CALCULUS



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MIT & University of Wisconsin-Stevens Point Calculus

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Licensing

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CHAPTER OVERVIEW

1: Sequences and Series

The topic of infinite series may seem unrelated to differential and integral calculus. In fact, an infinite series whose terms involve powers of a variable is a powerful tool that we can use to express functions as “infinite polynomials.” We can use infinite series to evaluate complicated functions, approximate definite integrals, and create new functions. In addition, infinite series are used to solve differential equations that model physical behavior, from tiny electronic circuits to Earth-orbiting satellites.

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Thumbnail: For the alternating harmonic series, the odd terms S_{2k+1} in the sequence of partial sums are decreasing and bounded below. The even terms S_{2k} are increasing and bounded above.

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1.1: Prelude to Sequence and Series

The Koch snowflake is constructed from an infinite number of nonoverlapping equilateral triangles. Consequently, we can express its area as a sum of infinitely many terms. How do we add an infinite number of terms? Can a sum of an infinite number of terms be finite? To answer these questions, we need to introduce the concept of an infinite series, a sum with infinitely many terms. Having defined the necessary tools, we will be able to calculate the area of the Koch snowflake.

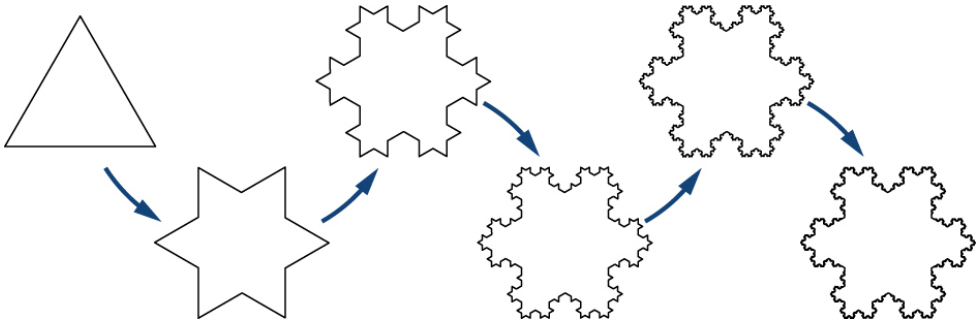


Figure 1.1.1: The Koch snowflake is constructed by using an iterative process. Starting with an equilateral triangle, at each step of the process the middle third of each line segment is removed and replaced with an equilateral triangle pointing outward.

The topic of infinite series may seem unrelated to differential and integral calculus. In fact, an infinite series whose terms involve powers of a variable is a powerful tool that we can use to express functions as “infinite polynomials.” We can use infinite series to evaluate complicated functions, approximate definite integrals, and create new functions. In addition, infinite series are used to solve differential equations that model physical behavior, from tiny electronic circuits to Earth-orbiting satellites.

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1.2: Sequences

Learning Objectives

- Find the formula for the general term of a sequence.
- Calculate the limit of a sequence if it exists.
- Determine the convergence or divergence of a given sequence.

In this section, we introduce sequences and define what it means for a sequence to converge or diverge. We show how to find limits of sequences that converge, often by using the properties of limits for functions discussed earlier. We close this section with the Monotone Convergence Theorem, a tool we can use to prove that certain types of sequences converge.

Terminology of Sequences

To work with this new topic, we need some new terms and definitions. First, an infinite sequence is an ordered list of numbers of the form

$$a_1, a_2, a_3, \ldots, a_n, \ldots.$$

Each of the numbers in the sequence is called a term. The symbol n is called the index variable for the sequence. We use the notation

$$\{a_n\}_{n=1}^{\infty},$$

or simply $\{a_n\}$, to denote this sequence. A similar notation is used for sets, but a sequence is an ordered list, whereas a set is not ordered. Because a particular number a_n exists for each positive integer n, we can also define a sequence as a function whose domain is the set of positive integers.

Let's consider the infinite, ordered list

$$2, 4, 8, 16, 32, \ldots.$$

This is a sequence in which the first, second, and third terms are given by $a_1 = 2, a_2 = 4,$ and $a_3 = 8.$ You can probably see that the terms in this sequence have the following pattern:

$$a_1 = 2^1,\ a_2 = 2^2,\ a_3 = 2^3,\ a_4 = 2^4 \text{ and } a_5 = 2^5.$$

Assuming this pattern continues, we can write the nth term in the sequence by the explicit formula $a_n = 2^n$. Using this notation, we can write this sequence as

$$\{2^n\}_{n=1}^{\infty}$$

or

$$\{2^n\}.$$

Alternatively, we can describe this sequence in a different way. Since each term is twice the previous term, this sequence can be defined recursively by expressing the nth term a_n in terms of the previous term a_{n-1}. In particular, we can define this sequence as the sequence $\{a_n\}$ where $a_1 = 2$ and for all $n \geq 2$, each term an is defined by the **recurrence relation**

$$a_n = 2a_{n-1}. \tag{1.2.1}$$

Definition: infinite sequence

An **infinite sequence** $\{a_n\}$ is an ordered list of numbers of the form

$$a_1,\ a_2,\ \ldots,\ a_n,\ \ldots.$$

The subscript n is called the **index variable** of the sequence. Each number a_n is a term of the sequence. Sometimes sequences are defined by **explicit formulas,** in which case $a_n = f(n)$ for some function $f(n)$ defined over the positive integers. In other cases, sequences are defined by using a **recurrence relation**. In a recurrence relation, one term (or more) of the sequence is given explicitly, and subsequent terms are defined in terms of earlier terms in the sequence.

Note that the index does not have to start at $n = 1$ but could start with other integers. For example, a sequence given by the explicit formula $a_n = f(n)$ could start at $n = 0$, in which case the sequence would be

$$a_0,\ a_1,\ a_2, \ldots .$$

Similarly, for a sequence defined by a recurrence relation, the term a_0 may be given explicitly, and the terms a_n for $n \geq 1$ may be defined in terms of a_{n-1}. Since a sequence $\{a_n\}$ has exactly one value for each positive integer n, it can be described as a function whose domain is the set of positive integers. As a result, it makes sense to discuss the graph of a sequence. The graph of a sequence $\{a_n\}$ consists of all points (n, a_n) for all positive integers n. Figure shows the graph of 2^n.

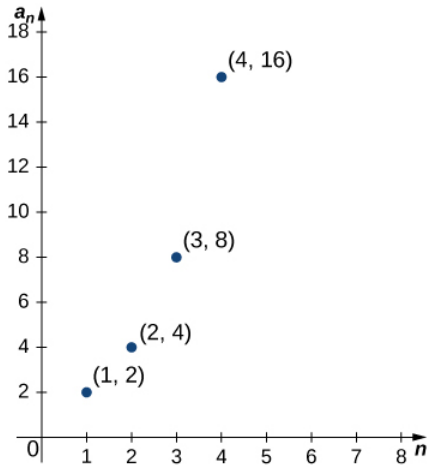


Figure 1.2.1: The plotted points are a graph of the sequence $\{2^n\}$.

Two types of sequences occur often and are given special names: arithmetic sequences and geometric sequences. In an arithmetic sequence, the difference between every pair of consecutive terms is the same. For example, consider the sequence

$$3,\ 7,\ 11,\ 15, 1\,9, \ldots$$

You can see that the difference between every consecutive pair of terms is 4. Assuming that this pattern continues, this sequence is an arithmetic sequence. It can be described by using the recurrence relation

$$\begin{cases} a_1 = 3 \\ a_n = a_{n-1} + 4, \text{ for } n \geq 2 \end{cases}.$$

Note that

$$a_2 = 3 + 4$$

$$a_3 = 3 + 4 + 4 = 3 + 2 \cdot 4$$

$$a_4 = 3 + 4 + 4 + 4 = 3 + 3 \cdot 4.$$

Thus the sequence can also be described using the explicit formula

$$a_n = 3 + 4(n-1) = 4n - 1.$$

In general, an arithmetic sequence is any sequence of the form $a_n = cn + b$.

In a geometric sequence**,** the ratio of every pair of consecutive terms is the same. For example, consider the sequence

$$2,\ -\frac{2}{3},\ \frac{2}{9},\ -\frac{2}{27},\ \frac{2}{81}, \ldots .$$

We see that the ratio of any term to the preceding term is $-\frac{1}{3}$. Assuming this pattern continues, this sequence is a geometric sequence. It can be defined recursively as

$$a_1 = 2$$

$$a_n = -\frac{1}{3} \cdot a_{n-1}, \text{ for } n \geq 2.$$

Alternatively, since

$$\begin{aligned} a_2 &= -\frac{1}{3} \cdot 2 \\ a_3 &= \left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)(2) = \left(-\frac{1}{3}\right)^2 \cdot 2 \\ a_4 &= \left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)(2) = \left(-\frac{1}{3}\right)^3 \cdot 2, \end{aligned}$$

we see that the sequence can be described by using the explicit formula

$$a_n = 2\left(-\frac{1}{3}\right)^{n-1}.$$

The sequence $\{2^n\}$ that we discussed earlier is a geometric sequence, where the ratio of any term to the previous term is 2. In general, a geometric sequence is any sequence of the form $a_n = cr^n$.

Example 1.2.1: Finding Explicit Formulas

For each of the following sequences, find an explicit formula for the nth term of the sequence.

a. $-\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, -\frac{5}{6}, \ldots$

b. $\frac{3}{4}, \frac{9}{7}, \frac{27}{10}, \frac{81}{13}, \frac{243}{16}, \ldots$.

Solution

a. First, note that the sequence is alternating from negative to positive. The odd terms in the sequence are negative, and the even terms are positive. Therefore, the nth term includes a factor of $(-1)^n$. Next, consider the sequence of numerators $1, 2, 3, \ldots$ and the sequence of denominators $2, 3, 4, \ldots$. We can see that both of these sequences are arithmetic sequences. The nth term in the sequence of numerators is n, and the nth term in the sequence of denominators is $n+1$. Therefore, the sequence can be described by the explicit formula

$$a_n = \frac{(-1)^n n}{n+1}. \tag{1.2.2}$$

b. The sequence of numerators $3, 9, 27, 81, 243, \ldots$ is a geometric sequence. The numerator of the nth term is 3^n The sequence of denominators $4, 7, 10, 13, 16, \ldots$ is an arithmetic sequence. The denominator of the nth term is $4 + 3(n-1) = 3n+1$. Therefore, we can describe the sequence by the explicit formula $a_n = \frac{3^n}{3n+1.}$

Exercise 1.2.1

Find an explicit formula for the nth term of the sequence $\left\{\frac{1}{5}, -\frac{1}{7}, \frac{1}{9}, -\frac{1}{11}, \ldots\right\}$.

Hint

The denominators form an arithmetic sequence.

Answer

$$a_n = \frac{(-1)^{n+1}}{3+2n}$$

Example 1.2.2: Defined by Recurrence Relations

For each of the following recursively defined sequences, find an explicit formula for the sequence.

a. $a_1 = 2, a_n = -3a_{n-1}$ for $n \geq 2$

b. $a_1 = \left(\frac{1}{2}\right), a_n = a_{n-1} + \left(\frac{1}{2}\right)^n$ for $n \geq 2$

Solution

a. Writing out the first few terms, we have

$$\begin{aligned} a_1 &= 2 \\ a_2 &= -3a_1 = -3(2) \\ a_3 &= -3a_2 = (-3)^2 2 \\ a_4 &= -3a_3 = (-3)^3 2. \end{aligned}$$

In general,

$$a_n = 2(-3)^{n-1}.$$

b. Write out the first few terms:

$$\begin{aligned} a_1 &= \frac{1}{2} \\ a_2 &= a_1 + \left(\frac{1}{2}\right)^2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} \\ a_3 &= a_2 + \left(\frac{1}{2}\right)^3 = \frac{3}{4} + \frac{1}{8} = \frac{7}{8} \\ a_4 &= a_3 + \left(\frac{1}{2}\right)^4 = \frac{7}{8} + \frac{1}{16} = \frac{15}{16} \quad . \end{aligned}$$

From this pattern, we derive the explicit formula

$$a_n = \frac{2^n - 1}{2^n} = 1 - \frac{1}{2^n} \ .$$

Exercise 1.2.2

Find an explicit formula for the sequence defined recursively such that $a_1 = -4$ and $a_n = a_{n-1} + 6$.

Hint

This is an arithmetic sequence.

Answer

$a_n = 6n - 10$

Limit of a Sequence

A fundamental question that arises regarding infinite sequences is the behavior of the terms as n gets larger. Since a sequence is a function defined on the positive integers, it makes sense to discuss the limit of the terms as $n \to \infty$. For example, consider the following four sequences and their different behaviors as $n \to \infty$ (Figure 1.2.2):

a. $\{1 + 3n\} = \{4, 7, 10, 13, \ldots\}$.The terms $1 + 3n$ become arbitrarily large as $n \to \infty$. In this case, we say that $1 + 3n \to \infty$ as $n \to \infty$.

b. $\left\{1 - \left(\frac{1}{2}\right)^n\right\} = \left\{\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16} \ldots\right\}$. The terms $1 - \left(\frac{1}{2}\right)^n \to 1$ as $n \to \infty$.

c. $\{(-1)^n\} = \{-1, 1, -1, 1, \ldots\}$.The terms alternate but do not approach one single value as $n \to \infty$.

d. $\left\{\dfrac{(-1)^n}{n}\right\} = \left\{-1, \dfrac{1}{2}, -\dfrac{1}{3}, \dfrac{1}{4}, \ldots\right\}$. The terms alternate for this sequence as well, but $\dfrac{(-1)^n}{n} \to 0$ as $n \to \infty$.

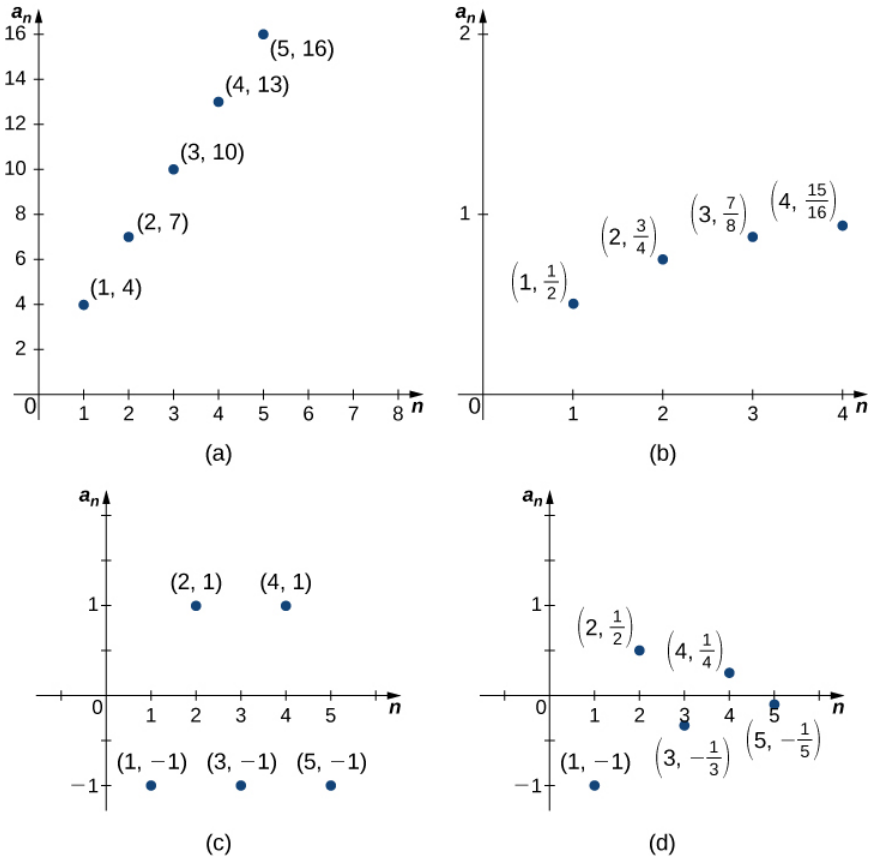


Figure 1.2.2: (a) The terms in the sequence become arbitrarily large as $n \to \infty$. (b) The terms in the sequence approach 1 as $n \to \infty$. (c) The terms in the sequence alternate between 1 and -1 as $n \to \infty$. (d) The terms in the sequence alternate between positive and negative values but approach 0 as $n \to \infty$.

From these examples, we see several possibilities for the behavior of the terms of a sequence as $n \to \infty$. In two of the sequences, the terms approach a finite number as $n \to \infty$. In the other two sequences, the terms do not. If the terms of a sequence approach a finite number L as $n \to \infty$, we say that the sequence is a convergent sequence and the real number L is the limit of the sequence. We can give an informal definition here.

Definition: convergent and divergent sequences

Given a sequence a_n, if the terms an become arbitrarily close to a finite number L as n becomes sufficiently large, we say $\{a_n\}$ is a **convergent sequence** and L is the **limit of the sequence.** In this case, we write

$$\lim_{n\to\infty} a_n = L. \tag{1.2.3}$$

If a sequence $\{a_n\}$ is not convergent, we say it is a **divergent sequence.**

From Figure, we see that the terms in the sequence $\left\{1 - \left(\dfrac{1}{2}\right)^n\right\}$ are becoming arbitrarily close to 1 as n becomes very large. We conclude that $\left\{1 - \left(\dfrac{1}{2}\right)^n\right\}$ is a convergent sequence and its limit is 1. In contrast, from Figure, we see that the terms in the sequence $1 + 3n$ are not approaching a finite number as n becomes larger. We say that $\{1 + 3n\}$ is a divergent sequence.

In the informal definition for the limit of a sequence, we used the terms “arbitrarily close” and “sufficiently large.” Although these phrases help illustrate the meaning of a converging sequence, they are somewhat vague. To be more precise, we now present the more formal definition of limit for a sequence and show these ideas graphically in Figure.

Definition: Convergence

A sequence $\{a_n\}$ converges to a real number L if for all $\varepsilon > 0$, there exists an integer N such that for all $n \geq N$ $|a_n - L| < \varepsilon$. The number L is the limit of the sequence and we write

$$\lim_{n\to\infty} a_n = L \text{ or } a_n \to L. \tag{1.2.4}$$

In this case, we say the sequence $\{a_n\}$ is a convergent sequence. If a sequence does not converge, it is a divergent sequence, and we say the limit does not exist.

We remark that the convergence or divergence of a sequence $\{a_n\}$ depends only on what happens to the terms a_n as $n \to \infty$. Therefore, if a finite number of terms $b_1, b_2, \ldots, b_N$ are placed before a_1 to create a new sequence

$$b_1,\ b_2,\ \ldots,\ b_N,\ a_1,\ a_2,\ \ldots,$$

this new sequence will converge if $\{a_n\}$ converges and diverge if $\{a_n\}$ diverges. Further, if the sequence $\{a_n\}$ converges to L, this new sequence will also converge to L.

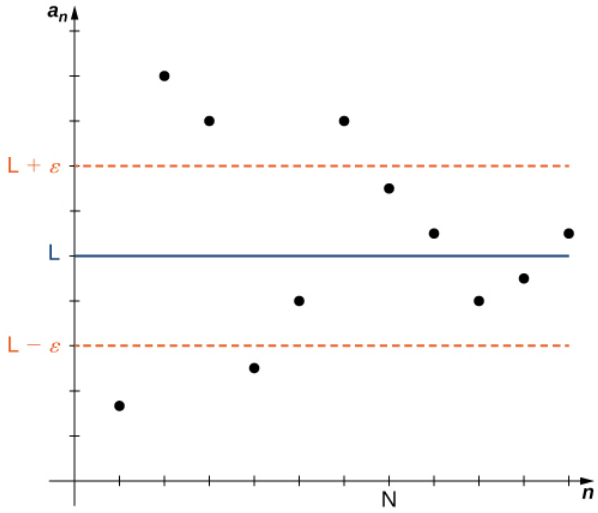


Figure 1.2.3: As n increases, the terms a_n become closer to L. For values of $n \geq N$, the distance between each point (n, a_n) and the line $y = L$ is less than ε.

As defined above, if a sequence does not converge, it is said to be a divergent sequence. For example, the sequences $\{1+3n\}$ and $\{(-1)^n\}$ shown in Figure diverge. However, different sequences can diverge in different ways. The sequence $\{(-1)^n\}$ diverges because the terms alternate between 1 and -1, but do not approach one value as $n \to \infty$. On the other hand, the sequence $\{1+3n\}$ diverges because the terms $1+3n \to \infty$ as $n \to \infty$. We say the sequence $\{1+3n\}$ diverges to infinity and write $\lim_{n\to\infty}(1+3n) = \infty$. It is important to recognize that this notation does not imply the limit of the sequence $\{1+3n\}$ exists. The sequence is, in fact, divergent. Writing that the limit is infinity is intended only to provide more information about why the sequence is divergent. A sequence can also diverge to negative infinity. For example, the sequence $\{-5n+2\}$ diverges to negative infinity because $-5n+2 \to -\infty$ as $n \to -\infty$. We write this as $\lim_{n\to\infty}(-5n+2) = \to -\infty$.

Because a sequence is a function whose domain is the set of positive integers, we can use properties of limits of functions to determine whether a sequence converges. For example, consider a sequence $\{a_n\}$ and a related function f defined on all positive real numbers such that $f(n) = a_n$ for all integers $n \geq 1$. Since the domain of the sequence is a subset of the domain of f, if $\lim_{x\to\infty} f(x)$ exists, then the sequence converges and has the same limit. For example, consider the sequence $\left\{\frac{1}{n}\right\}$ and the related function $f(x) = \frac{1}{x}$. Since the function f defined on all real numbers $x > 0$ satisfies $f(x) = \frac{1}{x} \to 0$ as $x \to \infty$, the sequence $\left\{\frac{1}{n}\right\}$ must satisfy $\frac{1}{n} \to 0$ as $n \to \infty$.

Limit of a Sequence Defined by a Function

Consider a sequence $\{a_n\}$ such that $a_n = f(n)$ for all $n \geq 1$. If there exists a real number L such that

$$\lim_{x\to\infty} f(x) = L, \tag{1.2.5}$$

then $\{a_n\}$ converges and

$$\lim_{n\to\infty} a_n = L. \tag{1.2.6}$$

We can use this theorem to evaluate $\lim_{n\to\infty} r^n$ for $0 \leq r \leq 1$. For example, consider the sequence $\{(1/2)^n\}$ and the related exponential function $f(x) = (1/2)^x$. Since $\lim_{x\to\infty}(1/2)^x = 0$, we conclude that the sequence $\{(1/2)^n\}$ converges and its limit is 0. Similarly, for any real number r such that $0 \leq r < 1$, $\lim_{x\to\infty} r^x = 0$, and therefore the sequence $\{r^n\}$ converges. On the other hand, if $r = 1$, then $\lim_{x\to\infty} r^x = 1$, and therefore the limit of the sequence $\{1^n\}$ is 1. If $r > 1$, $\lim_{x\to\infty} r^x = \infty$, and therefore we cannot apply this theorem. However, in this case, just as the function r^x grows without bound as $n \to \infty$, the terms r^n in the sequence become arbitrarily large as $n \to \infty$, and we conclude that the sequence $\{r^n\}$ diverges to infinity if $r > 1$.

We summarize these results regarding the geometric sequence r^n:

$$r^n \to 0 \text{ if } 0 < r < 1$$

$$r^n \to 1 \text{ if } r = 1$$

$$r^n \to \infty \text{ if } r > 1.$$

Later in this section we consider the case when $r < 0$.

We now consider slightly more complicated sequences. For example, consider the sequence $\{(2/3)^n + (1/4)^n\}$. The terms in this sequence are more complicated than other sequences we have discussed, but luckily the limit of this sequence is determined by the limits of the two sequences $\{(2/3)^n\}$ and $\{(1/4)^n\}$. As we describe in the following algebraic limit laws, since $\{(2/3)^n\}$ and $\{1/4)^n\}$ both converge to 0, the sequence $\{(2/3)^n + (1/4)^n\}$ converges to $0 + 0 = 0$. Just as we were able to evaluate a limit involving an algebraic combination of functions f and g by looking at the limits of f and g (see Introduction to Limits), we are able to evaluate the limit of a sequence whose terms are algebraic combinations of a_n and b_n by evaluating the limits of $\{a_n\}$ and $\{b_n\}$.

Algebraic Limit Laws

Given sequences $\{a_n\}$ and $\{b_n\}$ and any real number c, if there exist constants A and B such that $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$, then

i. $\lim_{n\to\infty} c = c$

ii. $\lim_{n\to\infty} ca_n = c\lim_{n\to\infty} a_n = cA$

iii. $\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n = A \pm B$

iv. $\lim_{n\to\infty}(a_n \cdot b_n) = \left(\lim_{n\to\infty} a_n\right) \cdot \left(\lim_{n\to\infty} b_n\right) = A \cdot B$

v. $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{A}{B}$, provided $B \neq 0$ and each $b_n \neq 0$.

Proof

We prove part iii.

Let $\epsilon > 0$. Since $\lim_{n\to\infty} a_n = A$, there exists a constant positive integer N_1 such that for all $n \geq N_1$. Since $\lim_{n\to\infty} b_n = B$, there exists a constant N_2 such that $|b_n - B| < \varepsilon/2$ for all $n \geq N_2$. Let N be the largest of N_1 and N_2. Therefore, for all $n \geq N$, $|(a_n + b_n) - (A + B)| \leq |a_n - A| + |b_n - B| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

□

The algebraic limit laws allow us to evaluate limits for many sequences. For example, consider the sequence $a_n = \dfrac{1}{n^2}$. As shown earlier, $\lim\limits_{n\to\infty} \dfrac{1}{n} = 0$. Similarly, for any positive integer k, we can conclude that

$$\lim_{n\to\infty} \frac{1}{n^k} = 0. \tag{1.2.7}$$

In the next example, we make use of this fact along with the limit laws to evaluate limits for other sequences.

Example 1.2.3: Determining Convergence and Finding Limits

For each of the following sequences, determine whether or not the sequence converges. If it converges, find its limit.

a. $\left\{5 - \dfrac{3}{n^2}\right\}$

b. $\left\{\dfrac{3n^4 - 7n^2 + 5}{6 - 4n^4}\right\}$

c. $\left\{\dfrac{2^n}{n^2}\right\}$

d. $\left\{\left(1 + \dfrac{4}{n}\right)^n\right\}$

Solution

a. We know that $\lim\limits_{n\to\infty} \dfrac{1}{n} = 0$. Using this fact, we conclude that

$$\lim_{n\to\infty} \frac{1}{n^2} = \lim_{n\to\infty} \frac{1}{n} . \lim_{n\to\infty} \frac{1}{n} = 0.$$

Therefore,

$$\lim_{n\to\infty} \left(5 - \frac{3}{n^2}\right) = \lim_{n\to\infty} 5 - 3 \lim_{n\to\infty} \frac{1}{n^2} = 5 - 3.0 = 5.$$

The sequence converges and its limit is 5.

b. By factoring n^4 out of the numerator and denominator and using the limit laws above, we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{3n^4 - 7n^2 + 5}{6 - 4n^4} &= \lim_{n\to\infty} \frac{3 - \dfrac{7}{n^2} + \dfrac{5}{n^4}}{\dfrac{6}{n^4} - 4} \\
&= \frac{\lim_{n\to\infty}(3 - \frac{7}{n^2} + \frac{5}{n^4})}{\lim_{n\to\infty}(\frac{6}{n^4} - 4)} \\
&= \frac{\lim_{n\to\infty}(3) - \lim_{n\to\infty} \dfrac{7}{n^2} + \lim_{n\to\infty} \dfrac{5}{n^4}}{\lim_{n\to\infty} \dfrac{6}{n^4} - \lim_{n\to\infty}(4)} \\
&= \frac{\lim_{n\to\infty}(3) - 7 \cdot \lim_{n\to\infty} \dfrac{1}{n^2} + 5 \cdot \lim_{n\to\infty} \dfrac{1}{n^4}}{6 \cdot \lim_{n\to\infty} \dfrac{1}{n^4} - \lim_{n\to\infty}(4)} \\
&= \frac{3 - 7 \cdot 0 + 5 \cdot 0}{6 \cdot 0 - 4} = -\frac{3}{4}.
\end{aligned}$$

The sequence converges and its limit is $-3/4$.

c. Consider the related function $f(x) = 2^x/x^2$ defined on all real numbers $x > 0$. Since $2^x \to \infty$ and $x^2 \to \infty$ as $x \to \infty$, apply L'Hôpital's rule and write

$$\begin{aligned} \lim_{x\to\infty} \frac{2^x}{x^2} &= \lim_{x\to\infty} \frac{2^x \ln 2}{2x} && \text{Take the derivatives of the numerator and denominator.} \\ &= \lim_{x\to\infty} \frac{2^x (\ln 2)^2}{2} && \text{Take the derivatives again.} \\ &= \infty. \end{aligned}$$

We conclude that the sequence diverges.

d. Consider the function $f(x) = \left(1 + \frac{4}{x}\right)^x$ defined on all real numbers $x > 0$. This function has the indeterminate form 1^∞ as $x \to \infty$. Let

$$y = \lim_{x\to\infty} \left(1 + \frac{4}{x}\right)^x.$$

Now taking the natural logarithm of both sides of the equation, we obtain

$$\ln(y) = \ln\left[\lim_{x\to\infty} \left(1 + \frac{4}{x}\right)^x\right].$$

Since the function $f(x) = \ln(x)$ is continuous on its domain, we can interchange the limit and the natural logarithm. Therefore,

$$\ln(y) = \lim_{x\to\infty} \left[\ln\left(1 + \frac{4}{x}\right)^x\right].$$

Using properties of logarithms, we write

$$\lim_{x\to\infty} \left[\ln\left(1 + \frac{4}{x}\right)^x\right] = \lim_{x\to\infty} x \ln\left(1 + \frac{4}{x}\right).$$

Since the right-hand side of this equation has the indeterminate form $\infty \cdot 0$, rewrite it as a fraction to apply L'Hôpital's rule. Write

$$\lim_{x\to\infty} x \ln\left(1 + \frac{4}{x}\right) = \lim_{x\to\infty} \frac{\ln(1 + 4/x)}{1/x}.$$

Since the right-hand side is now in the indeterminate form 0/0, we are able to apply L'Hôpital's rule. We conclude that

$$\lim_{x\to\infty} \frac{\ln(1 + 4/x)}{1/x} = \lim_{x\to\infty} \frac{4}{1 + 4/x} = 4.$$

Therefore, $\ln(y) = 4$ and $y = e^4$. Therefore, since $\lim_{x\to\infty} \left(1 + \frac{4}{x}\right)^x = e^4$, we can conclude that the sequence $\left\{\left(1 + \frac{4}{n}\right)^n\right\}$ converges to e^4.





Exercise 1.2.3

Consider the sequence $\left\{(5n^2+1)/e^n\right\}$. Determine whether or not the sequence converges. If it converges, find its limit.

Hint

Use L'Hôpital's rule.

Answer

The sequence converges, and its limit is 0

Recall that if f is a continuous function at a value L, then $f(x) \to f(L)$ as $x \to L$. This idea applies to sequences as well. Suppose a sequence $a_n \to L$, and a function f is continuous at L. Then $f(a_n) \to f(L)$. This property often enables us to find limits for

complicated sequences. For example, consider the sequence $\sqrt{5-\frac{3}{n^2}}$. From Example a. we know the sequence $5-\frac{3}{n^2} \to 5$. Since $\sqrt{x}$ is a continuous function at $x=5$,

$$\lim_{n\to\infty}\sqrt{5-\frac{3}{n^2}} = \sqrt{\lim_{n\to\infty}(5-\frac{3}{n^2})} = \sqrt{5}.$$

Continuous Functions Defined on Convergent Sequences

Consider a sequence $\{a_n\}$ and suppose there exists a real number L such that the sequence $\{a_n\}$ converges to L. Suppose f is a continuous function at L. Then there exists an integer N such that f is defined at all values an for $n \geq N$, and the sequence $\{f(a_n)\}$ converges to $f(L)$ (Figure 1.2.4).

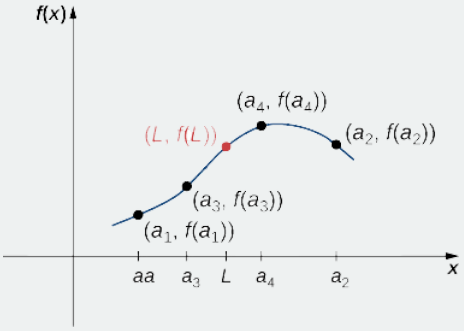


Figure 1.2.4: Because f is a continuous function as the inputs $a_1, a_2, a_3, \ldots$ approach L, the outputs $f(a_1), f(a_2), f(a_3), \ldots$ approach $f(L)$.

Proof

Let $\epsilon > 0$. Since f is continuous at L, there exists $\delta > 0$ such that $|f(x)-f(L)| < \varepsilon$ if $|x-L| < \delta$. Since the sequence $\{a_n\}$ converges to L, there exists N such that $|a_n - L| < \delta$ for all $n \geq N$. Therefore, for all $n \geq N$, $|a_n - L| < \delta$, which implies $|f(a_n) - f(L)| < \varepsilon$. We conclude that the sequence $\{f(a_n)\}$ converges to $f(L)$.

□

Example 1.2.4: Limits Involving Continuous Functions Defined on Convergent Sequences

Determine whether the sequence $\{\cos(3/n^2)\}$ converges. If it converges, find its limit.

Solution:

Since the sequence $\{3/n^2\}$ converges to 0 and $\cos x$ is continuous at $x=0$, we can conclude that the sequence $\{\cos(3/n^2)\}$ converges and

$$\lim_{n\to\infty}\cos\left(\frac{3}{n^2}\right) = \cos 0 = 1.$$

Exercise 1.2.4

Determine if the sequence $\left\{\sqrt{\frac{2n+1}{3n+5}}\right\}$ converges. If it converges, find its limit.

Hint

Consider the sequence $\left\{\frac{2n+1}{3n+5}\right\}$.

Answer

The sequence converges, and its limit is $\sqrt{2/3}$.

Another theorem involving limits of sequences is an extension of the Squeeze Theorem for limits discussed in Introduction to Limits.

Squeeze Theorem for Sequences

Consider sequences $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$. Suppose there exists an integer N such that

$$a_n \le b_n \le c_n \text{ for all } n \ge N.$$

If there exists a real number L such that

$$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n, \tag{1.2.8}$$

then $\{b_n\}$ converges and $\lim_{n\to\infty} b_n = L$ (Figure 1.2.5).

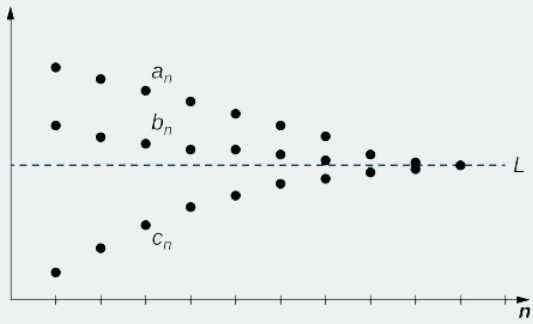


Figure 1.2.5: Each term bn satisfies $a_n \le b_n \le c_n$ and the sequences $\{a_n\}$ and $\{c_n\}$ converge to the same limit, so the sequence $\{b_n\}$ must converge to the same limit as well.

Proof

Let $\varepsilon > 0$. Since the sequence $\{a_n\}$ converges to L, there exists an integer N_1 such that $|a_n - L| < \varepsilon$ for all $n \ge N_1$. Similarly, since $\{c_n\}$ converges to L, there exists an integer N_2 such that $|c_n - L| < \varepsilon$ for all $n \ge N_2$. By assumption, there exists an integer N such that $a_n \le b_n \le c_n$ for all $n \ge N$. Let M be the largest of N_1, N_2, and N. We must show that $|b_n - L| < \varepsilon$ for all $n \ge M$. For all $n \ge M$,

$$-\varepsilon < -|a_n - L| \le a_n - L \le b_n - L \le c_n - L \le |c_n - L| < \varepsilon$$

Therefore, $-\varepsilon < b_n - L < \varepsilon$, and we conclude that $|b_n - L| < \varepsilon$ for all $n \ge M$, and we conclude that the sequence b_n converges to L.

□

Example 1.2.5: Using the Squeeze Theorem

Use the Squeeze Theorem to find the limit of each of the following sequences.

a. $\left\{\dfrac{\cos n}{n^2}\right\}$

b. $\left\{\left(-\dfrac{1}{2}\right)^n\right\}$

Solution

a. Since $-1 \le \cos n \le 1$ for all integers n, we have

$$-\frac{1}{n^2} \le \frac{\cos n}{n^2} \le \frac{1}{n^2}.$$

Since $-1/n^2 \to 0$ and $1/n^2 \to 0$, we conclude that $\cos n/n^2 \to 0$ as well.

b. Since

$$-\frac{1}{2^n} \le \left(-\frac{1}{2}\right)^n \le \frac{1}{2^n}$$

for all positive integers n, $-1/2^n \to 0$ and $1/2^n \to 0$, we can conclude that $(-1/2)^n \to 0$.

Exercise 1.2.5

Find $\lim_{n\to\infty} \dfrac{2n - \sin n}{n}$.

Hint

Use the fact that $-1 \leq \sin n \leq 1$.

Answer

2

Using the idea from Example 1.2.5 b we conclude that $r^n \to 0$ for any real number r such that $-1 < r < 0$. If $r < -1$, the sequence r^n diverges because the terms oscillate and become arbitrarily large in magnitude. If $r = -1$, the sequence $r^n = (-1)^n$ diverges, as discussed earlier. Here is a summary of the properties for geometric sequences.

$$r^n \to 0 \text{ if } |r| < 1$$

$$r^n \to 1 \text{ if } r = 1$$

$$r^n \to \infty \text{ if } r > 1$$

$$\{r^n\} \text{ diverges if } r \leq -1$$

Bounded Sequences

We now turn our attention to one of the most important theorems involving sequences: the Monotone Convergence Theorem. Before stating the theorem, we need to introduce some terminology and motivation. We begin by defining what it means for a sequence to be bounded.

Definition: Bound Sequences

A sequence $\{a_n\}$ is **bounded above** if there exists a real number M such that

$$a_n \leq M$$

for all positive integers n.

A sequence $\{a_n\}$ is **bounded below** if there exists a real number m such that

$$m \leq a_n$$

for all positive integers n.

A sequence $\{a_n\}$ is a **bounded sequence** if it is bounded above and bounded below.

If a sequence is not bounded, it is an **unbounded sequence.**

For example, the sequence $\{1/n\}$ is bounded above because $1/n \leq 1$ for all positive integers n. It is also bounded below because $1/n \geq 0$ for all positive integers n. Therefore, $\{1/n\}$ is a bounded sequence. On the other hand, consider the sequence $\{2^n\}$. Because $2^n \geq 2$ for all $n \geq 1$, the sequence is bounded below. However, the sequence is not bounded above. Therefore, $\{2^n\}$ is an unbounded sequence.

We now discuss the relationship between boundedness and convergence. Suppose a sequence $\{a_n\}$ is unbounded. Then it is not bounded above, or not bounded below, or both. In either case, there are terms an that are arbitrarily large in magnitude as n gets larger. As a result, the sequence $\{a_n\}$ cannot converge. Therefore, being bounded is a necessary condition for a sequence to converge.

Convergent Sequences Are Bounded

If a sequence $\{a_n\}$ converges, then it is bounded.

Note that a sequence being bounded is not a sufficient condition for a sequence to converge. For example, the sequence $\{(-1)^n\}$ is bounded, but the sequence diverges because the sequence oscillates between 1 and -1 and never approaches a finite number. We now discuss a sufficient (but not necessary) condition for a bounded sequence to converge.

Consider a bounded sequence $\{a_n\}$. Suppose the sequence $\{a_n\}$ is increasing. That is, $a_1 \leq a_2 \leq a_3 \ldots$. Since the sequence is increasing, the terms are not oscillating. Therefore, there are two possibilities. The sequence could diverge to infinity, or it could converge. However, since the sequence is bounded, it is bounded above and the sequence cannot diverge to infinity. We conclude that $\{a_n\}$ converges. For example, consider the sequence

$$\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\right\}. \tag{1.2.9}$$

Since this sequence is increasing and bounded above, it converges. Next, consider the sequence

$$\left\{2, 0, 3, 0, 4, 0, 1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, \ldots\right\}. \tag{1.2.10}$$

Even though the sequence is not increasing for all values of n, we see that $-1/2 < -1/3 < -1/4 < \cdots$. Therefore, starting with the eighth term, $a_8 = -1/2$, the sequence is increasing. In this case, we say the sequence is eventually increasing. Since the sequence is bounded above, it converges. It is also true that if a sequence is decreasing (or eventually decreasing) and bounded below, it also converges.

Definition

A sequence $\{a_n\}$ is increasing for all $n \geq n_0$ if

$$a_n \leq a_{n+1} \text{ for all } n \geq n_0.$$

A sequence $\{a_n\}$ is decreasing for all $n \geq n_0$ if

$$a_n \geq a_{n+1} \text{ for all } n \geq n_0.$$

A sequence $\{a_n\}$ is a *monotone sequence* for all $n \geq n_0$ if it is increasing for all $n \geq n_0$ or decreasing for all $n \geq n_0$.

We now have the necessary definitions to state the Monotone Convergence Theorem, which gives a sufficient condition for convergence of a sequence.

Definition: Monotone Convergence Theorem

If $\{a_n\}$ is a bounded sequence and there exists a positive integer n_0 such that $\{a_n\}$ is monotone for all $n \geq n_0$, then $\{a_n\}$ converges.

The proof of this theorem is beyond the scope of this text. Instead, we provide a graph to show intuitively why this theorem makes sense (Figure 1.2.6).

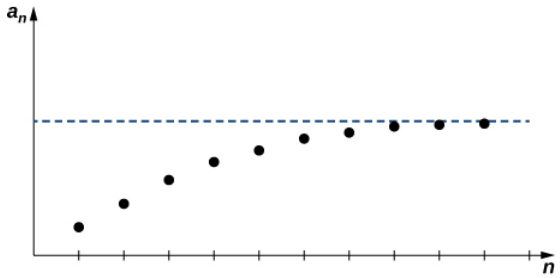


Figure 1.2.6: Since the sequence $\{a_n\}$ is increasing and bounded above, it must converge.

In the following example, we show how the Monotone Convergence Theorem can be used to prove convergence of a sequence.

Example 1.2.6: Using the Monotone Convergence Theorem

For each of the following sequences, use the Monotone Convergence Theorem to show the sequence converges and find its limit.

a. $\left\{\dfrac{4^n}{n!}\right\}$

b. $\{a_n\}$ defined recursively such that

$$a_1 = 2 \text{ and } a_{n+1} = \frac{a_n}{2} + \frac{1}{2a_n} \text{ for all } n \geq 2.$$

Solution

a. Writing out the first few terms, we see that

$$\left\{\frac{4^n}{n!}\right\} = \left\{4,\ 8,\ \frac{32}{3},\ \frac{32}{3},\ \frac{128}{15},\ \ldots\right\}.$$

At first, the terms increase. However, after the third term, the terms decrease. In fact, the terms decrease for all $n \geq 3$. We can show this as follows.

$$a_{n+1} = \frac{4^{n+1}}{(n+1)!} = \frac{4}{n+1} \cdot \frac{4^n}{n!} = \frac{4}{n+1} \cdot a_n \leq a_n \quad \text{if } n \geq 3.$$

Therefore, the sequence is decreasing for all $n \geq 3$. Further, the sequence is bounded below by 0 because $4n/n! \geq 0$ for all positive integers n. Therefore, by the Monotone Convergence Theorem, the sequence converges.

To find the limit, we use the fact that the sequence converges and let $L = \lim\limits_{n\to\infty} a_n$. Now note this important observation. Consider $\lim\limits_{n\to\infty} a_{n+1}$. Since

$$\{a_{n+1}\} = \{a_2,\ a_3,\ a_4,\ \ldots\},$$

the only difference between the sequences $\{a_{n+1}\}$ and $\{a_n\}$ is that $\{a_{n+1}\}$ omits the first term. Since a finite number of terms does not affect the convergence of a sequence,

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n = L.$$

Combining this fact with the equation

$$a_{n+1} = \frac{4}{n+1} a_n$$

and taking the limit of both sides of the equation

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{4}{n+1} a_n\,,$$

we can conclude that

$$L = 0 \cdot L = 0.$$

b. Writing out the first several terms,

$$\left\{2,\ \frac{5}{4},\ \frac{41}{40},\ \frac{3281}{3280},\ \ldots\right\}.$$

we can conjecture that the sequence is decreasing and bounded below by 1. To show that the sequence is bounded below by 1, we can show that

$$\frac{a_n}{2} + \frac{1}{2a_n} \geq 1.$$

To show this, first rewrite

$$\frac{a_n}{2} + \frac{1}{2a_n} = \frac{a_n^2 + 1}{2a_n}\ .$$

Since $a_1 > 0$ and a_2 is defined as a sum of positive terms, $a_2 > 0$. Similarly, all terms $a_n > 0$. Therefore,

$$\frac{a^2 n+1}{2a_n} \geq 1$$

if and only if

$$a_n^2 + 1 \geq 2a_n \, .$$

Rewriting the inequality $a_n^2 + 1 \geq 2a_n$ as $a_n^2 - 2a_n + 1 \geq 0$, and using the fact that

$$a_n^2 - 2a_n + 1 = (a_n - 1)^2 \geq 0$$

because the square of any real number is nonnegative, we can conclude that

$$\frac{a^n}{2} + \frac{1}{2a_n} \geq 1.$$

To show that the sequence is decreasing, we must show that $a_{n+1} \leq a_n$ for all $n \geq 1$. Since $1 \leq a_n^2$, it follows that

$$a_n^2 + 1 \leq 2a_n^2 \, .$$

Dividing both sides by $2a_n$, we obtain

$$\frac{a_n}{2} + \frac{1}{2a_n} \leq a_n.$$

Using the definition of a_{n+1}, we conclude that

$$a_{n+1} = \frac{a_n}{2} + \frac{1}{2a_n} \leq a_n \ .$$

Since $\{a_n\}$ is bounded below and decreasing, by the Monotone Convergence Theorem, it converges.

To find the limit, let $L = \lim\limits_{n\to\infty} a_n$. Then using the recurrence relation and the fact that $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} a_{n+1}$, we have

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} (\frac{a_n}{2} + \frac{1}{2a_n}) \, ,$$

and therefore

$$L = \frac{L}{2} + \frac{1}{2L} \ .$$

Multiplying both sides of this equation by $2L$, we arrive at the equation

$$2L^2 = L^2 + 1 \, .$$

Solving this equation for L, we conclude that $L^2 = 1$, which implies $L = \pm 1$. Since all the terms are positive, the limit $L = 1$.

Exercise 1.2.6

Consider the sequence $\{a_n\}$ defined recursively such that $a_1 = 1$, $a_n = a_{n-1}/2$. Use the Monotone Convergence Theorem to show that this sequence converges and find its limit.

Hint

Show the sequence is decreasing and bounded below.

Answer

0.

Definition: Fibonacci Numbers

The *Fibonacci numbers* are defined recursively by the sequence $\{F_n\}$ where $F_0 = 0$, $F_1 = 1$ and for $n \geq 2$,

$$F_n = F_{n-1} + F_{n-2}.$$

Here we look at properties of the Fibonacci numbers.

1. Write out the first twenty Fibonacci numbers.

2. Find a closed formula for the Fibonacci sequence by using the following steps.

   a. Consider the recursively defined sequence x_n where $x_0 = c$ and $x_{n+1} = ax_n$. Show that this sequence can be described by the closed formula $x_n = ca^n$ for all $n \geq 0$.

   b. Using the result from part a. as motivation, look for a solution of the equation

   $F_n = F_{n-1} + F_{n-2}$

   of the form $F_n = c\lambda^n$. Determine what two values for λ will allow F_n to satisfy this equation.

   c. Consider the two solutions from part b.: λ_1 and λ_2. Let $F_n = c_1\lambda_1^n + c_2\lambda_2^n$. Use the initial conditions F_0 and F_1 to determine the values for the constants c_1 and c_2 and write the closed formula F_n.

3. Use the answer in 2 c. to show that

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2}.$$

The number $\phi = (1+\sqrt{5})/2$ is known as the **golden ratio** (Figure and Figure).



Figure 1.2.7: The seeds in a sunflower exhibit spiral patterns curving to the left and to the right. The number of spirals in each direction is always a Fibonacci number—always. (credit: modification of work by Esdras Calderan, Wikimedia Commons)

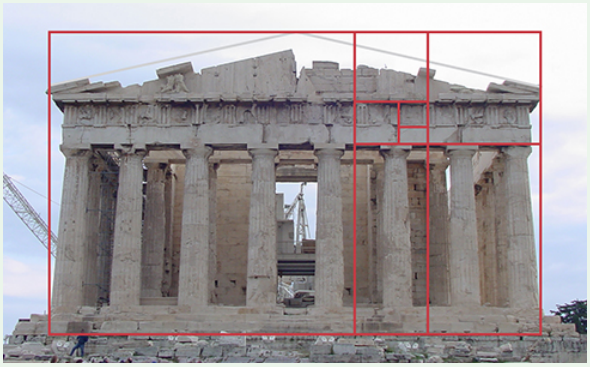


Figure 1.2.8: The proportion of the golden ratio appears in many famous examples of art and architecture. The ancient Greek temple known as the Parthenon was designed with these proportions, and the ratio appears again in many of the smaller details. (credit: modification of work by TravelingOtter, Flickr).

Key Concepts

- To determine the convergence of a sequence given by an explicit formula $a_n = f(n)$, we use the properties of limits for functions.
- If $\{a_n\}$ and $\{b_n\}$ are convergent sequences that converge to A and B, respectively, and c is any real number, then the sequence $\{ca_n\}$converges to $c \cdot A$, the sequences $\{a_n \pm b_n\}$ converge to $A \pm B$, the sequence $\{a_n \cdot b_n\}$ converges to $A \cdot B$, and the sequence $\{a_n/b_n\}$ converges to A/B, provided $B \neq 0$.
- If a sequence is bounded and monotone, then it converges, but not all convergent sequences are monotone.
- If a sequence is unbounded, it diverges, but not all divergent sequences are unbounded.
- The geometric sequence $\{r^n\}$ converges if and only if $|r| < 1$ or $r = 1$.

Glossary

arithmetic sequence
a sequence in which the difference between every pair of consecutive terms is the same is called an arithmetic sequence

bounded above
a sequence $\{a_n\}$ is bounded above if there exists a constant M such that $a_n \leq M$ for all positive integers n

bounded below
a sequence $\{a_n\}$ is bounded below if there exists a constant M such that $M \leq a_n$ for all positive integers n

bounded sequence
a sequence $\{a_n\}$ is bounded if there exists a constant M such that $|a_n| \leq M$ for all positive integers n

convergent sequence
a convergent sequence is a sequence $\{a_n\}$ for which there exists a real number L such that a_n is arbitrarily close to L as long as n is sufficiently large

divergent sequence
a sequence that is not convergent is divergent

explicit formula
a sequence may be defined by an explicit formula such that $a_n = f(n)$

geometric sequence
a sequence $\{a_n\}$ in which the ratio a_{n+1}/a_n is the same for all positive integers n is called a geometric sequence

index variable
the subscript used to define the terms in a sequence is called the index

limit of a sequence
the real number L to which a sequence converges is called the limit of the sequence

monotone sequence
an increasing or decreasing sequence

recurrence relation
a recurrence relation is a relationship in which a term a_n in a sequence is defined in terms of earlier terms in the sequence

sequence
an ordered list of numbers of the form $a_1, a_2, a_3, \ldots$ is a sequence

term
the number a_n in the sequence $\{a_n\}$ is called the nth term of the sequence

unbounded sequence
a sequence that is not bounded is called unbounded

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1.2E: Exercises for Section 1.2

In exercises 1 - 4, find the first six terms of each sequence, starting with $n=1$.

1) $a_n = 1+(-1)^n$ for $n \geq 1$

Answer

$a_n = 0$ if n is odd and $a_n = 2$ if n is even

2) $a_n = n^2 - 1$ for $n \geq 1$

3) $a_1 = 1$ and $a_n = a_{n-1} + n$ for $n \geq 2$

Answer

$a_n = 1, 3, 6, 10, 15, 21, \ldots$

4) $a_1 = 1, a_2 = 1$ and $a_n + 2 = a_n + a_{n+1}$ for $n \geq 1$

5) Find an explicit formula for a_n where $a_1 = 1$ and $a_n = a_{n-1} + n$ for $n \geq 2$.

Answer

$$a_n = \frac{n(n+1)}{2}$$

6) Find a formula a_n for the nth term of the arithmetic sequence whose first term is $a_1 = 1$ such that $a_{n+1} - a_n = 17$ for $n \geq 1$.

7) Find a formula a_n for the nth term of the arithmetic sequence whose first term is $a_1 = -3$ such that $a_{n+1} - a_n = 4$ for $n \geq 1$.

Answer

$a_n = 4n - 7$

8) Find a formula a_n for the nth term of the geometric sequence whose first term is $a_1 = 1$ such that $\frac{a_{n+1}}{a_n} = 10$ for $n \geq 1$.

9) Find a formula a_n for the nth term of the geometric sequence whose first term is $a_1 = 3$ such that $\frac{a_{n+1}}{a_n} = 1/10$ for $n \geq 1$.

Answer

$a_n = 3.10^{1-n} = 30.10^{-n}$

10) Find an explicit formula for the nth term of the sequence whose first several terms are $0, 3, 8, 15, 24, 35, 48, 63, 80, 99, \ldots$. (Hint: First add one to each term.)

11) Find an explicit formula for the nth term of the sequence satisfying $a_1 = 0$ and $a_n = 2a_{n-1} + 1$ for $n \geq 2$.

Answer

$a_n = 2^{n-1} - 1$

In exercises 12 and 13, find a formula for the general term a_n of each of the following sequences.

12) $1, 0, -1, 0, 1, 0, -1, 0, \ldots$ (Hint: Find where $\sin x$ takes these values)

13) $1, -1/3, 1/5, -1/7, \ldots$

Answer

$$a_n = \frac{(-1)^{n-1}}{2n-1}$$

In exercises 14-18, find a function $f(n)$ that identifies the nth term a_n of the following recursively defined sequences, as $a_n = f(n)$.

14) $a_1 = 1$ and $a_{n+1} = -a_n$ for $n \geq 1$

15) $a_1 = 2$ and $a_{n+1} = 2a_n$ for $n \geq 1$

Answer

$f(n) = 2^n$

16) $a_1 = 1$ and $a_{n+1} = (n+1)a_n$ for $n \geq 1$

17) $a_1 = 2$ and $a_{n+1} = (n+1)a_n/2$ for $n \geq 1$

Answer

$$f(n) = \frac{n!}{2^{n-2}}$$

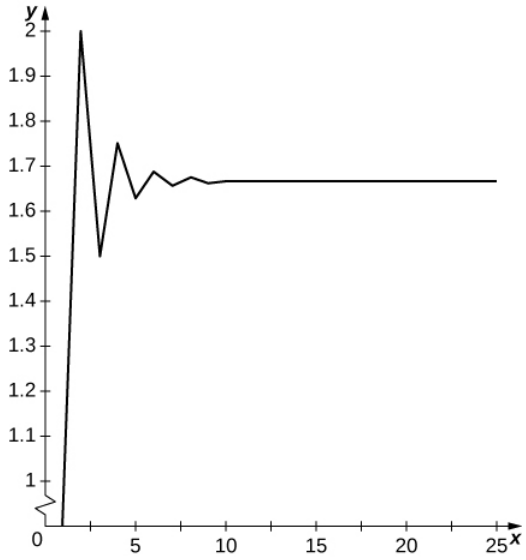
18) $a_1 = 1$ and $a_{n+1} = a_n/2^n$ for $n \geq 1$

In exercises 19 - 22, plot the first N terms of the given sequence. State whether the graphical evidence suggests that the sequence converges or diverges.

19) [T] $a_1 = 1, a_2 = 2$, and for $n \geq 2, a_n = \frac{1}{2}(a_{n-1} + a_{n-2})$; $N = 30$

Answer

Terms oscillate above and below $5/3$ and appear to converge to $5/3$.

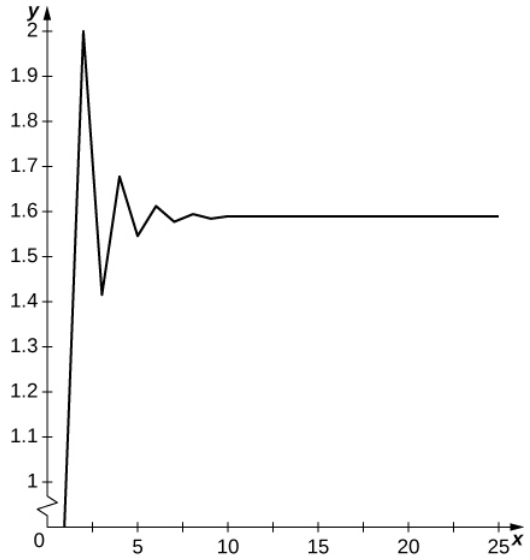


20) [T] $a_1 = 1, a_2 = 2, a_3 = 3$ and for $n \geq 4, a_n = \frac{1}{3}(a_{n-1} + a_{n-2} + a_{n-3}), N = 30$

21) [T] $a_1 = 1, a_2 = 2$, and for $n \geq 3, a_n = \sqrt{a_{n-1}a_{n-2}}; N = 30$

Answer

Terms oscillate above and below $y \approx 1.57..$ and appear to converge to a limit.



22) [T] $a_1 = 1, a_2 = 2, a_3 = 3$, and for $n \geq 4, a_n = \sqrt{a_{n-1}a_{n-2}a_{n-3}}; N = 30$

In exercises 23 - 16, suppose that $\lim_{n\to\infty} a_n = 1$, $\lim_{n\to\infty} b_n = -1$, and $0 < -b_n < a_n$ for all n.

Using this information, evaluate each of the following limits, state that the limit does not exist, or state that there is not enough information to determine whether the limit exists.

23) $\lim_{n\to\infty} 3a_n - 4b_n$

Answer

$\lim_{n\to\infty} 3a_n - 4b_n = 7$

24) $\lim_{n\to\infty} \frac{1}{2}b_n - \frac{1}{2}a_n$

25) $\lim_{n\to\infty} \frac{a_n + b_n}{a_n - b_n}$

Answer

$\lim_{n\to\infty} \frac{a_n + b_n}{a_n - b_n} = 0$

26) $\lim_{n\to\infty} \frac{a_n - b_n}{a_n + b_n}$

In exercises 27 - 30, find the limit of each of the following sequences, using L'Hôpital's rule when appropriate.

27) $\frac{n^2}{2^n}$

Answer

$\lim_{n\to\infty} \frac{n^2}{2^n} = 0$

28) $\frac{(n-1)^2}{(n+1)^2}$

29) $\frac{\sqrt{n}}{\sqrt{n+1}}$

Answer

$$\lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n+1}} \quad = \quad 1$$

30) $n^{1/n}$ (Hint: $n^{1/n}=e^{\frac{1}{n}\ln n}$)

In exercises 31 - 37, state whether each sequence is bounded and whether it is eventually monotone, increasing, or decreasing.

31) $n/2^n, n\geq 2$

Answer

bounded, decreasing for $n\geq 1$

32) $\ln\left(1+\frac{1}{n}\right)$

33) $\sin n$

Answer

bounded, not monotone

34) $\cos(n^2)$

35) $n^{1/n}, \quad n\geq 3$

Answer

bounded, decreasing

36) $n^{-1/n}, \quad n\geq 3$

37) $\tan n$

Answer

not monotone, not bounded

In exercises 38 - 39, determine whether the given sequence has a limit. If it does, find the limit.

38) $a_1=\sqrt{2}, a_2=\sqrt{2\sqrt{2}}. a_3=\sqrt{2\sqrt{2\sqrt{2}}}$ etc.

39) $a_1=3, a_n=\sqrt{2a_{n-1}}, n=2,3,\ldots.$

Answer

a_n is decreasing and bounded below by 2. The limit a must satisfy $a=\sqrt{2a}$ so $a=2$, independent of the initial value.

Use the Squeeze Theorem to find the limit of each sequence in exercises 40 - 43.

40) $n\sin(1/n)$

41) $\frac{\cos(1/n)-1}{1/n}$

Answer

0

42) $a_n=\frac{n!}{n^n}$

43) $a_n=\sin n\sin(1/n)$

Answer

0 since $|\sin x| \leq |x|$ and $|\sin x| \leq 1$ so $-\frac{1}{n} \leq a_n \leq \frac{1}{n}$) .

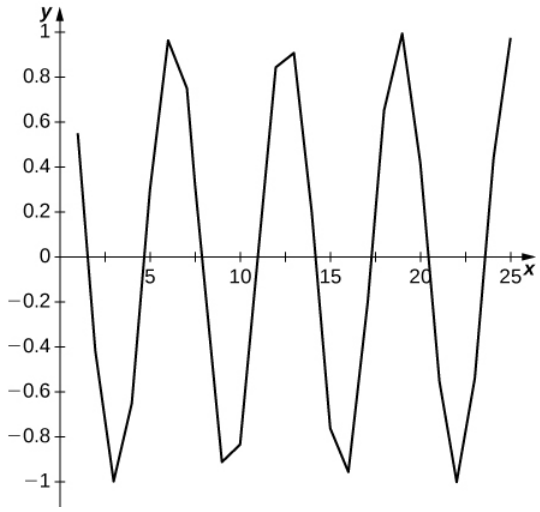
For the sequences in exercises 44 and 45, plot the first 25 terms of the sequence and state whether the graphical evidence suggests that the sequence converges or diverges.

44) [T] $a_n = \sin n$

45) [T] $a_n = \cos n$

Answer

Graph oscillates and suggests no limit.



In exercises 46 - 52, determine the limit of the sequence or show that the sequence diverges. If it converges, find its limit.

46) $a_n = \tan^{-1}(n^2)$

47) $a_n = (2n)^{1/n} - n^{1/n}$

Answer

$n^{1/n} \to 1$ and $2^{1/n} \to 1$, so $a_n \to 0$

48) $a_n = \dfrac{\ln(n^2)}{\ln(2n)}$

49) $a_n = \left(1 - \frac{2}{n}\right)^n$

Answer

Since $(1+1/n)^n \to e$, one has $(1-2/n)^n \approx (1+k)^{-2k} \to e^{-2}$ as $k \to \infty$.

50) $a_n = \ln\left(\dfrac{n+2}{n^2-3}\right)$

51) $a_n = \dfrac{2^n + 3^n}{4^n}$

Answer

$2^n + 3^n \leq 2 \cdot 3^n$ and $3^n/4^n \to 0$ as $n \to \infty$, so $a_n \to 0$ as $n \to \infty$.

52) $a_n = \dfrac{(1000)^n}{n!}$

https://math.libretexts.org/@go/page/83970

53) $a_n = \dfrac{(n!)^2}{(2n)!}$

Answer

$\dfrac{a_{n+1}}{a_n} = n!/(n+1)(n+2)\cdots(2n) = \dfrac{1\cdot 2\cdot 3\cdots n}{(n+1)(n+2)\cdots(2n)} < 1/2^n$. In particular, $a_{n+1}/a_n \le 1/2$, so $a_n \to 0$ as $n \to \infty$.

Newton's method seeks to approximate a solution $f(x) = 0$ that starts with an initial approximation x_0 and successively defines a sequence $x_{n+1} = x_n - \dfrac{f(x_n)}{f\,'(x_n)}$. For the given choice of f and x_0, write out the formula for x_{n+1}. If the sequence appears to converge, give an exact formula for the solution x, then identify the limit x accurate to four decimal places and the smallest n such that x_n agrees with x up to four decimal places.

54) [T] $f(x) = x^2 - 2, \quad x_0 = 1$

55) [T] $f(x) = (x-1)^2 - 2, \quad x_0 = 2$

Answer

$x_{n+1} = x_n - ((x_n - 1)^2 - 2)/2(x_n - 1);\ x = 1 + \sqrt{2},\ x \approx 2.4142,\ n = 5$

56) [T] $f(x) = e^x - 2, \quad x_0 = 1$

57) [T] $f(x) = \ln x - 1, \quad x_0 = 2$

Answer

$x_{n+1} = x_n - x_n(\ln(x_n) - 1);\ x = e,\ x \approx 2.7183,\ n = 5$

58) [T] Suppose you start with one liter of vinegar and repeatedly remove 0.1 L, replace with water, mix, and repeat.

a. Find a formula for the concentration after n steps.

b. After how many steps does the mixture contain less than 10% vinegar?

59) [T] A lake initially contains 2000 fish. Suppose that in the absence of predators or other causes of removal, the fish population increases by 6% each month. However, factoring in all causes, 150 fish are lost each month.

a. Explain why the fish population after n months is modeled by $P_n = 1.06P_{n-1} - 150$ with $P_0 = 2000$.

b. How many fish will be in the pond after one year?

Answer

a. Without losses, the population would obey $P_n = 1.06P_{n-1}$. The subtraction of 150 accounts for fish losses.
b. After 12 months, we have $P_{12} \approx 1494$.

60) [T] A bank account earns 5% interest compounded monthly. Suppose that $\$1000$ is initially deposited into the account, but that $\$10$ is withdrawn each month.

a. Show that the amount in the account after n months is $A_n = (1 + .05/12)A_{n-1} - 10;\ A_0 = 1000$.

b. How much money will be in the account after 1 year?

c. Is the amount increasing or decreasing?

d. Suppose that instead of $\$10$, a fixed amount d dollars is withdrawn each month. Find a value of d such that the amount in the account after each month remains $\$1000$.

e. What happens if d is greater than this amount?

61) [T] A student takes out a college loan of $\$10,000$ at an annual percentage rate of 6%, compounded monthly.

a. If the student makes payments of $\$100$ per month, how much does the student owe after 12 months?

b. After how many months will the loan be paid off?

Answer

a. The student owes $9383 after 12 months.

b. The loan will be paid in full after 139 months or eleven and a half years.

62) [T] Consider a series combining geometric growth and arithmetic decrease. Let $a_1 = 1$. Fix $a > 1$ and $0 < b < a$. Set $a_{n+1} = a.\, a_n - b$. Find a formula for a_{n+1} in terms of a_n, a, and b and a relationship between a and b such that a_n converges.

63) [T] The binary representation $x = 0.b_1 b_2 b_3 \ldots$ of a number x between 0 and 1 can be defined as follows. Let $b_1 = 0$ if $x < 1/2$ and $b_1 = 1$ if $1/2 \le x < 1$. Let $x_1 = 2x - b_1$. Let $b_2 = 0$ if $x_1 < 1/2$ and $b_2 = 1$ if $1/2 \le x < 1$. Let $x_2 = 2x_1 - b_2$ and in general, $x_n = 2x_{n-1} - b_n$ and $b_{n-}1 = 0$ if $x_n < 1/2$ and $b_{n-1} = 1$ if $1/2 \le x_n < 1$. Find the binary expansion of $1/3$.

Answer

$b_1 = 0, x_1 = 2/3, b_2 = 1, x_2 = 4/3 - 1 = 1/3,$ so the pattern repeats, and $1/3 = 0.010101\ldots$.

64) [T] To find an approximation for π, set $a_0 = \sqrt{2+1}, a_1 = \sqrt{2+a_0}$, and, in general, $a_{n+1} = \sqrt{2+a_n}$. Finally, set $p_n = 3.2^n\sqrt{2-a_n}$. Find the first ten terms of p_n and compare the values to π.

For the following two exercises, assume that you have access to a computer program or Internet source that can generate a list of zeros and ones of any desired length. Pseudo-random number generators (PRNGs) play an important role in simulating random noise in physical systems by creating sequences of zeros and ones that appear like the result of flipping a coin repeatedly. One of the simplest types of PRNGs recursively defines a random-looking sequence of N integers $a_1, a_2, \ldots, a_N$ by fixing two special integers (K and M and letting a_{n+1} be the remainder after dividing $K.\, a_n$ into M, then creates a bit sequence of zeros and ones whose n^{th} term b_n is equal to one if a_n is odd and equal to zero if a_n is even. If the bits b_n are pseudo-random, then the behavior of their average $(b_1 + b_2 + \cdots + b_N)/N$ should be similar to behavior of averages of truly randomly generated bits.

65) [T] Starting with $K = 16,807$ and $M = 2,147,483,647$, using ten different starting values of a_1, compute sequences of bits b_n up to $n = 1000$, and compare their averages to ten such sequences generated by a random bit generator.

Answer

For the starting values $a_1 = 1, a_2 = 2, \ldots, a_1 = 10,$ the corresponding bit averages calculated by the method indicated are $0.5220, 0.5000, 0.4960, 0.4870, 0.4860, 0.4680, 0.5130, 0.5210, 0.5040,$ and 0.4840. Here is an example of ten corresponding averages of strings of 1000 bits generated by a random number generator:
$0.4880, 0.4870, 0.5150, 0.5490, 0.5130, 0.5180, 0.4860, 0.5030, 0.5050,$ and 0.4980. There is no real pattern in either type of average. The random-number-generated averages range between 0.4860 and 0.5490, a range of 0.0630, whereas the calculated PRNG bit averages range between 0.4680 and 0.5220, a range of 0.0540.

66) [T] Find the first 1000 digits of π using either a computer program or Internet resource. Create a bit sequence b_n by letting $b_n = 1$ if the nth digit of π is odd and $b_n = 0$ if the nth digit of π is even. Compute the average value of b_n and the average value of $d_n = |b_{n+1} - b_n|, n = 1, \ldots, 999$. Does the sequence b_n appear random? Do the differences between successive elements of b_n appear random?

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1.3: Infinite Series

Learning Objectives

- Explain the meaning of the sum of an infinite series.
- Calculate the sum of a geometric series.
- Evaluate a telescoping series.

We have seen that a sequence is an ordered set of terms. If you add these terms together, you get a series. In this section we define an infinite series and show how series are related to sequences. We also define what it means for a series to converge or diverge. We introduce one of the most important types of series: the geometric series. We will use geometric series in the next chapter to write certain functions as polynomials with an infinite number of terms. This process is important because it allows us to evaluate, differentiate, and integrate complicated functions by using polynomials that are easier to handle. We also discuss the harmonic series, arguably the most interesting divergent series because it just fails to converge.

Sums and Series

An infinite series is a sum of infinitely many terms and is written in the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots.$$

But what does this mean? We cannot add an infinite number of terms in the same way we can add a finite number of terms. Instead, the value of an infinite series is defined in terms of the limit of partial sums. A partial sum of an infinite series is a finite sum of the form

$$\sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \cdots + a_k.$$

To see how we use partial sums to evaluate infinite series, consider the following example. Suppose oil is seeping into a lake such that 1000 gallons enters the lake the first week. During the second week, an additional 500 gallons of oil enters the lake. The third week, 250 more gallons enters the lake. Assume this pattern continues such that each week half as much oil enters the lake as did the previous week. If this continues forever, what can we say about the amount of oil in the lake? Will the amount of oil continue to get arbitrarily large, or is it possible that it approaches some finite amount? To answer this question, we look at the amount of oil in the lake after k weeks. Letting S_k denote the amount of oil in the lake (measured in thousands of gallons) after k weeks, we see that

$$\begin{aligned}
S_1 &= 1 \\
S_2 &= 1 + 0.5 = 1 + \tfrac{1}{2} \\
S_3 &= 1 + 0.5 + 0.25 = 1 + \tfrac{1}{2} + \tfrac{1}{4} \\
S_4 &= 1 + 0.5 + 0.25 + 0.125 = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} \\
S_5 &= 1 + 0.5 + 0.25 + 0.125 + 0.0625 = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16}.
\end{aligned}$$

Looking at this pattern, we see that the amount of oil in the lake (in thousands of gallons) after k weeks is

$$S_k = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^{k-1}} = \sum_{n=1}^{k} \left(\frac{1}{2}\right)^{n-1}.$$

We are interested in what happens as $k \to \infty$. Symbolically, the amount of oil in the lake as $k \to \infty$ is given by the infinite series

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots.$$

At the same time, as $k \to \infty$, the amount of oil in the lake can be calculated by evaluating $\lim\limits_{k\to\infty} S_k$. Therefore, the behavior of the infinite series can be determined by looking at the behavior of the sequence of partial sums S_k. If the sequence of partial sums S_k converges, we say that the infinite series converges, and its sum is given by $\lim\limits_{k\to\infty} S_k$. If the sequence S_k diverges, we say the infinite series diverges. We now turn our attention to determining the limit of this sequence S_k.

First, simplifying some of these partial sums, we see that

$$S_1 = 1$$

$$S_2 = 1 + \tfrac{1}{2} = \tfrac{3}{2}$$
$$S_3 = 1 + \tfrac{1}{2} + \tfrac{1}{4} = \tfrac{7}{4}$$
$$S_4 = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} = \tfrac{15}{8}$$
$$S_5 = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} = \tfrac{31}{16}.$$

Plotting some of these values in Figure, it appears that the sequence S_k could be approaching 2.

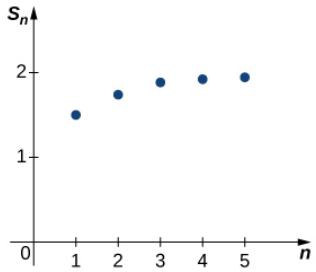


Figure 1.3.1: The graph shows the sequence of partial sums S_k. It appears that the sequence is approaching the value 2.

Let's look for more convincing evidence. In the following table, we list the values of S_k for several values of k.

k	5	10	15	20
S_k	1.9375	1.998	1.999939	1.999998

These data supply more evidence suggesting that the sequence S_k converges to 2. Later we will provide an analytic argument that can be used to prove that $\lim_{k\to\infty} S_k = 2$. For now, we rely on the numerical and graphical data to convince ourselves that the sequence of partial sums does actually converge to 2. Since this sequence of partial sums converges to 2, we say the infinite series converges to 2 and write

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} = 2.$$

Returning to the question about the oil in the lake, since this infinite series converges to 2, we conclude that the amount of oil in the lake will get arbitrarily close to 2000 gallons as the amount of time gets sufficiently large.

This series is an example of a geometric series. We discuss geometric series in more detail later in this section. First, we summarize what it means for an infinite series to converge.

Definition

An **infinite series** is an expression of the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots. \tag{1.3.1}$$

For each positive integer k, the sum

$$S_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \cdots + a_k \tag{1.3.2}$$

is called the kth **partial sum** of the infinite series. The partial sums form a sequence S_k. If the sequence of partial sums converges to a real number S, the infinite series converges. If we can describe the **convergence of a series** to S, we call S the sum of the series, and we write

$$\sum_{n=1}^{\infty} a_n = S. \tag{1.3.3}$$

If the sequence of partial sums diverges, we have the **divergence of a series.**

Note that the index for a series need not begin with $n = 1$ but can begin with any value. For example, the series

$$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n-1}$$

can also be written as

$$\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{n} \text{ or } \sum_{n=5}^{\infty}\left(\frac{1}{2}\right)^{n-5}.$$

Often it is convenient for the index to begin at 1, so if for some reason it begins at a different value, we can re-index by making a change of variables. For example, consider the series

$$\sum_{n=2}^{\infty}\frac{1}{n^2}.$$

By introducing the variable $m=n-1$, so that $n=m+1,$ we can rewrite the series as

$$\sum_{m=1}^{\infty}\frac{1}{(m+1)^2}.$$

Example 1.3.1: Evaluating Limits of Sequences of Partial Sums

For each of the following series, use the sequence of partial sums to determine whether the series converges or diverges.

a. $\sum_{n=1}^{\infty}\frac{n}{n+1}$

b. $\sum_{n=1}^{\infty}(-1)^n$

c. $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$

Solution

a. The sequence of partial sums S_k satisfies

$$S_1=\frac{1}{2}$$
$$S_2=\frac{1}{2}+\frac{2}{3}$$
$$S_3=\frac{1}{2}+\frac{2}{3}+\frac{3}{4}$$
$$S_4=\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}\ .$$

Notice that each term added is greater than $1/2$. As a result, we see that

$$S_1=\frac{1}{2}$$
$$S_2=\frac{1}{2}+\frac{2}{3}>\frac{1}{2}+\frac{1}{2}=2\left(\frac{1}{2}\right)$$
$$S_3=\frac{1}{2}+\frac{2}{3}+\frac{3}{4}>\frac{1}{2}+\frac{1}{2}+\frac{1}{2}=3\left(\frac{1}{2}\right)$$
$$S_4=\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}>\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}=4\left(\frac{1}{2}\right).$$

From this pattern we can see that $S_k>k\left(\frac{1}{2}\right)$ for every integer k. Therefore, S_k is unbounded and consequently, diverges. Therefore, the infinite series $\sum_{n=1}^{\infty}\frac{n}{n+1}$ diverges.

b. The sequence of partial sums S_k satisfies

$$S_1=-1$$
$$S_2=-1+1=0$$

$$S_3 = -1+1-1 = -1$$
$$S_4 = -1+1-1+1 = 0.$$

From this pattern we can see the sequence of partial sums is

$$S_k = -1, 0, -1, 0, \ldots.$$

Since this sequence diverges, the infinite series $\sum_{n=1}^{\infty}(-1)^n$ diverges.

c. The sequence of partial sums S_k satisfies

$$S_1 = \frac{1}{1\cdot 2} = \frac{1}{2}$$
$$S_2 = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$$
$$S_3 = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{3}{4}$$
$$S_4 = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} = \frac{4}{5}$$
$$S_5 = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \frac{1}{5\cdot 6} = \frac{5}{6}.$$

From this pattern, we can see that the kth partial sum is given by the explicit formula

$$S_k = \frac{k}{k+1}$$

.

Since $k/(k+1) \to 1,$ we conclude that the sequence of partial sums converges, and therefore the infinite series converges to 1. We have

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

Exercise 1.3.1

Determine whether the series $\sum_{n=1}^{\infty} \frac{n+1}{n}$ converges or diverges.

Hint

Look at the sequence of partial sums.

Answer

The series diverges because the kth partial sum $S_k > k$.

The Harmonic Series

A useful series to know about is the **harmonic series.** The harmonic series is defined as

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots. \tag{1.3.4}$$

This series is interesting because it diverges, but it diverges very slowly. By this we mean that the terms in the sequence of partial sums S_k approach infinity, but do so very slowly. We will show that the series diverges, but first we illustrate the slow growth of the terms in the sequence S_k in the following table.

k	10	100	1000	10,00	100,000	1,000,000
S_k	2.92897	5.18738	7.48547	9.78761	12.09015	14.39273

Even after $1,000,000$ terms, the partial sum is still relatively small. From this table, it is not clear that this series actually diverges. However, we can show analytically that the sequence of partial sums diverges, and therefore the series diverges.

To show that the sequence of partial sums diverges, we show that the sequence of partial sums is unbounded. We begin by writing the first several partial sums:

$$S_1 = 1$$
$$S_2 = 1 + \frac{1}{2}$$
$$S_3 = 1 + \frac{1}{2} + \frac{1}{3}$$
$$S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \quad .$$

Notice that for the last two terms in S_4,

$$\frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4}$$

Therefore, we conclude that

$$S_4 > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{1}{2} + \frac{1}{2} = 1 + 2\left(\frac{1}{2}\right).$$

Using the same idea for S_8, we see that

$$S_8 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 \quad .$$
$$+ 3\left(\frac{1}{2}\right)$$

From this pattern, we see that $S_1 = 1, S_2 = 1 + 1/2, S_4 > 1 + 2(1/2),$ and $S_8 > 1 + 3(1/2)$. More generally, it can be shown that $S_{2^j} > 1 + j(1/2)$ for all $j > 1$. Since $1 + j(1/2) \to \infty,$ we conclude that the sequence S_k is unbounded and therefore diverges. In the previous section, we stated that convergent sequences are bounded. Consequently, since S_k is unbounded, it diverges. Thus, the harmonic series diverges.

Algebraic Properties of Convergent Series

Since the sum of a convergent infinite series is defined as a limit of a sequence, the algebraic properties for series listed below follow directly from the algebraic properties for sequences.

Note 1.3.1: Algebraic Properties of Convergent Series

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be convergent series. Then the following algebraic properties hold.

i. The series $\sum_{n=1}^{\infty}(a_n + b_n)$ converges, and $\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$. (Sum Rule)

ii. The series $\sum_{n=1}^{\infty}(a_n - b_n)$ converges, and $\sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$. (Difference Rule)

iii. For any real number c, the series $\sum_{n=1}^{\infty} ca_n$ converges, and $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$. (Constant Multiple Rule)

Example 1.3.2: Using Algebraic Properties of Convergent Series

Evaluate $\sum_{n=1}^{\infty}\left[\frac{3}{n(n+1)} + \left(\frac{1}{2}\right)^{n-2}\right]$.

Solution

We showed earlier that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

and

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} = 2.$$

Since both of those series converge, we can apply the properties of Note 1.3.1 to evaluate

$$\sum_{n=1}^{\infty} \left[\frac{3}{n(n+1)} + \left(\frac{1}{2}\right)^{n-2}\right].$$

Using the sum rule, write

$$\sum_{n=1}^{\infty} \left[\frac{3}{n(n+1)} + \left(\frac{1}{2}\right)^{n-2}\right] = \sum_{n=1}^{\infty} \frac{3}{n(n+1)} + \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-2}.$$

Then, using the constant multiple rule and the sums above, we can conclude that

$$\sum_{n=1}^{\infty} \frac{3}{n(n+1)} + \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-2} = 3\sum_{n=1}^{\infty} \frac{1}{n(n+1)} + \left(\frac{1}{2}\right)^{-1} \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} = 3(1) + \left(\frac{1}{2}\right)^{-1}(2) = 3 + 2(2) = 7.$$

Exercise 1.3.2

Evaluate $\sum_{n=1}^{\infty} \frac{5}{2^{n-1}}$.

Hint

Rewrite as $\sum_{n=1}^{\infty} 5\left(\frac{1}{2}\right)^{n-1}$.

Answer

10

Geometric Series

A **geometric series** is any series that we can write in the form

$$a + ar + ar^2 + ar^3 + \cdots = \sum_{n=1}^{\infty} ar^{n-1}. \tag{1.3.5}$$

Because the ratio of each term in this series to the previous term is r, the number r is called the ratio. We refer to a as the initial term because it is the first term in the series. For example, the series

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

is a geometric series with initial term $a = 1$ and ratio $r = 1/2$.

In general, when does a geometric series converge? Consider the geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} \tag{1.3.6}$$

when $a > 0$. Its sequence of partial sums S_k is given by

$$S_k = \sum_{n=1}^{k} ar^{n-1} = a + ar + ar^2 + \cdots + ar^{k-1}. \tag{1.3.7}$$

Consider the case when $r = 1$. In that case,

$$S_k = a + a(1) + a(1)^2 + \cdots + a(1)^{k-1} = ak. \tag{1.3.8}$$

Since $a > 0$, we know $ak \to \infty$ as $k \to \infty$. Therefore, the sequence of partial sums is unbounded and thus diverges. Consequently, the infinite series diverges for $r = 1$. For $r \neq 1$, to find the limit of S_k, multiply Equation by $1 - r$. Doing so, we see that

$$\begin{aligned}(1-r)S_k = a(1-r)(1+r+r^2+r^3+\cdots+r^{k-1}) &= a[(1+r+r^2+r^3+\cdots+r^{k-1}) - (r+r^2+r^3+\cdots+r^k)] \\ &= a(1-r^k).\end{aligned} \tag{1.3.9}$$

All the other terms cancel out.

Therefore,

$$S_k = \frac{a(1-r^k)}{1-r} \text{ for } r \neq 1.$$

From our discussion in the previous section, we know that the geometric sequence $r^k \to 0$ if $|r| < 1$ and that r^k diverges if $|r| > 1$ or $r = \pm 1$. Therefore, for $|r| < 1$, $S_k \to \dfrac{a}{1-r}$ and we have

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \text{ if } |r| < 1.$$

If $|r| \geq 1$, S_k diverges, and therefore

$$\sum_{n=1}^{\infty} ar^{n-1} \text{ diverges if } |r| \geq 1.$$

Definitions: Diverging and Converging Series

A geometric series is a series of the form

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots. \tag{1.3.10}$$

If $|r| < 1$, the series **converges**, and

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \text{ for } |r| < 1. \tag{1.3.11}$$

If $|r| \geq 1$, the series **diverges**.

Geometric series sometimes appear in slightly different forms. For example, sometimes the index begins at a value other than $n = 1$ or the exponent involves a linear expression for n other than $n - 1$. As long as we can rewrite the series in the form given by Equation, it is a geometric series. For example, consider the series

$$\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^{n+2}.$$

To see that this is a geometric series, we write out the first several terms:

$$\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^{n+2} = \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \left(\frac{2}{3}\right)^4 + \cdots = \frac{4}{9} + \frac{4}{9} \cdot \left(\frac{2}{3}\right) + \frac{4}{9} \cdot \left(\frac{2}{3}\right)^2 + \cdots.$$

We see that the initial term is $a = 4/9$ and the ratio is $r = 2/3$. Therefore, the series can be written as

$$\sum_{n=1}^{\infty} \frac{4}{9} \cdot \left(\frac{2}{3}\right)^{n-1}.$$

Since $r = 2/3 < 1$, this series converges, and its sum is given by

$$\sum_{n=1}^{\infty} \frac{4}{9} \cdot \left(\frac{2}{3}\right)^{n-1} = \frac{4/9}{1 - 2/3} = \frac{4}{3}.$$

Example 1.3.3: Determining Convergence or Divergence of a Geometric Series

Determine whether each of the following geometric series converges or diverges, and if it converges, find its sum.

a. $\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{4^{n-1}}$

b. $\sum_{n=1}^{\infty} e^{2n}$

Solution

a. Writing out the first several terms in the series, we have

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{4^{n-1}} &= \frac{(-3)^2}{4^0} + \frac{(-3)^3}{4} + \frac{(-3)^4}{4^2} + \cdots \\ &= (-3)^2 + (-3)^2 \cdot \left(\frac{-3}{4}\right) + (-3)^2 \cdot \left(\frac{-3}{4}\right)^2 + \cdots \\ &= 9 + 9 \cdot \left(\frac{-3}{4}\right) + 9 \cdot \left(\frac{-3}{4}\right)^2 + \cdots.\end{aligned}$$

The initial term $a = -3$ and the ratio $r = -3/4$. Since $|r| = 3/4 < 1$, the series converges to

$$\frac{9}{1-(-3/4)} = \frac{9}{7/4} = \frac{36}{7}.$$

b. Writing this series as

$$e^2 \sum_{n=1}^{\infty} (e^2)^{n-1}$$

we can see that this is a geometric series where $r = e^2 > 1$. Therefore, the series diverges.



Exercise 1.3.3

Determine whether the series $\sum_{n=1}^{\infty} \left(\frac{-2}{5}\right)^{n-1}$ converges or diverges. If it converges, find its sum.

Hint

$r = -2/5$

Answer

$5/7$

We now turn our attention to a nice application of geometric series. We show how they can be used to write repeating decimals as fractions of integers.

Example 1.3.4: Writing Repeating Decimals as Fractions of Integers

Use a geometric series to write $3.2\bar{6}$ as a fraction of integers.

Solution

Since $3.2\bar{6} = 3.262626\ldots$, first we write

$$\begin{aligned} 3.262626\ldots &= 3 + \frac{26}{100} + \frac{26}{1000} + \frac{26}{100,000} + \cdots \\ &= 3 + \frac{26}{10^2} + \frac{26}{10^4} + \frac{26}{10^6} + \cdots. \end{aligned}$$

Ignoring the term 3, the rest of this expression is a geometric series with initial term $a = 26/10^2$ and ratio $r = 1/10^2$. Therefore, the sum of this series is

$$\frac{26/10^2}{1-(1/10^2)} = \frac{26/10^2}{99/10^2} = \frac{26}{99}.$$

Thus,

$$3.262626\ldots = 3 + \tfrac{26}{99} = \tfrac{323}{99}.$$

Exercise 1.3.4

Write $5.2\bar{7}$ as a fraction of integers.

Hint

By expressing this number as a series, find a geometric series with initial term $a = 7/100$ and ratio $r = 1/10$.

Answer

$475/90$

Example 1.3.5: Finding the Area of the Koch Snowflake

Define a sequence of figures $\{F_n\}$ recursively as follows (Figure 1.3.2). Let F_0 be an equilateral triangle with sides of length 1. For $n \geq 1$, let F_n be the curve created by removing the middle third of each side of F_{n-1} and replacing it with an equilateral triangle pointing outward. The limiting figure as $n \to \infty$ is known as **Koch's snowflake.**

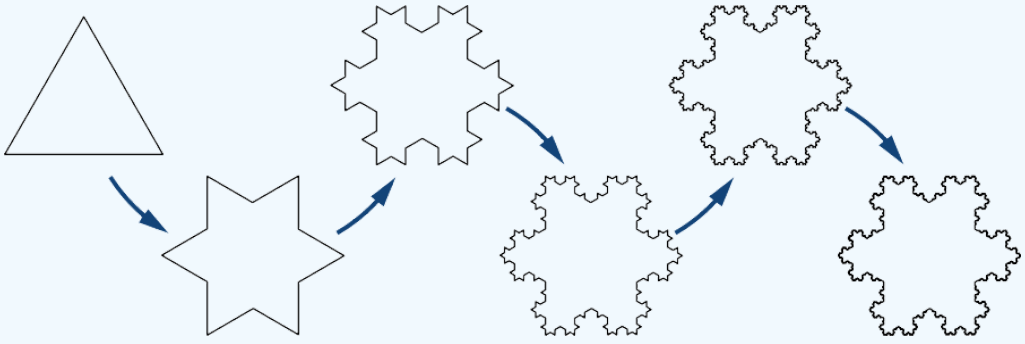
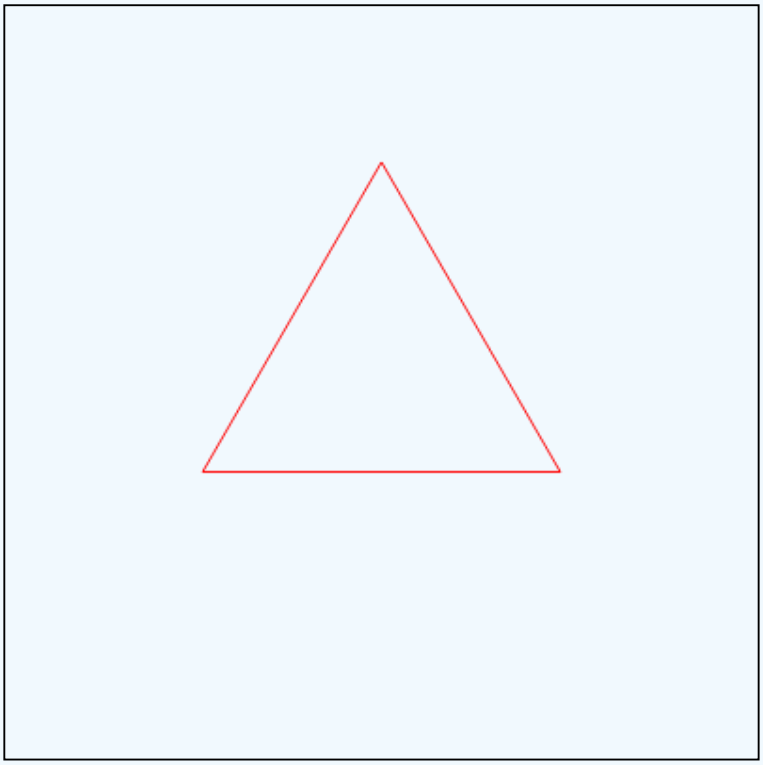


Figure 1.3.2: The first four figures, F_0, F_1, F_2, and F_3, in the construction of the Koch snowflake.

Koch's Snowflake Maker

By clicking on Next or Previous, you can visualize Koch's Snowflake for increased or decreased values of n.



Next Previous n = 1

a. Find the length L_n of the perimeter of F_n. Evaluate $\lim_{n\to\infty} L_n$ to find the length of the perimeter of Koch's snowflake.

b. Find the area A_n of figure F_n. Evaluate $\lim_{n\to\infty} A_n$ to find the area of Koch's snowflake.

Solution

a. Let N_n denote the number of sides of figure F_n. Since F_0 is a triangle, $N_0 = 3$. Let ln denote the length of each side of F_n. Since F_0 is an equilateral triangle with sides of length $l_0 = 1$, we now need to determine N_1 and l_1. Since F_1 is created by removing the middle third of each side and replacing that line segment with two line segments, for each side of F_0, we get four sides in F_1. Therefore, the number of sides for F_1 is

$$N_1 = 4 \cdot 3 .$$

Since the length of each of these new line segments is $1/3$ the length of the line segments in F_0, the length of the line segments for F_1 is given by

$$l_1 = \tfrac{1}{3} \cdot 1 = \tfrac{1}{3} .$$

Similarly, for F_2, since the middle third of each side of F_1 is removed and replaced with two line segments, the number of sides in F_2 is given by

$$N_2 = 4N_1 = 4(4 \cdot 3) = 4^2 \cdot 3.$$

Since the length of each of these sides is $1/3$ the length of the sides of F_1, the length of each side of figure F_2 is given by

$$l_2 = \tfrac{1}{3} \cdot l_1 = \tfrac{1}{3} \cdot \tfrac{1}{3} = \left(\tfrac{1}{3}\right)^2 .$$

More generally, since F_n is created by removing the middle third of each side of F_{n-1} and replacing that line segment with two line segments of length $\frac{1}{3} l_{n-1}$ in the shape of an equilateral triangle, we know that $N_n = 4N_{n-1}$ and $l_n = \dfrac{l_{n-1}}{3}$. Therefore, the number of sides of figure F_n is

$$N_n = 4^n \cdot 3$$

and the length of each side is

$$l_n = \left(\frac{1}{3}\right)^n. \tag{1.3.12}$$

Therefore, to calculate the perimeter of F_n, we multiply the number of sides N_n and the length of each side l_n. We conclude that the perimeter of F_n is given by

$$L_n = N_n \cdot l_n = 3 \cdot \left(\frac{4}{3}\right)^n \tag{1.3.13}$$

Therefore, the length of the perimeter of Koch's snowflake is

$$L = \lim_{n\to\infty} L_n = \infty. \tag{1.3.14}$$

b. Let T_n denote the area of each new triangle created when forming F_n. For $n = 0, T_0$ is the area of the original equilateral triangle. Therefore, $T_0 = A_0 = \sqrt{3}/4$. For $n \geq 1$, since the lengths of the sides of the new triangle are $1/3$ the length of the sides of F_{n-1}, we have

$$T_n = \left(\frac{1}{3}\right)^2 \cdot T_{n-1} = \frac{1}{9} \cdot T_{n-1}. \tag{1.3.15}$$

Therefore, $T_n = \left(\frac{1}{9}\right)^n \cdot \frac{\sqrt{3}}{4}$. Since a new triangle is formed on each side of F_{n-1},

$$A_n = A_{n-1} + N_{n-1} \cdot T_n = A_{n-1} + (3 \cdot 4_{n-1}) \cdot \left(\frac{1}{9}\right)^n \cdot \frac{\sqrt{3}}{4} = A_{n-1} + \frac{3}{4} \cdot \left(\frac{4}{9}\right)^n \cdot \frac{\sqrt{3}}{4}. \tag{1.3.16}$$

Writing out the first few terms A_0, A_1, A_2, we see that

$$A_0 = \tfrac{\sqrt{3}}{4}$$

$$A_1 = A_0 + \tfrac{3}{4} \cdot \left(\tfrac{4}{9}\right) \cdot \tfrac{\sqrt{3}}{4} = \tfrac{\sqrt{3}}{4} + \tfrac{3}{4} \cdot \left(\tfrac{4}{9}\right) \cdot \tfrac{\sqrt{3}}{4} = \tfrac{\sqrt{3}}{4}\left[1 + \tfrac{3}{4} \cdot \left(\tfrac{4}{9}\right)\right]$$

$$A_2 = A_1 + \tfrac{3}{4} \cdot \left(\tfrac{4}{9}\right)^2 \cdot \tfrac{\sqrt{3}}{4} = \tfrac{\sqrt{3}}{4}\left[1 + \tfrac{3}{4} \cdot \left(\tfrac{4}{9}\right)\right] + \tfrac{3}{4} \cdot \left(\tfrac{4}{9}\right)^2 \cdot \tfrac{\sqrt{3}}{4} = \tfrac{\sqrt{3}}{4}\left[1 + \tfrac{3}{4} \cdot \left(\tfrac{4}{9}\right) + \tfrac{3}{4} \cdot \left(\tfrac{4}{9}\right)^2\right] \quad .$$

More generally,

$$A_n = \tfrac{\sqrt{3}}{4}\left[1 + \tfrac{3}{4}\left(\tfrac{4}{9} + \left(\tfrac{4}{9}\right)^2 + \cdots + \left(\tfrac{4}{9}\right)^n\right)\right] \quad .$$

Factoring $4/9$ out of each term inside the inner parentheses, we rewrite our expression as

$$A_n = \tfrac{\sqrt{3}}{4}\left[1 + \tfrac{1}{3}\left(1 + \tfrac{4}{9} + \left(\tfrac{4}{9}\right)^2 + \cdots + \left(\tfrac{4}{9}\right)^{n-1}\right)\right] \quad .$$

The expression $1 + \left(\frac{4}{9}\right) + \left(\frac{4}{9}\right)^2 + \cdots + \left(\frac{4}{9}\right)^{n-1}$ is a geometric sum. As shown earlier, this sum satisfies

$$1 + \tfrac{4}{9} + \left(\tfrac{4}{9}\right)^2 + \cdots + \left(\tfrac{4}{9}\right)^{n-1} = \frac{1 - (4/9)^n}{1 - (4/9)}.$$

Substituting this expression into the expression above and simplifying, we conclude that

$$A_n = \frac{\sqrt{3}}{4}\left[1 + \frac{1}{3}\left(\frac{1 - (4/9)^n}{1 - (4/9)}\right)\right] = \frac{\sqrt{3}}{4}\left[\frac{8}{5} - \frac{3}{5}\left(\frac{4}{9}\right)^n\right].$$

Therefore, the area of Koch's snowflake is

$$A = \lim_{n\to\infty} A_n = \frac{2\sqrt{3}}{5}.$$

Analysis

The Koch snowflake is interesting because it has finite area, yet infinite perimeter. Although at first this may seem impossible, recall that you have seen similar examples earlier in the text. For example, consider the region bounded by the curve $y = 1/x^2$ and the x-axis on the interval $[1, \infty)$. Since the improper integral

$$\int_1^\infty \frac{1}{x^2}\, dx$$

converges, the area of this region is finite, even though the perimeter is infinite.

Telescoping Series

Consider the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. We discussed this series in Example, showing that the series converges by writing out the first several partial sums $S_1, S_2, \ldots, S_6$ and noticing that they are all of the form $S_k = \frac{k}{k+1}$. Here we use a different technique to show that this series converges. By using partial fractions, we can write

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

Therefore, the series can be written as

$$\sum_{n=1}^{\infty} \left[\frac{1}{n} - \frac{1}{n+1}\right] = \left(1 + \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots.$$

Writing out the first several terms in the sequence of partial sums S_k, we see that

$$\begin{aligned} S_1 &= 1 - \tfrac{1}{2} \\ S_2 &= \left(1 - \tfrac{1}{2}\right) + \left(\tfrac{1}{2} - \tfrac{1}{3}\right) = 1 - \tfrac{1}{3} \\ S_3 &= \left(1 - \tfrac{1}{2}\right) + \left(\tfrac{1}{2} - \tfrac{1}{3}\right) + \left(\tfrac{1}{3} - \tfrac{1}{4}\right) = 1 - \tfrac{1}{4} \quad . \end{aligned}$$

In general,

$$S_k = \left(1 - \tfrac{1}{2}\right) + \left(\tfrac{1}{2} - \tfrac{1}{3}\right) + \left(\tfrac{1}{3} - \tfrac{1}{4}\right) + \cdots + \left(\tfrac{1}{k} - \tfrac{1}{k+1}\right) = 1 - \frac{1}{k+1} \quad .$$

We notice that the middle terms cancel each other out, leaving only the first and last terms. In a sense, the series collapses like a spyglass with tubes that disappear into each other to shorten the telescope. For this reason, we call a series that has this property a telescoping series. For this series, since $S_k = 1 - 1/(k+1)$ and $1/(k+1) \to 0$ as $k \to \infty$, the sequence of partial sums converges to 1, and therefore the series converges to 1.

Definition

A **telescoping series** is a series in which most of the terms cancel in each of the partial sums, leaving only some of the first terms and some of the last terms.

For example, any series of the form

$$\sum_{n=1}^{\infty} [b_n - b_{n+1}] = (b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) + \cdots$$

is a telescoping series. We can see this by writing out some of the partial sums. In particular, we see that

$$\begin{aligned} S_1 &= b_1 - b_2 \\ S_2 &= (b_1 - b_2) + (b_2 - b_3) = b_1 - b_3 \\ S_3 &= (b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) = b_1 - b_4. \end{aligned}$$

In general, the *kth* partial sum of this series is

$$S_k = b_1 - b_{k+1} \ .$$

Since the *kth* partial sum can be simplified to the difference of these two terms, the sequence of partial sums S_k will converge if and only if the sequence b_{k+1} converges. Moreover, if the sequence b_{k+1} converges to some finite number B, then the sequence of partial sums converges to $b_1 - B$, and therefore

$$\sum_{n=1}^{\infty} [b_n - b_{n+1}] = b_1 - B.$$

In the next example, we show how to use these ideas to analyze a telescoping series of this form.

Example 1.3.6: Evaluating a Telescoping Series

Determine whether the telescoping series

$$\sum_{n=1}^{\infty}\left[\cos\left(\frac{1}{n}\right)-\cos\left(\frac{1}{n+1}\right)\right]$$

converges or diverges. If it converges, find its sum.

Solution

By writing out terms in the sequence of partial sums, we can see that

$$\begin{aligned} S_1 &= \cos(1)-\cos(\tfrac{1}{2}) \\ S_2 &= (\cos(1)-\cos(\tfrac{1}{2}))+(\cos(\tfrac{1}{2})-\cos(\tfrac{1}{3})) = \cos(1)-\cos(\tfrac{1}{3}) \\ S_3 &= (\cos(1)-\cos(\tfrac{1}{2}))+(\cos(\tfrac{1}{2})-\cos(\tfrac{1}{3}))+(\cos(\tfrac{1}{3})-\cos(\tfrac{1}{4})) \\ &= \cos(1)-\cos(\tfrac{1}{4}). \end{aligned}$$

In general,

$$S_k = \cos(1)-\cos\left(\tfrac{1}{k+1}\right).$$

Since $1/(k+1)\to 0$ as $k\to\infty$ and $\cos x$ is a continuous function, $\cos(1/(k+1))\to\cos(0)=1$. Therefore, we conclude that $S_k\to\cos(1)-1$. The telescoping series converges and the sum is given by

$$\sum_{n=1}^{\infty}\left[\cos\left(\frac{1}{n}\right)-\cos\left(\frac{1}{n+1}\right)\right]=\cos(1)-1.$$



Exercise 1.3.5

Determine whether $\sum_{n=1}^{\infty}[e^{1/n}-e^{1/(n+1)}]$ converges or diverges. If it converges, find its sum.

Hint

Write out the sequence of partial sums to see which terms cancel.

Answer

$e-1$

Euler's Constant

We have shown that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. Here we investigate the behavior of the partial sums S_k as $k \to \infty$. In particular, we show that they behave like the natural logarithm function by showing that there exists a constant γ such that

$$\sum_{n=1}^{k} \left(\frac{1}{n} - \ln k \right) \to \gamma \text{ as } k \to \infty.$$

This constant γ is known as **Euler's constant.**

1. Let $T_k = \sum_{n=1}^{k} \left(\frac{1}{n} - \ln k \right)$. Evaluate T_k for various values of k.

2. For T_k as defined in part 1. show that the sequence T_k converges by using the following steps.

   a. Show that the sequence T_k is monotone decreasing. (Hint: Show that $\ln(1+1/k) > 1/(k+1)$)

   b. Show that the sequence T_k is bounded below by zero. (Hint: Express $\ln k$ as a definite integral.)

   c. Use the Monotone Convergence Theorem to conclude that the sequence T_k converges. The limit γ is Euler's constant.

3. Now estimate how far T_k is from γ for a given integer k. Prove that for $k \geq 1, 0 < T_k - \gamma \leq 1/k$ by using the following steps.

   a. Show that $\ln(k+1) - \ln k < 1/k$.

   b. Use the result from part a. to show that for any integer k,

   $$T_k - T_{k+1} < \frac{1}{k} - \frac{1}{k+1}.$$

   c. For any integers k and j such that $j > k$, express $T_k - T_j$ as a telescoping sum by writing

   $$T_k - T_j = (T_k - T_{k+1}) + (T_{k+1} - T_{k+2}) + (T_{k+2} - T_{k+3}) + \cdots + (T_{j-1} - T_j).$$

   Use the result from part b. combined with this telescoping sum to conclude that

   $$T_k - T_j < \frac{1}{k} - \frac{1}{j}.$$

a. Apply the limit to both sides of the inequality in part c. to conclude that

$$T_k - \gamma \leq \frac{1}{k}.$$

e. Estimate γ to an accuracy of within **0.001**.

Key Concepts

- Given the infinite series

  $$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

  and the corresponding sequence of partial sums S_k where

  $$S_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \cdots + a_k \quad ,$$

  the series converges if and only if the sequence S_k converges.

- The geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ converges if $|r| < 1$ and diverges if $|r| \geq 1$. For $|r| < 1$,

  $$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}.$$

- The harmonic series

$$\sum_{n=1}^{\infty}\frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+\cdots$$

diverges.

- A series of the form $\sum_{n=1}^{\infty}[b_n-b_{n+1}]=[b_1-b_2]+[b_2-b_3]+[b_3-b_4]+\cdots+[b_n-b_{n+1}]+\cdots$ is a telescoping series. The kth partial sum of this series is given by $S_k=b_1-b_{k+1}$. The series will converge if and only if $\lim_{k\to\infty}b_{k+1}$ exists. In that case,

$$\sum_{n=1}^{\infty}[b_n-b_{n+1}]=b_1-\lim_{k\to\infty}(b_{k+1})\ .$$

Key Equations

- **Harmonic series**

$$\sum_{n=1}^{\infty}\frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$$

- **Sum of a geometric series**

$$\sum_{n=1}^{\infty}ar^{n-1}=\frac{a}{1-r}\ \text{ for } |r|<1$$

Glossary

convergence of a series

a series converges if the sequence of partial sums for that series converges

divergence of a series

a series diverges if the sequence of partial sums for that series diverges

geometric series

a geometric series is a series that can be written in the form

$$\sum_{n=1}^{\infty}ar^{n-1}=a+ar+ar^2+ar^3+\cdots$$

harmonic series

the harmonic series takes the form

$$\sum_{n=1}^{\infty}\frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+\cdots$$

infinite series

an infinite series is an expression of the form

$$a_1+a_2+a_3+\cdots=\sum_{n=1}^{\infty}a_n$$

partial sum

the kth partial sum of the infinite series $\sum_{n=1}^{\infty}a_n$ is the finite sum

$$S_k=\sum_{n=1}^{k}a_n=a_1+a_2+a_3+\cdots+a_k$$

telescoping series

a telescoping series is one in which most of the terms cancel in each of the partial sums

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1.3E: Exercises for Section 1.3

In exercises 1 - 4, use sigma notation to write each expressions as an infinite series.

1) $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$

Answer

$$\sum_{n=1}^{\infty}\frac{1}{n}$$

2) $1-1+1-1+\cdots$

3) $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$

Answer

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$$

4) $\sin 1+\sin\frac{1}{2}+\sin\frac{1}{3}+\sin\frac{1}{4}+\cdots$

In exercises 5 - 8, compute the first four partial sums $S_1,\ldots,S_4$ for the series having n^{th} term a_n starting with $n=1$ as follows.

5) $a_n=n$

Answer

$1,3,6,10$

6) $a_n=1/n$

7) $a_n=\sin\frac{n\pi}{2}$

Answer

$1,1,0,0$

8) $a_n=(-1)^n$

In exercises 9 - 12, compute the general term a_n of the series with the given partial sum S_n. If the sequence of partial sums converges, find its limit S.

9) $S_n=1-\frac{1}{n},\quad n\geq 2$

Answer

$a_n=S_n-S_{n-1}=\dfrac{1}{n-1}-\dfrac{1}{n}.$ Since $S=\lim\limits_{n\to\infty}S_n=\lim\limits_{n\to\infty}\left(1-\dfrac{1}{n}\right)=1,$ the series converges to $S=1.$

10) $S_n=\dfrac{n(n+1)}{2},\quad n\geq 1$

11) $S_n=\sqrt{n},\quad n\geq 2$

Answer

$a_n=S_n-S_{n-1}=\sqrt{n}-\sqrt{n-1}=\dfrac{1}{\sqrt{n-1}+\sqrt{n}}.$

The series diverges because the partial sums are unbounded.

That is, $\lim\limits_{n\to\infty}S_n=\lim\limits_{n\to\infty}\sqrt{n}=\infty.$

12) $S_n=2-\dfrac{n+2}{2^n},\quad n\geq 1$

For each series in exercises 13 - 16, use the sequence of partial sums to determine whether the series converges or diverges.

13) $\sum_{n=1}^{\infty}\dfrac{n}{n+2}$

Answer

$S_1=1/3,$

$S_2=1/3+2/4>1/3+1/3=2/3,$

$S_3 = 1/3 + 2/4 + 3/5 > 3 \cdot (1/3) = 1$.
In general $S_k > k/3$, so the series diverges.
Note that the nth Term Test for Divergence could also be used to prove that this series diverges.

14) $\sum_{n=1}^{\infty}(1-(-1)^n))$

15) $\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)}$ (Hint: Use a partial fraction decomposition like that for $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$.)

Answer

$S_1 = 1/(2\cdot 3) = 1/6 = 2/3 - 1/2,$
$S_2 = 1/(2\cdot 3) + 1/(3\cdot 4) = 2/12 + 1/12 = 1/4 = 3/4 - 1/2,$
$S_3 = 1/(2\cdot 3) + 1/(3\cdot 4) + 1/(4\cdot 5) = 10/60 + 5/60 + 3/60 = 3/10 = 4/5 - 1/2,$
$S_4 = 1/(2\cdot 3) + 1/(3\cdot 4) + 1/(4\cdot 5) + 1/(5\cdot 6) = 10/60 + 5/60 + 3/60 + 2/60 = 1/3 = 5/6 - 1/2.$

The pattern is $S_k = \frac{k+1}{k+2} - \frac{1}{2}$.

Then $\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{k+1}{k+2} - \frac{1}{2}\right) = \frac{1}{2}$, so the series converges to $1/2$.

16) $\sum_{n=1}^{\infty}\frac{1}{2n+1}$ (Hint: Follow the reasoning for $\sum_{n=1}^{\infty}\frac{1}{n}$.)

Suppose that $\sum_{n=1}^{\infty} a_n = 1$, that $\sum_{n=1}^{\infty} b_n = -1$, that $a_1 = 2$, and $b_1 = -3$. Use this information to find the sum of the indicated series in exercises 17 - 20.

17) $\sum_{n=1}^{\infty}(a_n + b_n)$

Answer

$$\sum_{n=1}^{\infty}(a_n+b_n) \quad = \quad \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n \quad = \quad 1 + (-1) \quad = \quad 0$$

18) $\sum_{n=1}^{\infty}(a_n - 2b_n)$

19) $\sum_{n=2}^{\infty}(a_n - b_n)$

Answer

$$\sum_{n=2}^{\infty}(a_n-b_n) \quad = \quad \sum_{n=2}^{\infty} a_n - \sum_{n=2}^{\infty} b_n \quad = \quad \left(\sum_{n=1}^{\infty} a_n - a_1\right) - \left(\sum_{n=1}^{\infty} b_n - b_1\right) \quad = \quad (1-2) - (-1-(-3)) = -1-2 \quad = \quad -3$$

20) $\sum_{n=1}^{\infty}(3a_{n+1} - 4b_{n+1})$

In exercises 21 - 26, state whether the given series converges or diverges and explain why.

