.5.1.

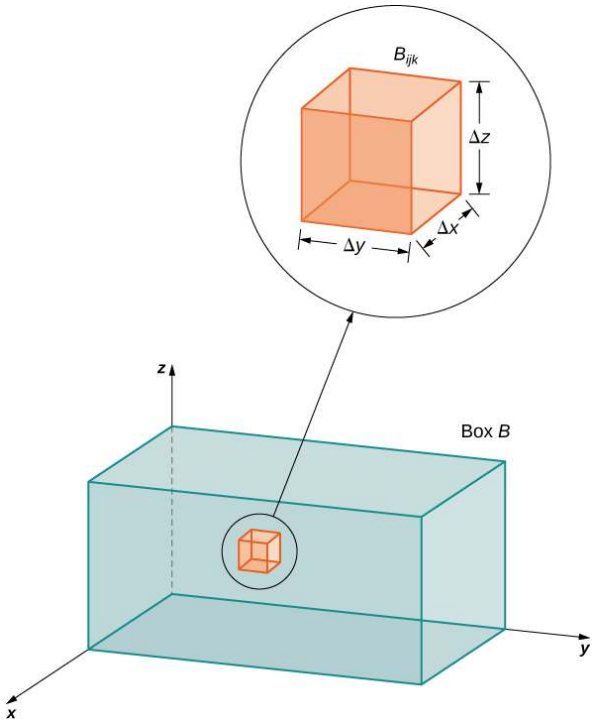


Figure 2.5.1: A rectangular box in $\mathbb{R}^3$ divided into subboxes by planes parallel to the coordinate planes.

For each i, j, and k, consider a sample point $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$ in each sub-box B_{ijk}. We see that its volume is $\Delta V = \Delta x \Delta y \Delta z$. Form the triple Riemann sum

$$\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\,\Delta x \Delta y \Delta z. \tag{2.5.6}$$

We define the triple integral in terms of the limit of a triple Riemann sum, as we did for the double integral in terms of a double Riemann sum.

Definition: The triple integral

The triple integral of a function $f(x, y, z)$ over a rectangular box B is defined as

$$\lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\,\Delta x \Delta y \Delta z = \iiint_B f(x, y, z)\,dV \tag{2.5.7}$$

if this limit exists.

When the triple integral exists on B the function $f(x, y, z)$ is said to be integrable on B. Also, the triple integral exists if $f(x, y, z)$ is continuous on B. Therefore, we will use continuous functions for our examples. However, continuity is sufficient but not necessary; in other words, f is bounded on B and continuous except possibly on the boundary of B. The sample point $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$ can be any point in the rectangular sub-box B_{ijk} and all the properties of a double integral apply to a triple integral. Just as the double integral has many practical applications, the triple integral also has many applications, which we discuss in later sections.

Now that we have developed the concept of the triple integral, we need to know how to compute it. Just as in the case of the double integral, we can have an iterated triple integral, and consequently, a version of **Fubini's theorem** for triple integrals exists.

Fubini's Theorem for Triple Integrals

If $f(x, y, z)$ is continuous on a rectangular box $B = [a, b] \times [c, d] \times [e, f]$, then

$$\iiint_B f(x, y, z)\,dV = \int_e^f \int_c^d \int_a^b f(x, y, z)\,dx\,dy\,dz. \tag{2.5.8}$$

This integral is also equal to any of the other five possible orderings for the iterated triple integral.

For a, b, c, d, e and f real numbers, the iterated triple integral can be expressed in six different orderings:

$$\int_e^f \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz = \int_e^f \left(\int_c^d \left(\int_a^b f(x,y,z)\,dx \right) dy \right) dz \tag{2.5.9}$$

$$= \int_c^d \left(\int_e^f \left(\int_a^b f(x,y,z)\,dx \right) dz \right) dy \tag{2.5.10}$$

$$= \int_a^b \left(\int_e^f \left(\int_c^d f(x,y,z)\,dy \right) dz \right) dx \tag{2.5.11}$$

$$= \int_e^f \left(\int_a^b \left(\int_c^d f(x,y,z)\,dy \right) dx \right) dz \tag{2.5.12}$$

$$= \int_c^d \left(\int_a^b \left(\int_c^d f(x,y,z)\,dz \right) dx \right) dy \tag{2.5.13}$$

$$= \int_a^b \left(\int_c^d \left(\int_e^f f(x,y,z)\,dz \right) dy \right) dx \tag{2.5.14}$$

For a rectangular box, the order of integration does not make any significant difference in the level of difficulty in computation. We compute triple integrals using Fubini's Theorem rather than using the Riemann sum definition. We follow the order of integration in the same way as we did for double integrals (that is, from inside to outside).

Example 2.5.1: Evaluating a Triple Integral

Evaluate the triple integral

$$\int_{z=0}^{z=1} \int_{y=2}^{y=4} \int_{x=-1}^{x=5} (x + yz^2)\,dx\,dy\,dz.$$

Solution

The order of integration is specified in the problem, so integrate with respect to x first, then y, and then z.

$$\begin{aligned}
&\int_{z=0}^{z=1} \int_{y=2}^{y=4} \int_{x=-1}^{x=5} (x + yz^2)\,dx\,dy\,dz \\
&= \int_{z=0}^{z=1} \int_{y=2}^{y=4} \left[\frac{x^2}{2} + xyz^2 \Big|_{x=-1}^{x=5} \right] dy\,dz \quad \text{Integrate with respect to } x. \\
&= \int_{z=0}^{z=1} \int_{y=2}^{y=4} \left[12 + 6yz^2 \right] dy\,dz \quad \text{Evaluate.} \\
&= \int_{z=0}^{z=1} \left[12y + 6\frac{y^2}{2} z^2 \Big|_{y=2}^{y=4} \right] dz \quad \text{Integrate with respect to } y. \\
&= \int_{z=0}^{z=1} \left[24 + 36z^2 \right] dz \quad \text{Evaluate.} \\
&= \left[24z + 36\frac{z^3}{3} \right]_{z=0}^{z=1} \quad \text{Integrate with respect to } z. \\
&= 36. \quad \text{Evaluate.}
\end{aligned}$$

Example 2.5.2: Evaluating a Triple Integral

Evaluate the triple integral

$$\iiint_B x^2 yz\,dV \tag{2.5.15}$$

where $B = \{(x,y,z) \mid -2 \le x \le 1,\ 0 \le y \le 3,\ 1 \le z \le 5\}$ as shown in Figure 2.5.2.

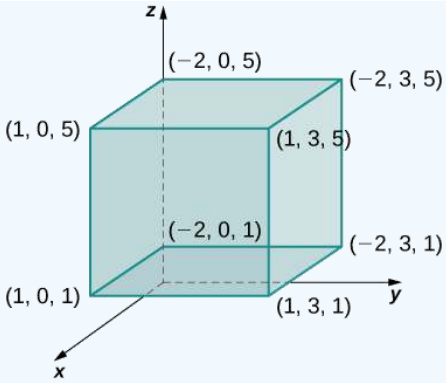


Figure 2.5.2: Evaluating a triple integral over a given rectangular box.

Solution

The order is not specified, but we can use the iterated integral in any order without changing the level of difficulty. Choose, say, to integrate y first, then x, and then z.

$$\begin{aligned}\iiint_B x^2yz\,dV &= \int_1^5\int_{-2}^1\int_0^3 [x^2yz]\,dy\,dx\,dz \\ &= \int_1^5\int_{-2}^1 \left[x^2\frac{y^3}{3}z\Big|_0^3\right] dx\,dz \\ &= \int_1^5\int_{-2}^1 \frac{y}{2}x^2z\,dx\,dz \\ &= \int_1^5 \left[\frac{9}{2}\frac{x^3}{3}z\Big|_{-2}^1\right] dz = \int_1^5 \frac{27}{2}z\,dz \\ &= \frac{27}{2}\frac{z^2}{2}\Big|_1^5 = 162.\end{aligned}$$

Now try to integrate in a different order just to see that we get the same answer. Choose to integrate with respect to x first, then z, then y

$$\begin{aligned}\iiint_B x^2yz\,dV &= \int_0^3\int_1^5\int_{-2}^1 [x^2yz]\,dx\,dz\,dy \\ &= \int_0^3\int_1^5 \left[\frac{x^3}{3}yz\Big|_{-2}^1\right] dz\,dy \\ &= \int_0^3\int_1^5 3yz\,dz\,dy \\ &= \int_0^3 \left[3y\frac{z^2}{2}\Big|_1^5\right] dy \\ &= \int_0^3 36y\,dy \\ &= 36\frac{y^2}{2}\Big|_0^3 = 18(9-0) = 162.\end{aligned}$$

? Exercise 2.5.1

Evaluate the triple integral

$$\iiint_B z\sin x\cos y\,dV$$

where $B = \{(x,y,z)\,|\,0 \le x \le \pi,\ \frac{3\pi}{2} \le y \le 2\pi,\ 1 \le z \le 3\}$.

Hint

Follow the steps in the previous example.

Answer

$$\iint_B z \sin x \cos y \, dV = 8$$

Triple Integrals over a General Bounded Region

We now expand the definition of the triple integral to compute a triple integral over a more **general bounded region** E in $\mathbb{R}^3$. The general bounded regions we will consider are of three types. First, let D be the bounded region that is a projection of E onto the xy-plane. Suppose the region E in $\mathbb{R}^3$ has the form

$$E = \{(x, y, z) \,|\, (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\}. \tag{2.5.16}$$

For two functions $z = u_1(x, y)$ and $u_2(x, y)$, such that $u_1(x, y) \leq u_2(x, y)$ for all (x, y) in D as shown in the following figure.

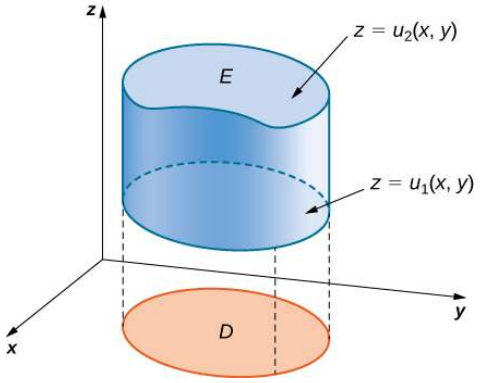


Figure 2.5.3: We can describe region E as the space between $u_1(x, y)$ and $u_2(x, y)$ above the projection D of E onto the xy-plane.

Triple Integral over a General Region

The triple integral of a continuous function $f(x, y, z)$ over a general three-dimensional region

$$E = \{(x, y, z) \,|\, (x, y) \in D, \, u_1(x, y) \leq z \leq u_2(x, y)\} \tag{2.5.17}$$

in $\mathbb{R}^3$, where D is the projection of E onto the xy-plane, is

$$\iiint_E f(x, y, z) \, dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z) \, dz \right] dA. \tag{2.5.18}$$

Similarly, we can consider a general bounded region D in the xy-plane and two functions $y = u_1(x, z)$ and $y = u_2(x, z)$ such that $u_1(x, z) \leq u_2(x, z)$ for all $9x, z)$ in D. Then we can describe the solid region E in $\mathbb{R}^3$ as

$$E = \{(x, y, z) \,|\, (x, z) \in D, \, u_1(x, z) \leq z \leq u_2(x, z)\} \tag{2.5.19}$$

where D is the projection of E onto the xy-plane and the triple integral is

$$\iiint_E f(x, y, z) \, dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x, y, z) \, dy \right] dA. \tag{2.5.20}$$

Finally, if D is a general bounded region in the xy-plane and we have two functions $x = u_1(y, z)$ and $x = u_2(y, z)$ such that $u_1(y, z) \leq u_2(y, z)$ for all (y, z) in D, then the solid region E in $\mathbb{R}^3$ can be described as

$$E = \{(x, y, z) \,|\, (y, z) \in D, \, u_1(y, z) \leq z \leq u_2(y, z)\} \tag{2.5.21}$$

where D is the projection of E onto the xy-plane and the triple integral is

$$\iiint_E f(x, y, z) \, dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x, y, z) \, dx \right] dA. \tag{2.5.22}$$

Note that the region D in any of the planes may be of Type I or Type II as described in previously. If D in the xy-plane is of Type I (Figure 2.5.4), then

$$E = \{(x, y, z) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x),\ u_1(x, y) \le z \le u_2(x, y)\}. \tag{2.5.23}$$

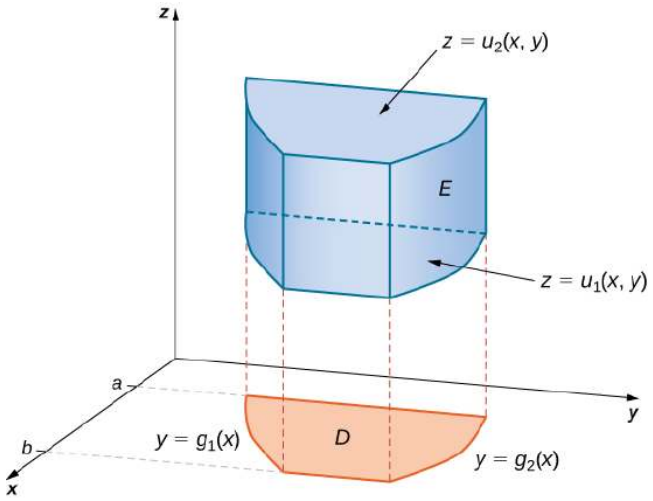


Figure 2.5.4: A box E where the projection D in the xy-plane is of Type I.

Then the triple integral becomes

$$\iiint_E f(x, y, z)\, dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz\, dy\, dx. \tag{2.5.24}$$

If D in the xy-plane is of Type II (Figure 2.5.5), then

$$E = \{(x, y, z) \mid c \le x \le d, h_1(x) \le y \le h_2(x),\ u_1(x, y) \le z \le u_2(x, y)\}. \tag{2.5.25}$$

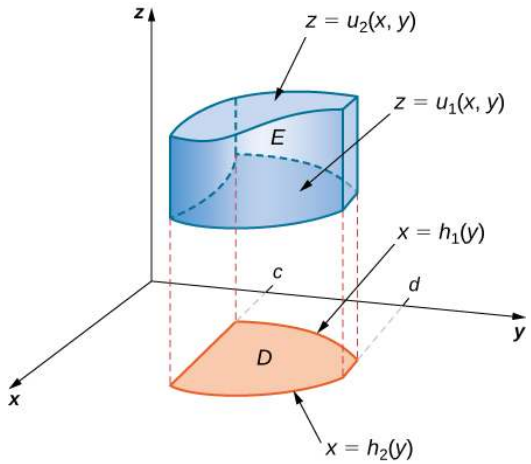


Figure 2.5.5: A box E where the projection D in the xy-plane is of Type II.

Then the triple integral becomes

$$\iiint_E f(x, y, z)\, dV = \int_{y=c}^{y=d} \int_{x=h_1(y)}^{x=h_2(y)} \int_{z=u_1(x,y)}^{z=u_2(x,y)} f(x, y, z)\, dz\, dx\, dy. \tag{2.5.26}$$

✓ Example 2.5.3A: Evaluating a Triple Integral over a General Bounded Region

Evaluate the triple integral of the function $f(x, y, z) = 5x - 3y$ over the solid tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$.

Solution

Figure 2.5.6 shows the solid tetrahedron E and its projection D on the xy-plane.

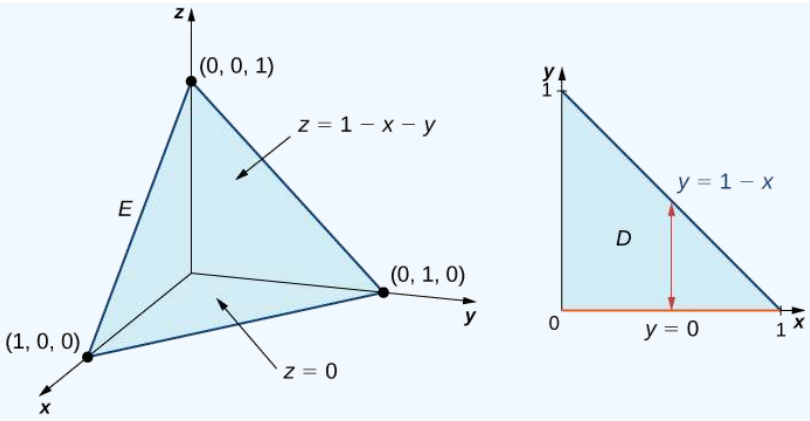


Figure 2.5.6: The solid E has a projection D on the xy-plane of Type I.

We can describe the solid region tetrahedron as

$$E = \{(x, y, z) \,|\, 0 \le x \le 1,\ 0 \le y \le 1 - x,\ 0 \le z \le 1 - x - y\}.$$

Hence, the triple integral is

$$\iiint_E f(x, y, z)\, dV = \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} \int_{z=0}^{z=1-x-y} (5x - 3y)\, dz\, dy\, dx.$$

To simplify the calculation, first evaluate the integral $\int_{z=0}^{z=1-x-y} (5x - 3y)\, dz$. We have

$$\int_{z=0}^{z=1-x-y} (5x - 3y)\, dz = (5x - 3y)z \Big|_{z=0}^{z=1-x-y} = (5x - 3y)(1 - x - y).$$

Now evaluate the integral

$$\int_{y=0}^{y=1-x} (5x - 3y)(1 - x - y)\, dy,$$

obtaining

$$\int_{y=0}^{y=1-x} (5x - 3y)(1 - x - y)\, dy = \frac{1}{2}(x - 1)^2(6x - 1).$$

Finally evaluate

$$\int_{x=0}^{x=1} \frac{1}{2}(x - 1)^2(6x - 1)\, dx = \frac{1}{12}.$$

Putting it all together, we have

$$\iiint_E f(x, y, z)\, dV = \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} \int_{z=0}^{z=1-x-y} (5x - 3y)\, dz\, dy\, dx = \frac{1}{12}.$$

Just as we used the double integral

$$\iint_D 1\, dA \tag{2.5.27}$$

to find the area of a general bounded region D we can use

$$\iiint_E 1\, dV \tag{2.5.28}$$

to find the volume of a general solid bounded region E. The next example illustrates the method.

Example 2.5.3B: Finding a Volume by Evaluating a Triple Integral

Find the volume of a right pyramid that has the square base in the xy-plane $[-1,1] \times [-1,1]$ and vertex at the point $(0,0,1)$ as shown in the following figure.

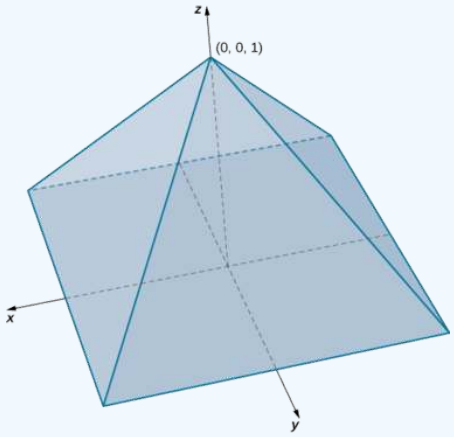


Figure 2.5.7: Finding the volume of a pyramid with a square base.

Solution

In this pyramid the value of z changes from 0 to 1 and at each height z the cross section of the pyramid for any value of z is the square

$$[-1+z,\, 1-z] \times [-1+z,\, 1-z].$$

Hence, the volume of the pyramid is

$$\iiint_E 1\, dV$$

where

$$E = \{(x,y,z) \,|\, 0 \le z \le 1,\, -1+z \le y \le 1-z,\, -1+z \le x \le 1-z\}.$$

Thus, we have

$$\begin{aligned} \iiint_E 1\, dV &= \int_{z=0}^{z=1} \int_{y=-1+z}^{y=1-z} \int_{x=-1+z}^{x=1-z} 1\, dx\, dy\, dz \\ &= \int_{z=0}^{z=1} \int_{y=-1+z}^{y=1-z} (2-2z)\, dy\, dz \\ &= \int_{z=0}^{z=1} (2-2z)^2\, dz = \frac{4}{3}. \end{aligned}$$

Hence, the volume of the pyramid is $\frac{4}{3}$ cubic units.



? Exercise 2.5.3

Consider the solid sphere $E = \{(x, y, z) \mid x^2 + y^2 + z^2 = 9\}$. Write the triple integral

$$\iiint_E f(x, y, z)\, dV$$

for an arbitrary function f as an iterated integral. Then evaluate this triple integral with $f(x, y, z) = 1$. Notice that this gives the volume of a sphere using a triple integral.

Hint

Follow the steps in the previous example. Use symmetry.

Answer

$$\iiint_E 1\, dV = 8 \int_{x=-3}^{x=3} \int_{y=-\sqrt{9-z^2}}^{y=\sqrt{9-z^2}} \int_{z=-\sqrt{9-x^2-y^2}}^{z=\sqrt{9-x^2-y^2}} 1\, dz\, dy\, dx$$
$$= 36\pi \text{ cubic units.}$$

Changing the Order of Integration

As we have already seen in double integrals over general bounded regions, changing the order of the integration is done quite often to simplify the computation. With a triple integral over a rectangular box, the order of integration does not change the level of difficulty of the calculation. However, with a triple integral over a general bounded region, choosing an appropriate order of integration can simplify the computation quite a bit. Sometimes making the change to polar coordinates can also be very helpful. We demonstrate two examples here.

✓ Example 2.5.4: Changing the Order of Integration

Consider the iterated integral

$$\int_{x=0}^{x=1} \int_{y=0}^{y=x^2} \int_{z=0}^{z=y} f(x, y, z)\, dz\, dy\, dx. \tag{2.5.29}$$

The order of integration here is first with respect to z, then y, and then x. Express this integral by changing the order of integration to be first with respect to x, then z, and then y. Verify that the value of the integral is the same if we let $f(x, y, z) = xyz$.

Solution

The best way to do this is to sketch the region E and its projections onto each of the three coordinate planes. Thus, let

$$E = \{(x, y, z) \,|\, 0 \le x \le 1,\ 0 \le y \le x^2,\ 0 \le z \le y\}.$$

and

$$\int_{x=0}^{x=1} \int_{y=0}^{y=x^2} \int_{z=0}^{z=x^2} f(x, y, z)\, dz\, dy\, dx = \iiint_E f(x, y, z)\, dV.$$

We need to express this triple integral as

$$\int_{y=c}^{y=d} \int_{z=v_1(y)}^{z=v_2(y)} \int_{x=u_1(y,z)}^{x=u_2(y,z)} f(x, y, z)\, dx\, dz\, dy.$$

Knowing the region E we can draw the following projections (Figure 2.5.8):

on the xy-plane is $D_1 = \{(x, y) \,|\, 0 \le x \le 1,\ 0 \le y \le x^2\} = \{(x, y) \,|\, 0 \le y \le 1,\ \sqrt{y} \le x \le 1\}$,

on the yz-plane is $D_2 = \{(y, z) \,|\, 0 \le y \le 1,\ 0 \le z \le y^2\}$, and

on the xz-plane is $D_3 = \{(x, z) \,|\, 0 \le x \le 1,\ 0 \le z \le x^2\}$.

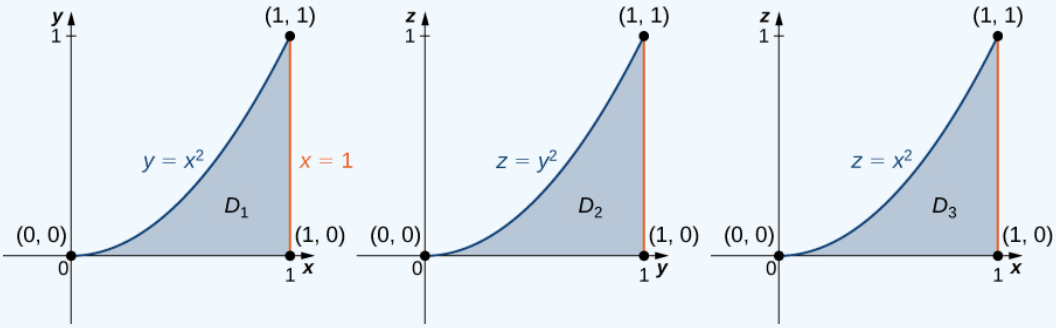


Figure 2.5.8. The three cross sections of E on the three coordinate planes.

Now we can describe the same region E as $\{(x, y, z) \,|\, 0 \le y \le 1,\ 0 \le z \le y^2,\ \sqrt{y} \le x \le 1\}$, and consequently, the triple integral becomes

$$\begin{aligned} &\int_{y=c}^{y=d} \int_{z=v_1(y)}^{z=v_2(y)} \int_{x=u_1(y,z)}^{x=u_2(y,z)} f(x, y, z)\, dx\, dz\, dy \\ &= \int_{y=0}^{y=1} \int_{z=0}^{z=x^2} \int_{x=\sqrt{y}}^{x=1} f(x, y, z)\, dx\, dz\, dy \end{aligned} \tag{2.5.30}$$

Now assume that $f(x, y, z) = xyz$ in each of the integrals. Then we have

$$\int_{x=0}^{x=1} \int_{y=0}^{y=x^2} \int_{z=0}^{z=y^2} xyz\, dz\, dy\, dx$$

$$= \int_{x=0}^{x=1} \int_{y=0}^{y=x^2} \left[xy \frac{z^2}{2} \Big|_{z=0}^{z=y^2} \right] dy\, dx = \int_{x=0}^{x=1} \int_{y=0}^{y=x^2} \left(x \frac{y^5}{2} \right) dy\, dx = \int_{x=0}^{x=1} \left[x \frac{y^6}{12} \Big|_{y=0}^{y=x^2} \right] dx$$

$$= \int_{x=0}^{x=1} \frac{x^{13}}{12} dx = \frac{x^{14}}{168} \Big|_{x=0}^{x=1} = \frac{1}{168},$$

$$\int_{y=0}^{y=1} \int_{z=0}^{z=y^2} \int_{x=\sqrt{y}}^{x=1} xyz\, dx\, dz\, dy$$

$$= \int_{y=0}^{y=1} \int_{z=0}^{z=y^2} \left[yz \frac{x^2}{2} \Big|_{\sqrt{y}}^{1} \right] dz\, dy = \int_{y=0}^{y=1} \int_{z=0}^{z=y^2} \left(\frac{yz}{2} - \frac{y^2 z}{2} \right) dz\, dy$$

$$= \int_{y=0}^{y=1} \left[\frac{yz^2}{4} - \frac{y^2 z^2}{4} \Big|_{z=0}^{z=y^2} \right] dy = \int_{y=0}^{y=1} \left(\frac{y^5}{4} - \frac{y^6}{4} \right) dy = \left(\frac{y^6}{24} - \frac{y^7}{28} \right) \Big|_{y=0}^{y=1} = \frac{1}{168}.$$

The answers match.

https://math.libretexts.org/@go/page/84736



? Exercise 2.5.4

Write five different iterated integrals equal to the given integral

$$\int_{z=0}^{z=4}\int_{y=0}^{y=4-z}\int_{x=0}^{x=\sqrt{y}} f(x,y,z)\,dx\,dy\,dz.$$

Hint

Follow the steps in the previous example, using the region E as $\{(x,y,z)\mid 0\le z\le 4,\ 0\le y\le 4-z,\ 0\le x\le\sqrt{y}\}$, and describe and sketch the projections onto each of the three planes, five different times.

Answer

$$(i)\ \int_{z=0}^{z=4}\int_{x=0}^{x=\sqrt{4-z}}\int_{y=x^2}^{y=4-z} f(x,y,z)\,dy\,dx\,dz,\ (ii)\ \int_{y=0}^{y=4}\int_{z=0}^{z=4-y}\int_{x=0}^{x=\sqrt{y}} f(x,y,z)\,dx\,dz\,dy,\ (iii)\ \int_{y=0}^{y=4}\int_{x=0}^{x=\sqrt{y}}\int_{z=0}^{Z=4-y} f(x,y,z)\,dz\,dx\,dy,$$

$$(iv)\ \int_{x=0}^{x=2}\int_{y=x^2}^{y=4}\int_{z=0}^{z=4-y} f(x,y,z)\,dz\,dy\,dx,\ (v)\ \int_{x=0}^{x=2}\int_{z=0}^{z=4-x^2}\int_{y=x^2}^{y=4-z} f(x,y,z)\,dy\,dz\,dx$$

✓ Example 2.5.5: Changing Integration Order and Coordinate Systems

Evaluate the triple integral

$$\iiint_E \sqrt{x^2+z^2}\,dV,$$

where E is the region bounded by the paraboloid $y=x^2+z^2$ (Figure 2.5.9) and the plane $y=4$.

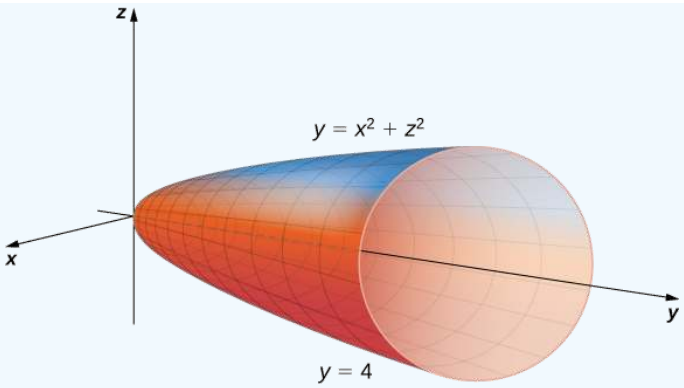


Figure 2.5.9. Integrating a triple integral over a paraboloid.

Solution

The projection of the solid region E onto the xy-plane is the region bounded above by $y = 4$ and below by the parabola $y = x^2$ as shown.

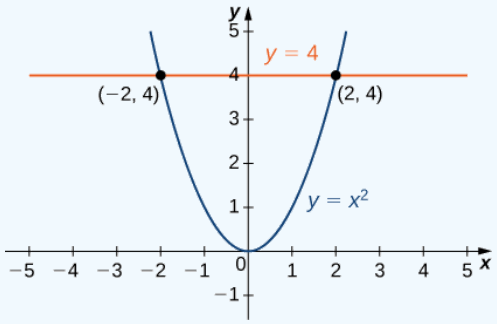


Figure 2.5.10. Cross section in the xy-plane of the paraboloid in Figure 2.5.9.

Thus, we have

$$E = \{(x, y, z) \mid -2 \le x \le 2,\ x^2 \le y \le 4,\ -\sqrt{y - x^2} \le z\sqrt{y - x^2}\}.$$

The triple integral becomes

$$\iiint_E \sqrt{x^2 + z^2}\, dV = \int_{x=-2}^{x=2} \int_{y=x^2}^{y=4} \int_{z=-\sqrt{y-x^2}}^{z=\sqrt{y-x^2}} \sqrt{x^2 + z^2}\, dz\, dy\, dx.$$

This expression is difficult to compute, so consider the projection of E onto the xz-plane. This is a circular disc $x^2 + z^2 \le 4$. So we obtain

$$\begin{aligned}\iiint_E \sqrt{x^2 + z^2}\, dV &= \int_{x=-2}^{x=2} \int_{y=x^2}^{y=4} \int_{z=-\sqrt{y-x^2}}^{z=\sqrt{y-x^2}} \sqrt{x^2 + z^2}\, dz\, dy\, dx \\ &= \int_{x=-2}^{x=2} \int_{z=-\sqrt{4-x^2}}^{z=\sqrt{4-x^2}} \int_{y=x^2+z^2}^{y=4} \sqrt{x^2 + z^2}\, dy\, dz\, dx.\end{aligned}$$

Here the order of integration changes from being first with respect to z then y and then x to being first with respect to y then to z and then to x. It will soon be clear how this change can be beneficial for computation. We have

$$\int_{x=-2}^{x=2} \int_{z=-\sqrt{4-x^2}}^{z=\sqrt{4-x^2}} \int_{y=x^2+z^2}^{y=4} \sqrt{x^2 + z^2}\, dy\, dz\, dx = \int_{x=-2}^{x=2} \int_{z=-\sqrt{4-x^2}}^{z=\sqrt{4-x^2}} (4 - x^2 - z^2)\sqrt{x^2 + z^2}\, dz\, dx.$$

Now use the polar substitution $x = r\cos\theta$, $z = r\sin\theta$, and $dz\, dx = r\, dr\, d\theta$ in the xz-plane. This is essentially the same thing as when we used polar coordinates in the xy-plane, except we are replacing y by z. Consequently the limits of integration change and we have, by using $r^2 = x^2 + z^2$,

$$\int_{x=-2}^{x=2}\int_{z=-\sqrt{4-x^2}}^{z=\sqrt{4-x^2}}(4-x^2-z^2)\sqrt{x^2+z^2}\,dz\,dx=\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=2}(4-r^2)rr\,dr\,d\theta$$
$$=\int_0^{2\pi}\left[\frac{4r^3}{3}-\frac{r^5}{5}\Big|_0^2\right]d\theta=\int_0^{2\pi}\frac{64}{15}\,d\theta=\frac{128\pi}{15}$$

Average Value of a Function of Three Variables

Recall that we found the average value of a function of two variables by evaluating the double integral over a region on the plane and then dividing by the area of the region. Similarly, we can find the average value of a function in three variables by evaluating the triple integral over a solid region and then dividing by the volume of the solid.

Average Value of a Function of Three Variables

If $f(x,y,z)$ is integrable over a solid bounded region E with positive volume $V(E)$, then the average value of the function is

$$f_{ave}=\frac{1}{V(E)}\iiint_E f(x,y,z)\,dV. \tag{2.5.31}$$

Note that the volume is

$$V(E)=\iiint_E 1\,dV. \tag{2.5.32}$$

Example 2.5.6: Finding an Average Temperature

The temperature at a point (x,y,z) of a solid E bounded by the coordinate planes and the plane $x+y+z=1$ is $T(x,y,z)=(xy+8z+20)$ °C. Find the average temperature over the solid.

Solution

Use the theorem given above and the triple integral to find the numerator and the denominator. Then do the division. Notice that the plane $x+y+z=1$ has intercepts $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. The region E looks like

$$E=\{(x,y,z)\,|\,0\le x\le 1,\ 0\le y\le 1-x,\ 0\le z\le 1-x-y\}.$$

Hence the triple integral of the temperature is

$$\iiint_E f(x,y,z)\,dV=\int_{x=0}^{x=1}\int_{y=0}^{y=1-x}\int_{z=0}^{z=1-x-y}(xy+8z+20)\,dz\,dy\,dx=\frac{147}{40}.$$

The volume evaluation is

$$V(E)=\iiint_E 1\,dV=\int_{x=0}^{x=1}\int_{y=0}^{y=1-x}\int_{z=0}^{z=1-x-y}1\,dz\,dy\,dx=\frac{1}{6}.$$

Hence the average value is

$$T_{ave}=\frac{147/40}{1/6}=\frac{6(147)}{40}=\frac{441}{20}\text{ °C}$$

.

Exercise 2.5.6

Find the average value of the function $f(x,y,z)=xyz$ over the cube with sides of length 4 units in the first octant with one vertex at the origin and edges parallel to the coordinate axes.

Hint

Follow the steps in the previous example.

Answer

$f_{ave} = 8$

Key Concepts

- To compute a triple integral we use Fubini's theorem, which states that if $f(x, y, z)$ is continuous on a rectangular box $B = [a, b] \times [c, d] \times [e, f]$, then

$$\iiint_B f(x, y, z)\, dV = \int_e^f \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz$$

  and is also equal to any of the other five possible orderings for the iterated triple integral.
- To compute the volume of a general solid bounded region E we use the triple integral

$$V(E) = \iiint_E 1\, dV.$$

- Interchanging the order of the iterated integrals does not change the answer. As a matter of fact, interchanging the order of integration can help simplify the computation.
- To compute the average value of a function over a general three-dimensional region, we use

$$f_{ave} = \frac{1}{V(E)} \iiint_E f(x, y, z)\, dV.$$

Key Equations

- **Triple integral**

$$\lim_{l,m,n\to\infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\, \Delta x \Delta y \Delta z = \iiint_B f(x, y, z)\, dV$$

Glossary

triple integral

the triple integral of a continuous function $f(x, y, z)$ over a rectangular solid box B is the limit of a Riemann sum for a function of three variables, if this limit exists

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2.5E: Exercises

In exercises 1 - 4, evaluate the triple integrals over the rectangular solid box B.

1. $\iiint_B (2x + 3y^2 + 4z^3)\, dV$, where $B = \{(x, y, z) \mid 0 \le x \le 1,\ 0 \le y \le 2,\ 0 \le z \le 3\}$

Answer

192

2. $\iiint_B (xy + yz + xz)\, dV$, where $B = \{(x, y, z) \mid 1 \le x \le 2,\ 0 \le y \le 2,\ 1 \le z \le 3\}$
3. $\iiint_B (x \cos y + z)\, dV$, where $B = \{(x, y, z) \mid 0 \le x \le 1,\ 0 \le y \le \pi,\ -1 \le z \le 1\}$

Answer

0

4. $\iiint_B (z \sin x + y^2)\, dV$, where $B = \{(x, y, z) \mid 0 \le x \le \pi,\ 0 \le y \le 1,\ -1 \le z \le 2\}$

In exercises 5 - 8, change the order of integration by integrating first with respect to z, then x, then y.

5. $\int_0^1 \int_1^2 \int_2^3 (x^2 + \ln y + z)\, dx\, dy\, dz$

Answer

$$\int_0^1 \int_1^2 \int_2^3 (x^2 + \ln y + z)\, dx\, dy\, dz = \frac{35}{6} + 2 \ln 2$$

6. $\int_0^1 \int_{-1}^1 \int_0^3 (ze^x + 2y)\, dx\, dy\, dz$
7. $\int_{-1}^2 \int_1^3 \int_0^4 \left(x^2 z + \frac{1}{y}\right) dx\, dy\, dz$

Answer

$$\int_{-1}^2 \int_1^3 \int_0^4 \left(x^2 z + \frac{1}{y}\right) dx\, dy\, dz = 64 + 12 \ln 3$$

8. $\int_1^2 \int_{-2}^{-1} \int_0^1 \frac{x + y}{z}\, dx\, dy\, dz$

9. Let F, G, and H be continuous functions on $[a, b]$, $[c, d]$, and $[e, f]$, respectively, where a, b, c, d, e, and f are real numbers such that $a < b$, $c < d$, and $e < f$. Show that

$$\int_a^b \int_c^d \int_e^f F(x)\, G(y)\, H(z)\, dz\, dy\, dx = \left(\int_a^b F(x)\, dx\right) \left(\int_c^d G(y)\, dy\right) \left(\int_e^f H(z)\, dz\right).$$

10. Let F, G, and H be differential functions on $[a, b]$, $[c, d]$, and $[e, f]$, respectively, where a, b, c, d, e, and f are real numbers such that $a < b$, $c < d$, and $e < f$. Show that

$$\int_a^b \int_c^d \int_e^f F'(x)\, G'(y)\, H'(z)\, dz\, dy\, dx = [F(b) - F(a)]\, [G(d) - G(c)]\, H(f) - H(e)].$$

In exercises 11 - 14, evaluate the triple integrals over the bounded region

$E = \{(x, y, z) \mid a \le x \le b,\ h_1(x) \le y \le h_2(x),\ e \le z \le f\}$.

11. $\iiint_E (2x + 5y + 7z)\, dV$, where $E = \{(x, y, z) \mid 0 \le x \le 1,\ 0 \le y \le -x + 1,\ 1 \le z \le 2\}$

Answer

$\frac{77}{12}$

12. $\iiint_E (y \ln x + z)\, dV$, where $E = \{(x, y, z) \,|\, 1 \le x \le e,\ 0 \le y \ln x,\ 0 \le z \le 1\}$

13. $\iiint_E (\sin x + \sin y)\, dV$, where $E = \{(x, y, z) \,|\, 0 \le x \le \frac{\pi}{2},\ -\cos x \le y \cos x,\ -1 \le z \le 1\}$

Answer

2

14. $\iiint_E (xy + yz + xz) dV$ where $E = \{(x, y, z) \,|\, 0 \le x \le 1,\ -x^2 \le y \le x^2,\ 0 \le z \le 1\}$

In exercises 15 - 18, evaluate the triple integrals over the indicated bounded region E.

15. $\iiint_E (x + 2yz)\, dV$, where $E = \{(x, y, z) \,|\, 0 \le x \le 1,\ 0 \le y \le x,\ 0 \le z \le 5 - x - y\}$

Answer

$\frac{430}{120}$

16. $\iiint_E (x^3 + y^3 + z^3)\, dV$, where $E = \{(x, y, z) \,|\, 0 \le x \le 2,\ 0 \le y \le 2x,\ 0 \le z \le 4 - x - y\}$

17. $\iiint_E y\, dV$, where $E = \{(x, y, z) \,|\, -1 \le x \le 1,\ -\sqrt{1 - x^2} \le y \le \sqrt{1 - x^2},\ 0 \le z \le 1 - x^2 - y^2\}$

Answer

0

18. $\iiint_E x\, dV$, where $E = \{(x, y, z) \,|\, -2 \le x \le 2,\ -4\sqrt{1 - x^2} \le y \le \sqrt{4 - x^2},\ 0 \le z \le 4 - x^2 - y^2\}$

In exercises 19 - 22, evaluate the triple integrals over the bounded region E of the form

$E = \{(x, y, z) \,|\, g_1(y) \le x \le g_2(y),\ c \le y \le d,\ e \le z \le f\}$.

19. $\iiint_E x^2\, dV$, where $E = \{(x, y, z) \,|\, 1 - y^2 \le x \le y^2 - 1,\ -1 \le y \le 1,\ 1 \le z \le 2\}$

Answer

$-\frac{64}{105}$

20. $\iiint_E (y + \sin x)\, dV$, where $E = \{(x, y, z) \,|\, -y^4 \le x \le y^4,\ 0 \le y \le 2,\ 0 \le z \le 4\}$

21. $\iiint_E (x - yz)\, dV$, where $E = \{(x, y, z) \,|\, -y^6 \le x \le \sqrt{y},\ 0 \le y \le 1x,\ -1 \le z \le 1\}$

Answer

$\frac{11}{26}$

22. $\iiint_E z\, dV$, where $E = \{(x, y, z) \,|\, 2 - 2y \le x \le 2 + \sqrt{y},\ 0 \le y \le 1x,\ 2 \le z \le 3\}$

In exercises 23 - 26, evaluate the triple integrals over the bounded region

$E = \{(x, y, z) \,|\, g_1(y) \le x \le g_2(y),\ c \le y \le d,\ u_1(x, y) \le z \le u_2(x, y)\}$

23. $\iiint_E z\, dV$, where $E = \{(x, y, z) \,|\, -y \le x \le y,\ 0 \le y \le 1,\ 0 \le z \le 1 - x^4 - y^4\}$

Answer

$\frac{113}{450}$

24. $\iiint_E (xz + 1)\, dV$, where $E = \{(x, y, z) \,|\, 0 \le x \le \sqrt{y},\ 0 \le y \le 2,\ 0 \le z \le 1 - x^2 - y^2\}$

25. $\iiint_E (x - z)\, dV$, where $E = \{(x, y, z) \,|\, -\sqrt{1 - y^2} \le x \le y,\ 0 \le y \le \frac{1}{2}x,\ 0 \le z \le 1 - x^2 - y^2\}$

Answer

$\frac{1}{160}(6\sqrt{3}-41)$

26. $\iiint_E (x+y)\,dV$, where $E=\{(x,y,z)\,|\,0\le x\le\sqrt{1-y^2},\ 0\le y\le 1x,\ 0\le z\le 1-x\}$

In exercises 27 - 30, evaluate the triple integrals over the bounded region

$E=\{(x,y,z)\,|\,(x,y)\in D,\ u_1(x,y)x\le z\le u_2(x,y)\}$, where D is the projection of E onto the xy-plane.

27. $\iint_D \left(\int_1^2 (x+y)\,dz\right)\,dA$, where $D=\{(x,y)\,|\,x^2+y^2\le 1\}$

Answer

$\frac{3\pi}{2}$

28. $\iint_D \left(\int_1^3 x(z+1)\,dz\right)\,dA$, where $D=\{(x,y)\,|\,x^2-y^2\ge 1,\ x\le\sqrt{5}\}$

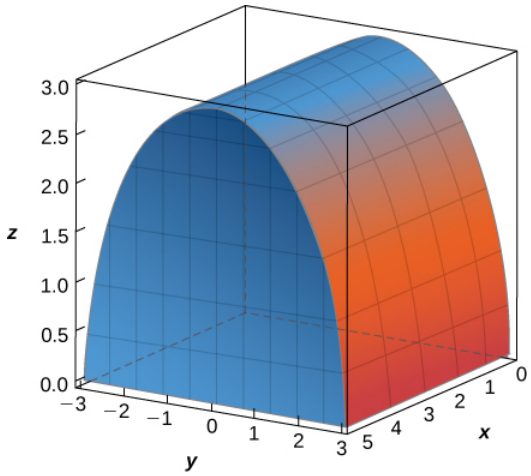
29. $\iint_D \left(\int_0^{10-x-y} (x+2z)\,dz\right)\,dA$, where $D=\{(x,y)\,|\,y\ge 0,\ x\ge 0,\ x+y\le 10\}$

Answer

1250

30. $\iint_D \left(\int_0^{4x^2+4y^2} y\,dz\right)\,dA$, where $D=\{(x,y)\,|\,x^2+y^2\le 4,\ y\ge 1,\ x\ge 0\}$

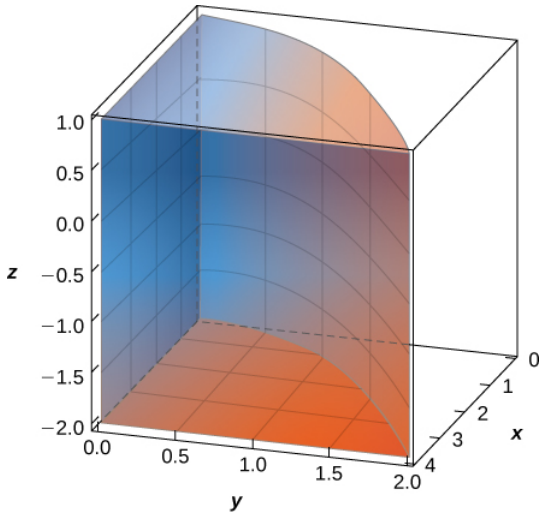
31. The solid E bounded by $y^2+z^2=9$, $z=0$, and $x=5$ is shown in the following figure. Evaluate the integral $\iiint_E z\,dV$ by integrating first with respect to z, then y, and then x.



Answer

$\int_0^5\int_{-3}^3\int_0^{\sqrt{9-y^2}} z\,dz\,dy\,dx=90$

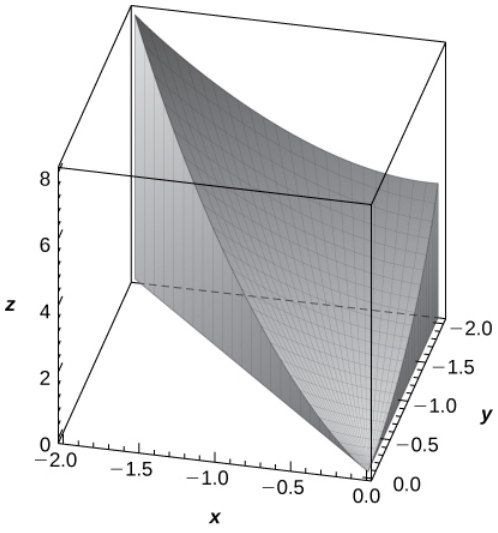
32. The solid E bounded by $y=\sqrt{x}$, $x=4$, $y=0$, and $z=1$ is given in the following figure. Evaluate the integral $\iiint_E xyz\,dV$ by integrating first with respect to x, then y, and then z.



33. [T] The volume of a solid E is given by the integral $\int_{-2}^{0}\int_{x}^{0}\int_{0}^{x^2+y^2} dz\,dy\,dx$. Use a computer algebra system (CAS) to graph E and find its volume. Round your answer to two decimal places.

Answer

$V \approx 5.33$ units^3



34. [T] The volume of a solid E is given by the integral $\int_{-1}^{0}\int_{-x^3}^{0}\int_{0}^{1+\sqrt{x^2+y^2}} dz\,dy\,dx$. Use a CAS to graph E and find its volume V. Round your answer to two decimal places.

In exercises 35 - 38, use two circular permutations of the variables x, y, and z to write new integrals whose values equal the value of the original integral. A circular permutation of x, y, and z is the arrangement of the numbers in one of the following orders: y, z, and x or z, x, and y.

35. $\int_{0}^{1}\int_{1}^{3}\int_{2}^{4}(x^2z^2+1)\,dx\,dy\,dz$

Answer

$\int_{0}^{1}\int_{1}^{3}\int_{2}^{4}(y^2z^2+1)\,dz\,dx\,dy;$
$\int_{0}^{1}\int_{1}^{3}\int_{2}^{4}(x^2y^2+1)\,dy\,dz\,dx$

36. $\int_0^3 \int_0^1 \int_0^{-x+1} (2x+5y+7z)dy\,dx\,dz$

37. $\int_0^1 \int_{-y}^{y} \int_0^{1-x^4-y^4} \ln x \; dz\,dx\,dy$

38. $\int_{-1}^1 \int_0^1 \int_{-y^6}^{\sqrt{y}} (x+yz)\,dx\,dy\,dz$

39. Set up the integral that gives the volume of the solid E bounded by $y^2 = x^2 + z^2$ and $y = a^2$, where $a > 0$.

Answer

$$V = \int_{-a}^{a} \int_{-\sqrt{a^2-z^2}}^{\sqrt{a^2-z^2}} \int_{\sqrt{x^2+z^2}}^{a^2} dy\,dx\,dz$$

40. Set up the integral that gives the volume of the solid E bounded by $x = y^2 + z^2$ and $x = a^2$, where $a > 0$.

Average Value of a Function

41. Find the average value of the function $f(x,y,z) = x + y + z$ over the parallelepiped determined by $x + 0$, $x = 1$, $y = 0$, $y = 3$, $z = 0$, and $z = 5$.

Answer

$\frac{9}{2}$

42. Find the average value of the function $f(x,y,z) = xyz$ over the solid $E = [0,1] \times [0,1] \times [0,1]$ situated in the first octant.

Finding Volumes using Triple Integrals

43. Find the volume of the solid E that lies under the plane $x + y + z = 9$ and whose projection onto the xy-plane is bounded by $x = sqrty - 1$, $x = 0$, and $x + y = 7$.

Answer

$\frac{156}{5}$ units^3

44. Find the volume of the solid E that lies under the plane $2x + y + z = 8$ and whose projection onto the xy-plane is bounded by $x = sin^{-1}y$, $y = 0$, and $x = \frac{\pi}{2}$.

45. Consider the pyramid with the base in the xy-plane of $[-2,2] \times [-2,2]$ and the vertex at the point $(0,0,8)$.

a. Show that the equations of the planes of the lateral faces of the pyramid are $4y + z = 8$, $4y - z = -8$, $4x + z = 8$, and $-4x + z = 8$.

b. Find the volume of the pyramid.

Answer

a. Answers may vary;

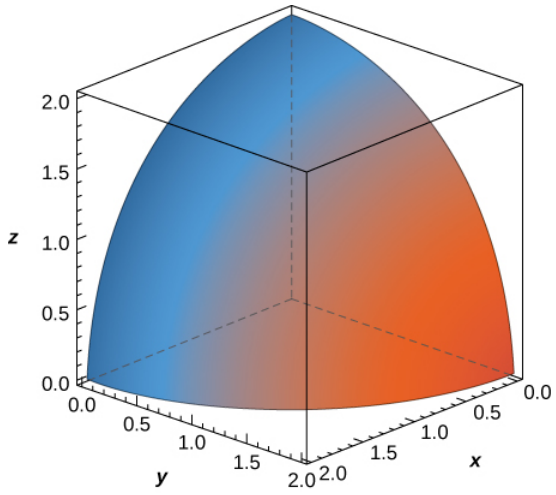
b. $\frac{128}{3}$ units^3

46. Consider the pyramid with the base in the xy-plane of $[-3,3] \times [-3,3]$ and the vertex at the point $(0,0,9)$.

a. Show that the equations of the planes of the side faces of the pyramid are $3y + z = 9$, $3y + z = 9$, $y = 0$ and $x = 0$.

b. Find the volume of the pyramid.

47. The solid E bounded by the sphere of equation $x^2 + y^2 + z^2 = r^2$ with $r > 0$ and located in the first octant is represented in the following figure.



a. Write the triple integral that gives the volume of E by integrating first with respect to z, then with y, and then with x.

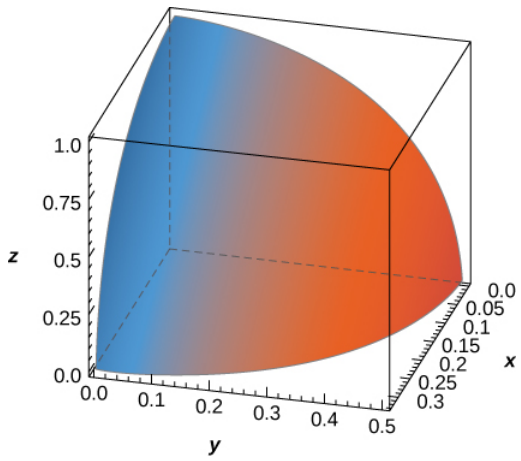
b. Rewrite the integral in part a. as an equivalent integral in five other orders.

Answer

a. $\int_0^4 \int_0^{\sqrt{r^2-x^2}} \int_0^{\sqrt{r^2-x^2-y^2}} dz\,dy\,dx$

b. $\int_0^2 \int_0^{\sqrt{r^2-x^2}} \int_0^{\sqrt{r^2-x^2-y^2}} dz\,dx\,dy$,
$\int_0^r \int_0^{\sqrt{r^2-x^2}} \int_0^{\sqrt{r^2-x^2-y^2}} dy\,dx\,dz$,
$\int_0^r \int_0^{\sqrt{r^2-x^2}} \int_0^{\sqrt{r^2-x^2-y^2}} dy\,dz\,dx$,
$\int_0^r \int_0^{\sqrt{r^2-x^2}} \int_0^{\sqrt{r^2-x^2-y^2}} dx\,dy\,dz$,
$\int_0^r \int_0^{\sqrt{r^2-x^2}} \int_0^{\sqrt{r^2-x^2-y^2}} dx\,dz\,dy$

48. The solid E bounded by the sphere of equation $9x^2 + 4y^2 + z^2 = 1$ and located in the first octant is represented in the following figure.



a. Write the triple integral that gives the volume of E by integrating first with respect to z then with y and then with x.

b. Rewrite the integral in part a. as an equivalent integral in five other orders.

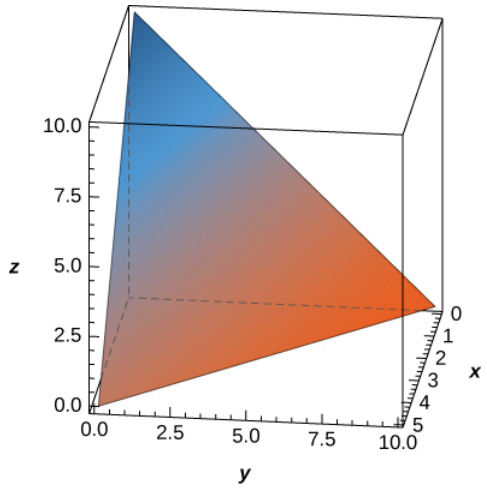
49. Find the volume of the prism with vertices $(0,0,0)$, $(2,0,0)$, $(2,3,0)$, $(0,3,0)$, $(0,0,1)$, and $(2,0,1)$.

Answer

3 units^3

50. Find the volume of the prism with vertices $(0,0,0)$, $(4,0,0)$, $(4,6,0)$, $(0,6,0)$, $(0,0,1)$, and $(4,0,1)$.

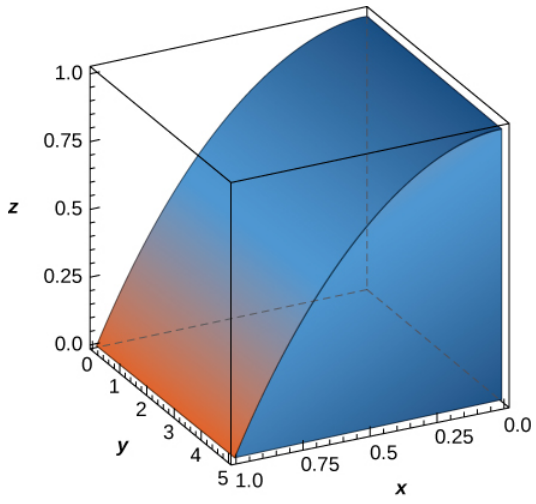
51. The solid E bounded by $z = 10 - 2x - y$ and situated in the first octant is given in the following figure. Find the volume of the solid.



Answer

$\frac{250}{3}$ units^3

52. The solid E bounded by $z = 1 - x^2$ and situated in the first octant is given in the following figure. Find the volume of the solid.



Approximating Triple Integrals

53. The midpoint rule for the triple integral $\iiint_B f(x,y,z)\,dV$ over the rectangular solid box B is a generalization of the midpoint rule for double integrals. The region B is divided into subboxes of equal sizes and the integral is approximated by the triple Riemann sum

$$\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(\bar{x}_i, \bar{y}_j, \bar{z}_k)\Delta V,$$

where $(\bar{x}_i, \bar{y}_j, \bar{z}_k)$ is the center of the box B_{ijk} and ΔV is the volume of each subbox. Apply the midpoint rule to approximate

$$\iiint_B x^2\,dV$$

over the solid $B = \{(x, y, z) \mid 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\}$ by using a partition of eight cubes of equal size. Round your answer to three decimal places.

Answer

$\iiint_B f(x, y, z)\,dV \approx \frac{5}{16} \approx 0.313$ units^3

54. [T]

a. Apply the midpoint rule to approximate $\iiint_B e^{-x^2}\,dV$ over the solid $B = \{(x, y, z) \mid 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\}$ by using a partition of eight cubes of equal size. Round your answer to three decimal places.

b. Use a CAS to improve the above integral approximation in the case of a partition of n^3 cubes of equal size, where $n = 3, 4, \ldots, 10$.

Applications

55. Suppose that the temperature in degrees Celsius at a point (x, y, z) of a solid E bounded by the coordinate planes and the plane $x + y + z = 5$ is given by:

$$T(x, y, z) = xz + 5z + 10$$

Find the average temperature over the solid.

Answer

17.5° C

56. Suppose that the temperature in degrees Fahrenheit at a point (x, y, z) of a solid E bounded by the coordinate planes and the plane $x + y + z = 5$ is given by:

$$T(x, y, z) = x + y + xy$$

Find the average temperature over the solid.

57. Show that the volume of a right square pyramid of height h and side length a is $v = \dfrac{ha^2}{3}$ by using triple integrals.

58. Show that the volume of a regular right hexagonal prism of edge length a is $\dfrac{3a^3\sqrt{3}}{2}$ by using triple integrals.

59. Show that the volume of a regular right hexagonal pyramid of edge length a is $\dfrac{a^3\sqrt{3}}{2}$ by using triple integrals.

60. If the charge density at an arbitrary point (x, y, z) of a solid E is given by the function $\rho(x, y, z)$, then the total charge inside the solid is defined as the triple integral $\iiint_E \rho(x, y, z)\,dV$. Assume that the charge density of the solid E enclosed by the paraboloids $x = 5 - y^2 - z^2$ and $x = y^2 + z^2 - 5$ is equal to the distance from an arbitrary point of E to the origin. Set up the integral that gives the total charge inside the solid E.

Answer

Total Charge inside the Solid E $= \int_{-\sqrt{5}}^{\sqrt{5}} \int_{-\sqrt{5-y^2}}^{\sqrt{5-y^2}} \int_{y^2+z^2-5}^{5-y^2-z^2} \sqrt{x^2 + y^2 + z^2}\,dx\,dz\,dy$

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2.6: Triple Integrals in Cylindrical and Spherical Coordinates

Learning Objectives

- Evaluate a triple integral by changing to cylindrical coordinates.
- Evaluate a triple integral by changing to spherical coordinates.

Earlier in this chapter we showed how to convert a double integral in rectangular coordinates into a double integral in polar coordinates in order to deal more conveniently with problems involving circular symmetry. A similar situation occurs with triple integrals, but here we need to distinguish between cylindrical symmetry and spherical symmetry. In this section we convert triple integrals in rectangular coordinates into a triple integral in either cylindrical or spherical coordinates.

Also recall the chapter prelude, which showed the opera house l'Hemisphèric in Valencia, Spain. It has four sections with one of the sections being a theater in a five-story-high sphere (ball) under an oval roof as long as a football field. Inside is an IMAX screen that changes the sphere into a planetarium with a sky full of 9000 twinkling stars. Using triple integrals in spherical coordinates, we can find the volumes of different geometric shapes like these.

Review of Cylindrical Coordinates

As we have seen earlier, in two-dimensional space $\mathbb{R}^2$ a point with rectangular coordinates (x, y) can be identified with (r, θ) in polar coordinates and vice versa, where $x = r\cos\theta$, $y = r\sin\theta$, $r^2 = x^2 + y^2$ and $\tan\theta = \left(\frac{y}{x}\right)$ are the relationships between the variables.

In three-dimensional space $\mathbb{R}^3$ a point with rectangular coordinates (x, y, z) can be identified with cylindrical coordinates (r, θ, z) and vice versa. We can use these same conversion relationships, adding z as the vertical distance to the point from the $(xy$-plane as shown in 2.6.1.

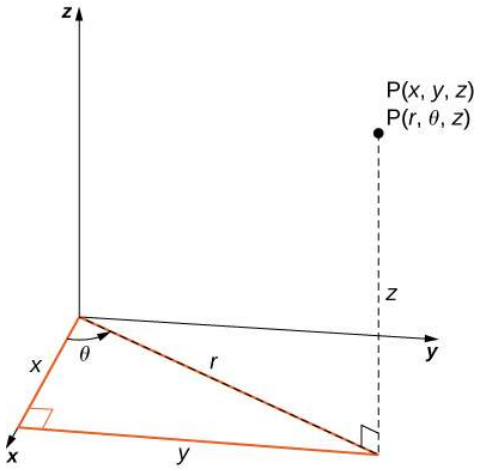


Figure 2.6.1: Cylindrical coordinates are similar to polar coordinates with a vertical z coordinate added.

To convert from rectangular to cylindrical coordinates, we use the conversion

- $x = r\cos\theta$
- $y = r\sin\theta$
- $z = z$

To convert from cylindrical to rectangular coordinates, we use

- $r^2 = x^2 + y^2$ and
- $\theta = \tan^{-1}\left(\frac{y}{x}\right)$
- $z = z$

Note that that z-coordinate remains the same in both cases.

In the two-dimensional plane with a rectangular coordinate system, when we say $x = k$ (constant) we mean an unbounded vertical line parallel to the y-axis and when $y = l$ (constant) we mean an unbounded horizontal line parallel to the x-axis. With the polar coordinate system, when we say $r = c$ (constant), we mean a circle of radius c units and when $\theta = \alpha$ (constant) we mean an infinite ray making an angle α with the positive x-axis.

Similarly, in three-dimensional space with rectangular coordinates (x, y, z) the equations $x = k$, $y = l$ and $z = m$ where k, l and m are constants, represent unbounded planes parallel to the yz-plane, xz-plane and xy-plane, respectively. With cylindrical coordinates (r, θ, z), by $r = c$, $\theta = \alpha$, and $z = m$, where c, α, and m are constants, we mean an unbounded vertical cylinder with the z-axis as its radial axis; a plane making a constant angle α with the xy-plane; and an unbounded horizontal plane parallel to the xy-plane, respectively. This means that the circular cylinder $x^2 + y^2 = c^2$ in rectangular coordinates can be represented simply as $r = c$ in cylindrical coordinates. (Refer to Cylindrical and Spherical Coordinates for more review.)

Integration in Cylindrical Coordinates

Triple integrals can often be more readily evaluated by using cylindrical coordinates instead of rectangular coordinates. Some common equations of surfaces in rectangular coordinates along with corresponding equations in cylindrical coordinates are listed in Table **2.6.1**. These equations will become handy as we proceed with solving problems using triple integrals.

Table **2.6.1**: Equations of Some Common Shapes

	Circular cylinder	**Circular cone**	**Sphere**	**Paraboloid**
Rectangular	$x^2 + y^2 = c^2$	$z^2 = c^2(x^2 + y^2)$	$x^2 + y^2 + z^2 = c^2$	$z = c(x^2 + y^2)$
Cylindrical	$r = c$	$z = cr$	$r^2 + z^2 = c^2$	$z = cr^2$

As before, we start with the simplest bounded region B in $\mathbb{R}^3$ to describe in cylindrical coordinates, in the form of a cylindrical box, $B = \{(r, \theta, z) | a \le r \le b,\ \alpha \le \theta \le \beta,\ c \le z \le d\}$ (Figure **2.6.2**). Suppose we divide each interval into l, m, and n subdivisions such that $\Delta r = \frac{b-a}{l}$, $\Delta\theta = \frac{\beta-\alpha}{m}$, and $\Delta z = \frac{d-c}{n}$. Then we can state the following definition for a triple integral in cylindrical coordinates.

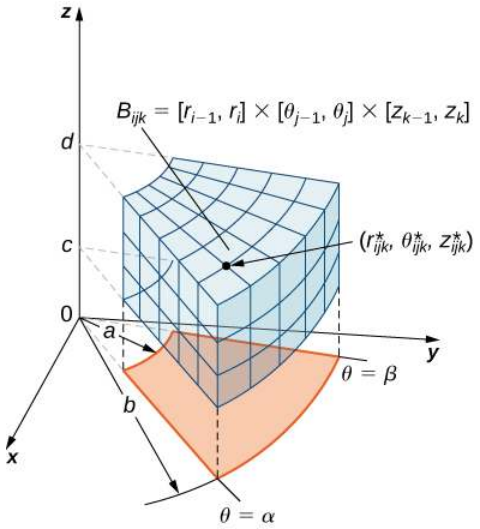


Figure **2.6.2**: A cylindrical box B described by cylindrical coordinates.

DEFINITION: triple integral in cylindrical coordinates

Consider the cylindrical box (expressed in cylindrical coordinates)

$$B = \{(r, \theta, z) | a \le r \le b,\ \alpha \le \theta \le \beta,\ c \le z \le d\}. \tag{2.6.1}$$

If the function $f(r, \theta, z)$ is continuous on B and if $(r^*_{ijk}, \theta^*_{ijk}, z^*_{ijk})$ is any sample point in the cylindrical subbox $B_{ijk} = |r_{i-1}, r_i| \times |\theta_{j-1}, \theta_j| \times |z_{k-1}, k_i|$ (Figure **2.6.2**), then we can define the *triple integral in cylindrical coordinates* as the limit of a triple Riemann sum, provided the following limit exists:

$$\lim_{l,m,n\to\infty}\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(r^*_{ijk},\theta^*_{ijk},z^*_{ijk})\Delta r\Delta\theta\Delta z. \tag{2.6.2}$$

Note that if $g(x,y,z)$ is the function in rectangular coordinates and the box B is expressed in rectangular coordinates, then the triple integral

$$\iiint_B g(x,y,z)dV \tag{2.6.3}$$

is equal to the triple integral

$$\iiint_B g(r\cos\theta,\, r\sin\theta,\, z)r\,dr\,d\theta\,dz \tag{2.6.4}$$

and we have

$$\iiint_B g(x,y,z)dV = \iiint_B g(r\cos\theta,\, r\sin\theta,\, z)r\,dr\,d\theta\,dz = \iiint_B f(r,\theta\, z)r\,dr\,d\theta\,dz. \tag{2.6.5}$$

As mentioned in the preceding section, all the properties of a double integral work well in triple integrals, whether in rectangular coordinates or cylindrical coordinates. They also hold for iterated integrals. To reiterate, in cylindrical coordinates, Fubini's theorem takes the following form:

Theorem: Fubini's Theorem in Cylindrical Coordinates

Suppose that $g(x,y,z)$ is continuous on a rectangular box B which when described in cylindrical coordinates looks like $B = \{(r,\theta,z)|a\le r\le b,\, \alpha\le\theta\le\beta,\, c\le z\le d\}$.

Then $g(x,y,z) = g(r\cos\theta, r\sin\theta, z) = f(r,\theta,z)$ and

$$\iiint_B g(x,y,z)dV = \int_c^d\int_\beta^\alpha\int_a^b f(r,\theta,z)r\,dr\,d\theta\,dz. \tag{2.6.6}$$

The iterated integral may be replaced equivalently by any one of the other five iterated integrals obtained by integrating with respect to the three variables in other orders.

Cylindrical coordinate systems work well for solids that are symmetric around an axis, such as cylinders and cones. Let us look at some examples before we define the triple integral in cylindrical coordinates on general cylindrical regions.

Example 2.6.1: Evaluating a Triple Integral over a Cylindrical Box

Evaluate the triple integral

$$\iiint_B (zr\sin\theta)r\,dr\,d\theta\,dz \tag{2.6.7}$$

where the cylindrical box B is $B = \{(r,\theta,z)|0\le r\le 2,\, 0\le\theta\le\pi/2,\, 0,\le z\le 4\}$.

Solution

As stated in Fubini's theorem, we can write the triple integral as the iterated integral

$$\iiint_B (zr\sin\theta)r\,dr\,d\theta\,dz = \int_{\theta=0}^{\theta=\pi/2}\int_{r=0}^{r=2}\int_{z=0}^{z=4}(zr\sin\theta)r\,dz\,dr\,d\theta. \tag{2.6.8}$$

The evaluation of the iterated integral is straightforward. Each variable in the integral is independent of the others, so we can integrate each variable separately and multiply the results together. This makes the computation much easier:

$$\int_{\theta=0}^{\theta=\pi/2}\int_{r=0}^{r=2}\int_{z=0}^{z=4}(zr\sin\theta)r\,dz\,dr\,d\theta=\left(\int_0^{\pi/2}\sin\theta\,d\theta\right)\left(\int_0^2 r^2dr\right)\left(\int_0^4 z\,dz\right) \tag{2.6.9}$$
$$=\left(-\cos\theta\Big|_0^{\pi/2}\right)\left(\frac{r^3}{3}\Big|_0^2\right)\left(\frac{z^2}{2}\Big|_0^4\right)=\frac{64}{3}.$$



? Exercise 2.6.1:

Evaluate the triple integral

$$\int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=1}\int_{z=0}^{z=4}rz\sin\theta r\,dz\,dr\,d\theta. \tag{2.6.10}$$

Hint

Follow the same steps as in the previous example.

Answer

8

If the cylindrical region over which we have to integrate is a general solid, we look at the projections onto the coordinate planes. Hence the triple integral of a continuous function $f(r,\theta,z)$ over a general solid region $E=\{(r,\theta,z)|(r,\theta)\in D, u_1(r,\theta)\le z\le u_2(r,\theta)\}$ in $\mathbb{R}^3$ where D is the projection of E onto the $r\theta$-plane, is

$$\iiint_E f(r,\theta,z)r\,dr\,d\theta\,dz=\iint_D\left[\int_{u_1(r,\theta)}^{u_2(r,\theta)}f(r,\theta,z)dz\right]r\,dr\,d\theta. \tag{2.6.11}$$

In particular, if $D=\{(r,\theta)|G_1(\theta)\le r\le g_2(\theta),\alpha\le\theta\le\beta\}$, then we have

$$\iiint_E f(r,\theta,z)r\,dr\,d\theta=\int_{\theta=\alpha}^{\theta=\beta}\int_{r=g_1(\theta)}^{r=g_2(\theta)}\int_{z=u_1(r,\theta)}^{z=u_2(r,\theta)}f(r,\theta,z)r\,dz\,dr\,d\theta. \tag{2.6.12}$$

Similar formulas exist for projections onto the other coordinate planes. We can use polar coordinates in those planes if necessary.

Example 2.6.2: Setting up a Triple Integral in Cylindrical Coordinates over a General Region

Consider the region E inside the right circular cylinder with equation $r = 2 \sin \theta$, bounded below by the $r\theta$-plane and bounded above by the sphere with radius 4 centered at the origin (Figure 15.5.3). Set up a triple integral over this region with a function $f(r, \theta, z)$ in cylindrical coordinates.

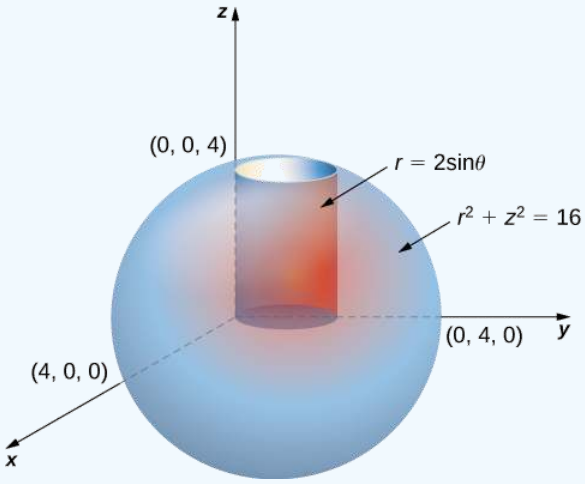


Figure 2.6.3: Setting up a triple integral in cylindrical coordinates over a cylindrical region.

Solution

First, identify that the equation for the sphere is $r^2 + z^2 = 16$. We can see that the limits for z are from 0 to $z = \sqrt{16 - r^2}$. Then the limits for r are from 0 to $r = 2 \sin \theta$. Finally, the limits for θ are from 0 to π. Hence the region is $E = \{(r, \theta, z) | 0 \leq \theta \leq \pi, \ 0 \leq r \leq 2 \sin \theta, \ 0 \leq z \leq \sqrt{16 - r^2}\}$. Therefore, the triple integral is

$$\iiint_E f(r, \theta, z) r \, dz \, dr \, d\theta = \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=2 \sin \theta} \int_{z=0}^{z=\sqrt{16-r^2}} f(r, \theta, z) r \, dz \, dr \, d\theta. \tag{2.6.13}$$

Exercise 2.6.2:

Consider the region inside the right circular cylinder with equation $r = 2 \sin \theta$ bounded below by the $r\theta$-plane and bounded above by $z = 4 - y$. Set up a triple integral with a function $f(r, \theta, z)$ in cylindrical coordinates.

Hint

Analyze the region, and draw a sketch.

Answer

$$\iiint_E f(r, \theta, z) r \, dz \, dr \, d\theta = \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=2 \sin \theta} \int_{z=0}^{z=4-r \sin \theta} f(r, \theta, z) r \, dz \, dr \, d\theta. \tag{2.6.14}$$

Example 2.6.3: Setting up a Triple Integral in Two Ways

Let E be the region bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the paraboloid $z = 2 - x^2 - y^2$. (Figure 15.5.4). Set up a triple integral in cylindrical coordinates to find the volume of the region, using the following orders of integration:

a. $dz \, dr \, d\theta$

b. $dr \, dz \, d\theta$

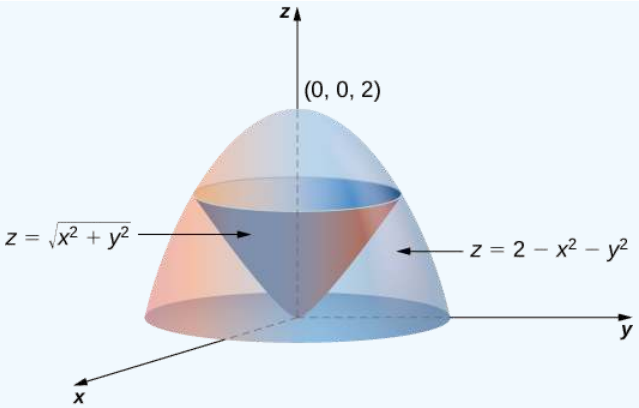


Figure 2.6.4: Setting up a triple integral in cylindrical coordinates over a conical region.

Solution

a. The cone is of radius 1 where it meets the paraboloid. Since $z = 2 - x^2 - y^2 = 2 - r^2$ and $z = \sqrt{x^2 + y^2} = r^2$ (assuming r is nonnegative), we have $2 - r^2 = r$. Solving, we have $r^2 + r - 2 = (r + 2)(r - 1) = 0$. Since $r \geq 0$, we have $r = 1$. Therefore $z = 1$. So the intersection of these two surfaces is a circle of radius 1 in the plane $z = 1$. The cone is the lower bound for z and the paraboloid is the upper bound. The projection of the region onto the xy-plane is the circle of radius 1 centered at the origin.

Thus, we can describe the region as $E = \{(r, \theta, z) | 0 \leq \theta \leq 2\pi,\ 0 \leq r \leq 1,\ r \leq z \leq 2 - r^2\}$.

Hence the integral for the volume is

$$V = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \int_{z=r}^{z=2-r^2} r\, dz\, dr\, d\theta. \tag{2.6.15}$$

b. We can also write the cone surface as $r = z$ and the paraboloid as $r^2 = 2 - z$. The lower bound for r is zero, but the upper bound is sometimes the cone and the other times it is the paraboloid. The plane $z = 1$ divides the region into two regions. Then the region can be described as

$$\begin{aligned} E = &\{(r, \theta, z) | 0 \leq \theta \leq 2\pi,\ 0 \leq z \leq 1,\ 0 \leq r \leq z\} \\ \cup &\{(r, \theta, z) | 0 \leq \theta \leq 2\pi,\ 1 \leq z \leq 2,\ 0 \leq r \leq \sqrt{2 - z}\}. \end{aligned} \tag{2.6.16}$$

Now the integral for the volume becomes

$$V = \int_{\theta=0}^{\theta=2\pi} \int_{z=0}^{z=1} \int_{r=0}^{r=z} r\, dr\, dz\, d\theta + \int_{\theta=0}^{\theta=2\pi} \int_{z=1}^{z=2} \int_{r=0}^{r=\sqrt{2-z}} r\, dr\, dz\, d\theta. \tag{2.6.17}$$

? Exercise 2.6.3:

Redo the previous example with the order of integration $d\theta\, dz\, dr$.

Hint

Note that θ is independent of r and z.

Answer

$E = \{(r, \theta, z) | 0 \leq \theta \leq 2\pi,\ 0 \leq z \leq 1,\ 0 \leq r \leq 2 - z^2\}$ and

$$V = \int_{r=0}^{r=1} \int_{z=0}^{z=2-r^2} \int_{\theta=0}^{\theta=2\pi} r\, d\theta\, dz\, dr. \tag{2.6.18}$$

✓ Example 2.6.4: Finding a Volume with Triple Integrals in Two Ways

Let E be the region bounded below by the $r\theta$-plane, above by the sphere $x^2 + y^2 + z^2 = 4$, and on the sides by the cylinder $x^2 + y^2 = 1$ (Figure 15.5.5). Set up a triple integral in cylindrical coordinates to find the volume of the region using the following

orders of integration, and in each case find the volume and check that the answers are the same:

a. $dz\,dr\,d\theta$

b. $dr\,dz\,d\theta$.

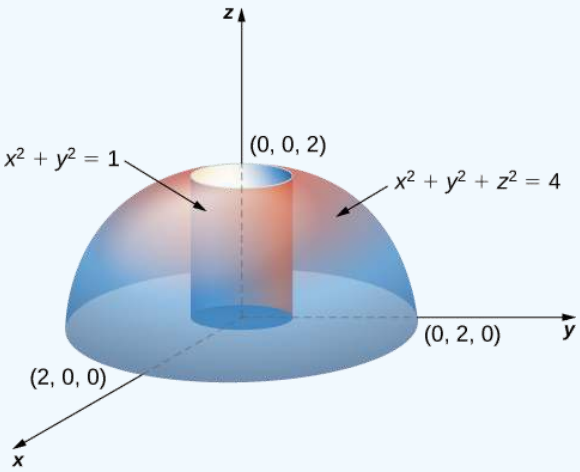


Figure **2.6.5**: Finding a cylindrical volume with a triple integral in cylindrical coordinates.

Solution

a. Note that the equation for the sphere is

$$x^2 + y^2 + z^2 = 4 \text{ or } r^2 + z^2 = 4 \tag{2.6.19}$$

and the equation for the cylinder is

$$x^2 + y^2 = 1 \text{ or } r^2 = 1. \tag{2.6.20}$$

Thus, we have for the region E

$$E = \{(r, \theta, z) | 0 \le z \le \sqrt{4 - r^2},\ 0 \le r \le 1,\ 0 \le \theta \le 2\pi\} \tag{2.6.21}$$

Hence the integral for the volume is

$$V(E) = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \int_{z=0}^{z=\sqrt{4-r^2}} r\,dz\,dr\,d\theta \tag{2.6.22}$$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \left[\rangle rz|_{z=0}^{z=\sqrt{4-r^2}}\right] dr\,d\theta = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \left(r\sqrt{4-r^2}\right) dr\,d\theta \tag{2.6.23}$$

$$= \int_0^{2\pi} \left(\frac{8}{3} - \sqrt{3}\right) d\theta = 2\pi\left(\frac{8}{3} - \sqrt{3}\right) \text{ cubic units.} \tag{2.6.24}$$

b. Since the sphere is $x^2 + y^2 + z^2 = 4$, which is $r^2 + z^2 = 4$, and the cylinder is $x^2 + y^2 = 1$, which is $r^2 = 1$, we have $1 + z^2 = 4$, that is, $z^2 = 3$. Thus we have two regions, since the sphere and the cylinder intersect at $(1, \sqrt{3})$ in the rz-plane

$$E_1 = \{(r, \theta, z) | 0 \le r \le \sqrt{4 - r^2},\ \sqrt{3} \le z \le 2,\ 0 \le \theta \le 2\pi\} \tag{2.6.25}$$

and

$$E_2 = \{(r, \theta, z) | 0 \le r \le 1,\ 0 \le z \le \sqrt{3},\ 0 \le \theta \le 2\pi\}. \tag{2.6.26}$$

Hence the integral for the volume is

$$V(E) = \int_{\theta=0}^{\theta=2\pi} \int_{z=\sqrt{3}}^{z=2} \int_{r=0}^{r=\sqrt{4-r^2}} r\,dr\,dz\,d\theta + \int_{\theta=0}^{\theta=2\pi} \int_{z=0}^{z=\sqrt{3}} \int_{r=0}^{r=1} r\,dr\,dz\,d\theta \tag{2.6.27}$$

$$= \sqrt{3}\pi + \left(\frac{16}{3} - 3\sqrt{3}\right)\pi = 2\pi\left(\frac{8}{3} - \sqrt{3}\right) \text{ cubic units.} \tag{2.6.28}$$

? Exercise 2.6.4

Redo the previous example with the order of integration $d\theta\, dz\, dr$.

Hint

A figure can be helpful. Note that θ is independent of r and z.

Answer

$E_2 = \{(r, \theta, z) | 0 \le \theta \le 2\pi,\ 0 \le r \le 1,\ r \le z \le \sqrt{4 - r^2}\}$ and

$$V = \int_{r=0}^{r=1} \int_{z=r}^{z=\sqrt{4-r^2}} \int_{\theta=0}^{\theta=2\pi} r\, d\theta\, dz\, dr. \tag{2.6.29}$$

Review of Spherical Coordinates

In three-dimensional space $\mathbb{R}^3$ in the spherical coordinate system, we specify a point P by its distance ρ from the origin, the polar angle θ from the positive x-axis (same as in the cylindrical coordinate system), and the angle φ from the positive z-axis and the line OP (Figure 2.6.6). Note that $\rho > 0$ and $0 \le \varphi \le \pi$. (Refer to Cylindrical and Spherical Coordinates for a review.) Spherical coordinates are useful for triple integrals over regions that are symmetric with respect to the origin.

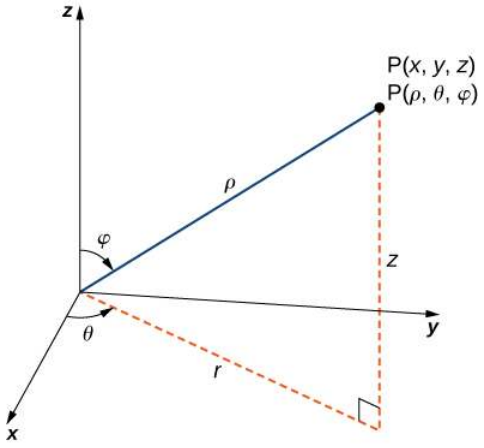


Figure 2.6.6: The spherical coordinate system locates points with two angles and a distance from the origin.

Recall the relationships that connect rectangular coordinates with spherical coordinates.

From spherical coordinates to rectangular coordinates:

$$x = \rho \sin\varphi \cos\theta,\ y = \rho \sin\varphi \sin\theta,\ and\ z = \rho \cos\varphi. \tag{2.6.30}$$

From rectangular coordinates to spherical coordinates:

$$\rho^2 = x^2 + y^2 + z^2,\ \tan\theta = \frac{y}{x},\ \varphi = \arccos\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right). \tag{2.6.31}$$

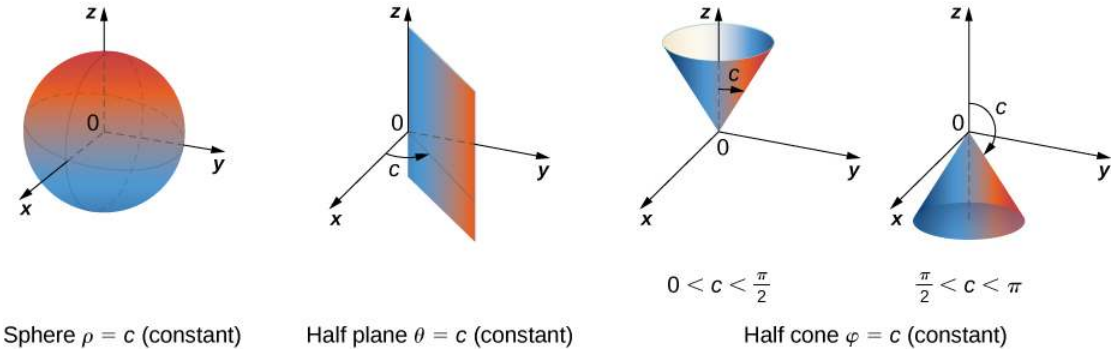
Other relationships that are important to know for conversions are

- $r = \rho \sin\varphi$
- $\theta = \theta$ These equations are used to convert from spherical coordinates to cylindrical coordinates.
- $z = \rho \cos\varphi$

and

- $\rho = \sqrt{r^2 + z^2}$
- $\theta = \theta$ These equations are used to convert from cylindrical coordinates to spherical coordinates.
- $\varphi = \arccos\left(\frac{z}{\sqrt{r^2+z^2}}\right)$

2.6.7 shows a few solid regions that are convenient to express in spherical coordinates.



Sphere $\rho = c$ (constant) Half plane $\theta = c$ (constant) Half cone $\varphi = c$ (constant)

Figure **2.6.7**: Spherical coordinates are especially convenient for working with solids bounded by these types of surfaces. (The letter c indicates a constant.)

Integration in Spherical Coordinates

We now establish a triple integral in the spherical coordinate system, as we did before in the cylindrical coordinate system. Let the function $f(\rho, \theta, \varphi)$ be continuous in a bounded spherical box, $B = \{(\rho, \theta, \varphi) | a \leq \rho \leq b,\ \alpha \leq \theta \leq \beta,\ \gamma \leq \varphi \leq \psi\}$. We then divide each interval into l, m, n and n subdivisions such that $\Delta\rho = \frac{b-a}{l}$, $\Delta\theta = \frac{\beta-\alpha}{m}$. $\Delta\varphi = \frac{\psi-\gamma}{n}$. Now we can illustrate the following theorem for triple integrals in spherical coordinates with $(\rho^*_{ijk}, \theta^*_{ijk}, \varphi^*_{ijk})$ being any sample point in the spherical subbox B_{ijk}. For the volume element of the subbox ΔV in spherical coordinates, we have $\Delta V = (\Delta\rho)\,(\rho\Delta\varphi)\,(\rho \sin\varphi\,\Delta\theta)$, as shown in the following figure.

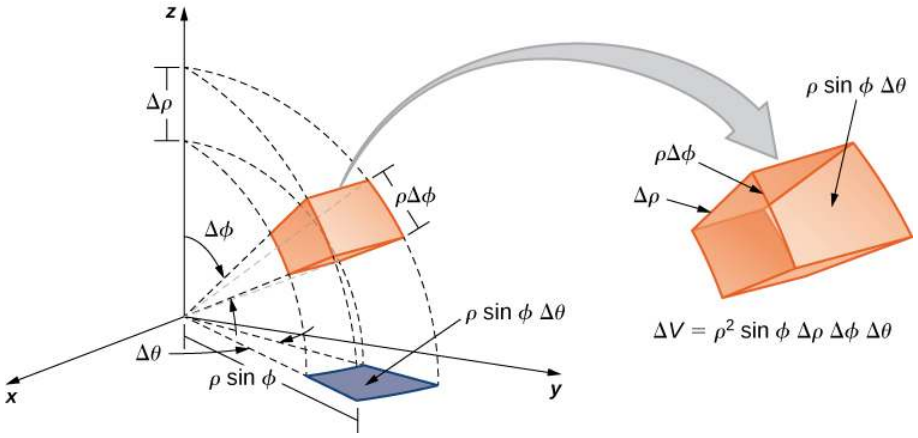


Figure **2.6.8**: The volume element of a box in spherical coordinates.

Definition: triple integral in spherical coordinates

The **triple integral in spherical coordinates** is the limit of a triple Riemann sum,

$$\lim_{l,m,n\to\infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(\rho^*_{ijk}, \theta^*_{ijk}, \varphi^*_{ijk})(\rho^*_{ijk})^2 \sin\varphi \Delta\rho\Delta\theta\Delta\varphi \tag{2.6.32}$$

provided the limit exists.

As with the other multiple integrals we have examined, all the properties work similarly for a triple integral in the spherical coordinate system, and so do the iterated integrals. Fubini's theorem takes the following form.

Theorem: Fubini's Theorem for Spherical Coordinates

If $f(\rho, \theta, \varphi)$ is continuous on a spherical solid box $B = [a, b] \times [\alpha, \beta] \times [\gamma, \psi]$, then

$$\iiint_B f(\rho, \theta, \varphi)\, \rho^2 \sin\varphi d\rho\, d\varphi\, d\theta = \int_{\varphi=\gamma}^{\varphi=\psi} \int_{\theta=\alpha}^{\theta=\beta} \int_{\rho=a}^{\rho=b} f(\rho, \theta, \varphi)\, \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta. \tag{2.6.33}$$

This iterated integral may be replaced by other iterated integrals by integrating with respect to the three variables in other orders.

As stated before, spherical coordinate systems work well for solids that are symmetric around a point, such as spheres and cones. Let us look at some examples before we consider triple integrals in spherical coordinates on general spherical regions.

Example 2.6.5: Evaluating a Triple Integral in Spherical Coordinates

Evaluate the iterated triple integral

$$\int_{\theta=0}^{\theta=2\pi}\int_{\varphi=0}^{\varphi=\pi/2}\int_{\rho=0}^{\rho=1}\rho^2\sin\,\varphi\,d\rho\,d\varphi\,d\theta. \tag{2.6.34}$$

Solution

As before, in this case the variables in the iterated integral are actually independent of each other and hence we can integrate each piece and multiply:

$$\begin{aligned}\int_0^{2\pi}\int_0^{\pi/2}\int_0^1\rho^2\sin\,\varphi\,d\rho\,d\varphi\,d\theta &= \int_0^{2\pi}d\theta\int_0^{\pi/2}\sin\,\varphi\,d\varphi\int_0^1\rho^2 d\rho \\ &= (2\pi)\,(1)\left(\frac{1}{3}\right)=\frac{2\pi}{3}\end{aligned} \tag{2.6.35}$$

The concept of triple integration in spherical coordinates can be extended to integration over a general solid, using the projections onto the coordinate planes. Note that dV and dA mean the increments in volume and area, respectively. The variables V and A are used as the variables for integration to express the integrals.

The triple integral of a continuous function $f(\rho,\theta,\varphi)$ over a general solid region

$$E=\{(\rho,\theta,\varphi)|(\rho,\theta)\in D, u_1(\rho,\theta)\le\varphi\le u_2(\rho,\theta)\} \tag{2.6.36}$$

in $\mathbb{R}^3$, where D is the projection of E onto the $\rho\theta$-plane, is

$$\iiint_E f(\rho,\theta,\varphi)dV=\iint_D\left[\int_{u_1(\rho,\theta)}^{u_2(\rho,\theta)}f(\rho,\theta,\varphi)\,d\varphi\right]\,dA. \tag{2.6.37}$$

In particular, if $D=\{(\rho,\theta)|g_1(\theta)\le\rho\le g_2(\theta),\,\alpha\le\theta\le\beta\}$, the we have

$$\iiint_E f(\rho,\theta,\varphi)dV=\int_\alpha^\beta\int_{g_1(\theta)}^{g_2(\theta)}\int_{u_1(\rho,\theta)}^{u_2(\rho,\theta)}f(\rho,\theta,\varphi)\rho^2\sin\,\varphi\,d\varphi\,d\rho\,d\theta. \tag{2.6.38}$$

Similar formulas occur for projections onto the other coordinate planes.

Example 2.6.6: Setting up a Triple Integral in Spherical Coordinates

Set up an integral for the volume of the region bounded by the cone $z=\sqrt{3(x^2+y^2)}$ and the hemisphere $z=\sqrt{4-x^2-y^2}$ (see the figure below).

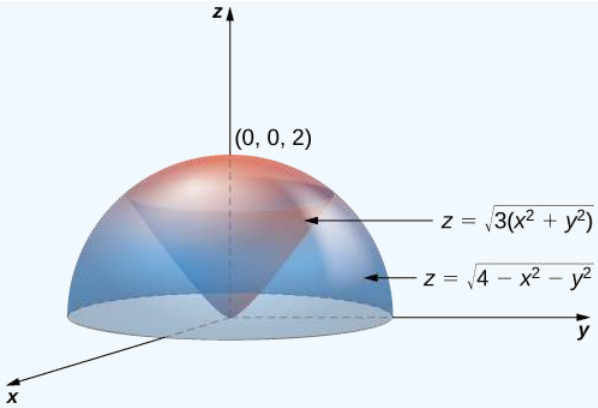


Figure 2.6.9: A region bounded below by a cone and above by a hemisphere.

Solution

Using the conversion formulas from rectangular coordinates to spherical coordinates, we have:

For the cone: $z = \sqrt{3(x^2 + y^2)}$ or $\rho \cos \varphi = \sqrt{3}\rho \sin \varphi$ or $\tan \varphi = \frac{1}{\sqrt{3}}$ or $\varphi = \frac{\pi}{6}$.

For the sphere: $z = \sqrt{4 - x^2 - y^2}$ or $z^2 + x^2 + y^2 = 4$ or $\rho^2 = 4$ or $\rho = 2$.

Thus, the triple integral for the volume is

$$V(E) = \int_{\theta=0}^{\theta=2\pi} \int_{\varphi=0}^{\varphi=\pi/6} \int_{\rho=0}^{\rho=2} \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta. \tag{2.6.39}$$



Exercise 2.6.5

Set up a triple integral for the volume of the solid region bounded above by the sphere $\rho = 2$ and bounded below by the cone $\varphi = \pi/3$.

Hint

Follow the steps of the previous example.

Answer

$$V(E) = \int_{\theta=0}^{\theta=2\pi} \int_{\varphi=0}^{\varphi=\pi/3} \int_{\rho=0}^{\rho=2} \rho^2 \sin\,\varphi \, d\rho \, d\varphi \, d\theta \tag{2.6.40}$$

Example 2.6.7: Interchanging Order of Integration in Spherical Coordinates

Let E be the region bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the sphere $z = x^2 + y^2 + z^2$ (Figure 15.5.10). Set up a triple integral in spherical coordinates and find the volume of the region using the following orders of integration:

a. $d\rho \, d\phi \, d\theta$
b. $d\varphi \, d\rho \, d\theta$

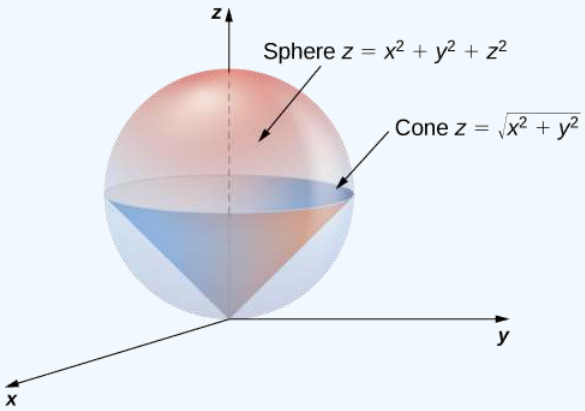


Figure 2.6.10:. A region bounded below by a cone and above by a sphere.

Solution

a. Use the conversion formulas to write the equations of the sphere and cone in spherical coordinates.

For the sphere:

$$x^2 + y^2 + z^2 = z \tag{2.6.41}$$
$$\rho^2 = \rho \cos\,\varphi \tag{2.6.42}$$
$$\rho = \cos\,\varphi. \tag{2.6.43}$$

For the cone:

$$z = \sqrt{x^2 + y^2} \tag{2.6.44}$$
$$\rho \cos\,\varphi = \sqrt{\rho^2 \sin^2\,\varphi \cos^2\theta + \rho^2 \sin^2\,\varphi \sin^2\theta} \tag{2.6.45}$$
$$\rho \cos\,\varphi = \sqrt{\rho^2 \sin^2\varphi\,(\cos^2\theta + \sin^2\theta)} \tag{2.6.46}$$
$$\rho \cos\,\varphi = \rho \sin\,\varphi \tag{2.6.47}$$
$$\cos\,\varphi = \sin\,\varphi \tag{2.6.48}$$
$$\varphi = \pi/4. \tag{2.6.49}$$

Hence the integral for the volume of the solid region E becomes

$$V(E) = \int_{\theta=0}^{\theta=2\pi} \int_{\varphi=0}^{\varphi=\pi/4} \int_{\rho=0}^{\rho=\cos\,\varphi} \rho^2 \sin\,\varphi \, d\rho \, d\varphi \, d\theta. \tag{2.6.50}$$

b. Consider the $\varphi\rho$-plane. Note that the ranges for φ and ρ (from part a.) are

$$0 \le \rho\sqrt{2}/2 \text{and} \sqrt{2} \le \rho 1 \tag{2.6.51}$$
$$0 \le \varphi \le \pi/4 0 \le \rho \le \cos\,\varphi \tag{2.6.52}$$

The curve $\rho = \cos\,\varphi$ meets the line $\varphi = \pi/4$ at the point $(\pi/4, \sqrt{2}/2)$. Thus, to change the order of integration, we need to use two pieces:

$$0 \le \rho \le \sqrt{2}/2,\ 0 \le \varphi \le \pi/4 \tag{2.6.53}$$

and

$$\sqrt{2}/2 \leq \rho \leq 1,\ 0 \leq \varphi \leq \cos^{-1}\rho. \tag{2.6.54}$$

Hence the integral for the volume of the solid region E becomes

$$\begin{aligned} V(E) = &\int_{\theta=0}^{\theta=2\pi}\int_{\rho=0}^{\rho=\sqrt{2}/2}\int_{\varphi=0}^{\varphi=\pi/4} \rho^2 \sin\,\varphi\, d\varphi\, d\rho\, d\theta \\ &+ \int_{\theta=0}^{\theta=2\pi}\int_{\rho=\sqrt{2}/2}^{\rho=1}\int_{\varphi=0}^{\varphi=\cos^{-1}\rho} \rho^2 \sin\,\varphi\, d\varphi\, d\rho\, d\theta \end{aligned} \tag{2.6.55}$$

In each case, the integration results in $V(E) = \frac{\pi}{8}$.

Before we end this section, we present a couple of examples that can illustrate the conversion from rectangular coordinates to cylindrical coordinates and from rectangular coordinates to spherical coordinates.

Example 2.6.8: Converting from Rectangular Coordinates to Cylindrical Coordinates

Convert the following integral into cylindrical coordinates:

$$\int_{y=-1}^{y=1}\int_{x=0}^{x=\sqrt{1-y^2}}\int_{z=x^2+y^2}^{z=\sqrt{x^2+y^2}} xyz\, dz\, dx\, dy. \tag{2.6.56}$$

Solution

The ranges of the variables are

$$-1 \leq y \leq y \tag{2.6.57}$$

$$0 \leq x \leq \sqrt{1-y^2} \tag{2.6.58}$$

$$x^2+y^2 \leq z \leq \sqrt{x^2+y^2}. \tag{2.6.59}$$

The first two inequalities describe the right half of a circle of radius 1. Therefore, the ranges for θ and r are

$$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \text{ and } 0 \leq r \leq 1. \tag{2.6.60}$$

The limits of z are $r^2 \leq z \leq r$, hence

$$\begin{aligned} &\int_{y=-1}^{y=1}\int_{x=0}^{x=\sqrt{1-y^2}}\int_{z=x^2+y^2}^{z=\sqrt{x^2+y^2}} xyz\, dz\, dx\, dy \\ &= \int_{\theta=-\pi/2}^{\theta=\pi/2}\int_{r=0}^{r=1}\int_{z=r^2}^{z=r} r(r\cos\theta)(r\sin\theta)\, z\, dz\, dr\, d\theta. \end{aligned} \tag{2.6.61}$$

Example 2.6.9: Converting from Rectangular Coordinates to Spherical Coordinates

Convert the following integral into spherical coordinates:

$$\int_{y=0}^{y=3}\int_{x=0}^{x=\sqrt{9-y^2}}\int_{z=\sqrt{x^2+y^2}}^{z=\sqrt{18-x^2-y^2}} (x^2+y^2+z^2)dz\, dx\, dy. \tag{2.6.62}$$

Solution

The ranges of the variables are

$$0 \leq y \leq 3 \tag{2.6.63}$$

$$0 \leq x \leq \sqrt{9-y^2} \tag{2.6.64}$$

$$\sqrt{x^2+y^2} \leq z \leq \sqrt{18-x^2-y^2}. \tag{2.6.65}$$

The first two ranges of variables describe a quarter disk in the first quadrant of the xy-plane. Hence the range for θ is $0 \leq \theta \leq \frac{\pi}{2}$.

The lower bound $z = \sqrt{x^2 + y^2}$ is the upper half of a cone and the upper bound $z = \sqrt{18 - x^2 - y^2}$ is the upper half of a sphere. Therefore, we have $0 \le \rho \le \sqrt{18}$, which is $0 \le \rho \le 3\sqrt{2}$.

For the ranges of φ we need to find where the cone and the sphere intersect, so solve the equation

$$r^2 + z^2 = 18 \tag{2.6.66}$$
$$(\sqrt{x^2 + y^2})^2 + z^2 = 18 \tag{2.6.67}$$
$$z^2 + z^2 = 18 \tag{2.6.68}$$
$$2z^2 = 18 \tag{2.6.69}$$
$$z^2 = 9 \tag{2.6.70}$$
$$z = 3. \tag{2.6.71}$$

This gives

$$3\sqrt{2}\cos\varphi = 3 \tag{2.6.72}$$
$$\cos\varphi = \frac{1}{\sqrt{2}} \tag{2.6.73}$$
$$\varphi = \frac{\pi}{4}. \tag{2.6.74}$$

Putting this together, we obtain

$$\int_{y=0}^{y=3}\int_{x=0}^{x=\sqrt{9-y^2}}\int_{z=\sqrt{x^2+y^2}}^{z=\sqrt{18-x^2-y^2}}(x^2+y^2+z^2)\,dz\,dx\,dy \tag{2.6.75}$$
$$= \int_{\varphi=0}^{\varphi=\pi/4}\int_{\theta=0}^{\theta=\pi/2}\int_{\rho=0}^{\rho=3\sqrt{2}}\rho^4\sin\varphi\,d\rho\,d\theta\,d\varphi.$$

? Exercise 2.6.6:

Use rectangular, cylindrical, and spherical coordinates to set up triple integrals for finding the volume of the region inside the sphere $x^2 + y^2 + z^2 = 4$ but outside the cylinder $x^2 + y^2 = 1$.

Answer: Rectangular

$$\int_{x=-2}^{x=2}\int_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}}\int_{z=-\sqrt{4-x^2-y^2}}^{z=\sqrt{4-x^2-y^2}}dz\,dy\,dx - \int_{x=-1}^{x=1}\int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}}\int_{z=-\sqrt{4-x^2-y^2}}^{z=\sqrt{4-x^2-y^2}}dz\,dy\,dx. \tag{2.6.76}$$

Answer: Cylindrical

$$\int_{\theta=0}^{\theta=2\pi}\int_{r=1}^{r=2}\int_{z=-\sqrt{4-r^2}}^{z=\sqrt{4-r^2}}r\,dz\,dr\,d\theta. \tag{2.6.77}$$

Answer: Spherical

$$\int_{\varphi=\pi/6}^{\varphi=5\pi/6}\int_{\theta=0}^{\theta=2\pi}\int_{\rho=\csc\varphi}^{\rho=2}\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi. \tag{2.6.78}$$

Now that we are familiar with the spherical coordinate system, let's find the volume of some known geometric figures, such as spheres and ellipsoids.

✓ Example 2.6.10: Chapter Opener: Finding the Volume of l'Hemisphèric

Find the volume of the spherical planetarium in l'Hemisphèric in Valencia, Spain, which is five stories tall and has a radius of approximately 50 ft, using the equation $x^2 + y^2 + z^2 = r^2$.



Figure 2.6.11: (credit: modification of work by Javier Yaya Tur, Wikimedia Commons)

Solution

We calculate the volume of the ball in the first octant, where $x \leq 0$, $y \leq 0$, and $z \leq 0$, using spherical coordinates, and then multiply the result by 8 for symmetry. Since we consider the region D as the first octant in the integral, the ranges of the variables are

$$0 \leq \varphi \leq \frac{\pi}{2}, 0 \leq \rho \leq r, 0 \leq \theta \leq \frac{\pi}{2}. \tag{2.6.79}$$

Therefore,

$$V = \iiint_D dx\, dy\, dz = 8 \int_{\theta=0}^{\theta=\pi/2} \int_{\rho=0}^{\rho=\pi} \int_{\varphi=0}^{\varphi=\pi/2} \rho^2 \sin\theta \, d\varphi \, d\rho \, d\varphi \tag{2.6.80}$$

$$= 8 \int_{\varphi=0}^{\varphi=\pi/2} d\varphi \int_{\rho=0}^{\rho=r} \rho^2 d\rho \int_{\theta=0}^{\theta=\pi/2} \sin\theta \, d\theta \tag{2.6.81}$$

$$= 8 \left(\frac{\pi}{2}\right) \left(\frac{r^3}{3}\right) (1) \tag{2.6.82}$$

$$= \frac{4}{3}\pi r^3. \tag{2.6.83}$$

This exactly matches with what we knew. So for a sphere with a radius of approximately 50 ft, the volume is $\frac{4}{3}\pi(50)^3 \approx 523,600$ ft^3.

For the next example we find the volume of an ellipsoid.

Example 2.6.11: Finding the Volume of an Ellipsoid

Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Solution

We again use symmetry and evaluate the volume of the ellipsoid using spherical coordinates. As before, we use the first octant $x \leq 0$, $y \leq 0$, and $z \leq 0$ and then multiply the result by 8.

In this case the ranges of the variables are

$$0 \leq \varphi \leq \frac{\pi}{2}\, 0 \leq \rho \leq 1, \text{ and } 0 \leq \theta \leq \frac{\pi}{2}. \tag{2.6.84}$$

Also, we need to change the rectangular to spherical coordinates in this way:

$$x = a\rho \cos\varphi \sin\theta, y = b\rho \sin\varphi \sin\theta, \text{ and } z = cp \cos\theta. \tag{2.6.85}$$

Then the volume of the ellipsoid becomes

$$V = \iiint_D dx\,dy\,dz \tag{2.6.86}$$

$$= 8\int_{\theta=0}^{\theta=\pi/2}\int_{\rho=0}^{\rho=1}\int_{\varphi=0}^{\varphi=\pi/2} abc\,\rho^2\sin\theta\,d\varphi\,d\rho\,d\theta \tag{2.6.87}$$

$$\tag{2.6.88}$$

$$= 8abc\int_{\varphi=0}^{\varphi=\pi/2} d\varphi\int_{\rho=0}^{\rho=1}\rho^2 d\rho\int_{\theta=0}^{\theta=\pi/2}\sin\theta\,d\theta \tag{2.6.89}$$

$$= 8abc\left(\frac{\pi}{2}\right)\left(\frac{1}{3}\right)(1) \tag{2.6.90}$$

$$= \frac{4}{3}\pi abc. \tag{2.6.91}$$

Example 2.6.12: Finding the Volume of the Space Inside an Ellipsoid and Outside a Sphere

Find the volume of the space inside the ellipsoid $\frac{x^2}{75^2}+\frac{y^2}{80^2}+\frac{z^2}{90^2}=1$ and outside the sphere $x^2+y^2+z^2=50^2$.

Solution

This problem is directly related to the l'Hemisphèric structure. The volume of space inside the ellipsoid and outside the sphere might be useful to find the expense of heating or cooling that space. We can use the preceding two examples for the volume of the sphere and ellipsoid and then substract.

First we find the volume of the ellipsoid using $a=75$ ft, $b=80$ ft, and $c=90$ ft in the result from Example. Hence the volume of the ellipsoid is

$$V_{ellipsoid} = \frac{4}{3}\pi(75)(80)(90) \approx 2,262,000\ ft^3. \tag{2.6.92}$$

From Example, the volume of the sphere is

$$V_{sphere} \approx 523,600\ ft^3. \tag{2.6.93}$$

Therefore, the volume of the space inside the ellipsoid $\frac{x^2}{75^2}+\frac{y^2}{80^2}+\frac{z^2}{90^2}=1$ and outside the sphere $x^2+y^2+z^2=50^2$ is approximately

$$V_{Hemispheric} = V_{ellipsoid} - V_{sphere} = 1,738,400\ ft^3. \tag{2.6.94}$$

Student Project: Hot air balloons

Hot air ballooning is a relaxing, peaceful pastime that many people enjoy. Many balloonist gatherings take place around the world, such as the Albuquerque International Balloon Fiesta. The Albuquerque event is the largest hot air balloon festival in the world, with over 500 balloons participating each year.



Figure 2.6.12: Balloons lift off at the 2001 Albuquerque International Balloon Fiesta. (credit: David Herrera, Flickr)

As the name implies, hot air balloons use hot air to generate lift. (Hot air is less dense than cooler air, so the balloon floats as long as the hot air stays hot.) The heat is generated by a propane burner suspended below the opening of the basket. Once the balloon takes off, the pilot controls the altitude of the balloon, either by using the burner to heat the air and ascend or by using a vent near the top of the balloon to release heated air and descend. The pilot has very little control over where the balloon goes, however—balloons are at the mercy of the winds. The uncertainty over where we will end up is one of the reasons balloonists are attracted to the sport.

In this project we use triple integrals to learn more about hot air balloons. We model the balloon in two pieces. The top of the balloon is modeled by a half sphere of radius 28

feet. The bottom of the balloon is modeled by a frustum of a cone (think of an ice cream cone with the pointy end cut off). The radius of the large end of the frustum is 28 feet and the radius of the small end of the frustum is 28 feet. A graph of our balloon model and a cross-sectional diagram showing the dimensions are shown in the following figure.

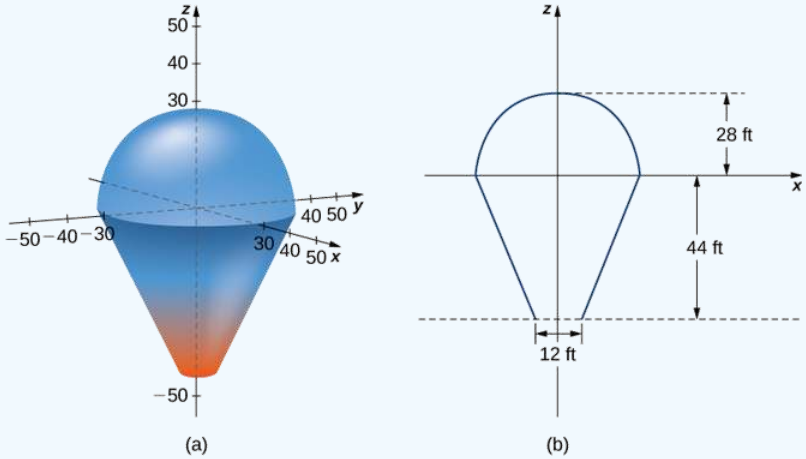


Figure **2.6.13**: (a) Use a half sphere to model the top part of the balloon and a frustum of a cone to model the bottom part of the balloon. (b) A cross section of the balloon showing its dimensions.

We first want to find the volume of the balloon. If we look at the top part and the bottom part of the balloon separately, we see that they are geometric solids with known volume formulas. However, it is still worthwhile to set up and evaluate the integrals we would need to find the volume. If we calculate the volume using integration, we can use the known volume formulas to check our answers. This will help ensure that we have the integrals set up correctly for the later, more complicated stages of the project.

1. Find the volume of the balloon in two ways.

a. Use triple integrals to calculate the volume. Consider each part of the balloon separately. (Consider using spherical coordinates for the top part and cylindrical coordinates for the bottom part.)

b. Verify the answer using the formulas for the volume of a sphere, $V = \frac{4}{3}\pi r^3$, and for the volume of a cone, $V = \frac{1}{3}\pi r^2 h$.

In reality, calculating the temperature at a point inside the balloon is a tremendously complicated endeavor. In fact, an entire branch of physics (thermodynamics) is devoted to studying heat and temperature. For the purposes of this project, however, we are going to make some simplifying assumptions about how temperature varies from point to point within the balloon. Assume that just prior to liftoff, the temperature (in degrees Fahrenheit) of the air inside the balloon varies according to the function

$$T_0(r,\theta,z) = \frac{z-r}{10} + 210. \tag{2.6.95}$$

2. What is the average temperature of the air in the balloon just prior to liftoff? (Again, look at each part of the balloon separately, and do not forget to convert the function into spherical coordinates when looking at the top part of the balloon.)

Now the pilot activates the burner for 10 seconds. This action affects the temperature in a 12-foot-wide column 20 feet high, directly above the burner. A cross section of the balloon depicting this column in shown in the following figure

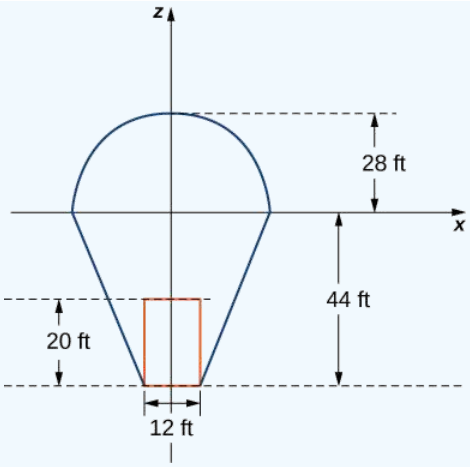


Figure 2.6.14: Activating the burner heats the air in a 20-foot-high, 12-foot-wide column directly above the burner.