21) $\sum_{n=1}^{\infty}\frac{1}{n+1000}$ (Hint: Rewrite using a change of index.)

Answer

The series diverges, $\sum_{n=1001}^{\infty}\frac{1}{n}$

22) $\sum_{n=1}^{\infty}\frac{1}{n+10^{80}}$ (Hint: Rewrite using a change of index.)

23) $1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \cdots$

Answer

This is a convergent geometric series, since $r = \frac{1}{10} < 1$

24) $1 + \frac{e}{\pi} + \frac{e^2}{\pi^2} + \frac{e^3}{\pi^3} + \cdots$

25) $1 + \frac{\pi}{e^2} + \frac{\pi^2}{e^4} + \frac{\pi^3}{e^6} + \frac{\pi^4}{e^8} + \cdots$

Answer

This is a convergent geometric series, since $r = \pi/e^2 < 1$

26) $1 - \sqrt{\frac{\pi}{3}} + \sqrt{\frac{\pi^2}{9}} - \sqrt{\frac{\pi^3}{27}} + \cdots$

For each a_n in exercises 27 - 30, write its sum as a geometric series of the form $\sum_{n=1}^{\infty} ar^n$. State whether the series converges and if it does, find the exact value of its sum.

27) $a_1 = -1$ and $\dfrac{a_n}{a_{n+1}} = -5$ for $n \geq 1$.

Answer

$\sum_{n=1}^{\infty} 5 \cdot (-1/5)^n$, converges to $-5/6$

28) $a_1 = 2$ and $\dfrac{a_n}{a_{n+1}} = 1/2$ for $n \geq 1$.

29) $a_1 = 10$ and $\dfrac{a_n}{a_{n+1}} = 10$ for $n \geq 1$.

Answer

$\sum_{n=1}^{\infty} 100 \cdot (1/10)^n$,converges to $\frac{100}{9}$

30) $a_1 = \frac{1}{10}$ and $a_n/a_{n+1} = -10$ for $n \geq 1$.

In exercises 31 - 34, use the identity $\dfrac{1}{1-y} = \sum_{n=0}^{\infty} y^n$ (which is true for $|y| < 1$) to express each function as a geometric series in the indicated term.

31) $\dfrac{x}{1+x}$ in x

Answer

$x \sum_{n=0}^{\infty} (-x)^n = \sum_{n=1}^{\infty} (-1)^{n-1} x^n$

32) $\dfrac{\sqrt{x}}{1 - x^{3/2}}$ in $\sqrt{x}$

33) $\dfrac{1}{1 + \sin^2 x}$ in $\sin x$

Answer

$\sum_{n=0}^{\infty} (-1)^n \sin^{2n}(x)$

34) $\sec^2 x$ in $\sin x$

In exercises 35 - 38, evaluate the telescoping series or state whether the series diverges.

35) $\sum_{n=1}^{\infty} 2^{1/n} - 2^{1/(n+1)}$

Answer

$S_k = 2 - 2^{1/(k+1)} \to 1$ as $k \to \infty$.

36) $\sum_{n=1}^{\infty} \frac{1}{n^{13}} - \frac{1}{(n+1)^{13}}$

37) $\sum_{n=1}^{\infty} (\sqrt{n} - \sqrt{n+1})$

Answer

$S_k = 1 - \sqrt{k+1}$ diverges

38) $\sum_{n=1}^{\infty} (\sin n - \sin(n+1))$

Express each series in exercises 39 - 42 as a telescoping sum and evaluate its nth partial sum.

39) $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$

Answer

$\sum_{n=1}^{\infty} [\ln n - \ln(n+1)],$
$S_k = -\ln(k+1)$

40) $\sum_{n=1}^{\infty} \frac{2n+1}{(n^2+n)^2}$ (Hint: Factor denominator and use partial fractions.)

41) $\sum_{n=2}^{\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{(\ln n)\ln(n+1)}$

Answer

$a_n = \frac{1}{\ln n} - \frac{1}{\ln(n+1)}$ and $S_k = \frac{1}{\ln(2)} - \frac{1}{\ln(k+1)} \to \frac{1}{\ln(2)}$

42) $\sum_{n=1}^{\infty} \frac{(n+2)}{n(n+1)2^{n+1}}$ (Hint: Look at $1/(n2^n)$.

A general telescoping series is one in which all but the first few terms cancel out after summing a given number of successive terms.

43) Let $a_n = f(n) - 2f(n+1) + f(n+2)$, in which $f(n) \to 0$ as $n \to \infty$. Find $\sum_{n=1}^{\infty} a_n$.

Answer

$\sum_{n=1}^{\infty} a_n = f(1) - f(2)$

44) $a_n = f(n) - f(n+1) - f(n+2) + f(n+3)$, in which $f(n) \to 0$ as $n \to \infty$. Find $\sum_{n=1}^{\infty} a_n$.

45) Suppose that $a_n = c_0 f(n) + c_1 f(n+1) + c_2 f(n+2) + c_3 f(n+3) + c_4 f(n+4)$, where $f(n) \to 0$ as $n \to \infty$. Find a condition on the coefficients $c_0, \ldots, c_4$ that make this a general telescoping series.

Answer

$c_0 + c_1 + c_2 + c_3 + c_4 = 0$

46) Evaluate $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$ (Hint: $\frac{1}{n(n+1)(n+2)} = \frac{1}{2n} - \frac{1}{n+1} + \frac{1}{2(n+2)}$)

47) Evaluate $\sum_{n=2}^{\infty} \frac{2}{n^3 - n}$.

Answer

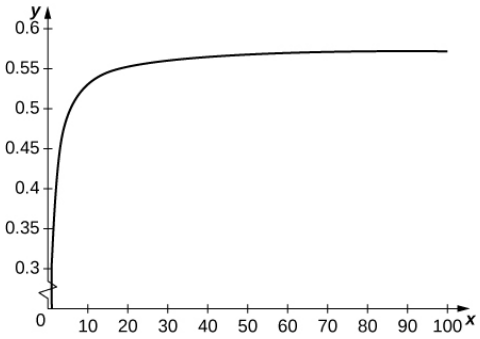
$\frac{2}{n^3-1} = \frac{1}{n-1} - \frac{2}{n} + \frac{1}{n+1},$
$S_n = (1 - 1 + 1/3) + (1/2 - 2/3 + 1/4) + (1/3 - 2/4 + 1/5) + (1/4 - 2/5 + 1/6) + \cdots = 1/2$

48) Find a formula for $\sum_{n=1}^{\infty} \frac{1}{n(n+N)}$ where N is a positive integer.

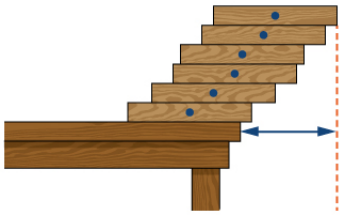
49) [T] Define a sequence $t_k = \sum_{n=1}^{k-1} (1/k) - \ln k$. Use the graph of $1/x$ to verify that t_k is increasing. Plot t_k for $k = 1 \ldots 100$ and state whether it appears that the sequence converges.

Answer

t_k converges to $0.57721\ldots$ $t_k$is a sum of rectangles of height $1/k$over the interval $[k, k+1]$ which lie above the graph of $1/x$.



50) [T] Suppose that N equal uniform rectangular blocks are stacked one on top of the other, allowing for some overhang. Archimedes' law of the lever implies that the stack of N blocks is stable as long as the center of mass of the top $(N-1)$ blocks lies at the edge of the bottom block. Let x denote the position of the edge of the bottom block, and think of its position as relative to the center of the next-to-bottom block. This implies that $(N-1)x = \left(\frac{1}{2} - x\right)$ or $x = 1/(2N)$. Use this expression to compute the maximum overhang (the position of the edge of the top block over the edge of the bottom block.) See the following figure.



Each of the following infinite series converges to the given multiple of π or $1/\pi$.

In each case, find the minimum value of N such that the Nth partial sum of the series accurately approximates the left-hand side to the given number of decimal places, and give the desired approximate value. Up to 15 decimals place, $\pi = 3.141592653589793\ldots$.

51) [T] $\pi = -3 + \sum_{n=1}^{\infty} \frac{n 2^n n!^2}{(2n)!}$, error < 0.0001

Answer

$N = 22,$
$S_N = 6.1415$

52) [T] $\frac{\pi}{2} = \sum_{k=0}^{\infty} \frac{k!}{(2k+1)!!} = \sum_{k=0}^{\infty} \frac{2^k k!^2}{(2k+1)!}$, error $< 10^{-4}$

53) [T] $\frac{9801}{2\pi} = \frac{4}{9801} \sum_{k=0}^{\infty} \frac{(4k)!(1103 + 26390k)}{(k!)^4 396^{4k}}$, error $< 10^{-12}$

Answer

$N = 3,$
$S_N = 1.559877597243667\ldots$

54) [T] $\frac{1}{12\pi} = \sum_{k=0}^{\infty} \frac{(-1)^k (6k)!(13591409 + 545140134k)}{(3k)!(k!)^3 640320^{3k+3/2}}$, error $< 10^{-15}$

55) [T] A fair coin is one that has probability $1/2$ of coming up heads when flipped.

a. What is the probability that a fair coin will come up tails n times in a row?

b. Find the probability that a coin comes up heads for the first time after an even number of coin flips.

Answer

a. The probability of any given ordered sequence of outcomes for n coin flips is $1/2^n$.

b. The probability of coming up heads for the first time on the n th flip is the probability of the sequence $TT\ldots TH$ which is $1/2^n$. The probability of coming up heads for the first time on an even flip is $\sum_{n=1}^{\infty} 1/2^{2n}$ or $1/3$.

56) [T] Find the probability that a fair coin is flipped a multiple of three times before coming up heads.

57) [T] Find the probability that a fair coin will come up heads for the second time after an even number of flips.

Answer

$5/9$

58) [T] Find a series that expresses the probability that a fair coin will come up heads for the second time on a multiple of three flips.

59) [T] The expected number of times that a fair coin will come up heads is defined as the sum over $n = 1, 2, \ldots$ of n times the probability that the coin will come up heads exactly n times in a row, or $\frac{n}{2^{n+1}}$. Compute the expected number of consecutive times that a fair coin will come up heads.

Answer

$E = \sum_{n=1}^{\infty} \frac{n}{2^{n+1}} = 1,$ as can be shown using summation by parts

60) [T] A person deposits $\$10$ at the beginning of each quarter into a bank account that earns 4 annual interest compounded quarterly (four times a year).

a. Show that the interest accumulated after n quarters is $\$10(\frac{1.01^{n+1}-1}{0.01} - n)$.

b. Find the first eight terms of the sequence.

c. How much interest has accumulated after 2 years?

61) [T] Suppose that the amount of a drug in a patient's system diminishes by a multiplicative factor $r < 1$ each hour. Suppose that a new dose is administered every N hours. Find an expression that gives the amount $A(n)$ in the patient's system after n hours for each n in terms of the dosage d and the ratio r. (Hint: Write $n = mN + k$, where $0 \le k < N$, and sum over values from the different doses administered.)

Answer

The part of the first dose after n hours is dr^n, the part of the second dose is dr^{n-N}, and, in general, the part remaining of the mth dose is dr^{n-mN}, so $A(n) = \sum_{l=0}^{m} dr^{n-lN} = \sum_{l=0}^{m} dr^{k+(m-l)N} = \sum_{q=0}^{m} dr^{k+qN} = dr^k \sum_{q=0}^{m} r^{Nq} = dr^k \frac{1-r^{(m+1)N}}{1-r^N},$ where $n = k + mN$.

62) [T] A certain drug is effective for an average patient only if there is at least 1 mg per kg in the patient's system, while it is safe only if there is at most 2 mg per kg in an average patient's system. Suppose that the amount in a patient's system diminishes by a multiplicative factor of 0.9 each hour after a dose is administered. Find the maximum interval N of hours between doses, and corresponding dose range d (in mg/kg) for this N that will enable use of the drug to be both safe and effective in the long term.

63) Suppose that $a_n \ge 0$ is a sequence of numbers. Explain why the sequence of partial sums of a_n is increasing.

Answer

$S_{N+1} = a_{N+1} + S_N \ge S_N$

64) [T] Suppose that a_n is a sequence of positive numbers and the sequence S_n of partial sums of a_n is bounded above. Explain why $\sum_{n=1}^{\infty} a_n$ converges. Does the conclusion remain true if we remove the hypothesis $a_n \ge 0$?

65) [T] Suppose that $a_1 = S_1 = 1$ and that, for given numbers $S > 1$ and $0 < k < 1$, one defines $a_{n+1} = k(S - S_n)$ and $S_{n+1} = a_{n+1} + S_n$. Does S_n converge? If so, to what? (Hint: First argue that $S_n < S$ for all n and S_n is increasing.)

Answer

Since $S > 1, a_2 > 0,$ and since $k < 1, S_2 = 1 + a_2 < 1 + (S - 1) = S$. If $S_n > S$ for some n, then there is a smallest n. For this $n, S > S_{n-1}$, so $S_n = S_{n-1} + k(S - S_{n-1}) = kS + (1 - k)S_{n-1} < S$, a contradiction. Thus $S_n < S$ and $a_{n+1} > 0$ for all n, so S_n is

increasing and bounded by S. Let $S_* = \lim S_n$. If $S_* < S$, then $\delta = k(S - S_*) > 0$, but we can find n such that $S_* - S_n < \delta/2$, which implies that $S_{n+1} = S_n + k(S - S_n) > S_* + \delta/2$, contradicting that Sn is increasing to S_*. Thus $S_n \to S$.

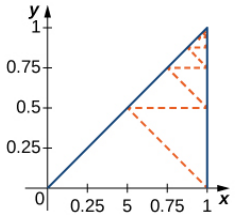
66) [T] A version of **von Bertalanffy growth** can be used to estimate the age of an individual in a homogeneous species from its length if the annual increase in year $n+1$ satisfies $a_{n+1} = k(S - S_n)$, with S_n as the length at year n, S as a limiting length, and k as a relative growth constant. If $S_1 = 3, S = 9,$ and $k = 1/2$, numerically estimate the smallest value of n such that $S_n \geq 8$. Note that $S_{n+1} = S_n + a_{n+1}$. Find the corresponding n when $k = 1/4$.

67) [T] Suppose that $\sum_{n=1}^{\infty} a_n$ is a convergent series of positive terms. Explain why $\lim_{N\to\infty} \sum_{n=N+1}^{\infty} a_n = 0$.

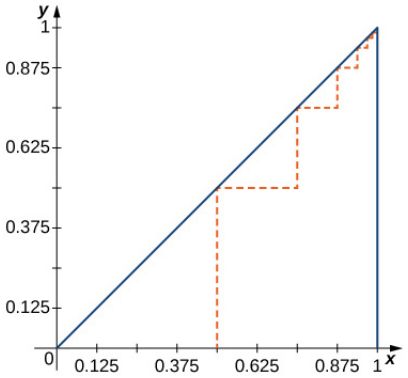
Answer

Let $S_k = \sum_{n=1}^{k} a_n$ and $S_k \to L$. Then S_k eventually becomes arbitrarily close to L, which means that $L - S_N = \sum_{n=N+1}^{\infty} a_n$ becomes arbitrarily small as $N \to \infty$.

68) [T] Find the length of the dashed zig-zag path in the following figure.



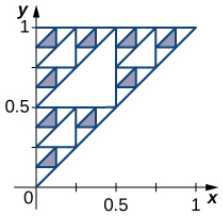
69) [T] Find the total length of the dashed path in the following figure.



Answer

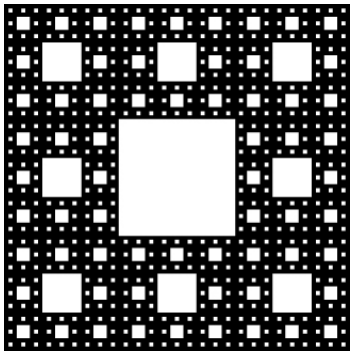
$L = \left(1 + \frac{1}{2}\right) \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{3}{2}$.

70) [T] The **Sierpinski triangle** is obtained from a triangle by deleting the middle fourth as indicated in the first step, by deleting the middle fourths of the remaining three congruent triangles in the second step, and in general deleting the middle fourths of the remaining triangles in each successive step. Assuming that the original triangle is shown in the figure, find the areas of the remaining parts of the original triangle after N steps and find the total length of all of the boundary triangles after N steps.



71) [T] The Sierpinski gasket is obtained by dividing the unit square into nine equal sub-squares, removing the middle square, then doing the same at each stage to the remaining sub-squares. The figure shows the remaining set after four iterations. Compute the total area removed after N stages, and compute the length the total perimeter of the remaining set after N stages.

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Answer

At stage one a square of area $1/9$ is removed, at stage 2 one removes 8 squares of area $1/9^2$, at stage three one removes 8^2 squares of area $1/9^3$, and so on. The total removed area after N stages is $\sum_{n=0}^{N-1} \frac{8^N}{9^{N+1}} = \frac{1}{8} \cdot \frac{1-(8/9)^N}{1-8/9} \to 1$ as $N \to \infty$. The total perimeter is $4 + 4\sum_{n=0}^{\infty} \frac{8^N}{3^{N+1}} \to \infty$.

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1.4: The Divergence and Integral Tests

Learning Objectives

- Use the divergence test to determine whether a series converges or diverges.
- Use the integral test to determine the convergence of a series.
- Estimate the value of a series by finding bounds on its remainder term.

In the previous section, we determined the convergence or divergence of several series by explicitly calculating the limit of the sequence of partial sums S_k. In practice, explicitly calculating this limit can be difficult or impossible. Luckily, several tests exist that allow us to determine convergence or divergence for many types of series. In this section, we discuss two of these tests: the divergence test and the integral test. We will examine several other tests in the rest of this chapter and then summarize how and when to use them.

Divergence Test

For a series $\sum_{n=1}^{\infty} a_n$ to converge, the nth term a_n must satisfy $a_n \to 0$ as $n \to \infty$. Therefore, from the algebraic limit properties of sequences,

$$\begin{aligned} \lim_{k\to\infty} a_k &= \lim_{k\to\infty} (S_k - S_{k-1}) \\ &= \lim_{k\to\infty} S_k - \lim_{k\to\infty} S_{k-1} \\ &= S - S = 0. \end{aligned}$$

Therefore, if $\sum_{n=1}^{\infty} a_n$ converges, the nth term $a_n \to 0$ as $n \to \infty$. An important consequence of this fact is the following statement:

$$\text{If } a_n \nrightarrow 0 \text{ as } n \to \infty, \sum_{n=1}^{\infty} a_n \text{ diverges.}$$

This test is known as the divergence test because it provides a way of proving that a series diverges.

Definition: The Divergence Test

If $\lim_{n\to\infty} a_n = c \neq 0$ or $\lim_{n\to\infty} a_n$ does not exist, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

It is important to note that the converse of this theorem is not true. That is, if $\lim_{n\to\infty} a_n = 0$, we cannot make any conclusion about the convergence of $\sum_{n=1}^{\infty} a_n$.

For example, $\lim_{n\to 0} \frac{1}{n} = 0$, but the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. In this section and the remaining sections of this chapter, we show many more examples of such series. Consequently, although we can use the divergence test to show that a series diverges, we cannot use it to prove that a series converges. Specifically, if $a_n \to 0$, the divergence test is inconclusive.

Example 1.4.1: Using the divergence test

For each of the following series, apply the divergence test. If the divergence test proves that the series diverges, state so. Otherwise, indicate that the divergence test is inconclusive.

a. $\sum_{n=1}^{\infty} \frac{n}{3n-1}$

b. $\sum_{n=1}^{\infty} \frac{1}{n^3}$

c. $\sum_{n=1}^{\infty} e^{1/n^2}$

Solution

a. Since $\lim_{n\to\infty} \frac{n}{3n-1} = \frac{1}{3} \neq 0$, by the divergence test, we can conclude that $\sum_{n=1}^{\infty} \frac{n}{3n-1}$ diverges.

b. Since $\lim_{n\to\infty} \frac{1}{n^3} = 0$, the divergence test is inconclusive.

c. Since $\lim_{n\to\infty} e^{1/n^2} = 1 \neq 0$, by the divergence test, the series $\sum_{n=1}^{\infty} e^{1/n^2}$ diverges.



Exercise 1.4.1

What does the divergence test tell us about the series $\sum_{n=1}^{\infty} \cos(1/n^2)$?

Hint

Look at $\lim_{n\to\infty} \cos(1/n^2)$.

Answer

The series diverges.

Integral Test

In the previous section, we proved that the harmonic series diverges by looking at the sequence of partial sums S_k and showing that $S_{2^k} > 1 + k/2$ for all positive integers k. In this section we use a different technique to prove the divergence of the harmonic series. This technique is important because it is used to prove the divergence or convergence of many other series. This test, called the **integral test,** compares an infinite sum to an improper integral. It is important to note that this test can only be applied when we are considering a series whose terms are all positive.

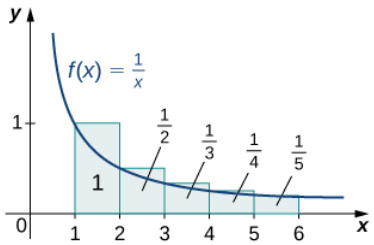


Figure 1.4.1: *The sum of the areas of the rectangles is greater than the area between the curve $f(x)=1/x$ and the x-axis for $x \geq 1$. Since the area bounded by the curve is infinite (as calculated by an improper integral), the sum of the areas of the rectangles is also infinite.*

To illustrate how the integral test works, use the harmonic series as an example. In Figure 1.4.1, we depict the harmonic series by sketching a sequence of rectangles with areas $1, 1/2, 1/3, 1/4, \ldots$ along with the function $f(x)=1/x$. From the graph, we see that

$$\sum_{n=1}^{k} \frac{1}{n} = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{k} > \int_1^{k+1} \frac{1}{x}\,dx. \tag{1.4.1}$$

Therefore, for each k, the kth partial sum S_k satisfies

$$\begin{aligned} S_k = \sum_{n=1}^{k} \frac{1}{n} &> \int_1^{k+1} \frac{1}{x}\,dx = \ln x\big|_1^{k+1} \\ &= \ln(k+1)-\ln(1) \\ &= \ln(k+1). \end{aligned}$$

Since $\lim\limits_{k\to\infty} \ln(k+1)=\infty$, we see that the sequence of partial sums S_k is unbounded. Therefore, S_k diverges, and, consequently, the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ also diverges.

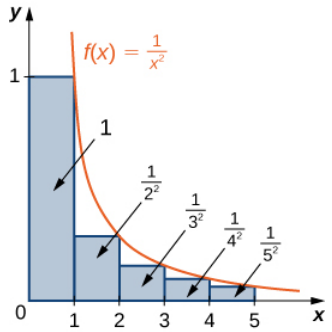


Figure 1.4.2: The sum of the areas of the rectangles is less than the sum of the area of the first rectangle and the area between the curve $f(x)=1/x^2$ and the x-axis for $x \geq 1$. Since the area bounded by the curve is finite, the sum of the areas of the rectangles is also finite.

Now consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$. We show how an integral can be used to prove that this series converges. In Figure 1.4.2, we sketch a sequence of rectangles with areas $1, 1/2^2, 1/3^2, \ldots$ along with the function $f(x)=\frac{1}{x^2}$. From the graph we see that

$$\sum_{n=1}^{k} \frac{1}{n^2} = 1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{k^2} < 1+\int_1^{k} \frac{1}{x^2}\,dx. \tag{1.4.2}$$

Therefore, for each k, the kth partial sum S_k satisfies

$$S_k = \sum_{n=1}^{k} \frac{1}{n^2} < 1 + \int_1^k \frac{1}{x^2}\,dx = 1 - \frac{1}{x}\Big|_1^k$$
$$= 1 - \frac{1}{k} + 1$$
$$= 2 - \frac{1}{k} < 2.$$

We conclude that the sequence of partial sums S_k is bounded. We also see that S_k is an increasing sequence:

$$S_k = S_{k-1} + \frac{1}{k^2} \tag{1.4.3}$$

for $k \geq 2$.

Since S_k is increasing and bounded, by the **Monotone Convergence Theorem**, it converges. Therefore, the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

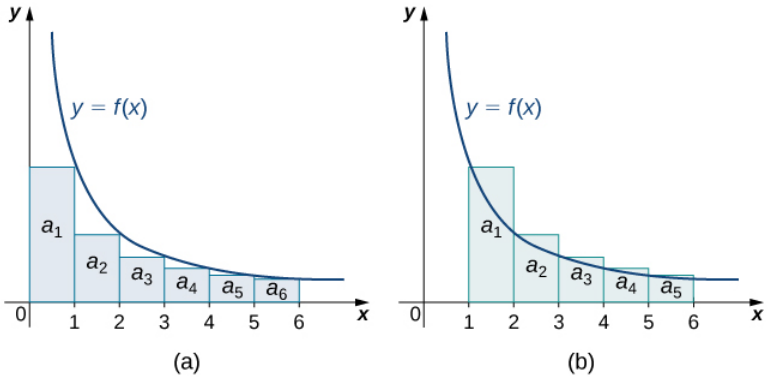


Figure 1.4.3: (a) If we can inscribe rectangles inside a region bounded by a curve $y = f(x)$ and the x-axis, and the area bounded by those curves for $x \geq 1$ is finite, then the sum of the areas of the rectangles is also finite. (b) If a set of rectangles circumscribes the region bounded by $y = f(x)$ and the x axis for $x \geq 1$ and the region has infinite area, then the sum of the areas of the rectangles is also infinite.

We can extend this idea to prove convergence or divergence for many different series. Suppose $\sum_{n=1}^{\infty} a_n$ is a series with positive terms a_n such that there exists a continuous, positive, decreasing function f where $f(n) = a_n$ for all positive integers. Then, as in Figure 1.4.3*a*, for any integer k, the kth partial sum S_k satisfies

$$S_k = a_1 + a_2 + a_3 + \cdots + a_k < a_1 + \int_1^k f(x)\,dx < 1 + \int_1^{\infty} f(x)\,dx. \tag{1.4.4}$$

Therefore, if $\int_1^{\infty} f(x)\,dx$ converges, then the sequence of partial sums S_k is bounded. Since S_k is an increasing sequence, if it is also a bounded sequence, then by the Monotone Convergence Theorem, it converges. We conclude that if $\int_1^{\infty} f(x)\,dx$ converges, then the series $\sum_{n=1}^{\infty} a_n$ also converges. On the other hand, from Figure 1.4.3*b*, for any integer k, the kth partial sum S_k satisfies

$$S_k = a_1 + a_2 + a_3 + \cdots + a_k > \int_1^{k+1} f(x)\,dx. \tag{1.4.5}$$

If

$$\lim_{k\to\infty} \int_1^{k+1} f(x)\,dx = \infty, \tag{1.4.6}$$

then S_k is an unbounded sequence and therefore diverges. As a result, the series $\sum_{n=1}^{\infty} a_n$ also diverges. Since f is a positive function, if $\int_1^{\infty} f(x)\,dx$ diverges, then

$$\lim_{k\to\infty} \int_1^{k+1} f(x)\,dx = \infty. \tag{1.4.7}$$

We conclude that if $\int_1^{\infty} f(x)\,dx$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Definition: The Integral Test

Suppose $\sum_{n=1}^{\infty} a_n$ is a series with positive terms a_n. Suppose there exists a function f and a positive integer N such that the following three conditions are satisfied:

i. f is continuous,
ii. f is decreasing, and
iii. $f(n) = a_n$ for all integers $n \geq N$.

Then

$$\sum_{n=1}^{\infty} a_n$$

and

$$\int_N^{\infty} f(x)\,dx$$

both converge or both diverge (Figure 1.4.3).

You will be shown a graph and the result of integrating from 1 to ∞. Your task is to decide whether the integral test gives convergence or divergence of the corresponding sum or if the integral test cannot be used. Click on the Start button to begin.

Diverges Converges Integral Test Cannot be Used

Start

Although convergence of $\int_N^\infty f(x)\,dx$ implies convergence of the related series $\sum_{n=1}^\infty a_n$, it does not imply that the value of the integral and the series are the same. They may be different, and often are. For example,

$$\sum_{n=1}^\infty \left(\frac{1}{e}\right)^n = \frac{1}{e} + \left(\frac{1}{e}\right)^2 + \left(\frac{1}{e}\right)^3 + \cdots \tag{1.4.8}$$

is a geometric series with initial term $a = 1/e$ and ratio $r = 1/e$, which converges to

$$\frac{1/e}{1-(1/e)} = \frac{1/e}{(e-1)/e} = \frac{1}{e-1}. \tag{1.4.9}$$

However, the related integral $\int_1^\infty (1/e)^x\,dx$ satisfies

$$\int_1^\infty \left(\frac{1}{e}\right)^x dx = \int_1^\infty e^{-x}\,dx = \lim_{b\to\infty}\int_1^b e^{-x}\,dx = \lim_{b\to\infty} -e^{-x}\Big|_1^b = \lim_{b\to\infty}\left[-e^{-b}+e^{-1}\right] = \frac{1}{e}. \tag{1.4.10}$$

Example 1.4.2: Using the Integral Test

For each of the following series, use the integral test to determine whether the series converges or diverges.

a. $\sum_{n=1}^\infty \frac{1}{n^3}$

b. $\sum_{n=1}^\infty \frac{1}{\sqrt{2n-1}}$

Solution

a. Compare

$$\sum_{n=1}^\infty \frac{1}{n^3} \text{ and } \int_1^\infty \frac{1}{x^3}\,dx.$$

We have

$$\int_1^\infty \frac{1}{x^3}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^3}\,dx = \lim_{b\to\infty}\left[-\frac{1}{2x^2}\Big|_1^b\right] = \lim_{b\to\infty}\left[-\frac{1}{2b^2}+\frac{1}{2}\right] = \frac{1}{2}.$$

Thus the integral $\int_1^\infty \frac{1}{x^3}\,dx$ converges, and therefore so does the series

$$\sum_{n=1}^\infty \frac{1}{n^3}.$$

b. Compare

$$\sum_{n=1}^\infty \frac{1}{\sqrt{2n-1}} \text{ and } \int_1^\infty \frac{1}{\sqrt{2x-1}}\,dx.$$

Since

$$\int_1^\infty \frac{1}{\sqrt{2x-1}}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{\sqrt{2x-1}}\,dx = \lim_{b\to\infty}\sqrt{2x-1}\Big|_1^b = \lim_{b\to\infty}\left[\sqrt{2b-1}-1\right] = \infty,$$

the integral $\int_1^\infty \frac{1}{\sqrt{2x-1}}\,dx$ diverges, and therefore

$$\sum_{n=1}^\infty \frac{1}{\sqrt{2n-1}}$$

diverges.





Exercise 1.4.2

Use the integral test to determine whether the series $\sum_{n=1}^{\infty} \frac{n}{3n^2+1}$ converges or diverges.

Hint

Compare to the integral $\int_1^{\infty} \frac{x}{3x^2+1}\,dx$.

Answer

The series diverges.

The p-Series

The harmonic series $\sum_{n=1}^{\infty} 1/n$ and the series $\sum_{n=1}^{\infty} 1/n^2$ are both examples of a type of series called a p-series.

Definition: p-series

For any real number p, the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \tag{1.4.11}$$

is called a *p-series*.

We know the p-series converges if $p = 2$ and diverges if $p = 1$. What about other values of p? In general, it is difficult, if not impossible, to compute the exact value of most p-series. However, we can use the tests presented thus far to prove whether a p-series converges or diverges.

If $p < 0$, then $1/n^p \to \infty$, and if $p = 0$, then $1/n^p \to 1$. Therefore, by the divergence test,

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \tag{1.4.12}$$

diverges if $p \leq 0$.

If $p > 0$, then $f(x) = 1/x^p$ is a positive, continuous, decreasing function. Therefore, for $p > 0$, we use the integral test, comparing

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \tag{1.4.13}$$

and

$$\int_1^{\infty} \frac{1}{x^p}\, dx. \tag{1.4.14}$$

We have already considered the case when $p = 1$. Here we consider the case when $p > 0, p \neq 1$. For this case,

$$\int_1^{\infty} \frac{1}{x^p}\, dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^p}\, dx = \lim_{b\to\infty} \frac{1}{1-p} x^{1-p}\,|_1^b = \lim_{b\to\infty} \frac{1}{1-p}[b^{1-p} - 1]. \tag{1.4.15}$$

Because

$$b^{1-p} \to 0 \text{ if } p > 1 \text{ and } b^{1-p} \to \infty \text{ if } p < 1,$$

we conclude that

$$\int_1^{\infty} \frac{1}{x^p}\, dx = \begin{cases} \frac{1}{p-1}, \text{if } p > 1 \\ \infty, \text{if } p < 1. \end{cases} \tag{1.4.16}$$

Therefore, $\sum_{n=1}^{\infty} 1/n^p$ converges if $p > 1$ and diverges if $0 < p < 1$.

In summary,

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \begin{cases} \text{converges if } p > 1 \\ \text{diverges if } p \leq 1 \end{cases} \tag{1.4.17}$$

.

Example 1.4.3: Testing for Convergence of p-series

For each of the following series, determine whether it converges or diverges.

a. $\sum_{n=1}^{\infty} \frac{1}{n^4}$

b. $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$

Solution

a. This is a p-series with $p = 4 > 1$,so the series converges.
b. Since $p = 2/3 < 1$, the series diverges.



Exercise 1.4.3

Does the series $\sum_{n=1}^{\infty} \frac{1}{n^{5/4}}$ converge or diverge?

Hint

$p = 5/4$

Answer

The series converges.

Estimating the Value of a Series

Suppose we know that a series $\sum_{n=1}^{\infty} a_n$ converges and we want to estimate the sum of that series. Certainly we can approximate that sum using any finite sum $\sum_{n=1}^{N} a_n$ where N is any positive integer. The question we address here is, for a convergent series $\sum_{n=1}^{\infty} a_n$, how good is the approximation $\sum_{n=1}^{N} a_n$?

More specifically, if we let

$$R_N = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n \tag{1.4.18}$$

be the remainder when the sum of an infinite series is approximated by the Nth partial sum, how large is R_N? For some types of series, we are able to use the ideas from the integral test to estimate R_N.

Note 1.4.1: Remainder Estimate from the Integral Test

Suppose $\sum_{n=1}^{\infty} a_n$ is a convergent series with positive terms. Suppose there exists a function f satisfying the following three conditions:

i. f is continuous,
ii. f is decreasing, and
iii. $f(n) = a_n$ for all integers $n \geq 1$.

Let S_N be the Nth partial sum of $\sum_{n=1}^{\infty} a_n$. For all positive integers N,

$$S_N + \int_{N+1}^{\infty} f(x)\,dx < \sum_{n=1}^{\infty} a_n < S_N + \int_{N}^{\infty} f(x)\,dx. \tag{1.4.19}$$

In other words, the remainder $R_N = \sum_{n=1}^{\infty} a_n - S_N = \sum_{n=N+1}^{\infty} a_n$ satisfies the following estimate:

$$\int_{N+1}^{\infty} f(x)\,dx < R_N < \int_{N}^{\infty} f(x)\,dx. \tag{1.4.20}$$

This is known as the **remainder estimate.**

We illustrate Note 1.4.1 in Figure 1.4.4. In particular, by representing the remainder $R_N = a_{N+1} + a_{N+2} + a_{N+3} + \cdots$ as the sum of areas of rectangles, we see that the area of those rectangles is bounded above by $\int_{N}^{\infty} f(x)\,dx$ and bounded below by $\int_{N+1}^{\infty} f(x)\,dx$. In other words,

$$R_N = a_{N+1} + a_{N+2} + a_{N+3} + \cdots > \int_{N+1}^{\infty} f(x)\,dx \tag{1.4.21}$$

and

$$R_N = a_{N+1} + a_{N+2} + a_{N+3} + \cdots < \int_{N}^{\infty} f(x)\,dx. \tag{1.4.22}$$

We conclude that

$$\int_{N+1}^{\infty} f(x)\,dx < R_N < \int_{N}^{\infty} f(x)\,dx. \tag{1.4.23}$$

Since

$$\sum_{n=1}^{\infty} a_n = S_N + R_N, \tag{1.4.24}$$

where S_N is the Nth partial sum, we conclude that

$$S_N + \int_{N+1}^{\infty} f(x)\,dx < \sum_{n=1}^{\infty} a_n < S_N + \int_{N}^{\infty} f(x)\,dx. \tag{1.4.25}$$

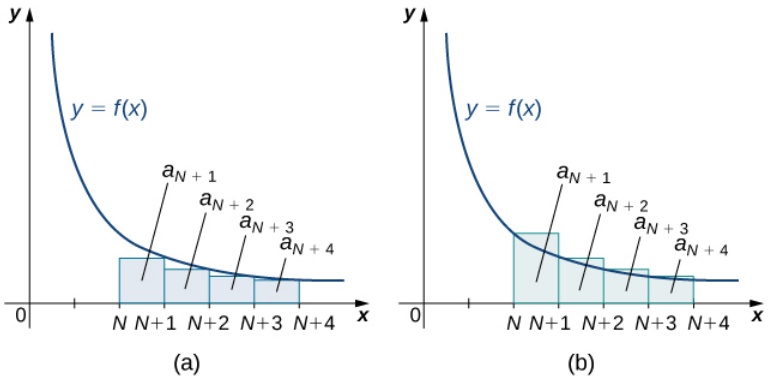


Figure 1.4.4: Given a continuous, positive, decreasing function f and a sequence of positive terms a_n such that $a_n = f(n)$ for all positive integers n, (a) the areas $a_{N+1} + a_{N+2} + a_{N+3} + \cdots < \int_N^\infty f(x)\,dx$, or (b) the areas $a_{N+1} + a_{N+2} + a_{N+3} + \cdots > \int_{N+1}^\infty f(x)\,dx$. Therefore, the integral is either an overestimate or an underestimate of the error.

Example 1.4.4: Estimating the Value of a Series

Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$.

a. Calculate $S_{10} = \sum_{n=1}^{10} \frac{1}{n^3}$ and estimate the error.

b. Determine the least value of N necessary such that S_N will estimate $\sum_{n=1}^{\infty} \frac{1}{n^3}$ to within 0.001.

Solution

a. Using a calculating utility, we have

$$S_{10} = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots + \frac{1}{10^3} \approx 1.19753. \tag{1.4.26}$$

By the remainder estimate, we know

$$R_N < \int_N^\infty \frac{1}{x^3}\,dx. \tag{1.4.27}$$

We have

$$\int_{10}^\infty \frac{1}{x^3}\,dx = \lim_{b\to\infty} \int_{10}^b \frac{1}{x^3}\,dx = \lim_{b\to\infty} \left[-\frac{1}{2x^2}\right]_N^b = \lim_{b\to\infty} \left[-\frac{1}{2b^2} + \frac{1}{2N^2}\right] = \frac{1}{2N^2}. \tag{1.4.28}$$

Therefore, the error is $R_{10} < 1/2(10)^2 = 0.005$.

b. Find N such that $R_N < 0.001$. In part a. we showed that $R_N < 1/2N^2$. Therefore, the remainder $R_N < 0.001$ as long as $1/2N^2 < 0.001$. That is, we need $2N^2 > 1000$. Solving this inequality for N, we see that we need $N > 22.36$. To ensure that the remainder is within the desired amount, we need to round up to the nearest integer. Therefore, the minimum necessary value is $N = 23$.

Exercise 1.4.4

For $\sum_{n=1}^{\infty} \frac{1}{n^4}$, calculate S_5 and estimate the error R_5.

Hint

Use the remainder estimate $R_N < \int_N^\infty \frac{1}{x^4}\,dx$.

Answer

$S_5 \approx 1.09035, R_5 < 0.00267$

Key Concepts

- If $\lim_{n\to\infty} a_n \neq 0,$ then the series $\sum_{n=1}^{\infty} a_n$ diverges.
- If $\lim_{n\to\infty} a_n = 0,$ the series $\sum_{n=1}^{\infty} a_n$ may converge or diverge.
- If $\sum_{n=1}^{\infty} a_n$ is a series with positive terms a_n and f is a continuous, decreasing function such that $f(n) = a_n$ for all positive integers n, then

$$\sum_{n=1}^{\infty} a_n \tag{1.4.29}$$

and

$$\int_1^\infty f(x)\,dx \tag{1.4.30}$$

either both converge or both diverge. Furthermore, if $\sum_{n=1}^{\infty} a_n$ converges, then the Nth partial sum approximation S_N is accurate up to an error R_N where $\int_{N+1}^\infty f(x)\,dx < R_N < \int_N^\infty f(x)\,dx$.

- The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Key Equations

- **Divergence test**

  If $a_n \nrightarrow 0$ as $n \to \infty$, $\sum_{n=1}^{\infty} a_n$ diverges.

- **p-series**

  $$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \begin{cases} \text{converges, if } p > 1 \\ \text{diverges, if } p \leq 1 \end{cases}$$

- **Remainder estimate from the integral test**

  $$\int_{N+1}^\infty f(x)\,dx < R_N < \int_N^\infty f(x)\,dx$$

Glossary

divergence test

if $\lim_{n\to\infty} a_n \neq 0,$ then the series $\sum_{n=1}^{\infty} a_n$ diverges

integral test

for a series $\sum_{n=1}^{\infty} a_n$ with positive terms a_n, if there exists a continuous, decreasing function f such that $f(n) = a_n$ for all positive integers n, then

$$\sum_{n=1}^{\infty} a_n \tag{1.4.31}$$

and

$$\int_1^{\infty} f(x)\,dx \tag{1.4.32}$$

either both converge or both diverge

_p_-series

a series of the form $\sum_{n=1}^{\infty} 1/n^p$

remainder estimate

for a series $\sum_{n=}^{\infty} 1a_n$ with positive terms a_n and a continuous, decreasing function f such that $f(n) = a_n$ for all positive integers n, the remainder $R_N = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n$ satisfies the following estimate:

$$\int_{N+1}^{\infty} f(x)\,dx < R_N < \int_N^{\infty} f(x)\,dx \tag{1.4.33}$$

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1.4E: Exercises for Section 1.4

Divergence Test Problems

Consider the sequence for each series in exercises 1 - 14, if the divergence test applies, either state that $\lim_{n\to\infty} a_n$ does not exist or find $\lim_{n\to\infty} a_n$. If the divergence test does not apply, state why.

1) $\sum_{n=1}^{\infty} \frac{n}{n+2}$

2) $\sum_{n=1}^{\infty} \frac{n}{5n^2-3}$

Answer

$\lim_{n\to\infty} a_n = 0$. The Divergence Test does not apply.

3) $\sum_{n=1}^{\infty} \frac{n}{\sqrt{3n^2+2n+1}}$

4) $\sum_{n=1}^{\infty} \frac{(2n+1)(n-1)}{(n+1)^2}$

Answer

$\lim_{n\to\infty} a_n = 2$. So the series diverges by the nth-Term Test for Divergence.

5) $\sum_{n=1}^{\infty} \frac{(2n+1)^{2n}}{(3n^2+1)^n}$

6) $\sum_{n=1}^{\infty} \frac{2^n}{3^{n/2}}$

Answer

$\lim_{n\to\infty} a_n = \infty$ (does not exist). So the series diverges by the nth-Term Test for Divergence.

7) $\sum_{n=1}^{\infty} \frac{2^n+3^n}{10^{n/2}}$

8) $\sum_{n=1}^{\infty} e^{-2/n}$

Answer

$\lim_{n\to\infty} a_n = 1$. So the series diverges by the nth-Term Test for Divergence.

9) $\sum_{n=1}^{\infty} \cos n$

10) $\sum_{n=1}^{\infty} \tan n$

Answer

$\lim_{n\to\infty} a_n$ does not exist. So the series diverges by the nth-Term Test for Divergence.

11) $\sum_{n=1}^{\infty} \frac{1-\cos^2(1/n)}{\sin^2(2/n)}$

12) $\sum_{n=1}^{\infty} \left(1-\frac{1}{n}\right)^{2n}$

Answer

$\lim_{n\to\infty} a_n = 1/e^2$. So the series diverges by the nth-Term Test for Divergence.

13) $\sum_{n=1}^{\infty} \frac{\ln n}{n}$

14) $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{\sqrt{n}}$

Answer

$\lim_{n\to\infty} a_n = 0$. The Divergence Test does not apply.

p-Series Problems & Integral Test Problems

In exercises 15 - 20, state whether the given p-series converges.

15) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

16) $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$

Answer

The series converges, since $p = 3/2 > 1$.

17) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2}}$

18) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^4}}$

Answer

The series converges, since $p = 4/3 > 1$.

19) $\sum_{n=1}^{\infty} \frac{n^e}{n^\pi}$

20) $\sum_{n=1}^{\infty} \frac{n^\pi}{n^{2e}}$

Answer

The series converges, since $p = 2e - \pi > 1$.

In exercises 21 - 27, use the integral test to determine whether the following sums converge.

21) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+5}}$

22) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n+5}}$

Answer

The series diverges by the Integral Test since $\int_1^{\infty} \frac{dx}{(x+5)^{1/3}}$ can be shown to diverge.

23) $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

24) $\sum_{n=1}^{\infty} \frac{n}{1+n^2}$

Answer

The series diverges by the Integral Test since $\int_1^{\infty} \frac{x}{1+x^2}\, dx$ can be shown to diverge.

25) $\sum_{n=1}^{\infty} \frac{e^n}{1+e^{2n}}$

26) $\sum_{n=1}^{\infty} \frac{2n}{1+n^4}$

Answer

The series converges by the Integral Test since $\int_1^{\infty} \frac{2x}{1+x^4}\, dx$ converges.

27) $\sum_{n=2}^{\infty} \frac{1}{n \ln^2 n}$

Express the sums in exercises 28 - 31 as p-series and determine whether each converges.

28) $\sum_{n=1}^{\infty} 2^{-\ln n}$ $\left(\text{Hint: } 2^{-\ln n} = \frac{1}{n^{\ln 2}}.\right)$

Answer

$2^{-\ln n} = 1/n^{\ln 2}$. Since $p = \ln 2 < 1$, this series diverges by the p-series test.

29) $\sum_{n=1}^{\infty} 3^{-\ln n}$ $\left(\text{Hint: } 3^{-\ln n} = \frac{1}{n^{\ln 3}}.\right)$

30) $\sum_{n=1}^{n} 2^{-2 \ln n}$

Answer

$2^{-2 \ln n} = 1/n^{2 \ln 2}$. Since $p = 2 \ln 2 - 1 < 1$, this series diverges by the p-series test.

31) $\sum_{n=1}^{\infty} n 3^{-2 \ln n}$

In exercises 32 - 35, use the estimate $R_N \leq \int_N^{\infty} f(t)\, dt$ to find a bound for the remainder $R_N = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n$ where $a_n = f(n)$.

32) $\sum_{n=1}^{1000} \frac{1}{n^2}$

Answer

$$R_{1000} \le \int_{1000}^{\infty} \frac{dt}{t^2} = \lim_{b\to\infty} -\frac{1}{t}\Big|_{1000}^{b} = \lim_{b\to\infty}\left(-\frac{1}{b}+\frac{1}{1000}\right) = 0.001$$

33) $\displaystyle\sum_{n=1}^{1000} \frac{1}{n^3}$

34) $\displaystyle\sum_{n=1}^{1000} \frac{1}{1+n^2}$

Answer

$$R_{1000} \le \int_{1000}^{\infty} \frac{dt}{1+t^2} = \lim_{b\to\infty}\left(\tan^{-1} b - \tan^{-1}(1000)\right) = \pi/2 - \tan^{-1}(1000) \approx 0.000999$$

35) $\displaystyle\sum_{n=1}^{100} \frac{n}{2^n}$

[T] In exercises 36 - 40, find the minimum value of N such that the remainder estimate $\displaystyle\int_{N+1}^{\infty} f(x)\,dx < R_N < \int_{N}^{\infty} f(x)\,dx$ guarantees that $\displaystyle\sum_{n=1}^{N} a_n$ estimates $\displaystyle\sum_{n=1}^{\infty} a_n$, accurate to within the given error.

36) $a_n = \dfrac{1}{n^2}$, error $< 10^{-4}$

Answer

$$R_N < \int_{N}^{\infty} \frac{dx}{x^2} = 1/N, \text{ for } N > 10^4$$

37) $a_n = \dfrac{1}{n^{1.1}}$, error $< 10^{-4}$

38) $a_n = \dfrac{1}{n^{1.01}}$, error $< 10^{-4}$

Answer

$$R_N < \int_{N}^{\infty} \frac{dx}{x^{1.01}} = 100N^{-0.01}, \text{ for } N > 10^{600}$$

39) $a_n = \dfrac{1}{n\ln^2 n}$, error $< 10^{-3}$

40) $a_n = \dfrac{1}{1+n^2}$, error $< 10^{-3}$

Answer

$$R_N < \int_{N}^{\infty} \frac{dx}{1+x^2} = \pi/2 - \tan^{-1}(N), \text{ for } N > \tan(\pi/2 - 10^{-3}) \approx 1000$$

In exercises 41 - 45, find a value of N such that R_N is smaller than the desired error. Compute the corresponding sum $\displaystyle\sum_{n=1}^{N} a_n$ and compare it to the given estimate of the infinite series.

41) $a_n = \dfrac{1}{n^{11}}$, error $< 10^{-4}$, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{11}} = 1.000494\ldots$

42) $a_n = \dfrac{1}{e^n}$, error $< 10^{-5}$, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{e^n} = \frac{1}{e-1} = 0.581976\ldots$

Answer

$R_N < \int_N^\infty \frac{dx}{e^x} = e^{-N}$, for $N > 5\ln(10)$, okay if $N = 12$; $\sum_{n=1}^{12} e^{-n} = 0.581973\ldots$Estimate agrees with $1/(e-1)$ to five decimal places.

43) $a_n = \frac{1}{e^{n^2}}$, error $< 10^{-5}$, $\sum_{n=1}^{\infty} \frac{1}{e^{n^2}} = 0.40488139857\ldots$

44) $a_n = \frac{1}{n^4}$, error $< 10^{-4}$, $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} = 1.08232\ldots$

Answer

$R_N < \int_N^\infty \frac{dx}{x^4} = 4/N^3$, for $N > (4.10^4)^{1/3}$, okay if $N = 35$; $\sum_{n=1}^{35} \frac{1}{n^4} = 1.08231\ldots$.Estimate agrees with the sum to four decimal places.

45) $a_n = \frac{1}{n^6}$, error $< 10^{-6}$, $\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945} = 1.01734306\ldots$,

46) Find the limit as $n \to \infty$ of $\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n}$. $\left(\text{Hint: Compare to } \int_n^{2n} \frac{1}{t}\, dt.\right)$

Answer

$\ln(2)$

47) Find the limit as $n \to \infty$ of $\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{3n}$

The next few exercises are intended to give a sense of applications in which partial sums of the harmonic series arise.

48) In certain applications of probability, such as the so-called Watterson estimator for predicting mutation rates in population genetics, it is important to have an accurate estimate of the number $H_k = (1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k})$. Recall that $T_k = H_k - \ln k$ is decreasing. Compute $T = \lim_{k\to\infty} T_k$ to four decimal places.

$\left(\text{Hint: } \frac{1}{k+1} < \int_k^{k+1} \frac{1}{x}\, dx.\right)$

Answer

$T = 0.5772\ldots$

49) [T] Complete sampling with replacement, sometimes called the **coupon collector's problem**, is phrased as follows: Suppose you have N unique items in a bin. At each step, an item is chosen at random, identified, and put back in the bin. The problem asks what is the expected number of steps $E(N)$ that it takes to draw each unique item at least once. It turns out that $E(N) = N.H_N = N\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N}\right)$. Find $E(N)$ for $N = 10, 20$, and 50.

50) [T] The simplest way to shuffle cards is to take the top card and insert it at a random place in the deck, called top random insertion, and then repeat. We will consider a deck to be randomly shuffled once enough top random insertions have been made that the card originally at the bottom has reached the top and then been randomly inserted. If the deck has n cards, then the probability that the insertion will be below the card initially at the bottom (call this card B) is $1/n$. Thus the expected number of top random insertions before B is no longer at the bottom is n. Once one card is below B, there are two places below B and the probability that a randomly inserted card will fall below B is $2/n$. The expected number of top random insertions before this happens is $n/2$. The two cards below B are now in random order. Continuing this way, find a formula for the expected number of top random insertions needed to consider the deck to be randomly shuffled.

Answer

The expected number of random insertions to get B to the top is $n+n/2+n/3+\cdots+n/(n-1)$. Then one more insertion puts B back in at random. Thus, the expected number of shuffles to randomize the deck is $n(1+1/2+\cdots+1/n)$.

51) Suppose a scooter can travel 100 km on a full tank of fuel. Assuming that fuel can be transferred from one scooter to another but can only be carried in the tank, present a procedure that will enable one of the scooters to travel $100H_N$ km, where $H_N=1+1/2+\cdots+1/N$.

52) Show that for the remainder estimate to apply on $[N,\infty)$ it is sufficient that $f(x)$ be decreasing on $[N,\infty)$, but f need not be decreasing on $[1,\infty)$.

Answer

Set $b_n=a_{n+N}$ and $g(t)=f(t+N)$ such that f is decreasing on $[t,\infty)$.

53) [T] Use the remainder estimate and integration by parts to approximate $\displaystyle\sum_{n=1}^{\infty}\frac{n}{e^n}$ within an error smaller than 0.0001.

54) Does $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^p}$ converge if p is large enough? If so, for which p?

Answer

The series converges for $p>1$ by integral test using change of variable.

55) [T] Suppose a computer can sum one million terms per second of the divergent series $\displaystyle\sum_{n=1}^{N}\frac{1}{n}$. Use the integral test to approximate how many seconds it will take to add up enough terms for the partial sum to exceed 100.

56) [T] A fast computer can sum one million terms per second of the divergent series $\displaystyle\sum_{n=2}^{N}\frac{1}{n\ln n}$. Use the integral test to approximate how many seconds it will take to add up enough terms for the partial sum to exceed 100.

Answer

$N=e^{e^{100}}\approx e^{10^{43}}$ terms are needed.

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1.5: Comparison Tests

Learning Objectives

- Use the comparison test to test a series for convergence.
- Use the limit comparison test to determine convergence of a series.

We have seen that the integral test allows us to determine the convergence or divergence of a series by comparing it to a related improper integral. In this section, we show how to use comparison tests to determine the convergence or divergence of a series by comparing it to a series whose convergence or divergence is known. Typically these tests are used to determine convergence of series that are similar to geometric series or p-series.

Comparison Test

In the preceding two sections, we discussed two large classes of series: geometric series and p-series. We know exactly when these series converge and when they diverge. Here we show how to use the convergence or divergence of these series to prove convergence or divergence for other series, using a method called the **comparison test.**

For example, consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}. \tag{1.5.1}$$

This series looks similar to the convergent series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \tag{1.5.2}$$

Since the terms in each of the series are positive, the sequence of partial sums for each series is monotone increasing. Furthermore, since

$$0 < \frac{1}{n^2+1} < \frac{1}{n^2} \tag{1.5.3}$$

for all positive integers n, the kth partial sum S_k of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ satisfies

$$S_k = \sum_{n=1}^{k} \frac{1}{n^2+1} < \sum_{n=1}^{k} \frac{1}{n^2} < \sum_{n=1}^{\infty} \frac{1}{n^2}. \tag{1.5.4}$$

(See Figure 1.5.1a and Table 1.5.1.) Since the series on the right converges, the sequence S_k is bounded above. We conclude that S_k is a monotone increasing sequence that is bounded above. Therefore, by the Monotone Convergence Theorem, S_k converges, and thus

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1} \tag{1.5.5}$$

converges.

Similarly, consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n-1/2}. \tag{1.5.6}$$

This series looks similar to the divergent series

$$\sum_{n=1}^{\infty} \frac{1}{n}. \tag{1.5.7}$$

The sequence of partial sums for each series is monotone increasing and

$$\frac{1}{n-1/2} > \frac{1}{n} > 0 \tag{1.5.8}$$

for every positive integer n. Therefore, the kth partial sum S_k of

$$\sum_{n=1}^{\infty} \frac{1}{n-1/2} \tag{1.5.9}$$

satisfies

$$S_k = \sum_{n=1}^{k} \frac{1}{n-1/2} > \sum_{n=1}^{k} \frac{1}{n}. \tag{1.5.10}$$

(See Figure 1.5.1$n$and Table 1.5.1). Since the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges to infinity, the sequence of partial sums $\sum_{n=1}^{k} \frac{1}{n}$ is unbounded. Consequently, S_k is an unbounded sequence, and therefore diverges. We conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n-1/2} \tag{1.5.11}$$

diverges.

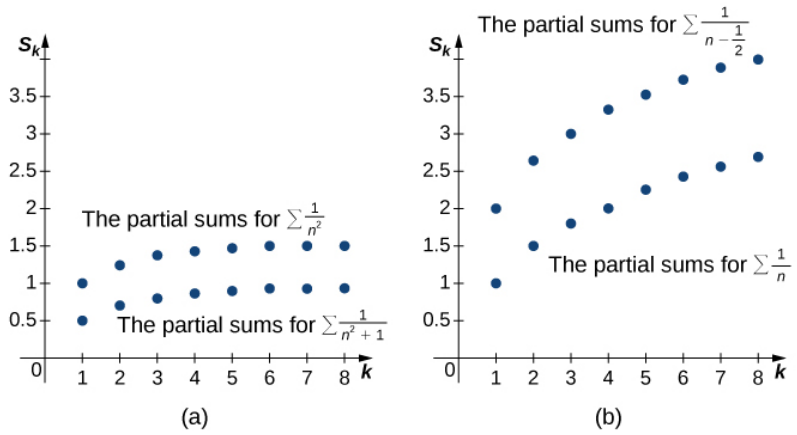


Figure 1.5.1: (a) Each of the partial sums for the given series is less than the corresponding partial sum for the converging $p-series$. (b) Each of the partial sums for the given series is greater than the corresponding partial sum for the diverging harmonic series.

Table 1.5.1: Comparing a series with a p-series ($p = 2$)

k	1	2	3	4	5	6	7	8
$\sum_{n=1}^{k} \frac{1}{n^2+1}$	0.5	0.7	0.8	0.8588	0.8973	0.9243	0.9443	0.9597
$\sum_{n=1}^{k} \frac{1}{n^2}$	1	1.25	1.3611	1.4236	1.4636	1.4914	1.5118	1.5274

Table 1.5.2: Comparing a series with the harmonic series

k	1	2	3	4	5	6	7	8
$\sum_{n=1}^{k} \frac{1}{n-1/2}$	2	2.6667	3.0667	3.3524	3.5746	3.7564	3.9103	4.0436
$\sum_{n=1}^{k} \frac{1}{n}$	1	1.5	1.8333	2.0933	2.2833	2.45	2.5929	2.7179

Comparison Test

i. Suppose there exists an integer N such that $0 \le a_n \le b_n$ for all $n \ge N$. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

ii. Suppose there exists an integer N such that $a_n \ge b_n \ge 0$ for all $n \ge N$. If $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Proof

We prove part i. The proof of part ii. is the contrapositive of part i. Let S_k be the sequence of partial sums associated with $\sum_{n=1}^{\infty} a_n$, and let $L = \sum_{n=1}^{\infty} b_n$. Since the terms $a_n \ge 0$,

$$S_k = a_1 + a_2 + \cdots + a_k \le a_1 + a_2 + \cdots + a_k + a_{k+1} = S_{k+1}.$$

Therefore, the sequence of partial sums is increasing. Further, since $a_n \le b_n$ for all $n \ge N$, then

$$\sum_{n=N}^{k} a_n \le \sum_{n=N}^{k} b_n \le \sum_{n=1}^{\infty} b_n = L.$$

Therefore, for all $k \ge 1$,

$$S_k = (a_1 + a_2 + \cdots + a_{N-1}) + \sum_{n=N}^{k} a_n \le (a_1 + a_2 + \cdots + a_{N-1}) + L.$$

Since $a_1 + a_2 + \cdots + a_{N-1}$ is a finite number, we conclude that the sequence S_k is bounded above. Therefore, S_k is an increasing sequence that is bounded above. By the Monotone Convergence Theorem, we conclude that S_k converges, and therefore the series $\sum_{n=1}^{\infty} a_n$ converges.

□

To use the comparison test to determine the convergence or divergence of a series $\sum_{n=1}^{\infty} a_n$, it is necessary to find a suitable series with which to compare it. Since we know the convergence properties of geometric series and p-series, these series are often used. If there exists an integer N such that for all $n \ge N$, each term an is less than each corresponding term of a known convergent series, then $\sum_{n=1}^{\infty} a_n$ converges. Similarly, if there exists an integer N such that for all $n \ge N$, each term an is greater than each corresponding term of a known divergent series, then $\sum_{n=1}^{\infty} a_n$ diverges.

Example 1.5.1: Using the Comparison Test

For each of the following series, use the comparison test to determine whether the series converges or diverges.

a. $\sum_{n=1}^{\infty} = \dfrac{1}{n^3 + 3n + 1}$

b. $\sum_{n=1}^{\infty} = \dfrac{1}{2^n + 1}$

c. $\sum_{n=2}^{\infty} = \dfrac{1}{\ln n}$

Solution

a. Compare to $\sum_{n=1}^{\infty} \dfrac{1}{n^3}$. Since $\sum_{n=1}^{\infty} \dfrac{1}{n^3}$ is a p-series with $p = 3$, it converges. Further,

$$\frac{1}{n^3+3n+1} < \frac{1}{n^3}$$

for every positive integer n. Therefore, we can conclude that $\sum_{n=1}^{\infty} \frac{1}{n^3+3n+1}$ converges.

b. Compare to $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$. Since $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ is a geometric series with $r = \frac{1}{2}$ and $\left|\frac{1}{2}\right| < 1$, it converges. Also,

$$\frac{1}{2^n+1} < \frac{1}{2^n}$$

for every positive integer n. Therefore, we see that $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ converges.

c. Compare to $\sum_{n=2}^{\infty} \frac{1}{n}$. Since

$$\frac{1}{\ln n} > \frac{1}{n}$$

for every integer $n \geq 2$ and $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges, we have that $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ diverges.

Exercise 1.5.1

Use the comparison test to determine if the series $\sum_{n=1}^{\infty} \frac{n}{n^3+n+1}$ converges or diverges.

Hint

Find a value p such that $\frac{n}{n^3+n+1} \leq \frac{1}{n^p}$.

Answer

The series converges.





Limit Comparison Test

The comparison test works nicely if we can find a comparable series satisfying the hypothesis of the test. However, sometimes finding an appropriate series can be difficult. Consider the series

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1}. \tag{1.5.12}$$

It is natural to compare this series with the convergent series

$$\sum_{n=2}^{\infty} \frac{1}{n^2}. \tag{1.5.13}$$

However, this series does not satisfy the hypothesis necessary to use the comparison test because

$$\frac{1}{n^2-1} > \frac{1}{n^2} \tag{1.5.14}$$

for all integers $n \geq 2$. Although we could look for a different series with which to compare $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$, instead we show how we can use the **limit comparison test** to compare

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1} \tag{1.5.15}$$

and

$$\sum_{n=2}^{\infty} \frac{1}{n^2}. \tag{1.5.16}$$

Let us examine the idea behind the limit comparison test. Consider two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$. with positive terms a_n and b_n and evaluate

$$\lim_{n\to\infty} \frac{a_n}{b_n}. \tag{1.5.17}$$

If

$$\lim_{n\to\infty}\frac{a_n}{b_n}=L\neq 0, \tag{1.5.18}$$

then, for n sufficiently large, $a_n\approx Lb_n$. Therefore, either both series converge or both series diverge. For the series $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n^2-1}$ and $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n^2}$, we see that

$$\lim_{n\to\infty}\frac{1/(n^2-1)}{1/n^2}=\lim_{n\to\infty}\frac{n^2}{n^2-1}=1. \tag{1.5.19}$$

Since $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n^2}$ converges, we conclude that

$$\sum_{n=2}^{\infty}\frac{1}{n^2-1} \tag{1.5.20}$$

converges.

The limit comparison test can be used in two other cases. Suppose

$$\lim_{n\to\infty}\frac{a_n}{b_n}=0. \tag{1.5.21}$$

In this case, a_n/b_n is a bounded sequence. As a result, there exists a constant M such that $a_n\leq Mb_n$. Therefore, if $\displaystyle\sum_{n=1}^{\infty}b_n$ converges, then $\displaystyle\sum_{n=1}^{\infty}a_n$ converges. On the other hand, suppose

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\infty. \tag{1.5.22}$$

In this case, a_n/b_n is an unbounded sequence. Therefore, for every constant M there exists an integer N such that $a_n\geq Mb_n$ for all $n\geq N$. Therefore, if $\displaystyle\sum_{n=1}^{\infty}b_n$ diverges, then $\displaystyle\sum_{n=1}^{\infty}a_n$ diverges as well.

Limit Comparison Test

Let $a_n, b_n\geq 0$ for all $n\geq 1$.

i. If $\displaystyle\lim_{n\to\infty}\frac{a_n}{b_n}=L\neq 0,$ then $\displaystyle\sum_{n=1}^{\infty}a_n$ and $\displaystyle\sum_{n=1}^{\infty}b_n$ both converge or both diverge.

ii. If $\displaystyle\lim_{n\to\infty}\frac{a_n}{b_n}=0$ and $\displaystyle\sum_{n=1}^{\infty}b_n$ converges, then $\displaystyle\sum_{n=1}^{\infty}a_n$ converges.

iii. If $\displaystyle\lim_{n\to\infty}\frac{a_n}{b_n}=\infty$ and $\displaystyle\sum_{n=1}^{\infty}b_n$ diverges, then $\displaystyle\sum_{n=1}^{\infty}a_n$ diverges.

Note that if $\dfrac{a_n}{b_n}\to 0$ and $\displaystyle\sum_{n=1}^{\infty}b_n$ diverges, the limit comparison test gives no information. Similarly, if $\dfrac{a_n}{b_n}\to\infty$ and $\displaystyle\sum_{n=1}^{\infty}b_n$ converges, the test also provides no information. For example, consider the two series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ and $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$. These series are both p-series with $p=\frac{1}{2}$ and $p=2$, respectively. Since $p=\frac{1}{2}<1,$ the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ diverges. On the other hand, since $p=2>1$,

the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. However, suppose we attempted to apply the limit comparison test, using the convergent **p−series** $\sum_{n=1}^{\infty} \frac{1}{n^3}$ as our comparison series. First, we see that

$$\frac{1/\sqrt{n}}{1/n^3} = \frac{n^3}{\sqrt{n}} = n^{5/2} \to \infty \text{ as } n \to \infty. \tag{1.5.23}$$

Similarly, we see that

$$\frac{1/n^2}{1/n^3} = n \to \infty \text{ as } n \to \infty. \tag{1.5.24}$$

Therefore, if $\frac{a_n}{b_n} \to \infty$ when $\sum_{n=1}^{\infty} b_n$ converges, we do not gain any information on the convergence or divergence of $\sum_{n=1}^{\infty} a_n$.

Example 1.5.2: Using the Limit Comparison Test

For each of the following series, use the limit comparison test to determine whether the series converges or diverges. If the test does not apply, say so.

a. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+1}$

b. $\sum_{n=1}^{\infty} \frac{2^n+1}{3^n}$

c. $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}$

Solution

a. Compare this series to $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$. Calculate

$$\lim_{n\to\infty} \frac{1/(\sqrt{n}+1)}{1/\sqrt{n}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}+1} = \lim_{n\to\infty} \frac{1/\sqrt{n}}{1+1/\sqrt{n}} = 1.$$

By the limit comparison test, since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges, then $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+1}$ diverges.

b. Compare this series to $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$. We see that

$$\lim_{n\to\infty} \frac{(2^n+1)/3^n}{2^n/3^n} = \lim_{n\to\infty} \frac{2^n+1}{3^n} \cdot \frac{3^n}{2^n} = \lim_{n\to\infty} \frac{2^n+1}{2^n} = \lim_{n\to\infty} \left[1+\left(\tfrac{1}{2}\right)^n\right] = 1.$$

Therefore,

$$\lim_{n\to\infty} \frac{(2^n+1)/3^n}{2^n/3^n} = 1.$$

Since $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$ converges, we conclude that $\sum_{n=1}^{\infty} \frac{2^n+1}{3^n}$ converges.

c. Since $\ln n < n$, compare with $\sum_{n=1}^{\infty} \frac{1}{n}$. We see that

$$\lim_{n\to\infty} \frac{\ln n/n^2}{1/n} = \lim_{n\to\infty} \frac{\ln n}{n^2} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{\ln n}{n}.$$

In order to evaluate $\lim_{n\to\infty} \ln n/n$, evaluate the limit as $x \to \infty$ of the real-valued function $\ln(x)/x$. These two limits are equal, and making this change allows us to use L'Hôpital's rule. We obtain

$$\lim_{x\to\infty} \frac{lnx}{x} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

Therefore, $\lim_{n\to\infty} \frac{\ln n}{n} = 0$, and, consequently,

$$\lim_{n\to\infty} \frac{(\ln n)/n^2}{1/n} = 0.$$

Since the limit is 0 but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, the limit comparison test does not provide any information.

Compare with $\sum_{n=1}^{\infty} \frac{1}{n^2}$ instead. In this case,

$$\lim_{n\to\infty} \frac{(\ln n)/n^2}{1/n^2} = \lim_{n\to\infty} \frac{\ln n}{n^2} \cdot \frac{n^2}{1} = \lim_{n\to\infty} \ln n = \infty.$$

Since the limit is ∞ but $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, the test still does not provide any information.

So now we try a series between the two we already tried. Choosing the series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$, we see that

$$\lim_{n\to\infty} \frac{(\ln n)/n^2}{1/n^{3/2}} = \lim_{n\to\infty} \frac{\ln n}{n^2} \cdot \frac{n^{3/2}}{1} = \lim_{n\to\infty} \frac{\ln n}{\sqrt{n}} \ .$$

As above, in order to evaluate $\lim_{n\to\infty} \frac{\ln n}{\sqrt{n}}$, evaluate the limit as $x \to \infty$ of the real-valued function $\frac{\ln n}{\sqrt{n}}$. Using L'Hôpital's rule,

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = \lim_{x\to\infty} \frac{2\sqrt{x}}{x} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0 \ .$$

Since the limit is 0 and $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges, we can conclude that $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ converges.





Exercise 1.5.2

Use the limit comparison test to determine whether the series $\sum_{n=1}^{\infty} \frac{5^n}{3^n+2}$ converges or diverges.

Hint

Compare with a geometric series.

Answer

The series diverges.

Key Concepts

- The comparison tests are used to determine convergence or divergence of series with positive terms.
- When using the comparison tests, a series $\sum_{n=1}^{\infty} a_n$ is often compared to a geometric or p-series.

Glossary

comparison test

If $0 \le a_n \le b_n$ for all $n \ge N$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges; if $a_n \ge b_n \ge 0$ for all $n \ge N$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

limit comparison test

Suppose $a_n, b_n \ge 0$ for all $n \ge 1$. If $\lim_{n\to\infty} a_n/b_n \to L \ne 0$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge; if $\lim_{n\to\infty} a_n/b_n \to 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. If $\lim_{n\to\infty} a_n/b_n \to \infty$, and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

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1.5E: Exercises for Section 1.5

Use the Comparison Test to determine whether each series in exercises 1 - 13 converges or diverges.

1) $\sum_{n=1}^{\infty} a_n$ where $a_n = \dfrac{2}{n(n+1)}$

2) $\sum_{n=1}^{\infty} a_n$ where $a_n = \dfrac{1}{n(n+1/2)}$

Answer

Converges by comparison with $\dfrac{1}{n^2}$.

3) $\sum_{n=1}^{\infty} \dfrac{1}{2(n+1)}$

4) $\sum_{n=1}^{\infty} \dfrac{1}{2n-1}$

Answer

Since $2n-1 < 2n$, we have $\dfrac{1}{2n} < \dfrac{1}{2n-1} \rightarrow \sum_{n=1}^{\infty} \dfrac{1}{2n} < \sum_{n=1}^{\infty} \dfrac{1}{2n-1}$.

But the series on the left is just half of the harmonic series, since $\sum_{n=1}^{\infty} \dfrac{1}{2n} = \frac{1}{2} \sum_{n=1}^{\infty} \dfrac{1}{n}$. This clearly diverges, so the given series also diverges, by the Comparison Test.

5) $\sum_{n=2}^{\infty} \dfrac{1}{(n \ln n)^2}$

6) $\sum_{n=1}^{\infty} \dfrac{n!}{(n+2)!}$

Answer

$a_n = 1/(n+1)(n+2) < 1/n^2$. Converges by comparison with p-series, $p = 2 > 1$.

7) $\sum_{n=1}^{\infty} \dfrac{1}{n!}$

8) $\sum_{n=1}^{\infty} \dfrac{\sin(1/n)}{n}$

Answer

$\sin(1/n) \leq 1/n$, so converges by comparison with p-series, $p = 2 > 1$.

9) $\sum_{n=1}^{\infty} \dfrac{\sin^2 n}{n^2}$

10) $\sum_{n=1}^{\infty} \dfrac{\sin(1/n)}{\sqrt{n}}$

Answer

$\sin(1/n) \leq 1$, so converges by comparison with p-series, $p = 3/2 > 1$.

11) $\sum_{n=1}^{\infty} \frac{n^{1.2}-1}{n^{2.3}+1}$

12) $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{n}$

Answer

Since $\sqrt{n+1}-\sqrt{n}=1/(\sqrt{n+1}+\sqrt{n}) \leq 2/\sqrt{n}$, series converges by comparison with p-series for $p=1.5>1$.

13) $\sum_{n=1}^{\infty} \frac{\sqrt[4]{n}}{\sqrt[3]{n^4+n^2}}$

Use the Limit Comparison Test to determine whether each series in exercises 14 - 28 converges or diverges.

14) $\sum_{n=1}^{\infty} \left(\frac{\ln n}{n}\right)^2$

Answer

Converges by limit comparison with p-series for $p>1$.

15) $\sum_{n=1}^{\infty} \left(\frac{\ln n}{n^{0.6}}\right)^2$

16) $\sum_{n=1}^{\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{n}$

Answer

Converges by limit comparison with p-series, $p=2>1$.

17) $\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n^2}\right)$

18) $\sum_{n=1}^{\infty} \frac{1}{4^n-3^n}$

Answer

Converges by limit comparison with 4^{-n}.

19) $\sum_{n=1}^{\infty} \frac{1}{n^2-n\sin n}$

20) $\sum_{n=1}^{\infty} \frac{1}{e^{(1.1)n}-3^n}$

Answer

Converges by limit comparison with $1/e^{1.1n}$.

21) $\sum_{n=1}^{\infty} \frac{1}{e^{(1.01)n}-3^n}$

22) $\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$

Answer

Diverges by limit comparison with harmonic series.

23) $\sum_{n=1}^{\infty} \frac{1}{2^{1+1/n}} n^{1+1/n}$

24) $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \sin\left(\frac{1}{n}\right)\right)$

Answer

Converges by limit comparison with p-series, $p = 3 > 1$.

25) $\sum_{n=1}^{\infty} \left(1 - \cos\left(\frac{1}{n}\right)\right)$

26) $\sum_{n=1}^{\infty} \frac{1}{n}\left(\tan^{-1} n - \frac{\pi}{2}\right)$

Answer

Converges by limit comparison with p-series, $p = 3 > 1$.

27) $\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)^{n.n}$ (Hint: $\left(1 - \frac{1}{n}\right)^{n} \to 1/e$.)

28) $\sum_{n=1}^{\infty} \left(1 - e^{-1/n}\right)$ (Hint: $1/e \approx (1 - 1/n)^n$, so $1 - e^{-1/n} \approx 1/n$.)

Answer

Diverges by limit comparison with $1/n$.

29) Does $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^p}$ converge if p is large enough? If so, for which p?

30) Does $\sum_{n=1}^{\infty} \left(\frac{\ln n}{n}\right)^p$ converge if p is large enough? If so, for which p?

Answer

Converges for $p > 1$ by comparison with a p series for slightly smaller p.

31) For which p does the series $\sum_{n=1}^{\infty} 2^{pn}/3^n$ converge?

32) For which $p > 0$ does the series $\sum_{n=1}^{\infty} \frac{n^p}{2^n}$ converge?

Answer

Converges for all $p > 0$.

33) For which $r > 0$ does the series $\sum_{n=1}^{\infty} \frac{r^{n^2}}{2^n}$ converge?

34) For which $r > 0$ does the series $\sum_{n=1}^{\infty} \frac{2^n}{r^{n^2}}$ converge?

Answer

Converges for all $r > 1$. If $r > 1$ then $r^n > 4$, say, once $n > \ln(2)/\ln(r)$ and then the series converges by limit comparison with a geometric series with ratio $1/2$

35) Find all values of p and q such that $\sum_{n=1}^{\infty} \frac{n^p}{(n!)^q}$ converges.

36) Does $\sum_{n=1}^{\infty} \frac{\sin^2(nr/2)}{n}$ converge or diverge? Explain.

Answer

The numerator is equal to 1 when n is odd and 0 when n is even, so the series can be rewritten $\sum_{n=1}^{\infty} \frac{1}{2n+1}$, which diverges by limit comparison with the harmonic series.

37) Explain why, for each n, at least one of $|\sin n|, |\sin(n+1)|, \ldots, |\sin(n+6)|$ is larger than $1/2$. Use this relation to test convergence of $\sum_{n=1}^{\infty} \frac{|\sin n|}{\sqrt{n}}$.

38) Suppose that $a_n \geq 0$ and $b_n \geq 0$ and that $\sum_{n=1}^{\infty} a_n^2$ and $\sum_{n=1}^{\infty} b_n^2$ converge. Prove that $\sum_{n=1}^{\infty} a_n b_n$ converges and $\sum_{n=1}^{\infty} a_n b_n \leq \frac{1}{2}\left(\sum_{n=1}^{\infty} a_n^2 + \sum_{n=1}^{\infty} b_n^2\right)$.

Answer

$(a-b)^2 = a^2 - 2ab + b^2$ or $a^2 + b^2 \geq 2ab$, so convergence follows from comparison of $2a_nb_n$ with $a_n^2 + b_n^2$. Since the partial sums on the left are bounded by those on the right, the inequality holds for the infinite series.

39) Does $\sum_{n=1}^{\infty} 2^{-\ln \ln n}$ converge? (Hint: Write $2^{\ln \ln n}$ as a power of $\ln n$.)

40) Does $\sum_{n=1}^{\infty} (\ln n)^{-\ln n}$ converge? (Hint: Use $t = e^{\ln(t)}$ to compare to a $p-$series.)

Answer

$(\ln n)^{-\ln n} = e^{-\ln(n)\ln\ln(n)}$. If n is sufficiently large, then $\ln \ln n > 2$, so $(\ln n)^{-\ln n} < 1/n^2$, and the series converges by comparison to a $p-$series.

41) Does $\sum_{n=2}^{\infty} (\ln n)^{-\ln \ln n}$ converge? (Hint: Compare a_n to $1/n$.)

42) Show that if $a_n \geq 0$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n^2$ converges. If $\sum_{n=1}^{\infty} a_n^2$ converges, does $\sum_{n=1}^{\infty} a_n$ necessarily converge?

Answer

$a_n \to 0$, so $a_n^2 \leq |a_n|$ for large n. Convergence follows from limit comparison. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, but $\sum_{n=1}^{\infty} \frac{1}{n}$ does not, so the fact that $\sum_{n=1}^{\infty} a_n^2$ converges does not imply that $\sum_{n=1}^{\infty} a_n$ converges.

43) Suppose that $a_n > 0$ for all n and that $\sum_{n=1}^{\infty} a_n$ converges. Suppose that b_n is an arbitrary sequence of zeros and ones. Does $\sum_{n=1}^{\infty} a_n b_n$ necessarily converge?

44) Suppose that $a_n > 0$ for all n and that $\sum_{n=1}^{\infty} a_n$ diverges. Suppose that b_n is an arbitrary sequence of zeros and ones with infinitely many terms equal to one. Does $\sum_{n=1}^{\infty} a_n b_n$ necessarily diverge?

Answer

No. $\sum_{n=1}^{\infty} 1/n$ diverges. Let $b_k = 0$ unless $k = n^2$ for some n. Then $\sum_k b_k/k = \sum 1/k^2$ converges.

45) Complete the details of the following argument: If $\sum_{n=1}^{\infty} \frac{1}{n}$ converges to a finite sum s, then $\frac{1}{2}s = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots$ and $s - \frac{1}{2}s = 1 + \frac{1}{3} + \frac{1}{5} + \cdots$. Why does this lead to a contradiction?

46) Show that if $a_n \geq 0$ and $\sum_{n=1}^{\infty} a_n^2$ converges, then $\sum_{n=1}^{\infty} \sin^2(a_n)$ converges.

Answer

$|\sin t| \leq |t|$, so the result follows from the comparison test.

47) Suppose that $a_n/b_n \to 0$ in the comparison test, where $a_n \geq 0$ and $b_n \geq 0$. Prove that if $\sum b_n$ converges, then $\sum a_n$ converges.

48) Let b_n be an infinite sequence of zeros and ones. What is the largest possible value of $x = \sum_{n=1}^{\infty} b_n/2^n$?

Answer

By the comparison test, $x = \sum_{n=1}^{\infty} b_n/2^n \leq \sum_{n=1}^{\infty} 1/2^n = 1$.

49) Let d_n be an infinite sequence of digits, meaning d_n takes values in $\{0, 1, \ldots, 9\}$. What is the largest possible value of $x = \sum_{n=1}^{\infty} d_n/10^n$ that converges?

50) Explain why, if $x > 1/2$, then x cannot be written $x = \sum_{n=2}^{\infty} \frac{b_n}{2^n}$ $(b_n = 0 \text{ or } 1,\ b_1 = 0)$.

Answer

If $b_1 = 0$, then, by comparison, $x \leq \sum_{n=2}^{\infty} 1/2^n = 1/2$.

51) [T] Evelyn has a perfect balancing scale, an unlimited number of 1-kg weights, and one each of $1/2$-kg, $1/4$-kg, $1/8$-kg, and so on weights. She wishes to weigh a meteorite of unspecified origin to arbitrary precision. Assuming the scale is big enough, can she do it? What does this have to do with infinite series?

52) [T] Robert wants to know his body mass to arbitrary precision. He has a big balancing scale that works perfectly, an unlimited collection of 1-kg weights, and nine each of 0.1-kg, 0.01-kg, 0.001-kg, and so on weights. Assuming the scale is big enough, can he do this? What does this have to do with infinite series?

Answer

Yes. Keep adding 1-kg weights until the balance tips to the side with the weights. If it balances perfectly, with Robert standing on the other side, stop. Otherwise, remove one of the 1-kg weights, and add 0.1-kg weights one at a time. If it balances after adding some of these, stop. Otherwise if it tips to the weights, remove the last 0.1-kg weight. Start adding 0.01-kg weights. If it balances, stop. If it tips to the side with the weights, remove the last 0.01-kg weight that was added.

Continue in this way for the 0.001-kg weights, and so on. After a finite number of steps, one has a finite series of the form $A+\sum_{n=1}^{N} s_n/10^n$ where A is the number of full kg weights and d_n is the number of $1/10^n$-kg weights that were added. If at some state this series is Robert's exact weight, the process will stop. Otherwise it represents the Nth partial sum of an infinite series that gives Robert's exact weight, and the error of this sum is at most $1/10^N$.

53) The series $\sum_{n=1}^{\infty} \frac{1}{2n}$ is half the harmonic series and hence diverges. It is obtained from the harmonic series by deleting all terms in which n is odd. Let $m>1$ be fixed. Show, more generally, that deleting all terms $1/n$ where $n=mk$ for some integer k also results in a divergent series.

54) In view of the previous exercise, it may be surprising that a subseries of the harmonic series in which about one in every five terms is deleted might converge. A depleted harmonic series is a series obtained from $\sum_{n=1}^{\infty} \frac{1}{n}$ by removing any term $1/n$ if a given digit, say 9, appears in the decimal expansion of n. Argue that this depleted harmonic series converges by answering the following questions.

a. How many whole numbers n have d digits?

b. How many d-digit whole numbers $h(d)$. do not contain 9 as one or more of their digits?

c. What is the smallest d-digit number $m(d)$?

d. Explain why the deleted harmonic series is bounded by $\sum_{d=1}^{\infty} \frac{h(d)}{m(d)}$.

e. Show that $\sum_{d=1}^{\infty} \frac{h(d)}{m(d)}$ converges.

Answer

a. $10^d-10^{d-1}<10^d$

b. $h(d)<9^d$

c. $m(d)=10^{d-1}+1$

d. Group the terms in the deleted harmonic series together by number of digits. $h(d)$ bounds the number of terms, and each term is at most $\frac{1}{m(d)}$.

Then $\sum_{d=1}^{\infty} h(d)/m(d) \le \sum_{d=1}^{\infty} 9^d/(10)^{d-1} \le 90$. One can actually use comparison to estimate the value to smaller than 80. The actual value is smaller than 23.

55) Suppose that a sequence of numbers $a_n>0$ has the property that $a_1=1$ and $a_{n+1}=\frac{1}{n+1}S_n$, where $S_n=a_1+\cdots+a_n$. Can you determine whether $\sum_{n=1}^{\infty} a_n$ converges? (Hint: S_n is monotone.)

56) Suppose that a sequence of numbers $a_n>0$ has the property that $a_1=1$ and $a_{n+1}=\frac{1}{(n+1)^2}S_n$, where $S_n=a_1+\cdots+a_n$. Can you determine whether $\sum_{n=1}^{\infty} a_n$ converges? (Hint: $S_2=a_2+a_1=a_2+S_1=a_2+1=1+1/4=(1+1/4)S_1, S_3=\frac{1}{3^2}S_2+S_2=(1+1/9)S_2$ $=(1+1/9)(1+1/4)S_1$, etc. Look at $\ln(S_n)$, and use $\ln(1+t)\le t, t>0$.)

Answer

Continuing the hint gives $S_N = (1+1/N^2)(1+1/(N-1)^2 \ldots (1+1/4))$. Then $\ln(S_N) = \ln(1+1/N^2) + \ln(1+1/(N-1)^2) + \cdots + \ln(1+1/4)$. Since $\ln(1+t)$ is bounded by a constant times t, when $0 < t < 1$ one has $\ln(S_N) \leq C \sum_{n=1}^{N} \frac{1}{n^2}$, which converges by comparison to the p-series for $p = 2$.

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1.6: Alternating Series

Learning Objectives

- Use the alternating series test to test an alternating series for convergence.
- Estimate the sum of an alternating series.
- Explain the meaning of absolute convergence and conditional convergence.

So far in this chapter, we have primarily discussed series with positive terms. In this section we introduce alternating series—those series whose terms alternate in sign. We will show in a later chapter that these series often arise when studying power series. After defining alternating series, we introduce the alternating series test to determine whether such a series converges.

The Alternating Series Test

A series whose terms alternate between positive and negative values is an **alternating series.** For example, the series

$$\sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^n = -\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\ldots \tag{1.6.1}$$

and

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots \tag{1.6.2}$$

are both alternating series.

Definition: Alternating Series

Any series whose terms alternate between positive and negative values is called an *alternating series*. An alternating series can be written in the form

$$\sum_{n=1}^{\infty}(-1)^{n+1}b_n = b_1-b_2+b_3-b_4+\ldots \tag{1.6.3}$$

or

$$\sum_{n-1}^{\infty}(-1)^{n}b_n = -b_1+b_2-b_3+b_4-\ldots \tag{1.6.4}$$

Where $b_n \geq 0$ for all positive integers n.

Series (1), shown in Equation 1.6.1, is a *geometric series*. Since $|r| = |-1/2| < 1,$ the series converges. Series (2), shown in Equation 1.6.2, is called the *alternating harmonic series*. We will show that whereas the harmonic series diverges, the alternating harmonic series converges. To prove this, we look at the sequence of partial sums $\{S_k\}$ (Figure 1).

Proof

Consider the odd terms S_{2k+1} for $k \geq 0$. Since $1/(2k+1) < 1/2k$,

$$S_{2k+1} = S_{2k-1}-\frac{1}{2k}+\frac{1}{2k+1} < S_{2k-1}. \tag{1.6.5}$$

Therefore, $\{S_{2k+1}\}$ is a decreasing sequence. Also,

$$S_{2k+1} = \left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\ldots+\left(\frac{1}{2k-1}-\frac{1}{2k}\right)+\frac{1}{2k+1} > 0. \tag{1.6.6}$$

Therefore, $\{S_{2k+1}\}$ is bounded below. Since $\{S_{2k+1}\}$ is a decreasing sequence that is bounded below, by the Monotone Convergence Theorem, $\{S_{2k+1}\}$ converges. Similarly, the even terms $\{S_{2k}\}$ form an increasing sequence that is bounded above because

$$S_{2k} = S_{2k-2}+\frac{1}{2k-1}-\frac{1}{2k} > S_{2k-2} \tag{1.6.7}$$

and

$$S_{2k} = 1 + \left(-\frac{1}{2} + \frac{1}{3}\right) + \ldots + \left(-\frac{1}{2k-2} + \frac{1}{2k-1}\right) - \frac{1}{2k} < 1. \tag{1.6.8}$$

Therefore, by the Monotone Convergence Theorem, the sequence $\{S_{2k}\}$ also converges. Since

$$S_{2k+1} = S_{2k} + \frac{1}{2k+1}, \tag{1.6.9}$$

we know that

$$\lim_{k\to\infty} S_{2k+1} = \lim_{k\to\infty} S_{2k} + \lim_{k\to\infty} \frac{1}{2k+1}. \tag{1.6.10}$$

Letting $S = \lim_{k\to\infty} S_{2k+1}$ and using the fact that $1/(2k+1) \to 0$, we conclude that $\lim_{k\to\infty} S_{2k} = S$. Since the odd terms and the even terms in the sequence of partial sums converge to the same limit S, it can be shown that the sequence of partial sums converges to S, and therefore the alternating harmonic series converges to S.

It can also be shown that $S = \ln 2$, and we can write

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + a \ldots = \ln(2). \tag{1.6.11}$$

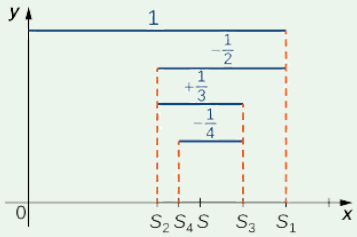


Figure 1.6.1: For the alternating harmonic series, the odd terms S_{2k+1} in the sequence of partial sums are decreasing and bounded below. The even terms S_{2k} are increasing and bounded above.

□

More generally, any alternating series of form (3) (Equation 1.6.3) or (4) (Equation 1.6.4) converges as long as $b_1 \geq b_2 \geq b_3 \geq \cdots$ and $b_n \to 0$ (Figure 2). The proof is similar to the proof for the alternating harmonic series.

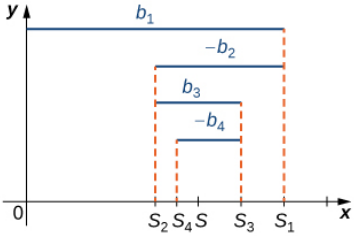


Figure 1.6.2: For an alternating series $b_1 - b_2 + b_3 - \cdots$ in which $b_1 > b_2 > b_3 > \cdots$, the odd terms S_{2k+1} in the sequence of partial sums are decreasing and bounded below. The even terms S_{2k} are increasing and bounded above.

Alternating Series Test

An alternating series of the form

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n \tag{1.6.12}$$

or

$$\sum_{n=1}^{\infty} (-1)^{n} b_n \tag{1.6.13}$$

converges if

i. $0 \leq b_{n+1} \leq b_n$ for all $n \geq 1$ and
ii. $\lim_{n\to\infty} b_n = 0$.

This is known as the *alternating series test*.

We remark that this theorem is true more generally as long as there exists some integer N such that $0 \le b_{n+1} \le b_n$ for all $n \ge N$.

Example 1.6.1: Convergence of Alternating Series

For each of the following alternating series, determine whether the series converges or diverges.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$

b. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n+1}$

Solution

a. Since $\dfrac{1}{(n+1)^2} < \dfrac{1}{n^2}$ and $\dfrac{1}{n^2} \to 0,$ the series converges.

b. Since $n/(n+1) \nrightarrow 0$ as $n \to \infty$, we cannot apply the alternating series test. Instead, we use the nth term test for divergence. Since $\displaystyle\lim_{n\to\infty} \frac{n}{n+1} = 1 \ne 0,$ the series diverges.

Exercise 1.6.1

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{2^n}$ converges or diverges.

Hint

Is $\left\{\frac{n}{2^n}\right\}$ decreasing? What is $\displaystyle\lim_{n\to\infty} \frac{n}{2^n}$?

Answer

The series converges.

Remainder of an Alternating Series

It is difficult to explicitly calculate the sum of most alternating series, so typically the sum is approximated by using a partial sum. When doing so, we are interested in the amount of error in our approximation. Consider an alternating series

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n \tag{1.6.14}$$

satisfying the hypotheses of the alternating series test. Let S denote the sum of this series and S_k be the corresponding sequence of partial sums. From Figure 1.6.2, we see that for any integer $N \ge 1$, the remainder R_N satisfies

$$|R_N| = |S - S_N| \le |S_{N+1} - S_N| = b_{n+1}. \tag{1.6.15}$$

Remainders in Alternating Series

Consider an alternating series of the form

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n \tag{1.6.16}$$

or

$$\sum_{n=1}^{\infty} (-1)^{n} b_n \tag{1.6.17}$$

that satisfies the hypotheses of the alternating series test. Let S denote the sum of the series and S_N denote the Nth partial sum. For any integer $N \ge 1$, the remainder $R_N = S - S_N$ satisfies

$$|R_N| \le b_{N+1}. \tag{1.6.18}$$

In other words, if the conditions of the alternating series test apply, then the error in approximating the infinite series by the Nth partial sum S_N is in magnitude at most the size of the next term b_{N+1}.

Example 1.6.2: Estimating the Remainder of an Alternating Series

Consider the alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}. \tag{1.6.19}$$

Use the remainder estimate to determine a bound on the error R_{10} if we approximate the sum of the series by the partial sum S_{10}.

Solution

From the theorem stated above,

$$|R_{10}| \leq b_{11} = \frac{1}{11^2} \approx 0.008265. \tag{1.6.20}$$



Exercise 1.6.2

Find a bound for R_{20} when approximating $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ by S_{20}.

Hint

$|R_{20}| \leq b_{21}$

Answer

0.04762

Absolute and Conditional Convergence

Consider a series $\sum_{n=1}^{\infty} a_n$ and the related series $\sum_{n=1}^{\infty} |a_n|$. Here we discuss possibilities for the relationship between the convergence of these two series. For example, consider the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$. The series whose terms are the absolute value of

these terms is the harmonic series, since $\sum_{n=1}^{\infty}\left|\frac{(-1)^{n+1}}{n}\right|=\sum_{n=1}^{\infty}\frac{1}{n}$. Since the alternating harmonic series converges, but the harmonic series diverges, we say the alternating harmonic series exhibits conditional convergence.

By comparison, consider the series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$. The series whose terms are the absolute values of the terms of this series is the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$. Since both of these series converge, we say the series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$ exhibits absolute convergence.

Definition: Absolute Convergence

A series $\sum_{n=1}^{\infty}a_n$ exhibits *absolute convergence* if $\sum_{n=1}^{\infty}|a_n|$ converges. A series $\sum_{n=1}^{\infty}a_n$ exhibits *conditional convergence* if $\sum_{n=1}^{\infty}a_n$ converges but $\sum_{n=1}^{\infty}|a_n|$ diverges.

As shown by the alternating harmonic series, a series $\sum_{n=1}^{\infty}a_n$ may converge, but $\sum_{n=1}^{\infty}|a_n|$ may diverge. In the following theorem, however, we show that if $\sum_{n=1}^{\infty}|a_n|$ converges, then $\sum_{n=1}^{\infty}a_n$ converges.

Absolute Convergence Implies Convergence

If $\sum_{n=1}^{\infty}|a_n|$ converges, then $\sum_{n=1}^{\infty}a_n$ converges.

Proof

Suppose that $\sum_{n=1}^{\infty}|a_n|$ converges. We show this by using the fact that $a_n=|a_n$ or $a_n=-|a_n|$ and therefore $|a_n|+a_n=2|a_n|$ or $|a_n|+a_n=0$. Therefore, $0\le|a_n|+a_n\le 2|a_n|$. Consequently, by the comparison test, since $2\sum_{n=1}^{\infty}|a_n|$ converges, the series

$$\sum_{n=1}^{\infty}(|a_n|+a_n) \tag{1.6.21}$$

converges. By using the algebraic properties for convergent series, we conclude that

$$\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}(|a_n|+a_n)-\sum_{n=1}^{\infty}|a_n| \tag{1.6.22}$$

converges.

□

Example 1.6.3: Absolute versus Conditional Convergence

For each of the following series, determine whether the series converges absolutely, converges conditionally, or diverges.

a. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{3n+1}$

b. $\sum_{n=1}^{\infty}\frac{\cos(n)}{n^2}$

Solution

a. We can see that

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^{n+1}}{3n+1}\right|=\sum_{n=1}^{\infty}\frac{1}{3n+1}$$

diverges by using the limit comparison test with the harmonic series. In fact,

$$\lim_{n\to\infty}\frac{1/(3n+1)}{1/n}=\frac{1}{3}.$$

Therefore, the series does not converge absolutely. However, since

$$\frac{1}{3(n+1)+1}<\frac{1}{3n+1}\text{ and }\frac{1}{3n+1}\to 0,$$

the series converges. We can conclude that $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{3n+1}$ converges conditionally.

b. Noting that $|\cos n|\le 1$, to determine whether the series converges absolutely, compare

$$\sum_{n=1}^{\infty}\left|\frac{\cos n}{n^2}\right|$$

with the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$. Since $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, by the comparison test, $\sum_{n=1}^{\infty}\left|\frac{\cos n}{n^2}\right|$ converges, and therefore $\sum_{n=1}^{\infty}\frac{\cos n}{n^2}$ converges absolutely.







Exercise 1.6.3

Determine whether the series $\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{n}{2n^3+1}$ converges absolutely, converges conditionally, or diverges.

Hint

Check for absolute convergence first.

Answer

The series converges absolutely.

To see the difference between absolute and conditional convergence, look at what happens when we rearrange the terms of the alternating harmonic series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$. We show that we can rearrange the terms so that the new series diverges. Certainly if we rearrange the terms of a finite sum, the sum does not change. When we work with an infinite sum, however, interesting things can happen.

Begin by adding enough of the positive terms to produce a sum that is larger than some real number $M = 10$ For example, let $M = 10$, and find an integer k such that

$$1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2k-1} > 10 \tag{1.6.23}$$

(We can do this because the series $\sum_{n=1}^{\infty}\frac{1}{2n-1}$ diverges to infinity.) Then subtract $1/2$. Then add more positive terms until the sum reaches 100. That is, find another integer $j > k$ such that

$$(1+\frac{1}{3}+\cdots+\frac{1}{2k-1}-\frac{1}{2}+\frac{1}{2k+1}+\ldots+\frac{1}{2j+1} > 100. \tag{1.6.24}$$

Then subtract $1/4$. Continuing in this way, we have found a way of rearranging the terms in the alternating harmonic series so that the sequence of partial sums for the rearranged series is unbounded and therefore diverges.

The terms in the alternating harmonic series can also be rearranged so that the new series converges to a different value. In Example, we show how to rearrange the terms to create a new series that converges to $3\ln(2)/2$. We point out that the alternating harmonic series can be rearranged to create a series that converges to any real number r; however, the proof of that fact is beyond the scope of this text.

In general, any series $\sum_{n=1}^{\infty}a_n$ that converges conditionally can be rearranged so that the new series diverges or converges to a different real number. A series that converges absolutely does not have this property. For any series $\sum_{n=1}^{\infty}a_n$ that converges absolutely, the value of $\sum_{n=1}^{\infty}a_n$ is the same for any rearrangement of the terms. This result is known as the **Riemann Rearrangement Theorem**, which is beyond the scope of this book.

Example 1.6.4: Rearranging Series

Use the fact that

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots=\ln 2 \tag{1.6.25}$$

to rearrange the terms in the alternating harmonic series so the sum of the rearranged series is $3\ln(2)/2$.

Solution

Let

$$\sum_{n=1}^{\infty}a_n = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\cdots. \tag{1.6.26}$$

Since $\sum_{n=1}^{\infty}a_n = \ln(2)$, by the algebraic properties of convergent series,

$$\sum_{n=1}^{\infty}\frac{1}{2}a_n = \frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\cdots=\frac{1}{2}\sum_{n=1}^{\infty}a_n=\frac{\ln 2}{2}. \tag{1.6.27}$$

Now introduce the series $\sum_{n=1}^{\infty}b_n$ such that for all $n \geq 1, b_{2n-1} = 0$ and $b_{2n} = a_n/2$. Then

$$\sum_{n=1}^{\infty}b_n = 0+\frac{1}{2}+0-\frac{1}{4}+0+\frac{1}{6}+0-\frac{1}{8}+\cdots=\frac{\ln 2}{2}. \tag{1.6.28}$$

Then using the algebraic limit properties of convergent series, since $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge, the series $\sum_{n=1}^{\infty}(a_n+b_n)$ converges and

$$\sum_{n=1}^{\infty}(a_n+b_n)=\sum_{n=1}^{\infty}a_n+\sum_{n=1}^{\infty}b_n=\ln 2+\frac{\ln 2}{2}=\frac{3\ln 2}{2}. \tag{1.6.29}$$

Now adding the corresponding terms, a_n and b_n, we see that

$$\begin{aligned}\sum_{n=1}^{\infty}(a_n+b_n)&=(1+0)+\left(-\frac{1}{2}+\frac{1}{2}\right)+\left(\frac{1}{3}+0\right)+\left(-\frac{1}{4}-14\right)+\left(\frac{1}{5}+0\right)+\left(-\frac{1}{6}+\frac{1}{6}\right)+\left(\frac{1}{7}+0\right)\\&\quad+\left(\frac{1}{8}-\frac{1}{8}\right)+\cdots=1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\cdots.\end{aligned} \tag{1.6.30}$$

We notice that the series on the right side of the equal sign is a rearrangement of the alternating harmonic series. Since $\sum_{n=1}^{\infty}(a_n+b_n)=3\ln(2)/2,$ we conclude that

$$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\cdots=\frac{3\ln(2)}{2}. \tag{1.6.31}$$

Therefore, we have found a rearrangement of the alternating harmonic series having the desired property.

Key Concepts

- For an alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}b_n$, if $b_{k+1}\le b_k$ for all k and $b_k\to 0$ as $k\to\infty,$ the alternating series converges.
- If $\sum_{n=1}^{\infty}|a_n|$ converges, then $\sum_{n=1}^{\infty}a_n$ converges.

Key Equations

- **Alternating series**

$$\sum_{n=1}^{\infty}(-1)^{n+1}b_n=b_1-b_2+b_3-b_4+\cdots \quad \text{or}$$

$$\sum_{n=1}^{\infty}(-1)^{n}b_n=-b_1+b_2-b_3+b_4-\cdots$$

Glossary

absolute convergence

if the series $\sum_{n=1}^{\infty}|a_n|$ converges, the series $\sum_{n=1}^{\infty}a_n$ is said to converge absolutely

alternating series

a series of the form $\sum_{n=1}^{\infty}(-1)^{n+1}b_n$ or $\sum_{n=1}^{\infty}(-1)^{n}b_n$, where $b_n\ge 0$, is called an alternating series

alternating series test

for an alternating series of either form, if $b_{n+1}\le b_n$ for all integers $n\ge 1$ and $b_n\to 0$, then an alternating series converges

conditional convergence

if the series $\sum_{n=1}^{\infty}a_n$ converges, but the series $\sum_{n=1}^{\infty}|a_n|$ diverges, the series $\sum_{n=1}^{\infty}a_n$ is said to converge conditionally

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1.6E: Exercises for Section 1.6

In exercises 1 - 30, state whether each of the following series converges absolutely, conditionally, or not at all.

1) $\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{n}{n+3}$

2) $\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{\sqrt{n}+1}{\sqrt{n}+3}$

Answer

This series diverges by the divergence test. Terms do not tend to zero.

3) $\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{1}{\sqrt{n+3}}$

4) $\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{\sqrt{n+3}}{n}$

Answer

Converges conditionally by alternating series test, since $\sqrt{n+3}\,/n$ is decreasing and its limit is 0. Does not converge absolutely by comparison with p-series, $p=1/2$.

5) $\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{1}{n!}$

6) $\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{3^n}{n!}$

Answer

Converges absolutely by limit comparison to $3^n/4^n$, for example.

7) $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\dfrac{n-1}{n}\right)^n$

8) $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\dfrac{n+1}{n}\right)^n$

Answer

Diverges by divergence test since $\lim\limits_{n\to\infty}|a_n|=e$ and not 0.

9) $\sum_{n=1}^{\infty}(-1)^{n+1}\sin^2 n$

10) $\sum_{n=1}^{\infty}(-1)^{n+1}\cos^2 n$

Answer

Diverges by the divergence test, since its terms do not tend to zero. The limit of the sequence of its terms does not exist.

11) $\sum_{n=1}^{\infty}(-1)^{n+1}\sin^2(1/n)$

12) $\sum_{n=1}^{\infty}(-1)^{n+1}\cos^2(1/n)$

Answer

$\lim_{n\to\infty} \cos^2(1/n) = 1$. Diverges by divergence test.

13) $\sum_{n=1}^{\infty}(-1)^{n+1}\ln(1/n)$

14) $\sum_{n=1}^{\infty}(-1)^{n+1}\ln(1+\frac{1}{n})$

Answer

Converges by alternating series test.

15) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{1+n^4}$

16) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^e}{1+n^\pi}$

Answer

Converges conditionally by alternating series test. Does not converge absolutely by limit comparison with p-series, $p = \pi - e$

Solution:

17) $\sum_{n=1}^{\infty}(-1)^{n+1}2^{1/n}$

18) $\sum_{n=1}^{\infty}(-1)^{n+1}n^{1/n}$

Answer

Diverges; terms do not tend to zero.

19) $\sum_{n=1}^{\infty}(-1)^{n}(1-n^{1/n})$ (Hint: $n^{1/n} \approx 1+\ln(n)/n$ for large n.)

20) $\sum_{n=1}^{\infty}(-1)^{n+1}n\left(1-\cos\left(\frac{1}{n}\right)\right)$ (Hint: $\cos(1/n) \approx 1-1/n^2$ for large n.)

Answer

Converges by alternating series test. Does not converge absolutely by limit comparison with harmonic series.

21) $\sum_{n=1}^{\infty}(-1)^{n+1}(\sqrt{n+1}-\sqrt{n})$ (Hint: Rationalize the numerator.)

22) $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right)$ (Hint: Cross-multiply then rationalize numerator.)

Answer

Converges absolutely by limit comparison with p-series, $p = 3/2$, after applying the hint.

23) $\sum_{n=1}^{\infty}(-1)^{n+1}(\ln(n+1)-\ln n)$

24) $\sum_{n=1}^{\infty}(-1)^{n+1}n(\tan^{-1}(n+1)-\tan^{-1}n)$ (Hint: Use Mean Value Theorem.)

Answer

Converges by alternating series test since $n(\tan^{-1}(n+1)-\tan^{-1} n)$ is decreasing to zero for large n.Does not converge absolutely by limit comparison with harmonic series after applying hint.

25) $\sum_{n=1}^{\infty}(-1)^{n+1}((n+1)^2-n^2)$

26) $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{1}{n}-\frac{1}{n+1}\right)$

Answer

Converges absolutely, since $a_n=\frac{1}{n}-\frac{1}{n+1}$ are terms of a telescoping series.

27) $\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{n}$

28) $\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{n^{1/n}}$

Answer

Terms do not tend to zero. Series diverges by divergence test.

29) $\sum_{n=1}^{\infty}\frac{1}{n}\sin(\frac{n\pi}{2})$

30) $\sum_{n=1}^{\infty}\sin(n\pi/2)\sin(1/n)$

Answer

Converges by alternating series test. Does not converge absolutely by limit comparison with harmonic series.

In exercises 31 - 36, use the estimate $|R_N|\le b_{N+1}$ to find a value of N that guarantees that the sum of the first N terms of the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}b_n$ differs from the infinite sum by at most the given error. Calculate the partial sum S_N for this N.

31) [T] $b_n=1/n$, error $<10^{-5}$

32) [T] $b_n=1/\ln(n), n\ge 2,$ error $<10^{-1}$

Answer

$\ln(N+1)>10, N+1>e^{10}, N\ge 22026; S_{22026}=0.0257\ldots$

33) [T] $b_n=1/\sqrt{n}$, error $<10^{-3}$

34) [T] $b_n=1/2^n$, error $<10^{-6}$

Answer

$2^{N+1}>10^6$ or $N+1>6\ln(10)/\ln(2)=19.93.$ or $N\ge 19; S_{19}=0.333333969\ldots$

35) [T] $b_n=ln(1+\frac{1}{n}),$ error $<10^{-3}$

36) [T] $b_n=1/n^2$, error $<10^{-6}$

Answer

$(N+1)^2>10^6$ or $N>999; S_{1000}\approx 0.822466.$

For exercises 37 - 45, indicate whether each of the following statements is true or false. If the statement is false, provide an example in which it is false.

37) If $b_n \geq 0$ is decreasing and $\lim_{n\to\infty} b_n = 0$, then $\sum_{n=1}^{\infty}(b_{2n-1} - b_{2n})$ converges absolutely.

38) If $b_n \geq 0$ is decreasing, then $\sum_{n=1}^{\infty}(b_{2n-1} - b_{2n})$ converges absolutely.

Answer

True. b_n need not tend to zero since if $c_n = b_n - \lim b_n$, then $c_{2n-1} - c_{2n} = b_{2n-1} - b_{2n}$.

39) If $b_n \geq 0$ and $\lim_{n\to\infty} b_n = 0$ then $\sum_{n=1}^{\infty}(\frac{1}{2}(b_{3n-2} + b_{3n-1}) - b_{3n})$ converges.

40) If $b_n \geq 0$ is decreasing and $\sum_{n=1}^{\infty}(b_{3n-2} + b_{3n-1} - b_{3n})$ converges then $\sum_{n=1}^{\infty} b_{3n-2}$ converges.

Answer

True. $b_{3n-1} - b_{3n} \geq 0,$ so convergence of $\sum b_{3n-2}$ follows from the comparison test.

41) If $b_n \geq 0$ is decreasing and $\sum_{n=1}^{\infty}(-1)^{n-1} b_n$ converges conditionally but not absolutely, then b_n does not tend to zero.

42) Let $a_n^+ = a_n$ if $a_n \geq 0$ and $a_n^- = -a_n$ if $a_n < 0$. (Also, $a_n^+ = 0$ if $a_n < 0$ and $a_n^- = 0$ if $a_n \geq 0$.) If $\sum_{n=1}^{\infty} a_n$ converges conditionally but not absolutely, then neither $\sum_{n=1}^{\infty} a_n^+$ nor $\sum_{n=1}^{\infty} a_n^-$ converge.

Answer

True. If one converges, then so must the other, implying absolute convergence.

43) Suppose that a_n is a sequence of positive real numbers and that $\sum_{n=1}^{\infty} a_n$ converges.

44) Suppose that b_n is an arbitrary sequence of ones and minus ones. Does $\sum_{n=1}^{\infty} a_n b_n$ necessarily converge?

45) Suppose that a_n is a sequence such that $\sum_{n=1}^{\infty} a_n b_n$ converges for every possible sequence b_n of zeros and ones. Does $\sum_{n=1}^{\infty} a_n$ converge absolutely?

Answer

Yes. Take $b_n = 1$ if $a_n \geq 0$ and $b_n = 0$ if $a_n < 0$. Then $\sum_{n=1}^{\infty} a_n b_n = \sum_{n:a_n \geq 0} a_n$ converges. Similarly, one can show $\sum_{n:a_n<0} a_n$ converges. Since both series converge, the series must converge absolutely.

In exercises 46 - 49, the series do not satisfy the hypotheses of the alternating series test as stated. In each case, state which hypothesis is not satisfied. State whether the series converges absolutely.

46) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin^2 n}{n}$

47) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos^2 n}{n}$

Answer

Not decreasing. Does not converge absolutely.

48) $1+\frac{1}{2}-\frac{1}{3}-\frac{1}{4}+\frac{1}{5}+\frac{1}{6}-\frac{1}{7}-\frac{1}{8}+\cdots$

49) $1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}+\frac{1}{8}-\frac{1}{9}+\cdots$

Answer

Not alternating. Can be expressed as $\sum_{n=1}^{\infty}\left(\frac{1}{3n-2}+\frac{1}{3n-1}-\frac{1}{3n}\right)$, which diverges by comparison with $\sum_{n=1}^{\infty}\frac{1}{3n-2}$.

50) Show that the alternating series $1-\frac{1}{2}+\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}+\frac{1}{4}-\frac{1}{8}+\cdots$ does not converge. What hypothesis of the alternating series test is not met?

51) Suppose that $\sum a_n$ converges absolutely. Show that the series consisting of the positive terms a_n also converges.

Answer

Let $a_n^+ = a_n$ if $a_n \geq 0$ and $a_n^+ = 0$ if $a_n < 0$. Then $a_n^+ \leq |a_n|$ for all n so the sequence of partial sums of a_n^+ is increasing and bounded above by the sequence of partial sums of $|a_n|$, which converges; hence, $\sum_{n=1}^{\infty} a_n^+$ converges.

52) Show that the alternating series $\frac{2}{3}-\frac{3}{5}+\frac{4}{7}-\frac{5}{9}+\cdots$ does not converge. What hypothesis of the alternating series test is not met?

53) The formula $\cos\theta = 1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\frac{\theta^6}{6!}+\cdots$ will be derived in the next chapter. Use the remainder $|R_N| \leq b_{N+1}$ to find a bound for the error in estimating $\cos\theta$ by the fifth partial sum $1-\theta^2/2!+\theta^4/4!-\theta^6/6!+\theta^8/8!$ for $\theta = 1, \theta = \pi/6,$ and $\theta = \pi$.

Answer

For $N = 5$ one has $\left|R_N\right| \; b_6 = \theta^{10}/10!$. When $\theta = 1, R_5 \leq 1/10! \approx 2.75 \times 10^{-7}$. When $\theta = \pi/6$, $R_5 \leq (\pi/6)^{10}/10! \approx 4.26 \times 10^{-10}$. When $\theta = \pi, R_5 \leq \pi^{10}/10! = 0.0258.$

54) The formula $\sin\theta = \theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\frac{\theta^7}{7!}+\cdots$ will be derived in the next chapter. Use the remainder $|R_N| \leq b_{N+1}$ to find a bound for the error in estimating $\sin\theta$ by the fifth partial sum $\theta-\theta^3/3!+\theta^5/5!-\theta^7/7!+\theta^9/9!$ for $\theta = 1, \theta = \pi/6,$ and $\theta = \pi$.

55) How many terms in $\cos\theta = 1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\frac{\theta^6}{6!}+\cdots$ are needed to approximate $\cos 1$ accurate to an error of at most 0.00001?

Answer

Let $b_n = 1/(2n-2)!$. Then $R_N \leq 1/(2N)! < 0.00001$ when $(2N)! > 10^5$ or $N = 5$ and $1-\frac{1}{2!}+\frac{1}{4!}-\frac{1}{6!}+\frac{1}{8!} = 0.540325\ldots$, whereas $\cos 1 = 0.5403023\ldots$

56) How many terms in $\sin\theta = \theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\frac{\theta^7}{7!}+\cdots$ are needed to approximate $\sin 1$ accurate to an error of at most 0.00001?

57) Sometimes the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ converges to a certain fraction of an absolutely convergent series $\sum_{n=1}^{\infty} b_n$ at a faster rate. Given that $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$, find $S = 1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots$. Which of the series $6\sum_{n=1}^{\infty}\frac{1}{n^2}$ and $S\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}$ gives a better estimation of π^2 using 1000 terms?

Answer

Let $T=\sum \frac{1}{n^2}$. Then $T-S=\frac{1}{2}T$, so $S=T/2$.

$\sqrt{6\times\sum_{n=1}^{1000}1/n^2}=3.140638\ldots;\sqrt{\frac{1}{2}\times\sum_{n=1}^{1000}(-1)^{n-1}/n^2}=3.141591\ldots;\pi=3.141592\ldots$The alternating series is more accurate for 1000terms.

The alternating series in exercises 58 & 59 converge to given multiples of π. Find the value of N predicted by the remainder estimate such that the Nth partial sum of the series accurately approximates the left-hand side to within the given error. Find the minimum N for which the error bound holds, and give the desired approximate value in each case. Up to 15 decimals places, $\pi=3.141592653589793\ldots$.

58) [T] $\frac{\pi}{4}=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$, error <0.0001

59) [T] $\frac{\pi}{\sqrt{12}}=\sum_{k=0}^{\infty}\frac{(-3)^{-k}}{2k+1}$, error <0.0001

Answer

$N=6, S_N=0.9068$

60) [T] The series $\sum_{n=0}^{\infty}\frac{\sin(x+\pi n)}{x+\pi n}$ plays an important role in signal processing. Show that $\sum_{n=0}^{\infty}\frac{\sin(x+\pi n)}{x+\pi n}$ converges whenever $0<x<\pi$. (Hint: Use the formula for the sine of a sum of angles.)

61) [T] If $\sum_{n=1}^{N}(-1)^{n-1}\frac{1}{n}\to ln2$, what is $1+\frac{1}{3}+\frac{1}{5}-\frac{1}{2}-\frac{1}{4}-\frac{1}{6}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\frac{1}{8}-\frac{1}{10}-\frac{1}{12}+\cdots$?

Answer

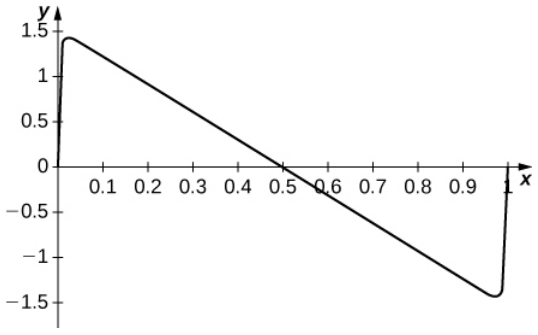
$\ln(2)$. The nth partial sum is the same as that for the alternating harmonic series.

62) [T] Plot the series $\sum_{n=1}^{100}\frac{\cos(2\pi nx)}{n}$ for $0\le x<1$. Explain why $\sum_{n=1}^{100}\frac{\cos(2\pi nx)}{n}$ diverges when $x=0,1$. How does the series behave for other x?

63) [T] Plot the series $\sum_{n=1}^{100}\frac{\sin(2\pi nx)}{n}$ for $0\le x<1$ and comment on its behavior

Answer

The series jumps rapidly near the endpoints. For x away from the endpoints, the graph looks like $\pi(1/2-x)$.



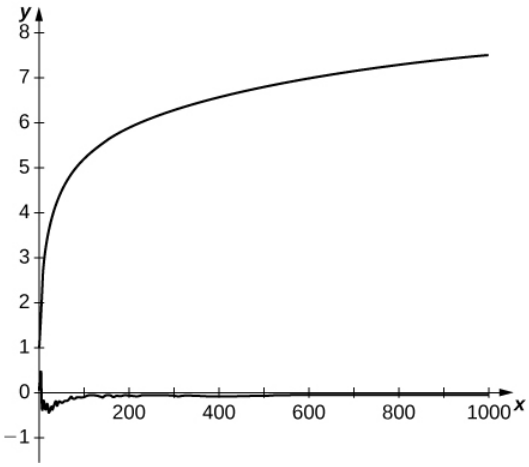
64) [T] Plot the series $\sum_{n=1}^{100}\frac{\cos(2\pi nx)}{n^2}$ for $0\le x<1$ and describe its graph.

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65) [T] The alternating harmonic series converges because of cancelation among its terms. Its sum is known because the cancelation can be described explicitly. A random harmonic series is one of the form $\sum_{n=1}^{\infty} \frac{S_n}{n}$, where s_n is a randomly generated sequence of $\pm 1's$ in which the values ± 1 are equally likely to occur. Use a random number generator to produce 1000 random $\pm 1's$ and plot the partial sums $S_N = \sum_{n=1}^{N} \frac{s_n}{n}$ of your random harmonic sequence for $N = 1$ to 1000. Compare to a plot of the first 1000 partial sums of the harmonic series.

Answer

Here is a typical result. The top curve consists of partial sums of the harmonic series. The bottom curve plots partial sums of a random harmonic series.



66) [T] Estimates of $\sum_{n=1}^{\infty} \frac{1}{n^2}$ can be accelerated by writing its partial sums as $\sum_{n=1}^{N} \frac{1}{n^2} = \sum_{n=1}^{N} \frac{1}{n(n+1)} + \sum_{n=1}^{N} \frac{1}{n^2(n+1)}$ and recalling that $\sum_{n=1}^{N} \frac{1}{n(n+1)} = 1 - \frac{1}{N+1}$ converges to one as $N \to \infty$. Compare the estimate of $\pi^2/6$ using the sums $\sum_{n=1}^{1000} \frac{1}{n^2}$ with the estimate using $1 + \sum_{n=1}^{1000} \frac{1}{n^2(n+1)}$.

67) [T] The **Euler transform** rewrites $S = \sum_{n=0}^{\infty} (-1)^n b_n$ as $S = \sum_{n=0}^{\infty} (-1)^n 2^{-n-1} \sum_{m=0}^{n} \binom{n}{m} b_{n-m}$. For the alternating harmonic series, it takes the form $\ln(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \sum_{n=1}^{\infty} \frac{1}{n2^n}$. Compute partial sums of $\sum_{n=1}^{\infty} \frac{1}{n2^n}$ until they approximate $\ln(2)$ accurate to within 0.0001. How many terms are needed? Compare this answer to the number of terms of the alternating harmonic series are needed to estimate $\ln(2)$.

Answer

By the alternating series test, $|S_n - S| \leq b_{n+1}$, so one needs 10^4 terms of the alternating harmonic series to estimate $\ln(2)$ to within 0.0001 The first 10 partial sums of the series $\sum_{n=1}^{\infty} \frac{1}{n2^n}$ are (up to four decimals)

$0.5000, 0.6250, 0.6667, 0.6823, 0.6885, 0.6911, 0.6923, 0.6928, 0.6930, 0.6931$, and the tenth partial sum is within 0.0001 of $\ln(2) = 0.6931\ldots$.

68) [T] In the text it was stated that a conditionally convergent series can be rearranged to converge to any number. Here is a slightly simpler, but similar, fact. If $a_n \geq 0$ is such that $a_n \to 0$ as $n \to \infty$ but $\sum_{n=1}^{\infty} a_n$ diverges, then, given any number A there is

a sequence s_n of $\pm 1's$ such that $\sum_{n=1}^{\infty} a_n s_n \to A$. Show this for $A > 0$ as follows.

a. Recursively define s_n by $s_n = 1$ if $S_{n-1} = \sum_{k=1}^{n-1} a_k s_k < A$ and $s_n = -1$ otherwise.

b. Explain why eventually $S_n \geq A$, and for any m larger than this n, $A - a_m \leq S_m \leq A + a_m$.

c. Explain why this implies that $S_n \to A$ as $n \to \infty$.

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1.7: Ratio and Root Tests

Learning Objectives

- Use the ratio test to determine absolute convergence of a series.
- Use the root test to determine absolute convergence of a series.
- Describe a strategy for testing the convergence of a given series.

In this section, we prove the last two series convergence tests: the ratio test and the root test. These tests are particularly nice because they do not require us to find a comparable series. The ratio test will be especially useful in the discussion of power series in the next chapter. Throughout this chapter, we have seen that no single convergence test works for all series. Therefore, at the end of this section we discuss a strategy for choosing which convergence test to use for a given series.

Ratio Test

Consider a series $\sum_{n=1}^{\infty} a_n$. From our earlier discussion and examples, we know that $\lim_{n\to\infty} a_n = 0$ is not a sufficient condition for the series to converge. Not only do we need $a_n \to 0$, but we need $a_n \to 0$ quickly enough. For example, consider the series $\sum_{n=1}^{\infty} \frac{1}{n}$ and the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. We know that $\frac{1}{n} \to 0$ and $\frac{1}{n^2} \to 0$. However, only the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. The series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges because the terms in the sequence $\left\{\frac{1}{n}\right\}$ do not approach zero fast enough as $n \to \infty$. Here we introduce the **ratio test,** which provides a way of measuring how fast the terms of a series approach zero.

Ratio Test

Let $\sum_{n=1}^{\infty} a_n$ be a series with nonzero terms. Let

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|. \tag{1.7.1}$$

i. If $0 \le \rho < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

ii. If $\rho > 1$ or $\rho = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.

iii. If $\rho = 1$, the test does not provide any information.

Proof

Let $\sum_{n=1}^{\infty} a_n$ be a series with nonzero terms.

We begin with the proof of part i. In this case, $\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$. Since $0 \le \rho < 1$, there exists R such that $0 \le \rho < R < 1$. Let $\varepsilon = R - \rho > 0$. By the definition of limit of a sequence, there exists some integer N such that

$$\left|\left|\frac{a_{n+1}}{a_n}\right| - \rho\right| < \varepsilon, \text{ for all } n \ge N. \tag{1.7.2}$$

Therefore,

$$\left|\frac{a_{n+1}}{a_n}\right| < \rho + \varepsilon = R, \text{ for all } n \ge N \tag{1.7.3}$$

and, thus,

$$|a_{N+1}| < R|a_N|$$
$$|a_{N+2}| < R\,|a_{N+1}| < R^2\,|a_N|$$
$$|a_{N+3}| < R\,|a_{N+2}| < R^2\,|a_{N+1}| < R^3\,|a_N|$$

$$|a_{N+4}|<R\,|a_{N+3}|<R^2\,|a_{N+2}|<R^3\,|a_{N+1}|<R^4\,|a_N|$$

$$\vdots$$

Since $R<1$, the geometric series

$$R\,|a_N|+R^2\,|a_N|+R^3\,|a_N|+\cdots \tag{1.7.4}$$

converges. Given the inequalities above, we can apply the comparison test and conclude that the series

$$|a_{N+1}|+|a_{N+2}|+|a_{N+3}|+|a_{N+4}|+\cdots \tag{1.7.5}$$

converges. Therefore, since

$$\sum_{n=1}^{\infty}|a_n|=\sum_{n=1}^{N}|a_n|+\sum_{n=N+1}^{\infty}|a_n| \tag{1.7.6}$$

where $\sum_{n=1}^{N}|a_n|$ is a finite sum and $\sum_{n=N+1}^{\infty}|a_n|$ converges, we conclude that $\sum_{n=1}^{\infty}|a_n|$ converges.

For part ii.

$$\rho=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|>1. \tag{1.7.7}$$

Since $\rho>1$, there exists R such that $\rho>R>1$. Let $\varepsilon=\rho-R>0$. By the definition of the limit of a sequence, there exists an integer N such that

$$\left|\left|\frac{a_{n+1}}{a_n}\right|-\rho\right|<\varepsilon,\ \text{for all } n\geq N. \tag{1.7.8}$$

Therefore,

$$R=\rho-\varepsilon<\left|\frac{a_{n+1}}{a_n}\right|,\ \text{for all } n\geq N, \tag{1.7.9}$$

and, thus,

$$|a_{N+1}|>R|a_N|$$

$$|a_{N+2}|>R\,|a_{N+1}|>R^2\,|a_N|$$

$$|a_{N+3}|>R\,|a_{N+2}|>R^2\,|a_{N+1}|>R^3\,|a_N|$$

$$|a_{N+4}|>R\,|a_{N+3}|>R^2\,|a_{N+2}|>R^3\,|a_{N+1}|>R^4\,|a_N|.$$

Since $R>1$, the geometric series

$$R\,|a_N|+R^2\,|a_N|+R^3\,|a_N|+\cdots \tag{1.7.10}$$

diverges. Applying the comparison test, we conclude that the series

$$|a_{N+1}|+|a_{N+2}|+|a_{N+3}|+\cdots \tag{1.7.11}$$

diverges, and therefore the series $\sum_{n=1}^{\infty}|a_n|$ diverges.

For part iii. we show that the test does not provide any information if $\rho=1$ by considering the $p-series$ $\sum_{n=1}^{\infty}\frac{1}{n^p}$. For any real number p,

$$\rho=\lim_{n\to\infty}\frac{1/(n+1)^p}{1/n^p}=\lim_{n\to\infty}\frac{n^p}{(n+1)^p}=1. \tag{1.7.12}$$

However, we know that if $p\leq 1$, the **p−series** $\sum_{n=1}^{\infty}\frac{1}{n^p}$ diverges, whereas $\sum_{n=1}^{\infty}\frac{1}{n^p}$ converges if $p>1$.

□

The ratio test is particularly useful for series whose terms contain factorials or exponential, where the ratio of terms simplifies the expression. The ratio test is convenient because it does not require us to find a comparative series. The drawback is that the test sometimes does not provide any information regarding convergence.



Example 1.7.1: Using the Ratio Test

For each of the following series, use the ratio test to determine whether the series converges or diverges.

a. $\sum_{n=1}^{\infty} \frac{2^n}{n!}$

b. $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

c. $\sum_{n=1}^{\infty} \frac{(-1)^n (n!)^2}{(2n)!}$

Solution

a. From the ratio test, we can see that

$$\rho = \lim_{n\to\infty} \frac{2^{n+1}/(n+1)!}{2^n/n!} = \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n}. \tag{1.7.13}$$

Since $(n+1)! = (n+1)\cdot n!$,

$$\rho = \lim_{n\to\infty} \frac{2}{n+1} = 0. \tag{1.7.14}$$

Since $\rho < 1$, the series converges.

b. We can see that

$$\rho = \lim_{n\to\infty} \frac{(n+1)^{n+1}/(n+1)!}{n^n/n!} = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = \lim_{n\to\infty} (\frac{n+1}{n})^n = \lim_{n\to\infty} (1+\frac{1}{n})^n = e. \tag{1.7.15}$$

Since $\rho > 1$, the series diverges.

c. Since

$$\left| \frac{(-1)^{n+1}((n+1)!)^2/(2(n+1))!}{(-1)^n (n!)^2/(2n)!} \right| = \frac{(n+1)!(n+1)!}{(2n+2)!} \cdot \frac{(2n)!}{n!n!} = \frac{(n+1)(n+1)}{(2n+2)(2n+1)} \tag{1.7.16}$$

we see that

$$\rho = \lim_{n\to\infty} \frac{(n+1)(n+1)}{(2n+2)(2n+1)} = \frac{1}{4}. \tag{1.7.17}$$

Since $\rho < 1$, the series converges.





Exercise 1.7.1

Use the ratio test to determine whether the series $\sum_{n=1}^{\infty} \frac{n^3}{3^n}$ converges or diverges.

Hint

Evaluate $\lim_{n\to\infty} \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3}$.

Answer

The series converges.

Root Test

The approach of the **root test** is similar to that of the ratio test. Consider a series $\sum_{n=1}^{\infty} a_n$ such that $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \rho$ for some real number ρ. Then for N sufficiently large, $|a_N| \approx \rho N$. Therefore, we can approximate $\sum_{n=N}^{\infty} |a_n|$ by writing

$$|a_N| + |a_{N+1}| + |a_{N+2}| + \cdots \approx \rho^N + \rho^{N+1} + \rho^{N+2} + \cdots. \tag{1.7.18}$$

The expression on the right-hand side is a geometric series. As in the ratio test, the series $\sum_{n=1}^{\infty} a_n$ converges absolutely if $0 \le \rho < 1$ and the series diverges if $\rho \ge 1$. If $\rho = 1$, the test does not provide any information. For example, for any p-series, $\sum_{n=1}^{\infty} \frac{1}{n^p}$, we see that

$$\rho = \lim_{n\to\infty} \sqrt[n]{\left|\frac{1}{n^p}\right|} = \lim_{n\to\infty} \frac{1}{n^{p/n}} \tag{1.7.19}$$

.

To evaluate this limit, we use the natural logarithm function. Doing so, we see that

$$\ln \rho = \ln(\lim_{n\to\infty} \tfrac{1}{n^{p/n}}) = \lim_{n\to\infty} \ln(\tfrac{1}{n})^{p/n} = \lim_{n\to\infty} \tfrac{p}{n} \cdot \ln(\tfrac{1}{n}) = \lim_{n\to\infty} \tfrac{p \ln(1/n)}{n}.$$

Using L'Hôpital's rule, it follows that $\ln \rho = 0$, and therefore $\rho = 1$ for all p. However, we know that the p-series only converges if $p > 1$ and diverges if $p < 1$.

Root Test

Consider the series $\sum_{n=1}^{\infty} a_n$. Let

$$\rho = \lim_{n\to\infty} \sqrt[n]{|a_n|} \tag{1.7.20}$$

.

i. If $0 \le \rho < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

ii. If $\rho > 1$ or $\rho = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.

iii. If $\rho = 1$, the test does not provide any information.

The root test is useful for series whose terms involve exponentials. In particular, for a series whose terms a_n satisfy $|a_n| = (b_n)^n$, then $\sqrt[n]{|a_n|} = b_n$ and we need only evaluate $\lim_{n\to\infty} b_n$.

Example 1.7.2: Using the Root Test

For each of the following series, use the root test to determine whether the series converges or diverges.

a. $\sum_{n=1}^{\infty} \frac{(n^2+3n)^n}{(4n^2+5)^n}$

b. $\sum_{n=1}^{\infty} \frac{n^n}{(\ln(n))^n}$

Solution

a. To apply the root test, we compute

$$\rho = \lim_{n\to\infty} \sqrt[n]{(n^2+3n)^n/(4n^2+5)^n} = \lim_{n\to\infty} \frac{n^2+3n}{4n^2+5} = \frac{1}{4}. \tag{1.7.21}$$

Since $\rho < 1$, the series converges absolutely.

b. We have

$$\rho = \lim_{n\to\infty} \sqrt[n]{n^n/(\ln n)^n} = \lim_{n\to\infty} \frac{n}{\ln n} = \infty \quad \text{by L'Hôpital's rule.} \tag{1.7.22}$$

Since $\rho = \infty$, the series diverges.



Exercise 1.7.2

Use the root test to determine whether the series $\sum_{n=1}^{\infty} 1/n^n$ converges or diverges.

Hint

Evaluate $\lim_{n\to\infty} \sqrt[n]{\frac{1}{n^n}}$.

Answer

The series converges.

Choosing a Convergence Test

At this point, we have a long list of convergence tests. However, not all tests can be used for all series. When given a series, we must determine which test is the best to use. Here is a strategy for finding the best test to apply.

Problem-Solving Strategy: Choosing a Convergence Test for a Series

Consider a series $\sum_{n=1}^{\infty} a_n$. In the steps below, we outline a strategy for determining whether the series converges.

1. Is $\sum_{n=1}^{\infty} a_n$ a familiar series? For example, is it the harmonic series (which diverges) or the alternating harmonic series (which converges)? Is it a **p−series** or geometric series? If so, check the power p or the ratio r to determine if the series converges.
2. Is it an alternating series? Are we interested in absolute convergence or just convergence? If we are just interested in whether the series converges, apply the alternating series test. If we are interested in absolute convergence, proceed to step 3, considering the series of absolute values $\sum_{n=1}^{\infty} |a_n|$.
3. Is the series similar to a **p−series** or geometric series? If so, try the comparison test or limit comparison test.
4. Do the terms in the series contain a factorial or power? If the terms are powers such that $a_n = (b_n)^n$, try the root test first. Otherwise, try the ratio test first.
5. Use the divergence test. If this test does not provide any information, try the integral test.

Visit this website for more information on testing series for convergence, plus general information on sequences and series.

Example 1.7.3: Using Convergence Tests

For each of the following series, determine which convergence test is the best to use and explain why. Then determine if the series converges or diverges. If the series is an alternating series, determine whether it converges absolutely, converges conditionally, or diverges.

a. $\sum_{n=1}^{\infty} \dfrac{n^2+2n}{n^3+3n^2+1}$

b. $\sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}(3n+1)}{n!}$

c. $\sum_{n=1}^{\infty} \dfrac{e^n}{n^3}$

d. $\sum_{n=1}^{\infty} \dfrac{3^n}{(n+1)^n}$

Solution

a. Step 1. The series is not a **p–series** or geometric series.

Step 2. The series is not alternating.

Step 3. For large values of n, we approximate the series by the expression

$$\frac{n^2+2n}{n^3+3n^2+1} \approx \frac{n^2}{n^3} = \frac{1}{n}.$$

Therefore, it seems reasonable to apply the comparison test or limit comparison test using the series $\sum_{n=1}^{\infty} 1/n$. Using the limit comparison test, we see that

$$\lim_{n\to\infty} \frac{(n^2+2n)/(n^3+3n^2+1)}{1/n} = \lim_{n\to\infty} \frac{n^3+2n^2}{n^3+3n^2+1} = 1.$$

Since the series $\sum_{n=1}^{\infty} 1/n$

diverges, this series diverges as well.

b. Step 1.The series is not a familiar series.

Step 2. The series is alternating. Since we are interested in absolute convergence, consider the series

$$\sum_{n=1}^{\infty} \frac{3n}{(n+1)!}.$$

Step 3. The series is not similar to a p-series or geometric series.

Step 4. Since each term contains a factorial, apply the ratio test. We see that

$$\lim_{n\to\infty}\frac{(3(n+1))/(n+1)!}{(3n+1)/n!}=\lim_{n\to\infty}\frac{3n+3}{(n+1)!}\cdot\frac{n!}{3n+1}=\lim_{n\to\infty}\frac{3n+3}{(n+1)(3n+1)}=0.$$

Therefore, this series converges, and we conclude that the original series converges absolutely, and thus converges.

c. Step 1. The series is not a familiar series.

Step 2. It is not an alternating series.

Step 3. There is no obvious series with which to compare this series.

Step 4. There is no factorial. There is a power, but it is not an ideal situation for the root test.

Step 5. To apply the divergence test, we calculate that

$$\lim_{n\to\infty}\frac{e^n}{n^3}=\infty.$$

Therefore, by the divergence test, the series diverges.

d. Step 1. This series is not a familiar series.

Step 2. It is not an alternating series.

Step 3. There is no obvious series with which to compare this series.

Step 4. Since each term is a power of n,we can apply the root test. Since

$$\lim_{n\to\infty}\sqrt[n]{(\frac{3}{n+1})^n}=\lim_{n\to\infty}\frac{3}{n+1}=0,$$

by the root test, we conclude that the series converges.

Exercise 1.7.3

For the series $\sum_{n=1}^{\infty}\frac{2^n}{3^n+n}$, determine which convergence test is the best to use and explain why.

Hint

The series is similar to the geometric series $\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n$.

Answer

The comparison test because $\frac{2^n}{3^n+n}<\frac{2^n}{3^n}$ for all positive integers n. The limit comparison test could also be used.

In Table, we summarize the convergence tests and when each can be applied. Note that while the comparison test, limit comparison test, and integral test require the series $\sum_{n=1}^{\infty}a_n$ to have nonnegative terms, if $\sum_{n=1}^{\infty}a_n$ has negative terms, these tests can be applied to $\sum_{n=1}^{\infty}|a_n|$ to test for absolute convergence.

Summary of Convergence Tests

Series or Test	Conclusions	Comments
Divergence Test For any series $\sum_{n=1}^{\infty}a_n$, evaluate $\lim_{n\to\infty}a_n$.	If $\lim_{n\to\infty}a_n=0$, the test is inconclusive.	This test cannot prove convergence of a series.
	If $\lim_{n\to\infty}a_n\neq 0$, the series diverges.	
Geometric Series $\sum_{n=1}^{\infty}ar^{n-1}$	If $\lvert r\rvert<1$, the series converges to $a/(1-r)$.	Any geometric series can be reindexed to be written in the form $a+ar+ar^2+\cdots$, where a is the initial term and r is the ratio.
	If $\lvert r\rvert\geq 1$, the series diverges.	
p-Series	If $p>1$, the series converges.	For $p=1$, we have the harmonic series

Series or Test	Conclusions	Comments
$\sum_{n=1}^{\infty} \frac{1}{n^p}$	If $p \le 1$, the series diverges.	$\sum_{n=1}^{\infty} 1/n$.
Comparison Test For $\sum_{n=1}^{\infty} a_n$ with nonnegative terms, compare with a known series $\sum_{n=1}^{\infty} b_n$.	If $a_n \le b_n$ for all $n \ge N$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. If $a_n \ge b_n$ for all $n \ge N$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.	Typically used for a series similar to a geometric or p-series. It can sometimes be difficult to find an appropriate series.
Limit Comparison Test For $\sum_{n=1}^{\infty} a_n$ with positive terms, compare with a series $\sum_{n=1}^{\infty} b_n$ by evaluating $L = \lim_{n\to\infty} \frac{a_n}{b_n}$.	If L is a real number and $L \neq 0$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge. If $L = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. If $L = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.	Typically used for a series similar to a geometric or p-series. Often easier to apply than the comparison test.
Integral Test If there exists a positive, continuous, decreasing function f such that $a_n = f(n)$ for all $n \ge N$, evaluate $\int_N^{\infty} f(x)dx$.	$\int_N^{\infty} f(x)dx$ and $\sum_{n=1}^{\infty} a_n$ both converge or both diverge.	Limited to those series for which the corresponding function can be easily integrated.
Alternating Series $\sum_{n=1}^{\infty}(-1)^{n+1}b_n$ or $\sum_{n=1}^{\infty}(-1)^n b_n$	If $b_{n+1} \le b_n$ for all $n \ge 1$ and $b_n \to 0$, then the series converges.	Only applies to alternating series.
Ratio Test For any series $\sum_{n=1}^{\infty} a_n$ with nonzero terms, let $\rho = \lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert$	If $0 \le \rho < 1$, the series converges absolutely. If $\rho > 1$ or $\rho = \infty$, the series diverges. If $\rho = 1$, the test is inconclusive.	Often used for series involving factorials or exponentials.
Root Test For any series $\sum_{n=1}^{\infty} a_n$, let $\rho = \lim_{n\to\infty} \sqrt[n]{\lvert a_n \rvert}$.	If $0 \le \rho < 1$, the series converges absolutely. If $\rho > 1$ or $\rho = \infty$, the series diverges. If $\rho = 1$, the test is inconclusive.	Often used for series where $\lvert a_n \rvert = (b_n)^n$.

Series Converging to π and $1/\pi$

Dozens of series exist that converge to π or an algebraic expression containing π. Here we look at several examples and compare their rates of convergence. By rate of convergence, we mean the number of terms necessary for a partial sum to be within a certain amount of the actual value. The series representations of π in the first two examples can be explained using Maclaurin series, which are discussed in the next chapter. The third example relies on material beyond the scope of this text.

1. The series

$$\pi = 4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1} = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \cdots \tag{1.7.23}$$

was discovered by Gregory and Leibniz in the late $1600s$. This result follows from the Maclaurin series for $f(x) = \tan^{-1} x$. We will discuss this series in the next chapter.

a. Prove that this series converges.

b. Evaluate the partial sums S_n for $n = 10, 20, 50, 100$.

c. Use the remainder estimate for alternating series to get a bound on the error R_n.

d. What is the smallest value of N that guarantees $|R_N| < 0.01$? Evaluate S_N.

2. The series

$$\pi = 6\sum_{n=0}^{\infty}\frac{(2n)!}{2^{4n+1}(n!)^2(2n+1)} = 6\left(\frac{1}{2} + \frac{1}{2\cdot 3}\left(\frac{1}{2}\right)^3 + \frac{1\cdot 3}{2\cdot 4\cdot 5}\cdot\left(\frac{1}{2}\right)^5 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7}\left(\frac{1}{2}\right)^7 + \cdots\right) \tag{1.7.24}$$

has been attributed to Newton in the late $1600s$. The proof of this result uses the Maclaurin series for $f(x) = \sin^{-1} x$.

a. Prove that the series converges.

b. Evaluate the partial sums S_n for $n = 5, 10, 20$.

c. Compare S_n to π for $n = 5, 10, 20$ and discuss the number of correct decimal places.

3. The series

$$\frac{1}{\pi} = \frac{\sqrt{8}}{9801}\sum_{n=0}^{\infty}\frac{(4n)!(1103 + 26390n)}{(n!)^4 396^{4n}} \tag{1.7.25}$$

was discovered by **Ramanujan** in the early $1900s$. William Gosper, Jr., used this series to calculate π to an accuracy of more than 17 million digits in the $mid - 1980s$. At the time, that was a world record. Since that time, this series and others by Ramanujan have led mathematicians to find many other series representations for π and $1/\pi$.

a. Prove that this series converges.

b. Evaluate the first term in this series. Compare this number with the value of π from a calculating utility. To how many decimal places do these two numbers agree? What if we add the first two terms in the series?

c. Investigate the life of Srinivasa Ramanujan $(1887–1920)$and write a brief summary. Ramanujan is one of the most fascinating stories in the history of mathematics. He was basically self-taught, with no formal training in mathematics, yet he contributed in highly original ways to many advanced areas of mathematics.

Key Concepts

- For the ratio test, we consider

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|. \tag{1.7.26}$$

If $\rho < 1$, the series $\sum_{n=1}^{\infty} a_n$ converges absolutely. If $\rho > 1$, the series diverges. If $\rho = 1$, the test does not provide any information. This test is useful for series whose terms involve factorials.

- For the root test, we consider

$$\rho = \lim_{n\to\infty} \sqrt[n]{|a_n|} \tag{1.7.27}$$

.

If $\rho < 1$, the series $\sum_{n=1}^{\infty} a_n$ converges absolutely. If $\rho > 1$, the series diverges. If $\rho = 1$, the test does not provide any information. The root test is useful for series whose terms involve powers.

- For a series that is similar to a geometric series or **p−series,** consider one of the comparison tests.

Glossary

ratio test

for a series $\sum_{n=1}^{\infty} a_n$ with nonzero terms, let $\rho = \lim_{n\to\infty} |a_{n+1}/a_n|$; if $0 \le \rho < 1$, the series converges absolutely; if $\rho > 1$, the series diverges; if $\rho = 1$, the test is inconclusive

root test

for a series $\sum_{n=1}^{\infty} a_n$, let $\rho = \lim_{n\to\infty} \sqrt[n]{|a_n|}$; if $0 \le \rho < 1$, the series converges absolutely; if $\rho > 1$, the series diverges; if $\rho = 1$, the test is inconclusive

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1.7E: Exercises for Section 1.7

In exercises 1 - 11, use the ratio test to determine whether each series $\sum_{n=1}^{\infty} a_n$ converges or diverges. State if the ratio test is inconclusive.

1) $\sum_{n=1}^{\infty} \frac{1}{n!}$

Answer

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0$. Converges by the Ratio Test.

2) $\sum_{n=1}^{\infty} \frac{10^n}{n!}$

3) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

Answer

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{2}\left(\frac{n+1}{n}\right)^2 = \frac{1}{2} < 1.$ Converges by the Ratio Test.

4) $\sum_{n=1}^{\infty} \frac{n^{10}}{2^n}$

5) $\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!}$

Answer

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{27} < 1.$ Converges by the Ratio Test.

6) $\sum_{n=1}^{\infty} \frac{2^{3n}(n!)^3}{(3n)!}$

7) $\sum_{n=1}^{\infty} \frac{(2n)!}{n^{2n}}$

Answer

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{4}{e^2} < 1.$ Converges by the Ratio Test.

8) $\sum_{n=1}^{\infty} \frac{(2n)!}{(2n)^n}$

9) $\sum_{n=1}^{\infty} \frac{n!}{(n/e)^n}$

Answer

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1.$ Ratio test is inconclusive.

10) $\sum_{n=1}^{\infty} \frac{(2n)!}{(n/e)^{2n}}$

11) $\sum_{n=1}^{\infty} \frac{(2^n n!)^2}{(2n)^{2n}}$

Answer

$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \frac{1}{e^2} < 1$. Converges by the Ratio Test.

In exercises 12 - 21, use the root test to determine whether $\sum_{n=1}^{\infty} a_n$ converges, where a_n is as follows.

12) $a_k = \left(\frac{k-1}{2k+3}\right)^k$

13) $a_k = \left(\frac{2k^2-1}{k^2+3}\right)^k$

Answer

$\lim_{k\to\infty} (a_k)^{1/k} = 2 > 1$. Diverges by the Root Test.

14) $a_n = \frac{(\ln n)^{2n}}{n^n}$

15) $a_n = n/2^n$

Answer

$\lim_{n\to\infty} (a_n)^{1/n} = 1/2 < 1$. Converges by the Root Test.

16) $a_n = n/e^n$

17) $a_k = \frac{k^e}{e^k}$

Answer

$\lim_{k\to\infty} (a_k)^{1/k} = 1/e < 1$. Converges by the Root Test.

18) $a_k = \frac{\pi^k}{k^\pi}$

19) $a_n = \left(\frac{1}{e} + \frac{1}{n}\right)^n$

Answer

$\lim_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty} \frac{1}{e} + \frac{1}{n} = \frac{1}{e} < 1$. Converges by the Root Test.

20) $a_k = \frac{1}{(1+\ln k)^k}$

21) $a_n = \frac{(\ln(1+\ln n))^n}{(\ln n)^n}$

Answer

$\lim_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty} \frac{(\ln(1+\ln n))}{(\ln n)} = 0$ by L'Hôpital's rule. Converges by the Root Test.

In exercises 22 - 28, use either the ratio test or the root test as appropriate to determine whether the series $\sum_{k=1}^{\infty} a_k$ with given terms a_k converges, or state if the test is inconclusive.

22) $a_k = \dfrac{k!}{1 \cdot 3 \cdot 5 \cdots (2k-1)}$

23) $a_k = \dfrac{2 \cdot 4 \cdot 6 \cdots 2k}{(2k)!}$

Answer

$\lim_{k\to\infty} \dfrac{a_{k+1}}{a_k} = \lim_{k\to\infty} \dfrac{1}{2k+1} = 0$. Converges by the Ratio Test.

24) $a_k = \dfrac{1 \cdot 4 \cdot 7 \cdots (3k-2)}{3^k k!}$

25) $a_n = \left(1 - \dfrac{1}{n}\right)^{n^2}$

Answer

$\lim_{n\to\infty} (a_n)^{1/n} = 1/e$. Converges by the Root Test.

26) $a_k = \left(\dfrac{1}{k+1} + \dfrac{1}{k+2} + \cdots + \dfrac{1}{2k}\right)^k$ $\left(\text{Hint: Compare } a_k^{1/k} \text{ to } \int_k^{2k} \dfrac{dt}{t}.\right)$

27) $a_k = \left(\dfrac{1}{k+1} + \dfrac{1}{k+2} + \cdots + \dfrac{1}{3k}\right)^k$

Answer

$\lim_{k\to\infty} a_k^{1/k} = \ln(3) > 1$. Diverges by the Root Test.

28) $a_n = \left(n^{1/n} - 1\right)^n$

In exercises 29 - 30, use the ratio test to determine whether $\sum_{n=1}^{\infty} a_n$ converges, or state if the ratio test is inconclusive.

29) $\sum_{n=1}^{\infty} \dfrac{3^{n^2}}{2^{n^3}}$

Answer

$\lim_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \dfrac{3^{2n+1}}{2^{3n^2+3n+1}} = 0$. Converges by the Ratio Test.

30) $\sum_{n=1}^{\infty} \dfrac{2^{n^2}}{n^n n!}$

In exercise 31, use the root and limit comparison tests to determine whether $\sum_{n=1}^{\infty} a_n$ converges.

31) $\sum_{n=1}^{\infty} \dfrac{1}{x_n^n}$ where $x_{n+1} = \frac{1}{2}x_n + \dfrac{1}{x_n}, x_1 = 1$ (Hint: Find limit of x_n.)

Answer

Converges by the Root Test and Limit Comparison Test since $\lim_{n\to\infty} x_n = \sqrt{2}$.

In exercises 32 - 43, use an appropriate test to determine whether the series converges.

32) $\sum_{n=1}^{\infty} \dfrac{n+1}{n^3+n^2+n+1}$

33) $\sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}(n+1)}{n^3+3n^2+3n+1}$

Answer

Converges absolutely by limit comparison with p**−series,** $p=2$.

34) $\sum_{n=1}^{\infty} \dfrac{(n+1)^2}{n^3+(1.1)^n}$

35) $\sum_{n=1}^{\infty} \dfrac{(n-1)^n}{(n+1)^n}$

Answer

$\lim_{n\to\infty} a_n = 1/e^2 \neq 0$. Series diverges by the Divergence Test.

36) $a_n = \left(1+\dfrac{1}{n^2}\right)^n$ $\left(\text{Hint: } \left(1+\dfrac{1}{n^2}\right)^{n^2} \approx e.\right)$

37) $a_k = 1/2^{\sin^2 k}$

Answer

Terms do not tend to zero: $a_k \geq 1/2$, since $\sin^2 x \leq 1$.

38) $a_k = 2^{-\sin(1/k)}$

39) $a_n = 1/\binom{n+2}{n}$ where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Answer

$a_n = \dfrac{2}{(n+1)(n+2)}$, which converges by comparison with p**−series** for $p=2$.

40) $a_k = 1/\binom{2k}{k}$

41) $a_k = 2^k/\binom{3k}{k}$

Answer

$a_k = \dfrac{2^k 1\cdot 2\cdots k}{(2k+1)(2k+2)\cdots 3k} \leq (2/3)^k$ converges by comparison with geometric series.

42) $a_k = \left(\dfrac{k}{k+\ln k}\right)^k$ $\left(\text{Hint: } a_k = \left(1+\dfrac{\ln k}{k}\right)^{-(k/\ln k)\ln k} \approx e^{-\ln k}.\right)$

43) $a_k = \left(\dfrac{k}{k+\ln k}\right)^{2k}$ $\left(\text{Hint: } a_k = \left(1+\dfrac{\ln k}{k}\right)^{-(k/\ln k)\ln k^2}.\right)$

Answer

$a_k \approx e^{-\ln k^2} = 1/k^2$. Series converges by limit comparison with p**−series**, $p=2$.

The series in exercises 44 - 47 converge by the ratio test. Use summation by parts, $\sum_{k=1}^{n} a_k(b_{k+1}-b_k) = [a_{n+1}b_{n+1}-a_1b_1] - \sum_{k=1}^{n} b_{k+1}(a_{k+1}-a_k)$ **, to find the sum of the given series.**

https://math.libretexts.org/@go/page/83980

44) $\sum_{k=1}^{\infty} \frac{k}{2^k}$ (Hint: Take $a_k = k$ and $b_k = 2^{1-k}$.)

45) $\sum_{k=1}^{\infty} \frac{k}{c^k}$, where $c > 1$ (Hint: Take $a_k = k$ and $b_k = c^{1-k}/(c-1)$.)

Answer

If $b_k = c^{1-k}/(c-1)$ and $a_k = k$, then $b_{k+1} - b_k = -c^{-k}$ and $\sum_{n=1}^{\infty} \frac{k}{c^k} = a_1 b_1 + \frac{1}{c-1} \sum_{k=1}^{\infty} c^{-k} = \frac{c}{(c-1)^2}$.

46) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

47) $\sum_{n=1}^{\infty} \frac{(n+1)^2}{2^n}$

Answer

$6+4+1=11$

The kth term of each of the following series has a factor x^k. Find the range of x for which the ratio test implies that the series converges.

48) $\sum_{k=1}^{\infty} \frac{x^k}{k^2}$

49) $\sum_{k=1}^{\infty} \frac{x^{2k}}{k^2}$

Answer

$|x| \le 1$

50) $\sum_{k=1}^{\infty} \frac{x^{2k}}{3^k}$

51) $\sum_{k=1}^{\infty} \frac{x^k}{k!}$

Answer

$|x| < \infty$

52) Does there exist a number p such that $\sum_{n=1}^{\infty} \frac{2^n}{n^p}$ converges?

53) Let $0 < r < 1$. For which real numbers p does $\sum_{n=1}^{\infty} n^p r^n$ converge?

Answer

All real numbers p by the Ratio Test.

54) Suppose that $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = p$. For which values of p must $\sum_{n=1}^{\infty} 2^n a_n$ converge?

55) Suppose that $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = p$. For which values of $r > 0$ is $\sum_{n=1}^{\infty} r^n a_n$ guaranteed to converge?

Answer

$r < 1/p$

56) Suppose that $\left|\frac{a_{n+1}}{a_n}\right| \le (n+1)^p$ for all $n = 1, 2, \ldots$ where p is a fixed real number. For which values of p is $\sum_{n=1}^{\infty} n! a_n$ guaranteed to converge?

57) For which values of $r > 0$, if any, does $\sum_{n=1}^{\infty} r^{\sqrt{n}}$ converge? $\left(\text{Hint: } sum_{n=1}^{\infty} a_n = \sum_{k=1}^{\infty} \sum_{n=k^2}^{(k+1)^2-1} a_n. \right)$

Answer

$0 < r < 1$. Note that the ratio and root tests are inconclusive. Using the hint, there are $2k$ terms $r^{\sqrt{n}}$ for $k^2 \le n < (k+1)^2$, and for $r < 1$ each term is at least r^k. Thus, $\sum_{n=1}^{\infty} r^{\sqrt{n}} = \sum_{k=1}^{\infty} \sum_{n=k^2}^{(k+1)^2-1} r^{\sqrt{n}} \ge \sum_{k=1}^{\infty} 2k r^k$, which converges by the ratio test for $r < 1$. For $r \ge 1$ the series diverges by the divergence test.

58) Suppose that $\left|\frac{a_{n+2}}{a_n}\right| \le r < 1$ for all n. Can you conclude that $\sum_{n=1}^{\infty} a_n$ converges?

59) Let $a_n = 2^{-[n/2]}$ where $[x]$ is the greatest integer less than or equal to x. Determine whether $\sum_{n=1}^{\infty} a_n$ converges and justify your answer.

Answer

One has $a_1 = 1, a_2 = a_3 = 1/2, \ldots a_{2n} = a_{2n+1} = 1/2^n$. The ratio test does not apply because $a_{n+1}/a_n = 1$ if n is even. However, $a_{n+2}/a_n = 1/2$, so the series converges according to the previous exercise. Of course, the series is just a duplicated geometric series.

The following *advanced exercises* use a generalized ratio test to determine convergence of some series that arise in particular applications when tests in this chapter, including the ratio and root test, are not powerful enough to determine their convergence. The test states that if $\lim_{n\to\infty} \frac{a_{2n}}{a_n} < 1/2$, then $\sum a_n$ converges, while if $\lim_{n\to\infty} \frac{a_{2n+1}}{a_n} > 1/2$, then $\sum a_n$ diverges.

60) Let $a_n = \frac{1}{4}\frac{3}{6}\frac{5}{8}\cdots\frac{2n-1}{2n+2} = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n(n+1)!}$. Explain why the ratio test cannot determine convergence of $\sum_{n=1}^{\infty} a_n$. Use the fact that $1 - 1/(4k)$ is increasing k to estimate $\lim_{n\to\infty} \frac{a_{2n}}{a_n}$.

61) Let $a_n = \frac{1}{1+x}\frac{2}{2+x}\cdots\frac{n}{n+x}\frac{1}{n} = \frac{(n-1)!}{(1+x)(2+x)\cdots(n+x).}$ Show that $a_{2n}/a_n \le e^{-x/2}/2$. For which $x > 0$ does the generalized ratio test imply convergence of $\sum_{n=1}^{\infty} a_n$? (Hint: Write $2a_{2n}/a_n$ as a product of n factors each smaller than $1/(1 + x/(2n)).$)

Answer

$a_{2n}/a_n = \frac{1}{2}\cdot\frac{n+1}{n+1+x}\frac{n+2}{n+2+x}\cdots\frac{2n}{2n+x}$. The inverse of the kth factor is $(n+k+x)/(n+k) > 1 + x/(2n)$ so the product is less than $(1 + x/(2n))^{-n} \approx e^{-x/2}$. Thus for $x > 0$, $\frac{a_{2n}}{a_n} \le \frac{1}{2}e^{-x/2}$. The series converges for $x > 0$.

62) Let $a_n = \frac{n^{\ln n}}{(\ln n)^n}$. Show that $\frac{a_{2n}}{a_n} \to 0$ as $n \to \infty$.

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1.8: Chapter 1 Review Exercises

***True or False?* Justify your answer with a proof or a counterexample.**

1) If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges.

Answer

false

2) If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

3) If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Answer

true

4) If $\sum_{n=1}^{\infty} 2^n a_n$ converges, then $\sum_{n=1}^{\infty} (-2)^n a_n$ converges.

Is the sequence bounded, monotone, and convergent or divergent? If it is convergent, find the limit.

5) $a_n = \dfrac{3+n^2}{1-n}$

Answer

unbounded, not monotone, divergent

6) $a_n = \ln\left(\frac{1}{n}\right)$

7) $a_n = \dfrac{\ln(n+1)}{\sqrt{n+1}}$

Answer

bounded, monotone, convergent, 0

8) $a_n = \dfrac{2^{n+1}}{5^n}$

9) $a_n = \dfrac{\ln(\cos n)}{n}$

Answer

unbounded, not monotone, divergent

Is the series convergent or divergent?

10) $\sum_{n=1}^{\infty} \dfrac{1}{n^2+5n+4}$

11) $\sum_{n=1}^{\infty} \ln\left(\dfrac{n+1}{n}\right)$

Answer

diverges

12) $\sum_{n=1}^{\infty} \frac{2^n}{n^4}$

13) $\sum_{n=1}^{\infty} \frac{e^n}{n!}$

Answer

converges

14) $\sum_{n=1}^{\infty} n^{-(n+1/n)}$

Is the series convergent or divergent? If convergent, is it absolutely convergent?

15) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

Answer

converges, but not absolutely

16) $\sum_{n=1}^{\infty} \frac{(-1)^n n!}{3^n}$

17) $\sum_{n=1}^{\infty} \frac{(-1)^n n!}{n^n}$

Answer

converges absolutely

18) $\sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{2}\right)$

19) $\sum_{n=1}^{\infty} \cos(\pi n) e^{-n}$

Answer

converges absolutely

Evaluate.

20) $\sum_{n=1}^{\infty} \frac{2^{n+4}}{7^n}$

21) $\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)}$

Answer

$\frac{1}{2}$

22) A legend from India tells that a mathematician invented chess for a king. The king enjoyed the game so much he allowed the mathematician to demand any payment. The mathematician asked for one grain of rice for the first square on the chessboard, two grains of rice for the second square on the chessboard, and so on. Find an exact expression for the total payment (in grains of rice) requested by the mathematician. Assuming there are $30,000$ grains of rice in 1 pound, and 2000 pounds in 1 ton, how many tons of rice did the mathematician attempt to receive?

The following problems consider a simple population model of the housefly, which can be exhibited by the recursive formula $x_{n+1} = bx_n$, where x_n is the population of houseflies at generation n, and b is the average number of offspring per housefly who

survive to the next generation. Assume a starting population x_0.

23) Find $\lim_{n\to\infty} x_n$ if $b > 1,\ b < 1$, and $b = 1$.

Answer

$\infty,\ 0,\ x_0$

24) Find an expression for $S_n = \sum_{i=0}^{n} x_i$ in terms of b and x_0. What does it physically represent?

25) If $b = \frac{3}{4}$ and $x_0 = 100$, find S_{10} and $\lim_{n\to\infty} S_n$

Answer

$S_{10} \approx 383, \quad \lim_{n\to\infty} S_n = 400$

26) For what values of b will the series converge and diverge? What does the series converge to?

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CHAPTER OVERVIEW

2: Power Series

A power series (in one variable) is an infinite series. Any polynomial can be easily expressed as a power series around any center c, although most of the coefficients will be zero since a power series has infinitely many terms by definition. One can view power series as being like "polynomials of infinite degree," although power series are not polynomials. The content in this Textmap's chapter is complemented by Guichard's Calculus Textmap.

2.1: Prelude to Power Series
2.2: Power Series and Functions
2.2E: Exercises for Section 2.2
2.3: Properties of Power Series
2.3E: Exercises for Section 2.3
2.4: Taylor and Maclaurin Series
2.4E: Exercises for Section 2.4
2.5: Working with Taylor Series
2.5E: Exercises for Section 2.5
2.6: Chapter 2 Review Exercises

Thumbnail: The graph shows the function $y = sinx$ and the Maclaurin polynomials p_1, p_3 and p_5. (CC BY-SA 3.0; OpenStax).

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2.1: Prelude to Power Series

When winning a lottery, sometimes an individual has an option of receiving winnings in one lump-sum payment or receiving smaller payments over fixed time intervals. For example, you might have the option of receiving 20 million dollars today or receiving 1.5 million dollars each year for the next 20 years. Which is the better deal? Certainly 1.5 million dollars over 20 years is equivalent to 30 million dollars. However, receiving the 20 million dollars today would allow you to invest the money.



Figure *[Math Processing Error]*: If you win a lottery, do you get more money by taking a lump-sum payment or by accepting fixed payments over time? (credit: modification of work by Robert Huffstutter, Flickr)

Alternatively, what if you were guaranteed to receive 1 million dollars every year indefinitely (extending to your heirs) or receive 20 million dollars today. Which would be the better deal? To answer these questions, you need to know how to use infinite series to calculate the value of periodic payments over time in terms of today's dollars.

An infinite series of the form

[Math Processing Error]

is known as a **power series**. Since the terms contain the variable *[Math Processing Error]*, power series can be used to define functions. They can be used to represent given functions, but they are also important because they allow us to write functions that cannot be expressed any other way than as "infinite polynomials." In addition, power series can be easily differentiated and integrated, thus being useful in solving differential equations and integrating complicated functions. An infinite series can also be truncated, resulting in a finite polynomial that we can use to approximate functional values. Power series have applications in a variety of fields, including physics, chemistry, biology, and economics. As we will see in this chapter, representing functions using power series allows us to solve mathematical problems that cannot be solved with other techniques.

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2.2: Power Series and Functions

Learning Objectives

- Identify a power series and provide examples of them.
- Determine the radius of convergence and interval of convergence of a power series.
- Use a power series to represent a function.

A power series is a type of series with terms involving a variable. More specifically, if the variable is x, then all the terms of the series involve powers of x. As a result, a power series can be thought of as an infinite polynomial. Power series are used to represent common functions and also to define new functions. In this section we define power series and show how to determine when a power series converges and when it diverges. We also show how to represent certain functions using power series.

Form of a Power Series

A series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots, \tag{2.2.1}$$

where x is a variable and the coefficients c_n are constants, is known as a **power series.** The series

$$1 + x + x^2 + \ldots = \sum_{n=0}^{\infty} x^n \tag{2.2.2}$$

is an example of a power series. Since this series is a geometric series with ratio $r = |x|$, we know that it converges if $|x| < 1$ and diverges if $|x| \geq 1$.

Definition 2.2.1: Power Series

A series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots \tag{2.2.3}$$

is a power series centered at $x = 0$. A series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \ldots \tag{2.2.4}$$

is a *power series* centered at $x = a$.

To make this definition precise, we stipulate that $x^0 = 1$ and $(x-a)^0 = 1$ even when $x = 0$ and $x = a$, respectively.

The series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \tag{2.2.5}$$

and

$$\sum_{n=0}^{\infty} n! x^n = 1 + x + 2! x^2 + 3! x^3 + \ldots \tag{2.2.6}$$

are both power series centered at $x = 0$. The series

$$\sum_{n=0}^{\infty} \frac{(x-2)^n}{(n+1)3^n} = 1 + \frac{x-2}{2 \cdot 3} + \frac{(x-2)^2}{3 \cdot 3^2} + \frac{(x-2)^3}{4 \cdot 3^3} + \ldots \tag{2.2.7}$$

is a power series centered at $x = 2$.

Convergence of a Power Series

Since the terms in a power series involve a variable x, the series may converge for certain values of x and diverge for other values of x. For a power series centered at $x = a$, the value of the series at $x = a$ is given by c_0. Therefore, a power series always converges at its center. Some power series converge only at that value of x. Most power series, however, converge for more than one value of x. In that case, the power series either converges for all real numbers x or converges for all x in a finite interval. For example, the geometric series $\sum_{n=0}^{\infty} x^n$ converges for all x in the interval $(-1, 1)$, but diverges for all x outside that interval. We now summarize these three possibilities for a general power series.

Note 2.2.1: Convergence of a Power Series

Consider the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$. The series satisfies exactly one of the following properties:

i. The series converges at $x = a$ and diverges for all $x \neq a$.
ii. The series converges for all real numbers x.
iii. There exists a real number $R > 0$ such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$. At the values x where |x−a|=R, the series may converge or diverge.

Proof

Suppose that the power series is centered at $a = 0$. (For a series centered at a value of a other than zero, the result follows by letting $y = x - a$ and considering the series

$$\sum_{n=1}^{\infty} c_n y^n.$$

We must first prove the following fact:

If there exists a real number $d \neq 0$ such that $\sum_{n=0}^{\infty} c_n d^n$ converges, then the series $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely for all x such that $|x| < |d|$.

Since $\sum_{n=0}^{\infty} c_n d^n$ converges, the nth term $c_n d^n \to 0$ as $n \to \infty$. Therefore, there exists an integer N such that $|c_n d^n| \leq 1$ for all $n \geq N$. Writing

$$|c_n x^n| = |c_n d^n| \left|\frac{x}{d}\right|^n,$$

we conclude that, for all n≥N,

$$|c_n x^n| \leq \left|\frac{x}{d}\right|^n.$$

The series

$$\sum_{n=N}^{\infty} \left|\frac{x}{d}\right|^n$$

is a geometric series that converges if $|\frac{x}{d}| < 1$. Therefore, by the comparison test, we conclude that $\sum_{n=N}^{\infty} c_n x^n$ also converges for $|x| < |d|$. Since we can add a finite number of terms to a convergent series, we conclude that $\sum_{n=0}^{\infty} c_n x^n$ converges for $|x| < |d|$.

With this result, we can now prove the theorem. Consider the series

$$\sum_{n=0}^{\infty} a_n x^n$$

and let S be the set of real numbers for which the series converges. Suppose that the set $S = 0$. Then the series falls under case i.

Suppose that the set S is the set of all real numbers. Then the series falls under case ii. Suppose that $S \neq 0$ and S is not the set of real numbers. Then there exists a real number $x* \neq 0$ such that the series does not converge. Thus, the series cannot converge for any x such that $|x| > |x*|$. Therefore, the set S must be a bounded set, which means that it must have a smallest upper bound. (This fact follows from the **Least Upper Bound Property** for the real numbers, which is beyond the scope of this text and is covered in real analysis courses.) Call that smallest upper bound R. Since $S \neq 0$, the number $R > 0$. Therefore, the series converges for all x such that $|x| < R$, and the series falls into case iii.

□

If a series $\sum_{n=0}^{\infty} c_n(x-a)^n$ falls into case iii. of Note, then the series converges for all x such that $|x-a| < R$ for some $R > 0$, and diverges for all x such that $|x-a| > R$. The series may converge or diverge at the values x where $|x-a| = R$. The set of values x for which the series $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges is known as the interval of convergence. Since the series diverges for all values x where $|x-a| > R$, the length of the interval is $2R$, and therefore, the radius of the interval is R. The value R is called the radius of convergence. For example, since the series $\sum_{n=0}^{\infty} x^n$ converges for all values x in the interval $(-1, 1)$ and diverges for all values x such that $|x| \geq 1$, the interval of convergence of this series is $(-1, 1)$. Since the length of the interval is 2, the radius of convergence is 1.

Definition: radius of convergence

Consider the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$. The set of real numbers x where the series converges is the interval of convergence. If there exists a real number $R > 0$ such that the series converges for $|x-a| < R$ and diverges for $|x-a| > R$, then R is the radius of convergence. If the series converges only at $x = a$, we say the radius of convergence is $R = 0$. If the series converges for all real numbers x, we say the radius of convergence is $R = \infty$ (Figure 2.2.1).

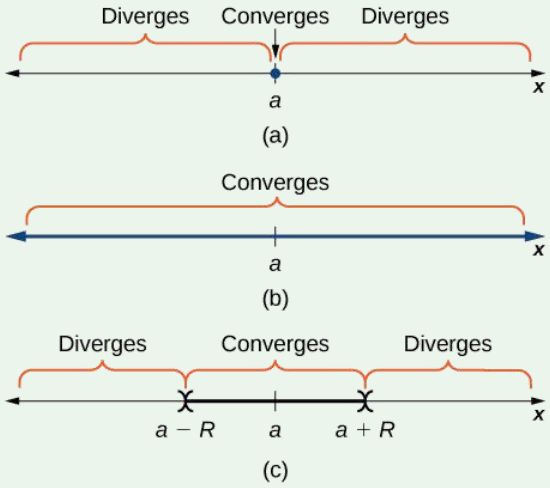


Figure 2.2.1: For a series $\sum_{n=0}^{\infty} c_n(x-a)^n$ graph (a) shows a radius of convergence at $R = 0$, graph (b) shows a radius of convergence at $R = \infty$, and graph (c) shows a radius of convergence at R. For graph (c) we note that the series may or may not converge at the endpoints $x = a + R$ and $x = a - R$.

To determine the interval of convergence for a power series, we typically apply the ratio test. In Example 2.2.1, we show the three different possibilities illustrated in Figure 2.2.1.

Example 2.2.1: Finding the Interval and Radius of Convergence

For each of the following series, find the interval and radius of convergence.

a. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

b. $\sum_{n=0}^{\infty} n! x^n$

c. $\sum_{n=0}^{\infty} \frac{(x-2)^n}{(n+1)3^n}$

Solution

a. To check for convergence, apply the ratio test. We have

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| \\
&= \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| \\
&= \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)\cdot n!} \cdot \frac{n!}{x^n} \right| \\
&= \lim_{n\to\infty} \left| \frac{x}{n+1} \right| \\
&= |x| \lim_{n\to\infty} \frac{1}{n+1} \\
&= 0 < 1
\end{aligned}$$

for all values of x. Therefore, the series converges for all real numbers x. The interval of convergence is $(-\infty, \infty)$ and the radius of convergence is $R = \infty$.

b. Apply the ratio test. For $x \neq 0$, we see that

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \left| \frac{(n+1)! x^{n+1}}{n! x^n} \right| \\
&= \lim_{n\to\infty} |(n+1)x| \\
&= |x| \lim_{n\to\infty} (n+1) \\
&= \infty.
\end{aligned}$$

Therefore, the series diverges for all $x \neq 0$. Since the series is centered at $x = 0$, it must converge there, so the series converges only for $x \neq 0$. The interval of convergence is the single value $x = 0$ and the radius of convergence is $R = 0$.

c. In order to apply the ratio test, consider

$$\rho = \lim_{n\to\infty} \left| \frac{\frac{(x-2)^{n+1}}{(n+2)3^{n+1}}}{\frac{(x-2)^n}{(n+1)3^n}} \right|$$
$$= \lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{(n+2)3^{n+1}} \cdot \frac{(n+1)3^n}{(x-2)^n} \right|$$
$$= \lim_{n\to\infty} \left| \frac{(x-2)(n+1)}{3(n+2)} \right|$$
$$= \frac{|x-2|}{3}.$$

The ratio $\rho < 1$ if $|x-2| < 3$. Since $|x-2| < 3$ implies that $-3 < x-2 < 3$, the series converges absolutely if $-1 < x < 5$. The ratio $\rho > 1$ if $|x-2| > 3$. Therefore, the series diverges if $x < -1$ or $x > 5$. The ratio test is inconclusive if $\rho = 1$. The ratio $\rho = 1$ if and only if $x = -1$ or $x = 5$. We need to test these values of x separately. For $x = -1$, the series is given by

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots.$$

Since this is the alternating harmonic series, it converges. Thus, the series converges at $x = -1$. For $x = 5$, the series is given by

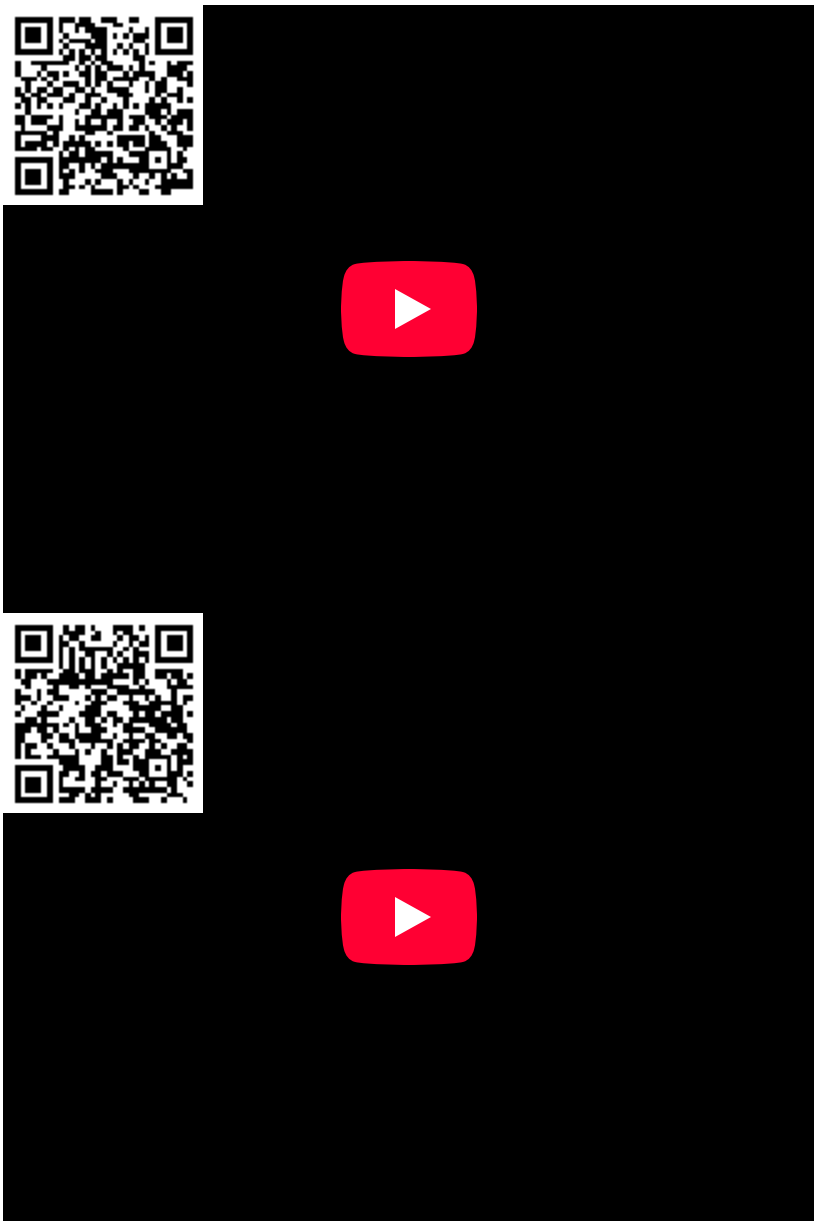
$$\sum_{n=0}^{\infty} \frac{1}{n+1} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots.$$

This is the harmonic series, which is divergent. Therefore, the power series diverges at $x = 5$. We conclude that the interval of convergence is $[-1, 5)$ and the radius of convergence is $R = 3$.









Exercise 2.2.1

Find the interval and radius of convergence for the series

$$\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}.$$

Hint

Apply the ratio test to check for absolute convergence.

Answer

The interval of convergence is $[-1, 1)$. The radius of convergence is $R = 1$.

Representing Functions as Power Series

Being able to represent a function by an "infinite polynomial" is a powerful tool. Polynomial functions are the easiest functions to analyze, since they only involve the basic arithmetic operations of addition, subtraction, multiplication, and division. If we can represent a complicated function by an infinite polynomial, we can use the polynomial representation to differentiate or integrate it. In addition, we can use a truncated version of the polynomial expression to approximate values of the function. So, the question is, when can we represent a function by a power series?

Consider again the geometric series

$$1+x+x^2+x^3+\ldots=\sum_{n=0}^{\infty}x^n. \tag{2.2.8}$$

Recall that the geometric series

$$a+ar+ar^2+ar^3+\ldots \tag{2.2.9}$$

converges if and only if $|r|<1$. In that case, it converges to $\dfrac{a}{1-r}$. Therefore, if $|x|<1$, the series in Example 2.2.1 converges to $\dfrac{1}{1-x}$ and we write

$$1+x+x^2+x^3+\ldots=\frac{1}{1-x}\,for|x|<1. \tag{2.2.10}$$

As a result, we are able to represent the function $f(x)=\dfrac{1}{1-x}$ by the power series

$$1+x+x^2+x^3+\ldots when|x|<1. \tag{2.2.11}$$

We now show graphically how this series provides a representation for the function $f(x)=\dfrac{1}{1-x}$ by comparing the graph of f with the graphs of several of the partial sums of this infinite series.

Example 2.2.2: Graphing a Function and Partial Sums of its Power Series

Sketch a graph of $f(x)=\dfrac{1}{1-x}$ and the graphs of the corresponding partial sums $S_N(x)=\sum_{n=0}^{N}x^n$ for $N=2,4,6$ on the interval $(-1,1)$. Comment on the approximation S_N as N increases.

Solution

From the graph in Figure you see that as N increases, S_N becomes a better approximation for $f(x)=\dfrac{1}{1-x}$ for x in the interval $(-1,1)$.

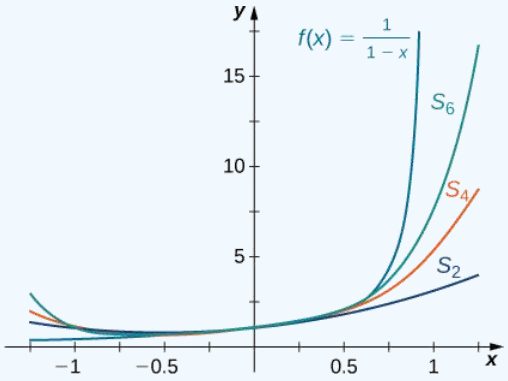
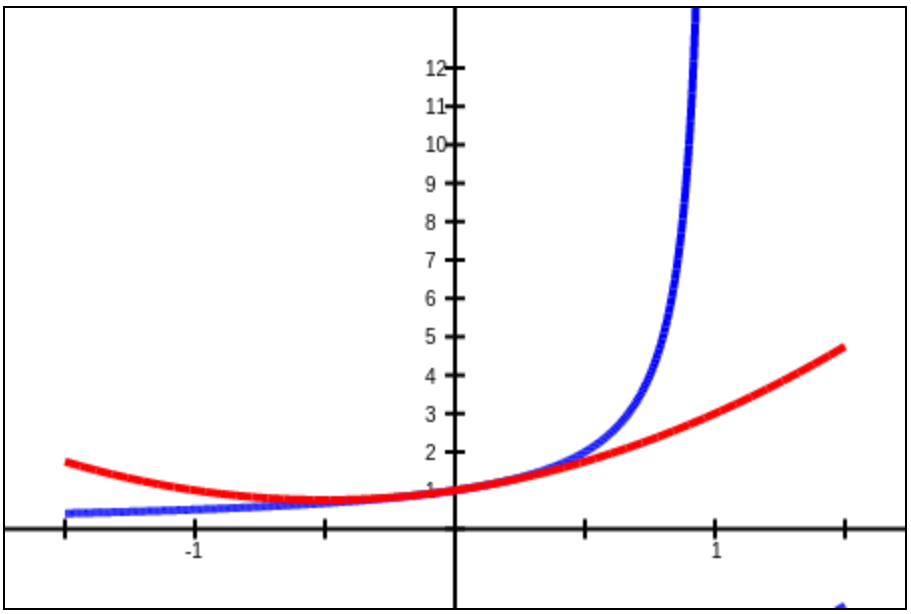


Figure 2.2.2: The graph shows a function and three approximations of it by partial sums of a power series.

Visualize the Power Series Approximation of $f(x) = \frac{1}{1-x}$

Move the slider to see how well $\sum_{n=0}^{N} x^n$ approximates $f(x) = \frac{1}{1-x}$.



N = 2

$S_2(x) = 1 + x + x^2$

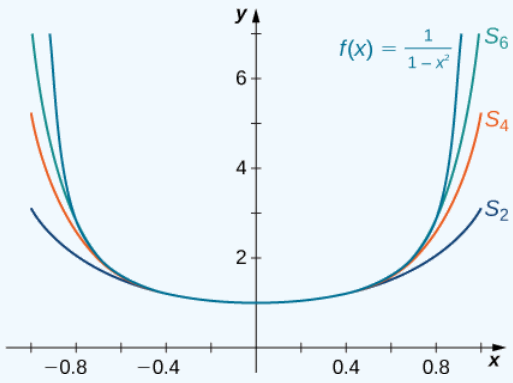
Exercise 2.2.2

Sketch a graph of $f(x) = \dfrac{1}{1-x^2}$ and the corresponding partial sums $S_N(x) = \sum_{n=0}^{N} x^{2n}$ for $N = 2, 4, 6$ on the interval $(-1, 1)$.

Hint

$$S_N(x) = 1 + x^2 + \ldots + x^{2N} = \frac{1 - x^{2(N+1)}}{1 - x^2}$$

Answer



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Next we consider functions involving an expression similar to the sum of a geometric series and show how to represent these functions using power series.

Example 2.2.3: Representing a Function with a Power Series

Use a power series to represent each of the following functions f. Find the interval of convergence.

a. $f(x)=\dfrac{1}{1+x^3}$

b. $f(x)=\dfrac{x^2}{4-x^2}$

Solution

a. You should recognize this function f as the sum of a geometric series, because

$$\frac{1}{1+x^3}=\frac{1}{1-(-x^3)}.$$

Using the fact that, for $|r|<1$, $\dfrac{a}{1-r}$ is the sum of the geometric series

$$\sum_{n=0}^{\infty} ar^n = a+ar+ar^2+\ldots,$$

we see that, for $|-x^3|<1$,

$$\begin{aligned}\frac{1}{1+x^3}&=\frac{1}{1-(-x^3)}\\ &=\sum_{n=0}^{\infty}(-x^3)^n\\ &=1-x^3+x^6-x^9+\ldots.\end{aligned}$$

Since this series converges if and only if $|-x^3|<1$, the interval of convergence is $(-1,1)$, and we have

$$\frac{1}{1+x^3}=1-x^3+x^6-x^9+\ldots for|x|<1.$$

b. This function is not in the exact form of a sum of a geometric series. However, with a little algebraic manipulation, we can relate f to a geometric series. By factoring 4 out of the two terms in the denominator, we obtain

$$\begin{aligned}\frac{x^2}{4-x^2}&=\frac{x^2}{4(\frac{1-x^2}{4})}\\ &=\frac{x^2}{4(1-(\frac{x}{2})^2)}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\frac{x^2}{4-x^2}&=\frac{x^2}{4(1-(\frac{x}{2})^2)}\\ &=\frac{\frac{x^2}{4}}{1-(\frac{x}{2})^2}\\ &=\sum_{n=0}^{\infty}\frac{x^2}{4}(\frac{x}{2})^{2n}.\end{aligned}$$

The series converges as long as $|(\frac{x}{2})^2| < 1$ (note that when $|(\frac{x}{2})^2| = 1$ the series does not converge). Solving this inequality, we conclude that the interval of convergence is $(-2, 2)$ and

$$\begin{aligned} \frac{x^2}{4-x^2} &= \sum_{n=0}^{\infty} \frac{x^{2n+2}}{4^{n+1}} \\ &= \frac{x^2}{4} + \frac{x^4}{4^2} + \frac{x^6}{4^3} + \dots \end{aligned}$$

for $|x| < 2$.





Exercise 2.2.3

Represent the function $f(x) = \dfrac{x^3}{2-x}$ using a power series and find the interval of convergence.

Hint

Rewrite f in the form $f(x) = \dfrac{g(x)}{1-h(x)}$ for some functions g and h.

Answer

$\displaystyle\sum_{n=0}^{\infty} \frac{x^{n+3}}{2^{n+1}}$ with interval of convergence $(-2, 2)$

In the remaining sections of this chapter, we will show ways of deriving power series representations for many other functions, and how we can make use of these representations to evaluate, differentiate, and integrate various functions.

Key Concepts

- For a power series centered at $x = a$, one of the following three properties hold:
  - i. The power series converges only at $x = a$. In this case, we say that the radius of convergence is $R = 0$.
  - ii. The power series converges for all real numbers x. In this case, we say that the radius of convergence is $R = \infty$.
  - iii. There is a real number R such that the series converges for $|x - a| < R$ and diverges for $|x - a| > R$. In this case, the radius of convergence is R.
- If a power series converges on a finite interval, the series may or may not converge at the endpoints.
- The ratio test may often be used to determine the radius of convergence.
- The geometric series $\displaystyle\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x| < 1$ allows us to represent certain functions using geometric series.

Key Equations

- **Power series centered at $x = 0$**

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots n$$

- **Power series centered at $x = a$**

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \ldots$$

Glossary

interval of convergence

the set of real numbers x for which a power series converges

power series

a series of the form $\displaystyle\sum_{n=0}^{\infty} c_n x^n$ is a power series centered at $x = 0$; a series of the form $\displaystyle\sum_{n=0}^{\infty} c_n (x-a)^n$ is a power series centered at $x = a$

radius of convergence

if there exists a real number $R > 0$ such that a power series centered at $x = a$ converges for $|x - a| < R$ and diverges for $|x - a| > R$, then R is the radius of convergence; if the power series only converges at $x = a$, the radius of convergence is $R = 0$; if the power series converges for all real numbers x, the radius of convergence is $R = \infty$

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2.2E: Exercises for Section 2.2

In exercises 1 - 4, state whether each statement is true, or give an example to show that it is false.

1) If $\sum_{n=1}^{\infty} a_n x^n$ converges, then $a_n x^n \to 0$ as $n \to \infty$.

Answer

True. If a series converges then its terms tend to zero.

2) $\sum_{n=1}^{\infty} a_n x^n$ converges at $x = 0$ for any real numbers a_n.

3) Given any sequence a_n, there is always some $R > 0$, possibly very small, such that $\sum_{n=1}^{\infty} a_n x^n$ converges on $(-R, R)$.

Answer

False. It would imply that $a_n x^n \to 0$ for $|x| < R$. If $a_n = n^n$, then $a_n x^n = (nx)^n$ does not tend to zero for any $x \neq 0$.

4) If $\sum_{n=1}^{\infty} a_n x^n$ has radius of convergence $R > 0$ and if $|b_n| \leq |a_n|$ for all n, then the radius of convergence of $\sum_{n=1}^{\infty} b_n x^n$ is greater than or equal to R.

5) Suppose that $\sum_{n=0}^{\infty} a_n (x-3)^n$ converges at $x = 6$. At which of the following points must the series also converge? Use the fact that if $\sum a_n (x-c)^n$ converges at x, then it converges at any point closer to c than x.

a. $x = 1$

b. $x = 2$

c. $x = 3$

d. $x = 0$

e. $x = 5.99$

f. $x = 0.000001$

Answer

It must converge on $(0, 6]$ and hence at: a. $x = 1$; b. $x = 2$; c. $x = 3$; d. $x = 0$; e. $x = 5.99$; and f. $x = 0.000001$

6) Suppose that $\sum_{n=0}^{\infty} a_n (x+1)^n$ converges at $x = -2$. At which of the following points must the series also converge? Use the fact that if $\sum a_n (x-c)^n$ converges at x, then it converges at any point closer to c than x.

a. $x = 2$

b. $x = -1$

c. $x = -3$

d. $x = 0$

e. $x = 0.99$

f. $x = 0.000001$

In the following exercises, suppose that $\left|\frac{a_{n+1}}{a_n}\right| \to 1$ as $n \to \infty$. Find the radius of convergence for each series.

7) $\sum_{n=0}^{\infty} a_n 2^n x^n$

Answer

$$\left|\frac{a_{n+1}2^{n+1}x^{n+1}}{a_n 2^n x^n}\right| = 2|x|\left|\frac{a_{n+1}}{a_n}\right| \to 2|x| \text{ so } R = \tfrac{1}{2}$$

8) $\sum_{n=0}^{\infty} \frac{a_n x^n}{2^n}$

9) $\sum_{n=0}^{\infty} \frac{a_n \pi^n x^n}{e^n}$

Answer

$$\left|\frac{a_{n+1}(\frac{\pi}{e})^{n+1}x^{n+1}}{a_n(\frac{\pi}{e})^n x^n}\right| = \frac{\pi|x|}{e}\left|\frac{a_{n+1}}{a_n}\right| \to \frac{\pi|x|}{e} \text{ so } R = \tfrac{e}{\pi}$$

10) $\sum_{n=0}^{\infty} \frac{a_n(-1)^n x^n}{10^n}$

11) $\sum_{n=0}^{\infty} a_n(-1)^n x^{2n}$

Answer

$$\left|\frac{a_{n+1}(-1)^{n+1}x^{2n+2}}{a_n(-1)^n x^{2n}}\right| = |x^2|\left|\frac{a_{n+1}}{a_n}\right| \to |x^2| \text{ so } R = 1$$

12) $\sum_{n=0}^{\infty} a_n(-4)^n x^{2n}$

In exercises 13 - 22, find the radius of convergence R and interval of convergence for $\sum a_n x^n$ with the given coefficients a_n.

13) $\sum_{n=1}^{\infty} \frac{(2x)^n}{n}$

Answer

$a_n = \frac{2^n}{n}$ so $\frac{a_{n+1}x}{a_n} \to 2x$. so $R = \frac{1}{2}$. When $x = \frac{1}{2}$ the series is harmonic and diverges. When $x = -\frac{1}{2}$ the series is alternating harmonic and converges. The interval of convergence is $I = \left[-\frac{1}{2}, \frac{1}{2}\right)$.

14) $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{\sqrt{n}}$

15) $\sum_{n=1}^{\infty} \frac{nx^n}{2^n}$

Answer

$a_n = \frac{n}{2^n}$ so $\frac{a_{n+1}x}{a_n} \to \frac{x}{2}$ so $R = 2$. When $x = \pm 2$ the series diverges by the divergence test. The interval of convergence is $I = (-2, 2)$.

16) $\sum_{n=1}^{\infty} \frac{nx^n}{e^n}$

17) $\sum_{n=1}^{\infty} \frac{n^2 x^n}{2^n}$

Answer

$a_n = \frac{n^2}{2^n}$ so $R = 2$. When $x = \pm 2$ the series diverges by the divergence test. The interval of convergence is $I = (-2, 2)$.

18) $\sum_{k=1}^{\infty} \frac{k^e x^k}{e^k}$

19) $\sum_{k=1}^{\infty} \frac{\pi^k x^k}{k^\pi}$

Answer

$a_k = \frac{\pi^k}{k^\pi}$ so $R = \frac{1}{\pi}$. When $x = \pm \frac{1}{\pi}$ the series is an absolutely convergent p-series. The interval of convergence is $I = \left[-\frac{1}{\pi}, \frac{1}{\pi}\right]$.

20) $\sum_{n=1}^{\infty} \frac{x^n}{n!}$

21) $\sum_{n=1}^{\infty} \frac{10^n x^n}{n!}$

Answer

$a_n = \frac{10^n}{n!}, \frac{a_{n+1} x}{a_n} = \frac{10x}{n+1} \to 0 < 1$ so the series converges for all x by the ratio test and $I = (-\infty, \infty)$.

22) $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{\ln(2n)}$

In exercises 23 - 28, find the radius of convergence of each series.

23) $\sum_{k=1}^{\infty} \frac{(k!)^2 x^k}{(2k)!}$

Answer

$a_k = \frac{(k!)^2}{(2k)!}$ so $\frac{a_{k+1}}{a_k} = \frac{(k+1)^2}{(2k+2)(2k+1)} \to \frac{1}{4}$ so $R = 4$

24) $\sum_{n=1}^{\infty} \frac{(2n)! x^n}{n^{2n}}$

25) $\sum_{k=1}^{\infty} \frac{k!}{1 \cdot 3 \cdot 5 \cdots (2k-1)} x^k$

Answer

$a_k = \frac{k!}{1 \cdot 3 \cdot 5 \cdots (2k-1)}$ so $\frac{a_{k+1}}{a_k} = \frac{k+1}{2k+1} \to \frac{1}{2}$ so $R = 2$

26) $\sum_{k=1}^{\infty} \frac{2 \cdot 4 \cdot 6 \cdots 2k}{(2k)!} x^k$

27) $\sum_{n=1}^{\infty} \frac{x^n}{\binom{2n}{n}}$ where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Answer

$$a_n = \frac{1}{\binom{2n}{n}} \text{ so } \frac{a_{n+1}}{a_n} = \frac{\left((n+1)!\right)^2}{(2n+2)!}\frac{2n!}{(n!)^2} = \frac{(n+1)^2}{(2n+2)(2n+1)} \to \frac{1}{4} \text{ so } R = 4$$

28) $\sum_{n=1}^{\infty} \sin^2 n x^n$

In exercises 29 - 32, use the ratio test to determine the radius of convergence of each series.

29) $\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!} x^n$

Answer

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^3}{(3n+3)(3n+2)(3n+1)} \to \frac{1}{27} \text{ so } R = 27$$

30) $\sum_{n=1}^{\infty} \frac{2^{3n}(n!)^3}{(3n)!} x^n$

31) $\sum_{n=1}^{\infty} \frac{n!}{n^n} x^n$

Answer

$$a_n = \frac{n!}{n^n} \text{ so } \frac{a_{n+1}}{a_n} = \frac{(n+1)!}{n!}\frac{n^n}{(n+1)^{n+1}} = \left(\frac{n}{n+1}\right)^n \to \frac{1}{e} \text{ so } R = e$$

32) $\sum_{n=1}^{\infty} \frac{(2n)!}{n^{2n}} x^n$

In the following exercises, given that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ with convergence in $(-1,1)$, find the power series for each function with the given center a, and identify its interval of convergence.

33) $f(x) = \frac{1}{x}; a = 1$ (Hint: $\frac{1}{x} = \frac{1}{1-(1-x)}$)

Answer

$$f(x) = \sum_{n=0}^{\infty} (1-x)^n \text{ on } I = (0,2)$$

34) $f(x) = \frac{1}{1-x^2}; a = 0$

35) $f(x) = \frac{x}{1-x^2}; a = 0$

Answer

$$\sum_{n=0}^{\infty} x^{2n+1} \text{ on } I = (-1,1)$$

36) $f(x) = \frac{1}{1+x^2}; a = 0$

37) $f(x) = \frac{x^2}{1+x^2}; a = 0$

Answer

$\sum_{n=0}^{\infty}(-1)^n x^{2n+2}$ on $I=(-1,1)$

38) $f(x)=\dfrac{1}{2-x}; a=1$

39) $f(x)=\dfrac{1}{1-2x}; a=0.$

Answer

$\sum_{n=0}^{\infty}2^n x^n$ on $\left(-\frac{1}{2},\frac{1}{2}\right)$

40) $f(x)=\dfrac{1}{1-4x^2}; a=0$

41) $f(x)=\dfrac{x^2}{1-4x^2}; a=0$

Answer

$\sum_{n=0}^{\infty}4^n x^{2n+2}$ on $\left(-\frac{1}{2},\frac{1}{2}\right)$

42) $f(x)=\dfrac{x^2}{5-4x+x^2}; a=2$

Use the result of exercise 43 to find the radius of convergence of the given series in the subsequent exercises (44 - 47).

43) Explain why, if $|a_n|^{1/n}\to r>0,$ then $|a_n x^n|^{1/n}\to |x|r<1$ whenever $|x|<\frac{1}{r}$ and, therefore, the radius of convergence of $\sum_{n=1}^{\infty}a_n x^n$ is $R=\frac{1}{r}$.

Answer

$|a_n x^n|^{1/n}=|a_n|^{1/n}|x|\to |x|r$ as $n\to\infty$ and $|x|r<1$ when $|x|<\frac{1}{r}$. Therefore, $\sum_{n=1}^{\infty}a_n x^n$ converges when $|x|<\frac{1}{r}$ by the nth root test.

44) $\sum_{n=1}^{\infty}\dfrac{x^n}{n^n}$

45) $\sum_{k=1}^{\infty}\left(\dfrac{k-1}{2k+3}\right)^k x^k$

Answer

$a_k=\left(\dfrac{k-1}{2k+3}\right)^k$ so $(a_k)^{1/k}\to\frac{1}{2}<1$ so $R=2$

46) $\sum_{k=1}^{\infty}(\dfrac{2k^2-1}{k^2+3})^k x^k$

47) $\sum_{n=1}^{\infty}a_n=(n^{1/n}-1)^n x^n$

Answer

$a_n=(n^{1/n}-1)^n$ so $(a_n)^{1/n}\to 0$ so $R=\infty$

48) Suppose that $p(x)=\sum_{n=0}^{\infty} a_n x^n$ such that $a_n=0$ if n is even. Explain why $p(x)=p(-x)$.

49) Suppose that $p(x)=\sum_{n=0}^{\infty} a_n x^n$ such that $a_n=0$ if n is odd. Explain why $p(x)=-p(-x)$.

Answer

We can rewrite $p(x)=\sum_{n=0}^{\infty} a_{2n+1}x^{2n+1}$ and $p(x)=p(-x)$ since $x^{2n+1}=-(-x)^{2n+1}$.

50) Suppose that $p(x)=\sum_{n=0}^{\infty} a_n x^n$ converges on $(-1,1]$. Find the interval of convergence of $p(Ax)$.

51) Suppose that $p(x)=\sum_{n=0}^{\infty} a_n x^n$ converges on $(-1,1]$. Find the interval of convergence of $p(2x-1)$.

Answer

If $x\in[0,1]$, then $y=2x-1\in[-1,1]$ so $p(2x-1)=p(y)=\sum_{n=0}^{\infty} a_n y^n$ converges.

In the following exercises, suppose that $p(x)=\sum_{n=0}^{\infty} a_n x^n$ satisfies $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=1$ where $a_n\geq 0$ for each n. State whether each series converges on the full interval $(-1,1)$, or if there is not enough information to draw a conclusion. Use the comparison test when appropriate.

52) $\sum_{n=0}^{\infty} a_n x^{2n}$

53) $\sum_{n=0}^{\infty} a_{2n} x^{2n}$

Answer

Converges on $(-1,1)$ by the ratio test

54) $\sum_{n=0}^{\infty} a_{2n} x^n$ (Hint:$x=\pm\sqrt{x^2}$)

55) $\sum_{n=0}^{\infty} a_{n^2} x^{n^2}$ (Hint: Let $b_k=a_k$ if $k=n^2$ for some n, otherwise $b_k=0$.)

Answer

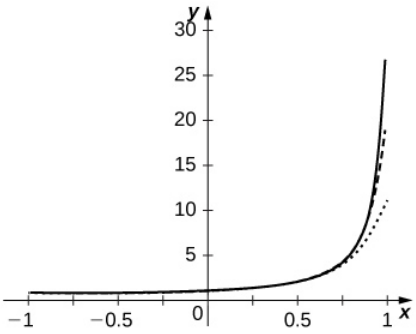
Consider the series $\sum b_k x^k$ where $b_k=a_k$ if $k=n^2$ and $b_k=0$ otherwise. Then $b_k\leq a_k$ and so the series converges on $(-1,1)$ by the comparison test.

56) Suppose that $p(x)$ is a polynomial of degree N. Find the radius and interval of convergence of $\sum_{n=1}^{\infty} p(n)x^n$.

57) [T] Plot the graphs of $\frac{1}{1-x}$ and of the partial sums $S_N=\sum_{n=0}^{N} x^n$ for $n=10,20,30$ on the interval $[-0.99,0.99]$ Comment on the approximation of $\frac{1}{1-x}$ by S_N near $x=-1$ and near $x=1$ as N increases.

Answer

The approximation is more accurate near $x = -1$. The partial sums follow $\dfrac{1}{1-x}$ more closely as N increases but are never accurate near $x = 1$ since the series diverges there.

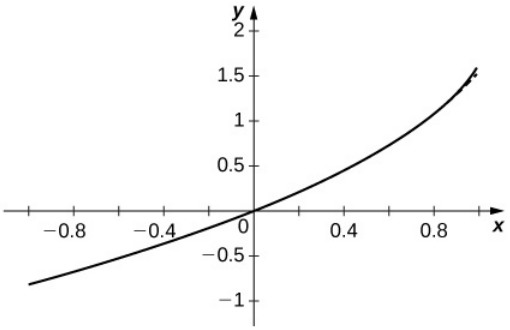


58) [T] Plot the graphs of $-\ln(1-x)$ and of the partial sums $S_N = \displaystyle\sum_{n=1}^{N} \frac{x^n}{n}$ for $n = 10, 50, 100$ on the interval $[-0.99, 0.99]$ Comment on the behavior of the sums near $x = -1$ and near $x = 1$ as N increases.

59) [T] Plot the graphs of the partial sums $S_n = \displaystyle\sum_{n=1}^{N} \frac{x^n}{n^2}$ for $n = 10, 50, 100$ on the interval $[-0.99, 0.99]$ Comment on the behavior of the sums near $x = -1$ and near $x = 1$ as N increases.

Answer

The approximation appears to stabilize quickly near both $x = \pm 1$.



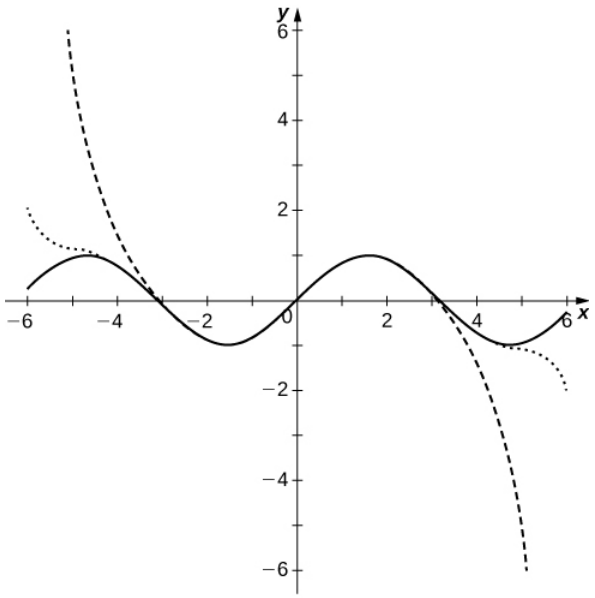
60) [T] Plot the graphs of the partial sums $S_N = \displaystyle\sum_{n=1}^{N} (\sin n) x^n$ for $n = 10, 50, 100$ on the interval $[-0.99, 0.99]$ Comment on the behavior of the sums near $x = -1$ and near $x = 1$ as N increases.

61) [T] Plot the graphs of the partial sums $S_N = \displaystyle\sum_{n=0}^{N} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ for $n = 3, 5, 10$ on the interval $[-2\pi, 2\pi]$ Comment on how these plots approximate $\sin x$ as N increases.

Answer

The polynomial curves have roots close to those of $\sin x$ up to their degree and then the polynomials diverge from $\sin x$.

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62) [T] Plot the graphs of the partial sums $S_N = \sum_{n=0}^{N} (-1)^n \frac{x^{2n}}{(2n)!}$ for $n = 3, 5, 10$ on the interval $[-2\pi, 2\pi]$. Comment on how these plots approximate $\cos x$ as N increases.

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2.3: Properties of Power Series

Learning Objectives

- Combine power series by addition or subtraction.
- Create a new power series by multiplication by a power of the variable or a constant, or by substitution.
- Multiply two power series together.
- Differentiate and integrate power series term-by-term.

In the preceding section on power series and functions we showed how to represent certain functions using power series. In this section we discuss how power series can be combined, differentiated, or integrated to create new power series. This capability is particularly useful for a couple of reasons. First, it allows us to find power series representations for certain elementary functions, by writing those functions in terms of functions with known power series. For example, given the power series representation for *[Math Processing Error]*, we can find a power series representation for *[Math Processing Error]*. Second, being able to create power series allows us to define new functions that cannot be written in terms of elementary functions. This capability is particularly useful for solving differential equations for which there is no solution in terms of elementary functions.

Combining Power Series

If we have two power series with the same interval of convergence, we can add or subtract the two series to create a new power series, also with the same interval of convergence. Similarly, we can multiply a power series by a power of *[Math Processing Error]* or evaluate a power series at *[Math Processing Error]* for a positive integer *[Math Processing Error]* to create a new power series. Being able to do this allows us to find power series representations for certain functions by using power series representations of other functions. For example, since we know the power series representation for *[Math Processing Error]*, we can find power series representations for related functions, such as

[Math Processing Error]

and

[Math Processing Error]

In Theorem *[Math Processing Error]*, we state results regarding addition or subtraction of power series, composition of a power series, and multiplication of a power series by a power of the variable. For simplicity, we state the theorem for power series centered at *[Math Processing Error]*. Similar results hold for power series centered at *[Math Processing Error]*.

Theorem *[Math Processing Error]*: Combining Power Series

Suppose that the two power series *[Math Processing Error]* and *[Math Processing Error]* converge to the functions *[Math Processing Error]* and *[Math Processing Error]*, respectively, on a common interval *[Math Processing Error]*.

i. The power series *[Math Processing Error]* converges to *[Math Processing Error]* on *[Math Processing Error]*.
ii. For any integer *[Math Processing Error]* and any real number *[Math Processing Error]*, the power series *[Math Processing Error]* converges to *[Math Processing Error]* on *[Math Processing Error]*.
iii. For any integer *[Math Processing Error]* and any real number *[Math Processing Error]*, the series *[Math Processing Error]* converges to *[Math Processing Error]* for all *[Math Processing Error]* such that *[Math Processing Error]* is in *[Math Processing Error]*.

Proof

We prove *[Math Processing Error]*. in the case of the series *[Math Processing Error]*. Suppose that *[Math Processing Error]* and *[Math Processing Error]* converge to the functions *[Math Processing Error]* and *[Math Processing Error]*, respectively, on the interval *[Math Processing Error]*. Let *[Math Processing Error]* be a point in *[Math Processing Error]* and let *[Math Processing Error]* and *[Math Processing Error]* denote the *[Math Processing Error]* partial sums of the series *[Math Processing Error]* and *[Math Processing Error]*, respectively. Then the sequence *[Math Processing Error]* converges to *[Math Processing Error]* and the sequence *[Math Processing Error]* converges to *[Math Processing Error]*. Furthermore, the *[Math Processing Error]* partial sum of *[Math Processing Error]* is

[Math Processing Error]

Because

[Math Processing Error]

we conclude that the series *[Math Processing Error]* converges to *[Math Processing Error]*

□

We examine products of power series in a later theorem. First, we show several applications of Theorem *[Math Processing Error]* and how to find the interval of convergence of a power series given the interval of convergence of a related power series.

✔ Example *[Math Processing Error]*: Combining Power Series

Suppose that *[Math Processing Error]* is a power series whose interval of convergence is *[Math Processing Error]*, and suppose that *[Math Processing Error]* is a power series whose interval of convergence is *[Math Processing Error]*

a. Find the interval of convergence of the series *[Math Processing Error]*
b. Find the interval of convergence of the series *[Math Processing Error]*

Solution

a. Since the interval *[Math Processing Error]* is a common interval of convergence of the series *[Math Processing Error]* and *[Math Processing Error]*, the interval of convergence of the series *[Math Processing Error]* is *[Math Processing Error]*.
b. Since *[Math Processing Error]* is a power series centered at zero with radius of convergence *[Math Processing Error]* it converges for all *[Math Processing Error]* in the interval *[Math Processing Error]* By Theorem *[Math Processing Error]*, the series *[Math Processing Error]* converges if *[Math Processing Error]* is in the interval *[Math Processing Error]*. Therefore, the series converges for all *[Math Processing Error]* in the interval *[Math Processing Error]*

? Exercise *[Math Processing Error]*

Suppose that *[Math Processing Error]* has an interval of convergence of *[Math Processing Error]*. Find the interval of convergence of *[Math Processing Error]*.

Hint

Find the values of *[Math Processing Error]* such that *[Math Processing Error]* is in the interval *[Math Processing Error]*

Answer

Interval of convergence is *[Math Processing Error]*

In the next example, we show how to use Theorem *[Math Processing Error]* and the power series for a function *[Math Processing Error]* to construct power series for functions related to *[Math Processing Error]*. Specifically, we consider functions related to the function *[Math Processing Error]* and we use the fact that

[Math Processing Error]

for *[Math Processing Error]*

✔ Example *[Math Processing Error]*: Constructing Power Series from Known Power Series

Use the power series representation for *[Math Processing Error]* combined with Theorem *[Math Processing Error]* to construct a power series for each of the following functions. Find the interval of convergence of the power series.

a. *[Math Processing Error]*
b. *[Math Processing Error]*

Solution

a. First write *[Math Processing Error]* as

[Math Processing Error]

Using the power series representation for [Math Processing Error] and parts ii. and iii. of Theorem [Math Processing Error], we find that a power series representation for [Math Processing Error] is given by

[Math Processing Error]

Since the interval of convergence of the series for [Math Processing Error] is [Math Processing Error], the interval of convergence for this new series is the set of real numbers [Math Processing Error] such that [Math Processing Error]. Therefore, the interval of convergence is [Math Processing Error]

b. To find the power series representation, use partial fractions to write [Math Processing Error] as the sum of two fractions. We have

[Math Processing Error]

First, using part ii. of Theorem [Math Processing Error], we obtain

[Math Processing Error]

Then, using parts ii. and iii. of Theorem [Math Processing Error], we have

[Math Processing Error]

Since we are combining these two power series, the interval of convergence of the difference must be the smaller of these two intervals. Using this fact and part i. of Theorem [Math Processing Error], we have

[Math Processing Error]

where the interval of convergence is [Math Processing Error].

? Exercise [Math Processing Error]

Use the series for [Math Processing Error] on [Math Processing Error] to construct a series for [Math Processing Error] Determine the interval of convergence.

Hint

Use partial fractions to rewrite [Math Processing Error] as the difference of two fractions.

Answer

[Math Processing Error]. The interval of convergence is [Math Processing Error].

In Example [Math Processing Error], we showed how to find power series for certain functions. In Example [Math Processing Error] we show how to do the opposite: given a power series, determine which function it represents.

✔ Example [Math Processing Error]: Finding the Function Represented by a Given Power Series

Consider the power series [Math Processing Error] Find the function [Math Processing Error] represented by this series. Determine the interval of convergence of the series.

Solution

Writing the given series as

[Math Processing Error]

we can recognize this series as the power series for

[Math Processing Error]

Since this is a geometric series, the series converges if and only if [Math Processing Error] Therefore, the interval of convergence is [Math Processing Error]

? Exercise *[Math Processing Error]*

Find the function represented by the power series *[Math Processing Error]*.

Determine its interval of convergence.

Hint

Write *[Math Processing Error]*.

Answer

[Math Processing Error] The interval of convergence is *[Math Processing Error]*.

Recall the questions posed in the chapter opener about which is the better way of receiving payouts from lottery winnings. We now revisit those questions and show how to use series to compare values of payments over time with a lump sum payment today. We will compute how much future payments are worth in terms of today's dollars, assuming we have the ability to invest winnings and earn interest. The value of future payments in terms of today's dollars is known as the *present value* of those payments.

✓ Example *[Math Processing Error]*: Present Value of Future Winnings

Suppose you win the lottery and are given the following three options:

- Receive 20 million dollars today;
- Receive 1.5 million dollars per year over the next 20 years; or
- Receive 1 million dollars per year indefinitely (being passed on to your heirs).

Which is the best deal, assuming that the annual interest rate is 5%? We answer this by working through the following sequence of questions.

a. How much is the 1.5 million dollars received annually over the course of 20 years worth in terms of today's dollars, assuming an annual interest rate of 5%?
b. Use the answer to part a. to find a general formula for the **present value** of payments of *[Math Processing Error]* dollars received each year over the next n years, assuming an average annual interest rate *[Math Processing Error]*.
c. Find a formula for the present value if annual payments of *[Math Processing Error]* dollars continue indefinitely, assuming an average annual interest rate *[Math Processing Error]*.
d. Use the answer to part c. to determine the present value of 1 million dollars paid annually indefinitely.
e. Use your answers to parts a. and d. to determine which of the three options is best.



Figure *[Math Processing Error]*: (credit: modification of work by Robert Huffstutter, Flickr)

Solution

a. Consider the payment of 1.5 million dollars made at the end of the first year. If you were able to receive that payment today instead of one year from now, you could invest that money and earn 5% interest. Therefore, the present value of that money *[Math Processing Error]* satisfies *[Math Processing Error]* **million dollars.** We conclude that

[Math Processing Error] **million dollars.**

Similarly, consider the payment of 1.5 million dollars made at the end of the second year. If you were able to receive that payment today, you could invest that money for two years, earning 5% interest, compounded annually. Therefore, the present value of that money *[Math Processing Error]* satisfies *[Math Processing Error]* **million dollars**. We conclude that

[Math Processing Error] **million dollars.**

The value of the future payments today is the sum of the present values *[Math Processing Error]* of each of those annual payments. The present value *[Math Processing Error]* satisfies

[Math Processing Error].

Therefore,

[Math Processing Error] **million dollars.**

b. Using the result from part a. we see that the present value *[Math Processing Error]* of *[Math Processing Error]* dollars paid annually over the course of *[Math Processing Error]* years, assuming an annual interest rate *[Math Processing Error]*, is given by

[Math Processing Error] **dollars.**

c. Using the result from part b. we see that the present value of an annuity that continues indefinitely is given by the infinite series

[Math Processing Error]

We can view the present value as a power series in *[Math Processing Error]*, which converges as long as *[Math Processing Error]*. Since *[Math Processing Error]*, this series converges. Rewriting the series as

[Math Processing Error]

we recognize this series as the power series for

[Math Processing Error].

We conclude that the present value of this annuity is

[Math Processing Error]

d. From the result to part c. we conclude that the present value *[Math Processing Error]* of *[Math Processing Error]* **million dollars** paid out every year indefinitely, assuming an annual interest rate *[Math Processing Error]*, is given by

[Math Processing Error] **million dollars.**

e. From part a. we see that receiving $1.5 million dollars over the course of 20 years is worth $18.693 million dollars in today's dollars. From part d. we see that receiving $1 million dollars per year indefinitely is worth $20 million dollars in today's dollars. Therefore, either receiving a lump-sum payment of $20 million dollars today or receiving $1 million dollars indefinitely have the same present value.

Multiplication of Power Series

We can also create new power series by multiplying power series. Being able to multiply two power series provides another way of finding power series representations for functions. The way we multiply them is similar to how we multiply polynomials. For example, suppose we want to multiply

[Math Processing Error]

and

[Math Processing Error]

It appears that the product should satisfy

[Math Processing Error]

In Theorem *[Math Processing Error]*, we state the main result regarding multiplying power series, showing that if *[Math Processing Error]* and *[Math Processing Error]* converge on a common interval *[Math Processing Error]*, then we can multiply the series in this way, and the resulting series also converges on the interval *[Math Processing Error]*.

Theorem [Math Processing Error]: Multiplying Power Series

Suppose that the power series [Math Processing Error] and [Math Processing Error] converge to [Math Processing Error] and [Math Processing Error], respectively, on a common interval [Math Processing Error]. Let

[Math Processing Error]

Then

[Math Processing Error]

and

[Math Processing Error]

The series [Math Processing Error] is known as the *Cauchy product* of the series [Math Processing Error] and [Math Processing Error].

We omit the proof of this theorem, as it is beyond the level of this text and is typically covered in a more advanced course. We now provide an example of this theorem by finding the power series representation for

[Math Processing Error]

using the power series representations for

[Math Processing Error].

Example [Math Processing Error]: Multiplying Power Series

Multiply the power series representation

[Math Processing Error]

for [Math Processing Error] with the power series representation

[Math Processing Error]

for [Math Processing Error] to construct a power series for [Math Processing Error] on the interval [Math Processing Error].

Solution

We need to multiply

[Math Processing Error]

Writing out the first several terms, we see that the product is given by

[Math Processing Error]

Since the series for [Math Processing Error] and [Math Processing Error] both converge on the interval [Math Processing Error], the series for the product also converges on the interval [Math Processing Error].

Exercise [Math Processing Error]

Multiply the series [Math Processing Error] by itself to construct a series for [Math Processing Error]

Hint

Multiply the first few terms of [Math Processing Error]

Answer

[Math Processing Error]

Differentiating and Integrating Power Series

Consider a power series *[Math Processing Error]* that converges on some interval *[Math Processing Error]*, and let *[Math Processing Error]* be the function defined by this series. Here we address two questions about *[Math Processing Error]*.

- Is *[Math Processing Error]* differentiable, and if so, how do we determine the derivative *[Math Processing Error]*?
- How do we evaluate the indefinite integral *[Math Processing Error]*?

We know that, for a polynomial with a finite number of terms, we can evaluate the derivative by differentiating each term separately. Similarly, we can evaluate the indefinite integral by integrating each term separately. Here we show that we can do the same thing for convergent power series. That is, if

[Math Processing Error]

converges on some interval *[Math Processing Error]*, then

[Math Processing Error]

and

[Math Processing Error]

Evaluating the derivative and indefinite integral in this way is called **term-by-term differentiation of a power series** and **term-by-term integration of a power series**, respectively. The ability to differentiate and integrate power series term-by-term also allows us to use known power series representations to find power series representations for other functions. For example, given the power series for *[Math Processing Error]*, we can differentiate term-by-term to find the power series for *[Math Processing Error]*. Similarly, using the power series for *[Math Processing Error]*, we can integrate term-by-term to find the power series for *[Math Processing Error]*, an antiderivative of *[Math Processing Error]*. We show how to do this in Example *[Math Processing Error]* and Example *[Math Processing Error]*. First, we state Theorem *[Math Processing Error]*, which provides the main result regarding differentiation and integration of power series.

Theorem *[Math Processing Error]*: Term-by-Term Differentiation and Integration for Power Series

Suppose that the power series *[Math Processing Error]* converges on the interval *[Math Processing Error]* for some *[Math Processing Error]*. Let *[Math Processing Error]* be the function defined by the series

[Math Processing Error]

for *[Math Processing Error]*. Then *[Math Processing Error]* is differentiable on the interval *[Math Processing Error]* and we can find *[Math Processing Error]* by differentiating the series term-by-term:

[Math Processing Error]

for *[Math Processing Error]* Also, to find *[Math Processing Error]*, we can integrate the series term-by-term. The resulting series converges on *[Math Processing Error]* and we have

[Math Processing Error]

for *[Math Processing Error]*

The proof of this result is beyond the scope of the text and is omitted. Note that although Theorem *[Math Processing Error]* guarantees the same radius of convergence when a power series is differentiated or integrated term-by-term, it says nothing about what happens at the endpoints. It is possible that the differentiated and integrated power series have different behavior at the endpoints than does the original series. We see this behavior in the next examples.

Example *[Math Processing Error]*: Differentiating Power Series

a. Use the power series representation *[Math Processing Error]* for *[Math Processing Error]* to find a power series representation for *[Math Processing Error]* on the interval *[Math Processing Error]* Determine whether the resulting series converges at the endpoints.
b. Use the result of part a. to evaluate the sum of the series *[Math Processing Error]*.

Solution

a. Since *[Math Processing Error]* is the derivative of *[Math Processing Error]*, we can find a power series representation for *[Math Processing Error]* by differentiating the power series for *[Math Processing Error]* term-by-term. The result is

[Math Processing Error]

for *[Math Processing Error]*

Theorem *[Math Processing Error]* does not guarantee anything about the behavior of this series at the endpoints. Testing the endpoints by using the divergence test, we find that the series diverges at both endpoints *[Math Processing Error]*. Note that this is the same result found in Example *[Math Processing Error]*.

b. From part a. we know that

[Math Processing Error]

Therefore,

[Math Processing Error]

? Exercise *[Math Processing Error]*

Differentiate the series *[Math Processing Error]* term-by-term to find a power series representation for *[Math Processing Error]* on the interval *[Math Processing Error]*.

Hint

Write out the first several terms and apply the power rule.

Answer

[Math Processing Error]

✓ Example *[Math Processing Error]*: Integrating Power Series

For each of the following functions *[Math Processing Error]*, find a power series representation for *[Math Processing Error]* by integrating the power series for *[Math Processing Error]* and find its interval of convergence.

a. *[Math Processing Error]*
b. *[Math Processing Error]*

Solution:

a. For *[Math Processing Error]*,the derivative is *[Math Processing Error]*. We know that

[Math Processing Error]

for *[Math Processing Error]*. To find a power series for *[Math Processing Error]*, we integrate the series term-by-term.