Assume that after the pilot activates the burner for 10 seconds, the temperature of the air in the column described above *increases* according to the formula

$$H(r,\theta,z) = -2z - 48. \tag{2.6.96}$$

Then the temperature of the air in the column is given by

$$T_1(r,\theta,z) = \frac{z-r}{10} + 210 + (-2z - 48), \tag{2.6.97}$$

while the temperature in the remainder of the balloon is still given by

$$T_0(r,\theta,z) = \frac{z-r}{10} + 210. \tag{2.6.98}$$

3. Find the average temperature of the air in the balloon after the pilot has activated the burner for 10 seconds.

Key Concepts

- To evaluate a triple integral in cylindrical coordinates, use the iterated integral

$$\int_{\theta=\alpha}^{\theta=\beta}\int_{r=g_1(\theta)}^{r=g_2(\theta)}\int_{z=u_1(r,\theta)}^{u_2(r,\theta)} f(r,\theta,z)r\,dz\,dr\,d\theta.$$

- To evaluate a triple integral in spherical coordinates, use the iterated integral

$$\int_{\theta=\alpha}^{\theta=\beta}\int_{\rho=g_1(\theta)}^{\rho=g_2(\theta)}\int_{\varphi=u_1(r,\theta)}^{u_2(r,\theta)} f(\rho,\theta,\varphi)\,\rho^2\sin\varphi\,d\varphi\,d\rho\,d\theta.$$

Key Equations

- **Triple integral in cylindrical coordinates**

$$\iiint_B g(s,y,z)dV = \iiint_B g(r\cos\theta,\, r\sin\theta,\, z)r\,dr\,d\theta\,dz = \iiint_B f(r,\theta,z)r\,dr\,d\theta\,dz$$

- **Triple integral in spherical coordinates**

$$\iiint_B f(\rho,\theta,\varphi)\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta = \int_{\varphi=\gamma}^{\varphi=\psi}\int_{\theta=\alpha}^{\theta=\beta}\int_{\rho=a}^{\rho=b} f(\rho,\theta,\varphi)\rho^2\sin\,\varphi\,d\rho\,d\varphi\,d\theta$$

Glossary

triple integral in cylindrical coordinates

the limit of a triple Riemann sum, provided the following limit exists:

$$lim_{l,m,n\to\infty}\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n}f(r^*_{ijk},\theta^*_{ijk},s^*_{ijk})r^*_{ijk}\Delta r\Delta\theta\Delta z$$

triple integral in spherical coordinates

the limit of a triple Riemann sum, provided the following limit exists:

$$lim_{l,m,n\to\infty}\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n}f(\rho^*_{ijk},\theta^*_{ijk},\varphi^*_{ijk})(\rho^*_{ijk})^2\sin\,\varphi\Delta\rho\Delta\theta\Delta\varphi$$

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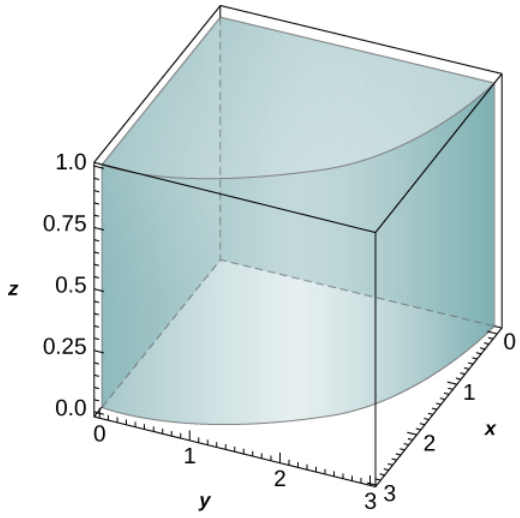
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2.6E: Exercises

In exercises 1 - 8, evaluate the triple integrals $\iiint_E f(x,y,z)\,dV$ over the solid E.

1. $f(x,y,z) = z, \quad B = \{(x,y,z) \mid x^2 + y^2 \le 9, \quad x \le 0, \quad y \le 0, \quad 0 \le z \le 1\}$

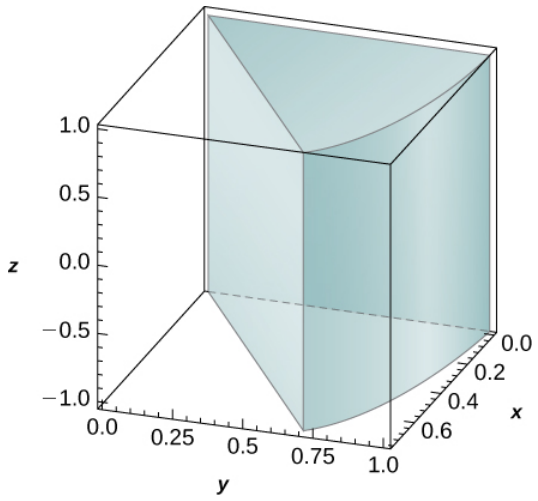


Answer

$\frac{9\pi}{8}$

2. $f(x,y,z) = xz^2, \; B = \{(x,y,z) \mid x^2 + y^2 \le 16, \; x \ge 0, \; y \le 0, \; -1 \le z \le 1\}$

3. $f(x,y,z) = xy, \; B = \{(x,y,z) \mid x^2 + y^2 \le 1, \; x \ge 0, \; x \ge y, \; -1 \le z \le 1\}$



Answer

$\frac{1}{8}$

4. $f(x,y,z) = x^2 + y^2, \; B = \{(x,y,z) \mid x^2 + y^2 \le 4, \; x \ge 0, \; x \le y, \; 0 \le z \le 3\}$

5. $f(x,y,z) = e^{\sqrt{x^2+y^2}}, \; B = \{(x,y,z) \mid 1 \le x^2 + y^2 \le 4, \; y \le 0, \; x \le y\sqrt{3}, \; 2 \le z \le 3\}$

Answer

$\frac{\pi e^2}{6}$

6. $f(x,y,z)=\sqrt{x^2+y^2},\ B=\{(x,y,z)\,|\,1\le x^2+y^2\le 9,\ y\le 0,\ 0\le z\le 1\}$

7. a. Let B be a cylindrical shell with inner radius a outer radius b, and height c where $0<a<b$ and $c>0$. Assume that a function F defined on B can be expressed in cylindrical coordinates as $F(x,y,z)=f(r)+h(z)$, where f and h are differentiable functions. If $\int_a^b \bar{f}(r)\,dr=0$ and $\bar{h}(0)=0$, where $\bar{f}$ and $\bar{h}$ are antiderivatives of f and h, respectively, show that $\iiint_B F(x,y,z)\,dV=2\pi c(b\bar{f}(b)-a\bar{f}(a))+\pi(b^2-a^2)\bar{h}(c)$.

b. Use the previous result to show that $\iiint_B \left(z+\sin\sqrt{x^2+y^2}\right)\,dx\,dy\,dz=6\pi^2(\pi-2)$, where B is a cylindrical shell with inner radius π outer radius 2π, and height 2.

8. a. Let B be a cylindrical shell with inner radius a outer radius b and height c where $0<a<b$ and $c>0$. Assume that a function F defined on B can be expressed in cylindrical coordinates as $F(x,y,z)=f(r)g(\theta)f(z)$, where f, g, and h are differentiable functions. If $\int_a^b \tilde{f}(r)\,dr=0$, where $\tilde{f}$ is an antiderivative of f, show that $\iiint_B F(x,y,z)\,dV=[b\tilde{f}(b)-a\tilde{f}(a)][\tilde{g}(2\pi)-\tilde{g}(0)][\tilde{h}(c)-\tilde{h}(0)]$, where $\tilde{g}$ and $\tilde{h}$ are antiderivatives of g and h, respectively.

b. Use the previous result to show that $\iiint_B z\sin\sqrt{x^2+y^2}\,dx\,dy\,dz=-12\pi^2$, where B is a cylindrical shell with inner radius π outer radius 2π, and height 2.

In exercises 9 - 12, the boundaries of the solid E are given in cylindrical coordinates.

a. Express the region E in cylindrical coordinates.

b. Convert the integral $\iiint_E f(x,y,z)\,dV$ to cylindrical coordinates.

9. E is bounded by the right circular cylinder $r=4\sin\theta$, the $r\theta$-plane, and the sphere $r^2+z^2=16$.

Answer

a. $E=\{(r,\theta,z)\,|\,0\le\theta\le\pi,\ 0\le r\le 4\sin\theta,\ 0\le z\le\sqrt{16-r^2}\}$

b. $\int_0^{\pi}\int_0^{4\sin\theta}\int_0^{\sqrt{16-r^2}} f(r,\theta,z)r\,dz\,dr\,d\theta$

10. E is bounded by the right circular cylinder $r=\cos\theta$, the $r\theta$-plane, and the sphere $r^2+z^2=9$.

11. E is located in the first octant and is bounded by the circular paraboloid $z=9-3r^2$, the cylinder $r=\sqrt{3}$, and the plane $r(\cos\theta+\sin\theta)=20-z$.

Answer

a. $E=\{(r,\theta,z)\,|\,0\le\theta\le\frac{\pi}{2},\ 0\le r\le\sqrt{3},\ 9-r^2\le z\le 10-r(\cos\theta+\sin\theta)\}$

b. $\int_0^{\pi/2}\int_0^{\sqrt{3}}\int_{9-r^2}^{10-r(\cos\theta+\sin\theta)} f(r,\theta,z)r\,dz\,dr\,d\theta$

12. E is located in the first octant outside the circular paraboloid $z=10-2r^2$ and inside the cylinder $r=\sqrt{5}$ and is bounded also by the planes $z=20$ and $\theta=\frac{\pi}{4}$.

In exercises 13 - 16, the function f and region E are given.

a. Express the region E and the function f in cylindrical coordinates.

b. Convert the integral $\iiint_B f(x,y,z)\,dV$ into cylindrical coordinates and evaluate it.

13. $f(x,y,z)=x^2+y^2,\ E=\{(x,y,z)\,|\,0\le x^2+y^2\le 9,\ x\ge 0,\ y\ge 0,\ 0\le z\le x+3\}$

Answer

a. $E=\{(r,\theta,z)\,|\,0\le r\le 3,\ 0\le\theta\le\frac{\pi}{2},\ 0\le z\le r\ \cos\theta+3\}$,
$f(r,\theta,z)=\frac{1}{r\ \cos\theta+3}$

b. $\int_0^3 \int_0^{\pi/2} \int_0^{r\ \cos\theta+3} \frac{r}{r\ \cos\theta+3}\, dz\ d\theta\ dr = \frac{9\pi}{4}$

14. $f(x,y,z) = x^2+y^2,\ E = \{(x,y,z) | 0 \le x^2+y^2 \le 4,\ y \ge 0,\ 0 \le z \le 3-x\}$

15. $f(x,y,z) = x,\ E = \{(x,y,z)\,|\,1 \le y^2+z^2 \le 9,\ 0 \le x \le 1-y^2-z^2\}$

Answer

a. $y = r\ \cos\theta,\ z = r\ \sin\theta,\ x = z,\ E = \{(r,\theta,z)\,|\,1 \le r \le 3,\ 0 \le \theta \le 2\pi,\ 0 \le z \le 1-r^2\},\ f(r,\theta,z) = z;$

b. $\int_1^3 \int_0^{2\pi} \int_0^{1-r^2} zr\ dz\ d\theta\ dr = \frac{356\pi}{3}$

16. $f(x,y,z) = y,\ E = \{(x,y,z)\,|\,1 \le x^2+z^2 \le 9,\ 0 \le y \le 1-x^2-z^2\}$

In exercises 17 - 24, find the volume of the solid E whose boundaries are given in rectangular coordinates.

17. E is above the xy-plane, inside the cylinder $x^2+y^2=1$, and below the plane $z=1$.

Answer

π

18. E is below the plane $z=1$ and inside the paraboloid $z=x^2+y^2$.

19. E is bounded by the circular cone $z=\sqrt{x^2+y^2}$ and $z=1$.

Answer

$\frac{\pi}{3}$

20. E is located above the xy-plane, below $z=1$, outside the one-sheeted hyperboloid $x^2+y^2-z^2=1$, and inside the cylinder $x^2+y^2=2$.

21. E is located inside the cylinder $x^2+y^2=1$ and between the circular paraboloids $z=1-x^2-y^2$ and $z=x^2+y^2$.

Answer

π

22. E is located inside the sphere $x^2+y^2+z^2=1$, above the xy-plane, and inside the circular cone $z=\sqrt{x^2+y^2}$.

23. E is located outside the circular cone $x^2+y^2=(z-1)^2$ and between the planes $z=0$ and $z=2$.

Answer

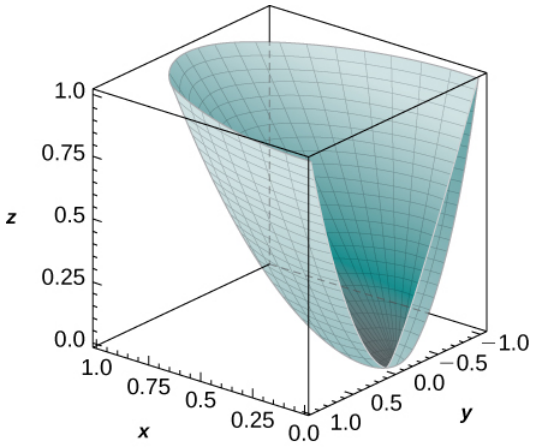
$\frac{4\pi}{3}$

24. E is located outside the circular cone $z=1-\sqrt{x^2+y^2}$, above the xy-plane, below the circular paraboloid, and between the planes $z=0$ and $z=2$.

25. [T] Use a computer algebra system (CAS) to graph the solid whose volume is given by the iterated integral in cylindrical coordinates $\int_{-\pi/2}^{\pi/2} \int_0^1 \int_{r^2}^r r\,dz\,dr\,d\theta$. Find the volume V of the solid. Round your answer to four decimal places.

Answer

$V = \frac{pi}{12} \approx 0.2618$



26. [T] Use a CAS to graph the solid whose volume is given by the iterated integral in cylindrical coordinates $\int_0^{\pi/2}\int_0^1\int_{r^4}^{r} r\,dz\,dr\,d\theta$. Find the volume E of the solid. Round your answer to four decimal places.

27. Convert the integral $\int_0^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{x^2+y^2}^{\sqrt{x^2+y^2}} xz\,dz\,dx\,dy$ into an integral in cylindrical coordinates.

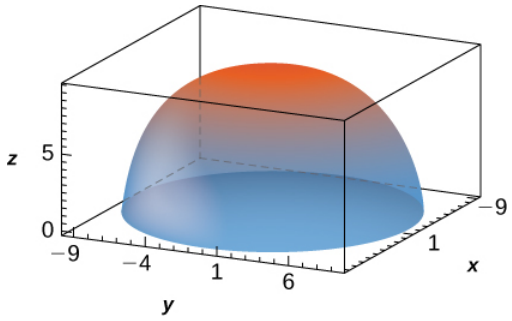
Answer

$$\int_0^1\int_0^{\pi}\int_{r^2}^{r} zr^2\cos\theta\,dz\,d\theta\,dr$$

28. Convert the integral $\int_0^2\int_0^y\int_0^1 (xy+z)\,dz\,dx\,dy$ into an integral in cylindrical coordinates.

In exercises 29 - 32, evaluate the triple integral $\iiint_B f(x,y,z)\,dV$ over the solid B.

29. $f(x,y,z)=1,\ B=\{(x,y,z)\,|\,x^2+y^2+z^2\le 90,\ z\ge 0\}$



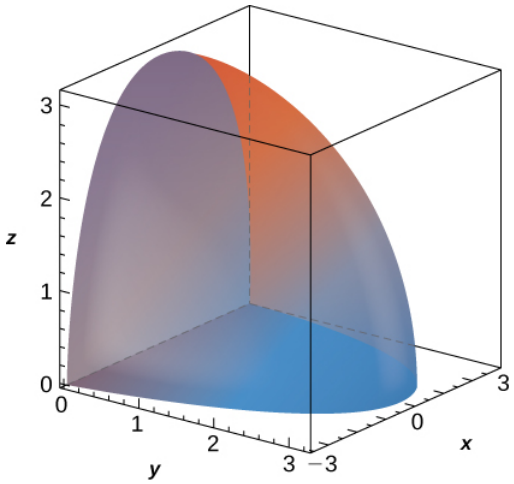
[Hide Solution]

Answer

$180\pi\sqrt{10}$

30. $f(x,y,z)=1-\sqrt{x^2+y^2+z^2},\ B=\{(x,y,z)\,|\,x^2+y^2+z^2\le 9,\ y\ge 0,\ z\ge 0\}$

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31. $f(x,y,z)=\sqrt{x^2+y^2},\ B$ is bounded above by the half-sphere $x^2+y^2+z^2=9$ with $z\geq 0$ and below by the cone $2z^2=x^2+y^2$.

Answer

$\frac{81\pi(\pi-2)}{16}$

32. $f(x,y,z)=\sqrt{x^2+y^2},\ B$ is bounded above by the half-sphere $x^2+y^2+z^2=16$ with $z\geq 0$ and below by the cone $2z^2=x^2+y^2$.

33. Show that if $F(\rho,\theta,\varphi)=f(\rho)g(\theta)h(\varphi)$ is a continuous function on the spherical box $B=\{(\rho,\theta,\varphi)\,|\,a\leq\rho\leq b,\ \alpha\leq\theta\leq\beta,\ \gamma\leq\varphi\leq\psi\}$, then $\iiint_B F\,dV=\left(\int_a^b \rho^2 f(\rho)\,dr\right)\left(\int_\alpha^\beta g(\theta)\,d\theta\right)\left(\int_\gamma^\psi h(\varphi)\,\sin\varphi\,d\varphi\right)$.

34. A function F is said to have spherical symmetry if it depends on the distance to the origin only, that is, it can be expressed in spherical coordinates as $F(x,y,z)=f(\rho)$, where $\rho=\sqrt{x^2+y^2+z^2}$. Show that $\iiint_B F(x,y,z)\,dV=2\pi\int_a^b \rho^2 f(\rho)\,d\rho$, where B is the region between the upper concentric hemispheres of radii a and b centered at the origin, with $0<a<b$ and F a spherical function defined on B.

Use the previous result to show that $\iiint_B (x^2+y^2+z^2)\sqrt{x^2+y^2+z^2}dV=21\pi$, where $B=\{(x,y,z)\,|\,1\leq x^2+y^2+z^2\leq 2,\ z\geq 0\}$.

35. Let B be the region between the upper concentric hemispheres of radii a and b centered at the origin and situated in the first octant, where $0<a<b$. Consider F a function defined on B whose form in spherical coordinates (ρ,θ,φ) is $F(x,y,z)=f(\rho)\cos\varphi$. Show that if $g(a)=g(b)=0$ and $\int_a^b h(\rho)\,d\rho=0$, then $\iiint_B F(x,y,z)\,dV=\frac{\pi^2}{4}[ah(a)-bh(b)]$, where g is an antiderivative of f and h is an antiderivative of g.

Use the previous result to show that $\iiint_B=\frac{z\cos\sqrt{x^2+y^2+z^2}}{\sqrt{x^2+y^2+z^2}}\,dV=\frac{3\pi^2}{2}$, where B is the region between the upper concentric hemispheres of radii π and 2π centered at the origin and situated in the first octant.

In exercises 36 - 39, the function f and region E are given.

a. Express the region E and function f in cylindrical coordinates.

b. Convert the integral $\iiint_B f(x,y,z)\,dV$ into cylindrical coordinates and evaluate it.

36. $f(x,y,z)=z;\ E=\{(x,y,z)\,|\,0\leq x^2+y^2+z^2\leq 1,\ z\geq 0\}$

37. $f(x,y,z)=x+y;\ E=\{(x,y,z)\,|\,1\leq x^2+y^2+z^2\leq 2,\ z\geq 0,\ y\geq 0\}$

Answer

a. $f(\rho,\theta,\varphi)=\rho\,\sin\varphi\,(\cos\theta+\sin\theta),\ E=\{(\rho,\theta,\varphi)\,|\,1\leq\rho\leq 2,\ 0\leq\theta\leq\pi,\ 0\leq\varphi\leq\frac{\pi}{2}\}$;

b. $\int_0^\pi\int_0^{\pi/2}\int_1^2 \rho^3\cos\varphi\,\sin\varphi\,d\rho\,d\varphi\,d\theta=\frac{15\pi}{8}$

38. $f(x,y,z)=2xy;\ E=\{(x,y,z)\,|\,\sqrt{x^2+y^2}\le z\le\sqrt{1-x^2-y^2},\ x\ge 0,\ y\ge 0\}$

39. $f(x,y,z)=z;\ E=\{(x,y,z)\,|\,x^2+y^2+z^2-2x\le 0,\ \sqrt{x^2+y^2}\le z\}$

Answer

a. $f(\rho,\theta,\varphi)=\rho\ \cos\varphi;\ E=\{(\rho,\theta,\varphi)\,|\,0\le\rho\le 2\ \cos\varphi,\ 0\le\theta\le\frac{\pi}{2},\ 0\le\varphi\le\frac{\pi}{4}\}$;

b. $\int_0^{\pi/2}\int_0^{\pi/4}\int_0^{2\ \cos\varphi}\rho^3\sin\varphi\ \cos\varphi\ d\rho\ d\varphi\ d\theta=\frac{7\pi}{24}$

In exercises 40 - 41, find the volume of the solid E whose boundaries are given in rectangular coordinates.

40. $E=\{(x,y,z)\,|\,\sqrt{x^2+y^2}\le z\le\sqrt{16-x^2-y^2},\ x\ge 0,\ y\ge 0\}$

41. $E=\{(x,y,z)\,|\,x^2+y^2+z^2-2z\le 0,\ \sqrt{x^2+y^2}\le z\}$

Answer

$\frac{\pi}{4}$

42. Use spherical coordinates to find the volume of the solid situated outside the sphere $\rho=1$ and inside the sphere $\rho=\cos\varphi$, with $\varphi\in[0,\frac{\pi}{2}]$.

43. Use spherical coordinates to find the volume of the ball $\rho\le 3$ that is situated between the cones $\varphi=\frac{\pi}{4}$ and $\varphi=\frac{\pi}{3}$.

Answer

$9\pi(\sqrt{2}-1)$

44. Convert the integral $\int_{-4}^{4}\int_{-\sqrt{16-y^2}}^{\sqrt{16-y^2}}\int_{-\sqrt{16-x^2-y^2}}^{\sqrt{16-x^2-y^2}}(x^2+y^2+z^2)\,dz\,dx\,dy$ into an integral in spherical coordinates.

45. Convert the integral $\int_0^4\int_0^{\sqrt{16-x^2}}\int_{-\sqrt{16-x^2-y^2}}^{\sqrt{16-x^2-y^2}}(x^2+y^2+z^2)^2\,dz\,dy\,dx$ into an integral in spherical coordinates.

Answer

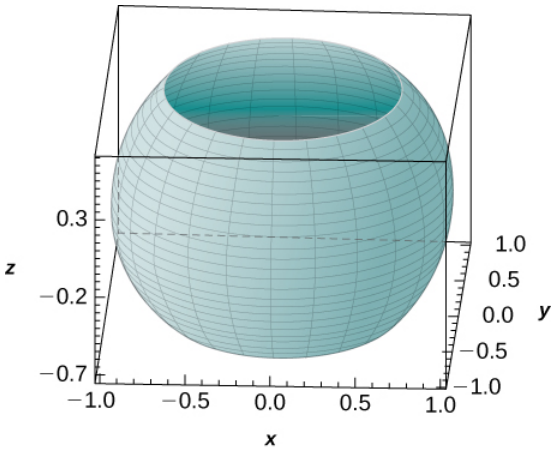
$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^4\rho^6\sin\varphi\,d\rho\,d\phi\,d\theta$

46. Convert the integral $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{16-x^2-y^2}}dz\,dy\,dx$ into an integral in spherical coordinates and evaluate it.

47. [T] Use a CAS to graph the solid whose volume is given by the iterated integral in spherical coordinates $\int_{\pi/2}^{\pi}\int_{5\pi}^{\pi/6}\int_0^2\rho^2\sin\varphi\ d\rho\ d\varphi\ d\theta$. Find the volume V of the solid. Round your answer to three decimal places.

Answer

$V=\frac{4\pi\sqrt{3}}{3}\approx 7.255$



48. [T] Use a CAS to graph the solid whose volume is given by the iterated integral in spherical coordinates as $\int_0^{2\pi}\int_{3\pi/4}^{\pi/4}\int_0^1 \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$. Find the volume V of the solid. Round your answer to three decimal places.

49. [T] Use a CAS to evaluate the integral $\iiint_E (x^2+y^2)\, dV$ where E lies above the paraboloid $z = x^2 + y^2$ and below the plane $z = 3y$.

Answer

$\frac{343\pi}{32}$

50. [T]

a. Evaluate the integral $\iiint_E e^{\sqrt{x^2+y^2+z^2}}\, dV$, where E is bounded by spheres $4x^2+4y^2+4z^2=1$ and $x^2+y^2+z^2=1$.

b. Use a CAS to find an approximation of the previous integral. Round your answer to two decimal places.

51. Express the volume of the solid inside the sphere $x^2+y^2+z^2=16$ and outside the cylinder $x^2+y^2=4$ as triple integrals in cylindrical coordinates and spherical coordinates, respectively.

Answer

$\int_0^{2\pi}\int_2^4\int_{-\sqrt{16-r^2}}^{\sqrt{16-r^2}} r\, dz\, dr\, d\theta$ and $\int_{\pi/6}^{5\pi/6}\int_0^{2\pi}\int_{2\csc\phi}^4 \rho^2 \sin\rho \, d\rho \, d\theta \, d\phi$

52. Express the volume of the solid inside the sphere $x^2+y^2+z^2=16$ and outside the cylinder $x^2+y^2=4$ that is located in the first octant as triple integrals in cylindrical coordinates and spherical coordinates, respectively.

53. The power emitted by an antenna has a power density per unit volume given in spherical coordinates by $p(\rho,\theta,\varphi) = \frac{P_0}{\rho^2}\cos^2\theta \, \sin^4\varphi$, where P_0 is a constant with units in watts. The total power within a sphere B of radius r meters is defined as $P = \iiint_B p(\rho,\theta,\varphi)\, dV$. Find the total power P.

Answer

$P = \frac{32P_0\pi}{3}$ watts

54. Use the preceding exercise to find the total power within a sphere B of radius 5 meters when the power density per unit volume is given by $p(\rho,\theta,\varphi) = \frac{30}{\rho^2}\cos^2\theta\sin^4\varphi$.

55. A charge cloud contained in a sphere B of radius r centimeters centered at the origin has its charge density given by $q(x,y,z) = k\sqrt{x^2+y^2+z^2}\,\frac{\mu C}{cm^3}$, where $k>0$. The total charge contained in B is given by $Q = \iiint_B q(x,y,z)\, dV$. Find the total charge Q.

Answer

$Q = kr^4\pi\mu C$

56. Use the preceding exercise to find the total charge cloud contained in the unit sphere if the charge density is $q(x, y, z) = 20\sqrt{x^2 + y^2 + z^2}\frac{\mu C}{cm^3}$.

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2.7: Calculating Centers of Mass and Moments of Inertia

Learning Objectives

- Use double integrals to locate the center of mass of a two-dimensional object.
- Use double integrals to find the moment of inertia of a two-dimensional object.
- Use triple integrals to locate the center of mass of a three-dimensional object.

We have already discussed a few applications of multiple integrals, such as finding areas, volumes, and the average value of a function over a bounded region. In this section we develop computational techniques for finding the center of mass and moments of inertia of several types of physical objects, using double integrals for a lamina (flat plate) and triple integrals for a three-dimensional object with variable density. The density is usually considered to be a constant number when the lamina or the object is homogeneous; that is, the object has uniform density.

Center of Mass in Two Dimensions

The center of mass is also known as the center of gravity if the object is in a uniform gravitational field. If the object has uniform density, the center of mass is the geometric center of the object, which is called the centroid. Figure **2.7.1** shows a point P as the center of mass of a lamina. The lamina is perfectly balanced about its center of mass.

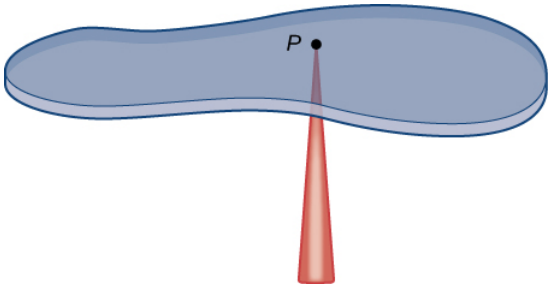


Figure **2.7.1**: A lamina is perfectly balanced on a spindle if the lamina's center of mass sits on the spindle.

To find the coordinates of the center of mass $P(\bar{x},\bar{y})$ of a lamina, we need to find the moment M_x of the lamina about the x-axis and the moment M_y about the y-axis. We also need to find the mass m of the lamina. Then

$$\bar{x} = \frac{M_y}{m} \tag{2.7.1}$$

and

$$\bar{y} = \frac{M_x}{m}. \tag{2.7.2}$$

Refer to Moments and Centers of Mass for the definitions and the methods of single integration to find the center of mass of a one-dimensional object (for example, a thin rod). We are going to use a similar idea here except that the object is a two-dimensional lamina and we use a double integral.

If we allow a constant density function, then $\bar{x} = \frac{M_y}{m}$ and $\bar{y} = \frac{M_x}{m}$ give the *centroid* of the lamina.

Suppose that the lamina occupies a region R in the xy-plane and let $\rho(x,y)$ be its density (in units of mass per unit area) at any point (x,y). Hence,

$$\rho(x,y) = \lim_{\Delta A \to 0} \frac{\Delta m}{\Delta A} \tag{2.7.3}$$

where Δm and ΔA are the mass and area of a small rectangle containing the point (x,y) and the limit is taken as the dimensions of the rectangle go to 0 (see the following figure).

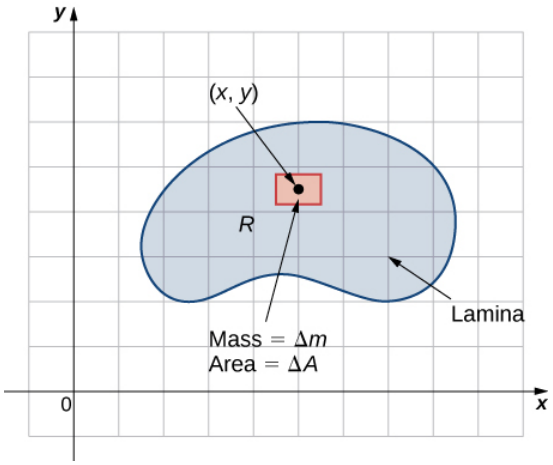


Figure **2.7.2**: The density of a lamina at a point is the limit of its mass per area in a small rectangle about the point as the area goes to zero.

Just as before, we divide the region R into tiny rectangles R_{ij} with area ΔA and choose (x_{ij}^*, y_{ij}^*) as sample points. Then the mass m_{ij} of each R_{ij} is equal to $\rho(x_{ij}^*, y_{ij}^*)\Delta A$ (Figure **2.7.2**). Let k and l be the number of subintervals in x and y respectively. Also, note that the shape might not always be rectangular but the limit works anyway, as seen in previous sections.

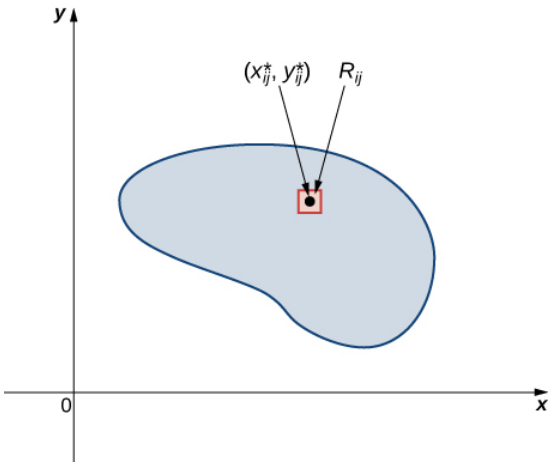


Figure **2.7.3**: Subdividing the lamina into tiny rectangles R_{ij} each containing a sample point (x_{ij}^*, y_{ij}^*).

Hence, the mass of the lamina is

$$m = \lim_{k,l\to\infty} \sum_{i=1}^{k} \sum_{j=1}^{l} m_{ij} = \lim_{k,l\to\infty} \sum_{i=1}^{k} \sum_{j=1}^{l} \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint_R \rho(x,y)dA. \tag{2.7.4}$$

Let's see an example now of finding the total mass of a triangular lamina.

Example 2.7.1: Finding the Total Mass of a Lamina

Consider a triangular lamina R with vertices $(0,0)$, $(0,3)$, $(3,0)$ and with density $\rho(x,y) = xy\, kg/m^2$. Find the total mass.

Solution

A sketch of the region R is always helpful, as shown in the following figure.

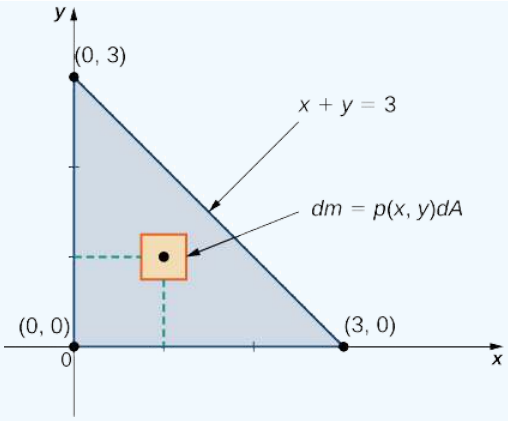


Figure 2.7.4: A lamina in the xy-plane with density $\rho(x,y) = xy$.

Using the expression developed for mass, we see that

$$m = \iint_R dm = \iint_R \rho(x,y)dA = \int_{x=0}^{x=3}\int_{y=0}^{y=3-x} xy\,dy\,dx = \int_{x=0}^{x=3}\left[x\frac{y^2}{2}\Big|_{y=0}^{y=3}\right]dx \tag{2.7.5}$$
$$= \int_{x=0}^{x=3}\frac{1}{2}x(3-x)^2dx = \left[\frac{9x^2}{4} - x^3 + \frac{x^4}{8}\right]\Big|_{x=0}^{x=3} = \frac{27}{8}.$$

The computation is straightforward, giving the answer $m = \frac{27}{8}\ kg$.

? Exercise 2.7.1

Consider the same region R as in the previous example, and use the density function $\rho(x,y) = \sqrt{xy}$. Find the total mass.

Answer

$\frac{9\pi}{8}\ kg$

Now that we have established the expression for mass, we have the tools we need for calculating moments and centers of mass. The moment M_z about the x-axis for R is the limit of the sums of moments of the regions R_{ij} about the x-axis. Hence

$$M_x = \lim_{k,l\to\infty}\sum_{i=1}^{k}\sum_{j=1}^{l}(y_{ij}^*)m_{ij} = \lim_{k,l\to\infty}\sum_{i=1}^{k}\sum_{j=1}^{l}(y_{ij}^*)\rho(x_{ij}^*, y_{ij}^*)\,\Delta A = \iint_R y\rho(x,y)\,dA \tag{2.7.6}$$

Similarly, the moment M_y about the y-axis for R is the limit of the sums of moments of the regions R_{ij} about the y-axis. Hence

$$M_y = \lim_{k,l\to\infty}\sum_{i=1}^{k}\sum_{j=1}^{l}(x_{ij}^*)m_{ij} = \lim_{k,l\to\infty}\sum_{i=1}^{k}\sum_{j=1}^{l}(x_{ij}^*)\rho(x_{ij}^*, y_{ij}^*)\,\Delta A = \iint_R x\rho(x,y)\,dA \tag{2.7.7}$$

✓ Example 2.7.2: Finding Moments

Consider the same triangular lamina R with vertices $(0,0)$, $(0,3)$, $(3,0)$ and with density $\rho(x,y) = xy$. Find the moments M_x and M_y.

Solution

Use double integrals for each moment and compute their values:

$$M_x = \iint_R y\rho(x,y)\,dA = \int_{x=0}^{x=3}\int_{y=0}^{y=3-x} xy^2\,dy\,dx = \frac{81}{20}, \tag{2.7.8}$$

$$M_y = \iint_R x\rho(x,y)\,dA = \int_{x=0}^{x=3}\int_{y=0}^{y=3-x} x^2y\,dy\,dx = \frac{81}{20}, \tag{2.7.9}$$

The computation is quite straightforward.

? Exercise 2.7.2

Consider the same lamina R as above and use the density function $\rho(x,y)=\sqrt{xy}$. Find the moments M_x and M_y.

Answer

$M_x=\frac{81\pi}{64}$ and $M_y=\frac{81\pi}{64}$

Finally we are ready to restate the expressions for the center of mass in terms of integrals. We denote the x-coordinate of the center of mass by $\bar{x}$ and the y-coordinate by $\bar{y}$. Specifically,

$$\bar{x}=\frac{M_y}{m}=\frac{\iint_R x\rho(x,y)\,dA}{\iint_R \rho(x,y)\,dA} \tag{2.7.10}$$

and

$$\bar{y}=\frac{M_x}{m}=\frac{\iint_R y\rho(x,y)\,dA}{\iint_R \rho(x,y)\,dA} \tag{2.7.11}$$

✓ Example 2.7.3: center of mass

Again consider the same triangular region R with vertices $(0,0)$, $(0,3)$, $(3,0)$ and with density function $\rho(x,y)=xy$. Find the center of mass.

Solution

Using the formulas we developed, we have

$$\bar{x}=\frac{M_y}{m}=\frac{\iint_R x\rho(x,y)\,dA}{\iint_R \rho(x,y)\,dA}=\frac{81/20}{27/8}=\frac{6}{5}, \tag{2.7.12}$$

$$\bar{y}=\frac{M_x}{m}=\frac{\iint_R y\rho(x,y)\,dA}{\iint_R \rho(x,y)\,dA}=\frac{81/20}{27/8}=\frac{6}{5}. \tag{2.7.13}$$

Therefore, the center of mass is the point $\left(\frac{6}{5},\frac{6}{5}\right)$.

Analysis

If we choose the density $\rho(x,y)$ instead to be uniform throughout the region (i.e., constant), such as the value 1 (any constant will do), then we can compute the centroid,

$$x_c=\frac{M_y}{m}=\frac{\iint_R x\,dA}{\iint_R dA}=\frac{9/2}{9/2}=1, \tag{2.7.14}$$

$$y_c=\frac{M_x}{m}=\frac{\iint_R y\,dA}{\iint_R dA}=\frac{9/2}{9/2}=1. \tag{2.7.15}$$

Notice that the center of mass $\left(\frac{6}{5},\frac{6}{5}\right)$ is not exactly the same as the centroid $(1,1)$ of the triangular region. This is due to the variable density of R If the density is constant, then we just use $\rho(x,y)=c$ (constant). This value cancels out from the formulas, so for a constant density, the center of mass coincides with the centroid of the lamina.

? Exercise 2.7.3

Again use the same region R as above and use the density function $\rho(x,y)=\sqrt{xy}$. Find the center of mass.

Answer

$$\bar{x} = \frac{M_y}{m} = \frac{81\pi/64}{9\pi/8} = \frac{9}{8} \text{ and } \bar{y} = \frac{M_x}{m} = \frac{81\pi}{9\pi/8} = \frac{0}{8}.$$

Once again, based on the comments at the end of Example **2.7.3**, we have expressions for the centroid of a region on the plane:

$$x_c = \frac{M_y}{m} = \frac{\iint_R x\,dA}{\iint_R dA} \text{ and } y_c = \frac{M_x}{m} = \frac{\iint_R y\,dA}{\iint_R dA}. \tag{2.7.16}$$

We should use these formulas and verify the centroid of the triangular region referred to in the last three examples.

Example 2.7.4: Finding Mass, Moments, and Center of Mass

Find the mass, moments, and the center of mass of the lamina of density $\rho(x, y) = x + y$ occupying the region R under the curve $y = x^2$ in the interval $0 \le x \le 2$ (see the following figure).

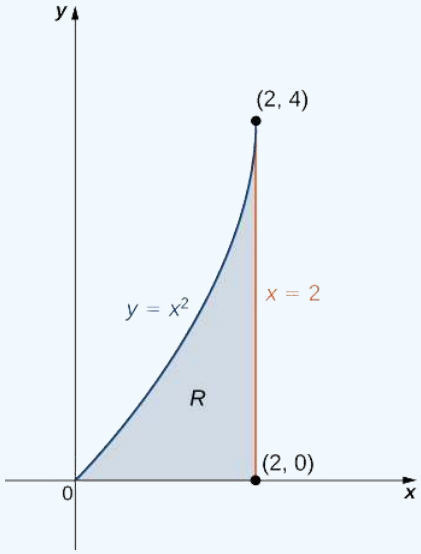


Figure **2.7.5**: Locating the center of mass of a lamina R with density $\rho(x, y) = x + y$.

Solution

First we compute the mass m. We need to describe the region between the graph of $y = x^2$ and the vertical lines $x = 0$ and $x = 2$:

$$m = \iint_R dm = \iint_R \rho(x, y)\,dA = \int_{x=0} x = 2 \int_{y=0}^{y=x^2} (x + y)dy\,dx \tag{2.7.17}$$

$$= \int_{x=0}^{x=2} \left[xy + \frac{y^2}{2} \Big|_{y=0}^{y=x^2} \right] dx$$

$$= \int_{x=0}^{x=2} \left[x^3 + \frac{x^4}{2} \right] dx = \left[\frac{x^4}{4} + \frac{x^5}{10} \right] \Big|_{x=0}^{x=2} = \frac{36}{5}. \tag{2.7.18}$$

Now compute the moments M_x and M_y:

$$M_x = \iint_R y\rho(x, y)\,dA = \int_{x=0}^{x=2} \int_{y=0}^{y=x^2} y(x + y)\,dy\,dx = \frac{80}{7}, \tag{2.7.19}$$

$$M_y = \iint_R x\rho(x, y)\,dA = \int_{x=0}^{x=2} \int_{y=0}^{y=x^2} x(x + y)\,dy\,dx = \frac{176}{15}. \tag{2.7.20}$$

Finally, evaluate the center of mass,

$$\bar{x} = \frac{M_y}{m} = \frac{\iint_R x\rho(x, y)\,dA}{\iint_R \rho(x, y)\,dA} = \frac{176/15}{36/5} = \frac{44}{27}, \tag{2.7.21}$$

$$\bar{y} = \frac{M_x}{m} = \frac{\iint_R y\rho(x,y)\,dA}{\iint_R \rho(x,y)\,dA} = \frac{80/7}{36/5} = \frac{100}{63}. \tag{2.7.22}$$

Hence the center of mass is $(\bar{x}, \bar{y}) = \left(\frac{44}{27}, \frac{100}{63}\right)$.

? Exercise 2.7.4

Calculate the mass, moments, and the center of mass of the region between the curves $y = x$ and $y = x^2$ with the density function $\rho(x,y) = x$ in the interval $0 \le x \le 1$.

Answer

$\bar{x} = \frac{M_y}{m} = \frac{1/20}{1/12} = \frac{3}{5}$ and $\bar{y} = \frac{M_x}{m} = \frac{1/24}{1/12} = \frac{1}{2}$

✓ Example 2.7.5: Finding a Centroid

Find the centroid of the region under the curve $y = e^x$ over the interval $1 \le x \le 3$ (Figure 2.7.6).

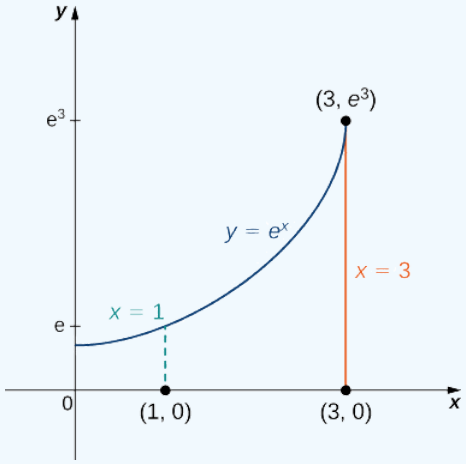


Figure 2.7.6: Finding a centroid of a region below the curve $y = e^x$.

Solution

To compute the centroid, we assume that the density function is constant and hence it cancels out:

$$x_c = \frac{M_y}{m} = \frac{\iint_R x\,dA}{\iint_R dA} \text{ and } y_c = \frac{M_x}{m} = \frac{\iint_R y\,dA}{\iint_R dA}, \tag{2.7.23}$$

$$\begin{aligned} x_c = \frac{M_y}{m} = \frac{\iint_R x\,dA}{\iint_R dA} &= \frac{\int_{x=1}^{x=3}\int_{y=0}^{y=e^x} x\,dy\,dx}{\int_{x=1}^{x=3}\int_{y=0}^{y=e^x} dy\,dx} \\ &= \frac{\int_{x=1}^{x=3} xe^x dx}{\int_{x=1}^{x=3} e^x dx} = \frac{2e^3}{e^3 - e} = \frac{2e^2}{e^2 - 1}, \end{aligned} \tag{2.7.24}$$

$$\begin{aligned} y_c = \frac{M_x}{m} = \frac{\iint_R y\,dA}{\iint_R dA} = \frac{\int_{x=1}^{x=3}\int_{y=0}^{y=e^x} y\,dy\,dx}{\int_{x=1}^{x=3}\int_{y=0}^{y=e^x} dy\,dx} &= \frac{\int_{x=1}^{x=3} \frac{e^{2x}}{2} dx}{\int_{x=1}^{x=3} e^x dx} \\ = \frac{\frac{1}{4}e^2(e^4 - 1)}{e(e^2 - 1)} &= \frac{1}{4}e(e^2 + 1). \end{aligned} \tag{2.7.25}$$

Thus the centroid of the region is

$$(x_c, y_c) = \left(\frac{2e^2}{e^2-1}, \frac{1}{4}e(e^2+1)\right). \tag{2.7.26}$$



Exercise 2.7.5

Calculate the centroid of the region between the curves $y = x$ and $y = \sqrt{x}$ with uniform density in the interval $0 \le x \le 1$.

Answer

$$x_c = \frac{M_y}{m} = \frac{1/15}{1/6} = \frac{2}{5} \text{ and } y_c = \frac{M_x}{m} = \frac{1/12}{1/6} = \frac{1}{2}$$

Moments of Inertia

For a clear understanding of how to calculate moments of inertia using double integrals, we need to go back to the general definition in Section 6.6. The moment of inertia of a particle of mass m about an axis is mr^2 where r is the distance of the particle from the axis. We can see from Figure *PageIndex3* that the moment of inertia of the subrectangle R_{ij} about the x-axis is $(y_{ij}^*)^2\rho(x_{ij}^*, y_{ij}^*)\Delta A$. Similarly, the moment of inertia of the subrectangle R_{ij} about the y-axis is $(x_{ij}^*)^2\rho(x_{ij}^*, y_{ij}^*)\Delta A$. The moment of inertia is related to the rotation of the mass; specifically, it measures the tendency of the mass to resist a change in rotational motion about an axis.

The moment of inertia I_x about the x-axis for the region R is the limit of the sum of moments of inertia of the regions R_{ij} about the x-axis. Hence

$$I_x = \lim_{k,l\to\infty} \sum_{i=1}^{k}\sum_{j=1}^{l} (y_{ij}^*)^2 m_{ij} = \lim_{k,l\to\infty} \sum_{i=1}^{k}\sum_{j=1}^{l} (y_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*)\,\Delta A = \iint_R y^2\rho(x,y)\,dA. \tag{2.7.27}$$

Similarly, the moment of inertia I_y about the y-axis for R is the limit of the sum of moments of inertia of the regions R_{ij} about the y -axis. Hence

$$I_y = \lim_{k,l\to\infty} \sum_{i=1}^{k}\sum_{j=1}^{l} (x_{ij}^*)^2 m_{ij} = \lim_{k,l\to\infty} \sum_{i=1}^{k}\sum_{j=1}^{l} (x_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*)\,\Delta A = \iint_R x^2\rho(x,y)\,dA. \tag{2.7.28}$$

Sometimes, we need to find the moment of inertia of an object about the origin, which is known as the polar moment of inertia. We denote this by I_0 and obtain it by adding the moments of inertia I_x and I_y. Hence

$$I_0 = I_x + I_y = \iint_R (x^2+y^2)\rho(x,y)\,dA. \tag{2.7.29}$$

All these expressions can be written in polar coordinates by substituting $x = r\cos\theta$, $y = r\sin\theta$, and $dA = r\,dr\,d\theta$. For example, $I_0 = \iint_R r^2\rho(r\cos\theta, r\sin\theta)\,dA$.

Example 2.7.6: Finding Moments of Inertia for a Triangular Lamina

Use the triangular region R with vertices $(0,0)$, $(2,2)$, and $(2,0)$ and with density $\rho(x,y) = xy$ as in previous examples. Find the moments of inertia.

Solution

Using the expressions established above for the moments of inertia, we have

$$I_x = \iint_R y^2\rho(x,y)\,dA = \int_{x=0}^{x=2}\int_{y=0}^{y=x} xy^3\,dy\,dx = \frac{8}{3}, \tag{2.7.30}$$

$$I_y = \iint_R x^2\rho(x,y)\,dA = \int_{x=0}^{x=2}\int_{y=0}^{y=x} x^3y\,dy\,dx = \frac{16}{3}, \tag{2.7.31}$$

$$I_0 = \iint_R (x^2+y^2)\rho(x,y)\,dA = \int_0^2\int_0^x (x^2+y^2)xy\,dy\,dx = I_x + I_y = 8 \tag{2.7.32}$$

Exercise 2.7.6

Again use the same region R as above and the density function $\rho(x,y) = \sqrt{xy}$. Find the moments of inertia.

Answer

$$I_x = \int_{x=0}^{x=2}\int_{y=0}^{y=x} y^2\sqrt{xy}\,dy\,dx = \frac{64}{35} \tag{2.7.33}$$

and

$$I_y = \int_{x=0}^{x=2}\int_{y=0}^{y=x} x^2\sqrt{xy}\,dy\,dx = \frac{64}{35}. \tag{2.7.34}$$

Also,

$$I_0 = \int_{x=0}^{x=2}\int_{y=0}^{y=x} (x^2+y^2)\sqrt{xy}\,dy\,dx = \frac{128}{21} \tag{2.7.35}$$



As mentioned earlier, the moment of inertia of a particle of mass m about an axis is mr^2 where r is the distance of the particle from the axis, also known as the **radius of gyration**.

Hence the radii of gyration with respect to the x-axis, the y-axis and the origin are

$$R_x = \sqrt{\frac{I_x}{m}},\ R_y = \sqrt{\frac{I_y}{m}},\ and\ R_0 = \sqrt{\frac{I_0}{m}}, \tag{2.7.36}$$

respectively. In each case, the radius of gyration tells us how far (perpendicular distance) from the axis of rotation the entire mass of an object might be concentrated. The moments of an object are useful for finding information on the balance and torque of the object about an axis, but radii of gyration are used to describe the distribution of mass around its centroidal axis. There are many applications in engineering and physics. Sometimes it is necessary to find the radius of gyration, as in the next example.

Example 2.7.7: Finding the Radius of Gyration for a Triangular Lamina

Consider the same triangular lamina R with vertices $(0,0)$, $(2,2)$, and $(2,0)$ and with density $\rho(x,y) = xy$ as in previous examples. Find the radii of gyration with respect to the x-axis the y-axis and the origin.

Solution

If we compute the mass of this region we find that $m = 2$. We found the moments of inertia of this lamina in Example 2.7.4. From these data, the radii of gyration with respect to the x-axis, y-axis and the origin are, respectively,

$$R_x = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{8/3}{2}} = \sqrt{\frac{8}{6}} = \frac{2\sqrt{3}}{3}, \tag{2.7.37}$$

$$R_y = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{16/3}{2}} = \sqrt{\frac{8}{3}} = \frac{2\sqrt{6}}{3}, \tag{2.7.38}$$

$$R_0 = \sqrt{\frac{I_0}{m}} = \sqrt{\frac{8}{2}} = \sqrt{4} = 2. \tag{2.7.39}$$

Exercise 2.7.7

Use the same region R from Example 2.7.7 and the density function $\rho(x,y) = \sqrt{xy}$. Find the radii of gyration with respect to the x-axis, the y-axis, and the origin.

Hint

Follow the steps shown in the previous example.

Answer

$R_x = \frac{6\sqrt{35}}{35}$, $R_y = \frac{6\sqrt{15}}{15}$, and $R_0 = \frac{4\sqrt{42}}{7}$.



Center of Mass and [redacted]

All the expressions of dou[redacted]

Definition

If we have a solid object Q with a density function $\rho(x,y,z)$ at any point (x,y,z) in space, then its mass is

$$m = \iiint_Q \rho(x,y,z)\,dV. \tag{2.7.40}$$

Its moments about the xy-plane the xz-plane and the yz-plane are

$$M_{xy} = \iiint_Q z\rho(x,y,z)\,dV,\ M_{xz} = \iiint_Q y\rho(x,y,z)\,dV,\ M_{yz} = \iiint_Q x\rho(x,y,z)\,dV. \tag{2.7.41}$$

If the center of mass of the object is the point $(\bar{x},\bar{y},\bar{z})$, then

$$\bar{x} = \frac{M_{yz}}{m},\ \bar{y} = \frac{M_{xz}}{m},\ \bar{z} = \frac{M_{xy}}{m}. \tag{2.7.42}$$

Also, if the solid object is homogeneous (with constant density), then the center of mass becomes the centroid of the solid. Finally, the moments of inertia about the yz-plane, xz-plane, and the xy-plane are

$$I_x = \iiint_Q (y^2+z^2)\,\rho(x,y,z)\,dV, \tag{2.7.43}$$

$$I_y = \iiint_Q (x^2+z^2)\,\rho(x,y,z)\,dV, \tag{2.7.44}$$

$$I_z = \iiint_Q (x^2+y^2)\,\rho(x,y,z)\,dV. \tag{2.7.45}$$

Example 2.7.8: Finding the Mass of a Solid

Suppose that Q is a solid region bounded by $x+2y+3z=6$ and the coordinate planes and has density $\rho(x,y,z)=x^2yz$. Find the total mass.

Solution

The region Q is a tetrahedron (Figure 2.7.7) meeting the axes at the points $(6,0,0)$, $(0,3,0)$, and $(0,0,2)$. To find the limits of integration, let $z=0$ in the slanted plane $z=\frac{1}{3}(6-x-2y)$. Then for x and y find the projection of Q onto the xy-plane, which is bounded by the axes and the line $x+2y=6$. Hence the mass is

$$m = \iiint_Q \rho(x,y,z)\,dV = \int_{x=0}^{x=6}\int_{y=0}^{y=1/2(6-x)}\int_{z=0}^{z=1/3(6-x-2y)} x^2yz\,dz\,dy\,dx = \frac{108}{35} \tag{2.7.46}$$

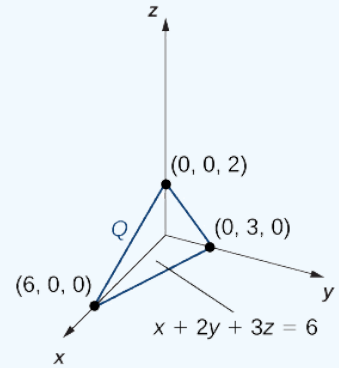


Figure 2.7.7: Finding the mass of a three-dimensional solid Q.

? Exercise 2.7.8

Consider the same region Q (Figure 2.7.7), and use the density function $\rho(x,y,z)=xy^2z$. Find the mass.

Hint

Follow the steps in the previous example.

Answer

$\frac{54}{35}=1.543$

✓ Example 2.7.9: Finding the Center of Mass of a Solid

Suppose Q is a solid region bounded by the plane $x+2y+3z=6$ and the coordinate planes with density $\rho(x,y,z)=x^2yz$ (see Figure 2.7.7). Find the center of mass using decimal approximation.

Solution

We have used this tetrahedron before and know the limits of integration, so we can proceed to the computations right away. First, we need to find the moments about the xy-plane, the xz-plane, and the yz-plane:

$$M_{xy}=\iiint_Q z\rho(x,y,z)\,dV \tag{2.7.47}$$
$$=\int_{x=0}^{x=6}\int_{y=0}^{y=\frac{1}{2}(6-x)}\int_{z=0}^{z=\frac{1}{3}(6-x-2y)} x^2yz^2\,dz\,dy\,dx=\frac{54}{35}\approx 1.543,$$

$$M_{xz}=\iiint_Q y\rho(x,y,z)\,dV \tag{2.7.48}$$
$$=\int_{x=0}^{x=6}\int_{y=0}^{y=\frac{1}{2}(6-x)}\int_{z=0}^{z=\frac{1}{3}(6-x-2y)} x^2y^2z\,dz\,dy\,dx=\frac{81}{35}\approx 2.314,$$

$$M_{yz}=\iiint_Q x\rho(x,y,z)\,dV \tag{2.7.49}$$
$$=\int_{x=0}^{x=6}\int_{y=0}^{y=\frac{1}{2}(6-x)}\int_{z=0}^{z=\frac{1}{3}(6-x-2y)} x^3yz\,dz\,dy\,dx=\frac{243}{35}\approx 6.943.$$

Hence the center of mass is

$$\bar{x}=\frac{M_{yz}}{m},\ \bar{y}=\frac{M_{xz}}{m},\ \bar{z}=\frac{M_{xy}}{m}, \tag{2.7.50}$$

$$\bar{x}=\frac{M_{yz}}{m}=\frac{243/35}{108/35}=\frac{243}{108}=2.25, \tag{2.7.51}$$

$$\bar{y}=\frac{M_{xz}}{m}=\frac{81/35}{108/35}=\frac{81}{108}=0.75, \tag{2.7.52}$$

$$\bar{z}=\frac{M_{xy}}{m}=\frac{54/35}{108/35}=\frac{54}{108}=0.5 \tag{2.7.53}$$

The center of mass for the tetrahedron Q is the point $(2.25,0.75,0.5)$.

? Exercise 2.7.9

Consider the same region Q (Figure 2.7.7) and use the density function $\rho(x,y,z)=xy^2z$. Find the center of mass.

Hint

Check that $M_{xy}=\frac{27}{35}$, $M_{xz}=\frac{243}{140}$, and $M_{yz}=\frac{81}{35}$. Then use m from a previous checkpoint question.

Answer

$\left(\frac{3}{2}, \frac{9}{8}, \frac{1}{2}\right)$

We conclude this section with an example of finding moments of inertia I_x, I_y, and I_z.

Example 2.7.10: Finding the Moments of Inertia of a Solid

Suppose that Q is a solid region and is bounded by $x + 2y + 3z = 6$ and the coordinate planes with density $\rho(x, y, z) = x^2yz$ (see Figure 2.7.7). Find the moments of inertia of the tetrahedron Q about the yz-plane, the xz-plane, and the xy-plane.

Solution

Once again, we can almost immediately write the limits of integration and hence we can quickly proceed to evaluating the moments of inertia. Using the formula stated before, the moments of inertia of the tetrahedron Q about the yz-plane, the xz-plane, and the xy-plane are

$$I_x = \iiint_Q (y^2 + z^2)\rho(x, y, z)\, dV, \tag{2.7.54}$$

$$I_y = \iiint_Q (x^2 + z^2)\rho(x, y, z)\, dV, \tag{2.7.55}$$

and

$$I_z = \iiint_Q (x^2 + y^2)\rho(x, y, z)\, dV \text{ with } \rho(x, y, z) = x^2yz. \tag{2.7.56}$$

Proceeding with the computations, we have

$$I_x = \iiint_Q (y^2 + z^2)x^2\rho(x, y, z)\, dV$$

$$= \int_{x=0}^{x=6} \int_{y=0}^{y=\frac{1}{2}(6-x)} \int_{z=0}^{z=\frac{1}{3}(6-x-2y)} (y^2 + z^2)x^2yz\, dz\, dy\, dx = \frac{117}{35} \approx 3.343,$$

$$I_y = \iiint_Q (x^2 + z^2)x^2\rho(x, y, z)\, dV$$

$$= \int_{x=0}^{x=6} \int_{y=0}^{y=\frac{1}{2}(6-x)} \int_{z=0}^{z=\frac{1}{3}(6-x-2y)} (x^2 + z^2)x^2yz\, dz\, dy\, dx = \frac{684}{35} \approx 19.543,$$

$$I_z = \iiint_Q (x^2 + y^2)x^2\rho(x, y, z)\, dV$$

$$= \int_{x=0}^{x=6} \int_{y=0}^{y=\frac{1}{2}(6-x)} \int_{z=0}^{z=\frac{1}{3}(6-x-2y)} (x^2 + y^2)x^2yz\, dz\, dy\, dx = \frac{729}{35} \approx 20.829.$$

Thus, the moments of inertia of the tetrahedron Q about the yz-plane, the xz-plane, and the xy-plane are $117/35$, $684/35$, and $729/35$, respectively.

Exercise 2.7.10

Consider the same region Q (Figure 2.7.7), and use the density function $\rho(x, y, z) = xy^2z$. Find the moments of inertia about the three coordinate planes.

Answer

The moments of inertia of the tetrahedron Q about the yz-plane, the xz-plane, and the xy-plane are $99/35$, $36/7$ and $243/35$, respectively.



Key Concepts

Finding the mass, center of mass, moments, and moments of inertia in double integrals:

- For a lamina R with a density function $\rho(x,y)$ at any point (x,y) in the plane, the mass is

$$m = \iint_R \rho(x,y)\,dA. \tag{2.7.57}$$

- The moments about the x-axis and y-axis are

$$M_x = \iint_R y\rho(x,y)\,dA \text{ and } M_y = \iint_R x\rho(x,y)\,dA. \tag{2.7.58}$$

- The center of mass is given by $\bar{x} = \dfrac{M_y}{m}$, $\bar{y} = \dfrac{M_x}{m}$.
- The center of mass becomes the centroid of the plane when the density is constant.
- The moments of inertia about the x-axis, y-axis, and the origin are

$$I_x = \iint_R y^2\rho(x,y)\,dA,\ I_y = \iint_R x^2\rho(x,y)\,dA, \tag{2.7.59}$$
$$\text{and } I_0 = I_x + I_y = \iint_R (x^2+y^2)\rho(x,y)\,dA.$$

Finding the mass, center of mass, moments, and moments of inertia in triple integrals:

- For a solid object Q with a density function $\rho(x,y,z)$ at any point (x,y,z) in space, the mass is

$$m = \iiint_Q \rho(x,y,z)\,dV. \tag{2.7.60}$$

- The moments about the xy-plane, the xz-plane, and the yz-plane are

$$M_{xy} = \iiint_Q z\rho(x,y,z)\,dV,\ M_{xz} = \iiint_Q y\rho(x,y,z)\,dV,\ M_{yz} = \iiint_Q x\rho(x,y,z)\,dV \tag{2.7.61}$$

- The center of mass is given by $\bar{x} = \dfrac{M_{yz}}{m}$, $\bar{y} = \dfrac{M_{xz}}{m}$, $\bar{z} = \dfrac{M_{xy}}{m}$.
- The center of mass becomes the centroid of the solid when the density is constant.
- The moments of inertia about the yz-plane, the xz-plane, and the xy-plane are

$$I_x = \iiint_Q (y^2 + z^2)\,\rho(x,y,z)\,dV,\ I_y = \iiint_Q (x^2 + z^2)\,\rho(x,y,z)\,dV, \qquad (2.7.62)$$
$$I_z = \iiint_Q (x^2 + y^2)\,\rho(x,y,z)\,dV.$$

Key Equations

- **Mass of a lamina**

$$m = \lim_{k,l\to\infty} \sum_{i=1}^{k}\sum_{j=1}^{l} m_{ij} = \lim_{k,l\to\infty} \sum_{i=1}^{k}\sum_{j=1}^{l} \rho(x_{ij}^*, y_{ij}^*)\,\Delta A = \iint_R \rho(x,y)\,dA \qquad (2.7.63)$$

- **Moment about the x-axis**

$$M_x = \lim_{k,l\to\infty} \sum_{i=1}^{k}\sum_{j=1}^{l} (y_{ij}^*) m_{ij} = \lim_{k,l\to\infty} \sum_{i=1}^{k}\sum_{j=1}^{l} (y_{ij}^*)\rho(x_{ij}^*, y_{ij}^*)\,\Delta A = \iint_R y\rho(x,y)\,dA \qquad (2.7.64)$$

- **Moment about the y-axis**

$$M_y = \lim_{k,l\to\infty} \sum_{i=1}^{k}\sum_{j=1}^{l} (x_{ij}^*) m_{ij} = \lim_{k,l\to\infty} \sum_{i=1}^{k}\sum_{j=1}^{l} (x_{ij}^*)\rho(x_{ij}^*, y_{ij}^*)\,\Delta A = \iint_R x\rho(x,y)\,dA \qquad (2.7.65)$$

- **Center of mass of a lamina**

$$\bar{x} = \frac{M_y}{m} = \frac{\iint_R x\rho(x,y)\,dA}{\iint_R \rho(x,y)\,dA} \text{ and } \bar{y} = \frac{M_x}{m} = \frac{\iint_R y\rho(x,y)\,dA}{\iint_R \rho(x,y)\,dA} \qquad (2.7.66)$$

Glossary

radius of gyration

the distance from an object's center of mass to its axis of rotation

Contributors and Attributions

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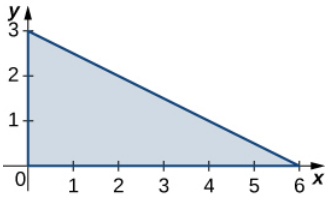
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2.7E: Exercises

In exercises 1 - 12, the region R occupied by a lamina is shown in a graph. Find the mass of R with the density function ρ.

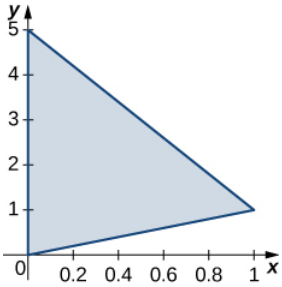
1. R is the triangular region with vertices $(0,0)$, $(0,3)$, and $(6,0)$; $\rho(x,y)=xy$.



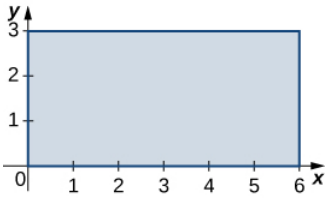
Answer

$\frac{27}{2}$

2. R is the triangular region with vertices $(0,0)$, $(1,1)$, and $(0,5)$; $\rho(x,y)=x+y$.



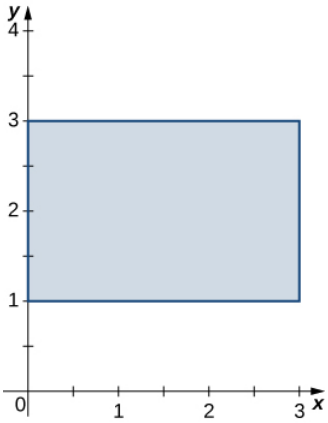
3. R is the rectangular region with vertices $(0,0)$, $(0,3)$, $(6,3)$ and $(6,0)$; $\rho(x,y)=\sqrt{xy}$.



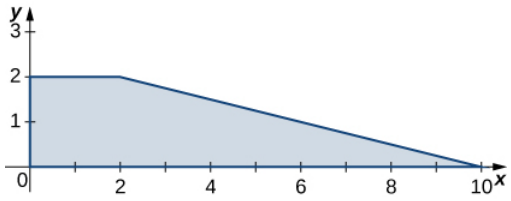
Answer

$24\sqrt{2}$

4. R is the rectangular region with vertices $(0,1)$, $(0,3)$, $(3,3)$ and $(3,1)$; $\rho(x,y)=x^2y$.



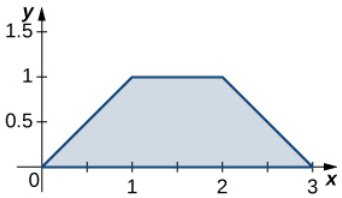
5. R is the trapezoidal region determined by the lines $y=-\frac{1}{4}x+\frac{5}{2}$, $y=0$, $y=2$, and $x=0$; $\rho(x,y)=3xy$.



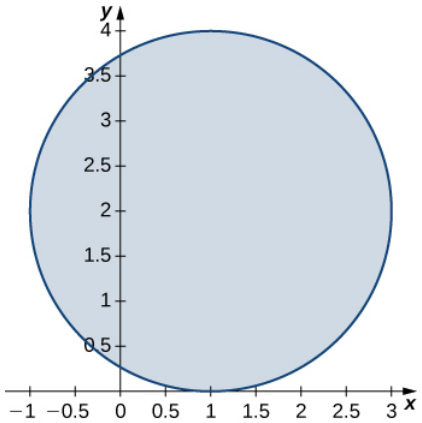
Answer

76

6. R is the trapezoidal region determined by the lines $y = 0,\ y = 1,\ y = x$ and $y = -x + 3;\ \rho(x, y) = 2x + y$.



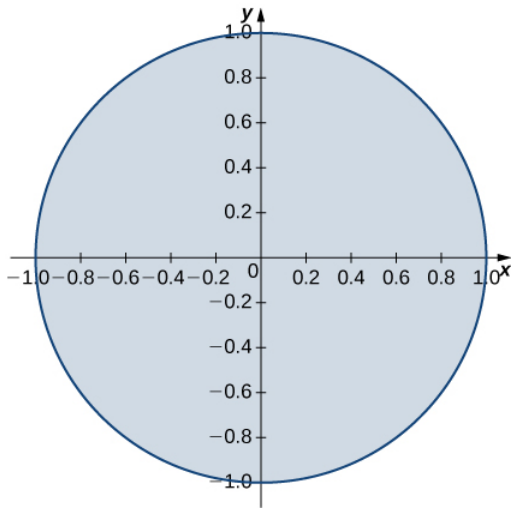
7. R is the disk of radius 2 centered at $(1, 2);\ \rho(x, y) = x^2 + y^2 - 2x - 4y + 5$.



Answer

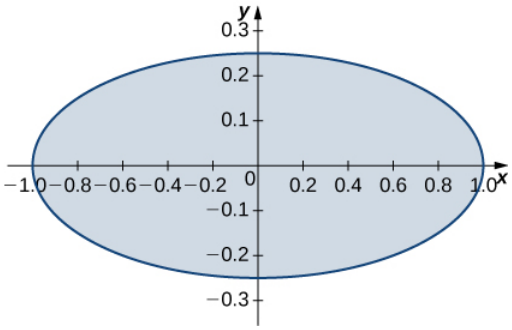
8π

8. R is the unit disk; $\rho(x, y) = 3x^4 + 6x^2y^2 + 3y^4$.



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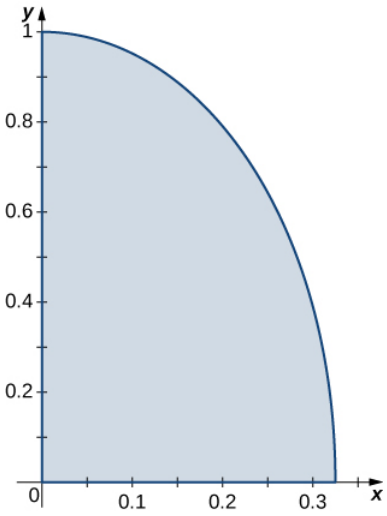
9. R is the region enclosed by the ellipse $x^2 + 4y^2 = 1;\ \rho(x, y) = 1$.



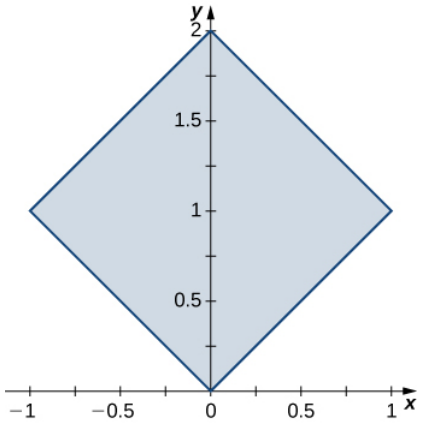
Answer

$\frac{\pi}{2}$

10. $R = \{(x, y) \mid 9x^2 + y^2 \leq 1,\ x \geq 0,\ y \geq 0\};\ \rho(x, y) = \sqrt{9x^2 + y^2}$.



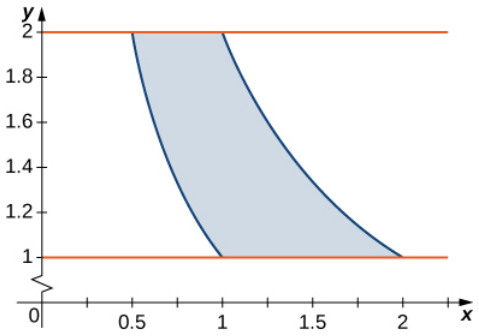
11. R is the region bounded by $y = x,\ y = -x,\ y = x + 2,\ y = -x + 2;\ \rho(x, y) = 1$.



Answer

2

12. R is the region bounded by $y = \frac{1}{x},\ y = \frac{2}{x},\ y = 1$, and $y = 2;\ \rho(x, y) = 4(x + y)$.



In exercises 13 - 24, consider a lamina occupying the region R and having the density function ρ given in the preceding group of exercises. Use a computer algebra system (CAS) to answer the following questions.

a. Find the moments M_x and M_y about the x-axis and y-axis, respectively.

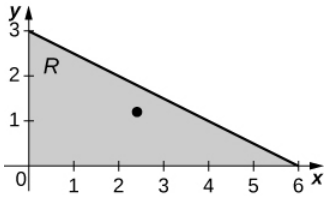
b. Calculate and plot the center of mass of the lamina.

c. [T] Use a CAS to locate the center of mass on the graph of R.

13. [T] R is the triangular region with vertices $(0,0)$, $(0,3)$, and $(6,0)$; $\rho(x,y)=xy$.

Answer

a. $M_x=\frac{81}{5}$, $M_y=\frac{162}{5}$;
b. $\bar{x}=\frac{12}{5}$, $\bar{y}=\frac{6}{5}$;
c.

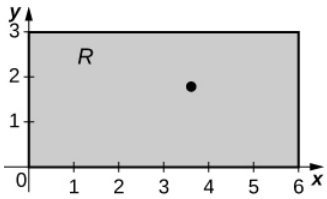


14. [T] R is the triangular region with vertices $(0,0)$, $(1,1)$, and $(0,5)$; $\rho(x,y)=x+y$.

15. [T] R is the rectangular region with vertices $(0,0)$, $(0,3)$, $(6,3)$, and $(6,0)$; $\rho(x,y)=\sqrt{xy}$.

Answer

a. $M_x=\frac{216\sqrt{2}}{5}$, $M_y=\frac{432\sqrt{2}}{5}$;
b. $\bar{x}=\frac{18}{5}$, $\bar{y}=\frac{9}{5}$;
c.



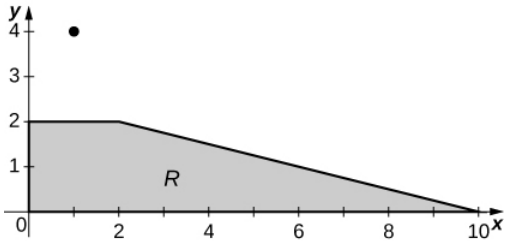
16. [T] R is the rectangular region with vertices $(0,1)$, $(0,3)$, $(3,3)$, and $(3,1)$; $\rho(x,y)=x^2y$.

17. [T] R is the trapezoidal region determined by the lines $y=-\frac{1}{4}x+\frac{5}{2}$, $y=0$, $y=2$, and $x=0$; $\rho(x,y)=3xy$.

Answer

a. $M_x=\frac{368}{5}$, $M_y=\frac{1552}{5}$;
b. $\bar{x}=\frac{92}{95}$, $\bar{y}=\frac{388}{95}$;

c.



18. [T] R is the trapezoidal region determined by the lines $y = 0,\ y = 1,\ y = x$, and $y = -x + 3;\ \rho(x, y) = 2x + y$.

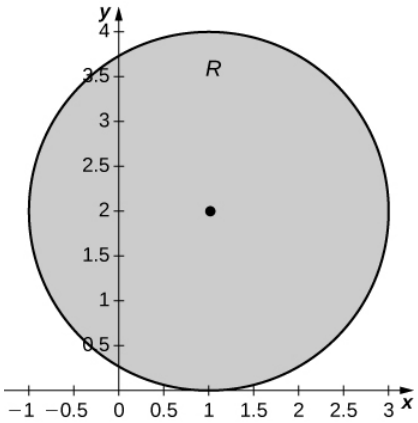
19. [T] R is the disk of radius 2 centered at $(1, 2);\ \rho(x, y) = x^2 + y^2 - 2x - 4y + 5$.

Answer

a. $M_x = 16\pi,\ M_y = 8\pi$;

b. $\bar{x} = 1,\ \bar{y} = 2$;

c.



20. [T] R is the unit disk; $\rho(x, y) = 3x^4 + 6x^2y^2 + 3y^4$.

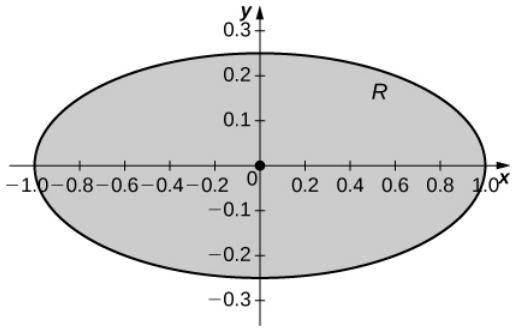
21. [T] R is the region enclosed by the ellipse $x^2 + 4y^2 = 1;\ \rho(x, y) = 1$.

Answer

a. $M_x = 0,\ M_y = 0)$;

b. $\bar{x} = 0,\ \bar{y} = 0$;

c.

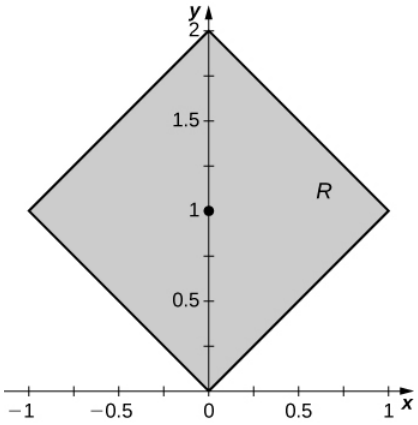


22. [T] $R = \{(x, y) \mid 9x^2 + y^2 \le 1,\ x \ge 0,\ y \ge 0\};\ \rho(x, y) = \sqrt{9x^2 + y^2}$.

23. [T] R is the region bounded by $y = x,\ y = -x,\ y = x + 2$, and $y = -x + 2;\ \rho(x, y) = 1$.

Answer

a. $M_x = 2,\ M_y = 0$);
b. $\bar{x} = 0,\ \bar{y} = 1$;
c.



24. [T] R is the region bounded by $y = \frac{1}{x},\ y = \frac{2}{x},\ y = 1$, and $y = 2;\ \rho(x, y) = 4(x + y)$.

In exercises 25 - 36, consider a lamina occupying the region R and having the density function ρ given in the first two groups of Exercises.

a. Find the moments of inertia I_x, I_y and I_0 about the x-axis, y-axis, and origin, respectively.

b. Find the radii of gyration with respect to the x-axis, y-axis, and origin, respectively.

25. R is the triangular region with vertices $(0, 0),\ (0, 3)$, and $(6, 0);\ \rho(x, y) = xy$.

Answer

a. $I_x = \frac{243}{10},\ I_y = \frac{486}{5}$, and $I_0 = \frac{243}{2}$;
b. $R_x = \frac{3\sqrt{5}}{5},\ R_y = \frac{6\sqrt{5}}{5}$, and $R_0 = 3$

26. R is the triangular region with vertices $(0, 0),\ (1, 1)$, and $(0, 5);\ \rho(x, y) = x + y$.

27. R is the rectangular region with vertices $(0, 0),\ (0, 3),\ (6, 3)$, and $(6, 0);\ \rho(x, y) = \sqrt{xy}$.

Answer

a. $I_x = \frac{2592\sqrt{2}}{7},\ I_y = \frac{648\sqrt{2}}{7}$, and $I_0 = \frac{3240\sqrt{2}}{7}$;
b. $R_x = \frac{6\sqrt{21}}{7},\ R_y = \frac{3\sqrt{21}}{7}$, and $R_0 = \frac{3\sqrt{106}}{7}$

28. R is the rectangular region with vertices $(0, 1),\ (0, 3),\ (3, 3)$, and $(3, 1);\ \rho(x, y) = x^2 y$.

29. R is the trapezoidal region determined by the lines $y = -\frac{1}{4}x + \frac{5}{2},\ y = 0,\ y = 2$, and x = 0; \space \rho (x,y) = 3xy\).

Answer

a. $I_x = 88,\ I_y = 1560$, and $I_0 = 1648$;
b. $R_x = \frac{\sqrt{418}}{19},\ R_y = \frac{\sqrt{7410}}{10}$, and $R_0 = \frac{2\sqrt{1957}}{19}$

30. R is the trapezoidal region determined by the lines $y = 0,\ y = 1,\ y = x$, and y = -x + 3; \space \rho (x,y) = 2x + y\).

31. R is the disk of radius 2 centered at $(1, 2);\ \rho(x, y) = x^2 + y^2 - 2x - 4y + 5$.

Answer

a. $I_x = \frac{128\pi}{3},\ I_y = \frac{56\pi}{3}$, and $I_0 = \frac{184\pi}{3}$;
b. $R_x = \frac{4\sqrt{3}}{3},\ R_y = \frac{\sqrt{21}}{2}$, and $R_0 = \frac{\sqrt{69}}{3}$

32. R is the unit disk; $\rho(x,y) = 3x^4 + 6x^2y^2 + 3y^4$.

33. R is the region enclosed by the ellipse $x^2 + 4y^2 = 1;\ \rho(x,y) = 1$.

Answer

a. $I_x = \frac{\pi}{32},\ I_y = \frac{\pi}{8}$, and $I_0 = \frac{5\pi}{32}$;
b. $R_x = \frac{1}{4},\ R_y = \frac{1}{2}$, and $R_0 = \frac{\sqrt{5}}{4}$

34. $R = \{(x,y) \mid 9x^2 + y^2 \le 1,\ x \ge 0,\ y \ge 0\};\ \rho(x,y) = \sqrt{9x^2 + y^2}$.

35. R is the region bounded by $y = x,\ y = -x,\ y = x + 2$, and $y = -x + 2;\ \rho(x,y) = 1$.

Answer

a. $I_x = \frac{7}{3},\ I_y = \frac{1}{3}$, and $I_0 = \frac{8}{3}$;
b. $R_x = \frac{\sqrt{42}}{6},\ R_y = \frac{\sqrt{6}}{6}$, and $R_0 = \frac{2\sqrt{3}}{3}$

36. R is the region bounded by $y = \frac{1}{x},\ y = \frac{2}{x},\ y = 1$, and $y = 2;\ \rho(x,y) = 4(x + y)$.

37. Let Q be the solid unit cube. Find the mass of the solid if its density ρ is equal to the square of the distance of an arbitrary point of Q to the xy-plane.

Answer

$m = \frac{1}{3}$

38. Let Q be the solid unit hemisphere. Find the mass of the solid if its density ρ is proportional to the distance of an arbitrary point of Q to the origin.

39. The solid Q of constant density 1 is situated inside the sphere $x^2 + y^2 + z^2 = 16$ and outside the sphere $x^2 + y^2 + z^2 = 1$. Show that the center of mass of the solid is not located within the solid.

40. Find the mass of the solid $Q = \{(x,y,z) \mid 1 \le x^2 + z^2 \le 25,\ y \le 1 - x^2 - z^2\}$ whose density is $\rho(x,y,z) = k$, where $k > 0$.

41. [T] The solid $Q = \{(x,y,z) \mid x^2 + y^2 \le 9,\ 0 \le z \le 1,\ x \ge 0,\ y \ge 0\}$ has density equal to the distance to the xy-plane. Use a CAS to answer the following questions.

a. Find the mass of Q.

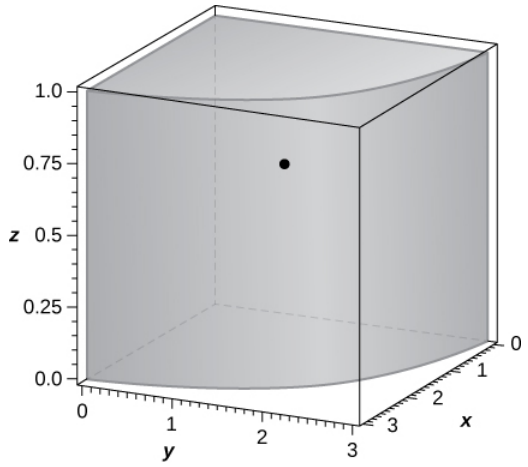
b. Find the moments $M_{xy},\ M_{xz}$ and M_{yz} about the xy-plane, xz-plane, and yz-plane, respectively.

c. Find the center of mass of Q.

d. Graph Q and locate its center of mass.

Answer

a. $m = \frac{9\pi}{4}$;
b. $M_{xy} = \frac{3\pi}{2},\ M_{xz} = \frac{81}{8},\ M_{yz} = \frac{81}{8}$;
c. $\bar{x} = \frac{9}{2\pi},\ \bar{y} = \frac{9}{2\pi},\ \bar{z} = \frac{2}{3}$;
d.



42. Consider the solid $Q = \{(x, y, z) \,|\, 0 \le x \le 1,\ 0 \le y \le 2,\ 0 \le z \le 3\}$ with the density function $\rho(x, y, z) = x + y + 1$.

a. Find the mass of Q.

b. Find the moments $M_{xy},\ M_{xz}$ and M_{yz} about the xy-plane, xz-plane, and yz-plane, respectively.

c. Find the center of mass of Q.

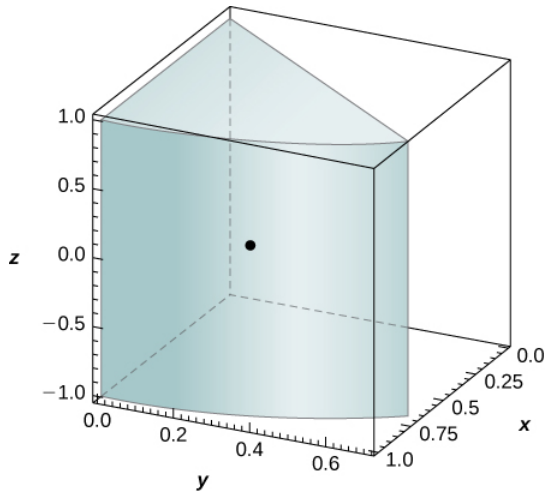
43. [T] The solid Q has the mass given by the triple integral $\int_{-1}^{1} \int_{0}^{\pi/4} \int_{0}^{1} r^2 \, dr \, d\theta \, dz$.

Use a CAS to answer the following questions.

- Show that the center of mass of Q is located in the xy-plane.
- Graph Q and locate its center of mass.

Answer

$\bar{x} = \frac{3\sqrt{2}}{2\pi}$, $\bar{y} = \frac{3(2-\sqrt{2})}{2\pi}$, $\bar{z} = 0$; 2. the solid Q and its center of mass are shown in the following figure.



44. The solid Q is bounded by the planes $x + 4y + z = 8,\ x = 0,\ y = 0$, and $z = 0$. Its density at any point is equal to the distance to the xz-plane. Find the moments of inertia of the solid about the xz-plane.

45. The solid Q is bounded by the planes $x + y + z = 3,\ x = 0,\ y = 0$, and $z = 0$. Its density is $\rho(x, y, z) = x + ay$, where $a > 0$. Show that the center of mass of the solid is located in the plane $z = \frac{3}{5}$ for any value of a.

46. Let Q be the solid situated outside the sphere $x^2 + y^2 + z^2 = z$ and inside the upper hemisphere $x^2 + y^2 + z^2 = R^2$, where $R > 1$. If the density of the solid is $\rho(x, y, z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$, find R such that the mass of the solid is $\frac{7\pi}{2}$.

47. The mass of a solid Q is given by $\int_0^2 \int_0^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{16-x^2-y^2}} (x^2+y^2+z^2)^n \, dz \, dy \, dx$, where n is an integer. Determine n such the mass of the solid is $(2-\sqrt{2})\pi$.

Answer

$n=-2$

48. Let Q be the solid bounded above the cone $x^2+y^2=z^2$ and below the sphere $x^2+y^2+z^2-4z=0$. Its density is a constant $k>0$. Find k such that the center of mass of the solid is situated 7 units from the origin.

49. The solid $Q=\{(x,y,z) \mid 0 \le x^2+y^2 \le 16,\ x \ge 0,\ y \ge 0,\ 0 \le z \le x\}$ has the density $\rho(x,y,z)=k$. Show that the moment M_{xy} about the xy-plane is half of the moment M_{yz} about the yz-plane.

50. The solid Q is bounded by the cylinder $x^2+y^2=a^2$, the paraboloid $b^2-z=x^2+y^2$, and the xy-plane, where $0<a<b$. Find the mass of the solid if its density is given by $\rho(x,y,z)=\sqrt{x^2+y^2}$.

51. Let Q be a solid of constant density k, where $k>0$, that is located in the first octant, inside the circular cone $x^2+y^2=9(z-1)^2$, and above the plane $z=0$. Show that the moment M_{xy} about the xy-plane is the same as the moment M_{yz} about the xz-plane.

52. The solid Q has the mass given by the triple integral $\int_0^1 \int_0^{\pi/2} \int_0^{r^3} (r^4+r) \, dz \, d\theta \, dr$.

a. Find the density of the solid in rectangular coordinates.

b. Find the moment M_{xy} about the xy-plane.

53. The solid Q has the moment of inertia I_x about the yz-plane given by the triple integral $\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_{\frac{1}{2}(x^2+y^2)}^{\sqrt{x^2+y^2}} (y^2+z^2)(x^2+y^2) \, dz \, dx \, dy$.

a. Find the density of Q.

b. Find the moment of inertia I_z about the xy-plane.

Answer

a. $\rho(x,y,z)=x^2+y^2$;
b. $\frac{16\pi}{7}$

54. The solid Q has the mass given by the triple integral $\int_0^{\pi/4} \int_0^{2 \sec\theta} \int_0^1 (r^3 \cos\theta \sin\theta + 2r) \, dz \, dr \, d\theta$.

a. Find the density of the solid in rectangular coordinates.

b. Find the moment M_{xz} about the xz-plane.

55. Let Q be the solid bounded by the xy-plane, the cylinder $x^2+y^2=a^2$, and the plane $z=1$, where $a>1$ is a real number. Find the moment M_{xy} of the solid about the xy-plane if its density given in cylindrical coordinates is $\rho(x,y,z)=\frac{d^2f}{dr^2}(r)$, where f is a differentiable function with the first and second derivatives continuous and differentiable on $(0,a)$.

Answer

$M_{xy}=\pi(f(0)-f(a)+af'(a))$

56. A solid Q has a volume given by $\iint_D \int_a^b dA \, dz$, where D is the projection of the solid onto the xy-plane and $a<b$ are real numbers, and its density does not depend on the variable z. Show that its center of mass lies in the plane $z=\frac{a+b}{2}$.

57. Consider the solid enclosed by the cylinder $x^2+z^2=a^2$ and the planes $y=b$ and $y=c$, where $a>0$ and $b<c$ are real numbers. The density of Q is given by $\rho(x,y,z)=f'(y)$, where f is a differential function whose derivative is continuous on (b,c). Show that if $f(b)=f(c)$, then the moment of inertia about the xz-plane of Q is null.

58. [T] The average density of a solid Q is defined as $\rho_{ave} = \frac{1}{V(Q)} \iiint_Q \rho(x,y,z)\, dV = \frac{m}{V(Q)}$, where $V(Q)$ and m are the volume and the mass of Q, respectively. If the density of the unit ball centered at the origin is $\rho(x,y,z) = e^{-x^2-y^2-z^2}$, use a CAS to find its average density. Round your answer to three decimal places.

59. Show that the moments of inertia I_x, I_y, and I_z about the yz-plane, xz-plane, and xy-plane, respectively, of the unit ball centered at the origin whose density is $\rho(x,y,z) = e^{-x^2-y^2-z^2}$ are the same. Round your answer to two decimal places.

Answer

$I_x = I_y = I_z \approx 0.84$

Contributors

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2.8: Change of Variables in Multiple Integrals

Learning Objectives

- Determine the image of a region under a given transformation of variables.
- Compute the Jacobian of a given transformation.
- Evaluate a double integral using a change of variables.
- Evaluate a triple integral using a change of variables.

Recall from Substitution Rule the method of integration by substitution. When evaluating an integral such as

$$\int_2^3 x(x^2-4)^5 dx, \tag{2.8.1}$$

we substitute $u = g(x) = x^2 - 4$. Then $du = 2x\,dx$ or $x\,dx = \frac{1}{2}du$ and the limits change to $u = g(2) = 2^2 - 4 = 0$ and $u = g(3) = 9 - 4 = 5$. Thus the integral becomes

$$\int_0^5 \frac{1}{2}u^5 du \tag{2.8.2}$$

and this integral is much simpler to evaluate. In other words, when solving integration problems, we make appropriate substitutions to obtain an integral that becomes much simpler than the original integral.

We also used this idea when we transformed double integrals in rectangular coordinates to polar coordinates and transformed triple integrals in rectangular coordinates to cylindrical or spherical coordinates to make the computations simpler. More generally,

$$\int_a^b f(x)dx = \int_c^d f(g(u))g'(u)du, \tag{2.8.3}$$

Where $x = g(u)$, $dx = g'(u)du$, and $u = c$ and $u = d$ satisfy $c = g(a)$ and $d = g(b)$.

A similar result occurs in double integrals when we substitute

- $x = f(r,\theta) = r\,\cos\,\theta$
- $y = g(r,\theta) = r\,\sin\,\theta$, and
- $dA = dx\,dy = r\,dr\,d\theta$.

Then we get

$$\iint_R f(x,y)dA = \iint_S (r\,\cos\,\theta,\, r\,\sin\,\theta)r\,dr\,d\theta \tag{2.8.4}$$

where the domain R is replaced by the domain S in polar coordinates. Generally, the function that we use to change the variables to make the integration simpler is called a transformation or mapping.

Planar Transformations

A planar transformation T is a function that transforms a region G in one plane into a region R in another plane by a change of variables. Both G and R are subsets of R^2. For example, Figure 2.8.1 shows a region G in the uv-plane transformed into a region R in the xy-plane by the change of variables $x = g(u,v)$ and $y = h(u,v)$, or sometimes we write $x = x(u,v)$ and $y = y(u,v)$. We shall typically assume that each of these functions has continuous first partial derivatives, which means g_u, g_v, h_u, and h_v exist and are also continuous. The need for this requirement will become clear soon.

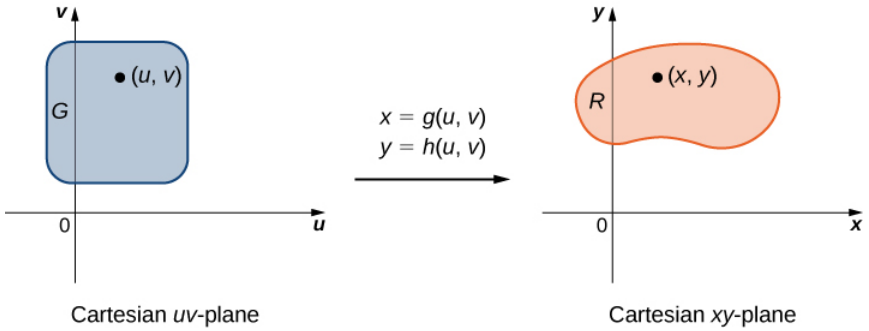


Figure 2.8.1: The transformation of a region G in the uv-plane into a region R in the xy-plane.

Definition: one-to-one transformation

A transformation $T: G \to R$, defined as $T(u,v) = (x,y)$, is said to be a one-to-one transformation if no two points map to the same image point.

To show that T s a one-to-one transformation, we assume $T(u_1,v_1) = T(u_2,v_2)$ and show that as a consequence we obtain$(u_1,v_1) = (u_2,v_2)$. If the transformation T is one-to-one in the domain G, then the inverse T^{-1} exists with the domain R such that $T^{-1} \circ T$ and $T \circ T^{-1}$ are identity functions.

Figure 2.8.2 shows the mapping $T(u,v) = (x,y)$ where x and y are related to u and v by the equations $x = g(u,v)$ and $y = h(u,v)$. The region G is the domain of T and the region R is the range of T, also known as the *image* of G under the transformation T.

Example 2.8.1A: Determining How the Transformation Works

Suppose a transformation T is defined as $T(r,\theta) = (x,y)$ where $x = r\cos\theta$, $y = r\sin\theta$. Find the image of the polar rectangle $G = \{(r,\theta) | 0 \le r \le 1,\ 0 \le \theta \le \pi/2\}$ in the $r\theta$-plane to a region R in the xy-plane. Show that T is a one-to-one transformation in G and find $T^{-1}(x,y)$.

Solution

Since r varies from 0 to 1 in the $r\theta$-plane, we have a circular disc of radius 0 to 1 in the xy-plane. Because θ varies from 0 to $\pi/2$ in the $r\theta$-plane, we end up getting a quarter circle of radius 1 in the first quadrant of the xy-plane (Figure 2.8.2). Hence R is a quarter circle bounded by $x^2 + y^2 = 1$ in the first quadrant.

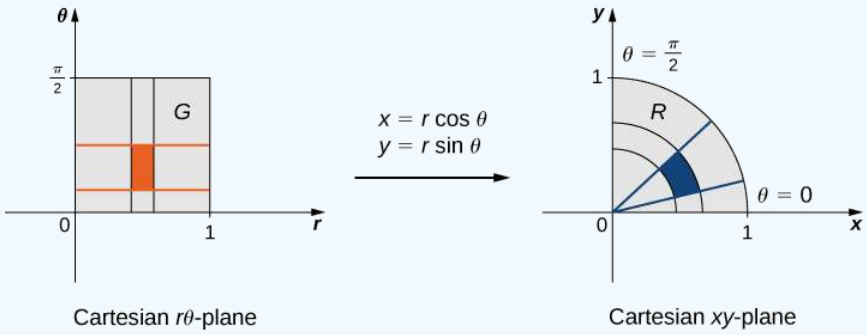


Figure 2.8.2: A rectangle in the $r\theta$-plane is mapped into a quarter circle in the xy-plane.

In order to show that T is a one-to-one transformation, assume $T(r_1,\theta_1) = T(r_2,\theta_2)$ and show as a consequence that $(r_1,\theta_1) = (r_2,\theta_2)$. In this case, we have

$$T(r_1,\theta_1) = T(r_2,\theta_2), \tag{2.8.5}$$

$$(x_1,y_1) = (x_1,y_1), \tag{2.8.6}$$

$$(r_1\cos\theta_1, r_1\sin\theta_1) = (r_2\cos\theta_2, r_2\sin\theta_2), \tag{2.8.7}$$

$$r_1\cos\theta_1 = r_2\cos\theta_2,\ r_1\sin\theta_1 = r_2\sin\theta_2. \tag{2.8.8}$$

Dividing, we obtain

$$\frac{r_1 \cos\theta_1}{r_1 \sin\theta_1} = \frac{r_2 \cos\theta_2}{r_2 \sin\theta_2} \tag{2.8.9}$$

$$\frac{\cos\theta_1}{\sin\theta_1} = \frac{\cos\theta_2}{\sin\theta_2} \tag{2.8.10}$$

$$\tan\theta_1 = \tan\theta_2 \tag{2.8.11}$$

$$\theta_1 = \theta_2 \tag{2.8.12}$$

since the tangent function is one-one function in the interval $0 \le \theta \le \pi/2$. Also, since $0 \le r \le 1$, we have $r_1 = r_2$, $\theta_1 = \theta_2$. Therefore, $(r_1, \theta_1) = (r_2, \theta_2)$ and T is a one-to-one transformation from G to R.

To find $T^{-1}(x, y)$ solve for r, θ in terms of x, y. We already know that $r^2 = x^2 + y^2$ and $\tan\theta = \frac{y}{x}$. Thus $T^{-1}(x, y) = (r, \theta)$ is defined as $r = \sqrt{x^2 + y^2}$ and $\tan^{-1}(\frac{y}{x})$.

Example 2.8.1B: Finding the Image under T

Let the transformation T be defined by $T(u, v) = (x, y)$ where $x = u^2 - v^2$ and $y = uv$. Find the image of the triangle in the uv-plane with vertices $(0, 0)$, $(0, 1)$, and $(1, 1)$.

Solution

The triangle and its image are shown in Figure 2.8.3. To understand how the sides of the triangle transform, call the side that joins $(0, 0)$ and $(0, 1)$ side A, the side that joins $(0, 0)$ and $(1, 1)$ side B, and the side that joins $(1, 1)$ and $(0, 1)$ side C.

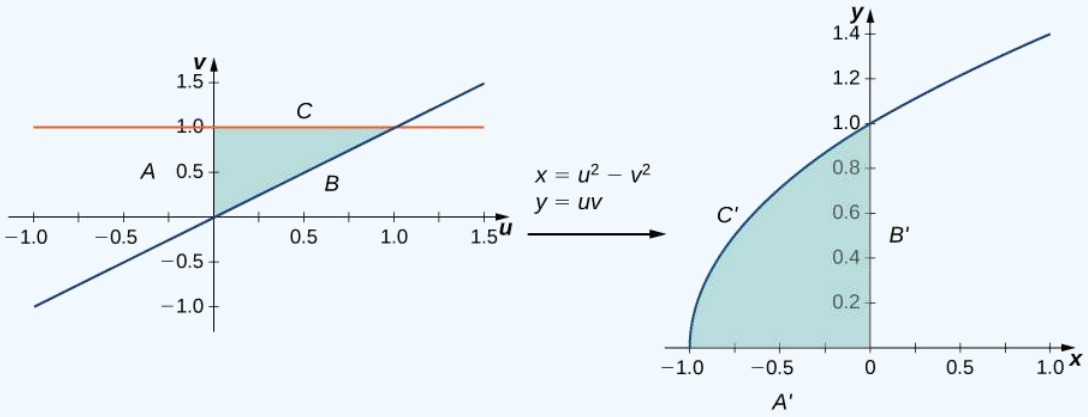


Figure 2.8.3: A triangular region in the uv-plane is transformed into an image in the xy-plane.

- For the side $A: u = 0,\ 0 \le v \le 1$ transforms to $x = -v^2,\ y = 0$ so this is the side A' that joins $(-1, 0)$ and $(0, 0)$.
- For the side $B: u = v,\ 0 \le u \le 1$ transforms to $x = 0,\ y = u^2$ so this is the side B' that joins $(0, 0)$ and $(0, 1)$.
- For the side $C: 0 \le u \le 1,\ v = 1$ transforms to $x = u^2 - 1,\ y = u$ (hence $x = y^2 - 1$ so this is the side C' that makes the upper half of the parabolic arc joining $(-1, 0)$ and $(0, 1)$.

All the points in the entire region of the triangle in the uv-plane are mapped inside the parabolic region in the xy-plane.

Exercise 2.8.1

Let a transformation T be defined as $T(u, v) = (x, y)$ where $x = u + v,\ y = 3v$. Find the image of the rectangle $G = \{(u, v) : 0 \le u \le 1,\ 0 \le v \le 2\}$ from the uv-plane after the transformation into a region R in the xy-plane. Show that T is a one-to-one transformation and find $T^{-1}(x, y)$.

Hint

Follow the steps of Example 2.8.1B.

Answer

$T^{-1}(x,y) = (u,v)$ where $u = \frac{3x-y}{3}$ and $v = \frac{y}{3}$

Jacobians

Recall that we mentioned near the beginning of this section that each of the component functions must have continuous first partial derivatives, which means that g_u, g_v, h_u and h_v exist and are also continuous. A transformation that has this property is called a C^1 transformation (here C denotes continuous). Let $T(u,v) = (g(u,v),\ h(u,v))$, where $x = g(u,v)$ and $y = h(u,v)$ be a one-to-one C^1 transformation. We want to see how it transforms a small rectangular region S, Δu units by Δv units, in the uv-plane (Figure 2.8.4).

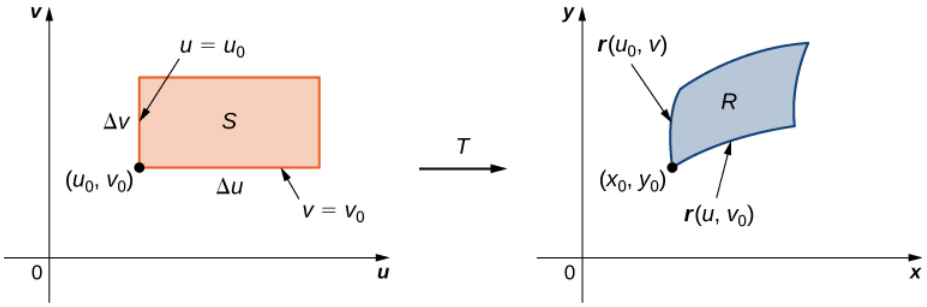


Figure 2.8.4: A small rectangle S in the uv-plane is transformed into a region R in the xy-plane.

Since $x = g(u,v)$ and $y = h(u,v)$, we have the position vector $r(u,v) = g(u,v)i + h(u,v)j$ of the image of the point (u,v). Suppose that $(u_0, v_0)$is the coordinate of the point at the lower left corner that mapped to $(x_0, y_0) = T(u_0, v_0)$ The line $v = v_0$ maps to the image curve with vector function $r(u, v_0)$, and the tangent vector at (x_0, y_0) to the image curve is

$$r_u = g_u(u_0,v_0)i + h_v(u_0,v_0)j = \frac{\partial x}{\partial u}i + \frac{\partial y}{\partial u}j. \tag{2.8.13}$$

Similarly, the line $u = u_0$ maps to the image curve with vector function $r(u_0, v)$, and the tangent vector at (x_0, y_0) to the image curve is

$$r_v = g_v(u_0,v_0)i + h_u(u_0,v_0)j = \frac{\partial x}{\partial v}i + \frac{\partial y}{\partial v}j. \tag{2.8.14}$$

Now, note that

$$r_u = \lim_{\Delta u \to 0} \frac{r(u_0+\Delta u, v_0) - r(u_0,v_0)}{\Delta u} \; so \; r(u_0+\Delta u, v_0) - r(u_0,v_0) \approx \Delta u r_u. \tag{2.8.15}$$

Similarly,

$$r_v = \lim_{\Delta v \to 0} \frac{r(u_0, v_0+\Delta v) - r(u_0,v_0)}{\Delta v} \; so \; r(u_0, v_0+\Delta v) - r(u_0,v_0) \approx \Delta v r_v. \tag{2.8.16}$$

This allows us to estimate the area ΔA of the image R by finding the area of the parallelogram formed by the sides $\Delta v r_v$ and $\Delta u r_u$. By using the cross product of these two vectors by adding the $\mathbf{k}$th component as $\mathbf{0}$, the area ΔA of the image R (refer to The Cross Product) is approximately $|\Delta u r_u \times \Delta v r_v| = |r_u \times r_v|\,\Delta u \Delta v$. In determinant form, the cross product is

$$r_u \times r_v = \begin{vmatrix} i & j & k \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} k = \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right) k \tag{2.8.17}$$

Since $|k| = 1$, we have

$\Delta A \approx |r_u \times r_v|\,\Delta u \Delta v = \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right) \Delta u \Delta v.$

Definition: Jacobian

The *Jacobian* of the C^1 transformation $T(u,v) = (g(u,v),\ h(u,v))$ is denoted by $J(u,v)$ and is defined by the 2×2 determinant

$$J(u,v)=\left|\frac{\partial(x,y)}{\partial(u,v)}\right|=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial y}{\partial u}\\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v}\end{vmatrix}=\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v}-\frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right). \tag{2.8.18}$$

Using the definition, we have

$$\Delta A\approx J(u,v)\Delta u\Delta v=\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\Delta u\Delta v. \tag{2.8.19}$$

Note that the Jacobian is frequently denoted simply by

$$J(u,v)=\frac{\partial(x,y)}{\partial(u,v)}. \tag{2.8.20}$$

Note also that

$$\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial y}{\partial u}\\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v}\end{vmatrix}=\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v}-\frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}.$$

Hence the notation $J(u,v)=\frac{\partial(x,y)}{\partial(u,v)}$ suggests that we can write the Jacobian determinant with partials of x in the first row and partials of y in the second row.

✔ Example 2.8.2A: Finding the Jacobian

Find the Jacobian of the transformation given in Example 2.8.1A.

Solution

The transformation in the example is $T(r,\theta)=(r\cos\theta,\ r\sin\theta)$ where $x=r\cos\theta$ and $y=r\sin\theta$. Thus the Jacobian is

$$J(r,\theta)=\frac{\partial(x,y)}{\partial(r,\theta)}=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta}\end{vmatrix}=\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}=r\cos^2\theta+r\sin^2\theta=r(\cos^2\theta+\sin^2\theta)=r.$$

✔ Example 2.8.2B: Finding the Jacobian

Find the Jacobian of the transformation given in Example 2.8.1B.

Solution

The transformation in the example is $T(u,v)=(u^2-v^2,uv)$ where $x=u^2-v^2$ and $y=uv$. Thus the Jacobian is

$$J(u,v)=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}2u & -2v\\ v & u\end{vmatrix}=2u^2+2v^2.$$

? Exercise 2.8.2

Find the Jacobian of the transformation given in the previous checkpoint: $T(u,v)=(u+v,2v)$.

Hint

Follow the steps in the previous two examples.

Answer

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2$$

Change of Variables for Double Integrals

We have already seen that, under the change of variables $T(u,v) = (x,y)$ where $x = g(u,v)$ and $y = h(u,v)$, a small region ΔA in the xy-plane is related to the area formed by the product $\Delta u \Delta v$ in the uv-plane by the approximation

$$\Delta A \approx J(u,v)\Delta u,\ \Delta v. \tag{2.8.21}$$

Now let's go back to the definition of double integral for a minute:

$$\iint_R f(x,y) fA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}, y_{ij})\Delta A. \tag{2.8.22}$$

Referring to Figure 2.8.5, observe that we divided the region S in the uv-plane into small subrectangles S_{ij} and we let the subrectangles R_{ij} in the xy-plane be the images of S_{ij} under the transformation $T(u,v) = (x,y)$.

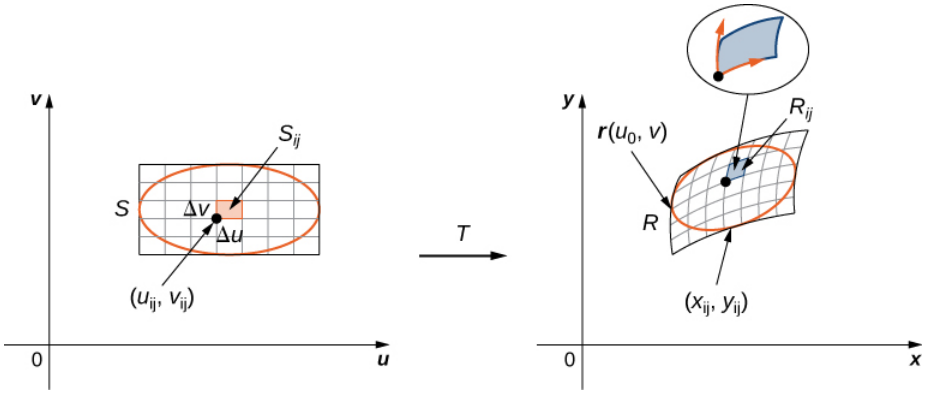


Figure 2.8.5: The subrectangles S_{ij} in the uv-plane transform into subrectangles R_{ij} in the xy-plane.

Then the double integral becomes

$$\iint_R = f(x,y)dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}, y_{ij})\Delta A \tag{2.8.23}$$

$$= \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(g(u_{ij}, v_{ij}),\ h(u_{ij}, v_{ij}))\,|J(u_{ij}, v_{ij})|\,\Delta u \Delta v.$$

Notice this is exactly the double Riemann sum for the integral

$$\iint_S f(g(u,v),\ h(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv. \tag{2.8.24}$$

Change of Variables for Double Integrals

Let $T(u,v) = (x,y)$ where $x = g(u,v)$ and $y = h(u,v)$ be a one-to-one C^1 transformation, with a nonzero Jacobian on the interior of the region S in the uv-plane it maps S into the region R in the xy-plane. If f is continuous on R, then

$$\iint_R f(x,y)dA = \iint_S f(g(u,v),\ h(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv. \tag{2.8.25}$$

With this theorem for double integrals, we can change the variables from (x,y) to (u,v) in a double integral simply by replacing

$$dA = dx\,dy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv \tag{2.8.26}$$

when we use the substitutions $x = g(u, v)$ and $y = h(u, v)$ and then change the limits of integration accordingly. This change of variables often makes any computations much simpler.

Example 2.8.3: Changing Variables from Rectangular to Polar Coordinates

Consider the integral

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\, dy\, dx. \tag{2.8.27}$$

Use the change of variables $x = r \cos \theta$ and $y = r \sin \theta$, and find the resulting integral.

Solution

First we need to find the region of integration. This region is bounded below by $y = 0$ and above by $y = \sqrt{2x - x^2}$ (Figure 2.8.6).

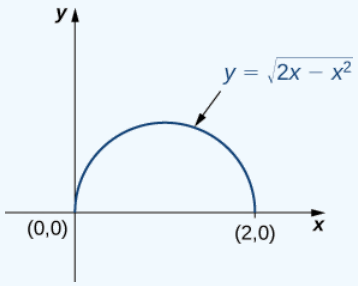


Figure 2.8.6: Changing a region from rectangular to polar coordinates.

Squaring and collecting terms, we find that the region is the upper half of the circle $x^2 + y^2 - 2x = 0$, that is $y^2 + (x-1)^2 = 1$. In polar coordinates, the circle is $r = 2 \cos \theta$ so the region of integration in polar coordinates is bounded by $0 \le r \le \cos \theta$ and $0 \le \theta \le \frac{\pi}{2}$.

The Jacobian is $J(r, \theta) = r$, as shown in Example 2.8.2A. Since $r \ge 0$, we have $|J(r, \theta)| = r$.

The integrand $\sqrt{x^2+y^2}$ changes to r in polar coordinates, so the double iterated integral is

$$\begin{aligned} \int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\, dy\, dx &= \int_0^{\pi/2} \int_0^{2\cos\theta} r\, |j(r,\theta)|\, dr\, d\theta \\ &= \int_0^{\pi/2} \int_0^{2\cos\theta} r^2\, dr\, d\theta. \end{aligned} \tag{2.8.28}$$



Exercise 2.8.3

Considering the integral $\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2+y^2)dy\,dx$, use the change of variables $x = r\cos\theta$ and $y = r\sin\theta$ and find the resulting integral.