[Math Processing Error]

Since *[Math Processing Error]* is an antiderivative of *[Math Processing Error]*, it remains to solve for the constant *[Math Processing Error]*. Since *[Math Processing Error]*, we have *[Math Processing Error]*. Therefore, a power series representation for *[Math Processing Error]* is

[Math Processing Error]

Theorem *[Math Processing Error]* does not guarantee anything about the behavior of this power series at the endpoints. However, checking the endpoints, we find that at *[Math Processing Error]* the series is the alternating harmonic series, which converges. Also, at *[Math Processing Error]*, the series is the harmonic series, which diverges. It is important to note that, even though this series converges at *[Math Processing Error]*, Theorem *[Math Processing Error]* does not guarantee that the series actually converges to *[Math Processing Error]*. In fact, the series does converge to *[Math Processing Error]*, but showing this fact requires more advanced techniques. (Abel's theorem, covered in more advanced texts, deals with this more technical point.) The interval of convergence is *[Math Processing Error]*.

b. The derivative of *[Math Processing Error]* is *[Math Processing Error]*. We know that

[Math Processing Error]

for [Math Processing Error]. To find a power series for [Math Processing Error], we integrate this series term-by-term.

[Math Processing Error]

Since [Math Processing Error], we have [Math Processing Error]. Therefore, a power series representation for [Math Processing Error] is

[Math Processing Error]

for [Math Processing Error]. Again, Theorem [Math Processing Error] does not guarantee anything about the convergence of this series at the endpoints. However, checking the endpoints and using the alternating series test, we find that the series converges at [Math Processing Error] and [Math Processing Error]. As discussed in part a., using Abel's theorem, it can be shown that the series actually converges to [Math Processing Error] and [Math Processing Error] at [Math Processing Error] and [Math Processing Error], respectively. Thus, the interval of convergence is [Math Processing Error].

? Exercise [Math Processing Error]

Integrate the power series [Math Processing Error] term-by-term to evaluate [Math Processing Error]

Hint

Use the fact that [Math Processing Error] is an antiderivative of [Math Processing Error].

Answer

[Math Processing Error]

Up to this point, we have shown several techniques for finding power series representations for functions. However, how do we know that these power series are unique? That is, given a function [Math Processing Error] and a power series for [Math Processing Error] at [Math Processing Error], is it possible that there is a different power series for [Math Processing Error] at [Math Processing Error] that we could have found if we had used a different technique? The answer to this question is no. This fact should not seem surprising if we think of power series as polynomials with an infinite number of terms. Intuitively, if

[Math Processing Error]

for all values [Math Processing Error] in some open interval [Math Processing Error] about zero, then the coefficients [Math Processing Error] should equal [Math Processing Error] for [Math Processing Error]. We now state this result formally.

Theorem [Math Processing Error]: Uniqueness of Power Series

Let [Math Processing Error] and [Math Processing Error] be two convergent power series such that

[Math Processing Error]

for all [Math Processing Error] in an open interval containing [Math Processing Error]. Then [Math Processing Error] for all [Math Processing Error].

Proof

Let

[Math Processing Error]

Then [Math Processing Error] By Theorem [Math Processing Error], we can differentiate both series term-by-term. Therefore,

[Math Processing Error]

and thus, [Math Processing Error] Similarly,

[Math Processing Error]

implies that *[Math Processing Error]* and therefore, *[Math Processing Error]*. More generally, for any integer *[Math Processing Error]* and consequently, *[Math Processing Error]* for all *[Math Processing Error]*

□

In this section we have shown how to find power series representations for certain functions using various algebraic operations, differentiation, or integration. At this point, however, we are still limited as to the functions for which we can find power series representations. Next, we show how to find power series representations for many more functions by introducing Taylor series.

Key Concepts

- Given two power series *[Math Processing Error]* and *[Math Processing Error]* that converge to functions *[Math Processing Error]* and *[Math Processing Error]* on a common interval *[Math Processing Error]*, the sum and difference of the two series converge to *[Math Processing Error]*, respectively, on *[Math Processing Error]*. In addition, for any real number *[Math Processing Error]* and integer *[Math Processing Error]*, the series *[Math Processing Error]* converges to *[Math Processing Error]* and the series *[Math Processing Error]* converges to *[Math Processing Error]* whenever *[Math Processing Error]* is in the interval *[Math Processing Error]*.
- Given two power series that converge on an interval *[Math Processing Error]* the Cauchy product of the two power series converges on the interval *[Math Processing Error]*.
- Given a power series that converges to a function *[Math Processing Error]* on an interval *[Math Processing Error]*, the series can be differentiated term-by-term and the resulting series converges to *[Math Processing Error]* on *[Math Processing Error]*. The series can also be integrated term-by-term and the resulting series converges to *[Math Processing Error]* on *[Math Processing Error]*.

Glossary

term-by-term differentiation of a power series

a technique for evaluating the derivative of a power series *[Math Processing Error]* by evaluating the derivative of each term separately to create the new power series *[Math Processing Error]*

term-by-term integration of a power series

a technique for integrating a power series *[Math Processing Error]* by integrating each term separately to create the new power series *[Math Processing Error]*

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2.3E: Exercises for Section 2.3

1) If $f(x)=\sum_{n=0}^{\infty}\frac{x^n}{n!}$ and $g(x)=\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{n!}$, find the power series of $\frac{1}{2}\big(f(x)+g(x)\big)$ and of $\frac{1}{2}\big(f(x)-g(x)\big)$.

Answer

$$\frac{1}{2}\big(f(x)+g(x)\big)=\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}\quad\text{and}\quad\frac{1}{2}\big(f(x)-g(x)\big)=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}\ .$$

2) If $C(x)=\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$ and $S(x)=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}$, find the power series of $C(x)+S(x)$ and of $C(x)-S(x)$.

In exercises 3 - 6, use partial fractions to find the power series of each function.

3) $\dfrac{4}{(x-3)(x+1)}$

Answer

$$\frac{4}{(x-3)(x+1)}=\frac{1}{x-3}-\frac{1}{x+1}=-\frac{1}{3(1-\frac{x}{3})}-\frac{1}{1-(-x)}=-\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n-\sum_{n=0}^{\infty}(-1)^nx^n=\sum_{n=0}^{\infty}\left((-1)^{n+1}-\frac{1}{3^{n+1}}\right)x^n$$

4) $\dfrac{3}{(x+2)(x-1)}$

5) $\dfrac{5}{(x^2+4)(x^2-1)}$

Answer

$$\frac{5}{(x^2+4)(x^2-1)}=\frac{1}{x^2-1}-\frac{1}{4}\frac{1}{1+\left(\frac{x}{2}\right)^2}=-\sum_{n=0}^{\infty}x^{2n}-\frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\left(\frac{x}{2}\right)^n=\sum_{n=0}^{\infty}\left((-1)+(-1)^{n+1}\frac{1}{2^{n+2}}\right)x^{2n}$$

6) $\dfrac{30}{(x^2+1)(x^2-9)}$

In exercises 7 - 10, express each series as a rational function.

7) $\sum_{n=1}^{\infty}\frac{1}{x^n}$

Answer

$$\frac{1}{x}\sum_{n=0}^{\infty}\frac{1}{x^n}=\frac{1}{x}\cdot\frac{1}{1-\frac{1}{x}}=\frac{1}{x-1}$$

8) $\sum_{n=1}^{\infty}\frac{1}{x^{2n}}$

9) $\sum_{n=1}^{\infty}\frac{1}{(x-3)^{2n-1}}$

Answer

$$\frac{1}{x-3}\cdot\frac{1}{1-\frac{1}{(x-3)^2}}=\frac{x-3}{(x-3)^2-1}$$

10) $\sum_{n=1}^{\infty}\left(\frac{1}{(x-3)^{2n-1}}-\frac{1}{(x-2)^{2n-1}}\right)$

Exercises 11 - 16 explore applications of *annuities*.

11) Calculate the present values P of an annuity in which \$10,000 is to be paid out annually for a period of 20 years, assuming interest rates of $r=0.03$, $r=0.05$, and $r=0.07$.

Answer

$P = P_1 + \cdots + P_{20}$ where $P_k = 10,000\dfrac{1}{(1+r)^k}$. Then $P = 10,000\displaystyle\sum_{k=1}^{20}\frac{1}{(1+r)^k} = 10,000\frac{1-(1+r)^{-20}}{r}$. When $r = 0.03,\ P \approx 10,000 \times 14.8775 = 148,775$When $r = 0.05,\ P \approx 10,000 \times 12.4622 = 124,622$When $r = 0.07,\ P \approx 105,940$

12) Calculate the present values P of annuities in which \$9,000 is to be paid out annually perpetually, assuming interest rates of $r = 0.03,\ r = 0.05$ and $r = 0.07$.

13) Calculate the annual payouts C to be given for 20 years on annuities having present value \$100,000 assuming respective interest rates of $r = 0.03,\ r = 0.05,$ and $r = 0.07$.

Answer

In general, $P = \dfrac{C(1-(1+r)^{-N})}{r}$ for N years of payouts, or $C = \dfrac{Pr}{1-(1+r)^{-N}}$. For $N = 20$ and $P = 100,000$ one has $C = 6721.57$when $r = 0.03;\ C = 8024.26$when $r = 0.05$; and $C \approx 9439.29$when $r = 0.07$.

14) Calculate the annual payouts C to be given perpetually on annuities having present value \$100,000 assuming respective interest rates of $r = 0.03,\ r = 0.05,$ and $r = 0.07$.

15) Suppose that an annuity has a present value $P = 1$ **million dollars.** What interest rate r would allow for perpetual annual payouts of \$50,000?

Answer

In general, $P = \dfrac{C}{r}$. Thus, $r = \dfrac{C}{P} = 5 \times \frac{10^4}{10^6} = 0.05$.

16) Suppose that an annuity has a present value $P = 10$ **million dollars.** What interest rate r would allow for perpetual annual payouts of \$100,000?

In exercises 17 - 20, express the sum of each power series in terms of geometric series, and then express the sum as a rational function.

17) $x + x^2 - x^3 + x^4 + x^5 - x^6 + \cdots$ (Hint: Group powers x^{3k}, x^{3k-1}, and x^{3k-2}.)

Answer

$$(x + x^2 - x^3)(1 + x^3 + x^6 + \cdots) = \frac{x + x^2 - x^3}{1 - x^3}$$

18) $x + x^2 - x^3 - x^4 + x^5 + x^6 - x^7 - x^8 + \cdots$ (Hint: Group powers x^{4k}, x^{4k-1}, etc.)

19) $x - x^2 - x^3 + x^4 - x^5 - x^6 + x^7 - \cdots$ (Hint: Group powers x^{3k}, x^{3k-1}, and x^{3k-2}.)

Answer

$$(x - x^2 - x^3)(1 + x^3 + x^6 + \cdots) = \frac{x - x^2 - x^3}{1 - x^3}$$

20) $\dfrac{x}{2} + \dfrac{x^2}{4} - \dfrac{x^3}{8} + \dfrac{x^4}{16} + \dfrac{x^5}{32} - \dfrac{x^6}{64} + \cdots$ (Hint: Group powers $\left(\dfrac{x}{2}\right)^{3k}$, $\left(\dfrac{x}{2}\right)^{3k-1}$, and $\left(\dfrac{x}{2}\right)^{3k-2}$.)

In exercises 21 - 24, find the power series of $f(x)g(x)$ given f and g as defined.

21) $f(x) = 2\displaystyle\sum_{n=0}^{\infty} x^n, g(x) = \sum_{n=0}^{\infty} nx^n$

Answer

$a_n = 2,\ b_n = n$ so $c_n = \displaystyle\sum_{k=0}^{n} b_k a_{n-k} = 2\sum_{k=0}^{n} k = (n)(n+1)$ and $f(x)g(x) = \displaystyle\sum_{n=1}^{\infty} n(n+1)x^n$

22) $f(x) = \displaystyle\sum_{n=1}^{\infty} x^n,\ g(x) = \sum_{n=1}^{\infty} \frac{1}{n}x^n$. Express the coefficients of $f(x)g(x)$ in terms of $H_n = \displaystyle\sum_{k=1}^{n}\frac{1}{k}$.

23) $f(x) = g(x) = \displaystyle\sum_{n=1}^{\infty}\left(\frac{x}{2}\right)^n$

Answer

$a_n = b_n = 2^{-n}$ so $c_n = \displaystyle\sum_{k=1}^{n} b_k a_{n-k} = 2^{-n}\sum_{k=1}^{n} 1 = \frac{n}{2^n}$ and $f(x)g(x) = \displaystyle\sum_{n=1}^{\infty} n\left(\frac{x}{2}\right)^n$

24) $f(x) = g(x) = \sum_{n=1}^{\infty} nx^n$

In exercises 25 - 26, differentiate the given series expansion of f term-by-term to obtain the corresponding series expansion for the derivative of f.

25) $f(x) = \dfrac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n$

Answer

The derivative of f is $-\dfrac{1}{(1+x)^2} = -\sum_{n=0}^{\infty}(-1)^n(n+1)x^n$.

26) $f(x) = \dfrac{1}{1-x^2} = \sum_{n=0}^{\infty} x^{2n}$

In exercises 27 - 28, integrate the given series expansion of f term-by-term from zero to x to obtain the corresponding series expansion for the indefinite integral of f.

27) $f(x) = \dfrac{2x}{(1+x^2)^2} = \sum_{n=1}^{\infty}(-1)^n(2n)x^{2n-1}$

Answer

The indefinite integral of f is $\dfrac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n}$.

28) $f(x) = \dfrac{2x}{1+x^2} = 2\sum_{n=0}^{\infty}(-1)^n x^{2n+1}$

In exercises 29 - 32, evaluate each infinite series by identifying it as the value of a derivative or integral of geometric series.

29) Evaluate $\sum_{n=1}^{\infty}\dfrac{n}{2^n}$ as $f'\left(\frac{1}{2}\right)$ where $f(x) = \sum_{n=0}^{\infty} x^n$.

Answer

$$f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x};\ f'(\frac{1}{2}) = \sum_{n=1}^{\infty}\frac{n}{2^{n-1}} = \frac{d}{dx}(1-x)^{-1}\Big|_{x=1/2} = \frac{1}{(1-x)^2}\Big|_{x=1/2} = 4 \quad \text{so} \sum_{n=1}^{\infty}\frac{n}{2^n} = 2.$$

30) Evaluate $\sum_{n=1}^{\infty}\dfrac{n}{3^n}$ as $f'\left(\frac{1}{3}\right)$ where $f(x) = \sum_{n=0}^{\infty} x^n$.

31) Evaluate $\sum_{n=2}^{\infty}\dfrac{n(n-1)}{2^n}$ as $f''\left(\frac{1}{2}\right)$ where $f(x) = \sum_{n=0}^{\infty} x^n$.

Answer

$$f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x};\ f''\left(\frac{1}{2}\right) = \sum_{n=2}^{\infty}\frac{n(n-1)}{2^{n-2}} = \frac{d^2}{dx^2}(1-x)^{-1}\Big|_{x=1/2} = \frac{2}{(1-x)^3}\Big|_{x=1/2} = 16 \quad \text{so} \sum_{n=2}^{\infty} n\frac{(n-1)}{2^n} = 4.$$

32) Evaluate $\sum_{n=0}^{\infty}\dfrac{(-1)^n}{n+1}$ as $\int_0^1 f(t)\,dt$ where $f(x) = \sum_{n=0}^{\infty}(-1)^n x^{2n} = \dfrac{1}{1+x^2}$.

In exercises 33 - 39, given that $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, use term-by-term differentiation or integration to find power series for each function centered at the given point.

33) $f(x) = \ln x$ centered at $x = 1$ (Hint: $x = 1-(1-x)$)

Answer

$$\int\sum(1-x)^n\,dx = \int\sum(-1)^n(x-1)^n\,dx = \sum\frac{(-1)^n(x-1)^{n+1}}{n+1}$$

34) $\ln(1-x)$ at $x = 0$

35) $\ln(1-x^2)$ at $x=0$

Answer

$$-\int_{t=0}^{x^2}\frac{1}{1-t}dt=-\sum_{n=0}^{\infty}\int_0^{x^2}t^n dx-\sum_{n=0}^{\infty}\frac{x^{2(n+1)}}{n+1}=-\sum_{n=1}^{\infty}\frac{x^{2n}}{n}$$

36) $f(x)=\dfrac{2x}{(1-x^2)^2}$ at $x=0$

37) $f(x)=\tan^{-1}(x^2)$ at $x=0$

Answer

$$\int_0^{x^2}\frac{dt}{1+t^2}=\sum_{n=0}^{\infty}(-1)^n\int_0^{x^2}t^{2n}dt=\sum_{n=0}^{\infty}(-1)^n\frac{t^{2n+1}}{2n+1}\,|_{t=0}^{x^2}=\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+2}}{2n+1}$$

38) $f(x)=\ln(1+x^2)$ at $x=0$

39) $f(x)=\displaystyle\int_0^x \ln t\,dt$ where $\ln(x)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(x-1)^n}{n}$

Answer

Term-by-term integration gives

$$\int_0^x \ln t\,dt=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(x-1)^{n+1}}{n(n+1)}=\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{1}{n}-\frac{1}{n+1}\right)(x-1)^{n+1}=(x-1)\ln x+\sum_{n=2}^{\infty}(-1)^n\frac{(x-1)^n}{n}=x\ln x-x.$$

40) [T] Evaluate the power series expansion $\ln(1+x)=\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$ at $x=1$ to show that $\ln(2)$ is the sum of the alternating harmonic series. Use the alternating series test to determine how many terms of the sum are needed to estimate $\ln(2)$ accurate to within 0.001, and find such an approximation.

41) [T] Subtract the infinite series of $\ln(1-x)$ from $\ln(1+x)$ to get a power series for $\ln\left(\dfrac{1+x}{1-x}\right)$. Evaluate at $x=\frac{1}{3}$. What is the smallest N such that the Nth partial sum of this series approximates $\ln(2)$ with an error less than 0.001?

Answer

We have $\ln(1-x)=-\displaystyle\sum_{n=1}^{\infty}\frac{x^n}{n}$ so $\ln(1+x)=\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$. Thus, $\ln\left(\dfrac{1+x}{1-x}\right)=\displaystyle\sum_{n=1}^{\infty}\left(1+(-1)^{n-1}\right)\frac{x^n}{n}=2\sum_{n=1}^{\infty}\frac{x^{2n-1}}{2n-1}$.

When $x=\frac{1}{3}$ we obtain $\ln(2)=2\displaystyle\sum_{n=1}^{\infty}\frac{1}{3^{2n-1}(2n-1)}$. We have $2\displaystyle\sum_{n=1}^{3}\frac{1}{3^{2n-1}(2n-1)}=0.69300\ldots$, while

$2\displaystyle\sum_{n=1}^{4}\frac{1}{3^{2n-1}(2n-1)}=0.69313\ldots$ and $\ln(2)=0.69314\ldots$; therefore, $N=4$.

In exercises 42 - 45, using a substitution if indicated, express each series in terms of elementary functions and find the radius of convergence of the sum.

42) $\displaystyle\sum_{k=0}^{\infty}(x^k-x^{2k+1})$

43) $\displaystyle\sum_{k=1}^{\infty}\frac{x^{3k}}{6k}$

Answer

$\displaystyle\sum_{k=1}^{\infty}\frac{x^k}{k}=-\ln(1-x)$ so $\displaystyle\sum_{k=1}^{\infty}\frac{x^{3k}}{6k}=-\frac{1}{6}\ln(1-x^3)$. The radius of convergence is equal to 1 by the ratio test.

44) $\displaystyle\sum_{k=1}^{\infty}(1+x^2)^{-k}$ using $y=\dfrac{1}{1+x^2}$

45) $\displaystyle\sum_{k=1}^{\infty}2^{-kx}$ using $y=2^{-x}$

Answer

If $y = 2^{-x}$, then $\sum_{k=1}^{\infty} y^k = \frac{y}{1-y} = \frac{2^{-x}}{1-2^{-x}} = \frac{1}{2^x - 1}$. If $a_k = 2^{-kx}$, then $\frac{a_{k+1}}{a_k} = 2^{-x} < 1$ when $x > 0$. So the series converges for all $x > 0$.

46) Show that, up to powers x^3 and y^3, $E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ satisfies $E(x+y) = E(x)E(y)$.

47) Differentiate the series $E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ term-by-term to show that $E(x)$ is equal to its derivative.

Answer

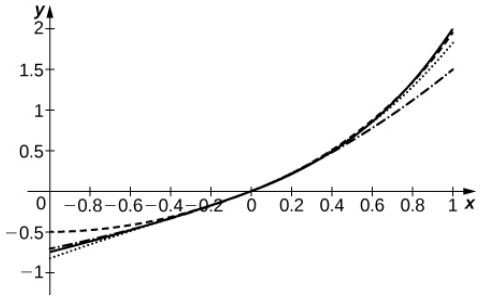
Answers will vary.

48) Show that if $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is a sum of even powers, that is, $a_n = 0$ if n is odd, then $F = \int_0^x f(t)\,dt$ is a sum of odd powers, while if I is a sum of odd powers, then F is a sum of even powers.

49) [T] Suppose that the coefficients an of the series $\sum_{n=0}^{\infty} a_n x^n$ are defined by the recurrence relation $a_n = \frac{a_{n-1}}{n} + \frac{a_{n-2}}{n(n-1)}$. For $a_0 = 0$ and $a_1 = 1$, compute and plot the sums $S_N = \sum_{n=0}^{N} a_n x^n$ for $N = 2, 3, 4, 5$ on $[-1, 1]$.

Answer

The solid curve is S_5. The dashed curve is S_2, dotted is S_3, and dash-dotted is S_4



50) [T] Suppose that the coefficients an of the series $\sum_{n=0}^{\infty} a_n x^n$ are defined by the recurrence relation $a_n = \frac{a_{n-1}}{\sqrt{n}} - \frac{a_{n-2}}{\sqrt{n(n-1)}}$. For $a_0 = 1$ and $a_1 = 0$, compute and plot the sums $S_N = \sum_{n=0}^{N} a_n x^n$ for $N = 2, 3, 4, 5$ on $[-1, 1]$.

51) [T] Given the power series expansion $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$, determine how many terms N of the sum evaluated at $x = -1/2$ are needed to approximate $\ln(2)$ accurate to within $1/1000$. Evaluate the corresponding partial sum $\sum_{n=1}^{N} (-1)^{n-1} \frac{x^n}{n}$.

Answer

When $x = -\frac{1}{2}$, $-\ln(2) = \ln\left(\frac{1}{2}\right) = -\sum_{n=1}^{\infty} \frac{1}{n2^n}$. Since $\sum_{n=11}^{\infty} \frac{1}{n2^n} < \sum_{n=11}^{\infty} \frac{1}{2^n} = \frac{1}{2^{10}}$, one has $\sum_{n=1}^{10} \frac{1}{n2^n} = 0.69306\ldots$ whereas $\ln(2) = 0.69314\ldots$; therefore, $N = 10$.

52) [T] Given the power series expansion $\tan^{-1}(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}$, use the alternating series test to determine how many terms N of the sum evaluated at $x = 1$ are needed to approximate $\tan^{-1}(1) = \frac{\pi}{4}$ accurate to within $1/1000$. Evaluate the corresponding partial sum $\sum_{k=0}^{N} (-1)^k \frac{x^{2k+1}}{2k+1}$.

53) [T] Recall that $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)=\frac{\pi}{6}$. Assuming an exact value of $\frac{1}{\sqrt{3}}$), estimate $\frac{\pi}{6}$ by evaluating partial sums $S_N\left(\frac{1}{\sqrt{3}}\right)$ of the power series expansion $\tan^{-1}(x)=\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{2k+1}$ at $x=\frac{1}{\sqrt{3}}$. What is the smallest number N such that $6S_N\left(\frac{1}{\sqrt{3}}\right)$ approximates π accurately to within 0.001? How many terms are needed for accuracy to within 0.00001?

Answer

$6S_N\left(\frac{1}{\sqrt{3}}\right)=2\sqrt{3}\sum_{n=0}^{N}(-1)^n\frac{1}{3^n(2n+1).}$ One has $\pi-6S_4\left(\frac{1}{\sqrt{3}}\right)=0.00101\ldots$ and $\pi-6S_5\left(\frac{1}{\sqrt{3}}\right)=0.00028\ldots$ so $N=5$ is the smallest partial sum with accuracy to within 0.001.Also, $\pi-6S_7\left(\frac{1}{\sqrt{3}}\right)=0.00002\ldots$ while $\pi-6S_8\left(\frac{1}{\sqrt{3}}\right)=-0.000007\ldots$ so $N=8$ is the smallest N to give accuracy to within 0.00001.

Contributors

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2.4: Taylor and Maclaurin Series

Learning Objectives

- Describe the procedure for finding a Taylor polynomial of a given order for a function.
- Explain the meaning and significance of Taylor's theorem with remainder.
- Estimate the remainder for a Taylor series approximation of a given function.

In the previous two sections we discussed how to find power series representations for certain types of functions—specifically, functions related to geometric series. Here we discuss power series representations for other types of functions. In particular, we address the following questions: Which functions can be represented by power series and how do we find such representations? If we can find a power series representation for a particular function f and the series converges on some interval, how do we prove that the series actually converges to f?

Overview of Taylor/Maclaurin Series

Consider a function f that has a power series representation at $x=a$. Then the series has the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots. \tag{2.4.1}$$

What should the coefficients be? For now, we ignore issues of convergence, but instead focus on what the series should be, if one exists. We return to discuss convergence later in this section. If the series Equation 2.4.1 is a representation for f at $x=a$, we certainly want the series to equal $f(a)$ at $x=a$. Evaluating the series at $x=a$, we see that

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(a-a) + c_2(a-a)^2 + \cdots = c_0. \tag{2.4.2}$$

Thus, the series equals $f(a)$ if the coefficient $c_0 = f(a)$. In addition, we would like the first derivative of the power series to equal $f'(a)$ at $x=a$. Differentiating Equation 2.4.2 term-by-term, we see that

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} c_n(x-a)^n\right) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \ldots. \tag{2.4.3}$$

Therefore, at $x=a$, the derivative is

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} c_n(x-a)^n\right) = c_1 + 2c_2(a-a) + 3c_3(a-a)^2 + \cdots = c_1. \tag{2.4.4}$$

Therefore, the derivative of the series equals $f'(a)$ if the coefficient $c_1 = f'(a)$. Continuing in this way, we look for coefficients c_n such that all the derivatives of the power series Equation 2.4.4 will agree with all the corresponding derivatives of f at $x=a$. The second and third derivatives of Equation 2.4.3 are given by

$$\frac{d^2}{dx^2}\left(\sum_{n=0}^{\infty} c_n(x-a)^n\right) = 2c_2 + 3\cdot 2c_3(x-a) + 4\cdot 3c_4(x-a)^2 + \ldots \tag{2.4.5}$$

and

$$\frac{d^3}{dx^3}\left(\sum_{n=0}^{\infty} c_n(x-a)^n\right) = 3\cdot 2c_3 + 4\cdot 3\cdot 2c_4(x-a) + 5\cdot 4\cdot 3c_5(x-a)^2 + \cdots. \tag{2.4.6}$$

Therefore, at $x=a$, the second and third derivatives

$$\frac{d^2}{dx^2}\left(\sum_{n=0}^{\infty} c_n(x-a)^n\right) = 2c_2 + 3\cdot 2c_3(a-a) + 4\cdot 3c_4(a-a)^2 + \cdots = 2c_2 \tag{2.4.7}$$

and

$$\frac{d^3}{dx^3}\left(\sum_{n=0}^{\infty} c_n(x-a)^n\right) = 3\cdot 2c_3 + 4\cdot 3\cdot 2c_4(a-a) + 5\cdot 4\cdot 3c_5(a-a)^2 + \cdots = 3\cdot 2c_3 \tag{2.4.8}$$

equal $f''(a)$ and $f'''(a)$, respectively, if $c_2 = \dfrac{f''(a)}{2}$ and $c_3 = \dfrac{f'''(a)}{3\cdot 2}$. More generally, we see that if f has a power series representation at $x=a$, then the coefficients should be given by $c_n = \dfrac{f^{(n)}(a)}{n!}$. That is, the series should be

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \tag{2.4.9}$$

This power series for f is known as the Taylor series for f at a. If $x = 0$, then this series is known as the Maclaurin series for f.

Definition 2.4.1: Maclaurin and Taylor series

If f has derivatives of all orders at $x = a$, then the *Taylor series* for the function f at a is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots \tag{2.4.10}$$

The Taylor series for f at 0 is known as the *Maclaurin series* for f.

Later in this section, we will show examples of finding Taylor series and discuss conditions under which the Taylor series for a function will converge to that function. Here, we state an important result. Recall that power series representations are unique. Therefore, if a function f has a power series at a, then it must be the Taylor series for f at a.

Uniqueness of Taylor Series

If a function f has a power series at a that converges to f on some open interval containing a, then that power series is the Taylor series for f at a.

The proof follows directly from that discussed previously.

To determine if a Taylor series converges, we need to look at its sequence of partial sums. These partial sums are finite polynomials, known as **Taylor polynomials.**

Taylor Polynomials

The nth partial sum of the Taylor series for a function f at a is known as the nth-degree Taylor polynomial. For example, the 0th, 1st, 2nd, and 3rd partial sums of the Taylor series are given by

$$\begin{aligned} p_0(x) &= f(a) \\ p_1(x) &= f(a) + f'(a)(x-a) \\ p_2(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 \\ p_3(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 \end{aligned}$$

respectively. These partial sums are known as the 0th, 1st, 2nd, and 3rd degree Taylor polynomials of f at a, respectively. If $x = a$, then these polynomials are known as *Maclaurin polynomials* for f. We now provide a formal definition of Taylor and Maclaurin polynomials for a function f.

Definition 2.4.2: Maclaurin polynomial

If f has n derivatives at $x = a$, then the nth-degree Taylor polynomial of f at a is

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n. \tag{2.4.11}$$

The nth-degree Taylor polynomial for f at 0 is known as the nth-degree Maclaurin polynomial for f.

We now show how to use this definition to find several Taylor polynomials for $f(x) = \ln x$ at $x = 1$.

Example 2.4.1: Finding Taylor Polynomials

Find the Taylor polynomials p_0, p_1, p_2 and p_3 for $f(x) = \ln x$ at $x = 1$. Use a graphing utility to compare the graph of f with the graphs of p_0, p_1, p_2 and p_3.

Solution

To find these Taylor polynomials, we need to evaluate f and its first three derivatives at $x = 1$.

$$f(x) = \ln x \; f(1) = 0$$

$$f'(x) = \frac{1}{x} \; f'(1) = 1$$

$$f''(x) = -\frac{1}{x^2} \quad f''(1) = -1$$

$$f'''(x) = \frac{2}{x^3} \quad f'''(1) = 2$$

Therefore,

$$\begin{aligned}
p_0(x) &= f(1) = 0, \\
p_1(x) &= f(1) + f'(1)(x-1) = x - 1, \\
p_2(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 = (x-1) - \frac{1}{2}(x-1)^2 \\
p_3(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3
\end{aligned}$$

The graphs of $y = f(x)$ and the first three Taylor polynomials are shown in Figure 2.4.1.

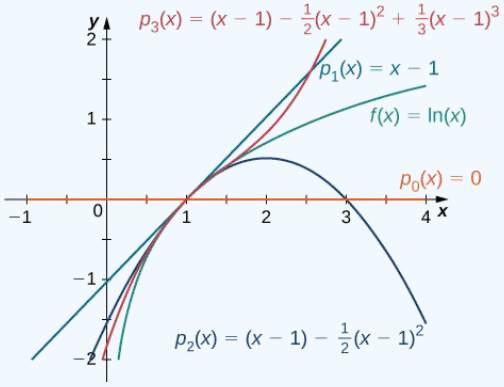
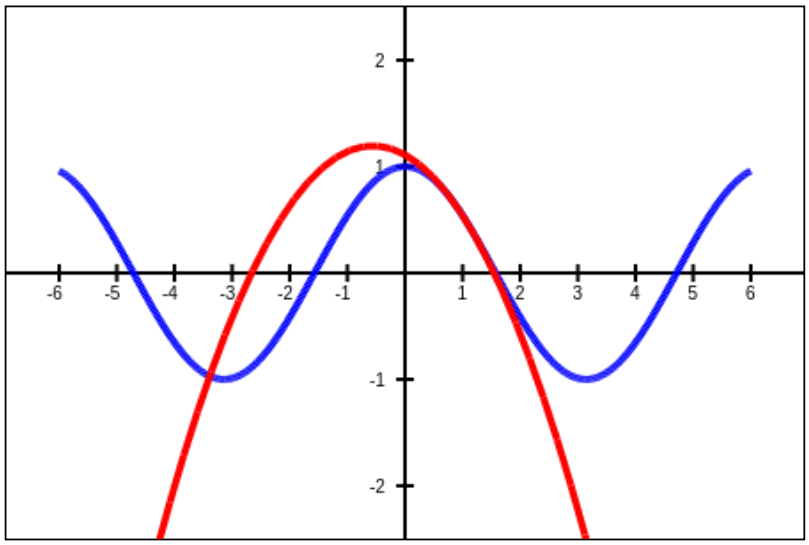


Figure 2.4.1: The function $y = \ln x$ and the Taylor polynomials p_0, p_1, p_2 and p_3 at $x = 1$ are plotted on this graph.

Below is the Nth degree Taylor polynomial for $f(x) = \cos(x)$ at $x = a$. Move the sliders to change N and a.



$p_2(x) = \cos(1.0)/0!(x - 1.0)^0 + -\sin(1.0)/1!(x - 1.0)^1 + -\cos(1.0)/2!(x - 1.0)^2$

N = 2

a = 1





Exercise 2.4.1

Find the Taylor polynomials p_0, p_1, p_2 and p_3 for $f(x) = \dfrac{1}{x^2}$ at $x = 1$.

Hint

Find the first three derivatives of f and evaluate them at $x = 1$.

Answer

$$p_0(x) = 1 \tag{2.4.12}$$

$$p_1(x) = 1 - 2(x-1) \tag{2.4.13}$$

$$p_2(x) = 1 - 2(x-1) + 3(x-1)^2 \tag{2.4.14}$$

$$p_3(x) = 1 - 2(x-1) + 3(x-1)^2 - 4(x-1)^3 \tag{2.4.15}$$

We now show how to find Maclaurin polynomials for e^x, $\sin x$, and $\cos x$. As stated above, Maclaurin polynomials are Taylor polynomials centered at zero.

Example 2.4.2: Finding Maclaurin Polynomials

For each of the following functions, find formulas for the Maclaurin polynomials p_0, p_1, p_2 and p_3. Find a formula for the nth-degree Maclaurin polynomial and write it using sigma notation. Use a graphing utility to compare the graphs of p_0, p_1, p_2 and p_3 with f.

a. $f(x) = e^x$
b. $f(x) = \sin x$
c. $f(x) = \cos x$

Solution

Since $f(x) = e^x$,we know that $f(x) = f'(x) = f''(x) = \cdots = f^{(n)}(x) = e^x$ for all positive integers n. Therefore,

$$f(0) = f'(0) = f''(0) = \cdots = f^{(n)}(0) = 1$$

for all positive integers n. Therefore, we have

$$p_0(x) = f(0) = 1,$$
$$p_1(x) = f(0) + f'(0)x = 1 + x,$$
$$p_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 = 1 + x + \frac{1}{2}x^2 \ ,$$
$$p_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 \ ,$$
$$p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} = \sum_{k=0}^{n} \frac{x^k}{k!} \ .$$

The function and the first three Maclaurin polynomials are shown in Figure 2.

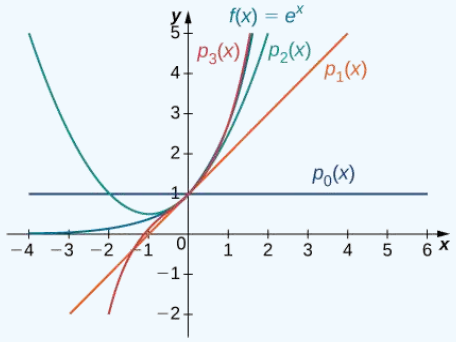


Figure 2.4.2: *The graph shows the function $y = e^x$ and the Maclaurin polynomials p_0, p_1, p_2 and p_3.*

b. For $f(x) = \sin x$, the values of the function and its first four derivatives at $x = 0$ are given as follows:

$$f(x) = \sin x \ f(0) = 0$$
$$f'(x) = \cos x \ f'(0) = 1$$
$$f''(x) = -\sin x \ f''(0) = 0$$
$$f'''(x) = -\cos x \ f'''(0) = -1$$
$$f^{(4)}(x) = \sin x \ f^{(4)}(0) = 0.$$

Since the fourth derivative is $\sin x$, the pattern repeats. That is, $f^{(2m)}(0) = 0$ and $f^{(2m+1)}(0) = (-1)^m$ for $m \geq 0$. Thus, we have

$$p_0(x) = 0,$$
$$p_1(x) = 0 + x = x,$$
$$p_2(x) = 0 + x + 0 = x,$$
$$p_3(x) = 0 + x + 0 - \frac{1}{3!}x^3 = x - \frac{x^3}{3!},$$
$$p_4(x) = 0 + x + 0 - \frac{1}{3!}x^3 + 0 = x - \frac{x^3}{3!} \ ,$$
$$p_5(x) = 0 + x + 0 - \frac{1}{3!}x^3 + 0 + \frac{1}{5!}x^5 = x - \frac{x^3}{3!} + \frac{x^5}{5!} \ ,$$

and for $m \geq 0$,

$$p_{2m+1}(x) = p_{2m+2}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^m \frac{x^{2m+1}}{(2m+1)!} = \sum_{k=0}^{m} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \quad .$$

Graphs of the function and its Maclaurin polynomials are shown in Figure 3.

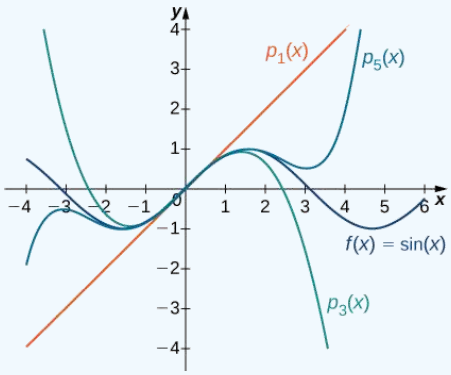


Figure 2.4.3: *The graph shows the function $y = \sin x$ and the Maclaurin polynomials p_1, p_3 and p_5.*

c. For $f(x) = \cos x$, the values of the function and its first four derivatives at $x = 0$ are given as follows:

$$f(x) = \cos x \; f(0) = 1$$
$$f'(x) = -\sin x \; f'(0) = 0$$
$$f''(x) = -\cos x \; f''(0) = -1$$
$$f'''(x) = \sin x \; f'''(0) = 0$$
$$f^{(4)}(x) = \cos x \; f^{(4)}(0) = 1.$$

Since the fourth derivative is $\sin x$, the pattern repeats. In other words, $f^{(2m)}(0) = (-1)^m$ and $f^{(2m+1)} = 0$ for $m \geq 0$. Therefore,

$$p_0(x) = 1,$$
$$p_1(x) = 1 + 0 = 1,$$
$$p_2(x) = 1 + 0 - \frac{1}{2!}x^2 = 1 - \frac{x^2}{2!} \quad ,$$
$$p_3(x) = 1 + 0 - \frac{1}{2!}x^2 + 0 = 1 - \frac{x^2}{2!} \quad ,$$
$$p_4(x) = 1 + 0 - \frac{1}{2!}x^2 + 0 + \frac{1}{4!}x^4 = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \quad ,$$
$$p_5(x) = 1 + 0 - \frac{1}{2!}x^2 + 0 + \frac{1}{4!}x^4 + 0 = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \quad ,$$

and for $n \geq 0$,

$$p_{2m}(x) = p_{2m+1}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^m \frac{x^{2m}}{(2m)!} = \sum_{k=0}^{m} (-1)^k \frac{x^{2k}}{(2k)!} \quad .$$

Graphs of the function and the Maclaurin polynomials appear in Figure 4.

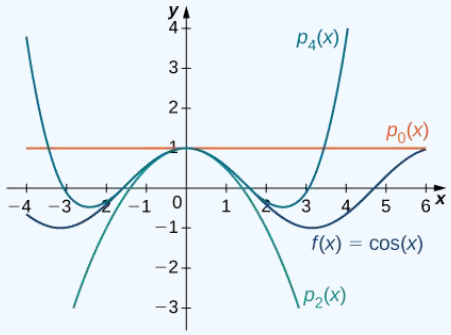


Figure 2.4.4: *The function $y = \cos x$ and the Maclaurin polynomials p_0, p_2 and p_4 are plotted on this graph.*

Exercise 2.4.2

Find formulas for the Maclaurin polynomials p_0, p_1, p_2 and p_3 for $f(x)=\dfrac{1}{1+x}$.

Find a formula for the nth-degree Maclaurin polynomial. Write your answer using sigma notation.

Hint

Evaluate the first four derivatives of f and look for a pattern.

Answer

$$p_0(x)=1;\ p_1(x)=1-x;\ p_2(x)=1-x+x^2;\ p_3(x)=1-x+x^2-x^3;\ p_n(x)=1-x+x^2-x^3+\cdots+(-1)^n x^n=\sum_{k=0}^{n}(-1)^k x^k$$

Taylor's Theorem with Remainder

Recall that the nth-degree Taylor polynomial for a function f at a is the nth partial sum of the Taylor series for f at a. Therefore, to determine if the Taylor series converges, we need to determine whether the sequence of Taylor polynomials p_n converges. However, not only do we want to know if the sequence of Taylor polynomials converges, we want to know if it converges to f. To answer this question, we define the remainder $R_n(x)$ as

$$R_n(x)=f(x)-p_n(x). \tag{2.4.16}$$

For the sequence of Taylor polynomials to converge to f, we need the remainder R_n to converge to zero. To determine if R_n converges to zero, we introduce **Taylor's theorem with remainder**. Not only is this theorem useful in proving that a Taylor series converges to its related function, but it will also allow us to quantify how well the nth-degree Taylor polynomial approximates the function.

Here we look for a bound on $|R_n|$. Consider the simplest case: $n=0$. Let p_0 be the 0th Taylor polynomial at a for a function f. The remainder R_0 satisfies

$$R_0(x)=f(x)-p_0(x)=f(x)-f(a).$$

If f is differentiable on an interval I containing a and x, then by the Mean Value Theorem there exists a real number c between a and x such that $f(x)-f(a)=f'(c)(x-a)$. Therefore,

$$R_0(x)=f'(c)(x-a). \tag{2.4.17}$$

Using the Mean Value Theorem in a similar argument, we can show that if f is n times differentiable on an interval I containing a and x, then the nth remainder R_n satisfies

$$R_n(x)=\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \tag{2.4.18}$$

for some real number c between a and x. It is important to note that the value c in the numerator above is not the center a, but rather an unknown value c between a and x. This formula allows us to get a bound on the remainder R_n. If we happen to know that $\left|f^{(n+1)}(x)\right|$ is bounded by some real number M on this interval I, then

$$|R_n(x)|\le\frac{M}{(n+1)!}|x-a|^{n+1} \tag{2.4.19}$$

for all x in the interval I.

We now state Taylor's theorem, which provides the formal relationship between a function f and its nth-degree Taylor polynomial $p_n(x)$. This theorem allows us to bound the error when using a Taylor polynomial to approximate a function value, and will be important in proving that a Taylor series for f converges to f.

Taylor's Theorem with Remainder

Let f be a function that can be differentiated $n+1$ times on an interval I containing the real number a. Let p_n be the nth-degree Taylor polynomial of f at a and let

$$R_n(x)=f(x)-p_n(x) \tag{2.4.20}$$

be the nth remainder. Then for each x in the interval I, there exists a real number c between a and x such that

$$R_n(x)=\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \tag{2.4.21}$$

.

If there exists a real number M such that $\left|f^{(n+1)}(x)\right|\le M$ for all $x\in I$, then

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \tag{2.4.22}$$

for all x in I.

Proof

Fix a point $x \in I$ and introduce the function g such that

$$g(t) = f(x) - f(t) - f'(t)(x-t) - \frac{f''(t)}{2!}(x-t)^2 - \cdots - \frac{f^{(n)}(t)}{n!}(x-t)^n - R_n(x)\frac{(x-t)^{n+1}}{(x-a)^{n+1}}. \tag{2.4.23}$$

We claim that g satisfies the criteria of Rolle's theorem. Since g is a polynomial function (in t), it is a differentiable function. Also, g is zero at $t=a$ and $t=x$ because

$$\begin{aligned} g(a) &= f(x) - f(a) - f'(a)(x-a) - \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n - R_n(x) \\ &= f(x) - p_n(x) - R_n(x) \\ &= 0, \\ g(x) &= f(x) - f(x) - 0 - \cdots - 0 \\ &= 0. \end{aligned}$$

Therefore, g satisfies Rolle's theorem, and consequently, there exists c between a and x such that $g'(c) = 0$. We now calculate g'. Using the product rule, we note that

$$\frac{d}{dt}\left[\frac{f^{(n)}(t)}{n!}(x-t)^n\right] = -\frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1} + \frac{f^{(n+1)}(t)}{n!}(x-t)^n. \tag{2.4.24}$$

Consequently,

$$\begin{aligned} g'(t) = -f'(t) + [f'(t) - f''(t)(x-t)] + \left[f''(t)(x-t) - \frac{f'''(t)}{2!}(x-t)^2\right] + \cdots \\ + \left[\frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1} - \frac{f^{(n+1)}(t)}{n!}(x-t)^n\right] + (n+1)R_n(x)\frac{(x-t)^n}{(x-a)^{n+1}} \end{aligned} \tag{2.4.25}$$

.

Notice that there is a telescoping effect. Therefore,

$$g'(t) = -\frac{f^{(n+1)}(t)}{n!}(x-t)^n + (n+1)R_n(x)\frac{(x-t)^n}{(x-a)^{n+1}} \tag{2.4.26}$$

.

By Rolle's theorem, we conclude that there exists a number c between a and x such that $g'(c) = 0$. Since

$$g'(c) = -\frac{f^{(n+1)}(c)}{n!}(x-c)^n + (n+1)R_n(x)\frac{(x-c)^n}{(x-a)^{n+1}} \tag{2.4.27}$$

we conclude that

$$-\frac{f^{(n+1)}(c)}{n!}(x-c)^n + (n+1)R_n(x)\frac{(x-c)^n}{(x-a)^{n+1}} = 0. \tag{2.4.28}$$

Adding the first term on the left-hand side to both sides of the equation and dividing both sides of the equation by $n+1$, we conclude that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \tag{2.4.29}$$

as desired. From this fact, it follows that if there exists M such that $\left|f^{(n+1)}(x)\right| \leq M$ for all x in I, then

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \tag{2.4.30}$$

.

□

Not only does Taylor's theorem allow us to prove that a Taylor series converges to a function, but it also allows us to estimate the accuracy of Taylor polynomials in approximating function values. We begin by looking at linear and quadratic approximations of $f(x)=\sqrt[3]{x}$ at $x=8$ and determine how accurate these approximations are at estimating $\sqrt[3]{11}$.

Example 2.4.3: Using Linear and Quadratic Approximations to Estimate Function Values

Consider the function $f(x)=\sqrt[3]{x}$.

a. Find the first and second Taylor polynomials for f at $x=8$. Use a graphing utility to compare these polynomials with f near $x=8$.
b. Use these two polynomials to estimate $\sqrt[3]{11}$.
c. Use Taylor's theorem to bound the error.

Solution:

a. For $f(x)=\sqrt[3]{x}$, the values of the function and its first two derivatives at $x=8$ are as follows:

$$f(x)=\sqrt[3]{x}, f(8)=2$$
$$f'(x)=\frac{1}{3x^{2/3}},\ f'(8)=\frac{1}{12}$$
$$f''(x)=\frac{-2}{9x^{5/3}},\ f''(8)=-\frac{1}{144.}$$

Thus, the first and second Taylor polynomials at $x=8$ are given by

$$\begin{aligned} p_1(x) &= f(8)+f'(8)(x-8) \\ &= 2+\frac{1}{12}(x-8) \\ p_2(x) &= f(8)+f'(8)(x-8)+\frac{f''(8)}{2!}(x-8)^2 \\ &= 2+\frac{1}{12}(x-8)-\frac{1}{288}(x-8)^2\ . \end{aligned}$$

The function and the Taylor polynomials are shown in Figure 4.

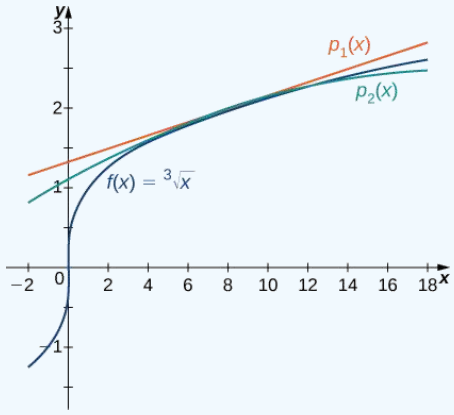


Figure 2.4.5: *The graphs of $f(x)=\sqrt[3]{x}$ and the linear and quadratic approximations $p_1(x)$ and $p_2(x)$*

b. Using the first Taylor polynomial at $x=8$, we can estimate

$$\sqrt[3]{11}\approx p_1(11)=2+\frac{1}{12}(11-8)=2.25. \tag{2.4.31}$$

Using the second Taylor polynomial at $x=8$, we obtain

$$\sqrt[3]{11}\approx p_2(11)=2+\frac{1}{12}(11-8)-\frac{1}{288}(11-8)^2=2.21875. \tag{2.4.32}$$

c. By Note, there exists a c in the interval $(8,11)$ such that the remainder when approximating $\sqrt[3]{11}$ by the first Taylor polynomial satisfies

$$R_1(11)=\frac{f''(c)}{2!}(11-8)^2. \tag{2.4.33}$$

We do not know the exact value of c, so we find an upper bound on $R_1(11)$ by determining the maximum value of f'' on the interval $(8,11)$. Since $f''(x)=-\frac{2}{9x^{5/3}}$, the largest value for $|f''(x)|$ on that interval occurs at $x=8$. Using the fact that $f''(8)=-\frac{1}{144}$, we obtain

$$|R_1(11)| \le \frac{1}{144 \cdot 2!}(11-8)^2 = 0.03125.$$

Similarly, to estimate $R_2(11)$, we use the fact that

$$R_2(11) = \frac{f'''(c)}{3!}(11-8)^3.$$

Since $f'''(x) = \dfrac{10}{27x^{8/3}}$, the maximum value of f''' on the interval $(8, 11)$ is $f'''(8) \approx 0.0014468$. Therefore, we have

$$|R_2(11)| \le \frac{0.0011468}{3!}(11-8)^3 \approx 0.0065104.$$





Exercise 2.4.3:

Find the first and second Taylor polynomials for $f(x) = \sqrt{x}$ at $x = 4$. Use these polynomials to estimate $\sqrt{6}$. Use Taylor's theorem to bound the error.

Hint

Evaluate $f(4)$, $f'(4)$,and $f''(4)$.

Answer

$p_1(x)=2+\frac{1}{4}(x-4); p_2(x)=2+\frac{1}{4}(x-4)-\frac{1}{64}(x-4)^2; p_1(6)=2.5; p_2(6)=2.4375;$

$|R_1(6)|\le 0.0625; |R_2(6)|\le 0.015625$

Example 2.4.4: Approximating $\sin x$ Using Maclaurin Polynomials

From Example b., the Maclaurin polynomials for $\sin x$ are given by

$$p_{2m+1}(x)=p_{2m+2}(x)=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots+(-1)^m\frac{x^{2m+1}}{(2m+1)!}$$

for $m=0,1,2,\ldots$.

a. Use the fifth Maclaurin polynomial for $\sin x$ to approximate $\sin\left(\frac{\pi}{18}\right)$ and bound the error.

b. For what values of x does the fifth Maclaurin polynomial approximate $\sin x$ to within 0.0001?

Solution

a.

The fifth Maclaurin polynomial is

$$p_5(x)=x-\frac{x^3}{3!}+\frac{x^5}{5!} \tag{2.4.34}$$

.

Using this polynomial, we can estimate as follows:

$$\sin\left(\frac{\pi}{18}\right)\approx p_5\left(\frac{\pi}{18}\right)=\frac{\pi}{18}-\frac{1}{3!}\left(\frac{\pi}{18}\right)^3+\frac{1}{5!}\left(\frac{\pi}{18}\right)^5\approx 0.173648. \tag{2.4.35}$$

To estimate the error, use the fact that the sixth Maclaurin polynomial is $p_6(x)=p_5(x)$ and calculate a bound on $R_6(\frac{\pi}{18})$. By Note, the remainder is

$$R_6\left(\frac{\pi}{18}\right)=\frac{f^{(7)}(c)}{7!}\left(\frac{\pi}{18}\right)^7 \tag{2.4.36}$$

for some c between 0 and $\frac{\pi}{18}$. Using the fact that $\left|f^{(7)}(x)\right|\le 1$ for all x, we find that the magnitude of the error is at most

$$\frac{1}{7!}\cdot\left(\frac{\pi}{18}\right)^7\le 9.8\times 10^{-10}. \tag{2.4.37}$$

b.

We need to find the values of x such that

$$\frac{1}{7!}|x|^7\le 0.0001. \tag{2.4.38}$$

Solving this inequality for x, we have that the fifth Maclaurin polynomial gives an estimate to within 0.0001as long as $|x|<0.907$.

Exercise 2.4.4

Use the fourth Maclaurin polynomial for $\cos x$ to approximate $\cos\left(\frac{\pi}{12}\right)$.

Hint

The fourth Maclaurin polynomial is $p_4(x)=1-\frac{x^2}{2!}+\frac{x^4}{4!}$.

Answer

0.96593

Now that we are able to bound the remainder $R_n(x)$, we can use this bound to prove that a Taylor series for f at a converges to f.

Representing Functions with Taylor and Maclaurin Series

We now discuss issues of convergence for Taylor series. We begin by showing how to find a Taylor series for a function, and how to find its interval of convergence.

Example 2.4.5: Finding a Taylor Series

Find the Taylor series for $f(x)=\dfrac{1}{x}$ at $x=1$. Determine the interval of convergence.

Solution

For $f(x)=\dfrac{1}{x}$, the values of the function and its first four derivatives at $x=1$ are

$$f(x)=\frac{1}{x}\ f(1)=1$$

$$f'(x)=-\frac{1}{x^2}\ f'(1)=-1$$

$$f''(x)=\frac{2}{x^3}\ f''(1)=2!$$

$$f'''(x)=-\frac{3\cdot 2}{x^4}\ f'''(1)=-3!$$

$$f^{(4)}(x)=\frac{4\cdot 3\cdot 2}{x^5}\ f^{(4)}(1)=4!.$$

That is, we have $f^{(n)}(1)=(-1)^n n!$ for all $n\geq 0$. Therefore, the Taylor series for f at $x=1$ is given by

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(1)}{n!}(x-1)^n=\sum_{n=0}^{\infty}(-1)^n(x-1)^n\ .$$

To find the interval of convergence, we use the ratio test. We find that

$$\frac{|a_{n+1}|}{|a_n|}=\frac{|(-1)^{n+1}(x-1)n^{+1}|}{|(-1)^n(x-1)^n|}=|x-1|\ .$$

Thus, the series converges if $|x-1|<1$. That is, the series converges for $0<x<2$. Next, we need to check the endpoints. At $x=2$, we see that

$$\sum_{n=0}^{\infty}(-1)^n(2-1)^n=\sum_{n=0}^{\infty}(-1)^n$$

diverges by the divergence test. Similarly, at $x=0$,

$$\sum_{n=0}^{\infty}(-1)^n(0-1)^n=\sum_{n=0}^{\infty}(-1)^{2n}=\sum_{n=0}^{\infty}1$$

diverges. Therefore, the interval of convergence is $(0,2)$.

Exercise 2.4.5

Find the Taylor series for $f(x)=\dfrac{1}{2}$ at $x=2$ and determine its interval of convergence.

Hint

$f^{(n)}(2)=\dfrac{(-1)^n n!}{2^{n+1}}$

Answer

$\dfrac{1}{2}\displaystyle\sum_{n=0}^{\infty}(\frac{2-x}{2})^n$. The interval of convergence is $(0,4)$.

We know that the Taylor series found in this example converges on the interval $(0,2)$, but how do we know it actually converges to f? We consider this question in more generality in a moment, but for this example, we can answer this question by writing

$$f(x)=\frac{1}{x}=\frac{1}{1-(1-x)}. \tag{2.4.39}$$

That is, f can be represented by the geometric series $\sum_{n=0}^{\infty}(1-x)^n$. Since this is a geometric series, it converges to $\frac{1}{x}$ as long as $|1-x|<1$. Therefore, the Taylor series found in Example does converge to $f(x)=\frac{1}{x}$ on $(0,2)$.

We now consider the more general question: if a Taylor series for a function f converges on some interval, how can we determine if it actually converges to f? To answer this question, recall that a series converges to a particular value if and only if its sequence of partial sums converges to that value. Given a Taylor series for f at a, the nth partial sum is given by the nth-degree Taylor polynomial p_n. Therefore, to determine if the Taylor series converges to f, we need to determine whether

$$\lim_{n\to\infty} p_n(x)=f(x).$$

Since the remainder $R_n(x)=f(x)-p_n(x)$, the Taylor series converges to f if and only if

$$\lim_{n\to\infty} R_n(x)=0.$$

We now state this theorem formally.

Convergence of Taylor Series

Suppose that f has derivatives of all orders on an interval I containing a. Then the Taylor series

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n \tag{2.4.40}$$

converges to $f(x)$ for all x in I if and only if

$$\lim_{n\to\infty} R_n(x)=0 \tag{2.4.41}$$

for all x in I.

With this theorem, we can prove that a Taylor series for f at a converges to f if we can prove that the remainder $R_n(x)\to 0$. To prove that $R_n(x)\to 0$, we typically use the bound

$$|R_n(x)|\le\frac{M}{(n+1)!}|x-a|^{n+1} \tag{2.4.42}$$

from Taylor's theorem with remainder.

In the next example, we find the Maclaurin series for e^x and $\sin x$ and show that these series converge to the corresponding functions for all real numbers by proving that the remainders $R_n(x)\to 0$ for all real numbers x.

Example 2.4.6: Finding Maclaurin Series

For each of the following functions, find the Maclaurin series and its interval of convergence. Use Note to prove that the Maclaurin series for f converges to f on that interval.

a. e^x
b. $\sin x$

Solution

a. Using the nth-degree Maclaurin polynomial for e^x found in Example a., we find that the Maclaurin series for e^x is given by

$$\sum_{n=0}^{\infty}\frac{x^n}{n!}.$$

To determine the interval of convergence, we use the ratio test. Since

$$\frac{|a_{n+1}|}{|a_n|}=\frac{|x|^{n+1}}{(n+1)!}\cdot\frac{n!}{|x|^n}=\frac{|x|}{n+1},$$

we have

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=\lim_{n\to\infty}\frac{|x|}{n+1}=0$$

for all x. Therefore, the series converges absolutely for all x, and thus, the interval of convergence is $(-\infty,\infty)$. To show that the series converges to e^x for all x, we use the fact that $f^{(n)}(x)=e^x$ for all $n\ge 0$ and e^x is an increasing function on $(-\infty,\infty)$. Therefore, for any real number b, the maximum value of e^x for all $|x|\le b$ is e^b. Thus,

$$|R_n(x)| \leq \frac{e^b}{(n+1)!}|x|^{n+1}.$$

Since we just showed that

$$\sum_{n=0}^{\infty} \frac{|x|^n}{n!}$$

converges for all x, by the divergence test, we know that

$$\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$$

for any real number x. By combining this fact with the squeeze theorem, the result is $\lim_{n\to\infty} R_n(x) = 0$.

b. Using the nth-degree Maclaurin polynomial for $\sin x$ found in Example b., we find that the Maclaurin series for $\sin x$ is given by

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

In order to apply the ratio test, consider

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|x|^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{|x|^{2n+1}} = \frac{|x|^2}{(2n+3)(2n+2)} \ .$$

Since

$$\lim_{n\to\infty} \frac{|x|^2}{(2n+3)(2n+2)} = 0$$

for all x, we obtain the interval of convergence as $(-\infty, \infty)$. To show that the Maclaurin series converges to $\sin x$, look at $R_n(x)$. For each x there exists a real number c between 0 and x such that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}.$$

Since $\left|f^{(n+1)}(c)\right| \leq 1$ for all integers n and all real numbers c, we have

$$|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}$$

for all real numbers x. Using the same idea as in part a., the result is $\lim_{n\to\infty} R_n(x) = 0$ for all x, and therefore, the Maclaurin series for $\sin x$ converges to $\sin x$ for all real x.

Exercise 2.4.6

Find the Maclaurin series for $f(x) = \cos x$. Use the ratio test to show that the interval of convergence is $(-\infty, \infty)$. Show that the Maclaurin series converges to $\cos x$ for all real numbers x.

Hint

Use the Maclaurin polynomials for $\cos x$.

Answer

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

By the ratio test, the interval of convergence is $(-\infty, \infty)$. Since $|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}$, the series converges to $\cos x$ for all real x.



Proving that e is Irrational

In this project, we use the Maclaurin polynomials for e^x to prove that e is irrational. The proof relies on supposing that e is rational and arriving at a contradiction. Therefore, in the following steps, we suppose $e = r/s$ for some integers r and s where $s \neq 0$.

1. Write the Maclaurin polynomials $p_0(x), p_1(x), p_2(x), p_3(x), p_4(x)$for e^x. Evaluate $p_0(1), p_1(1), p_2(1), p_3(1), p_4(1)$to estimate e.
2. Let $R_n(x)$ denote the remainder when using $p_n(x)$ to estimate e^x. Therefore, $R_n(x) = e^x - p_n(x)$, and $R_n(1) = e - p_n(1)$. Assuming that $e = \frac{r}{s}$ for integers r and s, evaluate $R_0(1), R_1(1), R_2(1), R_3(1), R_4(1)$.
3. Using the results from part 2, show that for each remainder $R_0(1), R_1(1), R_2(1), R_3(1), R_4(1)$, we can find an integer k such that $kR_n(1)$ is an integer for $n = 0, 1, 2, 3, 4$.
4. Write down the formula for the nth-degree Maclaurin polynomial $p_n(x)$ for e^x and the corresponding remainder $R_n(x)$. Show that $sn!R_n(1)$ is an integer.
5. Use Taylor's theorem to write down an explicit formula for $R_n(1)$. Conclude that $R_n(1) \neq 0$, and therefore, $sn!R_n(1) \neq 0$.
6. Use Taylor's theorem to find an estimate on $R_n(1)$. Use this estimate combined with the result from part 5 to show that $|sn!R_n(1)| < \frac{se}{n+1}$. Conclude that if n is large enough, then $|sn!R_n(1)| < 1$. Therefore, $sn!R_n(1)$ is an integer with magnitude less than 1. Thus, $sn!R_n(1) = 0$. But from part 5, we know that $sn!R_n(1) \neq 0$. We have arrived at a contradiction, and consequently, the original supposition that e is rational must be false.

Key Concepts

- Taylor polynomials are used to approximate functions near a value $x = a$. Maclaurin polynomials are Taylor polynomials at $x = 0$.
- The nth-degree Taylor polynomials for a function f are the partial sums of the Taylor series for f.
- If a function f has a power series representation at $x = a$, then it is given by its Taylor series at $x = a$.
- A Taylor series for f converges to f if and only if $\lim_{n\to\infty} R_n(x) = 0$ where $R_n(x) = f(x) - p_n(x)$.
- The Taylor series for e^x, $\sin x$, and $\cos x$ converge to the respective functions for all real x.

Key Equations

- **Taylor series for the function f at the point $x = a$**

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

Glossary

Maclaurin polynomial

a Taylor polynomial centered at 0; the nth-degree Taylor polynomial for f at 0 is the nth-degree Maclaurin polynomial for f

Maclaurin series

a Taylor series for a function f at $x = 0$ is known as a Maclaurin series for f

Taylor polynomials

the nth-degree Taylor polynomial for f at $x=a$ is $p_n(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n$

Taylor series

a power series at a that converges to a function f on some open interval containing a.

Taylor's theorem with remainder

for a function f and the nth-degree Taylor polynomial for f at $x=a$, the remainder $R_n(x)=f(x)-p_n(x)$ satisfies $R_n(x)=\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$

for some$c$ between x and a; if there exists an interval I containing a and a real number M such that $\left|f^{(n+1)}(x)\right|\leq M$ for all x in I, then $|R_n(x)|\leq\frac{M}{(n+1)!}|x-a|^{n+1}$

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2.4E: Exercises for Section 2.4

Taylor Polynomials

In exercises 1 - 8, find the Taylor polynomials of degree two approximating the given function centered at the given point.

1) $f(x) = 1 + x + x^2$ at $a = 1$

2) $f(x) = 1 + x + x^2$ at $a = -1$

Answer

$f(-1) = 1;\ f'(-1) = -1;\ f''(-1) = 2; \quad p_2(x) = 1 - (x+1) + (x+1)^2$

3) $f(x) = \cos(2x)$ at $a = \pi$

4) $f(x) = \sin(2x)$ at $a = \frac{\pi}{2}$

Answer

$f'(x) = 2\cos(2x);\ f''(x) = -4\sin(2x); \quad p_2(x) = -2(x - \frac{\pi}{2})$

5) $f(x) = \sqrt{x}$ at $a = 4$

6) $f(x) = \ln x$ at $a = 1$

Answer

$f'(x) = \dfrac{1}{x};\ f''(x) = -\dfrac{1}{x^2}; \quad p_2(x) = 0 + (x-1) - \frac{1}{2}(x-1)^2$

7) $f(x) = \dfrac{1}{x}$ at $a = 1$

8) $f(x) = e^x$ at $a = 1$

Answer

$p_2(x) = e + e(x-1) + \dfrac{e}{2}(x-1)^2$

Taylor Remainder Theorem

In exercises 9 - 14, verify that the given choice of n in the remainder estimate $|R_n| \le \dfrac{M}{(n+1)!}(x-a)^{n+1}$, where M is the maximum value of $\left|f^{(n+1)}(z)\right|$ on the interval between a and the indicated point, yields $|R_n| \le \frac{1}{1000}$. Find the value of the Taylor polynomial p_n of f at the indicated point.

9) [T] $\sqrt{10};\ a = 9,\ n = 3$

10) [T] $(28)^{1/3};\ a = 27,\ n = 1$

Answer

$\dfrac{d^2}{dx^2}x^{1/3} = -\dfrac{2}{9x^{5/3}} \ge -0.00092\ldots$ when $x \ge 28$ so the remainder estimate applies to the linear approximation

$x^{1/3} \approx p_1(27) = 3 + \dfrac{x-27}{27}$, which gives $(28)^{1/3} \approx 3 + \frac{1}{27} = 3.0\bar{3}7$, while $(28)^{1/3} \approx 3.03658.$

11) [T] $\sin(6);\ a = 2\pi,\ n = 5$

12) [T] $e^2;\ a = 0,\ n = 9$

Answer

Using the estimate $\dfrac{2^{10}}{10!} < 0.000283$we can use the Taylor expansion of order 9 to estimate e^x at $x = 2$. as

$e^2 \approx p_9(2) = 1 + 2 + \frac{2^2}{2} + \frac{2^3}{6} + \cdots + \frac{2^9}{9!} = 7.3887 \quad \ldots$ whereas $e^2 \approx 7.3891.$

13) [T] $\cos(\frac{\pi}{5});\ a=0,\ n=4$

14) [T] $\ln(2);\ a=1,\ n=1000$

Answer

Since $\frac{d^n}{dx^n}(\ln x)=(-1)^{n-1}\frac{(n-1)!}{x^n}$, $R_{1000}\approx\frac{1}{1001}$. One has $p_{1000}(1)=\sum_{n=1}^{1000}\frac{(-1)^{n-1}}{n}\approx 0.6936$ whereas $\ln(2)\approx 0.6931\cdots$.

Approximating Definite Integrals Using Taylor Series

15) Integrate the approximation $\sin t\approx t-\frac{t^3}{6}+\frac{t^5}{120}-\frac{t^7}{5040}$ evaluated at πt to approximate $\int_0^1\frac{\sin\pi t}{\pi t}\,dt$.

16) Integrate the approximation $e^x\approx 1+x+\frac{x^2}{2}+\cdots+\frac{x^6}{720}$ evaluated at $-x^2$ to approximate $\int_0^1 e^{-x^2}\,dx$.

Answer

$\int_0^1\left(1-x^2+\frac{x^4}{2}-\frac{x^6}{6}+\frac{x^8}{24}-\frac{x^{10}}{120}+\frac{x^{12}}{720}\right)dx=1-\frac{1^3}{3}+\frac{1^5}{10}-\frac{1^7}{42}+\frac{1^9}{9\cdot 24}-\frac{1^{11}}{120\cdot 11}+\frac{1^{13}}{720\cdot 13}\approx 0.74683$ whereas $\int_0^1 e^{-x^2}dx\approx 0.74682$.

More Taylor Remainder Theorem Problems

In exercises 17 - 20, find the smallest value of n such that the remainder estimate $|R_n|\le\frac{M}{(n+1)!}(x-a)^{n+1}$, where M is the maximum value of $\left|f^{(n+1)}(z)\right|$ on the interval between a and the indicated point, yields $|R_n|\le\frac{1}{1000}$ on the indicated interval.

17) $f(x)=\sin x$ on $[-\pi,\pi],\ a=0$

18) $f(x)=\cos x$ on $[-\frac{\pi}{2},\frac{\pi}{2}],\ a=0$

Answer

Since $f^{(n+1)}(z)$ is $\sin z$ or $\cos z$, we have $M=1$. Since $|x-0|\le\frac{\pi}{2}$, we seek the smallest n such that $\frac{\pi^{n+1}}{2^{n+1}(n+1)!}\le 0.001$. The smallest such value is $n=7$. The remainder estimate is $R_7\le 0.00092$.

19) $f(x)=e^{-2x}$ on $[-1,1], a=0$

20) $f(x)=e^{-x}$ on $[-3,3], a=0$

Answer

Since $f^{(n+1)}(z)=\pm e^{-z}$ one has $M=e^3$. Since $|x-0|\le 3$, one seeks the smallest n such that $\frac{3^{n+1}e^3}{(n+1)!}\le 0.001$. The smallest such value is $n=14$. The remainder estimate is $R_{14}\le 0.000220$.

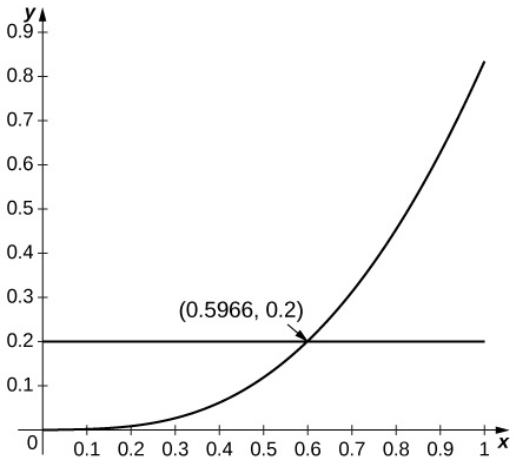
In exercises 21 - 24, the maximum of the right-hand side of the remainder estimate $|R_1|\le\frac{max|f''(z)|}{2}R^2$ on $[a-R,a+R]$ occurs at a or $a\pm R$. Estimate the maximum value of R such that $\frac{max|f''(z)|}{2}R^2\le 0.1$ on $[a-R,a+R]$ by plotting this maximum as a function of R.

21) [T] e^x approximated by $1+x,\ a=0$

22) [T] $\sin x$ approximated by $x,\ a=0$

Answer

Since $\sin x$ is increasing for small x and since $\frac{d^2}{dx^2}(\sin x)=-\sin x$, the estimate applies whenever $R^2\sin(R)\le 0.2$, which applies up to $R=0.596$.

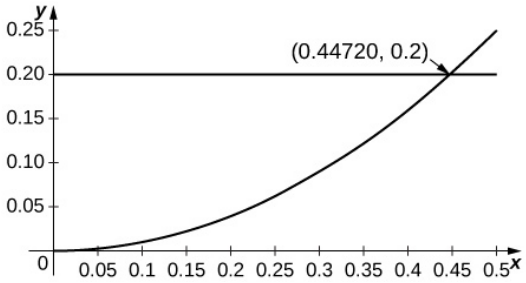


23) [T] $\ln x$ approximated by $x-1,\ a=1$

24) [T] $\cos x$ approximated by $1,\ a=0$

Answer

Since the second derivative of $\cos x$ is $-\cos x$ and since $\cos x$ is decreasing away from $x=0$, the estimate applies when $R^2\cos R\le 0.2$ or $R\le 0.447$.



Taylor Series

In exercises 25 - 35, find the Taylor series of the given function centered at the indicated point.

25) $f(x)=x^4$ at $a=-1$

26) $f(x)=1+x+x^2+x^3$ at $a=-1$

Answer

$(x+1)^3-2(x+1)^2+2(x+1)$

27) $f(x)=\sin x$ at $a=\pi$

28) $f(x)=\cos x$ at $a=2\pi$

Answer

Values of derivatives are the same as for $x=0$ so $\cos x=\sum_{n=0}^{\infty}(-1)^n\dfrac{(x-2\pi)^{2n}}{(2n)!}$

29) $f(x)=\sin x$ at $x=\frac{\pi}{2}$

30) $f(x)=\cos x$ at $x=\frac{\pi}{2}$

Answer

$\cos(\frac{\pi}{2})=0,\ -\sin(\frac{\pi}{2})=-1$ so $\cos x=\sum_{n=0}^{\infty}(-1)^{n+1}\dfrac{(x-\frac{\pi}{2})^{2n+1}}{(2n+1)!}$, which is also $-\cos(x-\frac{\pi}{2})$.

31) $f(x) = e^x$ at $a = -1$

32) $f(x) = e^x$ at $a = 1$

Answer

The derivatives are $f^{(n)}(1) = e$, so $e^x = e\sum_{n=0}^{\infty} \frac{(x-1)^n}{n!}$.

33) $f(x) = \frac{1}{(x-1)^2}$ at $a = 0$ (Hint: Differentiate the Taylor Series for $\frac{1}{1-x}$.)

34) $f(x) = \frac{1}{(x-1)^3}$ at $a = 0$

Answer

$$\frac{1}{(x-1)^3} = -\frac{1}{2}\frac{d^2}{dx^2}\left(\frac{1}{1-x}\right) = -\sum_{n=0}^{\infty}\left(\frac{(n+2)(n+1)x^n}{2}\right)$$

35) $F(x) = \int_0^x \cos(\sqrt{t})\,dt$; where $f(t) = \sum_{n=0}^{\infty}(-1)^n \frac{t^n}{(2n)!}$ at a=0 (Note: f is the Taylor series of $\cos(\sqrt{t})$.)

In exercises 36 - 44, compute the Taylor series of each function around $x = 1$.

36) $f(x) = 2 - x$

Answer

$2 - x = 1 - (x-1)$

37) $f(x) = x^3$

38) $f(x) = (x-2)^2$

Answer

$((x-1)-1)^2 = (x-1)^2 - 2(x-1) + 1$

39) $f(x) = \ln x$

40) $f(x) = \frac{1}{x}$

Answer

$$\frac{1}{1-(1-x)} = \sum_{n=0}^{\infty}(-1)^n(x-1)^n$$

41) $f(x) = \frac{1}{2x - x^2}$

42) $f(x) = \frac{x}{4x - 2x^2 - 1}$

Answer

$$x\sum_{n=0}^{\infty} 2^n(1-x)^{2n} = \sum_{n=0}^{\infty} 2^n(x-1)^{2n+1} + \sum_{n=0}^{\infty} 2^n(x-1)^{2n}$$

43) $f(x) = e^{-x}$

44) $f(x) = e^{2x}$

Answer

$$e^{2x} = e^{2(x-1)+2} = e^2\sum_{n=0}^{\infty}\frac{2^n(x-1)^n}{n!}$$

[T] In exercises 45 - 48, identify the value of x such that the given series $\sum_{n=0}^{\infty} a_n$ is the value of the Maclaurin series of $f(x)$ at x. Approximate the value of $f(x)$ using $S_{10} = \sum_{n=0}^{10} a_n$.

45) $\sum_{n=0}^{\infty} \frac{1}{n!}$

46) $\sum_{n=0}^{\infty} \frac{2^n}{n!}$

Answer

$x = e^2; \quad S_{10} = \frac{34,913}{4725} \approx 7.3889947$

47) $\sum_{n=0}^{\infty} \frac{(-1)^n (2\pi)^{2n}}{(2n)!}$

48) $\sum_{n=0}^{\infty} \frac{(-1)^n (2\pi)^{2n+1}}{(2n+1)!}$

Answer

$\sin(2\pi) = 0; \quad S_{10} = 8.27 \times 10^{-5}$

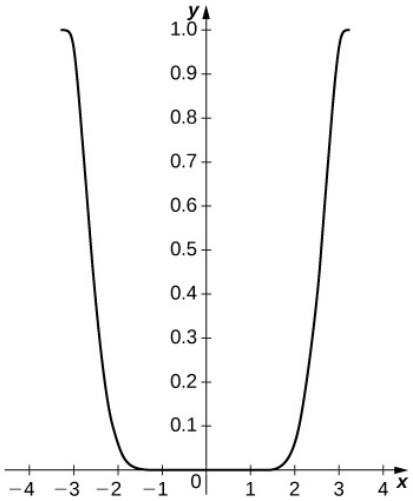
In exercises 49 - 52 use the functions $S_5(x) = x - \frac{x^3}{6} + \frac{x^5}{120}$ and $C_4(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}$ on $[-\pi, \pi]$.

49) [T] Plot $\sin^2 x - (S_5(x))^2$ on $[-\pi, \pi]$. Compare the maximum difference with the square of the Taylor remainder estimate for $\sin x$.

50) [T] Plot $\cos^2 x - (C_4(x))^2$ on $[-\pi, \pi]$. Compare the maximum difference with the square of the Taylor remainder estimate for $\cos x$.

Answer

The difference is small on the interior of the interval but approaches 1 near the endpoints. The remainder estimate is $|R_4| = \frac{\pi^5}{120} \approx 2.552$.

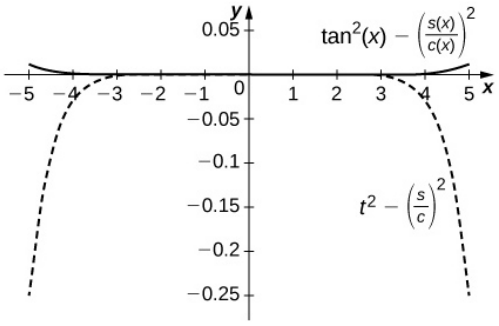


51) [T] Plot $|2S_5(x)C_4(x) - \sin(2x)|$ on $[-\pi, \pi]$.

52) [T] Compare $\dfrac{S_5(x)}{C_4(x)}$ on $[-1,1]$ to $\tan x$. Compare this with the Taylor remainder estimate for the approximation of $\tan x$ by $x+\dfrac{x^3}{3}+\dfrac{2x^5}{15}$.

Answer

The difference is on the order of 10^{-4} on $[-1,1]$ while the Taylor approximation error is around 0.1 near ± 1. The top curve is a plot of $\tan^2 x - \left(\dfrac{S_5(x)}{C_4(x)}\right)^2$ and the lower dashed plot shows $t^2 - \left(\dfrac{S_5}{C_4}\right)^2$.



53) [T] Plot $e^x - e_4(x)$ where $e_4(x) = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6} + \dfrac{x^4}{24}$ on $[0,2]$. Compare the maximum error with the Taylor remainder estimate.

54) (Taylor approximations and root finding.) Recall that Newton's method $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$ approximates solutions of $f(x) = 0$ near the input x_0.

a. If f and g are inverse functions, explain why a solution of $g(x) = a$ is the value $f(a)$ of f.

b. Let $p_N(x)$ be the Nth degree Maclaurin polynomial of e^x. Use Newton's method to approximate solutions of $p_N(x) - 2 = 0$ for $N = 4, 5, 6$.

c. Explain why the approximate roots of $p_N(x) - 2 = 0$ are approximate values of $\ln(2)$.

Answer

a. Answers will vary.
b. The following are the x_n values after 10 iterations of Newton's method to approximation a root of $p_N(x) - 2 = 0$: for $N = 4, x = 0.6939\ldots$;for $N = 5, x = 0.6932\ldots$;for $N = 6, x = 0.69315\ldots$; .(Note: $\ln(2) = 0.69314..$)
c. Answers will vary.

Evaluating Limits using Taylor Series

In exercises 55 - 58, use the fact that if $q(x) = \sum_{n=1}^{\infty} a_n (x-c)^n$ converges in an interval containing c, then $\lim_{x\to c} q(x) = a_0$ to evaluate each limit using Taylor series.

55) $\lim_{x\to 0} \dfrac{\cos x - 1}{x^2}$

56) $\lim_{x\to 0} \dfrac{\ln(1-x^2)}{x^2}$

Answer

$\dfrac{\ln(1-x^2)}{x^2} \to -1$

57) $\lim_{x\to 0} \dfrac{e^{x^2} - x^2 - 1}{x^4}$

58) $\lim\limits_{x\to 0^+} \dfrac{\cos(\sqrt{x})-1}{2x}$

Answer

$$\frac{\cos(\sqrt{x})-1}{2x} \approx \frac{(1-\frac{x}{2}+\frac{x^2}{4!}-\cdots)-1}{2x} \to -\frac{1}{4}$$

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2.5: Working with Taylor Series

Learning Objectives

- Write the terms of the binomial series.
- Recognize the Taylor series expansions of common functions.
- Recognize and apply techniques to find the Taylor series for a function.
- Use Taylor series to solve differential equations.
- Use Taylor series to evaluate nonelementary integrals.

In the preceding section, we defined Taylor series and showed how to find the Taylor series for several common functions by explicitly calculating the coefficients of the Taylor polynomials. In this section we show how to use those Taylor series to derive Taylor series for other functions. We then present two common applications of power series. First, we show how power series can be used to solve differential equations. Second, we show how power series can be used to evaluate integrals when the antiderivative of the integrand cannot be expressed in terms of elementary functions. In one example, we consider $\int e^{-x^2}\,dx$, an integral that arises frequently in probability theory.

The Binomial Series

Our first goal in this section is to determine the Maclaurin series for the function $f(x)=(1+x)^r$ for all real numbers r. The Maclaurin series for this function is known as the **binomial series.** We begin by considering the simplest case: r is a nonnegative integer. We recall that, for $r=0,1,2,3,4$, $f(x)=(1+x)^r$ can be written as

$$\begin{aligned} f(x)&=(1+x)^0=1,\\ f(x)&=(1+x)^1=1+x,\\ f(x)&=(1+x)^2=1+2x+x^2,\\ f(x)&=(1+x)^3=1+3x+3x^2+x^3\\ f(x)&=(1+x)^4=1+4x+6x^2+4x^3+x^4. \end{aligned} \tag{2.5.1}$$

The expressions on the right-hand side are known as binomial expansions and the coefficients are known as binomial coefficients. More generally, for any nonnegative integer r, the binomial coefficient of x^n in the binomial expansion of $(1+x)^r$ is given by

$$\binom{r}{n}=\frac{r!}{n!(r-n)!} \tag{2.5.2}$$

and

$$f(x)=(1+x)^r=\binom{r}{0}1+\binom{r}{1}x+\binom{r}{2}x^2+\binom{r}{3}x^3+\cdots+\binom{r}{r-1}x^{r-1}+\binom{r}{r}x^r=\sum_{n=0}^{r}\binom{r}{n}x^n. \tag{2.5.3}$$

For example, using this formula for $r=5$, we see that

$$\begin{aligned} f(x)&=(1+x)^5\\ &=\tbinom{5}{0}1+\tbinom{5}{1}x+\tbinom{5}{2}x^2+\tbinom{5}{3}x^3+\tbinom{5}{4}x^4+\tbinom{5}{5}x^5\\ &=\frac{5!}{0!5!}1+\frac{5!}{1!4!}x+\frac{5!}{2!3!}x^2+\frac{5!}{3!2!}x^3+\frac{5!}{4!1!}x^4+\frac{5!}{5!0!}x^5\\ &=1+5x+10x^2+10x^3+5x^4+x^5. \end{aligned}$$

We now consider the case when the exponent r

is any real number, not necessarily a nonnegative integer. If r is not a nonnegative integer, then $f(x)=(1+x)^r$ cannot be written as a finite polynomial. However, we can find a power series for f. Specifically, we look for the Maclaurin series for f. To do this, we find the derivatives of f and evaluate them at $x=0$.

$$\begin{aligned} f(x) &= (1+x)^r \\ f(0) &= 1 \\ f'(x) &= r(1+x)^{r-1} \\ f'(0) &= r \\ f''(x) &= r(r-1)(1+x)^{r-2} \\ f''(0) &= r(r-1) \\ f'''(x) &= r(r-1)(r-2)(1+x)^{r-3} \\ f'''(0) &= r(r-1)(r-2) \\ f(n)(x) &= r(r-1)(r-2)\cdots(r-n+1)(1+x)^{r-n} \\ f^{(n)}(0) &= r(r-1)(r-2)\cdots(r-n+1) \end{aligned}$$

We conclude that the coefficients in the binomial series are given by

$$\frac{f^{(n)}(0)}{n!} = \frac{r(r-1)(r-2)\cdots(r-n+1)}{n!}. \tag{2.5.4}$$

We note that if r is a nonnegative integer, then the $(r+1)st$ derivative $f^{(r+1)}$ is the zero function, and the series terminates. In addition, if r is a nonnegative integer, then Equation for the coefficients agrees with Equation for the coefficients, and the formula for the binomial series agrees with Equation for the finite binomial expansion. More generally, to denote the binomial coefficients for any real number r, we define

$$\binom{r}{n} = \frac{(r-1)(r-2)\cdots(r-n+1)}{n!}. \tag{2.5.5}$$

With this notation, we can write the binomial series for $(1+x)^r$ as

$$\sum_{n=0}^{\infty}\binom{r}{n}x^n = 1 + rx + \frac{r(r-1)}{2!}x^2 + \cdots + \frac{r(r-1)\cdots(r-n+1)}{n!}x^n + \cdots. \tag{2.5.6}$$

We now need to determine the interval of convergence for the binomial series Equation 2.5.6. We apply the ratio test. Consequently, we consider

$$\begin{aligned} \frac{|a_{n+1}|}{|a_n|} &= \frac{|r(r-1)(r-2)\cdots(r-n)||x||^{n+1}}{(n+1)!} \cdot \frac{n}{|r(r-1)(r-2)\cdots(r-n+1)||x|^n} \\ &= \frac{|r-n||x|}{|n+1|} \end{aligned}$$

.

Since

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = |x| < 1 \tag{2.5.7}$$

if and only if $|x| < 1$, we conclude that the interval of convergence for the binomial series is $(-1, 1)$. The behavior at the endpoints depends on r. It can be shown that for $r \geq 0$ the series converges at both endpoints; for $-1 < r < 0$, the series converges at $x = 1$ and diverges at $x = -1$; and for $r < -1$, the series diverges at both endpoints. The binomial series does converge to $(1+x)^r$ in $(-1, 1)$ for all real numbers r, but proving this fact by showing that the remainder $R_n(x) \to 0$ is difficult.

Definition: binomial series

For any real number r, the Maclaurin series for $f(x) = (1+x)^r$ is the binomial series. It converges to f for $|x| < 1$, and we write

$$(1+x)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^n = 1 + rx + \frac{r(r-1)}{2!}x^2 + \cdots + r\frac{(r-1)\cdots(r-n+1)}{n!}x^n + \cdots \tag{2.5.8}$$

for $|x| < 1$.

We can use this definition to find the binomial series for $f(x) = \sqrt{1+x}$ and use the series to approximate $\sqrt{1.5}$.

Example 2.5.1: Finding Binomial Series

a. Find the binomial series for $f(x) = \sqrt{1+x}$.
b. Use the third-order Maclaurin polynomial $p_3(x)$ to estimate $\sqrt{1.5}$. Use Taylor's theorem to bound the error. Use a graphing utility to compare the graphs of f and p_3.

Solution

a. Here $r = \frac{1}{2}$. Using the definition for the binomial series, we obtain

$$\begin{aligned}\sqrt{1+x} &= 1 + \frac{1}{2}x + \frac{(1/2)(-1/2)}{2!}x^2 + \frac{(1/2)(-1/2)(-3/2)}{3!}x^3 + \cdots \\ &= 1 + \frac{1}{2}x - \frac{1}{2!}\frac{1}{2^2}x^2 + \frac{1}{3!}\frac{1\cdot 3}{2^3}x^3 - \cdots + \frac{(-1)^{n+1}}{n!}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^n}x^n + \cdots \\ &= 1 + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n!}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^n}x^n.\end{aligned}$$

b. From the result in part a. the third-order Maclaurin polynomial is

$$p_3(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 \quad .$$

Therefore,

$$\sqrt{1.5} = \sqrt{1+0.5} \approx 1 + \frac{1}{2}(0.5) - \frac{1}{8}(0.5)^2 + \frac{1}{16}(0.5)^3 \approx 1.2266.$$

From Taylor's theorem, the error satisfies

$$R_3(0.5) = \frac{f^{(4)}(c)}{4!}(0.5)^4$$

for some c between 0 and 0.5. Since $f^{(4)}(x) = -\dfrac{15}{2^4(1+x)^{7/2}}$, and the maximum value of $\left|f^{(4)}(x)\right|$ on the interval $(0, 0.5)$ occurs at $x = 0$, we have

$$|R_3(0.5)| \le \frac{15}{4!2^4}(0.5)^4 \approx 0.00244.$$

The function and the Maclaurin polynomial p_3 are graphed in Figure.

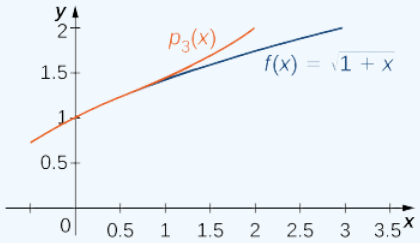


Figure 2.5.1: The third-order Maclaurin polynomial $p_3(x)$ provides a good approximation for $f(x) = \sqrt{1+x}$ for x near zero.



Exercise 2.5.1

Find the binomial series for $f(x)=\dfrac{1}{(1+x)^2}$.

Hint

Use the definition of binomial series for $r=-2$.

Answer

$\sum_{n=0}^{\infty}(-1)^n(n+1)x^n$

Common Functions Expressed as Taylor Series

At this point, we have derived Maclaurin series for exponential, trigonometric, and logarithmic functions, as well as functions of the form $f(x)=(1+x)^r$. In Table, we summarize the results of these series. We remark that the convergence of the Maclaurin series for $f(x)=ln(1+x)$ at the endpoint $x=1$ and the Maclaurin series for $f(x)=tan^{-1}x$ at the endpoints $x=1$ and $x=-1$ relies on a more advanced theorem than we present here. (Refer to Abel's theorem for a discussion of this more technical point.)

Maclaurin Series for Common Functions

Function	Maclaurin Series	Interval of Convergence
$f(x)=\dfrac{1}{1-x}$	$\sum_{n=0}^{\infty}x^n$	$-1<x<1$
$f(x)=e^x$	$\sum_{n=0}^{\infty}\dfrac{x^n}{n!}$	$-\infty<x<\infty$
$f(x)=\sin x$	$\sum_{n=0}^{\infty}(-1)^n\dfrac{x^{2n+1}}{(2n+1)!}$	$-\infty<x<\infty$
$f(x)=\cos x$	$\sum_{n=0}^{\infty}(-1)^n\dfrac{x^{2n}}{(2n)!}$	$-\infty<x<\infty$
$f(x)=\ln(1+x)$	$\sum_{n=0}^{\infty}(-1)^{n+1}\dfrac{x^n}{n}$	$-1<x<1$
$f(x)=tan^{-1}x$	$\sum_{n=0}^{\infty}(-1)^n\dfrac{x^{2n+1}}{2n+1}$	$-1<x<1$

Function	Maclaurin Series	Interval of Convergence
$f(x)=(1+x)^r$	$\sum_{n=0}^{\infty}\binom{r}{n}x^n$	$-1<x<1$



Earlier in the chapter, we showed how you could combine power series to create new power series. Here we use these properties, combined with the Maclaurin series in Table, to create Maclaurin series for other functions.

Example 2.5.2: Deriving Maclaurin Series from Known Series

Find the Maclaurin series of each of the following functions by using one of the series listed in Table.

a. $f(x)=\cos\sqrt{x}$
b. $f(x)=\sinh x$

Solution

a. Using the Maclaurin series for $\cos x$ we find that the Maclaurin series for $\cos\sqrt{x}$ is given by

$$\sum_{n=0}^{\infty}\frac{(-1)^n(\sqrt{x})^{2n}}{(2n)!}=\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{(2n)!}=1-\frac{x}{2!}+\frac{x^2}{4!}-\frac{x^3}{6!}+\frac{x^4}{8!}-\cdots.$$

This series converges to $cos\sqrt{x}$ for all x in the domain of $cos\sqrt{x}$; that is, for all $x\geq 0$.

b. To find the Maclaurin series for $sinhx$, we use the fact that

$$\sinh x=\frac{e^x-e^{-x}}{2}.$$

Using the Maclaurin series for e^x, we see that the nth term in the Maclaurin series for $sinhx$ is given by

$$\frac{x^n}{n!}-\frac{(-x)^n}{n!}.$$

For n even, this term is zero. For n odd, this term is $\frac{2x^n}{n!}$. Therefore, the Maclaurin series for $sinhx$ has only odd-order terms and is given by

$$\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}=x+\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots.$$





Exercise 2.5.2

Find the Maclaurin series for $\sin(x^2)$.

Hint

Use the Maclaurin series for $\sin x$.

Answer

$\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}$

We also showed previously in this chapter how power series can be differentiated term by term to create a new power series. In Example, we differentiate the binomial series for $\sqrt{1+x}$ term by term to find the binomial series for $\frac{1}{\sqrt{1+x}}$. Note that we could construct the binomial series for $\frac{1}{\sqrt{1+x}}$ directly from the definition, but differentiating the binomial series for $\sqrt{1+x}$ is an easier calculation.

Example 2.5.3: Differentiating a Series to Find a New Series

Use the binomial series for $\sqrt{1+x}$ to find the binomial series for $\frac{1}{\sqrt{1+x}}$.

Solution

The two functions are related by

$$\frac{d}{dx}\sqrt{1+x} = \frac{1}{2\sqrt{1+x}},$$

so the binomial series for $\frac{1}{\sqrt{1+x}}$ is given by

$$\frac{1}{\sqrt{1+x}} = 2\frac{d}{dx}\sqrt{1+x} = 1+\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n}x^n.$$

Exercise 2.5.3

Find the binomial series for $f(x) = \frac{1}{(1+x)^{3/2}}$

Hint

Differentiate the series for $\frac{1}{\sqrt{1+x}}$

Answer

$\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n}x^n$

In this example, we differentiated a known Taylor series to construct a Taylor series for another function. The ability to differentiate power series term by term makes them a powerful tool for solving differential equations. We now show how this is accomplished.

Solving Differential Equations with Power Series

Consider the differential equation

$$y'(x) = y. \tag{2.5.9}$$

Recall that this is a first-order separable equation and its solution is $y = Ce^x$. This equation is easily solved using techniques discussed earlier in the text. For most differential equations, however, we do not yet have analytical tools to solve them. Power series are an extremely useful tool for solving many types of differential equations. In this technique, we look for a solution of the form $y = \sum_{n=0}^{\infty} c_n x^n$ and determine what the coefficients would need to be. Shifting the index of summation is an important tool used in solving differential equations with power series.



Here is an example of showing that a given power series is a solution to a particular differential equation.



In the next example, we consider an initial-value problem involving $y' = y$ to illustrate the technique.

Example 2.5.4: Power Series Solution of a Differential Equation

Use power series to solve the initial-value problem

$$y' = y, y(0) = 3. \tag{2.5.10}$$

Solution

Suppose that there exists a power series solution

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots.$$

Differentiating this series term by term, we obtain

$$y' = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots.$$

If y satisfies the differential equation, then

$$c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots = c_1 + 2c_2 x + 3c_3 x^2 + 4c_3 x^3 + \cdots.$$

Using the uniqueness of power series representations, we know that these series can only be equal if their coefficients are equal. Therefore,

$$c_0 = c_1,$$
$$c_1 = 2c_2,$$
$$c_2 = 3c_3,$$
$$c_3 = 4c_4,$$
$$\vdots .$$

Using the initial condition $y(0) = 3$ combined with the power series representation

$$y(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots ,$$

we find that $c_0 = 3$. We are now ready to solve for the rest of the coefficients. Using the fact that $c_0 = 3$, we have

$$c_1 = c_0 = 3 = \frac{3}{1!} ,$$
$$c_2 = \frac{c_1}{2} = \frac{3}{2} = \frac{3}{2!},$$
$$c_3 = \frac{c_2}{3} = \frac{3}{3 \cdot 2} = \frac{3}{3!},$$
$$c_4 = \frac{c_3}{4} = \frac{3}{4 \cdot 3 \cdot 2} = \frac{3}{4!}.$$

Therefore,

$$y = 3[1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 \frac{1}{4!}x^4 + \cdots] = 3 \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad .$$

You might recognize

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

as the Taylor series for e^x. Therefore, the solution is $y = 3e^x$.

Exercise 2.5.4

Use power series to solve $y' = 2y, y(0) = 5$.

Hint

The equations for the first several coefficients c_n will satisfy $c_0 = 2c_1, c_1 = 2 \cdot 2c_2, c_2 = 2 \cdot 3c_3, \ldots .$ In general, for all $n \geq 0, c_n = 2(n+1)C_{n+1}$.

Answer

$y = 5e^{2x}$

We now consider an example involving a differential equation that we cannot solve using previously discussed methods. This differential equation

$$y' - xy = 0 \tag{2.5.11}$$

is known as **Airy's equation**. It has many applications in mathematical physics, such as modeling the diffraction of light. Here we show how to solve it using power series.

Example 2.5.5: Power Series Solution of Airy's Equation

Use power series to solve

$$y'' - xy = 0 \tag{2.5.12}$$

with the initial conditions $y(0) = a$ and $y'(0) = b$.

Solution

We look for a solution of the form

$$y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots \quad .$$

Differentiating this function term by term, we obtain

$$y' = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots,$$
$$y'' = 2 \cdot 1 c_2 + 3 \cdot 2 c_3 x + 4 \cdot 3 c_4 x^2 + \cdots.$$

If y satisfies the equation $y'' = xy$, then

$$2 \cdot 1 c_2 + 3 \cdot 2 c_3 x + 4 \cdot 3 c_4 x^2 + \cdots = x(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots).$$

Using [link] on the uniqueness of power series representations, we know that coefficients of the same degree must be equal. Therefore,

$$2 \cdot 1 c_2 = 0,$$
$$3 \cdot 2 c_3 = c_0,$$
$$4 \cdot 3 c_4 = c_1,$$
$$5 \cdot 4 c_5 = c_2,$$
$$\vdots .$$

More generally, for $n \geq 3$, we have $n \cdot (n-1) c_n = c_{n-3}$. In fact, all coefficients can be written in terms of c_0 and c_1. To see this, first note that $c_2 = 0$. Then

$$c_3 = \frac{c_0}{3 \cdot 2},$$
$$c_4 = \frac{c_1}{4 \cdot 3}.$$

For c_5, c_6, c_7, we see that

$$c_5 = \frac{c_2}{5 \cdot 4} = 0\,,$$
$$c_6 = \frac{c_3}{6 \cdot 5} = \frac{c_0}{6 \cdot 5 \cdot 3 \cdot 2}\,,$$
$$c_7 = \frac{c_4}{7 \cdot 6} = \frac{c_1}{7 \cdot 6 \cdot 4 \cdot 3}\,.$$

Therefore, the series solution of the differential equation is given by

$$y = c_0 + c_1 x + 0 \cdot x^2 + \frac{c_0}{3 \cdot 2} x^3 + \frac{c_1}{4 \cdot 3} x^4 + 0 \cdot x^5 + \frac{c_0}{6 \cdot 5 \cdot 3 \cdot 2} x^6 + \frac{c_1}{7 \cdot 6 \cdot 4 \cdot 3} x^7 + \cdots.$$

The initial condition $y(0) = a$ implies $c_0 = a$. Differentiating this series term by term and using the fact that $y'(0) = b$, we conclude that $c_1 = b$.

Therefore, the solution of this initial-value problem is

$$y = a\left(1 + \frac{x^3}{3 \cdot 2} + \frac{x}{6 \cdot 5 \cdot 3 \cdot 2} + \cdots\right) + b\left(x + \frac{x^4}{4 \cdot 3} + \frac{x^7}{7 \cdot 6 \cdot 4 \cdot 3} + \cdots\right).$$

Exercise 2.5.5

Use power series to solve $y'' + x^2 y = 0$ with the initial condition $y(0) = a$ and $y'(0) = b$.

Hint

The coefficients satisfy $c_0 = a, c_1 = b, c_2 = 0, c_3 = 0,$ and for $n \geq 4, n(n-1)c_n = -c_{n-4}$.

Answer

$$y = a(1 - \frac{x^4}{3 \cdot 4} + \frac{x^8}{3 \cdot 4 \cdot 7 \cdot 8} - \cdots) + b(x - \frac{x^5}{4 \cdot 5} + \frac{x^9}{4 \cdot 5 \cdot 8 \cdot 9} - \cdots)$$

Evaluating Nonelementary Integrals

Solving differential equations is one common application of power series. We now turn to a second application. We show how power series can be used to evaluate integrals involving functions whose antiderivatives cannot be expressed using elementary functions.

One integral that arises often in applications in probability theory is $\int e^{-x^2} dx$. Unfortunately, the antiderivative of the integrand e^{-x^2} is not an elementary function. By elementary function, we mean a function that can be written using a finite number of algebraic combinations or compositions of exponential, logarithmic, trigonometric, or power functions. We remark that the term "elementary function" is not synonymous with noncomplicated function. For example, the function $f(x) = \sqrt{x^2 - 3x} + e^{x^3} - sin(5x + 4)$ is an elementary function, although not a particularly simple-looking function. Any integral of the form $\int f(x)dx$ where the antiderivative of f cannot be written as an elementary function is considered a **nonelementary integral.**

Nonelementary integrals cannot be evaluated using the basic integration techniques discussed earlier. One way to evaluate such integrals is by expressing the integrand as a power series and integrating term by term. We demonstrate this technique by considering $\int e^{-x^2} dx$.

Example 2.5.6: Using Taylor Series to Evaluate a Definite Integral

a. Express $\int e^{-x^2} dx$ as an infinite series.

b. Evaluate $\int_0^1 e^{-x^2} dx$ to within an error of 0.01.

Solution

a. The Maclaurin series for e^{-x^2} is given by

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots + (-1)^n \frac{x^{2n}}{n!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}.$$

Therefore,

$$\int e^{-x^2} dx = \int (1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots + (-1)^n \frac{x^{2n}}{n!} + \cdots) dx = C + x - \frac{x^3}{3} + \frac{x^5}{5.2!} - \frac{x^7}{7.3!} + \cdots$$
$$+ (-1)^n \frac{x^{2n+1}}{(2n+1)n!} + \cdots.$$

b. Using the result from part a. we have

$$\int_0^1 e^{-x^2} dx = 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} - \cdots.$$

The sum of the first four terms is approximately 0.74. By the alternating series test, this estimate is accurate to within an error of less than $\frac{1}{216} \approx 0.0046296 < 0.01$.



Exercise 2.5.6

Express $\int cos\sqrt{x}dx$ as an infinite series. Evaluate $\int_0^1 cos\sqrt{x}dx$ to within an error of 0.01.

Hint

Use the series found in Example.

Answer

$C+\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{x^n}{n(2n-2)!}$ The definite integral is approximately $0.514$to within an error of 0.01.

As mentioned above, the integral $\int e^{-x^2}dx$ arises often in probability theory. Specifically, it is used when studying data sets that are normally distributed, meaning the data values lie under a bell-shaped curve. For example, if a set of data values is normally distributed with mean μ and standard deviation σ, then the probability that a randomly chosen value lies between $x=a$ and $x=b$ is given by

$$\frac{1}{\sigma\sqrt{2\pi}}\int_a^b e^{-(x-\mu)^2/(2\sigma^2)}dx. \tag{2.5.13}$$

(See Figure.)

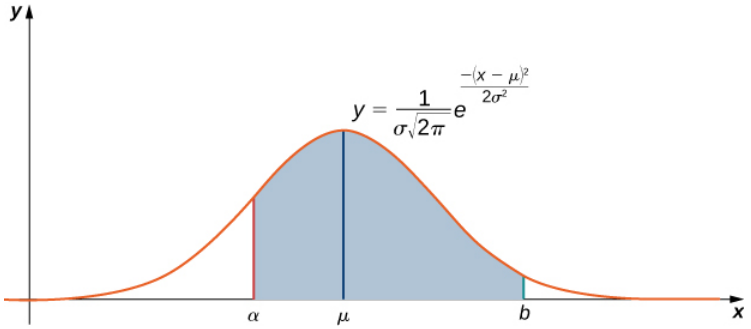


Figure 2.5.2: If data values are normally distributed with mean μ and standard deviation σ, the probability that a randomly selected data value is between a and b is the area under the curve $y=\dfrac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/(2\sigma^2)}$ between $x=a$ and $x=b$.

To simplify this integral, we typically let $z = \dfrac{x-\mu}{\sigma}$. This quantity z is known as the z score of a data value. With this simplification, integral Equation becomes

$$\frac{1}{\sqrt{2\pi}} \int_{(a-\mu)/\sigma}^{(b-\mu)/\sigma} e^{-z^2/2} dz. \tag{2.5.14}$$



In Example, we show how we can use this integral in calculating probabilities.

Example 2.5.7: Using Maclaurin Series to Approximate a Probability

Suppose a set of standardized test scores are normally distributed with mean $\mu = 100$ and standard deviation $\sigma = 50$. Use Equation and the first six terms in the Maclaurin series for $e^{-x^2/2}$ to approximate the probability that a randomly selected test score is between $x = 100$ and $x = 200$. Use the alternating series test to determine how accurate your approximation is.

Solution

Since $\mu = 100, \sigma = 50$, and we are trying to determine the area under the curve from $a = 100$ to $b = 200$, integral Equation becomes

$$\frac{1}{\sqrt{2\pi}} \int_0^2 e^{-z^2/2} dz.$$

The Maclaurin series for $e^{-x^2/2}$ is given by

$$e^{-x^2/2} = \sum_{n=0}^{\infty} \frac{(-\frac{x^2}{2})^n}{n!} = 1 - \frac{x^2}{2^1 \cdot 1!} + \frac{x^4}{2^2 \cdot 2!} - \frac{x^6}{2^3 \cdot 3!} + \cdots + (-1)^n \frac{x^{2n}}{2^n \cdot n}! + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2^n \cdot n!} \quad .$$

Therefore,

$$\frac{1}{\sqrt{2\pi}} \int e^{-z^2/2} dz = \frac{1}{\sqrt{2\pi}} \int (1 - \frac{z^2}{2^1 \cdot 1!} + \frac{z^4}{2^2 \cdot 2!} - \frac{z^6}{2^3 \cdot 3!} + \cdots + (-1)^n \frac{z^{2n}}{2^n \cdot n!} + \cdots) dz$$
$$= \frac{1}{\sqrt{2\pi}} (C + z - \frac{z^3}{3 \cdot 2^1 \cdot 1!} + \frac{z^5}{5 \cdot 2^2 \cdot 2!} - \frac{z^7}{7 \cdot 2^3 \cdot 3!} + \cdots + (-1)^n \frac{z^{2n+1}}{(2n+1) 2^n \cdot n!} + \cdots)$$
$$\frac{1}{\sqrt{2\pi}} \int_0^2 e^{-z^2/2} dz = \frac{1}{\sqrt{2\pi}} (2 - \frac{8}{6} + \frac{32}{40} - \frac{128}{336} + \frac{512}{3456} - \frac{2^{11}}{11 \cdot 2^5 \cdot 5!} + \cdots)$$

Using the first five terms, we estimate that the probability is approximately **0.4922.** By the alternating series test, we see that this estimate is accurate to within

$$\frac{1}{\sqrt{2\pi}}\frac{2^{13}}{13\cdot 2^6\cdot 6!}\approx 0.00546. \tag{2.5.15}$$

Analysis

If you are familiar with probability theory, you may know that the probability that a data value is within two standard deviations of the mean is approximately 95 Here we calculated the probability that a data value is between the mean and two standard deviations above the mean, so the estimate should be around 47.5. The estimate, combined with the bound on the accuracy, falls within this range.

Exercise 2.5.7

Use the first five terms of the Maclaurin series for $e^{-x^2/2}$ to estimate the probability that a randomly selected test score is between 100 and 150. Use the alternating series test to determine the accuracy of this estimate.

Hint

Evaluate $\int_0^1 e^{-z^2/2}dz$ using the first five terms of the Maclaurin series for $e^{-z^2/2}$.

Answer

The estimate is approximately 0.3414.This estimate is accurate to within 0.0000094.

Another application in which a nonelementary integral arises involves the period of a pendulum. The integral is

$$\int_0^{\pi/2}\frac{d\theta}{\sqrt{1-k^2sin^2\theta}} \tag{2.5.16}$$

.

An integral of this form is known as an **elliptic integral** of the first kind. Elliptic integrals originally arose when trying to calculate the arc length of an ellipse. We now show how to use power series to approximate this integral.

Example 2.5.8: Period of a Pendulum

The period of a pendulum is the time it takes for a pendulum to make one complete back-and-forth swing. For a pendulum with length L that makes a maximum angle θ_{max} with the vertical, its period T is given by

$$T=4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\frac{d\theta}{\sqrt{1-k^2sin^2\theta}}$$

where g is the acceleration due to gravity and $k=sin(\frac{\theta_{max}}{2})$ (see Figure). (We note that this formula for the period arises from a non-linearized model of a pendulum. In some cases, for simplification, a linearized model is used and $sin\theta$ is approximated by θ.)

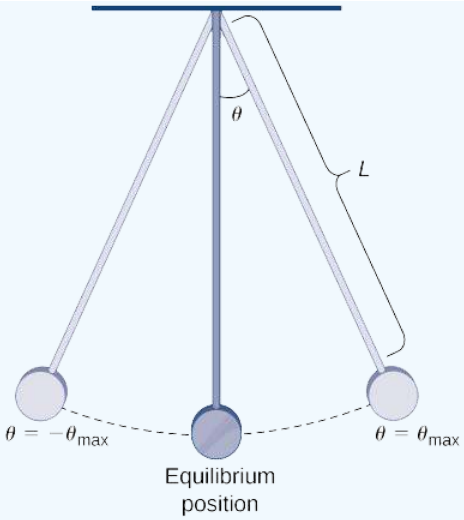


Figure 2.5.3: This pendulum has length L and makes a maximum angle θ_{max} with the vertical.

Use the binomial series

$$\frac{1}{\sqrt{1+x}} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n} x^n$$

to estimate the period of this pendulum. Specifically, approximate the period of the pendulum if

a. you use only the first term in the binomial series, and
b. you use the first two terms in the binomial series.