Hint

Follow the steps in the previous example.

Answer

$$\int_0^{\pi/2} \int_0^1 r^3 dr\,d\theta \tag{2.8.29}$$

Notice in the next example that the region over which we are to integrate may suggest a suitable transformation for the integration. This is a common and important situation.

Example 2.8.4: Changing Variables

Consider the integral

$$\iint_R (x-y)dy\,dx, \tag{2.8.30}$$

where R is the parallelogram joining the points $(1,2)$, $(3,4)$, $(4,3)$, and $(6,5)$ (Figure 2.8.7). Make appropriate changes of variables, and write the resulting integral.

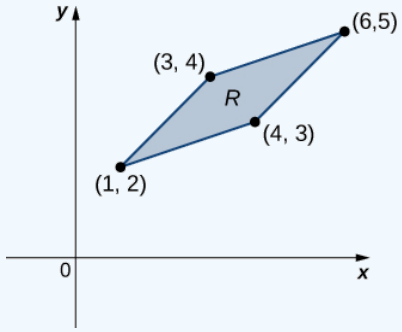


Figure 2.8.7: The region of integration for the given integral.

Solution

First, we need to understand the region over which we are to integrate. The sides of the parallelogram are $x-y+1$, $x-y-1=0$, $x-3y+5=0$ and $x-3y+9=0$ (Figure 2.8.8). Another way to look at them is $x-y=-1$, $x-y=1$, $x-3y=-5$, and $x-3y=9$.

Clearly the parallelogram is bounded by the lines $y=x+1$, $y=x-1$, $y=\frac{1}{3}(x+5)$, and $y=\frac{1}{3}(x+9)$.

Notice that if we were to make $u=x-y$ and $v=x-3y$, then the limits on the integral would be $-1\le u\le 1$ and $-9\le v\le -5$.

To solve for x and y, we multiply the first equation by 3 and subtract the second equation, $3u-v=(3x-3y)-(x-3y)=2x$. Then we have $x=\frac{3u-v}{2}$. Moreover, if we simply subtract the second equation from the first, we get $u-v=(x-y)-(x-3y)=2y$ and $y=\frac{u-v}{2}$.

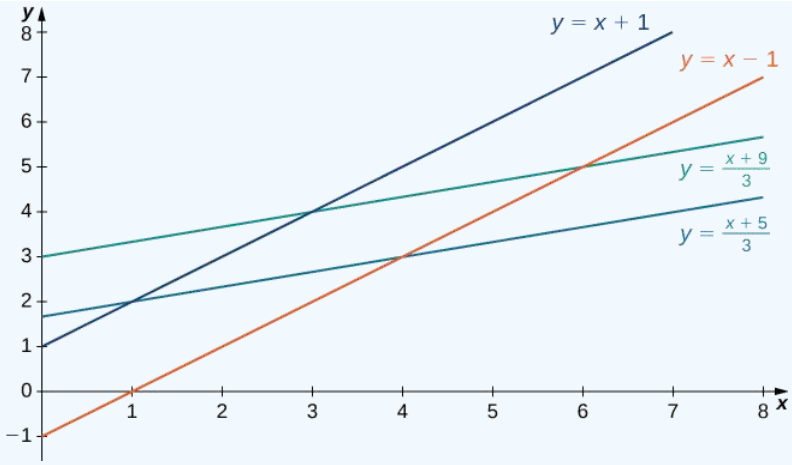


Figure **2.8.8**: A parallelogram in the xy-plane that we want to transform by a change in variables.

Thus, we can choose the transformation

$$T(u,v) = \left(\frac{3u-v}{2}, \frac{u-v}{2}\right) \tag{2.8.31}$$

and compute the Jacobian $J(u,v)$. We have

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 3/2 & -1/2 \\ 1/2 & -1/2 \end{vmatrix} = -\frac{3}{4} + \frac{1}{4} = -\frac{1}{2}$$

Therefore, $|J(u,v)| = \frac{1}{2}$. Also, the original integrand becomes

$$x - y = \frac{1}{2}[3u - v - u + v] = \frac{1}{2}[3u - u] = \frac{1}{2}[2u] = u.$$

Therefore, by the use of the transformation T, the integral changes to

$$\iint_R (x-y)\,dy\,dx = \int_{-9}^{-5}\int_{-1}^{1} J(u,v)u\,du\,dv = \int_{-9}^{-5}\int_{-1}^{1}\left(\frac{1}{2}\right)u\,du\,dv,$$

which is much simpler to compute.

? Exercise 2.8.4

Make appropriate changes of variables in the integral

$$\iint_R \frac{4}{(x-y)^2}\,dy\,dx,$$

where R is the trapezoid bounded by the lines $x-y=2$, $x-y=4$, $x=0$, and $y=0$. Write the resulting integral.

Hint

Follow the steps in the previous example.

Answer

$x = \frac{1}{2}(v+u)$ and $y = \frac{1}{2}(v-u)$

and

$$\int_2^4 \int_{-u}^{u} \left(\frac{1}{2}\right)\cdot\frac{4}{u^2}\,dv\,du.$$



We are ready to give a problem-solving strategy for change of variables.

Problem-Solving Strategy: Change of Variables

1. Sketch the region given by the problem in the xy-plane and then write the equations of the curves that form the boundary.
2. Depending on the region or the integrand, choose the transformations $x = g(u, v)$ and $y = h(u, v)$.
3. Determine the new limits of integration in the uv-plane.
4. Find the Jacobian $J(u, v)$.
5. In the integrand, replace the variables to obtain the new integrand.
6. Replace $dy\,dx$ or $dx\,dy$, whichever occurs, by $J(u, v)du\,dv$.

In the next example, we find a substitution that makes the integrand much simpler to compute.

Example 2.8.5: Evaluating an Integral

Using the change of variables $u = x - y$ and $v = x + y$, evaluate the integral

$$\iint_R (x - y)e^{x^2-y^2}\,dA, \tag{2.8.32}$$

where R is the region bounded by the lines $x + y = 1$ and $x + y = 3$ and the curves $x^2 - y^2 = -1$ and $x^2 - y^2 = 1$ (see the first region in Figure 2.8.9).

Solution

As before, first find the region R and picture the transformation so it becomes easier to obtain the limits of integration after the transformations are made (Figure 2.8.9).

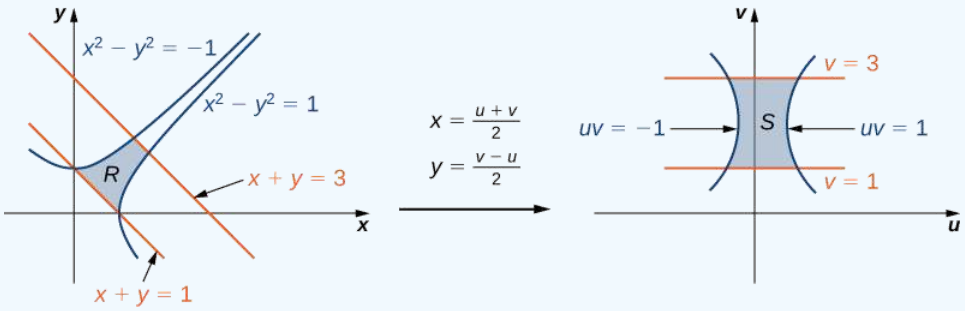


Figure 2.8.9: Transforming the region R into the region S to simplify the computation of an integral.

Given $u = x - y$ and $v = x + y$, we have $x = \frac{u+v}{2}$ and $y = \frac{v-u}{2}$ and hence the transformation to use is $T(u,v) = (\frac{u+v}{2}, \frac{v-u}{2})$. The lines $x + y = 1$ and $x + y = 3$ become $v = 1$ and $v = 3$, respectively. The curves $x^2 - y^2 = 1$ and $x^2 - y^2 = -1$ become $uv = 1$ and $uv = -1$, respectively.

Thus we can describe the region S (see the second region Figure 2.8.9) as

$$S = \left\{(u,v) | 1 \le v \le 3, \ \frac{-1}{v} \le u \le \frac{1}{v}\right\}.$$

The Jacobian for this transformation is

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{vmatrix} = \frac{1}{2}.$$

Therefore, by using the transformation T, the integral changes to

$$\iint_R (x-y)e^{x^2-y^2}\,dA = \frac{1}{2}\int_1^3 \int_{-1/v}^{1/v} ue^{uv}\,du\,dv.$$

Doing the evaluation, we have

$$\frac{1}{2}\int_1^3 \int_{-1/v}^{1/v} ue^{uv}\,du\,dv = \frac{2}{3e} \approx 0.245.$$

? Exercise 2.8.5

Using the substitutions $x = v$ and $y = \sqrt{u+v}$, evaluate the integral $\iint_R y \sin(y^2 - x)\,dA$, where R is the region bounded by the lines $y = \sqrt{x}$, $x = 2$ and $y = 0$.

Hint

Sketch a picture and find the limits of integration.

Answer

$\frac{1}{2}(\sin 2 - 2)$

Change of Variables for Triple Integrals

Changing variables in triple integrals works in exactly the same way. Cylindrical and spherical coordinate substitutions are special cases of this method, which we demonstrate here.

Suppose that G is a region in uvw-space and is mapped to D in xyz-space (Figure 2.8.10) by a one-to-one C^1 transformation $T(u,v,w) = (x,y,z)$ where $x = g(u,v,w)$, $y = h(u,v,w)$, and $z = k(u,v,w)$.

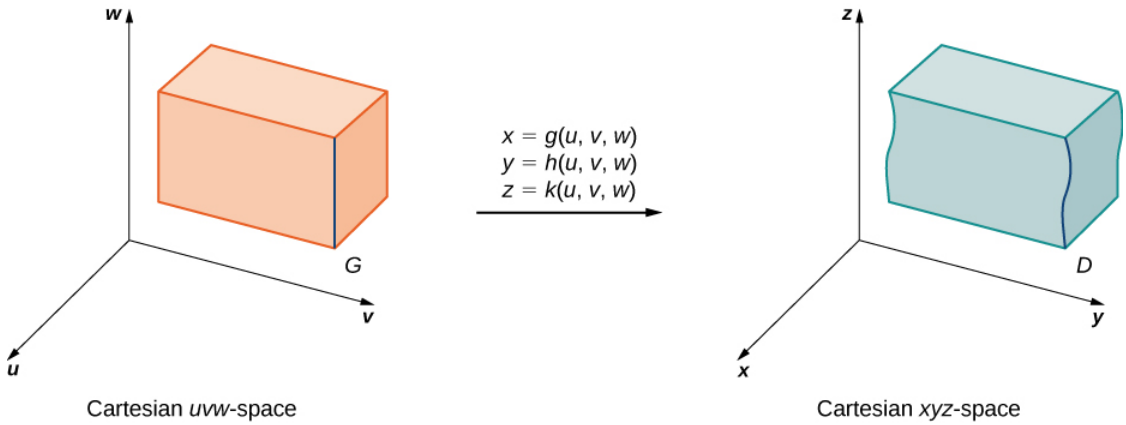


Figure 2.8.10: A region G in uvw-space mapped to a region D in xyz-space.

Then any function $F(x, y, z)$ defined on D can be thought of as another function $H(u, v, w)$ that is defined on G:

$$F(x, y, z) = F(g(u, v, w),\, h(u, v, w),\, k(u, v, w)) = H(u, v, w). \tag{2.8.33}$$

Now we need to define the Jacobian for three variables.

Definition: Jacobian determinant

The *Jacobian determinant* $J(u, v, w)$ in three variables is defined as follows:

$$J(u, v, w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix}. \tag{2.8.34}$$

This is also the same as

$$J(u, v, w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}. \tag{2.8.35}$$

The Jacobian can also be simply denoted as $\frac{\partial(x,y,z)}{\partial(u,v,w)}$.

With the transformations and the Jacobian for three variables, we are ready to establish the theorem that describes change of variables for triple integrals.

Change of Variables for Triple Integrals

Let $T(u, v, w) = (x, y, z)$ where $x = g(u, v, w)$, $y = h(u, v, w)$, and $z = k(u, v, w)$, be a one-to-one C^1 transformation, with a nonzero Jacobian, that maps the region G in the uvw-space into the region D in the xyz-space. As in the two-dimensional case, if F is continuous on D, then

$$\iiint_D F(x, y, z)dV = \iiint_G f(g(u, v, w)\, h(u, v, w),\, k(u, v, w)) \left|\frac{\partial(x, y, z)}{\partial(u, v, w)}\right| du\, dv\, dw \tag{2.8.36}$$

$$= \iiint_G H(u, v, w)\, |J(u, v, w)|\, du\, dv\, dw. \tag{2.8.37}$$

Let us now see how changes in triple integrals for cylindrical and spherical coordinates are affected by this theorem. We expect to obtain the same formulas as in Triple Integrals in Cylindrical and Spherical Coordinates.

Example 2.8.6A: Obtaining Formulas in Triple Integrals for Cylindrical and Spherical Coordinates

Derive the formula in triple integrals for

a. cylindrical and
b. spherical coordinates.

Solution

A.

For cylindrical coordinates, the transformation is $T(r,\theta,z)=(x,y,z)$ from the Cartesian $r\theta z$-space to the Cartesian xyz-space (Figure 2.8.11). Here $x=r\cos\theta$, $y=r\sin\theta$ and $z=z$. The Jacobian for the transformation is

$$J(r,\theta,z)=\frac{\partial(x,y,z)}{\partial(r,\theta,z)}=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial z}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial z}\\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial z}\end{vmatrix} \tag{2.8.38}$$

$$\begin{vmatrix}\cos\theta & -r\sin\theta & 0\\ \sin\theta & r\cos\theta & 0\\ 0 & 0 & 1\end{vmatrix}=r\cos^2\theta+r\sin^2\theta=r. \tag{2.8.39}$$

We know that $r\geq 0$, so $|J(r,\theta,z)|=r$. Then the triple integral is

$$\iiint_D f(x,y,z)dV=\iiint_G f(r\cos\theta,\ r\sin\theta,\ z)r\,dr\,d\theta\,dz. \tag{2.8.40}$$

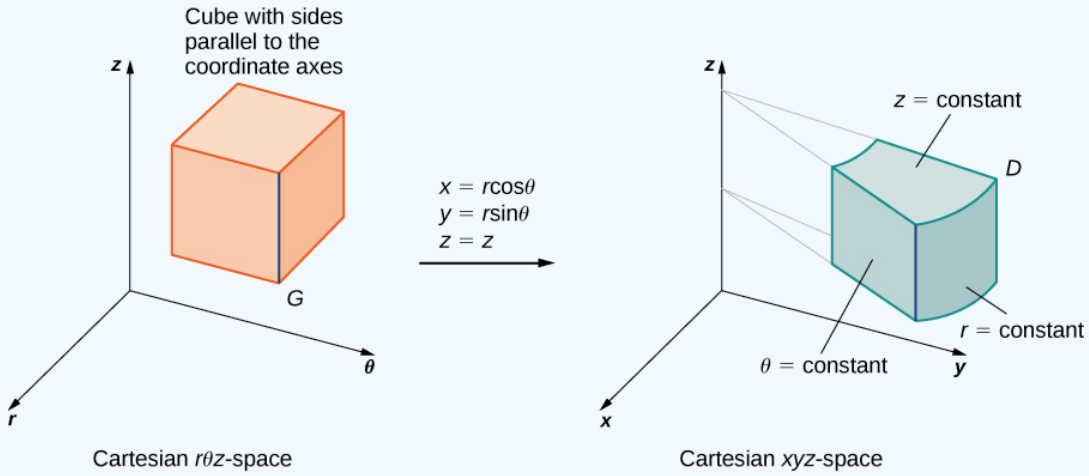


Figure 2.8.11: The transformation from rectangular coordinates to cylindrical coordinates can be treated as a change of variables from region G in $r\theta z$-space to region D in xyz-space.

B.

For spherical coordinates, the transformation is $T(\rho,\theta,\varphi)$ from the Cartesian $\rho\theta\varphi$-space to the Cartesian xyz-space (Figure 2.8.12). Here $x=\rho\sin\varphi\cos\theta$, $y=\rho\sin\varphi\sin\theta$, and $z=\rho\cos\varphi$. The Jacobian for the transformation is

$$J(\rho,\theta,\varphi)=\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)}=\begin{vmatrix}\frac{\partial x}{\partial\rho} & \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\varphi}\\ \frac{\partial y}{\partial\rho} & \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\varphi}\\ \frac{\partial z}{\partial\rho} & \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial\varphi}\end{vmatrix} \tag{2.8.41}$$

$$=\begin{vmatrix}\sin\varphi\cos\theta & -\rho\sin\varphi\sin\theta & \rho\cos\varphi\cos\theta\\ \sin\varphi\sin\theta & \rho\sin\varphi\cos\theta & \rho\cos\varphi\sin\theta\\ \cos\varphi & 0 & -\rho\sin\varphi\end{vmatrix}.$$

Expanding the determinant with respect to the third row:

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$$= \cos\varphi \begin{vmatrix} -\rho\sin\varphi\sin\theta & \rho\cos\varphi\cos\theta \\ \rho\sin\varphi\cos\theta & \rho\cos\varphi\sin\theta \end{vmatrix} - \rho\sin\varphi \begin{vmatrix} \sin\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\sin\varphi\cos\theta \end{vmatrix}$$

$$= \cos\varphi(-\rho^2\sin\varphi\cos\varphi\sin^2\theta - \rho^2\sin\varphi\cos\varphi\cos^2\theta)$$
$$-\rho\sin\varphi(\rho\sin^2\varphi\cos^2\theta + \rho\sin^2\varphi\sin^2\theta)$$

$$= -\rho^2\sin\varphi\cos^2\varphi(\sin^2\theta + \cos^2\theta) - \rho^2\sin\varphi\sin^2\varphi(\sin^2\theta + \cos^2\theta)$$

$$= -\rho^2\sin\varphi\cos^2\varphi - \rho^2\sin\varphi\sin^2\varphi$$

$$= -\rho\sin\varphi(\cos^2\varphi + \sin^2\varphi) = -\rho^2\sin\varphi.$$

Since $0 \le \varphi \le \pi$, we must have $\sin\varphi \ge 0$. Thus $|J(\rho,\theta,\varphi)| = |-\rho^2\sin\varphi| = \rho^2\sin\varphi$.

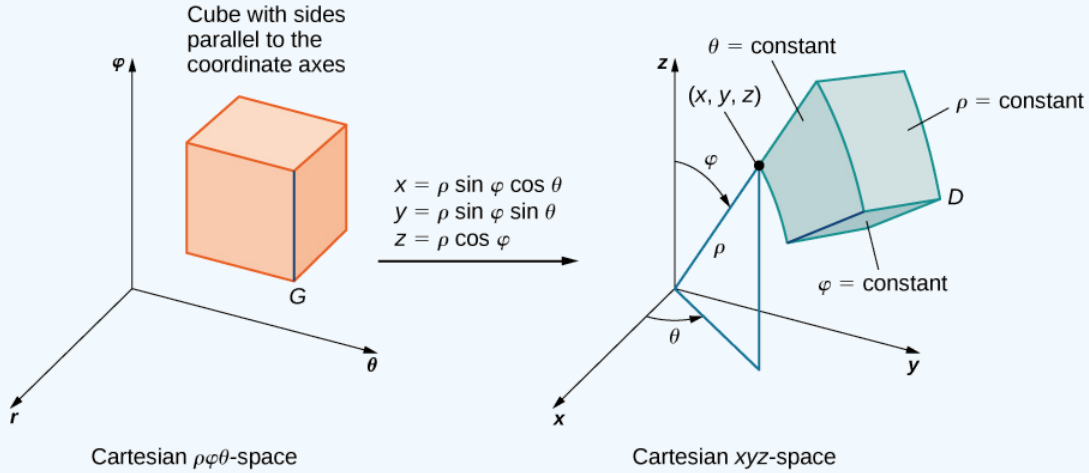


Figure 2.8.12: The transformation from rectangular coordinates to spherical coordinates can be treated as a change of variables from region G in $\rho\theta\varphi$-space to region D in xyz-space.

Then the triple integral becomes

$$\iiint_D f(x,y,z)dV = \iiint_G f(\rho\sin\varphi\cos\theta,\ \rho\sin\varphi\sin\theta,\rho\cos\varphi)\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta. \tag{2.8.42}$$

Let's try another example with a different substitution.

Example 2.8.6B: Evaluating a Triple Integral with a Change of Variables

Evaluate the triple integral

$$\int_0^3\int_0^4\int_{y/2}^{(y/2)+1}\left(x+\frac{z}{3}\right)dx\,dy\,dz \tag{2.8.43}$$

In xyz-space by using the transformation

$u = (2x-y)/2$, $v = y/2$, and $w = z/3$.

Then integrate over an appropriate region in uvw-space.

Solution

As before, some kind of sketch of the region G in xyz-space over which we have to perform the integration can help identify the region D in uvw-space (Figure 2.8.13). Clearly G in xyz-space is bounded by the planes $x = y/2$, $x = (y/2)+1$, $y = 0$, $y = 4$, $z = 0$, and $z = 4$. We also know that we have to use $u = (2x-y)/2$, $v = y/2$, and $w = z/3$ for the transformations. We need to solve for x, y and z. Here we find that $x = u+v$, $y = 2v$, and $z = 3w$.

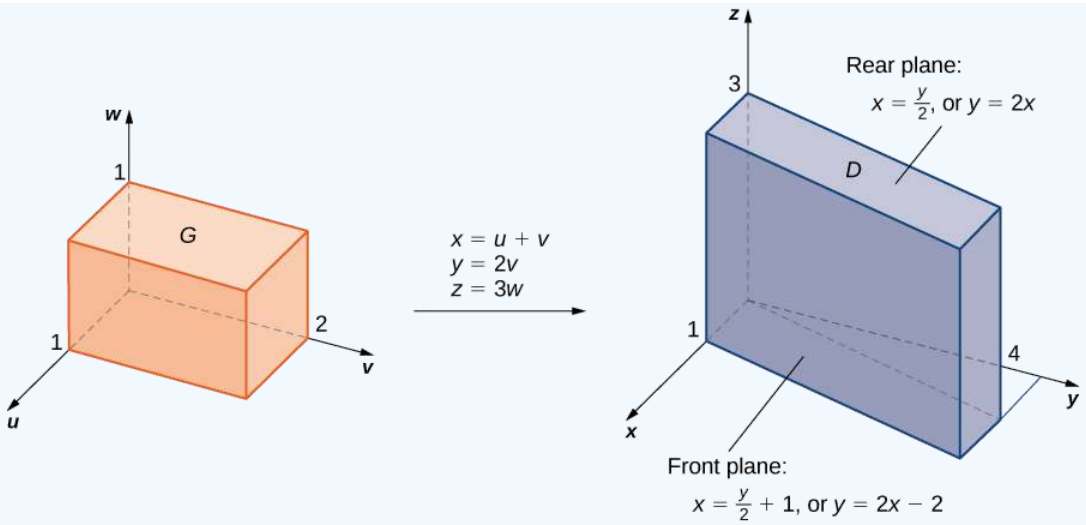


Figure 2.8.13: The region G in uvw-space is transformed to region D in xyz-space.

Using elementary algebra, we can find the corresponding surfaces for the region G and the limits of integration in uvw-space. It is convenient to list these equations in a table.

Equations in xyz for the region D	Corresponding equations in uvw for the region G	Limits for the integration in uvw
$x = y/2$	$u + v = 2v/2 = v$	$u = 0$
$x = y/2$	$u + v = (2v/2) + 1 = v + 1$	$u = 1$
$y = 0$	$2v = 0$	$v = 0$
$y = 4$	$2v = 4$	$v = 2$
$z = 0$	$3w = 0$	$w = 0$
$z = 3$	$3w = 3$	$w = 1$

Now we can calculate the Jacobian for the transformation:

$$J(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 6.$$

The function to be integrated becomes

$$f(x,y,z) = x + \frac{z}{3} = u + v + \frac{3w}{3} = u + v + w. \tag{2.8.44}$$

We are now ready to put everything together and complete the problem.

$$\int_0^3 \int_0^4 \int_{y/2}^{(y/2)+1} \left(x + \frac{z}{3}\right) dx\, dy\, dz = \int_0^1 \int_0^2 \int_0^1 (u + v + w)\, |J(u, v, w)|\, du\, dv\, dw$$
$$= \int_0^1 \int_0^2 \int_0^1 (u + v + w)\, |6|\, du\, dv\, dw$$
$$= 6 \int_0^1 \int_0^2 \int_0^1 (u + v + w)\, du\, dv\, dw$$
$$= 6 \int_0^1 \int_0^2 \left[\frac{u^2}{2} + vu + wu\right]_0^1 dv\, dw$$
$$= 6 \int_0^1 \int_0^2 \left(\frac{1}{2} + v + u\right) dv\, dw$$
$$= 6 \int_0^1 \left[\frac{1}{2}v + \frac{v^2}{2} + wv\right]_0^2 dw$$
$$= 6 \int_0^1 (3 + 2w)\, dw = 6\left[3w + w^2\right]_0^1 = 24.$$



Exercise 2.8.6

Let D be the region in xyz-space defined by $1 \le x \le 2,\ 0 \le xy \le 2$, and $0 \le z \le 1$.

Evaluate $\iiint_D (x^2y + 3xyz)\, dx\, dy\, dz$ by using the transformation $u = x$, $v = xy$, and $w = 3z$.

Hint

Make a table for each surface of the regions and decide on the limits, as shown in the example.

Answer

$$\int_0^3 \int_0^2 \int_1^2 \left(\frac{v}{3} + \frac{vw}{3u}\right) du\, dv\, dw = 2 + \ln 8 \tag{2.8.45}$$

Key Concepts

- A transformation T is a function that transforms a region G in one plane (space) into a region R. in another plane (space) by a change of variables.
- A transformation $T : G \to R$ defined as $T(u, v) = (x, y)$ (or $T(u, v, w) = (x, y, z)$)is said to be a one-to-one transformation if no two points map to the same image point.

- If f is continuous on R, then

$$\iint_R f(x,y)dA = \iint_S f(g(u,v),\, h(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv. \tag{2.8.46}$$

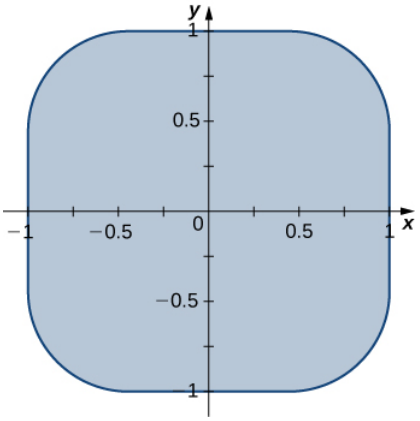
- If F is continuous on R, then

$$\begin{aligned}\iiint_R F(x,y,z)\, dV &= \iiint_G F(g(u,v,w),\, h(u,v,w),\, k(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\, dv\, dw \\ &= \iiint_G H(u,v,w)\, |J(u,v,w)|\, du\, dv\, dw.\end{aligned}$$

[T] Lamé ovals (or superellipses) are plane curves of equations $(\frac{x}{a})^n + (\frac{y}{b})^n = 1$, where a, b, and n are positive real numbers.

a. Use a CAS to graph the regions R bounded by Lamé ovals for $a = 1$, $b = 2$, $n = 4$ and $n = 6$ respectively.

b. Find the transformations that map the region R bounded by the Lamé oval $x^4 + y^4 = 1$ also called a squircle and graphed in the following figure, into the unit disk.



c. Use a CAS to find an approximation of the area $A(R)$ of the region R bounded by $x^4 + y^4 = 1$. Round your answer to two decimal places.

[T] Lamé ovals have been consistently used by designers and architects. For instance, Gerald Robinson, a Canadian architect, has designed a parking garage in a shopping center in Peterborough, Ontario, in the shape of a superellipse of the equation $(\frac{x}{a})^n + (\frac{y}{b})^n = 1$ with $\frac{a}{b} = \frac{9}{7}$ and $n = e$. Use a CAS to find an approximation of the area of the parking garage in the case $a = 900$ yards, $b = 700$ yards, and $n = 2.72$ yards.

[Hide Solution]

$A(R) \simeq 83,999.2$

Chapter Review Exercises

True or False? Justify your answer with a proof or a counterexample.

$$\int_a^b \int_c^d f(x,y)\, dy\, dx = \int_c^d \int_a^b f(x,y)\, dy\, dx \tag{2.8.47}$$

Fubini's theorem can be extended to three dimensions, as long as f is continuous in all variables.

[Hide solution]

True.

The integral

$$\int_0^{2\pi} \int_0^1 \int_0^1 dz\, dr\, d\theta \tag{2.8.48}$$

represents the volume of a right cone.

The Jacobian of the transformation for $x = u^2 - 2v$, $y = 3v - 2uv$ is given by $-4u^2 + 6u + 4v$.

[Hide Solution]

False.

Evaluate the following integrals.

$$\iint_R (5x^3y^2 - y^2)\, dA,\ R = \{(x,y) | 0 \le x \le 2,\ 1 \le y \le 4\} \tag{2.8.49}$$

$$\iint_D \frac{y}{3x^2+1} dA,\ D = \{(x,y) | 0 \le x \le 1,\ -x \le y \le x\} \tag{2.8.50}$$

[Hide Solution]

$\mathbf{0}$

$$\iint_D \sin(x^2+y^2) dA \tag{2.8.51}$$

where D is a disk of radius 2 centered at the origin

$$\int_0^1 \int_0^1 xye^{x^2} dx\, dy \tag{2.8.52}$$

[Hide Solution]

$\frac{1}{4}$

$$\int_{-1}^1 \int_0^z \int_0^{x-z} 6dy\, dx\, dz \tag{2.8.53}$$

$$\iiint_R 3y\, dV, \tag{2.8.54}$$

where $R = \{(x,y,z) | 0 \le x \le 1,\ 0 \le y \le x,\ 0 \le z \le \sqrt{9-y^2}\}$

[Hide Solution]

$\mathbf{1.475}$

$$\int_0^2 \int_0^{2\pi} \int_r^1 r\, dz\, d\theta\, dr \tag{2.8.55}$$

$$\int_0^{2\pi} \int_0^{\pi/2} \int_1^3 \rho^2 \sin(\varphi) d\rho\, d\varphi,\ d\theta \tag{2.8.56}$$

[Hide Solution]

$\frac{52}{3}\pi$

$$\int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} dz\, dy\, sx \tag{2.8.57}$$

For the following problems, find the specified area or volume.

The area of region enclosed by one petal of $r = \cos(4\theta)$.

[Hide Solution]

$\frac{\pi}{16}$

The volume of the solid that lies between the paraboloid $z = 2x^2 + 2y^2$ and the plane $z = 8$.

The volume of the solid bounded by the cylinder $x^2 + y^2 = 16$ and from $z = 1$ to $z + x = 2$.

[Hide Solution]

93.291

The volume of the intersection between two spheres of radius 1, the top whose center is $(0, 0, 0.25)$ and the bottom, which is centered at $(0, 0, 0)$.

For the following problems, find the center of mass of the region.

$\rho(x, y) = xy$ on the circle with radius 1 in the first quadrant only.

[Hide Solution]

$\left(\frac{8}{15}, \frac{8}{15}\right)$

$\rho(x, y) = (y + 1)\sqrt{x}$ in the region bounded by $y = e^x$, $y = 0$, and $x = 1$.

$\rho(x, y, z) = z$ on the inverted cone with radius 2 and height 2.

$\left(0, 0, \frac{8}{5}\right)$

The volume an ice cream cone that is given by the solid above $z = \sqrt{(x^2 + y^2)}$ and below $z^2 + x^2 + y^2 = z$.

The following problems examine Mount Holly in the state of Michigan. Mount Holly is a landfill that was converted into a ski resort. The shape of Mount Holly can be approximated by a right circular cone of height 1100 ft and radius 6000 ft.

If the compacted trash used to build Mount Holly on average has a density $400\ lb/ft^3$, find the amount of work required to build the mountain.

[Hide Solution]

$1.452\pi \times 10^{15}$ ft-lb

In reality, it is very likely that the trash at the bottom of Mount Holly has become more compacted with all the weight of the above trash. Consider a density function with respect to height: the density at the top of the mountain is still density $400\ lb/ft^3$ and the density increases. Every 100 feet deeper, the density doubles. What is the total weight of Mount Holly?

The following problems consider the temperature and density of Earth's layers.

[T] The temperature of Earth's layers is exhibited in the table below. Use your calculator to fit a polynomial of degree 3 to the temperature along the radius of the Earth. Then find the average temperature of Earth. (*Hint*: begin at 0 in the inner core and increase outward toward the surface)

Layer	Depth from center (km)	Temperature ^oC
Rocky Crust	0 to 40	0
Upper Mantle	40 to 150	870
Mantle	400 to 650	870
Inner Mantel	650 to 2700	870
Molten Outer Core	2890 to 5150	4300
Inner Core	5150 to 6378	7200

Source: http://www.enchantedlearning.com/sub...h/Inside.shtml

[Hide Solution]

$y = -1.238 \times 10^{-7}x^3 + 0.001196x^2 - 3.666x + 7208$; average temperature approximately 2800^oC

[T] The density of Earth's layers is displayed in the table below. Using your calculator or a computer program, find the best-fit quadratic equation to the density. Using this equation, find the total mass of Earth.

Layer	Depth from center (km)	Density (g/cm^3)
Inner Core	0	12.95
Outer Core	1228	11.05
Mantle	3488	5.00
Upper Mantle	6338	3.90
Crust	6378	2.55

Source: http://hyperphysics.phy-astr.gsu.edu...rthstruct.html

The following problems concern the Theorem of Pappus (see Moments and Centers of Mass for a refresher), a method for calculating volume using centroids. Assuming a region R, when you revolve around the x-axis the volume is given by $V_x = 2\pi A\bar{y}$, and when you revolve around the y-axis the volume is given by $V_y = 2\pi A\bar{x}$, where A is the area of R. Consider the region bounded by $x^2 + y^2 = 1$ and above $y = x + 1$.

Find the volume when you revolve the region around the x-axis.

[Hide Solution]

$\frac{\pi}{3}$

Find the volume when you revolve the region around the y-axis.

Glossary

Jacobian

the Jacobian $J(u, v)$ in two variables is a 2×2 determinant:

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix};$$

the Jacobian $J(u, v, w)$ in three variables is a 3×3 determinant:

$$J(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix}$$

one-to-one transformation

a transformation $T : G \to R$ defined as $T(u, v) = (x, y)$ is said to be one-to-one if no two points map to the same image point

planar transformation

a function T that transforms a region G in one plane into a region R in another plane by a change of variables

transformation

a function that transforms a region GG in one plane into a region RR in another plane by a change of variables

Contributors and Attributions

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2.8E: Exercises

In exercises 1 - 6, the function $T: S \to R,\ T(u,v) = (x,y)$ on the region $S = \{(u,v) \mid 0 \le u \le 1,\ 0 \le v \le 1\}$ bounded by the unit square is given, where $R \in R^2$ is the image of S under T.

a. Justify that the function T is a C^1 transformation.

b. Find the images of the vertices of the unit square S through the function T.

c. Determine the image R of the unit square S and graph it.

1. $x = 2u,\ y = 3v$

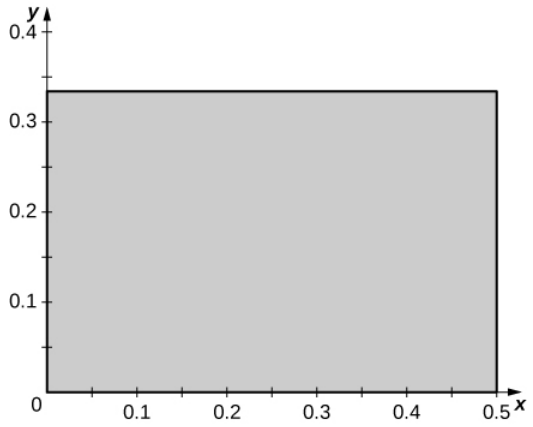
2. $x = \frac{u}{2},\ y = \frac{v}{3}$

Answer

a. $T(u,v) = (g(u,v),\ h(u,v),\ x = g(u,v) = \frac{u}{2}$ and $y = h(u,v) = \frac{v}{3}$. The functions g and h are continuous and differentiable, and the partial derivatives $g_u(u,v) = \frac{1}{2},\ g_v(u,v) = 0,\ h_u(u,v) = 0$ and $h_v(u,v) = \frac{1}{3}$ are continuous on S;

b. $T(0,0) = (0,0),\ T(1,0) = \left(\frac{1}{2}, 0\right),\ T(0,1) = \left(0, \frac{1}{3}\right)$, and $T(1,1) = \left(\frac{1}{2}, \frac{1}{3}\right)$;

c. R is the rectangle of vertices $(0,0),\ \left(0, \frac{1}{3}\right),\ \left(\frac{1}{2}, \frac{1}{3}\right)$, and $\left(0, \frac{1}{3}\right)$ in the xy-plane; the following figure.



3. $x = u - v,\ y = u + v$

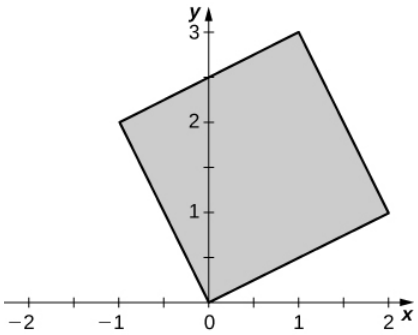
4. $x = 2u - v,\ y = u + 2v$

Answer

a. $T(u,v) = (g(u,v),\ h(u,v),\ x = g(u,v) = 2u - v$ and $y = h(u,v) = u + 2v$. The functions g and h are continuous and differentiable, and the partial derivatives $g_u(u,v) = 2,\ g_v(u,v) = -1,\ h_u(u,v) = 1$ and $h_v(u,v) = 2$ are continuous on S;

b. $T(0,0) = (0,0),\ T(1,0) = (2,1),\ T(0,1) = (-1,2)$, and $T(1,1) = (1,3)$;

c. R is the parallelogram of vertices $(0,0),\ (2,1)\ (1,3)$, and $(-1,2)$ in the xy-plane; the following figure.



5. $x = u^2,\ y = v^2$

6. $x = u^3,\ y = v^3$

Answer

a. $T(u,v) = (g(u,v),\ h(u,v),\ x = g(u,v) = u^3$ and $y = h(u,v) = v^3$. The functions g and h are continuous and differentiable, and the partial derivatives $g_u(u,v) = 3u^2,\ g_v(u,v) = 0,\ h_u(u,v) = 0$ and $h_v(u,v) = 3v^2$ are continuous on S;

b. $T(0,0) = (0,0),\ T(1,0) = (1,0),\ T(0,1) = (0,1)$, and $T(1,1) = (1,1)$;

c. R is the unit square in the xy-plane see the figure in the answer to the previous exercise.

In exercises 7 - 12, determine whether the transformations $T : S \to R$ are one-to-one or not.

7. $x = u^2,\ y = v^2$, where S is the rectangle of vertices $(-1,0),\ (1,0),\ (1,1)$, and $(-1,1)$.

8. $x = u^4,\ y = u^2 + v$, where S is the triangle of vertices $(-2,0),\ (2,0)$, and $(0,2)$.

Answer

T is not one-to-one: two points of S have the same image. Indeed, $T(-2,0) = T(2,0) = (16,4)$.

9. $x = 2u,\ y = 3v$, where S is the square of vertices $(-1,1),\ (-1,-1),\ (1,-1)$, and $(1,1)$.

10. $T(u,v) = (2u - v, u)$, where S is the triangle with vertices $(-1,1),\ (-1,-1)$, and $(1,-1)$.

Answer

T is one-to-one: We argue by contradiction. $T(u_1,v_1) = T(u_2,v_2)$ implies $2u_1 - v_1 = 2u_2 - v_2$ and $u_1 = u_2$. Thus, $u_1 = u + 2$ and $v_1 = v_2$.

11. $x = u + v + w,\ y = u + v,\ z = w$, where $S = R = R^3$.

12. $x = u^2 + v + w,\ y = u^2 + v,\ z = w$, where $S = R = R^3$.

Answer

T is not one-to-one: $T(1,v,w) = (-1,v,w)$

In exercises 13 - 18, the transformations $T : R \to S$ are one-to-one. Find their related inverse transformations $T^{-1} : R \to S$.

13. $x = 4u,\ y = 5v$, where $S = R = R^2$.

14. $x = u + 2v,\ y = -u + v$, where $S = R = R^2$.

Answer

$u = \frac{x-2y}{3},\ v = \frac{x+y}{3}$

15. $x = e^{2u+v},\ y = e^{u-v}$, where $S = R^2$ and $R = \{(x,y) \mid x > 0,\ y > 0\}$

16. $x = \ln u,\ y = \ln(uv)$, where $S = \{(u,v) \mid u > 0,\ v > 0\}$ and $R = R^2$.

Answer

$u = e^x,\ v = e^{-x+y}$

17. $x = u + v + w,\ y = 3v,\ z = 2w$, where $S = R = R^3$.

18. $x = u + v,\ y = v + w,\ z = u + w$, where $S = R = R^3$.

Answer

$u = \frac{x-y+z}{2},\ v = \frac{x+y-z}{2},\ w = \frac{-x+y+z}{2}$

In exercises 19 - 22, the transformation $T : S \to R,\ T(u,v) = (x,y)$ and the region $R \subset R^2$ are given. Find the region $S \subset R^2$.

19. $x = au,\ y = bv,\ R = \{(x,y) \mid x^2 + y^2 \le a^2b^2\}$ where $a, b > 0$

20. $x = au,\ y = bc,\ R = \{(x,y) \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1\}$, where $a, b > 0$

Answer

$S = \{(u,v) \mid u^2 + v^2 \le 1\}$

21. $x = \frac{u}{a},\ y = \frac{v}{b},\ z = \frac{w}{c},\ R = \{(x,y) \mid x^2 + y^2 + z^2 \le 1\}$, where $a, b, c > 0$

22. $x = au,\ y = bv,\ z = cw,\ R = \{(x,y) \mid \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} \le 1,\ z > 0\}$, where $a, b, c > 0$

Answer

$R = \{(u,v,w) \mid u^2 - v^2 - w^2 \le 1,\ w > 0\}$

In exercises 23 - 32, find the Jacobian J of the transformation.

23. $x = u + 2v,\ y = -u + v$

24. $x = \frac{u^3}{2},\ y = \frac{v}{u^2}$

Answer

$\frac{3}{2}$

25. $x = e^{2u-v},\ y = e^{u+v}$

26. $x = ue^v,\ y = e^{-v}$

Answer

-1

27. $x = u\ \cos(e^v),\ y = u\ \sin(e^v)$

28. $x = v\ \sin(u^2),\ y = v\ \cos(u^2)$

Answer

$2uv$

29. $x = u\ \cosh v,\ y = u\ \sinh v,\ z = w$

30. $x = v\ \cosh\left(\frac{1}{u}\right),\ y = v\ \sinh\left(\frac{1}{u}\right),\ z = u + w^2$

Answer

$\frac{v}{u^2}$

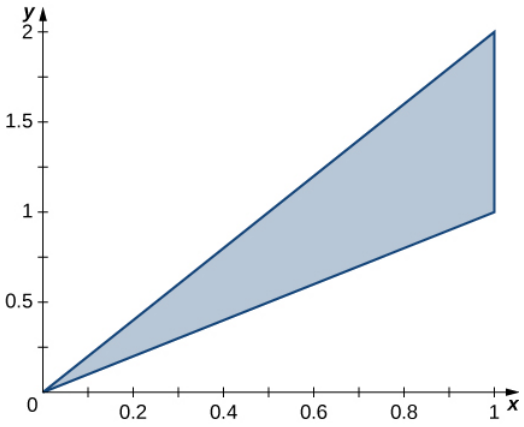
31. $x = u + v,\ y = v + w,\ z = u$

32. $x = u - v,\ y = u + v,\ z = u + v + w$

Answer

2

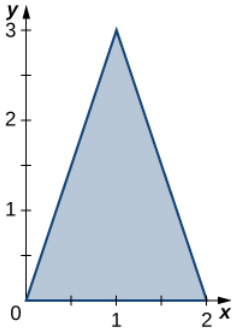
33. The triangular region R with the vertices $(0,0),\ (1,1)$, and $(1,2)$ is shown in the following figure.



a. Find a transformation $T : S \to R,\ T(u,v) = (x,y) = (au + bv + dv)$, where a, b, c, and d are real numbers with $ad - bc \neq 0$ such that $T^{-1}(0,0) = (0,0),\ T^{-1}(1,1) = (1,0)$, and $T^{-1}(1,2) = (0,1)$.

b. Use the transformation T to find the area $A(R)$ of the region R.

34. The triangular region R with the vertices $(0,0),\ (2,0)$, and $(1,3)$ is shown in the following figure.



a. Find a transformation $T : S \to R,\ T(u,v) = (x,y) = (au + bv + dv)$, where a, b, c, and d are real numbers with $ad - bc \neq 0$ such that $T^{-1}(0,0) = (0,0),\ T^{-1}(2,0) = (1,0)$, and $T^{-1}(1,3) = (0,1)$.

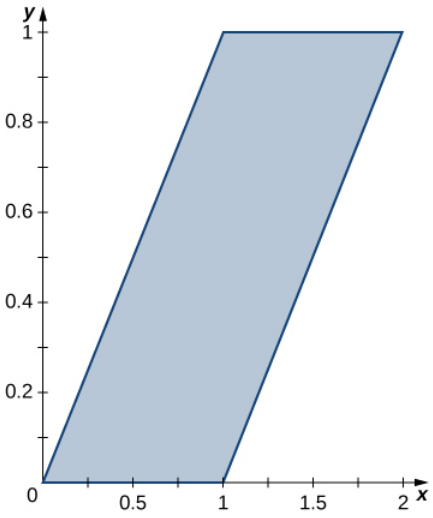
b. Use the transformation T to find the area $A(R)$ of the region R.

Answer

a. $T(u,v) = (2u + v,\ 3v)$

b. The area of R is $A(R) = \int_0^3 \int_{y/3}^{(6-y)/3} dx\,dy = \int_0^1 \int_0^{1-u} \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dv\,du = \int_0^1 \int_0^{1-u} 6\,dv\,du = 3.$

In exercises 35 - 36, use the transformation $u = y - x$, $v = y$, to evaluate the integrals on the parallelogram R of vertices $(0,0)$, $(1,0)$, $(2,1)$, and $(1,1)$ shown in the following figure.



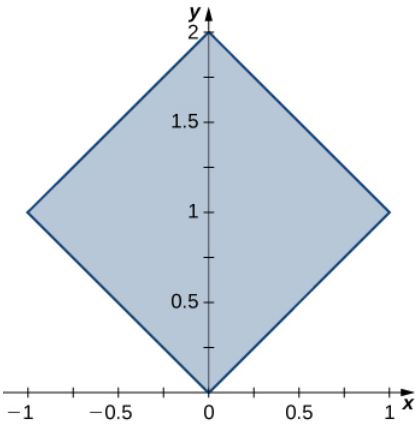
35. $\iint_R (y-x)\,dA$

36. $\iint_R (y^2-xy)\,dA$

Answer

$-\frac{1}{4}$

In exercises 37 - 38, use the transformation $y=x=u$, $x+y=v$ to evaluate the integrals on the square R determined by the lines $y=x$, $y=-x+2$, $y=x+2$, and $y=-x$ shown in the following figure.



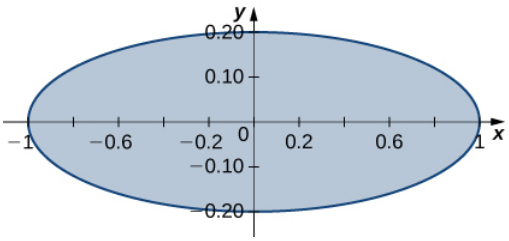
37. $\iint_R e^{x+y}\,dA$

38. $\iint_R \sin(x-y)\,dA$

Answer

$-1+\cos 2$

In exercises 39 - 40, use the transformation $x=u$, $5y=v$ to evaluate the integrals on the region R bounded by the ellipse $x^2+25y^2=1$ shown in the following figure.



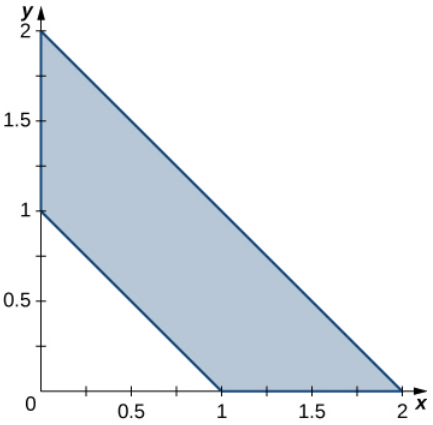
39. $\iint_R \sqrt{x^2+25y^2}\,dA$

40. $\iint_R (x^2+25y^2)^2\,dA$

Answer

$\frac{\pi}{15}$

In exercises 41 - 42, use the transformation $u=x+y$, $v=x-y$ to evaluate the integrals on the trapezoidal region R determined by the points $(1,0)$, $(2,0)$, $(0,2)$, and $(0,1)$ shown in the following figure.



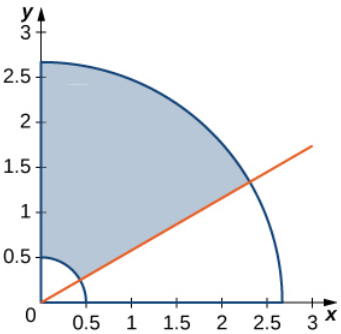
41. $\iint_R (x^2-2xy+y^2)\,e^{x+y}\,dA$

42. $\iint_R (x^3+3x^2y+3xy^2+y^3)\,dA$

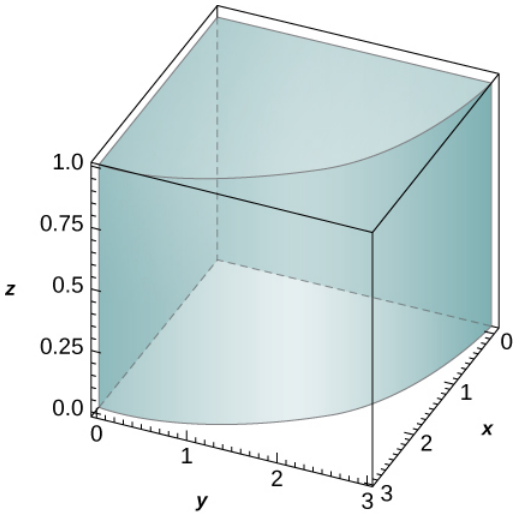
Answer

$\frac{31}{5}$

43. The circular annulus sector R bounded by the circles $4x^2+4y^2=1$ and $9x^2+9y^2=64$, the line $x=y\sqrt{3}$, and the y-axis is shown in the following figure. Find a transformation T from a rectangular region S in the $r\theta$-plane to the region R in the xy-plane. Graph S.



44. The solid R bounded by the circular cylinder $x^2 + y^2 = 9$ and the planes $z = 0,\ z = 1,\ x = 0$, and $y = 0$ is shown in the following figure. Find a transformation T from a cylindrical box S in $r\theta z$-space to the solid R in xyz-space.



Answer

$T(r, \theta, z) = (r\ \cos\theta,\ r\ \sin\theta,\ z);\ S = [0, 3] \times [0, \frac{\pi}{2}] \times [0, 1]$ in the $r\theta z$-space

45. Show that

$$\iint_R f\left(\sqrt{\frac{x^2}{3} + \frac{y^2}{3}}\right) dA = 2\pi\sqrt{15} \int_0^1 f(\rho)\rho\ d\rho,$$

where f is a continuous function on $[0, 1]$ and R is the region bounded by the ellipse $5x^2 + 3y^2 = 15$.

46. Show that

$$\iiint_R f\left(\sqrt{16x^2 + 4y^2 + z^2}\right) dV = \frac{\pi}{2} \int_0^1 f(\rho)\rho^2 d\rho,$$

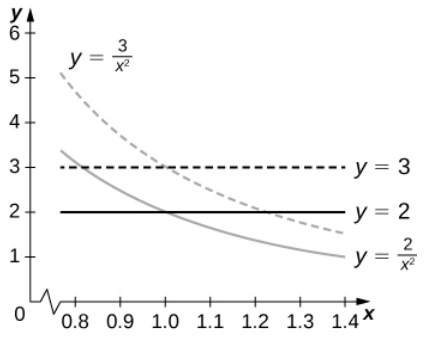
where f is a continuous function on $[0, 1]$ and R is the region bounded by the ellipsoid $16x^2 + 4y^2 + z^2 = 1$.

47. [T] Find the area of the region bounded by the curves $xy = 1,\ xy = 3,\ y = 2x$, and $y = 3x$ by using the transformation $u = xy$ and $v = \frac{y}{x}$. Use a computer algebra system (CAS) to graph the boundary curves of the region R.

48. [T] Find the area of the region bounded by the curves $x^2y = 2,\ x^2y = 3,\ y = x$, and $y = 2x$ by using the transformation $u = x^2y$ and $v = \frac{y}{x}$. Use a CAS to graph the boundary curves of the region R.

Answer

The area of R is $10 - 4\sqrt{6}$; the boundary curves of R are graphed in the following figure.



49. Evaluate the triple integral

$$\int_0^1 \int_1^2 \int_z^{z+1} (y+1)\, dx\, dy\, dz$$

by using the transformation $u = x - z,\ v = 3y$, and $w = \frac{z}{2}$.

50. Evaluate the triple integral

$$\int_0^2 \int_4^6 \int_{3z}^{3z+2} (5 - 4y)\, dx\, dy\, dz$$

by using the transformation $u = x - 3z,\ v = 4y$, and $w = z$.

Answer

8

51. A transformation $T : R^2 \to R^2,\ T(u,v) = (x,y)$ of the form $x = au + bv,\ y = cu + dv$, where a, b, c, and d are real numbers, is called linear. Show that a linear transformation for which $ad - bc \neq 0$ maps parallelograms to parallelograms.

52. A transformation $T_\theta : R^2 \to R^2,\ T_\theta(u,v) = (x,y)$ of the form $x = u\ \cos\theta - v\ \sin\theta,\ y = u\ \sin\theta + v\ \cos\theta$, is called a rotation angle θ. Show that the inverse transformation of T_θ satisfies $T_\theta^{-1} = T_{-\theta}$ where $T_{-\theta}$ is the rotation of angle $-\theta$.

53. [T] Find the region S in the uv-plane whose image through a rotation of angle $\frac{\pi}{4}$ is the region R enclosed by the ellipse $x^2 + 4y^2 = 1$. Use a CAS to answer the following questions.

a. Graph the region S.

b. Evaluate the integral $\iint_S e^{-2uv}\, du\, dv$. Round your answer to two decimal places.

54. [T] The transformations $T_i : \mathbb{R}^2 \to \mathbb{R}^2,\ i = 1, ..., 4$, defined by $T_1(u,v) = (u,-v),\ T_2(u,v) = (-u,v),\ T_3(u,v) = (-u,-v)$, and $T_4(u,v) = (v,u)$ are called reflections about the x-axis, y-axis origin, and the line $y = x$, respectively.

a. Find the image of the region $S = \{(u,v) \mid u^2 + v^2 - 2u - 4v + 1 \leq 0\}$ in the xy-plane through the transformation $T_1 \circ T_2 \circ T_3 \circ T_4$.

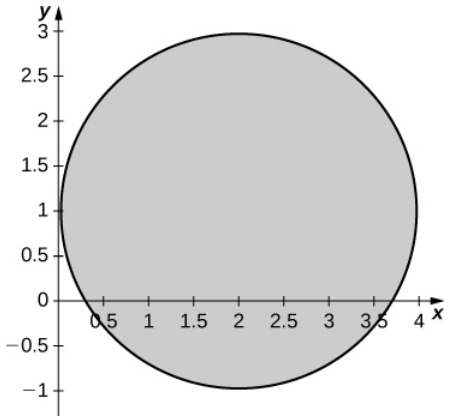
b. Use a CAS to graph R.

c. Evaluate the integral $\iint_S \sin(u^2)\, du\, dv$ by using a CAS. Round your answer to two decimal places.

Answer

a. $R = \{(x,y) \mid y^2 + x^2 - 2y - 4x + 1 \leq 0\}$;

b. R is graphed in the following figure;



c. 3.16

55. [T] The transformations $T_{k,1,1} : \mathbb{R}^3 \to \mathbb{R}^3,\ T_{k,1,1}(u,v,w) = (x,y,z)$ of the form $x = ku,\ y = v,\ z = w$, where $k \neq 1$ is a positive real number, is called a stretch if $k > 1$ and a compression if $0 < k < 1$ in the x-direction. Use a CAS to evaluate the integral

$\iiint_S e^{-(4x^2+9y^2+25z^2)}\,dx\,dy\,dz$ on the solid $S = \{(x, y, z) \mid 4x^2 + 9y^2 + 25z^2 \leq 1\}$ by considering the compression $T_{2,3,5}(u, v, w) = (x, y, z)$ defined by $x = \frac{u}{2},\ y = \frac{v}{3}$, and $z = \frac{w}{5}$. Round your answer to four decimal places.

56. [T] The transformation $T_{a,0} : \mathbb{R}^2 \to \mathbb{R}^2,\ T_{a,0}(u, v) = (u + av, v)$, where $a \neq 0$ is a real number, is called a shear in the x-direction. The transformation, $T_{b,0} : R^2 \to R^2,\ T_{o,b}(u, v) = (u, bu + v)$, where $b \neq 0$ is a real number, is called a shear in the y-direction.

a. Find transformations $T_{0,2} \circ T_{3,0}$.

b. Find the image R of the trapezoidal region S bounded by $u = 0,\ v = 0,\ v = 1$, and $v = 2 - u$ through the transformation $T_{0,2} \circ T_{3,0}$.

c. Use a CAS to graph the image R in the xy-plane.

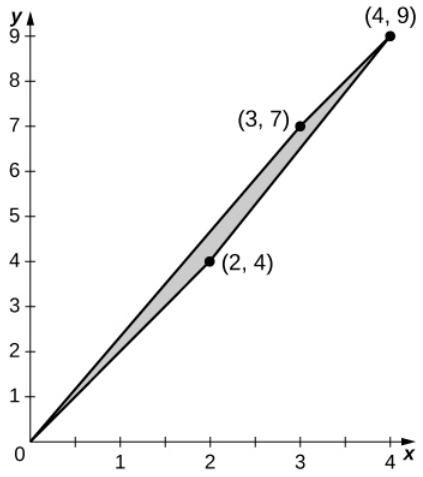
d. Find the area of the region R by using the area of region S.

Answer

a. $T_{0,2} \circ T_{3,0}(u, v) = (u + 3v, 2u + 7v)$;

b. The image S is the quadrilateral of vertices $(0, 0),\ (3, 7),\ (2, 4)$, and $(4, 9)$;

c. S is graphed in the following figure;



d. $\frac{3}{2}$

57. Use the transformation, $x = au,\ y = av,\ z = cw$ and spherical coordinates to show that the volume of a region bounded by the spheroid $\frac{x^2+y^2}{a^2} + \frac{z^2}{c^2} = 1$ is $\frac{4\pi a^2 c}{3}$.

58. Find the volume of a football whose shape is a spheroid $\frac{x^2+y^2}{a^2} + \frac{z^2}{c^2} = 1$ whose length from tip to tip is 11 inches and circumference at the center is 22 inches. Round your answer to two decimal places.

Answer

$\frac{2662}{3\pi} \approx 282.45\ in^3$

Contributors

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2.9: Chapter 2 Review Exercises

***True or False?* Justify your answer with a proof or a counterexample.**

1. $\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dy\,dx$

2. Fubini's theorem can be extended to three dimensions, as long as f is continuous in all variables.

Answer

True

3. The integral $\int_0^{2\pi} \int_0^1 \int_r^1 dz\,dr\,d\theta$ represents the volume of a right cone.

4. The Jacobian of the transformation for $x = u^2 - 2v,\ y = 3v - 2uv$ is given by $-4u^2 + 6u + 4v$.

Answer

False

Evaluate the following integrals.

5. $\iint_R (5x^3y^2 - y^2)\,dA$, where $R = \{(x,y) \mid 0 \le x \le 2,\ 1 \le y \le 4\}$

6. $\iint_D \frac{y}{3x^2+1}\,dA$, where $D = \{(x,y) \mid 0 \le x \le 1,\ -x \le y \le x\}$

Answer

0

7. $\iint_D \sin(x^2+y^2)\,dA$ where D is a disk of radius 2 centered at the origin.

8. $\int_0^1 \int_y^1 xye^{x^2}\,dx\,dy$

Answer

$\frac{1}{4}$

9. $\int_{-1}^1 \int_0^z \int_0^{x-z} 6\,dy\,dx\,dz$

10. $\iiint_R 3y\,dV$, where $R = \{(x,y,z) \mid 0 \le x \le 1,\ 0 \le y \le x,\ 0 \le z \le 9 - y^2\}$

Answer

1.475

11. $\int_0^2 \int_0^{2\pi} \int_r^1 r\,dz\,d\theta\,dr$

12. $\int_0^{2\pi} \int_0^{\pi/2} \int_1^3 \rho^2 \sin(\varphi)\,d\rho\,d\varphi\,d\theta$

Answer

$\frac{52\pi}{3}$

13. $\int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} dz\,dy\,dx$

For the following problems, find the specified area or volume.

14. The area of region enclosed by one petal of $r = \cos(4\theta)$.

Answer

$\frac{\pi}{16}$ units^3

15. The volume of the solid that lies between the paraboloid $z = 2x^2 + 2y^2$ and the plane $z = 8$.

16. The volume of the solid bounded by the cylinder $x^2 + y^2 = 16$ and from $z = 1$ to $z + x = 2$.

Answer

93.291 units^3

17. The volume of the intersection between two spheres of radius 1, the top whose center is $(0, 0, 0.25)$ and the bottom, which is centered at $(0, 0, 0)$.

For the following problems, find the center of mass of the region.

18. $\rho(x, y) = xy$ on the circle with radius 1 in the first quadrant only.

Answer

$(\frac{8}{15}, \frac{8}{15})$

19. $\rho(x, y) = (y + 1)\sqrt{x}$ in the region bounded by $y = e^x$, $y = 0$, and $x = 1$.

20. $\rho(x, y, z) = z$ on the inverted cone with radius 2 and height 2.

Answer

$(0, 0, \frac{8}{5})$

21. The volume an ice cream cone that is given by the solid above $z = \sqrt{x^2 + y^2}$ and below $z^2 + x^2 + y^2 = z$.

The following problems examine Mount Holly in the state of Michigan. Mount Holly is a landfill that was converted into a ski resort. The shape of Mount Holly can be approximated by a right circular cone of height 1100 ft and radius 6000 ft.

22. If the compacted trash used to build Mount Holly on average has a density 400 lb/ft^3, find the amount of work required to build the mountain.

Answer

$1.452\pi \times 10^{15}$ ft-lb

23. In reality, it is very likely that the trash at the bottom of Mount Holly has become more compacted with all the weight of the above trash. Consider a density function with respect to height: the density at the top of the mountain is still density 400 lb/ft^3, and the density increases. Every 100 feet deeper, the density doubles. What is the total weight of Mount Holly?

The following problems consider the temperature and density of Earth's layers.

24. **[T]** The temperature of Earth's layers is exhibited in the table below. Use your calculator to fit a polynomial of degree 3 to the temperature along the radius of the Earth. Then find the average temperature of Earth. (*Hint*: begin at 0 in the inner core and increase outward toward the surface)

Source: http://www.enchantedlearning.com/sub...h/Inside.shtml

Layer	Depth from center (km)	Temperature °C
Rocky Crust	0 to 40	0
Upper Mantle	40 to 150	870
Mantle	400 to 650	870
Inner Mantel	650 to 2700	870
Molten Outer Core	2890 to 5150	4300
Inner Core	5150 to 6378	7200

Answer

$y = -1.238 \times 10^{-7}x^3 + 0.001196x^2 - 3.666x + 7208$;
The average temperature is approximately 2800 °C.

25. **[T]** The density of Earth's layers is displayed in the table below. Using your calculator or a computer program, find the best-fit quadratic equation to the density. Using this equation, find the total mass of Earth.

Source: http://hyperphysics.phy-astr.gsu.edu...rthstruct.html

Layer	Depth from center (km)	Density ($\mathrm{g/cm^3}$)
Inner Core	0	12.95
Outer Core	1228	11.05
Mantle	3488	5.00
Upper Mantle	6338	3.90
Crust	6378	2.55

The following problems concern the Theorem of Pappus (see Moments and Centers of Mass for a refresher), a method for calculating volume using centroids. Assuming a region R, when you revolve around the x-axis the volume is given by $V_x = 2\pi A\bar{y}$, and when you revolve around the y-axis the volume is given by $V_y = 2\pi A\bar{x}$, where A is the area of R. Consider the region bounded by $x^2 + y^2 = 1$ and above $y = x + 1$.

26. Find the volume when you revolve the region around the x-axis.

Answer

$\frac{\pi}{3}\ \mathrm{units}^3$

27. Find the volume when you revolve the region around the y-axis.

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CHAPTER OVERVIEW

3: Vector Calculus

In this chapter, we learn to model new kinds of integrals over fields such as magnetic fields, gravitational fields, or velocity fields. We also learn how to calculate the work done on a charged particle traveling through a magnetic field, the work done on a particle with mass traveling through a gravitational field, and the volume per unit time of water flowing through a net dropped in a river. All these applications are based on the concept of a vector field. The content in this Textmap's chapter is complemented by Vector Calculus Modules in the Core and the Vector Calculus (UCD Mat 21D) Libretext.

3.1: Prelude to Vector Calculus
3.2: Vector Fields
3.2E: Exercises
3.3: Line Integrals
3.3E: Exercises
3.4: Conservative Vector Fields
3.4E: Exercises
3.5: Green's Theorem
3.5E: Exercises
3.6: Divergence and Curl
3.6E: Exercises
3.7: Surface Integrals
3.7E: Exercises
3.8: Stokes' Theorem
3.8E: Exercises
3.9: The Divergence Theorem
3.9E: Exercises
3.10: Chapter 3 Review Exercises

Thumbnail: Surface Σ with closed boundary $\partial\Sigma$. $\vec{F}$ could be the $\vec{E}$ or $\vec{B}$ fields. n is the unit normal. (Public Domain; Maschen).

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3.1: Prelude to Vector Calculus

Hurricanes are huge storms that can produce tremendous amounts of damage to life and property, especially when they reach land. Predicting where and when they will strike and how strong the winds will be is of great importance for preparing for protection or evacuation. Scientists rely on studies of rotational vector fields for their forecasts.



Figure **3.1.1**: Hurricanes form from rotating winds driven by warm temperatures over the ocean. Meteorologists forecast the motion of hurricanes by studying the rotating vector fields of their wind velocity. Shown is Cyclone Catarina in the South Atlantic Ocean in 2004, as seen from the International Space Station. (credit: modification of work by NASA)

In this chapter, we learn to model new kinds of integrals over fields such as magnetic fields, gravitational fields, or velocity fields. We also learn how to calculate the work done on a charged particle traveling through a magnetic field, the work done on a particle with mass traveling through a gravitational field, and the volume per unit time of water flowing through a net dropped in a river.

All these applications are based on the concept of a vector field, which we explore in this chapter. Vector fields have many applications because they can be used to model real fields such as electromagnetic or gravitational fields. A deep understanding of physics or engineering is impossible without an understanding of vector fields. Furthermore, vector fields have mathematical properties that are worthy of study in their own right. In particular, vector fields can be used to develop several higher-dimensional versions of the Fundamental Theorem of Calculus.

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3.2: Vector Fields

Learning Objectives

- Recognize a vector field in a plane or in space.
- Sketch a vector field from a given equation.
- Identify a conservative field and its associated potential function.

Vector fields are an important tool for describing many physical concepts, such as gravitation and electromagnetism, which affect the behavior of objects over a large region of a plane or of space. They are also useful for dealing with large-scale behavior such as atmospheric storms or deep-sea ocean currents. In this section, we examine the basic definitions and graphs of vector fields so we can study them in more detail in the rest of this chapter.

Examples of Vector Fields

How can we model the gravitational force exerted by multiple astronomical objects? How can we model the velocity of water particles on the surface of a river? Figure **3.2.1** gives visual representations of such phenomena.

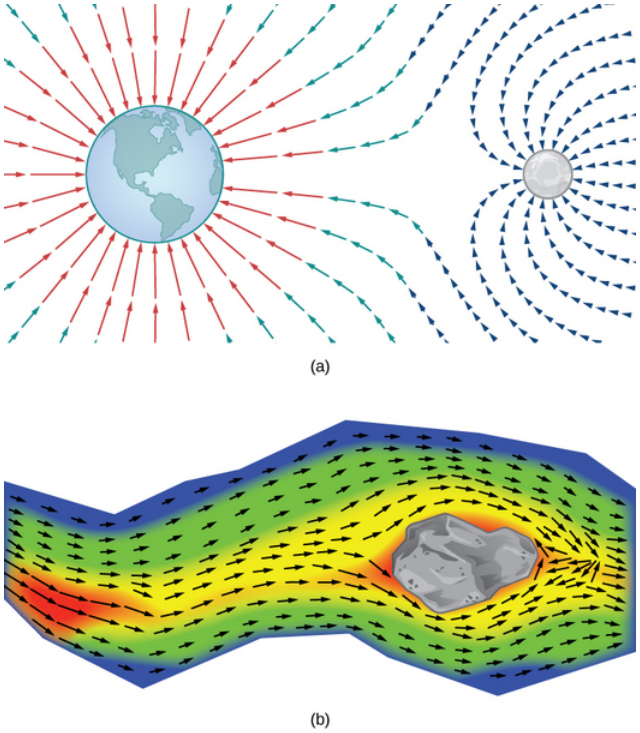


Figure **3.2.1** (a) The gravitational field exerted by two astronomical bodies on a small object. (b) The vector velocity field of water on the surface of a river shows the varied speeds of water. Red indicates that the magnitude of the vector is greater, so the water flows more quickly; blue indicates a lesser magnitude and a slower speed of water flow.

Figure **3.2.1*a*** shows a gravitational field exerted by two astronomical objects, such as a star and a planet or a planet and a moon. At any point in the figure, the vector associated with a point gives the net gravitational force exerted by the two objects on an object of unit mass. The vectors of largest magnitude in the figure are the vectors closest to the larger object. The larger object has greater mass, so it exerts a gravitational force of greater magnitude than the smaller object.

Figure **3.2.1*b*** shows the velocity of a river at points on its surface. The vector associated with a given point on the river's surface gives the velocity of the water at that point. Since the vectors to the left of the figure are small in magnitude, the water is flowing slowly on that part of the surface. As the water moves from left to right, it encounters some rapids around a rock. The speed of the water increases, and a whirlpool occurs in part of the rapids.

Each figure illustrates an example of a vector field. Intuitively, a vector field is a map of vectors. In this section, we study vector fields in $\mathbb{R}^2$ and $\mathbb{R}^3$.

DEFINITION: vector field

- A vector field $\vec{\mathbf{F}}$ in $\mathbb{R}^2$ is an assignment of a two-dimensional vector $\vec{\mathbf{F}}(x,y)$ to each point (x,y) of a subset D of $\mathbb{R}^2$. The subset D is the domain of the vector field.
- A vector field $\vec{\mathbf{F}}$ in $\mathbb{R}^3$ is an assignment of a three-dimensional vector $\vec{\mathbf{F}}(x,y,z)$ to each point (x,y,z) of a subset D of $\mathbb{R}^3$. The subset D is the domain of the vector field.

Vector Fields in $\mathbb{R}^2$

A vector field in $\mathbb{R}^2$ can be represented in either of two equivalent ways. The first way is to use a vector with components that are two-variable functions:

$$\vec{\mathbf{F}}(x,y) = \langle P(x,y), Q(x,y) \rangle \tag{3.2.1}$$

The second way is to use the standard unit vectors:

$$\vec{\mathbf{F}}(x,y) = P(x,y)\,\hat{\mathbf{i}} + Q(x,y)\,\hat{\mathbf{j}}. \tag{3.2.2}$$

A vector field is said to be *continuous* if its component functions are continuous.

Example 3.2.1: Finding a Vector Associated with a Given Point

Let $\vec{\mathbf{F}}(x,y) = (2y^2 + x - 4)\,\hat{\mathbf{i}} + \cos(x)\,\hat{\mathbf{j}}$ be a vector field in $\mathbb{R}^2$. Note that this is an example of a continuous vector field since both component functions are continuous. What vector is associated with point $(0,-1)$?

Solution

Substitute the point values for x and y:

$$\begin{aligned}\vec{\mathbf{F}}(0,-1) &= (2(-1)^2 + 0 - 4)\,\hat{\mathbf{i}} + \cos(0)\,\hat{\mathbf{j}} \\ &= -2\,\hat{\mathbf{i}} + \hat{\mathbf{j}}.\end{aligned}$$

Exercise 3.2.1

Let $\vec{\mathbf{G}}(x,y) = x^2 y\,\hat{\mathbf{i}} - (x+y)\,\hat{\mathbf{j}}$ be a vector field in $\mathbb{R}^2$. What vector is associated with the point $(-2,3)$?

Hint

Substitute the point values into the vector function.

Answer

$\vec{\mathbf{G}}(-2,3) = 12\hat{\mathbf{i}} - \hat{\mathbf{j}}$

Drawing a Vector Field

We can now represent a vector field in terms of its components of functions or unit vectors, but representing it visually by sketching it is more complex because the domain of a vector field is in $\mathbb{R}^2$, as is the range. Therefore the "graph" of a vector field in $\mathbb{R}^2$ lives in four-dimensional space. Since we cannot represent four-dimensional space visually, we instead draw vector fields in $\mathbb{R}^2$ in a plane itself. To do this, draw the vector associated with a given point at the point in a plane. For example, suppose the vector associated with point $(4,-1)$ is $\langle 3,1 \rangle$. Then, we would draw vector $\langle 3,1 \rangle$ at point $(4,-1)$.

We should plot enough vectors to see the general shape, but not so many that the sketch becomes a jumbled mess. If we were to plot the image vector at each point in the region, it would fill the region completely and is useless. Instead, we can choose points at the intersections of grid lines and plot a sample of several vectors from each quadrant of a rectangular coordinate system in $\mathbb{R}^2$.

There are two types of vector fields in $\mathbb{R}^2$ on which this chapter focuses: radial fields and rotational fields. Radial fields model certain gravitational fields and energy source fields, and rotational fields model the movement of a fluid in a vortex. In a radial field, all vectors either point directly toward or directly away from the origin. Furthermore, the magnitude of any vector depends only on its distance from the origin. In a radial field, the vector located at point (x, y) is perpendicular to the circle centered at the origin that contains point (x, y), and all other vectors on this circle have the same magnitude.

Example 3.2.2: Drawing a Radial Vector Field

Sketch the vector field $\vec{\mathbf{F}}(x, y) = \frac{x}{2}\hat{\mathbf{i}} + \frac{y}{2}\hat{\mathbf{j}}$.

Solution

To sketch this vector field, choose a sample of points from each quadrant and compute the corresponding vector. The following table gives a representative sample of points in a plane and the corresponding vectors.

Table 3.2.1

(x, y)	$\vec{\mathbf{F}}(x, y)$	(x, y)	$\vec{\mathbf{F}}(x, y)$	(x, y)	$\vec{\mathbf{F}}(x, y)$
$(1, 0)$	$\langle \frac{1}{2}, 0 \rangle$	$(2, 0)$	$\langle 1, 0 \rangle$	$(1, 1)$	$\langle \frac{1}{2}, \frac{1}{2} \rangle$
$(0, 1)$	$\langle 0, \frac{1}{2} \rangle$	$(0, 2)$	$\langle 0, 1 \rangle$	$(-1, 1)$	$\langle -\frac{1}{2}, \frac{1}{2} \rangle$
$(-1, 0)$	$\langle -\frac{1}{2}, 0 \rangle$	$(-2, 0)$	$\langle -1, 0 \rangle$	$(-1, -1)$	$\langle -\frac{1}{2}, -\frac{1}{2} \rangle$
$(0, -1)$	$\langle 0, -\frac{1}{2} \rangle$	$(0, -2)$	$\langle 0, -1 \rangle$	$(1, -1)$	$\langle \frac{1}{2}, -\frac{1}{2} \rangle$

Figure 3.2.2*a* shows the vector field. To see that each vector is perpendicular to the corresponding circle, Figure 3.2.2*b* shows circles overlain on the vector field.

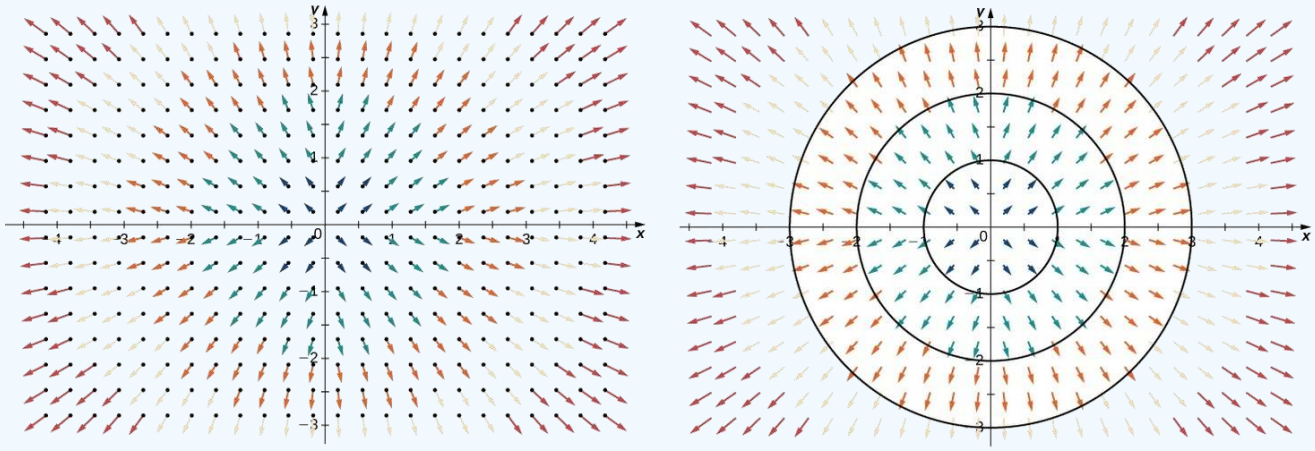


Figure 3.2.2: (a) A visual representation of the radial vector field $\vec{\mathbf{F}}(x, y) = \frac{x}{2}\hat{\mathbf{i}} + \frac{y}{2}\hat{\mathbf{j}}$. (b) The radial vector field $\vec{F}(x, y) = \frac{x}{2}\hat{\mathbf{i}} + \frac{y}{2}\hat{\mathbf{j}}$ with overlaid circles. Notice that each vector is perpendicular to the circle on which it is located.



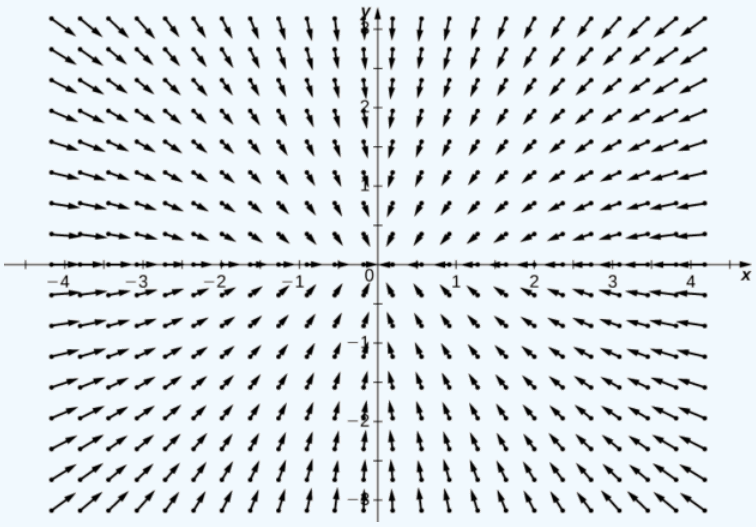
Exercise 3.2.2

Draw the radial field $\vec{\mathbf{F}}(x,y) = -\frac{x}{3}\hat{\mathbf{i}} - \frac{y}{3}\hat{\mathbf{j}}$.

Hint

Sketch enough vectors to get an idea of the shape.

Answer



In contrast to radial fields, in a *rotational field*, the vector at point (x,y) is tangent (not perpendicular) to a circle with radius $r = \sqrt{x^2+y^2}$. In a standard rotational field, all vectors point either in a clockwise direction or in a counterclockwise direction, and the magnitude of a vector depends only on its distance from the origin. Both of the following examples are clockwise rotational fields, and we see from their visual representations that the vectors appear to rotate around the origin.

Example 3.2.3: Drawing a Rotational Vector Field

Sketch the vector field $\vec{\mathbf{F}}(x, y) = \langle y, -x \rangle$.

Solution

Create a table (see the one that follows) using a representative sample of points in a plane and their corresponding vectors. Figure 3.2.3 shows the resulting vector field.

Table 3.2.2

(x, y)	$\vec{\mathbf{F}}(x, y)$	(x, y)	$\vec{\mathbf{F}}(x, y)$	(x, y)	$\vec{\mathbf{F}}(x, y)$
$(1, 0)$	$\langle 0, -1 \rangle$	$(2, 0)$	$\langle 0, -2 \rangle$	$(1, 1)$	$\langle 1, -1 \rangle$
$(0, 1)$	$\langle 1, 0 \rangle$	$(0, 2)$	$\langle 2, 0 \rangle$	$(-1, 1)$	$\langle 1, 1 \rangle$
$(-1, 0)$	$\langle 0, 1 \rangle$	$(-2, 0)$	$\langle 0, 2 \rangle$	$(-1, -1)$	$\langle -1, 1 \rangle$
$(0, -1)$	$\langle -1, 0 \rangle$	$(0, -2)$	$\langle -2, 0 \rangle$	$(1, -1)$	$\langle -1, -1 \rangle$

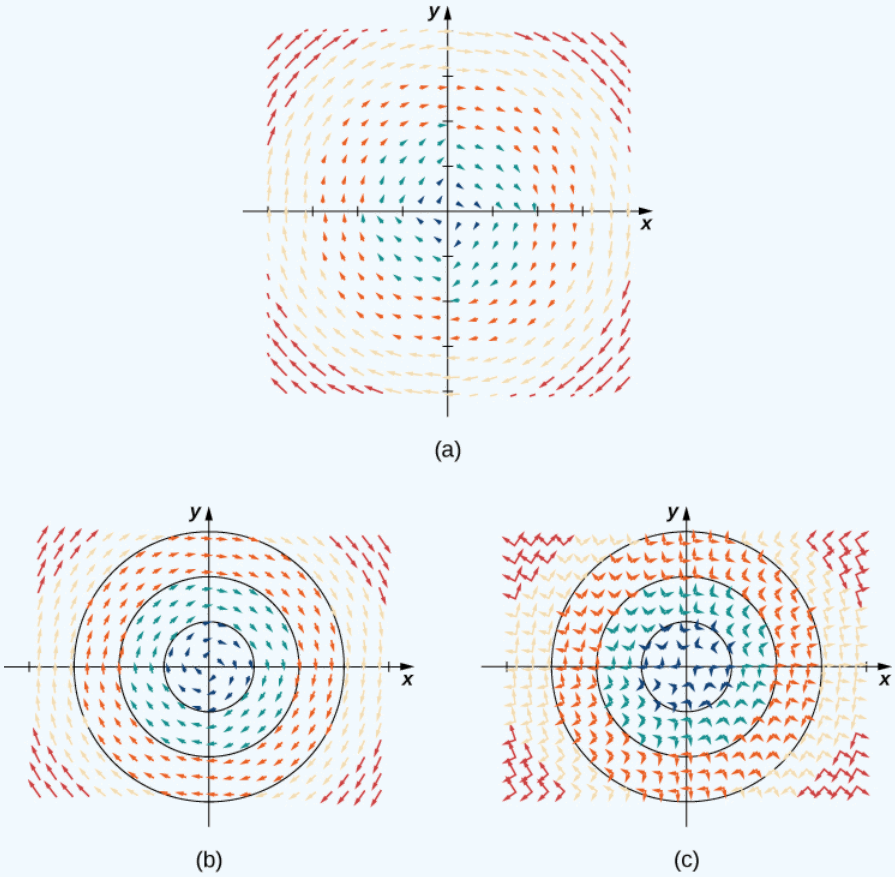


Figure 3.2.3: (a) A visual representation of vector field $\vec{\mathbf{F}}(x, y) = \langle y, -x \rangle$. (b) Vector field $\vec{\mathbf{F}}(x, y) = \langle y, -x \rangle$ with circles centered at the origin. (c) Vector $\vec{\mathbf{F}}(a, b)$ is perpendicular to radial vector $\langle a, b \rangle$ at point (a, b).

Analysis

Note that vector $\vec{\mathbf{F}}(a, b) = \langle b, -a \rangle$ points clockwise and is perpendicular to radial vector $\langle a, b \rangle$. (We can verify this assertion by computing the dot product of the two vectors: $\langle a, b \rangle \cdot \langle -b, a \rangle = -ab + ab = 0$.) Furthermore, vector $\langle b, -a \rangle$ has length $r = \sqrt{a^2 + b^2}$. Thus, we have a complete description of this rotational vector field: the vector associated with point (a, b) is the vector with length r tangent to the circle with radius r, and it points in the clockwise direction.

Sketches such as that in Figure **3.2.3** are often used to analyze major storm systems, including hurricanes and cyclones. In the northern hemisphere, storms rotate counterclockwise; in the southern hemisphere, storms rotate clockwise. (This is an effect caused by Earth's rotation about its axis and is called the Coriolis Effect.)



Figure **3.2.4**: (credit: modification of work by NASA)

Example 3.2.4: Sketching a Vector Field

Sketch vector field $\vec{\mathbf{F}}(x,y) = \frac{y}{x^2+y^2}\hat{\mathbf{i}}, -\frac{x}{x^2+y^2}\hat{\mathbf{j}}$.

Solution

To visualize this vector field, first note that the dot product $\vec{\mathbf{F}}(a,b) \cdot (a\,\hat{\mathbf{i}} + b\,\hat{\mathbf{j}})$ is zero for any point (a,b). Therefore, each vector is tangent to the circle on which it is located. Also, as $(a,b) \to (0,0)$, the magnitude of $\vec{\mathbf{F}}(a,b)$ goes to infinity. To see this, note that

$$||\vec{\mathbf{F}}(a,b)|| = \sqrt{\frac{a^2+b^2}{(a^2+b^2)^2}} = \sqrt{\frac{1}{a^2+b^2}}.$$

Since $\frac{1}{a^2+b^2} \to \infty$ as $(a,b) \to (0,0)$, then $||\vec{\mathbf{F}}(a,b)|| \to \infty$ as $(a,b) \to (0,0)$. This vector field looks similar to the vector field in Example **3.2.3**, but in this case the magnitudes of the vectors close to the origin are large. Table **3.2.3** shows a sample of points and the corresponding vectors, and Figure **3.2.5** shows the vector field. Note that this vector field models the whirlpool motion of the river in Figure **3.2.5**(b). The domain of this vector field is all of $\mathbb{R}^2$ except for point $(0,0)$.

Table **3.2.3**

(x,y)	$\vec{\mathbf{F}}(x,y)$	(x,y)	$\vec{\mathbf{F}}(x,y)$	(x,y)	$\vec{\mathbf{F}}(x,y)$
$(1,0)$	$\langle 0,-1\rangle$	$(2,0)$	$\langle 0,-\frac{1}{2}\rangle$	$(1,1)$	$\langle \frac{1}{2},-\frac{1}{2}\rangle$
$(0,1)$	$\langle 1,0\rangle$	$(0,2)$	$\langle \frac{1}{2},0\rangle$	$(-1,1)$	$\langle \frac{1}{2},\frac{1}{2}\rangle$
$(-1,0)$	$\langle 0,1\rangle$	$(-2,0)$	$\langle 0,\frac{1}{2}\rangle$	$(-1,-1)$	$\langle -\frac{1}{2},\frac{1}{2}\rangle$
$(0,-1)$	$\langle -1,0\rangle$	$(0,-2)$	$\langle -\frac{1}{2},0\rangle$	$(1,-1)$	$\langle -\frac{1}{2},-\frac{1}{2}\rangle$

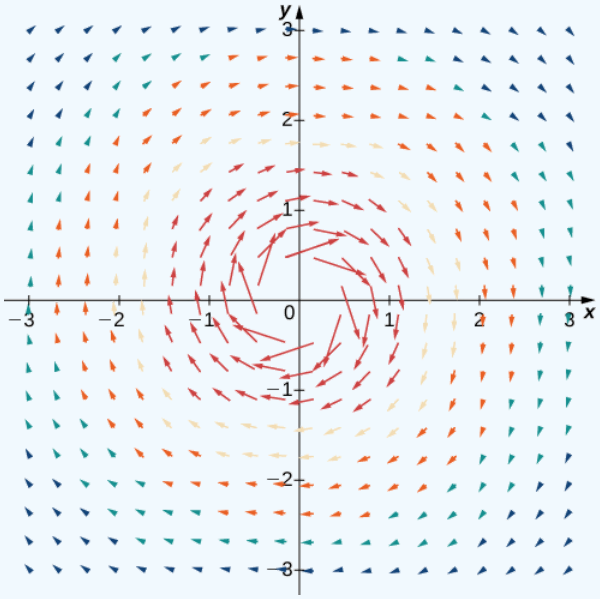


Figure 3.2.5: A visual representation of vector field $\vec{\mathbf{F}}(x,y) = \frac{y}{x^2+y^2}\hat{\mathbf{i}} - \frac{x}{x^2+y^2}\hat{\mathbf{j}}$. *This vector field could be used to model whirlpool motion of a fluid.*

Exercise 3.2.4

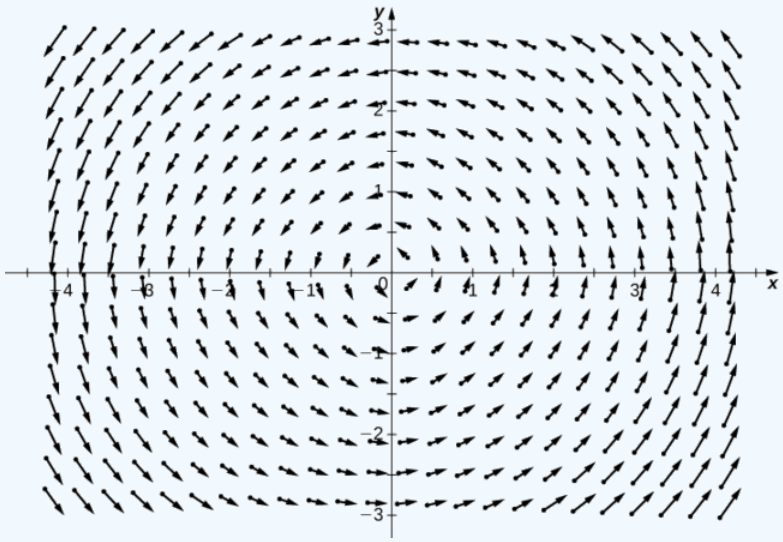
Sketch vector field $\vec{\mathbf{F}}(x,y) = \langle -2y, 2x \rangle$. Is the vector field radial, rotational, or neither?

Hint

Substitute enough points into $\vec{\mathbf{F}}$ to get an idea of the shape.

Answer

Rotational



Example 3.2.5: Velocity Field of a Fluid

Suppose that $\vec{\mathbf{v}}(x,y) = -\frac{2y}{x^2+y^2}\hat{\mathbf{i}} + \frac{2x}{x^2+y^2}\hat{\mathbf{j}}$ is the velocity field of a fluid. How fast is the fluid moving at point $(1,-1)$? (Assume the units of speed are meters per second.)

Solution

To find the velocity of the fluid at point $(1,-1)$, substitute the point into $\vec{\mathbf{v}}$:

$$\vec{\mathbf{v}}(1,-1) = \frac{-2(-1)}{1+1}\hat{\mathbf{i}} + \frac{2(1)}{1+1}\hat{\mathbf{j}} = \hat{\mathbf{i}} + \hat{\mathbf{j}}.$$

The speed of the fluid at $(1,-1)$ is the magnitude of this vector. Therefore, the speed is $||\hat{\mathbf{i}} + \hat{\mathbf{j}}|| = \sqrt{2}$ m/sec.

? Exercise 3.2.5

Vector field $\vec{\mathbf{v}}(x,y) = \langle 4|x|,\ 1\rangle$ models the velocity of water on the surface of a river. What is the speed of the water at point $(2,3)$? Use meters per second as the units.

Hint

Remember, speed is the magnitude of velocity.

Answer

$\sqrt{65}$ m/sec

We have examined vector fields that contain vectors of various magnitudes, but just as we have unit vectors, we can also have a unit vector field. A vector field $\vec{\mathbf{F}}$ is a **unit vector field** if the magnitude of each vector in the field is 1. In a unit vector field, the only relevant information is the direction of each vector.

✓ Example 3.2.6: A Unit Vector Field

Show that vector field $\vec{\mathbf{F}}(x,y) = \left\langle \frac{y}{\sqrt{x^2+y^2}}, -\frac{x}{\sqrt{x^2+y^2}} \right\rangle$ is a unit vector field.

Solution

To show that $\vec{\mathbf{F}}$ is a unit field, we must show that the magnitude of each vector is 1. Note that

$$\begin{aligned}\sqrt{\left(\frac{y}{\sqrt{x^2+y^2}}\right)^2 + \left(-\frac{x}{\sqrt{x^2+y^2}}\right)^2} &= \sqrt{\frac{y^2}{x^2+y^2} + \frac{x^2}{x^2+y^2}} \\ &= \sqrt{\frac{x^2+y^2}{x^2+y^2}} \\ &= 1\end{aligned}$$

Therefore, $\vec{\mathbf{F}}$ is a unit vector field.

? Exercise 3.2.6

Is vector field $\vec{\mathbf{F}}(x,y) = \langle -y,\ x\rangle$ a unit vector field?

Hint

Calculate the magnitude of $\vec{\mathbf{F}}$ at an arbitrary point (x,y).

Answer

No.

Why are unit vector fields important? Suppose we are studying the flow of a fluid, and we care only about the direction in which the fluid is flowing at a given point. In this case, the speed of the fluid (which is the magnitude of the corresponding velocity vector) is irrelevant, because all we care about is the direction of each vector. Therefore, the unit vector field associated with velocity is the field we would study.

If $\vec{\mathbf{F}} = \langle P, Q, R \rangle$ is a vector field, then the corresponding unit vector field is $\left\langle \frac{P}{\|\vec{\mathbf{F}}\|}, \frac{Q}{\|\vec{\mathbf{F}}\|}, \frac{R}{\|\vec{\mathbf{F}}\|} \right\rangle$. Notice that if $\vec{\mathbf{F}}(x, y) = \langle y, -x \rangle$ is the vector field from Example 3.2.6, then the magnitude of $\vec{\mathbf{F}}$ is $\sqrt{x^2 + y^2}$, and therefore the corresponding unit vector field is the field $\vec{\mathbf{G}}$ from the previous example.

If $\vec{\mathbf{F}}$ is a vector field, then the process of dividing $\vec{\mathbf{F}}$ by its magnitude to form unit vector field $\vec{\mathbf{F}} / \|\vec{\mathbf{F}}\|$ is called normalizing the field $\vec{\mathbf{F}}$.

Vector Fields in $\mathbb{R}^3$

We have seen several examples of vector fields in $\mathbb{R}^2$; let's now turn our attention to vector fields in $\mathbb{R}^3$. These vector fields can be used to model gravitational or electromagnetic fields, and they can also be used to model fluid flow or heat flow in three dimensions. A two-dimensional vector field can really only model the movement of water on a two-dimensional slice of a river (such as the river's surface). Since a river flows through three spatial dimensions, to model the flow of the entire depth of the river, we need a vector field in three dimensions.

The extra dimension of a three-dimensional field can make vector fields in $\mathbb{R}^3$ more difficult to visualize, but the idea is the same. To visualize a vector field in $\mathbb{R}^3$, plot enough vectors to show the overall shape. We can use a similar method to visualizing a vector field in $\mathbb{R}^2$ by choosing points in each octant.

Just as with vector fields in $\mathbb{R}^2$, we can represent vector fields in $\mathbb{R}^3$ with component functions. We simply need an extra component function for the extra dimension. We write either

$$\vec{\mathbf{F}}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle \tag{3.2.3}$$

or

$$\vec{\mathbf{F}}(x, y, z) = P(x, y, z)\hat{\mathbf{i}} + Q(x, y, z)\hat{\mathbf{j}} + R(x, y, z)\hat{\mathbf{k}}. \tag{3.2.4}$$

Example 3.2.7: Sketching a Vector Field in Three Dimensions

Describe vector field $\vec{\mathbf{F}}(x, y, z) = \langle 1, 1, z \rangle$.

Solution

For this vector field, the x- and y-components are constant, so every point in $\mathbb{R}^3$ has an associated vector with x- and y-components equal to one. To visualize $\vec{\mathbf{F}}$, we first consider what the field looks like in the xy-plane. In the xy-plane, $z = 0$. Hence, each point of the form $(a, b, 0)$ has vector $\langle 1, 1, 0 \rangle$ associated with it. For points not in the xy-plane but slightly above it, the associated vector has a small but positive z-component, and therefore the associated vector points slightly upward. For points that are far above the xy-plane, the z-component is large, so the vector is almost vertical. Figure 3.2.6 shows this vector field.

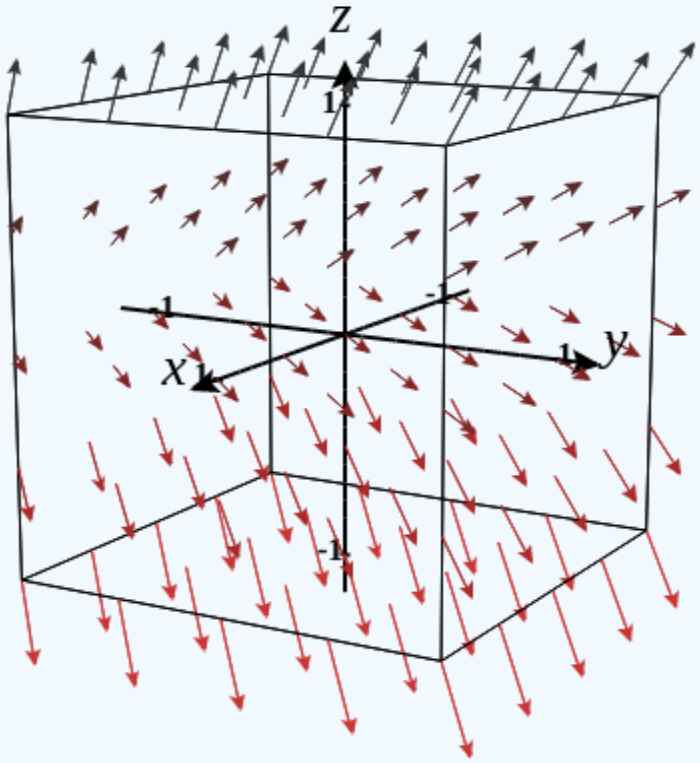


***Figure* 3.2.6:** *A visual representation of vector field* $\vec{\mathbf{F}}(x, y, z) = \langle 1, 1, z \rangle$.

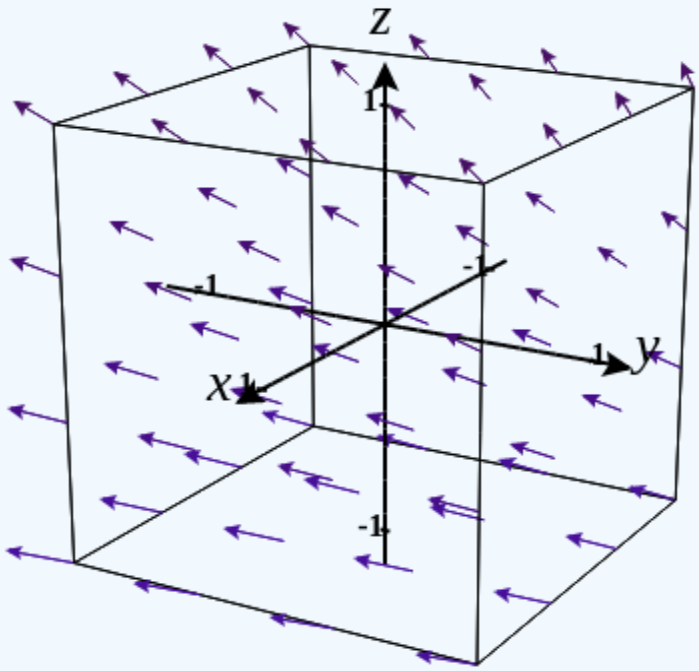
? Exercise 3.2.7

Sketch vector field $\vec{\mathbf{G}}(x, y, z) = \langle 2, \frac{z}{2}, 1 \rangle$.

Hint

Substitute enough points into the vector field to get an idea of the general shape.

Answer



In the next example, we explore one of the classic cases of a three-dimensional vector field: a gravitational field.

Example 3.2.8: Describing a Gravitational Vector Field

Newton's law of gravitation states that $\vec{\mathbf{F}} = -G\frac{m_1 m_2}{r^2}\hat{\mathbf{r}}$, where G is the universal gravitational constant. It describes the gravitational field exerted by an object (object 1) of mass m_1 located at the origin on another object (object 2) of mass m_2 located at point (x, y, z). Field $\vec{\mathbf{F}}$ denotes the gravitational force that object 1 exerts on object 2, r is the distance between the two objects, and $\hat{\mathbf{r}}$ indicates the unit vector from the first object to the second. The minus sign shows that the gravitational force attracts toward the origin; that is, the force of object 1 is attractive. Sketch the vector field associated with this equation.

Solution

Since object 1 is located at the origin, the distance between the objects is given by $r = \sqrt{x^2 + y^2 + z^2}$. The unit vector from object 1 to object 2 is $\hat{\mathbf{r}} = \frac{\langle x, y, z\rangle}{||\langle x, y, z\rangle||}$, and hence $\hat{\mathbf{r}} = \langle \frac{x}{r}, \frac{y}{r}, \frac{z}{r}\rangle$. Therefore, gravitational vector field $\vec{\mathbf{F}}$ exerted by object 1 on object 2 is

$$\vec{\mathbf{F}} = -Gm_1 m_2 \langle \frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3}\rangle.$$

This is an example of a radial vector field in $\mathbb{R}^3$.

Figure **3.2.7** shows what this gravitational field looks like for a large mass at the origin. Note that the magnitudes of the vectors increase as the vectors get closer to the origin.

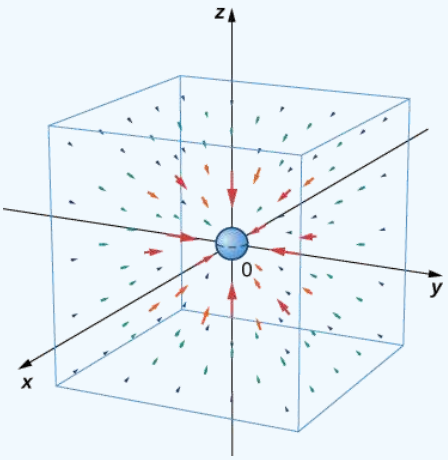


Figure 3.2.7: A visual representation of gravitational vector field $\vec{\mathbf{F}} = -Gm_1m_2\langle \frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3} \rangle$ for a large mass at the origin.

Exercise 3.2.8

The mass of asteroid 1 is 750,000 kg and the mass of asteroid 2 is 130,000 kg. Assume asteroid 1 is located at the origin, and asteroid 2 is located at $(15, -5, 10)$, measured in units of 10 to the eighth power kilometers. Given that the universal gravitational constant is $G = 6.67384 \times 10^{-11} m^3 kg^{-1} s^{-2}$, find the gravitational force vector that asteroid 1 exerts on asteroid 2.

Hint

Follow Example 3.2.8 and first compute the distance between the asteroids.

Answer

1.49063×10^{-18}, 4.96876×10^{-19}, 9.93752×10^{-19} N

Gradient Fields (Conservative Fields)

In this section, we study a special kind of vector field called a gradient field or a **conservative field**. These vector fields are extremely important in physics because they can be used to model physical systems in which energy is conserved. Gravitational fields and electric fields associated with a static charge are examples of gradient fields.

Recall that if f is a (scalar) function of x and y, then the gradient of f is

$$\text{grad } f = \vec{\nabla} f(x, y) = f_x(x, y)\hat{\mathbf{i}} + f_y(x, y)\hat{\mathbf{j}}. \tag{3.2.5}$$

We can see from the form in which the gradient is written that $\vec{\nabla} f$ is a vector field in $\mathbb{R}^2$. Similarly, if f is a function of x, y, and z, then the gradient of f is

$$\text{grad } f = \vec{\nabla} f(x, y, z) = f_x(x, y, z)\hat{\mathbf{i}} + f_y(x, y, z)\hat{\mathbf{j}} + f_z(x, y, z)\hat{\mathbf{k}}. \tag{3.2.6}$$

The gradient of a three-variable function is a vector field in $\mathbb{R}^3$. A gradient field is a vector field that can be written as the gradient of a function, and we have the following definition.

DEFINITION: Gradient Field

A vector field $\vec{\mathbf{F}}$ in $\mathbb{R}^2$ or in $\mathbb{R}^3$ is a gradient field if there exists a scalar function f such that $\vec{\nabla} f = \vec{\mathbf{F}}$.

Example 3.2.9: Sketching a Gradient Vector Field

Use technology to plot the gradient vector field of $f(x, y) = x^2 y^2$.

Solution

The gradient of f is $\vec{\nabla} f(x,y) = \langle 2xy^2,\ 2x^2y \rangle$. To sketch the vector field, use a computer algebra system such as Mathematica. Figure 3.2.8 shows $\vec{\nabla} f$.

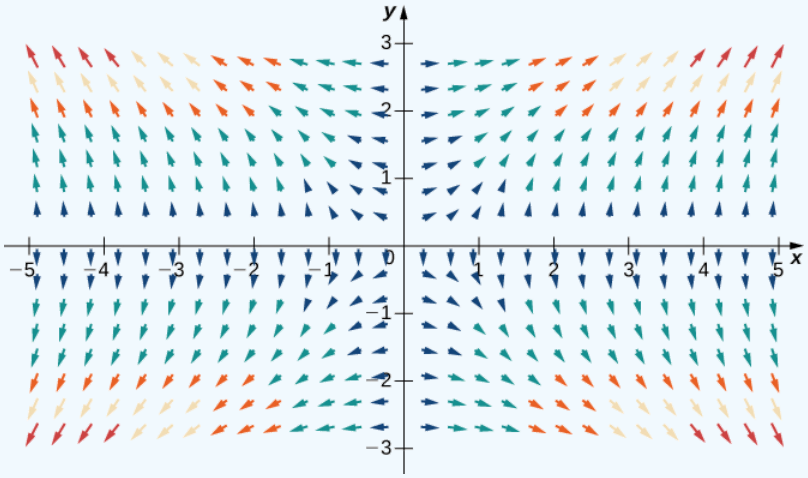


Figure 3.2.8: The gradient vector field is $\vec{\nabla} f$, where $f(x,y) = x^2y^2$.

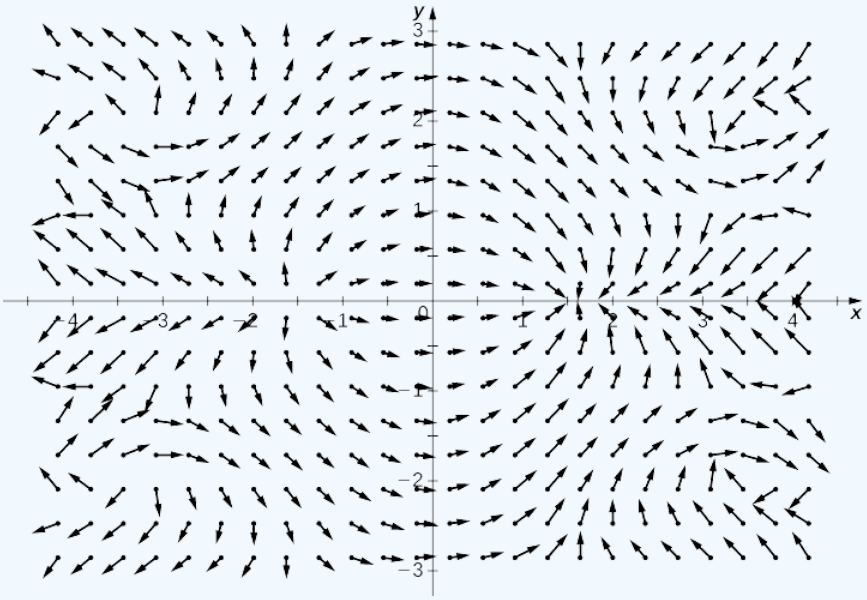
? Exercise 3.2.9

Use technology to plot the gradient vector field of $f(x,y) = \sin x \cos y$.

Hint

Find the gradient of f.

Answer



Consider the function $f(x,y) = x^2y^2$ from Example 3.2.9. Figure 3.2.9 shows the level curves of this function overlaid on the function's gradient vector field. The gradient vectors are perpendicular to the level curves, and the magnitudes of the vectors get larger as the level curves get closer together, because closely grouped level curves indicate the graph is steep, and the magnitude of the gradient vector is the largest value of the directional derivative. Therefore, you can see the local steepness of a graph by investigating the corresponding function's gradient field.

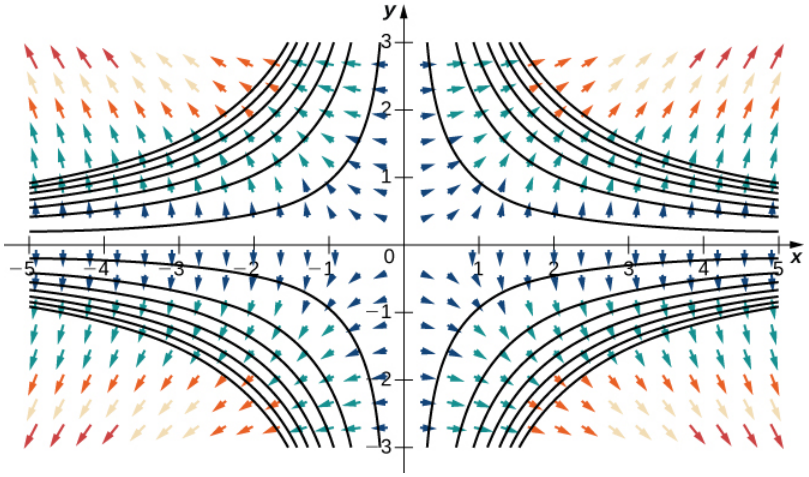


Figure 3.2.9: The gradient field of $f(x,y) = x^2y^2$ and several level curves of f. Notice that as the level curves get closer together, the magnitude of the gradient vectors increases.

As we learned earlier, a vector field $\vec{\mathbf{F}}$ is a conservative vector field, or a gradient field if there exists a scalar function f such that $\vec{\nabla} f = \vec{\mathbf{F}}$. In this situation, f is called a **potential function** for $\vec{\mathbf{F}}$. Conservative vector fields arise in many applications, particularly in physics. The reason such fields are called *conservative* is that they model forces of physical systems in which energy is conserved. We study conservative vector fields in more detail later in this chapter.

You might notice that, in some applications, a potential function f for $\vec{\mathbf{F}}$ is defined instead as a function such that $-\vec{\nabla} f = \vec{\mathbf{F}}$. This is the case for certain contexts in physics, for example.

Example 3.2.10: Verifying a Potential Function

Is $f(x,y,z) = x^2yz - \sin(xy)$ a potential function for vector field

$$\vec{\mathbf{F}}(x,y,z) = \langle 2xyz - y\cos(xy), x^2z - x\cos(xy), x^2y\rangle?$$

Solution

We need to confirm whether $\vec{\nabla} f = \vec{\mathbf{F}}$. We have

$$f_x(x,y) = 2xyz - y\cos(xy)$$
$$f_y(x,y) = x^2z - x\cos(xy).$$
$$f_z(x,y) = x^2y$$

Therefore, $\vec{\nabla} f = \vec{\mathbf{F}}$ and f is a potential function for $\vec{\mathbf{F}}$.

Exercise 3.2.10

Is $f(x,y,z) = x^2\cos(yz) + y^2z^2$ a potential function for $\vec{\mathbf{F}}(x,y,z) = \langle 2x\cos(yz), -x^2z\sin(yz) + 2yz^2, y^2\rangle$?

Hint

Compute the gradient of f.

Answer

No

Example 3.2.11: Verifying a Potential Function

The velocity of a fluid is modeled by field $\vec{\mathbf{v}}(x,y) = \langle xy, \frac{x^2}{2} - y\rangle$. Verify that $f(x,y) = \frac{x^2y}{2} - \frac{y^2}{2}$ is a potential function for $\vec{\mathbf{v}}$.

Solution

To show that f is a potential function, we must show that $\vec{\nabla} f = \vec{v}$. Note that $f_x(x,y) = xy$ and $f_y(x,y) = \frac{x^2}{2} - y$. Therefore, $\vec{\nabla} f(x,y) = \langle xy, \frac{x^2}{2} - y \rangle$ and f is a potential function for $\vec{v}$ (Figure 3.2.10).

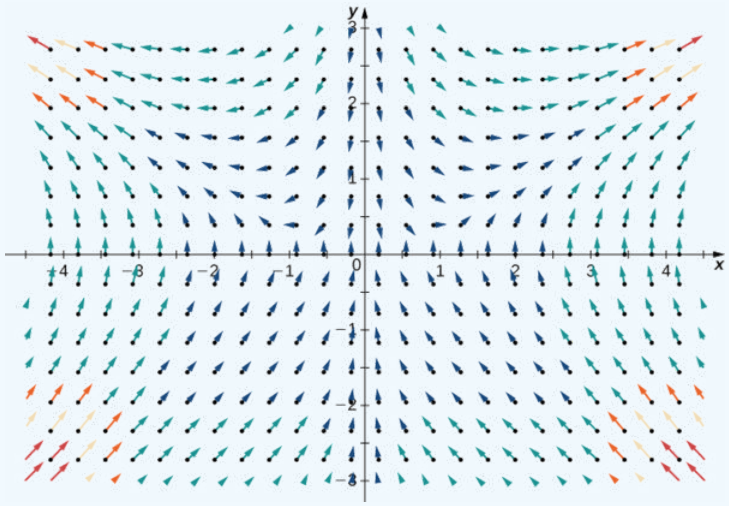


Figure 3.2.10: Velocity field $\vec{v}(x,y)$ has a potential function and is a conservative field.

? Exercise 3.2.11

Verify that $f(x,y) = x^2y^2 + x$ is a potential function for velocity field $\vec{v}(x,y) = \langle 3x^2y^2 + 1, 2x^3y \rangle$.

Hint

Calculate the gradient.

Answer

$\vec{\nabla} f(x,y) = \vec{v}(x,y)$

If $\vec{F}$ is a conservative vector field, then there is at least one potential function f such that $\vec{\nabla} f = \vec{F}$. But, could there be more than one potential function? If so, is there any relationship between two potential functions for the same vector field? Before answering these questions, let's recall some facts from single-variable calculus to guide our intuition. Recall that if $k(x)$ is an integrable function, then k has infinitely many antiderivatives. Furthermore, if $\vec{F}$ and $\vec{G}$ are both antiderivatives of k, then $\vec{F}$ and $\vec{G}$ **differ only by a constant**. That is, there is some number C such that $\vec{F}(x) = \vec{G}(x) + C$.

Now let $\vec{F}$ be a conservative vector field and let f and g be potential functions for $\vec{F}$. Since the gradient is like a derivative, $\vec{F}$ being conservative means that $\vec{F}$ is "integrable" with "antiderivatives" f and g. Therefore, if the analogy with single-variable calculus is valid, we expect there is some constant C such that $f(x) = g(x) + C$. The next theorem says that this is indeed the case.

To state the next theorem with precision, we need to assume the domain of the vector field is connected and open. To be connected means if P_1 and P_2 are any two points in the domain, then you can walk from P_1 to P_2 along a path that stays entirely inside the domain.

UNIQUENESS OF POTENTIAL FUNCTIONS

Let $\vec{F}$ be a conservative vector field on an open and connected domain and let f and g be functions such that $\vec{\nabla} f = \vec{F}$ and $\vec{\nabla} g = \vec{F}$. Then, there is a constant C such that $f = g + C$.

Proof

Since f and g are both potential functions for $\vec{F}$, then $\vec{\nabla}(f - g) = \vec{\nabla} f - \vec{\nabla} g = \vec{F} - \vec{F} = \vec{0}$. Let $h = f - g$, then we have $\vec{\nabla} h = \vec{0}$.We would like to show that h is a constant function.

Assume h is a function of x and y (the logic of this proof extends to any number of independent variables). Since $\vec{\nabla} h = \vec{0}$, we have $h_x(x,y) = 0$ and $h_y(x,y) = 0$. The expression $h_x(x,y) = 0$ implies that h is a constant function with respect to x—that is, $h(x,y) = k_1(y)$ for some function k_1. Similarly, $h_y(x,y) = 0$ implies $h(x,y) = k_2(x)$ for some function k_2. Therefore, function h depends only on y and also depends only on x. Thus, $h(x,y) = C$ for some constant C on the connected domain of $\vec{\mathbf{F}}$. Note that we really do need connectedness at this point; if the domain of $\vec{\mathbf{F}}$ came in two separate pieces, then k could be a constant C_1 on one piece but could be a different constant C_2 on the other piece. Since $f - g = h = C$, we have that $f = g + C$, as desired.

□

Conservative vector fields also have a special property called the ***cross-partial property***. This property helps test whether a given vector field is conservative.

THE CROSS-PARTIAL PROPERTY OF CONSERVATIVE VECTOR FIELDS

Let $\vec{\mathbf{F}}$ be a vector field in two or three dimensions such that the component functions of $\vec{\mathbf{F}}$ have continuous second-order mixed-partial derivatives on the domain of $\vec{\mathbf{F}}$.

If $\vec{\mathbf{F}}(x,y) = \langle P(x,y), Q(x,y) \rangle$ is a conservative vector field in $\mathbb{R}^2$, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}. \tag{3.2.7}$$

If $\vec{\mathbf{F}}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$ is a conservative vector field in $\mathbb{R}^3$, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$
$$\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$$
$$\frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}.$$

Proof

Since $\vec{\mathbf{F}}$ is conservative, there is a function $f(x,y)$ such that $\vec{\nabla} f = \vec{\mathbf{F}}$. Therefore, by the definition of the gradient, $f_x = P$ and $f_y = Q$. By Clairaut's theorem, $f_{xy} = f_{yx}$, But, $f_{xy} = P_y$ and $f_{yx} = Q_x$, and thus $P_y = Q_x$.

□

Clairaut's theorem gives a fast proof of the cross-partial property of conservative vector fields in $\mathbb{R}^3$, just as it did for vector fields in $\mathbb{R}^2$.

The Cross-Partial Property of Conservative Vector Fields shows that most vector fields are not conservative. The cross-partial property is difficult to satisfy in general, so most vector fields won't have equal cross-partials.

Show that rotational vector field $\vec{\mathbf{F}}(x,y) = \langle y, -x \rangle$ is not conservative.

Solution

Let $P(x,y) = y$ and $Q(x,y) = -x$. If $\vec{\mathbf{F}}$ is conservative, then the cross-partials would be equal—that is, P_y would equal Q_x.Therefore, to show that $\vec{\mathbf{F}}$ is not conservative, check that $P_y \neq Q_x$. Since $P_y = 1$ and $Q_x = -1$, the vector field is not conservative.

? Exercise 3.2.12

Show that vector field $\vec{\mathbf{F}}(x,y) = xy\,\hat{\mathbf{i}} - x^2 y\,\hat{\mathbf{j}}$ is not conservative.

Hint

Check the cross-partials.

Answer

$P_y(x,y) = x$ and $Q_x(x,y) = -2xy$. Since $P_y(x,y) \neq Q_x(x,y)$, $\vec{\mathbf{F}}$ is not conservative.

Example 3.2.13: Showing a Vector Field Is Not Conservative

Is vector field $\vec{\mathbf{F}}(x,y,z) = \langle 7, -2, x^3 \rangle$ conservative?

Solution

Let $P(x,y,z) = 7$, $Q(x,y,z) = -2$, and $R(x,y,z) = x^3$. If $\vec{\mathbf{F}}$ is conservative, then all three cross-partial equations will be satisfied —that is, if $\vec{\mathbf{F}}$ is conservative, then P_y would equal Q_x, Q_z would equal R_y, and R_x would equal P_z. Note that

$$P_y = Q_x = R_y = Q_z = 0$$

so the first two necessary equalities hold. However, $R_x(x,y,z) = x^3$ and $P_z(x,y,z) = 0$ so $R_x \neq P_z$. Therefore, $\vec{\mathbf{F}}$ is not conservative.

Exercise 3.2.13

Is vector field $\vec{\mathbf{G}}(x,y,z) = \langle y, x, xyz \rangle$ conservative?

Hint

Check the cross-partials.

Answer

No

We conclude this section with a word of warning: The Cross-Partial Property of Conservative Vector Fields says that if $\vec{\mathbf{F}}$ is conservative, then $\vec{\mathbf{F}}$ has the cross-partial property. The theorem does *not* say that, if $\vec{\mathbf{F}}$ has the cross-partial property, then $\vec{\mathbf{F}}$ is conservative (the converse of an implication is not logically equivalent to the original implication). In other words, The Cross-Partial Property of Conservative Vector Fields can only help determine that a field is not conservative; it does not let you conclude that a vector field is conservative.

For example, consider vector field $\vec{\mathbf{F}}(x,y) = \langle x^2 y, \frac{x^3}{3} \rangle$. This field has the cross-partial property, so it is natural to try to use The Cross-Partial Property of Conservative Vector Fields to conclude this vector field is conservative. However, this is a misapplication of the theorem. We learn later how to conclude that $\vec{\mathbf{F}}$ is conservative.

Key Concepts

- A vector field assigns a vector $\vec{\mathbf{F}}(x,y)$ to each point (x,y) in a subset D of $\mathbb{R}^2$ or $\mathbb{R}^3$. $\vec{\mathbf{F}}(x,y,z)$ to each point (x,y,z) in a subset D of $\mathbb{R}^3$.
- Vector fields can describe the distribution of vector quantities such as forces or velocities over a region of the plane or of space. They are in common use in such areas as physics, engineering, meteorology, oceanography.
- We can sketch a vector field by examining its defining equation to determine relative magnitudes in various locations and then drawing enough vectors to determine a pattern.
- A vector field $\vec{\mathbf{F}}$ is called conservative if there exists a scalar function f such that $\vec{\nabla} f = \vec{\mathbf{F}}$.

Key Equations

- **Vector field in $\mathbb{R}^2$**
  $\vec{\mathbf{F}}(x,y) = \langle P(x,y), Q(x,y) \rangle$
  or
  $\vec{\mathbf{F}}(x,y) = P(x,y)\,\hat{\mathbf{i}} + Q(x,y)\,\hat{\mathbf{j}}$
- **Vector field in $\mathbb{R}^3$**
  $\vec{\mathbf{F}}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$

or
$\vec{\mathbf{F}}(x,y,z) = P(x,y,z)\,\hat{\mathbf{i}} + Q(x,y,z)\,\hat{\mathbf{j}} + R(x,y,z)\,\hat{\mathbf{k}}$

Glossary

conservative field

a vector field for which there exists a scalar function f such that $\vec{\nabla} f = \vec{\mathbf{F}}$

gradient field

a vector field $\vec{\mathbf{F}}$ for which there exists a scalar function f such that $\vec{\nabla} f = \vec{\mathbf{F}}$; in other words, a vector field that is the gradient of a function; such vector fields are also called *conservative*

potential function

a scalar function f such that $\vec{\nabla} f = \vec{\mathbf{F}}$

radial field

a vector field in which all vectors either point directly toward or directly away from the origin; the magnitude of any vector depends only on its distance from the origin

rotational field

a vector field in which the vector at point (x,y) is tangent to a circle with radius $r = \sqrt{x^2+y^2}$; in a rotational field, all vectors flow either clockwise or counterclockwise, and the magnitude of a vector depends only on its distance from the origin

unit vector field

a vector field in which the magnitude of every vector is 1

vector field

measured in $\mathbb{R}^2$, an assignment of a vector $\vec{\mathbf{F}}(x,y)$ to each point (x,y) of a subset D of $\mathbb{R}^2$; in $\mathbb{R}^3$, an assignment of a vector $\vec{\mathbf{F}}(x,y,z)$ to each point (x,y,z) of a subset D of $\mathbb{R}^3$

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3.2E: Exercises

1. The domain of vector field $\vec{\mathbf{F}} = \vec{\mathbf{F}}(x, y)$ is a set of points (x, y) in a plane, and the range of $\vec{\mathbf{F}}$ is a set of *what* in the plane?

For exercises 2 - 4, determine whether the statement is *true or false*.

2. Vector field $\vec{\mathbf{F}} = \langle 3x^2, 1 \rangle$ is a gradient field for both $\phi_1(x, y) = x^3 + y$ and $\phi_2(x, y) = y + x^3 + 100$.

3. Vector field $\vec{\mathbf{F}} = \dfrac{\langle y, x \rangle}{\sqrt{x^2 + y^2}}$ is constant in direction and magnitude on a unit circle.

4. Vector field $\vec{\mathbf{F}} = \dfrac{\langle y, x \rangle}{\sqrt{x^2 + y^2}}$ is neither a radial field nor a rotation field.

For exercises 5 - 13, describe each vector field by drawing some of its vectors.

5. [T] $\vec{\mathbf{F}}(x, y) = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}$

6. [T] $\vec{\mathbf{F}}(x, y) = -y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}}$

7. [T] $\vec{\mathbf{F}}(x, y) = x\,\hat{\mathbf{i}} - y\,\hat{\mathbf{j}}$

8. [T] $\vec{\mathbf{F}}(x, y) = \hat{\mathbf{i}} + \hat{\mathbf{j}}$

9. [T] $\vec{\mathbf{F}}(x, y) = 2x\,\hat{\mathbf{i}} + 3y\,\hat{\mathbf{j}}$

10. [T] $\vec{\mathbf{F}}(x, y) = 3\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}}$

11. [T] $\vec{\mathbf{F}}(x, y) = y\,\hat{\mathbf{i}} + \sin x\,\hat{\mathbf{j}}$

12. [T] $\vec{\mathbf{F}}(x, y, z) = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$

13. [T] $\vec{\mathbf{F}}(x, y, z) = 2x\,\hat{\mathbf{i}} - 2y\,\hat{\mathbf{j}} - 2z\,\hat{\mathbf{k}}$

14. [T] $\vec{\mathbf{F}}(x, y, z) = yz\,\hat{\mathbf{i}} - xz\,\hat{\mathbf{j}}$

For exercises 15 - 20, find the gradient vector field of each function f.

15. $f(x, y) = x \sin y + \cos y$

16. $f(x, y, z) = ze^{-xy}$

17. $f(x, y, z) = x^2y + xy + y^2z$

18. $f(x, y) = x^2 \sin(5y)$

19. $f(x, y) = \ln(1 + x^2 + 2y^2)$

20. $f(x, y, z) = x \cos\left(\frac{y}{z}\right)$

21. What is vector field $\vec{\mathbf{F}}(x, y)$ with a value at (x, y) that is of unit length and points toward $(1, 0)$?

For exercises 22 - 24, write formulas for the vector fields with the given properties.

22. All vectors are parallel to the x-axis and all vectors on a vertical line have the same magnitude.

23. All vectors point toward the origin and have constant length.

24. All vectors are of unit length and are perpendicular to the position vector at that point.

25. Give a formula $\vec{\mathbf{F}}(x, y) = M(x, y)\,\hat{\mathbf{i}} + N(x, y)\,\hat{\mathbf{j}}$ for the vector field in a plane that has the properties that $\vec{\mathbf{F}} = \vec{\mathbf{0}}$ at $(0, 0)$ and that at any other point (a, b), $\vec{\mathbf{F}}$ is tangent to circle $x^2 + y^2 = a^2 + b^2$ and points in the clockwise direction with magnitude $\|\vec{\mathbf{F}}\| = \sqrt{a^2 + b^2}$.

26. Is vector field $\vec{\mathbf{F}}(x, y) = (P(x, y), Q(x, y)) = (\sin x + y)\,\hat{\mathbf{i}} + (\cos y + x)\,\hat{\mathbf{j}}$ a gradient field?

27. Find a formula for vector field $\vec{\mathbf{F}}(x, y) = M(x, y)\,\hat{\mathbf{i}} + N(x, y)\,\hat{\mathbf{j}}$ given the fact that for all points (x, y), $\vec{\mathbf{F}}$ points toward the origin and $\|\vec{\mathbf{F}}\| = \dfrac{10}{x^2 + y^2}$.

For exercises 28 - 29, assume that an electric field in the xy-plane caused by an infinite line of charge along the x-axis is a gradient field with potential function $V(x,y) = c\ln\left(\frac{r_0}{\sqrt{x^2+y^2}}\right)$, where $c > 0$ is a constant and r_0 is a reference distance at which the potential is assumed to be zero.

28. Find the components of the electric field in the x- and y-directions, where $\vec{\mathbf{E}}(x,y) = -\vec{\nabla}V(x,y)$.

29. Show that the electric field at a point in the xy-plane is directed outward from the origin and has magnitude $\|\vec{\mathbf{E}}\| = \frac{c}{r}$, where $r = \sqrt{x^2+y^2}$.

A ***flow line*** (or *streamline*) of a vector field $\vec{\mathbf{F}}$ is a curve $\vec{\mathbf{r}}(t)$ such that $d\vec{\mathbf{r}}/dt = \vec{\mathbf{F}}(\vec{\mathbf{r}}(t))$. If $\vec{\mathbf{F}}$ represents the velocity field of a moving particle, then the flow lines are paths taken by the particle. Therefore, flow lines are tangent to the vector field.

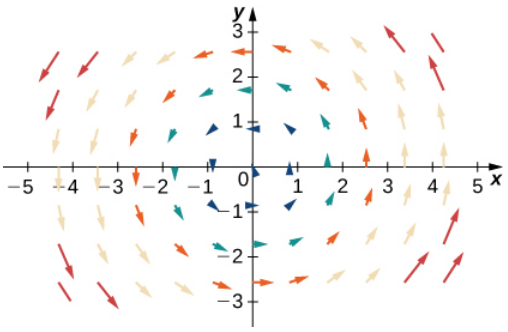
For exercises 30 and 31, show that the given curve $\vec{\mathbf{c}}(t)$ is a flow line of the given velocity vector field $\vec{\mathbf{F}}(x,y,z)$.

30. $\vec{\mathbf{c}}(t) = \langle e^{2t}, \ln|t|, \frac{1}{t}\rangle$, $t \neq 0$; $\quad \vec{\mathbf{F}}(x,y,z) = \langle 2x, z, -z^2\rangle$

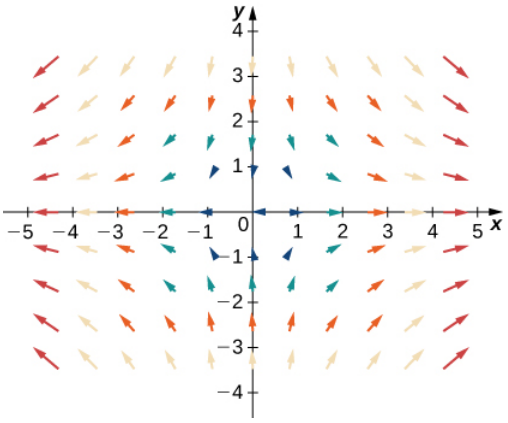
31. $\vec{\mathbf{c}}(t) = \langle \sin t, \cos t, e^t\rangle$; $\quad \vec{\mathbf{F}}(x,y,z) = \langle y, -x, z\rangle$

For exercises 32 - 34, let $\vec{\mathbf{F}} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}$, $\vec{\mathbf{G}} = -y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}}$, and $\vec{\mathbf{H}} = x\,\hat{\mathbf{i}} - y\,\hat{\mathbf{j}}$. Match $\vec{\mathbf{F}}$, $\vec{\mathbf{G}}$, and $\vec{\mathbf{H}}$ with their graphs.

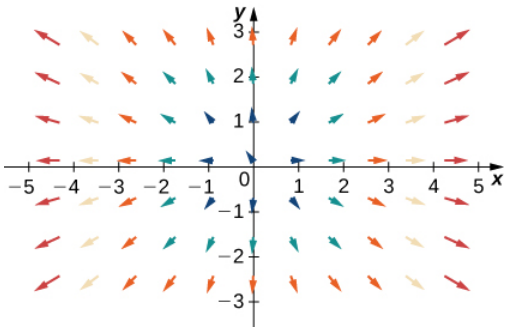
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33.



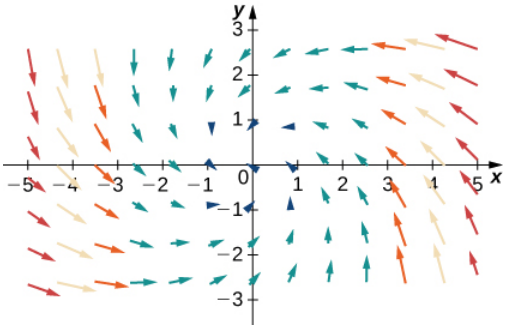
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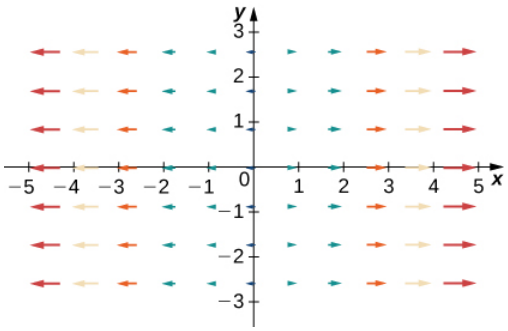
For exercises 35 - 38, let $\vec{\mathbf{F}} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}$, $\vec{\mathbf{G}} = -y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}}$, and $\vec{\mathbf{H}} = -x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}$. Match the vector fields with their graphs in (I)−(IV).

a. $\vec{\mathbf{F}} + \vec{\mathbf{G}}$
b. $\vec{\mathbf{F}} + \vec{\mathbf{H}}$
c. $\vec{\mathbf{G}} + \vec{\mathbf{H}}$
d. $-\vec{\mathbf{F}} + \vec{\mathbf{G}}$

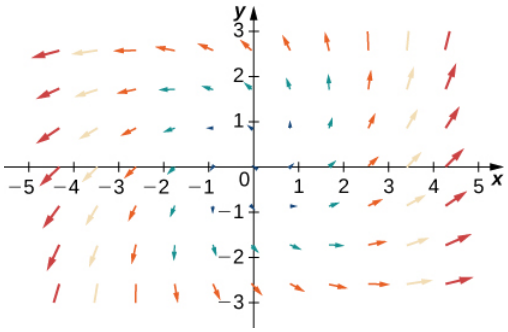
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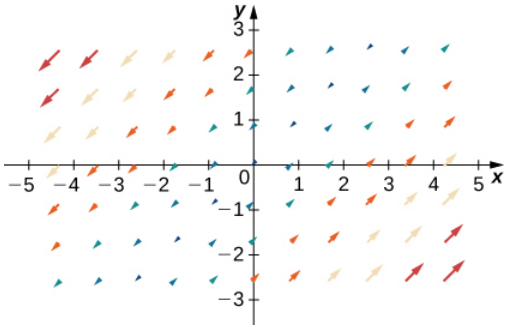
36.



37.



38.



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3.3: Line Integrals

Learning Objectives

- Calculate a scalar line integral along a curve.
- Calculate a vector line integral along an oriented curve in space.
- Use a line integral to compute the work done in moving an object along a curve in a vector field.
- Describe the flux and circulation of a vector field.

We are familiar with single-variable integrals of the form $\int_a^b f(x)\,dx$, where the domain of integration is an interval $[a,b]$. Such an interval can be thought of as a curve in the xy-plane, since the interval defines a line segment with endpoints $(a,0)$ and $(b,0)$—in other words, a line segment located on the x-axis. Suppose we want to integrate over *any* curve in the plane, not just over a line segment on the x-axis. Such a task requires a new kind of integral, called a *line integral.*

Line integrals have many applications to engineering and physics. They also allow us to make several useful generalizations of the Fundamental Theorem of Calculus. And, they are closely connected to the properties of vector fields, as we shall see.

Scalar Line Integrals

A line integral gives us the ability to integrate multivariable functions and vector fields over arbitrary curves in a plane or in space. There are two types of line integrals: scalar line integrals and vector line integrals. Scalar line integrals are integrals of a scalar function over a curve in a plane or in space. Vector line integrals are integrals of a vector field over a curve in a plane or in space. Let's look at scalar line integrals first.

A scalar line integral is defined just as a single-variable integral is defined, except that for a scalar line integral, the integrand is a function of more than one variable and the domain of integration is a curve in a plane or in space, as opposed to a curve on the x-axis.

For a scalar line integral, we let C be a smooth curve in a plane or in space and let ff be a function with a domain that includes C. We chop the curve into small pieces. For each piece, we choose point P in that piece and evaluate f at P. (We can do this because all the points in the curve are in the domain of f.) We multiply $f(P)$ by the arc length of the piece Δs, add the product $f(P)\Delta s$ over all the pieces, and then let the arc length of the pieces shrink to zero by taking a limit. The result is the scalar line integral of the function over the curve.

For a formal description of a scalar line integral, let C be a smooth curve in space given by the parameterization $\vec{\mathbf{r}}(t) = \langle x(t), y(t), z(t)\rangle$, $a \le t \le b$. Let $f(x,y,z)$ be a function with a domain that includes curve C. To define the line integral of the function f over C, we begin as most definitions of an integral begin: we chop the curve into small pieces. Partition the parameter interval $[a,b]$ into n subintervals $[t_{i-1}, t_i]$ of equal width for $1 \le i \le n$, where $t_0 = a$ and $t_n = b$ (Figure 3.3.1). Let t_i^* be a value in the ith interval $[t_{i-1}, t_i]$. Denote the endpoints of $\vec{\mathbf{r}}(t_0)$, $\vec{\mathbf{r}}(t_1)$,…, $\vec{\mathbf{r}}(t_n)$ by P_0,…, P_n. Points P_i divide curve C into n pieces C_1, C_2,…, C_n,with lengths Δs_1, Δs_2,…, Δs_n, respectively. Let P_i^* denote the endpoint of $\vec{\mathbf{r}}(t_i^*)$ for $1 \le i \le n$. Now, we evaluate the function f at point P_i^* for $1 \le i \le n$. Note that P_i^* is in piece C_1, and therefore P_i^* is in the domain of f. Multiply $f(P_i^*)$ by the length Δs_1 of C_1, which gives the area of the "sheet" with base C_1, and height $f(P_i^*)$. This is analogous to using rectangles to approximate area in a single-variable integral. Now, we form the sum $\sum_{i=1}^{n} f(P_i^*)\,\Delta s_i$.

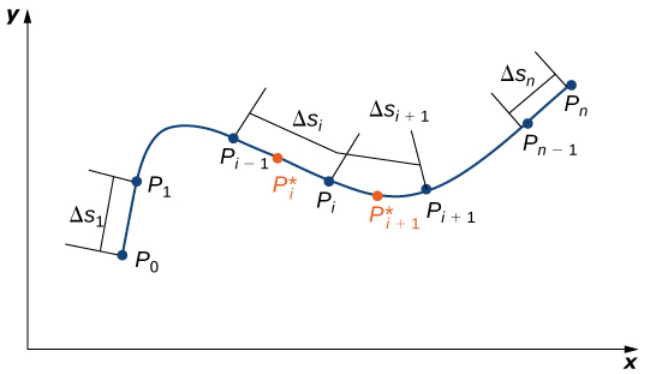


Figure 3.3.1: Curve C has been divided into n pieces, and a point inside each piece has been chosen.

Note the similarity of this sum versus a Riemann sum; in fact, this definition is a generalization of a Riemann sum to arbitrary curves in space. Just as with Riemann sums and integrals of form $\int_a^b g(x)\,dx$, we define an integral by letting the width of the pieces of the curve shrink to zero by taking a limit. The result is the scalar line integral of f along C.

You may have noticed a difference between this definition of a scalar line integral and a single-variable integral. In this definition, the arc lengths Δs_1, Δs_2,…, Δs_n aren't necessarily the same; in the definition of a single-variable integral, the curve in the x-axis is partitioned into pieces of equal length. This difference does not have any effect in the limit. As we shrink the arc lengths to zero, their values become close enough that any small difference becomes irrelevant.

DEFINITION: scalar line integral

Let f be a function with a domain that includes the smooth curve C that is parameterized by $\vec{\mathbf{r}}(t) = \langle x(t), y(t), z(t)\rangle$, $a \le t \le b$. The *scalar line integral* of f along C is

$$\int_C f(x,y,z)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(P_i^*)\,\Delta s_i \tag{3.3.1}$$

if this limit exists (t_i^* and Δs_i are defined as in the previous paragraphs). If C is a planar curve, then C can be represented by the parametric equations $x = x(t)$, $y = y(t)$, and $a \le t \le b$. If C is smooth and $f(x,y)$ is a function of two variables, then the scalar line integral of f along C is defined similarly as

$$\int_C f(x,y)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(P_i^*)\,\Delta s_i, \tag{3.3.2}$$

if this limit exists.

If f is a continuous function on a smooth curve C, then $\int_C f\,ds$ always exists. Since $\int_C f\,ds$ is defined as a limit of Riemann sums, the continuity of f is enough to guarantee the existence of the limit, just as the integral $\int_a^b g(x)\,dx$ exists if g is continuous over $[a,b]$.

Before looking at how to compute a line integral, we need to examine the geometry captured by these integrals. Suppose that $f(x,y) \ge 0$ for all points (x,y) on a smooth planar curve C. Imagine taking curve C and projecting it "up" to the surface defined by $f(x,y)$, thereby creating a new curve C' that lies in the graph of $f(x,y)$ (Figure 3.3.2). Now we drop a "sheet" from C' down to the xy-plane. The area of this sheet is $\int_C f(x,y)ds$. If $f(x,y) \le 0$ for some points in C, then the value of $\int_C f(x,y)\,ds$ is the area above the xy-plane less the area below the xy-plane. (Note the similarity with integrals of the form $\int_a^b g(x)\,dx$.)

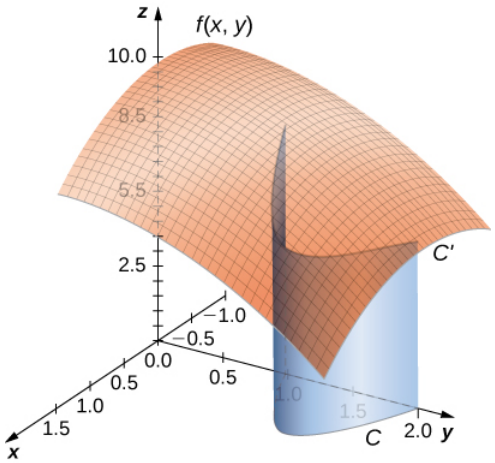


Figure **3.3.2**: The area of the blue sheet is $\int_C f(x,y)\,ds$.

From this geometry, we can see that line integral $\int_C f(x,y)\,ds$ does not depend on the parameterization $\vec{\mathbf{r}}(t)$ of C. As long as the curve is traversed exactly once by the parameterization, the area of the sheet formed by the function and the curve is the same. This same kind of geometric argument can be extended to show that the line integral of a three-variable function over a curve in space does not depend on the parameterization of the curve.

✔ Example 3.3.1: Finding the Value of a Line Integral

Find the value of integral $\int_C 2\,ds$, where C is the upper half of the unit circle.

Solution

The integrand is $f(x,y) = 2$. Figure **3.3.3** shows the graph of $f(x,y) = 2$, curve C, and the sheet formed by them. Notice that this sheet has the same area as a rectangle with width π and length 2. Therefore, $\int_C 2\,ds = 2\pi \text{ units}^2$.

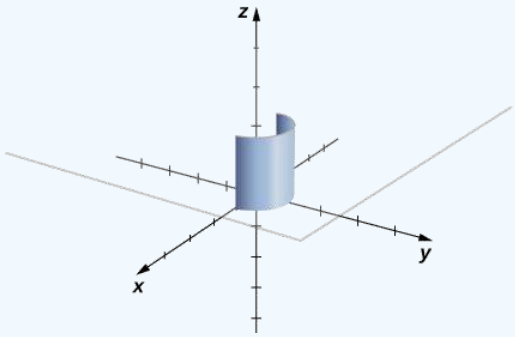


Figure **3.3.3**: The sheet that is formed by the upper half of the unit circle in a plane and the graph of $f(x,y) = 2$.

To see that $\int_C 2\,ds = 2\pi$ using the definition of line integral, we let $\vec{\mathbf{r}}(t)$ be a parameterization of C. Then, $f(\vec{\mathbf{r}}(t_i)) = 2$ for any number t_i in the domain of $\vec{\mathbf{r}}$. Therefore,

$$\begin{aligned}\int_C f\,ds &= \lim_{n\to\infty}\sum_{i=1}^{n} f(\vec{\mathbf{r}}(t_i^*))\,\Delta s_i \\ &= \lim_{n\to\infty}\sum_{i=1}^{n} 2\,\Delta s_i \\ &= 2\lim_{n\to\infty}\sum_{i=1}^{n} \Delta s_i \\ &= 2(\text{length of } C) \\ &= 2\pi \text{ units}^2.\end{aligned}$$

? Exercise 3.3.1

Find the value of $\int_C (x+y)\,ds$, where C is the curve parameterized by $x=t$, $y=t$, $0\le t\le 1$.

Hint

Find the shape formed by C and the graph of function $f(x,y)=x+y$.

Answer

$\sqrt{2}$

Note that in a scalar line integral, the integration is done with respect to arc length s, which can make a scalar line integral difficult to calculate. To make the calculations easier, we can translate $\int_C f\,ds$ to an integral with a variable of integration that is t.

Let $\vec{\mathbf{r}}(t)=\langle x(t),y(t),z(t)\rangle$ for $a\le t\le b$ be a parameterization of C. Since we are assuming that C is smooth, $\vec{\mathbf{r}}\,'(t)=\langle x'(t),y'(t),z'(t)\rangle$ is continuous for all t in $[a,b]$. In particular, $x'(t)$, $y'(t)$, and $z'(t)$ exist for all t in $[a,b]$. According to the arc length formula, we have

$$\text{length}(C_i)=\Delta s_i=\int_{t_{i-1}}^{t_i}\|\vec{\mathbf{r}}\,'(t)\|\,dt. \tag{3.3.3}$$

If width $\Delta t_i=t_i-t_{i-1}$ is small, then function $\int_{t_{i-1}}^{t_i}\|\vec{\mathbf{r}}\,'(t)\|\,dt \approx \|\vec{\mathbf{r}}\,'(t_i^*)\|\,\Delta t_i$, $\|\vec{\mathbf{r}}\,'(t)\|$ is almost constant over the interval $[t_{i-1},t_i]$.Therefore,

$$\int_{t_{i-1}}^{t_i}\|\vec{\mathbf{r}}\,'(t)\|\,dt \approx \|\vec{\mathbf{r}}\,'(t_i^*)\|\,\Delta t_i, \tag{3.3.4}$$

and we have

$$\sum_{i=1}^{n} f(\vec{\mathbf{r}}(t_i^*))\,\Delta s_i \approx \sum_{i=1}^{n} f(\vec{\mathbf{r}}(t_i^*))\|\vec{\mathbf{r}}\,'(t_i^*)\|\,\Delta t_i. \tag{3.3.5}$$

See Figure 3.3.4.

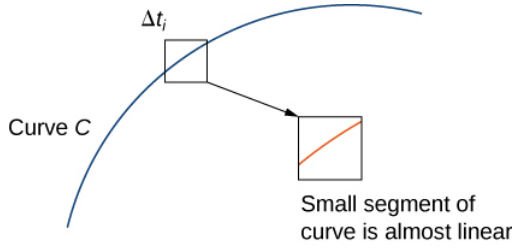


Figure 3.3.4: If we zoom in on the curve enough by making Δt_i very small, then the corresponding piece of the curve is approximately linear.

Note that

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(\vec{\mathbf{r}}(t_i^*))\|\vec{\mathbf{r}}\,'(t_i^*)\|\,\Delta t_i=\int_a^b f(\vec{\mathbf{r}}(t))\|\vec{\mathbf{r}}\,'(t)\|\,dt. \tag{3.3.6}$$

In other words, as the widths of intervals $[t_{i-1},t_i]$ shrink to zero, the sum $\sum_{i=1}^{n} f(\vec{\mathbf{r}}(t_i^*))\|\vec{\mathbf{r}}\,'(t_i^*)\|\,\Delta t_i$ converges to the integral $\int_a^b f(\vec{\mathbf{r}}(t))\|\vec{\mathbf{r}}\,'(t)\|\,dt$. Therefore, we have the following theorem.

Theorem: EVALUATING A SCALAR LINE INTEGRAL

Let f be a continuous function with a domain that includes the smooth curve C with parameterization $\vec{\mathbf{r}}(t)$, $a \le t \le b$. Then

$$\int_C f\,ds = \int_a^b f(\vec{\mathbf{r}}(t))\|\vec{\mathbf{r}}'(t)\|\,dt. \tag{3.3.7}$$

Although we have labeled Equation 3.3.4 as an equation, it is more accurately considered an approximation because we can show that the left-hand side of Equation 3.3.4 approaches the right-hand side as $n \to \infty$. In other words, letting the widths of the pieces shrink to zero makes the right-hand sum arbitrarily close to the left-hand sum. Since

$$\|\vec{\mathbf{r}}'(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}, \tag{3.3.8}$$

we obtain the following theorem, which we use to compute scalar line integrals.

Theorem: Scalar Line Integral Calculation

Let f be a continuous function with a domain that includes the smooth curve C with parameterization $\vec{\mathbf{r}}(t) = \langle x(t), y(t), z(t)\rangle$, $a \le t \le b$. Then

$$\int_C f(x,y,z)\,ds = \int_a^b f(\vec{\mathbf{r}}(t))\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\,dt. \tag{3.3.9}$$

Similarly,

$$\int_C f(x,y)\,ds = \int_a^b f(\vec{\mathbf{r}}(t))\sqrt{(x'(t))^2 + (y'(t))^2}\,dt \tag{3.3.10}$$

if C is a planar curve and f is a function of two variables.

Note that a consequence of this theorem is the equation $ds = \|\vec{\mathbf{r}}'(t)\|\,dt$. In other words, the change in arc length can be viewed as a change in the t-domain, scaled by the magnitude of vector $\vec{\mathbf{r}}'(t)$.

Example 3.3.2: Evaluating a Line Integral

Find the value of integral $\int_C (x^2 + y^2 + z)\,ds$, where C is part of the helix parameterized by $\vec{\mathbf{r}}(t) = \langle \cos t, \sin t, t\rangle$, $0 \le t \le 2\pi$.

Solution

To compute a scalar line integral, we start by converting the variable of integration from arc length s to t. Then, we can use Equation 3.3.1 to compute the integral with respect to t. Note that

$$f(\vec{\mathbf{r}}(t)) = \cos^2 t + \sin^2 t + t = 1 + t$$

and

$$\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} = \sqrt{(-\sin(t))^2 + \cos^2(t) + 1} = \sqrt{2}.$$

Therefore,

$$\int_C (x^2 + y^2 + z)\,ds = \int_0^{2\pi} (1 + t)\sqrt{2}\,dt.$$

Notice that Equation 3.3.1 translated the original difficult line integral into a manageable single-variable integral. Since

$$\begin{aligned}\int_0^{2\pi} (1 + t)\sqrt{2}\,dt &= \left[\sqrt{2}t + \frac{\sqrt{2}t^2}{2}\right]_0^{2\pi} \\ &= 2\sqrt{2}\pi + 2\sqrt{2}\pi^2,\end{aligned}$$

we have

$$\int_C (x^2 + y^2 + z)\, ds = 2\sqrt{2}\pi + 2\sqrt{2}\pi^2.$$



? Exercise 3.3.2

Evaluate $\int_C (x^2 + y^2 + z) ds$, where C is the curve with parameterization $\vec{\mathbf{r}}(t) = \langle \sin(3t), \cos(3t) \rangle$, $0 \leq t \leq \frac{\pi}{4}$.

Hint

Use the two-variable version of scalar line integral definition (Equation 3.3.2).

Answer

$$\frac{1}{3} + \frac{\sqrt{2}}{6} + \frac{3\pi}{4} \tag{3.3.11}$$

✓ Example 3.3.3: Independence of Parameterization

Find the value of integral $\int_C (x^2 + y^2 + z)\, ds$, where C is part of the helix parameterized by $\vec{\mathbf{r}}(t) = \langle \cos(2t), \sin(2t), 2t \rangle$, $0 \leq t \leq \pi$. Notice that this function and curve are the same as in the previous example; the only difference is that the curve has been reparameterized so that time runs twice as fast.

Solution

As with the previous example, we use Equation 3.3.1 to compute the integral with respect to t. Note that $f(\vec{\mathbf{r}}(t)) = \cos^2(2t) + \sin^2(2t) + 2t = 2t + 1$ and

$$\begin{aligned} \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} &= \sqrt{(-\sin t + \cos t + 4)} \\ &= 22 \end{aligned}$$

so we have

$$\begin{aligned}\int_C (x^2+y^2+z)ds &= 2\sqrt{2}\int_0^{\pi}(1+2t)dt \\ &= 2\sqrt{2}\Big[t+t^2\Big]_0^{\pi} \\ &= 2\sqrt{2}(\pi+\pi^2).\end{aligned}$$

Notice that this agrees with the answer in the previous example. Changing the parameterization did not change the value of the line integral. Scalar line integrals are independent of parameterization, as long as the curve is traversed exactly once by the parameterization.

? Exercise 3.3.3

Evaluate line integral $\int_C (x^2+yz)\,ds$, where C is the line with parameterization $\vec{\mathbf{r}}(t) = \langle 2t, 5t, -t\rangle$, $0 \le t \le 10$. Reparameterize C with parameterization $s(t) = \langle 4t, 10t, -2t\rangle$, $0 \le t \le 5$, recalculate line integral $\int_C (x^2+yz)\,ds$, and notice that the change of parameterization had no effect on the value of the integral.

Hint

Use Equation 3.3.1.

Answer

Both line integrals equal $-\dfrac{1000\sqrt{30}}{3}$.

Now that we can evaluate line integrals, we can use them to calculate arc length. If $f(x,y,z) = 1$, then

$$\begin{aligned}\int_C f(x,y,z)\,ds &= \lim_{n\to\infty}\sum_{i=1}^{n} f(t_i^*)\,\Delta s_i \\ &= \lim_{n\to\infty}\sum_{i=1}^{n}\Delta s_i \\ &= \lim_{n\to\infty}\text{length}(C) \\ &= \text{length}(C).\end{aligned}$$

Therefore, $\int_C 1\,ds$ is the arc length of C.

✓ Example 3.3.4: Calculating Arc Length

A wire has a shape that can be modeled with the parameterization $\vec{\mathbf{r}}(t) = \langle \cos t, \sin t, \frac{2}{3}t^{3/2}\rangle$, $0 \le t \le 4\pi$. Find the length of the wire.

Solution

The length of the wire is given by $\int_C 1\,ds$, where C is the curve with parameterization $\vec{\mathbf{r}}$. Therefore,

$$\begin{aligned}\text{The length of the wire} &= \int_C 1\,ds \\ &= \int_0^{4\pi} ||\,\vec{\mathbf{r}}'(t)\,||\,dt \\ &= \int_0^{4\pi} \sqrt{(-\sin t)^2 + \cos^2 t + t}dt \\ &= \int_0^{4\pi} \sqrt{1+t}dt \\ &= \frac{2(1+t)^{\frac{3}{2}}}{3}\Bigg|_0^{4\pi} \\ &= \frac{2}{3}\left((1+4\pi)^{3/2} - 1\right).\end{aligned}$$

? Exercise 3.3.4

Find the length of a wire with parameterization $\vec{\mathbf{r}}(t) = \langle 3t+1, 4-2t, 5+2t\rangle$, $0 \le t \le 4$.

Hint

Find the line integral of one over the corresponding curve.

Answer

$4\sqrt{17}$

Vector Line Integrals

The second type of line integrals are vector line integrals, in which we integrate along a curve through a vector field. For example, let

$$\vec{\mathbf{F}}(x,y,z) = P(x,y,z)\,\hat{\mathbf{i}} + Q(x,y,z)\,\hat{\mathbf{j}} + R(x,y,z)\,\hat{\mathbf{k}} \tag{3.3.12}$$

be a continuous vector field in $\mathbb{R}^3$ that represents a force on a particle, and let C be a smooth curve in $\mathbb{R}^3$ contained in the domain of $\vec{\mathbf{F}}$. How would we compute the work done by $\vec{\mathbf{F}}$ in moving a particle along C?

To answer this question, first note that a particle could travel in two directions along a curve: a forward direction and a backward direction. The work done by the vector field depends on the direction in which the particle is moving. Therefore, we must specify a direction along curve C; such a specified direction is called an orientation of a curve. The specified direction is the positive direction along C; the opposite direction is the negative direction along C. When C has been given an orientation, C is called an oriented curve (Figure 3.3.5). The work done on the particle depends on the direction along the curve in which the particle is moving.

A closed curve is one for which there exists a parameterization $\vec{\mathbf{r}}(t)$, $a \le t \le b$, such that $\vec{\mathbf{r}}(a) = \vec{\mathbf{r}}(b)$, and the curve is traversed exactly once. In other words, the parameterization is one-to-one on the domain (a,b).

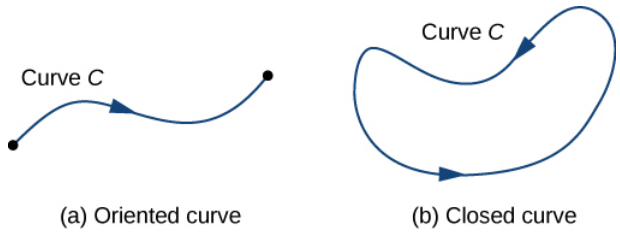


Figure 3.3.5: (a) An oriented curve between two points. (b) A closed oriented curve.

Let $\vec{\mathbf{r}}(t)$ be a parameterization of C for $a \le t \le b$ such that the curve is traversed exactly once by the particle and the particle moves in the positive direction along C. Divide the parameter interval $[a,b]$ into n subintervals $[t_{i-1}, t_i]$, $0 \le i \le n$, of equal width. Denote the endpoints of $r(t_0)$, $r(t_1)$,…, $r(t_n)$ by P_0,…,P_n. Points P_i divide C into n pieces. Denote the length of the piece from P_{i-1} to P_i by

Δs_i. For each i, choose a value t_i^* in the subinterval $[t_{i-1}, t_i]$. Then, the endpoint of $\vec{\mathbf{r}}(t_i^*)$ is a point in the piece of C between P_{i-1} and P_i (Figure 3.3.6). If Δs_i is small, then as the particle moves from P_{i-1} to P_i along C, it moves approximately in the direction of $\vec{\mathbf{T}}(P_i)$, the unit tangent vector at the endpoint of $\vec{\mathbf{r}}(t_i^*)$. Let P_i^* denote the endpoint of $\vec{\mathbf{r}}(t_i^*)$. Then, the work done by the force vector field in moving the particle from P_{i-1} to P_i is $\vec{\mathbf{F}}(P_i^*) \cdot (\Delta s_i \vec{\mathbf{T}}(P_i^*))$, so the total work done along C is

$$\sum_{i=1}^{n} \vec{\mathbf{F}}(P_i^*) \cdot (\Delta s_i \vec{\mathbf{T}}(P_i^*)) = \sum_{i=1}^{n} \vec{\mathbf{F}}(P_i^*) \cdot \vec{\mathbf{T}}(P_i^*)\, \Delta s_i. \tag{3.3.13}$$

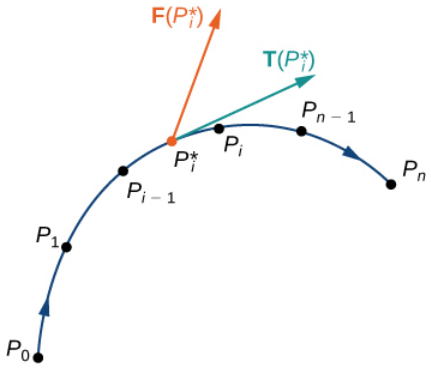


Figure 3.3.6: Curve C is divided into n pieces, and a point inside each piece is chosen. The dot product of any tangent vector in the ith piece with the corresponding vector $\vec{\mathbf{F}}$ is approximated by $\vec{\mathbf{F}}(P_i^*) \cdot \vec{\mathbf{T}}(P_i^*)$.



Letting the arc length of the pieces of C get arbitrarily small by taking a limit as $n \to \infty$ gives us the work done by the field in moving the particle along C. Therefore, the work done by $\vec{\mathbf{F}}$ in moving the particle in the positive direction along C is defined as

$$W = \int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds, \tag{3.3.14}$$

which gives us the concept of a vector line integral.

DEFINITION: Line Integral of a vector field

The vector line integral of vector field $\vec{\mathbf{F}}$ along oriented smooth curve C is

$$\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds = \lim_{n \to \infty} \sum_{i=1}^{n} \vec{\mathbf{F}}(P_i^*) \cdot \vec{\mathbf{T}}(P_i^*) \Delta s_i \tag{3.3.15}$$

if that limit exists.

With scalar line integrals, neither the orientation nor the parameterization of the curve matters. As long as the curve is traversed exactly once by the parameterization, the value of the line integral is unchanged. With vector line integrals, the orientation of the curve does matter. If we think of the line integral as computing work, then this makes sense: if you hike up a mountain, then the gravitational force of Earth does negative work on you. If you walk down the mountain by the exact same path, then Earth's gravitational force does positive work on you. In other words, reversing the path changes the work value from negative to positive in this case. Note that if C is an oriented curve, then we let $-C$ represent the same curve but with opposite orientation.

As with scalar line integrals, it is easier to compute a vector line integral if we express it in terms of the parameterization function $\vec{\mathbf{r}}$ and the variable t. To translate the integral $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}} ds$ in terms of t, note that unit tangent vector $\vec{\mathbf{T}}$ along C is given by $\vec{\mathbf{T}} = \dfrac{\vec{\mathbf{r}}'(t)}{\|\vec{\mathbf{r}}'(t)\|}$ (assuming $\|\vec{\mathbf{r}}'(t)\| \neq 0$). Since $ds = \|\vec{\mathbf{r}}'(t)\|\, dt$, as we saw when discussing scalar line integrals, we have

$$\vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds = \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \frac{\vec{\mathbf{r}}'(t)}{\|\vec{\mathbf{r}}'(t)\|} \|\vec{\mathbf{r}}'(t)\| dt = \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}'(t)\, dt. \tag{3.3.16}$$

Thus, we have the following formula for computing vector line integrals:

$$\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds = \int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}'(t)\, dt. \tag{3.3.17}$$

Because of Equation 3.3.17, we often use the notation $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ for the line integral $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds$.

If $\vec{\mathbf{r}}(t) = \langle x(t), y(t), z(t) \rangle$, then $\dfrac{d\vec{\mathbf{r}}}{dt}$ denotes vector $\langle x'(t), y'(t), z'(t) \rangle$, and $d\vec{\mathbf{r}} = \vec{\mathbf{r}}'(t)\, dt$.

Example 3.3.5: Evaluating a Vector Line Integral

Find the value of integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where C is the semicircle parameterized by $\vec{\mathbf{r}}(t) = \langle \cos t, \sin t \rangle$, $0 \leq t \leq \pi$ and $\vec{\mathbf{F}} = \langle -y, x \rangle$.

Solution

We can use Equation 3.3.17 to convert the variable of integration from s to t. We then have

$$\vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) = \langle -\sin t, \cos t \rangle \text{ and } \vec{\mathbf{r}}'(t) = \langle -\sin t, \cos t \rangle.$$

Therefore,

$$\begin{aligned} \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_0^\pi \langle -\sin t, \cos t \rangle \cdot \langle -\sin t, \cos t \rangle\, dt \\ &= \int_0^\pi \sin^2 t + \cos^2 t\, dt \\ &= \int_0^\pi 1\, dt = \pi. \end{aligned}$$

See Figure 3.3.7.

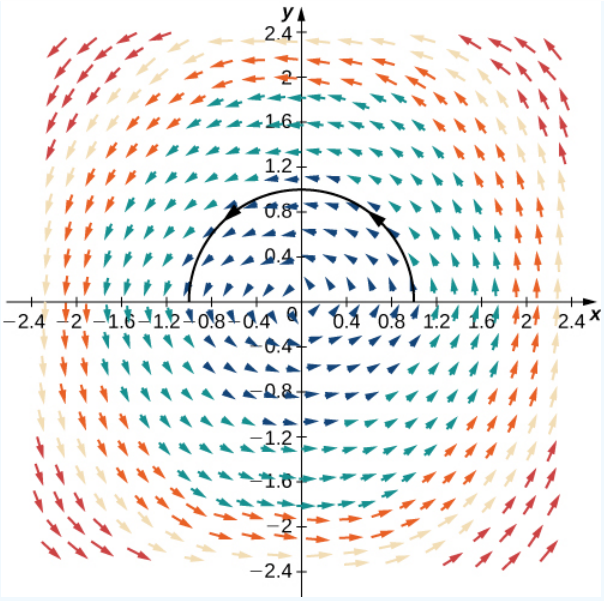


Figure 3.3.7: This figure shows curve $\vec{\mathbf{r}}(t) = \langle \cos t, \sin t \rangle$, $0 \le t \le \pi$ in vector field $\vec{\mathbf{F}} = \langle -y, x \rangle$.

Example 3.3.6: Reversing Orientation

Find the value of integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where C is the semicircle parameterized by $\vec{\mathbf{r}}(t) = \langle \cos(t + \pi), \sin t \rangle$, $0 \le t \le \pi$ and $\vec{\mathbf{F}} = \langle -y, x \rangle$.

Solution

Notice that this is the same problem as Example 3.3.5, except the orientation of the curve has been traversed. In this example, the parameterization starts at $\vec{\mathbf{r}}(0) = \langle -1, 0 \rangle$ and ends at $\vec{\mathbf{r}}(\pi) = \langle 1, 0 \rangle$. By Equation 3.3.17,

$$\begin{aligned} \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_0^\pi \langle -\sin t, \cos(t + \pi) \rangle \cdot \langle -\sin(t + \pi), \cos t \rangle dt \\ &= \int_0^\pi \langle -\sin t, -\cos t \rangle \cdot \langle \sin t, \cos t \rangle dt \\ &= \int_0^\pi (-\sin^2 t - \cos^2 t) dt \\ &= \int_0^\pi -1 dt \\ &= -\pi. \end{aligned}$$

Notice that this is the negative of the answer in Example 3.3.5. It makes sense that this answer is negative because the orientation of the curve goes against the "flow" of the vector field.

Let C be an oriented curve and let $-C$ denote the same curve but with the orientation reversed. Then, the previous two examples illustrate the following fact:

$$\int_{-C} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = -\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}. \tag{3.3.18}$$

That is, reversing the orientation of a curve changes the sign of a line integral.

Exercise 3.3.6

Let $\vec{\mathbf{F}} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}$ be a vector field and let C be the curve with parameterization $\langle t, t^2 \rangle$ for $0 \le t \le 2$. Which is greater: $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds$ or $\int_{-C} \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds$?

Hint

Imagine moving along the path and computing the dot product $\vec{\mathbf{F}} \cdot \vec{\mathbf{T}}$ as you go.

Answer

$$\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds \tag{3.3.19}$$

Another standard notation for integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ is $\int_C P\, dx + Q\, dy + R\, dz$. In this notation, P, Q, and R are functions, and we think of $d\vec{\mathbf{r}}$ as vector $\langle dx, dy, dz \rangle$. To justify this convention, recall that $d\vec{\mathbf{r}} = \vec{\mathbf{T}}\, ds = \vec{\mathbf{r}}'(t)\, dt = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle dt$. Therefore,

$$\vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \langle P, Q, R \rangle \cdot \langle dx, dy, dz \rangle = P\, dx + Q\, dy + R\, dz. \tag{3.3.20}$$

If $d\vec{\mathbf{r}} = \langle dx, dy, dz \rangle$, then $\frac{dr}{dt} = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$, which implies that $d\vec{\mathbf{r}} = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle dt$. Therefore

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C P\, dx + Q\, dy + R\, dz \tag{3.3.21}$$

$$= \int_a^b \left(P\big(\vec{\mathbf{r}}(t)\big) \frac{dx}{dt} + Q\big(\vec{\mathbf{r}}(t)\big) \frac{dy}{dt} + R\big(\vec{\mathbf{r}}(t)\big) \frac{dz}{dt} \right) dt. \tag{3.3.22}$$

Example 3.3.7: Finding the Value of an Integral of the Form $\int_C P\, dx + Q\, dy + R\, dz$

Find the value of integral $\int_C z\, dx + x\, dy + y\, dz$, where C is the curve parameterized by $\vec{\mathbf{r}}(t) = \langle t^2, \sqrt{t}, t \rangle$, $1 \le t \le 4$.

Solution

As with our previous examples, to compute this line integral we should perform a change of variables to write everything in terms of t. In this case, Equation 3.3.22 allows us to make this change:

$$\begin{aligned} \int_C z\, dx + x\, dy + y\, dz &= \int_1^4 \left(t(2t) + t^2 \left(\frac{1}{2\sqrt{t}} \right) + \sqrt{t} \right) dt \\ &= \int_1^4 \left(2t^2 + \frac{t^{3/2}}{2} + \sqrt{t} \right) dt \\ &= \left[\frac{2t^3}{3} + \frac{t^{5/2}}{5} + \frac{2t^{3/2}}{3} \right]_{t=1}^{t=4} \\ &= \frac{793}{15}. \end{aligned}$$

Exercise 3.3.7

Find the value of $\int_C 4x\, dx + z\, dy + 4y^2\, dz$, where C is the curve parameterized by $\vec{\mathbf{r}}(t) = \langle 4\cos(2t), 2\sin(2t), 3 \rangle$, $0 \le t \le \frac{\pi}{4}$.

Hint

Write the integral in terms of t using Equation 3.3.22.

Answer

-26

We have learned how to integrate smooth oriented curves. Now, suppose that C is an oriented curve that is not smooth, but can be written as the union of finitely many smooth curves. In this case, we say that C is a piecewise smooth curve. To be precise, curve C is piecewise smooth if C can be written as a union of n smooth curves C_1, C_2,…, C_n such that the endpoint of C_i is the starting point of C_{i+1} (Figure 3.3.8). When curves C_i satisfy the condition that the endpoint of C_i is the starting point of C_{i+1}, we write their union as $C_1 + C_2 + \cdots + C_n$.

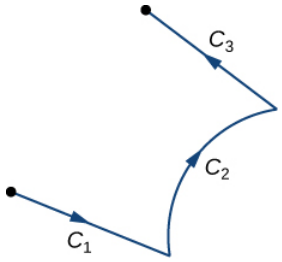


Figure **3.3.8**: The union of C_1, C_2, C_3 is a piecewise smooth curve.

The next theorem summarizes several key properties of vector line integrals.

Theorem: PROPERTIES OF VECTOR LINE INTEGRALS

Let $\vec{\mathbf{F}}$ and $\vec{\mathbf{G}}$ be continuous vector fields with domains that include the oriented smooth curve C. Then

1. $\displaystyle\int_C (\vec{\mathbf{F}} + \vec{\mathbf{G}}) \cdot d\vec{\mathbf{r}} = \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_C \vec{\mathbf{G}} \cdot d\vec{\mathbf{r}}$
2. $\displaystyle\int_C k\vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = k \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where k is a constant
3. $\displaystyle\int_{-C} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = -\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$
4. Suppose instead that C is a piecewise smooth curve in the domains of $\vec{\mathbf{F}}$ and $\vec{\mathbf{G}}$, where $C = C_1 + C_2 + \cdots + C_n$ and $C_1, C_2, \ldots, C_n$ are smooth curves such that the endpoint of C_i is the starting point of C_{i+1}. Then

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \cdots + \int_{C_n} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}. \tag{3.3.23}$$

Notice the similarities between these items and the properties of single-variable integrals. Properties i. and ii. say that line integrals are linear, which is true of single-variable integrals as well. Property iii. says that reversing the orientation of a curve changes the sign of the integral. If we think of the integral as computing the work done on a particle traveling along C, then this makes sense. If the particle moves backward rather than forward, then the value of the work done has the opposite sign. This is analogous to the equation $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$. Finally, if $[a_1, a_2]$, $[a_2, a_3]$,…, $[a_{n-1}, a_n]$ are intervals, then

$$\int_{a_1}^{a_n} f(x)\,dx = \int_{a_1}^{a_2} f(x)\,dx + \int_{a_1}^{a_3} f(x)\,dx + \cdots + \int_{a_{n-1}}^{a_n} f(x)\,dx, \tag{3.3.24}$$

which is analogous to property iv.

Example 3.3.8: Using Properties to Compute a Vector Line Integral

Find the value of integral $\displaystyle\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\,ds$, where C is the rectangle (oriented counterclockwise) in a plane with vertices $(0,0)$, $(2,0)$, $(2,1)$, and $(0,1)$, and where $\vec{\mathbf{F}} = \langle x - 2y, y - x \rangle$ (Figure **3.3.9**).

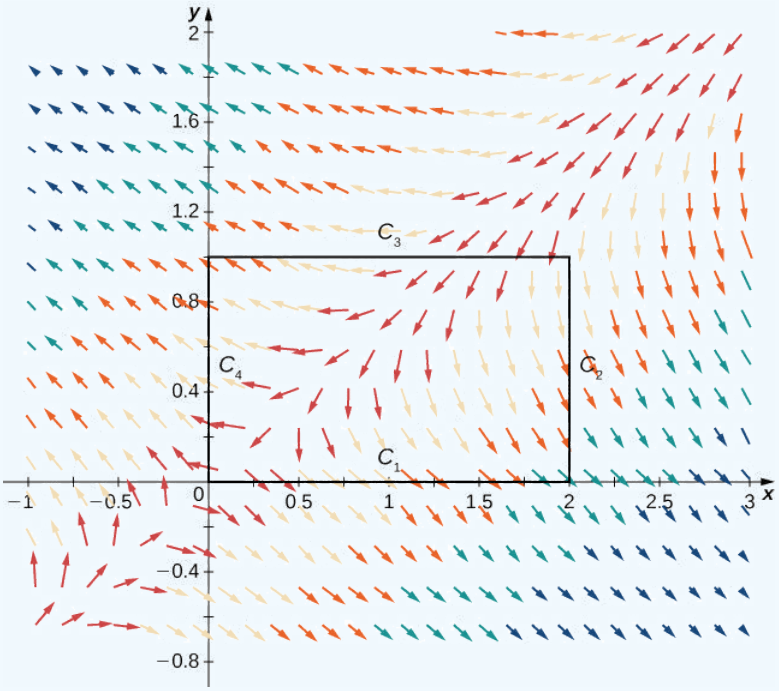


Figure 3.3.9: Rectangle and vector field for Example 3.3.8.

Solution

Note that curve C is the union of its four sides, and each side is smooth. Therefore C is piecewise smooth. Let C_1 represent the side from $(0,0)$ to $(2,0)$, let C_2 represent the side from $(2,0)$ to $(2,1)$, let C_3 represent the side from $(2,1)$ to $(0,1)$, and let C_4 represent the side from $(0,1)$ to $(0,0)$ (Figure 3.3.9). Then,

$$\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, dr = \int_{C_1} \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, dr + \int_{C_2} \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, dr + \int_{C_3} \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, dr + \int_{C_4} \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, dr. \tag{3.3.25}$$

We want to compute each of the four integrals on the right-hand side using Equation 3.3.1. Before doing this, we need a parameterization of each side of the rectangle. Here are four parameterizations (note that they traverse C counterclockwise):

$$\begin{aligned} C_1 &: \langle t, 0\rangle, 0 \le t \le 2 \\ C_2 &: \langle 2, t\rangle, 0 \le t \le 1 \\ C_3 &: \langle 2-t, 1\rangle, 0 \le t \le 2 \\ C_4 &: \langle 0, 1-t\rangle, 0 \le t \le 1. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_{C_1} \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, dr &= \int_0^2 \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}'(t)\, dt \\ &= \int_0^2 \langle t - 2(0), 0 - t\rangle \cdot \langle 1, 0\rangle\, dt = \int_0^2 t\, dt \\ &= \left[\tfrac{t^2}{2}\right]_0^2 = 2. \end{aligned}$$

Notice that the value of this integral is positive, which should not be surprising. As we move along curve C_1 from left to right, our movement flows in the general direction of the vector field itself. At any point along C_1, the tangent vector to the curve and the corresponding vector in the field form an angle that is less than 90°. Therefore, the tangent vector and the force vector have a positive dot product all along C_1, and the line integral will have positive value.

The calculations for the three other line integrals are done similarly:

$$\int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_0^1 \langle 2-2t, t-2\rangle \cdot \langle 0,1\rangle \, dt$$
$$= \int_0^1 (t-2)\, dt$$
$$= \left[\tfrac{t^2}{2} - 2t\right]_0^1 = -\frac{3}{2},$$

$$\int_{C_3} \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds = \int_0^2 \langle (2-t)-2, 1-(2-t)\rangle \cdot \langle -1,0\rangle \, dt$$
$$= \int_0^2 t\, dt = 2,$$

and

$$\int_{C_4} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_0^1 \langle -2(1-t), 1-t\rangle \cdot \langle 0,-1\rangle \, dt$$
$$= \int_0^1 (t-1)\, dt$$
$$= \left[\tfrac{t^2}{2} - t\right]_0^1 = -\frac{1}{2}.$$

Thus, we have $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 2$.

? Exercise 3.3.8

Calculate line integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where $\vec{\mathbf{F}}$ is vector field $\langle y^2, 2xy+1\rangle$ and C is a triangle with vertices $(0,0)$, $(4,0)$, and $(0,5)$, oriented counterclockwise.

Hint

Write the triangle as a union of its three sides, then calculate three separate line integrals.

Answer

0

Applications of Line Integrals

Scalar line integrals have many applications. They can be used to calculate the length or mass of a wire, the surface area of a sheet of a given height, or the electric potential of a charged wire given a linear charge density. Vector line integrals are extremely useful in physics. They can be used to calculate the work done on a particle as it moves through a force field, or the flow rate of a fluid across a curve. Here, we calculate the mass of a wire using a scalar line integral and the work done by a force using a vector line integral.

Suppose that a piece of wire is modeled by curve C in space. The mass per unit length (the linear density) of the wire is a continuous function $\rho(x,y,z)$. We can calculate the total mass of the wire using the scalar line integral $\int_C \rho(x,y,z)\, ds$. The reason is that mass is density multiplied by length, and therefore the density of a small piece of the wire can be approximated by $\rho(x^*,y^*,z^*)\, \Delta s$ for some point (x^*,y^*,z^*) in the piece. Letting the length of the pieces shrink to zero with a limit yields the line integral $\int_C \rho(x,y,z)\, ds$.

✓ Example 3.3.9: Calculating the Mass of a Wire

Calculate the mass of a spring in the shape of a curve parameterized by $\langle t, 2\cos t, 2\sin t\rangle$, $0 \le t \le \frac{\pi}{2}$, with a density function given by $\rho(x,y,z) = e^x + yz$ kg/m (Figure 3.3.10).

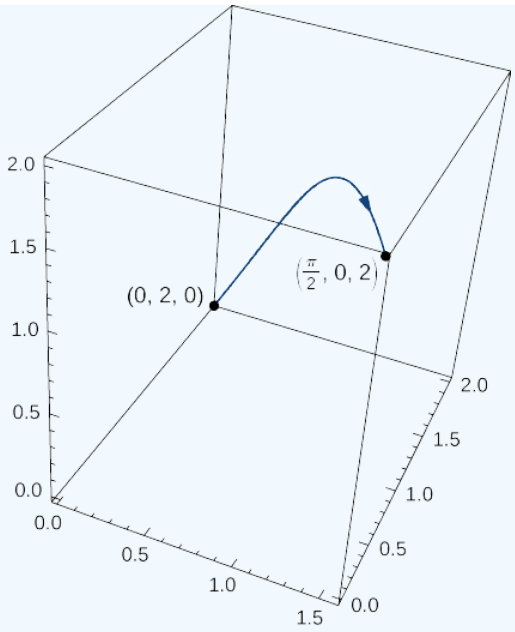


Figure **3.3.10**: The wire from Example **3.3.9**.

Solution

To calculate the mass of the spring, we must find the value of the scalar line integral $\int_C (e^x + yz)\, ds$, where C is the given helix. To calculate this integral, we write it in terms of t using Equation 3.3.1:

$$\begin{aligned}\int_C (e^x + yz)\, ds &= \int_0^{\frac{\pi}{2}} \left((e^t + 4\cos t \sin t)\sqrt{1 + (-2\cos t)^2 + (2\sin t)^2}\right) dt \\ &= \int_0^{\frac{\pi}{2}} \left((e^t + 4\cos t \sin t)\sqrt{5}\right) dt \\ &= \sqrt{5}\left[e^t + 2\sin^2 t\right]_{t=0}^{t=\pi/2} \\ &= \sqrt{5}(e^{\pi/2} + 1).\end{aligned}$$

Therefore, the mass is $\sqrt{5}(e^{\pi/2} + 1)$ kg.



? Exercise 3.3.9

Calculate the mass of a spring in the shape of a helix parameterized by $\vec{\mathbf{r}}(t) = \langle \cos t, \sin t, t \rangle$, $0 \le t \le 6\pi$, with a density function given by $\rho(x, y, z) = x + y + z$ kg/m.

Hint

Calculate the line integral of ρ over the curve with parameterization $\vec{\mathbf{r}}$.

Answer

$18\sqrt{2}\pi^2$ kg

When we first defined vector line integrals, we used the concept of work to motivate the definition. Therefore, it is not surprising that calculating the **work done by a vector field** representing a force is a standard use of vector line integrals. Recall that if an object moves along curve C in force field $\vec{\mathbf{F}}$, then the work required to move the object is given by $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$.

✓ Example 3.3.10: Calculating Work

How much work is required to move an object in vector force field $\vec{\mathbf{F}} = \langle yz, xy, xz \rangle$ along path $\vec{\mathbf{r}}(t) = \langle t^2, t, t^4 \rangle$, $0 \le t \le 1$? See Figure 3.3.11.

Solution

Let C denote the given path. We need to find the value of $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$. To do this, we use Equation 3.3.17 :

$$\begin{aligned} \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_0^1 \left(\langle t^5, t^3, t^6 \rangle \cdot \langle 2t, 1, 4t^3 \rangle \right) dt \\ &= \int_0^1 (2t^6 + t^3 + 4t^9)\, dt \\ &= \left[\frac{2t^7}{7} + \frac{t^4}{4} + \frac{2t^{10}}{5} \right]_{t=0}^{t=1} = \frac{131}{140} \text{ units of work.} \end{aligned}$$

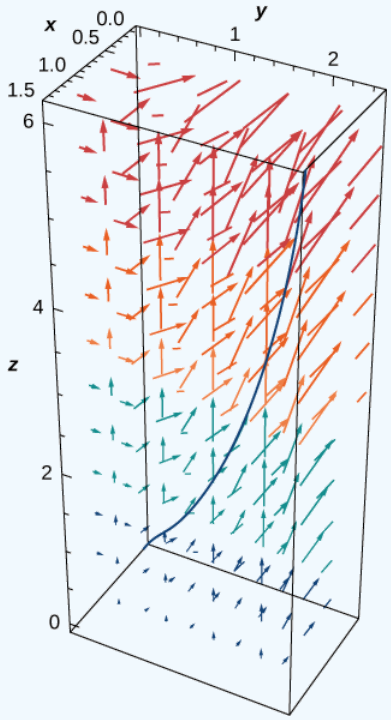


Figure 3.3.11: The curve and vector field for Example 3.3.10.



Flux

We close this section by discussing two key concepts related to line integrals: flux across a plane curve and circulation along a plane curve. Flux is used in applications to calculate fluid flow across a curve, and the concept of circulation is important for characterizing conservative gradient fields in terms of line integrals. Both these concepts are used heavily throughout the rest of this chapter. The idea of flux is especially important for Green's theorem, and in higher dimensions for Stokes' theorem and the divergence theorem.

Let C be a plane curve and let $\vec{\mathbf{F}}$ be a vector field in the plane. Imagine C is a membrane across which fluid flows, but C does not impede the flow of the fluid. In other words, C is an idealized membrane invisible to the fluid. Suppose $\vec{\mathbf{F}}$ represents the velocity field of the fluid. How could we quantify the rate at which the fluid is crossing C?

Recall that the line integral of $\vec{\mathbf{F}}$ along C is $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds$—in other words, the line integral is the dot product of the vector field with the unit tangential vector with respect to arc length. If we replace the unit tangential vector with unit normal vector $\vec{\mathbf{N}}(t)$ and instead compute integral $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds$, we determine the flux across C. To be precise, the definition of integral $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds$ is the same as integral $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds$, except the $\vec{\mathbf{T}}$ in the Riemann sum is replaced with $\vec{\mathbf{N}}$. Therefore, the flux across C is defined as

$$\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds = \lim_{n \to \infty} \sum_{i=1}^{n} \vec{\mathbf{F}}(P_i^*) \cdot \vec{\mathbf{N}}(P_i^*)\, \Delta s_i, \tag{3.3.26}$$

where P_i^* and Δs_i are defined as they were for integral $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds$. Therefore, a flux integral is an integral that is *perpendicular* to a vector line integral, because $\vec{\mathbf{N}}$ and $\vec{\mathbf{T}}$ are perpendicular vectors.

If $\vec{\mathbf{F}}$ is a velocity field of a fluid and C is a curve that represents a membrane, then the flux of $\vec{\mathbf{F}}$ across C is the quantity of fluid flowing across C per unit time, or the rate of flow.

More formally, let C be a plane curve parameterized by $\vec{\mathbf{r}}(t) = \langle x(t), y(t) \rangle$, $a \le t \le b$. Let $\vec{\mathbf{n}}(t) = \langle y'(t), -x'(t) \rangle$ be the vector that is normal to C at the endpoint of $\vec{\mathbf{r}}(t)$ and points to the right as we traverse C in the positive direction (Figure 3.3.12). Then, $\vec{\mathbf{N}}(t) = \dfrac{\vec{\mathbf{n}}(t)}{\|\vec{\mathbf{n}}(t)\|}$ is the unit normal vector to C at the endpoint of $\vec{\mathbf{r}}(t)$ that points to the right as we traverse C.

DEFINITION: flux

The *flux* of $\vec{\mathbf{F}}$ across C is line integral

$$\int_C \vec{\mathbf{F}} \cdot \frac{\vec{\mathbf{n}}(t)}{\|\vec{\mathbf{n}}(t)\|}\, ds. \tag{3.3.27}$$

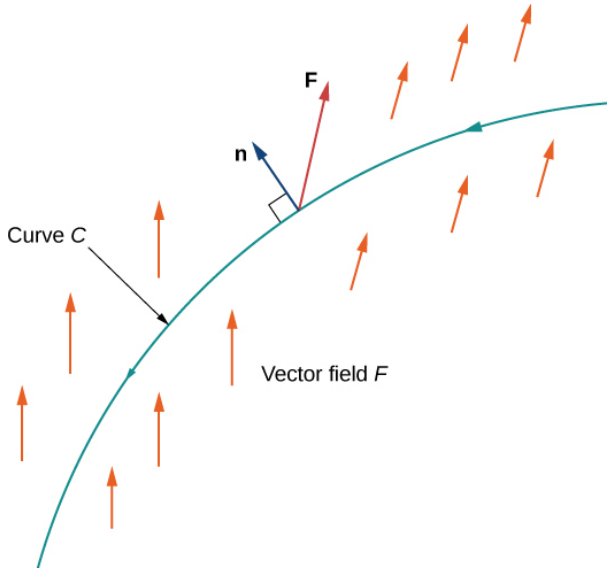


Figure 3.3.12: The flux of vector field $\vec{\mathbf{F}}$ *across curve* C *is computed by an integral similar to a vector line integral.*

We now give a formula for calculating the flux across a curve. This formula is analogous to the formula used to calculate a vector line integral (see Equation 3.3.17).

Theorem: CALCULATING FLUX ACROSS A CURVE

Let $\vec{\mathbf{F}}$ be a vector field and let C be a smooth curve with parameterization $r(t) = \langle x(t), y(t)\rangle$, $a \le t \le b$.Let $\vec{\mathbf{n}}(t) = \langle y'(t), -x'(t)\rangle$. The flux of $\vec{\mathbf{F}}$ across C is

$$\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds = \int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{n}}(t)\, dt. \tag{3.3.28}$$

Proof

Before deriving the formula, note that

$$\|\vec{\mathbf{n}}(t)\| = \|\langle y'(t), -x'(t)\rangle\| = \sqrt{(y'(t))^2 + (x'(t))^2} = \|\vec{\mathbf{r}}'(t)\|. \tag{3.3.29}$$

Therefore,

$$\begin{aligned}\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds &= \int_C \vec{\mathbf{F}} \cdot \frac{\vec{\mathbf{n}}(t)}{\|\vec{\mathbf{n}}(t)\|}\, ds \\ &= \int_a^b \vec{\mathbf{F}} \cdot \frac{\vec{\mathbf{n}}(t)}{\|\vec{\mathbf{n}}(t)\|} \|\vec{\mathbf{r}}'(t)\|\, dt \\ &= \int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{n}}(t)\, dt.\end{aligned}$$

□

Example 3.3.11: Flux across a Curve

Calculate the flux of $\vec{\mathbf{F}} = \langle 2x, 2y\rangle$ across a unit circle oriented counterclockwise (Figure 3.3.13).

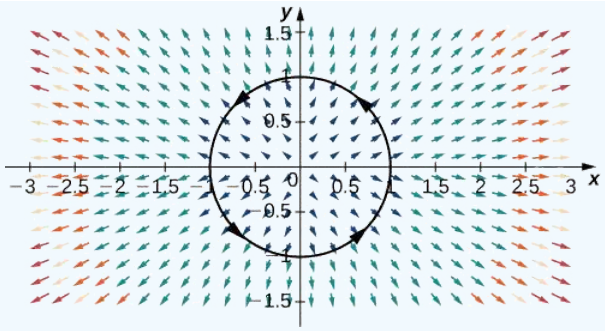


Figure 3.3.13: A unit circle in vector field $\vec{\mathbf{F}} = \langle 2x, 2y \rangle$.

Solution

To compute the flux, we first need a parameterization of the unit circle. We can use the standard parameterization $\vec{\mathbf{r}}(t) = \langle \cos t, \sin t \rangle$, $0 \leq t \leq 2\pi$. The normal vector to a unit circle is $\langle \cos t, \sin t \rangle$. Therefore, the flux is

$$\begin{aligned} \int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds &= \int_0^{2\pi} \langle 2\cos t, 2\sin t \rangle \cdot \langle \cos t, \sin t \rangle\, dt \\ &= \int_0^{2\pi} (2\cos^2 t + 2\sin^2 t)\, dt \\ &= 2\int_0^{2\pi} (\cos^2 t + \sin^2 t)\, dt \\ &= 2\int_0^{2\pi} dt = 4\pi. \end{aligned}$$

? Exercise 3.3.11

Calculate the flux of $\vec{\mathbf{F}} = \langle x + y, 2y \rangle$ across the line segment from $(0, 0)$ to $(2, 3)$, where the curve is oriented from left to right.

Hint

Use Equation 3.3.28.

Answer

$3/2$

Let $\vec{\mathbf{F}}(x, y) = \langle P(x, y), Q(x, y) \rangle$ be a two-dimensional vector field. Recall that integral $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds$ is sometimes written as $\int_C P\, dx + Q\, dy$. Analogously, flux $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds$ is sometimes written in the notation $\int_C -Q\, dx + P\, dy$, because the unit normal vector $\vec{\mathbf{N}}$ is perpendicular to the unit tangent $\vec{\mathbf{T}}$. Rotating the vector $d\vec{\mathbf{r}} = \langle dx, dy \rangle$ by 90° results in vector $\langle dy, -dx \rangle$. Therefore, the line integral in Example 3.3.8 can be written as $\int_C -2y\, dx + 2x\, dy$.

Circulation

Now that we have defined flux, we can turn our attention to circulation. The line integral of vector field $\vec{\mathbf{F}}$ along an oriented closed curve is called the circulation of $\vec{\mathbf{F}}$ along C. Circulation line integrals have their own notation: $\oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds$. The circle on the integral symbol denotes that C is "circular" in that it has no endpoints. Example 3.3.5 shows a calculation of circulation.

To see where the term *circulation* comes from and what it measures, let $\vec{\mathbf{v}}$ represent the velocity field of a fluid and let C be an oriented closed curve. At a particular point P, the closer the direction of $\vec{\mathbf{v}}(P)$ is to the direction of $\vec{\mathbf{T}}(P)$, the larger the value of the dot product $\vec{\mathbf{v}}(P) \cdot \vec{\mathbf{T}}(P)$. The maximum value of $\vec{\mathbf{v}}(P) \cdot \vec{\mathbf{T}}(P)$ occurs when the two vectors are pointing in the exact same direction; the minimum value of $\vec{\mathbf{v}}(P) \cdot \vec{\mathbf{T}}(P)$ occurs when the two vectors are pointing in opposite directions. Thus, the value of the circulation $\oint_C \vec{\mathbf{v}} \cdot \vec{\mathbf{T}}\, ds$ measures the tendency of the fluid to move in the direction of C.

Example 3.3.12: Calculating Circulation

Let $\vec{\mathbf{F}} = \langle -y, x \rangle$ be the vector field from Example 3.3.3 and let C represent the unit circle oriented counterclockwise. Calculate the circulation of $\vec{\mathbf{F}}$ along C.

Solution

We use the standard parameterization of the unit circle: $\vec{\mathbf{r}}(t) = \langle \cos t, \sin t \rangle$, $0 \leq t \leq 2\pi$. Then, $\vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) = \langle -\sin t, \cos t \rangle$ and $\vec{\mathbf{r}}'(t) = \langle -\sin t, \cos t \rangle$. Therefore, the circulation of $\vec{\mathbf{F}}$ along C is

$$\begin{aligned} \oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds &= \int_0^{2\pi} \langle -\sin t, \cos t \rangle \cdot \langle -\sin t, \cos t \rangle\, dt \\ &= \int_0^{2\pi} (\sin^2 t + \cos^2 t)\, dt \\ &= \int_0^{2\pi} dt = 2\pi \text{ units of work.} \end{aligned}$$

Notice that the circulation is positive. The reason for this is that the orientation of C "flows" with the direction of $\vec{\mathbf{F}}$. At any point along the circle, the tangent vector and the vector from $\vec{\mathbf{F}}$ form an angle of less than 90°, and therefore the corresponding dot product is positive.

In Example 3.3.12, what if we had oriented the unit circle clockwise? We denote the unit circle oriented clockwise by $-C$. Then

$$\oint_{-C} \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds = -\oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds = -2\pi \text{ units of work.} \tag{3.3.30}$$

Notice that the circulation is negative in this case. The reason for this is that the orientation of the curve flows against the direction of $\vec{\mathbf{F}}$.

Exercise 3.3.12

Calculate the circulation of $\vec{\mathbf{F}}(x, y) = \langle -\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \rangle$ along a unit circle oriented counterclockwise.

Hint

Use Equation 3.3.28.

Answer

2π units of work

Example 3.3.13: Calculating Work

Calculate the work done on a particle that traverses circle C of radius 2 centered at the origin, oriented counterclockwise, by field $\vec{\mathbf{F}}(x, y) = \langle -2, y \rangle$. Assume the particle starts its movement at $(1, 0)$.

Solution

The work done by $\vec{\mathbf{F}}$ on the particle is the circulation of $\vec{\mathbf{F}}$ along C: $\oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds$. We use the parameterization $\vec{\mathbf{r}}(t) = \langle 2\cos t, 2\sin t \rangle$, $0 \leq t \leq 2\pi$ for C. Then, $\vec{\mathbf{r}}'(t) = \langle -2\sin t, 2\cos t \rangle$ and $\vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) = \langle -2, 2\sin t \rangle$. Therefore, the circulation of $\vec{\mathbf{F}}$ along C is

$$\begin{aligned} \oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds &= \int_0^{2\pi} \langle -2, 2\sin t \rangle \cdot \langle -2\sin t, 2\cos t \rangle\, dt \\ &= \int_0^{2\pi} (4\sin t + 4\sin t \cos t)\, dt \\ &= \left[-4\cos t + 4\sin^2 t \right]_0^{2\pi} \\ &= \left(-4\cos(2\pi) + 2\sin^2(2\pi) \right) - \left(-4\cos(0) + 4\sin^2(0) \right) \\ &= -4 + 4 = 0 \text{ units of work.} \end{aligned}$$

The force field does zero work on the particle.

Notice that the circulation of $\vec{\mathbf{F}}$ along C is zero. Furthermore, notice that since $\vec{\mathbf{F}}$ is the gradient of $f(x,y) = -2x + \frac{y^2}{2}$, $\vec{\mathbf{F}}$ is conservative. We prove in a later section that under certain broad conditions, the circulation of a conservative vector field along a closed curve is zero.

? Exercise 3.3.14

Calculate the work done by field $\vec{\mathbf{F}}(x,y) = \langle 2x,\ 3y\rangle$ on a particle that traverses the unit circle. Assume the particle begins its movement at $(-1,\ 0)$.

Hint

Use Equation 3.3.28.

Answer

0 units of work

Key Concepts

- Line integrals generalize the notion of a single-variable integral to higher dimensions. The domain of integration in a single-variable integral is a line segment along the x-axis, but the domain of integration in a line integral is a curve in a plane or in space.
- If C is a curve, then the length of C is $\int_C ds$.
- There are two kinds of line integral: scalar line integrals and vector line integrals. Scalar line integrals can be used to calculate the mass of a wire; vector line integrals can be used to calculate the work done on a particle traveling through a field.
- Scalar line integrals can be calculated using Equation 3.3.1; vector line integrals can be calculated using Equation 3.3.17.
- Two key concepts expressed in terms of line integrals are flux and circulation. Flux measures the rate that a field crosses a given line; circulation measures the tendency of a field to move in the same direction as a given closed curve.

Key Equations

- **Calculating a scalar line integral**
  $$\int_C f(x,y,z)\,ds = \int_a^b f(\vec{\mathbf{r}}(t))\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\,dt$$
- **Calculating a vector line integral**
  $$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\,ds = \int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}'(t)\,dt$$
  or
  $$\int_C P\,dx + Q\,dy + R\,dz = \int_a^b \left(P\big(\vec{\mathbf{r}}(t)\big)\frac{dx}{dt} + Q\big(\vec{\mathbf{r}}(t)\big)\frac{dy}{dt} + R\big(\vec{\mathbf{r}}(t)\big)\frac{dz}{dt}\right) dt$$
- **Calculating flux**
  $$\int_C \vec{\mathbf{F}} \cdot \frac{\vec{\mathbf{n}}(t)}{\|\vec{\mathbf{n}}(t)\|}\,ds = \int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{n}}(t)\,dt$$

Glossary

circulation

the tendency of a fluid to move in the direction of curve C. If C is a closed curve, then the circulation of $\vec{\mathbf{F}}$ along C is line integral $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\,ds$, which we also denote $\oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\,ds$.

closed curve

a curve for which there exists a parameterization $\vec{\mathbf{r}}(t), a \le t \le b$, such that $\vec{\mathbf{r}}(a) = \vec{\mathbf{r}}(b)$, and the curve is traversed exactly once

flux

the rate of a fluid flowing across a curve in a vector field; the flux of vector field $\vec{\mathbf{F}}$ across plane curve C is line integral $\int_C \vec{\mathbf{F}} \cdot \frac{\vec{\mathbf{n}}(t)}{|\vec{\mathbf{n}}(t)|}\, ds$

line integral

the integral of a function along a curve in a plane or in space

orientation of a curve

the orientation of a curve C is a specified direction of C

piecewise smooth curve

an oriented curve that is not smooth, but can be written as the union of finitely many smooth curves

scalar line integral

the scalar line integral of a function f along a curve C with respect to arc length is the integral $\int_C f\, ds$, it is the integral of a scalar function f along a curve in a plane or in space; such an integral is defined in terms of a Riemann sum, as is a single-variable integral

vector line integral

the vector line integral of vector field $\vec{\mathbf{F}}$ along curve C is the integral of the dot product of $\vec{\mathbf{F}}$ with unit tangent vector $\vec{\mathbf{T}}$ of C with respect to arc length, $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds$; such an integral is defined in terms of a Riemann sum, similar to a single-variable integral

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3.3E: Exercises

1. *True or False?* Line integral $\int_C f(x,y)\,ds$ is equal to a definite integral if C is a smooth curve defined on $[a,b]$ and if function f is continuous on some region that contains curve C.

2. *True or False?* Vector functions $\vec{\mathbf{r}}_1 = t\,\hat{\mathbf{i}} + t^2\,\hat{\mathbf{j}}, \quad 0 \le t \le 1$, and $\vec{\mathbf{r}}_2 = (1-t)\,\hat{\mathbf{i}} + (1-t)^2\,\hat{\mathbf{j}}, \quad 0 \le t \le 1$, define the same oriented curve.

3. *True or False?* $\int_{-C}(P\,dx + Q\,dy) = \int_C (P\,dx - Q\,dy)$

4. *True or False?* A piecewise smooth curve C consists of a finite number of smooth curves that are joined together end to end.

5. *True or False?* If C is given by $x(t) = t, \quad y(t) = t, \quad 0 \le t \le 1$, then $\int_C xy\,ds = \int_0^1 t^2\,dt$.

For the following exercises, use a computer algebra system (CAS) to evaluate the line integrals over the indicated path.

6. [T] $\int_C (x+y)\,ds$

$C: x = t, y = (1-t), z = 0$ from $(0,1,0)$ to $(1,0,0)$

7. [T] $\int_C (x-y)ds$

$C: \vec{\mathbf{r}}(t) = 4t\,\hat{\mathbf{i}} + 3t\,\hat{\mathbf{j}}$ when $0 \le t \le 2$

8. [T] $\int_C (x^2+y^2+z^2)\,ds$

$C: \vec{\mathbf{r}}(t) = \sin t\,\hat{\mathbf{i}} + \cos t\,\hat{\mathbf{j}} + 8t\,\hat{\mathbf{k}}$ when $0 \le t \le \frac{\pi}{2}$

9. [T] Evaluate $\int_C xy^4\,ds$, where C is the right half of circle $x^2 + y^2 = 16$ and is traversed in the clockwise direction.

10. [T] Evaluate $\int_C 4x^3 ds$, where C is the line segment from $(-2,-1)$ to $(1,2)$.

For the following exercises, find the work done.

11. Find the work done by vector field $\vec{\mathbf{F}}(x,y,z) = x\,\hat{\mathbf{i}} + 3xy\,\hat{\mathbf{j}} - (x+z)\,\hat{\mathbf{k}}$ on a particle moving along a line segment that goes from $(1,4,2)$ to $(0,5,1)$.

12. Find the work done by a person weighing 150 lb walking exactly one revolution up a circular, spiral staircase of radius 3 ft if the person rises 10 ft.

13. Find the work done by force field $\vec{\mathbf{F}}(x,y,z) = -\frac{1}{2}x\,\hat{\mathbf{i}} - \frac{1}{2}y\,\hat{\mathbf{j}} + \frac{1}{4}\,\hat{\mathbf{k}}$ on a particle as it moves along the helix $\vec{\mathbf{r}}(t) = \cos t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}} + t\,\hat{\mathbf{k}}$ from point $(1,0,0)$ to point $(-1,0,3\pi)$.

14. Find the work done by vector field $\vec{\mathbf{F}}(x,y) = y\,\hat{\mathbf{i}} + 2x\,\hat{\mathbf{j}}$ in moving an object along path C, which joins points $(1,0)$ and $(0,1)$.

15. Find the work done by force $\vec{\mathbf{F}}(x,y) = 2y\,\hat{\mathbf{i}} + 3x\,\hat{\mathbf{j}} + (x+y)\,\hat{\mathbf{k}}$ in moving an object along curve $\vec{\mathbf{r}}(t) = \cos(t)\,\hat{\mathbf{i}} + \sin(t)\,\hat{\mathbf{j}} + 16\,\hat{\mathbf{k}}$, where $0 \le t \le 2\pi$.

16. Find the mass of a wire in the shape of a circle of radius 2 centered at (3, 4) with linear mass density $\rho(x,y) = y^2$.

For the following exercises, evaluate the line integrals.

17. Evaluate $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where $\vec{\mathbf{F}}(x,y) = -1\,\hat{\mathbf{j}}$, and C is the part of the graph of $y = \frac{1}{2}x^3 - x$ from $(2,2)$ to $(-2,-2)$.

18. Evaluate $\int_\gamma (x^2+y^2+z^2)^{-1} ds$, where γ is the helix $x = \cos t, y = \sin t, z = t$, with $0 \le t \le T$.

19. Evaluate $\int_C yz\,dx + xz\,dy + xy\,dz$ over the line segment from $(1,1,1)$ to $(3,2,0)$.

20. Let C be the line segment from point (0, 1, 1) to point (2, 2, 3). Evaluate line integral $\int_C y\,ds$.

21. [T] Use a computer algebra system to evaluate the line integral $\int_C y^2\,dx + x\,dy$, where C is the arc of the parabola $x = 4 - y^2$ from $(-5, -3)$ to $(0, 2)$.

22. [T] Use a computer algebra system to evaluate the line integral $\int_C (x + 3y^2)\,dy$ over the path C given by $x = 2t, y = 10t$, where $0 \le t \le 1$.

23. [T] Use a CAS to evaluate line integral $\int_C xy\,dx + y\,dy$ over path C given by $x = 2t, y = 10t$, where $0 \le t \le 1$.

24. Evaluate line integral $\int_C (2x - y)\,dx + (x + 3y)\,dy$, where C lies along the x-axis from $x = 0$ to $x = 5$.

26. [T] Use a CAS to evaluate $\int_C \frac{y}{2x^2 - y^2}\,ds$, where C is defined by the parametric equations $x = t, y = t$, for $1 \le t \le 5$.

27. [T] Use a CAS to evaluate $\int_C xy\,ds$, where C is defined by the parametric equations $x = t^2, y = 4t$, for $0 \le t \le 1$.

In the following exercises, find the work done by force field $\vec{\mathbf{F}}$ on an object moving along the indicated path.

28. $\vec{\mathbf{F}}(x, y) = -x\,\hat{\mathbf{i}} - 2y\,\hat{\mathbf{j}}$

$C : y = x^3$ from $(0, 0)$ to $(2, 8)$

29. $\vec{\mathbf{F}}(x, y) = 2x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}$

$<C$: counterclockwise around the triangle with vertices $(0, 0), (1, 0)$, and $(1, 1)$

30. $\vec{\mathbf{F}}(x, y, z) = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} - 5z\,\hat{\mathbf{k}}$

$C : \vec{\mathbf{r}}(t) = 2\cos t\,\hat{\mathbf{i}} + 2\sin t\,\hat{\mathbf{j}} + t\,\hat{\mathbf{k}},\ 0 \le t \le 2\pi$

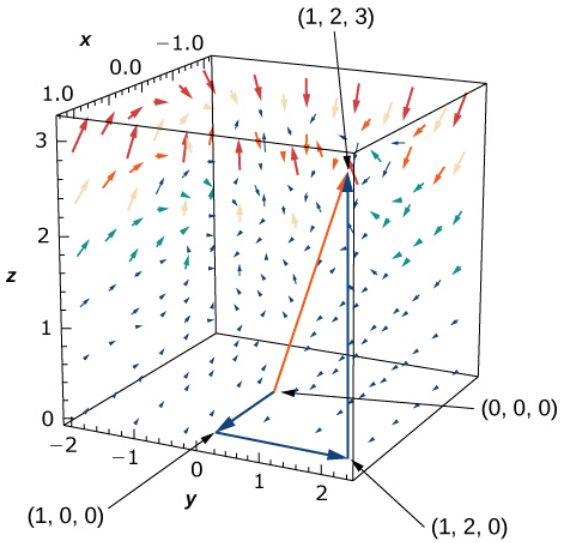
31. Let $\vec{\mathbf{F}}$ be vector field $\vec{\mathbf{F}}(x, y) = (y^2 + 2xe^y + 1)\,\hat{\mathbf{i}} + (2xy + x^2e^y + 2y)\,\hat{\mathbf{j}}$. Compute the work of integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where C is the path $\vec{\mathbf{r}}(t) = \sin t\,\hat{\mathbf{i}} + \cos t\,\hat{\mathbf{j}}, \quad 0 \le t \le \frac{\pi}{2}$.

32. Compute the work done by force $\vec{\mathbf{F}}(x, y, z) = 2x\,\hat{\mathbf{i}} + 3y\,\hat{\mathbf{j}} - z\,\hat{\mathbf{k}}$ along path $\vec{\mathbf{r}}(t) = t\,\hat{\mathbf{i}} + t^2\,\hat{\mathbf{j}} + t^3\,\hat{\mathbf{k}}$, where $0 \le t \le 1$.

33. Evaluate $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where $\vec{\mathbf{F}}(x, y) = \frac{1}{x + y}\,\hat{\mathbf{i}} + \frac{1}{x + y}\,\hat{\mathbf{j}}$ and C is the segment of the unit circle going counterclockwise from $(1, 0)$ to $(0, 1)$.

34. Force $\vec{\mathbf{F}}(x, y, z) = zy\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}} + z^2x\,\hat{\mathbf{k}}$ acts on a particle that travels from the origin to point $(1, 2, 3)$. Calculate the work done if the particle travels:

a. along the path $(0, 0, 0) \to (1, 0, 0) \to (1, 2, 0) \to (1, 2, 3)$ along straight-line segments joining each pair of endpoints;
b. along the straight line joining the initial and final points.
c. Is the work the same along the two paths?



35. Find the work done by vector field $\vec{\mathbf{F}}(x,y,z) = x\,\hat{\mathbf{i}} + 3xy\,\hat{\mathbf{j}} - (x+z)\,\hat{\mathbf{k}}$ on a particle moving along a line segment that goes from $(1,4,2)$ to $(0,5,1)$.

36. How much work is required to move an object in vector field $\vec{\mathbf{F}}(x,y) = y\,\hat{\mathbf{i}} + 3x\,\hat{\mathbf{j}}$ along the upper part of ellipse $\frac{x^2}{4} + y^2 = 1$ from $(2,0)$ to $(-2,0)$?

37. A vector field is given by $\vec{\mathbf{F}}(x,y) = (2x+3y)\,\hat{\mathbf{i}} + (3x+2y)\,\hat{\mathbf{j}}$. Evaluate the line integral of the field around a circle of unit radius traversed in a clockwise fashion.

38. Evaluate the line integral of scalar function xy along parabolic path $y = x^2$ connecting the origin to point $(1,1)$.

39. Find $\int_C y^2\,dx + (xy - x^2)\,dy$ along $C: y = 3x$ from $(0,0)$ to $(1,3)$.

40. Find $\int_C y^2\,dx + (xy - x^2)\,dy$ along $C: y^2 = 9x$ from $(0,0)$ to $(1,3)$.

For the following exercises, use a CAS to evaluate the given line integrals.

41. [T] Evaluate $\vec{\mathbf{F}}(x,y,z) = x^2z\,\hat{\mathbf{i}} + 6y\,\hat{\mathbf{j}} + yz^2\,\hat{\mathbf{k}}$, where C is represented by $\vec{\mathbf{r}}(t) = t\,\hat{\mathbf{i}} + t^2\,\hat{\mathbf{j}} + \ln t\,\hat{\mathbf{k}}, 1 \le t \le 3$.

42. [T] Evaluate line integral $\int_\gamma xe^y\,ds$ where, γ is the arc of curve $x = e^y$ from $(1,0)$ to $(e,1)$.

43. [T] Evaluate the integral $\int_\gamma xy^2\,ds$, where γ is a triangle with vertices $(0,1,2), (1,0,3)$, and $(0,-1,0)$.

44. [T] Evaluate line integral $\int_\gamma (y^2 - xy)\,dx$, where γ is curve $y = \ln x$ from $(1,0)$ toward $(e,1)$.

45. [T] Evaluate line integral $\int_\gamma xy^4\,ds$, where γ is the right half of circle $x^2 + y^2 = 16$.

46. [T] Evaluate $\int C\vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}, \int C\vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where $\vec{\mathbf{F}}(x,y,z) = x^2y\,\hat{\mathbf{i}} + (x-z)\,\hat{\mathbf{j}} + xyz\,\hat{\mathbf{k}}$ and

$C: \vec{\mathbf{r}}(t) = t\,\hat{\mathbf{i}} + t^2\,\hat{\mathbf{j}} + 2\,\hat{\mathbf{k}}, 0 \le t \le 1$.

47. Evaluate $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where $\vec{\mathbf{F}}(x,y) = 2x\sin y\,\hat{\mathbf{i}} + (x^2\cos y - 3y^2)\,\hat{\mathbf{j}}$ and

C is any path from $(-1,0)$ to $(5,1)$.

48. Find the line integral of $\vec{\mathbf{F}}(x,y,z) = 12x^2\,\hat{\mathbf{i}} - 5xy\,\hat{\mathbf{j}} + xz\,\hat{\mathbf{k}}$ over path C defined by $y = x^2, z = x^3$ from point $(0,0,0)$ to point $(2,4,8)$.

49. Find the line integral of $\int_C (1 + x^2y)\,ds$, where C is ellipse $\vec{\mathbf{r}}(t) = 2\cos t\,\hat{\mathbf{i}} + 3\sin t\,\hat{\mathbf{j}}$ from $0 \le t \le \pi$.

For the following exercises, find the flux.

50. Compute the flux of $\vec{\mathbf{F}} = x^2\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}$ across a line segment from $(0,0)$ to $(1,2)$.

51. Let $\vec{\mathbf{F}} = 5\,\hat{\mathbf{i}}$ and let C be curve $y = 0$, with $0 \le x \le 4$. Find the flux across C.

52. Let $\vec{\mathbf{F}} = 5\,\hat{\mathbf{j}}$ and let C be curve $y = 0$, with $0 \le x \le 4$. Find the flux across C.

53. Let $\vec{\mathbf{F}} = -y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}}$ and let $C : \vec{\mathbf{r}}(t) = \cos t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}}$ for $0 \le t \le 2\pi$. Calculate the flux across C.

54. Let $\vec{\mathbf{F}} = (x^2 + y^3)\,\hat{\mathbf{i}} + (2xy)\,\hat{\mathbf{j}}$. Calculate flux $\vec{\mathbf{F}}$ orientated counterclockwise across the curve $C : x^2 + y^2 = 9$.

Complete the rest of the exercises as stated.

55. Find the line integral of $\int_C z^2\,dx + y\,dy + 2y\,dz$, where C consists of two parts: C_1 and C_2. C_1 is the intersection of cylinder $x^2 + y^2 = 16$ and plane $z = 3$ from $(0,4,3)$ to $(-4,0,3)$. C_2 is a line segment from $(-4,0,3)$ to $(0,1,5)$.

56. A spring is made of a thin wire twisted into the shape of a circular helix $x = 2\cos t,\ y = 2\sin t,\ z = t$. Find the mass of two turns of the spring if the wire has a constant mass density of ρ grams per cm.

57. A thin wire is bent into the shape of a semicircle of radius a. If the linear mass density at point P is directly proportional to its distance from the line through the endpoints, find the mass of the wire.

58. An object moves in force field $\vec{\mathbf{F}}(x,y,z) = y^2\,\hat{\mathbf{i}} + 2(x+1)y\,\hat{\mathbf{j}}$ counterclockwise from point $(2,0)$ along elliptical path $x^2 + 4y^2 = 4$ to $(-2,0)$, and back to point $(2,0)$ along the x-axis. How much work is done by the force field on the object?

59. Find the work done when an object moves in force field $\vec{\mathbf{F}}(x,y,z) = 2x\,\hat{\mathbf{i}} - (x+z)\,\hat{\mathbf{j}} + (y-x)\,\hat{\mathbf{k}}$ along the path given by $\vec{\mathbf{r}}(t) = t^2\,\hat{\mathbf{i}} + (t^2 - t)\,\hat{\mathbf{j}} + 3\,\hat{\mathbf{k}},\ 0 \le t \le 1$.

60. If an inverse force field $\vec{\mathbf{F}}$ is given by $\vec{\mathbf{F}}(x,y,z) = \dfrac{k}{\|r\|^3} r$, where k is a constant, find the work done by $\vec{\mathbf{F}}$ as its point of application moves along the x-axis from $A(1,0,0)$ to $B(2,0,0)$.

61. David and Sandra plan to evaluate line integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ along a path in the xy-plane from $(0,0)$ to $(1,1)$. The force field is $\vec{\mathbf{F}}(x,y) = (x + 2y)\,\hat{\mathbf{i}} + (-x + y^2)\,\hat{\mathbf{j}}$. David chooses the path that runs along the x-axis from $(0,0)$ to $(1,0)$ and then runs along the vertical line $x = 1$ from $(1,0)$ to the final point $(1,1)$. Sandra chooses the direct path along the diagonal line $y = x$ from $(0,0)$ to $(1,1)$. Whose line integral is larger and by how much?

Contributors

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3.4: Conservative Vector Fields

Learning Objectives

- Describe simple and closed curves; define connected and simply connected regions.
- Explain how to find a potential function for a conservative vector field.
- Use the Fundamental Theorem for Line Integrals to evaluate a line integral in a vector field.
- Explain how to test a vector field to determine whether it is conservative.

In this section, we continue the study of conservative vector fields. We examine the Fundamental Theorem for Line Integrals, which is a useful generalization of the Fundamental Theorem of Calculus to line integrals of conservative vector fields. We also discover show how to test whether a given vector field is conservative, and determine how to build a potential function for a vector field known to be conservative.

Curves and Regions

Before continuing our study of conservative vector fields, we need some geometric definitions. The theorems in the subsequent sections all rely on integrating over certain kinds of curves and regions, so we develop the definitions of those curves and regions here. We first define two special kinds of curves: closed curves and simple curves. As we have learned, a closed curve is one that begins and ends at the same point. A simple curve is one that does not cross itself. A curve that is both closed and simple is a simple closed curve (Figure **3.4.1**).

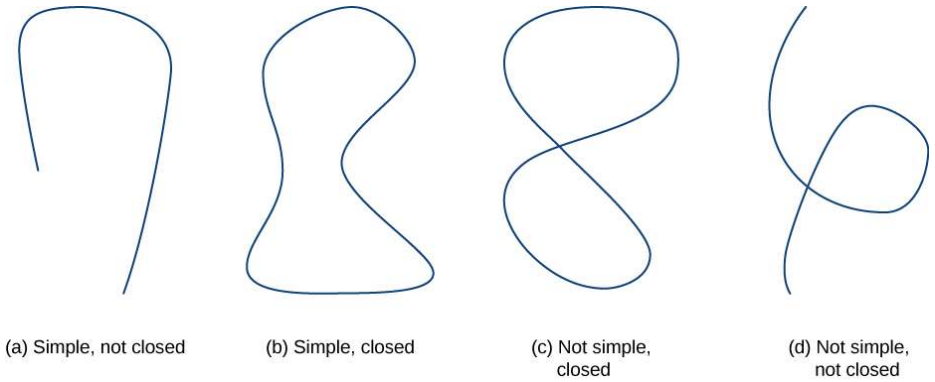


Figure **3.4.1**. Types of curves that are simple or not simple and closed or not closed.

DEFINITION: Closed Curves

Curve C is a *closed curve* if there is a parameterization $\vec{\mathbf{r}}(t)$, $a \leq t \leq b$ of C such that the parameterization traverses the curve exactly once and $\vec{\mathbf{r}}(a) = \vec{\mathbf{r}}(b)$. Curve C is a simple curve if C does not cross itself. That is, C is simple if there exists a parameterization $\vec{\mathbf{r}}(t)$, $a \leq t \leq b$ of C such that $\vec{\mathbf{r}}$ is one-to-one over (a, b). It is possible for $\vec{\mathbf{r}}(a) = \vec{\mathbf{r}}(b)$, meaning that the simple curve is also closed.

Example 3.4.1: Determining Whether a Curve Is Simple and Closed

Is the curve with parameterization $\vec{\mathbf{r}}(t) = \left\langle \cos t, \frac{\sin(2t)}{2} \right\rangle$, $0 \leq t \leq 2\pi$ a simple closed curve?

Solution

Note that $\vec{\mathbf{r}}(0) = \langle 1, 0 \rangle = \vec{\mathbf{r}}(2\pi)$; therefore, the curve is closed. The curve is not simple, however. To see this, note that $\vec{\mathbf{r}}\left(\frac{\pi}{2}\right) = \langle 0, 0 \rangle = \vec{\mathbf{r}}\left(\frac{3\pi}{2}\right)$, and therefore the curve crosses itself at the origin (Figure **3.4.2**).

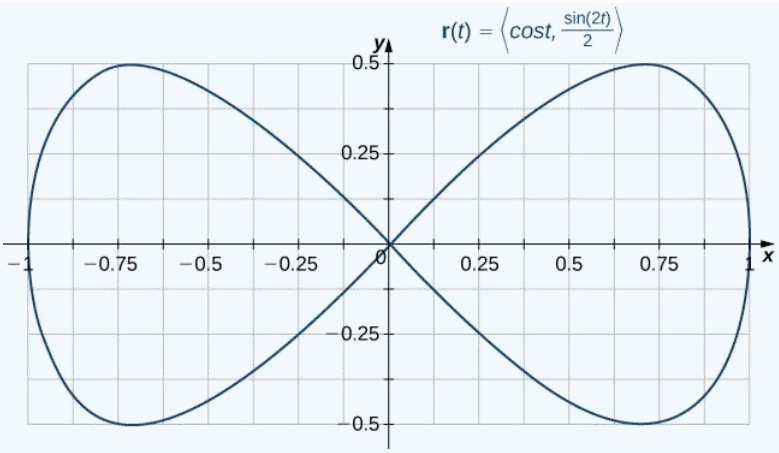


Figure **3.4.2.** *A curve that is closed but not simple.*

? Exercise 3.4.1

Is the curve given by parameterization $\vec{\mathbf{r}}(t) = \langle 2\cos t, 3\sin t\rangle$, $0 \le t \le 6\pi$, a simple closed curve?

Hint

Sketch the curve.

Answer

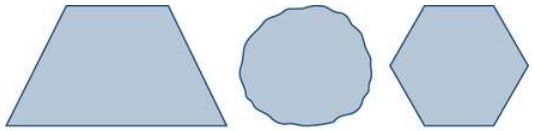
Yes

Many of the theorems in this chapter relate an integral over a region to an integral over the boundary of the region, where the region's boundary is a simple closed curve or a union of simple closed curves. To develop these theorems, we need two geometric definitions for regions: that of a connected region and that of a simply connected region. A connected region is one in which there is a path in the region that connects any two points that lie within that region. A simply connected region is a connected region that does not have any holes in it. These two notions, along with the notion of a simple closed curve, allow us to state several generalizations of the Fundamental Theorem of Calculus later in the chapter. These two definitions are valid for regions in any number of dimensions, but we are only concerned with regions in two or three dimensions.

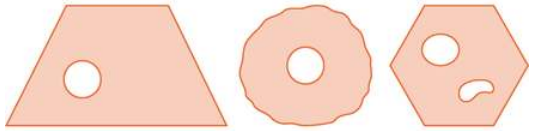
DEFINITION: connected regions

A region D is a *connected region* if, for any two points P_1 and P_2, there is a path from P_1 to P_2 with a trace contained entirely inside D. A region D is a simply connected region if D is connected for any simple closed curve C that lies inside D, and curve C can be shrunk continuously to a point while staying entirely inside D. In two dimensions, a region is simply connected if it is connected and has no holes.

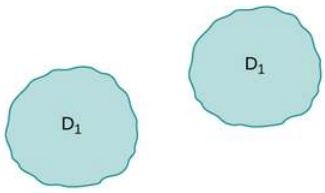
All simply connected regions are connected, but not all connected regions are simply connected (Figure **3.4.3**).



(a) Simply connected regions



(b) Connected regions that are not simply connected

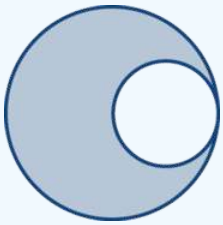


(c) A region that is not connected

Figure **3.4.3**: Not all connected regions are simply connected. (a) Simply connected regions have no holes. (b) Connected regions that are not simply connected may have holes but you can still find a path in the region between any two points. (c) A region that is not connected has some points that cannot be connected by a path in the region.

? Exercise 3.4.2

Is the region in the below image connected? Is the region simply connected?



Hint

Consider the definitions.

Answer

The region in the figure is connected. The region in the figure is not simply connected.

Fundamental Theorem for Line Integrals

Now that we understand some basic curves and regions, let's generalize the Fundamental Theorem of Calculus to line integrals. Recall that the Fundamental Theorem of Calculus says that if a function f has an antiderivative F, then the integral of f from a to b depends only on the values of F at a and at b—that is,

$$\int_a^b f(x)\,dx = F(b) - F(a). \tag{3.4.1}$$

If we think of the gradient as a derivative, then the same theorem holds for vector line integrals. We show how this works using a motivational example.

Example 3.4.2: Evaluating a Line Integral and the Antiderivatives of the Endpoints

Let $\vec{\mathbf{F}}(x,y) = \langle 2x, 4y\rangle$. Calculate $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where C is the line segment from $(0,0)$ to $(2,2)$ (Figure 3.4.4).

Solution

We use the method from the previous section to calculate $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$. Curve C can be parameterized by $\vec{\mathbf{r}}(t) = \langle 2t, 2t\rangle$, $0 \le t \le 1$. Then, $\vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) = \langle 4t, 8t\rangle$ and $\vec{\mathbf{r}}'(t) = \langle 2, 2\rangle$, which implies that

$$\begin{aligned}\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_0^1 \langle 4t, 8t\rangle \cdot \langle 2, 2\rangle dt \\ &= \int_0^1 (8t + 16t) dt = \int_0^1 24t dt \\ &= \left[12t^2\right]_0^1 = 12.\end{aligned}$$

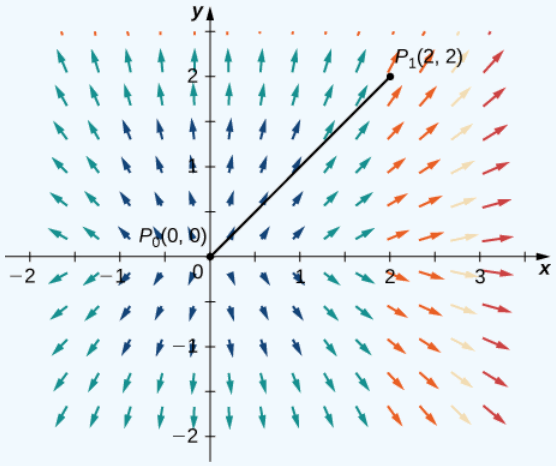


Figure 3.4.4: The value of line integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ *depends only on the value of the potential function of* $\vec{\mathbf{F}}$ *at the endpoints of the curve.*

Notice that $\vec{\mathbf{F}} = \vec{\nabla} f$, where $f(x,y) = x^2 + 2y^2$. If we think of the gradient as a derivative, then f is an "antiderivative" of $\vec{\mathbf{F}}$. In the case of single-variable integrals, the integral of derivative $g'(x)$ is $g(b) - g(a)$, where a is the start point of the interval of integration and b is the endpoint. If vector line integrals work like single-variable integrals, then we would expect integral $\vec{\mathbf{F}}$ to be $f(P_1) - f(P_0)$, where P_1 is the endpoint of the curve of integration and P_0 is the start point. Notice that this is the case for this example:

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C \vec{\nabla} f \cdot d\vec{\mathbf{r}} = 12$$

and

$$f(2,2) - f(0,0) = 4 + 8 - 0 = 12.$$

In other words, the integral of a "derivative" can be calculated by evaluating an "antiderivative" at the endpoints of the curve and subtracting, just as for single-variable integrals.

The following theorem says that, under certain conditions, what happened in the previous example holds for any gradient field. The same theorem holds for vector line integrals, which we call the **Fundamental Theorem for Line Integrals**.