Solution

We use the binomial series, replacing x with $-k^2 sin^2\theta$. Then we can write the period as

$$T = 4\sqrt{\frac{L}{g}} \int_0^{\pi/2} (1 + \frac{1}{2}k^2 sin^2\theta + \frac{1 \cdot 3}{2!2^2}k^4 sin^4\theta + \cdots) d\theta.$$

a. Using just the first term in the integrand, the first-order estimate is

$$T \approx 4\sqrt{\frac{L}{g}} \int_0^{\pi/2} d\theta = 2\pi\sqrt{\frac{L}{g}}\ .$$

If θ_{max} is small, then $k = sin(\frac{\theta_{max}}{2})$ is small. We claim that when k is small, this is a good estimate. To justify this claim, consider

$$\int_0^{\pi/2} (1 + \frac{1}{2}k^2 sin^2\theta + \frac{1 \cdot 3}{2!2^2}k^4 sin^4\theta + \cdots) d\theta.$$

Since $|sinx| \leq 1$, this integral is bounded by

$$\int_0^{\pi/2} (\frac{1}{2}k^2 + \frac{1.3}{2!2^2}k^4 + \cdots) d\theta < \frac{\pi}{2}(\frac{1}{2}k^2 + \frac{1 \cdot 3}{2!2^2}k^4 + \cdots).$$

Furthermore, it can be shown that each coefficient on the right-hand side is less than 1 and, therefore, that this expression is bounded by

$$\frac{\pi k^2}{2}(1 + k^2 + k^4 + \cdots) = \frac{\pi k^2}{2} \cdot \frac{1}{1 - k^2}\ ,$$

which is small for k small.

b. For larger values of θ_{max}, we can approximate T by using more terms in the integrand. By using the first two terms in the integral, we arrive at the estimate

$$T \approx 4\sqrt{fracLg}\int_0^{\pi/2}(1+\frac{1}{2}k^2sin^2\theta)d\theta = 2\pi\sqrt{\frac{L}{g}}(1+\frac{k^2}{4}).$$

The applications of Taylor series in this section are intended to highlight their importance. In general, Taylor series are useful because they allow us to represent known functions using polynomials, thus providing us a tool for approximating function values and estimating complicated integrals. In addition, they allow us to define new functions as power series, thus providing us with a powerful tool for solving differential equations.



Key Concepts

- The binomial series is the Maclaurin series for $f(x) = (1+x)^r$. It converges for $|x| < 1$.
- Taylor series for functions can often be derived by algebraic operations with a known Taylor series or by differentiating or integrating a known Taylor series.
- Power series can be used to solve differential equations.
- Taylor series can be used to help approximate integrals that cannot be evaluated by other means.

Glossary

binomial series

the Maclaurin series for $f(x) = (1+x)^r$; it is given by

$(1+x)^r = \sum_{n=0}^{\infty}\binom{r}{n}x^n = 1 + rx + \frac{r(r-1)}{2!}x^2 + \cdots + \frac{r(r-1)\cdots(r-n+1)}{n!}x^n + \cdots \qquad \text{for } |x| < 1$

nonelementary integral

an integral for which the antiderivative of the integrand cannot be expressed as an elementary function

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2.5E: Exercises for Section 2.5

In exercises 1 - 4, use appropriate substitutions to write down the Maclaurin series for the given binomial.

1) $(1-x)^{1/3}$

2) $(1+x^2)^{-1/3}$

Answer

$$(1+x^2)^{-1/3}=\sum_{n=0}^{\infty}\left(n^{-\frac{1}{3}}\right)x^{2n}$$

3) $(1-x)^{1.01}$

4) $(1-2x)^{2/3}$

Answer

$$(1-2x)^{2/3}=\sum_{n=0}^{\infty}(-1)^n 2^n\left(n^{\frac{2}{3}}\right)x^n$$

In exercises 5 - 12, use the substitution $(b+x)^r=(b+a)^r\left(1+\dfrac{x-a}{b+a}\right)^r$ **in the binomial expansion to find the Taylor series of each function with the given center.**

5) $\sqrt{x+2}$ at $a=0$

6) $\sqrt{x^2+2}$ at $a=0$

Answer

$$\sqrt{2+x^2}=\sum_{n=0}^{\infty}2^{(1/2)-n}\left(n^{\frac{1}{2}}\right)x^{2n};(|x^2|<2)$$

7) $\sqrt{x+2}$ at $a=1$

8) $\sqrt{2x-x^2}$ at $a=1$ (Hint: $2x-x^2=1-(x-1)^2$)

Answer

$$\sqrt{2x-x^2}=\sqrt{1-(x-1)^2}\text{ so }\sqrt{2x-x^2}=\sum_{n=0}^{\infty}(-1)^n\left(n^{\frac{1}{2}}\right)(x-1)^{2n}$$

9) $(x-8)^{1/3}$ at $a=9$

10) $\sqrt{x}$ at $a=4$

Answer

$$\sqrt{x}=2\sqrt{1+\tfrac{x-4}{4}}\text{ so }\sqrt{x}=\sum_{n=0}^{\infty}2^{1-2n}\left(n^{\frac{1}{2}}\right)(x-4)^n$$

11) $x^{1/3}$ at $a=27$

12) $\sqrt{x}$ at $x=9$

Answer

$$\sqrt{x}=\sum_{n=0}^{\infty}3^{1-3n}\left(n^{\frac{1}{2}}\right)(x-9)^n$$

In exercises 13 - 14, use the binomial theorem to estimate each number, computing enough terms to obtain an estimate accurate to an error of at most 1/1000.

13) [T] $(15)^{1/4}$ using $(16-x)^{1/4}$

14) [T] $(1001)^{1/3}$ using $(1000+x)^{1/3}$

Answer

$10(1+\dfrac{x}{1000})^{1/3}=\sum_{n=0}^{\infty}10^{1-3n}(\frac{1}{n})x^n$. Using, for example, a fourth-degree estimate at $x=1$ gives
$(1001)^{1/3}\approx 10\left(1+\left(1^{\frac{1}{3}}\right)10^{-3}+\left(2^{\frac{1}{3}}\right)10^{-6}+\left(3^{\frac{1}{3}}\right)10^{-9}+\left(3^{\frac{1}{3}}\right)10^{-12}\right)=10\left(1+\frac{1}{3.10^3}-\frac{1}{9.10^6}+\frac{5}{81.10^9}-\frac{10}{243.10^{12}}\right)=10.00333222\ldots$
whereas $(1001)^{1/3}=10.00332222839093$. Two terms would suffice for three-digit accuracy.

In exercises 15 - 18, use the binomial approximation $\sqrt{1-x} \approx 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} - \frac{5x^4}{128} - \frac{7x^5}{256}$ for $|x| < 1$ to approximate each number. Compare this value to the value given by a scientific calculator.

15) [T] $\frac{1}{\sqrt{2}}$ using $x = \frac{1}{2}$ in $(1-x)^{1/2}$

16) [T] $\sqrt{5} = 5 \times \frac{1}{\sqrt{5}}$ using $x = \frac{4}{5}$ in $(1-x)^{1/2}$

Answer

The approximation is 2.3152 the CAS value is $2.23\ldots$.

17) [T] $\sqrt{3} = \frac{3}{\sqrt{3}}$ using $x = \frac{2}{3}$ in $(1-x)^{1/2}$

18) [T] $\sqrt{6}$ using $x = \frac{5}{6}$ in $(1-x)^{1/2}$

Answer

The approximation is $2.583\ldots$; the CAS value is $2.449\ldots$.

19) Integrate the binomial approximation of $\sqrt{1-x}$ to find an approximation of $\int_0^x \sqrt{1-t}\, dt$.

20) [T] Recall that the graph of $\sqrt{1-x^2}$ is an upper semicircle of radius 1. Integrate the binomial approximation of $\sqrt{1-x^2}$ up to order 8 from $x = -1$ to $x = 1$ to estimate $\frac{\pi}{2}$.

Answer

$\sqrt{1-x^2} = 1 - \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^6}{16} - \frac{5x^8}{128} + \cdots$. Thus

$\int_{-1}^{1} \sqrt{1-x^2}\, dx = \left[x - \frac{x^3}{6} - \frac{x^5}{40} - \frac{x^7}{7 \cdot 16} - \frac{5x^9}{9 \cdot 128} + \cdots\right]\Big|_{-1}^{1} \approx 2 - \frac{1}{3} - \frac{1}{20} - \frac{1}{56} - \frac{10}{9 \cdot 128} + error = 1.590\ldots$ whereas $\frac{\pi}{2} = 1.570\ldots$

In exercises 21 - 24, use the expansion $(1+x)^{1/3} = 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 - \frac{10}{243}x^4 + \cdots$ to write the first five terms (not necessarily a quartic polynomial) of each expression.

21) $(1+4x)^{1/3};\ a = 0$

22) $(1+4x)^{4/3};\ a = 0$

Answer

$(1+x)^{4/3} = (1+x)(1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 - \frac{10}{243}x^4 + \cdots) = 1 + \frac{4x}{3} + \frac{2x^2}{9} - \frac{4x^3}{81} + \frac{5x^4}{243} + \cdots$

23) $(3+2x)^{1/3};\ a = -1$

24) $(x^2+6x+10)^{1/3};\ a = -3$

Answer

$(1+(x+3)^2)^{1/3} = 1 + \frac{1}{3}(x+3)^2 - \frac{1}{9}(x+3)^4 + \frac{5}{81}(x+3)^6 - \frac{10}{243}(x+3)^8 + \cdots$

25) Use $(1+x)^{1/3} = 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 - \frac{10}{243}x^4 + \cdots$ with $x = 1$ to approximate $2^{1/3}$.

26) Use the approximation $(1-x)^{2/3} = 1 - \frac{2x}{3} - \frac{x^2}{9} - \frac{4x^3}{81} - \frac{7x^4}{243} - \frac{14x^5}{729} + \cdots$ for $|x| < 1$ to approximate $2^{1/3} = 2.2^{-2/3}$.

Answer

Twice the approximation is $1.260\ldots$whereas $2^{1/3} = 1.2599\ldots$.

27) Find the 25th derivative of $f(x) = (1+x^2)^{13}$ at $x = 0$.

28) Find the 99th derivative of $f(x) = (1+x^4)^{25}$.

Answer

$f^{(99)}(0) = 0$

In exercises 29 - 36, find the Maclaurin series of each function.

29) $f(x) = xe^{2x}$

30) $f(x) = 2^x$

Answer

$\sum_{n=0}^{\infty} \frac{(\ln(2)x)^n}{n!}$

31) $f(x)=\dfrac{\sin x}{x}$

32) $f(x)=\dfrac{\sin(\sqrt{x})}{\sqrt{x}},(x>0),$

Answer

For $x>0$, $\sin(\sqrt{x})=\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{(2n+1)/2}}{\sqrt{x}(2n+1)!}=\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{(2n+1)!}$.

33) $f(x)=\sin(x^2)$

34) $f(x)=e^{x^3}$

Answer

$e^{x^3}=\displaystyle\sum_{n=0}^{\infty}\frac{x^{3n}}{n!}$

35) $f(x)=\cos^2 x$ using the identity $\cos^2 x=\frac{1}{2}+\frac{1}{2}\cos(2x)$

36) $f(x)=\sin^2 x$ using the identity $\sin^2 x=\frac{1}{2}-\frac{1}{2}\cos(2x)$

Answer

$\sin^2 x=-\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k 2^{2k-1}x^{2k}}{(2k)!}$

In exercises 37 - 44, find the Maclaurin series of $F(x)=\displaystyle\int_0^x f(t)\,dt$ by integrating the Maclaurin series of f term by term. If f is not strictly defined at zero, you may substitute the value of the Maclaurin series at zero.

37) $F(x)=\displaystyle\int_0^x e^{-t^2}\,dt;\ f(t)=e^{-t^2}=\sum_{n=0}^{\infty}(-1)^n\frac{t^{2n}}{n!}$

38) $F(x)=\tan^{-1}x;\ f(t)=\dfrac{1}{1+t^2}=\displaystyle\sum_{n=0}^{\infty}(-1)^n t^{2n}$

Answer

$\tan^{-1}x=\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{2k+1}$

39) $F(x)=\tanh^{-1}x;\ f(t)=\dfrac{1}{1-t^2}=\displaystyle\sum_{n=0}^{\infty}t^{2n}$

40) $F(x)=\sin^{-1}x;\ f(t)=\dfrac{1}{\sqrt{1-t^2}}=\displaystyle\sum_{k=0}^{\infty}\left(k^{\frac{1}{2}}\right)\frac{t^{2k}}{k!}$

Answer

$\sin^{-1}x=\displaystyle\sum_{n=0}^{\infty}\left(n^{\frac{1}{2}}\right)\frac{x^{2n+1}}{(2n+1)n!}$

41) $F(x)=\displaystyle\int_0^x\frac{\sin t}{t}\,dt;\ f(t)=\frac{\sin t}{t}=\sum_{n=0}^{\infty}(-1)^n\frac{t^{2n}}{(2n+1)!}$

42) $F(x)=\displaystyle\int_0^x\cos(\sqrt{t})\,dt;\ f(t)=\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{(2n)!}$

Answer

$F(x)=\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{(n+1)(2n)!}$

43) $F(x)=\displaystyle\int_0^x\frac{1-\cos t}{t^2}\,dt;\ f(t)=\frac{1-\cos t}{t^2}=\sum_{n=0}^{\infty}(-1)^n\frac{t^{2n}}{(2n+2)!}$

44) $F(x)=\displaystyle\int_0^x\frac{\ln(1+t)}{t}\,dt;\ f(t)=\sum_{n=0}^{\infty}(-1)^n\frac{t^n}{n+1}$

Answer

$$F(x)=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n^2}$$

In exercises 45 - 52, compute at least the first three nonzero terms (not necessarily a quadratic polynomial) of the Maclaurin series of f.

45) $f(x)=\sin\left(x+\frac{\pi}{4}\right)=\sin x\cos\left(\frac{\pi}{4}\right)+\cos x\sin\left(\frac{\pi}{4}\right)$

46) $f(x)=\tan x$

Answer

$$x+\frac{x^3}{3}+\frac{2x^5}{15}+\cdots$$

47) $f(x)=\ln(\cos x)$

48) $f(x)=e^x\cos x$

Answer

$$1+x-\frac{x^3}{3}-\frac{x^4}{6}+\cdots$$

49) $f(x)=e^{\sin x}$

50) $f(x)=\sec^2 x$

Answer

$$1+x^2+\frac{2x^4}{3}+\frac{17x^6}{45}+\cdots$$

51) $f(x)=\tanh x$

52) $f(x)=\dfrac{\tan\sqrt{x}}{\sqrt{x}}$ (see expansion for $\tan x$)

Answer

Using the expansion for $\tan x$ gives $1+\dfrac{x}{3}+\dfrac{2x^2}{15}$.

In exercises 53 - 56, find the radius of convergence of the Maclaurin series of each function.

53) $\ln(1+x)$

54) $\dfrac{1}{1+x^2}$

Answer

$\dfrac{1}{1+x^2}=\displaystyle\sum_{n=0}^{\infty}(-1)^n x^{2n}$ so $R=1$ by the ratio test.

55) $\tan^{-1}x$

56) $\ln(1+x^2)$

Answer

$\ln(1+x^2)=\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^{2n}$ so $R=1$ by the ratio test.

57) Find the Maclaurin series of $\sinh x=\dfrac{e^x-e^{-x}}{2}$.

58) Find the Maclaurin series of $\cosh x=\dfrac{e^x+e^{-x}}{2}$.

Answer

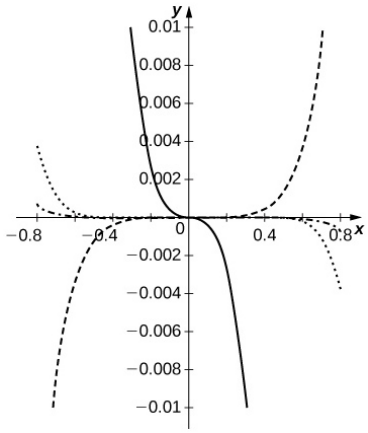
Add series of e^x and e^{-x} term by term. Odd terms cancel and $\cosh x=\displaystyle\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$.

59) Differentiate term by term the Maclaurin series of $\sinh x$ and compare the result with the Maclaurin series of $\cosh x$.

60) [T] Let $S_n(x)=\sum_{k=0}^{n}(-1)^k\frac{x^{2k+1}}{(2k+1)!}$ and $C_n(x)=\sum_{n=0}^{n}(-1)^k\frac{x^{2k}}{(2k)!}$ denote the respective Maclaurin polynomials of degree $2n+1$ of $\sin x$ and degree $2n$ of $\cos x$. Plot the errors $\frac{S_n(x)}{C_n(x)}-\tan x$ for $n=1,\ldots,5$ and compare them to $x+\frac{x^3}{3}+\frac{2x^5}{15}+\frac{17x^7}{315}-\tan x$ on $\left(-\frac{\pi}{4},\frac{\pi}{4}\right)$.

Answer

The ratio $\frac{S_n(x)}{C_n(x)}$ approximates $\tan x$ better than does $p_7(x)=x+\frac{x^3}{3}+\frac{2x^5}{15}+\frac{17x^7}{315}$ for $N\geq 3$. The dashed curves are $\frac{S_n}{C_n}-\tan x$ for $n=1,\ 2$. The dotted curve corresponds to $n=3$, and the dash-dotted curve corresponds to $n=4$. The solid curve is $p_7-\tan x$.



61) Use the identity $2\sin x\cos x=\sin(2x)$ to find the power series expansion of $\sin^2 x$ at $x=0$. (Hint: Integrate the Maclaurin series of $\sin(2x)$ term by term.)

62) If $y=\sum_{n=0}^{\infty}a_nx^n$, find the power series expansions of xy' and x^2y''.

Answer

By the term-by-term differentiation theorem, $y'=\sum_{n=1}^{\infty}na_nx^{n-1}$ so $y'=\sum_{n=1}^{\infty}na_nx^{n-1}xy'=\sum_{n=1}^{\infty}na_nx^n$, whereas $y'=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}$ so $xy''=\sum_{n=2}^{\infty}n(n-1)a_nx^n$.

63) [T] Suppose that $y=\sum_{k=0}^{\infty}a^kx^k$ satisfies $y'=-2xy$ and $y(0)=0$. Show that $a_{2k+1}=0$ for all k and that $a_{2k+2}=\frac{-a_{2k}}{k+1}$. Plot the partial sum S_{20} of y on the interval $[-4,4]$.

64) [T] Suppose that a set of standardized test scores is normally distributed with mean $\mu=100$ and standard deviation $\sigma=10$. Set up an integral that represents the probability that a test score will be between 90 and 110 and use the integral of the degree 10 Maclaurin polynomial of $\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ to estimate this probability.

Answer

The probability is $p=\frac{1}{\sqrt{2\pi}}\int_{(a-\mu)/\sigma}^{(b-\mu)/\sigma}e^{-x^2/2}\,dx$ where $a=90$ and $b=100$, that is,

$$p=\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}e^{-x^2/2}\,dx=\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}\sum_{n=0}^{5}(-1)^n\frac{x^{2n}}{2^n n!}\,dx=\frac{2}{\sqrt{2\pi}}\sum_{n=0}^{5}(-1)^n\frac{1}{(2n+1)2^n n!}\approx 0.6827.$$

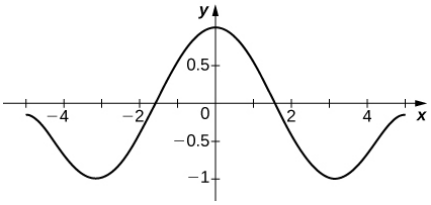
65) [T] Suppose that a set of standardized test scores is normally distributed with mean $\mu=100$ and standard deviation $\sigma=10$. Set up an integral that represents the probability that a test score will be between 70 and 130 and use the integral of the degree 50 Maclaurin polynomial of $\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ to estimate this probability.

66) [T] Suppose that $\sum_{n=0}^{\infty}a_nx^n$ converges to a function $f(x)$ such that $f(0)=1,\ f'(0)=0$, and $f''(x)=-f(x)$. Find a formula for a_n and plot the partial sum S_N for $N=20$ on $[-5,5]$.

Answer

https://math.libretexts.org/@go/page/83991

As in the previous problem one obtains $a_n = 0$ if n is odd and $a_n = -(n+2)(n+1)a_{n+2}$ if n is even, so $a_0 = 1$ leads to $a_{2n} = \dfrac{(-1)^n}{(2n)!}$.



67) [T] Suppose that $\sum_{n=0}^{\infty} a_n x^n$ converges to a function $f(x)$ such that $f(0) = 0,\ f'(0) = 1$, and $f''(x) = -f(x)$. Find a formula for an and plot the partial sum S_N for $N = 10$ on $[-5, 5]$.

68) Suppose that $\sum_{n=0}^{\infty} a_n x^n$ converges to a function y such that $y'' - y' + y = 0$ where $y(0) = 1$ and $y'(0) = 0$. Find a formula that relates $a_{n+2},\ a_{n+1}$, and an and compute $a_0, \ldots, a_5$.

Answer

$y'' = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$ and $y' = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$ so $y'' - y' + y = 0$ implies that $(n+2)(n+1)a_{n+2} - (n+1)a_{n+1} + a_n = 0$ or $a_n = \dfrac{a_{n-1}}{n} - \dfrac{a_{n-2}}{n(n-1)}$ for all $n \cdot y(0) = a_0 = 1$ and $y'(0) = a_1 = 0$, so $a_2 = \frac{1}{2},\ a_3 = \frac{1}{6}$, $a_4 = 0$, and $a_5 = -\frac{1}{120}$.

69) Suppose that $\sum_{n=0}^{\infty} a_n x^n$ converges to a function y such that $y'' - y' + y = 0$ where $y(0) = 0$ and $y'(0) = 1$. Find a formula that relates $a_{n+2},\ a_{n+1}$, and an and compute $a_1, \ldots, a_5$.

The error in approximating the integral $\int_a^b f(t)\,dt$ by that of a Taylor approximation $\int_a^b Pn(t)\,dt$ is at most $\int_a^b R_n(t)\,dt$. In exercises 70 - 71, the Taylor remainder estimate $R_n \le \frac{M}{(n+1)!}|x-a|^{n+1}$ guarantees that the integral of the Taylor polynomial of the given order approximates the integral of f with an error less than $\frac{1}{10}$.

a. Evaluate the integral of the appropriate Taylor polynomial and verify that it approximates the CAS value with an error less than $\frac{1}{100}$.

b. Compare the accuracy of the polynomial integral estimate with the remainder estimate.

70) [T] $\int_0^{\pi} \dfrac{\sin t}{t}\,dt;\quad P_s = 1 - \dfrac{x^2}{3!} + \dfrac{x^4}{5!} - \dfrac{x^6}{7!} + \dfrac{x^8}{9!}$ (You may assume that the absolute value of the ninth derivative of $\frac{\sin t}{t}$ is bounded by 0.1.)

Answer

a. (Proof)

b. We have $R_s \le \frac{0.1}{(9)!}\pi^9 \approx 0.0082 < 0.01$. We have $\int_0^{\pi}\left(1 - \dfrac{x^2}{3!} + \dfrac{x^4}{5!} - \dfrac{x^6}{7!} + \dfrac{x^8}{9!}\right)dx = \pi - \dfrac{\pi^3}{3\cdot 3!} + \dfrac{\pi^5}{5\cdot 5!} - \dfrac{\pi^7}{7\cdot 7!} + \dfrac{\pi^9}{9\cdot 9!} = 1.852\ldots$, whereas $\int_0^{\pi} \dfrac{\sin t}{t}\,dt = 1.85194..$, so the actual error is approximately 0.00006.

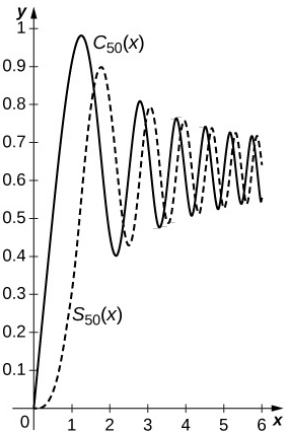
71) [T] $\int_0^2 e^{-x^2}\,dx;\ p_{11} = 1 - x^2 + \dfrac{x^4}{2} - \dfrac{x^6}{3!} + \cdots - \dfrac{x^{22}}{11!}$ (You may assume that the absolute value of the 23rd derivative of e^{-x^2} is less than 2×10^{14}.)

The following exercises (72-73) deal with *Fresnel integrals*.

72) The Fresnel integrals are defined by $C(x) = \int_0^x \cos(t^2)\,dt$ and $S(x) = \int_0^x \sin(t^2)\,dt$. Compute the power series of $C(x)$ and $S(x)$ and plot the sums $C_N(x)$ and $S_N(x)$ of the first $N = 50$ nonzero terms on $[0, 2\pi]$.

Answer

Since $\cos(t^2) = \sum_{n=0}^{\infty}(-1)^n \dfrac{t^{4n}}{(2n)!}$ and $\sin(t^2) = \sum_{n=0}^{\infty}(-1)^n \dfrac{t^{4n+2}}{(2n+1)!}$, one has $S(x) = \sum_{n=0}^{\infty}(-1)^n \dfrac{x^{4n+3}}{(4n+3)(2n+1)!}$ and $C(x) = \sum_{n=0}^{\infty}(-1)^n \dfrac{x^{4n+1}}{(4n+1)(2n)!}$. The sums of the first 50 nonzero terms are plotted below with $C_{50}(x)$ the solid curve and $S_{50}(x)$ the dashed curve.



73) [T] The Fresnel integrals are used in design applications for roadways and railways and other applications because of the curvature properties of the curve with coordinates $(C(t), S(t))$. Plot the curve (C_{50}, S_{50}) for $0 \le t \le 2\pi$, the coordinates of which were computed in the previous exercise.

74) Estimate $\int_0^{1/4} \sqrt{x - x^2}\, dx$ by approximating $\sqrt{1-x}$ using the binomial approximation $1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} - \frac{5x^4}{2128} - \frac{7x^5}{256}$.

Answer

$\int_0^{1/4} \sqrt{x}\left(1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} - \frac{5x^4}{128} - \frac{7x^5}{256}\right) dx = \frac{2}{3}2^{-3} - \frac{1}{2}\frac{2}{5}2^{-5} - \frac{1}{8}\frac{2}{7}2^{-7} - \frac{1}{16}\frac{2}{9}2^{-9} - \frac{5}{128}\frac{2}{11}2^{-11} - \frac{7}{256}\frac{2}{13}2^{-13} = 0.0767732\ldots$
whereas $\int_0^{1/4} \sqrt{x - x^2}\, dx = 0.076773.$

75) [T] Use Newton's approximation of the binomial $\sqrt{1-x^2}$ to approximate π as follows. The circle centered at $(\frac{1}{2}, 0)$ with radius $\frac{1}{2}$ has upper semicircle $y = \sqrt{x}\sqrt{1-x}$. The sector of this circle bounded by the x-axis between $x = 0$ and $x = \frac{1}{2}$ and by the line joining $(\frac{1}{4}, \frac{\sqrt{3}}{4})$ corresponds to $\frac{1}{6}$ of the circle and has area $\frac{\pi}{24}$. This sector is the union of a right triangle with height $\frac{\sqrt{3}}{4}$ and base $\frac{1}{4}$ and the region below the graph between $x = 0$ and $x = \frac{1}{4}$. To find the area of this region you can write $y = \sqrt{x}\sqrt{1-x} = \sqrt{x} \times (\text{binomial expansion of} \sqrt{1-x})$ and integrate term by term. Use this approach with the binomial approximation from the previous exercise to estimate π.

76) Use the approximation $T \approx 2\pi\sqrt{\frac{L}{g}}(1 + \frac{k^2}{4})$ to approximate the period of a pendulum having length 10 meters and maximum angle $\theta_{max} = \frac{\pi}{6}$ where $k = \sin\left(\frac{\theta_{max}}{2}\right)$. Compare this with the small angle estimate $T \approx 2\pi\sqrt{\frac{L}{g}}$.

Answer

$T \approx 2\pi\sqrt{\frac{10}{9.8}}\left(1 + \frac{\sin^2(\theta/12)}{4}\right) \approx 6.453$ seconds. The small angle estimate is $T \approx 2\pi\sqrt{\frac{10}{9.8}} \approx 6.347$. The relative error is around 2 percent.

77) Suppose that a pendulum is to have a period of 2 seconds and a maximum angle of $\theta_{max} = \frac{\pi}{6}$. Use $T \approx 2\pi\sqrt{\frac{L}{g}}\left(1 + \frac{k^2}{4}\right)$ to approximate the desired length of the pendulum. What length is predicted by the small angle estimate $T \approx 2\pi\sqrt{\frac{L}{g}}$?

78) Evaluate $\int_0^{\pi/2} \sin^4\theta\, d\theta$ in the approximation $T = 4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\left(1 + \frac{1}{2}k^2\sin^2\theta + \frac{3}{8}k^4\sin^4\theta + \cdots\right) d\theta$ to obtain an improved estimate for T.

Answer

$\int_0^{\pi/2} \sin^4\theta\, d\theta = \frac{3\pi}{16}$. Hence $T \approx 2\pi\sqrt{\frac{L}{g}}\left(1 + \frac{k^2}{4} + \frac{9}{256}k^4\right)$.

79) [T] An equivalent formula for the period of a pendulum with amplitude θ_{max} is $T(\theta_{max}) = 2\sqrt{2}\sqrt{\frac{L}{g}}\int_0^{\theta_{max}} \frac{d\theta}{\sqrt{\cos\theta - \cos(\theta_{max})}}$ where L is the pendulum length and g is the gravitational acceleration constant. When $\theta_{max} = \frac{\pi}{3}$ we get $\frac{1}{\sqrt{\cos t - 1/2}} \approx \sqrt{2}\left(1 + \frac{t^2}{2} + \frac{t^4}{3} + \frac{181t^6}{720}\right)$. Integrate this approximation to estimate $T(\frac{\pi}{3})$ in terms of L and g. Assuming $g = 9.806$ meters per second squared, find an approximate length L such that $T(\frac{\pi}{3}) = 2$ seconds.

Contributors

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2.6: Chapter 2 Review Exercises

True or False? In exercises 1 - 4, justify your answer with a proof or a counterexample.

1) If the radius of convergence for a power series $\sum_{n=0}^{\infty} a_n x^n$ is 5, then the radius of convergence for the series $\sum_{n=1}^{\infty} n a_n x^{n-1}$ is also 5.

Answer

True

2) Power series can be used to show that the derivative of e^x is e^x. (Hint: Recall that $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$.)

3) For small values of x, $\sin x \approx x$.

Answer

True

4) The radius of convergence for the Maclaurin series of $f(x) = 3^x$ is 3.

In exercises 5 - 8, find the radius of convergence and the interval of convergence for the given series.

5) $\sum_{n=0}^{\infty} n^2 (x-1)^n$

Answer

ROC: 1; IOC: $(0, 2)$

6) $\sum_{n=0}^{\infty} \frac{x^n}{n^n}$

7) $\sum_{n=0}^{\infty} \frac{3n x^n}{12^n}$

Answer

ROC: 12; IOC: $(-16, 8)$

8) $\sum_{n=0}^{\infty} \frac{2^n}{e^n} (x-e)^n$

In exercises 9 - 10, find the power series representation for the given function. Determine the radius of convergence and the interval of convergence for that series.

9) $f(x) = \frac{x^2}{x+3}$

Answer

$\sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}} x^n$; ROC: 3; IOC: $(-3, 3)$

10) $f(x) = \frac{8x+2}{2x^2-3x+1}$

In exercises 11 - 12, find the power series for the given function using term-by-term differentiation or integration.

11) $f(x) = \tan^{-1}(2x)$

Answer

integration: $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}(2x)^{2n+1}$

12) $f(x) = \frac{x}{(2+x^2)^2}$

In exercises 13 - 14, evaluate the Taylor series expansion of degree four for the given function at the specified point. What is the error in the approximation?

13) $f(x) = x^3 - 2x^2 + 4, \quad a = -3$

Answer

$p_4(x) = (x+3)^3 - 11(x+3)^2 + 39(x+3) - 41; \quad$ exact

14) $f(x) = e^{1/(4x)}, \quad a = 4$

In exercises 15 - 16, find the Maclaurin series for the given function.

15) $f(x) = \cos(3x)$

Answer

$\sum_{n=0}^{\infty} \frac{(-1)^n (3x)^{2n}}{2n!}$

16) $f(x) = \ln(x+1)$

In exercises 17 - 18, find the Taylor series at the given value.

17) $f(x) = \sin x, \quad a = \frac{\pi}{2}$

Answer

$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(x - \frac{\pi}{2}\right)^{2n}$

18) $f(x) = \frac{3}{x}, \quad a = 1$

In exercises 19 - 20, find the Maclaurin series for the given function.

19) $f(x) = e^{-x^2} - 1$

Answer

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} x^{2n}$

20) $f(x) = \cos x - x \sin x$

In exercises 21 - 23, find the Maclaurin series for $F(x) = \int_0^x f(t)\,dt$ by integrating the Maclaurin series of $f(x)$ term by term.

21) $f(x) = \frac{\sin x}{x}$

Answer

$F(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+1)!} x^{2n+1}$

22) $f(x) = 1 - e^x$

23) Use power series to prove **Euler's formula:** $e^{ix} = cosx + isinx$

Answer

Answers may vary.

Exercises 24 - 26 consider problems of *annuity payments*.

24) For annuities with a present value of $\$1$ million, calculate the annual payouts given over 25 years assuming interest rates of $1\%, 5\%$, and 10%.

25) A lottery winner has an annuity that has a present value of $\$10$ million. What interest rate would they need to live on perpetual annual payments of $\$250,000$?

Answer

2.5%

26) Calculate the necessary present value of an annuity in order to support annual payouts of $\$15,000$ given over 25 years assuming interest rates of $1\%, 5\%$,and 10%.

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CHAPTER OVERVIEW

3: Parametric Equations and Polar Coordinates

Parametric equations define a group of quantities as functions of one or more independent variables called parameters. Parametric equations are commonly used to express the coordinates of the points that make up a geometric object such as a curve or surface, in which case the equations are collectively called a parametric representation or parameterization. The polar coordinate system is a two-dimensional coordinate system in which each point on a plane is determined by a distance from a reference point and an angle from a reference direction. The reference point (analogous to the origin of a Cartesian system) is called the pole, and the ray from the pole in the reference direction is the polar axis. The distance from the pole is called the radial coordinate or radius, and the angle is called the angular coordinate, polar angle, or azimuth

3.1: Prelude to Parametric Equations and Polar Coordinates

3.2: Parametric Equations

3.2E: Exercises for Section 3.2

3.3: Calculus of Parametric Curves

3.3E: Exercises for Section 3.3

3.4: Polar Coordinates

3.4E: Exercises for Section 3.4

3.5: Area and Arc Length in Polar Coordinates

3.5E: Exercises for Section 3.5

3.6: Conic Sections

3.6E: Exercises for Section 3.6

3.7: Chapter 3 Review Exercises

Contributors and Attributions

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3.1: Prelude to Parametric Equations and Polar Coordinates

The **chambered nautilus** is a fascinating creature. This animal feeds on hermit crabs, fish, and other crustaceans. It has a hard outer shell with many chambers connected in a spiral fashion, and it can retract into its shell to avoid predators. When part of the shell is cut away, a perfect spiral is revealed, with chambers inside that are somewhat similar to growth rings in a tree.

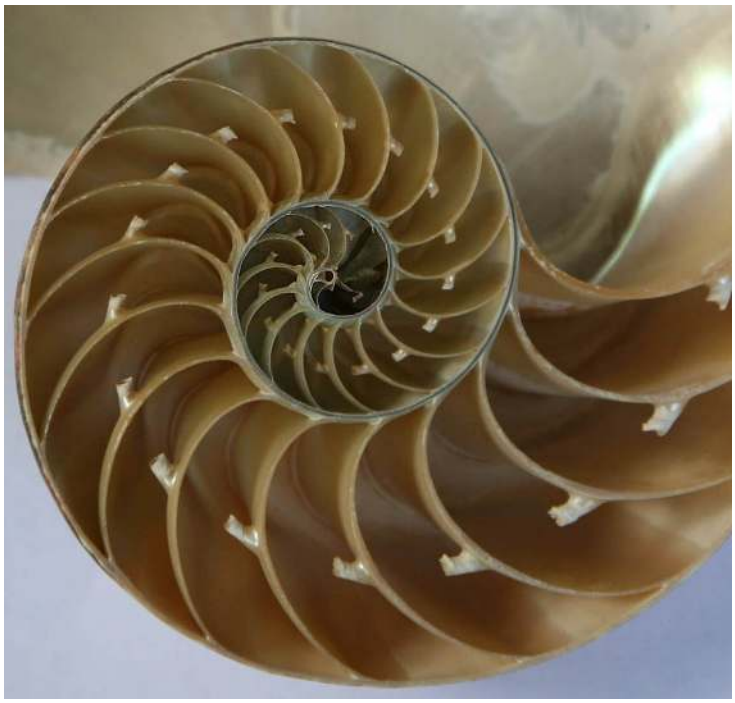


Figure 3.1.1: The chambered nautilus is a marine animal that lives in the tropical Pacific Ocean. Scientists think they have existed mostly unchanged for about 500 million years.(credit: modification of work by Jitze Couperus, Flickr)

The mathematical function that describes a spiral can be expressed using rectangular (or Cartesian) coordinates. However, if we change our coordinate system to something that works a bit better with circular patterns, the function becomes much simpler to describe. The polar coordinate system is well suited for describing curves of this type. How can we use this coordinate system to describe spirals and other radial figures?

In this chapter we also study parametric equations, which give us a convenient way to describe curves, or to study the position of a particle or object in two dimensions as a function of time. We will use parametric equations and polar coordinates for describing many topics later in this text.

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3.2: Parametric Equations

Learning Objectives

- Plot a curve described by parametric equations.
- Convert the parametric equations of a curve into the form $y = f(x)$.
- Recognize the parametric equations of basic curves, such as a line and a circle.
- Recognize the parametric equations of a cycloid.

In this section we examine parametric equations and their graphs. In the two-dimensional coordinate system, parametric equations are useful for describing curves that are not necessarily functions. The parameter is an independent variable that both x and y depend on, and as the parameter increases, the values of x and y trace out a path along a plane curve. For example, if the parameter is t (a common choice), then t might represent time. Then x and y are defined as functions of time, and $(x(t), y(t))$ can describe the position in the plane of a given object as it moves along a curved path.

Parametric Equations and Their Graphs

Consider the orbit of Earth around the Sun. Our year lasts approximately 365.25 days, but for this discussion we will use 365 days. On January 1 of each year, the physical location of Earth with respect to the Sun is nearly the same, except for leap years, when the lag introduced by the extra $\frac{1}{4}$ day of orbiting time is built into the calendar. We call January 1 "day 1" of the year. Then, for example, day 31 is January 31, day 59 is February 28, and so on.

The number of the day in a year can be considered a variable that determines Earth's position in its orbit. As Earth revolves around the Sun, its physical location changes relative to the Sun. After one full year, we are back where we started, and a new year begins. According to Kepler's laws of planetary motion, the shape of the orbit is elliptical, with the Sun at one focus of the ellipse. We study this idea in more detail in Conic Sections.

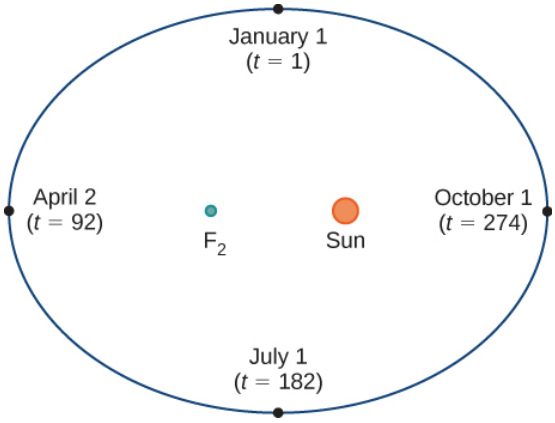


Figure 3.2.1: Earth's orbit around the Sun in one year.

Figure 3.2.1 depicts **Earth's orbit** around the Sun during one year. The point labeled F_2 is one of the foci of the ellipse; the other focus is occupied by the Sun. If we superimpose coordinate axes over this graph, then we can assign ordered pairs to each point on the ellipse (Figure 3.2.2). Then each x value on the graph is a value of position as a function of time, and each y value is also a value of position as a function of time. Therefore, each point on the graph corresponds to a value of Earth's position as a function of time.

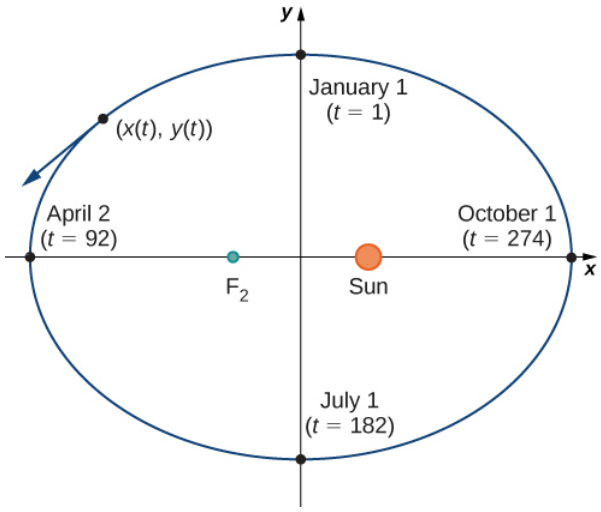


Figure 3.2.2: Coordinate axes superimposed on the orbit of Earth.

We can determine the functions for $x(t)$ and $y(t)$, thereby parameterizing the orbit of Earth around the Sun. The variable t is called an independent parameter and, in this context, represents time relative to the beginning of each year.

A curve in the (x, y) plane can be represented parametrically. The equations that are used to define the curve are called *parametric equations.*

Definition: Parametric Equations

If x and y are continuous functions of t on an interval I, then the equations

$$x = x(t) \tag{3.2.1}$$

and

$$y = y(t) \tag{3.2.2}$$

are called parametric equations and t is called the parameter. The set of points (x, y) obtained as t varies over the interval I is called the graph of the parametric equations. The graph of parametric equations is called a *parametric curve* or plane curve, and is denoted by C.

Notice in this definition that x and y are used in two ways. The first is as functions of the independent variable t. As t varies over the interval I, the functions $x(t)$ and $y(t)$ generate a set of ordered pairs (x, y). This set of ordered pairs generates the graph of the parametric equations. In this second usage, to designate the ordered pairs, x and y are variables. It is important to distinguish the variables x and y from the functions $x(t)$ and $y(t)$.

Example 3.2.1: Graphing a Parametrically Defined Curve

Sketch the curves described by the following parametric equations:

a. $x(t) = t - 1, \quad y(t) = 2t + 4, \quad \text{for } -3 \le t \le 2$
b. $x(t) = t^2 - 3, \quad y(t) = 2t + 1, \quad \text{for } -2 \le t \le 3$
c. $x(t) = 4\cos t, \quad y(t) = 4\sin t, \quad \text{for } 0 \le t \le 2\pi$

Solution

a. To create a graph of this curve, first set up a table of values. Since the independent variable in both $x(t)$ and $y(t)$ is t, let t appear in the first column. Then $x(t)$ and $y(t)$ will appear in the second and third columns of the table.

t	$x(t)$	$y(t)$
−3	−4	−2
−2	−3	0

t	$x(t)$	$y(t)$
−1	−2	2
0	−1	4
1	0	6
2	1	8

The second and third columns in this table provide a set of points to be plotted. The graph of these points appears in Figure 3.2.3. The arrows on the graph indicate the **orientation** of the graph, that is, the direction that a point moves on the graph as t varies from −3 to 2.

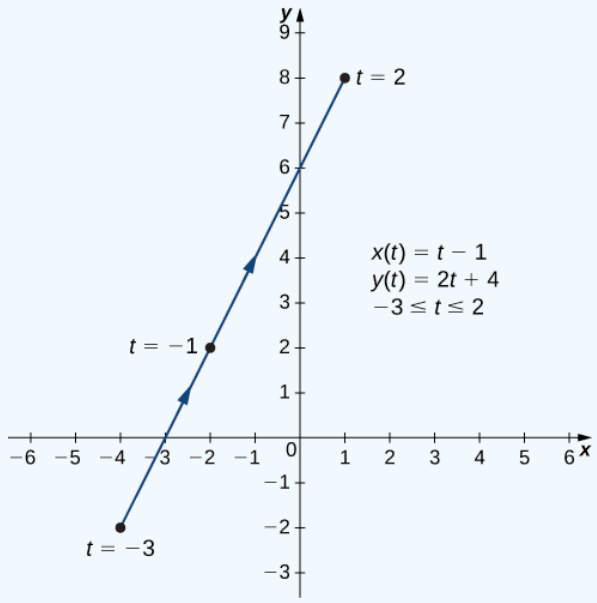


Figure 3.2.3: Graph of the plane curve described by the parametric equations in part a.

b. To create a graph of this curve, again set up a table of values.

t	$x(t)$	$y(t)$
−2	1	−3
−1	−2	−1
0	−3	1
1	−2	3
2	1	5
3	6	7

The second and third columns in this table give a set of points to be plotted (Figure 3.2.4). The first point on the graph (corresponding to $t = -2$) has coordinates $(1, -3)$, and the last point (corresponding to $t = 3$) has coordinates $(6, 7)$. As t progresses from -2 to 3, the point on the curve travels along a parabola. The direction the point moves is again called the orientation and is indicated on the graph.

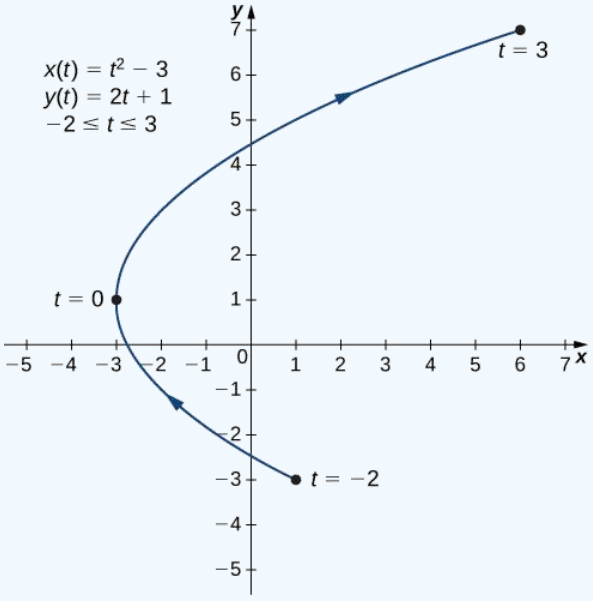


Figure 3.2.4: Graph of the plane curve described by the parametric equations in part b.

c. In this case, use multiples of $\pi/6$ for t and create another table of values:

t	$x(t)$	$y(t)$	t	$x(t)$	$y(t)$
0	4	0	$\frac{7\pi}{6}$	$-2\sqrt{3} \approx -3.5$	-2
$\frac{\pi}{6}$	$2\sqrt{3} \approx 3.5$	2	$\frac{4\pi}{3}$	-2	$-2\sqrt{3} \approx -3.5$
$\frac{\pi}{3}$	2	$2\sqrt{3} \approx 3.5$	$\frac{3\pi}{2}$	0	-4
$\frac{\pi}{2}$	0	4	$\frac{5\pi}{3}$	2	$-2\sqrt{3} \approx -3.5$
$\frac{2\pi}{3}$	-2	$2\sqrt{3} \approx 3.5$	$\frac{11\pi}{6}$	$2\sqrt{3} \approx 3.5$	-2
$\frac{5\pi}{6}$	$-2\sqrt{3} \approx -3.5$	2	2π	4	0
π	-4	0			

The graph of this plane curve appears in the following graph.

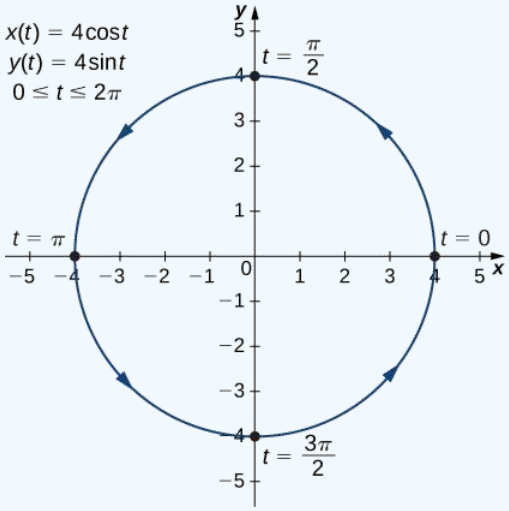


Figure 3.2.5: Graph of the plane curve described by the parametric equations in part c.

This is the graph of a circle with radius 4 centered at the origin, with a counterclockwise orientation. The starting point and ending points of the curve both have coordinates $(4, 0)$.

Exercise 3.2.1

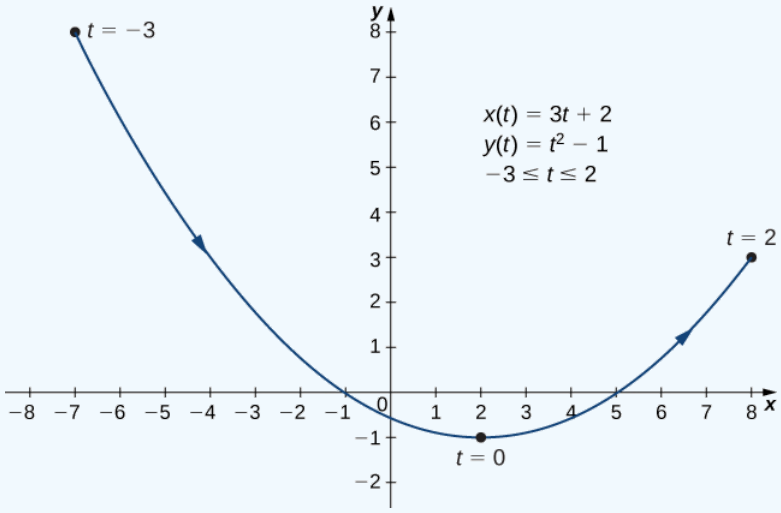
Sketch the curve described by the parametric equations

$$x(t) = 3t + 2, \quad y(t) = t^2 - 1, \quad \text{for } -3 \leq t \leq 2.$$

Hint

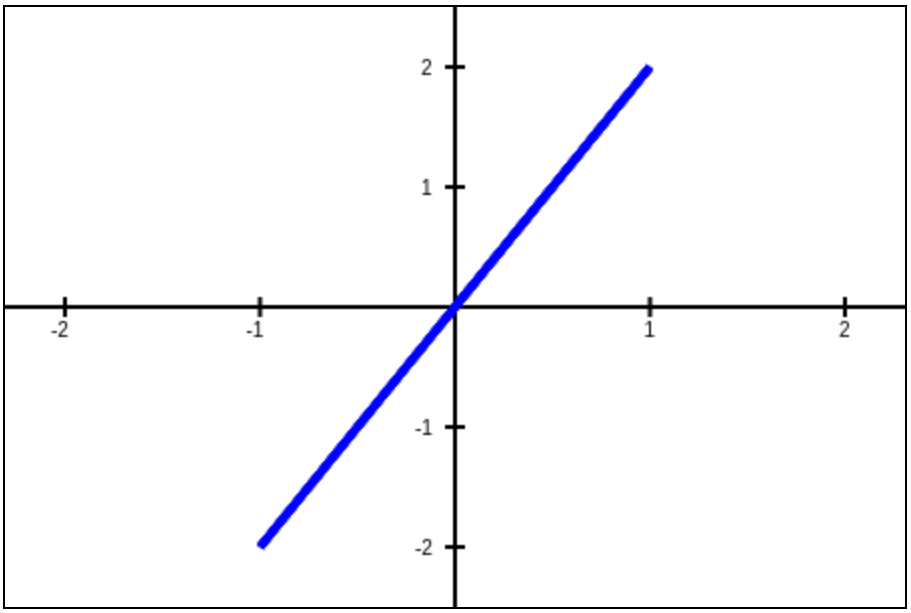
Make a table of values for $x(t)$ and $y(t)$ using t values from -3 to 2.

Answer



Visualizing a Parametric Curves As Time Elapses

Unlike curves that are described in the form $y = f(x)$ that just give the static path of an object, parametrically described curves, $x = x(t), y = y(t)$, give the dynamic motion of an object. The visualization below shows the graph of the static curve defined parametrically, but if you click on the "Move" button, you can observe its dynamic motion by observing the dynamic motion of an ant as it travels according to the motion described by the parametrically defined path. If you click on the Next button, it will take you to a different parametrically defined curve.



$x(t) = \sin(t), y(t) = 2\sin(t), 0 < t < 20$

Move Next

Eliminating the Parameter

To better understand the graph of a curve represented parametrically, it is useful to rewrite the two equations as a single equation relating the variables x and y. Then we can apply any previous knowledge of equations of curves in the plane to identify the curve. For example, the equations describing the plane curve in Example $3.2.1b$ are

$$x(t) = t^2 - 3 \tag{3.2.3}$$

$$y(t) = 2t + 1 \tag{3.2.4}$$

over the region $-2 \leq t \leq 3$.

Solving Equation 3.2.4 for t gives

$$t = \frac{y-1}{2}.$$

This can be substituted into Equation 3.2.3:

$$x = \left(\frac{y-1}{2}\right)^2 - 3 \tag{3.2.5}$$

$$= \frac{y^2 - 2y + 1}{4} - 3 \tag{3.2.6}$$

$$= \frac{y^2 - 2y - 11}{4}. \tag{3.2.7}$$

Equation 3.2.7 describes x as a function of y. These steps give an example of eliminating the parameter. The graph of this function is a parabola opening to the right (Figure $3.2.4$). Recall that the plane curve started at $(1, -3)$ and ended at $(6, 7)$. These terminations were due to the restriction on the parameter t.





Example 3.2.2: Eliminating the Parameter

Eliminate the parameter for each of the plane curves described by the following parametric equations and describe the resulting graph.

a. $x(t) = \sqrt{2t+4}, \quad y(t) = 2t+1, \quad \text{for } -2 \le t \le 6$
b. $x(t) = 4\cos t, \quad y(t) = 3\sin t, \quad \text{for } 0 \le t \le 2\pi$

Solution

a. To eliminate the parameter, we can solve either of the equations for t. For example, solving the first equation for t gives

$$x = \sqrt{2t+4}$$
$$x^2 = 2t+4$$
$$x^2 - 4 = 2t$$
$$t = \frac{x^2-4}{2}.$$

Note that when we square both sides it is important to observe that $x \geq 0$. Substituting $t = \frac{x^2-4}{2}$ into $y(t)$ yields

$$y(t) = 2t+1 \tag{3.2.8}$$
$$y = 2\left(\frac{x^2-4}{2}\right)+1 \tag{3.2.9}$$
$$y = x^2 - 4 + 1 \tag{3.2.10}$$
$$y = x^2 - 3. \tag{3.2.11}$$

This is the equation of a parabola opening upward. There is, however, a domain restriction because of the limits on the parameter t. When $t = -2$, $x = \sqrt{2(-2)+4} = 0$, and when $t = 6$, $x = \sqrt{2(6)+4} = 4$. The graph of this plane curve follows.

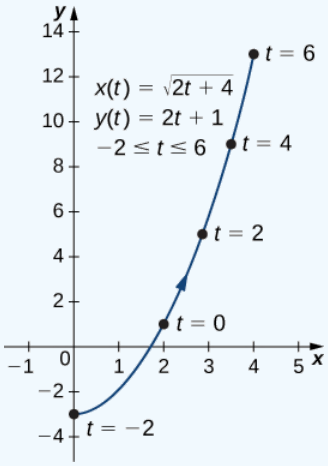


Figure 3.2.6: Graph of the plane curve described by the parametric equations in part a.

b. Sometimes it is necessary to be a bit creative in eliminating the parameter. The parametric equations for this example are

$$x(t) = 4\cos t$$

and

$$y(t) = 3\sin t$$

Solving either equation for t directly is not advisable because sine and cosine are not one-to-one functions. However, dividing the first equation by 4 and the second equation by 3 (and suppressing the t) gives us

$$\cos t = \frac{x}{4}$$

and

$$\sin t = \frac{y}{3}.$$

Now use the Pythagorean identity $\cos^2 t + \sin^2 t = 1$ and replace the expressions for $\sin t$ and $\cos t$ with the equivalent expressions in terms of x and y. This gives

$$\left(\frac{x}{4}\right)^2+\left(\frac{y}{3}\right)^2=1$$

$$\frac{x^2}{16}+\frac{y^2}{9}=1.$$

This is the equation of a horizontal ellipse centered at the origin, with semi-major axis 4 and semi-minor axis 3 as shown in the following graph.

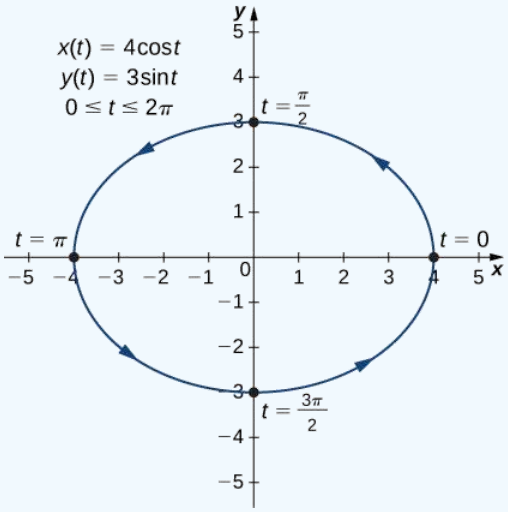


Figure 3.2.7: Graph of the plane curve described by the parametric equations in part b.

As t progresses from 0 to 2π, a point on the curve traverses the ellipse once, in a counterclockwise direction. Recall from the section opener that the orbit of Earth around the Sun is also elliptical. This is a perfect example of using parameterized curves to model a real-world phenomenon.

Exercise 3.2.2

Eliminate the parameter for the plane curve defined by the following parametric equations and describe the resulting graph.

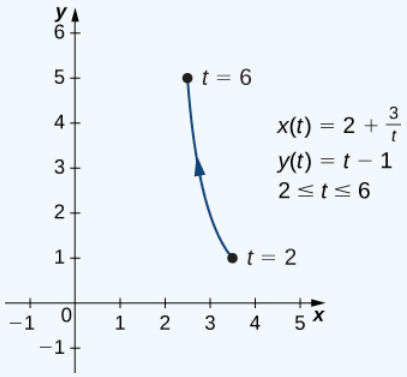
$$x(t)=2+\frac{3}{t},\quad y(t)=t-1,\quad \text{for } 2\le t\le 6$$

Hint

Solve one of the equations for t and substitute into the other equation.

Answer

$x=2+\frac{3}{y+1}$, or $y=-1+\frac{3}{x-2}$. This equation describes a portion of a rectangular hyperbola centered at $(2,-1)$.



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So far we have seen the method of eliminating the parameter, assuming we know a set of parametric equations that describe a plane curve. What if we would like to start with the equation of a curve and determine a pair of parametric equations for that curve? This is certainly possible, and in fact it is possible to do so in many different ways for a given curve. The process is known as **parameterization of a curve.**

Example 3.2.3: Parameterizing a Curve

Find two different pairs of parametric equations to represent the graph of $y = 2x^2 - 3$.

Solution

First, it is always possible to parameterize a curve by defining $x(t) = t$, then replacing x with t in the equation for $y(t)$. This gives the parameterization

$$x(t) = t, \quad y(t) = 2t^2 - 3.$$

Since there is no restriction on the domain in the original graph, there is no restriction on the values of t.

We have complete freedom in the choice for the second parameterization. For example, we can choose $x(t) = 3t - 2$. The only thing we need to check is that there are no restrictions imposed on x; that is, the range of $x(t)$ is all real numbers. This is the case for $x(t) = 3t - 2$. Now since $y = 2x^2 - 3$, we can substitute $x(t) = 3t - 2$ for x. This gives

$$y(t) = 2(3t-2)^2 - 2 = 2(9t^2 - 12t + 4) - 2 = 18t^2 - 24t + 8 - 2 = 18t^2 - 24t + 6.$$

Therefore, a second parameterization of the curve can be written as

$$x(t) = 3t - 2 \text{ and } y(t) = 18t^2 - 24t + 6.$$







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Exercise 3.2.3

Find two different sets of parametric equations to represent the graph of $y = x^2 + 2x$.

Hint

Follow the steps in Example 3.2.3 Remember we have freedom in choosing the parameterization for $x(t)$.

Answer

One possibility is $x(t) = t, \quad y(t) = t^2 + 2t.$ Another possibility is $x(t) = 2t - 3, \quad y(t) = (2t-3)^2 + 2(2t-3) = 4t^2 - 8t + 3.$ There are, in fact, an infinite number of possibilities.



Cycloids and Other Parametric Curves

Imagine going on a bicycle ride through the country. The tires stay in contact with the road and rotate in a predictable pattern. Now suppose a very determined ant is tired after a long day and wants to get home. So he hangs onto the side of the tire and gets a free ride. The path that this ant travels down a straight road is called a *cycloid* (Figure 3.2.8). A cycloid generated by a circle (or bicycle wheel) of radius a is given by the parametric equations

$$x(t) = a(t - \sin t), \quad y(t) = a(1 - \cos t).$$

To see why this is true, consider the path that the center of the wheel takes. The center moves along the x-axis at a constant height equal to the radius of the wheel. If the radius is a, then the coordinates of the center can be given by the equations

$$x(t) = at, \quad y(t) = a$$

for any value of t. Next, consider the ant, which rotates around the center along a circular path. If the bicycle is moving from left to right then the wheels are rotating in a clockwise direction. A possible parameterization of the circular motion of the ant (relative to the center of the wheel) is given by

$$x(t) = -a \sin t$$
$$y(t) = -a \cos t.$$

(The negative sign is needed to reverse the orientation of the curve. If the negative sign were not there, we would have to imagine the wheel rotating counterclockwise.) Adding these equations together gives the equations for the cycloid.

$$x(t) = a(t - \sin t)$$
$$y(t) = a(1 - \cos t)$$

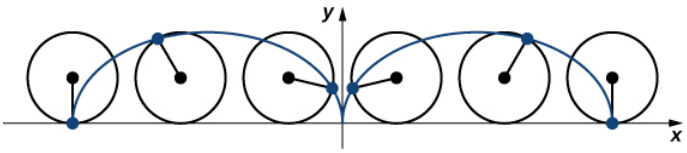


Figure 3.2.8: A wheel traveling along a road without slipping; the point on the edge of the wheel traces out a cycloid.

Now suppose that the bicycle wheel doesn't travel along a straight road but instead moves along the inside of a larger wheel, as in Figure 3.2.9. In this graph, the green circle is traveling around the blue circle in a counterclockwise direction. A point on the edge of the green circle traces out the red graph, which is called a *hypocycloid*.

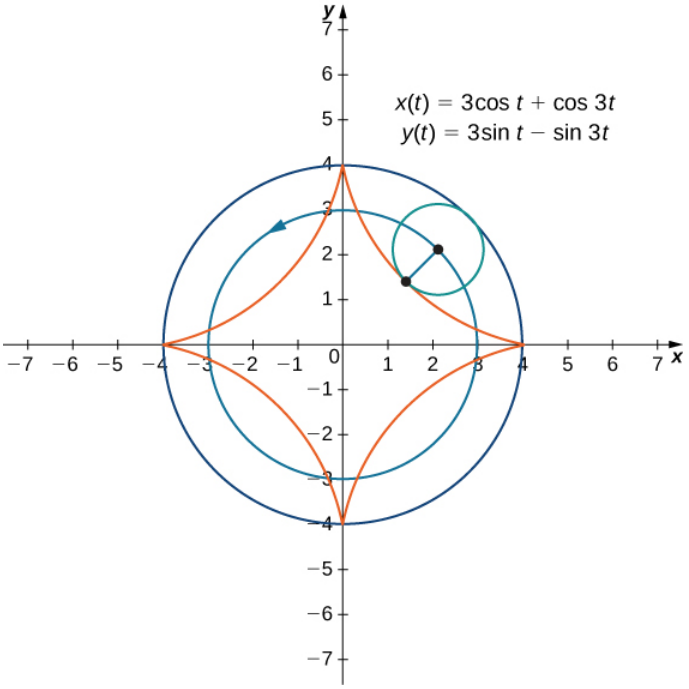


Figure 3.2.9: Graph of the hypocycloid described by the parametric equations shown.

The general parametric equations for a hypocycloid are

$$x(t) = (a-b)\cos t + b\cos(\frac{a-b}{b})t$$

$$y(t) = (a-b)\sin t - b\sin(\frac{a-b}{b})t.$$

These equations are a bit more complicated, but the derivation is somewhat similar to the equations for the cycloid. In this case we assume the radius of the larger circle is a and the radius of the smaller circle is b. Then the center of the wheel travels along a circle of radius $a-b$. This fact explains the first term in each equation above. The period of the second trigonometric function in both $x(t)$ and $y(t)$ is equal to $\dfrac{2\pi b}{a-b}$.

The ratio $\dfrac{a}{b}$ is related to the number of **cusps** on the graph (cusps are the corners or pointed ends of the graph), as illustrated in Figure 3.2.10. This ratio can lead to some very interesting graphs, depending on whether or not the ratio is rational. Figure 3.2.9 corresponds to $a = 4$ and $b = 1$. The result is a hypocycloid with four cusps. Figure 3.2.10 shows some other possibilities. The last two hypocycloids have irrational values for $\dfrac{a}{b}$. In these cases the hypocycloids have an infinite number of cusps, so they never return to their starting point. These are examples of what are known as **space-filling curves.**

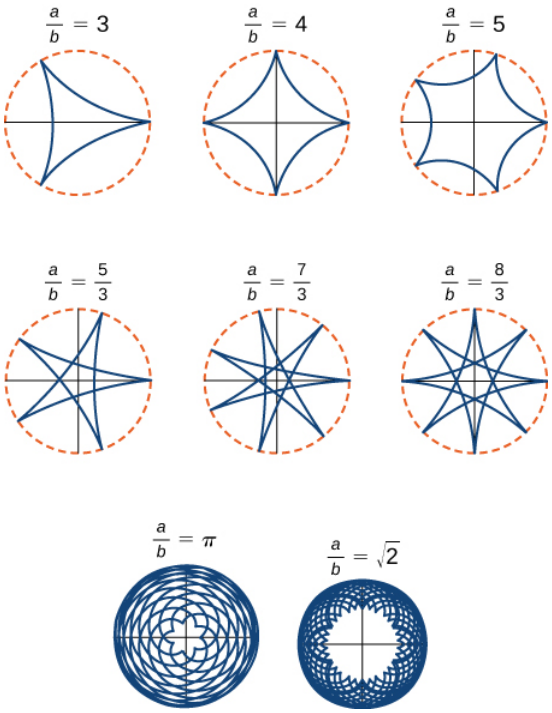


Figure 3.2.10: Graph of various hypocycloids corresponding to different values of a/b.

The Witch of Agnesi

Many plane curves in mathematics are named after the people who first investigated them, like the folium of Descartes or the spiral of Archimedes. However, perhaps the strangest name for a curve is the **witch of Agnesi.** Why a witch?

Maria Gaetana Agnesi (1718–1799) was one of the few recognized women mathematicians of eighteenth-century Italy. She wrote a popular book on analytic geometry, published in 1748, which included an interesting curve that had been studied by Fermat in 1630. The mathematician Guido Grandi showed in 1703 how to construct this curve, which he later called the "versoria," a Latin term for a rope used in sailing. Agnesi used the Italian term for this rope, "versiera," but in Latin, this same word means a "female goblin." When Agnesi's book was translated into English in 1801, the translator used the term "witch" for the curve, instead of rope. The name "witch of Agnesi" has stuck ever since.

The witch of Agnesi is a curve defined as follows: Start with a circle of radius a so that the points $(0, 0)$ and $(0, 2a)$ are points on the circle (Figure 3.2.11). Let O denote the origin. Choose any other point A on the circle, and draw the secant line OA. Let B denote the point at which the line OA intersects the horizontal line through $(0, 2a)$. The vertical line through B intersects the horizontal line through A at the point P. As the point A varies, the path that the point P travels is the witch of Agnesi curve for the given circle.

Witch of Agnesi curves have applications in physics, including modeling water waves and distributions of spectral lines. In probability theory, the curve describes the probability density function of the *Cauchy distribution*. In this project you will parameterize these curves.

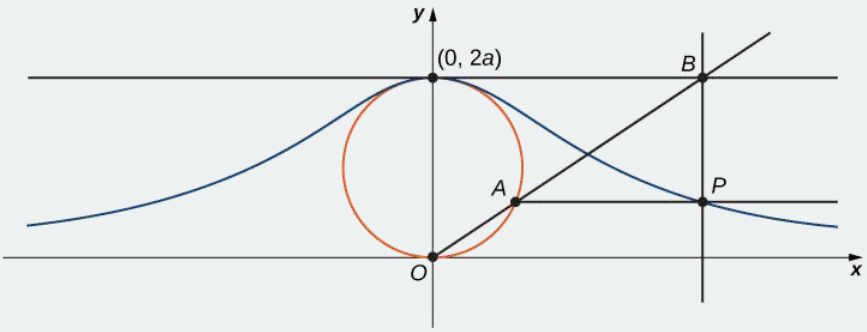


Figure 3.2.11: As the point A moves around the circle, the point P traces out the witch of Agnesi curve for the given circle.

1. On the figure, label the following points, lengths, and angle:

a. C is the point on the x-axis with the same x-coordinate as A.

b. x is the x-coordinate of P, and y is the y-coordinate of P.

c. E is the point $(0, a)$.

d. F is the point on the line segment OA such that the line segment EF is perpendicular to the line segment OA.

e. b is the distance from O to F.

f. c is the distance from F to A.

g. d is the distance from O to C.

h. θ is the measure of angle $\angle COA$.

The goal of this project is to parameterize the witch using θ as a parameter. To do this, write equations for x and y in terms of only θ.

2. Show that $d = \dfrac{2a}{\sin\theta}$.

3. Note that $x = d\cos\theta$. Show that $x = 2a\cot\theta$. When you do this, you will have parameterized the x-coordinate of the curve with respect to θ. If you can get a similar equation for y, you will have parameterized the curve.

4. In terms of θ, what is the angle $\angle EOA$?

5. Show that $b + c = 2a\cos\left(\frac{\pi}{2} - \theta\right)$.

6. Show that $y = 2a\cos\left(\frac{\pi}{2} - \theta\right)\sin\theta$.

7. Show that $y = 2a\sin^2\theta$. You have now parameterized the y-coordinate of the curve with respect to θ.

8. Conclude that a parameterization of the given witch curve is

$$x = 2a\cot\theta, \quad y = 2a\sin^2\theta, \quad \text{for } -\infty < \theta < \infty. \tag{3.2.12}$$

9. Use your parameterization to show that the given witch curve is the graph of the function $f(x) = \dfrac{8a^3}{x^2 + 4a^2}$.

Travels with My Ant: The Curtate and Prolate Cycloids

Earlier in this section, we looked at the parametric equations for a cycloid, which is the path a point on the edge of a wheel traces as the wheel rolls along a straight path. In this project we look at two different variations of the cycloid, called the curtate and prolate cycloids.

First, let's revisit the derivation of the parametric equations for a cycloid. Recall that we considered a tenacious ant trying to get home by hanging onto the edge of a bicycle tire. We have assumed the ant climbed onto the tire at the very edge, where the tire touches the ground. As the wheel rolls, the ant moves with the edge of the tire (Figure 3.2.12).

As we have discussed, we have a lot of flexibility when parameterizing a curve. In this case we let our parameter t represent the angle the tire has rotated through. Looking at Figure 3.2.12, we see that after the tire has rotated through an angle of t, the position of the center of the wheel, $C = (x_C, y_C)$, is given by

$$x_C = at \text{ and } y_C = a.$$

Furthermore, letting $A = (x_A, y_A)$ denote the position of the ant, we note that

$$x_C - x_A = a\sin t \text{ and } y_C - y_A = a\cos t$$

Then

$$x_A = x_C - a\sin t = at - a\sin t = a(t - \sin t) \tag{3.2.13}$$

$$y_A = y_C - a\cos t = a - a\cos t = a(1 - \cos t). \tag{3.2.14}$$

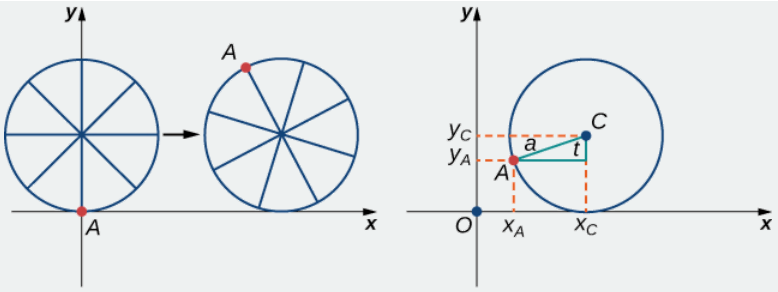


Figure 3.2.12: (a) The ant clings to the edge of the bicycle tire as the tire rolls along the ground. (b) Using geometry to determine the position of the ant after the tire has rotated through an angle of t.

Note that these are the same parametric representations we had before, but we have now assigned a physical meaning to the parametric variable t.

After a while the ant is getting dizzy from going round and round on the edge of the tire. So he climbs up one of the spokes toward the center of the wheel. By climbing toward the center of the wheel, the ant has changed his path of motion. The new path has less up-and-down motion and is called a **curtate cycloid** (Figure 3.2.13). As shown in the figure, we let b denote the distance along the spoke from the center of the wheel to the ant. As before, we let t represent the angle the tire has rotated through. Additionally, we let $C = (x_C, y_C)$ represent the position of the center of the wheel and $A = (x_A, y_A)$ represent the position of the ant.

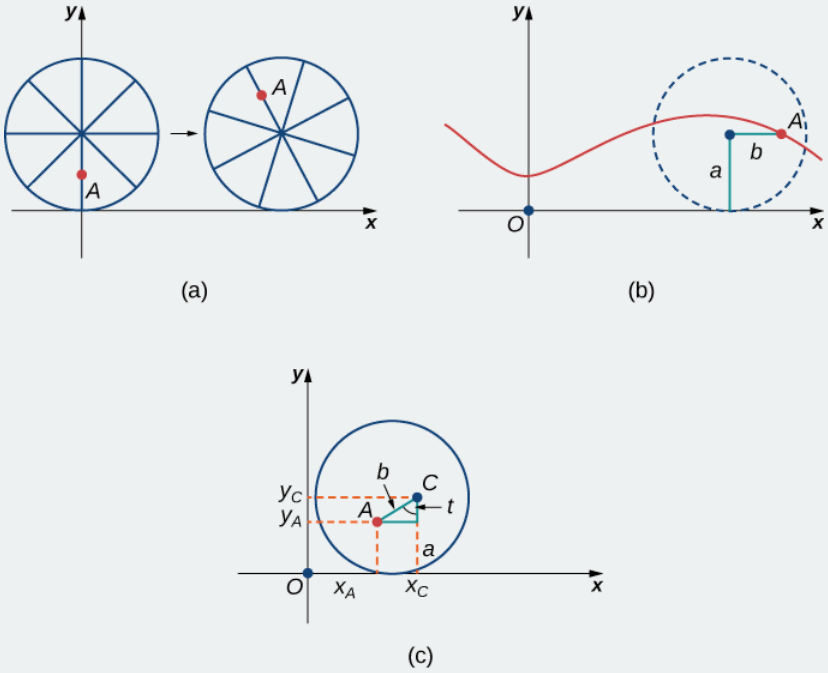


Figure 3.2.13: (a) The ant climbs up one of the spokes toward the center of the wheel. (b) The ant's path of motion after he climbs closer to the center of the wheel. This is called a curtate cycloid. (c) The new setup, now that the ant has moved closer to the center of the wheel.

1. What is the position of the center of the wheel after the tire has rotated through an angle of t?

2. Use geometry to find expressions for $x_C - x_A$ and for $y_C - y_A$.

3. On the basis of your answers to parts 1 and 2, what are the parametric equations representing the curtate cycloid?

Once the ant's head clears, he realizes that the bicyclist has made a turn, and is now traveling away from his home. So he drops off the bicycle tire and looks around. Fortunately, there is a set of train tracks nearby, headed back in the right direction. So the ant heads over to the train tracks to wait. After a while, a train goes by, heading in the right direction, and he manages to jump up and just catch the edge of the train wheel (without getting squished!).

The ant is still worried about getting dizzy, but the train wheel is slippery and has no spokes to climb, so he decides to just hang on to the edge of the wheel and hope for the best. Now, train wheels have a flange to keep the wheel running on the tracks. So, in this case, since the ant is hanging on to the very edge of the flange, the distance from the center of the wheel to the ant is actually greater than the radius of the wheel (Figure 3.2.14).

The setup here is essentially the same as when the ant climbed up the spoke on the bicycle wheel. We let b denote the distance from the center of the wheel to the ant, and we let t represent the angle the tire has rotated through. Additionally, we let $C = (x_C, y_C)$ represent the position of the center of the wheel and $A = (x_A, y_A)$ represent the position of the ant (Figure 3.2.14).

When the distance from the center of the wheel to the ant is greater than the radius of the wheel, his path of motion is called a prolate cycloid. A graph of a **prolate cycloid** is shown in the figure.

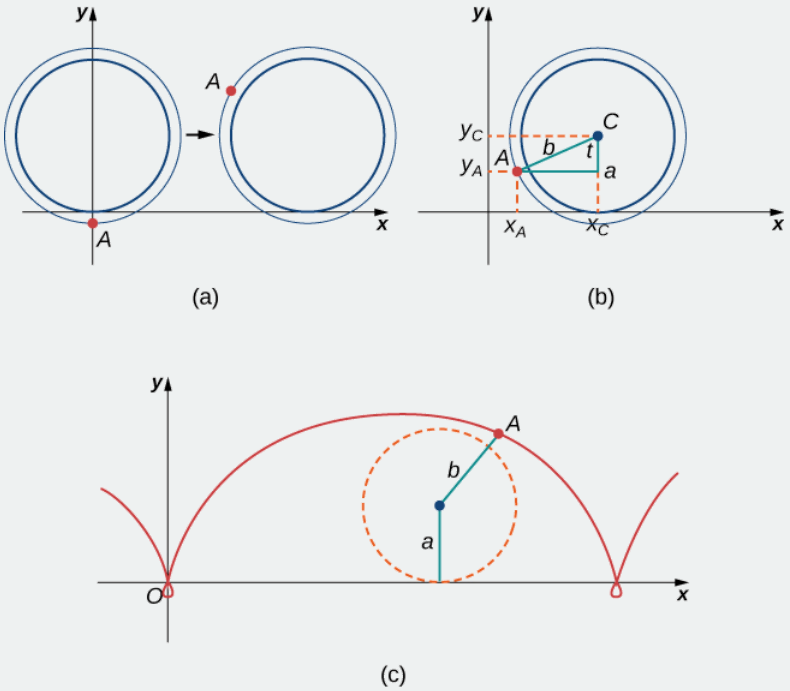


Figure 3.2.14: (a) The ant is hanging onto the flange of the train wheel. (b) The new setup, now that the ant has jumped onto the train wheel. (c) The ant travels along a prolate cycloid.

4. Using the same approach you used in parts 1– 3, find the parametric equations for the path of motion of the ant.

5. What do you notice about your answer to part 3 and your answer to part 4?

Notice that the ant is actually traveling backward at times (the “loops” in the graph), even though the train continues to move forward. He is probably going to be really dizzy by the time he gets home!

Key Concepts

- Parametric equations provide a convenient way to describe a curve. A parameter can represent time or some other meaningful quantity.
- It is often possible to eliminate the parameter in a parameterized curve to obtain a function or relation describing that curve.
- There is always more than one way to parameterize a curve.
- Parametric equations can describe complicated curves that are difficult or perhaps impossible to describe using rectangular coordinates.

Glossary

cycloid

the curve traced by a point on the rim of a circular wheel as the wheel rolls along a straight line without slippage

cusp

a pointed end or part where two curves meet

orientation

the direction that a point moves on a graph as the parameter increases

parameter

an independent variable that both x and y depend on in a parametric curve; usually represented by the variable t

parametric curve

the graph of the parametric equations $x(t)$ and $y(t)$ over an interval $a \leq t \leq b$ combined with the equations

parametric equations

the equations $x = x(t)$ and $y = y(t)$ that define a parametric curve

parameterization of a curve

rewriting the equation of a curve defined by a function $y = f(x)$ as parametric equations

Contributors and Attributions

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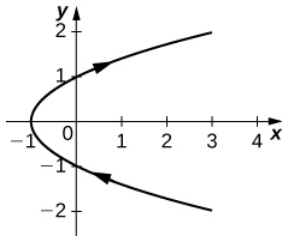
3.2E: Exercises for Section 3.2

In exercises 1 - 4, sketch the curves below by eliminating the parameter t. Give the orientation of the curve.

1) $x = t^2 + 2t, \quad y = t + 1$

Answer

Orientation: bottom to top

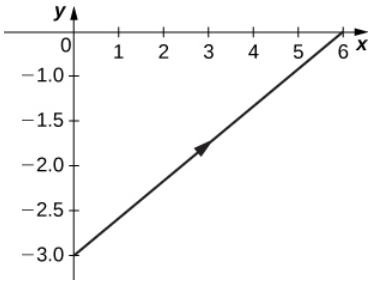


2) $x = \cos(t), \quad y = \sin(t), \quad \text{for } (0, 2\pi]$

3) $x = 2t + 4, \quad y = t - 1$

Answer

Orientation: left to right



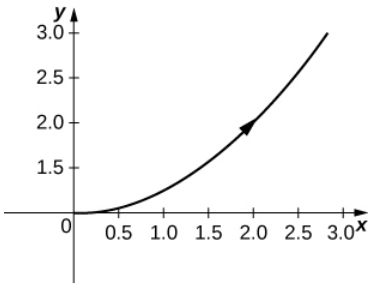
4) $x = 3 - t, \quad y = 2t - 3, \quad \text{for } 1.5 \le t \le 3$

In exercise 5, eliminate the parameter and sketch the graph.

5) $x = 2t^2, \quad y = t^4 + 1$

Answer

$$y = \frac{x^2}{4} + 1$$

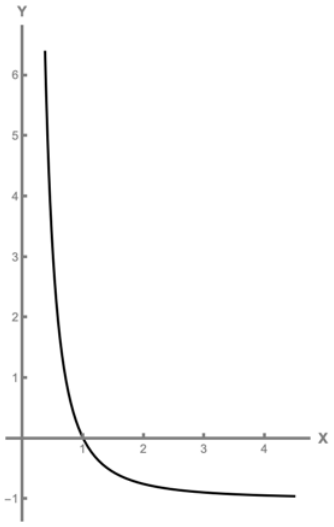


In exercises 6 - 9, use technology (CAS or calculator) to sketch the parametric equations.

6) [T] $x = t^2 + t, \quad y = t^2 - 1$

7) [T] $x = e^{-t}, \quad y = e^{2t} - 1$

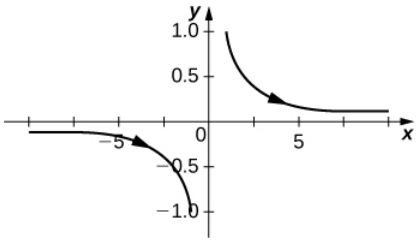
Answer



8) [T] $x = 3\cos t, \quad y = 4\sin t$

9) [T] $x = \sec t, \quad y = \cos t$

Answer

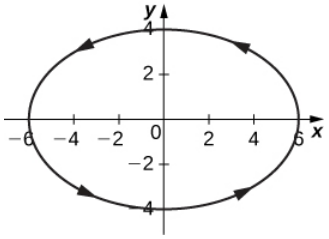


In exercises 10 - 20, sketch the parametric equations by eliminating the parameter. Indicate any asymptotes of the graph.

10) $x = e^t, \quad y = e^{2t} + 1$

11) $x = 6\sin(2\theta), \quad y = -4\cos(2\theta)$

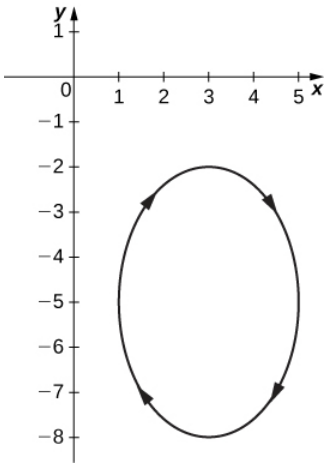
Answer



12) $x = \cos\theta, \quad y = 2\sin(2\theta)$

13) $x = 3 - 2\cos\theta, \quad y = -5 + 3\sin\theta$

Answer

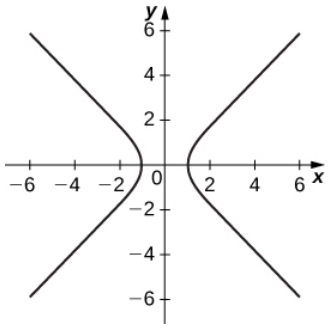


14) $x = 4 + 2\cos\theta, \quad y = -1 + \sin\theta$

15) $x = \sec t, \quad y = \tan t$

Answer

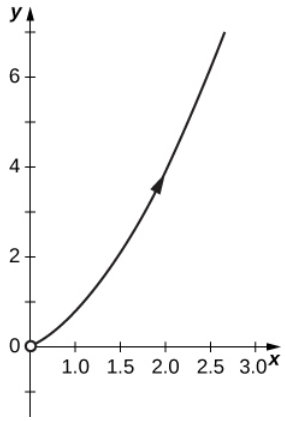
Asymptotes are $y = x$ and $y = -x$



16) $x = \ln(2t), \quad y = t^2$

17) $x = e^t, \quad y = e^{2t}$

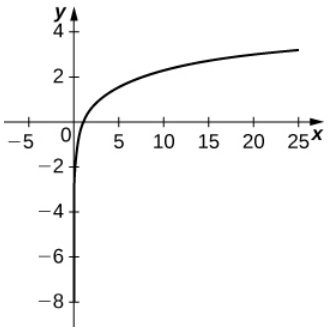
Answer



18) $x = e^{-2t}, \quad y = e^{3t}$

19) $x = t^3, \quad y = 3\ln t$

Answer



20) $x = 4\sec\theta, \quad y = 3\tan\theta$

In exercises 21 - 38, convert the parametric equations of a curve into rectangular form. No sketch is necessary. State the domain of the rectangular form.

21) $x = t^2 - 1, \quad y = \dfrac{t}{2}$

Answer

$x = 4y^2 - 1$; domain: $x \in [1, \infty)$.

22) $x = \dfrac{1}{\sqrt{t+1}}, \quad y = \dfrac{t}{1+t}, \quad \text{for } t > -1$

23) $x = 4\cos\theta, \quad y = 3\sin\theta, \quad \text{for } t \in (0, 2\pi]$

Answer

$\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$; domain $x \in [-4, 4]$.

24) $x = \cosh t, \quad y = \sinh t$

25) $x = 2t - 3, \quad y = 6t - 7$

Answer

$y = 3x + 2$; domain: all real numbers.

26) $x = t^2, \quad y = t^3$

27) $x = 1 + \cos t, \quad y = 3 - \sin t$

Answer

$(x-1)^2 + (y-3)^2 = 1$; domain: $x \in [0, 2]$.

28) $x = \sqrt{t}, \quad y = 2t + 4$

29) $x = \sec t, \quad y = \tan t, \quad \text{for } \pi \le t < \frac{3\pi}{2}$

Answer

$y = \sqrt{x^2 - 1}$; domain: $x \in (-\infty, -1]$.

30) $x = 2\cosh t, \quad y = 4\sinh t$

31) $x = \cos(2t), \quad y = \sin t$

Answer

$y^2 = \dfrac{1-x}{2}$; domain: $x \in [-1, 1]$.

32) $x = 4t + 3, \quad y = 16t^2 - 9$

33) $x = t^2, \quad y = 2\ln t, \quad \text{for } t \geq 1$

Answer

$y = \ln x$; domain: $x \in [1, \infty)$.

34) $x = t^3, \quad y = 3\ln t, \quad \text{for } t \geq 1$

35) $x = t^n, \quad y = n\ln t, \quad \text{for } t \geq 1, \text{ where } n \text{ is a natural number}$

Answer

$y = \ln x$; domain: $x \in (0, \infty)$.

36) $x = \ln(5t), \quad y = \ln(t^2) \text{ where } 1 \leq t \leq e$

37) $x = 2\sin(8t), \quad y = 2\cos(8t)$

Answer

$x^2 + y^2 = 4$; domain: $x \in [-2, 2]$.

38) $x = \tan t, \quad y = \sec^2 t - 1$

In exercises 39 - 48, the pairs of parametric equations represent lines, parabolas, circles, ellipses, or hyperbolas. Name the type of basic curve that each pair of equations represents.

39) $x = 3t + 4, \quad y = 5t - 2$

Answer

line

40) $x - 4 = 5t, \quad y + 2 = t$

41) $x = 2t + 1, \quad y = t^2 - 3$

Answer

parabola

42) $x = 3\cos t, \quad y = 3\sin t$

43) $x = 2\cos(3t), \quad y = 2\sin(3t)$

Answer

circle

44) $x = \cosh t, \quad y = \sinh t$

45) $x = 3\cos t, \quad y = 4\sin t$

Answer

ellipse

46) $x = 2\cos(3t), \quad y = 5\sin(3t)$

47) $x = 3\cosh(4t) \quad y = 4\sinh(4t)$

Answer

the right branch of a horizontally opening hyperbola

48) $x = 2\cosh t, \quad y = 2\sinh t$

49) Show that $x = h + r\cos\theta, \quad y = k + r\sin\theta$ represents the equation of a circle.

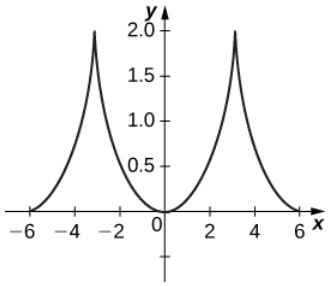
50) Use the equations in the preceding problem to find a set of parametric equations for a circle whose radius is 5 and whose center is $(-2, 3)$.

In exercises 51 - 53, use a graphing utility to graph the curve represented by the parametric equations and identify the curve from its equation.

51) [T] $x = \theta + \sin\theta, \quad y = 1 - \cos\theta$

Answer

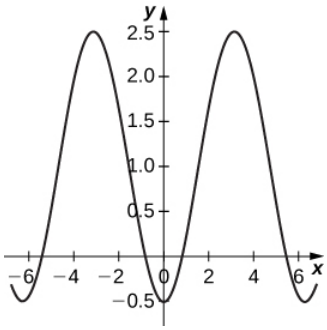
The equations represent a cycloid.



52) [T] $x = 2t - 2\sin t, \quad y = 2 - 2\cos t$

53) [T] $x = t - 0.5\sin t, \quad y = 1 - 1.5\cos t$

Answer



54) An airplane traveling horizontally at 100 m/s over flat ground at an elevation of 4000 meters must drop an emergency package on a target on the ground. The trajectory of the package is given by $x = 100t, \quad y = -4.9t^2 + 4000, \quad$ where $t \geq 0$ where the origin is the point on the ground directly beneath the plane at the moment of release. How many horizontal meters before the target should the package be released in order to hit the target?

55) The trajectory of a bullet is given by $x = v_0(\cos\alpha)t, \quad y = v_0(\sin\alpha)t - \frac{1}{2}gt^2$ where $v_0 = 500$ m/s, $g = 9.8 = 9.8\ \text{m/s}^2$, and $\alpha = 30$ degrees. When will the bullet hit the ground? How far from the gun will the bullet hit the ground?

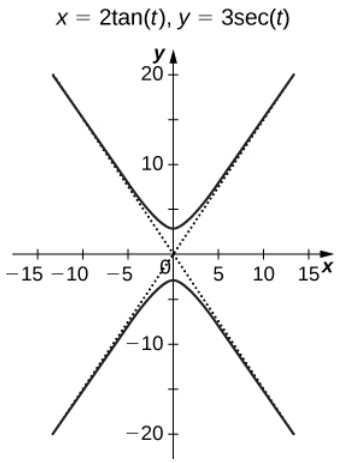
Answer

22,092 meters at approximately 51 seconds.

56) [T] Use technology to sketch the curve represented by $x = \sin(4t), \quad y = \sin(3t), \quad$ for $0 \leq t \leq 2\pi$.

57) [T] Use technology to sketch $x = 2\tan(t), \quad y = 3\sec(t), \quad$ for $-\pi < t < \pi$.

Answer



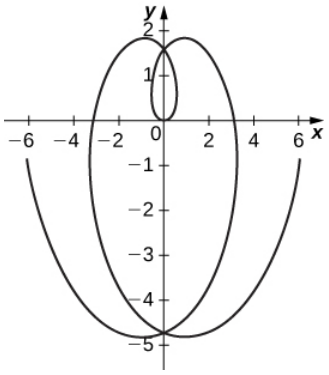
58) Sketch the curve known as an **epitrochoid**, which gives the path of a point on a circle of radius b as it rolls on the outside of a circle of radius a. The equations are

$x = (a+b)\cos t - c \cdot \cos\left[\frac{(a+b)t}{b}\right], \quad y = (a+b)\sin t - c \cdot \sin\left[\frac{(a+b)t}{b}\right]$.

Let $a = 1,\ b = 2,\ c = 1$.

59) [T] Use technology to sketch the spiral curve given by $x = t\cos(t), \quad y = t\sin(t)$ for $-2\pi \le t \le 2\pi$.

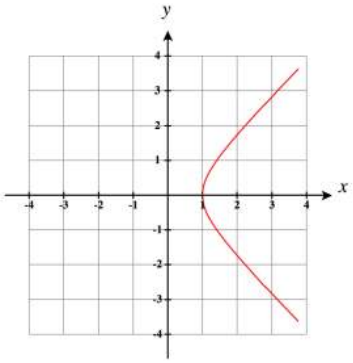
Answer



60) [T] Use technology to graph the curve given by the parametric equations $x = 2\cot(t), \quad y = 1 - \cos(2t), \quad$ for $-\pi/2 \le t \le \pi/2$. This curve is known as the witch of Agnesi.

61) [T] Sketch the curve given by parametric equations $x = \cosh(t), \quad y = \sinh(t),$ for $-2 \le t \le 2$.

Answer



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3.3: Calculus of Parametric Curves

Learning Objectives

- Determine derivatives and equations of tangents for parametric curves.
- Find the area under a parametric curve.
- Use the equation for arc length of a parametric curve.
- Apply the formula for surface area to a volume generated by a parametric curve.

Now that we have introduced the concept of a parameterized curve, our next step is to learn how to work with this concept in the context of calculus. For example, if we know a parameterization of a given curve, is it possible to calculate the slope of a tangent line to the curve? How about the arc length of the curve? Or the area under the curve?

Another scenario: Suppose we would like to represent the location of a baseball after the ball leaves a pitcher's hand. If the position of the baseball is represented by the plane curve $(x(t), y(t))$ then we should be able to use calculus to find the speed of the ball at any given time. Furthermore, we should be able to calculate just how far that ball has traveled as a function of time.

Derivatives of Parametric Equations

We start by asking how to calculate the slope of a line tangent to a parametric curve at a point. Consider the plane curve defined by the parametric equations

$$x(t) = 2t + 3 \tag{3.3.1}$$
$$y(t) = 3t - 4 \tag{3.3.2}$$

within $-2 \leq t \leq 3$.

The graph of this curve appears in Figure 3.3.1. It is a line segment starting at $(-1, -10)$ and ending at $(9, 5)$.

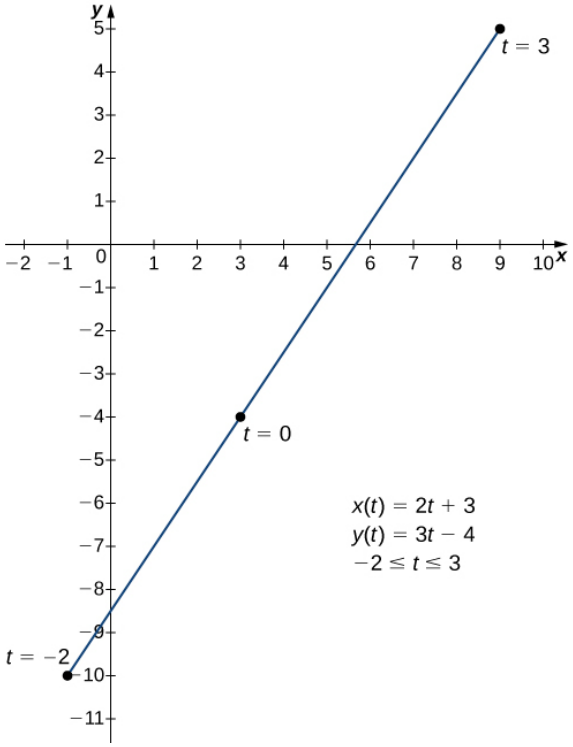


Figure 3.3.1: Graph of the line segment described by the given parametric equations.

We can eliminate the parameter by first solving Equation 3.3.1 for t:

$$x(t) = 2t + 3$$
$$x - 3 = 2t$$
$$t = \frac{x-3}{2}.$$

Substituting this into $y(t)$ (Equation 3.3.2), we obtain

$$y(t) = 3t - 4$$
$$y = 3\left(\frac{x-3}{2}\right) - 4$$
$$y = \frac{3x}{2} - \frac{9}{2} - 4$$
$$y = \frac{3x}{2} - \frac{17}{2}\ .$$

The slope of this line is given by $\frac{dy}{dx} = \frac{3}{2}$. Next we calculate $x'(t)$ and $y'(t)$. This gives $x'(t) = 2$ and $y'(t) = 3$. Notice that

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3}{2}.$$

This is no coincidence, as outlined in the following theorem.

Derivative of Parametric Equations

Consider the plane curve defined by the parametric equations $x = x(t)$ and $y = y(t)$. Suppose that $x'(t)$ and $y'(t)$ exist, and assume that $x'(t) \neq 0$. Then the derivative $\frac{dy}{dx}$ is given by

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)}. \tag{3.3.3}$$

Proof

This theorem can be proven using the Chain Rule. In particular, assume that the parameter t can be eliminated, yielding a differentiable function $y = F(x)$. Then $y(t) = F(x(t))$. Differentiating both sides of this equation using the Chain Rule yields

$$y'(t) = F'\big(x(t)\big)x'(t),$$

so

$$F'\big(x(t)\big) = \frac{y'(t)}{x'(t)}.$$

But $F'\big(x(t)\big) = \frac{dy}{dx}$, which proves the theorem.

□

Equation 3.3.3 can be used to calculate derivatives of plane curves, as well as critical points. Recall that a critical point of a differentiable function $y = f(x)$ is any point $x = x_0$ such that either $f'(x_0) = 0$ or $f'(x_0)$ does not exist. Equation 3.3.3 gives a formula for the slope of a tangent line to a curve defined parametrically regardless of whether the curve can be described by a function $y = f(x)$ or not.

Example 3.3.1: Finding the Derivative of a Parametric Curve

Calculate the derivative $\frac{dy}{dx}$ for each of the following parametrically defined plane curves, and locate any critical points on their respective graphs.

a. $x(t) = t^2 - 3, \quad y(t) = 2t - 1, \quad \text{for } -3 \leq t \leq 4$
b. $x(t) = 2t + 1, \quad y(t) = t^3 - 3t + 4, \quad \text{for } -2 \leq t \leq 2$
c. $x(t) = 5\cos t, \quad y(t) = 5\sin t, \quad \text{for } 0 \leq t \leq 2\pi$

Solution

a. To apply Equation 3.3.3, first calculate $x'(t)$ and $y'(t)$:

$$x'(t) = 2t$$
$$y'(t) = 2.$$

Next substitute these into the equation:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$
$$\frac{dy}{dx} = \frac{2}{2t}$$
$$\frac{dy}{dx} = \frac{1}{t}.$$

This derivative is undefined when $t = 0$. Calculating $x(0)$ and $y(0)$ gives $x(0) = (0)^2 - 3 = -3$ and $y(0) = 2(0) - 1 = -1$, which corresponds to the point $(-3, -1)$ on the graph. The graph of this curve is a parabola opening to the right, and the point $(-3, -1)$ is its vertex as shown.

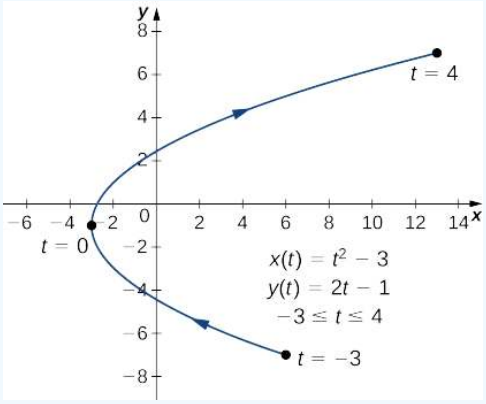


Figure 3.3.2: Graph of the parabola described by parametric equations in part a.

b. To apply Equation 3.3.3, first calculate $x'(t)$ and $y'(t)$:

$$x'(t) = 2$$
$$y'(t) = 3t^2 - 3.$$

Next substitute these into the equation:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$
$$\frac{dy}{dx} = \frac{3t^2 - 3}{2}.$$

This derivative is zero when $t = \pm 1$. When $t = -1$ we have

$$x(-1) = 2(-1) + 1 = -1 \text{ and } y(-1) = (-1)^3 - 3(-1) + 4 = -1 + 3 + 4 = 6,$$

which corresponds to the point $(-1, 6)$ on the graph. When $t = 1$ we have

$$x(1) = 2(1) + 1 = 3 \text{ and } y(1) = (1)^3 - 3(1) + 4 = 1 - 3 + 4 = 2,$$

which corresponds to the point $(3, 2)$ on the graph. The point $(3, 2)$ is a relative minimum and the point $(-1, 6)$ is a relative maximum, as seen in the following graph.

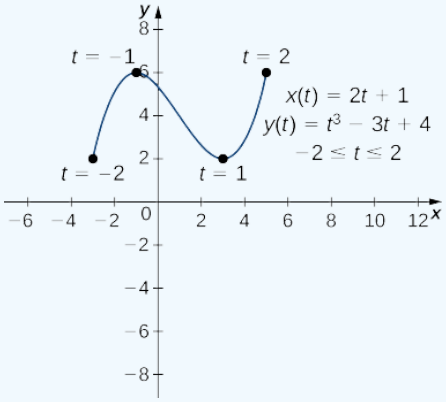


Figure 3.3.3: Graph of the curve described by parametric equations in part b.

c. To apply Equation 3.3.3, first calculate $x'(t)$ and $y'(t)$:

$$x'(t) = -5\sin t$$
$$y'(t) = 5\cos t.$$

Next substitute these into the equation:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$
$$\frac{dy}{dx} = \frac{5\cos t}{-5\sin t}$$
$$\frac{dy}{dx} = -\cot t.$$

This derivative is zero when $\cos t = 0$ and is undefined when $\sin t = 0$. This gives $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, and 2π as critical points for t. Substituting each of these into $x(t)$ and $y(t)$, we obtain

t	$x(t)$	$y(t)$
0	5	0
$\frac{\pi}{2}$	0	5
π	−5	0
$\frac{3\pi}{2}$	0	−5
2π	5	0

These points correspond to the sides, top, and bottom of the circle that is represented by the parametric equations (Figure 3.3.4). On the left and right edges of the circle, the derivative is undefined, and on the top and bottom, the derivative equals zero.

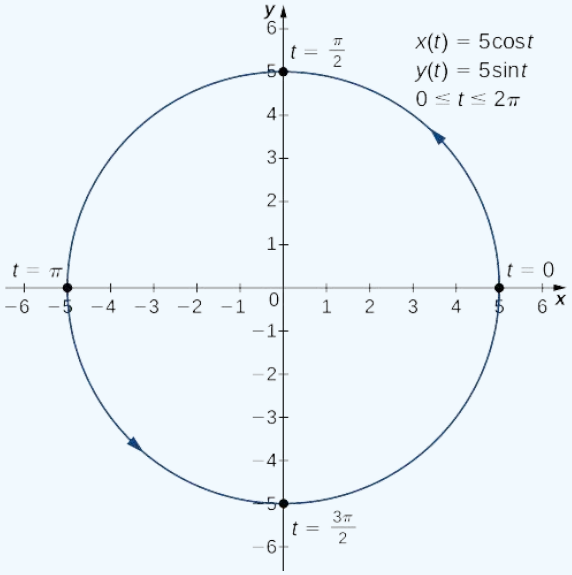


Figure 3.3.4: Graph of the curve described by parametric equations in part c.

Exercise 3.3.1

Calculate the derivative dy/dx for the plane curve defined by the equations

$$x(t) = t^2 - 4t, \quad y(t) = 2t^3 - 6t, \quad \text{for } -2 \le t \le 3$$

and locate any critical points on its graph.

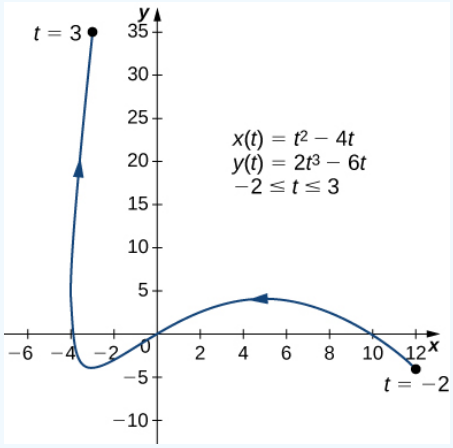
Hint

Calculate $x'(t)$ and $y'(t)$ and use Equation 3.3.3

Answer

$x'(t) = 2t - 4$ and $y'(t) = 6t^2 - 6$, so $\dfrac{dy}{dx} = \dfrac{6t^2 - 6}{2t - 4} = \dfrac{3t^2 - 3}{t - 2}$.

This expression is undefined when $t = 2$ and equal to zero when $t = \pm 1$.



Example 3.3.2: Finding a Tangent Line

Find the equation of the tangent line to the curve defined by the equations

$$x(t) = t^2 - 3, \quad y(t) = 2t - 1, \quad \text{for } -3 \le t \le 4 \tag{3.3.4}$$

when $t = 2$.

https://math.libretexts.org/@go/page/83997

Solution

First find the slope of the tangent line using Equation 3.3.3, which means calculating $x'(t)$ and $y'(t)$:

$$x'(t) = 2t$$
$$y'(t) = 2.$$

Next substitute these into the equation:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$
$$\frac{dy}{dx} = \frac{2}{2t}$$
$$\frac{dy}{dx} = \frac{1}{t}.$$

When $t = 2$, $\frac{dy}{dx} = \frac{1}{2}$, so this is the slope of the tangent line. Calculating $x(2)$ and $y(2)$ gives

$$x(2) = (2)^2 - 3 = 1 \text{ and } y(2) = 2(2) - 1 = 3,$$

which corresponds to the point $(1, 3)$ on the graph (Figure 3.3.5). Now use the point-slope form of the equation of a line to find the equation of the tangent line:

$$y - y_0 = m(x - x_0)$$
$$y - 3 = \frac{1}{2}(x - 1)$$
$$y - 3 = \frac{1}{2}x - \frac{1}{2}$$
$$y = \frac{1}{2}x + \frac{5}{2}.$$

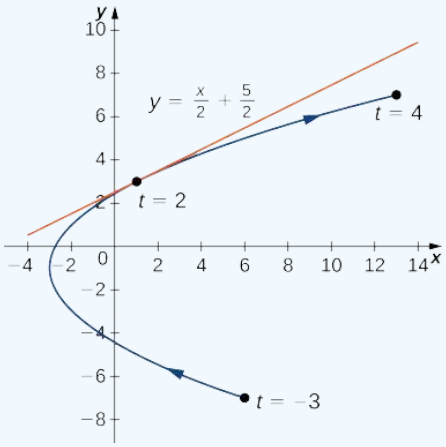


Figure 3.3.5: Tangent line to the parabola described by the given parametric equations when $t = 2$.



Exercise 3.3.2

Find the equation of the tangent line to the curve defined by the equations

$$x(t) = t^2 - 4t, \quad y(t) = 2t^3 - 6t, \quad \text{for } -2 \le t \le 6 \quad \text{when } t = 5\,.$$

Hint

Calculate $x\,'(t)$ and $y\,'(t)$ and use Equation 3.3.3

Answer

The equation of the tangent line is $y = 24x + 100$.



Second-Order Derivatives

Our next goal is to see how to take the second derivative of a function defined parametrically. The second derivative of a function $y = f(x)$ is defined to be the derivative of the first derivative; that is,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right]. \tag{3.3.5}$$

Since

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt},$$

we can replace the y on both sides of Equation 3.3.5 with $\frac{dy}{dx}$. This gives us

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{(d/dt)(dy/dx)}{dx/dt}. \tag{3.3.6}$$

If we know dy/dx as a function of t, then this formula is straightforward to apply

Example 3.3.3: Finding a Second Derivative

Calculate the second derivative d^2y/dx^2 for the plane curve defined by the parametric equations $x(t) = t^2 - 3, \quad y(t) = 2t - 1, \quad \text{for } -3 \le t \le 4.$

Solution

From Example 3.3.1 we know that $\frac{dy}{dx} = \frac{2}{2t} = \frac{1}{t}$. Using Equation 3.3.6, we obtain

$$\frac{d^2y}{dx^2} = \frac{(d/dt)(dy/dx)}{dx/dt} = \frac{(d/dt)(1/t)}{2t} = \frac{-t^{-2}}{2t} = -\frac{1}{2t^3} \quad .$$

Exercise 3.3.3

Calculate the second derivative d^2y/dx^2 for the plane curve defined by the equations

$$x(t) = t^2 - 4t, \quad y(t) = 2t^3 - 6t, \quad \text{for } -2 \le t \le 3$$

and locate any critical points on its graph.

Hint

Start with the solution from the previous exercise, and use Equation 3.3.6

Answer

$\frac{d^2y}{dx^2} = \frac{3t^2 - 12t + 3}{2(t-2)^3}$. Critical points $(5, 4)$, $(-3, -4)$,and $(-4, 6)$.

Integrals Involving Parametric Equations

Now that we have seen how to calculate the derivative of a plane curve, the next question is this: How do we find the area under a curve defined parametrically? Recall the cycloid defined by these parametric equations

$$\begin{aligned} x(t) &= t - \sin t \\ y(t) &= 1 - \cos t. \end{aligned}$$

Suppose we want to find the area of the shaded region in the following graph.

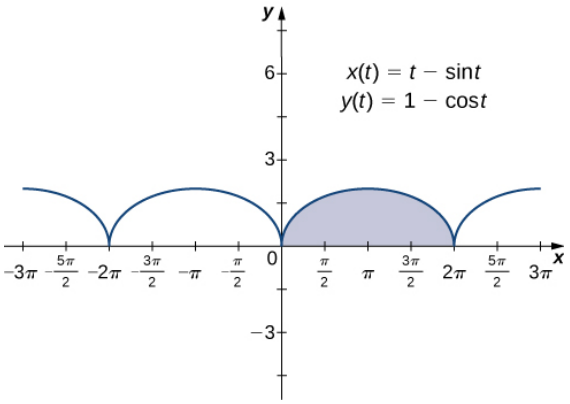


Figure 3.3.6: Graph of a cycloid with the arch over $[0, 2\pi]$ highlighted.

To derive a formula for the area under the curve defined by the functions

$$x = x(t)$$
$$y = y(t)$$

where $a \le t \le b$.

We assume that $x(t)$ is differentiable and start with an equal partition of the interval $a \le t \le b$. Suppose $t_0 = a < t_1 < t_2 < \cdots < t_n = b$ and consider the following graph.

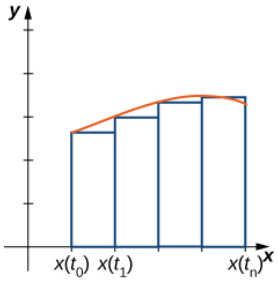


Figure 3.3.7: Approximating the area under a parametrically defined curve.