Theorem: THE FUNDAMENTAL THEOREM FOR LINE INTEGRALS

Let C be a piecewise smooth curve with parameterization $\vec{\mathbf{r}}(t)$, $a \le t \le b$. Let f be a function of two or three variables with first-order partial derivatives that exist and are continuous on C. Then,

$$\int_C \vec{\nabla} f \cdot d\vec{\mathbf{r}} = f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(a)). \tag{3.4.2}$$

Proof

First,

$$\int_C \vec{\nabla} f \cdot d\vec{\mathbf{r}} = \int_a^b \vec{\nabla} f(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}'(t)\, dt.$$

By the chain rule,

$$\frac{d}{dt}(f(\vec{\mathbf{r}}(t)) = \vec{\nabla} f(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}'(t)$$

Therefore, by the Fundamental Theorem of Calculus,

$$\begin{aligned} \int_C \vec{\nabla} f \cdot d\vec{\mathbf{r}} &= \int_a^b \vec{\nabla} f(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}'(t) dt \\ &= \int_a^b \frac{d}{dt}(f(\vec{\mathbf{r}}(t)) dt \\ &= \left[f(\vec{\mathbf{r}}(t)) \right]_{t=a}^{t=b} \\ &= f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(a)). \end{aligned}$$

□

We know that if $\vec{\mathbf{F}}$ is a conservative vector field, there is a potential function f such that $\vec{\nabla} f = \vec{\mathbf{F}}$. Therefore

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C \vec{\nabla} f \cdot d\vec{\mathbf{r}} = f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(a)). \tag{3.4.3}$$

In other words, just as with the Fundamental Theorem of Calculus, computing the line integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where $\vec{\mathbf{F}}$ is conservative, is a two-step process:

1. Find a potential function ("antiderivative") f for $\vec{\mathbf{F}}$ and
2. Compute the value of f at the endpoints of C and calculate their difference $f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(a))$.

Keep in mind, however, there is one major difference between the Fundamental Theorem of Calculus and the Fundamental Theorem for Line Integrals:
A function of one variable that is continuous must have an antiderivative. However, a vector field, even if it is continuous, does not need to have a potential function.

Example 3.4.3: Applying the Fundamental Theorem

Calculate integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where $\vec{\mathbf{F}}(x, y, z) = \langle 2x \ln y, \frac{x^2}{y} + z^2, 2yz \rangle$ and C is a curve with parameterization $\vec{\mathbf{r}}(t) = \langle t^2, t, t \rangle$, $1 \leq t \leq e$

a. without using the Fundamental Theorem of Line Integrals and
b. using the Fundamental Theorem of Line Integrals.

Solution

1. First, let's calculate the integral without the Fundamental Theorem for Line Integrals and instead use the method we learned in the previous section:

$$\begin{aligned}\int_C \vec{\mathbf{F}}\cdot d\mathbf{r} &= \int_1^e \vec{\mathbf{F}}(\vec{\mathbf{r}}(t))\cdot\vec{\mathbf{r}}'(t)\,dt\\ &= \int_1^e \langle 2t^2\ln t, \frac{t^4}{t}+t^2, 2t^2\rangle\cdot\langle 2t,1,1\rangle\,dt\\ &= \int_1^e (4t^3\ln t + t^3 + 3t^2)\,dt\\ &= \int_1^e 4t^3\ln t\,dt + \int_1^e (t^3+3t^2)\,dt\\ &= \int_1^e 4t^3\ln t\,dt + \left[\frac{t^4}{4}+t^3\right]_1^e\\ &= \int_1^e 4t^3\ln t\,dt + \frac{e^4}{4}+e^3-\frac{1}{4}-1\\ &= \int_1^e 4t^3\ln t\,dt + \frac{e^4}{4}+e^3-\frac{5}{4}\end{aligned}$$

Integral $\int_1^e t^3\ln t\,dt$ requires integration by parts. Let $u=\ln t$ and $dv=t^3$. Then $u=\ln t$, $dv=t^3$ and

$$du=\frac{1}{t}\,dt,\quad v=\frac{t^4}{4}.$$

Therefore,

$$\begin{aligned}\int_1^e t^3\ln t\,dt &= \left[\frac{t^4}{4}\ln t\right]_1^e - \frac{1}{4}\int_1^e t^3\,dt\\ &= \frac{e^4}{4}-\frac{1}{4}\left(\frac{e^4}{4}-\frac{1}{4}\right).\end{aligned}$$

Thus,

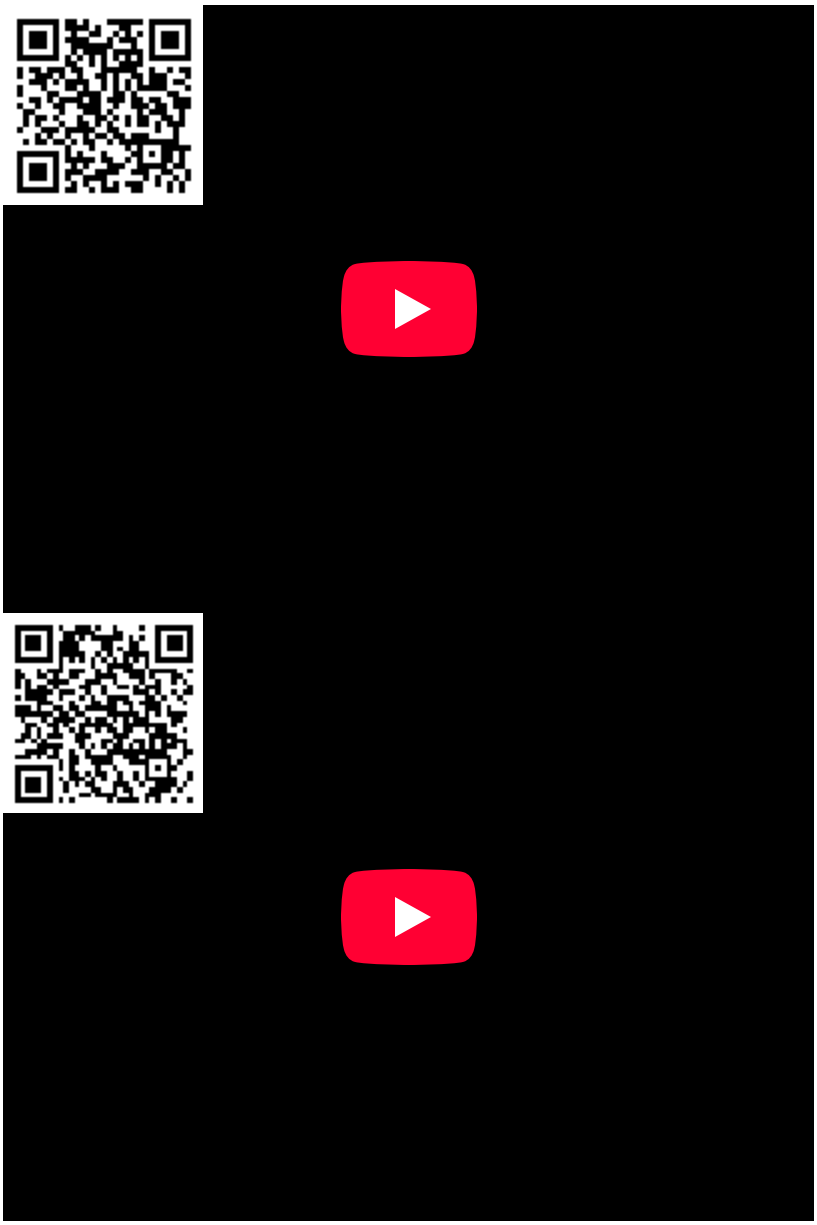
$$\begin{aligned}\int_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} &= 4\int_1^e t^3\ln t\,dt \quad + \quad \frac{e^4}{4}+e^3-\frac{5}{4}\\ &= 4\left(\frac{e^4}{4}-\frac{1}{4}\left(\frac{e^4}{4}-\frac{1}{4}\right)\right)+\frac{e^4}{4}+e^3-\frac{5}{4}\\ &= e^4-\frac{e^4}{4}+\frac{1}{4}+\frac{e^4}{4}+e^3-\frac{5}{4}\\ &= e^4+e^3-1.\end{aligned}$$

2. Given that $f(x,y,z)=x^2\ln y+yz^2$ is a potential function for $\vec{\mathbf{F}}$, let's use the Fundamental Theorem for Line Integrals to calculate the integral. Note that

$$\begin{aligned}\int_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} &= \int_C \vec{\nabla} f\cdot d\vec{\mathbf{r}}\\ &= f(\vec{\mathbf{r}}(e))-f(\vec{\mathbf{r}}(1))\\ &= f(e^2,e,e)-f(1,1,1)\\ &= e^4+e^3-1.\end{aligned}$$

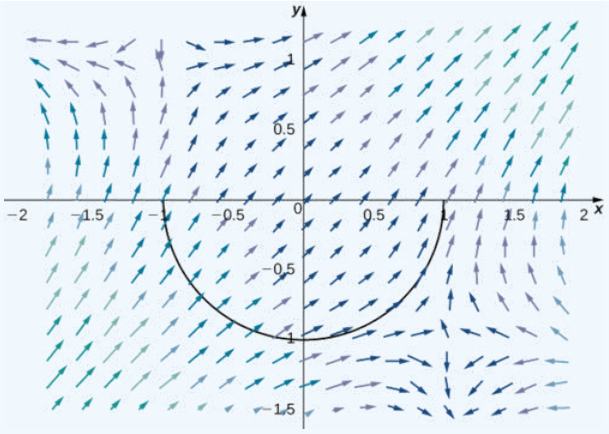
This calculation is much more straightforward than the calculation we did in (a). As long as we have a potential function, calculating a line integral using the Fundamental Theorem for Line Integrals is much easier than calculating without the theorem.

Example **3.4.3** illustrates a nice feature of the Fundamental Theorem of Line Integrals: it allows us to calculate more easily many vector line integrals. As long as we have a potential function, calculating the line integral is only a matter of evaluating the potential function at the endpoints and subtracting.



Exercise 3.4.3

Given that $f(x,y) = (x-1)^2y + (y+1)^2x$ is a potential function for $\vec{\mathbf{F}}(x,y) = \langle 2xy - 2y + (y+1)^2, (x-1)^2 + 2yx + 2x\rangle$, calculate integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where C is the lower half of the unit circle oriented counterclockwise.



Hint

The Fundamental Theorem for Line Intervals says this integral depends only on the value of f at the endpoints of C.

Answer

2

The Fundamental Theorem for Line Integrals has two important consequences. The first consequence is that if $\vec{\mathbf{F}}$ is conservative and C is a closed curve, then the circulation of $\vec{\mathbf{F}}$ along C is zero—that is, $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 0$. To see why this is true, let f be a potential function for $\vec{\mathbf{F}}$. Since C is a closed curve, the terminal point $\vec{\mathbf{r}}(b)$ of C is the same as the initial $\vec{\mathbf{r}}(a)$ of C—that is, $\vec{\mathbf{r}}(a) = \vec{\mathbf{r}}(b)$. Therefore, by the Fundamental Theorem for Line Integrals,

$$\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \oint_C \vec{\nabla} f \cdot d\vec{\mathbf{r}} \tag{3.4.4}$$

$$= f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(a)) \tag{3.4.5}$$

$$= f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(b)) \tag{3.4.6}$$

$$= 0. \tag{3.4.7}$$

Recall that the reason a conservative vector field $\vec{\mathbf{F}}$ is called "conservative" is because such vector fields model forces in which energy is conserved. We have shown gravity to be an example of such a force. If we think of vector field $\vec{\mathbf{F}}$ in integral $\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ as a gravitational field, then the equation $\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 0$ follows. If a particle travels along a path that starts and ends at the same place, then the work done by gravity on the particle is zero.

The second important consequence of the Fundamental Theorem for Line Integrals (Equation 3.4.2) is that line integrals of conservative vector fields are independent of path—meaning, they depend only on the endpoints of the given curve, and do not depend on the path between the endpoints.

DEFINITION: Path Independence

Let $\vec{\mathbf{F}}$ be a vector field with domain D; it is independent of path (or path independent) if

$$\int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \tag{3.4.8}$$

for any paths C_1 and C_2 in D with the same initial and terminal points.

The second consequence is stated formally in the following theorem.

Theorem: CONSERVATIVE FIELDS

If $\vec{\mathbf{F}}$ is a *conservative vector field*, then $\vec{\mathbf{F}}$ is independent of path.

Proof

Let D denote the domain of $\vec{\mathbf{F}}$ and let C_1 and C_2 be two paths in D with the same initial and terminal points (Figure 3.4.5). Call the initial point P_1 and the terminal point P_2. Since $\vec{\mathbf{F}}$ is conservative, there is a potential function f for $\vec{\mathbf{F}}$. By the Fundamental Theorem for Line Integrals,

$$\int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = f(P_2) - f(P_1) = \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}.$$

Therefore, $\int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ and $\vec{\mathbf{F}}$ is independent of path.

□

To visualize what independence of path means, imagine three hikers climbing from base camp to the top of a mountain. Hiker 1 takes a steep route directly from camp to the top. Hiker 2 takes a winding route that is not steep from camp to the top. Hiker 3 starts by taking the steep route but halfway to the top decides it is too difficult for him. Therefore he returns to camp and takes the non-steep path to the top. All three hikers are traveling along paths in a gravitational field. Since gravity is a force in which energy is conserved, the gravitational field is conservative. By independence of path, the total amount of work done by gravity on each of the hikers is the same because they all started in the same place and ended in the same place. The work done by the hikers includes other factors such as friction and muscle movement, so the total amount of energy each one expended is not the same, but the net energy expended against gravity is the same for all three hikers.

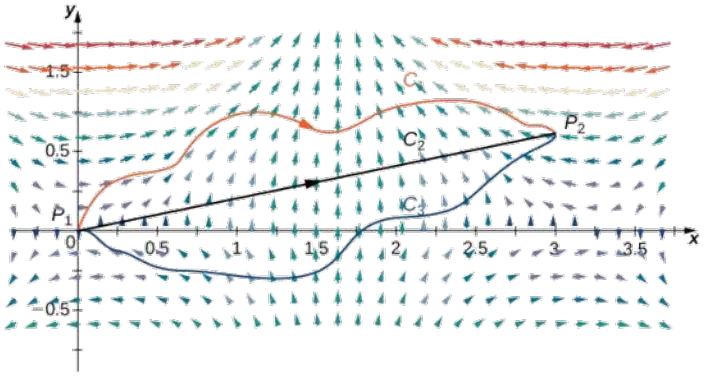


Figure 3.4.4: The vector field is conservative, and therefore independent of path.

We have shown that if $\vec{\mathbf{F}}$ is conservative, then $\vec{\mathbf{F}}$ is independent of path. It turns out that if the domain of $\vec{\mathbf{F}}$ is open and connected, then the converse is also true. That is, if $\vec{\mathbf{F}}$ is independent of path and the domain of $\vec{\mathbf{F}}$ is open and connected, then $\vec{\mathbf{F}}$ is conservative. Therefore, the set of conservative vector fields on open and connected domains is precisely the set of vector fields independent of path.

Theorem: THE PATH INDEPENDENCE TEST FOR CONSERVATIVE FIELDS

If $\vec{\mathbf{F}}$ is a continuous vector field that is independent of path and the domain D of $\vec{\mathbf{F}}$ is open and connected, then $\vec{\mathbf{F}}$ is conservative.

Proof

We prove the theorem for vector fields in $\mathbb{R}^2$. The proof for vector fields in $\mathbb{R}^3$ is similar. To show that $\vec{\mathbf{F}} = \langle P, Q \rangle$ is conservative, we must find a potential function f for $\vec{\mathbf{F}}$. To that end, let X be a fixed point in D. For any point (x, y) in D, let C be a path from X to (x, y). Define $f(x, y)$ by $f(x, y) = \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$. (Note that this definition of f makes sense only because $\vec{\mathbf{F}}$ is independent of path. If $\vec{\mathbf{F}}$ was not independent of path, then it might be possible to find another path C' from X to (x, y) such that $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \neq \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, and in such a case $f(x, y)$ would not be a function.) We want to show that f has the property $\vec{\nabla} f = \vec{\mathbf{F}}$.

Since domain D is open, it is possible to find a disk centered at (x, y) such that the disk is contained entirely inside D. Let (a, y) with $a < x$ be a point in that disk. Let C be a path from X to (x, y) that consists of two pieces: C_1 and C_2. The first piece, C_1, is

any path from C to (a,y) that stays inside D; C_2 is the horizontal line segment from (a,y) to (x,y) (Figure 3.4.6). Then

$$f(x,y) = \int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}.$$

The first integral does not depend on x, so

$$f_x(x,y) = \frac{\partial}{\partial x} \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}.$$

If we parameterize C_2 by $\vec{\mathbf{r}}(t) = \langle t, y \rangle$, $a \le t \le x$, then

$$\begin{aligned} f_x(x,y) &= \frac{\partial}{\partial x} \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \\ &= \frac{\partial}{\partial x} \int_a^x \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}'(t)\, dt \\ &= \frac{\partial}{\partial x} \int_a^x \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \frac{d}{dt}(\langle t, y \rangle)\, dt \\ &= \frac{\partial}{\partial x} \int_a^x \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \langle 1, 0 \rangle\, dt \\ &= \frac{\partial}{\partial x} \int_a^x P(t,y)\, dt. \end{aligned}$$

By the Fundamental Theorem of Calculus (part 1),

$$f_x(x,y) = \frac{\partial}{\partial x} \int_a^x P(t,y)\, dt = P(x,y).$$

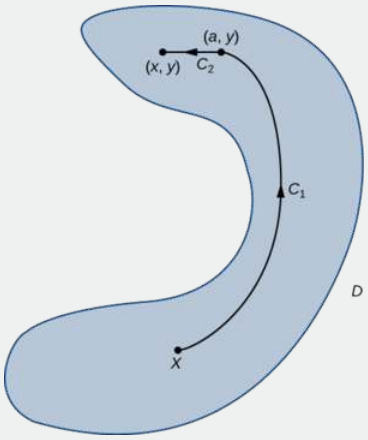


Figure 3.4.6. Here, C_1 is any path from C to (a,y) that stays inside D, and C_2 is the horizontal line segment from (a,y) to (x,y).

A similar argument using a vertical line segment rather than a horizontal line segment shows that $f_y(x,y) = Q(x,y)$.

Therefore $\vec{\nabla} f = \vec{\mathbf{F}}$ and $\vec{\mathbf{F}}$ is conservative.

□

We have spent a lot of time discussing and proving the theorems above, but we can summarize them simply: a vector field $\vec{\mathbf{F}}$ on an open and connected domain is conservative if and only if it is independent of path. This is important to know because conservative vector fields are extremely important in applications, and these theorems give us a different way of viewing what it means to be conservative using path independence.

✔ Example 3.4.4: Showing That a Vector Field Is Not Conservative

Use path independence to show that vector field $\vec{\mathbf{F}}(x,y) = \langle x^2 y, y + 5 \rangle$ is not conservative.

Solution

We can indicate that $\vec{\mathbf{F}}$ is not conservative by showing that $\vec{\mathbf{F}}$ is not path independent. We do so by giving two different paths, C_1 and C_2, that both start at $(0,0)$ and end at $(1,1)$, and yet $\int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \neq \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$.

Let C_1 be the curve with parameterization $\vec{\mathbf{r}}_1(t) = \langle t,\ t \rangle$, $0 \leq t \leq 1$ and let C_2 be the curve with parameterization $\vec{\mathbf{r}}_2(t) = \langle t,\ t^2 \rangle$, $0 \leq t \leq 1$ (Figure 3.4.7.). Then

$$\begin{aligned}\int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_0^1 \vec{\mathbf{F}}(\vec{\mathbf{r}}_1(t)) \cdot \vec{\mathbf{r}}_1{}'(t)\, dt \\ &= \int_0^1 \langle t^3, t+5 \rangle \cdot \langle 1,1 \rangle\, dt = \int_0^1 (t^3 + t + 5)\, dt \\ &= \left[\frac{t^4}{4} + \frac{t^2}{2} + 5t\right]_0^1 = \frac{23}{4}\end{aligned}$$

and

$$\begin{aligned}\int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_0^1 \vec{\mathbf{F}}(\vec{\mathbf{r}}_2(t)) \cdot \vec{\mathbf{r}}_2{}'(t)\, dt \\ &= \int_0^1 \langle t^4, t^2+5 \rangle \cdot \langle 1, 2t \rangle\, dt = \int_0^1 (t^4 + 2t^3 + 10t)\, dt \\ &= \left[\frac{t^5}{5} + \frac{t^4}{2} + 5t^2\right]_0^1 = \frac{57}{10}.\end{aligned}$$

Since $\int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \neq \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, the value of a line integral of $\vec{\mathbf{F}}$ depends on the path between two given points. Therefore, $\vec{\mathbf{F}}$ is not independent of path, and $\vec{\mathbf{F}}$ is not conservative.

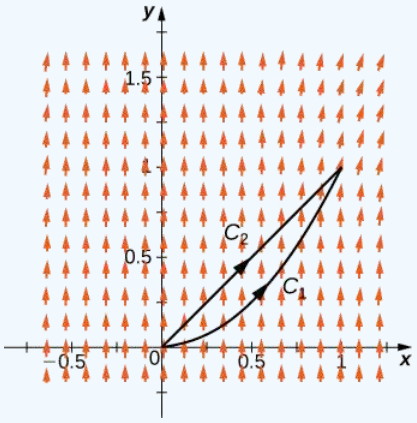


Figure 3.4.7: Curves C_1 and C_2 are both oriented from left to right.

? Exercise 3.4.4

Show that $\vec{\mathbf{F}}(x,y) = \langle xy,\ x^2y^2 \rangle$ is not path independent by considering the line segment from $(0,0)$ to $(0,2)$ and the piece of the graph of $y = \dfrac{x^2}{2}$ that goes from $(0,0)$ to $(0,2)$.

Hint

Calculate the corresponding line integrals.

Answer

If C_1 and C_2 represent the two curves, then

$$\int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \neq \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}.$$

Conservative Vector Fields and Potential Functions

As we have learned, the Fundamental Theorem for Line Integrals says that if $\vec{\mathbf{F}}$ is conservative, then calculating $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ has two steps: first, find a potential function f for $\vec{\mathbf{F}}$ and, second, calculate $f(P_1) - f(P_0)$, where P_1 is the endpoint of C and P_0 is the starting point. To use this theorem for a conservative field $\vec{\mathbf{F}}$, we must be able to find a potential function f for $\vec{\mathbf{F}}$. Therefore, we must answer the following question: Given a conservative vector field $\vec{\mathbf{F}}$, how do we find a function f such that $\vec{\nabla} f = \vec{\mathbf{F}}$? Before giving a general method for finding a potential function, let's motivate the method with an example.

Example 3.4.5: Finding a Potential Function

Find a potential function for $\vec{\mathbf{F}}(x, y) = \langle 2xy^3, 3x^2y^2 + \cos(y) \rangle$, thereby showing that $\vec{\mathbf{F}}$ is conservative.

Solution

Suppose that $f(x, y)$ is a potential function for $\vec{\mathbf{F}}$. Then, $\vec{\nabla} f = \vec{\mathbf{F}}$, and therefore

$$f_x(x, y) = 2xy^3 \quad \text{and} \quad f_y(x, y) = 3x^2y^2 + \cos y.$$

Integrating the equation $f_x(x, y) = 2xy^3$ with respect to x yields the equation

$$f(x, y) = x^2y^3 + h(y).$$

Notice that since we are integrating a two-variable function with respect to x, we must add a constant of integration that is a constant with respect to x, but may still be a function of y. The equation $f(x, y) = x^2y^3 + h(y)$ can be confirmed by taking the partial derivative with respect to x:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2y^3) + \frac{\partial}{\partial x}(h(y)) = 2xy^3 + 0 = 2xy^3.$$

Since f is a potential function for $\vec{\mathbf{F}}$,

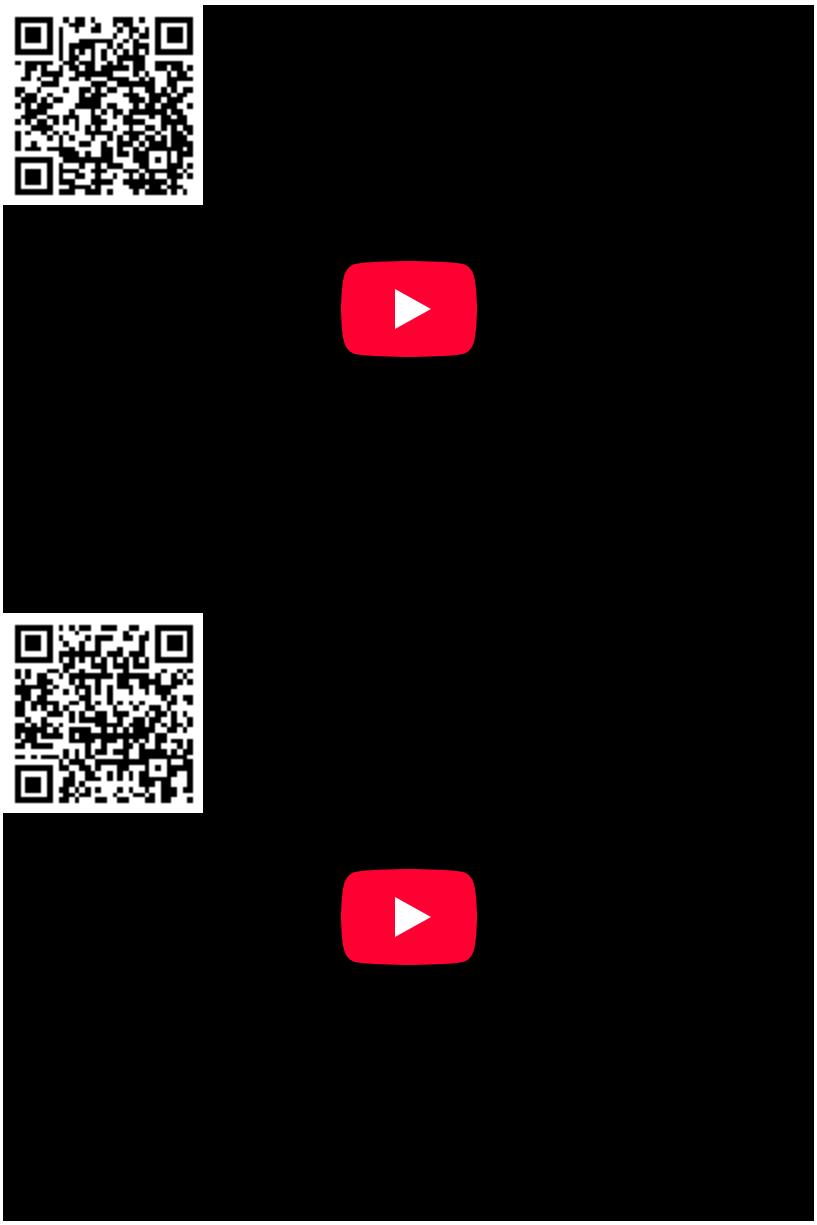
$$f_y(x, y) = 3x^2y^2 + \cos(y),$$

and therefore

$$3x^2y^2 + g'(y) = 3x^2y^2 + \cos(y).$$

This implies that $h'(y) = \cos y$, so $h(y) = \sin y + C$. Therefore, *any* function of the form $f(x, y) = x^2y^3 + \sin(y) + C$ is a potential function. Taking, in particular, $C = 0$ gives the potential function $f(x, y) = x^2y^3 + \sin(y)$.

To verify that f is a potential function, note that $\vec{\nabla} f(x, y) = \langle 2xy^3, 3x^2y^2 + \cos y \rangle = \vec{\mathbf{F}}$.



Exercise 3.4.5

Find a potential function for $\vec{\mathbf{F}}(x,y) = \langle e^x y^3 + y, 3e^x y^2 + x \rangle$.

Hint

Follow the steps in Example 3.4.5.

Answer

$f(x,y) = e^x y^3 + xy$

The logic of the previous example extends to finding the potential function for any conservative vector field in $\mathbb{R}^2$. Thus, we have the following problem-solving strategy for finding potential functions:

PROBLEM-SOLVING STRATEGY: FINDING A POTENTIAL FUNCTION FOR A CONSERVATIVE VECTOR FIELD $\vec{\mathbf{F}}(x,y) = \langle P(x,y), Q(x,y)\rangle$

1. Integrate P with respect to x. This results in a function of the form $g(x,y) + h(y)$, where $h(y)$ is unknown.
2. Take the partial derivative of $g(x,y) + h(y)$ with respect to y, which results in the function $g_y(x,y) + h'(y)$.
3. Use the equation $g_y(x,y) + h'(y) = Q(x,y)$ to find $h'(y)$.
4. Integrate $h'(y)$ to find $h(y)$.
5. Any function of the form $f(x,y) = g(x,y) + h(y) + C$, where C is a constant, is a potential function for $\vec{\mathbf{F}}$.

We can adapt this strategy to find potential functions for vector fields in $\mathbb{R}^3$, as shown in the next example.

Example 3.4.6: Finding a Potential Function in $\mathbb{R}^3$

Find a potential function for $F(x,y,z) = \langle 2xy, x^2 + 2yz^3, 3y^2z^2 + 2z\rangle$, thereby showing that $\vec{\mathbf{F}}$ is conservative.

Solution

Suppose that f is a potential function. Then, $\vec{\nabla} f = \vec{\mathbf{F}}$ and therefore $f_x(x,y,z) = 2xy$. Integrating this equation with respect to x yields the equation $f(x,y,z) = x^2y + g(y,z)$ for some function g. Notice that, in this case, the constant of integration with respect to x is a function of y and z.

Since f is a potential function,

$$x^2 + 2yz^3 = f_y(x,y,z) = x^2 + g_y(y,z).$$

Therefore,

$$g_y(y,z) = 2yz^3.$$

Integrating this function with respect to y yields

$$g(y,z) = y^2z^3 + h(z)$$

for some function $h(z)$ of z alone. (Notice that, because we know that g is a function of only y and z, we do not need to write $g(y,z) = y^2z^3 + h(x,z)$.) Therefore,

$$f(x,y,z) = x^2y + g(y,z) = x^2y + y^2z^3 + h(z).$$

To find f, we now must only find h. Since f is a potential function,

$$3y^2z^2 + 2z = g_z(y,z) = 3y^2z^2 + h'(z).$$

This implies that $h'(z) = 2z$, so $h(z) = z^2 + C$. Letting $C = 0$ gives the potential function

$$f(x,y,z) = x^2y + y^2z^3 + z^2.$$

To verify that f is a potential function, note that $\vec{\nabla} f(x,y,z) = \langle 2xy, x^2 + 2yz^3, 3y^2z^2 + 2z\rangle = \vec{\mathbf{F}}(x,y,z)$.

Exercise 3.4.6

Find a potential function for $\vec{\mathbf{F}}(x,y,z) = \langle 12x^2, \cos y \cos z, 1 - \sin y \sin z\rangle$.

Hint

Following Example **3.4.6**, begin by integrating with respect to x.

Answer

$f(x,y,z) = 4x^3 + \sin y \cos z + z$

We can apply the process of finding a potential function to a gravitational force. Recall that, if an object has unit mass and is located at the origin, then the gravitational force in $\mathbb{R}^2$ that the object exerts on another object of unit mass at the point (x,y) is

given by vector field

$$\vec{\mathbf{F}}(x,y) = -G\left\langle \frac{x}{(x^2+y^2)^{3/2}}, \frac{y}{(x^2+y^2)^{3/2}} \right\rangle,$$

where G is the universal gravitational constant. In the next example, we build a potential function for $\vec{\mathbf{F}}$, thus confirming what we already know: that gravity is conservative.

Example 3.4.7: Finding a Potential Function

Find a potential function f for $\vec{\mathbf{F}}(x,y) = -G\left\langle \frac{x}{(x^2+y^2)^{3/2}}, \frac{y}{(x^2+y^2)^{3/2}} \right\rangle$.

Solution

Suppose that f is a potential function. Then, $\vec{\nabla} f = \vec{\mathbf{F}}$ and therefore

$$f_x(x,y) = \frac{-Gx}{(x^2+y^2)^{3/2}}.$$

To integrate this function with respect to x, we can use u-substitution. If $u = x^2 + y^2$, then $\frac{du}{2} = x\,dx$, so

$$\begin{aligned}\int \frac{-Gx}{(x^2+y^2)^{3/2}}\,dx &= \int \frac{-G}{2u^{3/2}}\,du \\ &= \frac{G}{\sqrt{u}} + h(y) \\ &= \frac{G}{\sqrt{x^2+y^2}} + h(y)\end{aligned}$$

for some function $h(y)$. Therefore,

$$f(x,y) = \frac{G}{\sqrt{x^2+y^2}} + h(y).$$

Since f is a potential function for $\vec{\mathbf{F}}$,

$$f_y(x,y) = \frac{-Gy}{(x^2+y^2)^{3/2}}$$

.

Since $f(x,y) = \frac{G}{\sqrt{x^2+y^2}} + h(y)$, $f_y(x,y)$ also equals $\frac{-Gy}{(x^2+y^2)^{3/2}} + h'(y)$.

Therefore,

$$\frac{-Gy}{(x^2+y^2)^{3/2}} + h'(y) = \frac{-Gy}{(x^2+y^2)^{3/2}},$$

which implies that $h'(y) = 0$. Thus, we can take $h(y)$ to be any constant; in particular, we can let $h(y) = 0$. The function

$$f(x,y) = \frac{G}{\sqrt{x^2+y^2}}$$

is a potential function for the gravitational field $\vec{\mathbf{F}}$. To confirm that f is a potential function, note that

$$\begin{aligned}\vec{\nabla} f(x,y) &= \langle -\frac{1}{2}\frac{G}{(x^2+y^2)^{3/2}}(2x), -\frac{1}{2}\frac{G}{(x^2+y^2)^{3/2}}(2y)\rangle \\ &= \langle \frac{-Gx}{(x^2+y^2)^{3/2}}, \frac{-Gy}{(x^2+y^2)^{3/2}}\rangle \\ &= \vec{\mathbf{F}}(x,y).\end{aligned}$$

Exercise 3.4.7

Find a potential function f for the three-dimensional gravitational force $\vec{\mathbf{F}}(x,y,z) = \left\langle \frac{-Gx}{(x^2+y^2+z^2)^{3/2}}, \frac{-Gy}{(x^2+y^2+z^2)^{3/2}}, \frac{-Gz}{(x^2+y^2+z^2)^{3/2}} \right\rangle$.

Hint

Follow the Problem-Solving Strategy.

Answer

$$f(x,y,z) = \frac{G}{\sqrt{x^2+y^2+z^2}}$$

Testing a Vector Field

Until now, we have worked with vector fields that we know are conservative, but if we are not told that a vector field is conservative, we need to be able to test whether it is conservative. Recall that, if $\vec{\mathbf{F}}$ is conservative, then $\vec{\mathbf{F}}$ has the cross-partial property (see The Cross-Partial Property of Conservative Vector Fields). That is, if $\vec{\mathbf{F}} = \langle P, Q, R \rangle$ is conservative, then $P_y = Q_x$, $P_z = R_x$, and $Q_z = R_y$. So, if $\vec{\mathbf{F}}$ has the cross-partial property, then is $\vec{\mathbf{F}}$ conservative? If the domain of $\vec{\mathbf{F}}$ is open and simply connected, then the answer is yes.

Theorem: THE CROSS-PARTIAL TEST FOR CONSERVATIVE FIELDS

If $\vec{\mathbf{F}} = \langle P, Q, R \rangle$ is a vector field on an open, simply connected region D and $P_y = Q_x$, $P_z = R_x$, and $Q_z = R_y$ throughout D, then $\vec{\mathbf{F}}$ is conservative.

Although a proof of this theorem is beyond the scope of the text, we can discover its power with some examples. Later, we see why it is necessary for the region to be simply connected.

Combining this theorem with the cross-partial property, we can determine whether a given vector field is conservative:

Theorem: CROSS-PARTIAL PROPERTY OF CONSERVATIVE FIELDS

Let $\vec{\mathbf{F}} = \langle P, Q, R \rangle$ be a vector field on an open, simply connected region D. Then $P_y = Q_x$, $P_z = R_x$, and $Q_z = R_y$ throughout D if and only if $\vec{\mathbf{F}}$ is conservative.

The version of this theorem in $\mathbb{R}^2$ is also true. If $\vec{\mathbf{F}}(x,y) = \langle P, Q \rangle$ is a vector field on an open, simply connected domain in $\mathbb{R}^2$, then $\vec{\mathbf{F}}$ is conservative if and only if $P_y = Q_x$.

Example 3.4.8: Determining Whether a Vector Field Is Conservative

Determine whether vector field $\vec{\mathbf{F}}(x,y,z) = \langle xy^2z, x^2yz, z^2 \rangle$ is conservative.

Solution

Note that the domain of $\vec{\mathbf{F}}$ is all of $\mathbb{R}^2$ and $\mathbb{R}^3$ is simply connected. Therefore, we can use **The Cross-Partial Property of Conservative Vector Fields** to determine whether $\vec{\mathbf{F}}$ is conservative. Let

$$P(x,y,z) = xy^2z$$

$$Q(x,y,z) = x^2yz$$

and

$$R(x,y,z) = z^2.$$

Since $Q_z(x,y,z) = x^2y$ and $R_y(x,y,z) = 0$, the vector field is not conservative.

Example 3.4.9: Determining Whether a Vector Field Is Conservative

Determine vector field $\vec{\mathbf{F}}(x,y) = \langle x\ln(y), \frac{x^2}{2y}\rangle$ is conservative.

Solution

Note that the domain of $\vec{\mathbf{F}}$ is the part of $\mathbb{R}^2$ in which $y > 0$. Thus, the domain of $\vec{\mathbf{F}}$ is part of a plane above the x-axis, and this domain is simply connected (there are no holes in this region and this region is connected). Therefore, we can use **The Cross-Partial Property of Conservative Vector Fields** to determine whether $\vec{\mathbf{F}}$ is conservative. Let

$$P(x,y) = x\ln(y) \quad \text{and} \quad Q(x,y) = \frac{x^2}{2y}.$$

Then $P_y(x,y) = \frac{x}{y} = Q_x(x,y)$ and thus $\vec{\mathbf{F}}$ is conservative.



Exercise 3.4.8

Determine whether $\vec{\mathbf{F}}(x,y) = \langle \sin x\cos y, \cos x \sin y\rangle$ is conservative.

Hint

Use **The Cross-Partial Property of Conservative Vector Fields** from the previous section.

Answer

It is conservative.

When using **The Cross-Partial Property of Conservative Vector Fields**, it is important to remember that a theorem is a tool, and like any tool, it can be applied only under the right conditions. In the case of **The Cross-Partial Property of Conservative Vector Fields**, the theorem can be applied only if the domain of the vector field is simply connected.

To see what can go wrong when misapplying the theorem, consider the vector field from Example 3.4.4:

$$\vec{\mathbf{F}}(x,y) = \frac{y}{x^2+y^2}\,\hat{\mathbf{i}} + \frac{-x}{x^2+y^2}\,\hat{\mathbf{j}}. \tag{3.4.9}$$

This vector field satisfies the cross-partial property, since

$$\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)=\frac{(x^2+y^2)-y(2y)}{(x^2+y^2)^2}=\frac{x^2-y^2}{(x^2+y^2)^2} \tag{3.4.10}$$

and

$$\frac{\partial}{\partial x}\left(\frac{-x}{x^2+y^2}\right)=\frac{-(x^2+y^2)+x(2x)}{(x^2+y^2)^2}=\frac{x^2-y^2}{(x^2+y^2)^2}. \tag{3.4.11}$$

Since $\vec{\mathbf{F}}$ satisfies the cross-partial property, we might be tempted to conclude that $\vec{\mathbf{F}}$ is conservative. However, $\vec{\mathbf{F}}$ is not conservative. To see this, let

$$\vec{\mathbf{r}}(t)=\langle\cos t,\sin t\rangle,\quad 0\le t\le\pi \tag{3.4.12}$$

be a parameterization of the upper half of a unit circle oriented counterclockwise (denote this C_1) and let

$$\vec{\mathbf{s}}(t)=\langle\cos t,-\sin t\rangle,\quad 0\le t\le\pi \tag{3.4.13}$$

be a parameterization of the lower half of a unit circle oriented clockwise (denote this C_2). Notice that C_1 and C_2 have the same starting point and endpoint. Since $\sin^2 t+\cos^2 t=1$,

$$\vec{\mathbf{F}}(\vec{\mathbf{r}}(t))\cdot\vec{\mathbf{r}}'(t)=\langle\sin(t),-\cos(t)\rangle\cdot\langle-\sin(t),\cos(t)\rangle=-1 \tag{3.4.14}$$

and

$$\vec{\mathbf{F}}(\vec{\mathbf{s}}(t))\cdot\vec{\mathbf{s}}'(t)=\langle-\sin t,-\cos t\rangle\cdot\langle-\sin t,-\cos t\rangle=\sin^2 t+\cos^2 t=1. \tag{3.4.15}$$

Therefore,

$$\int_{C_1}\vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}=\int_0^{\pi}-1\,dt=-\pi \tag{3.4.16}$$

and

$$\int_{C_2}\vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}=\int_0^{\pi}1\,dt=\pi. \tag{3.4.17}$$

Thus, C_1 and C_2 have the same starting point and endpoint, but $\int_{C_1}\vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}\neq\int_{C_2}\vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}$. Therefore, $\vec{\mathbf{F}}$ is not independent of path and $\vec{\mathbf{F}}$ is not conservative.

To summarize: $\vec{\mathbf{F}}$ satisfies the cross-partial property and yet $\vec{\mathbf{F}}$ is not conservative. What went wrong? Does this contradict **The Cross-Partial Property of Conservative Vector Fields**? The issue is that the domain of $\vec{\mathbf{F}}$ is all of $\mathbb{R}^2$ except for the origin. In other words, the domain of $\vec{\mathbf{F}}$ has a hole at the origin, and therefore the domain is not simply connected. Since the domain is not simply connected, **The Cross-Partial Property of Conservative Vector Fields** does not apply to $\vec{\mathbf{F}}$.

We close this section by looking at an example of the usefulness of the Fundamental Theorem for Line Integrals. Now that we can test whether a vector field is conservative, we can always decide whether the Fundamental Theorem for Line Integrals can be used to calculate a vector line integral. If we are asked to calculate an integral of the form $\int_C\vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}$, then our first question should be: Is $\vec{\mathbf{F}}$ conservative? If the answer is yes, then we should find a potential function and use the Fundamental Theorem for Line Integrals to calculate the integral. If the answer is no, then the Fundamental Theorem for Line Integrals cannot help us and we have to use other methods, such as using the method from the previous section (using $\vec{\mathbf{F}}(\vec{\mathbf{r}}(t))$ and $\vec{\mathbf{r}}'(t)$).

✔ Example 3.4.10: Using the Fundamental Theorem for Line Integrals

Calculate line integral $\int_C\vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}$, where $\vec{\mathbf{F}}(x,y,z)=\langle 2xe^yz+e^xz,\ x^2e^yz,\ x^2e^y+e^x\rangle$ and C is any smooth curve that goes from the origin to $(1,1,1)$.

Solution

Before trying to compute the integral, we need to determine whether $\vec{\mathbf{F}}$ is conservative and whether the domain of $\vec{\mathbf{F}}$ is simply connected. The domain of $\vec{\mathbf{F}}$ is all of $\mathbb{R}^3$, which is connected and has no holes. Therefore, the domain of $\vec{\mathbf{F}}$ is simply connected. Let

$$P(x,y,z) = 2xe^y z + e^x z, \quad Q(x,y,z) = x^2 e^y z, \quad \text{and} \quad R(x,y,z) = x^2 e^y + e^x$$

so that $\vec{\mathbf{F}}(x,y,z) = \langle P, Q, R \rangle$. Since the domain of $\vec{\mathbf{F}}$ is simply connected, we can check the cross partials to determine whether $\vec{\mathbf{F}}$ is conservative. Note that

$$\begin{aligned} P_y(x,y,z) &= 2xe^y z = Q_x(x,y,z) \\ P_z(x,y,z) &= 2xe^y + e^x = R_x(x,y,z) \\ Q_z(x,y,z) &= x^2 e^y = R_y(x,y,z). \end{aligned}$$

Therefore, $\vec{\mathbf{F}}$ is conservative.

To evaluate $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ using the Fundamental Theorem for Line Integrals, we need to find a potential function f for $\vec{\mathbf{F}}$. Let f be a potential function for $\vec{\mathbf{F}}$. Then, $\vec{\nabla} f = \vec{\mathbf{F}}$, and therefore $f_x(x,y,z) = 2xe^y z + e^x z$. Integrating this equation with respect to x gives $f(x,y,z) = x^2 e^y z + e^x z + h(y,z)$ for some function h. Differentiating this equation with respect to y gives $x^2 e^y z + h_y(y,z) = Q(x,y,z) = x^2 e^y z$, which implies that $h_y(y,z) = 0$. Therefore, h is a function of z only, and $f(x,y,z) = x^2 e^y z + e^x z + h(z)$. To find h, note that $f_z = x^2 e^y + e^x + h'(z) = R = x^2 e^y + e^x$. Therefore, $h'(z) = 0$ and we can take $h(z) = 0$. A potential function for $\vec{\mathbf{F}}$ is $f(x,y,z) = x^2 e^y z + e^x z$.

Now that we have a potential function, we can use the Fundamental Theorem for Line Integrals to evaluate the integral. By the theorem,

$$\begin{aligned} \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_C \vec{\nabla} f \cdot d\vec{\mathbf{r}} \\ &= f(1,1,1) - f(0,0,0) \\ &= 2e. \end{aligned}$$

Analysis

Notice that if we hadn't recognized that $\vec{\mathbf{F}}$ is conservative, we would have had to parameterize C and use the method from the previous section. Since curve C is unknown, using the Fundamental Theorem for Line Integrals is much simpler.



? Exercise 3.4.9

Calculate integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where $\vec{\mathbf{F}}(x,y) = \langle \sin x \sin y, 5 - \cos x \cos y \rangle$ and C is a semicircle with starting point $(0,\pi)$ and endpoint $(0,-\pi)$.

Hint

Use the Fundamental Theorem for Line Integrals.

Answer

-10π

Example 3.4.11: Work Done on a Particle

Let $\vec{\mathbf{F}}(x,y) = \langle 2xy^2, 2x^2y\rangle$ be a force field. Suppose that a particle begins its motion at the origin and ends its movement at any point in a plane that is not on the x-axis or the y-axis. Furthermore, the particle's motion can be modeled with a smooth parameterization. Show that $\vec{\mathbf{F}}$ does positive work on the particle.

Solution

We show that $\vec{\mathbf{F}}$ does positive work on the particle by showing that $\vec{\mathbf{F}}$ is conservative and then by using the Fundamental Theorem for Line Integrals.

To show that $\vec{\mathbf{F}}$ is conservative, suppose $f(x,y)$ were a potential function for $\vec{\mathbf{F}}$. Then, $\vec{\nabla} f(x,y) = \vec{\mathbf{F}}(x,y) = \langle 2xy^2, 2x^2y\rangle$ and therefore $f_x(x,y) = 2xy^2$ and $f_y(x,y) = 2x^2y$. The equation $fx(x,y) = 2xy^2$ implies that $f(x,y) = x^2y^2 + h(y)$. Deriving both sides with respect to y yields $f_y(x,y) = 2x^2y + h'(y)$. Therefore, $h'(y) = 0$ and we can take $h(y) = 0$.

If $f(x,y) = x^2y^2$, then note that $\vec{\nabla} f(x,y) = \langle 2xy^2, 2x^2y\rangle = \vec{\mathbf{F}}$, and therefore f is a potential function for $\vec{\mathbf{F}}$.

Let (a,b) be the point at which the particle stops is motion, and let C denote the curve that models the particle's motion. The work done by $\vec{\mathbf{F}}$ on the particle is $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$. By the Fundamental Theorem for Line Integrals,

$$\begin{aligned}\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_C \nabla f \cdot d\vec{\mathbf{r}} \\ &= f(a,b) - f(0,0) \\ &= a^2b^2.\end{aligned}$$

Since $a \neq 0$ and $b \neq 0$, by assumption, $a^2b^2 > 0$. Therefore, $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} > 0$, and $\vec{\mathbf{F}}$ does positive work on the particle.

Analysis

Notice that this problem would be much more difficult without using the Fundamental Theorem for Line Integrals. To apply the tools we have learned, we would need to give a curve parameterization and use the method from the previous section. Since the path of motion C can be as exotic as we wish (as long as it is smooth), it can be very difficult to parameterize the motion of the particle.



? Exercise 3.4.10

Let $\vec{\mathbf{F}}(x,y) = \langle 4x^3y^4, 4x^4y^3 \rangle$, and suppose that a particle moves from point $(4,4)$ to $(1,1)$ along any smooth curve. Is the work done by $\vec{\mathbf{F}}$ on the particle positive, negative, or zero?

Hint

Use the Fundamental Theorem for Line Integrals.

Answer

Negative

Key Concepts

- The theorems in this section require curves that are closed, simple, or both, and regions that are connected or simply connected.
- The line integral of a conservative vector field can be calculated using the Fundamental Theorem for Line Integrals. This theorem is a generalization of the Fundamental Theorem of Calculus in higher dimensions. Using this theorem usually makes the calculation of the line integral easier.
- Conservative fields are independent of path. The line integral of a conservative field depends only on the value of the potential function at the endpoints of the domain curve.
- Given vector field $\vec{\mathbf{F}}$, we can test whether $\vec{\mathbf{F}}$ is conservative by using the cross-partial property. If $\vec{\mathbf{F}}$ has the cross-partial property and the domain is simply connected, then $\vec{\mathbf{F}}$ is conservative (and thus has a potential function). If $\vec{\mathbf{F}}$ is conservative, we can find a potential function by using the Problem-Solving Strategy.
- The circulation of a conservative vector field on a simply connected domain over a closed curve is zero.

Key Equations

- **Fundamental Theorem for Line Integrals**
  $\int_C \vec{\nabla} f \cdot d\vec{\mathbf{r}} = f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(a))$
- **Circulation of a conservative field over curve C that encloses a simply connected region**
  $\oint_C \vec{\nabla} f \cdot d\vec{\mathbf{r}} = 0$

Glossary

closed curve

a curve that begins and ends at the same point

connected region

a region in which any two points can be connected by a path with a trace contained entirely inside the region

Fundamental Theorem for Line Integrals

the value of line integral $\int_C \vec{\nabla} f \cdot d\vec{\mathbf{r}}$ depends only on the value of f at the endpoints of C: $\int_C \vec{\nabla} f \cdot d\vec{\mathbf{r}} = f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(a))$

independence of path

a vector field $\vec{\mathbf{F}}$ has path independence if $\int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ for any curves C_1 and C_2 in the domain of $\vec{\mathbf{F}}$ with the same initial points and terminal points

simple curve

a curve that does not cross itself

simply connected region

a region that is connected and has the property that any closed curve that lies entirely inside the region encompasses points that are entirely inside the region

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3.4E: Exercises

1. *True* or *False?* If vector field $\vec{\mathbf{F}}$ is conservative on the open and connected region D, then line integrals of $\vec{\mathbf{F}}$ are path independent on D, regardless of the shape of D.

2. *True* or *False?* Function $\vec{\mathbf{r}}(t) = \vec{\mathbf{a}} + t(\vec{\mathbf{b}} - \vec{\mathbf{a}})$, where $0 \le t \le 1$, parameterizes the straight-line segment from $\vec{\mathbf{a}}$ to $\vec{\mathbf{b}}$.

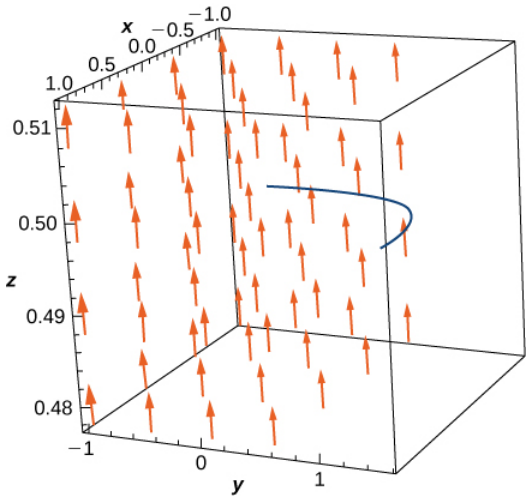
3. *True* or *False?* Vector field $\vec{\mathbf{F}}(x, y, z) = (y \sin z)\,\hat{\mathbf{i}} + (x \sin z)\,\hat{\mathbf{j}} + (xy \cos z)\,\hat{\mathbf{k}}$ is conservative.

4. *True* or *False?* Vector field $\vec{\mathbf{F}}(x, y, z) = y\,\hat{\mathbf{i}} + (x + z)\,\hat{\mathbf{j}} - y\,\hat{\mathbf{k}}$ is conservative.

5. Justify the Fundamental Theorem of Line Integrals for $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ in the case when $\vec{\mathbf{F}}(x, y) = (2x + 2y)\,\hat{\mathbf{i}} + (2x + 2y)\,\hat{\mathbf{j}}$ and C is a portion of the positively oriented circle $x^2 + y^2 = 25$ from $(5, 0)$ to $(3, 4)$.

6. [T] Find $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where $\vec{\mathbf{F}}(x, y) = (ye^{xy} + \cos x)\,\hat{\mathbf{i}} + \left(xe^{xy} + \frac{1}{y^2+1}\right)\hat{\mathbf{j}}$ and C is a portion of curve $y = \sin x$ from $x = 0$ to $x = \frac{\pi}{2}$.

7. [T] Evaluate line integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where $\vec{\mathbf{F}}(x, y) = (e^x \sin y - y)\,\hat{\mathbf{i}} + (e^x \cos y - x - 2)\,\hat{\mathbf{j}}$, and C is the path given by $\vec{\mathbf{r}}(t) = (t^3 \sin \frac{\pi t}{2})\,\hat{\mathbf{i}} - (\frac{\pi}{2}\cos(\frac{\pi t}{2} + \frac{\pi}{2}))\,\hat{\mathbf{j}}$ for $0 \le t \le 1$.



For the following exercises, determine whether the vector field is conservative and, if it is, find the potential function.

8. $\vec{\mathbf{F}}(x, y) = 2xy^3\,\hat{\mathbf{i}} + 3y^2x^2\,\hat{\mathbf{j}}$

9. $\vec{\mathbf{F}}(x, y) = (-y + e^x \sin y)\,\hat{\mathbf{i}} + ((x + 2)e^x \cos y)\,\hat{\mathbf{j}}$

10. $\vec{\mathbf{F}}(x, y) = (e^{2x} \sin y)\,\hat{\mathbf{i}} + (e^{2x} \cos y)\,\hat{\mathbf{j}}$

11. $\vec{\mathbf{F}}(x, y) = (6x + 5y)\,\hat{\mathbf{i}} + (5x + 4y)\,\hat{\mathbf{j}}$

12. $\vec{\mathbf{F}}(x, y) = (2x \cos(y) - y \cos(x))\,\hat{\mathbf{i}} + (-x^2 \sin(y) - \sin(x))\,\hat{\mathbf{j}}$

13. $\vec{\mathbf{F}}(x, y) = (ye^x + \sin(y))\,\hat{\mathbf{i}} + (e^x + x \cos(y))\,\hat{\mathbf{j}}$

For the following exercises, evaluate the line integrals using the Fundamental Theorem of Line Integrals.

14. $\oint_C (y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}}) \cdot d\vec{\mathbf{r}}$, where C is any path from $(0, 0)$ to $(2, 4)$

15. $\oint_C (2y\,dx + 2x\,dy)$, where C is the line segment from $(0, 0)$ to $(4, 4)$

16. [T] $\oint_C \left[\arctan\frac{y}{x} - \frac{xy}{x^2 + y^2}\right] dx + \left[\frac{x^2}{x^2 + y^2} + e^{-y}(1 - y)\right] dy$, where C is any smooth curve from $(1, 1)$ to $(-1, 2)$.

17. Find the conservative vector field for the potential function $f(x, y) = 5x^2 + 3xy + 10y^2$.

For the following exercises, determine whether the vector field is conservative and, if so, find a potential function.

18. $\vec{\mathbf{F}}(x,y) = (12xy)\,\hat{\mathbf{i}} + 6(x^2+y^2)\,\hat{\mathbf{j}}$

19. $\vec{\mathbf{F}}(x,y) = (e^x \cos y)\,\hat{\mathbf{i}} + 6(e^x \sin y)\,\hat{\mathbf{j}}$

20. $\vec{\mathbf{F}}(x,y) = (2xye^{x^2y})\,\hat{\mathbf{i}} + 6(x^2e^{x^2y})\,\hat{\mathbf{j}}$

21. $\vec{\mathbf{F}}(x,y,z) = (ye^z)\,\hat{\mathbf{i}} + (xe^z)\,\hat{\mathbf{j}} + (xye^z)\,\hat{\mathbf{k}}$

22. $\vec{\mathbf{F}}(x,y,z) = (\sin y)\,\hat{\mathbf{i}} - (x\cos y)\,\hat{\mathbf{j}} + \hat{\mathbf{k}}$

23. $\vec{\mathbf{F}}(x,y,z) = \dfrac{1}{y}\,\hat{\mathbf{i}} - \dfrac{x}{y^2}\,\hat{\mathbf{j}} + (2z-1)\,\hat{\mathbf{k}}$

24. $\vec{\mathbf{F}}(x,y,z) = 3z^2\,\hat{\mathbf{i}} - \cos y\,\hat{\mathbf{j}} + 2xz\,\hat{\mathbf{k}}$

25. $\vec{\mathbf{F}}(x,y,z) = (2xy)\,\hat{\mathbf{i}} + (x^2+2yz)\,\hat{\mathbf{j}} + y^2\,\hat{\mathbf{k}}$

For the following exercises, determine whether the given vector field is conservative and find a potential function.

26. $\vec{\mathbf{F}}(x,y) = (e^x \cos y)\,\hat{\mathbf{i}} + 6(e^x \sin y)\,\hat{\mathbf{j}}$

27. $\vec{\mathbf{F}}(x,y) = (2xye^{x^2y})\,\hat{\mathbf{i}} + (x^2e^{x^2y})\,\hat{\mathbf{j}}$

For the following exercises, evaluate the integral using the Fundamental Theorem of Line Integrals.

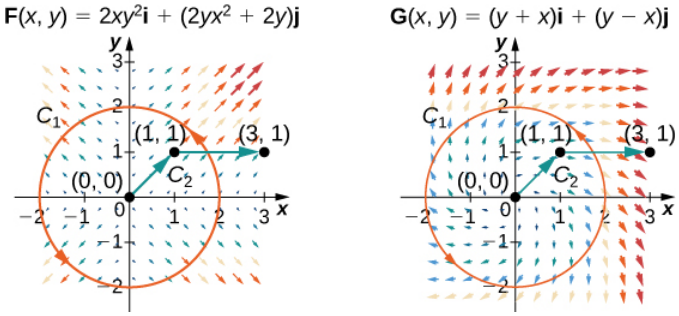
28. Evaluate $\int_C \vec{\nabla} f \cdot d\vec{\mathbf{r}}$, where $f(x,y,z) = \cos(\pi x) + \sin(\pi y) - xyz$ and C is any path that starts at $(1,12,2)$ and ends at $(2,1,-1)$.

29. [T] Evaluate $\int_C \vec{\nabla} f \cdot d\vec{\mathbf{r}}$, where $f(x,y) = xy + e^x$ and C is a straight line from $(0,0)$ to $(2,1)$.

30. [T] Evaluate $\int_C \vec{\nabla} f \cdot d\vec{\mathbf{r}}$, where $f(x,y) = x^2y - x$ and C is any path in a plane from (1, 2) to (3, 2).

31. Evaluate $\int_C \vec{\nabla} f \cdot d\vec{\mathbf{r}}$, where $f(x,y,z) = xyz^2 - yz$ and C has initial point $(1,2,3)$ and terminal point $(3,5,2)$.

For the following exercises, let $\vec{\mathbf{F}}(x,y) = 2xy^2\,\hat{\mathbf{i}} + (2yx^2+2y)\,\hat{\mathbf{j}}$ and $G(x,y) = (y+x)\,\hat{\mathbf{i}} + (y-x)\,\hat{\mathbf{j}}$, and let C_1 be the curve consisting of the circle of radius 2, centered at the origin and oriented counterclockwise, and C_2 be the curve consisting of a line segment from $(0,0)$ to $(1,1)$ followed by a line segment from $(1,1)$ to $(3,1)$.



32. Calculate the line integral of $\vec{\mathbf{F}}$ over C_1.

33. Calculate the line integral of $\vec{\mathbf{G}}$ over C_1.

34. Calculate the line integral of $\vec{\mathbf{F}}$ over C_2.

35. Calculate the line integral of $\vec{\mathbf{G}}$ over C_2.

36. [T] Let $\vec{\mathbf{F}}(x,y,z) = x^2\,\hat{\mathbf{i}} + z\sin(yz)\,\hat{\mathbf{j}} + y\sin(yz)\,\hat{\mathbf{k}}$. Calculate $\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where C is a path from $A = (0,0,1)$ to $B = (3,1,2)$.

37. [T] Find line integral $\oint_C \vec{\mathbf{F}} \cdot dr$ of vector field $\vec{\mathbf{F}}(x,y,z) = 3x^2z\,\hat{\mathbf{i}} + z^2\,\hat{\mathbf{j}} + (x^3+2yz)\,\hat{\mathbf{k}}$ along curve C parameterized by $\vec{\mathbf{r}}(t) = \left(\frac{\ln t}{\ln 2}\right)\hat{\mathbf{i}} + t^{3/2}\,\hat{\mathbf{j}} + t\cos(\pi t), 1 \le t \le 4$.

https://math.libretexts.org/@go/page/84752

For exercises 38 - 40, show that the following vector fields are conservative. Then calculate $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ for the given curve.

38. $\vec{\mathbf{F}}(x,y) = (xy^2 + 3x^2y)\,\hat{\mathbf{i}} + (x+y)x^2\,\hat{\mathbf{j}}$; C is the curve consisting of line segments from $(1,1)$ to $(0,2)$ to $(3,0)$.

39. $\vec{\mathbf{F}}(x,y) = \dfrac{2x}{y^2+1}\,\hat{\mathbf{i}} - \dfrac{2y(x^2+1)}{(y^2+1)^2}\,\hat{\mathbf{j}}$; C is parameterized by $x = t^3 - 1,\ y = t^6 - t$, for $0 \le t \le 1$.

40. [T] $\vec{\mathbf{F}}(x,y) = [\cos(xy^2) - xy^2\sin(xy^2)]\,\hat{\mathbf{i}} - 2x^2y\sin(xy^2)\,\hat{\mathbf{j}}$; C is the curve $\langle e^t, e^{t+1}\rangle$, for $-1 \le t \le 0$.

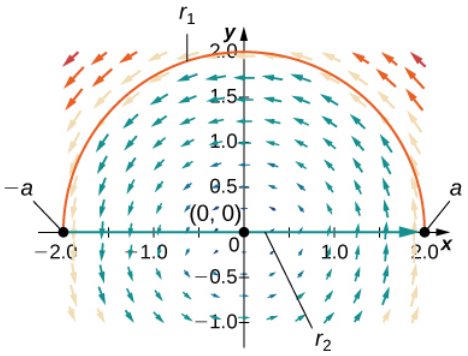
41. The mass of Earth is approximately $6 \times 10^{27} g$ and that of the Sun is 330,000 times as much. The gravitational constant is $6.7 \times 10^{-8} cm^3/s^2 \cdot g$. The distance of Earth from the Sun is about $1.5 \times 10^{12} cm$. Compute, approximately, the work necessary to increase the distance of Earth from the Sun by $1\ cm$.

42. [T] Let $\vec{\mathbf{F}}(x,y,z) = (e^x \sin y)\,\hat{\mathbf{i}} + (e^x \cos y)\,\hat{\mathbf{j}} + z^2\,\hat{\mathbf{k}}$. Evaluate the integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where $\vec{\mathbf{r}}(t) = \langle \sqrt{t}, t^3, e^{\sqrt{t}}\rangle$, for $0 \le t \le 1$.

43. [T] Let $C : [1,2] \to \mathbb{R}^2$ be given by $x = e^{t-1}, y = \sin\left(\frac{\pi}{t}\right)$. Use a computer to compute the integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C 2x\cos y\,dx - x^2 \sin y\,dy$, where $\vec{\mathbf{F}}(x,y) = (2x\cos y)\,\hat{\mathbf{i}} - (x^2\sin y)\,\hat{\mathbf{j}}$.

44. [T] Use a computer algebra system to find the mass of a wire that lies along the curve $\vec{\mathbf{r}}(t) = (t^2 - 1)\,\hat{\mathbf{j}} + 2t\,\hat{\mathbf{k}}$, where $0 \le t \le 1$, if the density is given by $d(t) = \dfrac{3}{2}t$.

45. Find the circulation and flux of field $\vec{\mathbf{F}}(x,y) = -y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}}$ around and across the closed semicircular path that consists of semicircular arch $\vec{\mathbf{r}}_1(t) = (a\cos t)\,\hat{\mathbf{i}} + (a\sin t)\,\hat{\mathbf{j}}$, $0 \le t \le \pi$, followed by line segment $\vec{\mathbf{r}}_2(t) = t\,\hat{\mathbf{i}}$, $-a \le t \le a$.



46. Compute $\int_C \cos x \cos y\,dx - \sin x \sin y\,dy$, where $\vec{\mathbf{r}}(t) = \langle t, t^2\rangle$, $0 \le t \le 1$.

47. Complete the proof of the theorem titled THE PATH INDEPENDENCE TEST FOR CONSERVATIVE FIELDS by showing that $f_y = Q(x,y)$.

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3.5: Green's Theorem

Learning Objectives

- Apply the circulation form of Green's theorem.
- Apply the flux form of Green's theorem.
- Calculate circulation and flux on more general regions.

In this section, we examine Green's theorem, which is an extension of the Fundamental Theorem of Calculus to two dimensions. Green's theorem has two forms: a circulation form and a flux form, both of which require region D in the double integral to be simply connected. However, we will extend Green's theorem to regions that are not simply connected.

Put simply, Green's theorem relates a line integral around a simply closed plane curve C and a double integral over the region enclosed by C. The theorem is useful because it allows us to translate difficult line integrals into more simple double integrals, or difficult double integrals into more simple line integrals.

Extending the Fundamental Theorem of Calculus

Recall that the Fundamental Theorem of Calculus says that

$$\int_a^b F'(x)\,dx = F(b) - F(a). \tag{3.5.1}$$

As a geometric statement, this equation says that the integral over the region below the graph of $F'(x)$ and above the line segment $[a, b]$ depends only on the value of F at the endpoints a and b of that segment. Since the numbers a and b are the boundary of the line segment $[a, b]$, the theorem says we can calculate integral $\int_a^b F'(x)\,dx$ based on information about the boundary of line segment $[a, b]$ (Figure 3.5.1). The same idea is true of the Fundamental Theorem for Line Integrals:

$$\int_C \vec{\nabla} f \cdot d\vec{\mathbf{r}} = f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(a)). \tag{3.5.2}$$

When we have a potential function (an "antiderivative"), we can calculate the line integral based solely on information about the boundary of curve C.

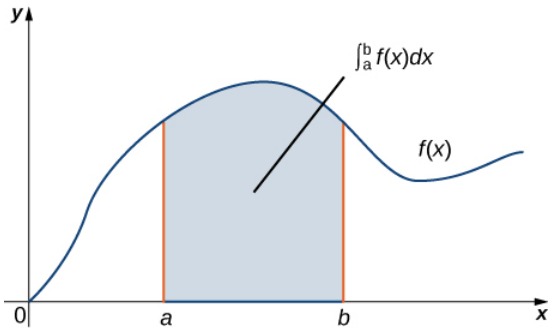


Figure 3.5.1: The Fundamental Theorem of Calculus says that the integral over line segment $[a, b]$ depends only on the values of the antiderivative at the endpoints of $[a, b]$.

Green's theorem takes this idea and extends it to calculating double integrals. Green's theorem says that we can calculate a double integral over region D based solely on information about the boundary of D. Green's theorem also says we can calculate a line integral over a simple closed curve C based solely on information about the region that C encloses. In particular, Green's theorem connects a double integral over region D to a line integral around the boundary of D.

Circulation Form of Green's Theorem

The first form of Green's theorem that we examine is the circulation form. This form of the theorem relates the vector line integral over a simple, closed plane curve C to a double integral over the region enclosed by C. Therefore, the circulation of a vector field along a simple closed curve can be transformed into a double integral and vice versa.

GREEN'S THEOREM (CIRCULATION FORM)

Let D be an open, simply connected region with a boundary curve C that is a piecewise smooth, simple closed curve oriented counterclockwise (Figure **3.5.1**). Let $\vec{\mathbf{F}} = \langle P, Q \rangle$ be a vector field with component functions that have continuous partial derivatives on D. Then,

$$\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \oint_C P\,dx + Q\,dy \tag{3.5.3}$$

$$= \iint_D (Q_x - P_y)\,dA. \tag{3.5.4}$$

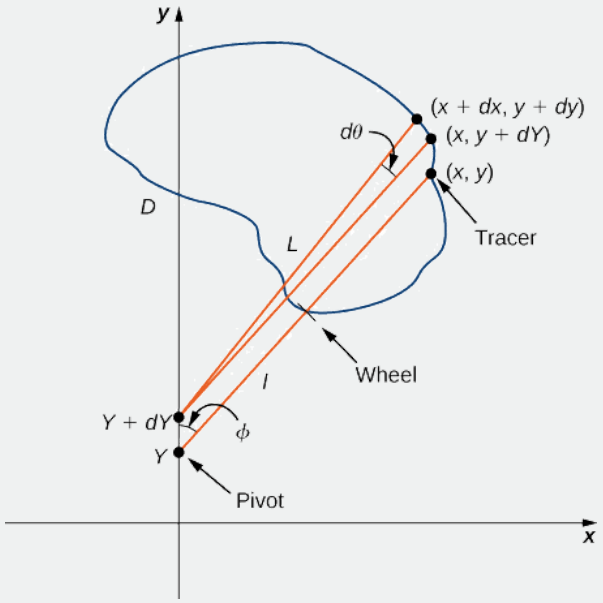


Figure **3.5.2**: The circulation form of Green's theorem relates a line integral over curve C to a double integral over region D.

Notice that Green's theorem can be used only for a two-dimensional vector field $\vec{\mathbf{F}}$. If $\vec{\mathbf{F}}$ is a three-dimensional field, then Green's theorem does not apply. Since

$$\int_C P\,dx + Q\,dy = \int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\,ds \tag{3.5.5}$$

this version of Green's theorem is sometimes referred to as the ***tangential form* of Green's theorem.**

The proof of Green's theorem is rather technical, and beyond the scope of this text. Here we examine a proof of the theorem in the special case that D is a rectangle. For now, notice that we can quickly confirm that the theorem is true for the special case in which $\vec{\mathbf{F}} = \langle P, Q \rangle$ is conservative. In this case,

$$\oint_C P\,dx + Q\,dy = 0 \tag{3.5.6}$$

because the circulation is zero in conservative vector fields. $\vec{\mathbf{F}}$ satisfies the cross-partial condition, so $P_y = Q_x$. Therefore,

$$\iint_D (Q_x - P_y)\,dA = \int_D 0\,dA = 0 = \oint_C P\,dx + Q\,dy \tag{3.5.7}$$

which confirms Green's theorem in the case of **conservative vector fields**.

Proof

Let's now prove that the circulation form of Green's theorem is true when the region D is a rectangle. Let D be the rectangle $[a, b] \times [c, d]$ oriented counterclockwise. Then, the boundary C of D consists of four piecewise smooth pieces C_1, C_2, C_3, and C_4 (Figure **3.5.3**). We parameterize each side of D as follows:

$$C_1 : \vec{\mathbf{r}}_1(t) = \langle t, c \rangle,\ a \le t \le b$$

$$C_2 : \vec{\mathbf{r}}_2(t) = \langle b, t \rangle,\ c \le t \le d$$

$$-C_3 : \vec{\mathbf{r}}_3(t) = \langle t, d \rangle,\ a \le t \le b$$

$$-C_4 : \vec{\mathbf{r}}_4(t) = \langle a, t \rangle,\ c \le t \le d.$$

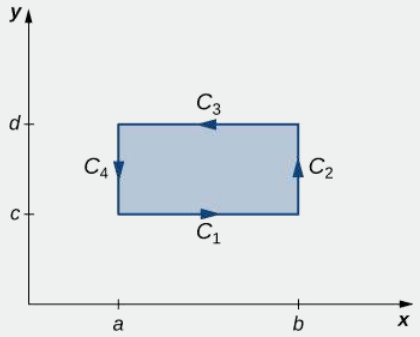


Figure 3.5.3: Rectangle D is oriented counterclockwise.

Then,

$$\begin{aligned}
\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{C_3} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{C_4} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \\
&= \int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} - \int_{-C_3} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} - \int_{-C_4} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \\
&= \int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}_1(t)) \cdot \vec{\mathbf{r}}_1'(t)\, dt + \int_c^d \vec{\mathbf{F}}(\vec{\mathbf{r}}_2(t)) \cdot \vec{\mathbf{r}}_2'(t)\, dt \\
&\quad - \int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}_3(t)) \cdot \vec{\mathbf{r}}_3'(t)\, dt - \int_c^d \vec{\mathbf{F}}(\vec{\mathbf{r}}_4(t)) \cdot \vec{\mathbf{r}}_4'(t)\, dt \\
&= \int_a^b P(t,c)\, dt + \int_c^d Q(b,t)\, dt - \int_a^b P(t,d)\, dt - \int_c^d Q(a,t)\, dt \\
&= \int_a^b (P(t,c) - P(t,d))\, dt + \int_c^d (Q(b,t) - Q(a,t))\, dt \\
&= -\int_a^b (P(t,d) - P(t,c))\, dt + \int_c^d (Q(b,t) - Q(a,t))\, dt.
\end{aligned}$$

By the Fundamental Theorem of Calculus,

$$P(t,d) - P(t,c) = \int_c^d \frac{\partial}{\partial y} P(t,y) dy$$

and

$$Q(b,t) - Q(a,t) = \int_a^b \frac{\partial}{\partial x} Q(x,t)\, dx.$$

Therefore,

$$\begin{aligned}
&-\int_a^b (P(t,d) - P(t,c))\, dt + \int_c^d (Q(b,t) - Q(a,t))\, dt \\
&= -\int_a^b \int_c^d \frac{\partial}{\partial y} P(t,y)\, dy\, dt + \int_c^d \int_a^b \frac{\partial}{\partial x} Q(x,t)\, dx\, dt.
\end{aligned}$$

But,

$$-\int_a^b \int_c^d \frac{\partial}{\partial y} P(t,y)\,dy\,dt + \int_c^d \int_a^b \frac{\partial}{\partial x} Q(x,t)\,dx\,dt = -\int_a^b \int_c^d \frac{\partial}{\partial y} P(x,y)\,dy\,dx + \int_c^d \int_a^b \frac{\partial}{\partial x} Q(x,y)\,dx\,dy$$
$$= \int_a^b \int_c^d (Q_x - P_y)\,dy\,dx$$
$$= \iint_D (Q_x - P_y)\,dA.$$

Therefore, $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_D (Q_x - P_y)\,dA$ and we have proved Green's theorem in the case of a rectangle.

□

To prove Green's theorem over a general region D, we can decompose D into many tiny rectangles and use the proof that the theorem works over rectangles. The details are technical, however, and beyond the scope of this text.

Example 3.5.1: Applying Green's Theorem over a Rectangle

Calculate the line integral

$$\oint_C x^2 y dx + (y-3) dy,$$

where C is a rectangle with vertices $(1,1)$, $(4,1)$, $(4,5)$, and $(1,5)$ oriented counterclockwise.

Solution

Let $\vec{\mathbf{F}}(x,y) = \langle P(x,y), Q(x,y)\rangle = \langle x^2 y, y-3\rangle$. Then, $Q_x(x,y) = 0$ and $P_y(x,y) = x^2$. Therefore, $Q_x - P_y = -x^2$.

Let D be the rectangular region enclosed by C (Figure 3.5.4). By Green's theorem,

$$\oint_C x^2 y dx + (y-3)\,dy = \iint_D (Q_x - P_y)\,dA$$
$$= \iint_D -x^2\,dA = \int_1^5 \int_1^4 -x^2\,dx\,dy$$
$$= \int_1^5 -21\,dy = -84.$$

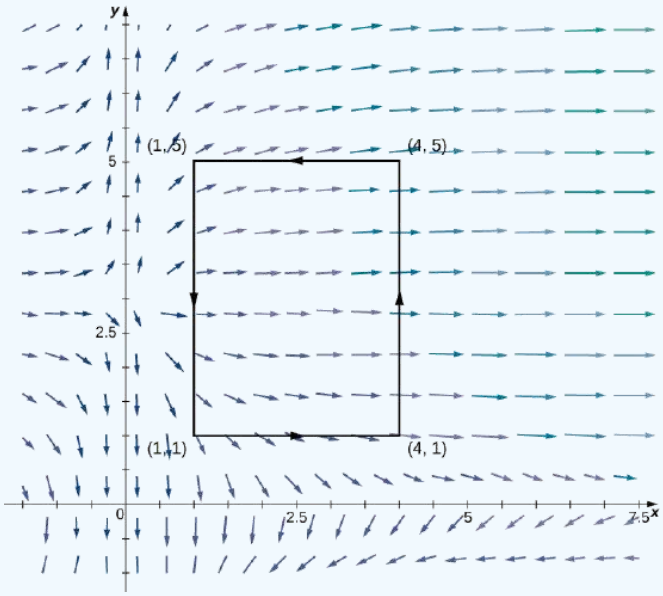


Figure 3.5.4: The line integral over the boundary of the rectangle can be transformed into a double integral over the rectangle.

Analysis

If we were to evaluate this line integral without using Green's theorem, we would need to parameterize each side of the rectangle, break the line integral into four separate line integrals, and use the methods from the section titled **Line Integrals** to evaluate each integral. Furthermore, since the vector field here is not conservative, we cannot apply the Fundamental Theorem for Line Integrals. Green's theorem makes the calculation much simpler.



Example 3.5.2: Applying Green's Theorem to Calculate Work

Calculate the work done on a particle by force field

$$\vec{\mathbf{F}}(x,y) = \langle y + \sin x, e^y - x \rangle$$

as the particle traverses circle $x^2 + y^2 = 4$ exactly once in the counterclockwise direction, starting and ending at point $(2,0)$.

Solution

Let C denote the circle and let D be the disk enclosed by C. The work done on the particle is

$$W = \oint_C (y + \sin x)\,dx + (e^y - x)\,dy.$$

As with Example 3.5.1, this integral can be calculated using tools we have learned, but it is easier to use the double integral given by Green's theorem (Figure 3.5.5).

Let $\vec{\mathbf{F}}(x,y) = \langle P(x,y), Q(x,y) \rangle = \langle y + \sin x, e^y - x \rangle$. Then, $Q_x = -1$ and $P_y = 1$. Therefore, $Q_x - P_y = -2$.

By Green's theorem,

$$\begin{aligned} W &= \oint_C (y + \sin(x))dx + (e^y - x)\,dy \\ &= \iint_D (Q_x - P_y)\,dA \\ &= \iint_D -2\,dA \\ &= -2(area(D)) = -2\pi(2^2) = -8\pi. \end{aligned}$$

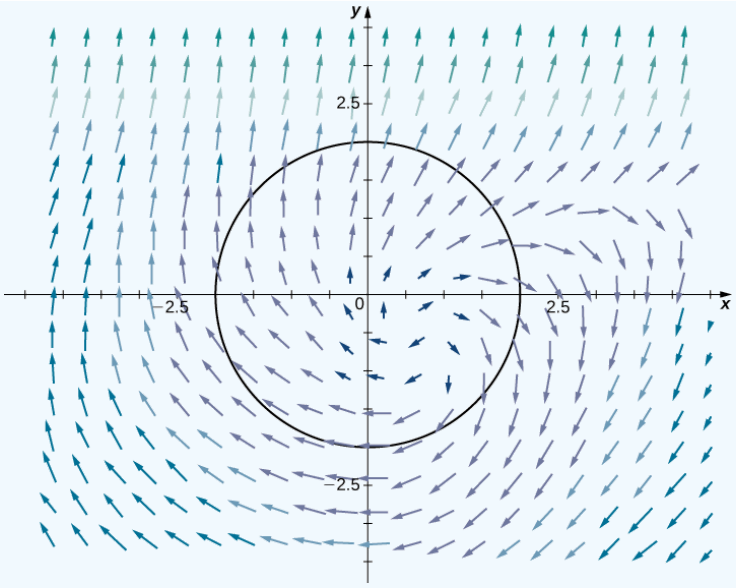


Figure 3.5.5: The line integral over the boundary circle can be transformed into a double integral over the disk enclosed by the circle.



? Exercise 3.5.2

Use Green's theorem to calculate line integral

$$\oint_C \sin(x^2)\,dx + (3x - y)\,dy.$$

where C is a right triangle with vertices $(-1, 2)$, $(4, 2)$, and $(4, 5)$ oriented counterclockwise.

Hint

Transform the line integral into a double integral.

Answer

$\frac{45}{2}$

In the preceding two examples, the double integral in Green's theorem was easier to calculate than the line integral, so we used the theorem to calculate the line integral. In the next example, the double integral is more difficult to calculate than the line integral, so we use Green's theorem to translate a double integral into a line integral.

Example 3.5.3: Applying Green's Theorem over an Ellipse

Calculate the area enclosed by ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (Figure 3.5.6).

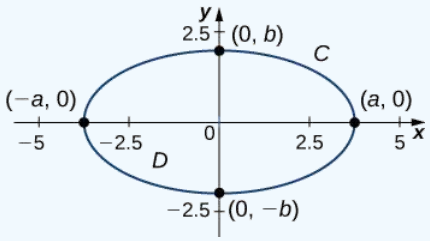


Figure 3.5.6: Ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is denoted by C.

Solution

Let C denote the ellipse and let D be the region enclosed by C. Recall that ellipse C can be parameterized by

- $x=a\cos t$,
- $y=b\sin t$,
- $0\le t\le 2\pi$.

Calculating the area of D is equivalent to computing double integral $\iint_D dA$. To calculate this integral without Green's theorem, we would need to divide D into two regions: the region above the x-axis and the region below. The area of the ellipse is

$$\int_{-a}^{a}\int_{0}^{\sqrt{b^2-(bx/a)^2}} dy\,dx+\int_{-a}^{a}\int_{-\sqrt{b^2-(bx/a)^2}}^{0} dy\,dx.$$

These two integrals are not straightforward to calculate (although when we know the value of the first integral, we know the value of the second by symmetry). Instead of trying to calculate them, we use Green's theorem to transform $\iint_D dA$ into a line integral around the boundary C.

Consider vector field

$$F(x,y)=\langle P,Q\rangle=\langle -\frac{y}{2},\frac{x}{2}\rangle.$$

Then, $Q_x=\frac{1}{2}$ and $P_y=-\frac{1}{2}$, and therefore $Q_x-P_y=1$. Notice that $\vec{\mathbf{F}}$ was chosen to have the property that $Q_x-P_y=1$. Since this is the case, Green's theorem transforms the line integral of $\vec{\mathbf{F}}$ over C into the double integral of 1 over D.

By Green's theorem,

$$\begin{aligned}\iint_D dA &= \iint_D (Q_x-P_y)\,dA\\ &= \int_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}=\frac{1}{2}\int_C -y\,dx+x\,dy\\ &= \frac{1}{2}\int_0^{2\pi} -b\sin t(-a\sin t)+a(\cos t)b\cos t\,dt\\ &= \frac{1}{2}\int_0^{2\pi} ab\cos^2 t+ab\sin^2 t\,dt\\ &= \frac{1}{2}\int_0^{2\pi} ab\,dt=\pi ab.\end{aligned}$$

Therefore, the area of the ellipse is πab units^2.

In Example 3.5.3, we used vector field $\vec{\mathbf{F}}(x,y) = \langle P, Q\rangle = \langle -\frac{y}{2}, \frac{x}{2}\rangle$ to find the area of any ellipse. The logic of the previous example can be extended to derive a formula for the area of any region D. Let D be any region with a boundary that is a simple closed curve C oriented counterclockwise. If $F(x,y) = \langle P, Q\rangle = \langle -\frac{y}{2}, \frac{x}{2}\rangle$, then $Q_x - P_y = 1$. Therefore, by the same logic as in Example 3.5.3,

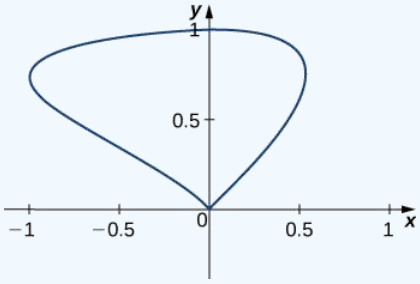
$$\text{area of } D = \iint_D dA = \frac{1}{2}\oint_C -ydx + xdy. \tag{3.5.8}$$

It's worth noting that if $F = \langle P, Q\rangle$ is any vector field with $Q_x - P_y = 1$, then the logic of the previous paragraph works. So. Equation 3.5.8 is not the only equation that uses a vector field's mixed partials to get the area of a region.



? Exercise 3.5.3

Find the area of the region enclosed by the curve with parameterization $r(t) = \langle \sin t \cos t, \sin t\rangle$, $0 \le t \le \pi$.



Hint

Use Equation 3.5.8.

Answer

$\frac{4}{3}$

Flux Form of Green's Theorem

The circulation form of Green's theorem relates a double integral over region D to line integral $\oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}} ds$, where C is the boundary of D. The flux form of Green's theorem relates a double integral over region D to the flux across boundary C. The flux of a fluid

across a curve can be difficult to calculate using the flux line integral. This form of Green's theorem allows us to translate a difficult flux integral into a double integral that is often easier to calculate.

GREEN'S THEOREM (FLUX FORM)

Let D be an open, simply connected region with a boundary curve C that is a piecewise smooth, simple closed curve that is oriented counterclockwise (Figure 3.5.7). Let $\vec{\mathbf{F}} = \langle P, Q \rangle$ be a vector field with component functions that have continuous partial derivatives on an open region containing D. Then,

$$\oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds = \iint_D P_x + Q_y\, dA. \tag{3.5.9}$$

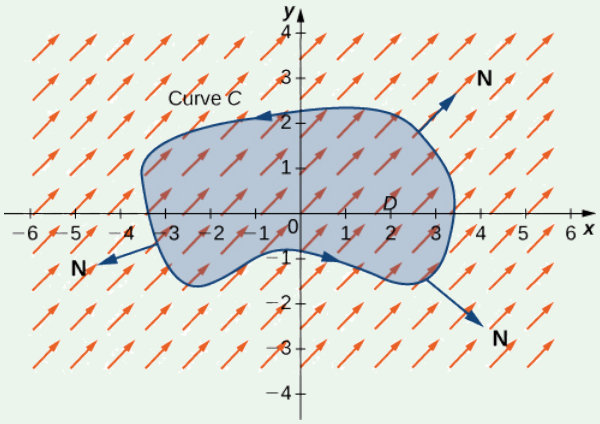


Figure 3.5.7: The flux form of Green's theorem relates a double integral over region D to the flux across curve C.

Because this form of Green's theorem contains unit normal vector $\vec{\mathbf{N}}$, it is sometimes referred to as the *normal form* of Green's theorem.

Proof

Recall that $\oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds = \oint_C -Q\, dx + P\, dy$. Let $M = -Q$ and $N = P$. By the circulation form of Green's theorem,

$$\begin{aligned} \oint_C -Q\, dx + P\, dy &= \oint_C M\, dx + N\, dy \\ &= \iint_D N_x - M_y\, dA \\ &= \iint_D P_x - (-Q)_y\, dA \\ &= \iint_D P_x + Q_y\, dA. \end{aligned}$$

□

Example 3.5.4A: Applying Green's Theorem for Flux across a Circle

Let C be a circle of radius r centered at the origin (Figure 3.5.8) and let $\vec{\mathbf{F}}(x, y) = \langle x, y \rangle$. Calculate the flux across C.

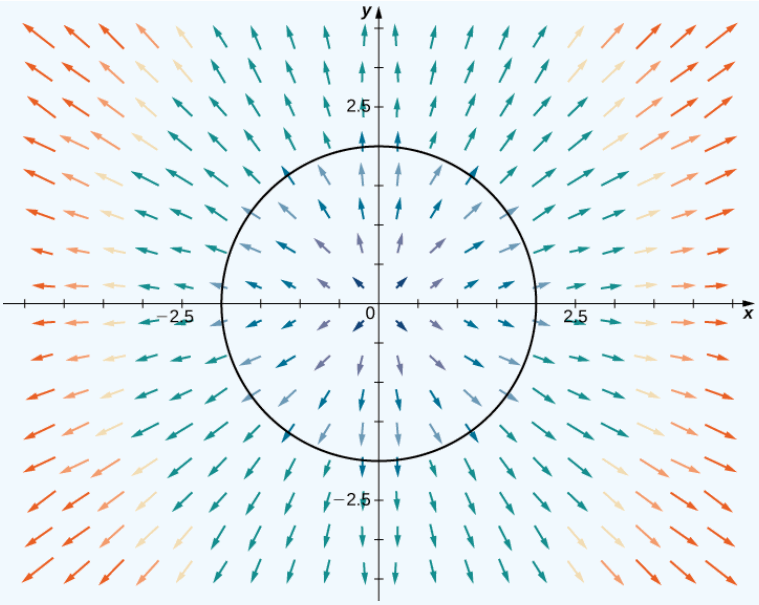


Figure 3.5.8: Curve C is a circle of radius r centered at the origin.

Solution

Let D be the disk enclosed by C. The flux across C is $\oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds$. We could evaluate this integral using tools we have learned, but Green's theorem makes the calculation much more simple. Let $P(x,y) = x$ and $Q(x,y) = y$ so that $\vec{\mathbf{F}} = \langle P, Q\rangle$. Note that $P_x = 1 = Q_y$, and therefore $P_x + Q_y = 2$. By Green's theorem,

$$\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds = \iint_D 2\, dA = 2 \iint_D dA.$$

Since $\iint_D dA$ is the area of the circle, $\iint_D dA = \pi r^2$. Therefore, the flux across C is $2\pi r^2$.

Example 3.5.4B: Applying Green's Theorem for Flux across a Triangle

Let S be the triangle with vertices $(0,0)$, $(1,0)$, and $(0,3)$ oriented clockwise (Figure 3.5.9). Calculate the flux of $\vec{\mathbf{F}}(x,y) = \langle P(x,y), Q(x,y)\rangle = \langle x^2 + e^y, x + y\rangle$ across S.

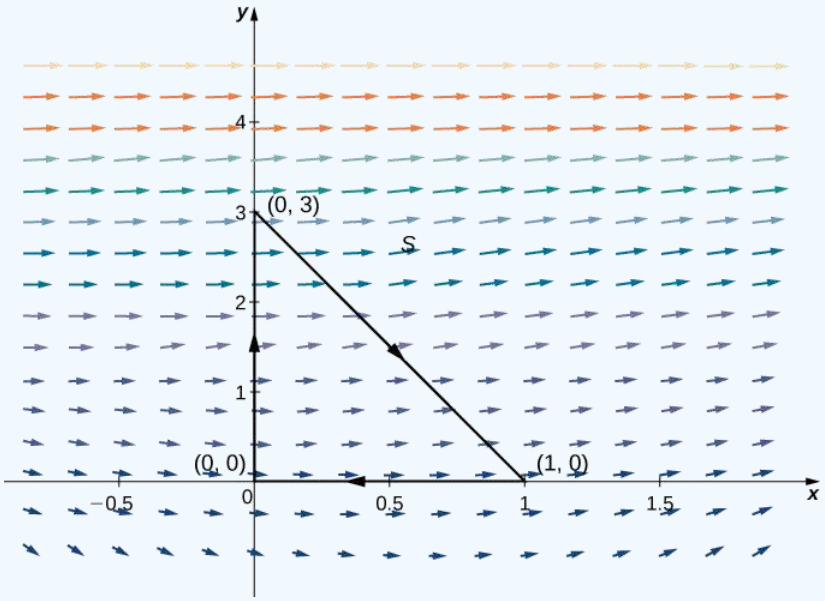


Figure 3.5.9: Curve S is a triangle with vertices $(0,0)$, $(1,0)$, and $(0,3)$ oriented clockwise.

Solution

To calculate the flux without Green's theorem, we would need to break the flux integral into three line integrals, one integral for each side of the triangle. Using Green's theorem to translate the flux line integral into a single double integral is much more simple.

Let D be the region enclosed by S. Note that $P_x = 2x$ and $Q_y = 1$; therefore, $P_x + Q_y = 2x + 1$. Green's theorem applies only to simple closed curves oriented counterclockwise, but we can still apply the theorem because $\oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds = -\oint_{-S} \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds$ and $-S$ is oriented counterclockwise. By Green's theorem, the flux is

$$\begin{aligned} \oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds &= \oint_{-S} \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds \\ &= -\iint_D (P_x + Q_y)\, dA \\ &= -\iint_D (2x+1)\, dA. \end{aligned}$$

Notice that the top edge of the triangle is the line $y = -3x + 3$. Therefore, in the iterated double integral, the y-values run from $y = 0$ to $y = -3x + 3$, and we have

$$\begin{aligned} -\iint_D (2x+1)\, dA &= -\int_0^1 \int_0^{-3x+3} (2x+1)\, dy\, dx \\ &= -\int_0^1 (2x+1)(-3x+3)\, dx \\ &= -\int_0^1 (-6x^2 + 3x + 3)\, dx \\ &= -\left[-2x^3 + \frac{3x^2}{2} + 3x\right]_0^1 \\ &= -\frac{5}{2}. \end{aligned}$$

Exercise 3.5.4

Calculate the flux of $\vec{\mathbf{F}}(x, y) = \langle x^3, y^3 \rangle$ across a unit circle oriented counterclockwise.

Hint

Apply Green's theorem and use polar coordinates.

Answer

$\frac{3\pi}{2}$

Example 3.5.5: Applying Green's Theorem for Water Flow across a Rectangle

Water flows from a spring located at the origin. The velocity of the water is modeled by vector field $\vec{\mathbf{v}}(x, y) = \langle 5x + y, x + 3y \rangle$ m/sec. Find the amount of water per second that flows across the rectangle with vertices $(-1, -2)$, $(1, -2)$, $(1, 3)$,and $(-1, 3)$, oriented counterclockwise (Figure 3.5.10).

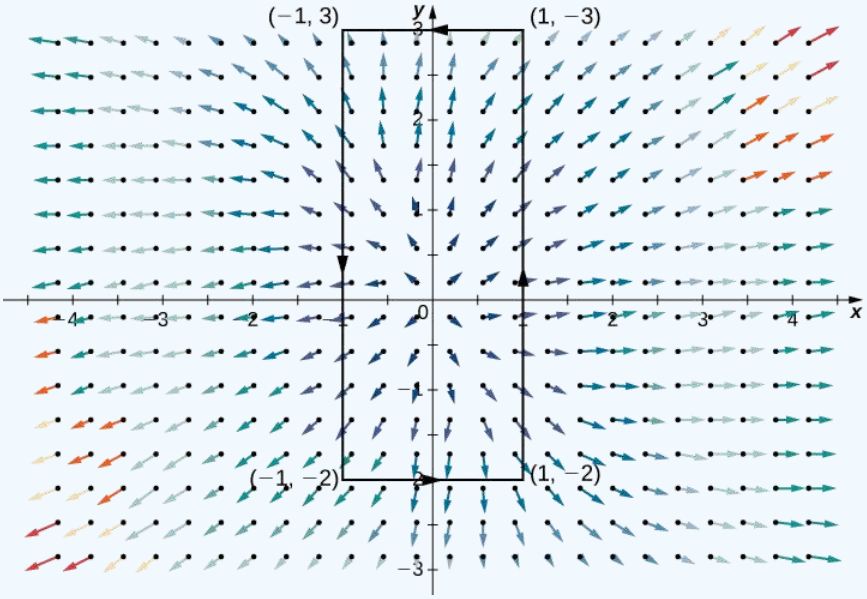


Figure 3.5.10: Water flows across the rectangle with vertices $(-1,-2)$, $(1,-2)$, $(1,3)$,and $(-1,3)$, oriented counterclockwise.

Solution

Let C represent the given rectangle and let D be the rectangular region enclosed by C. To find the amount of water flowing across C, we calculate flux $\int_C \vec{\mathbf{v}} \cdot d\vec{\mathbf{r}}$. Let $P(x,y) = 5x + y$ and $Q(x,y) = x + 3y$ so that $\vec{\mathbf{v}} = \langle P, Q\rangle$. Then, $P_x = 5$ and $Q_y = 3$. By Green's theorem,

$$\begin{aligned}\int_C \vec{\mathbf{v}} \cdot d\vec{\mathbf{r}} &= \iint_D (P_x + Q_y)\, dA \\ &= \iint_D 8\, dA \\ &= 8(\text{area of } D) = 80.\end{aligned}$$

Therefore, the water flux is 80 m^2/sec.

Recall that if vector field $\vec{\mathbf{F}}$ is conservative, then $\vec{\mathbf{F}}$ does no work around closed curves—that is, the circulation of $\vec{\mathbf{F}}$ around a closed curve is zero. In fact, if the domain of $\vec{\mathbf{F}}$ is simply connected, then $\vec{\mathbf{F}}$ is conservative if and only if the circulation of $\vec{\mathbf{F}}$ around any closed curve is zero. If we replace "circulation of $\vec{\mathbf{F}}$" with "flux of $\vec{\mathbf{F}}$," then we get a definition of a source-free vector field. The following statements are all equivalent ways of defining a source-free field $\vec{\mathbf{F}} = \langle P, Q\rangle$ on a simply connected domain (note the similarities with properties of conservative vector fields):

1. The flux $\oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds$ across any closed curve C is zero.
2. If C_1 and C_2 are curves in the domain of $\vec{\mathbf{F}}$ with the same starting points and endpoints, then $\int_{C_1} \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds = \int_{C_2} \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds$. In other words, flux is independent of path.
3. There is a stream function $g(x,y)$ for $\vec{\mathbf{F}}$. A stream function for $\vec{\mathbf{F}} = \langle P, Q\rangle$ is a function g such that $P = g_y$ and $Q = -g_x$.Geometrically, $\vec{\mathbf{F}} = \langle a, b\rangle$ is tangential to the level curve of g at (a,b). Since the gradient of g is perpendicular to the level curve of g at (a,b), stream function g has the property $\vec{\mathbf{F}}(a,b) \cdot \vec{\nabla} g(a,b) = 0$ for any point (a,b) in the domain of g. (Stream functions play the same role for source-free fields that potential functions play for conservative fields.)
4. $P_x + Q_y = 0$

Example 3.5.6: Finding a Stream Function

Verify that rotation vector field $\vec{\mathbf{F}}(x,y) = \langle y, -x\rangle$ is source free, and find a stream function for $\vec{\mathbf{F}}$.

Solution

Note that the domain of $\vec{\mathbf{F}}$ is all of $\mathbb{R}^2$, which is simply connected. Therefore, to show that $\vec{\mathbf{F}}$ is source free, we can show any of items 1 through 4 from the previous list to be true. In this example, we show that item 4 is true. Let $P(x,y) = y$ and $Q(x,y) = -x$. Then $P_x + 0 = Q_y$, and therefore $P_x + Q_y = 0$. Thus, $\vec{\mathbf{F}}$ is source free.

To find a stream function for $\vec{\mathbf{F}}$, proceed in the same manner as finding a potential function for a conservative field. Let g be a stream function for $\vec{\mathbf{F}}$. Then $g_y = y$, which implies that

$$g(x,y) = \frac{y^2}{2} + h(x).$$

Since $-g_x = Q = -x$, we have $h'(x) = x$. Therefore,

$$h(x) = \frac{x^2}{2} + C.$$

Letting $C = 0$ gives stream function

$$g(x,y) = \frac{x^2}{2} + \frac{y^2}{2}.$$

To confirm that g is a stream function for $\vec{\mathbf{F}}$, note that $g_y = y = P$ and $-g_x = -x = Q$.

Notice that source-free rotation vector field $\vec{\mathbf{F}}(x,y) = \langle y, -x \rangle$ is perpendicular to conservative radial vector field $\vec{\nabla} g = \langle x, y \rangle$ (Figure 3.5.11).

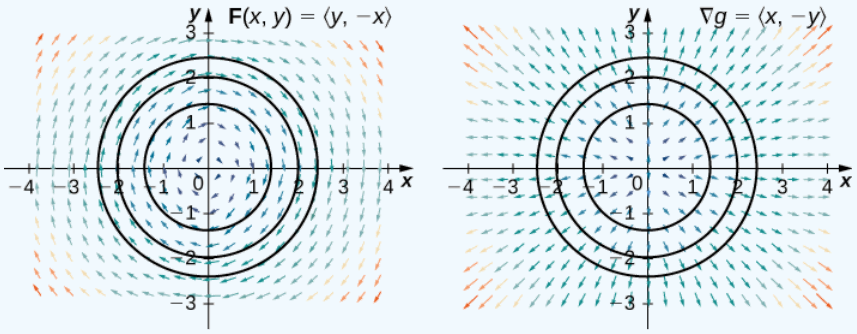


Figure 3.5.11: (a) In this image, we see the three-level curves of g and vector field $\vec{\mathbf{F}}$. Note that the $\vec{\mathbf{F}}$ vectors on a given level curve are tangent to the level curve. (b) In this image, we see the three-level curves of g and vector field $\vec{\nabla} g$. The gradient vectors are perpendicular to the corresponding level curve. Therefore, $\vec{\mathbf{F}}(a,b) \cdot \vec{\nabla} g(a,b) = 0$ for any point in the domain of g.

? Exercise 3.5.6

Find a stream function for vector field $\vec{\mathbf{F}}(x,y) = \langle x \sin y, \cos y \rangle$.

Hint

Follow the outline provided in the previous example.

Answer

$g(x,y) = -x \cos y$

Vector fields that are both conservative and source free are important vector fields. One important feature of conservative and source-free vector fields on a simply connected domain is that any potential function f of such a field satisfies Laplace's equation $f_{xx} + f_{yy} = 0$. Laplace's equation is foundational in the field of partial differential equations because it models such phenomena as gravitational and magnetic potentials in space, and the velocity potential of an ideal fluid. A function that satisfies Laplace's equation is called a *harmonic* function. Therefore any potential function of a conservative and source-free vector field is harmonic.

To see that any potential function of a conservative and source-free vector field on a simply connected domain is harmonic, let f be such a potential function of vector field $\vec{\mathbf{F}} = \langle P, Q \rangle$. Then, $f_x = P$ and $f_x = Q$ because $\vec{\nabla} f = \vec{\mathbf{F}}$. Therefore, $f_{xx} = P_x$ and $f_{yy} = Q_y$. Since $\vec{\mathbf{F}}$ is source free, $f_{xx} + f_{yy} = P_x + Q_y = 0$, and we have that f is harmonic.

Example 3.5.7: Satisfying Laplace's Equation

For vector field $\vec{\mathbf{F}}(x,y) = \langle e^x \sin y, e^x \cos y\rangle$, verify that the field is both conservative and source free, find a potential function for $\vec{\mathbf{F}}$, and verify that the potential function is harmonic.

Solution

Let $P(x,y) = e^x \sin y$ and $Q(x,y) = e^x \cos y$. Notice that the domain of $\vec{\mathbf{F}}$ is all of two-space, which is simply connected. Therefore, we can check the cross-partials of $\vec{\mathbf{F}}$ to determine whether $\vec{\mathbf{F}}$ is conservative. Note that $P_y = e^x \cos y = Q_x$, so $\vec{\mathbf{F}}$ is conservative. Since $P_x = e^x \sin y$ and $Q_y = e^x \sin y$, $P_x + Q_y = 0$ and the field is source free.

To find a potential function for $\vec{\mathbf{F}}$, let f be a potential function. Then, $\vec{\nabla} f = \vec{\mathbf{F}}$, so $f_x(x,y) = e^x \sin y$. Integrating this equation with respect to x gives $f(x,y) = e^x \sin y + h(y)$. Since $f_y(x,y) = e^x \cos y$, differentiating f with respect to y gives $e^x \cos y = e^x \cos y + h'(y)$. Therefore, we can take $h(y) = 0$, and $f(x,y) = e^x \sin y$ is a potential function for f.

To verify that f is a harmonic function, note that $f_{xx}(x,y) = \dfrac{\partial}{\partial x}(e^x \sin y) = e^x \sin y$ and $f_{yy}(x,y) = \dfrac{\partial}{\partial x}(e^x \cos y) = -e^x \sin y$. Therefore, $f_{xx} + f_{yy} = 0$, and f satisfies Laplace's equation.

Exercise 3.5.7

Is the function $f(x,y) = e^{x+5y}$ harmonic?

Hint

Determine whether the function satisfies Laplace's equation.

Answer

No

Green's Theorem on General Regions

Green's theorem, as stated, applies only to regions that are simply connected—that is, Green's theorem as stated so far cannot handle regions with holes. Here, we extend Green's theorem so that it does work on regions with finitely many holes (Figure 3.5.12).

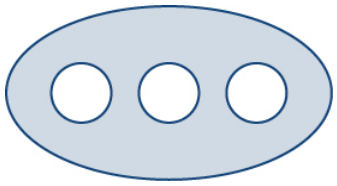


Figure 3.5.12: Green's theorem, as stated, does not apply to a nonsimply connected region with three holes like this one.

Before discussing extensions of Green's theorem, we need to go over some terminology regarding the boundary of a region. Let D be a region and let C be a component of the boundary of D. We say that C is *positively oriented* if, as we walk along C in the direction of orientation, region D is always on our left. Therefore, the counterclockwise orientation of the boundary of a disk is a positive orientation, for example. Curve C is *negatively oriented* if, as we walk along C in the direction of orientation, region D is always on our right. The clockwise orientation of the boundary of a disk is a negative orientation, for example.

Let D be a region with finitely many holes (so that D has finitely many boundary curves), and denote the boundary of D by ∂D (Figure 3.5.13). To extend Green's theorem so it can handle D, we divide region D into two regions, D_1 and D_2 (with respective boundaries ∂D_1 and ∂D_2), in such a way that $D = D_1 \cup D_2$ and neither D_1 nor D_2 has any holes (Figure 3.5.13).

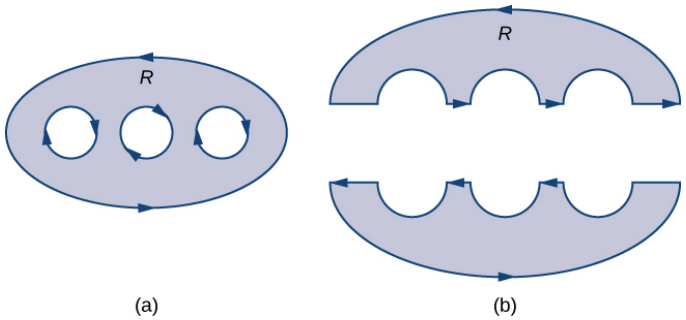


Figure **3.5.13**: (a) Region D with an oriented boundary has three holes. (b) Region D split into two simply connected regions has no holes.

Assume the boundary of D is oriented as in the figure, with the inner holes given a negative orientation and the outer boundary given a positive orientation. The boundary of each simply connected region D_1 and D_2 is positively oriented. If $\vec{\mathbf{F}}$ is a vector field defined on D, then Green's theorem says that

$$\oint_{\partial D} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \oint_{\partial D_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \oint_{\partial D_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \tag{3.5.10}$$

$$= \iint_{D_1} Q_x - P_y \, dA + \iint_{D_2} Q_x - P_y \, dA \tag{3.5.11}$$

$$= \iint_{D} (Q_x - P_y) \, dA. \tag{3.5.12}$$

Therefore, Green's theorem still works on a region with holes.

To see how this works in practice, consider annulus D in Figure **3.5.14** and suppose that $F = \langle P, Q \rangle$ is a vector field defined on this annulus. Region D has a hole, so it is not simply connected. Orient the outer circle of the annulus counterclockwise and the inner circle clockwise (Figure **3.5.14**) so that, when we divide the region into D_1 and D_2, we are able to keep the region on our left as we walk along a path that traverses the boundary. Let D_1 be the upper half of the annulus and D_2 be the lower half. Neither of these regions has holes, so we have divided D into two simply connected regions.

We label each piece of these new boundaries as P_i for some i, as in Figure **3.5.14**. If we begin at P and travel along the oriented boundary, the first segment is P_1, then P_2, P_3, and P_4. Now we have traversed D_1 and returned to P. Next, we start at P again and traverse D_2. Since the first piece of the boundary is the same as P_4 in D_1, but oriented in the opposite direction, the first piece of D_2 is $-P_4$. Next, we have P_5, then $-P_2$, and finally P_6.

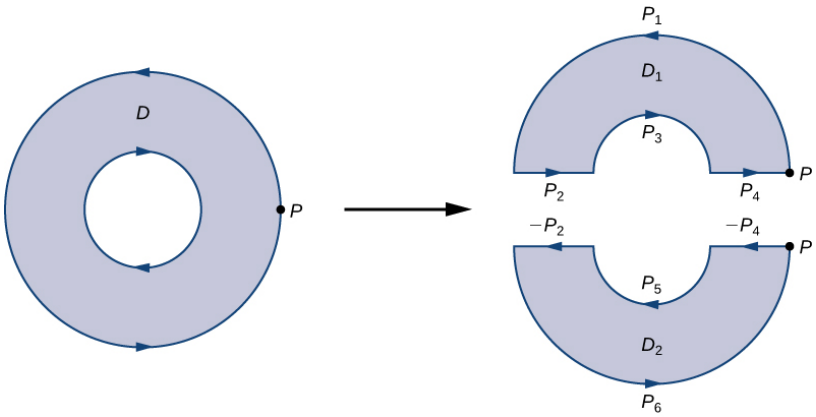


Figure **3.5.14**: Breaking the annulus into two separate regions gives us two simply connected regions. The line integrals over the common boundaries cancel out.

Figure **3.5.14** shows a path that traverses the boundary of D. Notice that this path traverses the boundary of region D_1, returns to the starting point, and then traverses the boundary of region D_2. Furthermore, as we walk along the path, the region is always on our left. Notice that this traversal of the P_i paths covers the entire boundary of region D. If we had only traversed one portion of the boundary of D, then we cannot apply Green's theorem to D.

The boundary of the upper half of the annulus, therefore, is $P_1 \cup P_2 \cup P_3 \cup P_4$ and the boundary of the lower half of the annulus is $-P_4 \cup P_5 \cup -P_2 \cup P_6$. Then, Green's theorem implies

$$\oint_{\partial D} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{P_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{P_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{P_3} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{P_4} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{-P_4} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \tag{3.5.13}$$

$$+ \int_{P_5} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{-P_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{P_6} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$$

$$= \int_{P_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{P_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{P_3} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{P_4} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{P_4} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \tag{3.5.14}$$

$$+ \int_{P_5} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{-P_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{P_6} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$$

$$= \int_{P_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{P_3} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{P_5} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{P_6} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \tag{3.5.15}$$

$$= \oint_{\partial D_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \oint_{\partial D_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \tag{3.5.16}$$

$$= \iint_{D_1} (Q_x - P_y)\, dA + \iint_{D_2} (Q_x - P_y)\, dA \tag{3.5.17}$$

$$= \iint_{D} (Q_x - P_y)\, dA. \tag{3.5.18}$$

Therefore, we arrive at the equation found in Green's theorem—namely,

$$\oint_{\partial D} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_{D} (Q_x - P_y)\, dA. \tag{3.5.19}$$

The same logic implies that the flux form of Green's theorem can also be extended to a region with finitely many holes:

$$\oint_{C} F \cdot N\, ds = \iint_{D} (P_x + Q_y)\, dA. \tag{3.5.20}$$

Example 3.5.8A: Using Green's Theorem on a Region with Holes

Calculate the integral

$$\oint_{\partial D} (\sin x - \frac{y^3}{3})dx + (\frac{y^3}{3} + \sin y)dy, \tag{3.5.21}$$

where D is the annulus given by the polar inequalities $1 \le r \le 2$, $0 \le \theta \le 2\pi$.

Solution

Although D is not simply connected, we can use the extended form of Green's theorem to calculate the integral. Since the integration occurs over an annulus, we convert to polar coordinates:

$$\begin{aligned} \oint_{\partial D} (\sin x - \frac{y^3}{3})\, dx + (\frac{x^3}{3} + \sin y)\, dy &= \iint_{D} (Q_x - P_y)\, dA \\ &= \iint_{D} (x^2 + y^2)\, dA \\ &= \int_0^{2\pi} \int_1^2 r^3\, dr d\theta = \int_0^{2\pi} \frac{15}{4}\, d\theta \\ &= \frac{15\pi}{2}. \end{aligned}$$

Example 3.5.8B: Using the Extended Form of Green's Theorem

Let $\vec{\mathbf{F}} = \langle P, Q \rangle = \langle \frac{y}{x^2+y^2}, -\frac{x}{x^2+y^2} \rangle$ and let C be any simple closed curve in a plane oriented counterclockwise. What are the possible values of $\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$?

Solution

We use the extended form of Green's theorem to show that $\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ is either 0 or -2π—that is, no matter how crazy curve C is, the line integral of $\vec{\mathbf{F}}$ along C can have only one of two possible values. We consider two cases: the case when C

encompasses the origin and the case when C does not encompass the origin.

Case 1: C Does Not Encompass the Origin

In this case, the region enclosed by C is simply connected because the only hole in the domain of $\vec{\mathbf{F}}$ is at the origin. We showed in our discussion of cross-partials that $\vec{\mathbf{F}}$ satisfies the cross-partial condition. If we restrict the domain of $\vec{\mathbf{F}}$ just to C and the region it encloses, then $\vec{\mathbf{F}}$ with this restricted domain is now defined on a simply connected domain. Since $\vec{\mathbf{F}}$ satisfies the cross-partial property on its restricted domain, the field $\vec{\mathbf{F}}$ is conservative on this simply connected region and hence the circulation $\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ is zero.

Case 2: C Does Encompass the Origin

In this case, the region enclosed by C is not simply connected because this region contains a hole at the origin. Let C_1 be a circle of radius a centered at the origin so that C_1 is entirely inside the region enclosed by C (Figure 3.5.15). Give C_1 a clockwise orientation.

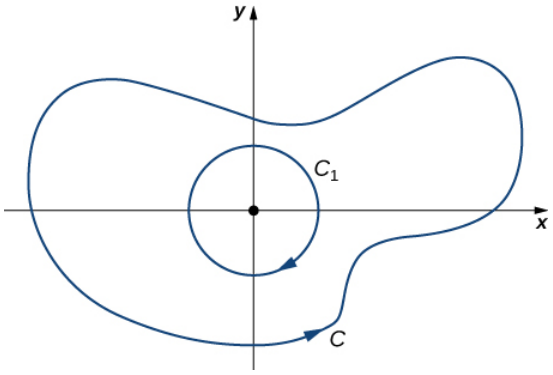


Figure 3.5.15: Choose circle C_1 centered at the origin that is contained entirely inside C.

Let D be the region between C_1 and C, and C is orientated counterclockwise. By the extended version of Green's theorem,

$$\begin{aligned}\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \iint_D Qx\_P_y \, dA \\ &= \iint_D -\frac{y^2 - x^2}{(x^2+y^2)^2} + \frac{y^2 - x^2}{(x^2+y^2)^2} dA \\ &= 0,\end{aligned}$$

and therefore

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = -\int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}.$$

Since C_1 is a specific curve, we can evaluate $\int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$. Let

$$x = a\cos t, \quad y = a\sin t, \quad 0 \le t \le 2\pi$$

be a parameterization of C_1. Then,

$$\begin{aligned}\int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_0^{2\pi} F(r(t)) \cdot r'(t) dt \\ &= \int_0^{2\pi} \langle -\frac{\sin(t)}{a}, -\frac{\cos(t)}{a} \rangle \cdot \langle -a\sin(t), -a\cos(t) \rangle dt \\ &= \int_0^{2\pi} \sin^2(t) + \cos^2(t) dt \\ &= \int_0^{2\pi} dt = 2\pi.\end{aligned}$$

Therefore, $\int_C F \cdot ds = -2\pi$.



? Exercise 3.5.8

Calculate integral $\oint_{\partial D} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where D is the annulus given by the polar inequalities $2 \leq r \leq 5$, $0 \leq \theta \leq 2\pi$, and $F(x, y) = \langle x^3, 5x + e^y \sin y \rangle$.

Hint

Use the extended version of Green's theorem.

Answer

105π

MEASURING AREA FROM A BOUNDARY: THE PLANIMETER

Imagine you are a doctor who has just received a magnetic resonance image of your patient's brain. The brain has a tumor (Figure **3.5.16**). How large is the tumor? To be precise, what is the area of the red region? The red cross-section of the tumor has an irregular shape, and therefore it is unlikely that you would be able to find a set of equations or inequalities for the region and then be able to calculate its area by conventional means. You could approximate the area by chopping the region into tiny squares (a Riemann sum approach), but this method always gives an answer with some error.

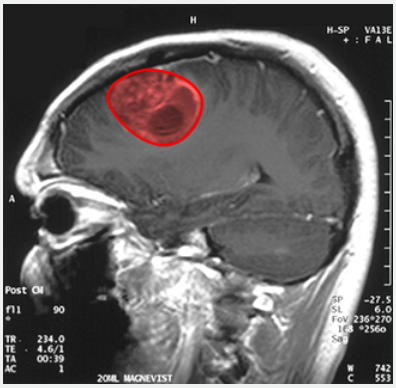


Figure **3.5.16**: This magnetic resonance image of a patient's brain shows a tumor, which is highlighted in red. (credit: modification of work by Christaras A, Wikimedia Commons)

Instead of trying to measure the area of the region directly, we can use a device called a *rolling planimeter* to calculate the area of the region exactly, simply by measuring its boundary. In this project you investigate how a planimeter works, and you use

Green's theorem to show the device calculates area correctly.

A rolling planimeter is a device that measures the area of a planar region by tracing out the boundary of that region (Figure **3.5.17**). To measure the area of a region, we simply run the tracer of the planimeter around the boundary of the region. The planimeter measures the number of turns through which the wheel rotates as we trace the boundary; the area of the shape is proportional to this number of wheel turns. We can derive the precise proportionality equation using Green's theorem. As the tracer moves around the boundary of the region, the tracer arm rotates and the roller moves back and forth (but does not rotate).

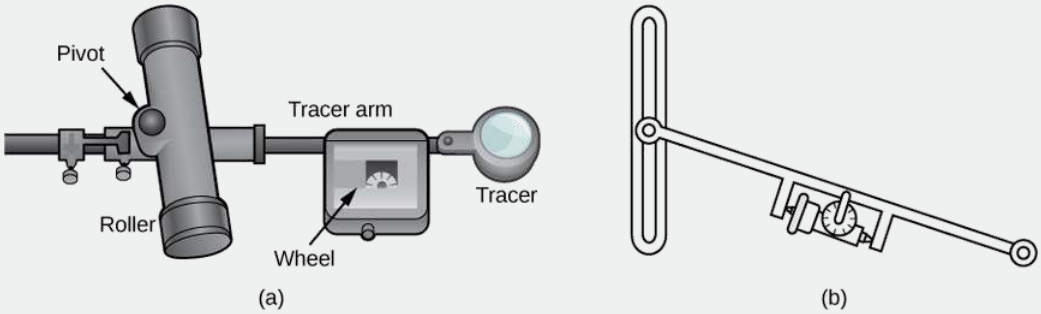


Figure **3.5.17**: (a) A rolling planimeter. The pivot allows the tracer arm to rotate. The roller itself does not rotate; it only moves back and forth. (b) An interior view of a rolling planimeter. Notice that the wheel cannot turn if the planimeter is moving back and forth with the tracer arm perpendicular to the roller.

Let C denote the boundary of region D, the area to be calculated. As the tracer traverses curve C, assume the roller moves along the y-axis (since the roller does not rotate, one can assume it moves along a straight line). Use the coordinates (x, y) to represent points on boundary C, and coordinates $(0, Y)$ to represent the position of the pivot. As the planimeter traces C, the pivot moves along the y-axis while the tracer arm rotates on the pivot.

Watch a short animation of a planimeter in action.

Begin the analysis by considering the motion of the tracer as it moves from point (x, y) counterclockwise to point $(x + dx, y + dy)$ that is close to (x, y) (Figure **3.5.18**). The pivot also moves, from point $(0, Y)$ to nearby point $(0, Y + dY)$. How much does the wheel turn as a result of this motion? To answer this question, break the motion into two parts. First, roll the pivot along the y-axis from $(0, Y)$ to $(0, Y + dY)$ without rotating the tracer arm. The tracer arm then ends up at point $(x, y + dY)$ while maintaining a constant angle ϕ with the x-axis. Second, rotate the tracer arm by an angle $d\theta$ without moving the roller. Now the tracer is at point $(x + dx, y + dy)$. Let ll be the distance from the pivot to the wheel and let L be the distance from the pivot to the tracer (the length of the tracer arm).

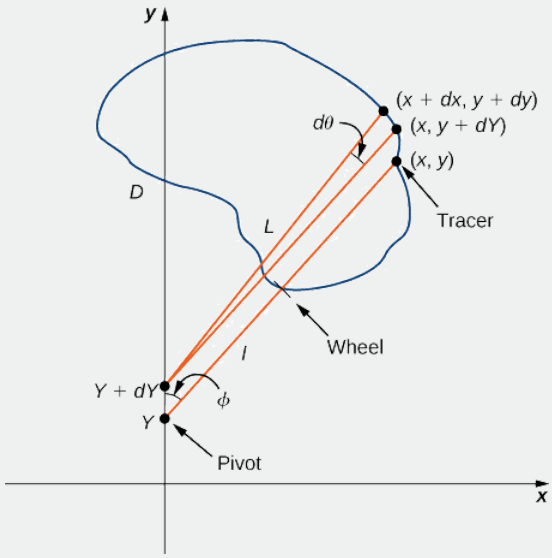


Figure **3.5.18**: Mathematical analysis of the motion of the planimeter.

1. Explain why the total distance through which the wheel rolls the small motion just described is $\sin\phi dY + l d\theta = \frac{x}{L} dY + l d\theta$.
2. Show that $\oint_C d\theta = 0$.

3. Use step 2 to show that the total rolling distance of the wheel as the tracer traverses curve C is

   Total wheel roll $= \frac{1}{L}\oint_C x dY$.

   Now that you have an equation for the total rolling distance of the wheel, connect this equation to Green's theorem to calculate area D enclosed by C.
4. Show that $x^2 + (y - Y)^2 = L^2$.
5. Assume the orientation of the planimeter is as shown in Figure 3.5.18. Explain why $Y \leq y$, and use this inequality to show there is a unique value of Y for each point (x, y): $Y = y = \sqrt{L^2 - x^2}$.
6. Use step 5 to show that $dY = dy + \frac{x}{L^2 - x^2}dx$.
7. Use Green's theorem to show that $\oint_C \frac{x}{L^2 - x^2}dx = 0$.
8. Use step 7 to show that the total wheel roll is

$$\text{Total wheel roll} \quad = \quad 1L\oint_C x\, dy. \tag{3.5.22}$$

   It took a bit of work, but this equation says that the variable of integration Y in step 3 can be replaced with y.
9. Use Green's theorem to show that the area of D is $\oint_C x dy$. The logic is similar to the logic used to show that the area of $D = 12\oint_C -y\, dx + x\, dy$.
10. Conclude that the area of D equals the length of the tracer arm multiplied by the total rolling distance of the wheel.

You now know how a planimeter works and you have used Green's theorem to justify that it works. To calculate the area of a planar region D, use a planimeter to trace the boundary of the region. The area of the region is the length of the tracer arm multiplied by the distance the wheel rolled.

Key Concepts

- Green's theorem relates the integral over a connected region to an integral over the boundary of the region. Green's theorem is a version of the Fundamental Theorem of Calculus in one higher dimension.
- Green's Theorem comes in two forms: a circulation form and a flux form. In the circulation form, the integrand is $\vec{\mathbf{F}} \cdot \vec{\mathbf{T}}$. In the flux form, the integrand is $\vec{\mathbf{F}} \cdot \vec{\mathbf{N}}$.
- Green's theorem can be used to transform a difficult line integral into an easier double integral, or to transform a difficult double integral into an easier line integral.
- A vector field is source free if it has a stream function. The flux of a source-free vector field across a closed curve is zero, just as the circulation of a conservative vector field across a closed curve is zero.

Key Equations

- **Green's theorem, circulation form**
  $\oint_C P\, dx + Q\, dy = \iint_D Q_x - P_y\, dA$, where C is the boundary of D
- **Green's theorem, flux form**
  $\oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds = \iint_D P_x + Q_y\, dA$, where C is the boundary of D
- **Green's theorem, extended version**
  $\oint_{\partial D} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_D Q_x - P_y\, dA$

Glossary

Green's theorem

relates the integral over a connected region to an integral over the boundary of the region

stream function

if $\vec{\mathbf{F}} = \langle P, Q \rangle$ is a source-free vector field, then stream function g is a function such that $P = g_y$ and $Q = -g_x$

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3.5E: Exercises

For the following exercises, evaluate the line integrals by applying Green's theorem.

1. $\int_C 2xy\,dx + (x+y)\,dy$, where C is the path from $(0,0)$ to $(1,1)$ along the graph of $y = x^3$ and from $(1,1)$ to $(0,0)$ along the graph of $y = x$ oriented in the counterclockwise direction

2. $\int_C 2xy\,dx + (x+y)\,dy$, where C is the boundary of the region lying between the graphs of $y = 0$ and $y = 4 - x^2$ oriented in the counterclockwise direction

3. $\int_C 2\arctan\left(\frac{y}{x}\right)\,dx + \ln(x^2+y^2)\,dy$, where C is defined by $x = 4 + 2\cos\theta,\ y = 4\sin\theta$ oriented in the counterclockwise direction

4. $\int_C \sin x\cos y\,dx + (xy + \cos x\sin y)\,dy$, where C is the boundary of the region lying between the graphs of $y = x$ and $y = \sqrt{x}$ oriented in the counterclockwise direction

5. $\int_C xy\,dx + (x+y)\,dy$, where C is the boundary of the region lying between the graphs of $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$ oriented in the counterclockwise direction

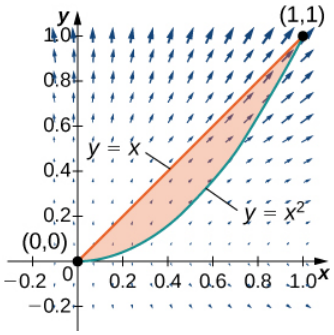
6. $\oint_C (-y\,dx + x\,dy)$, where C consists of line segment C_1 from $(-1,0)$ to $(1,0)$, followed by the semicircular arc C_2 from $(1,0)$ back to $(-1,0)$

For the following exercises, use Green's theorem.

7. Let C be the curve consisting of line segments from $(0,0)$ to $(1,1)$ to $(0,1)$ and back to $(0,0)$. Find the value of $\int_C xy\,dx + \sqrt{y^2+1}\,dy$.

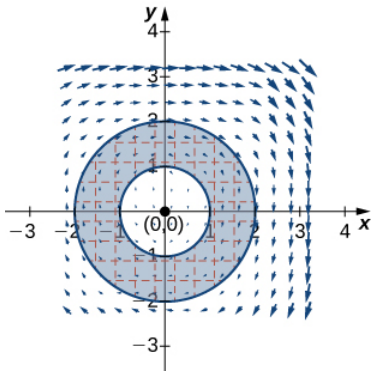
8. Evaluate line integral $\int_C xe^{-2x}\,dx + (x^4 + 2x^2y^2)\,dy$, where C is the boundary of the region between circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$, and is a positively oriented curve.

9. Find the counterclockwise circulation of field $\vec{\mathbf{F}}(x,y) = xy\,\hat{\mathbf{i}} + y^2\,\hat{\mathbf{j}}$ around and over the boundary of the region enclosed by curves $y = x^2$ and $y = x$ in the first quadrant and oriented in the counterclockwise direction.



10. Evaluate $\oint_C y^3\,dx - x^3y^2\,dy$, where C is the positively oriented circle of radius 2 centered at the origin.

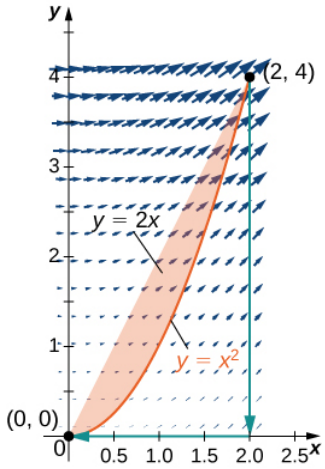
11. Evaluate $\oint_C y^3\,dx - x^3\,dy$, where C includes the two circles of radius 2 and radius 1 centered at the origin, both with positive orientation.



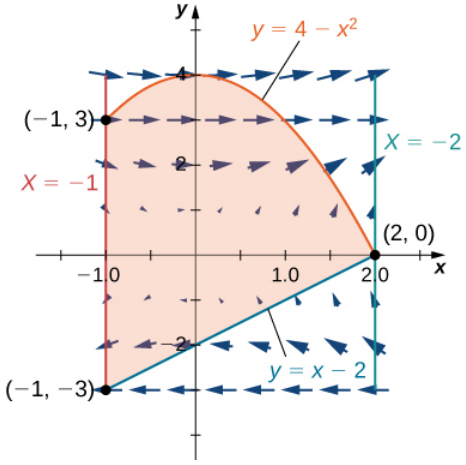
12. Calculate $\oint_C -x^2y\,dx + xy^2\,dy$, where C is a circle of radius 2 centered at the origin and oriented in the counterclockwise direction.

13. Calculate integral $\oint_C 2[y + x\sin(y)]\,dx + [x^2\cos(y) - 3y^2]\,dy$ along triangle C with vertices $(0,0)$, $(1,0)$ and $(1,1)$, oriented counterclockwise, using Green's theorem.

14. Evaluate integral $\oint_C (x^2 + y^2)\,dx + 2xy\,dy$, where C is the curve that follows parabola $y = x^2$ from $(0,0)$, $(2,4)$, then the line from $(2,4)$ to $(2,0)$, and finally the line from $(2,0)$ to $(0,0)$.



15. Evaluate line integral $\oint_C (y - \sin(y)\cos(y))\,dx + 2x\sin^2(y)\,dy$, where C is oriented in a counterclockwise path around the region bounded by $x = -1$, $x = 2$, $y = 4 - x^2$, and $y = x - 2$.



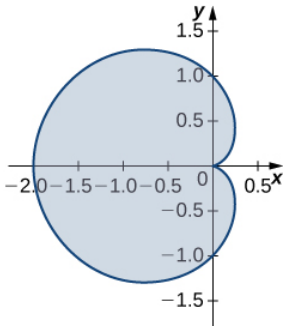
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For the following exercises, use Green's theorem to find the area.

16. Find the area between ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and circle $x^2 + y^2 = 25$.

17. Find the area of the region enclosed by parametric equation

$$\vec{\mathbf{p}}(\theta) = (\cos(\theta) - \cos^2(\theta))\,\hat{\mathbf{i}} + (\sin(\theta) - \cos(\theta)\sin(\theta))\,\hat{\mathbf{j}} \text{ for } 0 \le \theta \le 2\pi.$$



18. Find the area of the region bounded by hypocycloid $\vec{\mathbf{r}}(t) = \cos^3(t)\,\hat{\mathbf{i}} + \sin^3(t)\,\hat{\mathbf{j}}$. The curve is parameterized by $t \in [0, 2\pi]$.

19. Find the area of a pentagon with vertices $(0, 4)$, $(4, 1)$, $(3, 0)$, $(-1, -1)$, and $(-2, 2)$.

20. Use Green's theorem to evaluate $\int_{C^+} (y^2 + x^3)\,dx + x^4\,dy$, where C^+ is the perimeter of square $[0, 1] \times [0, 1]$ oriented counterclockwise.

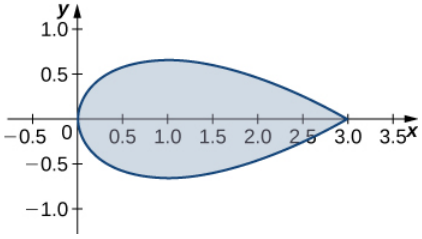
21. Use Green's theorem to prove the area of a disk with radius a is $A = \pi a^2$ units^2.

22. Use Green's theorem to find the area of one loop of a four-leaf rose $r = 3\sin 2\theta$. (*Hint*: $x\,dy - y\,dx = r^2\,d\theta$).

23. Use Green's theorem to find the area under one arch of the cycloid given by the parametric equations: $x = t - \sin t,\ y = 1 - \cos t,\ t \ge 0$.

24. Use Green's theorem to find the area of the region enclosed by curve

$$\vec{\mathbf{r}}(t) = t^2\,\hat{\mathbf{i}} + \left(\frac{t^3}{3} - t\right)\hat{\mathbf{j}}, \text{ for } -\sqrt{3} \le t \le \sqrt{3}.$$



25. [T] Evaluate Green's theorem using a computer algebra system to evaluate the integral $\int_C xe^y\,dx + e^x\,dy$, where C is the circle given by $x^2 + y^2 = 4$ and is oriented in the counterclockwise direction.

26. Evaluate $\int_C (x^2y - 2xy + y^2)\,ds$, where C is the boundary of the unit square $0 \le x \le 1,\ 0 \le y \le 1$, traversed counterclockwise.

27. Evaluate $\int_C \frac{-(y+2)\,dx + (x-1)\,dy}{(x-1)^2 + (y+2)^2}$, where C is any simple closed curve with an interior that does not contain point $(1, -2)$ traversed counterclockwise.

28. Evaluate $\int_C \frac{x\,dx + y\,dy}{x^2 + y^2}$, where C is any piecewise, smooth simple closed curve enclosing the origin, traversed counterclockwise.

For the following exercises, use Green's theorem to calculate the work done by force $\vec{\mathbf{F}}$ on a particle that is moving counterclockwise around closed path C.

29. $\vec{\mathbf{F}}(x,y) = xy\,\hat{\mathbf{i}} + (x+y)\,\hat{\mathbf{j}}, \quad C: x^2+y^2=4$

30. $\vec{\mathbf{F}}(x,y) = (x^{3/2}-3y)\,\hat{\mathbf{i}} + (6x+5\sqrt{y})\,\hat{\mathbf{j}}, \quad C$: boundary of a triangle with vertices $(0,0)$, $(5,0)$, and $(0,5)$

31. Evaluate $\int_C (2x^3-y^3)\,dx + (x^3+y^3)\,dy$, where C is a unit circle oriented in the counterclockwise direction.

32. A particle starts at point $(-2,0)$, moves along the x-axis to $(2,0)$, and then travels along semicircle $y=\sqrt{4-x^2}$ to the starting point. Use Green's theorem to find the work done on this particle by force field $\vec{\mathbf{F}}(x,y) = x\,\hat{\mathbf{i}} + (x^3+3xy^2)\,\hat{\mathbf{j}}$.

33. David and Sandra are skating on a frictionless pond in the wind. David skates on the inside, going along a circle of radius 2 in a counterclockwise direction. Sandra skates once around a circle of radius 3, also in the counterclockwise direction. Suppose the force of the wind at point (x,y) is $\vec{\mathbf{F}}(x,y) = (x^2y+10y)\,\hat{\mathbf{i}} + (x^3+2xy^2)\,\hat{\mathbf{j}}$. Use Green's theorem to determine who does more work.

34. Use Green's theorem to find the work done by force field $\vec{\mathbf{F}}(x,y) = (3y-4x)\,\hat{\mathbf{i}} + (4x-y)\,\hat{\mathbf{j}}$ when an object moves once counterclockwise around ellipse $4x^2+y^2=4$.

35. Use Green's theorem to evaluate line integral $\oint_C e^{2x}\sin 2y\,dx + e^{2x}\cos 2y\,dy$, where C is ellipse $9(x-1)^2+4(y-3)^2=36$ oriented counterclockwise.

36. Evaluate line integral $\oint_C y^2\,dx + x^2\,dy$, where C is the boundary of a triangle with vertices $(0,0)$, $(1,1)$, and $(1,0)$, with the counterclockwise orientation.

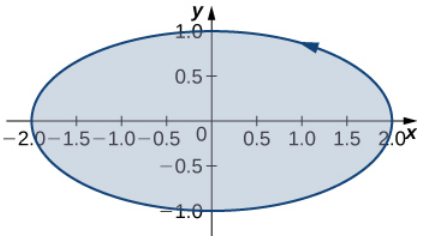
37. Use Green's theorem to evaluate line integral $\int_C \vec{\mathbf{h}}\cdot d\vec{\mathbf{r}}$ if $\vec{\mathbf{h}}(x,y) = e^y\,\hat{\mathbf{i}} - \sin \pi x\,\hat{\mathbf{j}}$, where C is a triangle with vertices $(1,0)$, $(0,1)$, and $(-1,0)$, traversed counterclockwise.

38. Use Green's theorem to evaluate line integral $\int_C \sqrt{1+x^3}\,dx + 2xy\,dy$ where C is a triangle with vertices $(0,0)$, $(1,0)$, and $(1,3)$ oriented clockwise.

39. Use Green's theorem to evaluate line integral $\int_C x^2y\,dx - xy^2\,dy$ where C is a circle $x^2+y^2=4$ oriented counterclockwise.

40. Use Green's theorem to evaluate line integral $\int_C \left(3y-e^{\sin x}\right)dx + \left(7x+\sqrt{y^4+1}\right)dy$ where C is circle $x^2+y^2=9$ oriented in the counterclockwise direction.

41. Use Green's theorem to evaluate line integral $\int_C (3x-5y)\,dx + (x-6y)\,dy$, where C is ellipse $\frac{x^2}{4}+y^2=1$ and is oriented in the counterclockwise direction.



42. Let C be a triangular closed curve from $(0,0)$ to $(1,0)$ to $(1,1)$ and finally back to $(0,0)$. Let $\vec{\mathbf{F}}(x,y) = 4y\,\hat{\mathbf{i}} + 6x^2\,\hat{\mathbf{j}}$. Use Green's theorem to evaluate $\oint_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}$.

43. Use Green's theorem to evaluate line integral $\oint_C y\,dx - x\,dy$, where C is circle $x^2+y^2=a^2$ oriented in the clockwise direction.

44. Use Green's theorem to evaluate line integral $\oint_C (y+x)\,dx + (x+\sin y)\,dy$, where C is any smooth simple closed curve joining the origin to itself oriented in the counterclockwise direction.

45. Use Green's theorem to evaluate line integral $\oint_C \left(y - \ln(x^2 + y^2)\right)\, dx + \left(2\arctan\frac{y}{x}\right)\, dy$, where C is the positively oriented circle $(x-2)^2 + (y-3)^2 = 1$.

46. Use Green's theorem to evaluate $\oint_C xy\, dx + x^3y^3\, dy$, where C is a triangle with vertices $(0,0)$, $(1,0)$, and $(1,2)$ with positive orientation.

47. Use Green's theorem to evaluate line integral $\int_C \sin y\, dx + x\cos y\, dy$, where C is ellipse $x^2 + xy + y^2 = 1$ oriented in the counterclockwise direction.

48. Let $\vec{\mathbf{F}}(x,y) = \left(\cos(x^5) - 13y^3\right)\hat{\mathbf{i}} + 13x^3\hat{\mathbf{j}}$. Find the counterclockwise circulation $\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where C is a curve consisting of the line segment joining $(-2,0)$ and $(-1,0)$, half circle $y = \sqrt{1-x^2}$, the line segment joining $(1,0)$ and $(2,0)$, and half circle $y = \sqrt{4-x^2}$.

49. Use Green's theorem to evaluate line integral $\int_C \sin(x^3)\, dx + 2ye^{x^2}\, dy$, where C is a triangular closed curve that connects the points $(0,0)$, $(2,2)$, and $(0,2)$ counterclockwise.

50. Let C be the boundary of square $0 \le x \le \pi,\ 0 \le y \le \pi$, traversed counterclockwise. Use Green's theorem to find $\int_C \sin(x+y)\, dx + \cos(x+y)\, dy$.

51. Use Green's theorem to evaluate line integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where $\vec{\mathbf{F}}(x,y) = \left(y^2 - x^2\right)\hat{\mathbf{i}} + \left(x^2 + y^2\right)\hat{\mathbf{j}}$, and C is a triangle bounded by $y = 0,\ x = 3$, and $y = x$, oriented counterclockwise.

52. Use Green's Theorem to evaluate integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where $\vec{\mathbf{F}}(x,y) = \left(xy^2\right)\hat{\mathbf{i}} + x\hat{\mathbf{j}}$, and C is a unit circle oriented in the counterclockwise direction.

53. Use Green's theorem in a plane to evaluate line integral $\oint_C \left(xy + y^2\right) dx + x^2\, dy$, where C is a closed curve of a region bounded by $y = x$ and $y = x^2$ oriented in the counterclockwise direction.

54. Calculate the outward flux of $\vec{\mathbf{F}}(x,y) = -x\hat{\mathbf{i}} + 2y\hat{\mathbf{j}}$ over a square with corners $(\pm 1, \pm 1)$, where the unit normal is outward pointing and oriented in the counterclockwise direction.

55. **[T]** Let C be circle $x^2 + y^2 = 4$ oriented in the counterclockwise direction. Evaluate $\oint_C \left[\left(3y - e^{\arctan x}\right) dx + \left(7x + \sqrt{y^4 + 1}\right) dy\right]$ using a computer algebra system.

56. Find the flux of field $\vec{\mathbf{F}}(x,y) = -x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$ across $x^2 + y^2 = 16$ oriented in the counterclockwise direction.

57. Let $\vec{\mathbf{F}} = \left(y^2 - x^2\right)\hat{\mathbf{i}} + \left(x^2 + y^2\right)\hat{\mathbf{j}}$, and let C be a triangle bounded by $y = 0,\ x = 3$, and $y = x$ oriented in the counterclockwise direction. Find the outward flux of $\vec{\mathbf{F}}$ through C.

58. **[T]** Let C be unit circle $x^2 + y^2 = 1$ traversed once counterclockwise. Evaluate $\int_C \left[-y^3 + \sin(xy) + xy\cos(xy)\right] dx + \left[x^3 + x^2\cos(xy)\right] dy$ by using a computer algebra system.

59. **[T]** Find the outward flux of vector field $\vec{\mathbf{F}}(x,y) = xy^2\hat{\mathbf{i}} + x^2y\hat{\mathbf{j}}$ across the boundary of annulus $R = \left\{(x,y) : 1 \le x^2 + y^2 \le 4\right\} = \left\{(r,\theta) : 1 \le r \le 2,\ 0 \le \theta \le 2\pi\right\}$ using a computer algebra system.

60. Consider region R bounded by parabolas $y = x^2$ and $x = y^2$. Let C be the boundary of R oriented counterclockwise. Use Green's theorem to evaluate $\oint_C \left(y + e^{\sqrt{x}}\right) dx + \left(2x + \cos(y^2)\right) dy$.

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3.6: Divergence and Curl

Learning Objectives

- Determine divergence from the formula for a given vector field.
- Determine curl from the formula for a given vector field.
- Use the properties of curl and divergence to determine whether a vector field is conservative.

In this section, we examine two important operations on a vector field: divergence and curl. They are important to the field of calculus for several reasons, including the use of curl and divergence to develop some higher-dimensional versions of the Fundamental Theorem of Calculus. In addition, curl and divergence appear in mathematical descriptions of fluid mechanics, electromagnetism, and elasticity theory, which are important concepts in physics and engineering. We can also apply curl and divergence to other concepts we already explored. For example, under certain conditions, a vector field is conservative if and only if its curl is zero.

In addition to defining curl and divergence, we look at some physical interpretations of them, and show their relationship to conservative and source-free vector fields.

Divergence

Divergence is an operation on a vector field that tells us how the field behaves toward or away from a point. Locally, the divergence of a vector field $\vec{\mathbf{F}}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ at a particular point P is a measure of the "outflowing-ness" of the vector field at P. If $\vec{\mathbf{F}}$ represents the velocity of a fluid, then the divergence of $\vec{\mathbf{F}}$ at P measures the net rate of change with respect to time of the amount of fluid flowing away from P (the tendency of the fluid to flow "out of" P). In particular, if the amount of fluid flowing into P is the same as the amount flowing out, then the divergence at P is zero.

Definition: divergence in $\mathbb{R}^3$

If $\vec{\mathbf{F}} = \langle P, Q, R \rangle$ is a vector field in $\mathbb{R}^3$ and P_x, Q_y, and R_z all exist, then the divergence of $\vec{\mathbf{F}}$ is defined by

$$\operatorname{div} F = P_x + Q_y + R_z \tag{3.6.1}$$

$$= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}. \tag{3.6.2}$$

Note the divergence of a vector field is not a vector field, but a scalar function. In terms of the gradient operator

$$\vec{\nabla} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \tag{3.6.3}$$

divergence can be written symbolically as the dot product

$$\operatorname{div} \vec{\mathbf{F}} = \vec{\nabla} \cdot \vec{\mathbf{F}}. \tag{3.6.4}$$

Note this is merely helpful notation, because the dot product of a vector of operators and a vector of functions is not meaningfully defined given our current definition of dot product.

If $\vec{\mathbf{F}} = \langle P, Q \rangle$ is a vector field in $\mathbb{R}^2$, and P_x and Q_y both exist, then the divergence of $\vec{\mathbf{F}}$ is defined similarly as

$$\begin{aligned} \operatorname{div} \vec{\mathbf{F}} &= P_x + Q_y \\ &= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \\ &= \vec{\nabla} \cdot \vec{\mathbf{F}}. \end{aligned}$$

To illustrate this point, consider the two vector fields in Figure 3.6.1. At any particular point, the amount flowing in is the same as the amount flowing out, so at every point the "outflowing-ness" of the field is zero. Therefore, we expect the divergence of both fields to be zero, and this is indeed the case, as

$$\text{div}(\langle 1,2\rangle) = \frac{\partial}{\partial x}(1) + \frac{\partial}{\partial y}(2) = 0 \tag{3.6.5}$$

and

$$\text{div}(\langle -y,x\rangle) = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x) = 0. \tag{3.6.6}$$

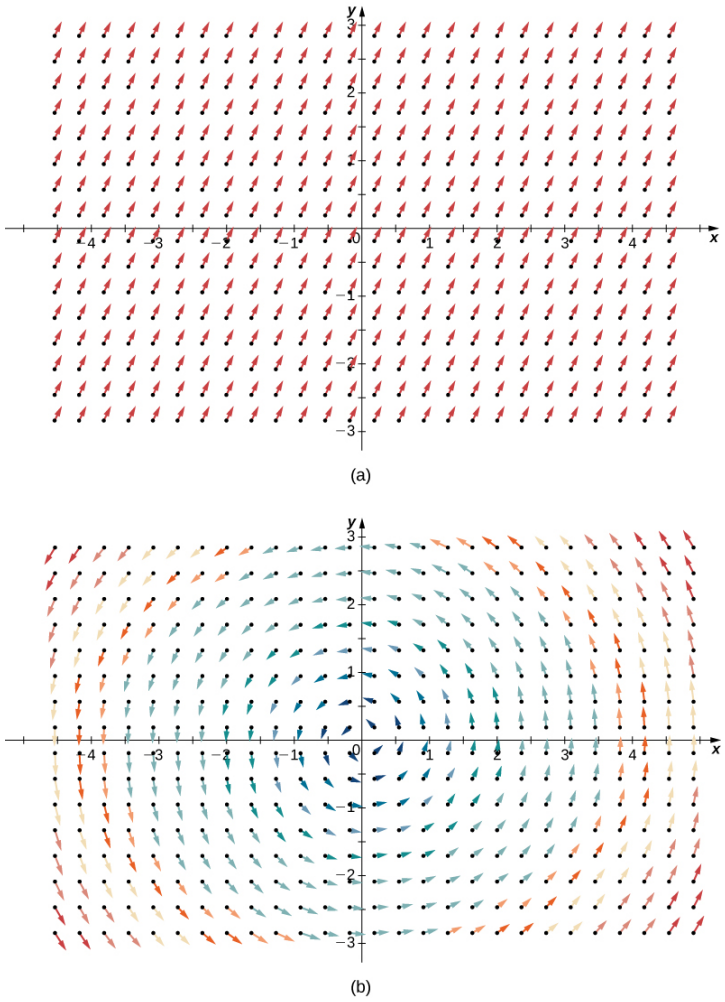


Figure 3.6.1: (a) Vector field $\langle 1,2\rangle$ has zero divergence. (b) Vector field $\langle -y,x\rangle$ also has zero divergence.

By contrast, consider radial vector field $\vec{\mathbf{R}}(x,y) = \langle -x,-y\rangle$ in Figure 3.6.2. At any given point, more fluid is flowing in than is flowing out, and therefore the "outgoingness" of the field is negative. We expect the divergence of this field to be negative, and this is indeed the case, as

$$\text{div}(\vec{\mathbf{R}}) = \frac{\partial}{\partial x}(-x) + \frac{\partial}{\partial y}(-y) = -2. \tag{3.6.7}$$

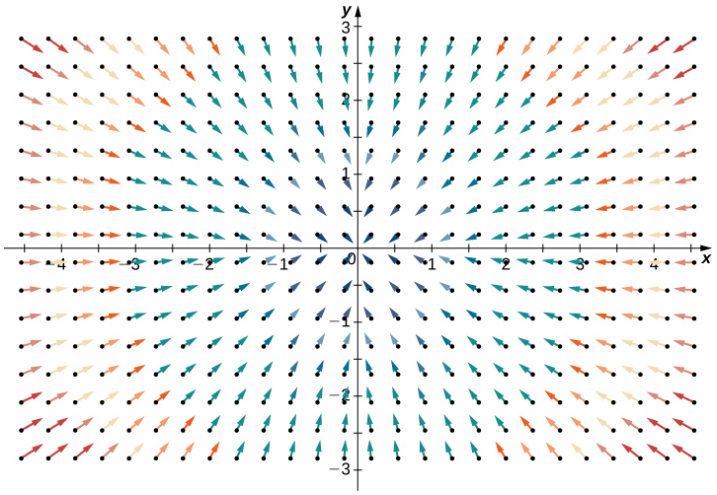


Figure 3.6.2: This vector field has negative divergence.

To get a global sense of what divergence is telling us, suppose that a vector field in $\mathbb{R}^2$ represents the velocity of a fluid. Imagine taking an elastic circle (a circle with a shape that can be changed by the vector field) and dropping it into a fluid. If the circle maintains its exact area as it flows through the fluid, then the divergence is zero. This would occur for both vector fields in Figure 3.6.1. On the other hand, if the circle's shape is distorted so that its area shrinks or expands, then the divergence is not zero. Imagine dropping such an elastic circle into the radial vector field in Figure 3.6.2 so that the center of the circle lands at point $(3,3)$. The circle would flow toward the origin, and as it did so the front of the circle would travel more slowly than the back, causing the circle to "scrunch" and lose area. This is how you can see a negative divergence.

Example 3.6.1: Calculating Divergence at a Point

If $\vec{\mathbf{F}}(x,y,z) = e^x\hat{i} + yz\hat{j} - yz^2\hat{k}$, then find the divergence of $\vec{\mathbf{F}}$ at $(0,2,-1)$.

Solution

The divergence of $\vec{\mathbf{F}}$ is

$$\frac{\partial}{\partial x}(e^x) + \frac{\partial}{\partial y}(yz) - \frac{\partial}{\partial z}(yz^2) = e^x + z - 2yz.$$

Therefore, the divergence at $(0,2,-1)$ is $e^0 - 1 + 4 = 4$. If $\vec{\mathbf{F}}$ represents the velocity of a fluid, then more fluid is flowing out than flowing in at point $(0,2,-1)$.



Exercise 3.6.1

Find $\operatorname{div} \vec{\mathbf{F}}$ for

$$\vec{\mathbf{F}}(x, y, z) = \langle xy,\ 5 - z^2,\ x^2 + y^2 \rangle.$$

Hint

Follow Example **3.6.1**.

Answer

$\operatorname{div} \vec{\mathbf{F}} = y$

One application for divergence occurs in physics, when working with magnetic fields. A magnetic field is a vector field that models the influence of electric currents and magnetic materials. Physicists use divergence in Gauss's law for magnetism, which states that if $\vec{\mathbf{B}}$ is a magnetic field, then $\vec{\nabla} \cdot \vec{\mathbf{B}} = 0$; in other words, the divergence of a magnetic field is zero.

Example 3.6.2: Determining Whether a Field Is Magnetic

Is it possible for $\vec{\mathbf{F}}(x, y) = \langle x^2 y,\ y - xy^2 \rangle$ to be a magnetic field?

Solution

If $\vec{\mathbf{F}}$ were magnetic, then its divergence would be zero. The divergence of $\vec{\mathbf{F}}$ is

$$\frac{\partial}{\partial x}(x^2 y) + \frac{\partial}{\partial y}(y - xy^2) = 2xy + 1 - 2xy = 1$$

and therefore $\vec{\mathbf{F}}$ cannot model a magnetic field (Figure **3.6.3**).

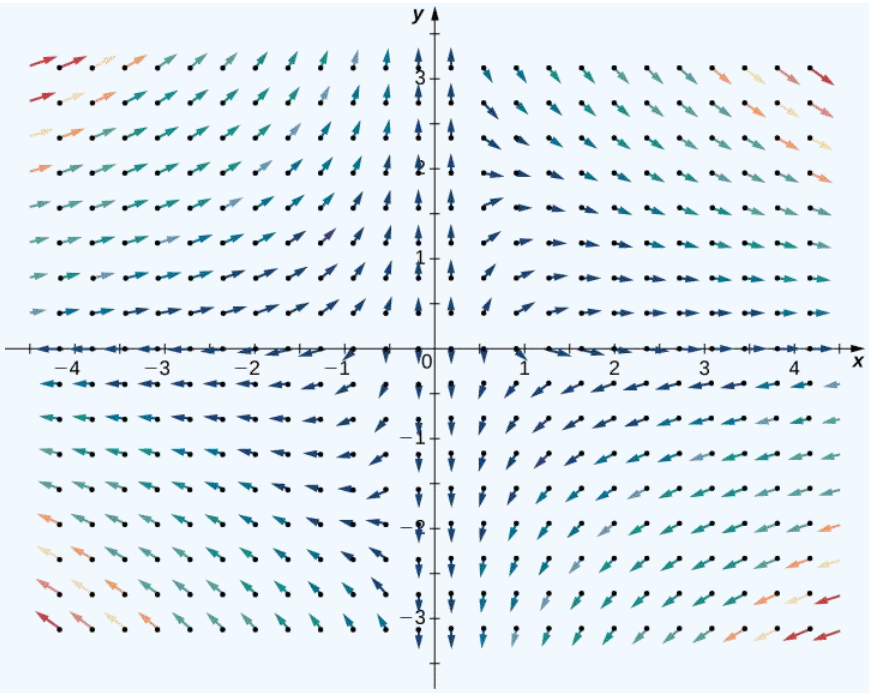


Figure **3.6.3**: The divergence of vector field $\vec{\mathbf{F}}(x,y) = \langle x^2y,\ y - xy^2\rangle$ is one, so it cannot model a magnetic field.

Another application for divergence is detecting whether a field is source free. Recall that a source-free field is a vector field that has a stream function; equivalently, a source-free field is a field with a flux that is zero along any closed curve. The next two theorems say that, under certain conditions, source-free vector fields are precisely the vector fields with zero divergence.

Theorem: Divergence of a Source-Free Vector Field

If $\vec{\mathbf{F}} = \langle P, Q\rangle$ is a source-free continuous vector field with differentiable component functions, then $\mathbf{div}\,\vec{\mathbf{F}} = 0$.

Proof

Since $\vec{\mathbf{F}}$ is source free, there is a function $g(x,y)$ with $g_y = P$ and $-g_x = Q$. Therefore, $\vec{\mathbf{F}} = \langle g_y, -g_x\rangle$ and $\mathbf{div}\,\vec{\mathbf{F}} = g_{yx} - g_{xy} = 0$ by Clairaut's theorem.

□

The converse of Divergence of a Source-Free Vector Field is true on simply connected regions, but the proof is too technical to include here. Thus, we have the following theorem, which can test whether a vector field in $\mathbb{R}^2$ is source free.

Theorem: Divergence Test for Source-Free Vector Fields

Let $\vec{\mathbf{F}} = \langle P, Q\rangle$ be a continuous vector field with differentiable component functions with a domain that is simply connected. Then, $\mathbf{div}\,\vec{\mathbf{F}} = 0$ if and only if $\vec{\mathbf{F}}$ is source free.

Example 3.6.3: Determining Whether a Field Is Source Free

Is field $\vec{\mathbf{F}}(x,y) = \langle x^2y,\ 5 - xy^2\rangle$ source free?

Solution

Note the domain of $\vec{\mathbf{F}}$ is $\mathbb{R}^2$ which is simply connected. Furthermore, $\vec{\mathbf{F}}$ is continuous with differentiable component functions. Therefore, we can use the Divergence Test for Source-Free Vector Fields to analyze $\vec{\mathbf{F}}$. The divergence of $\vec{\mathbf{F}}$ is

$$\frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(5 - xy^2) = 2xy - 2xy = 0.$$

Therefore, $\vec{\mathbf{F}}$ is source free by the Divergence Test for Source-Free Vector Fields.

? Exercise 3.6.2

Let $\vec{\mathbf{F}}(x, y) = \langle -ay, bx \rangle$ be a rotational field where a and b are positive constants. Is $\vec{\mathbf{F}}$ source free?

Hint

Calculate the divergence.

Answer

Yes

Recall that the flux form of Green's theorem says that

$$\oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds = \iint_D P_x + Q_y\, dA, \tag{3.6.8}$$

where C is a simple closed curve and D is the region enclosed by C. Since $P_x + Q_y = \operatorname{div} \vec{\mathbf{F}}$, Green's theorem is sometimes written as

$$\oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds = \iint_D \operatorname{div} \vec{\mathbf{F}}\, dA. \tag{3.6.9}$$

Therefore, Green's theorem can be written in terms of divergence. If we think of divergence as a derivative of sorts, then Green's theorem says the "derivative" of $\vec{\mathbf{F}}$ on a region can be translated into a line integral of $\vec{\mathbf{F}}$ along the boundary of the region. This is analogous to the Fundamental Theorem of Calculus, in which the derivative of a function f on a line segment $[a, b]$ can be translated into a statement about f on the boundary of $[a, b]$. Using divergence, we can see that Green's theorem is a higher-dimensional analog of the Fundamental Theorem of Calculus.

We can use all of what we have learned in the application of divergence. Let $\vec{\mathbf{v}}$ be a vector field modeling the velocity of a fluid. Since the divergence of $\vec{\mathbf{v}}$ at point P measures the "outflowing-ness" of the fluid at P, $\operatorname{div} v(P) > 0$ implies that more fluid is flowing out of P than flowing in. Similarly, $\operatorname{div} v(P) < 0$ implies the more fluid is flowing in to P than is flowing out, and $\operatorname{div} \vec{\mathbf{v}}(P) = 0$ implies the same amount of fluid is flowing in as flowing out.

✓ Example 3.6.4: Determining Flow of a Fluid

Suppose $\vec{\mathbf{v}}(x, y) = \langle -xy, y \rangle$, $y > 0$ models the flow of a fluid. Is more fluid flowing into point $(1, 4)$ than flowing out?

Solution

To determine whether more fluid is flowing into $(1, 4)$ than is flowing out, we calculate the divergence of $\vec{\mathbf{v}}$ at $(1, 4)$:

$$div(\vec{\mathbf{v}}) = \frac{\partial}{\partial x}(-xy) + \frac{\partial}{\partial y}(y) = -y + 1.$$

To find the divergence at $(1, 4)$ substitute the point into the divergence: $-4 + 1 = -3$. Since the divergence of $\vec{\mathbf{v}}$ at $(1, 4)$ is negative, more fluid is flowing in than flowing out (Figure 3.6.4).

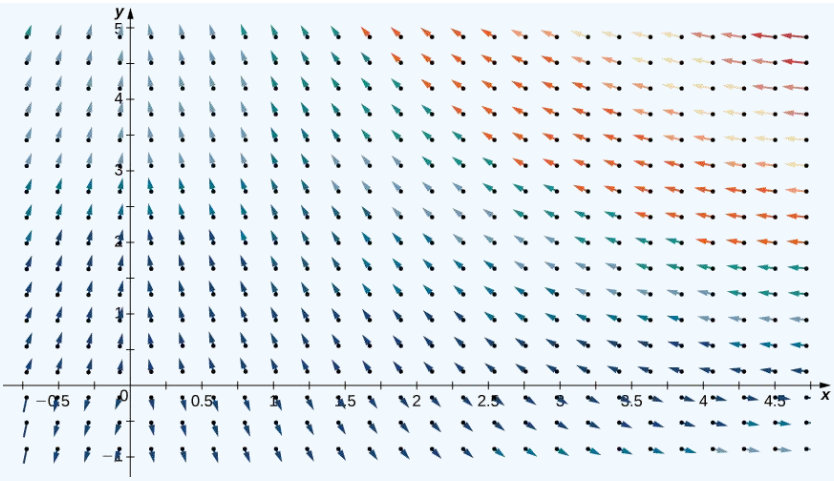


Figure 3.6.4: Vector field $\vec{\mathbf{v}}(x,y) = \langle -xy, y\rangle$ has negative divergence at $(1,4)$

? Exercise 3.6.3

For vector field $\vec{\mathbf{v}}(x,y) = \langle -xy, y\rangle$, $y > 0$, find all points P such that the amount of fluid flowing in to P equals the amount of fluid flowing out of P.

Hint

Find where the divergence is zero.

Answer

All points on line $y = 1$.

Curl

The second operation on a vector field that we examine is the curl, which measures the extent of rotation of the field about a point. Suppose that $\vec{\mathbf{F}}$ represents the velocity field of a fluid. Then, the curl of $\vec{\mathbf{F}}$ at point P is a vector that measures the tendency of particles near P to rotate about the axis that points in the direction of this vector. The magnitude of the curl vector at P measures how quickly the particles rotate around this axis. In other words, the curl at a point is a measure of the vector field's "spin" at that point. Visually, imagine placing a paddlewheel into a fluid at P, with the axis of the paddlewheel aligned with the curl vector (Figure 3.6.5). The curl measures the tendency of the paddlewheel to rotate.

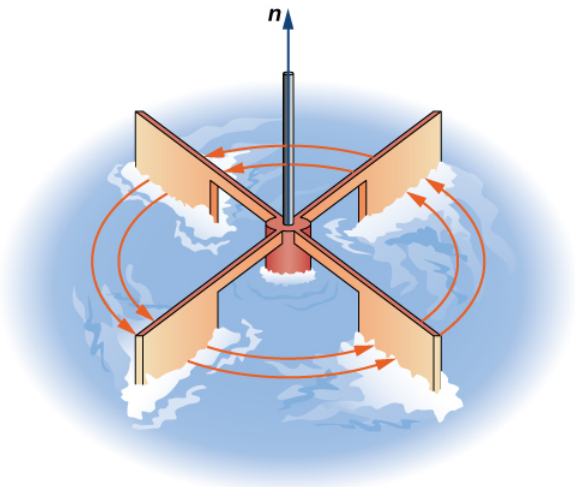


Figure 3.6.5: To visualize curl at a point, imagine placing a small paddlewheel into the vector field at a point.

Consider the vector fields in Figure 3.6.1. In part (a), the vector field is constant and there is no spin at any point. Therefore, we expect the curl of the field to be zero, and this is indeed the case. Part (b) shows a rotational field, so the field has spin. In particular, if you place a paddlewheel into a field at any point so that the axis of the wheel is perpendicular to a plane, the wheel rotates counterclockwise. Therefore, we expect the curl of the field to be nonzero, and this is indeed the case (the curl is $2\hat{\mathbf{k}}$).

To see what curl is measuring globally, imagine dropping a leaf into the fluid. As the leaf moves along with the fluid flow, the curl measures the tendency of the leaf to rotate. If the curl is zero, then the leaf doesn't rotate as it moves through the fluid.

Definition: Curl

If $\vec{\mathbf{F}} = \langle P, Q, R \rangle$ is a vector field in $\mathbb{R}^3$, and P_x, Q_y, and R_z all exist, then the curl of $\vec{\mathbf{F}}$ is defined by

$$\operatorname{curl} \vec{\mathbf{F}} = (R_y - Q_z)\hat{\mathbf{i}} + (P_z - R_x)\hat{\mathbf{j}} + (Q_x - P_y)\hat{\mathbf{k}} \tag{3.6.10}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{\mathbf{k}}. \tag{3.6.11}$$

Note that the curl of a vector field is a vector field, in contrast to divergence.

The definition of curl can be difficult to remember. To help with remembering, we use the notation $\vec{\nabla} \times \vec{\mathbf{F}}$ to stand for a "determinant" that gives the curl formula:

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}. \tag{3.6.12}$$

The determinant of this matrix is

$$(R_y - Q_z)\hat{\mathbf{i}} - (R_x - P_z)\hat{\mathbf{j}} + (Q_x - P_y)\hat{\mathbf{k}} \tag{3.6.13}$$
$$= (R_y - Q_z)\hat{\mathbf{i}} + (P_z - R_x)\hat{\mathbf{j}} + (Q_x - P_y)\hat{\mathbf{k}} = \operatorname{curl} \vec{\mathbf{F}}.$$

Thus, this matrix is a way to help remember the formula for curl. Keep in mind, though, that the word *determinant* is used very loosely. A determinant is not really defined on a matrix with entries that are three vectors, three operators, and three functions.

If $\vec{\mathbf{F}} = \langle P, Q \rangle$ is a vector field in $\mathbb{R}^2$, then the curl of $\vec{\mathbf{F}}$, by definition, is

$$\operatorname{curl} \vec{\mathbf{F}} = (Q_x - P_y)\hat{\mathbf{k}} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{\mathbf{k}}. \tag{3.6.14}$$

Example 3.6.5: Finding the Curl of a Three-Dimensional Vector Field

Find the curl of $\vec{\mathbf{F}}(P, Q, R) = \langle x^2 z, e^y + xz, xyz \rangle$.

Solution

The curl is

$$\begin{aligned} \operatorname{curl} f &= \vec{\nabla} \times \vec{\mathbf{F}} \\ &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} \\ &= (R_y - Q_z)\hat{\mathbf{i}} + (P_z - R_x)\hat{\mathbf{j}} + (Q_x - P_y)\hat{\mathbf{k}} \\ &= (xz - x)\hat{\mathbf{i}} + (x^2 - yz)\hat{\mathbf{j}} + z\hat{\mathbf{k}}. \end{aligned}$$

? Exercise 3.6.4

Find the curl of $\vec{\mathbf{F}} = \langle \sin x \cos z, \sin y \sin z, \cos x \cos y \rangle$ at point $\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right)$.

Hint

Find the determinant of matrix $\vec{\nabla} \times \vec{\mathbf{F}}$.

Answer

$-\hat{i}$

✓ Example 3.6.6: Finding the Curl of a Two-Dimensional Vector Field

Find the curl of $\vec{\mathbf{F}} = \langle P, Q \rangle = \langle y, 0 \rangle$.

Solution

Notice that this vector field consists of vectors that are all parallel. In fact, each vector in the field is parallel to the x-axis. This fact might lead us to the conclusion that the field has no spin and that the curl is zero. To test this theory, note that

$$\text{curl}\, \vec{\mathbf{F}} = (Q_x - P_y)\hat{\mathbf{k}} = -\hat{\mathbf{k}} \neq \vec{\mathbf{0}}. \tag{3.6.15}$$

Therefore, this vector field does have spin. To see why, imagine placing a paddlewheel at any point in the first quadrant (Figure 3.6.6). The larger magnitudes of the vectors at the top of the wheel cause the wheel to rotate. The wheel rotates in the clockwise (negative) direction, causing the coefficient of the curl to be negative.

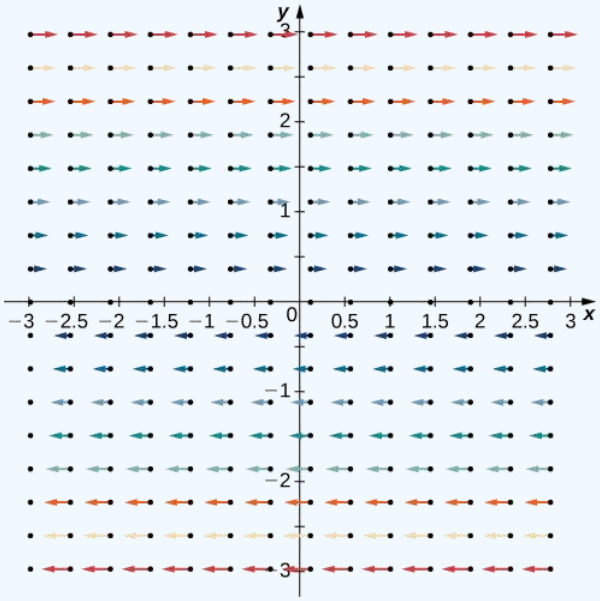


Figure 3.6.6: Vector field $\vec{\mathbf{F}}(x, y) = \langle y, 0 \rangle$ consists of vectors that are all parallel.

Note that if $\vec{\mathbf{F}} = \langle P, Q \rangle$ is a vector field in a plane, then $\text{curl}\, \vec{\mathbf{F}} \cdot \hat{\mathbf{k}} = (Q_x - P_y)\hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = Q_x - P_y$. Therefore, the circulation form of Green's theorem is sometimes written as

$$\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_D \text{curl}\, \vec{\mathbf{F}} \cdot \hat{\mathbf{k}}\, dA, \tag{3.6.16}$$

where C is a simple closed curve and D is the region enclosed by C. Therefore, the circulation form of Green's theorem can be written in terms of the curl. If we think of curl as a derivative of sorts, then Green's theorem says that the "derivative" of $\vec{\mathbf{F}}$ on a region can be translated into a line integral of $\vec{\mathbf{F}}$ along the boundary of the region. This is analogous to the Fundamental Theorem of Calculus, in which the derivative of a function f on line segment $[a, b]$ can be translated into a statement about f on the boundary of

$[a, b]$. Using curl, we can see the circulation form of Green's theorem is a higher-dimensional analog of the Fundamental Theorem of Calculus.

We can now use what we have learned about curl to show that gravitational fields have no "spin." Suppose there is an object at the origin with mass m_1 at the origin and an object with mass m_2. Recall that the gravitational force that object 1 exerts on object 2 is given by field

$$\vec{\mathbf{F}}(x,y,z) = -Gm_1m_2\left\langle \frac{x}{(x^2+y^2+z^2)^{3/2}}, \frac{y}{(x^2+y^2+z^2)^{3/2}}, \frac{z}{(x^2+y^2+z^2)^{3/2}} \right\rangle. \tag{3.6.17}$$

Example 3.6.7: Determining the Spin of a Gravitational Field

Show that a gravitational field has no spin.

Solution

To show that $\vec{\mathbf{F}}$ has no spin, we calculate its curl. Let

- $P(x,y,z) = \dfrac{x}{(x^2+y^2+z^2)^{3/2}}$,
- $Q(x,y,z) = \dfrac{y}{(x^2+y^2+z^2)^{3/2}}$, and
- $R(x,y,z) = \dfrac{z}{(x^2+y^2+z^2)^{3/2}}$.

Then,

$$\begin{aligned} \operatorname{curl}\vec{\mathbf{F}} &= -Gm_1m_2[(R_y - Q_z)\hat{\mathbf{i}} + (P_z - R_x)\hat{\mathbf{j}} + (Q_x - P_y)\hat{\mathbf{k}}] \\ &= -Gm_1m_2 \begin{pmatrix} \left(\dfrac{-3yz}{(x^2+y^2+z^2)^{5/2}} - \left(\dfrac{-3yz}{(x^2+y^2+z^2)^{5/2}}\right)\right)\hat{\mathbf{i}} \\ +\left(\dfrac{-3xz}{(x^2+y^2+z^2)^{5/2}} - \left(\dfrac{-3xz}{(x^2+y^2+z^2)^{5/2}}\right)\right)\hat{\mathbf{j}} \\ +\left(\dfrac{-3xy}{(x^2+y^2+z^2)^{5/2}} - \left(\dfrac{-3xy}{(x^2+y^2+z^2)^{5/2}}\right)\right)\hat{\mathbf{k}} \end{pmatrix} \\ &= \vec{\mathbf{0}}. \end{aligned}$$

Since the curl of the gravitational field is zero, the field has no spin.

Exercise 3.6.7

Field $\vec{\mathbf{v}}(x,y) = \langle -\dfrac{y}{x^2+y^2}, \dfrac{x}{x^2+y^2}\rangle$ models the flow of a fluid. Show that if you drop a leaf into this fluid, as the leaf moves over time, the leaf does not rotate.

Hint

Calculate the curl.

Answer

$\operatorname{curl}\vec{\mathbf{v}} = \vec{\mathbf{0}}$

Using Divergence and Curl

Now that we understand the basic concepts of divergence and curl, we can discuss their properties and establish relationships between them and conservative vector fields.

If $\vec{\mathbf{F}}$ is a vector field in $\mathbb{R}^3$ then the curl of $\vec{\mathbf{F}}$ is also a vector field in $\mathbb{R}^3$. Therefore, we can take the divergence of a curl. The next theorem says that the result is always zero. This result is useful because it gives us a way to show that some vector fields are not the curl of any other field. To give this result a physical interpretation, recall that divergence of a velocity field $\vec{\mathbf{v}}$ at point P measures

the tendency of the corresponding fluid to flow out of P. Since $\text{div}(\text{curl}\,\vec{\mathbf{v}}) = 0$, the net rate of flow in vector field $\text{curl}\,\vec{\mathbf{v}}$\) at any point is zero. Taking the curl of vector field $\vec{\mathbf{F}}$ eliminates whatever divergence was present in $\vec{\mathbf{F}}$.

Theorem: Divergence of the Curl

Let $\vec{\mathbf{F}} = \langle P, Q, R \rangle$ be a vector field in $\mathbb{R}^3$ such that the component functions all have continuous second-order partial derivatives. Then,

$$\text{div}(\text{curl}\,\vec{\mathbf{F}}) = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{\mathbf{F}}) = 0. \tag{3.6.18}$$

Proof

By the definitions of divergence and curl, and by Clairaut's theorem,

$$\begin{aligned} \text{div}(\text{curl}\,\vec{\mathbf{F}}) &= \text{div}[(R_y - Q_z)\,\hat{\mathbf{i}} + (P_z - R_x)\,\hat{\mathbf{j}} + (Q_x - P_y)\,\hat{\mathbf{k}}] \\ &= R_{yx} - Q_{xz} + P_{yz} - R_{yx} + Q_{xz} - P_{zy} \\ &= 0. \end{aligned}$$

□

Example 3.6.8: Showing That a Vector Field Is Not the Curl of Another

Show that $\vec{\mathbf{F}}(x, y, z) = e^x\,\hat{\mathbf{i}} + yz\,\hat{\mathbf{j}} + xz^2\,\hat{\mathbf{k}}$ is not the curl of another vector field. That is, show that there is no other vector $\vec{\mathbf{G}}$ with $\text{curl}\,\vec{\mathbf{G}} = \vec{\mathbf{F}}$.

Solution

Notice that the domain of $\vec{\mathbf{F}}$ is all of $\mathbb{R}^3$ and the second-order partials of $\vec{\mathbf{F}}$ are all continuous. Therefore, we can apply the previous theorem to $\vec{\mathbf{F}}$.

The divergence of $\vec{\mathbf{F}}$ is $e^x + z + 2xz$. If $\vec{\mathbf{F}}$ were the curl of vector field $\vec{\mathbf{G}}$, then $\text{div}\,\vec{\mathbf{F}} = \text{div}(\text{curl}\,\vec{\mathbf{G}}) = 0$. But, the divergence of $\vec{\mathbf{F}}$ is not zero, and therefore $\vec{\mathbf{F}}$ is not the curl of any other vector field.

Exercise 3.6.8

Is it possible for $\vec{\mathbf{G}}(x, y, z) = \langle \sin x, \cos y, \sin(xyz) \rangle$ to be the curl of a vector field?

Hint

Find the divergence of $\vec{\mathbf{G}}$.

Answer

No.

With the next two theorems, we show that if $\vec{\mathbf{F}}$ is a conservative vector field then its curl is zero, and if the domain of $\vec{\mathbf{F}}$ is simply connected then the converse is also true. This gives us another way to test whether a vector field is conservative.

Theorem: Curl of a Conservative Vector Field

If $\vec{\mathbf{F}} = \langle P, Q, R \rangle$ is conservative, then $\text{curl}\,\vec{\mathbf{F}} = \vec{\mathbf{0}}$.

Proof

Since conservative vector fields satisfy the cross-partials property, all the cross-partials of $\vec{\mathbf{F}}$ are equal. Therefore,

$$\begin{aligned}\text{curl}\,\vec{\mathbf{F}} &= (R_y - Q_z)\,\hat{\mathbf{i}} + (P_z - R_x)\,\hat{\mathbf{j}} + (Q_x - P_y)\,\hat{\mathbf{k}} \\ &= \vec{\mathbf{0}}.\end{aligned}$$

□

The same theorem is true for vector fields in a plane.

Since a conservative vector field is the gradient of a scalar function, the previous theorem says that $\text{curl}\,(\vec{\nabla} f) = \vec{\mathbf{0}}$ for any scalar function f. In terms of our curl notation, $\vec{\nabla} \times \vec{\nabla}(f) = \vec{\mathbf{0}}$. This equation makes sense because the cross product of a vector with itself is always the zero vector. Sometimes equation $\vec{\nabla} \times \vec{\nabla}(f) = \vec{\mathbf{0}}$ is simplified as $\vec{\nabla} \times \vec{\nabla} = \vec{\mathbf{0}}$.

Theorem: Curl Test for a Conservative Field

Let $\vec{\mathbf{F}} = \langle P, Q, R \rangle$ be a vector field in space on a simply connected domain. If $\text{curl}\,\vec{\mathbf{F}} = \vec{\mathbf{0}}$, then $\vec{\mathbf{F}}$ is conservative.

Proof

Since $\text{curl}\,\vec{\mathbf{F}} = \vec{\mathbf{0}}$, we have that $R_y = Q_z$, $P_z = R_x$, and $Q_x = P_y$. Therefore, $\vec{\mathbf{F}}$ satisfies the cross-partials property on a simply connected domain, and the Cross-Partial Property of Conservative Fields implies that $\vec{\mathbf{F}}$ is conservative.

□

The same theorem is also true in a plane. Therefore, if $\vec{\mathbf{F}}$ is a vector field in a plane or in space and the domain is simply connected, then $\vec{\mathbf{F}}$ is conservative if and only if $\text{curl}\,\vec{\mathbf{F}} = \vec{\mathbf{0}}$.

Example 3.6.9: Testing Whether a Vector Field Is Conservative

Use the curl to determine whether $\vec{\mathbf{F}}(x, y, z) = \langle yz, xz, xy \rangle$ is conservative.

Solution

Note that the domain of $\vec{\mathbf{F}}$ is all of $\mathbb{R}^3$ which is simply connected (Figure 3.6.7). Therefore, we can test whether $\vec{\mathbf{F}}$ is conservative by calculating its curl.

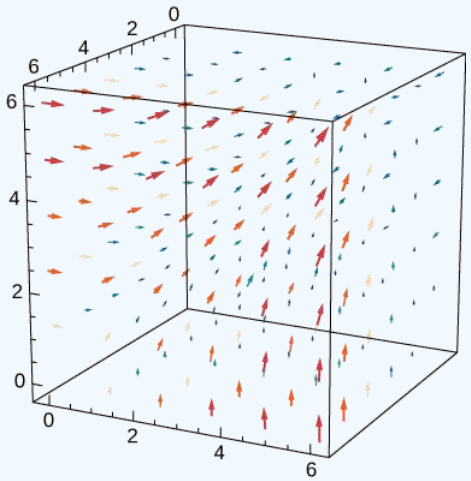


Figure 3.6.7: The curl of vector field $\vec{\mathbf{F}}(x, y, z) = \langle yz, xz, xy \rangle$ is zero.

The curl of $\vec{\mathbf{F}}$ is

$$\begin{aligned}&\left(\frac{\partial}{\partial y}xy - \frac{\partial}{\partial z}xz\right)\hat{\mathbf{i}} + \left(\frac{\partial}{\partial y}yz - \frac{\partial}{\partial z}xy\right)\hat{\mathbf{j}} + \left(\frac{\partial}{\partial y}xz - \frac{\partial}{\partial z}yz\right)\hat{\mathbf{k}} \\ &= (x - x)\,\hat{\mathbf{i}} + (y - y)\,\hat{\mathbf{j}} + (z - z)\,\hat{\mathbf{k}} = \vec{\mathbf{0}}.\end{aligned}$$

Thus, $\vec{\mathbf{F}}$ is conservative.

We have seen that the curl of a gradient is zero. What is the divergence of a gradient? If f is a function of two variables, then $\text{div}(\vec{\nabla} f) = \vec{\nabla} \cdot (\vec{\nabla} f) = f_{xx} + f_{yy}$. We abbreviate this "double dot product" as $\vec{\nabla}^2$. This operator is called the *Laplace operator*, and in this notation Laplace's equation becomes $\vec{\nabla}^2 f = 0$. Therefore, a harmonic function is a function that becomes zero after taking the divergence of a gradient.

Similarly, if f is a function of three variables then

$$\text{div}(\vec{\nabla} f) = \vec{\nabla} \cdot (\vec{\nabla} f) = f_{xx} + f_{yy} + f_{zz}. \tag{3.6.19}$$

Using this notation we get Laplace's equation for harmonic functions of three variables:

$$\vec{\nabla}^2 f = 0. \tag{3.6.20}$$

Harmonic functions arise in many applications. For example, the potential function of an electrostatic field in a region of space that has no static charge is harmonic.

Example 3.6.10: Finding a Potential Function

Is it possible for $f(x, y) = x^2 + x - y$ to be the potential function of an electrostatic field that is located in a region of $\mathbb{R}^2$ free of static charge?

Solution

If f were such a potential function, then f would be harmonic. Note that $f_{xx} = 2$ and $f_{yy} = 0$, and so $f_{xx} + f_{yy} \neq 0$. Therefore, f is not harmonic and f cannot represent an electrostatic potential.

Exercise 3.6.10

Is it possible for function $f(x, y) = x^2 - y^2 + x$ to be the potential function of an electrostatic field located in a region of $\mathbb{R}^2$ free of static charge?

Hint

Determine whether the function is harmonic.

Answer

Yes.

Key Concepts

- The divergence of a vector field is a scalar function. Divergence measures the "outflowing-ness" of a vector field. If $\vec{\mathbf{v}}$ is the velocity field of a fluid, then the divergence of $\vec{\mathbf{v}}$ at a point is the outflow of the fluid less the inflow at the point.
- The curl of a vector field is a vector field. The curl of a vector field at point P measures the tendency of particles at P to rotate about the axis that points in the direction of the curl at P.
- A vector field with a simply connected domain is conservative if and only if its curl is zero.

Key Equations

- **Curl**

$$\vec{\nabla} \times \vec{\mathbf{F}} = (R_y - Q_z)\hat{\mathbf{i}} + (P_z - R_x)\hat{\mathbf{j}} + (Q_x - P_y)\hat{\mathbf{k}}$$

- **Divergence**

$$\vec{\nabla} \cdot \vec{\mathbf{F}} = P_x + Q_y + R_z$$

- **Divergence of curl is zero**

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{\mathbf{F}}) = 0$$

- **Curl of a gradient is the zero vector**

$$\vec{\nabla} \times (\vec{\nabla} f) = 0$$

Glossary

curl

the curl of vector field $\vec{\mathbf{F}} = \langle P, Q, R \rangle$, denoted $\vec{\nabla} \times \vec{\mathbf{F}}$ is the "determinant" of the matrix

$$\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}.$$

and is given by the expression $(R_y - Q_z)\hat{\mathbf{i}} + (P_z - R_x)\hat{\mathbf{j}} + (Q_x - P_y)\hat{\mathbf{k}}$; it measures the tendency of particles at a point to rotate about the axis that points in the direction of the curl at the point

divergence

the divergence of a vector field $\vec{\mathbf{F}} = \langle P, Q, R \rangle$, denoted $\vec{\nabla} \times \vec{\mathbf{F}}$, is $P_x + Q_y + R_z$; it measures the "outflowing-ness" of a vector field

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3.6E: Exercises

For the following exercises, determine whether the statement is *True* or *False*.

1. If the coordinate functions of $\vec{\mathbf{F}} : \mathbb{R}^3 \to \mathbb{R}^3$ have continuous second partial derivatives, then $\textbf{curl}\,(\textbf{div}\,\vec{\mathbf{F}})$ equals zero.

2. $\vec{\nabla} \cdot (x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}) = 1$.

Answer

False

3. All vector fields of the form $\vec{\mathbf{F}}(x, y, z) = f(x)\,\hat{\mathbf{i}} + g(y)\,\hat{\mathbf{j}} + h(z)\,\hat{\mathbf{k}}$ are conservative.

4. If $\textbf{curl}\,\vec{\mathbf{F}} = \vec{\mathbf{0}}$, then $\vec{\mathbf{F}}$ is conservative.

Answer

True

5. If $\vec{\mathbf{F}}$ is a constant vector field then $\textbf{div}\,\vec{\mathbf{F}} = 0$.

6. If $\vec{\mathbf{F}}$ is a constant vector field then $\textbf{curl}\,\vec{\mathbf{F}} = \vec{\mathbf{0}}$.

Answer

True

For the following exercises, find the curl of $\vec{\mathbf{F}}$.

7. $\vec{\mathbf{F}}(x, y, z) = xy^2z^4\,\hat{\mathbf{i}} + (2x^2y + z)\,\hat{\mathbf{j}} + y^3z^2\,\hat{\mathbf{k}}$

8. $\vec{\mathbf{F}}(x, y, z) = x^2z\,\hat{\mathbf{i}} + y^2x\,\hat{\mathbf{j}} + (y + 2z)\,\hat{\mathbf{k}}$

Answer

$\textbf{curl}\,\vec{\mathbf{F}} = \hat{\mathbf{i}} + x^2\,\hat{\mathbf{j}} + y^2\,\hat{\mathbf{k}}$

9. $\vec{\mathbf{F}}(x, y, z) = 3xyz^2\,\hat{\mathbf{i}} + y^2 \sin z\,\hat{\mathbf{j}} + xe^{2z}\,\hat{\mathbf{k}}$

10. $\vec{\mathbf{F}}(x, y, z) = x^2yz\,\hat{\mathbf{i}} + xy^2z\,\hat{\mathbf{j}} + xyz^2\,\hat{\mathbf{k}}$

Answer

$\textbf{curl}\,\vec{\mathbf{F}} = (xz^2 - xy^2)\,\hat{\mathbf{i}} + (x^2y - yz^2)\,\hat{\mathbf{j}} + (y^2z - x^2z)\,\hat{\mathbf{k}}$

11. $\vec{\mathbf{F}}(x, y, z) = (x \cos y)\,\hat{\mathbf{i}} + xy^2\,\hat{\mathbf{j}}$

12. $\vec{\mathbf{F}}(x, y, z) = (x - y)\,\hat{\mathbf{i}} + (y - z)\,\hat{\mathbf{j}} + (z - x)\,\hat{\mathbf{k}}$

Answer

$\textbf{curl}\ \vec{\mathbf{F}} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$

13. $\vec{\mathbf{F}}(x, y, z) = xyz\,\hat{\mathbf{i}} + x^2y^2z^2\,\hat{\mathbf{j}} + y^2z^3\,\hat{\mathbf{k}}$

14. $\vec{\mathbf{F}}(x, y, z) = xy\,\hat{\mathbf{i}} + yz\,\hat{\mathbf{j}} + xz\,\hat{\mathbf{k}}$

Answer

$\textbf{curl}\ \vec{\mathbf{F}} = -y\,\hat{\mathbf{i}} - z\,\hat{\mathbf{j}} - x\,\hat{\mathbf{k}}$

15. $\vec{\mathbf{F}}(x, y, z) = x^2\,\hat{\mathbf{i}} + y^2\,\hat{\mathbf{j}} + z^2\,\hat{\mathbf{k}}$

16. $\vec{\mathbf{F}}(x, y, z) = ax\,\hat{\mathbf{i}} + by\,\hat{\mathbf{j}} + c\,\hat{\mathbf{k}}$ for constants a, b, c.

Answer

$$\text{curl } \vec{\mathbf{F}} = \vec{\mathbf{0}}$$

For the following exercises, find the divergence of $\vec{\mathbf{F}}$.

17. $\vec{\mathbf{F}}(x,y,z) = x^2z\,\hat{\mathbf{i}} + y^2x\,\hat{\mathbf{j}} + (y+2z)\,\hat{\mathbf{k}}$

18. $\vec{\mathbf{F}}(x,y,z) = 3xyz^2\,\hat{\mathbf{i}} + y^2\sin z\,\hat{\mathbf{j}} + xe^{2z}z\,\hat{\mathbf{k}}$

Answer

$\text{div}\,\vec{\mathbf{F}} = 3yz^2 + 2y\sin z + 2xe^{2z}$

19. $\vec{\mathbf{F}}(x,y) = (\sin x)\,\hat{\mathbf{i}} + (\cos y)\,\hat{\mathbf{j}}$

20. $\vec{\mathbf{F}}(x,y,z) = x^2\,\hat{\mathbf{i}} + y^2\,\hat{\mathbf{j}} + z^2\,\hat{\mathbf{k}}$

Answer

$\text{div}\,\vec{\mathbf{F}} = 2(x+y+z)$

21. $\vec{\mathbf{F}}(x,y,z) = (x-y)\,\hat{\mathbf{i}} + (y-z)\,\hat{\mathbf{j}} + (z-x)\,\hat{\mathbf{k}}$

22. $\vec{\mathbf{F}}(x,y) = \dfrac{x}{\sqrt{x^2+y^2}}\,\hat{\mathbf{i}} + \dfrac{y}{\sqrt{x^2+y^2}}\,\hat{\mathbf{j}}$

Answer

$\text{div}\,\vec{\mathbf{F}} = \dfrac{1}{\sqrt{x^2+y^2}}$

23. $\vec{\mathbf{F}}(x,y) = x\,\hat{\mathbf{i}} - y\,\hat{\mathbf{j}}$

24. $\vec{\mathbf{F}}(x,y,z) = ax\,\hat{\mathbf{i}} + by\,\hat{\mathbf{j}} + c\,\hat{\mathbf{k}}$ for constants a, b, c.

Answer

$\text{div}\,\vec{\mathbf{F}} = a + b$

25. $\vec{\mathbf{F}}(x,y,z) = xyz\,\hat{\mathbf{i}} + x^2y^2z^2\,\hat{\mathbf{j}} + y^2z^3\,\hat{\mathbf{k}}$

26. $\vec{\mathbf{F}}(x,y,z) = xy\,\hat{\mathbf{i}} + yz\,\hat{\mathbf{j}} + xz\,\hat{\mathbf{k}}$

Answer

$\text{div}\,\vec{\mathbf{F}} = x + y + z$

For exercises 27 & 28, determine whether each of the given scalar functions is harmonic.

27. $u(x,y,z) = e^{-x}(\cos y - \sin y)$

28. $w(x,y,z) = (x^2+y^2+z^2)^{-1/2}$

Answer

Harmonic

29. If $\vec{\mathbf{F}}(x,y,z) = 2\,\hat{\mathbf{i}} + 2xj + 3yk$ and $\vec{\mathbf{G}}(x,y,z) = x\,\hat{\mathbf{i}} - y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$, find $\text{curl}\,(\vec{\mathbf{F}} \times \vec{\mathbf{G}})$.

30. If $\vec{\mathbf{F}}(x,y,z) = 2\,\hat{\mathbf{i}} + 2xj + 3yk$ and $\vec{\mathbf{G}}(x,y,z) = x\,\hat{\mathbf{i}} - y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$, find $\text{div}\,(\vec{\mathbf{F}} \times \vec{\mathbf{G}})$.

Answer

$\text{div}\,(\vec{\mathbf{F}} \times \vec{\mathbf{G}}) = 2z + 3x$

31. Find $\text{div}\,\vec{\mathbf{F}}$, given that $\vec{\mathbf{F}} = \vec{\nabla} f$, where $f(x,y,z) = xy^3z^2$.

32. Find the divergence of $\vec{\mathbf{F}}$ for vector field $\vec{\mathbf{F}}(x,y,z) = (y^2+z^2)(x+y)\,\hat{\mathbf{i}} + (z^2+x^2)(y+z)\,\hat{\mathbf{j}} + (x^2+y^2)(z+x)\,\hat{\mathbf{k}}$.

Answer

$\text{div}\,\vec{\mathbf{F}} = 2r^2$

33. Find the divergence of $\vec{\mathbf{F}}$ for vector field $\vec{\mathbf{F}}(x,y,z) = f_1(y,z)\,\hat{\mathbf{i}} + f_2(x,z)\,\hat{\mathbf{j}} + f_3(x,y)\,\hat{\mathbf{k}}$.

For exercises 34 - 36, use $r = |\vec{\mathbf{r}}|$ and $\vec{\mathbf{r}}(x,y,z) = \langle x,y,z\rangle$.

34. Find the $\text{curl}\,\vec{\mathbf{r}}$

Answer

$\text{curl}\,\vec{\mathbf{r}} = \vec{\mathbf{0}}$

35. Find the $\text{curl}\,\dfrac{\vec{\mathbf{r}}}{r}$.

36. Find the $\text{curl}\,\dfrac{\vec{\mathbf{r}}}{r^3}$.

Answer

$\text{curl}\,\dfrac{\vec{\mathbf{r}}}{r^3} = \vec{\mathbf{0}}$

37. Let $\vec{\mathbf{F}}(x,y) = \dfrac{-y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}}}{x^2+y^2}$, where $\vec{\mathbf{F}}$ is defined on $\{(x,y) \in \mathbb{R} | (x,y) \neq (0,0)\}$. Find $\text{curl}\,\vec{\mathbf{F}}$.

For the following exercises, use a computer algebra system to find the curl of the given vector fields.

38. [T] $\vec{\mathbf{F}}(x,y,z) = \arctan\left(\dfrac{x}{y}\right)\,\hat{\mathbf{i}} + \ln\sqrt{x^2+y^2}\,\hat{\mathbf{j}} + \hat{\mathbf{k}}$

Answer

$\text{curl}\;\vec{\mathbf{F}} = \dfrac{2x}{x^2+y^2}\,\hat{\mathbf{k}}$

39. [T] $\vec{\mathbf{F}}(x,y,z) = \sin(x-y)\,\hat{\mathbf{i}} + \sin(y-z)\,\hat{\mathbf{j}} + \sin(z-x)\,\hat{\mathbf{k}}$

For the following exercises, find the divergence of $\vec{\mathbf{F}}$ at the given point.

40. $\vec{\mathbf{F}}(x,y,z) = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ at $(2,-1,3)$

Answer

$\text{div}\,\vec{\mathbf{F}} = 0$

41. $\vec{\mathbf{F}}(x,y,z) = xyz\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$ at $(1,2,3)$

42. $\vec{\mathbf{F}}(x,y,z) = e^{-xy}\,\hat{\mathbf{i}} + e^{xz}\,\hat{\mathbf{j}} + e^{yz}\,\hat{\mathbf{k}}$ at $(3,2,0)$

Answer

$\text{div}\,\vec{\mathbf{F}} = 2 - 2e^{-6}$

43. $\vec{\mathbf{F}}(x,y,z) = xyz\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$ at $(1,2,1)$

44. $\vec{\mathbf{F}}(x,y,z) = e^x \sin y\,\hat{\mathbf{i}} - e^x \cos y\,\hat{\mathbf{j}}$ at $(0,0,3)$

Answer

$\text{div}\,\vec{\mathbf{F}} = 0$

For exercises 45- 49, find the curl of $\vec{\mathbf{F}}$ at the given point.

45. $\vec{\mathbf{F}}(x,y,z) = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ at $(2,-1,3)$

46. $\vec{\mathbf{F}}(x,y,z) = xyz\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + x\,\hat{\mathbf{k}}$ at $(1,2,3)$

Answer

$\text{curl } \vec{\mathbf{F}} = \hat{\mathbf{j}} - 3\hat{\mathbf{k}}$

47. $\vec{\mathbf{F}}(x, y, z) = e^{-xy}\,\hat{\mathbf{i}} + e^{xz}\,\hat{\mathbf{j}} + e^{yz}\,\hat{\mathbf{k}}$ at $(3, 2, 0)$

48. $\vec{\mathbf{F}}(x, y, z) = xyz\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$ at $(1, 2, 1)$

Answer

$\text{curl } \vec{\mathbf{F}} = 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$

49. $\vec{\mathbf{F}}(x, y, z) = e^{x} \sin y\,\hat{\mathbf{i}} - e^{x} \cos y\,\hat{\mathbf{j}}$ at $(0, 0, 3)$

50. Let $\vec{\mathbf{F}}(x, y, z) = (3x^2y + az)\,\hat{\mathbf{i}} + x^3\,\hat{\mathbf{j}} + (3x + 3z^2)\,\hat{\mathbf{k}}$. For what value of a is $\vec{\mathbf{F}}$ conservative?

Answer

$a = 3$

51. Given vector field $\vec{\mathbf{F}}(x, y) = \dfrac{1}{x^2 + y^2}\langle -y, x\rangle$ on domain $D = \dfrac{\mathbb{R}^2}{\{(0, 0)\}} = \{(x, y) \in \mathbb{R}^2 | (x, y) \neq (0, 0)\}$, is $\vec{\mathbf{F}}$ conservative?

52. Given vector field $\vec{\mathbf{F}}(x, y) = \dfrac{1}{x^2 + y^2}\langle x, y\rangle$ on domain $D = \dfrac{\mathbb{R}^2}{\{(0, 0)\}}$, is $\vec{\mathbf{F}}$ conservative?

Answer

$\vec{\mathbf{F}}$ is conservative.

53. Find the work done by force field $\vec{\mathbf{F}}(x, y) = e^{-y}\,\hat{\mathbf{i}} - xe^{-y}\,\hat{\mathbf{j}}$ in moving an object from $P(0, 1)$ to $Q(2, 0)$. Is the force field conservative?

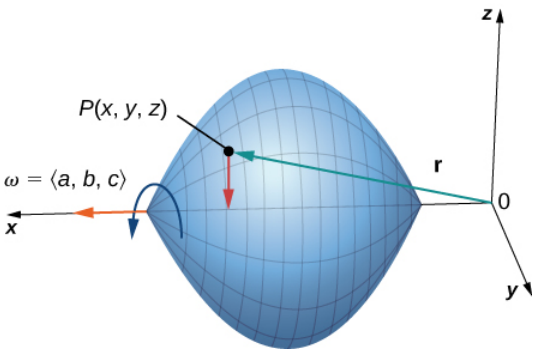
54. Compute divergence $\vec{\mathbf{F}}(x, y, z) = (\sinh x)\,\hat{\mathbf{i}} + (\cosh y)\,\hat{\mathbf{j}} - xyz\,\hat{\mathbf{k}}$.

Answer

$\text{div } \vec{\mathbf{F}} = \cosh x + \sinh y - xy$

55. Compute $\text{curl } \vec{\mathbf{F}} = (\sinh x)\,\hat{\mathbf{i}} + (\cosh y)\,\hat{\mathbf{j}} - xyz\,\hat{\mathbf{k}}$.

For the following exercises, consider a rigid body that is rotating about the x-axis counterclockwise with constant angular velocity $\vec{\omega} = \langle a, b, c\rangle$. If P is a point in the body located at $\vec{\mathbf{r}} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$, the velocity at P is given by vector field $\vec{\mathbf{F}} = \vec{\omega} \times \vec{\mathbf{r}}$.



56. Express $\vec{\mathbf{F}}$ in terms of $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ vectors.

Answer

$\vec{\mathbf{F}} = (bz - cy)\,\hat{\mathbf{i}} + (cx - az)\,\hat{\mathbf{j}} + (ay - bx)\,\hat{\mathbf{k}}$

57. Find $\text{div } F$.

58. Find $\text{curl } F$

Answer

$$\text{curl } \vec{\mathbf{F}} = 2\vec{\omega}$$

In the following exercises, suppose that $\vec{\nabla} \cdot \vec{\mathbf{F}} = 0$ and $\vec{\nabla} \cdot \vec{\mathbf{G}} = 0$.

59. Does $\vec{\mathbf{F}} + \vec{\mathbf{G}}$ necessarily have zero divergence?

60. Does $\vec{\mathbf{F}} \times \vec{\mathbf{G}}$ necessarily have zero divergence?

Answer

$\vec{\mathbf{F}} \times \vec{\mathbf{G}}$ does not have zero divergence.

In the following exercises, suppose a solid object in $\mathbb{R}^3$ has a temperature distribution given by $T(x, y, z)$. The heat flow vector field in the object is $\vec{\mathbf{F}} = -k\vec{\nabla}T$, where $k > 0$ is a property of the material. The heat flow vector points in the direction opposite to that of the gradient, which is the direction of greatest temperature decrease. The divergence of the heat flow vector is $\vec{\nabla} \cdot \vec{\mathbf{F}} = -k\vec{\nabla} \cdot \vec{\nabla}T = -k\vec{\nabla}^2 T$.

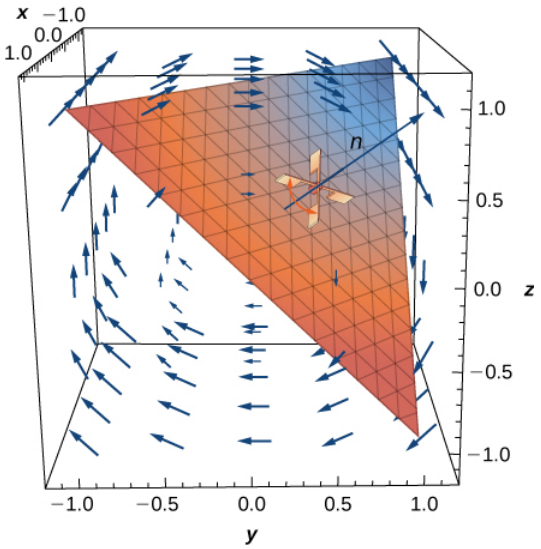
61. Compute the heat flow vector field.

62. Compute the divergence.

Answer

$\vec{\nabla} \cdot \vec{\mathbf{F}} = -200k[1 + 2(x^2 + y^2 + z^2)]e^{-x^2+y^2+z^2}$

63. [T] Consider rotational velocity field $\vec{\mathbf{v}} = \langle 0, 10z, -10y \rangle$. If a paddlewheel is placed in plane $x + y + z = 1$ with its axis normal to this plane, using a computer algebra system, calculate how fast the paddlewheel spins in revolutions per unit time.



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3.7: Surface Integrals

Learning Objectives

- Find the parametric representations of a cylinder, a cone, and a sphere.
- Describe the surface integral of a scalar-valued function over a parametric surface.
- Use a surface integral to calculate the area of a given surface.
- Explain the meaning of an oriented surface, giving an example.
- Describe the surface integral of a vector field.
- Use surface integrals to solve applied problems.

We have seen that a line integral is an integral over a path in a plane or in space. However, if we wish to integrate over a surface (a two-dimensional object) rather than a path (a one-dimensional object) in space, then we need a new kind of integral that can handle integration over objects in higher dimensions. We can extend the concept of a line integral to a surface integral to allow us to perform this integration.

Surface integrals are important for the same reasons that line integrals are important. They have many applications to physics and engineering, and they allow us to develop higher dimensional versions of the Fundamental Theorem of Calculus. In particular, surface integrals allow us to generalize Green's theorem to higher dimensions, and they appear in some important theorems we discuss in later sections.

Parametric Surfaces

A surface integral is similar to a line integral, except the integration is done over a surface rather than a path. In this sense, surface integrals expand on our study of line integrals. Just as with line integrals, there are two kinds of surface integrals: a surface integral of a scalar-valued function and a surface integral of a vector field.

However, before we can integrate over a surface, we need to consider the surface itself. Recall that to calculate a scalar or vector line integral over curve C, we first need to parameterize C. In a similar way, to calculate a surface integral over surface S, we need to parameterize S. That is, we need a working concept of a *parameterized surface* (or a *parametric surface*), in the same way that we already have a concept of a parameterized curve.

A parameterized surface is given by a description of the form

$$\vec{\mathbf{r}}(u,v) = \langle x(u,v),\, y(u,v),\, z(u,v)\rangle. \tag{3.7.1}$$

Notice that this parameterization involves two parameters, u and v, because a surface is two-dimensional, and therefore two variables are needed to trace out the surface. The parameters u and v vary over a region called the parameter domain, or parameter space—the set of points in the uv-plane that can be substituted into $\vec{\mathbf{r}}$. Each choice of u and v in the parameter domain gives a point on the surface, just as each choice of a parameter t gives a point on a parameterized curve. The entire surface is created by making all possible choices of u and v over the parameter domain.

Definition: Parameter Domain

Given a parameterization of surface

$$\vec{\mathbf{r}}(u,v) = \langle x(u,v),\, y(u,v),\, z(u,v)\rangle. \tag{3.7.2}$$

the *parameter domain* of the parameterization is the set of points in the uv-plane that can be substituted into $\vec{\mathbf{r}}$.

Example 3.7.1: Parameterizing a Cylinder

Describe surface S parameterized by

$$\vec{\mathbf{r}}(u,v) = \langle \cos u,\, \sin u,\, v\rangle,\ -\infty < u < \infty,\ -\infty < v < \infty.$$

Solution

To get an idea of the shape of the surface, we first plot some points. Since the parameter domain is all of $\mathbb{R}^2$, we can choose any value for u and v and plot the corresponding point. If $u = v = 0$, then $\vec{\mathbf{r}}(0,0) = \langle 1,0,0\rangle$, so point (1, 0, 0) is on S. Similarly, points $\vec{\mathbf{r}}(\pi,2) = (-1,0,2)$ and $\vec{\mathbf{r}}\left(\frac{\pi}{2},4\right) = (0,1,4)$ are on S.

Although plotting points may give us an idea of the shape of the surface, we usually need quite a few points to see the shape. Since it is time-consuming to plot dozens or hundreds of points, we use another strategy. To visualize S, we visualize two families of curves that lie on S. In the first family of curves we hold u constant; in the second family of curves we hold v constant. This allows us to build a "skeleton" of the surface, thereby getting an idea of its shape.

- Suppose that u is a constant K. Then the curve traced out by the parameterization is $\langle \cos K, \sin K, v\rangle$, which gives a vertical line that goes through point $(\cos K, \sin K, v\rangle$ in the xy-plane.
- Suppose that v is a constant K. Then the curve traced out by the parameterization is $\langle \cos u, \sin u, K\rangle$, which gives a circle in plane $z = K$ with radius 1 and center $(0,0,K)$.

If u is held constant, then we get vertical lines; if v is held constant, then we get circles of radius 1 centered around the vertical line that goes through the origin. Therefore the surface traced out by the parameterization is cylinder $x^2 + y^2 = 1$ (Figure 3.7.1).

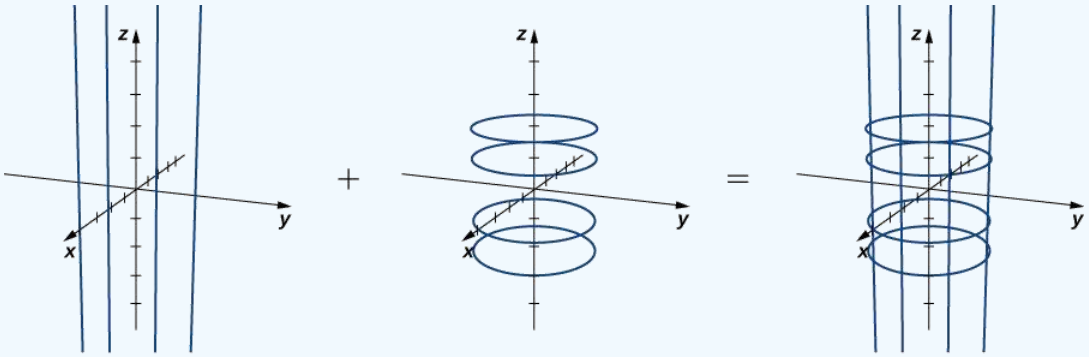


Figure 3.7.1: (a) Lines $\langle \cos K, \sin K, v\rangle$ for $K = 0, \frac{\pi}{2}, \pi$, and $\frac{3\pi}{2}$. (b) Circles $\langle \cos u, \sin u, K\rangle$ for $K = -2, -1, 1$, and 2 (c) The lines and circles together. As u and v vary, they describe a cylinder.

Notice that if $x = \cos u$ and $y = \sin u$, then $x^2 + y^2 = 1$, so points from S do indeed lie on the cylinder. Conversely, each point on the cylinder is contained in some circle $\langle \cos u, \sin u, k\rangle$ for some k, and therefore each point on the cylinder is contained in the parameterized surface (Figure 3.7.2).

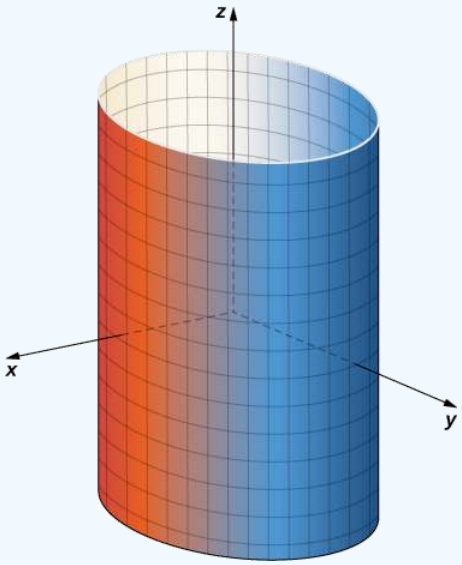


Figure 3.7.2: Cylinder $x^2 + y^2 = r^2$ has parameterization $\vec{\mathbf{r}}(u,v) = \langle r\cos u, r\sin u, v\rangle$, $0 \le u \le 2\pi$, $-\infty < v < \infty$.

Analysis

Notice that if we change the parameter domain, we could get a different surface. For example, if we restricted the domain to $0 \leq u \leq \pi,\ -\infty < v < 6$, then the surface would be a half-cylinder of height 6.



? Exercise 3.7.1

Describe the surface with parameterization

$$\vec{\mathbf{r}}(u, v) = \langle 2\cos u,\ 2\sin u,\ v\rangle,\ 0 \leq u \leq 2\pi,\ -\infty < v < \infty$$

Hint

Hold u and v constant, and see what kind of curves result.

Answer

Cylinder $x^2 + y^2 = 4$

It follows from Example 3.7.1 that we can parameterize all cylinders of the form $x^2 + y^2 = R^2$. If S is a cylinder given by equation $x^2 + y^2 = R^2$, then a parameterization of S is $\vec{\mathbf{r}}(u, v) = \langle R\cos u,\ R\sin u,\ v\rangle,\ 0 \leq u \leq 2\pi,\ -\infty < v < \infty$.

We can also find different types of surfaces given their parameterization, or we can find a parameterization when we are given a surface.



Example 3.7.2: Describing a Surface

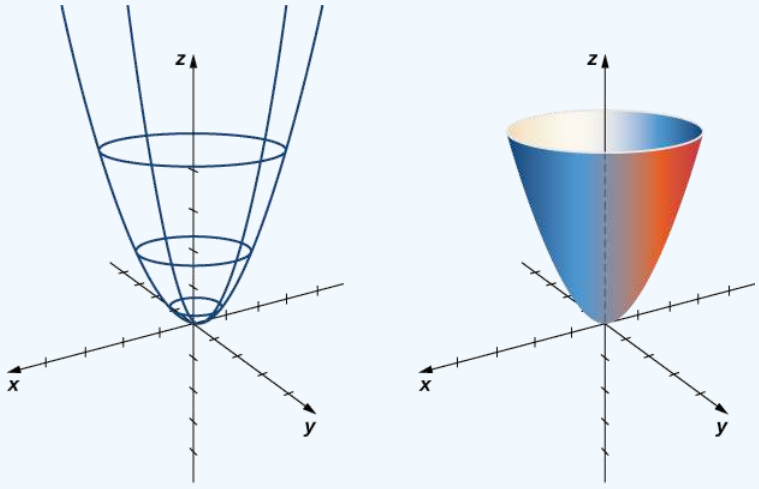
Describe surface S parameterized by $\vec{\mathbf{r}}(u,v) = \langle u\cos v,\ u\sin v,\ u^2\rangle$, $0 \le u < \infty$, $0 \le v < 2\pi$.

Solution

Notice that if u is held constant, then the resulting curve is a circle of radius u in plane $z = u$. Therefore, as u increases, the radius of the resulting circle increases. If v is held constant, then the resulting curve is a vertical parabola. Therefore, we expect the surface to be an elliptic paraboloid. To confirm this, notice that

$$\begin{aligned} x^2 + y^2 &= (u\cos v)^2 + (u\sin v)^2 \\ &= u^2\cos^2 v + u^2 sin^2 v \\ &= u^2 \\ &= z \end{aligned}$$

Therefore, the surface is the elliptic paraboloid $x^2 + y^2 = z$ (Figure 3.7.3).



(a) (b)

Figure 3.7.3: (a) Circles arise from holding u constant; the vertical parabolas arise from holding v constant. (b) An elliptic paraboloid results from all choices of u and v in the parameter domain.

Exercise 3.7.2

Describe the surface parameterized by $\vec{\mathbf{r}}(u,v) = \langle u\ \cos v,\ u\ \sin v,\ u\rangle$, $-\infty < u < \infty$, $0 \le v < 2\pi$.

Hint

Hold u constant and see what kind of curves result. Imagine what happens as u increases or decreases.

Answer

Cone $x^2 + y^2 = z^2$

Example 3.7.3: Finding a Parameterization

Give a parameterization of the cone $x^2 + y^2 = z^2$ lying on or above the plane $z = -2$.

Solution

The horizontal cross-section of the cone at height $z = u$ is circle $x^2 + y^2 = u^2$. Therefore, a point on the cone at height u has coordinates $(u\ \cos v,\ u\ \sin v,\ u)$ for angle v. Hence, a parameterization of the cone is $\vec{\mathbf{r}}(u,v) = \langle u\ \cos v,\ u\ \sin v,\ u\rangle$. Since we are not interested in the entire cone, only the portion on or above plane $z = -2$, the parameter domain is given by $-2 < u < \infty,\ 0 \le v < 2\pi$ (Figure 3.7.4).

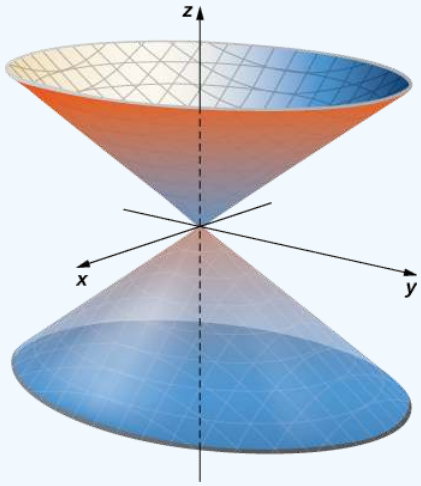


Figure Extra close brace or missing open brace: Cone $x^2 + y^2 = z^2$ has parametrization $\vec{\mathbf{r}}(u,v) = \langle u\ \cos v,\ u\ \sin v, u\rangle, u,\ -\infty < u < \infty,\ 0 \le v \le 2\pi$.

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Exercise 3.7.3

Give a parameterization for the portion of cone $x^2 + y^2 = z^2$ lying in the first octant.

Hint

Consider the parameter domain for this surface.

Answer

$\vec{\mathbf{r}}(u, v) = \langle u\cos v,\ u\sin v,\ u\rangle,\ 0 < u < \infty,\ 0 \le v < \dfrac{\pi}{2}$



We have discussed parameterizations of various surfaces, but two important types of surfaces need a separate discussion: spheres and graphs of two-variable functions. To parameterize a sphere, it is easiest to use spherical coordinates. The sphere of radius ρ centered at the origin is given by the parameterization

$$\vec{\mathbf{r}}(\phi, \theta) = \langle \rho\cos\theta\sin\phi,\ \rho\sin\theta\sin\phi,\ \rho\cos\phi\rangle,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi.$$

The idea of this parameterization is that as ϕ sweeps downward from the positive z-axis, a circle of radius $\rho \sin\phi$ is traced out by letting θ run from 0 to 2π. To see this, let ϕ be fixed. Then

$$\begin{aligned} x^2 + y^2 &= (\rho \cos\theta \sin\phi)^2 + (\rho \sin\theta \sin\phi)^2 \\ &= \rho^2 \sin^2\phi(\cos^2\theta + \sin^2\theta) \\ &= \rho^2 \sin^2\phi \\ &= (\rho \sin\phi)^2. \end{aligned}$$

This results in the desired circle (Figure 3.7.5).

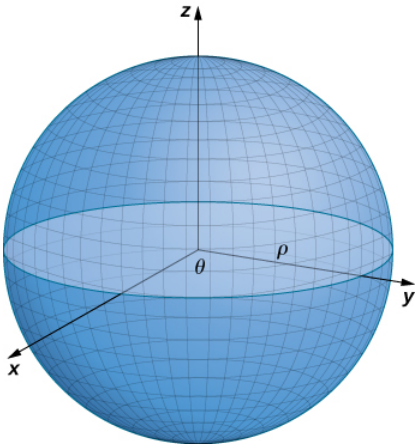


Figure 3.7.5: The sphere of radius ρ has parameterization $\vec{\mathbf{r}}(\phi,\theta) = \langle \rho \cos\theta \sin\phi,\ \rho \sin\theta \sin\phi,\ \rho \cos\phi \rangle$, $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$.

Finally, to parameterize the graph of a two-variable function, we first let $z = f(x,y)$ be a function of two variables. The simplest parameterization of the graph of f is $\vec{\mathbf{r}}(x,y) = \langle x, y, f(x,y) \rangle$, where x and y vary over the domain of f (Figure 3.7.6). For example, the graph of $f(x,y) = x^2 y$ can be parameterized by $\vec{\mathbf{r}}(x,y) = \langle x, y, x^2 y \rangle$, where the parameters x and y vary over the domain of f. If we only care about a piece of the graph of f - say, the piece of the graph over rectangle $[1,3] \times [2,5]$ - then we can restrict the parameter domain to give this piece of the surface:

$$\vec{\mathbf{r}}(x,y) = \langle x, y, x^2 y \rangle,\ 1 \le x \le 3,\ 2 \le y \le 5. \tag{3.7.3}$$

Similarly, if S is a surface given by equation $x = g(y,z)$ or equation $y = h(x,z)$, then a parameterization of S is $\vec{\mathbf{r}}(y,z) = \langle g(y,z),\ y, z \rangle$ or $\vec{\mathbf{r}}(x,z) = \langle x, h(x,z), z \rangle$, respectively. For example, the graph of paraboloid $2y = x^2 + z^2$ can be parameterized by $\vec{\mathbf{r}}(x,y) = \left\langle x, \dfrac{x^2+z^2}{2}, z \right\rangle$, $0 \le x < \infty$, $0 \le z < \infty$. Notice that we do not need to vary over the entire domain of y because x and z are squared.

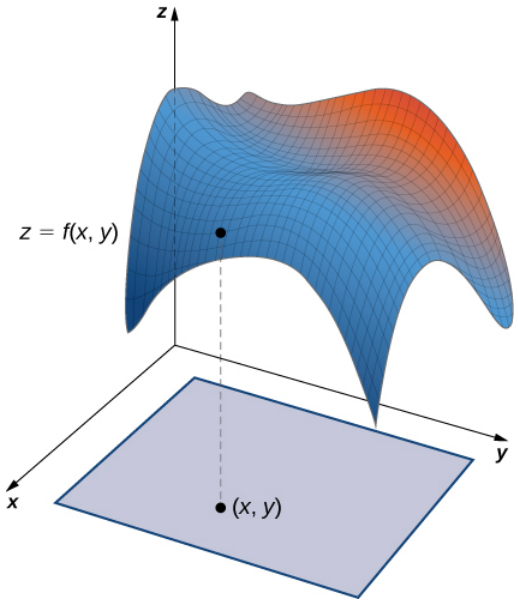


Figure **3.7.6**: The simplest parameterization of the graph of a function is $\vec{\mathbf{r}}(x,y) = \langle x, y, f(x,y)\rangle$.

Let's now generalize the notions of smoothness and regularity to a parametric surface. Recall that curve parameterization $\vec{\mathbf{r}}(t)$, $a \le t \le b$ is regular (or **smooth**) if $\vec{\mathbf{r}}'(t) \neq \vec{\mathbf{0}}$ for all t in $[a,b]$. For a curve, this condition ensures that the image of $\vec{\mathbf{r}}$ really is a curve, and not just a point. For example, consider curve parameterization $\vec{\mathbf{r}}(t) = \langle 1,2\rangle$, $0 \le t \le 5$. The image of this parameterization is simply point $(1,2)$, which is not a curve. Notice also that $\vec{\mathbf{r}}'(t) = \vec{\mathbf{0}}$. The fact that the derivative is the zero vector indicates we are not actually looking at a curve.

Analogously, we would like a notion of regularity (or smoothness) for surfaces so that a surface parameterization really does trace out a surface. To motivate the definition of regularity of a surface parameterization, consider the parameterization

$$\vec{\mathbf{r}}(u,v) = \langle 0, \cos v, 1\rangle,\ 0 \le u \le 1,\ 0 \le v \le \pi. \tag{3.7.4}$$

Although this parameterization appears to be the parameterization of a surface, notice that the image is actually a line (Figure **3.7.7**). How could we avoid parameterizations such as this? Parameterizations that do not give an actual surface? Notice that $\vec{\mathbf{r}}_u = \langle 0,0,0\rangle$ and $\vec{\mathbf{r}}_v = \langle 0, -\sin v, 0\rangle$, and the corresponding cross product is zero. The analog of the condition $\vec{\mathbf{r}}'(t) = \vec{\mathbf{0}}$ is that $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v$ is not zero for point (u,v) in the parameter domain, which is a regular parameterization.

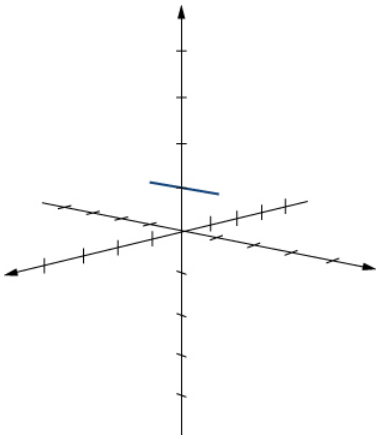


Figure **3.7.7**: The image of parameterization $\vec{\mathbf{r}}(u,v) = \langle 0, \cos v, 1\rangle$, $0 \le u \le 1$, $0 \le v \le \pi$ is a line.

DEFINITION: Regular Parameterization

Parameterization $\vec{\mathbf{r}}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$ is a *regular parameterization* if $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v$ is not zero for point (u,v) in the parameter domain.

If parameterization $\vec{r}$ is regular, then the image of $\vec{r}$ is a two-dimensional object, as a surface should be. Throughout this chapter, parameterizations $\vec{\mathbf{r}}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$are assumed to be regular.

Recall that curve parameterization $\vec{\mathbf{r}}(t)$, $a \le t \le b$ is smooth if $\vec{\mathbf{r}}'(t)$ is continuous and $\vec{\mathbf{r}}'(t) \neq \vec{\mathbf{0}}$ for all t in $[a,b]$. Informally, a curve parameterization is smooth if the resulting curve has no sharp corners. The definition of a smooth surface parameterization is similar. Informally, a surface parameterization is *smooth* if the resulting surface has no sharp corners.

DEFINITION: Smooth Surfaces

A surface parameterization $\vec{\mathbf{r}}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$ is *smooth* if vector $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v$ is not zero for any choice of u and v in the parameter domain.

A surface may also be *piecewise smooth* if it has smooth faces but also has locations where the directional derivatives do not exist.

Example 3.7.4: Identifying Smooth and Nonsmooth Surfaces

Which of the figures in Figure 3.7.8 is smooth?

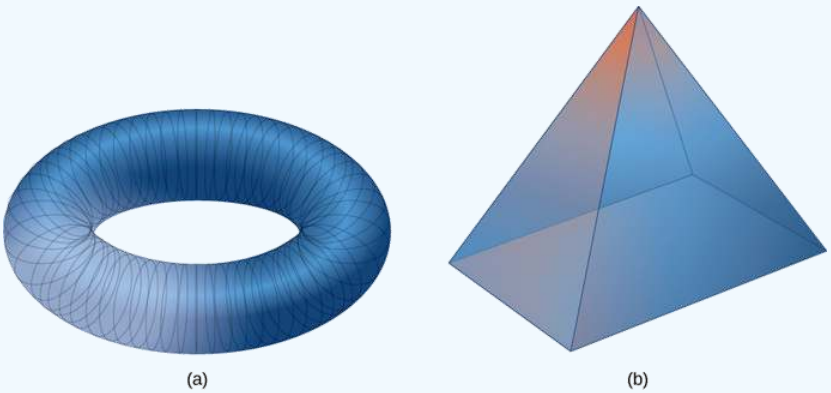


Figure 3.7.8: (a) This surface is smooth. (b) This surface is piecewise smooth.

Solution

The surface in Figure 3.7.8*a* can be parameterized by

$$\vec{\mathbf{r}}(u,v) = \langle (2+\cos v)\cos u,\ (2+\cos v)\sin u,\ \sin v\rangle,\ 0 \le u < 2\pi,\ 0 \le v < 2\pi$$

(we can use technology to verify). Notice that vectors

$$\vec{\mathbf{r}}_u = \langle -(2+\cos v)\sin u,\ (2+\cos v)\cos u, 0\rangle$$

and

$$\vec{\mathbf{r}}_v = \langle -\sin v\cos u,\ -\sin v\sin u,\ \cos v\rangle$$

exist for any choice of u and v in the parameter domain, and

$$\begin{aligned}\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -(2+\cos v)\sin u & (2+\cos v)\cos u & 0 \\ -\sin v\cos u & -\sin v\sin u & \cos v \end{vmatrix} \\ &= [(2+\cos v)\cos u\cos v]\hat{\mathbf{i}} + [2+\cos v)\sin u\cos v]\hat{\mathbf{j}} \\ &\quad + [(2+\cos v)\sin v\sin^2 u + (2+\cos v)\sin v\cos^2 u]\hat{\mathbf{k}} \\ &= [(2+\cos v)\cos u\cos v]\hat{\mathbf{i}} + [(2+\cos v)\sin u\cos v]\hat{\mathbf{j}} + [(2+\cos v)\sin v]\hat{\mathbf{k}}.\end{aligned}$$

The $\hat{\mathbf{k}}$ component of this vector is zero only if $v = 0$ or $v = \pi$. If $v = 0$ or $v = \pi$, then the only choices for u that make the $\hat{\mathbf{j}}$ component zero are $u = 0$ or $u = \pi$. But, these choices of u do not make the $\hat{\mathbf{i}}$ component zero. Therefore, $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v$ is not zero for any choice of u and v in the parameter domain, and the parameterization is smooth. Notice that the corresponding surface has no sharp corners.

In the pyramid in Figure 3.7.8*b*, the sharpness of the corners ensures that directional derivatives do not exist at those locations. Therefore, the pyramid has no smooth parameterization. However, the pyramid consists of four smooth faces, and thus this surface is piecewise smooth.

? Exercise 3.7.4

Is the surface parameterization $\vec{\mathbf{r}}(u,v) = \langle u^{2v}, v+1, \sin u\rangle$, $0 \le u \le 2$, $0 \le v \le 3$ smooth?

Hint

Investigate the cross product $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v$.

Answer

Yes



Surface Area of a Parametric Surface

Our goal is to define a surface integral, and as a first step we have examined how to parameterize a surface. The second step is to define the surface area of a parametric surface. The notation needed to develop this definition is used throughout the rest of this chapter.

Let S be a surface with parameterization $\vec{\mathbf{r}}(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v)\rangle$ over some parameter domain D. We assume here and throughout that the surface parameterization $\vec{\mathbf{r}}(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v)\rangle$ is continuously differentiable—meaning, each component function has continuous partial derivatives. Assume for the sake of simplicity that D is a rectangle (although the following material can be extended to handle nonrectangular parameter domains). Divide rectangle D into subrectangles D_{ij} with horizontal width Δu and vertical length Δv. Suppose that i ranges from 1 to m and j ranges from 1 to n so that D is subdivided into mn rectangles. This division of D into subrectangles gives a corresponding division of surface S into pieces S_{ij}. Choose point P_{ij} in each piece S_{ij}. Point P_{ij} corresponds to point (u_i, v_j) in the parameter domain.

Note that we can form a grid with lines that are parallel to the u-axis and the v-axis in the uv-plane. These grid lines correspond to a set of grid curves on surface S that is parameterized by $\vec{\mathbf{r}}(u,v)$. Without loss of generality, we assume that P_{ij} is located at the corner of two grid curves, as in Figure 3.7.9. If we think of $\vec{\mathbf{r}}$ as a mapping from the uv-plane to $\mathbb{R}^3$, the grid curves are the image of the grid lines under $\vec{\mathbf{r}}$. To be precise, consider the grid lines that go through point (u_i, v_j). One line is given by $x = u_i,\ y = v$; the other is given by $x = u,\ y = v_j$. In the first grid line, the horizontal component is held constant, yielding a vertical line through

(u_i, v_j). In the second grid line, the vertical component is held constant, yielding a horizontal line through (u_i, v_j). The corresponding grid curves are $\vec{\mathbf{r}}(u_i, v)$ and (u, v_j) and these curves intersect at point P_{ij}.