We use rectangles to approximate the area under the curve. The height of a typical rectangle in this parametrization is $y(x(\bar{t}_i))$ for some value $\bar{t}_i$ in the ith subinterval, and the width can be calculated as $x(t_i) - x(t_{i-1})$. Thus the area of the ith rectangle is given by

$$A_i = y(x(\bar{t}_i))(x(t_i) - x(t_{i-1})).$$

Then a Riemann sum for the area is

$$A_n = \sum_{i=1}^{n} y(x(\bar{t}_i))(x(t_i) - x(t_{i-1})).$$

Multiplying and dividing each area by $t_i - t_{i-1}$ gives

$$\begin{aligned} A_n &= \sum_{i=1}^{n} y(x(\bar{t}_i)) \left(\frac{x(t_i) - x(t_{i-1})}{t_i - t_{i-1}} \right) (t_i - t_{i-1}) \\ &= \sum_{i=1}^{n} y(x(\bar{t}_i)) \left(\frac{x(t_i) - x(t_{i-1})}{\Delta t} \right) \Delta t. \end{aligned}$$

Taking the limit as n approaches infinity gives

$$A = \lim_{n \to \infty} A_n = \int_a^b y(t) x'(t)\, dt.$$

This leads to the following theorem.

Area under a Parametric Curve

Consider the non-self-intersecting plane curve defined by the parametric equations

$$x = x(t), \quad y = y(t), \quad \text{for } a \le t \le b$$

and assume that $x(t)$ is differentiable. The area under this curve is given by

$$A = \int_a^b y(t)x'(t)\,dt. \tag{3.3.7}$$

Example 3.3.4: Finding the Area under a Parametric Curve

Find the area under the curve of the cycloid defined by the equations

$$x(t) = t - \sin t, \quad y(t) = 1 - \cos t, \quad \text{for } 0 \le t \le 2\pi. \tag{3.3.8}$$

Solution

Using Equation 3.3.7, we have

$$\begin{aligned} A &= \int_a^b y(t)x'(t)\,dt \\ &= \int_0^{2\pi} (1-\cos t)(1-\cos t)\,dt \\ &= \int_0^{2\pi} (1 - 2\cos t + \cos^2 t)\,dt \\ &= \int_0^{2\pi} \left(1 - 2\cos t + \frac{1+\cos(2t)}{2}\right) dt \\ &= \int_0^{2\pi} \left(\frac{3}{2} - 2\cos t + \frac{\cos(2t)}{2}\right) dt \\ &= \frac{3t}{2} - 2\sin t + \frac{\sin(2t)}{4}\Big|_0^{2\pi} \\ &= 3\pi \end{aligned}$$

Exercise 3.3.4

Find the area under the curve of the hypocycloid defined by the equations

$$x(t) = 3\cos t + \cos(3t), \quad y(t) = 3\sin t - \sin(3t), \quad \text{for } 0 \le t \le \pi.$$

Hint

Use Equation 3.3.7, along with the identities $\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$ and $\sin^2 t = \frac{1-\cos(2t)}{2}$.

Answer

$A = 3\pi$ (Note that the integral formula actually yields a negative answer. This is due to the fact that $x(t)$ is a decreasing function over the interval $[0, \pi]$; that is, the curve is traced from right to left.)

Arc Length of a Parametric Curve

In addition to finding the area under a parametric curve, we sometimes need to find the arc length of a parametric curve. In the case of a line segment, arc length is the same as the distance between the endpoints. If a particle travels from point A to point B along a curve, then the distance that particle travels is the arc length. To develop a formula for arc length, we start with an approximation by line segments as shown in the following graph.

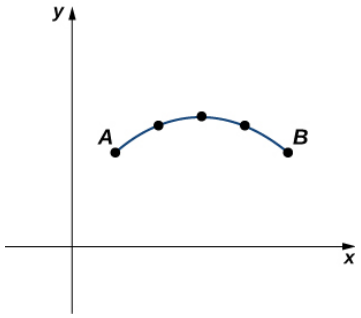


Figure 3.3.7: Approximation of a curve by line segments.

Given a plane curve defined by the functions $x = x(t), \quad y = y(t), \quad$ for $a \le t \le b$, we start by partitioning the interval $[a, b]$ into n equal subintervals: $t_0 = a < t_1 < t_2 < \cdots < t_n = b$. The width of each subinterval is given by $\Delta t = (b-a)/n$. We can calculate the length of each line segment:

$$d_1 = \sqrt{(x(t_1) - x(t_0))^2 + (y(t_1) - y(t_0))^2}$$

$$d_2 = \sqrt{(x(t_2) - x(t_1))^2 + (y(t_2) - y(t_1))^2}$$

etc.

Then add these up. We let s denote the exact arc length and s_n denote the approximation by n line segments:

$$s \approx \sum_{k=1}^{n} s_k = \sum_{k=1}^{n} \sqrt{(x(t_k) - x(t_{k-1}))^2 + (y(t_k) - y(t_{k-1}))^2} \, . \tag{3.3.9}$$

If we assume that $x(t)$ and $y(t)$ are differentiable functions of t, then the Mean Value Theorem applies, so in each subinterval $[t_{k-1}, t_k]$ there exist $\hat{t_k}$ and $\tilde{t_k}$ such that

$$x(t_k) - x(t_{k-1}) = x'(\hat{t_k})(t_k - t_{k-1}) = x'(\hat{t_k})\,\Delta t$$

$$y(t_k) - y(t_{k-1}) = y'(\tilde{t_k})(t_k - t_{k-1}) = y'(\tilde{t_k})\,\Delta t.$$

Therefore Equation 3.3.9 becomes

$$\begin{aligned} s \approx \sum_{k=1}^{n} s_k &= \sum_{k=1}^{n} \sqrt{(x'(\hat{t_k})\Delta t)^2 + (y'(\tilde{t_k})\Delta t)^2} \\ &= \sum_{k=1}^{n} \sqrt{(x'(\hat{t_k}))^2(\Delta t)^2 + (y'(\tilde{t_k}))^2(\Delta t)^2} \\ &= \sum_{k=1}^{n} \sqrt{(x'(\hat{t_k}))^2 + (y'(\tilde{t_k}))^2}\,\Delta t. \end{aligned}$$

This is a Riemann sum that approximates the arc length over a partition of the interval $[a, b]$. If we further assume that the derivatives are continuous and let the number of points in the partition increase without bound, the approximation approaches the exact arc length. This gives

$$\begin{aligned} s &= \lim_{n\to\infty} \sum_{k=1}^{n} s_k \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{(x'(\hat{t_k}))^2 + (y'(\tilde{t_k}))^2}\,\Delta t \\ &= \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\,dt. \end{aligned}$$

When taking the limit, the values of $\hat{t_k}$ and $\tilde{t_k}$ are both contained within the same ever-shrinking interval of width Δt, so they must converge to the same value.

We can summarize this method in the following theorem.

Arc Length of a Parametric Curve

Consider the plane curve defined by the parametric equations

$$x = x(t), \quad y = y(t), \quad \text{for } t_1 \le t \le t_2$$

and assume that $x(t)$ and $y(t)$ are differentiable functions of t. Then the arc length of this curve is given by

$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt. \tag{3.3.10}$$

At this point a side derivation leads to a previous formula for arc length. In particular, suppose the parameter can be eliminated, leading to a function $y = F(x)$. Then $y(t) = F(x(t))$ and the Chain Rule gives

$$y'(t) = F'\big(x(t)\big)x'(t).$$

Substituting this into Equation 3.3.10 gives

$$\begin{aligned} s &= \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(F'(x)\frac{dx}{dt}\right)^2}\, dt \\ &= \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 \left(1 + (F'(x))^2\right)}\, dt \\ &= \int_{t_1}^{t_2} x'(t)\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dt. \end{aligned}$$

Here we have assumed that $x'(t) > 0$, which is a reasonable assumption. The Chain Rule gives $dx = x'(t)\, dt$, and letting $a = x(t_1)$ and $b = x(t_2)$ we obtain the formula

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx,$$

which is the formula for arc length obtained in the Introduction to the Applications of Integration.

Example 3.3.5: Finding the Arc Length of a Parametric Curve

Find the arc length of the semicircle defined by the equations

$$x(t) = 3\cos t, \quad y(t) = 3\sin t, \quad \text{for } 0 \le t \le \pi.$$

Solution

The values $t = 0$ to $t = \pi$ trace out the blue curve in Figure 3.3.8. To determine its length, use Equation 3.3.10:

$$\begin{aligned} s &= \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= \int_0^{\pi} \sqrt{(-3\sin t)^2 + (3\cos t)^2}\, dt \\ &= \int_0^{\pi} \sqrt{9\sin^2 t + 9\cos^2 t}\, dt \\ &= \int_0^{\pi} \sqrt{9(\sin^2 t + \cos^2 t)}\, dt \\ &= \int_0^{\pi} 3\, dt = 3t\Big|_0^{\pi} \\ &= 3\pi \text{ units.} \end{aligned}$$

Note that the formula for the arc length of a semicircle is πr and the radius of this circle is 3. This is a great example of using calculus to derive a known formula of a geometric quantity.

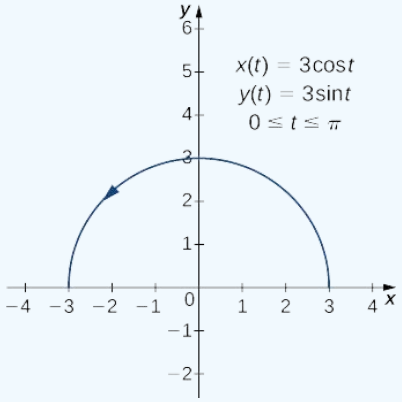


Figure 3.3.8: The arc length of the semicircle is equal to its radius times π.



Exercise 3.3.5

Find the arc length of the curve defined by the equations

$$x(t) = 3t^2, \quad y(t) = 2t^3, \quad \text{for } 1 \le t \le 3.$$

Hint

Use Equation 3.3.10

Answer

$s = 2(10^{3/2} - 2^{3/2}) \approx 57.589$ units

We now return to the problem posed at the beginning of the section about a baseball leaving a pitcher's hand. Ignoring the effect of air resistance (unless it is a curve ball!), the ball travels a parabolic path. Assuming the pitcher's hand is at the origin and the ball travels left to right in the direction of the positive x-axis, the parametric equations for this curve can be written as

$$x(t) = 140t, \quad y(t) = -16t^2 + 2t$$

where t represents time. We first calculate the distance the ball travels as a function of time. This distance is represented by the arc length. We can modify the arc length formula slightly. First rewrite the functions $x(t)$ and $y(t)$ using v as an independent variable, so as to eliminate any confusion with the parameter t:

$$x(v) = 140v, \quad y(v) = -16v^2 + 2v.$$

Then we write the arc length formula as follows:

$$\begin{aligned} s(t) &= \int_0^t \sqrt{(\frac{dx}{dv})^2 + (\frac{dy}{dv})^2}\, dv \\ &= \int_0^t \sqrt{140^2 + (-32v+2)^2}\, dv \end{aligned}$$

The variable v acts as a dummy variable that disappears after integration, leaving the arc length as a function of time t. To integrate this expression we can use a formula from Appendix A,

$$\int \sqrt{a^2+u^2}\, du = \frac{u}{2}\sqrt{a^2+u^2} + \frac{a^2}{2}\ln\left| u + \sqrt{a^2+u^2}\right| + C.$$

We set $a = 140$ and $u = -32v+2$. This gives $du = -32\, dv$, so $dv = -\frac{1}{32}\, du$. Therefore

$$\begin{aligned} \int \sqrt{140^2+(-32v+2)^2}\, dv &= -\frac{1}{32}\int \sqrt{a^2+u^2}\, du \\ &= -\frac{1}{32}\left[\frac{(-32v+2)}{2}\sqrt{140^2+(-32v+2)^2} + \frac{140^2}{2}\ln\left|(-32v+2) + \sqrt{140^2+(-32v+2)^2}\right| + C\right] \end{aligned}$$

and

$$\begin{aligned} s(t) &= -\frac{1}{32}\left[\frac{(-32t+2)}{2}\sqrt{140^2+(-32t+2)^2} + \frac{140^2}{2}\ln\left|(-32t+2) + \sqrt{140^2+(-32t+2)^2}\right|\right] \\ &\quad + \frac{1}{32}\left[\sqrt{140^2+2^2} + \frac{140^2}{2}\ln\left|2+\sqrt{140^2+2^2}\right|\right] \\ &= \left(\frac{t}{2} - \frac{1}{32}\right)\sqrt{1024t^2-128t+19604} - \frac{1225}{4}\ln\left|(-32t+2) + \sqrt{1024t^2-128t+19604}\right| + \frac{\sqrt{19604}}{32} + \frac{1225}{4}\ln \\ &\quad (2+\sqrt{19604}) \end{aligned}$$

This function represents the distance traveled by the ball as a function of time. To calculate the speed, take the derivative of this function with respect to t. While this may seem like a daunting task, it is possible to obtain the answer directly from the Fundamental Theorem of Calculus:

$$\frac{d}{dx}\int_a^x f(u)\, du = f(x).$$

Therefore

$$\begin{aligned} s'(t) &= \frac{d}{dt}\left[s(t)\right] \\ &= \frac{d}{dt}\left[\int_0^t \sqrt{140^2+(-32v+2)^2}\, dv\right] \\ &= \sqrt{140^2+(-32t+2)^2} \\ &= \sqrt{1024t^2-128t+19604} \\ &= 2\sqrt{256t^2-32t+4901}. \end{aligned}$$

One third of a second after the ball leaves the pitcher's hand, the distance it travels is equal to

$$
\begin{aligned}
s\left(\frac{1}{3}\right) &= \left(\frac{1/3}{2}-\frac{1}{32}\right)\sqrt{1024\left(\frac{1}{3}\right)^2-128\left(\frac{1}{3}\right)+19604} \\
&\quad -\frac{1225}{4}\ln\left|\left(-32\left(\frac{1}{3}\right)+2\right)+\sqrt{1024\left(\frac{1}{3}\right)^2-128\left(\frac{1}{3}\right)+19604}\right| \\
&\quad +\frac{\sqrt{19604}}{32}+\frac{1225}{4}\ln(2+\sqrt{19604}) \\
&\approx 46.69 \text{ feet.}
\end{aligned}
$$

This value is just over three quarters of the way to home plate. The speed of the ball is

$$s'\left(\tfrac{1}{3}\right) = 2\sqrt{256\left(\tfrac{1}{3}\right)^2-32\left(\tfrac{1}{3}\right)+4901} \approx 140.27 \text{ ft/s.}$$

This speed translates to approximately 95 mph—a major-league fastball.

Surface Area Generated by a Parametric Curve

Recall the problem of finding the surface area of a volume of revolution. In Curve Length and Surface Area, we derived a formula for finding the surface area of a volume generated by a function $y=f(x)$ from $x=a$ to $x=b$, revolved around the x-axis:

$$S = 2\pi\int_a^b f(x)\sqrt{1+(f'(x))^2}\,dx. \tag{3.3.11}$$

We now consider a volume of revolution generated by revolving a parametrically defined curve $x=x(t), \quad y=y(t), \quad$ for $a\le t\le b$ around the x-axis as shown in Figure 3.3.9.

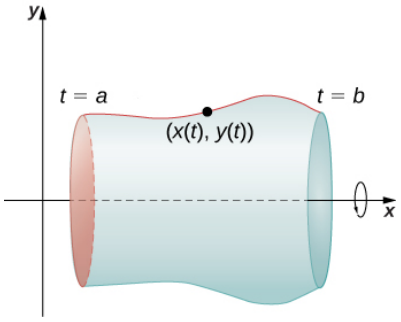


Figure 3.3.9: A surface of revolution generated by a parametrically defined curve.

The analogous formula for a parametrically defined curve is

$$S = 2\pi\int_a^b y(t)\sqrt{(x'(t))^2+(y'(t))^2}\,dt \tag{3.3.12}$$

provided that $y(t)$ is not negative on $[a,b]$.

Example 3.3.6: Finding Surface Area

Find the surface area of a sphere of radius r centered at the origin.

Solution

We start with the curve defined by the equations

$$x(t)=r\cos t, \quad y(t)=r\sin t, \quad \text{for } 0\le t\le\pi.$$

This generates an upper semicircle of radius r centered at the origin as shown in the following graph.

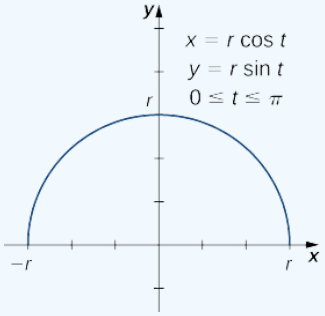


Figure 3.3.10: A semicircle generated by parametric equations.

When this curve is revolved around the x-axis, it generates a sphere of radius r. To calculate the surface area of the sphere, we use Equation 3.3.12:

$$\begin{aligned} S &= 2\pi \int_a^b y(t)\sqrt{(x'(t))^2 + (y'(t))^2}\, dt \\ &= 2\pi \int_0^\pi r\sin t \sqrt{(-r\sin t)^2 + (r\cos t)^2}\, dt \\ &= 2\pi \int_0^\pi r\sin t \sqrt{r^2\sin^2 t + r^2\cos^2 t}\, dt \\ &= 2\pi \int_0^\pi r\sin t \sqrt{r^2(\sin^2 t + \cos^2 t)}\, dt \\ &= 2\pi \int_0^\pi r^2 \sin t\, dt \\ &= 2\pi r^2 \left(-\cos t\Big|_0^\pi\right) \\ &= 2\pi r^2(-\cos\pi + \cos 0) \\ &= 4\pi r^2 \text{ units}^2. \end{aligned}$$

This is, in fact, the formula for the surface area of a sphere.



Exercise 3.3.6

Find the surface area generated when the plane curve defined by the equations

$$x(t) = t^3, \quad y(t) = t^2, \quad \text{for } 0 \le t \le 1$$

is revolved around the x-axis.

Hint

Use Equation 3.3.12 When evaluating the integral, use a u-substitution.

Answer

$$A = \frac{\pi(494\sqrt{13} + 128)}{1215} \text{ units}^2$$

Key Concepts

- The derivative of the parametrically defined curve $x = x(t)$ and $y = y(t)$ can be calculated using the formula $\frac{dy}{dx} = \frac{y'(t)}{x'(t)}$. Using the derivative, we can find the equation of a tangent line to a parametric curve.
- The area between a parametric curve and the x-axis can be determined by using the formula $A = \int_{t_1}^{t_2} y(t)x'(t)\,dt$.
- The arc length of a parametric curve can be calculated by using the formula

$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt.$$

- The surface area of a volume of revolution revolved around the x-axis is given by

$$S = 2\pi \int_a^b y(t)\sqrt{(x'(t))^2 + (y'(t))^2}\,dt.$$

- If the curve is revolved around the y-axis, then the formula is

$$S = 2\pi \int_a^b x(t)\sqrt{(x'(t))^2 + (y'(t))^2}\,dt.$$

Key Equations

- **Derivative of parametric equations**

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)}$$

- **Second-order derivative of parametric equations**

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{(d/dt)(dy/dx)}{dx/dt}$$

- **Area under a parametric curve**

$$A = \int_a^b y(t)x'(t)\,dt$$

- **Arc length of a parametric curve**

$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

- **Surface area generated by a parametric curve**

$$S = 2\pi \int_a^b y(t)\sqrt{(x'(t))^2 + (y'(t))^2}\,dt$$

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3.3E: Exercises for Section 3.3

In exercises 1 - 4, each set of parametric equations represents a line. Without eliminating the parameter, find the slope of each line.

1) $x = 3 + t, \quad y = 1 - t$

2) $x = 8 + 2t, \quad y = 1$

Answer

$m = 0$

3) $x = 4 - 3t, \quad y = -2 + 6t$

4) $x = -5t + 7, \quad y = 3t - 1$

Answer

$m = -\frac{3}{5}$

In exercises 5 - 9, determine the slope of the tangent line, then find the equation of the tangent line at the given value of the parameter.

5) $x = 3\sin t, \quad y = 3\cos t, \quad \text{for } t = \frac{\pi}{4}$

6) $x = \cos t, \quad y = 8\sin t, \quad \text{for } t = \frac{\pi}{2}$

Answer

Slope$= 0; y = 8.$

7) $x = 2t, \quad y = t^3, \quad \text{for } t = -1$

8) $x = t + \dfrac{1}{t}, \quad y = t - \dfrac{1}{t}, \quad \text{for } t = 1$

Answer

Slope is undefined; $x = 2$.

9) $x = \sqrt{t}, \quad y = 2t, \quad \text{for } t = 4$

In exercises 10 - 13, find all points on the curve that have the given slope.

10) $x = 4\cos t, \quad y = 4\sin t,$ slope = 0.5

Solution

$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{4\cos t}{-4\sin t} = -\cot t.$

Setting this derivative equal to 0.5, we obtain the equation, $\tan t = -2$.

$\tan t = -2 \implies \dfrac{y}{x} = -2 \implies y = -2x.$

Note also that this pair of parametric equations represents the circle $x^2 + y^2 = 16$.

By substitution, we find that this curve has a slope of 0.5 at the points:

$\left(\frac{4\sqrt{5}}{5}, \frac{-8\sqrt{5}}{5}\right)$ and $\left(\frac{-4\sqrt{5}}{5}, \frac{8\sqrt{5}}{5}\right)$.

11) $x = 2\cos t, \quad y = 8\sin t,$ slope$= -1$

12) $x = t + \dfrac{1}{t}, \quad y = t - \dfrac{1}{t},$ slope$= 1$

Answer

No points possible; undefined expression.

13) $x = 2 + \sqrt{t}, \quad y = 2 - 4t,$ slope= 0

In exercises 14 - 16, write the equation of the tangent line in Cartesian coordinates for the given parameter t.

14) $x = e^{\sqrt{t}}, \quad y = 1 - \ln t^2, \quad$ for $t = 1$

Answer

$y = -\left(\frac{4}{e}\right)x + 5$

15) $x = t \ln t, \quad y = \sin^2 t, \quad$ for $t = \frac{\pi}{4}$

16) $x = e^t, \quad y = (t-1)^2,$ at $(1, 1)$

Answer

$y = -2x + 3$

17) For $x = \sin(2t), \quad y = 2\sin t$ where $0 \le t < 2\pi$. Find all values of t at which a horizontal tangent line exists.

18) For $x = \sin(2t), \quad y = 2\sin t$ where $0 \le t < 2\pi$. Find all values of t at which a vertical tangent line exists.

Answer

A vertical tangent line exists at $t = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}$

19) Find all points on the curve $x = 4\cos(t), \quad y = 4\sin(t)$ that have the slope of $\frac{1}{2}$.

20) Find $\dfrac{dy}{dx}$ for $x = \sin(t), \quad y = \cos(t)$.

Answer

$\dfrac{dy}{dx} = -\tan(t)$

21) Find the equation of the tangent line to $x = \sin(t), \quad y = \cos(t)$ at $t = \frac{\pi}{4}$.

22) For the curve $x = 4t, \quad y = 3t - 2,$ find the slope and concavity of the curve at $t = 3$.

Answer

$\dfrac{dy}{dx} = \dfrac{3}{4}$ and $\dfrac{d^2y}{dx^2} = 0$, so the curve is neither concave up nor concave down at $t = 3$. Therefore the graph is linear and has a constant slope but no concavity.

23) For the parametric curve whose equation is $x = 4\cos\theta, \quad y = 4\sin\theta$, find the slope and concavity of the curve at $\theta = \frac{\pi}{4}$.

24) Find the slope and concavity for the curve whose equation is $x = 2 + \sec\theta, \quad y = 1 + 2\tan\theta$ at $\theta = \frac{\pi}{6}$.

Answer

$\dfrac{dy}{dx} = 4, \quad \dfrac{d^2y}{dx^2} = -4\sqrt{3};$ the curve is concave down at $\theta = \frac{\pi}{6}$.

25) Find all points on the curve $x = t + 4, \quad y = t^3 - 3t$ at which there are vertical and horizontal tangents.

26) Find all points on the curve $x = \sec\theta, \quad y = \tan\theta$ at which horizontal and vertical tangents exist.

Answer

No horizontal tangents. Vertical tangents at $(1, 0)$ and $(-1, 0)$.

In exercises 27 - 29, find d^2y/dx^2.

27) $x = t^4 - 1, \quad y = t - t^2$

28) $x = \sin(\pi t), \quad y = \cos(\pi t)$

Answer

$d^2y/dx^2 = -\sec^3(\pi t)$

29) $x = e^{-t}, \quad y = te^{2t}$

In exercises 30 - 31, find points on the curve at which tangent line is horizontal or vertical.

30) $x = t(t^2 - 3), \quad y = 3(t^2 - 3)$

Answer

Horizontal $(0, -9)$;
Vertical $(\pm 2, -6)$.

31) $x = \dfrac{3t}{1+t^3}, \quad y = \dfrac{3t^2}{1+t^3}$

In exercises 32 - 34, find dy/dx at the value of the parameter.

32) $x = \cos t, y = \sin t, \quad \text{for } t = \frac{3\pi}{4}$

Answer

$dy/dx = 1$

33) $x = \sqrt{t}, \quad y = 2t + 4, t = 9$

34) $x = 4\cos(2\pi s), \quad y = 3\sin(2\pi s), \quad \text{for } s = -\frac{1}{4}$

Answer

$dy/dx = 0$

In exercises 35 - 36, find d^2y/dx^2 at the given point without eliminating the parameter.

35) $x = \frac{1}{2}t^2, \quad y = \frac{1}{3}t^3, \quad \text{for } t = 2$

36) $x = \sqrt{t}, \quad y = 2t + 4, \quad \text{for } t = 1$

Answer

$d^2y/dx^2 = 4$

37) Find intervals for t on which the curve $x = 3t^2, \quad y = t^3 - t$ is concave up as well as concave down.

38) Determine the concavity of the curve $x = 2t + \ln t, \quad y = 2t - \ln t$.

Answer

Concave up on $t > 0$.

39) Sketch and find the area under one arch of the cycloid $x = r(\theta - \sin\theta), \quad y = r(1 - \cos\theta)$.

40) Find the area bounded by the curve $x = \cos t, \quad y = e^t, \quad \text{for } 0 \le t \le \frac{\pi}{2}$ and the lines $y = 1$ and $x = 0$.

Answer

1 unit^2

41) Find the area enclosed by the ellipse $x = a\cos\theta, \quad y = b\sin\theta, \quad \text{for } 0 \le \theta < 2\pi.$

42) Find the area of the region bounded by $x = 2\sin^2\theta, \quad y = 2\sin^2\theta\tan\theta$, for $0 \le \theta \le \frac{\pi}{2}$.

Answer

$\frac{3\pi}{2} \text{ units}^2$

In exercises 43 - 46, find the area of the regions bounded by the parametric curves and the indicated values of the parameter.

43) $x = 2\cot\theta, \quad y = 2\sin^2\theta, \quad \text{for } 0 \le \theta \le \pi$

44) [T] $x = 2a\cos t - a\cos(2t), \quad y = 2a\sin t - a\sin(2t), \quad \text{for } 0 \le t < 2\pi$

Answer

$6\pi a^2 \text{ units}^2$

45) [T] $x = a\sin(2t), \quad y = b\sin(t), \quad \text{for } 0 \le t < 2\pi$ (the "hourglass")

46) [T] $x = 2a\cos t - a\sin(2t), \quad y = b\sin t, \quad \text{for } 0 \le t < 2\pi$ (the "teardrop")

Answer

$2\pi ab \text{ units}^2$

In exercises 47 - 52, find the arc length of the curve on the indicated interval of the parameter.

47) $x = 4t + 3, \quad y = 3t - 2, \quad \text{for } 0 \le t \le 2$

48) $x = \frac{1}{3}t^3, \quad y = \frac{1}{2}t^2, \quad \text{for } 0 \le t \le 1$

Answer

$s = \frac{1}{3}(2\sqrt{2} - 1) \text{ units}$

49) $x = \cos(2t), \quad y = \sin(2t), \quad \text{for } 0 \le t \le \frac{\pi}{2}$

50) $x = 1 + t^2, \quad y = (1+t)^3, \quad \text{for } 0 \le t \le 1$

Answer

$s = 7.075 \text{ units}$

51) $x = e^t\cos t, \quad y = e^t\sin t, \quad \text{for } 0 \le t \le \frac{\pi}{2}$ (express answer as a decimal rounded to three places)

52) $x = a\cos^3\theta, \quad y = a\sin^3\theta$ on the interval $[0, 2\pi)$(the hypocycloid)

Answer

$s = 6a \text{ units}$

53) Find the length of one arch of the cycloid $x = 4(t - \sin t), \quad y = 4(1 - \cos t)$.

54) Find the distance traveled by a particle with position (x, y) as t varies in the given time interval: $x = \sin^2 t, \quad y = \cos^2 t, \quad \text{for } 0 \le t \le 3\pi$.

Answer

$6\sqrt{2} \text{ units}$

55) Find the length of one arch of the cycloid $x = \theta - \sin\theta, \quad y = 1 - \cos\theta$.

56) Show that the total length of the ellipse $x = 4\sin\theta, \quad y = 3\cos\theta$ is $L = 16\int_0^{\pi/2} \sqrt{1 - e^2\sin^2\theta}\, d\theta$, where $e = \frac{c}{a}$ and $c = \sqrt{a^2 - b^2}$.

57) Find the length of the curve $x = e^t - t, \quad y = 4e^{t/2}, \quad \text{for } -8 \le t \le 3$.

In exercises 58 - 59, find the area of the surface obtained by rotating the given curve about the x-axis.

58) $x = t^3, \quad y = t^2, \quad \text{for } 0 \le t \le 1$

Answer

$$\frac{2\pi(247\sqrt{13}+64)}{1215}\ \text{units}^2$$

59) $x = a\cos^3\theta, \quad y = a\sin^3\theta, \quad \text{for } 0 \le \theta \le \frac{\pi}{2}$

60) [T] Use a CAS to find the area of the surface generated by rotating $x = t + t^3, \quad y = t - \frac{1}{t^2}, \quad \text{for } 1 \le t \le 2$ about the x-axis. (Answer to three decimal places.)

Answer

$59.101\ \text{units}^2$

61) Find the surface area obtained by rotating $x = 3t^2, \quad y = 2t^3, \quad \text{for } 0 \le t \le 5$ about the y-axis.

62) Find the area of the surface generated by revolving $x = t^2, \quad y = 2t, \quad \text{for } 0 \le t \le 4$ about the x-axis.

Answer

$\frac{8\pi}{3}(17\sqrt{17}-1)\ \text{units}^2$

63) Find the surface area generated by revolving $x = t^2, \quad y = 2t^2, \quad \text{for } 0 \le t \le 1$ about the y-axis.

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3.4: Polar Coordinates

Learning Objectives

- Locate points in a plane by using polar coordinates.
- Convert points between rectangular and polar coordinates.
- Sketch polar curves from given equations.
- Convert equations between rectangular and polar coordinates.
- Identify symmetry in polar curves and equations.

The rectangular coordinate system (or Cartesian plane) provides a means of mapping points to ordered pairs and ordered pairs to points. This is called a *one-to-one mapping* from points in the plane to ordered pairs. The polar coordinate system provides an alternative method of mapping points to ordered pairs. In this section we see that in some circumstances, polar coordinates can be more useful than rectangular coordinates.

Defining Polar Coordinates

To find the coordinates of a point in the polar coordinate system, consider Figure 3.4.1. The point P has Cartesian coordinates (x, y). The line segment connecting the origin to the point P measures the distance from the origin to P and has length r. The angle between the positive x-axis and the line segment has measure θ. This observation suggests a natural correspondence between the coordinate pair (x, y) and the values r and θ. This correspondence is the basis of the **polar coordinate system**. Note that every point in the Cartesian plane has two values (hence the term ordered pair) associated with it. In the polar coordinate system, each point also has two values associated with it: r and θ.

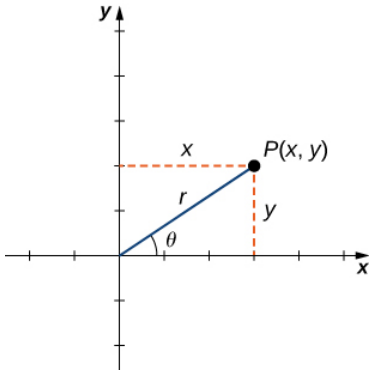


Figure 3.4.1: An arbitrary point in the Cartesian plane.

Using right-triangle trigonometry, the following equations are true for the point P:

$$\cos\theta = \frac{x}{r} \text{ so } x = r\cos\theta \tag{3.4.1}$$

$$\sin\theta = \frac{y}{r} \text{ so } y = r\sin\theta. \tag{3.4.2}$$

Furthermore,

$$r^2 = x^2 + y^2 \tag{3.4.3}$$

and

$$\tan\theta = \frac{y}{x}. \tag{3.4.4}$$

Each point (x, y) in the Cartesian coordinate system can therefore be represented as an ordered pair (r, θ) in the polar coordinate system. The first coordinate is called the **radial coordinate** and the second coordinate is called the **angular coordinate**. Every point in the plane can be represented in this form.

Note that the equation $\tan\theta = y/x$ has an infinite number of solutions for any ordered pair (x, y). However, if we restrict the solutions to values between 0 and 2π then we can assign a unique solution to the quadrant in which the original point (x, y) is

located. Then the corresponding value of r is positive, so $r^2 = x^2 + y^2$.

Converting Points between Coordinate Systems

Given a point P in the plane with Cartesian coordinates (x, y) and polar coordinates (r, θ), the following conversion formulas hold true:

$$x = r\cos\theta \tag{3.4.5}$$

$$y = r\sin\theta \tag{3.4.6}$$

and

$$r^2 = x^2 + y^2 \tag{3.4.7}$$

$$\tan\theta = \frac{y}{x} \tag{3.4.8}$$

These formulas can be used to convert from rectangular to polar or from polar to rectangular coordinates. Notice that Equation 3.4.7 is the *Pythagorean theorem*. (Figure 3.4.1).

Example 3.4.1: Converting between Rectangular and Polar Coordinates

Convert each of the following points into polar coordinates.

a. $(1, 1)$
b. $(-3, 4)$
c. $(0, 3)$
d. $(5\sqrt{3}, -5)$

Convert each of the following points into rectangular coordinates.

e. $(3, \pi/3)$
f. $(2, 3\pi/2)$
g. $(6, -5\pi/6)$

Solution

a. Use $x = 1$ and $y = 1$ in Equation 3.4.7:

$$\begin{aligned} r^2 &= x^2 + y^2 \\ &= 1^2 + 1^2 \\ r &= \sqrt{2} \end{aligned}$$

and via Equation 3.4.8

$$\begin{aligned} \tan\theta = \frac{y}{x} = \frac{1}{1} = 1 \\ \theta = \frac{\pi}{4}. \end{aligned}$$

Therefore this point can be represented as $(\sqrt{2}, \frac{\pi}{4})$ in polar coordinates.

b. Use $x = -3$ and $y = 4$ in Equation 3.4.7:

$$\begin{aligned} r^2 &= x^2 + y^2 = (-3)^2 + (4)^2 \\ r &= 5 \end{aligned}$$

and via Equation 3.4.8

$$\begin{aligned} \tan\theta = \frac{y}{x} = -\frac{4}{3} \\ \theta = \arctan(-\frac{4}{3}) + \pi \approx 2.21. \end{aligned}$$

Therefore this point can be represented as $(5, 2.21)$ in polar coordinates.

c. Use $x = 0$ and $y = 3$ in Equation 3.4.7:

$$r^2 = x^2 + y^2 = (3)^2 + (0)^2 = 9 + 0 \quad r = 3$$

and via Equation 3.4.8

$$\tan\theta = \frac{y}{x} = \frac{3}{0}\ .$$

Direct application of the second equation leads to division by zero. Graphing the point $(0, 3)$ on the rectangular coordinate system reveals that the point is located on the positive y-axis. The angle between the positive x-axis and the positive y-axis is $\frac{\pi}{2}$. Therefore this point can be represented as $(3, \frac{\pi}{2})$ in polar coordinates.

d. Use $x = 5\sqrt{3}$ and $y = -5$ in Equation 3.4.7:

$$r^2 = x^2 + y^2 = (5\sqrt{3})^2 + (-5)^2 = 75 + 25$$
$$r = 10$$

and via Equation 3.4.8

$$\tan\theta = \frac{y}{x} = \frac{-5}{5\sqrt{3}} = -\frac{\sqrt{3}}{3}$$
$$\theta = -\frac{\pi}{6}\ .$$

Therefore this point can be represented as $(10, -\frac{\pi}{6})$ in polar coordinates.

e. Use $r = 3$ and $\theta = \frac{\pi}{3}$ in Equation 3.4.5:

$$x = r\cos\theta = 3\cos(\frac{\pi}{3}) = 3(\frac{1}{2}) = \frac{3}{2}$$

and

$$y = r\sin\theta = 3\sin(\frac{\pi}{3}) = 3(\frac{\sqrt{3}}{2}) = \frac{3\sqrt{3}}{2}\ .$$

Therefore this point can be represented as $(\frac{3}{2}, \frac{3\sqrt{3}}{2})$ in rectangular coordinates.

f. Use $r = 2$ and $\theta = \frac{3\pi}{2}$ in Equation 3.4.5:

$$x = r\cos\theta = 2\cos(\frac{3\pi}{2}) = 2(0) = 0$$

and

$$y = r\sin\theta = 2\sin(\frac{3\pi}{2}) = 2(-1) = -2.$$

Therefore this point can be represented as $(0, -2)$ in rectangular coordinates.

g. Use $r = 6$ and $\theta = -\frac{5\pi}{6}$ in Equation 3.4.5:

$$x = r\cos\theta = 6\cos(-\frac{5\pi}{6}) = 6(-\frac{\sqrt{3}}{2}) = -3\sqrt{3}$$

and

$$y = r\sin\theta = 6\sin(-\frac{5\pi}{6}) = 6(-\frac{1}{2}) = -3\ .$$

Therefore this point can be represented as $(-3\sqrt{3}, -3)$ in rectangular coordinates.

Exercise 3.4.1

Convert $(-8, -8)$ into polar coordinates and $(4, \frac{2\pi}{3})$ into rectangular coordinates.

Hint

Use Equation 3.4.7and Equation 3.4.5 Make sure to check the quadrant when calculating θ.

Answer

$(8\sqrt{2}, \frac{5\pi}{4})$ and $(-2, 2\sqrt{3})$

The polar representation of a point is not unique. For example, the polar coordinates $(2, \frac{\pi}{3})$ and $(2, \frac{7\pi}{3})$ both represent the point $(1, \sqrt{3})$ in the rectangular system. Also, the value of r can be negative. Therefore, the point with polar coordinates $(-2, \frac{4\pi}{3})$ also represents the point $(1, \sqrt{3})$ in the rectangular system, as we can see by using Equation 3.4.5:

$$x = r\cos\theta = -2\cos(\frac{4\pi}{3}) = -2(-\frac{1}{2}) = 1 \tag{3.4.9}$$

and

$$y = r\sin\theta = -2\sin(\frac{4\pi}{3}) = -2(-\frac{\sqrt{3}}{2}) = \sqrt{3}. \tag{3.4.10}$$

Every point in the plane has an infinite number of representations in polar coordinates. However, each point in the plane has only one representation in the rectangular coordinate system.

Note that the polar representation of a point in the plane also has a visual interpretation. In particular, r is the directed distance that the point lies from the origin, and θ measures the angle that the line segment from the origin to the point makes with the positive x-axis. Positive angles are measured in a counterclockwise direction and negative angles are measured in a clockwise direction. The polar coordinate system appears in Figure 3.4.2.

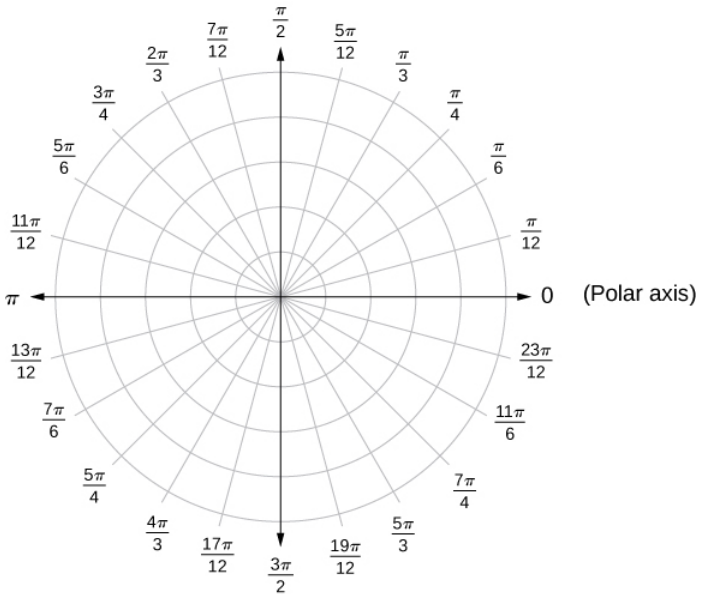


Figure 3.4.2: The polar coordinate system.

The line segment starting from the center of the graph going to the right (called the positive x-axis in the Cartesian system) is the **polar axis.** The center point is the **pole**, or origin, of the coordinate system, and corresponds to $r = 0$. The innermost circle shown in Figure 3.4.2 contains all points a distance of 1 unit from the pole, and is represented by the equation $r = 1$. Then $r = 2$ is the set of points 2 units from the pole, and so on. The line segments emanating from the pole correspond to fixed angles. To plot a point in

the polar coordinate system, start with the angle. If the angle is positive, then measure the angle from the polar axis in a counterclockwise direction. If it is negative, then measure it clockwise. If the value of r is positive, move that distance along the terminal ray of the angle. If it is negative, move along the ray that is opposite the terminal ray of the given angle.

Example 3.4.2: Plotting Points in the Polar Plane

Plot each of the following points on the polar plane.

a. $(2, \frac{\pi}{4})$

b. $(-3, \frac{2\pi}{3})$

c. $(4, \frac{5\pi}{4})$

Solution

The three points are plotted in Figure 3.4.3.

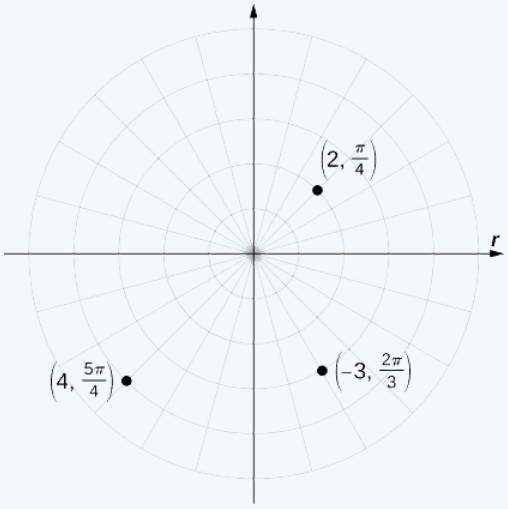


Figure 3.4.3: Three points plotted in the polar coordinate system.

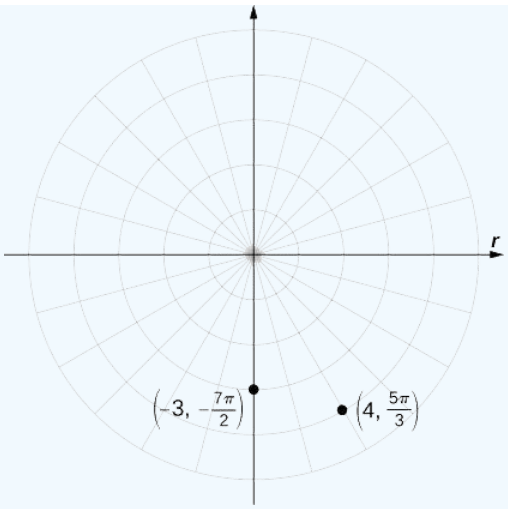
Exercise 3.4.2

Plot $(4, \frac{5\pi}{3})$ and $(-3, -\frac{7\pi}{2})$ on the polar plane.

Hint

Start with θ, then use r.

Answer



Polar Curves

Now that we know how to plot points in the polar coordinate system, we can discuss how to plot curves. In the rectangular coordinate system, we can graph a function $y = f(x)$ and create a curve in the Cartesian plane. In a similar fashion, we can graph a curve that is generated by a function $r = f(\theta)$.

The general idea behind graphing a function in polar coordinates is the same as graphing a function in rectangular coordinates. Start with a list of values for the independent variable (θ in this case) and calculate the corresponding values of the dependent variable r. This process generates a list of ordered pairs, which can be plotted in the polar coordinate system. Finally, connect the points, and take advantage of any patterns that may appear. The function may be periodic, for example, which indicates that only a limited number of values for the independent variable are needed.

Problem-Solving Strategy: Plotting a Curve in Polar Coordinates

1. Create a table with two columns. The first column is for θ, and the second column is for r.
2. Create a list of values for θ.
3. Calculate the corresponding r values for each θ.
4. Plot each ordered pair (r, θ) on the coordinate axes.
5. Connect the points and look for a pattern.

Example 3.4.3: Graphing a Function in Polar Coordinates

Graph the curve defined by the function $r = 4\sin\theta$. Identify the curve and rewrite the equation in rectangular coordinates.

Solution

Because the function is a multiple of a sine function, it is periodic with period 2π, so use values for θ between 0 and 2π. The result of steps 1–3 appear in the following table. Figure 3.4.4 shows the graph based on this table.

θ	$r = 4\sin\theta$	θ	$r = 4\sin\theta$
0	0	π	0
$\frac{\pi}{6}$	2	$\frac{7\pi}{6}$	-2
$\frac{\pi}{4}$	$2\sqrt{2} \approx 2.8$	$\frac{5\pi}{4}$	$-2\sqrt{2} \approx -2.8$
$\frac{\pi}{3}$	$2\sqrt{3} \approx 3.4$	$\frac{4\pi}{3}$	$-2\sqrt{3} \approx -3.4$
$\frac{\pi}{2}$	4	$\frac{3\pi}{2}$	-4

θ	$r = 4\sin\theta$	θ	$r = 4\sin\theta$
$\frac{2\pi}{3}$	$2\sqrt{3} \approx 3.4$	$\frac{5\pi}{3}$	$-2\sqrt{3} \approx -3.4$
$\frac{3\pi}{4}$	$2\sqrt{2} \approx 2.8$	$\frac{7\pi}{4}$	$-2\sqrt{2} \approx -2.8$
$\frac{5\pi}{6}$	2	$\frac{11\pi}{6}$	-2
		2π	0

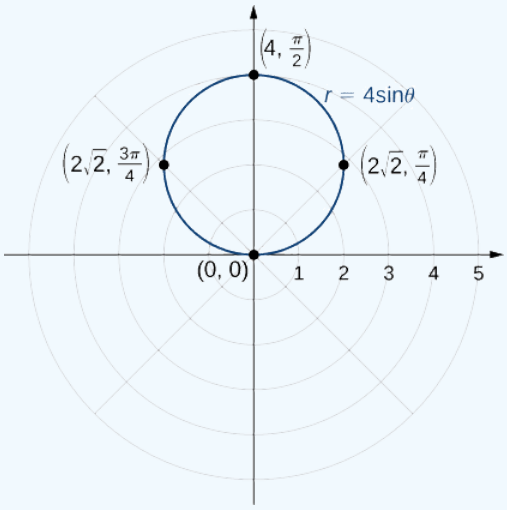


Figure 3.4.4: The graph of the function $r = 4\sin\theta$ is a circle.

This is the graph of a circle. The equation $r = 4\sin\theta$ can be converted into rectangular coordinates by first multiplying both sides by r. This gives the equation $r^2 = 4r\sin\theta$. Next use the facts that $r^2 = x^2 + y^2$ and $y = r\sin\theta$. This gives $x^2 + y^2 = 4y$. To put this equation into standard form, subtract $4y$ from both sides of the equation and complete the square:

$$\begin{aligned} x^2 + y^2 - 4y &= 0 \\ x^2 + (y^2 - 4y) &= 0 \\ x^2 + (y^2 - 4y + 4) &= 0 + 4 \\ x^2 + (y-2)^2 &= 4 \end{aligned}$$

This is the equation of a circle with radius 2 and center $(0, 2)$ in the rectangular coordinate system.

Exercise 3.4.3

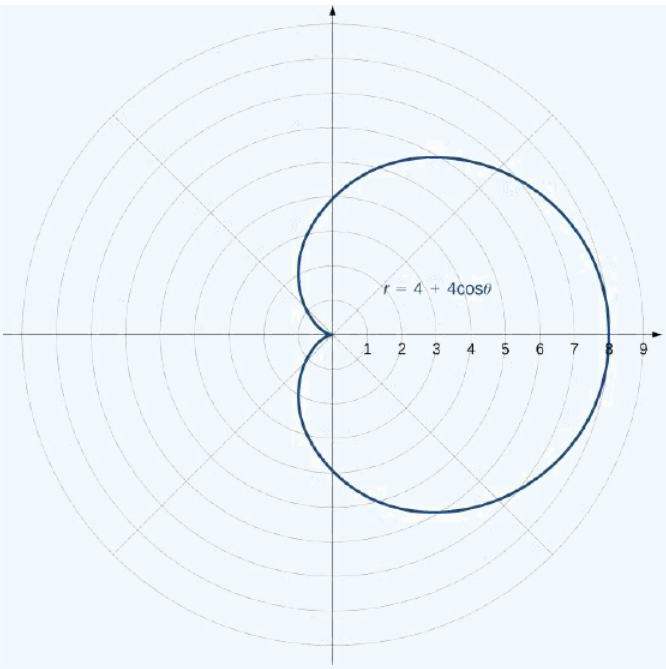
Create a graph of the curve defined by the function $r = 4 + 4\cos\theta$.

Hint

Follow the problem-solving strategy for creating a graph in polar coordinates.

Answer

The name of this shape is a cardioid, which we will study further later in this section.



The graph in Example 3.4.3 was that of a circle. The equation of the circle can be transformed into rectangular coordinates using the coordinate transformation formulas in Equation 3.4.5. Example 3.4.4 gives some more examples of functions for transforming from polar to rectangular coordinates.

Example 3.4.4: Transforming Polar Equations to Rectangular Coordinates

Rewrite each of the following equations in rectangular coordinates and identify the graph.

a. $\theta = \dfrac{\pi}{3}$
b. $r = 3$
c. $r = 6\cos\theta - 8\sin\theta$

Solution:

a. Take the tangent of both sides. This gives $\tan\theta = \tan(\pi/3) = \sqrt{3}$ .Since $\tan\theta = y/x$ we can replace the left-hand side of this equation by y/x. This gives $y/x = \sqrt{3}$, which can be rewritten as $y = x\sqrt{3}$. This is the equation of a straight line passing through the origin with slope $\sqrt{3}$. In general, any polar equation of the form $\theta = K$ represents a straight line through the pole with slope equal to $\tan K$.

b. First, square both sides of the equation. This gives $r^2 = 9$. Next replace r^2 with $x^2 + y^2$. This gives the equation $x^2 + y^2 = 9$, which is the equation of a circle centered at the origin with radius 3. In general, any polar equation of the form $r = k$ where *k* is a positive constant represents a circle of radius *k* centered at the origin. (Note: when squaring both sides of an equation it is possible to introduce new points unintentionally. This should always be taken into consideration. However, in this case we do not introduce new points. For example, $(-3, \dfrac{\pi}{3})$ is the same point as $(3, \dfrac{4\pi}{3})$.)

c. Multiply both sides of the equation by r. This leads to $r^2 = 6r\cos\theta - 8r\sin\theta$. Next use the formulas

$$r^2 = x^2 + y^2, x = r\cos\theta, y = r\sin\theta.$$

This gives

$$r^2 = 6(r\cos\theta) - 8(r\sin\theta)$$
$$x^2 + y^2 = 6x - 8y.$$

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To put this equation into standard form, first move the variables from the right-hand side of the equation to the left-hand side, then complete the square.

$$x^2+y^2=6x-8y$$
$$x^2-6x+y^2+8y=0$$
$$(x^2-6x)+(y^2+8y)=0$$
$$(x^2-6x+9)+(y^2+8y+16)=9+16$$
$$(x-3)^2+(y+4)^2=25.$$

This is the equation of a circle with center at $(3,-4)$ and radius 5. Notice that the circle passes through the origin since the center is 5 units away.



Exercise 3.4.4

Rewrite the equation $r=\sec\theta\tan\theta$ in rectangular coordinates and identify its graph.

Hint

Convert to sine and cosine, then multiply both sides by cosine.

Answer

$y=x^2$, which is the equation of a parabola opening upward.



We have now seen several examples of drawing graphs of curves defined by **polar equations.** A summary of some common curves is given in the tables below. In each equation, a and b are arbitrary constants.

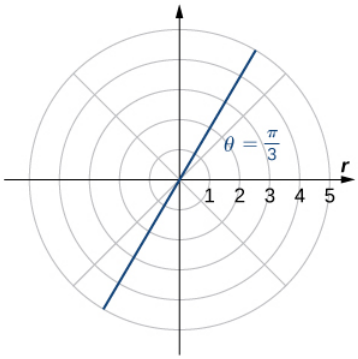
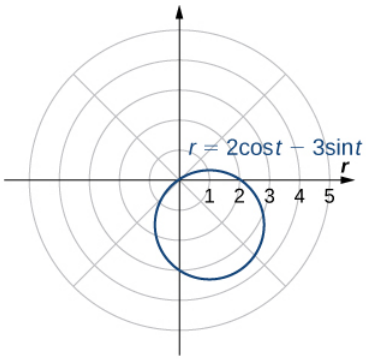
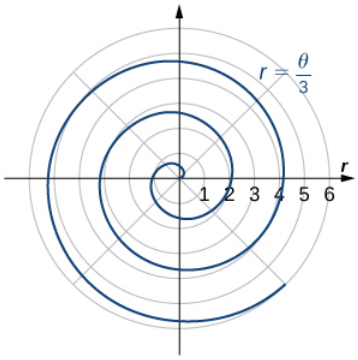
Name	Equation	Example
Line passing through the pole with slope tan K	$\theta = K$	
Circle	$r = a\cos\theta + b\sin\theta$	
Spiral	$r = a + b\theta$	

Figure 3.4.5

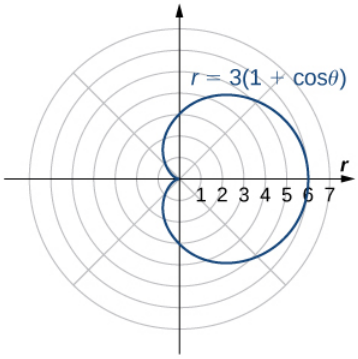
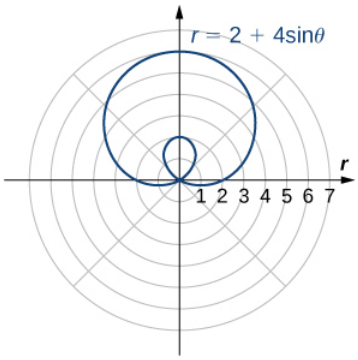
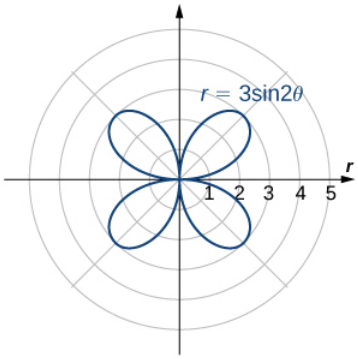
Name	Equation	Example
Cardioid	$r = a(1 + \cos\theta)$ $r = a(1 - \cos\theta)$ $r = a(1 + \sin\theta)$ $r = a(1 - \sin\theta)$	
Limaçon	$r = a\cos\theta + b$ $r = a\sin\theta + b$	
Rose	$r = a\cos(b\theta)$ $r = a\sin(b\theta)$	

Figure 3.4.6

A **cardioid** is a special case of a **limaçon** (pronounced “lee-mah-son”), in which $a = b$ or $a = -b$. The **rose i**s a very interesting curve. Notice that the graph of $r = 3\sin 2\theta$ has four petals. However, the graph of $r = 3\sin 3\theta$ has three petals as shown.

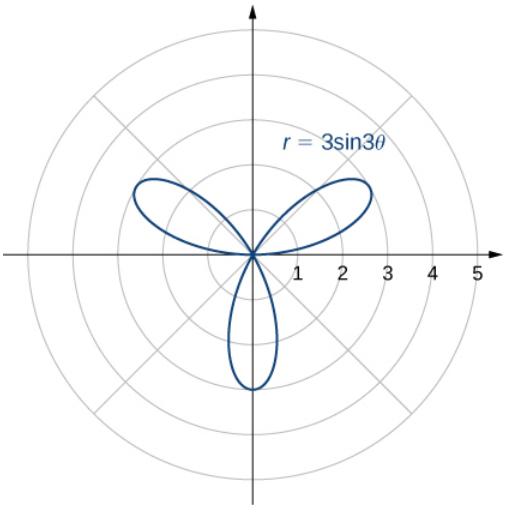
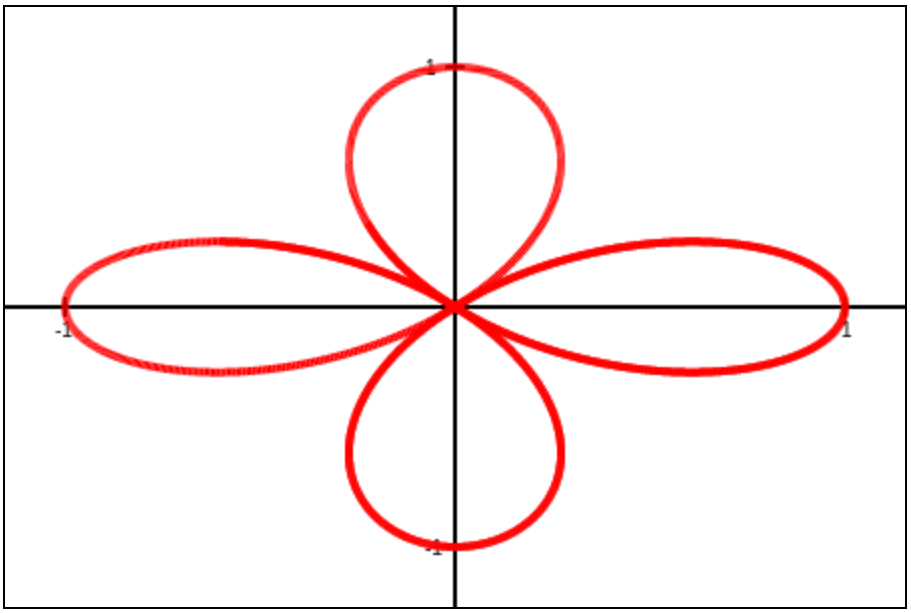


Figure 3.4.7: Graph of $r = 3 \sin 3\theta$.

If the coefficient of θ is even, the graph has twice as many petals as the coefficient. If the coefficient of θ is odd, then the number of petals equals the coefficient. You are encouraged to explore why this happens.

Exploring Graphs of Multiple Petaled Roses

Decide whether you want to graph a rose that comes from a sin or cos polar curve. Then move the slider to see how changing the factor in front of x changes the rose.



r = cos(2x)

sin cos

Even more interesting graphs emerge when the coefficient of θ is not an integer. For example, if it is rational, then the curve is closed; that is, it eventually ends where it started (Figure 3.4.8*a*). However, if the coefficient is irrational, then the curve never closes (Figure 3.4.8*b*). Although it may appear that the curve is closed, a closer examination reveals that the petals just above the positive x axis are slightly thicker. This is because the petal does not quite match up with the starting point.

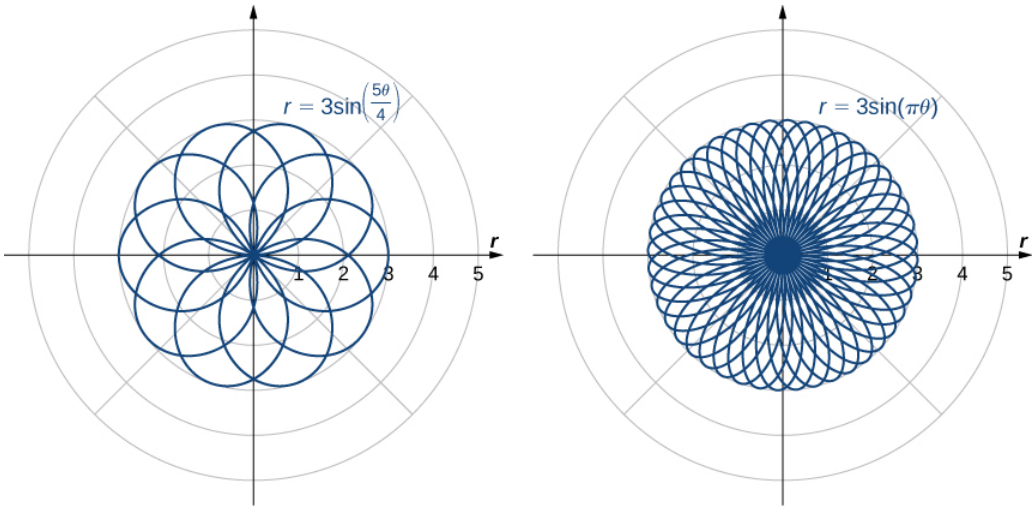


Figure 3.4.8: Polar rose graphs of functions with (a) rational coefficient and (b) irrational coefficient. Note that the rose in part (b) would actually fill the entire circle if plotted in full.

Since the curve defined by the graph of $r = 3\sin(\pi\theta)$ never closes, the curve depicted in Figure 3.4.8*b* is only a partial depiction. In fact, this is an example of a **space-filling curve.** A space-filling curve is one that in fact occupies a two-dimensional subset of the real plane. In this case the curve occupies the circle of radius 3 centered at the origin.

Example 3.4.5: Describing a Spiral

Recall the **chambered nautilus** introduced in the chapter prelude. This creature displays a spiral when half the outer shell is cut away. It is possible to describe a spiral using rectangular coordinates. Figure 3.4.9 shows a spiral in rectangular coordinates. How can we describe this curve mathematically?

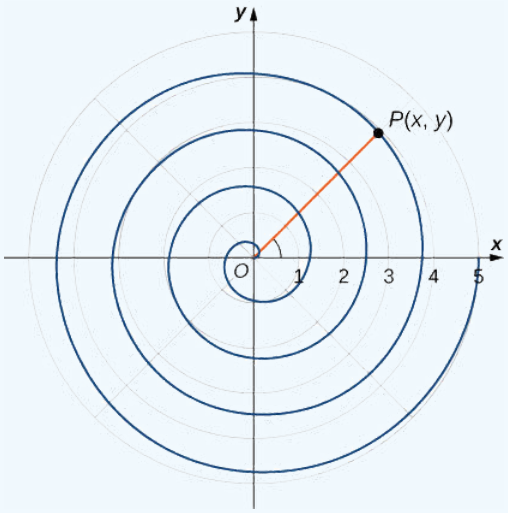


Figure 3.4.9: How can we describe a spiral graph mathematically?

Solution

As the point P travels around the spiral in a counterclockwise direction, its distance d from the origin increases. Assume that the distance d is a constant multiple k of the angle θ that the line segment OP makes with the positive x-axis. Therefore $d(P, O) = k\theta$, where O is the origin. Now use the distance formula and some trigonometry:

$$d(P, O) = k\theta$$

$$\sqrt{(x-0)^2 + (y-0)^2} = k\arctan\left(\frac{y}{x}\right)$$

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$$\sqrt{x^2+y^2} = k\arctan(\frac{y}{x})$$

$$\arctan(\frac{y}{x}) = \frac{\sqrt{x^2+y^2}}{k}$$

$$y = x\tan(\frac{\sqrt{x^2+y^2}}{k}).$$

Although this equation describes the spiral, it is not possible to solve it directly for either x or y. However, if we use polar coordinates, the equation becomes much simpler. In particular, $d(P,O) = r$, and θ is the second coordinate. Therefore the equation for the spiral becomes $r = k\theta$. Note that when $\theta = 0$ we also have $r = 0$, so the spiral emanates from the origin. We can remove this restriction by adding a constant to the equation. Then the equation for the spiral becomes $r = a + k\theta$ for arbitrary constants a and k. This is referred to as an **Archimedean spiral**, after the Greek mathematician Archimedes.

Another type of spiral is the logarithmic spiral, described by the function $r = a \cdot b^{\theta}$. A graph of the function $r = 1.2(1.25^{\theta})$ is given in Figure 3.4.10. This spiral describes the shell shape of the chambered nautilus.

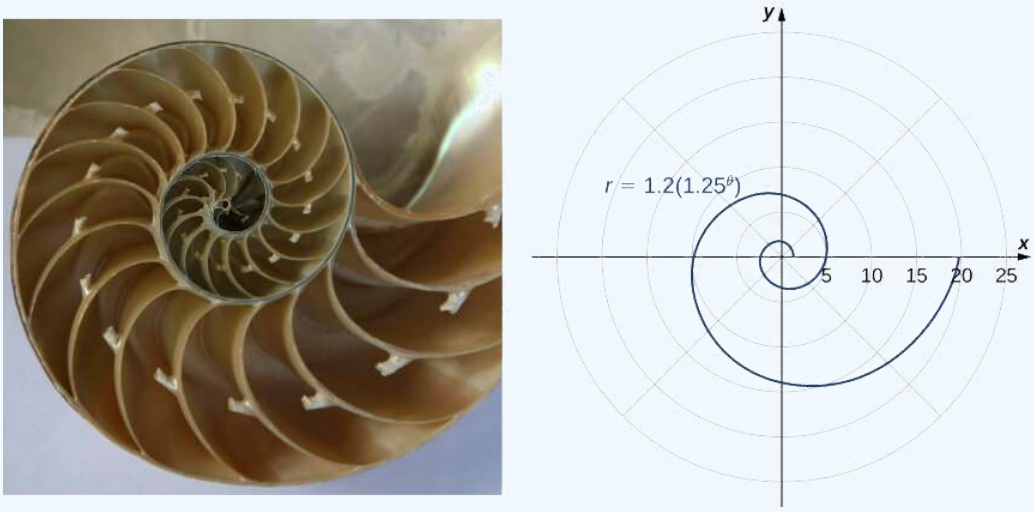


Figure 3.4.10: A logarithmic spiral is similar to the shape of the chambered nautilus shell. (credit: modification of work by Jitze Couperus, Flickr)

Suppose a curve is described in the polar coordinate system via the function $r = f(\theta)$. Since we have conversion formulas from polar to rectangular coordinates given by

$$x = r\cos\theta \tag{3.4.11}$$

$$y = r\sin\theta \tag{3.4.12}$$

,

it is possible to rewrite these formulas using the function

$$x = f(\theta)\cos\theta \tag{3.4.13}$$

$$y = f(\theta)\sin\theta. \tag{3.4.14}$$

This step gives a parameterization of the curve in rectangular coordinates using θ as the parameter. For example, the spiral formula $r = a + b\theta$ from Figure becomes

$$x = (a + b\theta)\cos\theta \tag{3.4.15}$$

$$y = (a + b\theta)\sin\theta. \tag{3.4.16}$$

Letting θ range from $-\infty$ to ∞ generates the entire spiral.

Symmetry in Polar Coordinates

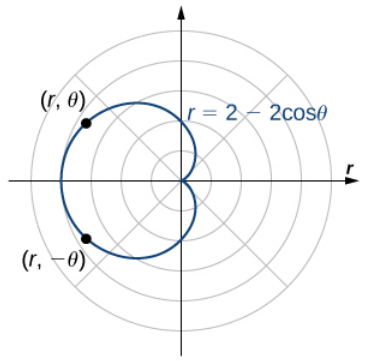
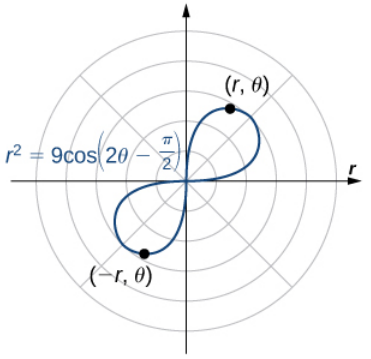
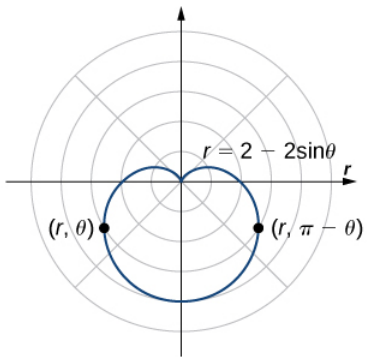
When studying **symmetry** of functions in rectangular coordinates (i.e., in the form $y = f(x)$), we talk about symmetry with respect to the *y*-axis and symmetry with respect to the origin. In particular, if $f(-x) = f(x)$ for all x in the domain of f, then f is an even function and its graph is symmetric with respect to the *y*-axis. If $f(-x) = -f(x)$ for all *x* in the domain of f, then f is an odd function and its graph is symmetric with respect to the origin. By determining which types of symmetry a graph exhibits, we can learn more about the shape and appearance of the graph. Symmetry can also reveal other properties of the function that generates the graph. Symmetry in polar curves works in a similar fashion.

Symmetry in Polar Curves and Equations

Consider a curve generated by the function $r = f(\theta)$ in polar coordinates.

i. The curve is symmetric about the polar axis if for every point (r, θ) on the graph, the point $(r, -\theta)$ is also on the graph. Similarly, the equation $r = f(\theta)$ is unchanged by replacing θ with $-\theta$.
ii. The curve is symmetric about the pole if for every point (r, θ) on the graph, the point $(r, \pi + \theta)$ is also on the graph. Similarly, the equation $r = f(\theta)$ is unchanged when replacing r with $-r$, or θ with $\pi + \theta$.
iii. The curve is symmetric about the vertical line $\theta = \frac{\pi}{2}$ if for every point (r, θ) on the graph, the point $(r, \pi - \theta)$ is also on the graph. Similarly, the equation $r = f(\theta)$ is unchanged when θ is replaced by $\pi - \theta$.

The following table shows examples of each type of symmetry.

Symmetry with respect to the polar axis: For every point (r, θ) on the graph, there is also a point reflected directly across the horizontal (polar) axis.	
Symmetry with respect to the pole: For every point (r, θ) on the graph, there is also a point on the graph that is reflected through the pole as well.	
Symmetry with respect to the vertical line $\theta = \frac{\pi}{2}$: For every point (r, θ) on the graph, there is also a point reflected directly across the vertical axis.	

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Example 3.4.6: Using Symmetry to Graph a Polar Equation

Find the symmetry of the rose defined by the equation $r = 3\sin(2\theta)$ and create a graph.

Solution

Suppose the point (r, θ) is on the graph of $r = 3\sin(2\theta)$.

i. To test for symmetry about the polar axis, first try replacing θ with $-\theta$. This gives $r = 3\sin(2(-\theta)) = -3\sin(2\theta)$. Since this changes the original equation, this test is not satisfied. However, returning to the original equation and replacing r with $-r$ and θ with $\pi - \theta$ yields

$$\begin{aligned} -r &= 3\sin(2(\pi - \theta)) \\ -r &= 3\sin(2\pi - 2\theta) \\ -r &= 3\sin(-2\theta) \\ -r &= -3\sin 2\theta. \end{aligned}$$

Multiplying both sides of this equation by -1 gives $r = 3\sin 2\theta$, which is the original equation. This demonstrates that the graph is symmetric with respect to the polar axis.

ii. To test for symmetry with respect to the pole, first replace r with $-r$, which yields $-r = 3\sin(2\theta)$. Multiplying both sides by -1 gives $r = -3\sin(2\theta)$, which does not agree with the original equation. Therefore the equation does not pass the test for this symmetry. However, returning to the original equation and replacing θ with $\theta + \pi$ gives

$$\begin{aligned} r &= 3\sin(2(\theta + \pi)) \\ &= 3\sin(2\theta + 2\pi) \\ &= 3(\sin 2\theta \cos 2\pi + \cos 2\theta \sin 2\pi) \\ &= 3\sin 2\theta. \end{aligned}$$

Since this agrees with the original equation, the graph is symmetric about the pole.

iii. To test for symmetry with respect to the vertical line $\theta = \dfrac{\pi}{2}$, first replace both r with $-r$ and θ with $-\theta$.

$$\begin{aligned} -r &= 3\sin(2(-\theta)) \\ -r &= 3\sin(-2\theta) \\ -r &= -3\sin 2\theta. \end{aligned}$$

Multiplying both sides of this equation by -1 gives $r = 3\sin 2\theta$, which is the original equation. Therefore the graph is symmetric about the vertical line $\theta = \dfrac{\pi}{2}$.

This graph has symmetry with respect to the polar axis, the origin, and the vertical line going through the pole. To graph the function, tabulate values of θ between 0 and $\pi/2$ and then reflect the resulting graph.

0	0
$\dfrac{\pi}{6}$	$\dfrac{3\sqrt{3}}{2} \approx 2.6$
$\dfrac{\pi}{4}$	3
$\dfrac{\pi}{3}$	$\dfrac{3\sqrt{3}}{2} \approx 2.6$
$\dfrac{\pi}{2}$	0

This gives one petal of the rose, as shown in the following graph.

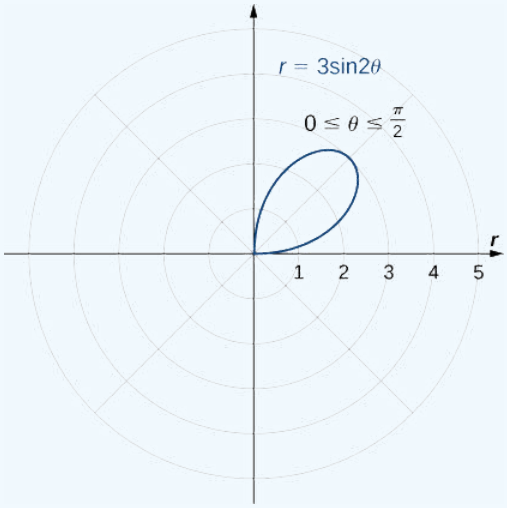


Figure 3.4.11: The graph of the equation between $\theta = 0$ and $\theta = \pi/2$.

Reflecting this image into the other three quadrants gives the entire graph as shown.

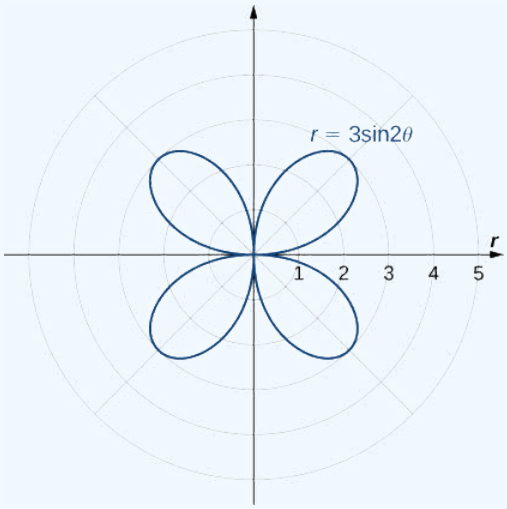


Figure 3.4.12: The entire graph of the equation is called a four-petaled rose.

Exercise 3.4.5Symmetry

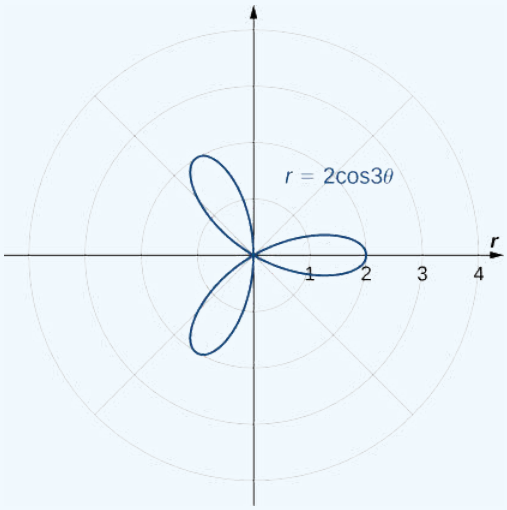
Determine the symmetry of the graph determined by the equation $r = 2\cos(3\theta)$ and create a graph.

Hint

Use Note.

Answer

Symmetric with respect to the polar axis.



Key Concepts

- The polar coordinate system provides an alternative way to locate points in the plane.
- Convert points between rectangular and polar coordinates using the formulas

$$x = r\cos\theta \text{ and } y = r\sin\theta \tag{3.4.17}$$

and

$$r = \sqrt{x^2 + y^2} \text{ and } \tan\theta = \frac{y}{x}. \tag{3.4.18}$$

- To sketch a polar curve from a given polar function, make a table of values and take advantage of periodic properties.
- Use the conversion formulas to convert equations between rectangular and polar coordinates.
- Identify symmetry in polar curves, which can occur through the pole, the horizontal axis, or the vertical axis.

Glossary

angular coordinate

θ the angle formed by a line segment connecting the origin to a point in the polar coordinate system with the positive radial (x) axis, measured counterclockwise

cardioid

a plane curve traced by a point on the perimeter of a circle that is rolling around a fixed circle of the same radius; the equation of a cardioid is $r = a(1 + \sin\theta)$ or $r = a(1 + \cos\theta)$

limaçon

the graph of the equation $r = a + b\sin\theta$ or $r = a + b\cos\theta$. If $a = b$ then the graph is a cardioid

polar axis

the horizontal axis in the polar coordinate system corresponding to $r \geq 0$

polar coordinate system

a system for locating points in the plane. The coordinates are r, the radial coordinate, and θ, the angular coordinate

polar equation

an equation or function relating the radial coordinate to the angular coordinate in the polar coordinate system

pole

the central point of the polar coordinate system, equivalent to the origin of a Cartesian system

radial coordinate

r the coordinate in the polar coordinate system that measures the distance from a point in the plane to the pole

rose

graph of the polar equation $r = a\cos 2\theta$ or $r = a\sin 2\theta$ for a positive constant a

space-filling curve

a curve that completely occupies a two-dimensional subset of the real plane

Contributors and Attributions

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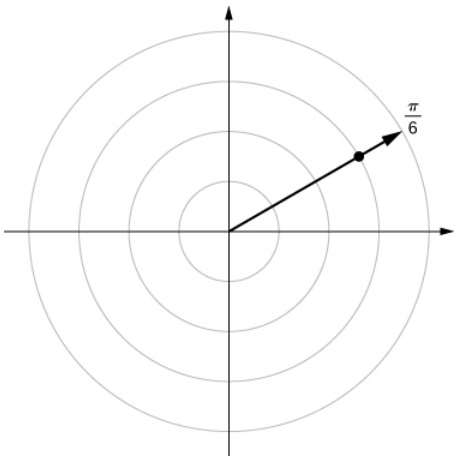
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3.4E: Exercises for Section 3.4

In exercises 1 - 7, plot the point whose polar coordinates are given by first constructing the angle θ and then marking off the distance r along the ray.

1) $\left(3, \frac{\pi}{6}\right)$

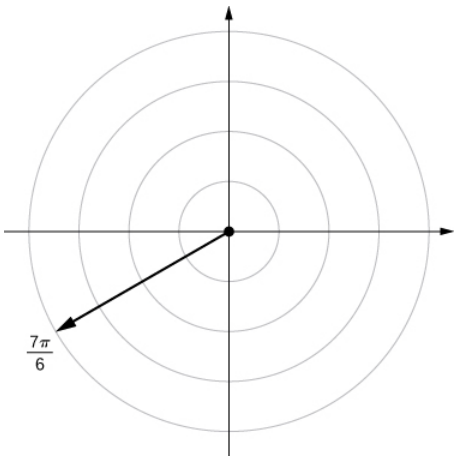
Answer



2) $\left(-2, \frac{5\pi}{3}\right)$

3) $\left(0, \frac{7\pi}{6}\right)$

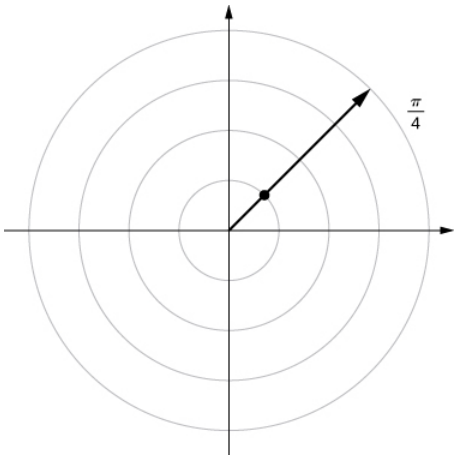
Answer



4) $\left(-4, \frac{3\pi}{4}\right)$

5) $\left(1, \frac{\pi}{4}\right)$

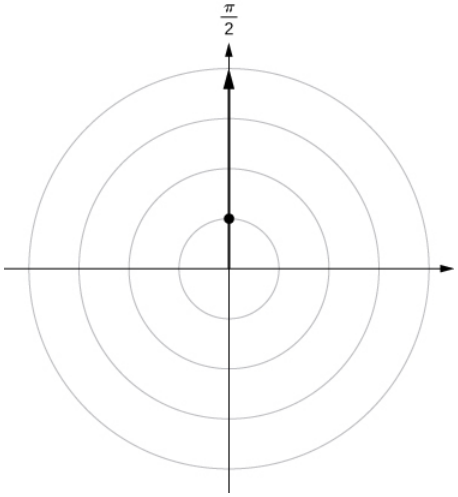
Answer



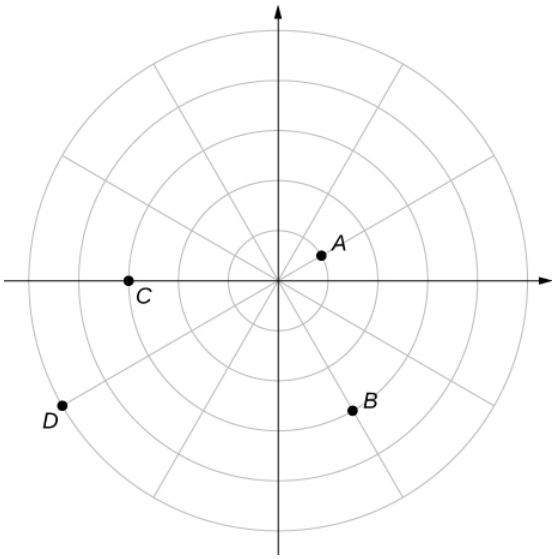
6) $\left(2, \frac{5\pi}{6}\right)$

7) $\left(1, \frac{\pi}{2}\right)$

Answer



In exercises 8 - 11, consider the polar graph below. Give two sets of polar coordinates for each point.



8) Coordinates of point A.

9) Coordinates of point B.

Answer

$B\left(3, \frac{-\pi}{3}\right)\ B\left(-3, \frac{2\pi}{3}\right)$

10) Coordinates of point C.

11) Coordinates of point D.

Answer

$D\left(5, \frac{7\pi}{6}\right)\ D\left(-5, \frac{\pi}{6}\right)$

In exercises 12 - 17, the rectangular coordinates of a point are given. Find two sets of polar coordinates for the point in $(0, 2\pi]$. Round to three decimal places.

12) $(2, 2)$

13) $(3, -4)$

Answer

$(5, -0.927),\ (-5, -0.927 + \pi)$

14) $(8, 15)$

15) $(-6, 8)$

Answer

$(10, -0.927),\ (-10, -0.927 + \pi)$

16) $(4, 3)$

17) $(3, -\sqrt{3})$

Answer

$(2\sqrt{3}, -0.524),\ (-2\sqrt{3}, -0.524 + \pi)$

In exercises 18 - 24, find rectangular coordinates for the given point in polar coordinates.

18) $\left(2, \frac{5\pi}{4}\right)$

19) $\left(-2, \frac{\pi}{6}\right)$

Answer

$(-\sqrt{3}, -1)$

20) $\left(5, \frac{\pi}{3}\right)$

21) $\left(1, \frac{7\pi}{6}\right)$

Answer

$\left(-\frac{\sqrt{3}}{2}, \frac{-1}{2}\right)$

22) $\left(-3, \frac{3\pi}{4}\right)$

23) $\left(0, \frac{\pi}{2}\right)$

Answer

$(0, 0)$

24) $(-4.5, 6.5)$

In exercises 25 - 29, determine whether the graphs of the polar equation are symmetric with respect to the x-axis, the y-axis, or the origin.

25) $r = 3\sin(2\theta)$

Answer

Symmetry with respect to the *x*-axis, *y*-axis, and origin.

26) $r^2 = 9\cos\theta$

27) $r = \cos\left(\frac{\theta}{5}\right)$

Answer

Symmetric with respect to *x*-axis only.

28) $r = 2\sec\theta$

29) $r = 1 + \cos\theta$

Answer

Symmetry with respect to *x*-axis only.

In exercises 30 - 33, describe the graph of each polar equation. Confirm each description by converting into a rectangular equation.

30) $r = 3$

31) $\theta = \frac{\pi}{4}$

Answer

Line $y = x$

32) $r = \sec\theta$

33) $r = \csc\theta$

Answer

$y = 1$

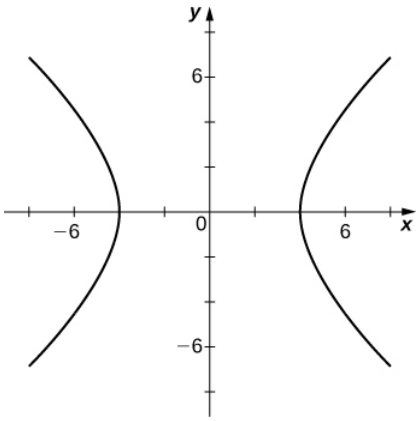
In exercises 34 - 36, convert the rectangular equation to polar form and sketch its graph.

34) $x^2 + y^2 = 16$

35) $x^2 - y^2 = 16$

Answer

Hyperbola; polar form $r^2\cos(2\theta) = 16$ or $r^2 = 16\sec\theta$.



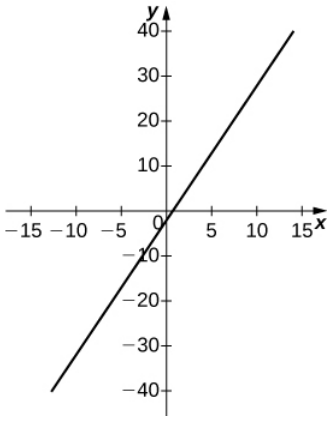
36) $x = 8$

In exercises 37 - 38, convert the rectangular equation to polar form and sketch its graph.

37) $3x - y = 2$

Answer

$r = \frac{2}{3\cos\theta - \sin\theta}$



38) $y^2 = 4x$

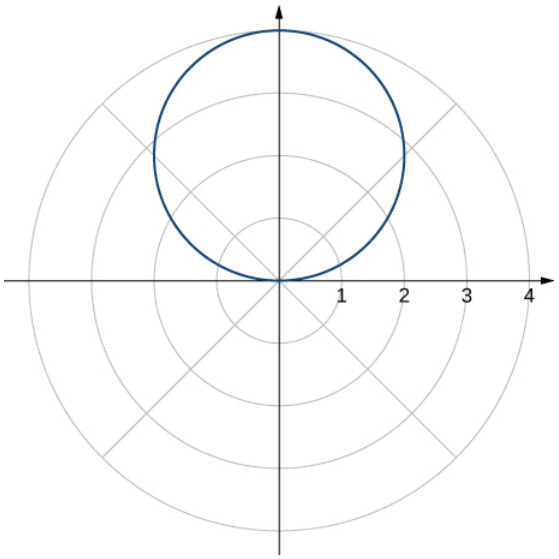
In exercises 39 - 43, convert the polar equation to rectangular form and sketch its graph.

39) $r = 4\sin\theta$

40) $x^2 + y^2 = 4y$

Answer

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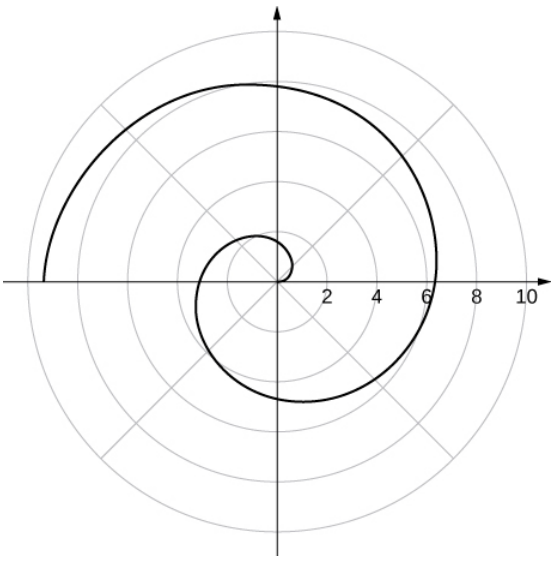


41) $r = 6\cos\theta$

42) $r = \theta$

Answer

$x \tan\sqrt{x^2+y^2} = y$



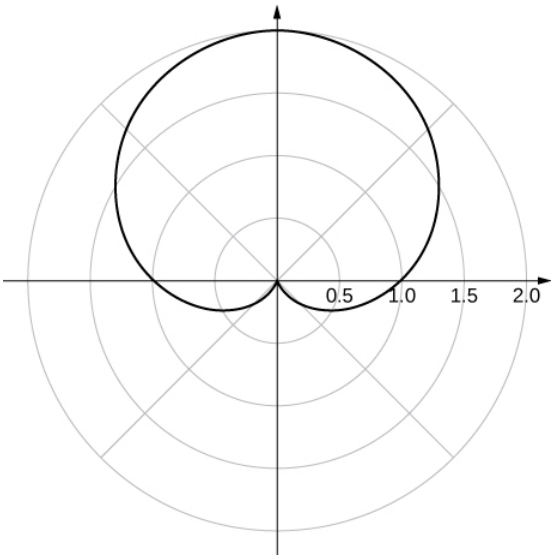
43) $r = \cot\theta\csc\theta$

In exercises 44 - 54, sketch a graph of the polar equation and identify any symmetry.

44) $r = 1+\sin\theta$

Answer

y-axis symmetry

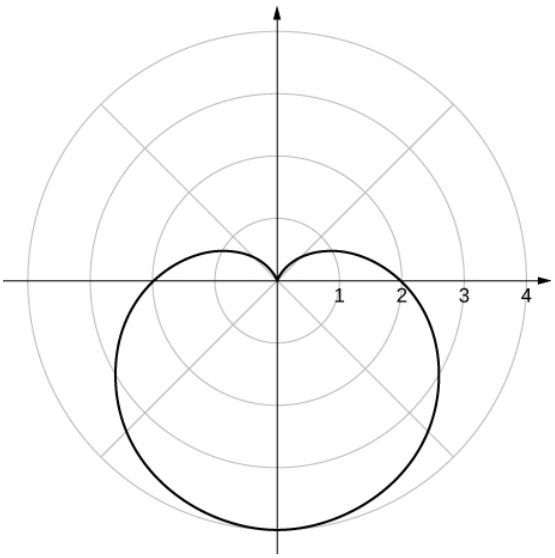


45) $r = 3 - 2\cos\theta$

46) $r = 2 - 2\sin\theta$

Answer

y-axis symmetry

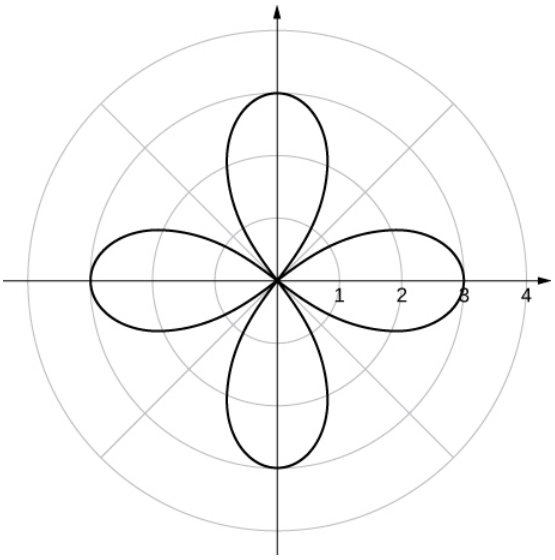


47) $r = 5 - 4\sin\theta$

48) $r = 3\cos(2\theta)$

Answer

x-and y-axis symmetry and symmetry about the pole

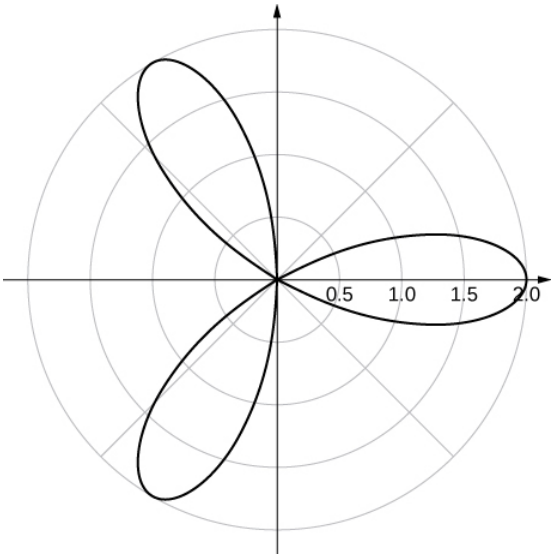


49) $r = 3\sin(2\theta)$

50) $r = 2\cos(3\theta)$

Answer

x-axis symmetry

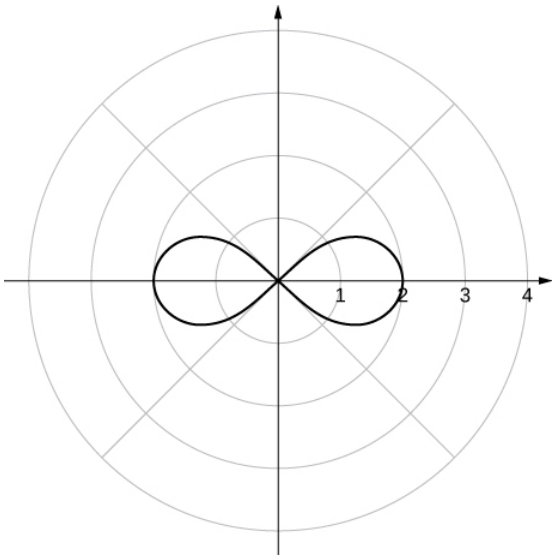


51) $r = 3\cos\left(\frac{\theta}{2}\right)$

52) $r^2 = 4\cos\left(\frac{2}{\theta}\right)$

Answer

x-and y-axis symmetry and symmetry about the pole

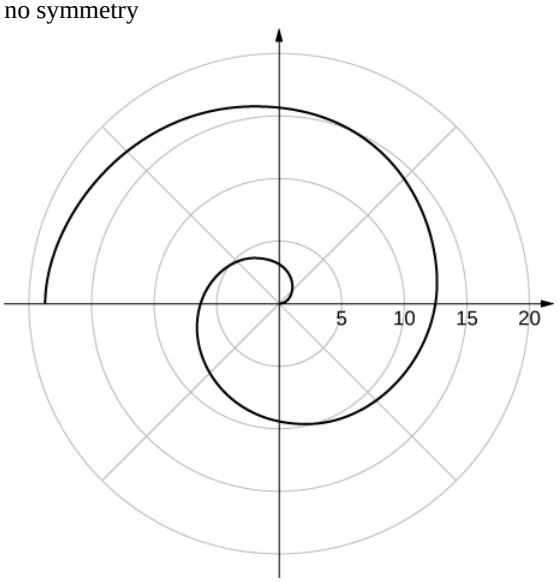


53) $r^2 = 4\sin\theta$

54) $r = 2\theta$

Answer

no symmetry

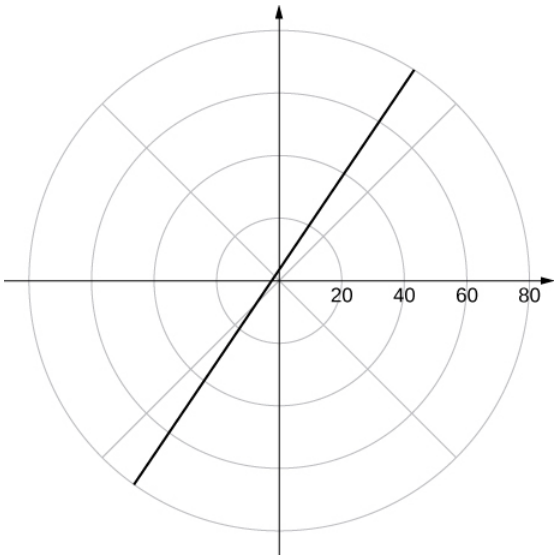


55) [T] The graph of $r = 2\cos(2\theta)\sec(\theta)$. is called a strophoid. Use a graphing utility to sketch the graph, and, from the graph, determine the asymptote.

56) [T] Use a graphing utility and sketch the graph of $r = \dfrac{6}{2\sin\theta - 3\cos\theta}$.

Answer

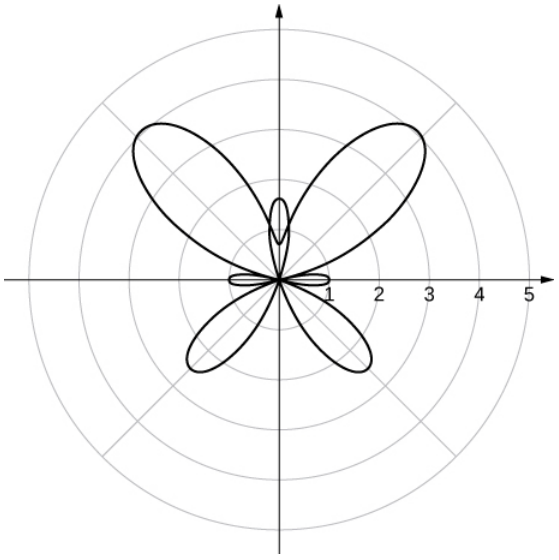
a line



57) [T] Use a graphing utility to graph $r = \frac{1}{1-\cos\theta}$.

58) [T] Use technology to graph $r = e^{\sin(\theta)} - 2\cos(4\theta)$.

Answer

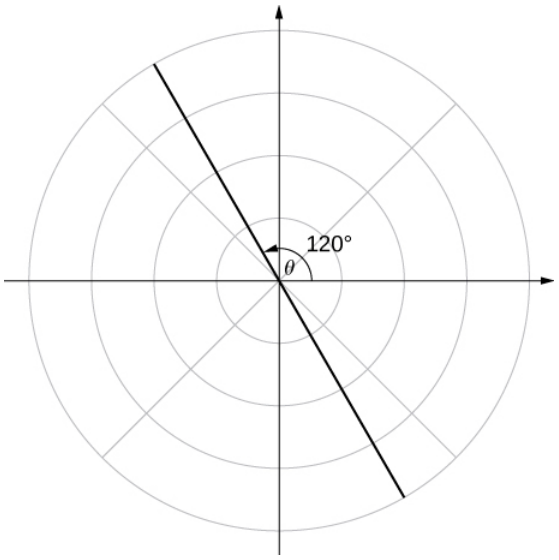


59) [T] Use technology to plot $r = \sin(\frac{3\theta}{7})$ (use the interval $0 \leq \theta \leq 14\pi$).

60) Without using technology, sketch the polar curve $\theta = \frac{2\pi}{3}$.

Answer

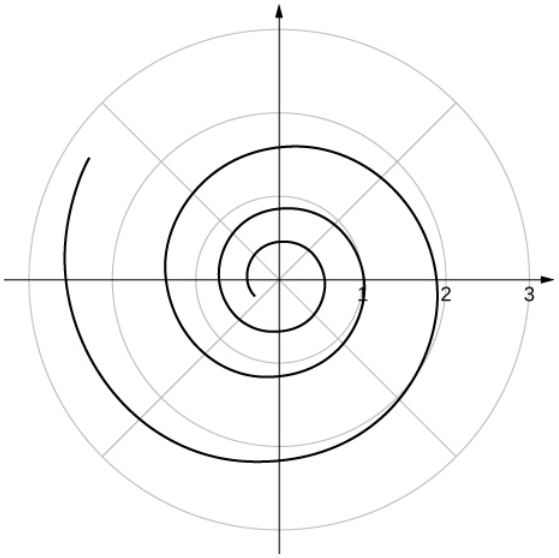
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61) [T] Use a graphing utility to plot $r = \theta \sin \theta$ for $-\pi \leq \theta \leq \pi$.

62) [T] Use technology to plot $r = e^{-0.1\theta}$ for $-10 \leq \theta \leq 10$.

Answer



63) [T] There is a curve known as the "Black Hole." Use technology to plot $r = e^{-0.01\theta}$ for $-100 \leq \theta \leq 100$.

64) [T] Use the results of the preceding two problems to explore the graphs of $r = e^{-0.001\theta}$ and $r = e^{-0.0001\theta}$ for $|\theta| > 100$.

Answer

Answers vary. One possibility is the spiral lines become closer together and the total number of spirals increases.

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3.5: Area and Arc Length in Polar Coordinates

Learning Objectives

- Apply the formula for area of a region in polar coordinates.
- Determine the arc length of a polar curve.

In the rectangular coordinate system, the definite integral provides a way to calculate the area under a curve. In particular, if we have a function $y = f(x)$ defined from $x = a$ to $x = b$ where $f(x) > 0$ on this interval, the area between the curve and the x-axis is given by

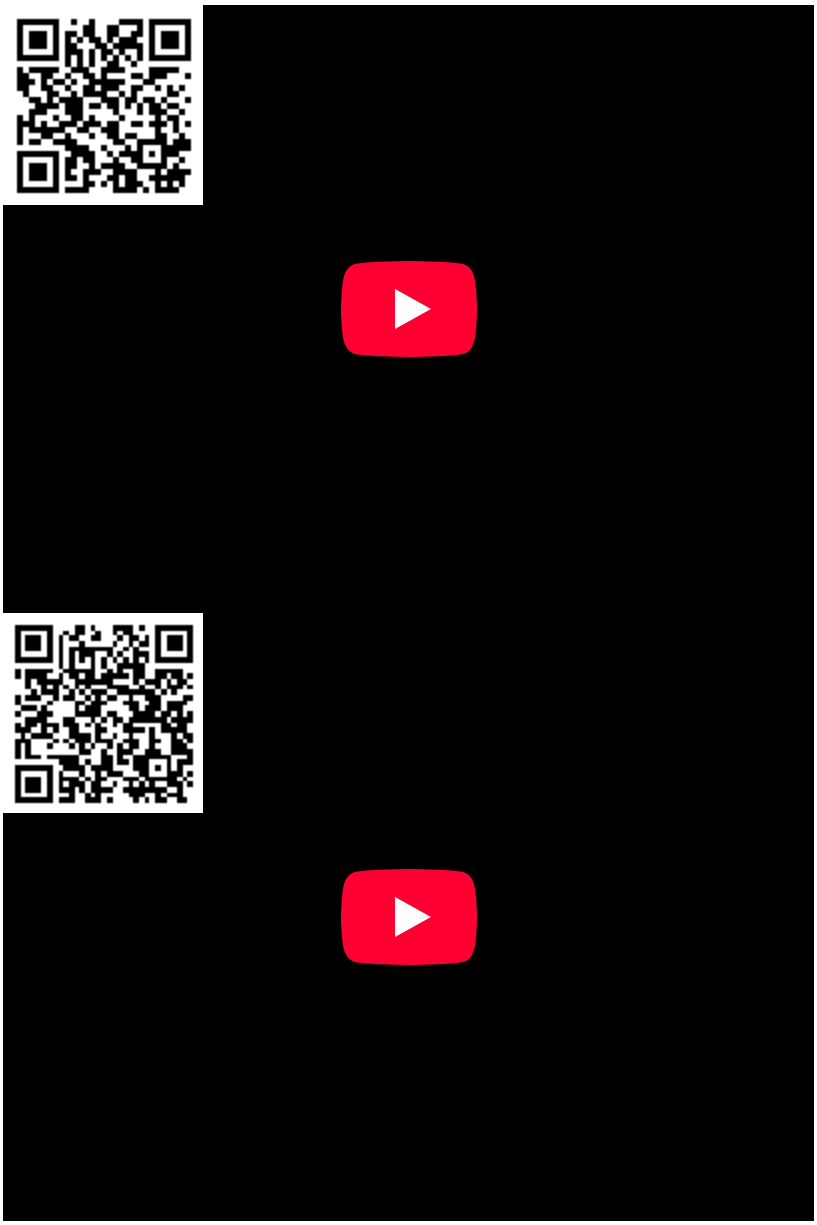
$$A = \int_a^b f(x)dx.$$

This fact, along with the formula for evaluating this integral, is summarized in the Fundamental Theorem of Calculus. Similarly, the arc length of this curve is given by

$$L = \int_a^b \sqrt{1 + (f'(x))^2}dx.$$

In this section, we study analogous formulas for area and arc length in the polar coordinate system.





Areas of Regions Bounded by Polar Curves

We have studied the formulas for area under a curve defined in rectangular coordinates and parametrically defined curves. Now we turn our attention to deriving a formula for the area of a region bounded by a polar curve. Recall that the proof of the Fundamental Theorem of Calculus used the concept of a Riemann sum to approximate the area under a curve by using rectangles. For polar curves we use the Riemann sum again, but the rectangles are replaced by sectors of a circle.

Consider a curve defined by the function $r = f(\theta)$, where $\alpha \leq \theta \leq \beta$. Our first step is to partition the interval $[\alpha, \beta]$ into n equal-width subintervals. The width of each subinterval is given by the formula $\Delta\theta = (\beta - \alpha)/n$, and the ith partition point θ_i is given by the formula $\theta_i = \alpha + i\Delta\theta$. Each partition point $\theta = \theta_i$ defines a line with slope $\tan\theta_i$ passing through the pole as shown in the following graph.

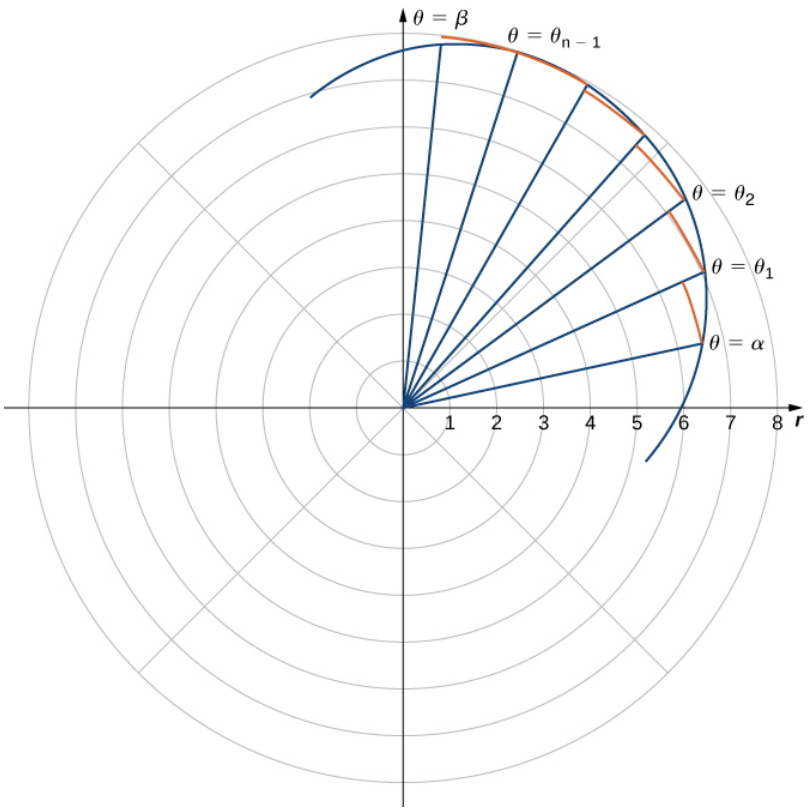


Figure 3.5.1: A partition of a typical curve in polar coordinates.

The line segments are connected by arcs of constant radius. This defines sectors whose areas can be calculated by using a geometric formula. The area of each sector is then used to approximate the area between successive line segments. We then sum the areas of the sectors to approximate the total area. This approach gives a Riemann sum approximation for the total area. The formula for the area of a sector of a circle is illustrated in the following figure.

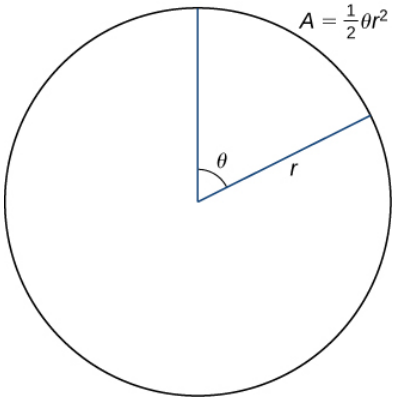


Figure 3.5.2: *The area of a sector of a circle is given by* $A = \frac{1}{2}\theta r^2$.

Recall that the area of a circle is $A = \pi r^2$. When measuring angles in radians, 360 degrees is equal to 2π radians. Therefore a fraction of a circle can be measured by the central angle θ. The fraction of the circle is given by $\frac{\theta}{2\pi}$, so the area of the sector is this fraction multiplied by the total area:

$$A = (\frac{\theta}{2\pi})\pi r^2 = \frac{1}{2}\theta r^2. \tag{3.5.1}$$

Since the radius of a typical sector in Figure 3.5.1 is given by $r_i = f(\theta_i)$, the area of the ith sector is given by

$$A_i = \frac{1}{2}(\Delta\theta)(f(\theta_i))^2. \tag{3.5.2}$$

Therefore a Riemann sum that approximates the area is given by

$$A_n = \sum_{i=1}^{n} A_i \approx \sum_{i=1}^{n} \frac{1}{2}(\Delta\theta)(f(\theta_i))^2. \tag{3.5.3}$$

We take the limit as $n \to \infty$ to get the exact area:

$$A = \lim_{n\to\infty} A_n = \frac{1}{2}\int_{\alpha}^{\beta} (f(\theta))^2 d\theta. \tag{3.5.4}$$

This gives the following theorem.

Area of a Region Bounded by a Polar Curve

Suppose f is continuous and nonnegative on the interval $\alpha \le \theta \le \beta$ with $0 < \beta - \alpha \le 2\pi$. The area of the region bounded by the graph of $r = f(\theta)$ between the radial lines $\theta = \alpha$ and $\theta = \beta$ is

$$A = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 d\theta \tag{3.5.5}$$

$$= \frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta. \tag{3.5.6}$$

Example 3.5.1: Finding an Area of a Polar Region

Find the area of one petal of the rose defined by the equation $r = 3\sin(2\theta)$.

Solution

The graph of $r = 3\sin(2\theta)$ follows.

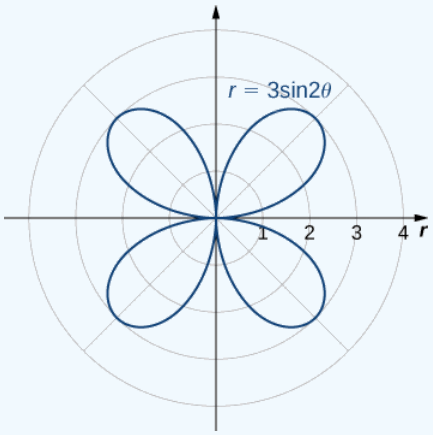


Figure 3.5.3: The graph of $r = 3\sin(2\theta)$.

When $\theta = 0$ we have $r = 3\sin(2(0)) = 0$. The next value for which $r = 0$ is $\theta = \pi/2$. This can be seen by solving the equation $3\sin(2\theta) = 0$ for θ. Therefore the values $\theta = 0$ to $\theta = \pi/2$ trace out the first petal of the rose. To find the area inside this petal, use Equation 3.5.6 with $f(\theta) = 3\sin(2\theta), \alpha = 0,$ and $\beta = \pi/2$:

$$A = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 d\theta$$

$$= \frac{1}{2}\int_{0}^{\pi/2} [3\sin(2\theta)]^2 d\theta$$

$$= \frac{1}{2}\int_{0}^{\pi/2} 9\sin^2(2\theta) d\theta.$$

To evaluate this integral, use the formula $\sin^2\alpha = (1 - \cos(2\alpha))/2$ with $\alpha = 2\theta$:

$$
\begin{aligned}
A &= \frac{1}{2}\int_0^{\pi/2} 9\sin^2(2\theta)d\theta \\
&= \frac{9}{2}\int_0^{\pi/2} \frac{(1-\cos(4\theta))}{2}d\theta \\
&= \frac{9}{4}(\int_0^{\pi/2} 1-\cos(4\theta)d\theta) \\
&= \frac{9}{4}(\theta - \frac{\sin(4\theta)}{4}\Big|_0^{\pi/2} \\
&= \frac{9}{4}(\frac{\pi}{2} - \frac{\sin 2\pi}{4}) - \frac{9}{4}(0 - \frac{\sin 4(0)}{4}) \\
&= \frac{9\pi}{8}
\end{aligned}
$$



Exercise 3.5.1

Find the area inside the cardioid defined by the equation $r = 1 - \cos\theta$.

Hint

Use Equation 3.5.6 Be sure to determine the correct limits of integration before evaluating.

Answer

$A = 3\pi/2$

Example 3.5.1 involved finding the area inside one curve. We can also use Equation 3.5.6 to find the area between two polar curves. However, we often need to find the points of intersection of the curves and determine which function defines the outer curve or the inner curve between these two points.

Example 3.5.2: Finding the Area between Two Polar Curves

Find the area outside the cardioid $r = 2 + 2\sin\theta$ and inside the circle $r = 6\sin\theta$.

Solution

First draw a graph containing both curves as shown.

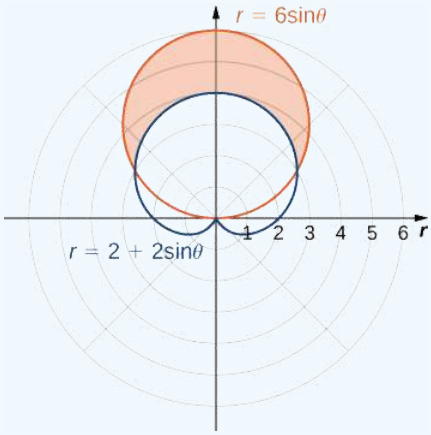


Figure 3.5.4: *The region between the curves* $r = 2 + 2\sin\theta$ *and* $r = 6\sin\theta$.

To determine the limits of integration, first find the points of intersection by setting the two functions equal to each other and solving for θ:

$$\begin{aligned} 6\sin\theta &= 2 + 2\sin\theta \\ 4\sin\theta &= 2 \\ \sin\theta &= \frac{1}{2} \end{aligned}$$

This gives the solutions $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$, which are the limits of integration. The circle $r = 3\sin\theta$ is the red graph, which is the outer function, and the cardioid $r = 2 + 2\sin\theta$ is the blue graph, which is the inner function. To calculate the area between the curves, start with the area inside the circle between $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$, then subtract the area inside the cardioid between $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$:

$$\begin{aligned} A &= \text{circle} - \text{cardioid} \\ &= \frac{1}{2}\int_{\pi/6}^{5\pi/6}[6\sin\theta]^2 d\theta - \frac{1}{2}\int_{\pi/6}^{5\pi/6}[2+2\sin\theta]^2 d\theta \\ &= \frac{1}{2}\int_{\pi/6}^{5\pi/6} 36\sin^2\theta\, d\theta - \frac{1}{2}\int_{\pi/6}^{5\pi/6} 4 + 8\sin\theta + 4\sin^2\theta\, d\theta \\ &= 18\int_{\pi/6}^{5\pi/6} \frac{1-\cos(2\theta)}{2} d\theta - 2\int_{\pi/6}^{5\pi/6} 1 + 2\sin\theta + \frac{1-\cos(2\theta)}{2} d\theta \\ &= 9[\theta - \frac{\sin(2\theta)}{2}]_{\pi/6}^{5\pi/6} - 2[\frac{3\theta}{2} - 2\cos\theta - \frac{\sin(2\theta)}{4}]_{\pi/6}^{5\pi/6} \\ &= 9(\frac{5\pi}{6} - \frac{\sin(10\pi/6)}{2}) - 9(\frac{\pi}{6} - \frac{\sin(2\pi/6)}{2}) - (3(\frac{5\pi}{6}) - 4\cos\frac{5\pi}{6} - \frac{\sin(10\pi/6)}{2}) \\ &+ (3(\frac{\pi}{6}) - 4\cos\frac{\pi}{6} - \frac{\sin(2\pi/6)}{2}) \\ &= 4\pi. \end{aligned}$$

Exercise 3.5.2

Find the area inside the circle $r = 4\cos\theta$ and outside the circle $r = 2$.

Hint

Use Equation 3.5.6and take advantage of symmetry.

Answer

$$A = \frac{4\pi}{3} + 2\sqrt{3}$$

In Example 3.5.2 we found the area inside the circle and outside the cardioid by first finding their intersection points. Notice that solving the equation directly for θ yielded two solutions: $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$. However, in the graph there are three intersection points. The third intersection point is the origin. The reason why this point did not show up as a solution is because the origin is on both graphs but for different values of θ. For example, for the cardioid we get

$$\begin{aligned} 2 + 2\sin\theta &= 0 \\ \sin\theta &= -1, \end{aligned}.$$

so the values for θ that solve this equation are $\theta = \frac{3\pi}{2} + 2n\pi$, where n is any integer. For the circle we get

$$6\sin\theta = 0. \tag{3.5.7}$$

The solutions to this equation are of the form $\theta = n\pi$ for any integer value of n. These two solution sets have no points in common. Regardless of this fact, the curves intersect at the origin. This case must always be taken into consideration.

Arc Length in Polar Curves

Here we derive a formula for the arc length of a curve defined in polar coordinates. In rectangular coordinates, the arc length of a parameterized curve $(x(t), y(t))$ for $a \le t \le b$ is given by

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt. \tag{3.5.8}$$

In polar coordinates we define the curve by the equation $r = f(\theta)$, where $\alpha \le \theta \le \beta$. In order to adapt the arc length formula for a polar curve, we use the equations

$$x = r\cos\theta = f(\theta)\cos\theta \tag{3.5.9}$$

and

$$y = r\sin\theta = f(\theta)\sin\theta, \tag{3.5.10}$$

and we replace the parameter t by θ. Then

$$\frac{dx}{d\theta} = f'(\theta)\cos\theta - f(\theta)\sin\theta \tag{3.5.11}$$

$$\frac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta. \tag{3.5.12}$$

We replace dt by $d\theta$, and the lower and upper limits of integration are α and β, respectively. Then the arc length formula becomes

$$
\begin{aligned}
L &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\
&= \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta \\
&= \int_\alpha^\beta \sqrt{(f'(\theta)\cos\theta - f(\theta)\sin\theta)^2 + (f'(\theta)\sin\theta + f(\theta)\cos\theta)^2}\, d\theta \\
&= \int_\alpha^\beta \sqrt{(f'(\theta))^2(\cos^2\theta + \sin^2\theta) + (f(\theta))^2(\cos^2\theta + \sin^2\theta)}\, d\theta \\
&= \int_\alpha^\beta \sqrt{(f'(\theta))^2 + (f(\theta))^2}\, d\theta \\
&= \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta
\end{aligned}
$$

This gives us the following theorem.

Arc Length of a Curve Defined by a Polar Function

Let f be a function whose derivative is continuous on an interval $\alpha \le \theta \le \beta$. The length of the graph of $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is

$$L = \int_\alpha^\beta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta \tag{3.5.13}$$

$$= \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta. \tag{3.5.14}$$

Example 3.5.3: Finding the Arc Length of a cardioid

Find the arc length of the cardioid $r = 2 + 2\cos\theta$.

Solution

When $\theta = 0, r = 2 + 2\cos 0 = 4$. Furthermore, as θ goes from 0 to 2π, the cardioid is traced out exactly once. Therefore these are the limits of integration. Using $f(\theta) = 2 + 2\cos\theta, \alpha = 0,$ and $\beta = 2\pi$, Equation 3.5.13 becomes

$$
\begin{aligned}
L &= \int_\alpha^\beta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta \\
&= \int_0^{2\pi} \sqrt{[2 + 2\cos\theta]^2 + [-2\sin\theta]^2}\, d\theta \\
&= \int_0^{2\pi} \sqrt{4 + 8\cos\theta + 4\cos^2\theta + 4\sin^2\theta}\, d\theta \\
&= \int_0^{2\pi} \sqrt{4 + 8\cos\theta + 4(\cos^2\theta + \sin^2\theta)}\, d\theta \\
&= \int_0^{2\pi} \sqrt{8 + 8\cos\theta}\, d\theta \\
&= 2\int_0^{2\pi} \sqrt{2 + 2\cos\theta}\, d\theta.
\end{aligned}
$$

Next, using the identity $\cos(2\alpha) = 2\cos^2\alpha - 1,$ add 1 to both sides and multiply by 2. This gives $2 + 2\cos(2\alpha) = 4\cos^2\alpha$. Substituting $\alpha = \theta/2$ gives $2 + 2\cos\theta = 4\cos^2(\theta/2)$, so the integral becomes

$$\begin{aligned} L &= 2\int_0^{2\pi} \sqrt{2+2\cos\theta}\, d\theta \\ &= 2\int_0^{2\pi} \sqrt{4\cos^2(\frac{\theta}{2})}\, d\theta \\ &= 4\int_0^{2\pi} |\cos(\frac{\theta}{2})|\, d\theta. \end{aligned}$$

The absolute value is necessary because the cosine is negative for some values in its domain. To resolve this issue, change the limits from 0 to π and double the answer. This strategy works because cosine is positive between 0 and $\frac{\pi}{2}$. Thus,

$$\begin{aligned} L &= 4\int_0^{2\pi} |\cos(\frac{\theta}{2})|\, d\theta \\ &= 8\int_0^{\pi} \cos(\frac{\theta}{2})\, d\theta \\ &= 8(2\sin(\frac{\theta}{2}))\Big|_0^{\pi} \\ &= 16 \end{aligned}$$



Exercise 3.5.3

Find the total arc length of $r = 3\sin\theta$.

Hint

Use Equation 3.5.13 To determine the correct limits, make a table of values.

Answer

$s = 3\pi$



Key Concepts

- The area of a region in polar coordinates defined by the equation $r = f(\theta)$ with $\alpha \leq \theta \leq \beta$ is given by the integral $A = \frac{1}{2}\int_{\alpha}^{\beta}[f(\theta)]^2 d\theta$.
- To find the area between two curves in the polar coordinate system, first find the points of intersection, then subtract the corresponding areas.
- The arc length of a polar curve defined by the equation $r = f(\theta)$ with $\alpha \leq \theta \leq \beta$ is given by the integral $L = \int_{\alpha}^{\beta}\sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta = \int_{\alpha}^{\beta}\sqrt{r^2 + (\frac{dr}{d\theta})^2}\, d\theta$.

Key Equations

- **Area of a region bounded by a polar curve**

$$A = \frac{1}{2}\int_{\alpha}^{\beta}[f(\theta)]^2 d\theta = \frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta$$

- **Arc length of a polar curve**

$$L = \int_{\alpha}^{\beta}\sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta = \int_{\alpha}^{\beta}\sqrt{r^2 + (\frac{dr}{d\theta})^2}\, d\theta$$

Contributors and Attributions

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3.5E: Exercises for Section 3.5

In exercises 1 -13, determine a definite integral that represents the area.

1) Region enclosed by $r = 4$

2) Region enclosed by $r = 3\sin\theta$

Answer

$$\frac{9}{2}\int_0^{\pi} \sin^2\theta\, d\theta$$

3) Region in the first quadrant within the cardioid $r = 1 + \sin\theta$

4) Region enclosed by one petal of $r = 8\sin(2\theta)$

Answer

$$32\int_0^{\pi/2} \sin^2(2\theta)\, d\theta$$

5) Region enclosed by one petal of $r = cos(3\theta)$

6) Region below the polar axis and enclosed by $r = 1 - \sin\theta$

Answer

$$\frac{1}{2}\int_{\pi}^{2\pi} (1-\sin\theta)^2\, d\theta$$

7) Region in the first quadrant enclosed by $r = 2 - \cos\theta$

8) Region enclosed by the inner loop of $r = 2 - 3\sin\theta$

Answer

$$\int_{\sin^{-1}(2/3)}^{\pi/2} (2-3\sin\theta)^2\, d\theta$$

9) Region enclosed by the inner loop of $r = 3 - 4\cos\theta$

10) Region enclosed by $r = 1 - 2\cos\theta$ and outside the inner loop

Answer

$$\int_0^{\pi} (1-2\cos\theta)^2\, d\theta - \int_0^{\pi/3} (1-2\cos\theta)^2\, d\theta$$

11) Region common to $r = 3\sin\theta$ and $r = 2 - \sin\theta$

12) Region common to $r = 2$ and $r = 4\cos\theta$

Answer

$$4\int_0^{\pi/3} d\theta + 16\int_{\pi/3}^{\pi/2} (\cos^2\theta)\, d\theta$$

13) Region common to $r = 3\cos\theta$ and $r = 3\sin\theta$

In exercises 14 -26, find the area of the described region.

14) Enclosed by $r = 6\sin\theta$

Answer

9π units^2

15) Above the polar axis enclosed by $r = 2 + \sin\theta$

16) Below the polar axis and enclosed by $r = 2 - \cos\theta$

Answer

$\frac{9\pi}{4}$ units^2

17) Enclosed by one petal of $r = 4\cos(3\theta)$

18) Enclosed by one petal of $r = 3\cos(2\theta)$

Answer

$\frac{9\pi}{8}$ units^2

19) Enclosed by $r = 1 + \sin\theta$

20) Enclosed by the inner loop of $r = 3 + 6\cos\theta$

Answer

$\frac{18\pi - 27\sqrt{3}}{2}$ units^2

21) Enclosed by $r = 2 + 4\cos\theta$ and outside the inner loop

22) Common interior of $r = 4\sin(2\theta)$ and $r = 2$

Answer

$\frac{4}{3}(4\pi - 3\sqrt{3})$ units^2

23) Common interior of $r = 3 - 2\sin\theta$ and $r = -3 + 2\sin\theta$

24) Common interior of $r = 6\sin\theta$ and $r = 3$

Answer

$\frac{3}{2}(4\pi - 3\sqrt{3})$ units^2

25) Inside $r = 1 + \cos\theta$ and outside $r = \cos\theta$

26) Common interior of $r = 2 + 2\cos\theta$ and $r = 2\sin\theta$

Answer

$(2\pi - 4)$ units^2

In exercises 27 - 30, find a definite integral that represents the arc length.

27) $r = 4\cos\theta$ on the interval $0 \le \theta \le \frac{\pi}{2}$

28) $r = 1 + \sin\theta$ on the interval $0 \le \theta \le 2\pi$

Answer

$$\int_0^{2\pi} \sqrt{(1+\sin\theta)^2 + \cos^2\theta}\; d\theta$$

29) $r = 2\sec\theta$ on the interval $0 \le \theta \le \frac{\pi}{3}$

30) $r = e^{\theta}$ on the interval $0 \le \theta \le 1$

Answer

$$\sqrt{2}\int_0^1 e^{\theta}\, d\theta$$

In exercises 31 - 35, find the length of the curve over the given interval.

31) $r = 6$ on the interval $0 \le \theta \le \frac{\pi}{2}$

32) $r = e^{3\theta}$ on the interval $0 \le \theta \le 2$

Answer

$\frac{\sqrt{10}}{3}(e^6 - 1)$ units

33) $r = 6\cos\theta$ on the interval $0 \le \theta \le \frac{\pi}{2}$

34) $r = 8 + 8\cos\theta$ on the interval $0 \le \theta \le \pi$

Answer

32 units

35) $r = 1 - \sin\theta$ on the interval $0 \le \theta \le 2\pi$

In exercises 36 - 40, use the integration capabilities of a calculator to approximate the length of the curve.

36) [T] $r = 3\theta$ on the interval $0 \le \theta \le \frac{\pi}{2}$

Answer

6.238units

37) [T] $r = \dfrac{2}{\theta}$ on the interval $\pi \le \theta \le 2\pi$

38) [T] $r = \sin^2\left(\frac{\theta}{2}\right)$ on the interval $0 \le \theta \le \pi$

Answer

2 units

39) [T] $r = 2\theta^2$ on the interval $0 \le \theta \le \pi$

40) [T] $r = \sin(3\cos\theta)$ on the interval $0 \le \theta \le \pi$

Answer

4.39units

In exercises 41 - 43, use the familiar formula from geometry to find the area of the region described and then confirm by using the definite integral.

41) $r = 3\sin\theta$ on the interval $0 \le \theta \le \pi$

42) $r = \sin\theta + \cos\theta$ on the interval $0 \le \theta \le \pi$

Answer

$A = \pi\left(\frac{\sqrt{2}}{2}\right)^2 = \frac{\pi}{2}$ units^2 and $\frac{1}{2}\int_0^{\pi}(1 + 2\sin\theta\cos\theta)\,d\theta = \frac{\pi}{2}$ units^2

43) $r = 6\sin\theta + 8\cos\theta$ on the interval $0 \le \theta \le \pi$

In exercises 44 - 46, use the familiar formula from geometry to find the length of the curve and then confirm using the definite integral.

44) $r = 3\sin\theta$ on the interval $0 \le \theta \le \pi$

Answer

$C = 2\pi\left(\frac{3}{2}\right) = 3\pi$ units and $\int_0^{\pi} 3\,d\theta = 3\pi$ units

45) $r = \sin\theta + \cos\theta$ on the interval $0 \le \theta \le \pi$

46) $r = 6\sin\theta + 8\cos\theta$ on the interval $0 \le \theta \le \pi$

Answer

$C = 2\pi(5) = 10\pi$ units and $\int_0^{\pi} 10\, d\theta = 10\pi$ units

47) Verify that if $y = r\sin\theta = f(\theta)\sin\theta$ then $\dfrac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta$.

In exercises 48 - 56, find the slope of a tangent line to a polar curve $r = f(\theta)$. Let $x = r\cos\theta = f(\theta)\cos\theta$ and $y = r\sin\theta = f(\theta)\sin\theta$, so the polar equation $r = f(\theta)$ is now written in parametric form.

48) Use the definition of the derivative $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta}$ and the product rule to derive the derivative of a polar equation.

Answer

$$\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

49) $r = 1 - \sin\theta;\ \left(\frac{1}{2}, \frac{\pi}{6}\right)$

50) $r = 4\cos\theta;\ \left(2, \frac{\pi}{3}\right)$

Answer

The slope is $\frac{1}{\sqrt{3}}$.

51) $r = 8\sin\theta;\ \left(4, \frac{5\pi}{6}\right)$

52) $r = 4 + \sin\theta;\ \left(3, \frac{3\pi}{2}\right)$

Answer

The slope is 0.

53) $r = 6 + 3\cos\theta;\ (3, \pi)$

54) $r = 4\cos(2\theta)$; tips of the leaves

Answer

At $(4, 0)$, the slope is undefined. At $\left(-4, \frac{\pi}{2}\right)$, the slope is 0.

55) $r = 2\sin(3\theta)$; tips of the leaves

56) $r = 2\theta;\ \left(\frac{\pi}{2}, \frac{\pi}{4}\right)$

Answer

The slope is undefined at $\theta = \frac{\pi}{4}$.

57) Find the points on the interval $-\pi \le \theta \le \pi$ at which the cardioid $r = 1 - \cos\theta$ has a vertical or horizontal tangent line.

58) For the cardioid $r = 1 + \sin\theta$, find the slope of the tangent line when $\theta = \frac{\pi}{3}$.

Answer

Slope = -1.

In exercises 59 - 62, find the slope of the tangent line to the given polar curve at the point given by the value of θ.

59) $r = 3\cos\theta,\ \theta = \frac{\pi}{3}$

60) $r = \theta,\ \theta = \frac{\pi}{2}$

Answer

Slope is $\frac{-2}{\pi}$.

61) $r = \ln\theta,\ \theta = e$

62) [T] Use technology: $r = 2 + 4\cos\theta$ at $\theta = \frac{\pi}{6}$

Answer

Calculator answer: -0.836.

In exercises 63 - 66, find the points at which the following polar curves have a horizontal or vertical tangent line.

63) $r = 4\cos\theta$

64) $r^2 = 4\cos(2\theta)$

Answer

Horizontal tangent at $\left(\pm\sqrt{2}, \frac{\pi}{6}\right)$, $\left(\pm\sqrt{2}, -\frac{\pi}{6}\right)$.

65) $r = 2\sin(2\theta)$

66) The cardioid $r = 1 + \sin\theta$

Answer

Horizontal tangents at $\frac{\pi}{2}$, $\frac{7\pi}{6}$, $\frac{11\pi}{6}$.
Vertical tangents at $\frac{\pi}{6}$, $\frac{5\pi}{6}$ and also at the pole $(0, 0)$.

67) Show that the curve $r = \sin\theta\tan\theta$ (called a **cissoid of Diocles**) has the line $x = 1$ as a vertical asymptote.

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3.6: Conic Sections

Learning Objectives

- Identify the equation of a parabola in standard form with given focus and directrix.
- Identify the equation of an ellipse in standard form with given foci.
- Identify the equation of a hyperbola in standard form with given foci.
- Recognize a parabola, ellipse, or hyperbola from its eccentricity value.
- Write the polar equation of a conic section with eccentricity e.
- Identify when a general equation of degree two is a parabola, ellipse, or hyperbola.

Conic sections have been studied since the time of the ancient Greeks, and were considered to be an important mathematical concept. As early as 320 BCE, such Greek mathematicians as Menaechmus, Appollonius, and Archimedes were fascinated by these curves. Appollonius wrote an entire eight-volume treatise on conic sections in which he was, for example, able to derive a specific method for identifying a conic section through the use of geometry. Since then, important applications of conic sections have arisen (for example, in astronomy), and the properties of conic sections are used in radio telescopes, satellite dish receivers, and even architecture. In this section we discuss the three basic conic sections, some of their properties, and their equations.

Conic sections get their name because they can be generated by intersecting a plane with a cone. A cone has two identically shaped parts called **nappes.** One nappe is what most people mean by "cone," having the shape of a party hat. A right circular cone can be generated by revolving a line passing through the origin around the y-axis as shown in Figure 3.6.1.

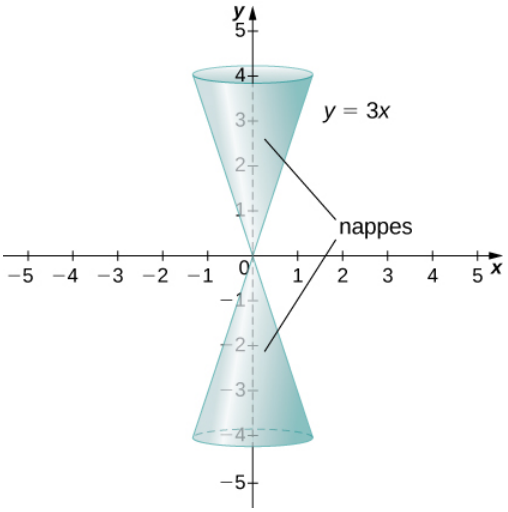


Figure 3.6.1: A cone generated by revolving the line $y = 3x$ around the y-axis.

Conic sections are generated by the intersection of a plane with a cone (Figure 3.6.2). If the plane is parallel to the axis of revolution (the y-axis), then the **conic section** is a hyperbola. If the plane is parallel to the generating line, the conic section is a parabola. If the plane is perpendicular to the axis of revolution, the conic section is a circle. If the plane intersects one nappe at an angle to the axis (other than **90°**), then the conic section is an ellipse.

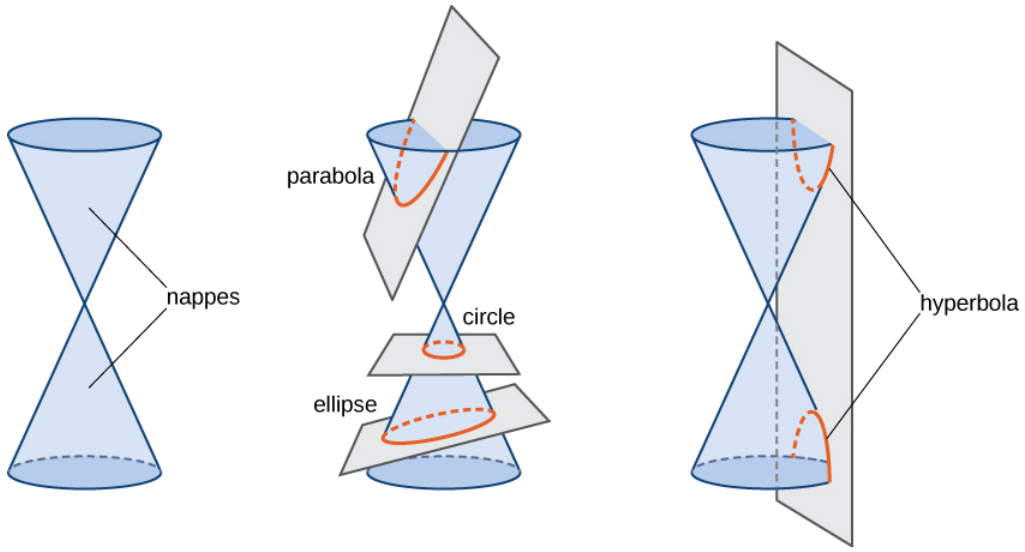


Figure 3.6.2: The four conic sections. Each conic is determined by the angle the plane makes with the axis of the cone.

Parabolas

A parabola is generated when a plane intersects a cone parallel to the generating line. In this case, the plane intersects only one of the nappes. A parabola can also be defined in terms of distances.

Definitions: The Focus, Directrix and Vertex

A parabola is the set of all points whose distance from a fixed point, called the *focus*, is equal to the distance from a fixed line, called the *directrix*. The point halfway between the focus and the directrix is called the *vertex* of the parabola.

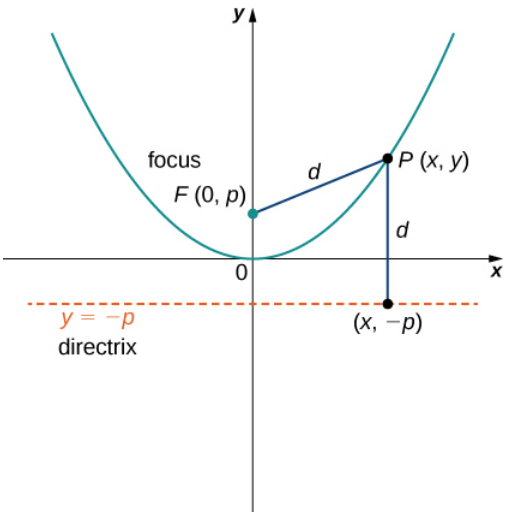


Figure 3.6.3: A typical parabola in which the distance from the focus to the vertex is represented by the variable p.

A graph of a typical parabola appears in Figure 3.6.3. Using this diagram in conjunction with the distance formula, we can derive an equation for a parabola. Recall the distance formula: Given point P with coordinates (x_1, y_1) and point Q with coordinates (x_2, y_2), the distance between them is given by the formula

$$d(P,Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\,. \tag{3.6.1}$$

Then from the definition of a parabola and Figure 3.6.3, we get

$$d(F,P) = d(P,Q) \tag{3.6.2}$$

$$\sqrt{(0-x)^2 + (p-y)^2} = \sqrt{(x-x)^2 + (-p-y)^2}\,. \tag{3.6.3}$$

Squaring both sides and simplifying yields

$$x^2+(p-y)^2=0^2+(-p-y)^2 \tag{3.6.4}$$
$$x^2+p^2-2py+y^2=p^2+2py+y^2 \tag{3.6.5}$$
$$x^2-2py=2py \tag{3.6.6}$$
$$x^2=4py. \tag{3.6.7}$$

Now suppose we want to relocate the vertex. We use the variables (h,k) to denote the coordinates of the vertex. Then if the focus is directly above the vertex, it has coordinates $(h,k+p)$ and the directrix has the equation $y=k-p$. Going through the same derivation yields the formula $(x-h)^2=4p(y-k)$. Solving this equation for y leads to the following theorem.

Equations for Parabolas: standard form

Given a parabola opening upward with vertex located at (h,k) and focus located at $(h,k+p)$, where p is a constant, the equation for the parabola is given by

$$y=\frac{1}{4p}(x-h)^2+k. \tag{3.6.8}$$

This is the *standard form* of a parabola.

We can also study the cases when the parabola opens down or to the left or the right. The equation for each of these cases can also be written in standard form as shown in the following graphs.

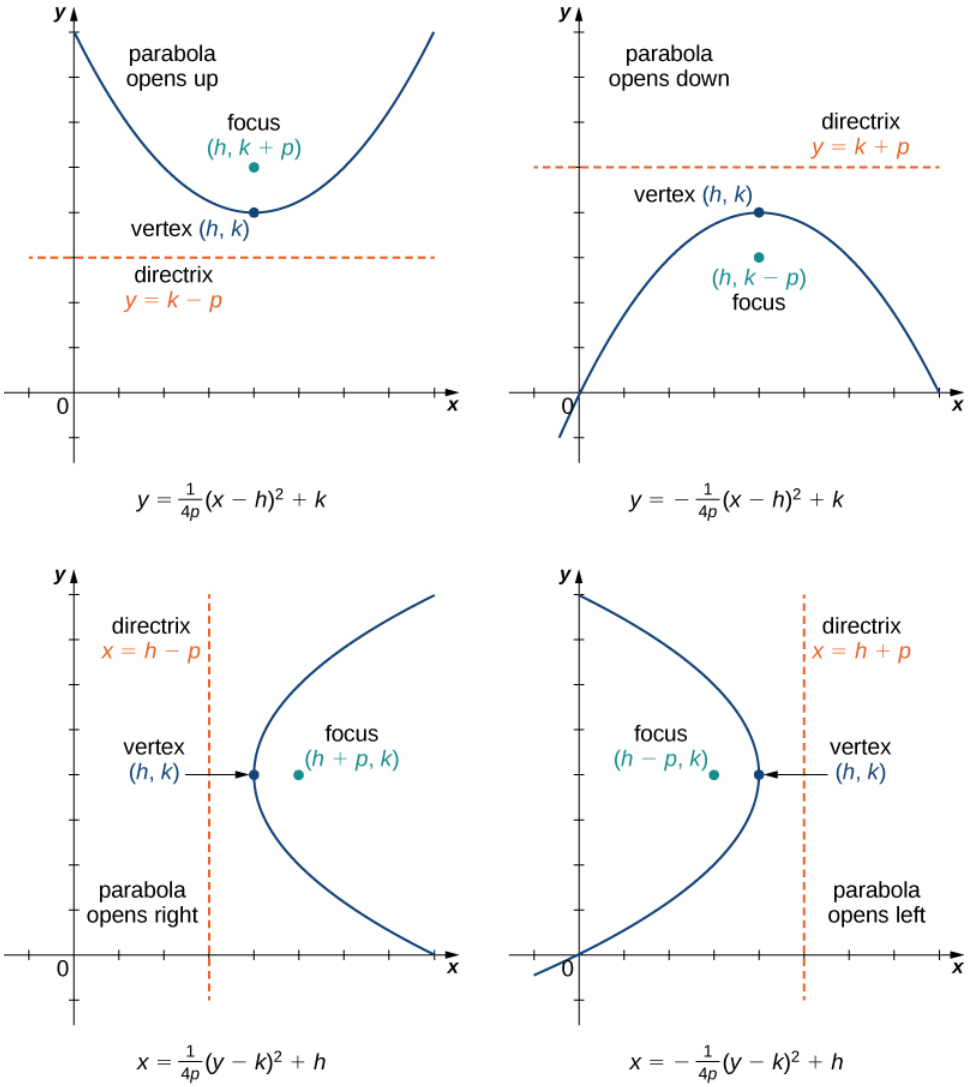


Figure 3.6.4: Four parabolas, opening in various directions, along with their equations in standard form.

In addition, the equation of a parabola can be written in the **general form**, though in this form the values of h, k, and p are not immediately recognizable. The general form of a parabola is written as

$$ax^2 + bx + cy + d = 0 \tag{3.6.9}$$

or

$$ay^2 + bx + cy + d = 0. \tag{3.6.10}$$

Equation 3.6.9 represents a parabola that opens either up or down. Equation 3.6.10 represents a parabola that opens either to the left or to the right. To put the equation into standard form, use the method of completing the square.

Example 3.6.1: Converting the Equation of a Parabola from General into Standard Form

Put the equation

$$x^2 - 4x - 8y + 12 = 0 \tag{3.6.11}$$

into standard form and graph the resulting parabola.

Solution

Since y is not squared in this equation, we know that the parabola opens either upward or downward. Therefore we need to solve this equation for y, which will put the equation into standard form. To do that, first add $8y$ to both sides of the equation:

$$8y = x^2 - 4x + 12. \tag{3.6.12}$$

The next step is to complete the square on the right-hand side. Start by grouping the first two terms on the right-hand side using parentheses:

$$8y = (x^2 - 4x) + 12. \tag{3.6.13}$$

Next determine the constant that, when added inside the parentheses, makes the quantity inside the parentheses a perfect square trinomial. To do this, take half the coefficient of x and square it. This gives $(\frac{-4}{2})^2 = 4$. Add 4 inside the parentheses and subtract 4 outside the parentheses, so the value of the equation is not changed:

$$8y = (x^2 - 4x + 4) + 12 - 4. \tag{3.6.14}$$

Now combine like terms and factor the quantity inside the parentheses:

$$8y = (x - 2)^2 + 8. \tag{3.6.15}$$

Finally, divide by 8:

$$y = \frac{1}{8}(x - 2)^2 + 1. \tag{3.6.16}$$

This equation is now in standard form. Comparing this to Equation gives $h = 2, k = 1$, and $p = 2$. The parabola opens up, with vertex at $(2, 1)$, focus at $(2, 3)$, and directrix $y = -1$. The graph of this parabola appears as follows.

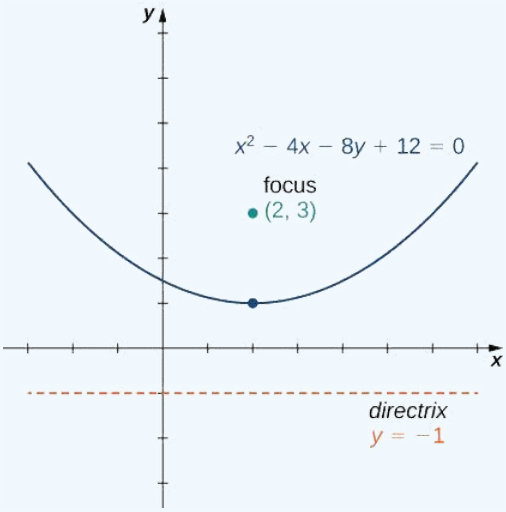


Figure 3.6.5: The parabola in Example 3.6.1.

Exercise 3.6.1

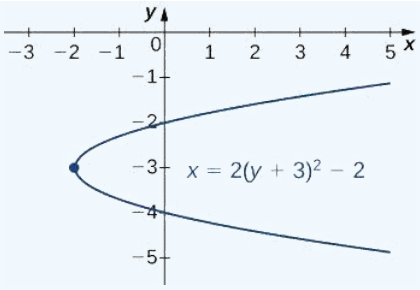
Put the equation $2y^2 - x + 12y + 16 = 0$ into standard form and graph the resulting parabola.

Hint

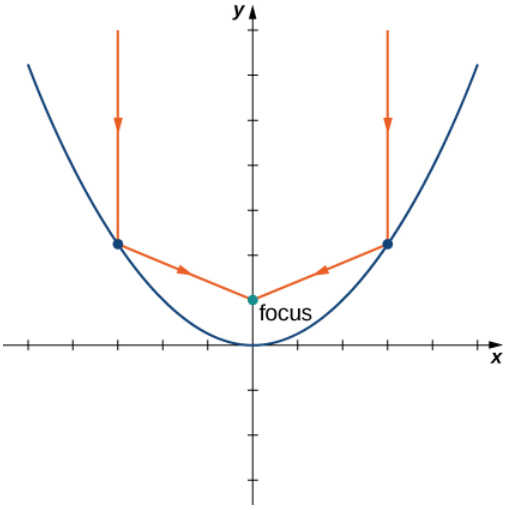
Solve for x. Check which direction the parabola opens.

Answer

$$x = 2(y + 3)^2 - 2 \tag{3.6.17}$$



The axis of symmetry of a vertical (opening up or down) parabola is a vertical line passing through the vertex. The parabola has an interesting reflective property. Suppose we have a satellite dish with a parabolic cross section. If a beam of electromagnetic waves, such as light or radio waves, comes into the dish in a straight line from a satellite (parallel to the axis of symmetry), then the waves reflect off the dish and collect at the focus of the parabola as shown.



Consider a parabolic dish designed to collect signals from a satellite in space. The dish is aimed directly at the satellite, and a receiver is located at the focus of the parabola. Radio waves coming in from the satellite are reflected off the surface of the parabola to the receiver, which collects and decodes the digital signals. This allows a small receiver to gather signals from a wide angle of sky. Flashlights and headlights in a car work on the same principle, but in reverse: the source of the light (that is, the light bulb) is located at the focus and the reflecting surface on the parabolic mirror focuses the beam straight ahead. This allows a small light bulb to illuminate a wide angle of space in front of the flashlight or car.

Ellipses

An ellipse can also be defined in terms of distances. In the case of an ellipse, there are two foci (plural of focus), and two directrices (plural of directrix). We look at the directrices in more detail later in this section.

Definition: Ellipse

An ellipse is the set of all points for which the sum of their distances from two fixed points (the foci) is constant.

A graph of a typical ellipse is shown in Figure 3.6.6. In this figure the foci are labeled as F and F'. Both are the same fixed distance from the origin, and this distance is represented by the variable c. Therefore the coordinates of F are $(c, 0)$ and the coordinates of F' are $(-c, 0)$. The points P and P' are located at the ends of the **major axis** of the ellipse, and have coordinates $(a, 0)$ and $(-a, 0)$, respectively. The major axis is always the longest distance across the ellipse, and can be horizontal or vertical. Thus, the length of the major axis in this ellipse is $2a$. Furthermore, P and P' are called the vertices of the ellipse. The points Q and Q' are located at the ends of the **minor axis** of the ellipse, and have coordinates $(0, b)$ and $(0, -b)$, respectively. The minor axis is the shortest distance across the ellipse. The minor axis is perpendicular to the major axis.

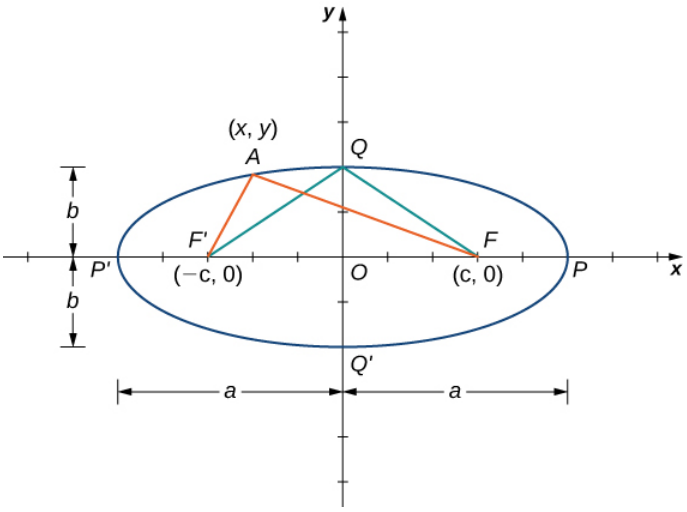


Figure 3.6.6: A typical ellipse in which the sum of the distances from any point on the ellipse to the foci is constant.

According to the definition of the ellipse, we can choose any point on the ellipse and the sum of the distances from this point to the two foci is constant. Suppose we choose the point P. Since the coordinates of point P are $(a, 0)$, the sum of the distances is

$$d(P, F) + d(P, F') = (a - c) + (a + c) = 2a. \tag{3.6.18}$$

Therefore the sum of the distances from an arbitrary point A with coordinates (x, y) is also equal to $2a$. Using the distance formula, we get

$$d(A, F) + d(A, F') = 2a. \tag{3.6.19}$$

$$\sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} = 2a \tag{3.6.20}$$

Subtract the second radical from both sides and square both sides:

$$\sqrt{(x-c)^2 + y^2} = 2a - \sqrt{(x+c)^2 + y^2} \tag{3.6.21}$$

$$(x-c)^2 + y^2 = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + (x+c)^2 + y^2 \tag{3.6.22}$$

$$x^2 - 2cx + c^2 + y^2 = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + x^2 + 2cx + c^2 + y^2 \tag{3.6.23}$$

$$-2cx = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + 2cx. \tag{3.6.24}$$

Now isolate the radical on the right-hand side and square again:

$$-2cx = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + 2cx \tag{3.6.25}$$

$$4a\sqrt{(x+c)^2 + y^2} = 4a^2 + 4cx \tag{3.6.26}$$

$$\sqrt{(x+c)^2 + y^2} = a + \frac{cx}{a} \tag{3.6.27}$$

$$(x+c)^2 + y^2 = a^2 + 2cx + \frac{c^2x^2}{a^2} \tag{3.6.28}$$

$$x^2 + 2cx + c^2 + y^2 = a^2 + 2cx + \frac{c^2x^2}{a^2} \tag{3.6.29}$$

$$x^2 + c^2 + y^2 = a^2 + \frac{c^2x^2}{a^2}. \tag{3.6.30}$$

Isolate the variables on the left-hand side of the equation and the constants on the right-hand side:

$$x^2 - \frac{c^2x^2}{a^2} + y^2 = a^2 - c^2 \tag{3.6.31}$$

$$\frac{(a^2 - c^2)x^2}{a^2} + y^2 = a^2 - c^2. \tag{3.6.32}$$

Divide both sides by $a^2 - c^2$. This gives the equation

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1. \tag{3.6.33}$$

If we refer back to Figure 3.6.6, then the length of each of the two green line segments is equal to a. This is true because the sum of the distances from the point Q to the foci F and F' is equal to $2a$, and the lengths of these two line segments are equal. This line segment forms a right triangle with hypotenuse length a and leg lengths b and c. From the Pythagorean theorem, $b^2 + c^2 = a^2$ and $b^2 = a^2 - c^2$. Therefore the equation of the ellipse becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{3.6.34}$$

Finally, if the center of the ellipse is moved from the origin to a point (h, k), we have the following standard form of an ellipse.

Equation of an Ellipse in Standard Form

Consider the ellipse with center (h, k), a horizontal major axis with length $2a$, and a vertical minor axis with length $2b$. Then the equation of this ellipse in standard form is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \tag{3.6.35}$$

and the foci are located at $(h \pm c, k)$, where $c^2 = a^2 - b^2$. The equations of the directrices are $x = h \pm \frac{a^2}{c}$.

If the major axis is vertical, then the equation of the ellipse becomes

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1 \tag{3.6.36}$$

and the foci are located at $(h, k \pm c)$, where $c^2 = a^2 - b^2$. The equations of the directrices in this case are $y = k \pm \frac{a^2}{c}$.

If the major axis is horizontal, then the ellipse is called horizontal, and if the major axis is vertical, then the ellipse is called vertical. The equation of an ellipse is in general form if it is in the form

$$Ax^2 + By^2 + Cx + Dy + E = 0, \tag{3.6.37}$$

where *A* and *B* are either both positive or both negative. To convert the equation from general to standard form, use the method of **completing the square**.

Example 3.6.2: Finding the Standard Form of an Ellipse

Put the equation

$$9x^2 + 4y^2 - 36x + 24y + 36 = 0 \tag{3.6.38}$$

into standard form and graph the resulting ellipse.

Solution

First subtract 36 from both sides of the equation:

$$9x^2 + 4y^2 - 36x + 24y = -36. \tag{3.6.39}$$

Next group the x terms together and the y terms together, and factor out the common factor:

$$(9x^2 - 36x) + (4y^2 + 24y) = -36 \tag{3.6.40}$$

$$9(x^2-4x)+4(y^2+6y)=-36. \tag{3.6.41}$$

We need to determine the constant that, when added inside each set of parentheses, results in a perfect square. In the first set of parentheses, take half the coefficient of x and square it. This gives $(\frac{-4}{2})^2=4$. In the second set of parentheses, take half the coefficient of y and square it. This gives $(\frac{6}{2})^2=9$. Add these inside each pair of parentheses. Since the first set of parentheses has a 9 in front, we are actually adding 36 to the left-hand side. Similarly, we are adding 36 to the second set as well. Therefore the equation becomes

$$9(x^2-4x+4)+4(y^2+6y+9)=-36+36+36 \tag{3.6.42}$$

$$9(x^2-4x+4)+4(y^2+6y+9)=36. \tag{3.6.43}$$

Now factor both sets of parentheses and divide by 36:

$$9(x-2)^2+4(y+3)^2=36 \tag{3.6.44}$$

$$\frac{9(x-2)^2}{36}+\frac{4(y+3)^2}{36}=1 \tag{3.6.45}$$

$$\frac{(x-2)^2}{4}+\frac{(y+3)^2}{9}=1. \tag{3.6.46}$$

The equation is now in standard form. Comparing this to Equation 3.6.36 gives $h=2, k=-3, a=3,$ and $b=2$. This is a vertical ellipse with center at $(2,-3)$, major axis 6, and minor axis 4. The graph of this ellipse appears as follows.

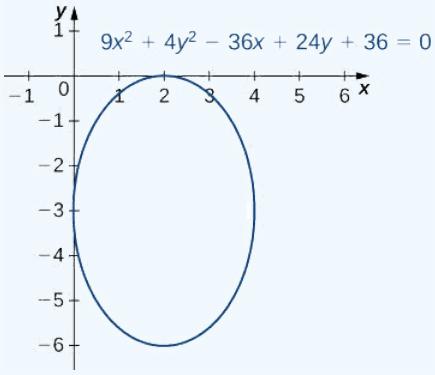


Figure 3.6.7: The ellipse in Example 3.6.2.

Exercise 3.6.2

Put the equation

$$9x^2+16y^2+18x-64y-71=0 \tag{3.6.47}$$

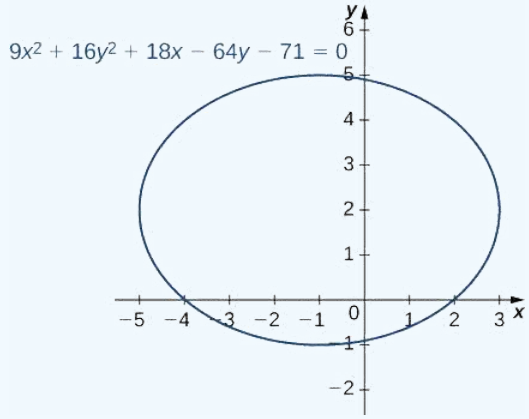
into standard form and graph the resulting ellipse.

Hint

Move the constant over and complete the square.

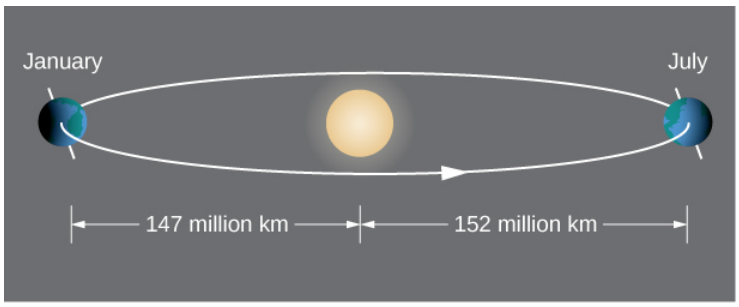
Answer

$$\frac{(x+1)^2}{16}+\frac{(y-2)^2}{9}=1 \tag{3.6.48}$$



According to Kepler's first law of planetary motion, the orbit of a planet around the Sun is an ellipse with the Sun at one of the foci as shown in Figure 3.6.8*A*. Because Earth's orbit is an ellipse, the distance from the Sun varies throughout the year. A commonly held misconception is that Earth is closer to the Sun in the summer. In fact, in summer for the northern hemisphere, Earth is farther from the Sun than during winter. The difference in season is caused by the tilt of Earth's axis in the orbital plane. Comets that orbit the Sun, such as Halley's Comet, also have elliptical orbits, as do moons orbiting the planets and satellites orbiting Earth.

Ellipses also have interesting reflective properties: A light ray emanating from one focus passes through the other focus after mirror reflection in the ellipse. The same thing occurs with a sound wave as well. The National Statuary Hall in the U.S. Capitol in Washington, DC, is a famous room in an elliptical shape as shown in Figure 3.6.8*B*. This hall served as the meeting place for the U.S. House of Representatives for almost fifty years. The location of the two foci of this semi-elliptical room are clearly identified by marks on the floor, and even if the room is full of visitors, when two people stand on these spots and speak to each other, they can hear each other much more clearly than they can hear someone standing close by. Legend has it that John Quincy Adams had his desk located on one of the foci and was able to eavesdrop on everyone else in the House without ever needing to stand. Although this makes a good story, it is unlikely to be true, because the original ceiling produced so many echoes that the entire room had to be hung with carpets to dampen the noise. The ceiling was rebuilt in 1902 and only then did the now-famous whispering effect emerge. Another famous whispering gallery—the site of many marriage proposals—is in Grand Central Station in New York City.



(a)



(b)

Figure 3.6.8: (a) Earth's orbit around the Sun is an ellipse with the Sun at one focus. (b) Statuary Hall in the U.S. Capitol is a whispering gallery with an elliptical cross section.

Hyperbolas

A hyperbola can also be defined in terms of distances. In the case of a hyperbola, there are two foci and two directrices. Hyperbolas also have two asymptotes.

Definition: hyperbola

A hyperbola is the set of all points where the difference between their distances from two fixed points (the foci) is constant.

A graph of a typical hyperbola appears as follows.

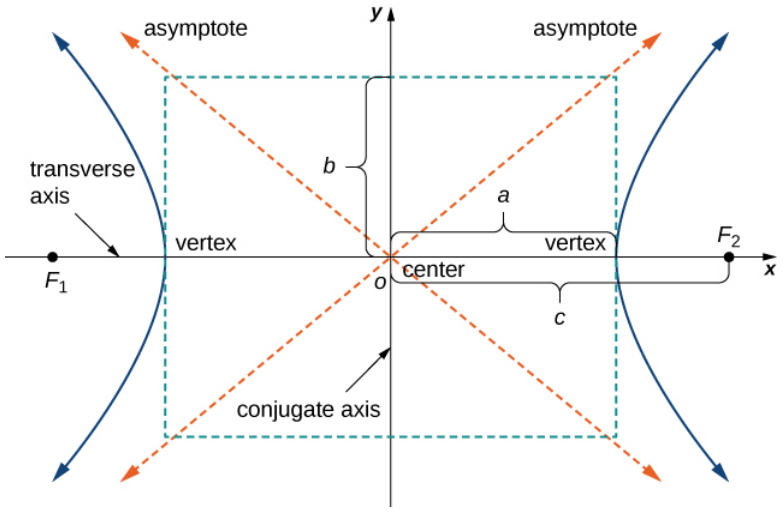


Figure 3.6.9: A typical hyperbola in which the difference of the distances from any point on the hyperbola to the foci is constant. The transverse axis is also called the major axis, and the conjugate axis is also called the minor axis.

The derivation of the equation of a hyperbola in standard form is virtually identical to that of an ellipse. One slight hitch lies in the definition: The difference between two numbers is always positive. Let P be a point on the hyperbola with coordinates (x, y). Then the definition of the hyperbola gives $|d(P, F_1) - d(P, F_2)| = constant$. To simplify the derivation, assume that P is on the right branch of the hyperbola, so the absolute value bars drop. If it is on the left branch, then the subtraction is reversed. The vertex of the right branch has coordinates $(a, 0)$, so

$$d(P, F_1) - d(P, F_2) = (c + a) - (c - a) = 2a. \tag{3.6.49}$$

This equation is therefore true for any point on the hyperbola. Returning to the coordinates (x, y) for P:

$$d(P, F_1) - d(P, F_2) = 2a \tag{3.6.50}$$

$$\sqrt{(x + c)^2 + y^2} - \sqrt{(x - c)^2 + y^2} = 2a. \tag{3.6.51}$$

Isolate the second radical and square both sides:

$$\sqrt{(x - c)^2 + y^2} = -2a + \sqrt{(x + c)^2 + y^2} \tag{3.6.52}$$

$$(x - c)^2 + y^2 = 4a^2 - 4a\sqrt{(x + c)^2 + y^2} + (x + c)^2 + y^2 \tag{3.6.53}$$

$$x^2 - 2cx + c^2 + y^2 = 4a^2 - 4a\sqrt{(x + c)^2 + y^2} + x^2 + 2cx + c^2 + y^2 \tag{3.6.54}$$

$$-2cx = 4a^2 - 4a\sqrt{(x + c)^2 + y^2} + 2cx. \tag{3.6.55}$$

Now isolate the radical on the right-hand side and square again:

$$-2cx = 4a^2 - 4a\sqrt{(x + c)^2 + y^2} + 2cx$$

$$-4a\sqrt{(x + c)^2 + y^2} = -4a^2 - 4cx$$

$$-\sqrt{(x + c)^2 + y^2} = -a - \frac{cx}{a}$$

$$(x + c)^2 + y^2 = a^2 + 2cx + \frac{c^2x^2}{a^2}$$

$$x^2 + 2cx + c^2 + y^2 = a^2 + 2cx + \frac{c^2x^2}{a^2}$$

$$x^2 + c^2 + y^2 = a^2 + \frac{c^2x^2}{a^2} \quad .$$

Isolate the variables on the left-hand side of the equation and the constants on the right-hand side:

$$x^2 - \frac{c^2x^2}{a^2} + y^2 = a^2 - c^2 \tag{3.6.56}$$

$$\frac{(a^2 - c^2)x^2}{a^2} + y^2 = a^2 - c^2. \tag{3.6.57}$$

Finally, divide both sides by $a^2 - c^2$. This gives the equation

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1. \tag{3.6.58}$$

We now define b so that $b^2 = c^2 - a^2$. This is possible because $c > a$. Therefore the equation of the hyperbola becomes

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \tag{3.6.59}$$

Finally, if the center of the hyperbola is moved from the origin to the point (h, k), we have the following standard form of a hyperbola.

Equation of a Hyperbola in Standard Form

Consider the hyperbola with center (h, k), a horizontal major axis, and a vertical minor axis. Then the equation of this hyperbola is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \tag{3.6.60}$$

and the foci are located at $(h \pm c, k)$, where $c^2 = a^2 + b^2$. The equations of the asymptotes are given by $y = k \pm \frac{b}{a}(x - h)$. The equations of the directrices are

$$x = h \pm \frac{a^2}{\sqrt{a^2 + b^2}} = h \pm \frac{a^2}{c} \tag{3.6.61}$$

If the major axis is vertical, then the equation of the hyperbola becomes

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1 \tag{3.6.62}$$

and the foci are located at $(h, k \pm c)$, where $c^2 = a^2 + b^2$. The equations of the asymptotes are given by $y = k \pm \frac{a}{b}(x - h)$. The equations of the directrices are

$$y = k \pm \frac{a^2}{\sqrt{a^2 + b^2}} = k \pm \frac{a^2}{c}. \tag{3.6.63}$$

If the major axis (transverse axis) is horizontal, then the hyperbola is called horizontal, and if the major axis is vertical then the hyperbola is called vertical. The equation of a hyperbola is in general form if it is in the form

$$Ax^2 + By^2 + Cx + Dy + E = 0, \tag{3.6.64}$$

where A and B have opposite signs. In order to convert the equation from general to standard form, use the method of completing the square.

Example 3.6.3: Finding the Standard Form of a Hyperbola

Put the equation $9x^2 - 16y^2 + 36x + 32y - 124 = 0$ into standard form and graph the resulting hyperbola. What are the equations of the asymptotes?

Solution

First add 124 to both sides of the equation:

$$9x^2-16y^2+36x+32y=124.$$

Next group the *x* terms together and the *y* terms together, then factor out the common factors:

$$(9x^2+36x)-(16y^2-32y)=124$$
$$9(x^2+4x)-16(y^2-2y)=124\ .$$

We need to determine the constant that, when added inside each set of parentheses, results in a perfect square. In the first set of parentheses, take half the coefficient of x and square it. This gives $(\frac{4}{2})^2=4$. In the second set of parentheses, take half the coefficient of y and square it. This gives $(\frac{-2}{2})^2=1$. Add these inside each pair of parentheses. Since the first set of parentheses has a 9 in front, we are actually adding 36 to the left-hand side. Similarly, we are subtracting 16 from the second set of parentheses. Therefore the equation becomes

$$9(x^2+4x+4)-16(y^2-2y+1)=124+36-16$$
$$9(x^2+4x+4)-16(y^2-2y+1)=144.$$

Next factor both sets of parentheses and divide by 144:

$$9(x+2)^2-16(y-1)^2=144$$
$$\frac{9(x+2)^2}{144}-\frac{16(y-1)^2}{144}=1$$
$$\frac{(x+2)^2}{16}-\frac{(y-1)^2}{9}=1.$$

The equation is now in standard form. Comparing this to Equation 3.6.60 gives $h=-2, k=1, a=4,$ and $b=3$. This is a horizontal hyperbola with center at $(-2,1)$ and asymptotes given by the equations $y=1\pm\frac{3}{4}(x+2)$. The graph of this hyperbola appears in Figure 3.6.10.

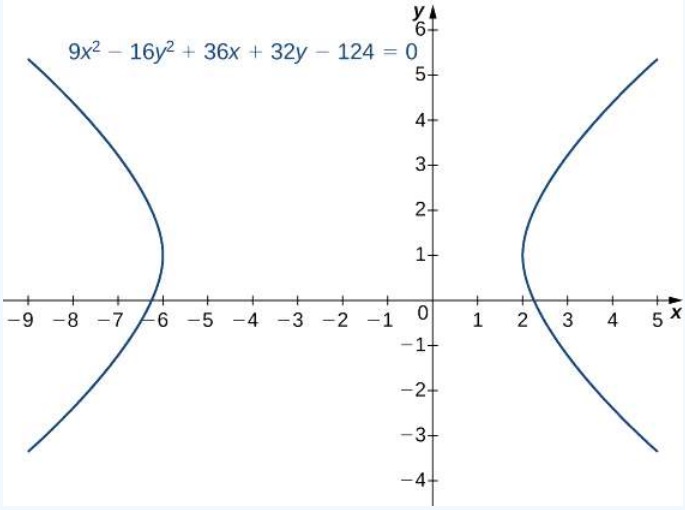


Figure 3.6.10: Graph of the hyperbola in Example 3.6.3.

Exercise 3.6.3

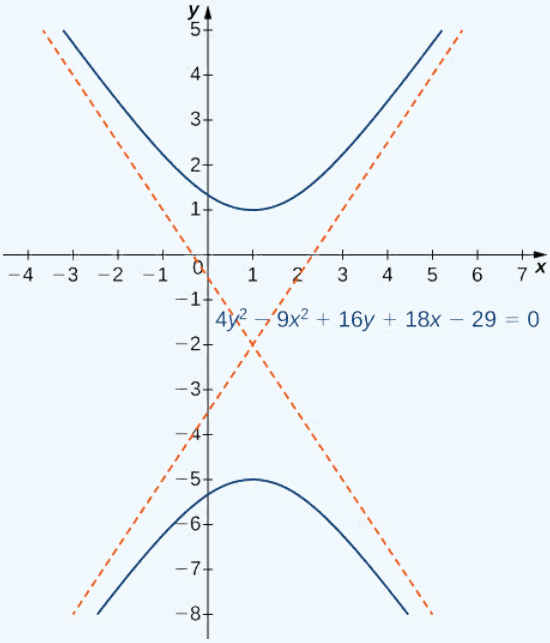
Put the equation $4y^2-9x^2+16y+18x-29=0$ into standard form and graph the resulting hyperbola. What are the equations of the asymptotes?

Hint

Move the constant over and complete the square. Check which direction the hyperbola opens

Answer

$\frac{(y+2)^2}{9} - \frac{(x-1)^2}{4} = 1$. This is a vertical hyperbola. Asymptotes $y = -2 \pm \frac{3}{2}(x-1)$.



Hyperbolas also have interesting reflective properties. A ray directed toward one focus of a hyperbola is reflected by a hyperbolic mirror toward the other focus. This concept is illustrated in Figure 3.6.11.

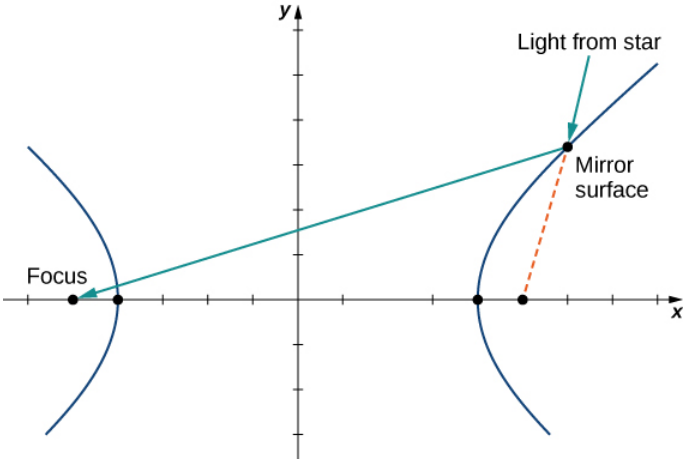


Figure 3.6.11: A hyperbolic mirror used to collect light from distant stars.

This property of the hyperbola has important applications. It is used in radio direction finding (since the difference in signals from two towers is constant along hyperbolas), and in the construction of mirrors inside telescopes (to reflect light coming from the parabolic mirror to the eyepiece). Another interesting fact about hyperbolas is that for a comet entering the solar system, if the speed is great enough to escape the Sun's gravitational pull, then the path that the comet takes as it passes through the solar system is hyperbolic.

Eccentricity and Directrix

An alternative way to describe a conic section involves the directrices, the foci, and a new property called eccentricity. We will see that the value of the eccentricity of a conic section can uniquely define that conic.

Definition: Eccentricity and Directrices

The *eccentricity* e of a conic section is defined to be the distance from any point on the conic section to its focus, divided by the perpendicular distance from that point to the nearest directrix. This value is constant for any conic section, and can define the conic section as well:

1. If $e = 1$, the conic is a parabola.
2. If $e < 1$, it is an ellipse.
3. If $e > 1$, it is a hyperbola.

The eccentricity of a circle is zero. The *directrix* of a conic section is the line that, together with the point known as the focus, serves to define a conic section. Hyperbolas and noncircular ellipses have two foci and two associated directrices. Parabolas have one focus and one directrix.

The three conic sections with their directrices appear in Figure 3.6.12

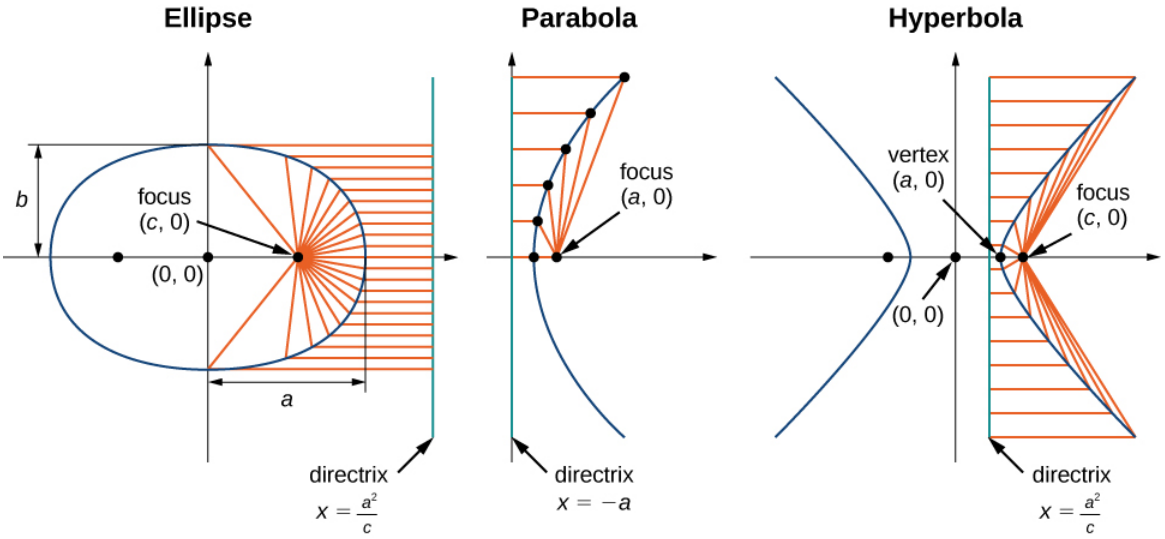


Figure 3.6.12: The three conic sections with their foci and directrices.

Recall from the definition of a parabola that the distance from any point on the parabola to the focus is equal to the distance from that same point to the directrix. Therefore, by definition, the eccentricity of a parabola must be 1. The equations of the directrices of a horizontal ellipse are $x = \pm\dfrac{a^2}{c}$. The right vertex of the ellipse is located at $(a, 0)$ and the right focus is $(c, 0)$. Therefore the distance from the vertex to the focus is $a - c$ and the distance from the vertex to the right directrix is $\dfrac{a^2}{c} - c$. This gives the eccentricity as

$$e = \frac{a-c}{\frac{a^2}{c} - a} = \frac{c(a-c)}{a^2 - ac} = \frac{c(a-c)}{a(a-c)} = \frac{c}{a}. \tag{3.6.65}$$

Since $c < a$, this step proves that the eccentricity of an ellipse is less than 1. The directrices of a horizontal hyperbola are also located at $x = \pm\dfrac{a^2}{c}$, and a similar calculation shows that the eccentricity of a hyperbola is also $e = \dfrac{c}{a}$. However in this case we have $c > a$, so the eccentricity of a hyperbola is greater than 1.

Example 3.6.4: Determining Eccentricity of a Conic Section

Determine the eccentricity of the ellipse described by the equation

$$\frac{(x-3)^2}{16} + \frac{(y+2)^2}{25} = 1.$$

Solution

From the equation we see that $a=5$ and $b=4$. The value of c can be calculated using the equation $a^2=b^2+c^2$ for an ellipse. Substituting the values of a and b and solving for c gives $c=3$. Therefore the eccentricity of the ellipse is $e=\dfrac{c}{a}=\dfrac{3}{5}=0.6$.

Exercise 3.6.4

Determine the eccentricity of the hyperbola described by the equation

$$\frac{(y-3)^2}{49}-\frac{(x+2)^2}{25}=1.$$

Hint

First find the values of a and b, then determine c using the equation $c^2=a^2+b^2$.

Answer

$$e=\frac{c}{a}=\frac{\sqrt{74}}{7}\approx 1.229$$

Polar Equations of Conic Sections

Sometimes it is useful to write or identify the equation of a conic section in polar form. To do this, we need the concept of the focal parameter. The **focal parameter** of a conic section p is defined as the distance from a focus to the nearest directrix. The following table gives the focal parameters for the different types of conics, where a is the length of the semi-major axis (i.e., half the length of the major axis), c is the distance from the origin to the focus, and e is the eccentricity. In the case of a parabola, a represents the distance from the vertex to the focus.

Table 3.6.1: Eccentricities and Focal Parameters of the Conic Sections

Conic	e	p
Ellipse	$0<e<1$	$\dfrac{a^2-c^2}{c}=\dfrac{a(1-e^2)}{c}$
Parabola	$e=1$	$2a$
Hyperbola	$e>1$	$\dfrac{c^2-a^2}{c}=\dfrac{a(e^2-1)}{c}$

Using the definitions of the focal parameter and eccentricity of the conic section, we can derive an equation for any conic section in polar coordinates. In particular, we assume that one of the foci of a given conic section lies at the pole. Then using the definition of the various conic sections in terms of distances, it is possible to prove the following theorem.

Polar Equation of Conic Sections

The polar equation of a conic section with focal parameter p is given by

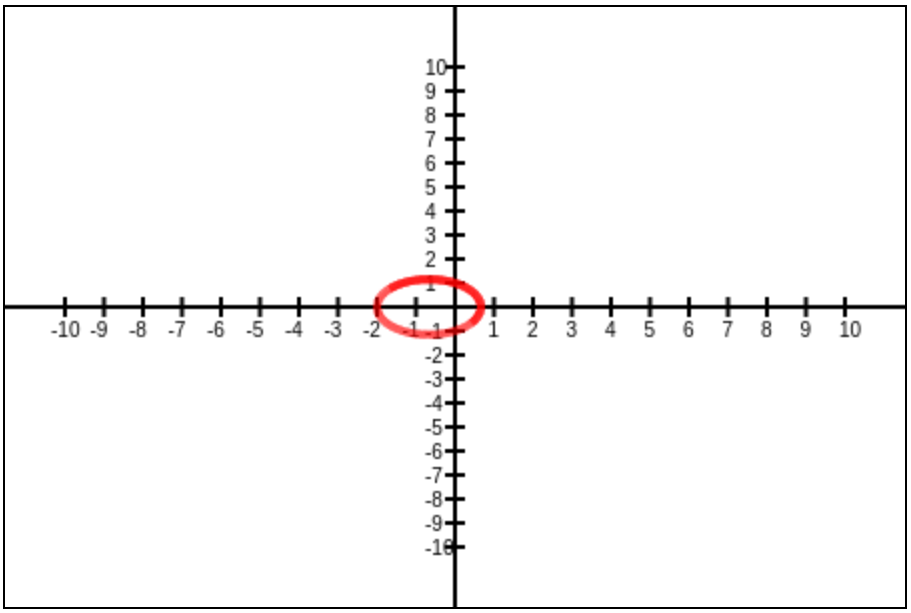
$$r=\frac{ep}{1\pm e\cos\theta}\quad\text{or}\quad r=\frac{ep}{1\pm e\sin\theta}.$$

In the equation on the left, the major axis of the conic section is horizontal, and in the equation on the right, the major axis is vertical. To work with a conic section written in polar form, first make the constant term in the denominator equal to 1. This can be done by dividing both the numerator and the denominator of the fraction by the constant that appears in front of the plus or minus in the denominator. Then the coefficient of the sine or cosine in the denominator is the eccentricity. This value identifies the conic. If cosine appears in the denominator, then the conic is horizontal. If sine appears, then the conic is vertical. If both appear then the axes are rotated. The center of the conic is not necessarily at the origin. The center is at the origin only if the conic is a circle (i.e., $e=0$).

Conic Sections in Polar Coordinates and Their Graphs

Move the sliders to change the coefficients and pick either sinθ or cosθ to change the graph of the conic section:

$r = \frac{ep}{1 \pm e \sin\theta}$ or $r = \frac{ep}{1 \pm e \cos\theta}$



ep = 1

e = 0.5

○ sin ◉ cos

Example 3.6.5: Graphing a Conic Section in Polar Coordinates

Identify and create a graph of the conic section described by the equation

$$r = \frac{3}{1 + 2\cos\theta}.$$

Solution

The constant term in the denominator is 1, so the eccentricity of the conic is 2. This is a hyperbola. The focal parameter p can be calculated by using the equation $ep = 3$. Since $e = 2$, this gives $p = \frac{3}{2}$. The cosine function appears in the denominator, so the hyperbola is horizontal. Pick a few values for θ and create a table of values. Then we can graph the hyperbola (Figure 3.6.13).

θ	r	θ	r
0	1	π	-3
$\frac{\pi}{4}$	$\frac{3}{1+\sqrt{2}} \approx 1.2426$	$\frac{5\pi}{4}$	$\frac{3}{1-\sqrt{2}} \approx -7.2426$

θ	r	θ	r
$\frac{\pi}{2}$	3	$\frac{3\pi}{2}$	3
$\frac{3\pi}{4}$	$\frac{3}{1-\sqrt{2}} \approx -7.2426$	$\frac{7\pi}{4}$	$\frac{3}{1+\sqrt{2}} \approx 1.2426$

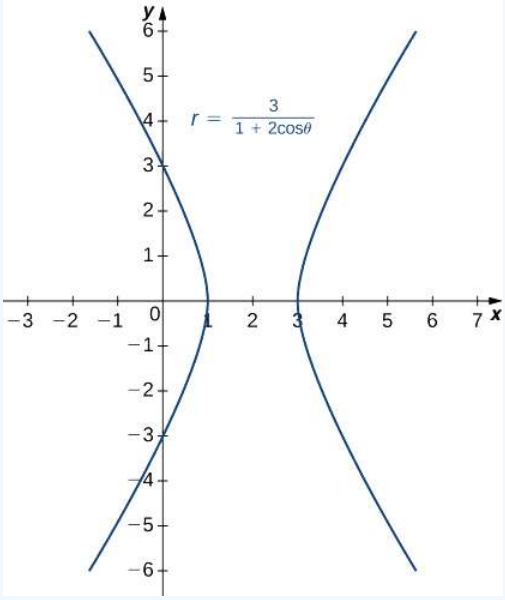


Figure 3.6.13: Graph of the hyperbola described in Example 3.6.5.

Exercise 3.6.5

Identify and create a graph of the conic section described by the equation

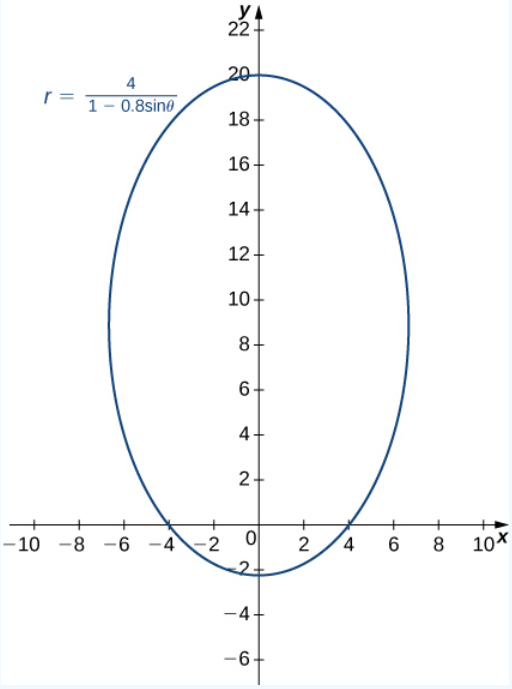
$$r = \frac{4}{1 - 0.8\sin\theta}.$$

Hint

First find the values of e and p, and then create a table of values.

Answer

Here $e = 0.8$ and $p = 5$. This conic section is an ellipse.



General Equations of Degree Two

A general equation of degree two can be written in the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0. \tag{3.6.66}$$

The graph of an equation of this form is a conic section. If $B \neq 0$ then the coordinate axes are rotated. To identify the conic section, we use the discriminant of the conic section $4AC - B^2$.

Identifying the Conic Section

One of the following cases must be true:

1. $4AC - B^2 > 0$. If so, the graph is an ellipse.
2. $4AC - B^2 = 0$. If so, the graph is a parabola.
3. $4AC - B^2 < 0$. If so, the graph is a hyperbola.

The simplest example of a second-degree equation involving a cross term is $xy = 1$. This equation can be solved for y to obtain $y = \dfrac{1}{x}$. The graph of this function is called a rectangular hyperbola as shown.

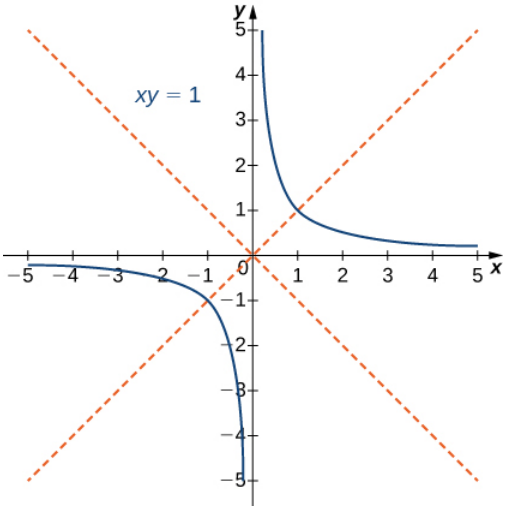


Figure 3.6.14: Graph of the equation $xy = 1$; The red lines indicate the rotated axes.

The asymptotes of this hyperbola are the x and y coordinate axes. To determine the angle θ of rotation of the conic section, we use the formula $\cot 2\theta = \frac{A-C}{B}$. In this case $A = C = 0$ and $B = 1$, so $\cot 2\theta = (0-0)/1 = 0$ and $\theta = 45^\circ$. The method for graphing a conic section with rotated axes involves determining the coefficients of the conic in the rotated coordinate system. The new coefficients are labeled A', B', C', D', E',and F', and are given by the formulas

$$A' = A\cos^2\theta + B\cos\theta\sin\theta + C\sin^2\theta \tag{3.6.67}$$
$$B' = 0 \tag{3.6.68}$$
$$C' = A\sin^2\theta - B\sin\theta\cos\theta + C\cos^2\theta \tag{3.6.69}$$
$$D' = D\cos\theta + E\sin\theta \tag{3.6.70}$$
$$E' = -D\sin\theta + E\cos\theta \tag{3.6.71}$$
$$F' = F. \tag{3.6.72}$$

Procedure: graphing a rotated conic

The procedure for graphing a rotated conic is the following:

1. Identify the conic section using the discriminant $4AC - B^2$.
2. Determine θ using the formula

$$\cot 2\theta = \frac{A-C}{B}. \tag{3.6.73}$$

3. Calculate A', B', C', D', E',and F'.
4. Rewrite the original equation using A', B', C', D', E',and F'.
5. Draw a graph using the rotated equation.

Example 3.6.6: Identifying a Rotated Conic

Identify the conic and calculate the angle of rotation of axes for the curve described by the equation

$$13x^2 - 6\sqrt{3}xy + 7y^2 - 256 = 0. \tag{3.6.74}$$

Solution

In this equation, $A = 13, B = -6\sqrt{3}, C = 7, D = 0, E = 0,$ and $F = -256$. The discriminant of this equation is

$$4AC - B^2 = 4(13)(7) - (-6\sqrt{3})^2 = 364 - 108 = 256. \tag{3.6.75}$$

Therefore this conic is an ellipse.

To calculate the angle of rotation of the axes, use Equation 3.6.73

$$\cot 2\theta = \frac{A-C}{B}. \tag{3.6.76}$$

This gives

$$\cot 2\theta = \frac{A-C}{B} = \frac{13-7}{-6\sqrt{3}} = -\frac{\sqrt{3}}{3}\ .$$

Therefore $2\theta = 120^o$ and $\theta = 60^o$, which is the angle of the rotation of the axes.

To determine the rotated coefficients, use the formulas given above:

$$\begin{aligned}
A' &= A\cos^2\theta + B\cos\theta\sin\theta + C\sin^2\theta \\
&= 13\cos^2 60 + (-6\sqrt{3})\cos 60\sin 60 + 7\sin^2 60 \\
&= 13(\frac{1}{2})^2 - 6\sqrt{3}(\frac{1}{2})(\frac{\sqrt{3}}{2}) + 7(\frac{\sqrt{3}}{2})^2 \\
&= 4, \\
B' &= 0 \\
C' &= A\sin^2\theta - B\sin\theta\cos\theta + C\cos^2\theta \\
&= 13\sin^2 60 + (6\sqrt{3})\sin 60\cos 60 + 7\cos^2 60 \\
&= 13(\frac{\sqrt{3}}{2})^2 + 6\sqrt{3}(\frac{\sqrt{3}}{2})(\frac{1}{2}) + 7(\frac{1}{2})^2 \\
&= 16, \\
D' &= D\cos\theta + E\sin\theta \\
&= (0)\cos 60 + (0)\sin 60 \\
&= 0, \\
E' &= -D\sin\theta + E\cos\theta \\
&= -(0)\sin 60 + (0)\cos 60 \\
&= 0 \\
F' &= F \\
&= -256.
\end{aligned}$$

The equation of the conic in the rotated coordinate system becomes

$$\begin{aligned}
4(x')^2 + 16(y')^2 &= 256 \\
\frac{(x')^2}{64} + \frac{(y')^2}{16} &= 1\ .
\end{aligned}$$

A graph of this conic section appears as follows.

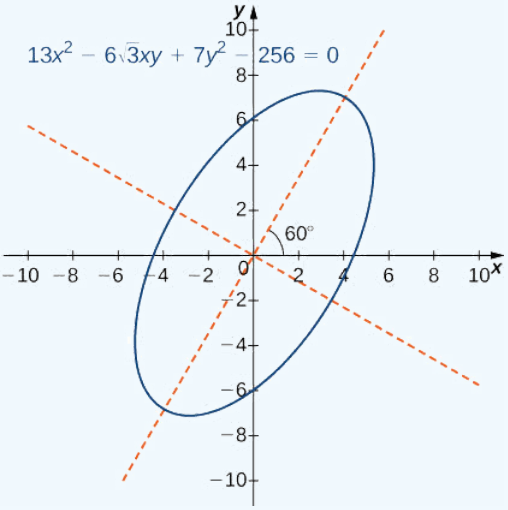


Figure 3.6.15: Graph of the ellipse described by the equation $13x^2 - 6\sqrt{3}xy + 7y^2 - 256 = 0$. The axes are rotated 60°. The red dashed lines indicate the rotated axes.

Exercise 3.6.6

Identify the conic and calculate the angle of rotation of axes for the curve described by the equation

$$3x^2 + 5xy - 2y^2 - 125 = 0. \tag{3.6.77}$$

Hint

Follow steps 1 and 2 of the five-step method outlined above

Answer

The conic is a hyperbola and the angle of rotation of the axes is $\theta = 22.5^\circ$.

Key Concepts

- The equation of a vertical parabola in standard form with given focus and directrix is $y = \frac{1}{4p}(x-h)^2 + k$ where p is the distance from the vertex to the focus and (h, k) are the coordinates of the vertex.
- The equation of a horizontal ellipse in standard form is $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ where the center has coordinates (h, k), the major axis has length *2a*, the minor axis has length *2b*, and the coordinates of the foci are $(h \pm c, k)$, where $c^2 = a^2 - b^2$.
- The equation of a horizontal hyperbola in standard form is $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ where the center has coordinates (h, k), the vertices are located at $(h \pm a, k)$, and the coordinates of the foci are $(h \pm c, k)$, where $c^2 = a^2 + b^2$.
- The eccentricity of an ellipse is less than 1, the eccentricity of a parabola is equal to 1, and the eccentricity of a hyperbola is greater than 1. The eccentricity of a circle is 0.
- The polar equation of a conic section with eccentricity *e* is $r = \frac{ep}{1 \pm ecos\theta}$ or $r = \frac{ep}{1 \pm esin\theta}$, where *p* represents the focal parameter.
- To identify a conic generated by the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$,first calculate the discriminant $D = 4AC - B^2$. If $D > 0$ then the conic is an ellipse, if $D = 0$ then the conic is a parabola, and if $D < 0$ then the conic is a hyperbola.

Glossary

conic section

a conic section is any curve formed by the intersection of a plane with a cone of two nappes

directrix

a directrix (plural: directrices) is a line used to construct and define a conic section; a parabola has one directrix; ellipses and hyperbolas have two

discriminant

the value $4AC - B^2$, which is used to identify a conic when the equation contains a term involving xy, is called a discriminant

focus

a focus (plural: foci) is a point used to construct and define a conic section; a parabola has one focus; an ellipse and a hyperbola have two

eccentricity

the eccentricity is defined as the distance from any point on the conic section to its focus divided by the perpendicular distance from that point to the nearest directrix

focal parameter

the focal parameter is the distance from a focus of a conic section to the nearest directrix

general form

an equation of a conic section written as a general second-degree equation

major axis

the major axis of a conic section passes through the vertex in the case of a parabola or through the two vertices in the case of an ellipse or hyperbola; it is also an axis of symmetry of the conic; also called the transverse axis

minor axis

the minor axis is perpendicular to the major axis and intersects the major axis at the center of the conic, or at the vertex in the case of the parabola; also called the conjugate axis

nappe

a nappe is one half of a double cone

standard form

an equation of a conic section showing its properties, such as location of the vertex or lengths of major and minor axes

vertex

a vertex is an extreme point on a conic section; a parabola has one vertex at its turning point. An ellipse has two vertices, one at each end of the major axis; a hyperbola has two vertices, one at the turning point of each branch

Contributors and Attributions

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3.6E: Exercises for Section 3.6

In exercises 1 - 8, determine the equation of the parabola using the information given.

1) Focus $(4,0)$ and directrix $x=-4$

Answer

$y^2=16x$

2) Focus $(0,-3)$ and directrix $y=3$

3) Focus $(0,0.5)$ and directrix $y=-0.5$

Answer

$x^2=2y$

4) Focus $(2,3)$ and directrix $x=-2$

5) Focus $(0,2)$ and directrix $y=4$

Answer

$x^2=-4(y-3)$

6) Focus $(-1,4)$ and directrix $x=5$

7) Focus $(-3,5)$ and directrix $y=1$

Answer

$(x+3)^2=8(y-3)$

8) Focus $\left(\frac{5}{2},-4\right)$ and directrix $x=\frac{7}{2}$

In exercises 9 - 16, determine the equation of the ellipse using the information given.

9) Endpoints of major axis at $(4,0),\ (-4,0)$ and foci located at $(2,0),\ (-2,0)$

Answer

$\frac{x^2}{16}+\frac{y^2}{12}=1$

10) Endpoints of major axis at $(0,5),\ (0,-5)$ and foci located at $(0,3),\ (0,-3)$

11) Endpoints of major axis at $(0,2),\ (0,-2)$ and foci located at $(3,0),\ (-3,0)$

Answer

$\frac{x^2}{13}+\frac{y^2}{4}=1$

12) Endpoints of major axis at $(-3,3),\ (7,3)$ and foci located at $(-2,3),\ (6,3)$

13) Endpoints of major axis at $(-3,5),\ (-3,-3)$ and foci located at $(-3,3),\ (-3,-1)$

Answer

$\frac{(y-1)^2}{16}+\frac{(x+3)^2}{12}=1$

14) Endpoints of major axis at $(0,0),\ (0,4)$ and foci located at $(5,2),\ (-5,2)$

15) Foci located at $(2,0),\ (-2,0)$ and eccentricity of $\frac{1}{2}$

Answer

$$\frac{x^2}{16}+\frac{y^2}{12}=1$$

16) Foci located at $(0,-3),\ (0,3)$ and eccentricity of $\frac{3}{4}$

In exercises 17 - 24, determine the equation of the hyperbola using the information given.

17) Vertices located at $(5,0),\ (-5,0)$ and foci located at $(6,0),\ (-6,0)$

Answer

$\frac{x^2}{25}-\frac{y^2}{11}=1$

18) Vertices located at $(0,2),\ (0,-2)$ and foci located at $(0,3),\ (0,-3)$

19) Endpoints of the conjugate axis located at $(0,3),\ (0,-3)$ and foci located $(4,0),\ (-4,0)$

Answer

$$\frac{x^2}{7}-\frac{y^2}{9}=1$$

20) Vertices located at $(0,1),\ (6,1)$ and focus located at $(8,1)$

21) Vertices located at $(-2,0),\ (-2,-4)$ and focus located at $(-2,-8)$

Answer

$$\frac{(y+2)^2}{4}-\frac{(x+2)^2}{32}=1$$

22) Endpoints of the conjugate axis located at $(3,2),\ (3,4)$ and focus located at $(3,7)$

23) Foci located at $(6,-0),\ (6,0)$ and eccentricity of 3

Answer

$$\frac{x^2}{4}-\frac{y^2}{32}=1$$

24) $(0,10),\ (0,-10)$ and eccentricity of 2.5

In exercises 25 - 30, consider the following polar equations of conics. Determine the eccentricity and identify the conic.

25) $r=\dfrac{-1}{1+\cos\theta}$

Answer

$e=1,$ parabola

26) $r=\dfrac{8}{2-\sin\theta}$

27) $r=\dfrac{5}{2+\sin\theta}$

Answer

$e=\frac{1}{2},$ ellipse

28) $r=\dfrac{5}{-1+2\sin\theta}$

29) $r=\dfrac{3}{2-6\sin\theta}$

Answer

$e = 3$, hyperbola

30) $r = \dfrac{3}{-4 + 3\sin\theta}$

In exercises 31 - 34, find a polar equation of the conic with focus at the origin and eccentricity and directrix as given.

31) Directrix: $x = 4;\ e = \frac{1}{5}$

Answer

$$r = \frac{4}{5 + \cos\theta}$$

32) Directrix: $x = -4;\ e = 5$

33) Directrix: $y = 2;\ e = 2$

Answer

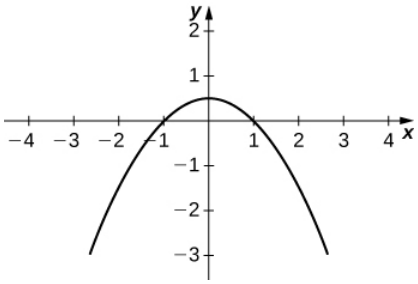
$$r = \frac{4}{1 + 2\sin\theta}$$

34) Directrix: $y = -2;\ e = \frac{1}{2}$

In exercises 35 - 51, sketch the graph of each conic.

35) $r = \dfrac{1}{1 + \sin\theta}$

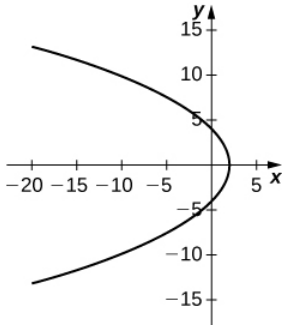
Answer



36) $r = \dfrac{1}{1 - \cos\theta}$

37) $r = \dfrac{4}{1 + \cos\theta}$

Answer

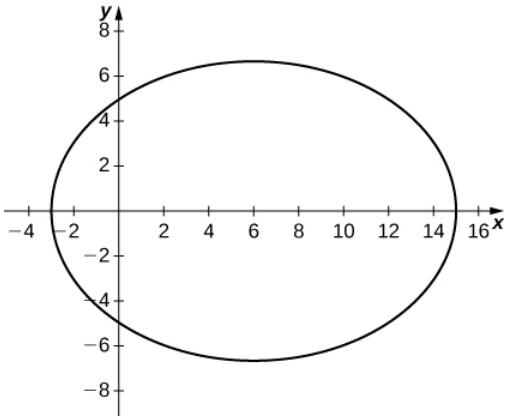


38) $r = \dfrac{10}{5 + 4\sin\theta}$

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39) $r = \dfrac{15}{3 - 2\cos\theta}$

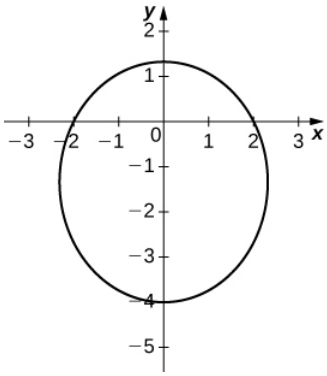
Answer



40) $r = \dfrac{32}{3 + 5\sin\theta}$

41) $r(2 + \sin\theta) = 4$

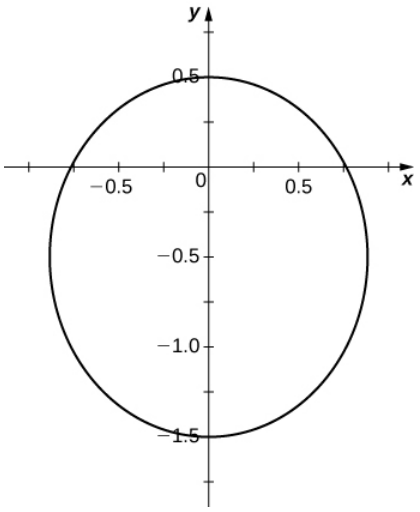
Answer



42) $r = \dfrac{3}{2 + 6\sin\theta}$

43) $r = \dfrac{3}{-4 + 2\sin\theta}$

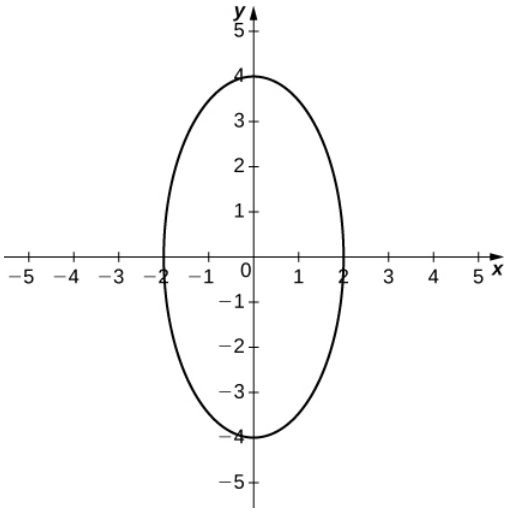
Answer



44) $\frac{x^2}{9}+\frac{y^2}{4}=1$

45) $\frac{x^2}{4}+\frac{y^2}{16}=1$

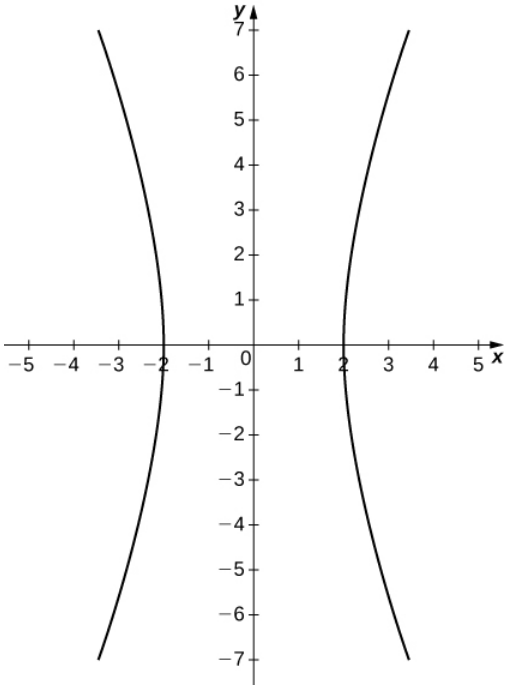
Answer



46) $4x^2+9y^2=36$

47) $25x^2-4y^2=100$

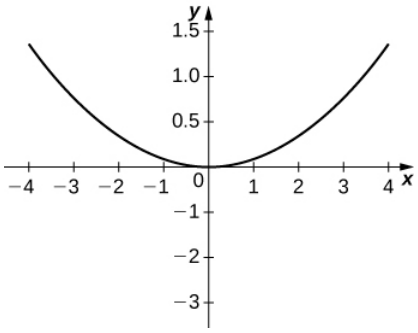
Answer



48) $\frac{x^2}{16} - \frac{y^2}{9} = 1$

49) $x^2 = 12y$

Answer

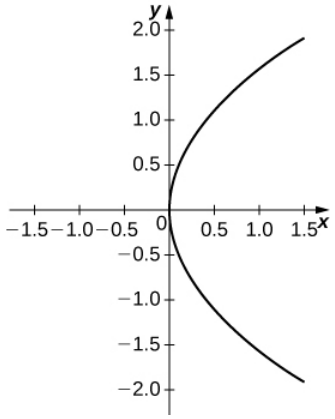


50) $y^2 = 20x$

51) $12x = 5y^2$

Answer

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For the equations in exercises 52 - 57, determine which of the conic sections is described.

52) $xy = 4$

53) $x^2 + 4xy - 2y^2 - 6 = 0$

Answer

Hyperbola

54) $x^2 + 2\sqrt{3}xy + 3y^2 - 6 = 0$

55) $x^2 - xy + y^2 - 2 = 0$

Answer

Ellipse

56) $34x^2 - 24xy + 41y^2 - 25 = 0$

57) $52x^2 - 72xy + 73y^2 + 40x + 30y - 75 = 0$

Answer

Ellipse

58) The mirror in an automobile headlight has a parabolic cross section, with the lightbulb at the focus. On a schematic, the equation of the parabola is given as $x^2 = 4y$. At what coordinates should you place the lightbulb?

59) A satellite dish is shaped like a paraboloid of revolution. The receiver is to be located at the focus. If the dish is 12 feet across at its opening and 4 feet deep at its center, where should the receiver be placed?

Answer

At the point 2.25 feet above the vertex.

60) Consider the satellite dish of the preceding problem. If the dish is 8 feet across at the opening and 2 feet deep, where should we place the receiver?

61) A searchlight is shaped like a paraboloid of revolution. A light source is located 1 foot from the base along the axis of symmetry. If the opening of the searchlight is 3 feet across, find the depth.

Answer

0.5625feet

62) Whispering galleries are rooms designed with elliptical ceilings. A person standing at one focus can whisper and be heard by a person standing at the other focus because all the sound waves that reach the ceiling are reflected to the other person. If a whispering gallery has a length of 120 feet and the foci are located 30 feet from the center, find the height of the ceiling at the center.

63) A person is standing 8 feet from the nearest wall in a whispering gallery. If that person is at one focus and the other focus is 80 feet away, what is the length and the height at the center of the gallery?

Answer

Length is 96 feet and height is approximately 26.53 feet.

In exercises 64 - 67, determine the polar equation form of the orbit given the length of the major axis and eccentricity for the orbits of the comets or planets. Distance is given in astronomical units (AU).

64) Halley's Comet: length of major axis = 35.88,eccentricity = 0.967

65) Hale-Bopp Comet: length of major axis = 525.91,eccentricity = 0.995

Answer

$$r = \frac{2.616}{1 + 0.995\cos\theta}$$

66) Mars: length of major axis = 3.049,eccentricity = 0.0934

67) Jupiter: length of major axis = 10.408,eccentricity = 0.0484

Answer

$$r = \frac{5.192}{1 + 0.0484\cos\theta}$$

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3.7: Chapter 3 Review Exercises

***True or False?* Justify your answer with a proof or a counterexample.**

1) The rectangular coordinates of the point $\left(4, \frac{5\pi}{6}\right)$ are $\left(2\sqrt{3}, -2\right)$.

2) The equations $x = \cosh(3t),\ y = 2\sinh(3t)$ represent a hyperbola.

Answer

True

3) The arc length of the spiral given by $r = \dfrac{\theta}{2}$ for $0 \le \theta \le 3\pi$ is $\frac{9}{4}\pi^3$ units.

4) Given $x = f(t)$ and $y = g(t)$, if $\dfrac{dx}{dy} = \dfrac{dy}{dx}$, then $f(t) = g(t) + C,$ where C is a constant.

Answer

False. Imagine $y = t + 1,\ x = -t + 1$.

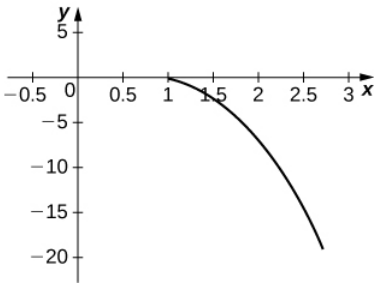
In exercises 5 -8, sketch the parametric curve and eliminate the parameter to find the Cartesian equation of the curve.

5) $x = 1 + t,\ y = t^2 - 1, \quad -1 \le t \le 1$

6) $x = e^t,\ y = 1 - e^{3t}, \quad 0 \le t \le 1$

Answer

$y = 1 - x^3$

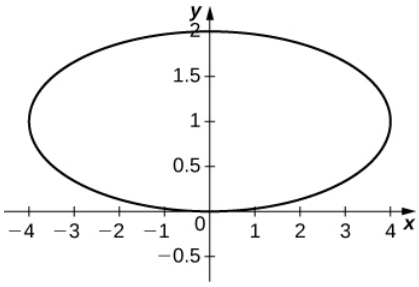


7) $x = \sin\theta,\ y = 1 - \csc\theta, \quad 0 \le \theta \le 2\pi$

8) $x = 4\cos\phi,\ y = 1 - \sin\phi, \quad 0 \le \phi \le 2\pi$

Answer

$\dfrac{x^2}{16} + (y-1)^2 = 1$



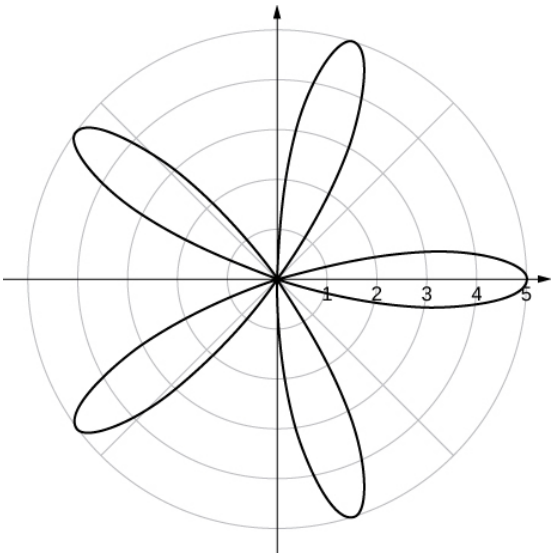
In exercises 9 - 10, sketch the polar curve and determine what type of symmetry exists, if any.

9) $r = 4\sin\left(\frac{\theta}{3}\right)$

10) $r = 5\cos(5\theta)$

Answer

Symmetric about polar axis



In exercises 11 - 12, find the polar equation for the curve given as a Cartesian equation.

11) $x + y = 5$

12) $y^2 = 4 + x^2$

Answer

$$r^2 = \frac{4}{\sin^2\theta - \cos^2\theta}$$

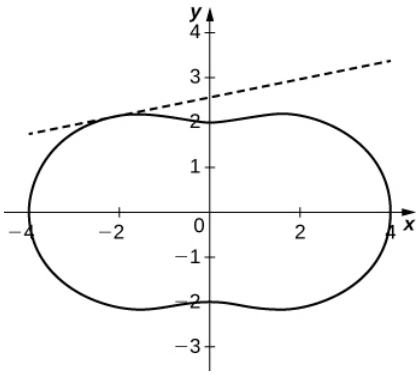
In exercises 13 - 14, find the equation of the tangent line to the given curve. Graph both the function and its tangent line.

13) $x = \ln(t),\ y = t^2 - 1,\ t = 1$

14) $r = 3 + \cos(2\theta),\ \theta = \frac{3\pi}{4}$

Answer

$y = \frac{3\sqrt{2}}{2} + \frac{1}{5}\left(x + \frac{3\sqrt{2}}{2}\right)$



15) Find $\frac{dy}{dx}$, $\frac{dx}{dy}$, and $\frac{d^2x}{dy^2}$ of $y = (2 + e^{-t}),\ x = 1 - \sin t$

In exercises 16 -17, find the area of the region.

16) $x = t^2,\ y = \ln(t),\quad 0 \le t \le e$

Answer

$\frac{e^2}{2}$ units^2

17) $r = 1 - \sin\theta$ in the first quadrant

In exercises 18 - 19, find the arc length of the curve over the given interval.

18) $x = 3t + 4,\ y = 9t - 2,\quad 0 \le t \le 3$

Answer

$9\sqrt{10}$ units

19) $r = 6\cos\theta,\quad 0 \le \theta \le 2\pi$. Check your answer by geometry.

In exercises 20 - 22, find the Cartesian equation describing the given shapes.

20) A parabola with focus $(2, -5)$ and directrix $x = 6$

Answer

$(y+5)^2 = -8x + 32$

21) An ellipse with a major axis length of 10 and foci at $(-7, 2)$ and $(1, 2)$

22) A hyperbola with vertices at $(3, -2)$ and $(-5, -2)$ and foci at $(-2, -6)$ and $(-2, 4)$

Answer

$\frac{(y+1)^2}{16} - \frac{(x+2)^2}{9} = 1$

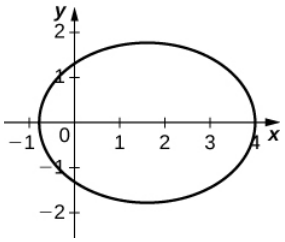
In exercises 23 - 25, determine the eccentricity and identify the conic. Sketch the conic.

23) $r = \frac{6}{1 + 3\cos\theta}$

24) $r = \frac{4}{3 - 2\cos\theta}$

Answer

$e = \frac{2}{3}$, ellipse



25) $r = \frac{7}{5 - 5\cos\theta}$

26) Determine the Cartesian equation describing the orbit of Pluto, the most eccentric orbit around the Sun. The length of the major axis is 39.26 AU and minor axis is 38.07 AU. What is the eccentricity?

Answer

$$\frac{y^2}{19.03^2}+\frac{x^2}{19.63^2}=1, \quad e=0.2447$$

27) The C/1980 E1 comet was observed in 1980. Given an eccentricity of 1.057 and a perihelion (point of closest approach to the Sun) of 3.364 AU, find the Cartesian equations describing the comet's trajectory. Are we guaranteed to see this comet again? (Hint: Consider the Sun at point $(0,0)$.)

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CHAPTER OVERVIEW

4: Vectors in Space

A quantity that has magnitude and direction is called a **vector**. Vectors have many real-life applications, including situations involving force or velocity. For example, consider the forces acting on a boat crossing a river. The boat's motor generates a force in one direction, and the current of the river generates a force in another direction. Both forces are vectors. We must take both the magnitude and direction of each force into account if we want to know where the boat will go.

4.1: Prelude to Vectors in Space
4.2: Vectors in the Plane
4.2E: Exercises for Section 4.2
4.3: Vectors in Three Dimensions
4.3E: Exercises for Section 4.3
4.4: The Dot Product
4.4E: Exercises for Section 4.4
4.5: The Cross Product
4.5E: Exercises for Section 4.5
4.6: Equations of Lines and Planes in Space
4.6E: Exercises for Section 4.6
4.7: Quadric Surfaces
4.7E: Exercises for Section 4.7
4.8: Cylindrical and Spherical Coordinates
4.8E: Exercises for Section 4.8
4.9: Chapter 4 Review Exercises

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4.1: Prelude to Vectors in Space

Modern astronomical observatories often consist of a large number of parabolic reflectors, connected by computers, used to analyze radio waves. Each dish focuses the incoming parallel beams of radio waves to a precise focal point, where they can be synchronized by computer. If the surface of one of the parabolic reflectors is described by the equation $\frac{x^2}{100} + \frac{y^2}{100} = \frac{z}{4}$, where is the focal point of the reflector?



Figure 4.1.1: The Karl G. Jansky Very Large Array, located in Socorro, New Mexico, consists of a large number of radio telescopes that can collect radio waves and collate them as if they were gathering waves over a huge area with no gaps in coverage. (credit: modification of work by CGP Grey, Wikimedia Commons)

We are now about to begin a new part of the calculus course, when we study functions of two or three independent variables in multidimensional space. Many of the computations are similar to those in the study of single-variable functions, but there are also a lot of differences. In this first chapter, we examine coordinate systems for working in three-dimensional space, along with vectors, which are a key mathematical tool for dealing with quantities in more than one dimension. Let's start here with the basic ideas and work our way up to the more general and powerful tools of mathematics in later chapters.

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4.2: Vectors in the Plane

Learning Objectives

- Describe a plane vector, using correct notation.
- Perform basic vector operations (scalar multiplication, addition, subtraction).
- Express a vector in component form.
- Explain the formula for the magnitude of a vector.
- Express a vector in terms of unit vectors.
- Give two examples of vector quantities.

When describing the movement of an airplane in flight, it is important to communicate two pieces of information: the direction in which the plane is traveling and the plane's speed. When measuring a force, such as the thrust of the plane's engines, it is important to describe not only the strength of that force, but also the direction in which it is applied. Some quantities, such as or force, are defined in terms of both size (also called magnitude) and direction. A quantity that has magnitude and direction is called a **vector**. In this text, we denote vectors by boldface letters, such as $\mathbf{v}$. Many times, we will also include an arrow or harpoon above the boldface letter, giving us $\vec{v}$ or $\vec{\mathbf{v}}$. This is how we will write vectors by hand, since it's hard to write in boldface.

Definition: Vector

A vector is a quantity that has both magnitude and direction.

Vector Representation

A vector in a plane is represented by a directed line segment (an arrow). The endpoints of the segment are called the **initial point** and the **terminal point** of the vector. An arrow from the initial point to the terminal point indicates the direction of the vector. The length of the line segment represents its **magnitude**. We use the notation $\|\vec{\mathbf{v}}\|$ to denote the magnitude of the vector $\vec{\mathbf{v}}$. A vector with an initial point and terminal point that are the same is called the **zero vector**, denoted $\vec{\mathbf{0}}$. The zero vector is the only vector without a direction, and by convention can be considered to have any direction convenient to the problem at hand.

Vectors with the same magnitude and direction are called equivalent vectors. We treat equivalent vectors as equal, even if they have different initial points. Thus, if $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ are equivalent, we write

$$\vec{\mathbf{v}} = \vec{\mathbf{w}}. \tag{4.2.1}$$

Definition: equivalent vectors

Vectors are said to be *equivalent vectors* if they have the same magnitude and direction.

The arrows in Figure $4.2.1(b)$ are equivalent. Each arrow has the same length and direction. A closely related concept is the idea of parallel vectors. Two vectors are said to be parallel if they have the same or opposite directions. We explore this idea in more detail later in the chapter. A vector is defined by its magnitude and direction, regardless of where its initial point is located.

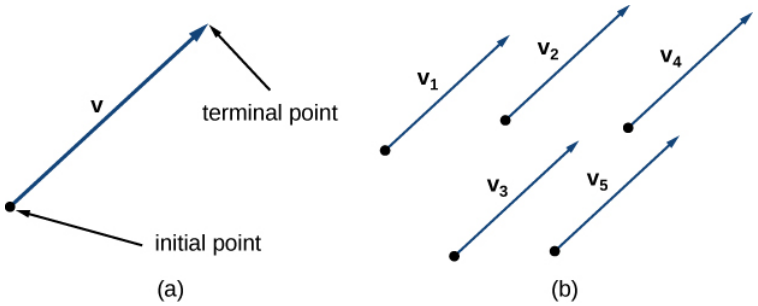


Figure 4.2.1: (a) A vector is represented by a directed line segment from its initial point to its terminal point. (b) Vectors $\vec{\mathbf{v}}_1$ through $\vec{\mathbf{v}}_5$ *are equivalent.*

The use of boldface, lowercase letters to name vectors is a common representation in print, but there are alternative notations. When writing the name of a vector by hand, for example, it is easier to sketch an arrow over the variable than to show it is a vector:

$\vec{v}$. When a vector has initial point P and terminal point Q, the notation $\overrightarrow{PQ}$ is useful because it indicates the direction and location of the vector.

Example 4.2.1: Sketching Vectors

Sketch a vector in the plane from initial point $P(1,1)$ to terminal point $Q(8,5)$.

Solution

See Figure 4.2.2. Because the vector goes from point P to point Q, we name it $\overrightarrow{PQ}$.

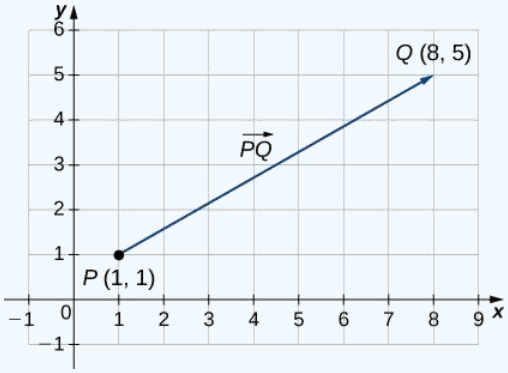


Figure 4.2.2: The vector with initial point $(1,1)$ and terminal point $(8,5)$ is named $\overrightarrow{PQ}$.

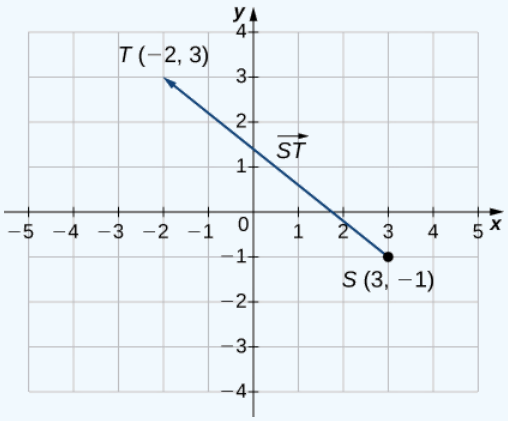
Exercise 4.2.1

Sketch the vector $\overrightarrow{ST}$ where S is point $(3,-1)$ and T is point $(-2,3)$.

Hint

The first point listed in the name of the vector is the initial point of the vector.

Answer



Combining Vectors

Vectors have many real-life applications, including situations involving force or velocity. For example, consider the forces acting on a boat crossing a river. The boat's motor generates a force in one direction, and the current of the river generates a force in another direction. Both forces are vectors. We must take both the magnitude and direction of each force into account if we want to know where the boat will go.

A second example that involves vectors is a quarterback throwing a football. The quarterback does not throw the ball parallel to the ground; instead, he aims up into the air. The velocity of his throw can be represented by a vector. If we know how hard he throws

the ball (magnitude—in this case, speed), and the angle (direction), we can tell how far the ball will travel down the field.