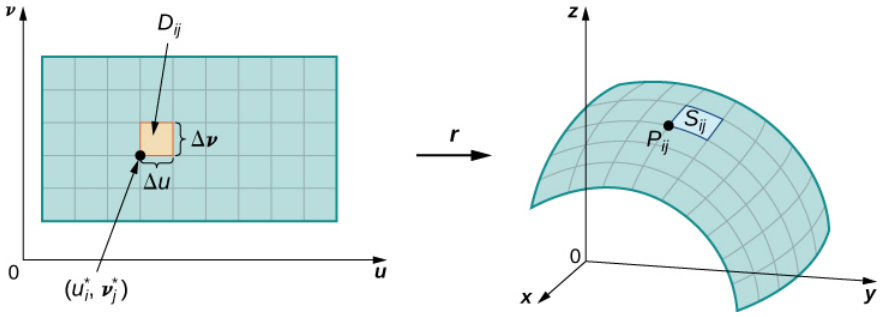


Figure 3.7.9: Grid lines on a parameter domain correspond to grid curves on a surface.

Now consider the vectors that are tangent to these grid curves. For grid curve $\vec{\mathbf{r}}(u_i, v)$, the tangent vector at P_{ij} is

$$\vec{\mathbf{t}}_v(P_{ij}) = \vec{\mathbf{r}}_v(u_i, v_j) = \langle x_v(u_i, v_j),\ y_v(u_i, v_j),\ z_v(u_i, v_j)\rangle. \tag{3.7.5}$$

For grid curve $\vec{\mathbf{r}}(u, v_j)$, the tangent vector at P_{ij} is

$$\vec{\mathbf{t}}_u(P_{ij}) = \vec{\mathbf{r}}_u(u_i, v_j) = \langle x_u(u_i, v_j),\ y_u(u_i, v_j),\ z_u(u_i, v_j)\rangle. \tag{3.7.6}$$

If vector $\vec{\mathbf{N}} = \vec{\mathbf{t}}_u(P_{ij}) \times \vec{\mathbf{t}}_v(P_{ij})$ exists and is not zero, then the tangent plane at P_{ij} exists (Figure 3.7.10). If piece S_{ij} is small enough, then the tangent plane at point P_{ij} is a good approximation of piece S_{ij}.

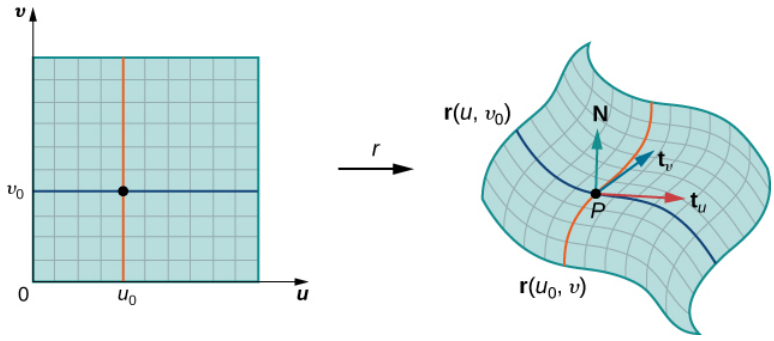


Figure 3.7.10: If the cross product of vectors $\vec{\mathbf{t}}_u$ and $\vec{\mathbf{t}}_v$ exists, then there is a tangent plane.

The tangent plane at P_{ij} contains vectors $\vec{\mathbf{t}}_u(P_{ij})$ and $\vec{\mathbf{t}}_v(P_{ij})$ and therefore the parallelogram spanned by $\vec{\mathbf{t}}_u(P_{ij})$ and $\vec{\mathbf{t}}_v(P_{ij})$ is in the tangent plane. Since the original rectangle in the uv-plane corresponding to S_{ij} has width Δu and length Δv, the parallelogram that we use to approximate S_{ij} is the parallelogram spanned by $\Delta u\vec{\mathbf{t}}_u(P_{ij})$ and $\Delta v\vec{\mathbf{t}}_v(P_{ij})$. In other words, we scale the tangent vectors by the constants Δu and Δv to match the scale of the original division of rectangles in the parameter domain. Therefore, the area of the parallelogram used to approximate the area of S_{ij} is

$$\Delta S_{ij} \approx \| (\Delta u\vec{\mathbf{t}}_u(P_{ij})) \times (\Delta v\vec{\mathbf{t}}_v(P_{ij})) \| = \| \vec{\mathbf{t}}_u(P_{ij}) \times \vec{\mathbf{t}}_v(P_{ij}) \| \Delta u \Delta v. \tag{3.7.7}$$

Varying point P_{ij} over all pieces S_{ij} and the previous approximation leads to the following definition of surface area of a parametric surface (Figure 3.7.11).

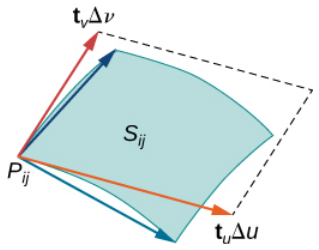


Figure 3.7.11: The parallelogram spanned by $\vec{\mathbf{t}}_u$ *and* $\vec{\mathbf{t}}_v$ *approximates the piece of surface* S_{ij}.

Definition: Smooth Parameterization of a Surface

Let $\vec{\mathbf{r}}(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v)\rangle$ with parameter domain D be a smooth parameterization of surface S. Furthermore, assume that S is traced out only once as (u,v) varies over D. The surface area of S is

$$\iint_D || \vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v || \, dA, \tag{3.7.8}$$

where $\vec{\mathbf{t}}_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle$

and

$$\vec{\mathbf{t}}_v = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle. \tag{3.7.9}$$

Example 3.7.5: Calculating Surface Area

Calculate the lateral surface area (the area of the "side," not including the base) of the right circular cone with height h and radius r.

Solution

Before calculating the surface area of this cone using Equation 3.7.8, we need a parameterization. We assume this cone is in $\mathbb{R}^3$ with its vertex at the origin (Figure 3.7.12). To obtain a parameterization, let α be the angle that is swept out by starting at the positive z-axis and ending at the cone, and let $k = \tan\alpha$. For a height value v with $0 \le v \le h$, the radius of the circle formed by intersecting the cone with plane $z = v$ is kv. Therefore, a parameterization of this cone is

$$\vec{\mathbf{s}}(u,v) = \langle kv\cos u,\ kv\sin u,\ v\rangle,\ 0 \le u < 2\pi,\ 0 \le v \le h.$$

The idea behind this parameterization is that for a fixed v-value, the circle swept out by letting u vary is the circle at height v and radius kv. As v increases, the parameterization sweeps out a "stack" of circles, resulting in the desired cone.

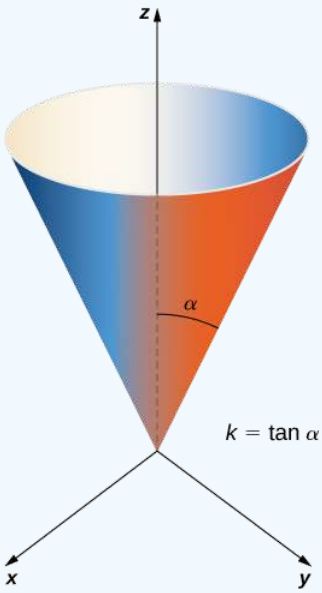


Figure 3.7.12: The right circular cone with radius $r = kh$ and height h has parameterization

$$\vec{\mathbf{s}}(u,v) = \langle kv\cos u,\ kv\sin u,\ v\rangle,\ 0 \le u < 2\pi,\ 0 \le v \le h. \tag{3.7.10}$$

With a parameterization in hand, we can calculate the surface area of the cone using Equation 3.7.8. The tangent vectors are $\vec{\mathbf{t}}_u = \langle -kv\sin u,\ kv\cos u,\ 0\rangle$ and $\vec{\mathbf{t}}_v = \langle k\cos u,\ k\sin u,\ 1\rangle$. Therefore,

$$\begin{aligned}\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -kv\sin u & kv\cos u & 0 \\ k\cos u & k\sin u & 1 \end{vmatrix} \\ &= \langle kv\,\cos u,\, kv\,\sin u,\, -k^2 v\,\sin^2 u - k^2 v\,\cos^2 u\rangle \\ &= \langle kv\,\cos u,\, kv\,\sin u,\, -k^2 v\rangle.\end{aligned}$$

The magnitude of this vector is

$$\begin{aligned}||\langle kv\,\cos u,\, kv\,\sin u,\, -k^2 v\rangle|| &= \sqrt{k^2v^2\cos^2 u + k^2v^2\sin^2 u + k^4v^2} \\ &= \sqrt{k^2v^2 + k^4v^2} \\ &= kv\sqrt{1+k^2}.\end{aligned}$$

By Equation 3.7.8, the surface area of the cone is

$$\begin{aligned}\iint_D ||\,\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v\,||\,dA &= \int_0^h \int_0^{2\pi} kv\sqrt{1+k^2}\,du\,dv \\ &= 2\pi k\sqrt{1+k^2}\int_0^h v\,dv \\ &= 2\pi k\sqrt{1+k^2}\left[\frac{v^2}{2}\right]_0^h \\ &= \pi k h^2\sqrt{1+k^2}.\end{aligned}$$

Since $k = \tan\alpha = r/h$,

$$\begin{aligned}\pi k h^2\sqrt{1+k^2} &= \pi\frac{r}{h}h^2\sqrt{1+\frac{r^2}{h^2}} \\ &= \pi r h\sqrt{1+\frac{r^2}{h^2}} \\ &= \pi r\sqrt{h^2 + h^2\left(\frac{r^2}{h^2}\right)} \\ &= \pi r\sqrt{h^2+r^2}.\end{aligned}$$

Therefore, the lateral surface area of the cone is $\pi r\sqrt{h^2+r^2}$.

Analysis

The surface area of a right circular cone with radius r and height h is usually given as $\pi r^2 + \pi r\sqrt{h^2+r^2}$. The reason for this is that the circular base is included as part of the cone, and therefore the area of the base πr^2 is added to the lateral surface area $\pi r\sqrt{h^2+r^2}$ that we found.



Exercise 3.7.5

Find the surface area of the surface with parameterization $\vec{\mathbf{r}}(u,v) = \langle u+v,\, u^2,\, 2v\rangle,\ 0 \le u \le 3,\ 0 \le v \le 2$.

Hint

Use Equation 3.7.8.

Answer

≈ 43.02

Example 3.7.6: Calculating Surface Area

Show that the surface area of the sphere $x^2+y^2+z^2=r^2$ is $4\pi r^2$.

Solution

The sphere has parameterization

$$r\cos\theta\sin\phi,\ r\sin\theta\sin\phi,\ r\cos\phi\rangle,\ 0 \le \theta < 2\pi,\ 0 \le \phi \le \pi.$$

The tangent vectors are

$$\vec{\mathbf{t}}_\theta = \langle -r\sin\theta\sin\phi,\ r\cos\theta\sin\phi,\ 0\rangle$$

and

$$\vec{\mathbf{t}}_\phi = \langle r\cos\theta\cos\phi,\ r\sin\theta\cos\phi,\ -r\sin\phi\rangle.$$

Therefore,

$$\begin{aligned}\vec{\mathbf{t}}_\phi \times \vec{\mathbf{t}}_\theta &= \langle r^2\cos\theta\sin^2\phi,\ r^2\sin\theta\sin^2\phi,\ r^2\sin^2\theta\sin\phi\cos\phi + r^2\cos^2\theta\sin\phi\cos\phi\rangle \\ &= \langle r^2\cos\theta\sin^2\phi,\ r^2\sin\theta\sin^2\phi,\ r^2\sin\phi\cos\phi\rangle.\end{aligned}$$

Now,

$$\begin{aligned} ||\vec{\mathbf{t}}_\phi \times \vec{\mathbf{t}}_\theta|| &= \sqrt{r^4 \sin^4\phi \cos^2\theta + r^4 \sin^4\phi \sin^2\theta + r^4 \sin^2\phi \cos^2\phi} \\ &= \sqrt{r^4 \sin^4\phi + r^4 \sin^2\phi \cos^2\phi} \\ &= r^2\sqrt{\sin^2\phi} \\ &= r \sin\phi. \end{aligned}$$

Notice that $\sin\phi \geq 0$ on the parameter domain because $0 \leq \phi < \pi$, and this justifies equation $\sqrt{\sin^2\phi} = \sin\phi$. The surface area of the sphere is

$$\int_0^{2\pi} \int_0^{\pi} r^2 \sin\phi \, d\phi \, d\theta = r^2 \int_0^{2\pi} 2 \, d\theta = 4\pi r^2.$$

We have derived the familiar formula for the surface area of a sphere using surface integrals.



? Exercise 3.7.6

Show that the surface area of cylinder $x^2 + y^2 = r^2$, $0 \leq z \leq h$ is $2\pi rh$. Notice that this cylinder does not include the top and bottom circles.

Hint

Use the standard parameterization of a cylinder and follow the previous example.

Answer

With the standard parameterization of a cylinder, Equation 3.7.8 shows that the surface area is $2\pi rh$.

In addition to parameterizing surfaces given by equations or standard geometric shapes such as cones and spheres, we can also parameterize surfaces of revolution. Therefore, we can calculate the surface area of a surface of revolution by using the same techniques. Let $y = f(x) \geq 0$ be a positive single-variable function on the domain $a \leq x \leq b$ and let S be the surface obtained by rotating f about the x-axis (Figure 3.7.13). Let θ be the angle of rotation. Then, S can be parameterized with parameters x and θ by

$$\vec{\mathbf{r}}(x, \theta) = \langle x, f(x) \cos\theta, f(x) \sin\theta \rangle, \, a \leq x \leq b, \, 0 \leq x \leq 2\pi. \tag{3.7.11}$$

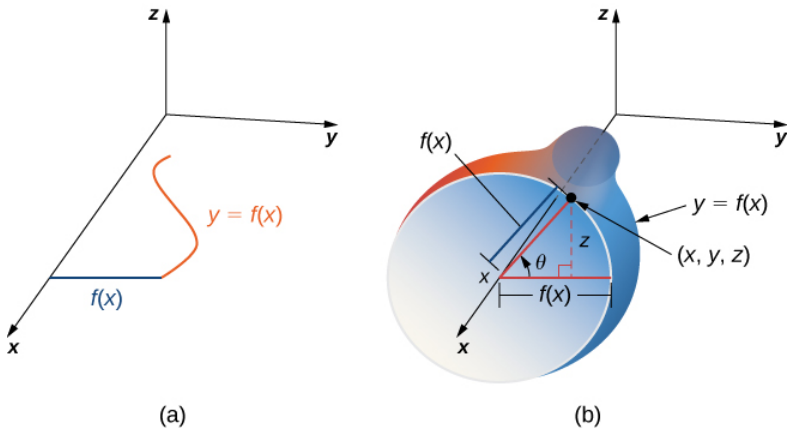


Figure 3.7.13: We can parameterize a surface of revolution by $\vec{\mathbf{r}}(x,\theta) = \langle x, f(x)\cos\theta, f(x)\sin\theta\rangle$, $a \le x \le b$, $0 \le x < 2\pi$.

Example 3.7.7: Calculating Surface Area

Find the area of the surface of revolution obtained by rotating $y = x^2$, $0 \le x \le b$ about the x-axis (Figure 3.7.14).

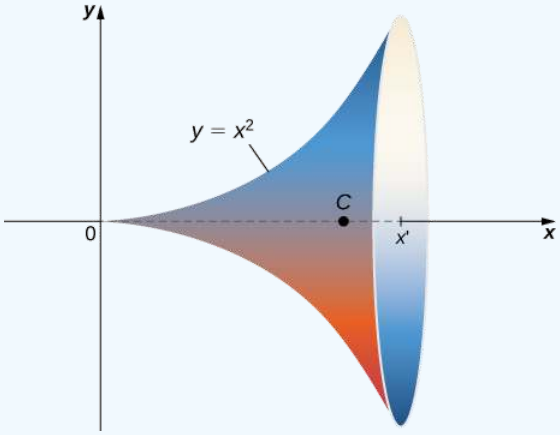


Figure 3.7.14: A surface integral can be used to calculate the surface area of this solid of revolution.

Solution

This surface has parameterization $\vec{\mathbf{r}}(x,\theta) = \langle x, x^2\cos\theta, x^2\sin\theta\rangle$, $0 \le x \le b$, $0 \le x < 2\pi$.

The tangent vectors are $\vec{\mathbf{t}}_x = \langle 1, 2x\cos\theta, 2x\sin\theta\rangle$ and $\vec{\mathbf{t}}_\theta = \langle 0, -x^2\sin\theta, -x^2\cos\theta\rangle$.

Therefore,

$$\begin{aligned}\vec{\mathbf{t}}_x \times \vec{\mathbf{t}}_\theta &= \langle 2x^3\cos^2\theta + 2x^3\sin^2\theta, -x^2\cos\theta, -x^2\sin\theta\rangle \\ &= \langle 2x^3, -x^2\cos\theta, -x^2\sin\theta\rangle\end{aligned}$$

and

$$\begin{aligned}\vec{\mathbf{t}}_x \times \vec{\mathbf{t}}_\theta &= \sqrt{4x^6 + x^4\cos^2\theta + x^4\sin^2\theta} \\ &= \sqrt{4x^6 + x^4} \\ &= x^2\sqrt{4x^2+1}\end{aligned}$$

The area of the surface of revolution is

$$\begin{aligned}\int_0^b \int_0^{2\pi} x^2\sqrt{4x^2+1}\,d\theta\,dx &= 2\pi\int_0^b x^2\sqrt{4x^2+1}\,dx \\ &= 2\pi\left[\frac{1}{64}\left(2\sqrt{4x^2+1}(8x^3+x)\sinh^{-1}(2x)\right)\right]_0^b \\ &= 2\pi\left[\frac{1}{64}\left(2\sqrt{4b^2+1}(8b^3+b)\sinh^{-1}(2b)\right)\right].\end{aligned}$$



? Exercise 3.7.7

Use Equation 3.7.8 to find the area of the surface of revolution obtained by rotating curve $y = \sin x$, $0 \le x \le \pi$ about the x-axis.

Hint

Use the parameterization of surfaces of revolution given before Example 3.7.7.

Answer

$2\pi(\sqrt{2} + \sinh^{-1}(1))$

Surface Integral of a Scalar-Valued Function

Now that we can parameterize surfaces and we can calculate their surface areas, we are able to define surface integrals. First, let's look at the surface integral of a scalar-valued function. Informally, the surface integral of a scalar-valued function is an analog of a scalar line integral in one higher dimension. The domain of integration of a scalar line integral is a parameterized curve (a one-dimensional object); the domain of integration of a scalar surface integral is a parameterized surface (a two-dimensional object). Therefore, the definition of a surface integral follows the definition of a line integral quite closely. For scalar line integrals, we chopped the domain curve into tiny pieces, chose a point in each piece, computed the function at that point, and took a limit of the corresponding Riemann sum. For scalar surface integrals, we chop the domain *region* (no longer a curve) into tiny pieces and proceed in the same fashion.

Let S be a piecewise smooth surface with parameterization $\vec{\mathbf{r}}(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v)\rangle$ with parameter domain D and let $f(x,y,z)$ be a function with a domain that contains S. For now, assume the parameter domain D is a rectangle, but we can extend the basic logic of how we proceed to any parameter domain (the choice of a rectangle is simply to make the notation more manageable). Divide rectangle D into subrectangles D_{ij} with horizontal width Δu and vertical length Δv. Suppose that i ranges from 1 to m and j ranges from 1 to n so that D is subdivided into mn rectangles. This division of D into subrectangles gives a corresponding division of S into pieces S_{ij}. Choose point P_{ij} in each piece S_{ij} evaluate P_{ij} at f, and multiply by area S_{ij} to form the Riemann sum

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij})\,\Delta S_{ij}.$$

To define a surface integral of a scalar-valued function, we let the areas of the pieces of S shrink to zero by taking a limit.

DEFINITION: Surface Integral of a Scalar-Valued Function

The *surface integral of a scalar-valued function* of f over a piecewise smooth surface S is

$$\iint_S f(x,y,z)dA = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij})\Delta S_{ij}. \tag{3.7.12}$$

Again, notice the similarities between this definition and the definition of a scalar line integral. In the definition of a line integral we chop a curve into pieces, evaluate a function at a point in each piece, and let the length of the pieces shrink to zero by taking the limit of the corresponding Riemann sum. In the definition of a surface integral, we chop a surface into pieces, evaluate a function at a point in each piece, and let the area of the pieces shrink to zero by taking the limit of the corresponding Riemann sum. Thus, a surface integral is similar to a line integral but in one higher dimension.

The definition of a scalar line integral can be extended to parameter domains that are not rectangles by using the same logic used earlier. The basic idea is to chop the parameter domain into small pieces, choose a sample point in each piece, and so on. The exact shape of each piece in the sample domain becomes irrelevant as the areas of the pieces shrink to zero.

Scalar surface integrals are difficult to compute from the definition, just as scalar line integrals are. To develop a method that makes surface integrals easier to compute, we approximate surface areas ΔS_{ij} with small pieces of a tangent plane, just as we did in the previous subsection. Recall the definition of vectors $\vec{\mathbf{t}}_u$ and $\vec{\mathbf{t}}_v$:

$$\vec{\mathbf{t}}_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle \text{ and } \vec{\mathbf{t}}_v = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle.$$

From the material we have already studied, we know that

$$\Delta S_{ij} \approx \| \vec{\mathbf{t}}_u(P_{ij}) \times \vec{\mathbf{t}}_v(P_{ij}) \|\, \Delta u\, \Delta v.$$

Therefore,

$$\iint_S f(x,y,z)\,dS \approx \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij})\,\| \vec{\mathbf{t}}_u(P_{ij}) \times \vec{\mathbf{t}}_v(P_{ij}) \|\, \Delta u\, \Delta v.$$

This approximation becomes arbitrarily close to $\lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij})\Delta S_{ij}$ as we increase the number of pieces S_{ij} by letting m and n go to infinity. Therefore, we have the following equation to calculate **scalar surface integrals**:

$$\iint_S f(x,y,z)\,dS = \iint_D f(\vec{\mathbf{r}}(u,v))\,\| \vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v \|\,dA. \tag{3.7.13}$$

Equation 3.7.13 allows us to calculate a surface integral by transforming it into a double integral. This equation for surface integrals is analogous to the equation for line integrals:

$$\iint_C f(x,y,z)\,ds = \int_a^b f(\vec{\mathbf{r}}(t))\,\| \vec{\mathbf{r}}'(t) \|\,dt.$$

In this case, vector $\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v$ is perpendicular to the surface, whereas vector $\vec{\mathbf{r}}'(t)$ is tangent to the curve.

Example 3.7.8: Calculating a Surface Integral

Calculate surface integral

$$\iint_S 5\,dS,$$

where S is the surface with parameterization $\vec{\mathbf{r}}(u,v) = \langle u, u^2, v\rangle$ for $0 \le u \le 2$ and $0 \le v \le u$.

Solution

Notice that this parameter domain$D$ is a triangle, and therefore the parameter domain is not rectangular. This is not an issue though, because Equation 3.7.13 does not place any restrictions on the shape of the parameter domain.

To use Equation 3.7.13 to calculate the surface integral, we first find vectors $\vec{\mathbf{t}}_u$ and $\vec{\mathbf{t}}_v$. Note that $\vec{\mathbf{t}}_u = \langle 1, 2u, 0\rangle$ and $\vec{\mathbf{t}}_v = \langle 0, 0, 1\rangle$. Therefore,

$$\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 2u & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle 2u, -1, 0\rangle$$

and

$$||\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v|| = \sqrt{1+4u^2}. \tag{3.7.14}$$

By Equation 3.7.13,

$$\begin{aligned} \iint_S 5\, dS &= 5 \iint_D u\sqrt{1+4u^2}\, dA \\ &= 5\int_0^2 \int_0^u \sqrt{1+4u^2}\, dv\, du = 5\int_0^2 u\sqrt{1+4u^2}\, du \\ &= 5\left[\frac{(1+4u^2)^{3/2}}{3}\right]_0^2 \\ &= \frac{5(17^{3/2}-1)}{3} \approx 115.15. \end{aligned}$$



Example 3.7.9: Calculating the Surface Integral of a Cylinder

Calculate surface integral

$$\iint_S (x+y^2)\, dS, \tag{3.7.15}$$

where S is cylinder $x^2+y^2=4$, $0 \le z \le 3$ (Figure 3.7.15).

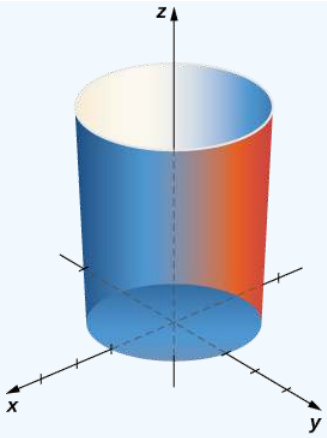


Figure 3.7.15: Integrating function $f(x,y,z) = x + y^2$ over a cylinder.

Solution

To calculate the surface integral, we first need a parameterization of the cylinder. A parameterization is $\vec{\mathbf{r}}(u,v) = \langle \cos u, \sin u, v \rangle, 0 \le u \le 2\pi, 0 \le v \le 3$.

The tangent vectors are $\vec{\mathbf{t}}_u = \langle \sin u, \cos u, 0 \rangle$ and $\vec{\mathbf{t}}_v = \langle 0, 0, 1 \rangle$. Then,

$$\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -\sin u & \cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle \cos u, \sin u, 0 \rangle \tag{3.7.16}$$

and $||\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v|| = \sqrt{\cos^2 u + \sin^2 u} = 1$. By Equation 3.7.13,

$$\begin{aligned} \iint_S f(x,y,z)dS &= \iint_D f(\vec{\mathbf{r}}(u,v)) \, ||\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v|| \, dA \\ &= \int_0^3 \int_0^{2\pi} (\cos u + \sin^2 u) \, du \, dv \\ &= \int_0^3 \left[\sin u + \frac{u}{2} - \frac{\sin(2u)}{4} \right]_0^{2\pi} dv \\ &= \int_0^3 \pi \, dv = 3\pi. \end{aligned}$$

? Exercise 3.7.8

Calculate

$$\iint_S (x^2 - z) \, dS, \tag{3.7.17}$$

where S is the surface with parameterization $\vec{\mathbf{r}}(u,v) = \langle v, u^2 + v^2, 1 \rangle$, $0 \le u \le 2$, $0 \le v \le 3$.

Hint

Use Equation 3.7.13.

Answer

24

✓ Example 3.7.10: Calculating the Surface Integral of a Piece of a Sphere

Calculate surface integral

$$\iint_S f(x,y,z) \, dS, \tag{3.7.18}$$

where $f(x,y,z) = z^2$ and S is the surface that consists of the piece of sphere $x^2 + y^2 + z^2 = 4$ that lies on or above plane $z = 1$ and the disk that is enclosed by intersection plane $z = 1$ and the given sphere (Figure 3.7.16).

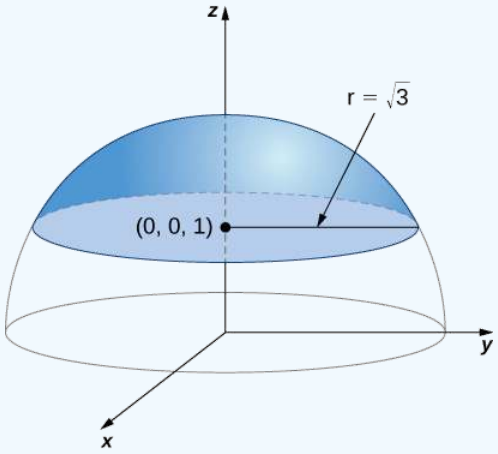


Figure 3.7.16: Calculating a surface integral over surface S.

Solution

Notice that S is not smooth but is piecewise smooth; S can be written as the union of its base S_1 and its spherical top S_2, and both S_1 and S_2 are smooth. Therefore, to calculate

$$\iint_S z^2 dS,$$

we write this integral as

$$\iint_{S_1} z^2\, dS + \iint_{S_2} z^2\, dS$$

and we calculate integrals

$$\iint_{S_1} z^2\, dS$$

and

$$\iint_{S_2} Z^2\, dS.$$

First, we calculate $\iint_{S_1} z^2\, dS$. To calculate this integral we need a parameterization of S_1. This surface is a disk in plane $z = 1$ centered at $(0,0,1)$. To parameterize this disk, we need to know its radius. Since the disk is formed where plane $z = 1$ intersects sphere $x^2 + y^2 + z^2 = 4$, we can substitute $z = 1$ into equation $x^2 + y^2 + z^2 = 4$:

$$x^2 + y^2 + 1 = 4 \Rightarrow x^2 + y^2 = 3.$$

Therefore, the radius of the disk is $\sqrt{3}$ and a parameterization of S_1 is $\vec{\mathbf{r}}(u,v) = \langle u\, \cos v,\, u\, \sin v,\, 1\rangle$, $0 \le u \le \sqrt{3}$, $0 \le v \le 2\pi$. The tangent vectors are $\vec{\mathbf{t}}_u = \langle \cos v,\, \sin v,\, 0\rangle$ and $\vec{\mathbf{t}}_v = \langle -u\, \sin v,\, u\, \cos v,\, 0\rangle$, and thus

$$\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 0 \end{vmatrix} = \langle 0,\, 0,\, u\cos^2 v + u\sin^2 v\rangle = \langle 0,0,u\rangle. \tag{3.7.19}$$

The magnitude of this vector is u. Therefore,

$$\begin{aligned}\iint_{S_1} z^2\,dS &= \int_0^{\sqrt{3}}\int_0^{2\pi} f(r(u,v))\,||\,t_u \times t_v\,||\,dv\,du \\ &= \int_0^{\sqrt{3}}\int_0^{2\pi} u\,dv\,du \\ &= 2\pi\int_0^{\sqrt{3}} u\,du \\ &= 2\pi\sqrt{3}.\end{aligned}$$

Now we calculate

$$\iint_{S_2} dS.$$

To calculate this integral, we need a parameterization of S_2. The parameterization of full sphere $x^2+y^2+z^2=4$ is

$$\vec{\mathbf{r}}(\phi,\theta) = \langle 2\cos\theta\sin\phi,\ 2\sin\theta\sin\phi,\ 2\cos\phi\rangle,\ 0\le\theta\le 2\pi, 0\le\phi\le\pi. \tag{3.7.20}$$

Since we are only taking the piece of the sphere on or above plane $z=1$, we have to restrict the domain of ϕ. To see how far this angle sweeps, notice that the angle can be located in a right triangle, as shown in Figure 3.7.17 (the $\sqrt{3}$ comes from the fact that the base of S is a disk with radius $\sqrt{3}$). Therefore, the tangent of ϕ is $\sqrt{3}$, which implies that ϕ is $\pi/6$. We now have a parameterization of S_2:

$\vec{\mathbf{r}}(\phi,\theta) = \langle 2\cos\theta\sin\phi,\ 2\sin\theta\sin\phi,\ 2\cos\phi\rangle,\ 0\le\theta\le 2\pi,\ 0\le\phi\le\pi/3.$

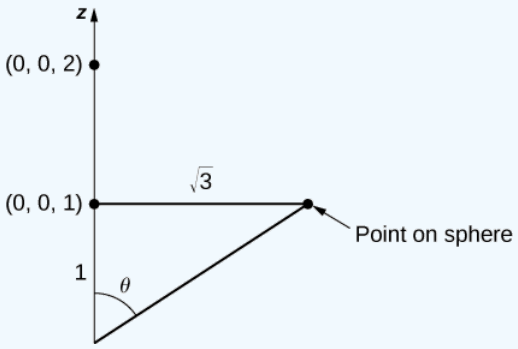


Figure 3.7.17: The maximum value of ϕ has a tangent value of $\sqrt{3}$.

The tangent vectors are $\vec{\mathbf{t}}_\phi = \langle 2\cos\theta\cos\phi,\ 2\sin\theta\cos\phi,\ -2\sin\phi\rangle$ and $\vec{\mathbf{t}}_\theta = \langle -2\sin\theta\sin\phi,\ u\cos\theta\sin\phi,\ 0\rangle$, and thus

$$\begin{aligned}\vec{\mathbf{t}}_\phi \times \vec{\mathbf{t}}_\theta &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2\cos\theta\cos\phi & 2\sin\theta\cos\phi & -2\sin\phi \\ -2\sin\theta\sin\phi & 2\cos\theta\sin\phi & 0 \end{vmatrix} \\ &= \langle 4\cos\theta\sin^2\phi,\ 4\sin\theta\sin^2\phi,\ 4\cos^2\theta\cos\phi\sin\phi + 4\sin^2\theta\cos\phi\sin\phi\rangle \\ &= \langle 4\cos\theta\sin^2\phi,\ 4\sin\theta\sin^2\phi,\ 4\cos\phi\sin\phi\rangle.\end{aligned}$$

The magnitude of this vector is

$$\begin{aligned}\vec{\mathbf{t}}_\phi \times \vec{\mathbf{t}}_\theta &= \sqrt{16\cos^2\theta\sin^4\phi + 16\sin^2\theta\sin^4\phi + 16\cos^2\phi\sin^2\phi} \\ &= 4\sqrt{\sin^4\phi + \cos^2\phi\sin^2\phi}.\end{aligned}$$

Therefore,

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$$\begin{aligned}\iint_{S_2} z\,dS &= \int_0^{\pi/6}\int_0^{2\pi} f(\vec{\mathbf{r}}(\phi,\theta))\,||\,\vec{\mathbf{t}}_\phi \times \vec{\mathbf{t}}_\theta\,||\,d\theta\,d\phi \\ &= \int_0^{\pi/6}\int_0^{2\pi} 16\cos^2\phi\sqrt{\sin^4\phi+\cos^2\phi\sin^2\phi}\,d\theta\,d\phi \\ &= 32\pi\int_0^{\pi/6}\cos^2\phi\sqrt{\sin^4\phi+\cos^2\phi\sin^2\phi}\,d\phi \\ &= 32\pi\int_0^{\pi/6}\cos^2\phi\sin\phi\sqrt{\sin^2\phi+\cos^2\phi}\,d\phi \\ &= 32\pi\int_0^{\pi/6}\cos^2\phi\sin\phi\,d\phi \\ &= 32\pi\left[-\frac{\cos^3\phi}{3}\right]_0^{\pi/6} \\ &= 32\pi\left[\frac{1}{3}-\frac{\sqrt{3}}{8}\right] = \frac{32\pi}{3}-4\sqrt{3}.\end{aligned}$$

Since

$$\iint_S z^2\,dS = \iint_{S_1} z^2\,dS + \iint_{S_2} z^2\,dS,$$

we have

$$\iint_S z^2\,dS = (2\pi-4)\sqrt{3}+\frac{32\pi}{3}.$$

Analysis

In this example we broke a surface integral over a piecewise surface into the addition of surface integrals over smooth subsurfaces. There were only two smooth subsurfaces in this example, but this technique extends to finitely many smooth subsurfaces.

? Exercise 3.7.9

Calculate line integral $\iint_S (x-y)\,dS$, where S is cylinder $x^2+y^2=1$, $0\le z\le 2$, including the circular top and bottom.

Hint

Break the integral into three separate surface integrals.

Answer

0

Scalar surface integrals have several real-world applications. Recall that scalar line integrals can be used to compute the mass of a wire given its density function. In a similar fashion, we can use scalar surface integrals to compute the mass of a sheet given its density function. If a thin sheet of metal has the shape of surface S and the density of the sheet at point (x,y,z) is $\rho(x,y,z)$ then mass m of the sheet is

$$m = \iint_S \rho(x,y,z)\,dS. \tag{3.7.21}$$

✓ Example 3.7.11: Calculating the Mass of a Sheet

A flat sheet of metal has the shape of surface $z=1+x+2y$ that lies above rectangle $0\le x\le 4$ and $0\le y\le 2$. If the density of the sheet is given by $\rho(x,y,z)=x^2yz$, what is the mass of the sheet?

Solution

Let S be the surface that describes the sheet. Then, the mass of the sheet is given by $m = \iint_S x^2yx\,dS$. To compute this surface integral, we first need a parameterization of S. Since S is given by the function $f(x,y) = 1 + x + 2y$, a parameterization of S is $\vec{\mathbf{r}}(x,y) = \langle x, y, 1 + x + 2y\rangle$, $0 \le x \le 4$, $0 \le y \le 2$.

The tangent vectors are $\vec{\mathbf{t}}_x = \langle 1, 0, 1\rangle$ and $\vec{\mathbf{t}}_y = \langle 1, 0, 2\rangle$. Therefore, $\vec{\mathbf{t}}_x + \vec{\mathbf{t}}_y = \langle -1, -2, 1\rangle$ and $||\vec{\mathbf{t}}_x \times \vec{\mathbf{t}}_y|| = \sqrt{6}$.

By the definition of the line integral (Section 16.2),

$$\begin{aligned} m &= \iint_S x^2yz\,dS \\ &= \sqrt{6}\int_0^4 \int_0^2 x^2y(1 + x + 2y)\,dy\,dx \\ &= \sqrt{6}\int_0^4 \frac{22x^2}{3} + 2x^3\,dx \\ &= \frac{2560\sqrt{6}}{9} \approx 696.74. \end{aligned}$$



? Exercise 3.7.10

A piece of metal has a shape that is modeled by paraboloid $z = x^2 + y^2$, $0 \le z \le 4$, and the density of the metal is given by $\rho(x,y,z) = z + 1$. Find the mass of the piece of metal.

Hint

The mass of a sheet is given by Equation 3.7.21. A useful parameterization of a paraboloid was given in a previous example.

Answer

$38.401\pi \approx 120.640$

Orientation of a Surface

Recall that when we defined a scalar line integral, we did not need to worry about an orientation of the curve of integration. The same was true for scalar surface integrals: we did not need to worry about an "orientation" of the surface of integration.

On the other hand, when we defined vector line integrals, the curve of integration needed an orientation. That is, we needed the notion of an oriented curve to define a vector line integral without ambiguity. Similarly, when we define a surface integral of a

vector field, we need the notion of an oriented surface. An oriented surface is given an “upward” or “downward” orientation or, in the case of surfaces such as a sphere or cylinder, an “outward” or “inward” orientation.

Let S be a smooth surface. For any point (x, y, z) on S, we can identify two unit normal vectors $\vec{\mathbf{N}}$ and $-\vec{\mathbf{N}}$. If it is possible to choose a unit normal vector $\vec{\mathbf{N}}$ at every point (x, y, z) on S so that $\vec{\mathbf{N}}$ varies continuously over S, then S is “*orientable*.” Such a choice of unit normal vector at each point gives the **orientation of a surface** S. If you think of the normal field as describing water flow, then the side of the surface that water flows toward is the “negative” side and the side of the surface at which the water flows away is the “positive” side. Informally, a choice of orientation gives S an “outer” side and an “inner” side (or an “upward” side and a “downward” side), just as a choice of orientation of a curve gives the curve “forward” and “backward” directions.

Closed surfaces such as spheres are orientable: if we choose the outward normal vector at each point on the surface of the sphere, then the unit normal vectors vary continuously. This is called the *positive orientation of the closed surface* (Figure 3.7.18). We also could choose the inward normal vector at each point to give an “inward” orientation, which is the negative orientation of the surface.

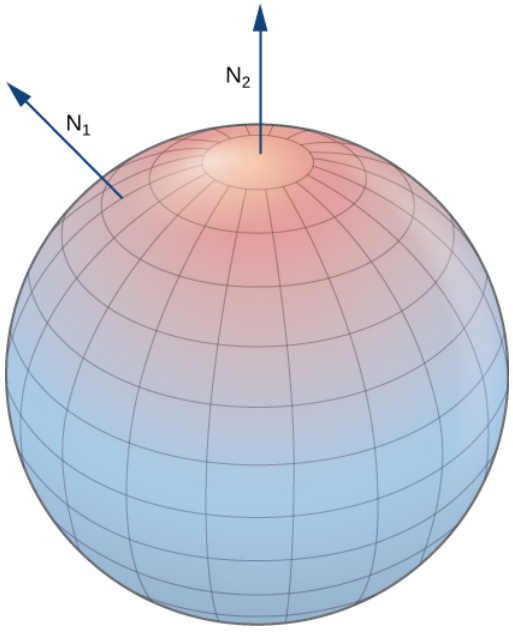


Figure 3.7.18: An oriented sphere with positive orientation.

A portion of the graph of any smooth function $z = f(x, y)$ is also orientable. If we choose the unit normal vector that points “above” the surface at each point, then the unit normal vectors vary continuously over the surface. We could also choose the unit normal vector that points “below” the surface at each point. To get such an orientation, we parameterize the graph of f in the standard way: $\vec{\mathbf{r}}(x, y) = \langle x,\ y,\ f(x, y)\rangle$, where x and y vary over the domain of f. Then, $\vec{\mathbf{t}}_x = \langle 1, 0, f_x\rangle$ and $\vec{\mathbf{t}}_y = \langle 0, 1, f_y\rangle$, and therefore the cross product $\vec{\mathbf{t}}_x \times \vec{\mathbf{t}}_y$ (which is normal to the surface at any point on the surface) is $\langle -f_x,\ -f_y,\ 1\rangle$Since the z-component of this vector is one, the corresponding unit normal vector points “upward,” and the upward side of the surface is chosen to be the “positive” side.

Let S be a smooth orientable surface with parameterization $\vec{\mathbf{r}}(u, v)$. For each point $\vec{\mathbf{r}}(a, b)$ on the surface, vectors $\vec{\mathbf{t}}_u$ and $\vec{\mathbf{t}}_v$ lie in the tangent plane at that point. Vector $\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v$ is normal to the tangent plane at $\vec{\mathbf{r}}(a, b)$ and is therefore normal to S at that point. Therefore, the choice of unit normal vector

$$\vec{\mathbf{N}} = \frac{\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v}{\|\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v\|} \tag{3.7.22}$$

gives an orientation of surface S.

Example 3.7.12:Choosing an Orientation

Give an orientation of cylinder $x^2 + y^2 = r^2$, $0 \le z \le h$.

Solution

This surface has parameterization $\vec{\mathbf{r}}(u, v) = \langle r\cos u,\ r\sin u,\ v\rangle$, $0 \le u < 2\pi$, $0 \le v \le h$.

The tangent vectors are $\vec{\mathbf{t}}_u = \langle -r\sin u,\ r\cos u,\ 0\rangle$ and $\vec{\mathbf{t}}_v = \langle 0, 0, 1\rangle$. To get an orientation of the surface, we compute the unit normal vector

$$\vec{\mathbf{N}} = \frac{\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v}{||\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v||}$$

In this case, $\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v = \langle r\cos u,\ r\sin u,\ 0\rangle$ and therefore

$$||\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v|| = \sqrt{r^2\cos^2 u + r^2\sin^2 u} = r.$$

An orientation of the cylinder is

$$\vec{\mathbf{N}}(u, v) = \frac{\langle r\cos u,\ r\sin u,\ 0\rangle}{r} = \langle \cos u,\ \sin u,\ 0\rangle.$$

Notice that all vectors are parallel to the xy-plane, which should be the case with vectors that are normal to the cylinder. Furthermore, all the vectors point outward, and therefore this is an outward orientation of the cylinder (Figure 3.7.19).



Figure 3.7.19: If all the vectors normal to a cylinder point outward, then this is an outward orientation of the cylinder.

Exercise 3.7.11

Give the "upward" orientation of the graph of $f(x, y) = xy$.

Hint

Parameterize the surface and use the fact that the surface is the graph of a function.

Answer

$$\vec{\mathbf{N}}(x, y) = \left\langle \frac{-y}{\sqrt{1 + x^2 + y^2}},\ \frac{-x}{\sqrt{1 + x^2 + y^2}},\ \frac{1}{\sqrt{1 + x^2 + y^2}} \right\rangle$$



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Since every curve has a “[redacted]wise and counterclockwise direction), it is possible to [redacted]s an oriented curve. This is not the case with surface[redacted]*norientable*. Essentially, a surface can be oriented if [redacted]a “downward” side. Some surfaces are twisted in such a fashion that there is no well-defined notion of an “inner” or “outer” side.

The classic example of a nonorientable surface is the Möbius strip. To create a Möbius strip, take a rectangular strip of paper, give the piece of paper a half-twist, and the glue the ends together (Figure **3.7.20**). Because of the half-twist in the strip, the surface has no “outer” side or “inner” side. If you imagine placing a normal vector at a point on the strip and having the vector travel all the way around the band, then (because of the half-twist) the vector points in the opposite direction when it gets back to its original position. Therefore, the strip really only has one side.

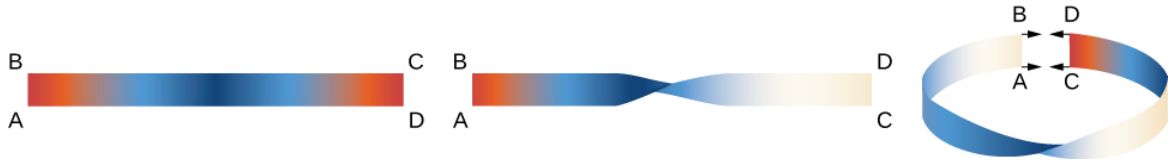


Figure **3.7.20**: The construction of a Möbius strip.

Since some surfaces are *nonorientable*, it is not possible to define a vector surface integral on all piecewise smooth surfaces. This is in contrast to vector line integrals, which can be defined on any piecewise smooth curve.

Surface Integral of a Vector Field

With the idea of orientable surfaces in place, we are now ready to define a surface integral of a vector field. The definition is analogous to the definition of the flux of a vector field along a plane curve. Recall that if $\vec{\mathbf{F}}$ is a two-dimensional vector field and C is a plane curve, then the definition of the flux of $\vec{\mathbf{F}}$ along C involved chopping C into small pieces, choosing a point inside each piece, and calculating $\vec{\mathbf{F}} \cdot \vec{\mathbf{N}}$ at the point (where $\vec{\mathbf{N}}$ is the unit normal vector at the point). The definition of a surface integral of a vector field proceeds in the same fashion, except now we chop surface S into small pieces, choose a point in the small (two-dimensional) piece, and calculate $\vec{\mathbf{F}} \cdot \vec{\mathbf{N}}$ at the point.

To place this definition in a real-world setting, let S be an oriented surface with unit normal vector $\vec{\mathbf{N}}$. Let $\vec{\mathbf{v}}$ be a velocity field of a fluid flowing through S, and suppose the fluid has density $\rho(x, y, z)$ Imagine the fluid flows through S, but S is completely permeable so that it does not impede the fluid flow (Figure **3.7.21**). The mass flux of the fluid is the rate of mass flow per unit area. The mass flux is measured in mass per unit time per unit area. How could we calculate the mass flux of the fluid across S?

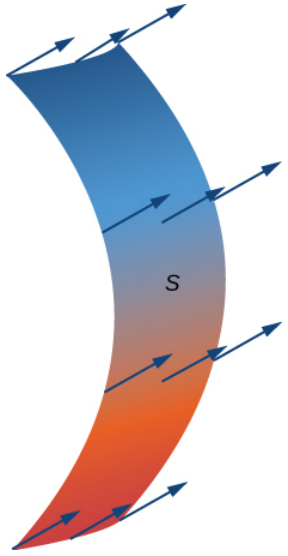


Figure **3.7.21**: Fluid flows across a completely permeable surface S.

The rate of flow, measured in mass per unit time per unit area, is $\rho\vec{\mathbf{N}}$. To calculate the mass flux across S, chop S into small pieces S_{ij}. If S_{ij} is small enough, then it can be approximated by a tangent plane at some point P in S_{ij}. Therefore, the unit normal vector

at P can be used to approximate $\vec{\mathbf{N}}(x,y,z)$ across the entire piece S_{ij} because the normal vector to a plane does not change as we move across the plane. The component of the vector ρv at P in the direction of $\vec{\mathbf{N}}$ is $\rho\vec{\mathbf{v}}\cdot\vec{\mathbf{N}}$ at P. Since S_{ij} is small, the dot product $\rho v \cdot N$ changes very little as we vary across S_{ij} and therefore $\rho\vec{\mathbf{v}}\cdot\vec{\mathbf{N}}$ can be taken as approximately constant across S_{ij}. To approximate the mass of fluid per unit time flowing across S_{ij} (and not just locally at point P), we need to multiply $(\rho\vec{\mathbf{v}}\cdot\vec{\mathbf{N}})(P)$ by the area of S_{ij}. Therefore, the mass of fluid per unit time flowing across S_{ij} in the direction of $\vec{\mathbf{N}}$ can be approximated by $(\rho\vec{\mathbf{v}}\cdot\vec{\mathbf{N}})\Delta S_{ij}$ where $\vec{\mathbf{N}}$, ρ and $\vec{\mathbf{v}}$ are all evaluated at P (Figure 3.7.22). This is analogous to the flux of two-dimensional vector field $\vec{\mathbf{F}}$ across plane curve C, in which we approximated flux across a small piece of C with the expression $(\vec{\mathbf{F}}\cdot\vec{\mathbf{N}})\,\Delta s$. To approximate the mass flux across S, form the sum

$$\sum_{i=1}^{m}\sum_{j=1}^{n}(\rho\vec{\mathbf{v}}\cdot\vec{\mathbf{N}})\Delta S_{ij}.$$

As pieces S_{ij} get smaller, the sum

$$\sum_{i=1}^{m}\sum_{j=1}^{n}(\rho\vec{\mathbf{v}}\cdot\vec{\mathbf{N}})\Delta S_{ij}$$

gets arbitrarily close to the mass flux. Therefore, the mass flux is

$$\iint_S \rho\vec{\mathbf{v}}\cdot\vec{\mathbf{N}}\,dS=\lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n}(\rho\vec{\mathbf{v}}\cdot\vec{\mathbf{N}})\Delta S_{ij}. \tag{3.7.23}$$

This is a surface integral of a vector field. Letting the vector field $\rho\vec{\mathbf{v}}$ be an arbitrary vector field $\vec{\mathbf{F}}$ leads to the following definition.

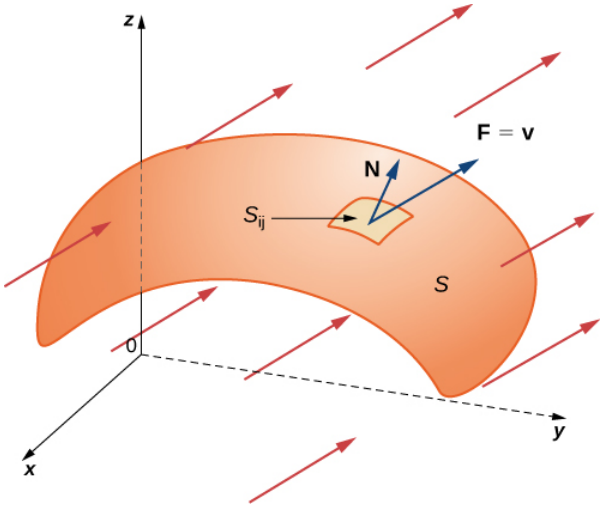


Figure 3.7.22: The mass of fluid per unit time flowing across S_{ij} in the direction of $\vec{\mathbf{N}}$ can be approximated by $(\rho\vec{\mathbf{v}}\cdot\vec{\mathbf{N}})\Delta S_{ij}$.

Definition: Surface Integrals

Let $\vec{\mathbf{F}}$ be a continuous vector field with a domain that contains oriented surface S with unit normal vector $\vec{\mathbf{N}}$. The surface integral of $\vec{\mathbf{F}}$ over S is

$$\iint_S \vec{\mathbf{F}}\cdot\vec{\mathbf{S}}=\iint_S \vec{\mathbf{F}}\cdot\vec{\mathbf{N}}\,dS. \tag{3.7.24}$$

Notice the parallel between this definition and the definition of vector line integral $\int_C \vec{\mathbf{F}}\cdot\vec{\mathbf{N}}\,dS$. A surface integral of a vector field is defined in a similar way to a flux line integral across a curve, except the domain of integration is a surface (a two-dimensional object) rather than a curve (a one-dimensional object). Integral $\iint_S \vec{\mathbf{F}}\cdot\vec{\mathbf{N}}\,dS$ is called the *flux of* $\vec{\mathbf{F}}$ *across* S, just as integral $\int_C \vec{\mathbf{F}}\cdot\vec{\mathbf{N}}\,dS$ is the flux of $\vec{\mathbf{F}}$ across curve C. A surface integral over a vector field is also called a **flux integral**.

Just as with vector line integrals, surface integral $\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, dS$ is easier to compute after surface S has been parameterized. Let $\vec{\mathbf{r}}(u,v)$ be a parameterization of S with parameter domain D. Then, the unit normal vector is given by $\vec{\mathbf{N}} = \dfrac{\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v}{||\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v||}$ and, from Equation 3.7.24, we have

$$\begin{aligned}\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, dS &= \iint_S \vec{\mathbf{F}} \cdot \frac{\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v}{||\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v||}\, dS \\ &= \iint_D \left(\vec{\mathbf{F}}(\vec{\mathbf{r}}(u,v)) \cdot \frac{\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v}{||\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v||} \right) ||\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v||\, dA \\ &= \iint_D (\vec{\mathbf{F}}(\vec{\mathbf{r}}(u,v)) \cdot (\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v))\, dA.\end{aligned}$$

Therefore, to compute a surface integral over a vector field we can use the equation

$$\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, dS = \iint_D (\vec{\mathbf{F}}(\vec{\mathbf{r}}(u,v)) \cdot (\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v))\, dA. \tag{3.7.25}$$

Example 3.7.13: Calculating a Surface Integral

Calculate the surface integral

$$\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, dS,$$

where $\vec{\mathbf{F}} = \langle -y, x, 0 \rangle$ and S is the surface with parameterization

$$\vec{\mathbf{r}}(u,v) = \langle u, v^2 - u, u + v \rangle,\ 0 \le u \le 3,\ 0 \le v \le 4.$$

Solution

The tangent vectors are $\vec{\mathbf{t}}_u = \langle 1, -1, 1 \rangle$ and $\vec{\mathbf{t}}_v = \langle 0, 2v, 1 \rangle$. Therefore,

$$\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v = \langle -1 - 2v, -1, 2v \rangle.$$

By Equation 3.7.25,

$$\begin{aligned}\iint_S \vec{\mathbf{F}} \cdot dS &= \int_0^4 \int_0^3 F(\vec{\mathbf{r}}(u,v)) \cdot (\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v)\, du\, dv \\ &= \int_0^4 \int_0^3 \langle u - v^2,\ u,\ 0 \rangle \cdot \langle -1 - 2v,\ -1,\ 2v \rangle\, du\, dv \\ &= \int_0^4 \int_0^3 [(u - v^2)(-1 - 2v) - u]\, du\, dv \\ &= \int_0^4 \int_0^3 (2v^3 + v^2 - 2uv - 2u)\, du\, dv \\ &= \int_0^4 \left[2v^3 u + v^2 u - v u^2 - u^2 \right]_0^3 dv \\ &= \int_0^4 (6v^3 + 3v^2 - 9v - 9)\, dv \\ &= \left[\frac{3v^4}{2} + v^3 - \frac{9v^2}{2} - 9v \right]_0^4 \\ &= 340.\end{aligned}$$

Therefore, the flux of $\vec{\mathbf{F}}$ across S is 340.



? Exercise 3.7.12

Calculate surface integral

$$\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, dS,$$

where $\vec{\mathbf{F}} = \langle 0, -z, y \rangle$ and S is the portion of the unit sphere in the first octant with outward orientation.

Hint

Use Equation 3.7.25.

Answer

0

✓ Example 3.7.14:Calculating Mass Flow Rate

Let $\vec{\mathbf{v}}(x, y, z) = \langle 2x,\ 2y,\ z \rangle$ represent a velocity field (with units of meters per second) of a fluid with constant density 80 kg/m^3. Let S be hemisphere $x^2 + y^2 + z^2 = 9$ with $z \leq 0$ such that S is oriented outward. Find the mass flow rate of the fluid across S.

Solution

A parameterization of the surface is

$$\vec{\mathbf{r}}(\phi, \theta) = \langle 3\cos\theta\sin\phi,\ 3\sin\theta\sin\phi,\ 3\cos\phi \rangle,\ 0 \leq \theta \leq 2\pi,\ 0 \leq \phi \leq \pi/2. \tag{3.7.26}$$

As in Example, the tangent vectors are $\vec{\mathbf{t}}_\theta = \langle -3\sin\theta\sin\phi,\ 3\cos\theta\sin\phi,\ 0 \rangle$ and $\vec{\mathbf{t}}_\phi = \langle 3\cos\theta\cos\phi,\ 3\sin\theta\cos\phi,\ -3\sin\phi \rangle$, and their cross product is

$$\vec{\mathbf{t}}_\phi \times \vec{\mathbf{t}}_\theta = \langle 9\cos\theta\sin^2\phi,\ 9\sin\theta\sin^2\phi,\ 9\sin\phi\cos\phi \rangle. \tag{3.7.27}$$

Notice that each component of the cross product is positive, and therefore this vector gives the outward orientation. Therefore we use the orientation

$\vec{\mathbf{N}} = \langle 9\cos\theta\sin^2\phi,\ 9\sin\theta\sin^2\phi,\ 9\sin\phi\cos\phi \rangle$

for the sphere.

By \label{surfaceI},

$$\begin{aligned}\iint_S \rho v \cdot dS &= 80 \int_0^{2\pi} \int_0^{\pi/2} v(r(\phi,\theta)) \cdot (t_\phi \times t_\theta)\, d\phi\, d\theta \\ &= 80 \int_0^{2\pi} \int_0^{\pi/2} \langle 6\cos\theta\sin\phi,\, 6\sin\theta\sin\phi,\, 3\cos\phi\rangle \\ &\quad \cdot \langle 9\cos\theta\sin^2\phi,\, 9\sin\theta\sin^2\phi,\, 9\sin\phi\cos\phi\rangle\, d\phi\, d\theta \\ &= 80 \int_0^{2\pi} \int_0^{\pi/2} 54\sin^3\phi + 27\cos^2\phi\sin\phi\, d\phi\, d\theta \\ &= 80 \int_0^{2\pi} \int_0^{\pi/2} 54(1-\cos^2\phi)\sin\phi + 27\cos^2\phi\sin\phi\, d\phi\, d\theta \\ &= 80 \int_0^{2\pi} \int_0^{\pi/2} 54\sin\phi - 27\cos^2\phi\sin\phi\, d\phi\, d\theta \\ &= 80 \int_0^{2\pi} \Big[-54\cos\phi + 9\cos^3\phi\Big]_{\phi=0}^{\phi=2\pi} d\theta \\ &= 80 \int_0^{2\pi} 45\, d\theta \\ &= 7200\pi.\end{aligned}$$

Therefore, the mass flow rate is $7200\pi\,\text{kg/sec/m}^2$.

? Exercise 3.7.13

Let $\vec{\mathbf{v}}(x,y,z) = \langle x^2+y^2,\, z,\, 4y\rangle$ m/sec represent a velocity field of a fluid with constant density 100 kg/m^3. Let S be the half-cylinder $\vec{\mathbf{r}}(u,v) = \langle \cos u,\, \sin u,\, v\rangle$, $0 \le u \le \pi$, $0 \le v \le 2$ oriented outward. Calculate the mass flux of the fluid across S.

Hint

Use \label{surfaceI}.

Answer

400 kg/sec/m

In Example **3.7.14**, we computed the mass flux, which is the rate of mass flow per unit area. If we want to find the flow rate (measured in volume per time) instead, we can use flux integral

$$\iint_S \vec{\mathbf{v}} \cdot \vec{\mathbf{N}}\, dS, \tag{3.7.28}$$

which leaves out the density. Since the flow rate of a fluid is measured in volume per unit time, flow rate does not take mass into account. Therefore, we have the following characterization of the flow rate of a fluid with velocity $\vec{\mathbf{v}}$ across a surface S:

$$\text{Flow rate of fluid across S} = \iint_S \vec{\mathbf{v}} \cdot dS.$$

To compute the flow rate of the fluid in Example, we simply remove the density constant, which gives a flow rate of $90\pi\, m^3/sec$.

Both mass flux and flow rate are important in physics and engineering. Mass flux measures how much mass is flowing across a surface; flow rate measures how much volume of fluid is flowing across a surface.

In addition to modeling fluid flow, surface integrals can be used to model heat flow. Suppose that the temperature at point (x,y,z) in an object is $T(x,y,z)$. Then the heat flow is a vector field proportional to the negative temperature gradient in the object. To be precise, the heat flow is defined as vector field $F = -k\nabla T$, where the constant k is the *thermal conductivity* of the substance from which the object is made (this constant is determined experimentally). The rate of heat flow across surface S in the object is given by the flux integral

$$\iint_S \vec{\mathbf{F}} \cdot dS = \iint_S -k\vec{\nabla} T \cdot dS. \tag{3.7.29}$$

Example 3.7.15: Calculating Heat Flow

A cast-iron solid cylinder is given by inequalities $x^2+y^2 \le 1$, $1 \le z \le 4$. The temperature at point (x,y,z) in a region containing the cylinder is $T(x,y,z) = (x^2+y^2)z$. Given that the thermal conductivity of cast iron is 55, find the heat flow across the boundary of the solid if this boundary is oriented outward.

Solution

Let S denote the boundary of the object. To find the heat flow, we need to calculate flux integral

$$\iint_S -k\vec{\nabla}T \cdot dS. \tag{3.7.30}$$

Notice that S is not a smooth surface but is piecewise smooth, since S is the union of three smooth surfaces (the circular top and bottom, and the cylindrical side). Therefore, we calculate three separate integrals, one for each smooth piece of S. Before calculating any integrals, note that the gradient of the temperature is $\vec{\nabla}T = \langle 2xz,\, 2yz,\, x^2+y^2\rangle$.

First we consider the circular bottom of the object, which we denote S_1. We can see that S_1 is a circle of radius 1 centered at point $(0,0,1)$ sitting in plane $z=1$. This surface has parameterization $\vec{\mathbf{r}}(u,v) = \langle v\cos u,\, v\sin u,\, 1\rangle$, $0 \le u < 2\pi$, $0 \le v \le 1$.

Therefore,

$\vec{\mathbf{t}}_u = \langle -v\sin u,\, v\cos u,\, 0\rangle$ and $\vec{\mathbf{t}}_v = \langle \cos u,\, v\sin u,\, 0\rangle$, and $\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v = \langle 0,\, 0,\, -v\sin^2 u - v\cos^2 u\rangle = \langle 0,\, 0,\, -v\rangle$.

Since the surface is oriented outward and S_1 is the bottom of the object, it makes sense that this vector points downward. By Equation, the heat flow across S_1 is

$$\begin{aligned}
\iint_{S_1} -k\vec{\nabla}T \cdot dS &= -55\int_0^{2\pi}\int_0^1 \vec{\nabla}T(u,v)\cdot(\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v)\,dv\,du \\
&= -55\int_0^{2\pi}\int_0^1 \langle 2v\cos u,\, 2v\sin u,\, v^2\cos^2 u + v^2\sin^2 u\rangle \cdot \langle 0,0,-v\rangle\,dv\,du \\
&= -55\int_0^{2\pi}\int_0^1 \langle 2v\cos u,\, 2v\sin u,\, v^2\rangle \cdot \langle 0,0,-v\rangle\,dv\,du \\
&= -55\int_0^{2\pi}\int_0^1 -v^3\,dv\,du \\
&= -55\int_0^{2\pi} -\frac{1}{4}\,du \\
&= \frac{55\pi}{2}.
\end{aligned}$$

Now let's consider the circular top of the object, which we denote S_2. We see that S_2 is a circle of radius 1 centered at point $(0,0,4)$, sitting in plane $z=4$. This surface has parameterization $\vec{\mathbf{r}}(u,v) = \langle v\cos u,\, v\sin u,\, 4\rangle$, $0 \le u < 2\pi$, $0 \le v \le 1$.

Therefore, $\vec{\mathbf{t}}_u = \langle -v\sin u,\, v\cos u,\, 0\rangle$ and $\vec{\mathbf{t}}_v = \langle \cos u,\, v\sin u,\, 0\rangle$, and $\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v = \langle 0,\, 0,\, -v\sin^2 u - v\cos^2 u\rangle = \langle 0,0,-v\rangle$.

Since the surface is oriented outward and S_1 is the top of the object, we instead take vector $\vec{\mathbf{t}}_v \times \vec{\mathbf{t}}_u = \langle 0,0,v\rangle$. By Equation, the heat flow across S_1 is

$$\begin{aligned}
\iint_{S_2} -k\vec{\nabla}T \cdot dS &= -55\int_0^{2\pi}\int_0^1 \vec{\nabla}T(u,v)\cdot(\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v)\,dv\,du \\
&= -55\int_0^{2\pi}\int_0^1 \langle 8v\cos u,\, 8v\sin u,\, v^2\cos^2 u + v^2\sin^2 u\rangle \cdot \langle 0,0,-v\rangle\,dv\,du \\
&= -55\int_0^{2\pi}\int_0^1 \langle 8v\cos u,\, 8v\sin u,\, v^2\rangle \cdot \langle 0,0,-v\rangle\,dv\,du \\
&= -55\int_0^{2\pi}\int_0^1 -v^3\,dv\,du = -55\int_0^{2\pi} -\frac{1}{4}\,du = -\frac{55\pi}{2}.
\end{aligned}$$

Last, let's consider the cylindrical side of the object. This surface has parameterization $\vec{\mathbf{r}}(u,v) = \langle \cos u,\, \sin u,\, v\rangle$, $0 \le u < 2\pi$, $1 \le v \le 4$. By Example, we know that $\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v = \langle \cos u,\, \sin u,\, 0\rangle$. By Equation,

$$\begin{aligned}\iint_{S_3} -k\vec{\nabla}T\cdot dS &= -55\int_0^{2\pi}\int_1^4 \vec{\nabla}T(u,v)\cdot(\vec{\mathbf{t}}_u\times\vec{\mathbf{t}}_v)\,dv\,du\\ &= -55\int_0^{2\pi}\int_1^4 \langle 2v\cos u,\, 2v\sin u,\, \cos^2 u+\sin^2 u\rangle\cdot\langle\cos u,\,\sin u,\,0\rangle\,dv\,du\\ &= -55\int_0^{2\pi}\int_0^1 \langle 2v\cos^2 u,\, 2v\sin u,\, 1\rangle\cdot\langle\cos u,\,\sin u,\,0\rangle\,dv\,du\\ &= -55\int_0^{2\pi}\int_0^1 (2v\cos^2 u+2v\sin^2 u)\,dv\,du\\ &= -55\int_0^{2\pi}\int_0^1 2v\,dv\,du\\ &= -55\int_0^{2\pi} du\\ &= -110\pi.\end{aligned}$$

Therefore, the rate of heat flow across S is

$$\frac{55\pi}{2}-\frac{55\pi}{2}-110\pi=-110\pi.$$

? Exercise 3.7.14

A cast-iron solid ball is given by inequality $x^2+y^2+z^2\le 1$. The temperature at a point in a region containing the ball is $T(x,y,z)=\frac{1}{3}(x^2+y^2+z^2)$. Find the heat flow across the boundary of the solid if this boundary is oriented outward.

Hint

Follow the steps of Example 3.7.15.

Answer

$-\frac{440\pi}{3}$

Key Concepts

- Surfaces can be parameterized, just as curves can be parameterized. In general, surfaces must be parameterized with two parameters.
- Surfaces can sometimes be oriented, just as curves can be oriented. Some surfaces, such as a Möbius strip, cannot be oriented.
- A surface integral is like a line integral in one higher dimension. The domain of integration of a surface integral is a surface in a plane or space, rather than a curve in a plane or space.
- The integrand of a surface integral can be a scalar function or a vector field. To calculate a surface integral with an integrand that is a function, use Equation. To calculate a surface integral with an integrand that is a vector field, use Equation.
- If S is a surface, then the area of S is

$$\iint_S dS. \tag{3.7.31}$$

Key Equations

- **Scalar surface integral**

$$\iint_S f(x,y,z)\,dS=\iint_D f(\vec{\mathbf{r}}(u,v))\,||\,\vec{\mathbf{t}}_u\times\vec{\mathbf{t}}_v\,||\,dA$$

- **Flux integral**

$$\iint_S \vec{\mathbf{F}}\cdot\vec{\mathbf{N}}\,dS=\iint_S \vec{\mathbf{F}}\cdot dS=\iint_D \vec{\mathbf{F}}(\vec{\mathbf{r}}(u,v))\cdot(\vec{\mathbf{t}}_u\times\vec{\mathbf{t}}_v)\,dA$$

Glossary

flux integral
another name for a surface integral of a vector field; the preferred term in physics and engineering

grid curves
curves on a surface that are parallel to grid lines in a coordinate plane

heat flow
a vector field proportional to the negative temperature gradient in an object

mass flux
the rate of mass flow of a fluid per unit area, measured in mass per unit time per unit area

orientation of a surface
if a surface has an "inner" side and an "outer" side, then an orientation is a choice of the inner or the outer side; the surface could also have "upward" and "downward" orientations

parameter domain (parameter space)
the region of the uv-plane over which the parameters u and v vary for parameterization $\vec{\mathbf{r}}(u,v) = \langle x(u,v),\, y(u,v),\, z(u,v)\rangle$

parameterized surface (parametric surface)
a surface given by a description of the form $\vec{\mathbf{r}}(u,v) = \langle x(u,v),\, y(u,v),\, z(u,v)\rangle$, where the parameters u and v vary over a parameter domain in the uv-plane

regular parameterization
parameterization $\vec{\mathbf{r}}(u,v) = \langle x(u,v),\, y(u,v),\, z(u,v)\rangle$ such that $r_u \times r_v$ is not zero for point (u,v) in the parameter domain

surface area
the area of surface S given by the surface integral

$$\iint_S dS \tag{3.7.32}$$

surface integral
an integral of a function over a surface

surface integral of a scalar-valued function
a surface integral in which the integrand is a scalar function

surface integral of a vector field
a surface integral in which the integrand is a vector field

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3.7E: Exercises

In exercises 1 - 4, determine whether the statements are *true or false*.

1. If surface S is given by $\{(x, y, z) : 0 \le x \le 1, 0 \le y \le 1, z = 10\}$, then $\iint_S f(x, y, z)\, dS = \int_0^1 \int_0^1 f(x, y, 10)\, dx\, dy$.

Answer

True

2. If surface S is given by $\{(x, y, z) : 0 \le x \le 1, 0 \le y \le 1, z = x\}$, then $\iint_S f(x, y, z)\, dS = \int_0^1 \int_0^1 f(x, y, x)\, dx\, dy$.

3. Surface $\vec{\mathbf{r}} = \langle v \cos u,\ v \sin u,\ v^2 \rangle$, for $0 \le u \le \pi$, $0 \le v \le 2$ is the same surface $\vec{\mathbf{r}} = \langle \sqrt{v} \cos 2u,\ \sqrt{v} \sin 2u,\ v \rangle$, for $0 \le u \le \frac{\pi}{2}$, $0 \le v \le 4$.

Answer

True

4. Given the standard parameterization of a sphere, normal vectors $\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v$ are outward normal vectors.

In exercises 5 - 10, find parametric descriptions for the following surfaces.

5. Plane $3x - 2y + z = 2$

Answer

$\vec{\mathbf{r}}(u, v) = \langle u,\ v,\ 2 - 3u + 2v \rangle$ for $-\infty \le u < \infty$ and $-\infty \le v < \infty$.

6. Paraboloid $z = x^2 + y^2$, for $0 \le z \le 9$.

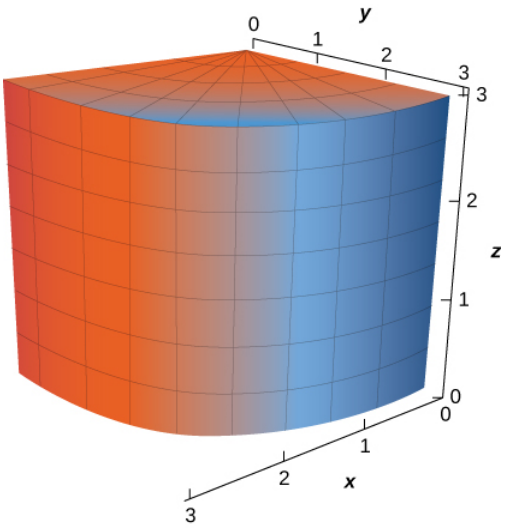
7. Plane $2x - 4y + 3z = 16$

Answer

$\vec{\mathbf{r}}(u, v) = \langle u,\ v,\ \frac{1}{3}(16 - 2u + 4v) \rangle$ for $|u| < \infty$ and $|v| < \infty$.

8. The frustum of cone $z^2 = x^2 + y^2$, for $2 \le z \le 8$

9. The portion of cylinder $x^2 + y^2 = 9$ in the first octant, for $0 \le z \le 3$



Answer

$\vec{\mathbf{r}}(u, v) = \langle 3 \cos u,\ 3 \sin u,\ v \rangle$ for $0 \le u \le \frac{\pi}{2}$, $0 \le v \le 3$

10. A cone with base radius r and height h, where r and h are positive constants.

For exercises 11 - 12, use a computer algebra system to approximate the area of the following surfaces using a parametric description of the surface.

11. [T] Half cylinder $\{(r,\theta,z) : r = 4,\ 0 \le \theta \le \pi,\ 0 \le z \le 7\}$

Answer

$A = 87.9646$

12. [T] Plane $z = 10 - z - y$ above square $|x| \le 2,\ |y| \le 2$

In exercises 13 - 15, let S be the hemisphere $x^2 + y^2 + z^2 = 4$, with $z \ge 0$, and evaluate each surface integral, in the counterclockwise direction.

13. $\iint_S z\,dS$

Answer

$\iint_S z\,dS = 8\pi$

14. $\iint_S (x - 2y)\,dS$

15. $\iint_S (x^2 + y^2)\,dS$

Answer

$\iint_S (x^2 + y^2)\,dS = 16\pi$

In exercises 16 - 18, evaluate $\int\int_S \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\,dS$ for vector field $\vec{\mathbf{F}}$ where $\vec{\mathbf{N}}$ is an outward normal vector to surface S.

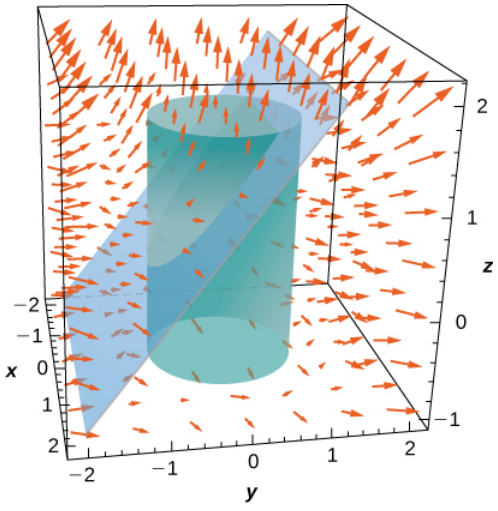
16. $\vec{\mathbf{F}}(x,y,z) = x\,\hat{\mathbf{i}} + 2y\,\hat{\mathbf{j}} = 3z\,\hat{\mathbf{k}}$, and S is that part of plane $15x - 12y + 3z = 6$ that lies above unit square $0 \le x \le 1,\ 0 \le y \le 1$.

17. $\vec{\mathbf{F}}(x,y) = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}$, and S is hemisphere $z = \sqrt{1 - x^2 - y^2}$.

Answer

$\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\,dS = \frac{4\pi}{3}$

18. $\vec{\mathbf{F}}(x,y,z) = x^2\,\hat{\mathbf{i}} + y^2\,\hat{\mathbf{j}} + z^2\,\hat{\mathbf{k}}$, and S is the portion of plane $z = y + 1$ that lies inside cylinder $x^2 + y^2 = 1$.



In exercises 19 - 20, approximate the mass of the homogeneous lamina that has the shape of given surface S. Round to four decimal places.

19. [T] S is surface $z = 4 - x - 2y$, with $z \geq 0$, $x \geq 0$, $y \geq 0$; $\xi = x$.

Answer

$m \approx 13.0639$

20. [T] S is surface $z = x^2 + y^2$, with $z \leq 1$; $\xi = z$.

21. [T] S is surface $x^2 + y^2 + x^2 = 5$, with $z \geq 1$; $\xi = \theta^2$.

Answer

$m \approx 228.5313$

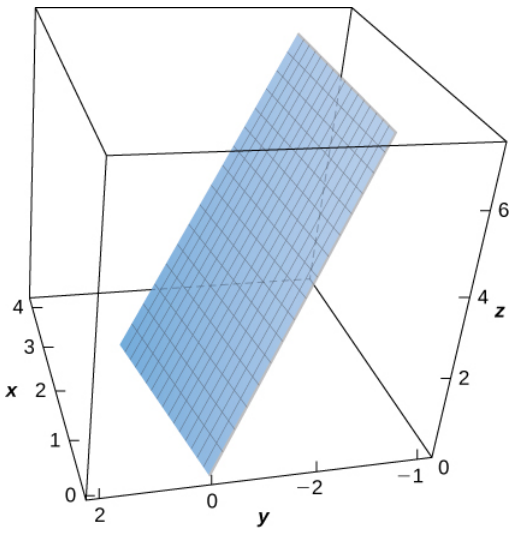
22. Evaluate $\iint_S (y^2 z\,\hat{\mathbf{i}} + y^3\,\hat{\mathbf{j}} + xz\hat{\mathbf{k}}) \cdot dS$, where S is the surface of cube $-1 \leq x \leq 1$, $-1 \leq y \leq 1$, and $0 \leq z \leq 2$ in a counterclockwise direction.

23. Evaluate surface integral $\iint_S g\,dS$, where $g(x, y, z) = xz + 2x^2 - 3xy$ and S is the portion of plane $2x - 3y + z = 6$ that lies over unit square $R : 0 \leq x \leq 1$, $0 \leq y \leq 1$.

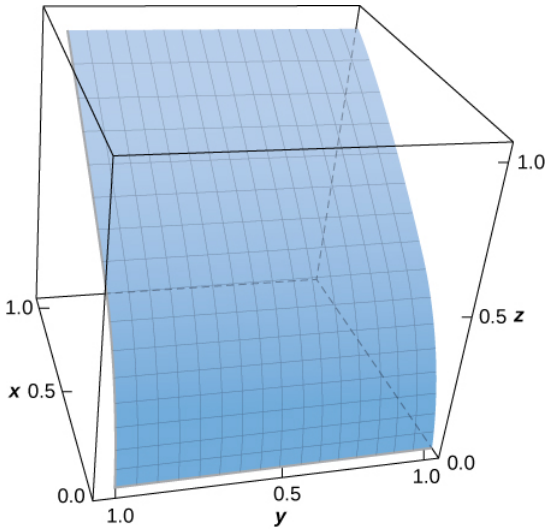
Answer

$\iint_S g\,dS = 3\sqrt{4}$

24. Evaluate $\iint_S (x + y + z)\,dS$, where S is the surface defined parametrically by $\vec{\mathbf{R}}(u, v) = (2u + v)\,\hat{\mathbf{i}} + (u - 2v)\,\hat{\mathbf{j}} + (u + 3v)\,\hat{\mathbf{k}}$ for $0 \leq u \leq 1$, and $0 \leq v \leq 2$.



25. [T] Evaluate $\iint_S (x - y^2 + z)\,dS$, where S is the surface defined parametrically by $\vec{\mathbf{R}}(u, v) = u^2\,\hat{\mathbf{i}} + v\,\hat{\mathbf{j}} + u\,\hat{\mathbf{k}}$ for $0 \leq u \leq 1$, $0 \leq v \leq 1$.



Answer

$$\iint_S (x^2 + y - z)\, dS \approx 0.9617$$

26. [T] Evaluate where S is the surface defined by $\vec{\mathbf{R}}(u, v) = u\,\hat{\mathbf{i}} - u^2\,\hat{\mathbf{j}} + v\hat{\mathbf{k}}$, $0 \le u \le 2$, $0 \le v \le 1$ for $0 \le u \le 1$, $0 \le v \le 2$.

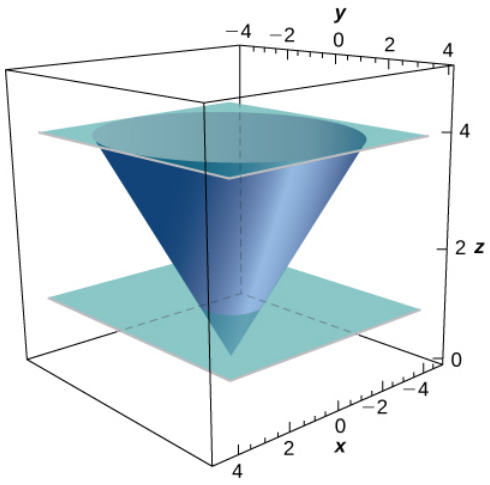
27. Evaluate $\iint_S (x^2 + y^2)\, dS$, where S is the surface bounded above hemisphere $z = \sqrt{1 - x^2 - y^2}$, and below by plane $z = 0$.

Answer

$$\iint_S (x^2 + y^2)\, dS = \frac{4\pi}{3}$$

28. Evaluate $\iint_S (x^2 + y^2 + z^2)\, dS$, where S is the portion of plane that lies inside cylinder $x^2 + y^2 = 1$.

29. [T] Evaluate $\iint_S x^2 z\, dS$, where S is the portion of cone $z^2 = x^2 + y^2$ that lies between planes $z = 1$ and $z = 4$.



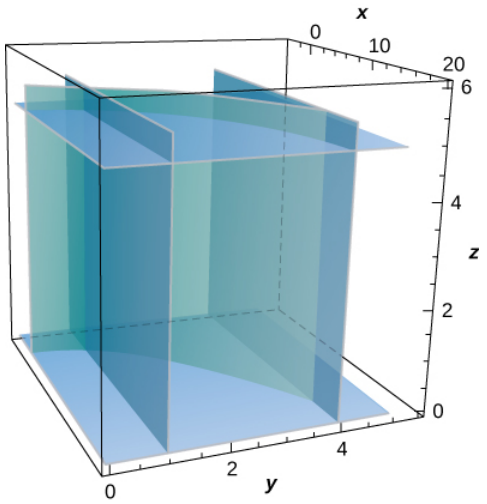
Answer

$$\text{div}\,\vec{\mathbf{F}} = a + b$$

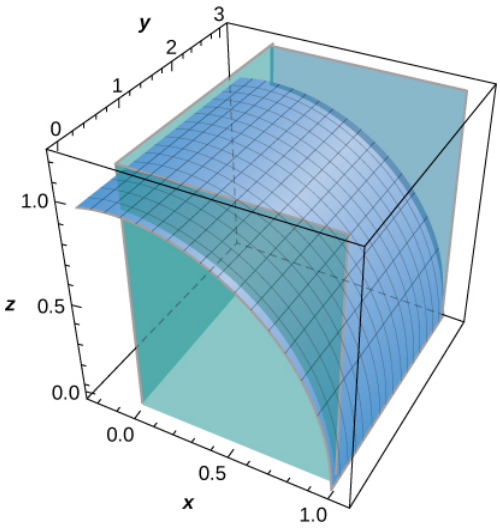
$$\iint_S x^2 z\, dS = \frac{1023\sqrt{2\pi}}{5}$$

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30. [T] Evaluate $\iint_S \frac{xz}{y}\,dS$, where S is the portion of cylinder $x = y^2$ that lies in the first octant between planes $z = 0$, $z = 5$, and $y = 4$.



31. [T] Evaluate $\iint_S (z + y)\,dS$, where S is the part of the graph of $z = \sqrt{1 - x^2}$ in the first octant between the xy-plane and plane $y = 3$.



Answer

$\iint_S (z + y)\,dS \approx 10.1$

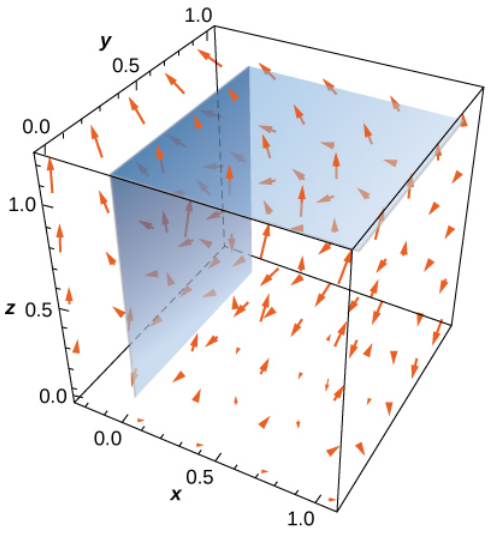
32. Evaluate $\iint_S xyz\,dS$ if S is the part of plane $z = x + y$ that lies over the triangular region in the xy-plane with vertices (0, 0, 0), (1, 0, 0), and (0, 2, 0).

33. Find the mass of a lamina of density $\xi(x, y, z) = z$ in the shape of hemisphere $z = (a^2 - x^2 - y^2)^{1/2}$.

Answer

$m = \pi a^3$

34. Compute $\int\int_S \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\,dS$, where $\vec{\mathbf{F}}(x, y, z) = x\,\hat{\mathbf{i}} - 5y\,\hat{\mathbf{j}} + 4z\,\hat{\mathbf{k}}$ and $\vec{\mathbf{N}}$ is an outward normal vector S, where S is the union of two squares $S_1 : x = 0,\ 0 \le y \le 1,\ 0 \le z \le 1$ and $S_2 : x = 0,\ 0 \le x \le 1,\ 0 \le y \le 1$.



35. Compute $\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, dS$, where $\vec{\mathbf{F}}(x,y,z) = xy\,\hat{\mathbf{i}} + z\,\hat{\mathbf{j}} + (x+y)\,\hat{\mathbf{k}}$ and $\vec{\mathbf{N}}$ is an outward normal vector S, where S is the triangular region cut off from plane $x+y+z=1$ by the positive coordinate axes.

Answer

$$\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, dS = \frac{13}{24}$$

36. Compute $\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, dS$, where $\vec{\mathbf{F}}(x,y,z) = 2yz\,\hat{\mathbf{i}} + (\tan^{-1} xz)\,\hat{\mathbf{j}} + e^{xy}\,\hat{\mathbf{k}}$ and $\vec{\mathbf{N}}$ is an outward normal vector S, where S is the surface of sphere $x^2+y^2+z^2=1$.

37. Compute $\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, dS$, where $\vec{\mathbf{F}}(x,y,z) = xyz\,\hat{\mathbf{i}} + xyz\,\hat{\mathbf{j}} + xyz\,\hat{\mathbf{k}}$ and $\vec{\mathbf{N}}$ is an outward normal vector S, where S is the surface of the five faces of the unit cube $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$ missing $z=0$.

Answer

$$\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, dS = \frac{3}{4}$$

For exercises 38 - 39, express the surface integral as an iterated double integral by using a projection on S on the yz-plane.

38. $\iint_S xy^2z^3\, dS$; S is the first-octant portion of plane $2x+3y+4z=12$.

39. $\iint_S (x^2-2y+z)\, dS$; S is the portion of the graph of $4x+y=8$ bounded by the coordinate planes and plane $z=6$.

Answer

$$\int_0^8 \int_0^6 \left(4-3y+\frac{1}{16}y^2+z\right)\left(\frac{1}{4}\sqrt{17}\right) dz\, dy$$

For exercises 40 - 41, express the surface integral as an iterated double integral by using a projection on S on the xz-plane.

40. $\iint_S xy^2z^3\, dS$; S is the first-octant portion of plane $2x+3y+4z=12$.

41. $\iint_S (x^2-2y+z)\, dS$; is the portion of the graph of $4x+y=8$ bounded by the coordinate planes and plane $z=6$.

Answer

$$\int_0^2 \int_0^6 \left[x^2-2(8-4x)+z\right]\sqrt{17}\, dz\, dx$$

42. Evaluate surface integral $\iint_S yz\, dS$, where S is the first-octant part of plane $x+y+z=\lambda$, where λ is a positive constant.

43. Evaluate surface integral $\iint_S (x^2z + y^2z)\, dS$, where S is hemisphere $x^2 + y^2 + z^2 = a^2$, $z \geq 0$.

Answer

$$\iint_S (x^2z + y^2z)\, dS = \frac{\pi a^5}{2}$$

44. Evaluate surface integral $\iint_S z\, dA$, where S is surface $z = \sqrt{x^2 + y^2}$, $0 \leq z \leq 2$.

45. Evaluate surface integral $\iint_S x^2yz\, dS$, where S is the part of plane $z = 1 + 2x + 3y$ that lies above rectangle $0 \leq x \leq 3$ and $0 \leq y \leq 2$.

Answer

$$\iint_S x^2yz\, dS = 171\sqrt{14}$$

46. Evaluate surface integral $\iint_S yz\, dS$, where S is plane $x + y + z = 1$ that lies in the first octant.

47. Evaluate surface integral $\iint_S yz\, dS$, where S is the part of plane $z = y + 3$ that lies inside cylinder $x^2 + y^2 = 1$.

Answer

$$\iint_S yz\, dS = \frac{\sqrt{2}\pi}{4}$$

For exercises 48 - 50, use geometric reasoning to evaluate the given surface integrals.

48. $\iint_S \sqrt{x^2 + y^2 + z^2}\, dS$, where S is surface $x^2 + y^2 + z^2 = 4$, $z \geq 0$

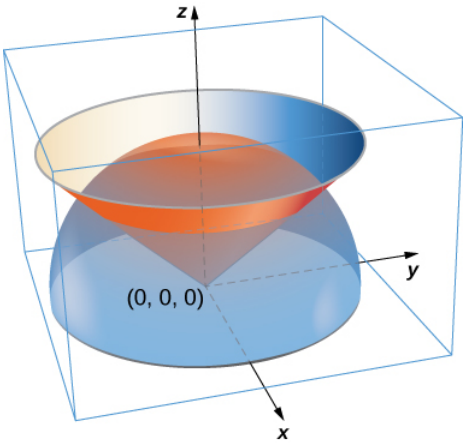
49. $\iint_S (x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}) \cdot dS$, where S is surface $x^2 + y^2 = 4$, $1 \leq z \leq 3$, oriented with unit normal vectors pointing outward

Answer

$$\iint_S (x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}) \cdot dS = 16\pi$$

50. $\iint_S (z\,\hat{\mathbf{k}}) \cdot dS$, where S is disc $x^2 + y^2 \leq 9$ on plane $z = 4$ oriented with unit normal vectors pointing upward

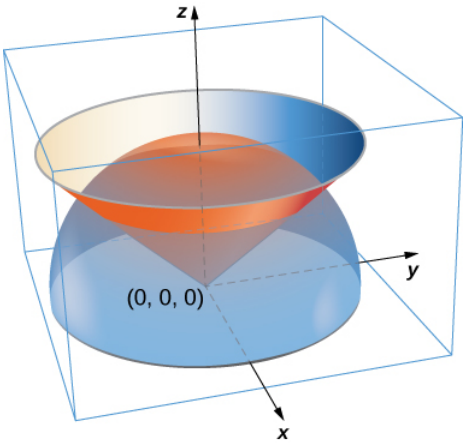
51. A lamina has the shape of a portion of sphere $x^2 + y^2 + z^2 = a^2$ that lies within cone $z = \sqrt{x^2 + y^2}$. Let S be the spherical shell centered at the origin with radius a, and let C be the right circular cone with a vertex at the origin and an axis of symmetry that coincides with the z-axis. Determine the mass of the lamina if $\delta(x, y, z) = x^2y^2z$.



Answer

$$m = \frac{\pi a^7}{192}$$

52. A lamina has the shape of a portion of sphere $x^2 + y^2 + z^2 = a^2$ that lies within cone $z = \sqrt{x^2 + y^2}$. Let S be the spherical shell centered at the origin with radius a, and let C be the right circular cone with a vertex at the origin and an axis of symmetry that coincides with the z-axis. Suppose the vertex angle of the cone is ϕ_0, with $0 \leq \phi_0 < \frac{\pi}{2}$. Determine the mass of that portion of the shape enclosed in the intersection of S and C. Assume $\delta(x, y, z) = x^2 y^2 z$.



53. A paper cup has the shape of an inverted right circular cone of height 6 in. and radius of top 3 in. If the cup is full of water weighing $62.5\, lb/ft^3$, find the magnitude of the total force exerted by the water on the inside surface of the cup.

Answer

$F \approx 4.57\, lb$

For exercises 54 - 55, the heat flow vector field for conducting objects is $\vec{\mathbf{F}} = -k\vec{\nabla}T$, where $T(x, y, z)$ is the temperature in the object and $k > 0$ is a constant that depends on the material. Find the outward flux of $\vec{\mathbf{F}}$ across the following surfaces S for the given temperature distributions and assume $k = 1$.

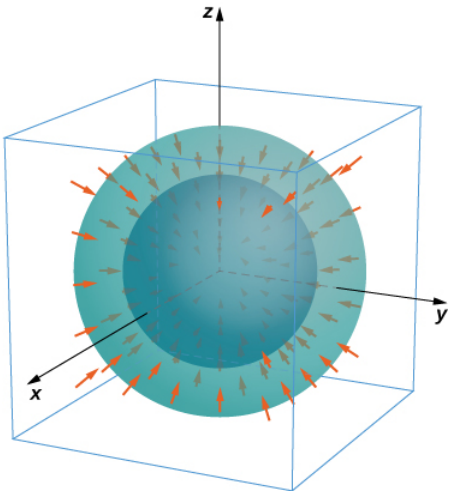
54. $T(x, y, z) = 100e^{-x-y}$; S consists of the faces of cube $|x| \leq 1$, $|y| \leq 1$, $|z| \leq 1$.

55. $T(x, y, z) = -\ln(x^2 + y^2 + z^2)$; S is sphere $x^2 + y^2 + z^2 = a^2$.

Answer

$8\pi a$

For exercises 56 - 57, consider the radial fields $\vec{\mathbf{F}} = \frac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{p/2}} = \frac{r}{|r|^p}$, where p is a real number. Let S consist of spheres A and B centered at the origin with radii $0 < a < b$. The total outward flux across S consists of the outward flux across the outer sphere B less the flux into S across inner sphere A.



56. Find the total flux across S with $p = 0$.

57. Show that for $p = 3$ the flux across S is independent of a and b.

Answer

The net flux is zero.

Contributors

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3.8: Stokes' Theorem

Learning Objectives

- Explain the meaning of Stokes' theorem.
- Use Stokes' theorem to evaluate a line integral.
- Use Stokes' theorem to calculate a surface integral.
- Use Stokes' theorem to calculate a curl.

In this section, we study Stokes' theorem, a higher-dimensional generalization of Green's theorem. This theorem, like the Fundamental Theorem for Line Integrals and Green's theorem, is a generalization of the Fundamental Theorem of Calculus to higher dimensions. Stokes' theorem relates a vector surface integral over surface S in space to a line integral around the boundary of S. Therefore, just as the theorems before it, Stokes' theorem can be used to reduce an integral over a geometric object S to an integral over the boundary of S. In addition to allowing us to translate between line integrals and surface integrals, Stokes' theorem connects the concepts of curl and circulation. Furthermore, the theorem has applications in fluid mechanics and electromagnetism. We use Stokes' theorem to derive Faraday's law, an important result involving electric fields.

Stokes' Theorem

Stokes' theorem says we can calculate the flux of $curl\,\vec{\mathbf{F}}$ across surface S by knowing information only about the values of $\vec{\mathbf{F}}$ along the boundary of S. Conversely, we can calculate the line integral of vector field $\vec{\mathbf{F}}$ along the boundary of surface S by translating to a double integral of the curl of $\vec{\mathbf{F}}$ over S.

Let S be an oriented smooth surface with unit normal vector $\vec{\mathbf{N}}$. Furthermore, suppose the boundary of S is a simple closed curve C. The orientation of S induces the positive orientation of C if, as you walk in the positive direction around C with your head pointing in the direction of $\vec{\mathbf{N}}$, the surface is always on your left. With this definition in place, we can state *Stokes' theorem*.

Theorem 3.8.1: Stokes' Theorem

Let S be a piecewise smooth oriented surface with a boundary that is a simple closed curve C with positive orientation (Figure 3.8.1). If $\vec{\mathbf{F}}$ is a vector field with component functions that have continuous partial derivatives on an open region containing S, then

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_S curl\,\vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}. \tag{3.8.1}$$

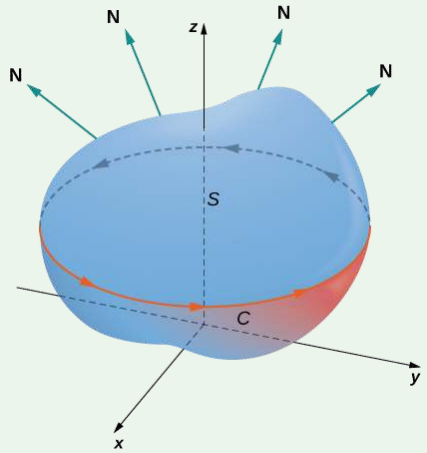


Figure 3.8.1: Stokes' theorem relates the flux integral over the surface to a line integral around the boundary of the surface. Note that the orientation of the curve is positive.

Suppose surface S is a flat region in the xy-plane with upward orientation. Then the unit normal vector is $\vec{\mathbf{k}}$ and surface integral

$$\iint_S curl\,\vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} \tag{3.8.2}$$

is actually the double integral

$$\iint_S curl\,\vec{\mathbf{F}} \cdot \vec{\mathbf{k}}\, dA. \tag{3.8.3}$$

In this special case, Stokes' theorem gives

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_S curl\,\vec{\mathbf{F}} \cdot \vec{\mathbf{k}}\, dA. \tag{3.8.4}$$

However, this is the flux form of Green's theorem, which shows us that Green's theorem is a special case of Stokes' theorem. Green's theorem can only handle surfaces in a plane, but Stokes' theorem can handle surfaces in a plane or in space.

The complete proof of Stokes' theorem is beyond the scope of this text. We look at an intuitive explanation for the truth of the theorem and then see proof of the theorem in the special case that surface S is a portion of a graph of a function, and S, the boundary of S, and $\vec{\mathbf{F}}$ are all fairly tame.

Proof

First, we look at an informal proof of the theorem. This proof is not rigorous, but it is meant to give a general feeling for why the theorem is true. Let S be a surface and let D be a small piece of the surface so that D does not share any points with the boundary of S. We choose D to be small enough so that it can be approximated by an oriented square E. Let D inherit its orientation from S, and give E the same orientation. This square has four sides; denote them E_l, E_r, E_u, and E_d for the left, right, up, and down sides, respectively. On the square, we can use the flux form of Green's theorem:

$$\int_{E_l+E_d+E_r+E_u} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_E curl\,\vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, d\vec{\mathbf{S}} = \iint_E curl\,\vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}. \tag{3.8.5}$$

To approximate the flux over the entire surface, we add the values of the flux on the small squares approximating small pieces of the surface (Figure 3.8.2).

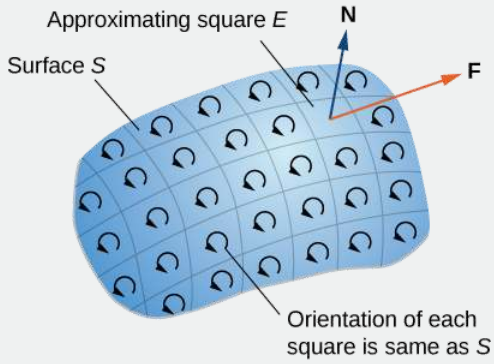


Figure 3.8.2: Chop the surface into small pieces. The pieces should be small enough that they can be approximated by a square.

By Green's theorem, the flux across each approximating square is a line integral over its boundary. Let F be an approximating square with an orientation inherited from S and with a right side E_l (so F is to the left of E). Let F_r denote the right side of F; then, $E_l = -F_r$. In other words, the right side of F is the same curve as the left side of E, just oriented in the opposite direction. Therefore,

$$\int_{E_l} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = -\int_{F_r} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}.$$

As we add up all the fluxes over all the squares approximating surface S, line integrals

$$\int_{E_l} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \tag{3.8.6}$$

and

$$\int_{F_r} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \tag{3.8.7}$$

cancel each other out. The same goes for the line integrals over the other three sides of E. These three line integrals cancel out with the line integral of the lower side of the square above E, the line integral over the left side of the square to the right of E, and the line integral over the upper side of the square below E (Figure 3.8.3). After all this cancelation occurs over all the approximating squares, the only line integrals that survive are the line integrals over sides approximating the boundary of S. Therefore, the sum of all the fluxes (which, by Green's theorem, is the sum of all the line integrals around the boundaries of approximating squares) can be approximated by a line integral over the boundary of S. In the limit, as the areas of the approximating squares go to zero, this approximation gets arbitrarily close to the flux.

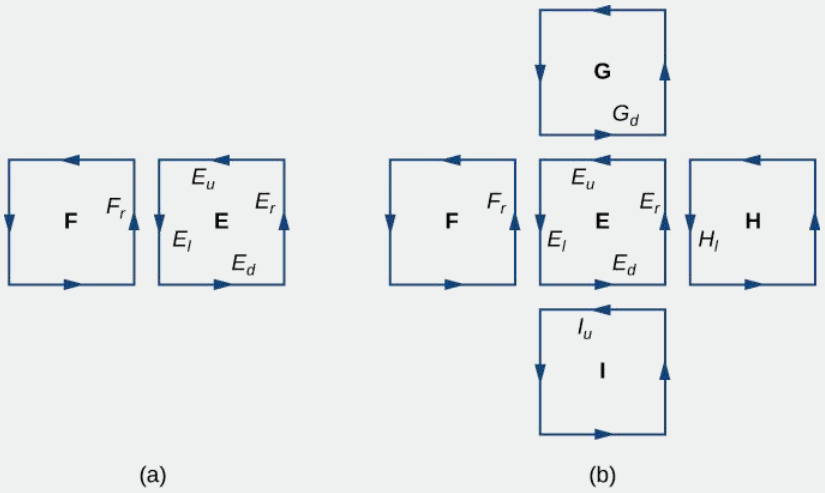


Figure 3.8.3: (a) The line integral along E_l cancels out the line integral along F_r because $E_l = -F_r$. (b) The line integral along any of the sides of E cancels out with the line integral along a side of an adjacent approximating square.

Let's now look at a rigorous proof of the theorem in the special case that S is the graph of function $z = f(x, y)$, where x and y vary over a bounded, simply connected region D of finite area (Figure 3.8.4). Furthermore, assume that f has continuous second-order partial derivatives. Let C denote the boundary of S and let C' denote the boundary of D. Then, D is the "shadow" of S in the plane and C' is the "shadow" of C. Suppose that S is oriented upward. The counterclockwise orientation of C is positive, as is the counterclockwise orientation of C'. Let $\vec{\mathbf{F}}(x, y, z) = \langle P, Q, R \rangle$ be a vector field with component functions that have continuous partial derivatives.

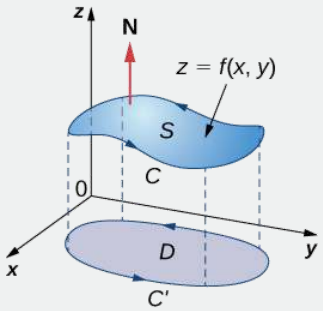


Figure 3.8.4: D is the "shadow," or projection, of S in the plane and C' is the projection of C.

We take the standard parameterization of $S: x = x,\ y = y,\ z = g(x, y)$. The tangent vectors are $\vec{\mathbf{t}}_x = \langle 1, 0, g_x \rangle$ and $\vec{\mathbf{t}}_y = \langle 0, 1, g_y \rangle$, and therefore $\vec{\mathbf{t}}_x \times \vec{\mathbf{t}}_y = \langle -g_x, -g_y, 1 \rangle$.

$$\iint_S curl\, \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_D [-(R_y - Q_z)z_x - (P_z - R_x)z_y + (Q_x - P_y)]\, dA,$$

where the partial derivatives are all evaluated at $(x, y, g(x, y))$, making the integrand depend on x and y only. Suppose $\langle x(t), y(t) \rangle$, $a \le t \le b$ is a parameterization of C'. Then, a parameterization of C is $\langle x(t), y(t), g(x(t), y(t)) \rangle$, $a \le t \le b$. Armed with

these parameterizations, the Chain rule, and Green's theorem, and keeping in mind that P, Q and R are all functions of x and y, we can evaluate line integral

$$\begin{aligned}\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_a^b (Px'(t) + Qy'(t) + Rz'(t))\, dt \\ &= \int_a^b \left[Px'(t) + Qy'(t) + R\left(\frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}\right)\right] dt \\ &= \int_a^b \left[\left(P + R\frac{\partial z}{\partial x}\right)x'(t) + \left(Q + R\frac{\partial z}{\partial y}\right)y'(t)\right] dt \\ &= \int_{C'} \left(P + R\frac{\partial z}{\partial x}\right) dx + \left(Q + R\frac{\partial z}{\partial y}\right) dy \\ &= \iint_D \left[\frac{\partial}{\partial x}\left(Q + R\frac{\partial z}{\partial y}\right) - \frac{\partial}{\partial y}\left(P + R\frac{\partial z}{\partial x}\right)\right] dA \\ &= \iint_D \left(\frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial R}{\partial x}\frac{\partial z}{\partial y} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} + R\frac{\partial^2 z}{\partial x \partial y}\right) \\ &\quad - \left(\frac{\partial P}{\partial y} + \frac{\partial P}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial y}\frac{\partial z}{\partial x} + R\frac{\partial^2 z}{\partial y \partial x}\right)\end{aligned}$$

By Clairaut's theorem,

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$$

Therefore, four of the terms disappear from this double integral, and we are left with

$$\iint_D [-(R_y - Q_z)Z_x - (P_z - R_x)z_y + (Q_x - P_y)]\, dA,$$

which equals

$$\iint_S curl\, \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}.$$

□

We have shown that Stokes' theorem is true in the case of a function with a domain that is a simply connected region of finite area. We can quickly confirm this theorem for another important case: when vector field $\vec{\mathbf{F}}$ is a conservative field. If $\vec{\mathbf{F}}$ is conservative, the curl of $\vec{\mathbf{F}}$ is zero, so

$$\iint_S curl\, \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = 0. \tag{3.8.8}$$

Since the boundary of S is a closed curve, the integral

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}. \tag{3.8.9}$$

is also zero.

Example 3.8.1: Verifying Stokes' Theorem for a Specific Case

Verify that Stokes' theorem is true for vector field $\vec{\mathbf{F}}(x, y) = \langle -z, x, 0\rangle$ and surface S, where S is the hemisphere, oriented outward, with parameterization $\vec{\mathbf{r}}(\phi, \theta) = \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi\rangle$, $0 \le \theta \le \pi$, $0 \le \phi \le \pi$ as shown in Figure 3.8.5.

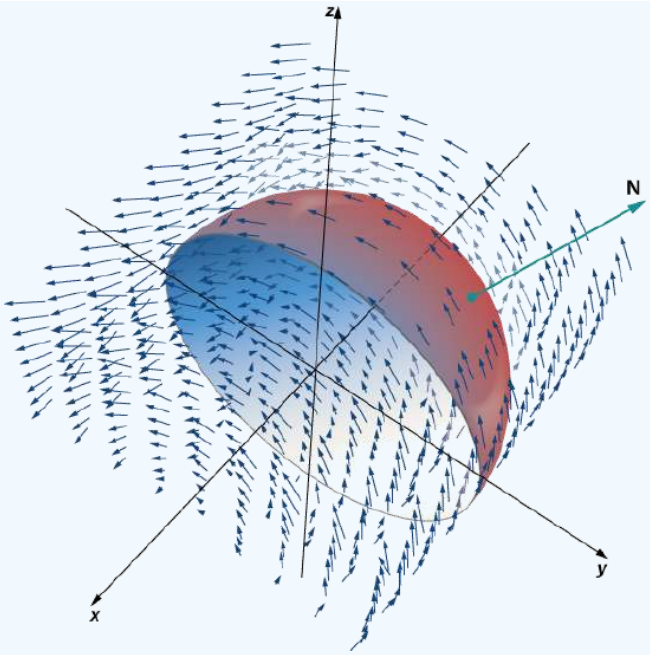


Figure **3.8.5**: Verifying Stokes' theorem for a hemisphere in a vector field.

Solution

Let C be the boundary of S. Note that C is a circle of radius 1, centered at the origin, sitting in plane $y = 0$. This circle has parameterization $\langle \cos t,\ 0,\ \sin t \rangle$, $0 \le t \le 2\pi$. the equation for scalar surface integrals

$$\begin{aligned} \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_0^{2\pi} \langle -\sin t,\ \cos t,\ 0 \rangle \cdot \langle -\sin t,\ 0,\ \cos t \rangle \, dt \\ &= \int_0^{2\pi} \sin^2 t \, dt \\ &= \pi. \end{aligned}$$

By the equation for vector line integrals,

$$\begin{aligned} \iint_S curl\, \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} &= \iint_D curl\, \vec{\mathbf{F}}(\vec{\mathbf{r}}(\phi, \theta)) \cdot (\vec{\mathbf{t}}_\phi \times \vec{\mathbf{t}}_\theta) \, dA \\ &= \iint_D \langle 0, -1, 1 \rangle \cdot \langle \cos\theta \sin^2\phi,\ \sin\theta \sin^2\phi,\ \sin\phi \cos\phi \rangle \, dA \\ &= \int_0^{\pi} \int_0^{\pi} (\sin\phi \cos\phi - \sin\theta \sin^2\phi) \, d\phi d\theta \\ &= \frac{\pi}{2} \int_0^{\pi} \sin\theta \, d\theta \\ &= \pi. \end{aligned}$$

Therefore, we have verified Stokes' theorem for this example.



? Exercise 3.8.1

Verify that Stokes' theorem is true for vector field $\vec{\mathbf{F}}(x,y,z) = \langle y, x, -z\rangle$ and surface S, where S is the upwardly oriented portion of the graph of $f(x,y) = x^2 y$ over a triangle in the xy-plane with vertices $(0,0)$, $(2,0)$, and $(0,2)$.

Hint

Calculate the double integral and line integral separately.

Answer

Both integrals give $-\frac{136}{45}$:

Applying Stokes' Theorem

Stokes' theorem translates between the flux integral of surface S to a line integral around the boundary of S. Therefore, the theorem allows us to compute surface integrals or line integrals that would ordinarily be quite difficult by translating the line integral into a surface integral or vice versa. We now study some examples of each kind of translation.

✓ Example 3.8.2: Calculating a Surface Integral

Calculate surface integral

$$\iint_S curl\, \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}},$$

where S is the surface, oriented outward, in Figure 3.8.6 and $\vec{\mathbf{F}} = \langle z,\ 2xy,\ x+y\rangle$.

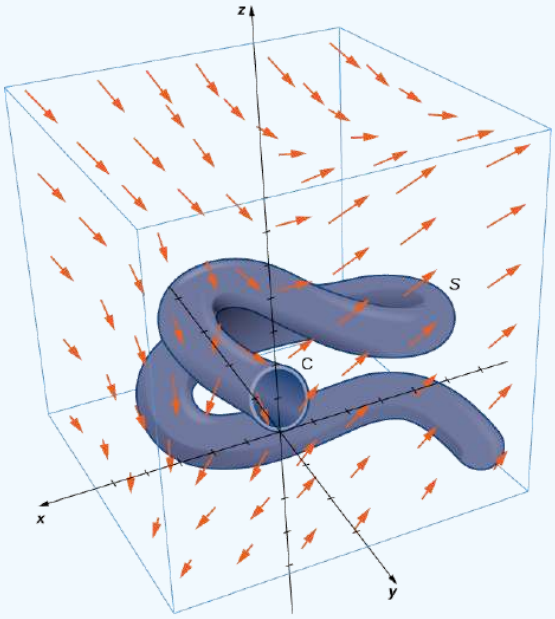


Figure 3.8.6: A complicated surface in a vector field.

Solution

Note that to calculate

$$\iint_S curl\,\vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$$

without using Stokes' theorem, we would need the equation for scalar surface integrals. Use of this equation requires a parameterization of S. Surface S is complicated enough that it would be extremely difficult to find a parameterization. Therefore, the methods we have learned in previous sections are not useful for this problem. Instead, we use Stokes' theorem, noting that the boundary C of the surface is merely a single circle with radius 1.

The curl of $\vec{\mathbf{F}}$ is $\langle 1, 1, 2y \rangle$. By Stokes' theorem,

$$\iint_S curl\,\vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}},$$

where C has parameterization $\langle \cos t,\ \sin t,\ 1 \rangle, 0 \le t \le 2\pi$. By the equation for vector line integrals,

$$\begin{aligned}
\iint_S curl\,F \cdot d\vec{\mathbf{S}} &= \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \\
&= \int_0^2 \langle 1,\ \sin t \cos t,\ \cos t + \sin t \rangle \cdot \langle -\sin t,\ \cos t,\ 0 \rangle\, dt \\
&= \int_0^{2\pi} (-\sin t + 2\ \sin t\ \cos^2 t)\, dt \\
&= \left[\cos t - \frac{2\cos^3 t}{3} \right]_0^{2\pi} \\
&= \cos(2\pi) - \frac{2\ \cos^3(2\pi)}{3} - \left(\cos(0) - \frac{2\ \cos^3(0)}{3} \right) \\
&= 0.
\end{aligned}$$

An amazing consequence of Stokes' theorem is that if S' is any other smooth surface with boundary C and the same orientation as S, then

$$\iint_S curl\,\vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 0 \tag{3.8.10}$$

because Stokes' theorem says the surface integral depends on the line integral around the boundary only.

In Example 3.8.2, we calculated a surface integral simply by using information about the boundary of the surface. In general, let S_1 and S_2 be smooth surfaces with the same boundary C and the same orientation. By Stokes' theorem,

$$\iint_{S_1} curl\,\vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_{S_2} curl\,\vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}. \tag{3.8.11}$$

Therefore, if

$$\iint_{S_1} curl\,\vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} \tag{3.8.12}$$

is difficult to calculate but

$$\iint_{S_2} curl\,\vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} \tag{3.8.13}$$

is easy to calculate, Stokes' theorem allows us to calculate the easier surface integral. In Example 3.8.2, we could have calculated

$$\iint_{S} curl\,\vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} \tag{3.8.14}$$

by calculating

$$\iint_{S'} curl\,\vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}, \tag{3.8.15}$$

where $\vec{\mathbf{S}}'$ is the disk enclosed by boundary curve C (a much more simple surface with which to work).

Equation 3.8.11 shows that flux integrals of curl vector fields are surface independent in the same way that line integrals of gradient fields are path independent. Recall that if $\vec{\mathbf{F}}$ is a two-dimensional conservative vector field defined on a simply connected domain, f is a *potential function* for $\vec{\mathbf{F}}$, and C is a curve in the domain of $\vec{\mathbf{F}}$, then

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \tag{3.8.16}$$

depends only on the endpoints of C. Therefore if C' is any other curve with the same starting point and endpoint as C (that is, C' has the same orientation as C), then

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{C'} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \tag{3.8.17}$$

In other words, the value of the integral depends on the boundary of the path only; it does not really depend on the path itself.

Analogously, suppose that S and S' are surfaces with the same boundary and same orientation, and suppose that $\vec{\mathbf{G}}$ is a three-dimensional vector field that can be written as the curl of another vector field $\vec{\mathbf{F}}$ (so that $\vec{\mathbf{F}}$ is like a "potential field" of $\vec{\mathbf{G}}$). By Equation 3.8.11,

$$\iint_S \vec{\mathbf{G}} \cdot d\vec{\mathbf{S}} = \iint_S curl\,\vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_{S'} curl\,\vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_{S'} \vec{\mathbf{G}} \cdot d\vec{\mathbf{S}}.$$

Therefore, the flux integral of $\vec{\mathbf{G}}$ does not depend on the surface, only on the boundary of the surface. Flux integrals of vector fields that can be written as the curl of a vector field are surface independent in the same way that line integrals of vector fields that can be written as the gradient of a scalar function are path independent.

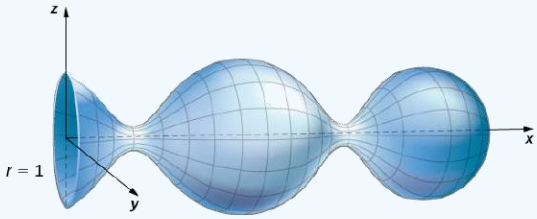


Exercise 3.8.1

Use Stokes' theorem to calculate surface integral

$$\iint_S curl\,\vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}, \tag{3.8.18}$$

where $\vec{\mathbf{F}} = \langle x, y, z\rangle$ and S is the surface as shown in the following figure.



Hint

Parameterize the boundary of S and translate to a line integral.

Answer

$-\pi$

Example 3.8.3: Calculating a Line Integral

Calculate the line integral

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}},$$

where $\vec{\mathbf{F}} = \langle xy,\ x^2 + y^2 + z^2,\ yz\rangle$ and C is the boundary of the parallelogram with vertices $(0,0,1)$, $(0,1,0)$, $(2,0,-1)$, and $(2,1,-2)$.

Solution

To calculate the line integral directly, we need to parameterize each side of the parallelogram separately, calculate four separate line integrals, and add the result. This is not overly complicated, but it is time-consuming.

By contrast, let's calculate the line integral using Stokes' theorem. Let S denote the surface of the parallelogram. Note that S is the portion of the graph of $z = 1 - x - y$ for (x, y) varying over the rectangular region with vertices $(0,0)$, $(0,1)$, $(2,0)$, and $(2,1)$ in the xy-plane. Therefore, a parameterization of S is $\langle x, y,\ 1 - x - y\rangle$, $0 \le x \le 2$, $0 \le y \le 1$. The curl of $\vec{\mathbf{F}}$ is $\langle -z,\ 0,\ x\rangle$,and Stokes' theorem and the equation for scalar surface integrals

$$\begin{aligned}\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \iint_S curl\, \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} \\ &= \int_0^2 \int_0^1 curl\, \vec{\mathbf{F}}(x,y) \cdot (\vec{\mathbf{t}}_x \times \vec{\mathbf{t}}_y)\, dy\, dx \\ &= \int_0^2 \int_0^1 \langle -(1-x-y),\, 0,\, x\rangle \cdot (\langle 1,\, 0, -1\rangle \times \langle 0,\, 1,\, -1\rangle)\, dy\, dx \\ &= \int_0^2 \int_0^1 \langle x+y-1,\, 0,\, x\rangle \cdot \langle 1,1,1\rangle\, dy\, dx \\ &= \int_0^2 \int_0^1 2x+y-1\, dy\, dx \\ &= 3.\end{aligned}$$

? Exercise 3.8.3

Use Stokes' theorem to calculate line integral

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}},$$

where $\vec{\mathbf{F}} = \langle z, x, y\rangle$ and C is the boundary of a triangle with vertices $(0,0,1)$, $(3,0,-2)$, and $(0,1,2)$.

Hint

This triangle lies in plane $z = 1 - x + y$.

Answer

$\frac{3}{2}$

Interpretation of Curl

In addition to translating between line integrals and flux integrals, Stokes' theorem can be used to justify the physical interpretation of curl that we have learned. Here we investigate the relationship between curl and circulation, and we use Stokes' theorem to state Faraday's law—an important law in electricity and magnetism that relates the curl of an electric field to the rate of change of a magnetic field.

Recall that if C is a closed curve and $\vec{\mathbf{F}}$ is a vector field defined on C, then the circulation of $\vec{\mathbf{F}}$ around C is line integral

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}. \tag{3.8.19}$$

If $\vec{\mathbf{F}}$ represents the velocity field of a fluid in space, then the circulation measures the tendency of the fluid to move in the direction of C.

Let $\vec{\mathbf{F}}$ be a continuous vector field and let D_r be a small disk of radius r with center P_0 (Figure 3.8.7). If D_r is small enough, then $(curl\, \vec{\mathbf{F}})(P) \approx (curl\, \vec{\mathbf{F}})(P_0)$ for all points P in D_r because the curl is continuous. Let C_r be the boundary circle of D_r: By Stokes' theorem,

$$\int_{C_r} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_{D_r} curl\, \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, d\vec{\mathbf{S}} \approx \iint_{D_r} (curl\, \vec{\mathbf{F}})(P_0) \cdot \vec{\mathbf{N}}(P_0)\, d\vec{\mathbf{S}}. \tag{3.8.20}$$

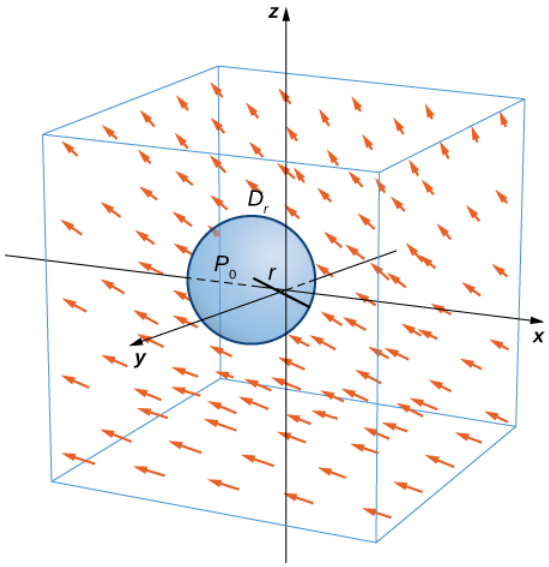


Figure **3.8.7**: Disk D_r is a small disk in a continuous vector field.

The quantity $(curl\,\vec{\mathbf{F}})(P_0)\cdot\vec{\mathbf{N}}(P_0)$ is constant, and therefore

$$\iint_{D_r}(curl\,\vec{\mathbf{F}})(P_0)\cdot\vec{\mathbf{N}}(P_0)\,d\vec{\mathbf{S}}=\pi r^2[(curl\,\vec{\mathbf{F}})(P_0)\cdot\vec{\mathbf{N}}(P_0)].$$

Thus

$$\int_{C_r}\vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}\approx\pi r^2[(curl\,\vec{\mathbf{F}})(P_0)\cdot\vec{\mathbf{N}}(P_0)],$$

and the approximation gets arbitrarily close as the radius shrinks to zero. Therefore Stokes' theorem implies that

$$(curl\,\vec{\mathbf{F}})(P_0)\cdot\vec{\mathbf{N}}(P_0)=\lim_{r\to0^+}\frac{1}{\pi r^2}\int_{C_r}\vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}.$$

This equation relates the curl of a vector field to the circulation. Since the area of the disk is πr^2, this equation says we can view the curl (in the limit) as the circulation per unit area. Recall that if $\vec{\mathbf{F}}$ is the velocity field of a fluid, then circulation

$$\oint_{C_r}\vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}=\oint_{C_r}\vec{\mathbf{F}}\cdot\vec{\mathbf{T}}\,ds \tag{3.8.21}$$

is a measure of the tendency of the fluid to move around C_r: The reason for this is that $\vec{\mathbf{F}}\cdot\vec{\mathbf{T}}$ is a component of $\vec{\mathbf{F}}$ in the direction of $\vec{\mathbf{T}}$, and the closer the direction of $\vec{\mathbf{F}}$ is to $\vec{\mathbf{T}}$, the larger the value of $\vec{\mathbf{F}}\cdot\vec{\mathbf{T}}$ (remember that if $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are vectors and $\vec{\mathbf{b}}$ is fixed, then the dot product $\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}$ is maximal when $\vec{\mathbf{a}}$ points in the same direction as $\vec{\mathbf{b}}$). Therefore, if $\vec{\mathbf{F}}$ is the velocity field of a fluid, then $curl\,\vec{\mathbf{F}}\cdot\vec{\mathbf{N}}$ is a measure of how the fluid rotates about axis $\vec{\mathbf{N}}$. The effect of the curl is largest about the axis that points in the direction of $\vec{\mathbf{N}}$, because in this case $curl\,\vec{\mathbf{F}}\cdot\vec{\mathbf{N}}$ is as large as possible.

To see this effect in a more concrete fashion, imagine placing a tiny paddlewheel at point P_0 (Figure **3.8.8**). The paddlewheel achieves its maximum speed when the axis of the wheel points in the direction of curl $\vec{\mathbf{F}}$. This justifies the interpretation of the curl we have learned: curl is a measure of the rotation in the vector field about the axis that points in the direction of the normal vector $\vec{\mathbf{N}}$, and Stokes' theorem justifies this interpretation.

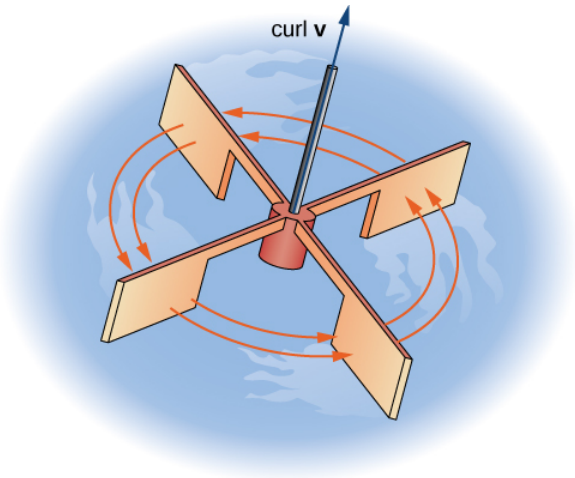


Figure **3.8.8**: To visualize curl at a point, imagine placing a tiny paddlewheel at that point in the vector field.

Now that we have learned about Stokes' theorem, we can discuss applications in the area of electromagnetism. In particular, we examine how we can use Stokes' theorem to translate between two equivalent forms of Faraday's law. Before stating the two forms of Faraday's law, we need some background terminology.

Let C be a closed curve that models a thin wire. In the context of electric fields, the wire may be moving over time, so we write $C(t)$ to represent the wire. At a given time t, curve $C(t)$ may be different from original curve C because of the movement of the wire, but we assume that $C(t)$ is a closed curve for all times t. Let $D(t)$ be a surface with $C(t)$ as its boundary, and orient $C(t)$ so that $D(t)$ has positive orientation. Suppose that $C(t)$is in a magnetic field $\vec{\mathbf{B}}(t)$ that can also change over time. In other words, $\vec{\mathbf{B}}$ has the form

$$\vec{\mathbf{B}}(x,y,z) = \langle P(x,y,z),\, Q(x,y,z),\, R(x,y,z)\rangle, \tag{3.8.22}$$

where P, Q, and R can all vary continuously over time. We can produce current along the wire by changing field $\vec{\mathbf{B}}(t)$ (this is a consequence of Ampere's law). Flux $\phi(t) = \iint_{D(t)} \vec{\mathbf{B}}(t)\cdot d\vec{\mathbf{S}}$ creates electric field $\vec{\mathbf{E}}(t)$ that does work. The integral form of Faraday's law states that

$$Work = \int_{C(t)} \vec{\mathbf{E}}(t)\cdot d\vec{\mathbf{r}} = -\frac{\partial \phi}{\partial t}. \tag{3.8.23}$$

In other words, the work done by $\vec{\mathbf{E}}$ is the line integral around the boundary, which is also equal to the rate of change of the flux with respect to time. The differential form of Faraday's law states that

$$curl\, \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}. \tag{3.8.24}$$

Using Stokes' theorem, we can show that the differential form of Faraday's law is a consequence of the integral form. By Stokes' theorem, we can convert the line integral in the integral form into surface integral

$$-\frac{\partial \phi}{\partial t} = \int_{C(t)} \vec{\mathbf{E}}(t)\cdot d\vec{\mathbf{r}} = \iint_{D(t)} curl\, \vec{\mathbf{E}}(t)\cdot d\vec{\mathbf{S}}. \tag{3.8.25}$$

Since

$$\phi(t) = \iint_{D(t)} B(t)\cdot d\vec{\mathbf{S}}, \tag{3.8.26}$$

then as long as the integration of the surface does not vary with time we also have

$$-\frac{\partial \phi}{\partial t} = \iint_{D(t)} -\frac{\partial \vec{\mathbf{B}}}{\partial t}\cdot d\vec{\mathbf{S}}. \tag{3.8.27}$$

Therefore,

$$\iint_{D(t)} -\frac{\partial \vec{\mathbf{B}}}{\partial t} \cdot d\vec{\mathbf{S}} = \iint_{D(t)} curl\, \vec{\mathbf{E}} \cdot d\vec{\mathbf{S}}. \tag{3.8.28}$$

To derive the differential form of Faraday's law, we would like to conclude that $curl\, \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}$: In general, the equation

$$\iint_{D(t)} -\frac{\partial \vec{\mathbf{B}}}{\partial t} \cdot d\vec{\mathbf{S}} = \iint_{D(t)} curl\, \vec{\mathbf{E}} \cdot d\vec{\mathbf{S}} \tag{3.8.29}$$

is not enough to conclude that $curl\, \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}$: The integral symbols do not simply "cancel out," leaving equality of the integrands. To see why the integral symbol does not just cancel out in general, consider the two single-variable integrals $\int_0^1 x\, dx$ and $\int_0^1 f(x)\, dx$, where

$$f(x) = \begin{cases} 1, & \text{if } 0 \le x \le 1/2 \\ 0, & \text{if } 1/2 \le x \le 1. \end{cases} \tag{3.8.30}$$

Both of these integrals equal $\frac{1}{2}$, so $\int_0^1 x\, dx = \int_0^1 f(x)\, dx$.

However, $x \neq f(x)$. Analogously, with our equation

$$\iint_{D(t)} -\frac{\partial \vec{\mathbf{B}}}{\partial t} \cdot d\vec{\mathbf{S}} = \iint_{D(t)} curl\, \vec{\mathbf{E}} \cdot d\vec{\mathbf{S}}, \tag{3.8.31}$$

we cannot simply conclude that $curl\, \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}$ just because their integrals are equal. However, in our context, equation

$$\iint_{D(t)} -\frac{\partial \vec{\mathbf{B}}}{\partial t} \cdot d\vec{\mathbf{S}} = \iint_{D(t)} curl\, \vec{\mathbf{E}} \cdot d\vec{\mathbf{S}} \tag{3.8.32}$$

is true for *any* region, however small (this is in contrast to the single-variable integrals just discussed). If $\vec{\mathbf{F}}$ and $\vec{\mathbf{G}}$ are three-dimensional vector fields such that

$$\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_S \vec{\mathbf{G}} \cdot d\vec{\mathbf{S}} \tag{3.8.33}$$

for any surface S, then it is possible to show that $\vec{\mathbf{F}} = \vec{\mathbf{G}}$ by shrinking the area of S to zero by taking a limit (the smaller the area of S, the closer the value of $\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$ to the value of $\vec{\mathbf{F}}$ at a point inside S). Therefore, we can let area $D(t)$ shrink to zero by taking a limit and obtain the differential form of Faraday's law:

$$curl\, \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}. \tag{3.8.34}$$

In the context of electric fields, the curl of the electric field can be interpreted as the negative of the rate of change of the corresponding magnetic field with respect to time.

✔ Example 3.8.4: Using Faraday's Law

Calculate the curl of electric field $\vec{\mathbf{E}}$ if the corresponding magnetic field is constant field $\vec{\mathbf{B}}(t) = \langle 1, -4, 2 \rangle$.

Solution

Since the magnetic field does not change with respect to time, $-\frac{\partial \vec{\mathbf{B}}}{\partial t} = \vec{\mathbf{0}}$. By Faraday's law, the curl of the electric field is therefore also zero.

Analysis

A consequence of Faraday's law is that the curl of the electric field corresponding to a constant magnetic field is always zero.

? Exercise 3.8.4

Calculate the curl of electric field $\vec{\mathbf{E}}$ if the corresponding magnetic field is $\vec{\mathbf{B}}(t) = \langle tx, ty, -2tz \rangle$, $0 \leq t < \infty$.

Hint

- Use the differential form of Faraday's law.
- Notice that the curl of the electric field does not change over time, although the magnetic field does change over time.

Answer

$curl\, \vec{\mathbf{E}} = \langle x, y, -2z \rangle$

Key Concepts

- Stokes' theorem relates a flux integral over a surface to a line integral around the boundary of the surface. Stokes' theorem is a higher dimensional version of Green's theorem, and therefore is another version of the Fundamental Theorem of Calculus in higher dimensions.
- Stokes' theorem can be used to transform a difficult surface integral into an easier line integral, or a difficult line integral into an easier surface integral.
- Through Stokes' theorem, line integrals can be evaluated using the simplest surface with boundary C.
- Faraday's law relates the curl of an electric field to the rate of change of the corresponding magnetic field. Stokes' theorem can be used to derive Faraday's law.

Key Equations

- **Stokes' theorem**

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_S curl\, \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$$

Glossary

Stokes' theorem
relates the flux integral over a surface S to a line integral around the boundary C of the surface S

surface independent
flux integrals of curl vector fields are surface independent if their evaluation does not depend on the surface but only on the boundary of the surface

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3.8E: Exercises

In exercises 1 - 6, without using Stokes' theorem, calculate directly both the flux of $curl\,\vec{\mathbf{F}}\cdot\vec{\mathbf{N}}$ over the given surface and the circulation integral around its boundary, assuming all are oriented clockwise.

1. $\vec{\mathbf{F}}(x,y,z)=y^2\,\hat{\mathbf{i}}+z^2\,\hat{\mathbf{j}}+x^2\,\hat{\mathbf{k}}$; S is the first-octant portion of plane $x+y+z=1$.

2. $\vec{\mathbf{F}}(x,y,z)=z\,\hat{\mathbf{i}}+x\,\hat{\mathbf{j}}+y\,\hat{\mathbf{k}}$; S is hemisphere $z=(a^2-x^2-y^2)^{1/2}$.

Answer

$$\iint_S (curl\,\vec{\mathbf{F}}\cdot\vec{\mathbf{N}})\,dS=\pi a^2$$

3. $\vec{\mathbf{F}}(x,y,z)=y^2\,\hat{\mathbf{i}}+2x\,\hat{\mathbf{j}}+5\,\hat{\mathbf{k}}$; S is hemisphere $z=(4-x^2-y^2)^{1/2}$.

4. $\vec{\mathbf{F}}(x,y,z)=z\,\hat{\mathbf{i}}+2x\,\hat{\mathbf{j}}+3y\,\hat{\mathbf{k}}$; S is upper hemisphere $z=\sqrt{9-x^2-y^2}$.

Answer

$$\iint_S (curl\,\vec{\mathbf{F}}\cdot\vec{\mathbf{N}})\,dS=18\pi$$

5. $\vec{\mathbf{F}}(x,y,z)=(x+2z)\,\hat{\mathbf{i}}+(y-x)\,\hat{\mathbf{j}}+(z-y)\,\hat{\mathbf{k}}$; S is a triangular region with vertices $(3,0,0)$, $(0,3/2,0)$, and $(0,0,3)$.

6. $\vec{\mathbf{F}}(x,y,z)=2y\,\hat{\mathbf{i}}+6z\,\hat{\mathbf{j}}+3x\,\hat{\mathbf{k}}$; S is a portion of paraboloid $z=4-x^2-y^2$ and is above the xy-plane.

Answer

$$\iint_S (curl\,\vec{\mathbf{F}}\cdot\vec{\mathbf{N}})\,dS=-8\pi$$

In exercises 7 - 9, use Stokes' theorem to evaluate $\iint_S (curl\,\vec{\mathbf{F}}\cdot\vec{\mathbf{N}})\,dS$ for the vector fields and surface.

7. $\vec{\mathbf{F}}(x,y,z)=xy\,\hat{\mathbf{i}}-z\,\hat{\mathbf{j}}$ and S is the surface of the cube $0\le x\le 1$, $0\le y\le 1$, $0\le z\le 1$, except for the face where $z=0$ and using the outward unit normal vector.

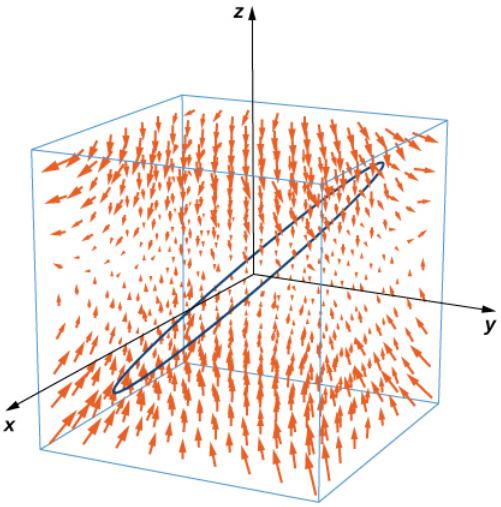
8. $\vec{\mathbf{F}}(x,y,z)=xy\,\hat{\mathbf{i}}+x^2\,\hat{\mathbf{j}}+z^2\,\hat{\mathbf{k}}$; and C is the intersection of paraboloid $z=x^2+y^2$ and plane $z=y$, and using the outward normal vector.

Answer

$$\iint_S (curl\,\vec{\mathbf{F}}\cdot\vec{\mathbf{N}})\,dS=0$$

9. $\vec{\mathbf{F}}(x,y,z)=4y\,\hat{\mathbf{i}}+z\,\hat{\mathbf{j}}+2y\,\hat{\mathbf{k}}$; and C is the intersection of sphere $x^2+y^2+z^2=4$ with plane $z=0$, and using the outward normal vector.

10. Use Stokes' theorem to evaluate $\int_C \left[2xy^2z\,dx+2x^2yz\,dy+(x^2y^2-2z)\,dz\right]$, where C is the curve given by $x=\cos t$, $y=\sin t$, $0\le t\le 2\pi$, traversed in the direction of increasing t.



Answer

$$\int_C \vec{\mathbf{F}} \cdot d\vec{S} = 0$$

11. [T] Use a computer algebraic system (CAS) and Stokes' theorem to approximate line integral $\int_C (y\,dx + z\,dy + x\,dz)$, where C is the intersection of plane $x + y = 2$ and surface $x^2 + y^2 + z^2 = 2(x + y)$, traversed counterclockwise viewed from the origin.

12. [T] Use a CAS and Stokes' theorem to approximate line integral $\int_C (3y\,dx + 2z\,dy - 5x\,dz)$, where C is the intersection of the xy-plane and hemisphere $z = \sqrt{1 - x^2 - y^2}$, traversed counterclockwise viewed from the top—that is, from the positive z-axis toward the xy-plane.

Answer

$$\int_C \vec{\mathbf{F}} \cdot d\vec{S} = -9.4248$$

13. [T] Use a CAS and Stokes' theorem to approximate line integral $\int_C [(1 + y)\,z\,dx + (1 + z)x\,dy + (1 + x)y\,dz]$, where C is a triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ oriented counterclockwise.

14. Use Stokes' theorem to evaluate $\iint_S curl\ \vec{\mathbf{F}} \cdot d\vec{S}$, where $\vec{\mathbf{F}}(x, y, z) = e^{xy} cos\ z\ \hat{\mathbf{i}} + x^2 z\ \hat{\mathbf{j}} + xy\ \hat{\mathbf{k}}$, and S is half of sphere $x = \sqrt{1 - y^2 - z^2}$, oriented out toward the positive x-axis.

Answer

$$\iint_S \vec{\mathbf{F}} \cdot d\vec{S} = 0$$

15. [T] Use a CAS and Stokes' theorem to evaluate $\iint_S (curl\ \vec{\mathbf{F}} \cdot \vec{\mathbf{N}})\,dS$, where $\vec{\mathbf{F}}(x, y, z) = x^2 y\,\hat{\mathbf{i}} + xy^2\,\hat{\mathbf{j}} + z^3\,\hat{\mathbf{k}}$ and C is the curve of the intersection of plane $3x + 2y + z = 6$ and cylinder $x^2 + y^2 = 4$, oriented clockwise when viewed from above.

16. [T] Use a CAS and Stokes' theorem to evaluate $\iint_S curl\ \vec{\mathbf{F}} \cdot d\vec{S}$, where $\vec{\mathbf{F}}(x, y, z) = \left(\sin(y + z) - yx^2 - \frac{y^3}{3}\right)\hat{\mathbf{i}} + x\ \cos(y + z)\,\hat{\mathbf{j}} + \cos(2y)\,\hat{\mathbf{k}}$ and S consists of the top and the four sides but not the bottom of the cube with vertices $(\pm 1, \pm 1, \pm 1)$, oriented outward.

Answer

$$\iint_S curl\ \vec{\mathbf{F}} \cdot d\vec{S} = 2.6667$$

17. [T] Use a CAS and Stokes' theorem to evaluate $\iint_S curl\ \vec{\mathbf{F}} \cdot d\vec{S}$, where $\vec{\mathbf{F}}(x, y, z) = z^2\,\hat{\mathbf{i}} + 3xy\,\hat{\mathbf{j}} + x^3 y^3\,\hat{\mathbf{k}}$ and S is the top part of $z = 5 - x^2 - y^2$ above plane $z = 1$ and S is oriented upward.

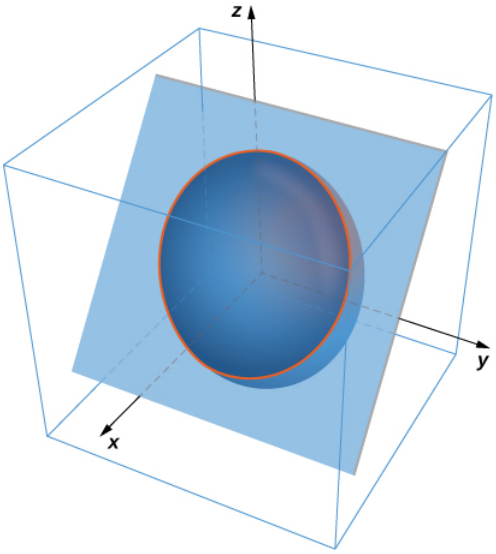
18. Use Stokes' theorem to evaluate $\iint_S (curl\, \vec{\mathbf{F}} \cdot \vec{\mathbf{N}})dS$, where $\vec{\mathbf{F}}(x,y,z) = z^2\,\hat{\mathbf{i}} + y^2\,\hat{\mathbf{j}} + x\,\hat{\mathbf{k}}$ and S is a triangle with vertices $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ with counterclockwise orientation.

Answer

$$\iint_S (curl\, \vec{\mathbf{F}} \cdot \vec{\mathbf{N}})dS = -\frac{1}{6}$$

19. Use Stokes' theorem to evaluate line integral $\int_C (z\,dx + x\,dy + y\,dz)$, where C is a triangle with vertices $(3,0,0)$, $(0,0,2)$, and $(0,6,0)$ traversed in the given order.

20. Use Stokes' theorem to evaluate $\int_C \left(\frac{1}{2}y^2\,dx + z\,dy + x\,dz\right)$, where C is the curve of intersection of plane $x + z = 1$ and ellipsoid $x^2 + 2y^2 + z^2 = 1$, oriented clockwise from the origin.



Answer

$$\int_C \left(\frac{1}{2}y^2\,dx + z\,dy + x\,dz\right) = -\frac{\pi}{4}$$

21. Use Stokes' theorem to evaluate $\iint_S (curl\, \vec{\mathbf{F}} \cdot \vec{\mathbf{N}})dS$, where $\vec{\mathbf{F}}(x,y,z) = x\,\hat{\mathbf{i}} + y^2\,\hat{\mathbf{j}} + ze^{xy}\,\hat{\mathbf{k}}$ and S is the part of surface $z = 1 - x^2 - 2y^2$ with $z \geq 0$, oriented counterclockwise.

22. Use Stokes' theorem for vector field $\vec{\mathbf{F}}(x,y,z) = z\,\hat{\mathbf{i}} + 3x\,\hat{\mathbf{j}} + 2z\hat{\mathbf{k}}$ where S is surface $z = 1 - x^2 - 2y^2$, $z \geq 0$, C is boundary circle $x^2 + y^2 = 1$, and S is oriented in the positive z-direction.

Answer

$$\iint_S (curl\, \vec{\mathbf{F}} \cdot \vec{\mathbf{N}})dS = -3\pi$$

23. Use Stokes' theorem for vector field $\vec{\mathbf{F}}(x,y,z) = -\frac{3}{2}y^2\,\hat{\mathbf{i}} - 2xy\,\hat{\mathbf{j}} + yz\hat{\mathbf{k}}$, where S is that part of the surface of plane $x + y + z = 1$ contained within triangle C with vertices $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, traversed counterclockwise as viewed from above.

24. A certain closed path C in plane $2x + 2y + z = 1$ is known to project onto unit circle $x^2 + y^2 = 1$ in the xy-plane. Let C be a constant and let $\vec{\mathbf{R}}(x,y,z) = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$. Use Stokes' theorem to evaluate $\int_C (c\,\hat{\mathbf{k}} \times \vec{\mathbf{R}}) \cdot dS$.

Answer

$$\int_C (c\,\hat{\mathbf{k}} \times \vec{\mathbf{R}}) \cdot dS = 2\pi c$$

25. Use Stokes' theorem and let C be the boundary of surface $z = x^2 + y^2$ with $0 \le x \le 2$ and $0 \le y \le 1$ oriented with upward facing normal. Define $\vec{\mathbf{F}}(x, y, z) = \left(\sin(x^3) + xz\right)\hat{\mathbf{i}} + (x - yz)\hat{\mathbf{j}} + \cos(z^4)\hat{\mathbf{k}}$ and evaluate $\int_C \vec{\mathbf{F}} \cdot d\vec{S}$.

26. Let S be hemisphere $x^2 + y^2 + z^2 = 4$ with $z \ge 0$, oriented upward. Let $\vec{\mathbf{F}}(x, y, z) = x^2 e^{yz}\hat{\mathbf{i}} + y^2 e^{xz}\hat{\mathbf{j}} + z^2 e^{xy}\hat{\mathbf{k}}$ be a vector field. Use Stokes' theorem to evaluate $\iint_S curl\, \vec{\mathbf{F}} \cdot d\vec{S}$.

Answer

$$\iint_S curl\, \vec{\mathbf{F}} \cdot d\vec{S} = 0$$

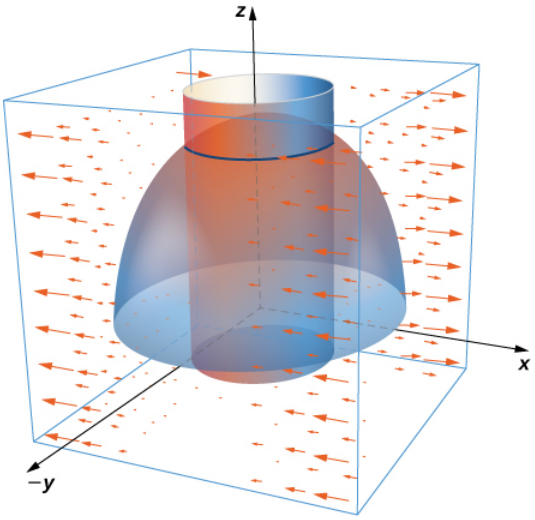
27. Let $\vec{\mathbf{F}}(x, y, z) = xy\hat{\mathbf{i}} + (e^{z^2} + y)\hat{\mathbf{j}} + (x + y)\hat{\mathbf{k}}$ and let S be the graph of function $y = \frac{x^2}{9} + \frac{z^2}{9} - 1$ with $z \le 0$ oriented so that the normal vector S has a positive y component. Use Stokes' theorem to compute integral $\iint_S curl\, \vec{\mathbf{F}} \cdot d\vec{S}$.

28. Use Stokes' theorem to evaluate $\oint \vec{\mathbf{F}} \cdot d\vec{S}$, where $\vec{\mathbf{F}}(x, y, z) = y\hat{\mathbf{i}} + z\hat{\mathbf{j}} + x\hat{\mathbf{k}}$ and C is a triangle with vertices $(0, 0, 0)$, $(2, 0, 0)$ and $0, -2, 2)$ oriented counterclockwise when viewed from above.

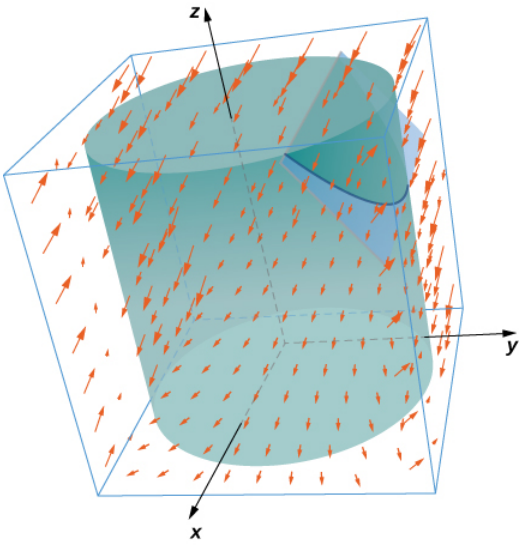
Answer

$$\oint \vec{\mathbf{F}} \cdot d\vec{S} = -4$$

29. Use the surface integral in Stokes' theorem to calculate the circulation of field $\vec{\mathbf{F}}$, $\vec{\mathbf{F}}(x, y, z) = x^2 y^3\hat{\mathbf{i}} + \hat{\mathbf{j}} + z\hat{\mathbf{k}}$ around C, which is the intersection of cylinder $x^2 + y^2 = 4$ and hemisphere $x^2 + y^2 + z^2 = 16$, $z \ge 0$, oriented counterclockwise when viewed from above.



30. Use Stokes' theorem to compute $\iint_S curl\, \vec{\mathbf{F}} \cdot d\vec{S}$. where $\vec{\mathbf{F}}(x, y, z) = \hat{\mathbf{i}} + xy^2\hat{\mathbf{j}} + xy^2\hat{\mathbf{k}}$ and S is a part of plane $y + z = 2$ inside cylinder $x^2 + y^2 = 1$ and oriented counterclockwise.



Answer

$$\iint_S curl\, \vec{\mathbf{F}} \cdot dS = 0$$

31. Use Stokes' theorem to evaluate $\iint_S curl\, \vec{\mathbf{F}} \cdot dS$, where $\vec{\mathbf{F}}(x, y, z) = -y^2\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}} + z^2\,\hat{\mathbf{k}}$ and S is the part of plane $x + y + z = 1$ in the positive octant and oriented counterclockwise $x \geq 0,\ y \geq 0,\ z \geq 0$.

32. Let $\vec{\mathbf{F}}(x, y, z) = xy\,\hat{\mathbf{i}} + 2z\,\hat{\mathbf{j}} - 2y\,\hat{\mathbf{k}}$ and let C be the intersection of plane $x + z = 5$ and cylinder $x^2 + y^2 = 9$, which is oriented counterclockwise when viewed from the top. Compute the line integral of $\vec{\mathbf{F}}$ over C using Stokes' theorem.

Answer

$$\iint_S curl\, \vec{\mathbf{F}} \cdot dS = -36\pi$$

33. [T] Use a CAS and let $\vec{\mathbf{F}}(x, y, z) = xy^2\,\hat{\mathbf{i}} + (yz - x)\,\hat{\mathbf{j}} + e^{yxz}\,\hat{\mathbf{k}}$. Use Stokes' theorem to compute the surface integral of curl $\vec{\mathbf{F}}$ over surface S with inward orientation consisting of cube $[0, 1] \times [0, 1] \times [0, 1]$ with the right side missing.

34. Let S be ellipsoid $\frac{x^2}{4} + \frac{y^2}{9} + z^2 = 1$ oriented counterclockwise and let $\vec{\mathbf{F}}$ be a vector field with component functions that have continuous partial derivatives.

Answer

$$\iint_S curl\, \vec{\mathbf{F}} \cdot \vec{\mathbf{N}} = 0$$

35. Let S be the part of paraboloid $z = 9 - x^2 - y^2$ with $z \geq 0$. Verify Stokes' theorem for vector field $\vec{\mathbf{F}}(x, y, z) = 3z\,\hat{\mathbf{i}} + 4x\,\hat{\mathbf{j}} + 2y\,\hat{\mathbf{k}}$.

36. [T] Use a CAS and Stokes' theorem to evaluate $\oint \vec{\mathbf{F}} \cdot dS$, if $\vec{\mathbf{F}}(x, y, z) = (3z - \sin x)\,\hat{\mathbf{i}} + (x^2 + e^y)\,\hat{\mathbf{j}} + (y^3 - \cos z)\,\hat{\mathbf{k}}$, where C is the curve given by $x = \cos t,\ y = \sin t,\ z = 1;\ 0 \leq t \leq 2\pi$.

Answer

$$\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 0$$

37. [T] Use a CAS and Stokes' theorem to evaluate $\vec{\mathbf{F}}(x, y, z) = 2y\,\hat{\mathbf{i}} + e^z\,\hat{\mathbf{j}} - \arctan x\,\hat{\mathbf{k}}$ with S as a portion of paraboloid $z = 4 - x^2 - y^2$ cut off by the xy-plane oriented counterclockwise.

38. [T] Use a CAS to evaluate $\iint_S curl(F) \cdot dS$, where $\vec{\mathbf{F}}(x, y, z) = 2z\,\hat{\mathbf{i}} + 3x\,\hat{\mathbf{j}} + 5y\,\hat{\mathbf{k}}$ and S is the surface parametrically by $\vec{\mathbf{r}}(r, \theta) = r\cos\theta\,\hat{\mathbf{i}} + r\sin\theta\,\hat{\mathbf{j}} + (4 - r^2)\,\hat{\mathbf{k}}\ (0 \leq \theta \leq 2\pi,\ 0 \leq r \leq 3)$.

Answer

$$\iint_S curl(F) \cdot dS = 84.8230$$

39. Let S be paraboloid $z = a(1 - x^2 - y^2)$, for $z \geq 0$, where $a > 0$ is a real number. Let $\vec{\mathbf{F}}(x, y, z) = \langle x - y,\ y + z,\ z - x \rangle$. For what value(s) of a (if any) does $\iint_S (\vec{\nabla} \times \vec{\mathbf{F}}) \cdot \vec{\mathbf{n}}\, dS$ have its maximum value?

For application exercises 40 - 41, the goal is to evaluate $A = \iint_S (\vec{\nabla} \times \vec{\mathbf{F}}) \cdot \vec{\mathbf{n}}\, dS$, where $\vec{\mathbf{F}} = \langle xz,\ -xz,\ xy \rangle$ and S is the upper half of ellipsoid $x^2 + y^2 + 8z^2 = 1$, where $z \geq 0$.

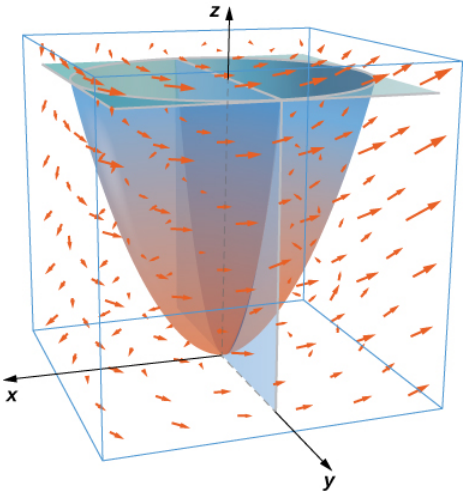
40. Evaluate a surface integral over a more convenient surface to find the value of A.

Answer

$$A = \iint_S (\vec{\nabla} \times \vec{\mathbf{F}}) \cdot \vec{\mathbf{n}}\, dS = 0$$

41. Evaluate A using a line integral.

42. Take paraboloid $z = x^2 + y^2$, for $0 \leq z \leq 4$, and slice it with plane $y = 0$. Let S be the surface that remains for $y \geq 0$, including the planar surface in the xz-plane. Let C be the semicircle and line segment that bounded the cap of S in plane $z = 4$ with counterclockwise orientation. Let $\vec{\mathbf{F}} = \langle 2z + y,\ 2x + z,\ 2y + x \rangle$. Evaluate $\iint_S (\vec{\nabla} \times \vec{\mathbf{F}}) \cdot \vec{\mathbf{n}}\, dS$.



Answer

$$\iint_S (\vec{\nabla} \times \vec{\mathbf{F}}) \cdot \vec{\mathbf{n}}\, dS = 2\pi$$

For exercises 43 - 45, let S be the disk enclosed by curve C : $\vec{\mathbf{r}}(t) = \langle \cos\varphi \cos t,\ \sin t,\ \sin\varphi \cos t \rangle$, for $0 \leq t \leq 2\pi$, where $0 \leq \varphi \leq \frac{\pi}{2}$ is a fixed angle.

43. What is the length of C in terms of φ?

44. What is the circulation of C of vector field $\vec{\mathbf{F}} = \langle -y,\ -z,\ x \rangle$ as a function of φ?

Answer

$$C = \pi(\cos\varphi - \sin\varphi)$$

45. For what value of φ is the circulation a maximum?

46. Circle C in plane $x + y + z = 8$ has radius 4 and center $(2, 3, 3)$. Evaluate $\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ for $\vec{\mathbf{F}} = \langle 0,\ -z,\ 2y \rangle$, where C has a counterclockwise orientation when viewed from above.

Answer

$$\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 48\pi$$

47. Velocity field $v = \langle 0, 1 - x^2, 0 \rangle$, for $|x| \leq 1$ and $|z| \leq 1$, represents a horizontal flow in the y-direction. Compute the curl of $\vec{\mathbf{v}}$ in a clockwise rotation.

48. Evaluate integral $\iint_S (\vec{\nabla} \times \vec{\mathbf{F}}) \cdot \vec{\mathbf{n}}\, dS$, where $\vec{\mathbf{F}} = -xz\hat{\mathbf{i}} + yz\hat{\mathbf{j}} + xye^z\hat{\mathbf{k}}$ and S is the cap of paraboloid $z = 5 - x^2 - y^2$ above plane $z = 3$, and $\vec{\mathbf{n}}$ points in the positive z-direction on S.

Answer

$$\iint_S (\vec{\nabla} \times \vec{\mathbf{F}}) \cdot \vec{\mathbf{n}} = 0$$

In exercises 49 - 50, use Stokes' theorem to find the circulation of the following vector fields around any smooth, simple closed curve C.

49. $\vec{\mathbf{F}} = \vec{\nabla}(x \sin ye^z)$

50. $\vec{\mathbf{F}} = \langle y^2z^3,\ z2xyz^3, 3xy^2z^2 \rangle$

Answer

0

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3.9: The Divergence Theorem

Learning Objectives

- Explain the meaning of the divergence theorem.
- Use the divergence theorem to calculate the flux of a vector field.
- Apply the divergence theorem to an electrostatic field.

We have examined several versions of the Fundamental Theorem of Calculus in higher dimensions that relate the integral around an oriented boundary of a domain to a "derivative" of that entity on the oriented domain. In this section, we state the divergence theorem, which is the final theorem of this type that we will study. The divergence theorem has many uses in physics; in particular, the divergence theorem is used in the field of partial differential equations to derive equations modeling heat flow and conservation of mass. We use the theorem to calculate flux integrals and apply it to electrostatic fields.

Overview of Theorems

Before examining the divergence theorem, it is helpful to begin with an overview of the versions of the Fundamental Theorem of Calculus we have discussed:

1. The Fundamental Theorem of Calculus:

$$\int_a^b f'(x)\,dx = f(b) - f(a). \tag{3.9.1}$$

   This theorem relates the integral of derivative f' over line segment $[a, b]$ along the x-axis to a difference of f evaluated on the boundary.

2. The Fundamental Theorem for Line Integrals:

$$\int_C \vec{\nabla} f \cdot d\vec{\mathbf{r}} = f(P_1) - f(P_0), \tag{3.9.2}$$

   where P_0 is the initial point of C and P_1 is the terminal point of C. The Fundamental Theorem for Line Integrals allows path C to be a path in a plane or in space, not just a line segment on the x-axis. If we think of the gradient as a derivative, then this theorem relates an integral of derivative ∇f over path C to a difference of f evaluated on the boundary of C.

3. Green's theorem, circulation form:

$$\iint_D (Q_x - P_y)\,dA = \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}. \tag{3.9.3}$$

   Since $Q_x - P_y = \text{curl } \vec{\mathbf{F}} \cdot \hat{\mathbf{k}}$ and curl is a derivative of sorts, Green's theorem relates the integral of derivative curl $\vec{\mathbf{F}}$ over planar region D to an integral of $\vec{\mathbf{F}}$ over the boundary of D.

4. Green's theorem, flux form:

$$\iint_D (P_x + Q_y)\,dA = \int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\,dS. \tag{3.9.4}$$

   Since $P_x + Q_y = \text{div } \vec{\mathbf{F}}$ and divergence is a derivative of sorts, the flux form of Green's theorem relates the integral of derivative div $\vec{\mathbf{F}}$ over planar region D to an integral of $\vec{\mathbf{F}}$ over the boundary of D.

5. Stokes' theorem:

$$\iint_S curl\, \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}. \tag{3.9.5}$$

   If we think of the curl as a derivative of sorts, then Stokes' theorem relates the integral of derivative curl $\vec{\mathbf{F}}$ over surface S (not necessarily planar) to an integral of $\vec{\mathbf{F}}$ over the boundary of S.

Stating the Divergence Theorem

The divergence theorem follows the general pattern of these other theorems. If we think of divergence as a derivative of sorts, then the divergence theorem relates a triple integral of derivative div $\vec{\mathbf{F}}$ over a solid to a flux integral of $\vec{\mathbf{F}}$ over the boundary of the solid. More specifically, the divergence theorem relates a flux integral of vector field $\vec{\mathbf{F}}$ over a closed surface S to a triple integral of the divergence of $\vec{\mathbf{F}}$ over the solid enclosed by S.

The Divergence Theorem

Let S be a piecewise, smooth closed surface that encloses solid E in space. Assume that S is oriented outward, and let $\vec{\mathbf{F}}$ be a vector field with continuous partial derivatives on an open region containing E (Figure 3.9.1). Then

$$\iiint_E \operatorname{div} \vec{\mathbf{F}}\, dV = \iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}. \tag{3.9.6}$$

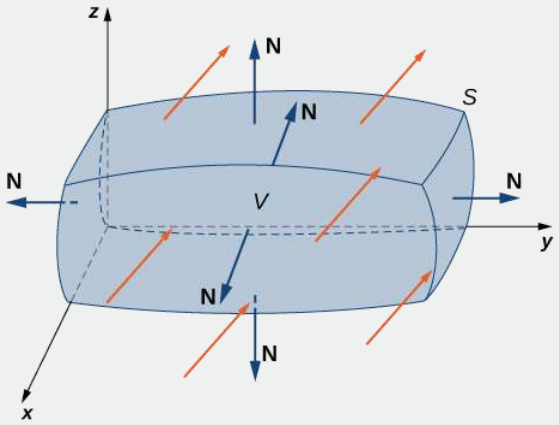


Figure 3.9.1: The divergence theorem relates a flux integral across a closed surface S to a triple integral over solid E enclosed by the surface.

Recall that the flux form of Green's theorem states that

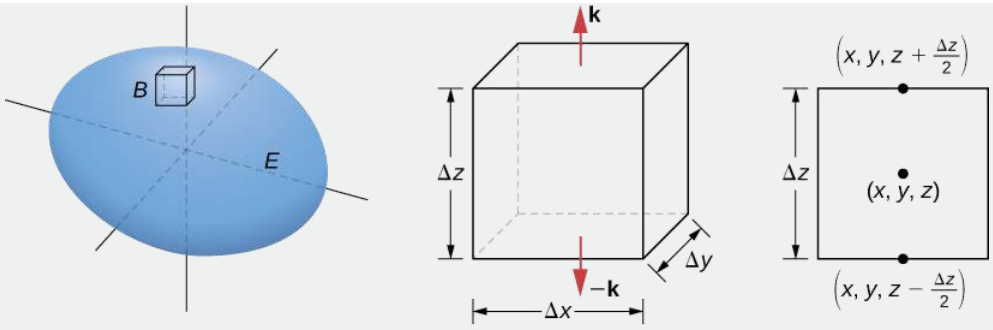
$$\iint_D \operatorname{div} \vec{\mathbf{F}}\, dA = \int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, dS. \tag{3.9.7}$$

Therefore, the divergence theorem is a version of Green's theorem in one higher dimension.

The proof of the divergence theorem is beyond the scope of this text. However, we look at an informal proof that gives a general feel for why the theorem is true, but does not prove the theorem with full rigor. This explanation follows the informal explanation given for why Stokes' theorem is true.

Proof

Let B be a small box with sides parallel to the coordinate planes inside E (Figure 3.9.2a). Let the center of B have coordinates (x, y, z) and suppose the edge lengths are Δx, Δy, and Δz. (Figure 3.9.1b). The normal vector out of the top of the box is $\hat{\mathbf{k}}$ and the normal vector out of the bottom of the box is $-\hat{\mathbf{k}}$. The dot product of $\vec{\mathbf{F}} = \langle P, Q, R \rangle$ with $\hat{\mathbf{k}}$ is R and the dot product with $-\hat{\mathbf{k}}$ is $-R$. The area of the top of the box (and the bottom of the box) ΔS is $\Delta x \Delta y$.



(a) (b) (c)

Figure 3.9.2: (a) A small box B inside surface E has sides parallel to the coordinate planes. (b) Box B has side lengths Δx, Δy, and Δz (c) If we look at the side view of B, we see that, since (x, y, z) is the center of the box, to get to the top of the box we must travel a vertical distance of $\frac{\Delta z}{2}$ up from (x, y, z). Similarly, to get to the bottom of the box we must travel a distance $\frac{\Delta z}{2}$ down from (x, y, z).

The flux out of the top of the box can be approximated by $R\left(x,\, y,\, z+\frac{\Delta z}{2}\right)\,\Delta x\,\Delta y$ (Figure 3.9.2c) and the flux out of the bottom of the box is $-R\left(x,\, y,\, z-\frac{\Delta z}{2}\right)\,\Delta x\,\Delta y$. If we denote the difference between these values as ΔR, then the net flux in the vertical direction can be approximated by $\Delta R\,\Delta x\,\Delta y$. However,

$$\Delta R\,\Delta x\,\Delta y = \left(\frac{\Delta R}{\Delta z}\right)\,\Delta x\,\Delta y\Delta z \approx \left(\frac{\partial R}{\partial z}\right)\,\Delta V.$$

Therefore, the net flux in the vertical direction can be approximated by $\left(\frac{\partial R}{\partial z}\right)\Delta V$. Similarly, the net flux in the x-direction can be approximated by $\left(\frac{\partial P}{\partial x}\right)\,\Delta V$ and the net flux in the y-direction can be approximated by $\left(\frac{\partial Q}{\partial y}\right)\,\Delta V$. Adding the fluxes in all three directions gives an approximation of the total flux out of the box:

$$\text{Total flux } \approx \left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)\Delta V = \operatorname{div}\vec{\mathbf{F}}\,\Delta V.$$

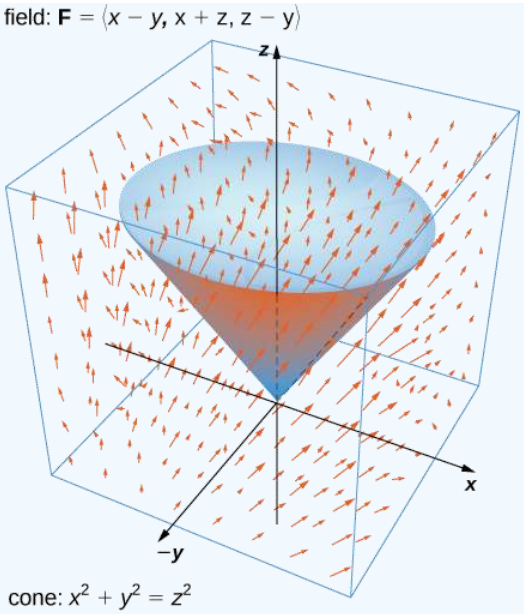
This approximation becomes arbitrarily close to the value of the total flux as the volume of the box shrinks to zero.

The sum of $\operatorname{div}\vec{\mathbf{F}}\,\Delta V$ over all the small boxes approximating E is approximately $\iiint_E \operatorname{div}\vec{\mathbf{F}}\,dV$. On the other hand, the sum of $\operatorname{div}\vec{\mathbf{F}}\,\Delta V$ over all the small boxes approximating E is the sum of the fluxes over all these boxes. Just as in the informal proof of Stokes' theorem, adding these fluxes over all the boxes results in the cancelation of a lot of the terms. If an approximating box shares a face with another approximating box, then the flux over one face is the negative of the flux over the shared face of the adjacent box. These two integrals cancel out. When adding up all the fluxes, the only flux integrals that survive are the integrals over the faces approximating the boundary of E. As the volumes of the approximating boxes shrink to zero, this approximation becomes arbitrarily close to the flux over S.

□

Example 3.9.1: Verifying the Divergence Theorem

Verify the divergence theorem for vector field $\vec{\mathbf{F}} = \langle x-y,\, x+z,\, z-y\rangle$ and surface S that consists of cone $x^2+y^2=z^2$, $0 \le z \le 1$, and the circular top of the cone (see the following figure). Assume this surface is positively oriented.



Solution

Let E be the solid cone enclosed by S. To verify the theorem for this example, we show that

$$\iiint_E \operatorname{div} \vec{\mathbf{F}}\, dV = \iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$$

by calculating each integral separately.

To compute the triple integral, note that $\operatorname{div} \vec{\mathbf{F}} = P_x + Q_y + R_z = 2$, and therefore the triple integral is

$$\begin{aligned}\iiint_E \operatorname{div} \vec{\mathbf{F}}\, dV &= 2 \iiint_E dV \\ &= 2\,(volume\ of\ E).\end{aligned}$$

The volume of a right circular cone is given by $\pi r^2 \frac{h}{3}$. In this case, $h = r = 1$. Therefore,

$$\iiint_E \operatorname{div} \vec{\mathbf{F}}\, dV = 2\,(volume\ of\ E) = \frac{2\pi}{3}.$$

To compute the flux integral, first note that S is piecewise smooth; S can be written as a union of smooth surfaces. Therefore, we break the flux integral into two pieces: one flux integral across the circular top of the cone and one flux integral across the remaining portion of the cone. Call the circular top S_1 and the portion under the top S_2. We start by calculating the flux across the circular top of the cone. Notice that S_1 has parameterization

$$\vec{\mathbf{r}}(u, v) = \langle u \cos v,\ u \sin v,\ 1\rangle,\ 0 \le u \le 1,\ 0 \le v \le 2\pi.$$

Then, the tangent vectors are $\vec{\mathbf{t}}_u = \langle \cos v,\ \sin v,\ 0\rangle$ and $\vec{\mathbf{t}}_v = \langle -u \sin v,\ u \cos v, 0\rangle$. Therefore, the flux across S_1 is

$$\begin{aligned}\iint_{S_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} &= \int_0^1 \int_0^{2\pi} \vec{\mathbf{F}}(\vec{\mathbf{r}}(u, v)) \cdot (\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v)\, dA \\ &= \int_0^1 \int_0^{2\pi} \langle u \cos v - u \sin v,\ u \cos v + 1,\ 1 - u \sin v\rangle \cdot \langle 0, 0, u\rangle\, dv\, du \\ &= \int_0^1 \int_0^{2\pi} u - u^2 \sin v\, dv du \\ &= \pi.\end{aligned}$$

We now calculate the flux over S_2. A parameterization of this surface is

$$\vec{\mathbf{r}}(u, v) = \langle u \cos v,\ u \sin v,\ u\rangle,\ 0 \le u \le 1,\ 0 \le v \le 2\pi.$$

The tangent vectors are $\vec{\mathbf{t}}_u = \langle \cos v,\ \sin v,\ 1\rangle$ and $\vec{\mathbf{t}}_v = \langle -u\sin v,\ u\cos v, 0\rangle$, so the cross product is

$$\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v = \langle -u\cos v,\ -u\sin v,\ u\rangle.$$

Notice that the negative signs on the x and y components induce the negative (or inward) orientation of the cone. Since the surface is positively oriented, we use vector $\vec{\mathbf{t}}_v \times \vec{\mathbf{t}}_u = \langle u\cos v,\ u\sin v,\ -u\rangle$ in the flux integral. The flux across S_2 is then

$$\begin{aligned}\iint_{S_2} \vec{\mathbf{F}}\cdot d\vec{\mathbf{S}} &= \int_0^1\int_0^{2\pi} \vec{\mathbf{F}}(\vec{\mathbf{r}}(u,v))\cdot(\vec{\mathbf{t}}_u\times\vec{\mathbf{t}}_v)\,dA\\ &= \int_0^1\int_0^{2\pi} \langle u\cos v - u\sin v,\ u\cos v + u,\ u - u\sin v\rangle\cdot\langle u\cos v,\ u\sin v,\ -u\rangle\,dv\,du\\ &= \int_0^1\int_0^{2\pi} u^2\cos^2 v + 2u^2\sin v - u^2\,dv\,du\\ &= -\frac{\pi}{3}\end{aligned}$$

The total flux across S is

$$\iint_S \vec{\mathbf{F}}\cdot d\vec{\mathbf{S}} = \iint_{S_1}\vec{\mathbf{F}}\cdot d\vec{\mathbf{S}} + \iint_{S_2}\vec{\mathbf{F}}\cdot d\vec{\mathbf{S}} = \frac{2\pi}{3} = \iiint_E \operatorname{div}\vec{\mathbf{F}}\,dV,$$

and we have verified the divergence theorem for this example.

? Exercise 3.9.1

Verify the divergence theorem for vector field $\vec{\mathbf{F}}(x,y,z) = \langle x+y+z,\ y,\ 2x-y\rangle$ and surface S given by the cylinder $x^2+y^2=1$, $0\le z\le 3$ plus the circular top and bottom of the cylinder. Assume that S is positively oriented.

Hint

Calculate both the flux integral and the triple integral with the divergence theorem and verify they are equal.

Answer

Both integrals equal 6π.

Recall that the divergence of continuous field $\vec{\mathbf{F}}$ at point P is a measure of the "outflowing-ness" of the field at P. If $\vec{\mathbf{F}}$ represents the velocity field of a fluid, then the divergence can be thought of as the rate per unit volume of the fluid flowing out less the rate per unit volume flowing in. The divergence theorem confirms this interpretation. To see this, let P be a point and let B_r be a ball of small radius r centered at P (Figure 3.9.3). Let S_r be the boundary sphere of B_r. Since the radius is small and $\vec{\mathbf{F}}$ is continuous, $\operatorname{div}\vec{\mathbf{F}}(Q) \approx \operatorname{div}\vec{\mathbf{F}}(P)$ for all other points Q in the ball. Therefore, the flux across S_r can be approximated using the divergence theorem:

$$\iint_{S_r}\vec{\mathbf{F}}\cdot d\vec{\mathbf{S}} = \iiint_{B_r}\operatorname{div}\vec{\mathbf{F}}\,dV \approx \iiint_{B_r}\operatorname{div}\vec{\mathbf{F}}(P)\,dV.$$

Since $divF(P)$ is a constant,

$$\iiint_{B_r}\operatorname{div}\vec{\mathbf{F}}(P)\,dV = \operatorname{div}\vec{\mathbf{F}}(P)\,V(B_r).$$

Therefore, flux

$$\iint_{S_r}\vec{\mathbf{F}}\cdot d\vec{\mathbf{S}} \tag{3.9.8}$$

can be approximated by $\vec{\mathbf{F}}(P)\,V(B_r)$. This approximation gets better as the radius shrinks to zero, and therefore

$$\operatorname{div}\vec{\mathbf{F}}(P) = \lim_{r\to 0}\frac{1}{V(B_r)}\iint_{S_r}\vec{\mathbf{F}}\cdot d\vec{\mathbf{S}}.$$

This equation says that the divergence at P is the net rate of outward flux of the fluid per unit volume.

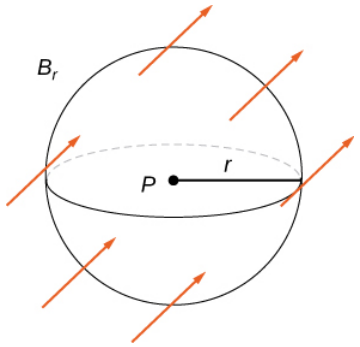


Figure 3.9.3: Ball B_r of small radius r centered at P.

Using the Divergence Theorem

The divergence theorem translates between the flux integral of closed surface S and a triple integral over the solid enclosed by S. Therefore, the theorem allows us to compute flux integrals or triple integrals that would ordinarily be difficult to compute by translating the flux integral into a triple integral and vice versa.

Example 3.9.2: Applying the Divergence Theorem

Calculate the surface integral

$$\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}},$$

where S is cylinder $x^2 + y^2 = 1$, $0 \le z \le 2$, including the circular top and bottom, and $\vec{\mathbf{F}} = \left\langle \frac{x^3}{3} + yz,\ \frac{y^3}{3} - \sin(xz),\ z - x - y \right\rangle$.

Solution

We could calculate this integral without the divergence theorem, but the calculation is not straightforward because we would have to break the flux integral into three separate integrals: one for the top of the cylinder, one for the bottom, and one for the side. Furthermore, each integral would require parameterizing the corresponding surface, calculating tangent vectors and their cross product..

By contrast, the divergence theorem allows us to calculate the single triple integral

$$\iiint_E \operatorname{div} \vec{\mathbf{F}}\, dV,$$

where E is the solid enclosed by the cylinder. Using the divergence theorem (Equation 3.9.6) and converting to cylindrical coordinates, we have

$$\begin{aligned}
\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} &= \iiint_E \operatorname{div} \vec{\mathbf{F}}\, dV, \\
&= \iiint_E (x^2 + y^2 + 1)\, dV \\
&= \int_0^{2\pi} \int_0^1 \int_0^2 (r^2 + 1)\, r\, dz\, dr\, d\theta \\
&= \frac{3}{2} \int_0^{2\pi} d\theta \\
&= 3\pi.
\end{aligned}$$

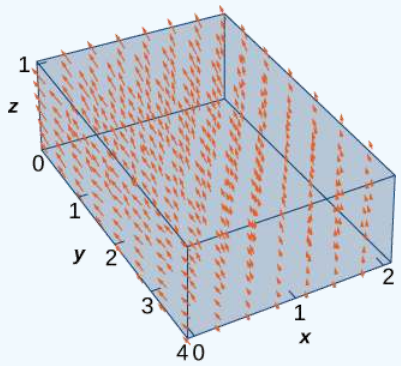


Exercise 3.9.2

Use the divergence theorem to calculate flux integral

$$\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}},$$

where S is the boundary of the box given by $0 \le x \le 2,\ 0 \le y \le 4,\ 0 \le z \le 1$ and $\vec{\mathbf{F}} = \langle x^2 + yz,\ y - z,\ 2x + 2y + 2z \rangle$ (see the following figure).



Hint

Calculate the corresponding triple integral.

Answer

40

Example 3.9.3: Applying the Divergence Theorem

Let $\vec{\mathbf{v}} = \langle -\frac{y}{z},\ \frac{x}{z},\ 0 \rangle$ be the velocity field of a fluid. Let C be the solid cube given by $1 \le x \le 4,\ 2 \le y \le 5,\ 1 \le z \le 4$, and let S be the boundary of this cube (see the following figure). Find the flow rate of the fluid across S.

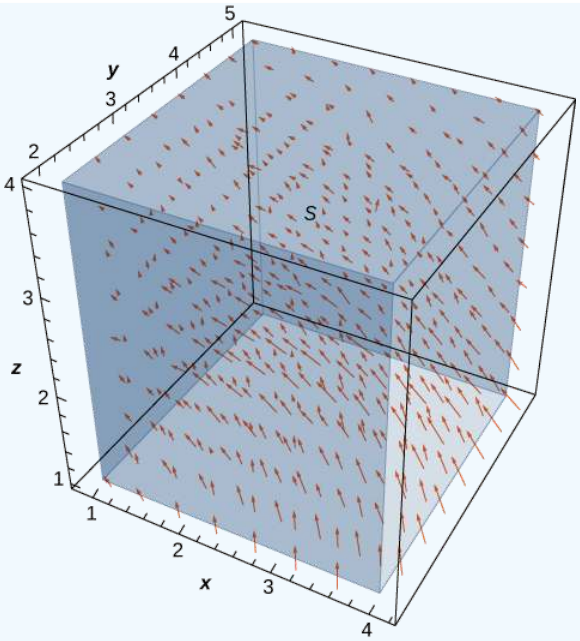


Figure 3.9.4: Vector field $\vec{\mathbf{v}} = \langle -\frac{y}{z}, \frac{x}{z}, 0 \rangle$.

Solution

The flow rate of the fluid across S is $\iint_S \vec{\mathbf{v}} \cdot d\vec{\mathbf{S}}$. Before calculating this flux integral, let's discuss what the value of the integral should be. Based on Figure 3.9.4, we see that if we place this cube in the fluid (as long as the cube doesn't encompass the origin), then the rate of fluid entering the cube is the same as the rate of fluid exiting the cube. The field is rotational in nature and, for a given circle parallel to the xy-plane that has a center on the z-axis, the vectors along that circle are all the same magnitude. That is how we can see that the flow rate is the same entering and exiting the cube. The flow into the cube cancels with the flow out of the cube, and therefore the flow rate of the fluid across the cube should be zero.

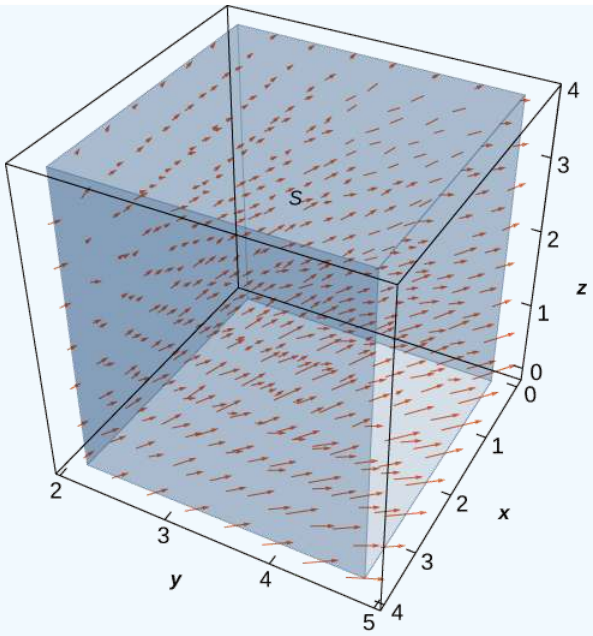
To verify this intuition, we need to calculate the flux integral. Calculating the flux integral directly requires breaking the flux integral into six separate flux integrals, one for each face of the cube. We also need to find tangent vectors, compute their cross product. However, using the divergence theorem makes this calculation go much more quickly:

$$\begin{aligned} \iint_S \vec{\mathbf{v}} \cdot d\vec{\mathbf{S}} &= \iiint_C \operatorname{div} \vec{\mathbf{v}} \, dV \\ &= \iiint_C 0 \, dV = 0. \end{aligned}$$

Therefore the flux is zero, as expected.

? Exercise 3.9.3

Let $\vec{\mathbf{v}} = \langle \frac{x}{z}, \frac{y}{z}, 0 \rangle$ be the velocity field of a fluid. Let C be the solid cube given by $1 \leq x \leq 4$, $2 \leq y \leq 5$, $1 \leq z \leq 4$, and let S be the boundary of this cube (see the following figure). Find the flow rate of the fluid across S.



Hint

Use the divergence theorem and calculate a triple integral

Answer

$9\ln(16)$

Example illustrates a remarkable consequence of the divergence theorem. Let S be a piecewise, smooth closed surface and let $\vec{\mathbf{F}}$ be a vector field defined on an open region containing the surface enclosed by S. If $\vec{\mathbf{F}}$ has the form $F = \langle f(y,z),\ g(x,z),\ h(x,y)\rangle$, then the divergence of $\vec{\mathbf{F}}$ is zero. By the divergence theorem, the flux of $\vec{\mathbf{F}}$ across S is also zero. This makes certain flux integrals incredibly easy to calculate. For example, suppose we wanted to calculate the flux integral $\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$ where S is a cube and

$$\vec{\mathbf{F}} = \langle \sin(y)\, e^{yz},\ x^2 z^2,\ \cos(xy)\, e^{\sin x}\rangle. \tag{3.9.9}$$

Calculating the flux integral directly would be difficult, if not impossible, using techniques we studied previously. At the very least, we would have to break the flux integral into six integrals, one for each face of the cube. But, because the divergence of this field is zero, the divergence theorem immediately shows that the flux integral is zero.

We can now use the divergence theorem to justify the physical interpretation of divergence that we discussed earlier. Recall that if $\vec{\mathbf{F}}$ is a continuous three-dimensional vector field and P is a point in the domain of $\vec{\mathbf{F}}$, then the divergence of $\vec{\mathbf{F}}$ at P is a measure of the "outflowing-ness" of $\vec{\mathbf{F}}$ at P. If $\vec{\mathbf{F}}$ represents the velocity field of a fluid, then the divergence of $\vec{\mathbf{F}}$ at P is a measure of the net flow rate out of point P (the flow of fluid out of P less the flow of fluid in to P). To see how the divergence theorem justifies this interpretation, let B_r be a ball of very small radius r with center P, and assume that B_r is in the domain of $\vec{\mathbf{F}}$. Furthermore, assume that B_r has a positive, outward orientation. Since the radius of B_r is small and $\vec{\mathbf{F}}$ is continuous, the divergence of $\vec{\mathbf{F}}$ is approximately constant on B_r. That is, ifv P' is any point in B_r, then $\operatorname{div}\vec{\mathbf{F}}(P) \approx \operatorname{div}\vec{\mathbf{F}}(P')$. Let S_r denote the boundary sphere of B_r. We can approximate the flux across S_r using the divergence theorem as follows:

$$\begin{aligned}\iint_{S_r} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} &= \iiint_{B_r} \operatorname{div}\vec{\mathbf{F}}\, dV \\ &\approx \iiint_{B_r} \operatorname{div}\vec{\mathbf{F}}(P)\, dV \\ &= \operatorname{div}\vec{\mathbf{F}}(P)\, V(B_r).\end{aligned}$$

As we shrink the radius r to zero via a limit, the quantity $\operatorname{div}\vec{\mathbf{F}}(P)\, V(B_r)$ gets arbitrarily close to the flux. Therefore,

$$\text{div}\,\vec{\mathbf{F}}(P) = \lim_{\tau \to 0} \frac{1}{V(B_\tau)} \iint_{S_\tau} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} \tag{3.9.10}$$

and we can consider the divergence at P as measuring the net rate of outward flux per unit volume at P. Since "outflowing-ness" is an informal term for the net rate of outward flux per unit volume, we have justified the physical interpretation of divergence we discussed earlier, and we have used the divergence theorem to give this justification.



Application to Electrostatic Fields

The divergence theorem has many applications in physics and engineering. It allows us to write many physical laws in both an integral form and a differential form (in much the same way that Stokes' theorem allowed us to translate between an integral and differential form of Faraday's law). Areas of study such as fluid dynamics, electromagnetism, and quantum mechanics have equations that describe the conservation of mass, momentum, or energy, and the divergence theorem allows us to give these equations in both integral and differential forms.

One of the most common applications of the divergence theorem is to **electrostatic fields**. An important result in this subject is **Gauss' law**. This law states that if S is a closed surface in electrostatic field $\vec{\mathbf{E}}$, then the flux of $\vec{\mathbf{E}}$ across S is the total charge enclosed by S (divided by an electric constant). We now use the divergence theorem to justify the special case of this law in which the electrostatic field is generated by a stationary point charge at the origin.

If (x, y, z) is a point in space, then the distance from the point to the origin is $r = \sqrt{x^2 + y^2 + z^2}$. Let $\vec{\mathbf{F}}_r$ denote radial vector field $\vec{\mathbf{F}}_r = \frac{1}{r^2}\left\langle \frac{x}{r}, \frac{y}{r}, \frac{z}{r} \right\rangle$.The vector at a given position in space points in the direction of unit radial vector $\left\langle \frac{x}{r}, \frac{y}{r}, \frac{z}{r} \right\rangle$ and is scaled by the quantity $1/r^2$. Therefore, the magnitude of a vector at a given point is inversely proportional to the square of the vector's distance from the origin. Suppose we have a stationary charge of q Coulombs at the origin, existing in a vacuum. The charge generates electrostatic field $\vec{\mathbf{E}}$ given by

$$\vec{\mathbf{E}} = \frac{q}{4\pi\epsilon_0}\vec{\mathbf{F}}_r, \tag{3.9.11}$$

where the approximation $\epsilon_0 = 8.854 \times 10^{-12}$ farad (F)/m is an electric constant. (The constant ϵ_0 is a measure of the resistance encountered when forming an electric field in a vacuum.) Notice that $\vec{\mathbf{E}}$ is a radial vector field similar to the gravitational field described in [link]. The difference is that this field points outward whereas the gravitational field points inward. Because

$$\vec{\mathbf{E}} = \frac{q}{4\pi\epsilon_0}\vec{\mathbf{F}}_r = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r^2}\left\langle \frac{x}{r}, \frac{y}{r}, \frac{z}{r} \right\rangle\right), \tag{3.9.12}$$

we say that electrostatic fields obey an inverse-square law. That is, the electrostatic force at a given point is inversely proportional to the square of the distance from the source of the charge (which in this case is at the origin). Given this vector field, we show that

the flux across closed surface S is zero if the charge is outside of S, and that the flux is $q/epsilon_0$ if the charge is inside of S. In other words, the flux across S is the charge inside the surface divided by constant ϵ_0. This is a special case of Gauss' law, and here we use the divergence theorem to justify this special case.

To show that the flux across S is the charge inside the surface divided by constant ϵ_0, we need two intermediate steps. First we show that the divergence of $\vec{\mathbf{F}}_\tau$ is zero and then we show that the flux of $\vec{\mathbf{F}}_\tau$ across any smooth surface S is either zero or 4π. We can then justify this special case of Gauss' law.

Example 3.9.4: The Divergence of F_τ is Zero

Verify that the divergence of $\vec{\mathbf{F}}_\tau$ is zero where $\vec{\mathbf{F}}_\tau$ is defined (away from the origin).

Solution

Since $\tau = \sqrt{x^2+y^2+z^2}$, the quotient rule gives us

$$\begin{aligned}\frac{\partial}{\partial x}\left(\frac{x}{\tau^3}\right) &= \frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right)\\ &= \frac{(x^2+y^2+z^2)^{3/2} - x\left[\frac{3}{2}(x^2+y^2+z^2)^{1/2}2x\right]}{(x^2+y^2+z^2)^3}\\ &= \frac{\tau^3 - 3x^2\tau}{\tau^6} = \frac{\tau^2-3x^2}{\tau^5}.\end{aligned}$$

Similarly,

$$\frac{\partial}{\partial y}\left(\frac{y}{\tau^3}\right) = \frac{\tau^2-3y^2}{\tau^5} \text{ and } \frac{\partial}{\partial z}\left(\frac{z}{\tau^3}\right) = \frac{\tau^2-3z^2}{\tau^5}.$$

Therefore,

$$\begin{aligned}\text{div } \vec{\mathbf{F}}_\tau &= \frac{\tau^2-3x^2}{\tau^5} + \frac{\tau^2-3y^2}{\tau^5} + \frac{\tau^2-3z^2}{\tau^5}\\ &= \frac{3\tau^2 - 3(x^2+y^2+z^2)}{\tau^5}\\ &= \frac{3\tau^2-3\tau^2}{\tau^5} = 0.\end{aligned}$$

Notice that since the divergence of $\vec{\mathbf{F}}_\tau$ is zero and $\vec{\mathbf{E}}$ is $\vec{\mathbf{F}}_\tau$ scaled by a constant, the divergence of electrostatic field $\vec{\mathbf{E}}$ is also zero (except at the origin).

Flux across a Smooth Surface

Let S be a connected, piecewise smooth closed surface and let $\vec{\mathbf{F}}_\tau = \frac{1}{\tau^2}\left\langle \frac{x}{\tau}, \frac{y}{\tau}, \frac{z}{\tau}\right\rangle$. Then,

$$\iint_S \vec{\mathbf{F}}_\tau \cdot d\vec{\mathbf{S}} = \begin{cases} 0, & \text{if } S \text{ does not encompass the origin} \\ 4\pi, & \text{if } S \text{ encompasses the origin.}\end{cases} \tag{3.9.13}$$

In other words, this theorem says that the flux of $\vec{\mathbf{F}}_\tau$ across any piecewise smooth closed surface S depends only on whether the origin is inside of S.

Proof

The logic of this proof follows the logic of [link], only we use the divergence theorem rather than Green's theorem.

First, suppose that S does not encompass the origin. In this case, the solid enclosed by S is in the domain of $\vec{\mathbf{F}}_\tau$, and since the divergence of $\vec{\mathbf{F}}_\tau$ is zero, we can immediately apply the divergence theorem and find that

$$\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} \tag{3.9.14}$$

is zero.

Now suppose that S does encompass the origin. We cannot just use the divergence theorem to calculate the flux, because the field is not defined at the origin. Let S_a be a sphere of radius a inside of S centered at the origin. The outward normal vector field on the sphere, in spherical coordinates, is

$$\vec{\mathbf{t}}_\phi \times \vec{\mathbf{t}}_\theta = \langle a^2 \cos\theta \sin^2\phi,\ a^2 \sin\theta \sin^2\phi,\ a^2 \sin\phi \cos\phi \rangle \tag{3.9.15}$$

(see [link]). Therefore, on the surface of the sphere, the dot product $\vec{\mathbf{F}}_\tau \cdot \vec{\mathbf{N}}$ (in spherical coordinates) is

$$\begin{aligned} \vec{\mathbf{F}}_\tau \cdot \vec{\mathbf{N}} &= \left\langle \frac{\sin\phi\cos\theta}{a^2}, \frac{\sin\phi\sin\theta}{a^2}, \frac{\cos\phi}{a^2} \right\rangle \cdot \langle a^2\cos\theta\sin^2\phi, a^2\sin\theta\sin^2\phi,\ a^2\sin\phi\cos\phi\rangle \\ &= \sin\phi(\langle \sin\phi\cos\theta,\ \sin\phi\sin\theta,\ \cos\phi\rangle \cdot \langle \sin\phi\cos\theta, \sin\phi\sin\theta,\ \cos\phi\rangle) \\ &= \sin\phi. \end{aligned}$$

The flux of $\vec{\mathbf{F}}_\tau$ across S_a is

$$\iint_{S_a} \vec{\mathbf{F}}_\tau \cdot \vec{\mathbf{N}} dS = \int_0^{2\pi}\int_0^{\pi} \sin\phi\, d\phi\, d\theta = 4\pi. \tag{3.9.16}$$

Now, remember that we are interested in the flux across S, not necessarily the flux across S_a. To calculate the flux across S, let E be the solid between surfaces S_a and S. Then, the boundary of E consists of S_a and S. Denote this boundary by $S - S_a$ to indicate that S is oriented outward but now S_a is oriented inward. We would like to apply the divergence theorem to solid E. Notice that the divergence theorem, as stated, can't handle a solid such as E because E has a hole. However, the divergence theorem can be extended to handle solids with holes, just as Green's theorem can be extended to handle regions with holes. This allows us to use the divergence theorem in the following way. By the divergence theorem,

$$\begin{aligned} \iint_{S-S_a} \vec{\mathbf{F}}_\tau \cdot d\vec{\mathbf{S}} &= \iint_S \vec{\mathbf{F}}_\tau \cdot d\vec{\mathbf{S}} - \iint_{S_a} \vec{\mathbf{F}}_\tau \cdot d\vec{\mathbf{S}} \\ &= \iiint_E \operatorname{div} \vec{\mathbf{F}}_\tau\, dV \\ &= \iiint_E 0\, dV = 0. \end{aligned}$$

Therefore,

$$\iint_S \vec{\mathbf{F}}_\tau \cdot d\vec{\mathbf{S}} = \iint_{S_a} \vec{\mathbf{F}}_\tau \cdot d\vec{\mathbf{S}} = 4\pi,$$

and we have our desired result.

□

Now we return to calculating the flux across a smooth surface in the context of electrostatic field $\vec{\mathbf{E}} = \frac{q}{4\pi\epsilon_0}\vec{\mathbf{F}}_\tau$ of a point charge at the origin. Let S be a piecewise smooth closed surface that encompasses the origin. Then

$$\begin{aligned} \iint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{S}} &= \iint_S \frac{q}{4\pi\epsilon_0}\vec{\mathbf{F}}_\tau \cdot d\vec{\mathbf{S}} \\ &= \frac{q}{4\pi\epsilon_0}\iint_S \vec{\mathbf{F}}_\tau \cdot d\vec{\mathbf{S}} \\ &= \frac{q}{\epsilon_0}. \end{aligned}$$

If S does not encompass the origin, then

$$\iint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{S}} = \frac{q}{4\pi\epsilon_0}\iint_S \vec{\mathbf{F}}_\tau \cdot d\vec{\mathbf{S}} = 0.$$

Therefore, we have justified the claim that we set out to justify: the flux across closed surface S is zero if the charge is outside of S, and the flux is q/ϵ_0 if the charge is inside of S.

This analysis works only if there is a single point charge at the origin. In this case, Gauss' law says that the flux of $\vec{\mathbf{E}}$ across S is the total charge enclosed by S. Gauss' law can be extended to handle multiple charged solids in space, not just a single point charge at the origin. The logic is similar to the previous analysis, but beyond the scope of this text. In full generality, Gauss' law states that if S is a piecewise smooth closed surface and Q is the total amount of charge inside of S, then the flux of $\vec{\mathbf{E}}$ across S is Q/ϵ_0.

Example 3.9.5: Using Gauss' law

Suppose we have four stationary point charges in space, all with a charge of 0.002 Coulombs (C). The charges are located at $(0,0,1)$, $(1,1,4)$, $(-1,0,0)$, and $(-2,-2,2)$. Let $\vec{\mathbf{E}}$ denote the electrostatic field generated by these point charges. If S is the sphere of radius 2 oriented outward and centered at the origin, then find

$$\iint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{S}}.$$

Solution

According to Gauss' law, the flux of $\vec{\mathbf{E}}$ across S is the total charge inside of S divided by the electric constant. Since S has radius 2, notice that only two of the charges are inside of S: the charge at $(0,1,1)$ and the charge at $(-1,0,0)$. Therefore, the total charge encompassed by S is 0.004 and, by Gauss' law,

$$\iint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{S}} = \frac{0.004}{8.854 \times 10^{-12}} \approx 4.418 \times 10^9\, V - m.$$

Exercise 3.9.4

Work the previous example for surface S that is a sphere of radius 4 centered at the origin, oriented outward.

Hint

Use Gauss' law.

Answer

$\approx 6.777 \times 10^9$

Key Concepts

- The divergence theorem relates a surface integral across closed surface S to a triple integral over the solid enclosed by S. The divergence theorem is a higher dimensional version of the flux form of Green's theorem, and is therefore a higher dimensional version of the Fundamental Theorem of Calculus.
- The divergence theorem can be used to transform a difficult flux integral into an easier triple integral and vice versa.
- The divergence theorem can be used to derive Gauss' law, a fundamental law in electrostatics.

Key Equations

- **Divergence theorem**

$$\iiint_E \operatorname{div} \vec{\mathbf{F}}\, dV = \iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$$

Glossary

divergence theorem
a theorem used to transform a difficult flux integral into an easier triple integral and vice versa

Gauss' law

if S is a piecewise, smooth closed surface in a vacuum and Q is the total stationary charge inside of S, then the flux of electrostatic field $\vec{\mathbf{E}}$ across S is Q/ϵ_0

inverse-square law

the electrostatic force at a given point is inversely proportional to the square of the distance from the source of the charge

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3.9E: Exercises

For exercises 1 - 9, use a computer algebraic system (CAS) and the divergence theorem to evaluate surface integral $\int_S \vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\, ds$ for the given choice of $\vec{\mathbf{F}}$ and the boundary surface S. For each closed surface, assume $\vec{\mathbf{N}}$ is the outward unit normal vector.

1. [T] $\vec{\mathbf{F}}(x,y,z) = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$; S is the surface of cube $0 \le x \le 1$, $0 \le y \le 1$, $0 < z \le 1$.

2. [T] $\vec{\mathbf{F}}(x,y,z) = (\cos yz)\,\hat{\mathbf{i}} + e^{xz}\,\hat{\mathbf{j}} + 3z^2\,\hat{\mathbf{k}}$; S is the surface of hemisphere $z = \sqrt{4 - x^2 - y^2}$ together with disk $x^2 + y^2 \le 4$ in the xy-plane.

Answer

$$\int_S \vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\, ds = 75.3982$$

3. [T] $\vec{\mathbf{F}}(x,y,z) = (x^2 + y^2 - x^2)\,\hat{\mathbf{i}} + x^2 y\,\hat{\mathbf{j}} + 3z\,\hat{\mathbf{k}}$; S is the surface of the five faces of unit cube $0 \le x \le 1$, $0 \le y \le 1$, $0 < z \le 1$.

4. [T] $\vec{\mathbf{F}}(x,y,z) = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$; S is the surface of paraboloid $z = x^2 + y^2$ for $0 \le z \le 9$.

Answer

$$\int_S \vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\, ds = 127.2345$$

5. [T] $\vec{\mathbf{F}}(x,y,z) = x^2\,\hat{\mathbf{i}} + y^2\,\hat{\mathbf{j}} + z^2\,\hat{\mathbf{k}}$; S is the surface of sphere $x^2 + y^2 + z^2 = 4$.

6. [T] $\vec{\mathbf{F}}(x,y,z) = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + (z^2 - 1)\,\hat{\mathbf{k}}$; S is the surface of the solid bounded by cylinder $x^2 + y^2 = 4$ and planes $z = 0$ and $z = 1$.

Answer

$$\int_S \vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\, ds = 37.699$$

7. [T] $\vec{\mathbf{F}}(x,y,z) = x^2\,\hat{\mathbf{i}} + y^2\,\hat{\mathbf{j}} + z^2\,\hat{\mathbf{k}}$; S is the surface bounded above by sphere $\rho = 2$ and below by cone $\varphi = \frac{\pi}{4}$ in spherical coordinates. (Think of S as the surface of an "ice cream cone.")

8. [T] $\vec{\mathbf{F}}(x,y,z) = x^3\,\hat{\mathbf{i}} + y^3\,\hat{\mathbf{j}} + 3a^2 z\,\hat{\mathbf{k}}$ ($constant\ a > 0$); S is the surface bounded by cylinder $x^2 + y^2 = a^2$ and planes $z = 0$ and $z = 1$.

Answer

$$\int_S \vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\, ds = \frac{9\pi a^4}{2}$$

9. [T] Surface integral $\iint_S \vec{\mathbf{F}} \cdot dS$, where S is the solid bounded by paraboloid $z = x^2 + y^2$ and plane $z = 4$, and $\vec{\mathbf{F}}(x,y,z) = (x + y^2 z^2)\,\hat{\mathbf{i}} + (y + z^2 x^2)\,\hat{\mathbf{j}} + (z + x^2 y^2)\,\hat{\mathbf{k}}$

10. Use the divergence theorem to calculate surface integral $\iint_S \vec{\mathbf{F}} \cdot dS$, where $\vec{\mathbf{F}}(x,y,z) = (e^{y^2}\,\hat{\mathbf{i}} + (y + \sin(z^2))\,\hat{\mathbf{j}} + (z - 1)\,\hat{\mathbf{k}}$ and S is upper hemisphere $x^2 + y^2 + z^2 = 1$, $z \ge 0$, oriented upward.

Answer

$$\iint_S \vec{\mathbf{F}} \cdot dS = \frac{\pi}{3}$$

11. Use the divergence theorem to calculate surface integral $\iint_S \vec{\mathbf{F}} \cdot dS$, where $\vec{\mathbf{F}}(x,y,z) = x^4\,\hat{\mathbf{i}} - x^3 z^2\,\hat{\mathbf{j}} + 4xy^2 z\,\hat{\mathbf{k}}$ and S is the surface bounded by cylinder $x^2 + y^2 = 1$ and planes $z = x + 2$ and $z = 0$.

12. Use the divergence theorem to calculate surface integral $\iint_S \vec{\mathbf{F}} \cdot dS$, when $\vec{\mathbf{F}}(x,y,z) = x^2 z^3\,\hat{\mathbf{i}} + 2xyz^3\,\hat{\mathbf{j}} + xz^4\,\hat{\mathbf{k}}$ and S is the surface of the box with vertices $(\pm 1, \pm 2, \pm 3)$.

Answer

$$\iint_S \vec{\mathbf{F}} \cdot dS = 0$$

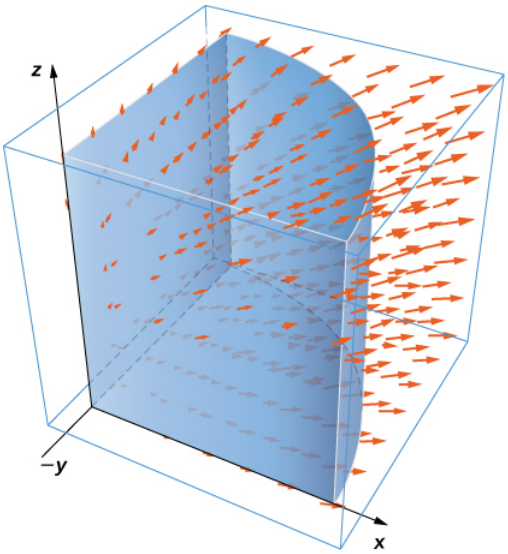
13. Use the divergence theorem to calculate surface integral $\iint_S \vec{\mathbf{F}} \cdot dS$, when $\vec{\mathbf{F}}(x, y, z) = z \tan^{-1}(y^2)\,\hat{\mathbf{i}} + z^3 \ln(x^2 + 1)\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$ and S is a part of paraboloid $x^2 + y^2 + z = 2$ that lies above plane $z = 1$ and is oriented upward.

14. [T] Use a CAS and the divergence theorem to calculate flux $\iint_S \vec{\mathbf{F}} \cdot dS$, where $\vec{\mathbf{F}}(x, y, z) = (x^3 + y^3)\,\hat{\mathbf{i}} + (y^3 + z^3)\,\hat{\mathbf{j}} + (z^3 + x^3)\,\hat{\mathbf{k}}$ and S is a sphere with center $(0, 0)$ and radius 2.

Answer

$$\iint_S \vec{\mathbf{F}} \cdot dS = 241.2743$$

15. Use the divergence theorem to compute the value of flux integral $\iint_S \vec{\mathbf{F}} \cdot dS$, where $\vec{\mathbf{F}}(x, y, z) = (y^3 + 3x)\,\hat{\mathbf{i}} + (xz + y)\,\hat{\mathbf{j}} + (z + x^4 \cos(x^2 y))\,\hat{\mathbf{k}}$ and S is the area of the region bounded by $x^2 + y^2 = 1$, $x \geq 0$, $y \geq 0$, and $0 \leq z \leq 1$.



16. Use the divergence theorem to compute flux integral $\iint_S \vec{\mathbf{F}} \cdot dS$, where $\vec{\mathbf{F}}(x, y, z) = y\,\hat{\mathbf{j}} - z\,\hat{\mathbf{k}}$ and S consists of the union of paraboloid $y = x^2 + z^2$, $0 \leq y \leq 1$, and disk $x^2 + z^2 \leq 1$, $y = 1$, oriented outward. What is the flux through just the paraboloid?

Answer

$$\iint_S \vec{\mathbf{F}} \cdot dS = -\pi$$

17. Use the divergence theorem to compute flux integral $\iint_S \vec{\mathbf{F}} \cdot dS$, where $\vec{\mathbf{F}}(x, y, z) = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z^4\,\hat{\mathbf{k}}$ and S is a part of cone $z = \sqrt{x^2 + y^2}$ beneath top plane $z = 1$ oriented downward.

18. Use the divergence theorem to calculate surface integral $\iint_S \vec{\mathbf{F}} \cdot dS$ for $\vec{\mathbf{F}}(x, y, z) = x^4\,\hat{\mathbf{i}} - x^3 z^2\,\hat{\mathbf{j}} + 4xy^2 z\,\hat{\mathbf{k}}$, where S is the surface bounded by cylinder $x^2 + y^2 = 1$ and planes $z = x + 2$ and $z = 0$.

Answer

$$\iint_S \vec{\mathbf{F}} \cdot dS = \frac{2\pi}{3}$$

19. Consider $\vec{\mathbf{F}}(x, y, z) = x^2\,\hat{\mathbf{i}} + xy\,\hat{\mathbf{j}} + (z + 1)\,\hat{\mathbf{k}}$. Let E be the solid enclosed by paraboloid $z = 4 - x^2 - y^2$ and plane $z = 0$ with normal vectors pointing outside E. Compute flux $\vec{\mathbf{F}}$ across the boundary of E using the divergence theorem.

In exercises 20 - 23, use a CAS along with the divergence theorem to compute the net outward flux for the fields across the given surfaces S.

20. [T] $\vec{\mathbf{F}} = \langle x, -2y, 3z\rangle$; S is sphere $\{(x, y, z) : x^2 + y^2 + z^2 = 6\}$.

Answer

$15\sqrt{6}\pi$

21. [T] $\vec{\mathbf{F}} = \langle x, 2y, z\rangle$; S is the boundary of the tetrahedron in the first octant formed by plane $x + y + z = 1$.

22. [T] $\vec{\mathbf{F}} = \langle y - 2x, x^3 - y, y^2 - z\rangle$; S is sphere $\{(x, y, z) : x^2 + y^2 + z^2 = 4\}$.

Answer

$-\frac{128}{3}\pi$

23. [T] $\vec{\mathbf{F}} = \langle x, y, z\rangle$; S is the surface of paraboloid $z = 4 - x^2 - y^2$, for $z \geq 0$, plus its base in the xy-plane.

For exercises 24 - 26, use a CAS and the divergence theorem to compute the net outward flux for the vector fields across the boundary of the given regions D.

24. [T] $\vec{\mathbf{F}} = \langle z - x, x - y, 2y - z\rangle$; D is the region between spheres of radius 2 and 4 centered at the origin.

Answer

-703.7168

25. [T] $\vec{\mathbf{F}} = \dfrac{\vec{\mathbf{r}}}{\|\vec{\mathbf{r}}\|} = \dfrac{\langle x, y, z\rangle}{\sqrt{x^2 + y^2 + z^2}}$; D is the region between spheres of radius 1 and 2 centered at the origin.

26. [T] $\vec{\mathbf{F}} = \langle x^2, -y^2, z^2\rangle$; D is the region in the first octant between planes $z = 4 - x - y$ and $z = 2 - x - y$.

Answer

20

27. Let $\vec{\mathbf{F}}(x, y, z) = 2x\,\hat{\mathbf{i}} - 3xy\,\hat{\mathbf{j}} + xz^2\,\hat{\mathbf{k}}$. Use the divergence theorem to calculate $\iint_S \vec{\mathbf{F}} \cdot d\vec{S}$, where S is the surface of the cube with corners at $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(1, 1, 0)$, $(0, 0, 1)$, $(1, 0, 1)$, $(0, 1, 1)$, and $(1, 1, 1)$, oriented outward.

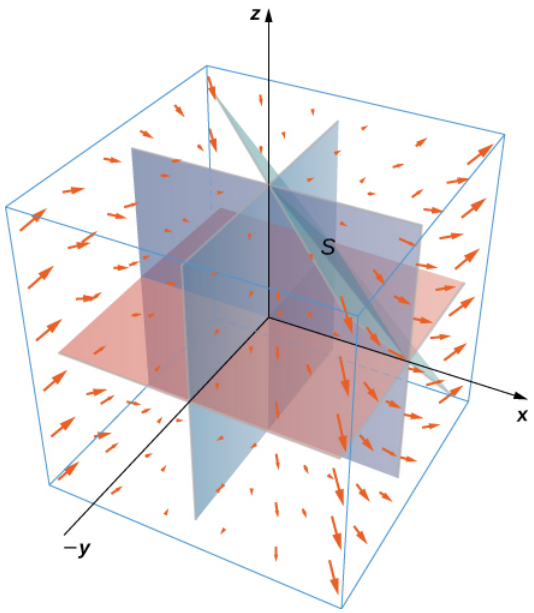
28. Use the divergence theorem to find the outward flux of field $\vec{\mathbf{F}}(x, y, z) = (x^3 - 3y)\,\hat{\mathbf{i}} + (2yz + 1)\,\hat{\mathbf{j}} + xyz\,\hat{\mathbf{k}}$ through the cube bounded by planes $x = \pm 1$, $y = \pm 1$, and $z = \pm 1$.

Answer

$\iint_S \vec{\mathbf{F}} \cdot d\vec{S} = 8$

29. Let $\vec{\mathbf{F}}(x, y, z) = 2x\,\hat{\mathbf{i}} - 3y\,\hat{\mathbf{j}} + 5z\,\hat{\mathbf{k}}$ and let S be hemisphere $z = \sqrt{9 - x^2 - y^2}$ together with disk $x^2 + y^2 \leq 9$ in the xy-plane. Use the divergence theorem.

30. Evaluate $\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\, dS$, where $\vec{\mathbf{F}}(x, y, z) = x^2\,\hat{\mathbf{i}} + xy\,\hat{\mathbf{j}} + x^3y^3\,\hat{\mathbf{k}}$ and S is the surface consisting of all faces except the tetrahedron bounded by plane $x + y + z = 1$ and the coordinate planes, with outward unit normal vector $\vec{\mathbf{N}}$.



Answer

$$\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\, dS = \frac{1}{8}$$

31. Find the net outward flux of field $\vec{\mathbf{F}} = \langle bz - cy,\ cx - az,\ ay - bx \rangle$ across any smooth closed surface in R^3 where a, b, and c are constants.

32. Use the divergence theorem to evaluate $\iint_S ||\vec{\mathbf{R}}||\vec{\mathbf{R}} \cdot \vec{\mathbf{n}}\, ds$, where $\vec{\mathbf{R}}(x, y, z) = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$ and S is sphere $x^2 + y^2 + z^2 = a^2$, with constant $a > 0$.

Answer

$$\iint_S ||\vec{\mathbf{R}}||\vec{\mathbf{R}} \cdot \vec{\mathbf{n}}\, ds = 4\pi a^4$$

33. Use the divergence theorem to evaluate $\iint_S \vec{\mathbf{F}} \cdot dS$, where $\vec{\mathbf{F}}(x, y, z) = y^2 z\,\hat{\mathbf{i}} + y^3\,\hat{\mathbf{j}} + xz\,\hat{\mathbf{k}}$ and S is the boundary of the cube defined by $-1 \le x \le 1$, $-1 \le y \le 1$, and $0 \le z \le 2$.

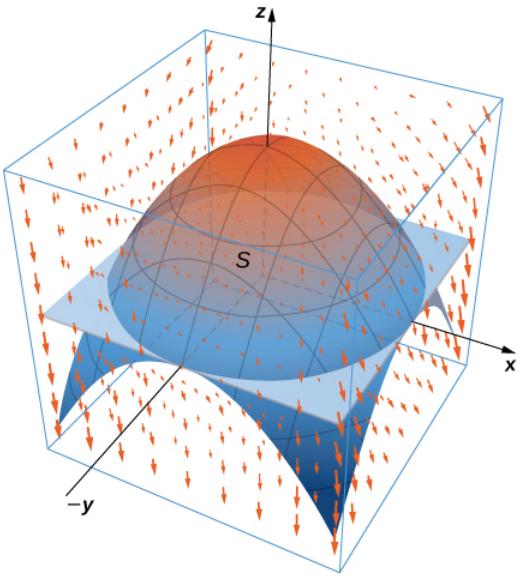
34. Let R be the region defined by $x^2 + y^2 + z^2 \le 1$. Use the divergence theorem to find $\iiint_R z^2\, dV$.

Answer

$$\iiint_R z^2 dV = \frac{4\pi}{15}$$

35. Let E be the solid bounded by the xy-plane and paraboloid $z = 4 - x^2 - y^2$ so that S is the surface of the paraboloid piece together with the disk in the xy-plane that forms its bottom. If $\vec{\mathbf{F}}(x, y, z) = (xz \sin(yz) + x^3)\,\hat{\mathbf{i}} + \cos(yz)\,\hat{\mathbf{j}} + (3zy^2 - e^{x^2+y^2})\,\hat{\mathbf{k}}$, find $\iint_S \vec{\mathbf{F}} \cdot dS$ using the divergence theorem.

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36. Let E be the solid unit cube with diagonally opposite corners at the origin and $(1,1,1)$, and faces parallel to the coordinate planes. Let S be the surface of E, oriented with the outward-pointing normal. Use a CAS to find $\iint_S \vec{\mathbf{F}} \cdot dS$ using the divergence theorem if $\vec{\mathbf{F}}(x,y,z) = 2xy\,\hat{\mathbf{i}} + 3ye^z\,\hat{\mathbf{j}} + x\sin z\,\hat{\mathbf{k}}$.

Answer

$$\iint_S \vec{\mathbf{F}} \cdot dS = 6.5759$$

37. Use the divergence theorem to calculate the flux of $\vec{\mathbf{F}}(x,y,z) = x^3\,\hat{\mathbf{i}} + y^3\,\hat{\mathbf{j}} + z^3\,\hat{\mathbf{k}}$ through sphere $x^2 + y^2 + z^2 = 1$.

38. Find $\iint_S \vec{\mathbf{F}} \cdot dS$, where $\vec{\mathbf{F}}(x,y,z) = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$ and S is the outwardly oriented surface obtained by removing cube $[1,2] \times [1,2] \times [1,2]$ from cube $[0,2] \times [0,2] \times [0,2]$.

Answer

$$\iint_S \vec{\mathbf{F}} \cdot dS = 21$$

39. Consider radial vector field $\vec{\mathbf{F}} = \dfrac{\vec{\mathbf{r}}}{\|\vec{\mathbf{r}}\|} = \dfrac{\langle x,y,z\rangle}{(x^2+y^2+z^2)^{1/2}}$. Compute the surface integral, where S is the surface of a sphere of radius a centered at the origin.

40. Compute the flux of water through parabolic cylinder $S : y = x^2$, from $0 \le x \le 2$, $0 \le z \le 3$, if the velocity vector is $\vec{\mathbf{F}}(x,y,z) = 3z^2\,\hat{\mathbf{i}} + 6\,\hat{\mathbf{j}} + 6xz\,\hat{\mathbf{k}}$.

Answer

$$\iint_S \vec{\mathbf{F}} \cdot dS = 72$$

41. [T] Use a CAS to find the flux of vector field $\vec{\mathbf{F}}(x,y,z) = z\,\hat{\mathbf{i}} + z\,\hat{\mathbf{j}} + \sqrt{x^2+y^2}\,\hat{\mathbf{k}}$ across the portion of hyperboloid $x^2 + y^2 = z^2 + 1$ between planes $z = 0$ and $z = \dfrac{\sqrt{3}}{3}$, oriented so the unit normal vector points away from the z-axis.

42. Use a CAS to find the flux of vector field $\vec{\mathbf{F}}(x,y,z) = (e^y + x)\,\hat{\mathbf{i}} + (3\cos(xz) - y)\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$ through surface S, where S is given by $z^2 = 4x^2 + 4y^2$ from $0 \le z \le 4$, oriented so the unit normal vector points downward.

Answer

$$\iint_S \vec{\mathbf{F}} \cdot dS = -33.5103$$

43. [T] Use a CAS to compute $\iint_S \vec{\mathbf{F}} \cdot d\vec{S}$, where $\vec{\mathbf{F}}(x,y,z) = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + 2z\,\hat{\mathbf{k}}$ and S is a part of sphere $x^2 + y^2 + z^2 = 2$ with $0 \le z \le 1$.

44. Evaluate $\iint_S \vec{\mathbf{F}} \cdot d\vec{S}$, where $\vec{\mathbf{F}}(x,y,z) = bxy^2\,\hat{\mathbf{i}} + bx^2y\,\hat{\mathbf{j}} + (x^2+y^2)z^2\,\hat{\mathbf{k}}$ and S is a closed surface bounding the region and consisting of solid cylinder $x^2 + y^2 \le a^2$ and $0 \le z \le b$.

Answer

$$\iint_S \vec{\mathbf{F}} \cdot d\vec{S} = \pi a^4 b^2$$

45. [T] Use a CAS to calculate the flux of $\vec{\mathbf{F}}(x,y,z) = (x^3 + y\,\sin z)\,\hat{\mathbf{i}} + (y^3 + z\,\sin x)\,\hat{\mathbf{j}} + 3z\,\hat{\mathbf{k}}$ across surface S, where S is the boundary of the solid bounded by hemispheres $z = \sqrt{4 - x^2 - y^2}$ and $z = \sqrt{1 - x^2 - y^2}$, and plane $z = 0$.

46. Use the divergence theorem to evaluate $\iint_S \vec{\mathbf{F}} \cdot d\vec{S}$, where $\vec{\mathbf{F}}(x,y,z) = xy\,\hat{\mathbf{i}} - \frac{1}{2}y^2\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$ and S is the surface consisting of three pieces: $z = 4 - 3x^2 - 3y^2$, $1 \le z \le 4$ on the top; $x^2 + y^2 = 1$, $0 \le z \le 1$ on the sides; and $z = 0$ on the bottom.

Answer

$$\iint_S \vec{\mathbf{F}} \cdot d\vec{S} = \frac{5}{2}\pi$$

47. [T] Use a CAS and the divergence theorem to evaluate $\iint_S \vec{\mathbf{F}} \cdot d\vec{S}$, where $\vec{\mathbf{F}}(x,y,z) = (2x + y\,\cos z)\,\hat{\mathbf{i}} + (x^2 - y)\,\hat{\mathbf{j}} + y^2 z\,\hat{\mathbf{k}}$ and S is sphere $x^2 + y^2 + z^2 = 4$ orientated outward.

48. Use the divergence theorem to evaluate $\iint_S \vec{\mathbf{F}} \cdot d\vec{S}$, where $\vec{\mathbf{F}}(x,y,z) = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$ and S is the boundary of the solid enclosed by paraboloid $y = x^2 + z^2 - 2$, cylinder $x^2 + z^2 = 1$, and plane $x + y = 2$, and S is oriented outward.

Answer

$$\iint_S \vec{\mathbf{F}} \cdot d\vec{S} = \frac{21\pi}{2}$$

For the following exercises, **Fourier's law of heat transfer** states that the heat flow vector $\vec{\mathbf{F}}$ at a point is proportional to the negative gradient of the temperature; that is, $\vec{\mathbf{F}} = -k\vec{\nabla}T$, which means that heat energy flows hot regions to cold regions. The constant $k > 0$ is called the *conductivity*, which has metric units of joules per meter per second-kelvin or watts per meter-kelvin. A temperature function for region D s given. Use the divergence theorem to find net outward heat flux $\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\, dS = -k \iint_S \vec{\nabla}T \cdot N\, dS$ across the boundary S of D, where $k = 1$.

49. $T(x,y,z) = 100 + x + 2y + z$;

$D = \{(x,y,z) : 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\}$

50. $T(x,y,z) = 100 + e^{-z}$;

$D = \{(x,y,z) : 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\}$

Answer

$-(1 - e^{-1})$

51. $T(x,y,z) = 100e^{-x^2-y^2-z^2}$; D is the sphere of radius a centered at the origin.

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3.10: Chapter 3 Review Exercises

***True or False?* Justify your answer with a proof or a counterexample.**

1. The vector field $\vec{\mathbf{F}}(x,y) = x^2y\,\hat{\mathbf{i}} + y^2x\,\hat{\mathbf{j}}$ is conservative.

Answer

False

2. For vector field $\vec{\mathbf{F}}(x,y) = P(x,y)\,\hat{\mathbf{i}} + Q(x,y)\,\hat{\mathbf{j}}$, if $P_y(x,y) = Q_x(x,y)$ in open region D, then $\int_{\partial D} P\,dx + Q\,dy = 0$.

3. The divergence of a vector field is a vector field.

Answer

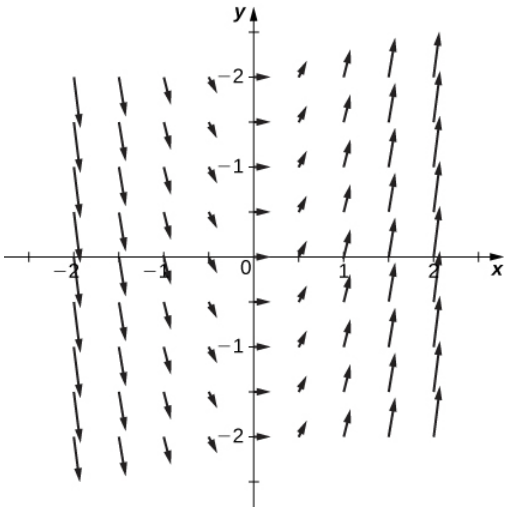
False

4. If $curl\,\vec{\mathbf{F}} = \vec{\mathbf{0}}$, then $\vec{\mathbf{F}}$ is a conservative vector field.

Draw the following vector fields.

5. $\vec{\mathbf{F}}(x,y) = \frac{1}{2}\,\hat{\mathbf{i}} + 2x\,\hat{\mathbf{j}}$

Answer



6. $\vec{\mathbf{F}}(x,y) = \sqrt{\dfrac{y\,\hat{\mathbf{i}} + 3x\,\hat{\mathbf{j}}}{x^2+y^2}}$

Are the following the vector fields conservative? If so, find the potential function $\vec{\mathbf{F}}$ such that $\vec{\mathbf{F}} = \vec{\nabla} f$.

7. $\vec{\mathbf{F}}(x,y) = y\,\hat{\mathbf{i}} + (x - 2e^y)\,\hat{\mathbf{j}}$

Answer

Conservative, $f(x,y) = xy - 2e^y$

8. $\vec{\mathbf{F}}(x,y) = (6xy)\,\hat{\mathbf{i}} + (3x^2 - ye^y)\,\hat{\mathbf{j}}$

9. $\vec{\mathbf{F}}(x,y) = (2xy + z^2)\,\hat{\mathbf{i}} + (x^2 + 2yz)\,\hat{\mathbf{j}} + (2xz + y^2)\,\hat{\mathbf{k}}$

Answer

Conservative, $f(x,y,z) = x^2y + y^2z + z^2x$

10. $\vec{\mathbf{F}}(x,y,z) = (e^x y)\,\hat{\mathbf{i}} + (e^x + z)\,\hat{\mathbf{j}} + (e^x + y^2)\,\hat{\mathbf{k}}$

Evaluate the following integrals.

11. $\int_C x^2\,dy + (2x - 3xy)\,dx$, along $C : y = \frac{1}{2}x$ from $(0,0)$ to $(4,2)$

Answer

$-\frac{16}{3}$

12. $\int_C y\,dx + xy^2\,dy$, where $C : x = \sqrt{t},\ y = t - 1,\ 0 \le t \le 1$

13. $\iint_S xy^2\,dS$, where S is the surface $z = x^2 - y,\ 0 \le x \le 1,\ 0 \le y \le 4$

Answer

$\frac{32\sqrt{2}}{9}(3\sqrt{3} - 1)$

Find the divergence and curl for the following vector fields.

14. $\vec{\mathbf{F}}(x,y,z) = 3xyz\,\hat{\mathbf{i}} + xye^x\,\hat{\mathbf{j}} - 3xy\,\hat{\mathbf{k}}$

15. $\vec{\mathbf{F}}(x,y,z) = e^x\,\hat{\mathbf{i}} + e^{xy}\,\hat{\mathbf{j}} - e^{xyz}\,\hat{\mathbf{k}}$

Answer

Divergence: $e^x + x\,e^{xy} + xy\,e^{xyz}$
Curl: $xze^{xyz}\,\hat{\mathbf{i}} - yze^{xyz}\,\hat{\mathbf{j}} + ye^{xy}\,\hat{\mathbf{k}}$

Use Green's theorem to evaluate the following integrals.

16. $\int_C 3xy\,dx + 2xy^2\,dy$, where C is a square with vertices $(0,0)$, $(0,2)$, $(2,2)$ and $(2,0)$.

17. $\oint_C 3y\,dx + (x + e^y)\,dy$, where C is a circle centered at the origin with radius 3.

Answer

-2π

Use Stokes' theorem to evaluate $\iint_S curl\,\vec{\mathbf{F}} \cdot dS$.

18. $\vec{\mathbf{F}}(x,y,z) = y\,\hat{\mathbf{i}} - x\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$, where S is the upper half of the unit sphere

19. $\vec{\mathbf{F}}(x,y,z) = y\,\hat{\mathbf{i}} + xyz\,\hat{\mathbf{j}} - 2zx\,\hat{\mathbf{k}}$, where S is the upward-facing paraboloid $z = x^2 + y^2$ lying in cylinder $x^2 + y^2 = 1$

Answer

$-\pi$

Use the divergence theorem to evaluate $\iint_S \vec{\mathbf{F}} \cdot dS$.

20. $\vec{\mathbf{F}}(x,y,z) = (x^3 y)\,\hat{\mathbf{i}} + (3y - e^x)\,\hat{\mathbf{j}} + (z + x)\,\hat{\mathbf{k}}$, over cube S defined by $-1 \le x \le 1,\ 0 \le y \le 2,\ 0 \le z \le 2$

21. $\vec{\mathbf{F}}(x,y,z) = (2xy)\,\hat{\mathbf{i}} + (-y^2)\,\hat{\mathbf{j}} + (2z^3)\,\hat{\mathbf{k}}$, where S is bounded by paraboloid $z = x^2 + y^2$ and plane $z = 2$

Answer

$31\pi/2$

22. Find the amount of work performed by a 50-kg woman ascending a helical staircase with radius 2 m and height 100 m. The woman completes five revolutions during the climb.

23. Find the total mass of a thin wire in the shape of a semicircle with radius $\sqrt{2}$, and a density function of $\rho(x,y) = y + x^2$.

Answer

$\sqrt{2}(2+\pi)$

24. Find the total mass of a thin sheet in the shape of a hemisphere with radius 2 for $z \geq 0$ with a density function $\rho(x,y,z) = x + y + z$.

25. Use the divergence theorem to compute the value of the flux integral over the unit sphere with $\vec{\mathbf{F}}(x,y,z) = 3z\hat{\mathbf{i}} + 2y\hat{\mathbf{j}} + 2x\hat{\mathbf{k}}$.

Answer

$2\pi/3$

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