A real number is often called a **scalar** in mathematics and physics. Unlike vectors, scalars are generally considered to have a magnitude only, but no direction. Multiplying a vector by a scalar changes the vector's magnitude. This is called scalar multiplication. Note that changing the magnitude of a vector does not indicate a change in its direction. For example, wind blowing from north to south might increase or decrease in speed while maintaining its direction from north to south.

Definition: Scalar Multiplication

The product $k\vec{\mathbf{v}}$ of a vector $\vec{\mathbf{v}}$ and a scalar k is a vector with a magnitude that is $|k|$ times the magnitude of $\vec{\mathbf{v}}$, and with a direction that is the same as the direction of $\vec{\mathbf{v}}$ if $k>0$, and opposite the direction of $\vec{\mathbf{v}}$ if $k<0$. This is called *scalar multiplication*. If $k=0$ or $\vec{\mathbf{v}}=\vec{\mathbf{0}}$, then $k\vec{\mathbf{v}}=\vec{\mathbf{0}}$.

As you might expect, if $k=-1$, we denote the product $k\vec{\mathbf{v}}$ as

$$k\vec{\mathbf{v}}=(-1)\vec{\mathbf{v}}=-\vec{\mathbf{v}}. \tag{4.2.2}$$

Note that $-\vec{\mathbf{v}}$ has the same magnitude as $\vec{\mathbf{v}}$, but has the opposite direction (Figure 4.2.3).

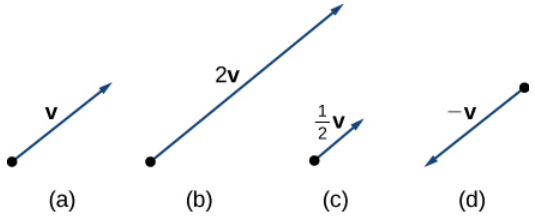


Figure 4.2.3: (a) The original vector $\vec{\mathbf{v}}$ has length n units. (b) The length of $2\vec{\mathbf{v}}$ equals $2n$ units. (c) The length of $\vec{\mathbf{v}}/2$ is $n/2$ units. (d) The vectors $\vec{\mathbf{v}}$ and $-\vec{\mathbf{v}}$ have the same length but opposite directions.

Another operation we can perform on vectors is to add them together in vector addition, but because each vector may have its own direction, the process is different from adding two numbers. The most common graphical method for adding two vectors is to place the initial point of the second vector at the terminal point of the first, as in Figure $4.2.4(a)$. To see why this makes sense, suppose, for example, that both vectors represent displacement. If an object moves first from the initial point to the terminal point of vector $\vec{\mathbf{v}}$, then from the initial point to the terminal point of vector $\vec{\mathbf{w}}$, the overall displacement is the same as if the object had made just one movement from the initial point to the terminal point of the vector $\vec{\mathbf{v}}+\vec{\mathbf{w}}$. For obvious reasons, this approach is called the **triangle method**. Notice that if we had switched the order, so that $\vec{\mathbf{w}}$ was our first vector and $\vec{\mathbf{v}}$ was our second vector, we would have ended up in the same place. (Again, see Figure $4.2.4(a)$.) Thus,

$$\vec{\mathbf{v}}+\vec{\mathbf{w}}=\vec{\mathbf{w}}+\vec{\mathbf{v}}. \tag{4.2.3}$$

A second method for adding vectors is called the **parallelogram method**. With this method, we place the two vectors so they have the same initial point, and then we draw a parallelogram with the vectors as two adjacent sides, as in Figure $4.2.4(b)$. The length of the diagonal of the parallelogram is the sum. Comparing Figure $4.2.4(b)$ and Figure $4.2.4(a)$, we can see that we get the same answer using either method. The vector $\vec{\mathbf{v}}+\vec{\mathbf{w}}$ is called the **vector sum.**

Definition: Vector Addition

The sum of two vectors $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ can be constructed graphically by placing the initial point of $\vec{\mathbf{w}}$ at the terminal point of $\vec{\mathbf{v}}$. Then, the vector sum, $\vec{\mathbf{v}}+\vec{\mathbf{w}}$, is the vector with an initial point that coincides with the initial point of $\vec{\mathbf{v}}$ and has a terminal point that coincides with the terminal point of $\vec{\mathbf{w}}$. This operation is known as *vector addition*.

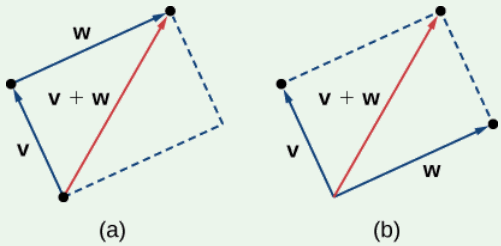


Figure 4.2.4: (a) When adding vectors by the triangle method, the initial point of $\vec{\mathbf{w}}$ is the terminal point of $\vec{\mathbf{v}}$. (b) When adding vectors by the parallelogram method, the vectors$\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ have the same initial point.

It is also appropriate here to discuss vector subtraction. We define $\vec{\mathbf{v}}-\vec{\mathbf{w}}$ as $\vec{\mathbf{v}}+(-\vec{\mathbf{w}})=\vec{\mathbf{v}}+(-1)\vec{\mathbf{w}}$. The vector $\vec{\mathbf{v}}-\vec{\mathbf{w}}$ is called the **vector difference**. Graphically, the vector $\vec{\mathbf{v}}-\vec{\mathbf{w}}$ is depicted by drawing a vector from the terminal point of $\vec{\mathbf{w}}$ to the terminal point of $\vec{\mathbf{v}}$ (Figure 4.2.5).

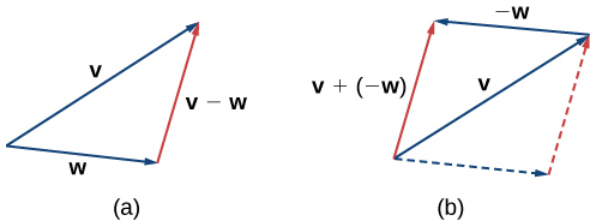


Figure 4.2.5: (a) The vector difference $\vec{\mathbf{v}}-\vec{\mathbf{w}}$ is depicted by drawing a vector from the terminal point of $\vec{\mathbf{w}}$ to the terminal point of $\vec{\mathbf{v}}$. (b) The vector$\vec{\mathbf{v}}-\vec{\mathbf{w}}$ is equivalent to the vector $\vec{\mathbf{v}}+(-\vec{\mathbf{w}})$.

In Figure 4.2.5(a), the initial point of $\vec{\mathbf{v}}+\vec{\mathbf{w}}$ is the initial point of $\vec{\mathbf{v}}$. The terminal point of $\vec{\mathbf{v}}+\vec{\mathbf{w}}$ is the terminal point of $\vec{\mathbf{w}}$. These three vectors form the sides of a triangle. It follows that the length of any one side is less than the sum of the lengths of the remaining sides. So we have

$$\|\vec{\mathbf{v}}+\vec{\mathbf{w}}\|\leq\|\vec{\mathbf{v}}\|+\|\vec{\mathbf{w}}\|. \tag{4.2.4}$$

This is known more generally as the **triangle inequality**. There is one case, however, when the resultant vector $\vec{\mathbf{u}}+\vec{\mathbf{v}}$ has the same magnitude as the sum of the magnitudes of $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$. This happens only when $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ have the same direction.

Example 4.2.2: Combining Vectors

Given the vectors v and w shown in Figure 4.2.6, sketch the vectors

a. $3\vec{\mathbf{w}}$
b. $\vec{\mathbf{v}}+\vec{\mathbf{w}}$
c. $2\vec{\mathbf{v}}-\vec{\mathbf{w}}$

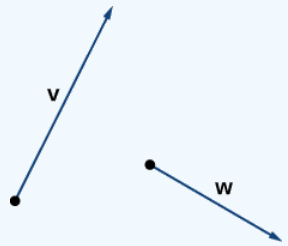
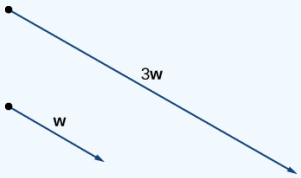


Figure 4.2.6: Vectors $\vec{\mathbf{v}}$ *and* $\vec{\mathbf{w}}$ *lie in the same plane.*

Solution

a. The vector $3\vec{\mathbf{w}}$ has the same direction as $\vec{\mathbf{w}}$; it is three times as long as $\vec{\mathbf{w}}$.



Vector $3\vec{\mathbf{w}}$ has the same direction as $\vec{\mathbf{w}}$ and is three times as long.

b. Use either addition method to find $\vec{\mathbf{v}}+\vec{\mathbf{w}}$.

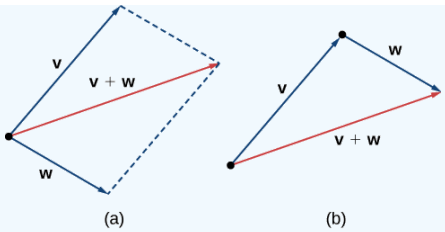


Figure 4.2.7: To find $\vec{\mathbf{v}}+\vec{\mathbf{w}}$, align the vectors at their initial points or place the initial point of one vector at the terminal point of the other. (a) The vector $\vec{\mathbf{v}}+\vec{\mathbf{w}}$ is the diagonal of the parallelogram with sides $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$. (b) The vector $\vec{\mathbf{v}}+\vec{\mathbf{w}}$ is the third side of a triangle formed with $\vec{\mathbf{w}}$ placed at the terminal point of $\vec{\mathbf{v}}$.

c. To find $2\vec{\mathbf{v}}-\vec{\mathbf{w}}$, we can first rewrite the expression as $2\vec{\mathbf{v}}+(-\vec{\mathbf{w}})$. Then we can draw the vector $-\vec{\mathbf{w}}$, then add it to the vector $2\vec{\mathbf{v}}$.

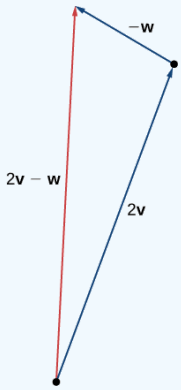


Figure 4.2.8: To find $2\vec{\mathbf{v}}-\vec{\mathbf{w}}$, *simply add* $2\vec{\mathbf{v}}+(-\vec{\mathbf{w}})$.



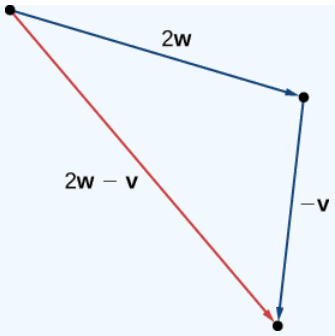
Exercise 4.2.2

Using vectors $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ from Example 4.2.2, sketch the vector $2\vec{\mathbf{w}}-\vec{\mathbf{v}}$.

Hint

First sketch vectors $2\vec{\mathbf{w}}$ and $-\vec{\mathbf{v}}$.

Answer



Vector Components

Working with vectors in a plane is easier when we are working in a coordinate system. When the initial points and terminal points of vectors are given in Cartesian coordinates, computations become straightforward.

Example 4.2.3: Comparing Vectors

Are $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ equivalent vectors?

a.

- $\vec{\mathbf{v}}$ has initial point $(3, 2)$ and terminal point $(7, 2)$
- $\vec{\mathbf{w}}$ has initial point $(1, -4)$ and terminal point $(1, 0)$

b.

- $\vec{\mathbf{v}}$ has initial point $(0, 0)$ and terminal point $(1, 1)$
- $\vec{\mathbf{w}}$ has initial point $(-2, 2)$ and terminal point $(-1, 3)$

Solution

a. The vectors are each 4 units long, but they are oriented in different directions. So $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ are not equivalent (Figure 4.2.9).

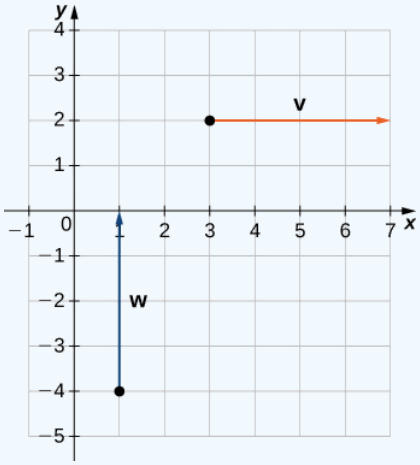


Figure 4.2.9: These vectors are not equivalent.

b. Based on Figure 4.2.10, and using a bit of geometry, it is clear these vectors have the same length and the same direction, so $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ are equivalent.

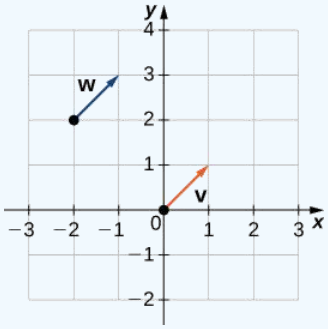
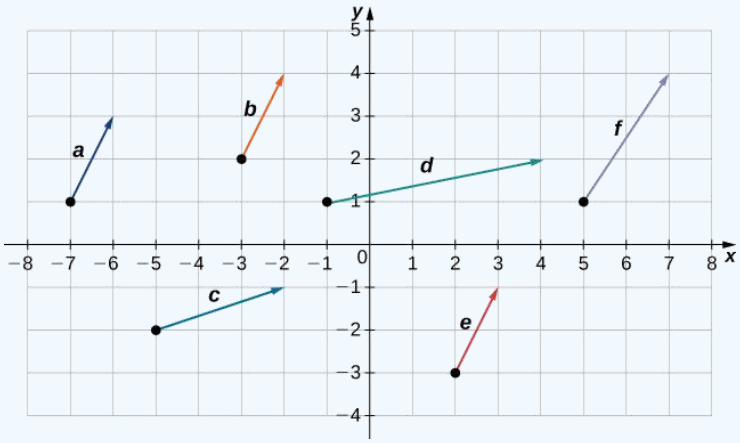


Figure 4.2.10: These vectors are equivalent.

Exercise 4.2.3

Which of the following vectors are equivalent?



Hint

Equivalent vectors have both the same magnitude and the same direction.

Answer

Vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, and $\vec{\mathbf{e}}$ are equivalent.

We have seen how to plot a vector when we are given an initial point and a terminal point. However, because a vector can be placed anywhere in a plane, it may be easier to perform calculations with a vector when its initial point coincides with the origin. We call a vector with its initial point at the origin a **standard-position vector.** Because the initial point of any vector in standard position is known to be $(0, 0)$, we can describe the vector by looking at the coordinates of its terminal point. Thus, if vector $\vec{\mathbf{v}}$ has its initial point at the origin and its terminal point at (x, y), we write the vector in component form as

$$\vec{\mathbf{v}} = \langle x, y \rangle. \tag{4.2.5}$$

When a vector is written in component form like this, the scalars x and y are called the **components** of $\vec{\mathbf{v}}$.

Definition: Vector components

The vector with initial point $(0, 0)$ and terminal point (x, y) can be written in component form as

$$\vec{\mathbf{v}} = \langle x, y \rangle. \tag{4.2.6}$$

The scalars x and y are called the components of $\vec{\mathbf{v}}$.

Recall that vectors are named with lowercase letters in bold type or by drawing an arrow over their name. We have also learned that we can name a vector by its component form, with the coordinates of its terminal point in angle brackets. However, when writing

the component form of a vector, it is important to distinguish between $\langle x, y \rangle$ and (x, y). The first ordered pair uses angle brackets to describe a vector, whereas the second uses parentheses to describe a point in a plane. The initial point of $\langle x, y \rangle$ is $(0, 0)$; the terminal point of $\langle x, y \rangle$ is (x, y).

When we have a vector not already in standard position, we can determine its component form in one of two ways. We can use a geometric approach, in which we sketch the vector in the coordinate plane, and then sketch an equivalent standard-position vector. Alternatively, we can find it algebraically, using the coordinates of the initial point and the terminal point. To find it algebraically, we subtract the x-coordinate of the initial point from the x-coordinate of the terminal point to get the x-component, and we subtract the y-coordinate of the initial point from the y-coordinate of the terminal point to get the y-component.

Rule: Component Form of a Vector

Let $\vec{\mathbf{v}}$ be a vector with initial point (x_i, y_i) and terminal point (x_t, y_t). Then we can express $\vec{\mathbf{v}}$ in component form as $\vec{\mathbf{v}} = \langle x_t - x_i, y_t - y_i \rangle$.

Example 4.2.4: Expressing Vectors in Component Form

Express vector $\vec{\mathbf{v}}$ with initial point $(-3, 4)$ and terminal point $(1, 2)$ in component form.

Solution:

a. Geometric

1. Sketch the vector in the coordinate plane (Figure 4.2.11).
2. The terminal point is 4 units to the right and 2 units down from the initial point.
3. Find the point that is 4 units to the right and 2 units down from the origin.
4. In standard position, this vector has initial point $(0, 0)$ and terminal point $(4, -2)$:

$$\vec{\mathbf{v}} = \langle 4, -2 \rangle.$$

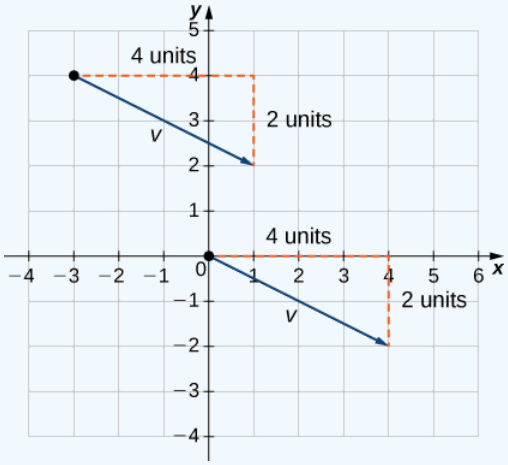


Figure 4.2.11: These vectors are equivalent.

b. Algebraic

In the first solution, we used a sketch of the vector to see that the terminal point lies 4 units to the right. We can accomplish this algebraically by finding the difference of the x-coordinates:

$$x_t - x_i = 1 - (-3) = 4.$$

Similarly, the difference of the y-coordinates shows the vertical length of the vector.

$$y_t - y_i = 2 - 4 = -2.$$

So, in component form,

$$\vec{\mathbf{v}} = \langle x_t - x_i, y_t - y_i \rangle = \langle 1 - (-3), 2 - 4 \rangle = \langle 4, -2 \rangle.$$

Exercise 4.2.4

Vector $\vec{\mathbf{w}}$ has initial point $(-4,-5)$ and terminal point $(-1,2)$. Express $\vec{\mathbf{w}}$ in component form.

Hint

You may use either geometric or algebraic method.

Answer

$\langle 3,7\rangle$

To find the magnitude of a vector, we calculate the distance between its initial point and its terminal point. The **magnitude** of vector $\vec{\mathbf{v}}=\langle x,y\rangle$ is denoted $\|\vec{\mathbf{v}}\|$, or $|\vec{\mathbf{v}}|$, and can be computed using the formula

$$\|\vec{\mathbf{v}}\|=\sqrt{x^2+y^2}. \tag{4.2.7}$$

Note that because this vector is written in component form, it is equivalent to a vector in standard position, with its initial point at the origin and terminal point (x,y). Thus, it suffices to calculate the magnitude of the vector in standard position. Using the distance formula to calculate the distance between initial point $(0,0)$ and terminal point (x,y), we have

$$\|\vec{\mathbf{v}}\|=\sqrt{(x-0)^2+(y-0)^2}=\sqrt{x^2+y^2}. \tag{4.2.8}$$

Based on this formula, it is clear that for any vector $\vec{\mathbf{v}}$, $\|\vec{\mathbf{v}}\|\geq 0$, and $\|\vec{\mathbf{v}}\|=0$ if and only if $\vec{\mathbf{v}}=\vec{\mathbf{0}}$.

The magnitude of a vector can also be derived using the Pythagorean theorem, as in the following figure.

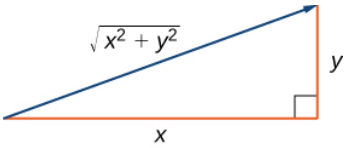


Figure 4.2.12: If you use the components of a vector to define a right triangle, the magnitude of the vector is the length of the triangle's hypotenuse.

We have defined scalar multiplication and vector addition geometrically. Expressing vectors in component form allows us to perform these same operations algebraically.

Definition: Scalar multiplication and Vector addition

Let $\vec{\mathbf{v}}=\langle x_1,y_1\rangle$ and $\vec{\mathbf{w}}=\langle x_2,y_2\rangle$ be vectors, and let k be a scalar.

- *Scalar multiplication:*

$$k\vec{\mathbf{v}}=\langle kx_1,ky_1\rangle \tag{4.2.9}$$

- *Vector addition:*

$$\vec{\mathbf{v}}+\vec{\mathbf{w}}=\langle x_1,y_1\rangle+\langle x_2,y_2\rangle=\langle x_1+x_2,y_1+y_2\rangle \tag{4.2.10}$$

Example 4.2.5: Performing Operations in Component Form

Let $\vec{\mathbf{v}}$ be the vector with initial point $(2,5)$ and terminal point $(8,13)$, and let $\vec{\mathbf{w}}=\langle -2,4\rangle$.

a. Express $\vec{\mathbf{v}}$ in component form and find $\|\vec{\mathbf{v}}\|$. Then, using algebra, find
b. $\vec{\mathbf{v}}+\vec{\mathbf{w}}$,
c. $3\vec{\mathbf{v}}$, and
d. $\vec{\mathbf{v}}-2\vec{\mathbf{w}}$.

Solution

a. To place the initial point of $\vec{\mathbf{v}}$ at the origin, we must translate the vector **2** units to the left and **5** units down (Figure 4.2.13). Using the algebraic method, we can express $\vec{\mathbf{v}}$ as $\vec{\mathbf{v}}=\langle 8-2,13-5\rangle=\langle 6,8\rangle$:

$$\|\vec{\mathbf{v}}\| = \sqrt{6^2+8^2} = \sqrt{36+64} = \sqrt{100} = 10 \ .$$

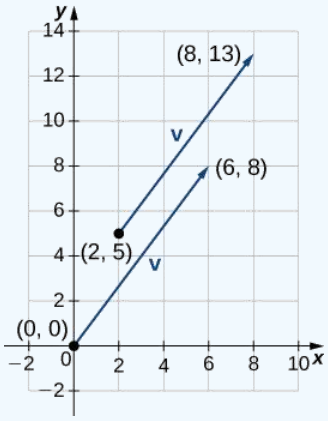


Figure 4.2.13: In component form, $\vec{\mathbf{v}} = \langle 6, 8 \rangle$.

b. To find $\vec{\mathbf{v}} + \vec{\mathbf{w}}$, add the x-components and the y-components separately:

$$\vec{\mathbf{v}} + \vec{\mathbf{w}} = \langle 6, 8 \rangle + \langle -2, 4 \rangle = \langle 4, 12 \rangle.$$

c. To find $3\vec{\mathbf{v}}$, multiply $\vec{\mathbf{v}}$ by the scalar $k = 3$:

$$3\vec{\mathbf{v}} = 3 \cdot \langle 6, 8 \rangle = \langle 3 \cdot 6, 3 \cdot 8 \rangle = \langle 18, 24 \rangle.$$

d. To find $\vec{\mathbf{v}} - 2\vec{\mathbf{w}}$, find $-2\vec{\mathbf{w}}$ and add it to $\vec{\mathbf{v}}$:

$$\vec{\mathbf{v}} - 2\vec{\mathbf{w}} = \langle 6, 8 \rangle - 2 \cdot \langle -2, 4 \rangle = \langle 6, 8 \rangle + \langle 4, -8 \rangle = \langle 10, 0 \rangle.$$

Exercise 4.2.5A

Let $\vec{\mathbf{a}} = \langle 7, 1 \rangle$ and let $\vec{\mathbf{b}}$ be the vector with initial point $(3, 2)$ and terminal point $(-1, -1)$.

a. Find $\|\vec{\mathbf{a}}\|$.
b. Express $\vec{\mathbf{b}}$ in component form.
c. Find $3\vec{\mathbf{a}} - 4\vec{\mathbf{b}}$.

Hint

Use the Pythagorean Theorem to find $\|\vec{\mathbf{a}}\|$. To find $3\vec{\mathbf{a}} - 4\vec{\mathbf{b}}$, start by finding the scalar multiples $3\vec{\mathbf{a}}$ and $-4\vec{\mathbf{b}}$.

Answer a

$\|\vec{\mathbf{a}}\| = 5\sqrt{2}$,

Answer b

$\vec{\mathbf{b}} = \langle -4, -3 \rangle$,

Answer c

$3\vec{\mathbf{a}} - 4\vec{\mathbf{b}} = \langle 37, 15 \rangle$

Now that we have established the basic rules of vector arithmetic, we can state the properties of vector operations. We will prove two of these properties. The others can be proved in a similar manner.

Properties of Vector Operations

Let $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$ be vectors in a plane. Let r and s be scalars.

i. *Commutative property*

$$\vec{\mathbf{u}} + \vec{\mathbf{v}} = \vec{\mathbf{v}} + \vec{\mathbf{u}} \tag{4.2.11}$$

ii. *Associative property*

$$(\vec{\mathbf{u}}+\vec{\mathbf{v}})+\vec{\mathbf{w}}=\vec{\mathbf{u}}+(\vec{\mathbf{v}}+\vec{\mathbf{w}}) \tag{4.2.12}$$

iii. *Additive identity property*

$$\vec{\mathbf{u}}+\vec{\mathbf{0}}=\vec{\mathbf{u}} \tag{4.2.13}$$

iv. *Additive inverse property*

$$\vec{\mathbf{u}}+(-\vec{\mathbf{u}})=\vec{\mathbf{0}} \tag{4.2.14}$$

v. *Associativity of scalar multiplication*

$$r(s\vec{\mathbf{u}})=(rs)\vec{\mathbf{u}} \tag{4.2.15}$$

vi. *Distributive property*

$$(r+s)\vec{\mathbf{u}}=r\vec{\mathbf{u}}+s\vec{\mathbf{u}} \tag{4.2.16}$$

vii. *Distributive property*

$$r(\vec{\mathbf{u}}+\vec{\mathbf{v}})=r\vec{\mathbf{u}}+r\vec{\mathbf{v}} \tag{4.2.17}$$

viii. *Identity and zero properties*

$$1\vec{\mathbf{u}}=\vec{\mathbf{u}},\ 0\vec{\mathbf{u}}=\vec{\mathbf{0}} \tag{4.2.18}$$

Proof of Commutative Property

Let $\vec{\mathbf{u}}=\langle x_1,y_1\rangle$ and $\vec{\mathbf{v}}=\langle x_2,y_2\rangle$. Apply the commutative property (Equation 4.2.11) for real numbers:

$$\begin{aligned}\vec{\mathbf{u}}+\vec{\mathbf{v}}&=\langle x_1+x_2,y_1+y_2\rangle\\&=\langle x_2+x_1,y_2+y_1\rangle\\&=\vec{\mathbf{v}}+\vec{\mathbf{u}}.\end{aligned}$$

□

Proof of Distributive Property

Apply the distributive property (Equation 4.2.16) for real numbers:

$$\begin{aligned}r(\vec{\mathbf{u}}+\vec{\mathbf{v}})&=r\cdot\langle x_1+x_2,y_1+y_2\rangle\\&=\langle r(x_1+x_2),r(y_1+y_2)\rangle\\&=\langle rx_1+rx_2,ry_1+ry_2\rangle\\&=\langle rx_1,ry_1\rangle+\langle rx_2,ry_2\rangle\\&=r\vec{\mathbf{u}}+r\vec{\mathbf{v}}.\end{aligned}$$

□

Exercise 4.2.5B

Prove the additive inverse property.

Hint

Use the component form of the vectors.

We have found the components of a vector given its initial and terminal points. In some cases, we may only have the magnitude and direction of a vector, not the points. For these vectors, we can identify the horizontal and vertical components using

trigonometry (Figure 4.2.14).

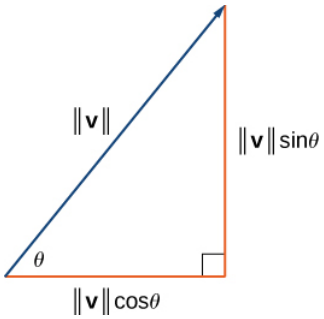


Figure 4.2.14: The components of a vector form the legs of a right triangle, with the vector as the hypotenuse.

Consider the angle θ formed by the vector $\vec{\mathbf{v}}$ and the positive x-axis. We can see from the triangle that the components of vector $\vec{\mathbf{v}}$ are $\langle \|\vec{\mathbf{v}}\| \cos\theta,\ \|\vec{\mathbf{v}}\| \sin\theta \rangle$. Therefore, given an angle and the magnitude of a vector, we can use the cosine and sine of the angle to find the components of the vector.

Example 4.2.6: Finding the Component Form of a Vector Using Trigonometry

Find the component form of a vector with magnitude 4 that forms an angle of -45° with the x-axis.

Solution

Let x and y represent the components of the vector (Figure 4.2.15). Then $x = 4\cos(-45^\circ) = 2\sqrt{2}$ and $y = 4\sin(-45^\circ) = -2\sqrt{2}$. The component form of the vector is $\langle 2\sqrt{2}, -2\sqrt{2} \rangle$.

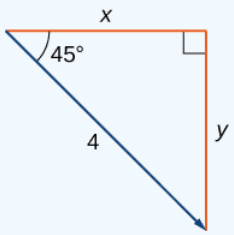


Figure 4.2.15: Use trigonometric ratios, $x = \|\vec{\mathbf{v}}\| \cos\theta$ *and* $y = \|\vec{\mathbf{v}}\| \sin\theta$, *to identify the components of the vector.*

Exercise 4.2.6

Find the component form of vector $\vec{\mathbf{v}}$ with magnitude **10** that forms an angle of 120° with the positive x-axis.

Hint

$x = \|\vec{\mathbf{v}}\| \cos\theta$ and $y = \|\vec{\mathbf{v}}\| \sin\theta$

Answer

$v = \langle -5, 5\sqrt{3} \rangle$

Unit Vectors

A **unit vector** is a vector with magnitude 1. For any nonzero vector $\vec{\mathbf{v}}$, we can use scalar multiplication to find a unit vector $\vec{\mathbf{u}}$ that has the same direction as $\vec{\mathbf{v}}$. To do this, we multiply the vector by the reciprocal of its magnitude:

$$\vec{\mathbf{u}} = \frac{1}{\|\vec{\mathbf{v}}\|}\vec{\mathbf{v}}. \tag{4.2.19}$$

Recall that when we defined scalar multiplication, we noted that $\|k\vec{\mathbf{v}}\| = |k| \cdot \|\vec{\mathbf{v}}\|$. For $\vec{\mathbf{u}} = \frac{1}{\|\vec{\mathbf{v}}\|}\vec{\mathbf{v}}$, it follows that $\|\vec{\mathbf{u}}\| = \frac{1}{\|\vec{\mathbf{v}}\|}(\|\vec{\mathbf{v}}\|) = 1$. We say that $\vec{\mathbf{u}}$ is the *unit vector in the direction of* $\vec{\mathbf{v}}$ (Figure 4.2.16). The process of using scalar multiplication to find a unit vector with a given direction is called **normalization.**



Figure 4.2.16: The vector $\vec{\mathbf{v}}$ *and associated unit vector* $\vec{\mathbf{u}} = \dfrac{1}{\|\vec{\mathbf{v}}\|}\vec{\mathbf{v}}$. *In this case,* $\|\vec{\mathbf{v}}\| > 1$.

Example 4.2.7: Finding a Unit Vector

Let $\vec{\mathbf{v}} = \langle 1, 2\rangle$.

a. Find a unit vector with the same direction as $\vec{\mathbf{v}}$.
b. Find a vector $\vec{\mathbf{w}}$ with the same direction as $\vec{\mathbf{v}}$ such that $\|\vec{\mathbf{w}}\| = 7$.

Solution:

a. First, find the magnitude of $\vec{\mathbf{v}}$, then divide the components of $\vec{\mathbf{v}}$ by the magnitude:

$$\|\vec{\mathbf{v}}\| = \sqrt{1^2 + 2^2} = \sqrt{1+4} = \sqrt{5}$$

$$\vec{\mathbf{u}} = \frac{1}{\|\vec{\mathbf{v}}\|}v = \frac{1}{\sqrt{5}}\langle 1, 2\rangle = \langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\rangle.$$

b. The vector $\vec{\mathbf{u}}$ is in the same direction as $\vec{\mathbf{v}}$ and $\|\vec{\mathbf{u}}\| = 1$. Use scalar multiplication to increase the length of $\vec{\mathbf{u}}$ without changing direction:

$$\vec{\mathbf{w}} = 7\vec{\mathbf{u}} = 7\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\rangle = \langle \frac{7}{\sqrt{5}}, \frac{14}{\sqrt{5}}\rangle.$$



Exercise 4.2.7

Let $\vec{\mathbf{v}} = \langle 9, 2\rangle$. Find a vector with magnitude 5 in the opposite direction as $\vec{\mathbf{v}}$.

Hint

First, find a unit vector in the same direction as $\vec{\mathbf{v}}$.

Answer

$$\langle -\frac{45}{\sqrt{85}}, -\frac{10}{\sqrt{85}}\rangle$$

We have seen how convenient it can be to write a vector in component form. Sometimes, though, it is more convenient to write a vector as a sum of a horizontal vector and a vertical vector. To make this easier, let's look at standard unit vectors. The **standard unit vectors** are the vectors $\hat{\mathbf{i}} = \langle 1, 0 \rangle$ and $\hat{\mathbf{j}} = \langle 0, 1 \rangle$ (Figure 4.2.17).

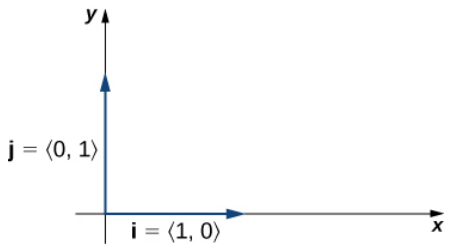


Figure 4.2.17: The standard unit vectors $\hat{\mathbf{i}}$ *and* $\hat{\mathbf{j}}$.

By applying the properties of vectors, it is possible to express any vector in terms of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ in what we call a *linear combination:*

$$\vec{\mathbf{v}} = \langle x, y \rangle = \langle x, 0 \rangle + \langle 0, y \rangle = x\langle 1, 0 \rangle + y\langle 0, 1 \rangle = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}. \tag{4.2.20}$$

Thus, $\vec{\mathbf{v}}$ is the sum of a horizontal vector with magnitude x, and a vertical vector with magnitude y, as in Figure 4.2.18.

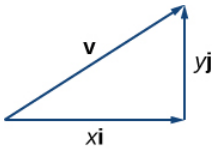


Figure 4.2.18: The vector $\vec{\mathbf{v}}$ *is the sum of* $x\hat{\mathbf{i}}$ *and* $y\hat{\mathbf{j}}$.

Example 4.2.8: Using Standard Unit Vectors

a. Express the vector $\vec{\mathbf{w}} = \langle 3, -4 \rangle$ in terms of standard unit vectors.
b. Vector $\vec{\mathbf{u}}$ is a unit vector that forms an angle of 60° with the positive x-axis. Use standard unit vectors to describe $\vec{\mathbf{u}}$.

Solution:

a. Resolve vector $\vec{\mathbf{w}}$ into a vector with a zero y-component and a vector with a zero x-component:

$$\vec{\mathbf{w}} = \langle 3, -4 \rangle = 3\hat{\mathbf{i}} - 4\hat{\mathbf{j}}.$$

b. Because $\vec{\mathbf{u}}$ is a unit vector, the terminal point lies on the unit circle when the vector is placed in standard position (Figure 4.2.19).

$$\begin{aligned} \vec{\mathbf{u}} &= \langle \cos 60^\circ, \sin 60^\circ \rangle \\ &= \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle \\ &= \frac{1}{2}\hat{\mathbf{i}} + \frac{\sqrt{3}}{2}\hat{\mathbf{j}}. \end{aligned}$$

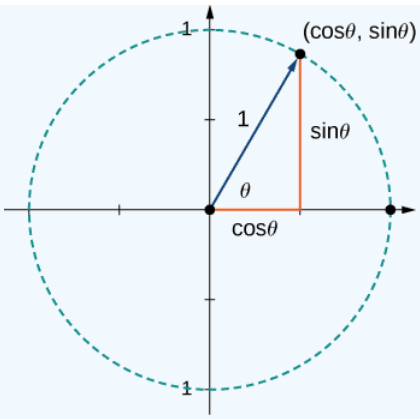


Figure 4.2.19: The terminal point of $\vec{\mathbf{u}}$ lies on the unit circle $(\cos\theta, \sin\theta)$.

Exercise 4.2.8

Let $\vec{\mathbf{a}} = \langle 16, -11 \rangle$ and let $\vec{\mathbf{b}}$ be a unit vector that forms an angle of 225° with the positive x-axis. Express $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ in terms of the standard unit vectors.

Hint

Use sine and cosine to find the components of $\vec{\mathbf{b}}$.

Answer

$$\vec{\mathbf{a}} = 16\hat{\mathbf{i}} - 11\hat{\mathbf{j}}, \quad \vec{\mathbf{b}} = -\frac{\sqrt{2}}{2}\hat{\mathbf{i}} - \frac{\sqrt{2}}{2}\hat{\mathbf{j}}$$

Applications of Vectors

Because vectors have both direction and magnitude, they are valuable tools for solving problems involving such applications as motion and **force.** Recall the boat example and the quarterback example we described earlier. Here we look at two other examples in detail.

Example 4.2.9A: Finding Resultant Force

Jane's car is stuck in the mud. Lisa and Jed come along in a truck to help pull her out. They attach one end of a tow strap to the front of the car and the other end to the truck's trailer hitch, and the truck starts to pull. Meanwhile, Jane and Jed get behind the car and push. The truck generates a horizontal force of **300** lb on the car. Jane and Jed are pushing at a slight upward angle and generate a force of **150** lb on the car. These forces can be represented by vectors, as shown in Figure 4.2.20. The angle between these vectors is **15°**. Find the resultant force (the vector sum) and give its magnitude to the nearest tenth of a pound and its direction angle from the positive x-axis.



Figure 4.2.20: Two forces acting on a car in different directions.

Solution

To find the effect of combining the two forces, add their representative vectors. First, express each vector in component form or in terms of the standard unit vectors. For this purpose, it is easiest if we align one of the vectors with the positive x-axis. The horizontal vector, then, has initial point $(0, 0)$ and terminal point $(300, 0)$. It can be expressed as $\langle 300, 0 \rangle$ or $300\hat{\mathbf{i}}$.

The second vector has magnitude 150 and makes an angle of 15° with the first, so we can express it as $\langle 150\cos(15^\circ), 150\sin(15^\circ)\rangle$,or $150\cos(15^\circ)\hat{\mathbf{i}} + 150\sin(15^\circ)\hat{\mathbf{j}}$. Then, the sum of the vectors, or resultant vector, is $\vec{\mathbf{r}} = \langle 300, 0\rangle + \langle 150\cos(15^\circ), 150\sin(15^\circ)\rangle$,and we have

$$\|\vec{\mathbf{r}}\| = \sqrt{(300 + 150\cos(15^\circ))^2 + (150\sin(15^\circ))^2} \approx 446.6.$$

The angle θ made by $\vec{\mathbf{r}}$ and the positive x-axis has $\tan\theta = \dfrac{150\sin 15^\circ}{(300 + 150\cos 15^\circ)} \approx 0.09$, so $\theta \approx \tan^{-1}(0.09) \approx 5^\circ$, which means the resultant force $\vec{\mathbf{r}}$ has an angle of 5° above the horizontal axis.

Example $4.2.9B$: Finding Resultant Velocity

An airplane flies due west at an airspeed of 425 mph. The wind is blowing from the northeast at 40 mph. What is the ground speed of the airplane? What is the bearing of the airplane?

Solution

Let's start by sketching the situation described (Figure 4.2.21).

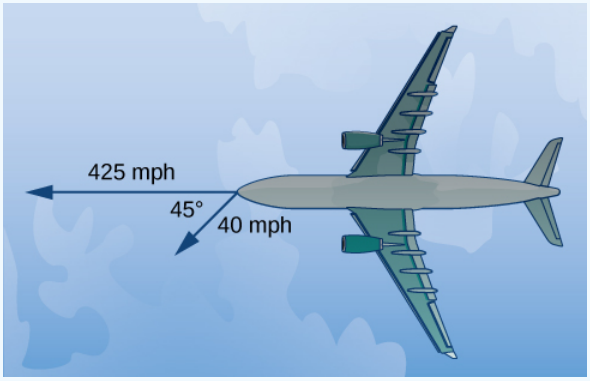


Figure 4.2.21: Initially, the plane travels due west. The wind is from the northeast, so it is blowing to the southwest. The angle between the plane's course and the wind is 45°. (Figure not drawn to scale.)

Set up a sketch so that the initial points of the vectors lie at the origin. Then, the plane's velocity vector is $\vec{\mathbf{p}} = -425\hat{\mathbf{i}}$. The vector describing the wind makes an angle of 225° with the positive x-axis:

$$\vec{\mathbf{w}} = \langle 40\cos(225^\circ), 40\sin(225^\circ)\rangle = \langle -\frac{40}{\sqrt{2}}, -\frac{40}{\sqrt{2}}\rangle = -\frac{40}{\sqrt{2}}\hat{\mathbf{i}} - \frac{40}{\sqrt{2}}\hat{\mathbf{j}}.$$

When the airspeed and the wind act together on the plane, we can add their vectors to find the resultant force:

$$\vec{\mathbf{p}} + \vec{\mathbf{w}} = -425\hat{\mathbf{i}} + (-\frac{40}{\sqrt{2}}\hat{\mathbf{i}} - \frac{40}{\sqrt{2}}\hat{\mathbf{j}}) = (-425 - \frac{40}{\sqrt{2}})\hat{\mathbf{i}} - \frac{40}{\sqrt{2}}\hat{\mathbf{j}}.$$

The magnitude of the resultant vector shows the effect of the wind on the ground speed of the airplane:

$$\|\vec{\mathbf{p}} + \vec{\mathbf{w}}\| = \sqrt{(-425 - \frac{40}{\sqrt{2}})^2 + (-\frac{40}{\sqrt{2}})^2} \approx 454.17\, mph$$

As a result of the wind, the plane is traveling at approximately 454 mph relative to the ground.

To determine the bearing of the airplane, we want to find the direction of the vector $\vec{\mathbf{p}} + \vec{\mathbf{w}}$:

$$\tan\theta = \frac{-\frac{40}{\sqrt{2}}}{(-425 - \frac{40}{\sqrt{2}})} \approx 0.06$$

$$\theta \approx 3.57^\circ.$$

The overall direction of the plane is 3.57° south of west.



Exercise 4.2.9

An airplane flies due north at an airspeed of 550 mph. The wind is blowing from the northwest at 50 mph. What is the ground speed of the airplane?

Hint

Sketch the vectors with the same initial point and find their sum.

Answer

Approximately 516 mph

Key Concepts

- Vectors are used to represent quantities that have both magnitude and direction.
- We can add vectors by using the parallelogram method or the triangle method to find the sum. We can multiply a vector by a scalar to change its length or give it the opposite direction.
- Subtraction of vectors is defined in terms of adding the negative of the vector.
- A vector is written in component form as $\vec{\mathbf{v}} = \langle x, y \rangle$.
- The magnitude of a vector is a scalar: $\|\vec{\mathbf{v}}\| = \sqrt{x^2 + y^2}$.
- A unit vector $\vec{\mathbf{u}}$ has magnitude 1 and can be found by dividing a vector by its magnitude: $\vec{\mathbf{u}} = \dfrac{1}{\|\vec{\mathbf{v}}\|}\vec{\mathbf{v}}$. The standard unit vectors are $\hat{\mathbf{i}} = \langle 1, 0 \rangle$ and $\hat{\mathbf{j}} = \langle 0, 1 \rangle$. A vector $\vec{\mathbf{v}} = \langle x, y \rangle$ can be expressed in terms of the standard unit vectors as $\vec{\mathbf{v}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$.
- Vectors are often used in physics and engineering to represent forces and velocities, among other quantities.

Glossary

component

a scalar that describes either the vertical or horizontal direction of a vector

equivalent vectors

vectors that have the same magnitude and the same direction

initial point

the starting point of a vector

magnitude
the length of a vector

normalization
using scalar multiplication to find a unit vector with a given direction

parallelogram method
a method for finding the sum of two vectors; position the vectors so they share the same initial point; the vectors then form two adjacent sides of a parallelogram; the sum of the vectors is the diagonal of that parallelogram

scalar
a real number

scalar multiplication
a vector operation that defines the product of a scalar and a vector

standard-position vector
a vector with initial point $(0, 0)$

standard unit vectors
unit vectors along the coordinate axes: $\hat{\mathbf{i}} = \langle 1, 0 \rangle$, $\hat{\mathbf{j}} = \langle 0, 1 \rangle$

terminal point
the endpoint of a vector

triangle inequality
the length of any side of a triangle is less than the sum of the lengths of the other two sides

triangle method
a method for finding the sum of two vectors; position the vectors so the terminal point of one vector is the initial point of the other; these vectors then form two sides of a triangle; the sum of the vectors is the vector that forms the third side; the initial point of the sum is the initial point of the first vector; the terminal point of the sum is the terminal point of the second vector

unit vector
a vector with magnitude 1

vector
a mathematical object that has both magnitude and direction

vector addition
a vector operation that defines the sum of two vectors

vector difference
the vector difference $\vec{\mathbf{v}} - \vec{\mathbf{w}}$ is defined as $\vec{\mathbf{v}} + (-\vec{\mathbf{w}}) = \vec{\mathbf{v}} + (-1)\vec{\mathbf{w}}$

vector sum
the sum of two vectors, $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$, can be constructed graphically by placing the initial point of $\vec{\mathbf{w}}$ at the terminal point of $\vec{\mathbf{v}}$; then the vector sum $\vec{\mathbf{v}} + \vec{\mathbf{w}}$ is the vector with an initial point that coincides with the initial point of $\vec{\mathbf{v}}$, and with a terminal point that coincides with the terminal point of $\vec{\mathbf{w}}$

zero vector
the vector with both initial point and terminal point $(0, 0)$

Contributors and Attributions

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4.2E: Exercises for Section 4.2

For exercises 1 - 10, consider points $P(-1,3)$, $Q(1,5)$,and $R(-3,7)$. Determine the requested vectors and express each of them

a. in component form and

b. by using standard unit vectors.

1) $\overrightarrow{PQ}$

Answer

a. $\overrightarrow{PQ} = \langle 2, 2 \rangle$
b. $\overrightarrow{PQ} = 2\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$

2) $\overrightarrow{PR}$

3) $\overrightarrow{QP}$

Answer

a. $\overrightarrow{QP} = \langle -2, -2 \rangle$
b. $\overrightarrow{QP} = -2\hat{\mathbf{i}} - 2\hat{\mathbf{j}}$

4) $\overrightarrow{RP}$

5) $\overrightarrow{PQ} + \overrightarrow{PR}$

Answer

a. $\overrightarrow{PQ} + \overrightarrow{PR} = \langle 0, 6 \rangle$
b. $\overrightarrow{PQ} + \overrightarrow{PR} = 6\hat{\mathbf{j}}$

6) $\overrightarrow{PQ} - \overrightarrow{PR}$

7) $2\overrightarrow{PQ} - 2\overrightarrow{PR}$

Answer

a. $2\overrightarrow{PQ} \rightarrow -2\overrightarrow{PR} = \langle 8, -4 \rangle$
b. $2\overrightarrow{PQ} - 2\overrightarrow{PR} = 8\hat{\mathbf{i}} - 4\hat{\mathbf{j}}$

8) $2\overrightarrow{PQ} + \frac{1}{2}\overrightarrow{PR}$

9) The unit vector in the direction of $\overrightarrow{PQ}$

Answer

a. $\left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$
b. $\frac{\sqrt{2}}{2}\hat{\mathbf{i}} + \frac{\sqrt{2}}{2}\hat{\mathbf{j}}$

10) The unit vector in the direction of $\overrightarrow{PR}$

11) A vector $\vec{\mathbf{v}}$ has initial point $(-1, -3)$ and terminal point $(2, 1)$. Find the unit vector in the direction of $\vec{\mathbf{v}}$. Express the answer in component form.

Answer

$\langle \frac{3}{5}, \frac{4}{5} \rangle$

12) A vector $\vec{\mathbf{v}}$ has initial point $(-2, 5)$ and terminal point $(3, -1)$. Find the unit vector in the direction of $\vec{\mathbf{v}}$. Express the answer in component form.

13) The vector $\vec{\mathbf{v}}$ has initial point $P(1, 0)$ and terminal point Q that is on the y-axis and above the initial point. Find the coordinates of terminal point Q such that the magnitude of the vector $\vec{\mathbf{v}}$ is $\sqrt{5}$.

Answer

$Q(0, 2)$

14) The vector $\vec{\mathbf{v}}$ has initial point $P(1, 1)$ and terminal point Q that is on the x-axis and left of the initial point. Find the coordinates of terminal point Q such that the magnitude of the vector $\vec{\mathbf{v}}$ is $\sqrt{10}$.

For exercises 15 and 16, use the given vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$.

a. Determine the vector sum $\vec{\mathbf{a}} + \vec{\mathbf{b}}$ and express it in both the component form and by using the standard unit vectors.

b. Find the vector difference $\vec{\mathbf{a}} - \vec{\mathbf{b}}$ and express it in both the component form and by using the standard unit vectors.

c. Verify that the vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, and $\vec{\mathbf{a}} + \vec{\mathbf{b}}$, and, respectively, $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, and $\vec{\mathbf{a}} - \vec{\mathbf{b}}$ satisfy the triangle inequality.

d. Determine the vectors $2\vec{\mathbf{a}}$, $-\vec{\mathbf{b}}$, and $2\vec{\mathbf{a}} - \vec{\mathbf{b}}$. Express the vectors in both the component form and by using standard unit vectors.

15) $\vec{\mathbf{a}} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}}, \quad \vec{\mathbf{b}} = \hat{\mathbf{i}} + 3\hat{\mathbf{j}}$

Answer

a. $\vec{\mathbf{a}} + \vec{\mathbf{b}} = \langle 3, 4 \rangle, \quad \vec{\mathbf{a}} + \vec{\mathbf{b}} = 3\hat{\mathbf{i}} + 4\hat{\mathbf{j}}$
b. $\vec{\mathbf{a}} - \vec{\mathbf{b}} = \langle 1, -2 \rangle, \quad \vec{\mathbf{a}} - \vec{\mathbf{b}} = \hat{\mathbf{i}} - 2\hat{\mathbf{j}}$
c. Answers will vary
d. $2\vec{\mathbf{a}} = \langle 4, 2 \rangle, \quad 2\vec{\mathbf{a}} = 4\hat{\mathbf{i}} + 2\hat{\mathbf{j}}, \quad -\vec{\mathbf{b}} = \langle -1, -3 \rangle, \quad -\vec{\mathbf{b}} = -\hat{\mathbf{i}} - 3\hat{\mathbf{j}}, \quad 2\vec{\mathbf{a}} - \vec{\mathbf{b}} = \langle 3, -1 \rangle, \quad 2\vec{\mathbf{a}} - \vec{\mathbf{b}} = 3\hat{\mathbf{i}} - \hat{\mathbf{j}}$

16) $\vec{\mathbf{a}} = 2\hat{\mathbf{i}}, \quad \vec{\mathbf{b}} = -2\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$

17) Let $\vec{\mathbf{a}}$ be a standard-position vector with terminal point $(-2, -4)$. Let $\vec{\mathbf{b}}$ be a vector with initial point $(1, 2)$ and terminal point $(-1, 4)$. Find the magnitude of vector $-3\vec{\mathbf{a}} + \vec{\mathbf{b}} - 4\hat{\mathbf{i}} + \hat{\mathbf{j}}$.

Answer

15

18) Let $\vec{\mathbf{a}}$ be a standard-position vector with terminal point at $(2, 5)$. Let $\vec{\mathbf{b}}$ be a vector with initial point $(-1, 3)$ and terminal point $(1, 0)$. Find the magnitude of vector $\vec{\mathbf{a}} - 3\vec{\mathbf{b}} + 14\hat{\mathbf{i}} - 14\hat{\mathbf{j}}$.

19) Let $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ be two nonzero vectors that are nonequivalent. Consider the vectors $\vec{\mathbf{a}} = 4\vec{\mathbf{u}} + 5\vec{\mathbf{v}}$ and $\vec{\mathbf{b}} = \vec{\mathbf{u}} + 2\vec{\mathbf{v}}$ defined in terms of $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$. Find the scalar λ such that vectors $\vec{\mathbf{a}} + \lambda\vec{\mathbf{b}}$ and $\vec{\mathbf{u}} - \vec{\mathbf{v}}$ are equivalent.

Answer

$\lambda = -3$

20) Let $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ be two nonzero vectors that are nonequivalent. Consider the vectors $\vec{\mathbf{a}} = 2\vec{\mathbf{u}} - 4\vec{\mathbf{v}}$ and $\vec{\mathbf{b}} = 3\vec{\mathbf{u}} - 7\vec{\mathbf{v}}$ defined in terms of $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$. Find the scalars α and β such that vectors $\alpha\vec{\mathbf{a}} + \beta\vec{\mathbf{b}}$ and $\vec{\mathbf{u}} - \vec{\mathbf{v}}$ are equivalent.

21) Consider the vector $\vec{\mathbf{a}}(t) = \langle \cos t, \sin t \rangle$ with components that depend on a real number t. As the number t varies, the components of $\vec{\mathbf{a}}(t)$ change as well, depending on the functions that define them.

a. Write the vectors $\vec{\mathbf{a}}(0)$ and $\vec{\mathbf{a}}(\pi)$ in component form.

b. Show that the magnitude $\|\vec{\mathbf{a}}(t)\|$ of vector $\vec{\mathbf{a}}(t)$ remains constant for any real number t.

c. As t varies, show that the terminal point of vector $\vec{\mathbf{a}}(t)$ describes a circle centered at the origin of radius 1.

Answer

$a.\ \vec{\mathbf{a}}(0) = \langle 1, 0\rangle, \quad \vec{\mathbf{a}}(\pi) = \langle -1, 0\rangle$

$b.$ Answers may vary

$c.$ Answers may vary

22) Consider vector $\vec{\mathbf{a}}(x) = \langle x, \sqrt{1-x^2}\,\rangle$ with components that depend on a real number $x \in [-1, 1]$. As the number x varies, the components of $\vec{\mathbf{a}}(x)$ change as well, depending on the functions that define them.

a. Write the vectors $\vec{\mathbf{a}}(0)$ and $\vec{\mathbf{a}}(1)$ in component form.

b. Show that the magnitude $\|\vec{\mathbf{a}}(x)\|$ of vector $\vec{\mathbf{a}}(x)$ remains constant for any real number x.

c. As x varies, show that the terminal point of vector $\vec{\mathbf{a}}(x)$ describes a circle centered at the origin of radius 1.

23) Show that vectors $\vec{\mathbf{a}}(t) = \langle \cos t, \sin t\rangle$ and $\vec{\mathbf{a}}(x) = \langle x, \sqrt{1-x^2}\,\rangle$ are equivalent for $x = 1$ and $t = 2k\pi$, where k is an integer.

Answer Answers may vary

24) Show that vectors $\vec{\mathbf{a}}(t) = \langle \cos t, \sin t\rangle$ and $\vec{\mathbf{a}}(x) = \langle x, \sqrt{1-x^2}\,\rangle$ are opposite for $x = 1$ and $t = \pi + 2k\pi$, where k is an integer.

For exercises 25-28, find a vector $\vec{\mathbf{v}}$ with the given magnitude and in the same direction as the vector $\vec{\mathbf{u}}$.

25) $\|\vec{\mathbf{v}}\| = 7, \quad \vec{\mathbf{u}} = \langle 3, 4\rangle$

Answer

$\vec{\mathbf{v}} = \langle \frac{21}{5}, \frac{28}{5}\rangle$

26) $\|\vec{\mathbf{v}}\| = 3, \quad \vec{\mathbf{u}} = \langle -2, 5\rangle$

27) $\|\vec{\mathbf{v}}\| = 7, \quad \vec{\mathbf{u}} = \langle 3, -5\rangle$

Answer

$\vec{\mathbf{v}} = \langle \frac{21\sqrt{34}}{34}, -\frac{35\sqrt{34}}{34}\rangle$

28) $\|\vec{\mathbf{v}}\| = 10, \quad \vec{\mathbf{u}} = \langle 2, -1\rangle$

For exercises 29-34, find the component form of vector $\vec{\mathbf{u}}$, given its magnitude and the angle the vector makes with the positive x-axis. Give exact answers when possible.

29) $\|\vec{\mathbf{u}}\| = 2, \theta = 30^\circ$

Answer

$\vec{\mathbf{u}} = \langle \sqrt{3}, 1\rangle$

30) $\|\vec{\mathbf{u}}\| = 6, \theta = 60^\circ$

31) $\|\vec{\mathbf{u}}\| = 5, \theta = \frac{\pi}{2}$

Answer

$\vec{\mathbf{u}} = \langle 0, 5\rangle$

32) $\|\vec{\mathbf{u}}\| = 8, \theta = \pi$

33) $\|\vec{\mathbf{u}}\| = 10, \theta = \frac{5\pi}{6}$

Answer

$\vec{\mathbf{u}} = \langle -5\sqrt{3}, 5\rangle$

34) $\|\vec{\mathbf{u}}\| = 50, \theta = \frac{3\pi}{4}$

For exercises 35 and 36, vector $\vec{\mathbf{u}}$ is given. Find the angle $\theta \in [0, 2\pi)$ that vector $\vec{\mathbf{u}}$ makes with the positive direction of the x-axis, in a counter-clockwise direction.

35) $\vec{\mathbf{u}} = 5\sqrt{2}\hat{\mathbf{i}} - 5\sqrt{2}\hat{\mathbf{j}}$

Answer

$\theta = \frac{7\pi}{4}$

36) $\vec{\mathbf{u}} = -\sqrt{3}\hat{\mathbf{i}} - \hat{\mathbf{j}}$

37) Let $\vec{\mathbf{a}} = \langle a_1, a_2 \rangle, \vec{\mathbf{b}} = \langle b_1, b_2 \rangle$, and $\vec{\mathbf{c}} = \langle c_1, c_2 \rangle$ be three nonzero vectors. If $a_1 b_2 - a_2 b_1 \neq 0$, then show there are two scalars, α and β, such that $\vec{\mathbf{c}} = \alpha\vec{\mathbf{a}} + \beta\vec{\mathbf{b}}$.

Answer Answers may vary

38) Consider vectors $\vec{\mathbf{a}} = \langle 2, -4 \rangle, \vec{\mathbf{b}} = \langle -1, 2 \rangle,$ and $\vec{\mathbf{c}} = \vec{\mathbf{0}}$ Determine the scalars α and β such that $\vec{\mathbf{c}} = \alpha\vec{\mathbf{a}} + \beta\vec{\mathbf{b}}$.

39) Let $P(x_0, f(x_0))$ be a fixed point on the graph of the differentiable function f with a domain that is the set of real numbers.

a. Determine the real number z_0 such that point $Q(x_0 + 1, z_0)$ is situated on the line tangent to the graph of f at point P.

b. Determine the unit vector $\vec{\mathbf{u}}$ with initial point P and terminal point Q.

Answer

$a. \quad z_0 = f(x_0) + f\,'(x_0); \quad b. \quad \vec{\mathbf{u}} = \frac{1}{\sqrt{1+[f\,'(x_0)]^2}}\langle 1, f\,'(x_0) \rangle$

40) Consider the function $f(x) = x^4$, where $x \in R$.

a. Determine the real number z_0 such that point $Q(2, z_0)$ s situated on the line tangent to the graph of f at point $P(1, 1)$.

b. Determine the unit vector $\vec{\mathbf{u}}$ with initial point P and terminal point Q.

41) Consider f and g two functions defined on the same set of real numbers D. Let $\vec{\mathbf{a}} = \langle x, f(x) \rangle$ and $\vec{\mathbf{b}} = \langle x, g(x) \rangle$ be two vectors that describe the graphs of the functions, where $x \in D$. Show that if the graphs of the functions f and g do not intersect, then the vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are not equivalent.

42) Find $x \in R$ such that vectors $\vec{\mathbf{a}} = \langle x, \sin x \rangle$ and $\vec{\mathbf{b}} = \langle x, \cos x \rangle$ are equivalent.

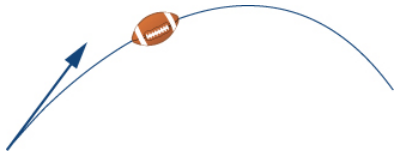
43) Calculate the coordinates of point D such that $ABCD$ is a parallelogram, with $A(1, 1), B(2, 4)$, and $C(7, 4)$.

Answer

$D(6, 1)$

44) Consider the points $A(2, 1), B(10, 6), C(13, 4)$ and $D(16, -2)$. Determine the component form of vector $\overrightarrow{AD}$.

45) The **speed** of an object is the magnitude of its related velocity vector. A football thrown by a quarterback has an initial speed of 70 mph and an angle of elevation of 30°. Determine the velocity vector in mph and express it in component form. (Round to two decimal places.)

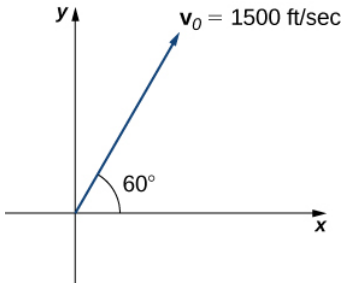


Answer

$\langle 60.62, 35 \rangle$

46) A baseball player throws a baseball at an angle of 30° with the horizontal. If the initial speed of the ball is 100 mph, find the horizontal and vertical components of the initial velocity vector of the baseball. (Round to two decimal places.)

47) A bullet is fired with an initial velocity of 1500 ft/sec at an angle of 60° with the horizontal. Find the horizontal and vertical components of the velocity vector of the bullet. (Round to two decimal places.)

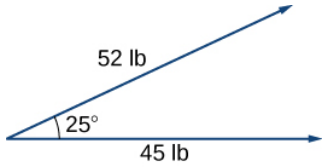


Answer

The horizontal and vertical components are 750 ft/sec and 1299.04ft/sec, respectively.

48) [T] A 65-kg sprinter exerts a force of 798 N at a 19° angle with respect to the ground on the starting block at the instant a race begins. Find the horizontal component of the force. (Round to two decimal places.)

49) [T] Two forces, a horizontal force of 45 lb and another of 52 lb, act on the same object. The angle between these forces is 25°. Find the magnitude and direction angle from the positive x-axis of the resultant force that acts on the object. (Round to two decimal places.)



Answer

The magnitude of resultant force is 94.71lb; the direction angle is 13.42°.

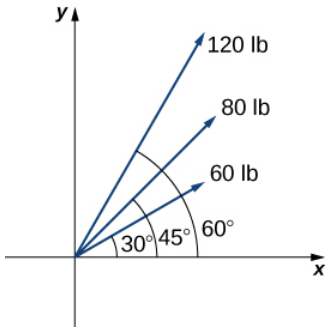
50) [T] Two forces, a vertical force of 26 lb and another of 45 lb, act on the same object. The angle between these forces is 55°. Find the magnitude and direction angle from the positive x-axis of the resultant force that acts on the object. (Round to two decimal places.)

51) [T] Three forces act on object. Two of the forces have the magnitudes 58 N and 27 N, and make angles 53° and 152°, respectively, with the positive x-axis. Find the magnitude and the direction angle from the positive x-axis of the third force such that the resultant force acting on the object is zero. (Round to two decimal places.)

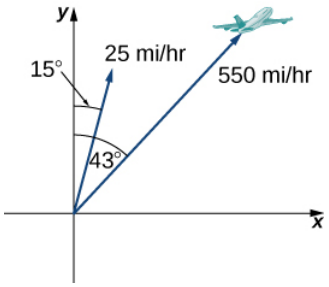
Answer

The magnitude of the third vector is 60.03N; the direction angle is 259.38°.

52) Three forces with magnitudes **80** lb, **120** lb, and **60** lb act on an object at angles of $45^\circ, 60^\circ$ and 30°, respectively, with the positive x-axis. Find the magnitude and direction angle from the positive x-axis of the resultant force. (Round to two decimal places.)



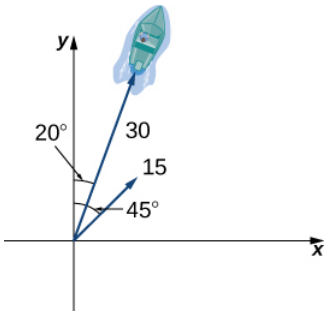
53) [T] An airplane is flying in the direction of 43° east of north (also abbreviated as $N43E$ at a speed of 550 mph. A wind with speed 25 mph comes from the southwest at a bearing of $N15E$. What are the ground speed and new direction of the airplane?



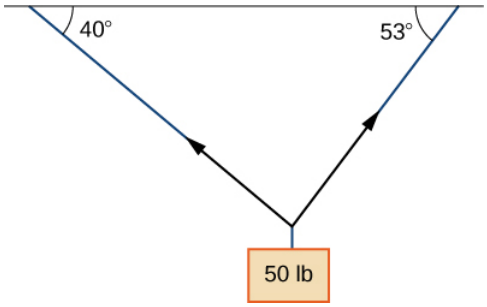
Answer

The new ground speed of the airplane is 572.19mph; the new direction is $N41.82E$.

54) [T] A boat is traveling in the water at 30 mph in a direction of $N20E$ (that is, 20° east of north). A strong current is moving at 15 mph in a direction of $N45E$. What are the new speed and direction of the boat?



55) [T] A 50-lb weight is hung by a cable so that the two portions of the cable make angles of 40° and 53°, respectively, with the horizontal. Find the magnitudes of the forces of tension $\vec{\mathbf{T}}_1$ and $\vec{\mathbf{T}}_2$ in the cables if the resultant force acting on the object is zero. (Round to two decimal places.)

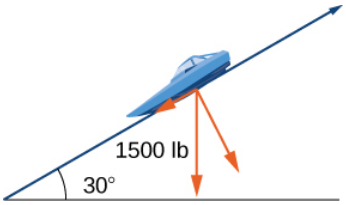


Answer

$\|\vec{\mathbf{T}}_1\| = 30.13\,lb, \quad \|\vec{\mathbf{T}}_2\| = 38.35\,lb$

56) [T] A 62-lb weight hangs from a rope that makes the angles of 29° and 61°, respectively, with the horizontal. Find the magnitudes of the forces of tension $\vec{\mathbf{T}}_1$ and $\vec{\mathbf{T}}_2$ in the cables if the resultant force acting on the object is zero. (Round to two decimal places.)

57) [T] A 1500-lb boat is parked on a ramp that makes an angle of 30° with the horizontal. The boat's weight vector points downward and is a sum of two vectors: a horizontal vector $\vec{\mathbf{v}}_1$ that is parallel to the ramp and a vertical vector $\vec{\mathbf{v}}_2$ that is perpendicular to the inclined surface. The magnitudes of vectors $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$ are the horizontal and vertical component, respectively, of the boat's weight vector. Find the magnitudes of $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$. (Round to the nearest integer.)

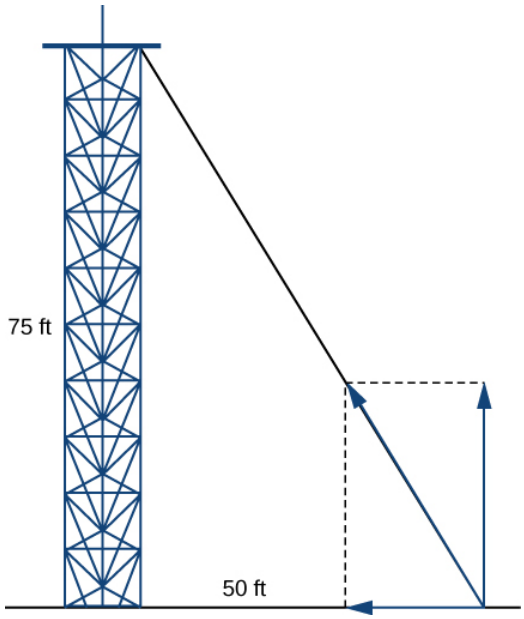


Answer

$\|\vec{\mathbf{v}}_1\| = 750\,lb, \quad \|\vec{\mathbf{v}}_2\| = 1299\,lb$

58) [T] An 85-lb box is at rest on a 26° incline. Determine the magnitude of the force parallel to the incline necessary to keep the box from sliding. (Round to the nearest integer.)

59) A guy-wire supports a pole that is 75 ft high. One end of the wire is attached to the top of the pole and the other end is anchored to the ground 50 ft from the base of the pole. Determine the horizontal and vertical components of the force of tension in the wire if its magnitude is 50 lb. (Round to the nearest integer.)



Answer

The two horizontal and vertical components of the force of tension are 28 lb and 42 lb, respectively.

60) A telephone pole guy-wire has an angle of elevation of 35° with respect to the ground. The force of tension in the guy-wire is 120 lb. Find the horizontal and vertical components of the force of tension. (Round to the nearest integer.)

Contributors

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4.3: Vectors in Three Dimensions

Learning Objectives

- Describe three-dimensional space mathematically.
- Locate points in space using coordinates.
- Write the distance formula in three dimensions.
- Write the equations for simple planes and spheres.
- Perform vector operations in $\mathbb{R}^3$.

Vectors are useful tools for solving two-dimensional problems. Life, however, happens in three dimensions. To expand the use of vectors to more realistic applications, it is necessary to create a framework for describing three-dimensional space. For example, although a two-dimensional map is a useful tool for navigating from one place to another, in some cases the topography of the land is important. Does your planned route go through the mountains? Do you have to cross a river? To appreciate fully the impact of these geographic features, you must use three dimensions. This section presents a natural extension of the two-dimensional Cartesian coordinate plane into three dimensions.

Three-Dimensional Coordinate Systems

As we have learned, the two-dimensional rectangular coordinate system contains two perpendicular axes: the horizontal x-axis and the vertical y-axis. We can add a third dimension, the z-axis, which is perpendicular to both the x-axis and the y-axis. We call this system the three-dimensional rectangular coordinate system. It represents the three dimensions we encounter in real life.

Definition: Three-dimensional Rectangular Coordinate System

The three-dimensional rectangular coordinate system consists of three perpendicular axes: the x-axis, the y-axis, and the z-axis. Because each axis is a number line representing all real numbers in $\mathbb{R}$, the three-dimensional system is often denoted by $\mathbb{R}^3$.

In Figure $4.3.1a$, the positive z-axis is shown above the plane containing the x- and y-axes. The positive x-axis appears to the left and the positive y-axis is to the right. A natural question to ask is: How was this arrangement determined? The system displayed follows the **right-hand rule**. If we take our right hand and align the fingers with the positive x-axis, then curl the fingers so they point in the direction of the positive y-axis, our thumb points in the direction of the positive z-axis (Figure $4.3.1b$). In this text, we always work with coordinate systems set up in accordance with the right-hand rule. Some systems do follow a left-hand rule, but the right-hand rule is considered the standard representation.

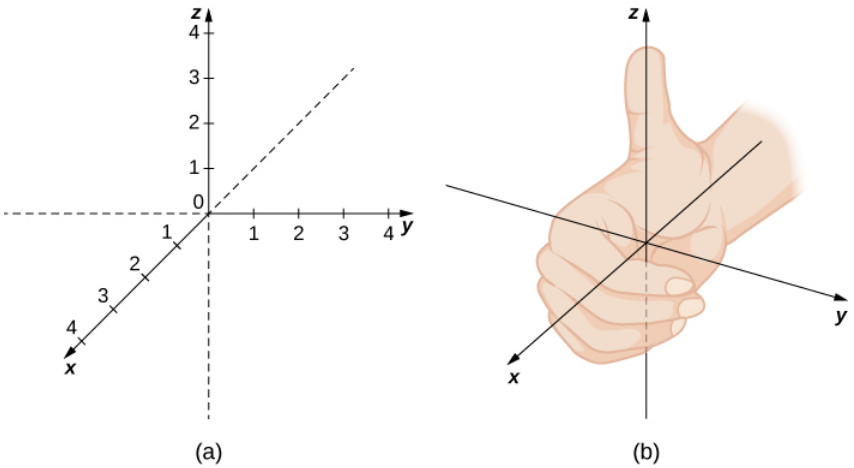


Figure 4.3.1: (a) We can extend the two-dimensional rectangular coordinate system by adding a third axis, the z-axis, that is perpendicular to both the x-axis and the y-axis. (b) The right-hand rule is used to determine the placement of the coordinate axes in the standard Cartesian plane.

In two dimensions, we describe a point in the plane with the coordinates (x, y). Each coordinate describes how the point aligns with the corresponding axis. In three dimensions, a new coordinate, z, is appended to indicate alignment with the z-axis: (x, y, z).

A point in space is identified by all three coordinates (Figure 4.3.2). To plot the point (x, y, z), go x units along the x-axis, then y units in the direction of the y-axis, then z units in the direction of the z-axis.

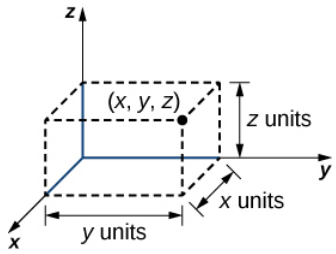
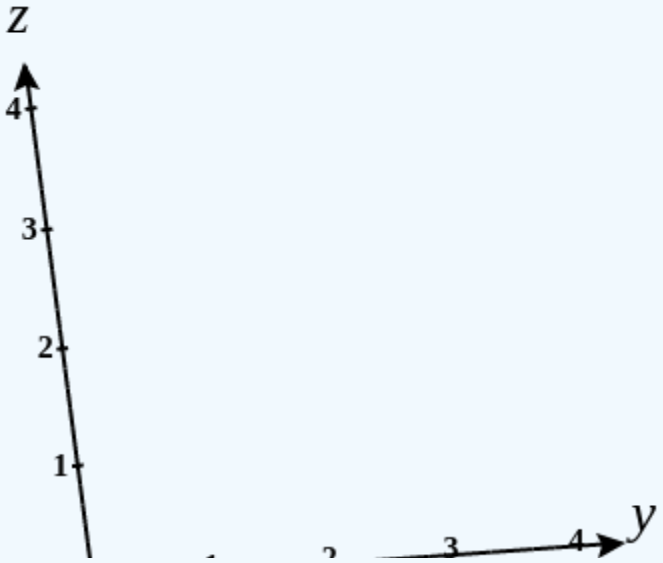


Figure 4.3.2: To plot the point (x, y, z) go x units along the x-axis, then y units in the direction of the y-axis, then z units in the direction of the z-axis.

Sketching the Vector <2,1,3>

Below is an example of sketching the vector <2,1,3>. Notice that to draw <2,1,3> you start at the origin, (0,0,0), move to (2,0,0), then move to (2,1,0), and then finally arrive at the point (2,1,3).



Start Drawing

Example 4.3.1: Locating Points in Space

Sketch the point $(1, -2, 3)$ in three-dimensional space.

Solution

To sketch a point, start by sketching three sides of a rectangular prism along the coordinate axes: one unit in the positive x direction, 2 units in the negative y direction, and 3 units in the positive z direction. Complete the prism to plot the point (Figure).

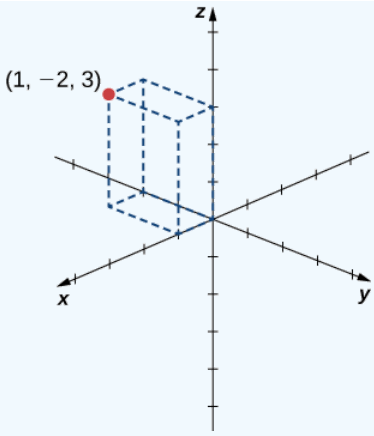


Figure 4.3.3: Sketching the point $(1, -2, 3)$.



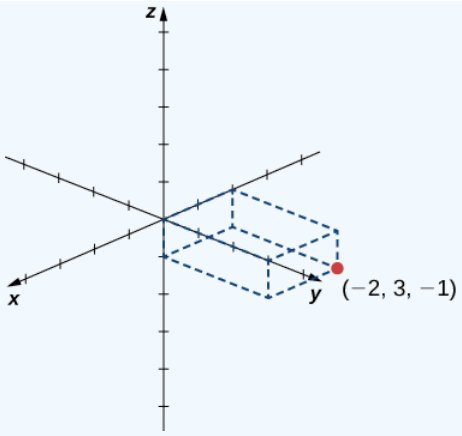
Exercise 4.3.1

Sketch the point $(-2, 3, -1)$ in three-dimensional space.

Hint

Start by sketching the coordinate axes. e.g., Figure 4.3.3 Then sketch a rectangular prism to help find the point in space.

Answer



In two-dimensional space, the coordinate plane is defined by a pair of perpendicular axes. These axes allow us to name any location within the plane. In three dimensions, we define **coordinate planes** by the coordinate axes, just as in two dimensions. There are three axes now, so there are three intersecting pairs of axes. Each pair of axes forms a coordinate plane: the xy-plane, the xz-plane, and the yz-plane (Figure 4.3.3). We define the xy-plane formally as the following set: $\{(x, y, 0) : x, y \in \mathbb{R}\}$. Similarly, the xz-plane and the yz-plane are defined as $\{(x, 0, z) : x, z \in \mathbb{R}\}$ and $\{(0, y, z) : y, z \in \mathbb{R}\}$, respectively.

To visualize this, imagine you're building a house and are standing in a room with only two of the four walls finished. (Assume the two finished walls are adjacent to each other.) If you stand with your back to the corner where the two finished walls meet, facing out into the room, the floor is the xy-plane, the wall to your right is the xz-plane, and the wall to your left is the yz-plane.

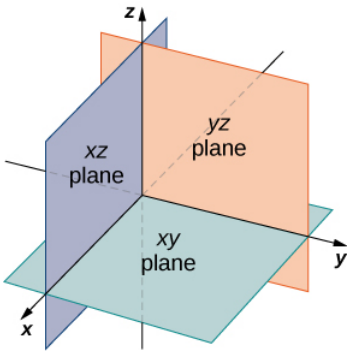


Figure 4.3.3: The plane containing the x- and y-axes is called the xy-plane. The plane containing the x- and z-axes is called the xz-plane, and the y- and z-axes define the yz-plane.

In two dimensions, the coordinate axes partition the plane into four quadrants. Similarly, the coordinate planes divide space between them into eight regions about the origin, called **octants.** The octants fill $\mathbb{R}^3$ in the same way that quadrants fill $\mathbb{R}^2$, as shown in Figure 4.3.4.

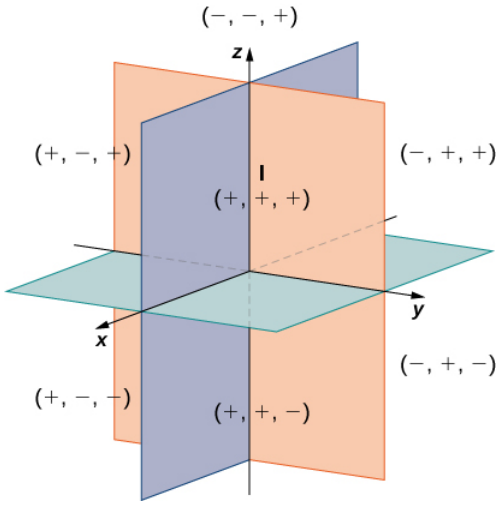


Figure 4.3.4: Points that lie in octants have three nonzero coordinates.

Most work in three-dimensional space is a comfortable extension of the corresponding concepts in two dimensions. In this section, we use our knowledge of circles to describe spheres, then we expand our understanding of vectors to three dimensions. To accomplish these goals, we begin by adapting the distance formula to three-dimensional space.

If two points lie in the same coordinate plane, then it is straightforward to calculate the distance between them. We know that the distance d between two points (x_1, y_1) and (x_2, y_2) in the xy-coordinate plane is given by the formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\,. \tag{4.3.1}$$

The formula for the distance between two points in space is a natural extension of this formula.

The Distance between Two Points in Space

The distance d between points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by the formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}\,. \tag{4.3.2}$$

The proof of this theorem is left as an exercise. (Hint: First find the distance d_1 between the points (x_1, y_1, z_1) and (x_2, y_2, z_1) as shown in Figure 4.3.5.)

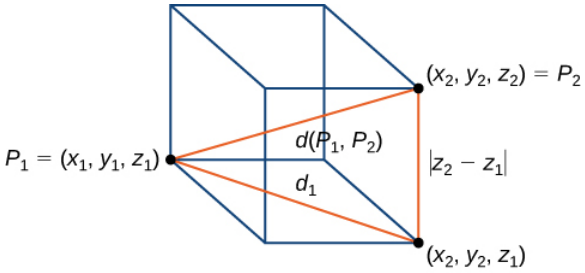


Figure 4.3.5: The distance between P_1 and P_2 is the length of the diagonal of the rectangular prism having P_1 and P_2 as opposite corners.

Example 4.3.2: Distance in Space

Find the distance between points $P_1 = (3, -1, 5)$ and $P_2 = (2, 1, -1)$.

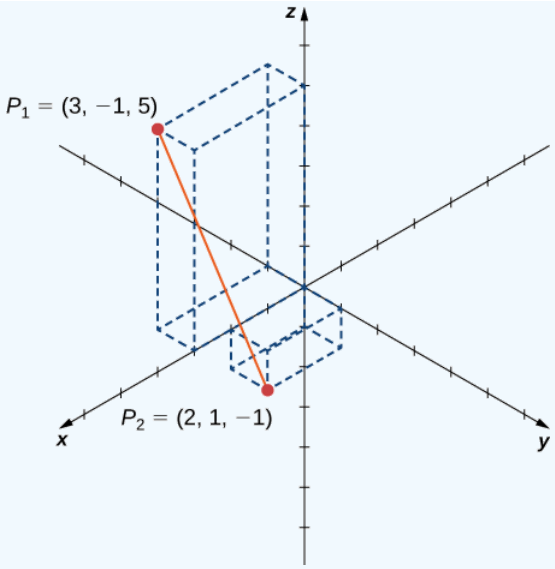


Figure 4.3.6: Find the distance between the two points.

Solution

Substitute values directly into the distance formula (Equation 4.3.2):

$$\begin{aligned} d(P_1, P_2) &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \\ &= \sqrt{(2-3)^2 + (1-(-1))^2 + (-1-5)^2} \\ &= \sqrt{(-1)^2 + 2^2 + (-6)^2} \\ &= \sqrt{41}. \end{aligned}$$



Exercise 4.3.2

Find the distance between points $P_1 = (1, -5, 4)$ and $P_2 = (4, -1, -1)$.

Hint

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Answer

$$5\sqrt{2}$$

Before moving on to the next section, let's get a feel for how $\mathbb{R}^3$ differs from $\mathbb{R}^2$. For example, in $\mathbb{R}^2$, lines that are not parallel must always intersect. This is not the case in $\mathbb{R}^3$. For example, consider the line shown in Figure 4.3.7. These two lines are not parallel, nor do they intersect.

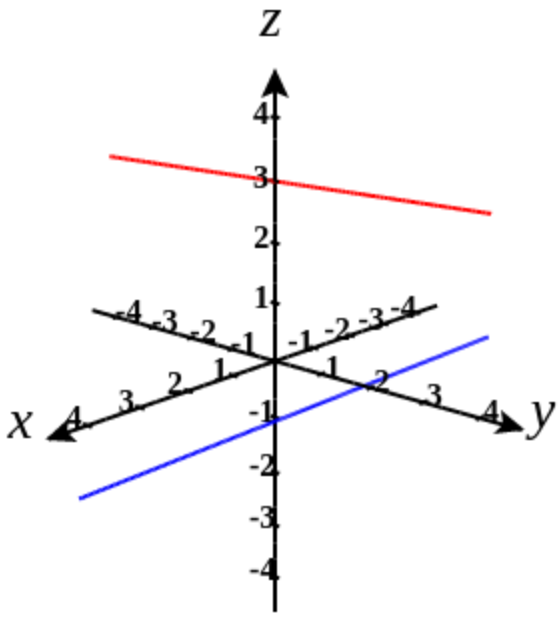


***Figure* 4.3.7:** *These two lines are not parallel, but still do not intersect.*

You can also have circles that are interconnected but have no points in common, as in Figure 4.3.8.

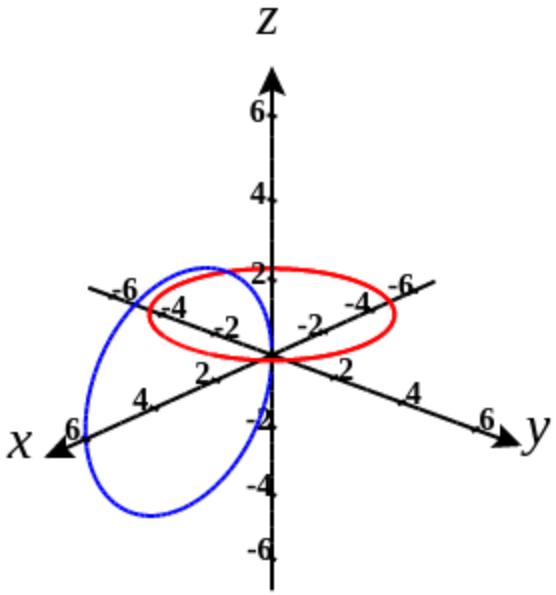


Figure* 4.3.8*: *These circles are interconnected, but have no points in common.*

We have a lot more flexibility working in three dimensions than we do if we stuck with only two dimensions.

Writing Equations in $\mathbb{R}^3$

Now that we can represent points in space and find the distance between them, we can learn how to write equations of geometric objects such as lines, planes, and curved surfaces in $\mathbb{R}^3$. First, we start with a simple equation. Compare the graphs of the equation $x = 0$ in $\mathbb{R}$, $\mathbb{R}^2$,and $\mathbb{R}^3$ (Figure 4.3.9). From these graphs, we can see the same equation can describe a point, a line, or a plane.

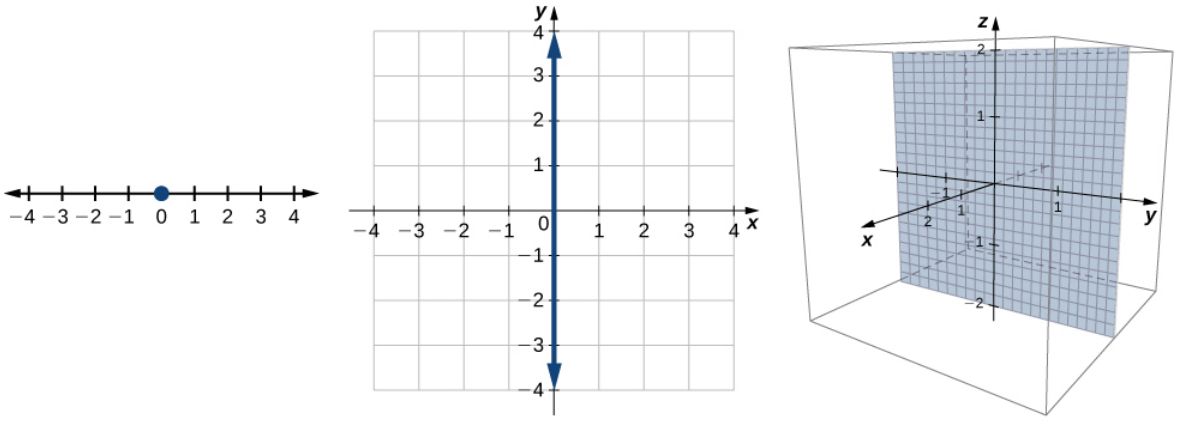


Figure 4.3.9: (a) In $\mathbb{R}$, the equation $x = 0$ describes a single point. (b) In $\mathbb{R}^2$, the equation $x = 0$ describes a line, the y-axis. (c) In $\mathbb{R}^3$, the equation $x = 0$ describes a plane, the yz-plane.

In space, the equation $x = 0$ describes all points $(0, y, z)$. This equation defines the yz-plane. Similarly, the xy-plane contains all points of the form $(x, y, 0)$. The equation $z = 0$ defines the xy-plane and the equation $y = 0$ describes the xz-plane (Figure 4.3.10).

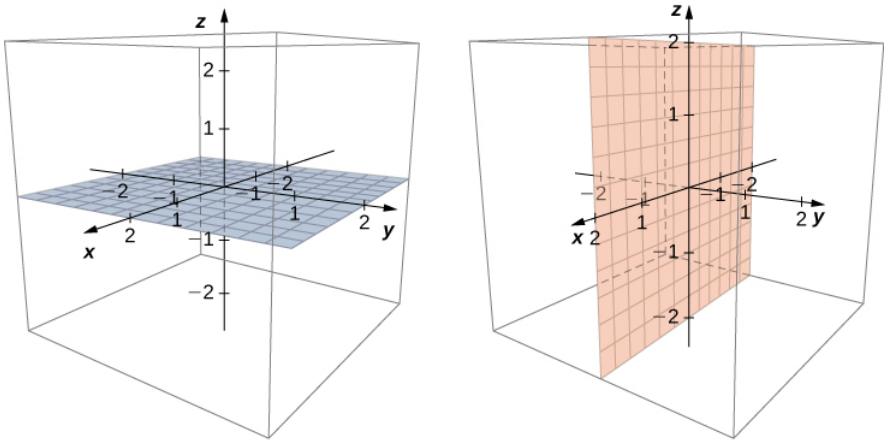


Figure 4.3.10: (a) In space, the equation $z = 0$ describes the xy-plane. (b) All points in the xz-plane satisfy the equation $y = 0$.

Understanding the equations of the coordinate planes allows us to write an equation for any plane that is parallel to one of the coordinate planes. When a plane is parallel to the xy-plane, for example, the z-coordinate of each point in the plane has the same constant value. Only the x- and y-coordinates of points in that plane vary from point to point.

Equations of Planes Parallel to Coordinate Planes

1. The plane in space that is parallel to the xy-plane and contains point (a, b, c) can be represented by the equation $z = c$.
2. The plane in space that is parallel to the xz-plane and contains point (a, b, c) can be represented by the equation $y = b$.
3. The plane in space that is parallel to the yz-plane and contains point (a, b, c) can be represented by the equation $x = a$.

Example 4.3.3: Writing Equations of Planes Parallel to Coordinate Planes

a. Write an equation of the plane passing through point $(3, 11, 7)$ that is parallel to the yz-plane.
b. Find an equation of the plane passing through points $(6, -2, 9)$, $(0, -2, 4)$,and $(1, -2, -3)$.

Solution

a. When a plane is parallel to the yz-plane, only the y- and z-coordinates may vary. The x-coordinate has the same constant value for all points in this plane, so this plane can be represented by the equation $x = 3$.
b. Each of the points $(6, -2, 9)$, $(0, -2, 4)$,and $(1, -2, -3)$ has the same y-coordinate. This plane can be represented by the equation $y = -2$.

Exercise 4.3.3

Write an equation of the plane passing through point $(1, -6, -4)$ that is parallel to the xy-plane.

Hint

If a plane is parallel to the xy-plane, the z-coordinates of the points in that plane do not vary.

Answer

$$z = -4$$



As we have seen, in $\mathbb{R}^2$ the equation $x = 5$ describes the vertical line passing through point $(5, 0)$. This line is parallel to the y-axis. In a natural extension, the equation $x = 5$ in $\mathbb{R}^3$ describes the plane passing through point $(5, 0, 0)$, which is parallel to the yz-plane. Another natural extension of a familiar equation is found in the equation of a sphere.

Definition: Sphere

A sphere is the set of all points in space equidistant from a fixed point, the center of the sphere (Figure 4.3.11), just as the set of all points in a plane that are equidistant from the center represents a circle. In a sphere, as in a circle, the distance from the center to a point on the sphere is called the *radius.*

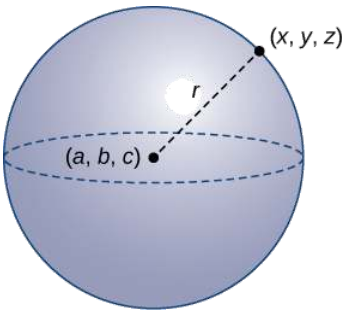


Figure 4.3.11: Each point (x, y, z) on the surface of a sphere is r units away from the center (a, b, c).

The equation of a circle is derived using the distance formula in two dimensions. In the same way, the equation of a sphere is based on the three-dimensional formula for distance.

Standard Equation of a Sphere

The sphere with center (a, b, c) and radius r can be represented by the equation

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2. \tag{4.3.3}$$

This equation is known as the *standard equation of a sphere*.

Example 4.3.4: Finding an Equation of a Sphere

Find the standard equation of the sphere with center $(10, 7, 4)$ and point $(-1, 3, -2)$, as shown in Figure 4.3.12

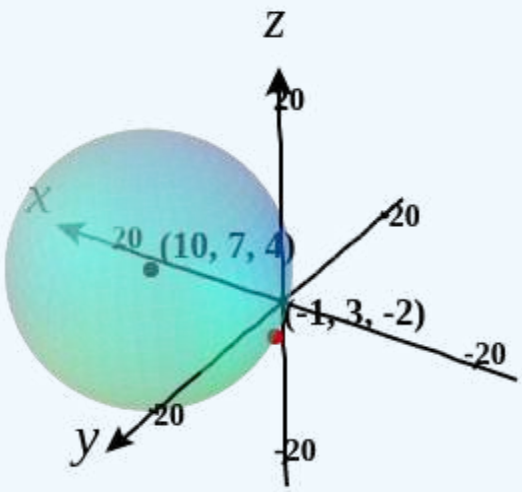


***Figure* 4.3.12:** *The sphere centered at* $(10, 7, 4)$ *containing point* $(-1, 3, -2)$.

Solution

Use the distance formula to find the radius r of the sphere:

$$\begin{aligned} r &= \sqrt{(-1-10)^2+(3-7)^2+(-2-4)^2} \\ &= \sqrt{(-11)^2+(-4)^2+(-6)^2} \\ &= \sqrt{173} \end{aligned}$$

The standard equation of the sphere is

$$(x-10)^2+(y-7)^2+(z-4)^2 = 173.$$

Exercise 4.3.4

Find the standard equation of the sphere with center $(-2, 4, -5)$ containing point $(4, 4, -1)$.

Hint

First use the distance formula to find the radius of the sphere.

Answer

$$(x+2)^2+(y-4)^2+(z+5)^2 = 52$$

Example 4.3.5: Finding the Equation of a Sphere

Let $P=(-5,2,3)$ and $Q=(3,4,-1)$, and suppose line segment $\overline{PQ}$ forms the diameter of a sphere (Figure 4.3.13). Find the equation of the sphere.

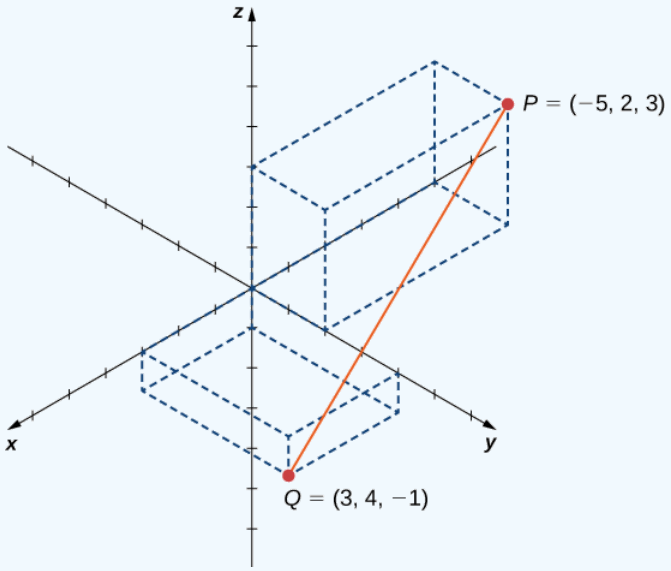


Figure 4.3.13: Line segment $\overline{PQ}$.

Solution:

Since $\overline{PQ}$ is a diameter of the sphere, we know the center of the sphere is the midpoint of $\overline{PQ}$.Then,

$$C=\left(\frac{-5+3}{2},\frac{2+4}{2},\frac{3+(-1)}{2}\right)=(-1,3,1).$$

Furthermore, we know the radius of the sphere is half the length of the diameter. This gives

$$\begin{aligned} r &= \frac{1}{2}\sqrt{(-5-3)^2+(2-4)^2+(3-(-1))^2} \\ &= \frac{1}{2}\sqrt{64+4+16} \\ &= \sqrt{21} \end{aligned}$$

Then, the equation of the sphere is $(x+1)^2+(y-3)^2+(z-1)^2=21$.

Exercise 4.3.5

Find the equation of the sphere with diameter $\overline{PQ}$, where $P=(2,-1,-3)$ and $Q=(-2,5,-1)$.

Hint

Find the midpoint of the diameter first.

Answer

$$x^2+(y-2)^2+(z+2)^2=14$$

Example 4.3.6: Graphing Other Equations in Three Dimensions

Describe the set of points that satisfies $(x-4)(z-2)=0$, and graph the set.

Solution

We must have either $x-4=0$ or $z-2=0$, so the set of points forms the two planes $x=4$ and $z=2$ (Figure 4.3.14).

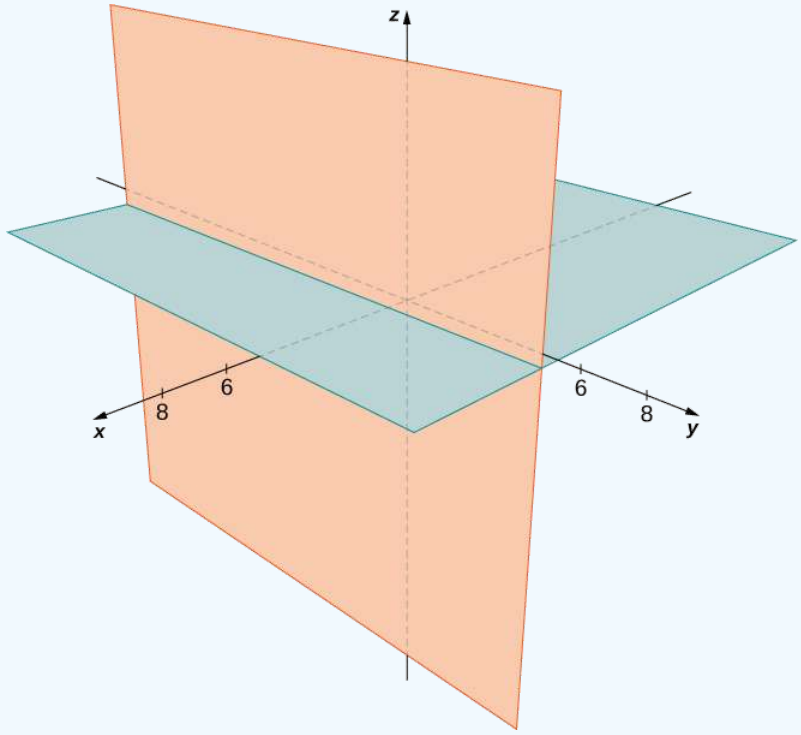


Figure 4.3.14: The set of points satisfying $(x-4)(z-2)=0$ forms the two planes $x=4$ and $z=2$.

Exercise 4.3.6

Describe the set of points that satisfies $(y+2)(z-3)=0$, and graph the set.

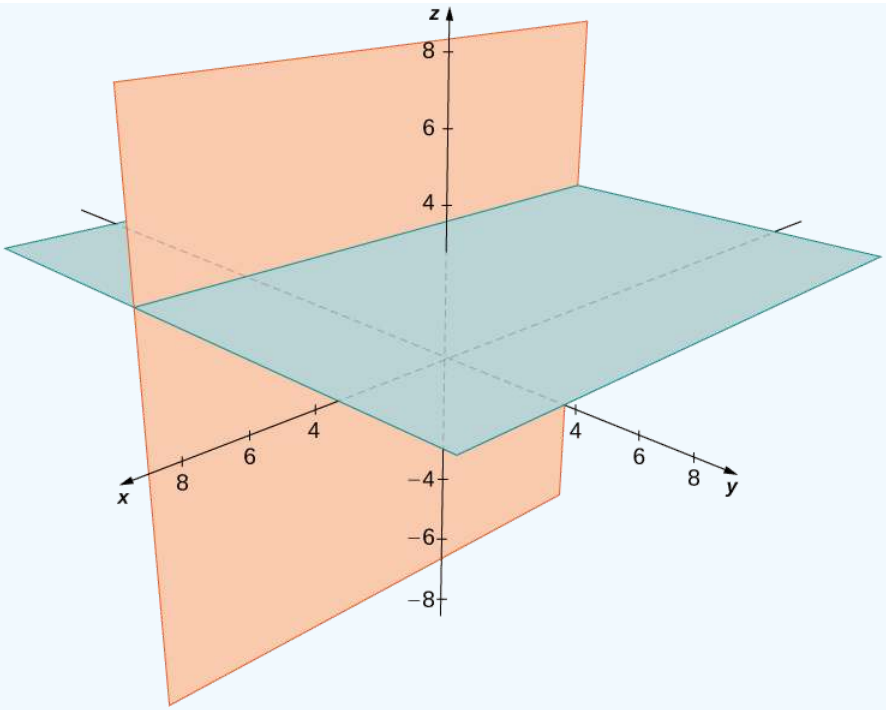
Hint

One of the factors must be zero.

Answer

The set of points forms the two planes $y=-2$ and $z=3$.

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Example 4.3.7: Graphing Other Equations in Three Dimensions

Describe the set of points in three-dimensional space that satisfies $(x-2)^2+(y-1)^2=4$, and graph the set.

Solution

The x- and y-coordinates form a circle in the xy-plane of radius 2, centered at $(2,1)$. Since there is no restriction on the z-coordinate, the three-dimensional result is a circular cylinder of radius 2 centered on the line with $x=2$ and $y=1$. The cylinder extends indefinitely in the z-direction (Figure 4.3.15).

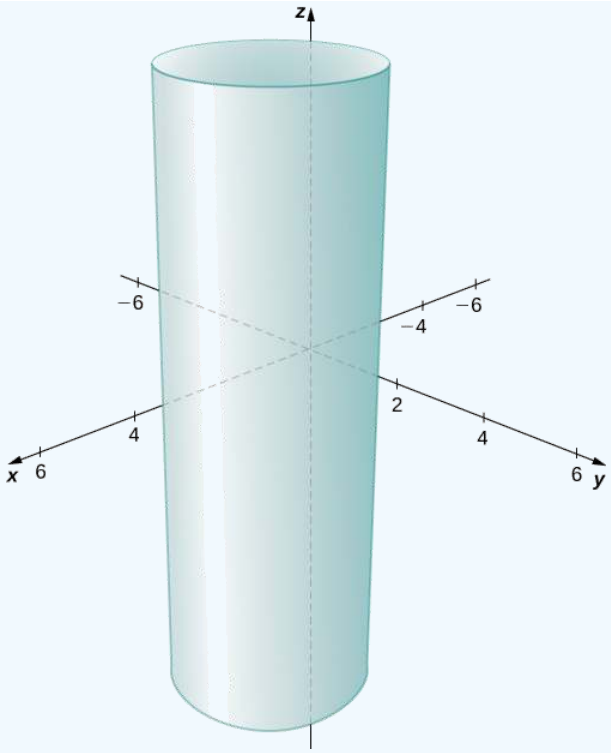


Figure 4.3.15: The set of points satisfying $(x-2)^2+(y-1)^2=4$. This is a cylinder of radius 2 centered on the line with $x=2$ and $y=1$.

Exercise 4.3.7

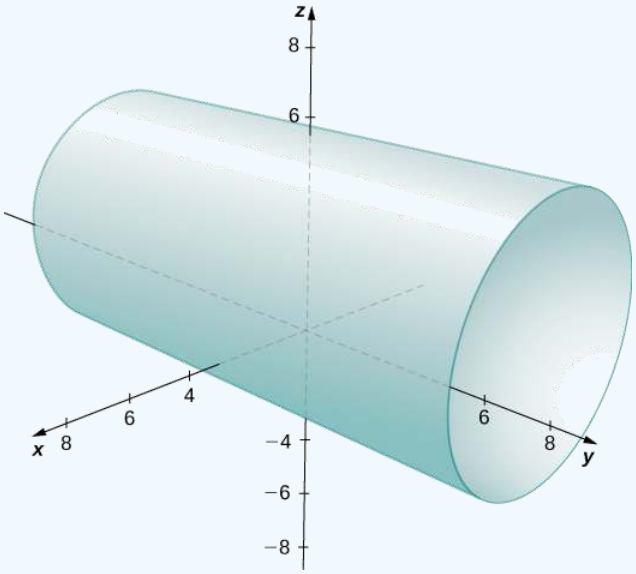
Describe the set of points in three dimensional space that satisfies $x^2+(z-2)^2=16$, and graph the surface.

Hint

Think about what happens if you plot this equation in two dimensions in the xz-plane.

Answer

A cylinder of radius 4 centered on the line with $x=0$ and $z=2$.



Working with Vectors in $\mathbb{R}^3$

Just like two-dimensional vectors, three-dimensional vectors are quantities with both magnitude and direction, and they are represented by directed line segments (arrows). With a three-dimensional vector, we use a three-dimensional arrow.

Three-dimensional vectors can also be represented in component form. The notation $\vec{\mathbf{v}} = \langle x, y, z \rangle$ is a natural extension of the two-dimensional case, representing a vector with the initial point at the origin, $(0, 0, 0)$, and terminal point (x, y, z). The zero vector is $\vec{\mathbf{0}} = \langle 0, 0, 0 \rangle$. So, for example, the three dimensional vector $\vec{\mathbf{v}} = \langle 2, 4, 1 \rangle$ is represented by a directed line segment from point $(0, 0, 0)$ to point $(2, 4, 1)$ (Figure 4.3.16).

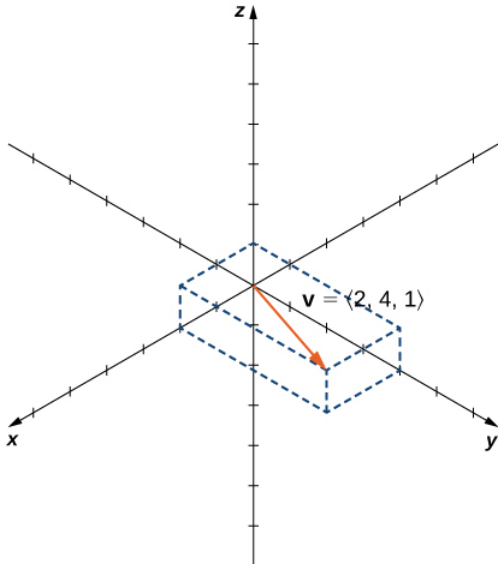


Figure 4.3.16: Vector $\vec{\mathbf{v}} = \langle 2, 4, 1 \rangle$ is represented by a directed line segment from point $(0, 0, 0)$ to point $(2, 4, 1)$.

Vector addition and scalar multiplication are defined analogously to the two-dimensional case. If $\vec{\mathbf{v}} = \langle x_1, y_1, z_1 \rangle$ and $\vec{\mathbf{w}} = \langle x_2, y_2, z_2 \rangle$ are vectors, and k is a scalar, then

$$\vec{\mathbf{v}} + \vec{\mathbf{w}} = \langle x_1 + x_2, y_1 + y_2, z_1 + z_2 \rangle \tag{4.3.4}$$

and

$$k\vec{\mathbf{v}} = \langle kx_1, ky_1, kz_1 \rangle. \tag{4.3.5}$$

If $k = -1$, then $k\vec{\mathbf{v}} = (-1)\vec{\mathbf{v}}$ is written as $-\vec{\mathbf{v}}$, and vector subtraction is defined by $\vec{\mathbf{v}} - \vec{\mathbf{w}} = \vec{\mathbf{v}} + (-\vec{\mathbf{w}}) = \vec{\mathbf{v}} + (-1)\vec{\mathbf{w}}$.

The standard unit vectors extend easily into three dimensions as well, $\hat{\mathbf{i}} = \langle 1, 0, 0 \rangle$, $\hat{\mathbf{j}} = \langle 0, 1, 0 \rangle$, and $\hat{\mathbf{k}} = \langle 0, 0, 1 \rangle$, and we use them in the same way we used the standard unit vectors in two dimensions. Thus, we can represent a vector in $\mathbb{R}^3$ in the following ways:

$$\vec{\mathbf{v}} = \langle x, y, z \rangle = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} \tag{4.3.6}$$

.

Example 4.3.8: Vector Representations

Let $\overrightarrow{PQ}$ be the vector with initial point $P = (3, 12, 6)$ and terminal point $Q = (-4, -3, 2)$ as shown in Figure 4.3.17. Express $\overrightarrow{PQ}$ in both component form and using standard unit vectors.

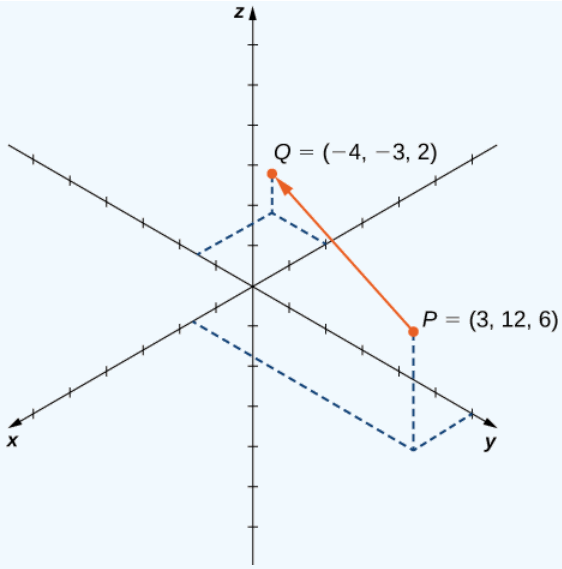


Figure 4.3.17: The vector with initial point $P=(3,12,6)$ and terminal point $Q=(-4,-3,2)$.

Solution

In component form,

$$\begin{aligned}\overrightarrow{PQ} &= \langle x_2-x_1, y_2-y_1, z_2-z_1\rangle \\ &= \langle -4-3, -3-12, 2-6\rangle \\ &= \langle -7, -15, -4\rangle.\end{aligned}$$

In standard unit form,

$$\overrightarrow{PQ} = -7\hat{\mathbf{i}} - 15\hat{\mathbf{j}} - 4\hat{\mathbf{k}}.$$

Exercise 4.3.8

Let $S=(3,8,2)$ and $T=(2,-1,3)$. Express $\overrightarrow{ST}$ in component form and in standard unit form.

Hint

Write $\overrightarrow{ST}$ in component form first. T is the terminal point of $\overrightarrow{ST}$.

Answer

$\overrightarrow{ST} = \langle -1, -9, 1\rangle = -\hat{\mathbf{i}} - 9\hat{\mathbf{j}} + \hat{\mathbf{k}}$

As described earlier, vectors in three dimensions behave in the same way as vectors in a plane. The geometric interpretation of vector addition, for example, is the same in both two- and three-dimensional space (Figure 4.3.18).

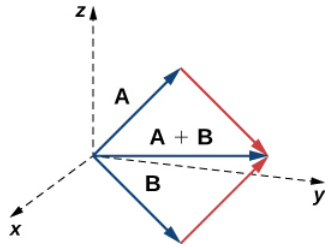


Figure 4.3.18: To add vectors in three dimensions, we follow the same procedures we learned for two dimensions.

We have already seen how some of the algebraic properties of vectors, such as vector addition and scalar multiplication, can be extended to three dimensions. Other properties can be extended in similar fashion. They are summarized here for our reference.

Properties of Vectors in Space

Let $\vec{\mathbf{v}} = \langle x_1, y_1, z_1 \rangle$ and $\vec{\mathbf{w}} = \langle x_2, y_2, z_2 \rangle$ be vectors, and let k be a scalar.

- **Scalar multiplication**:

$$k\vec{\mathbf{v}} = \langle kx_1, ky_1, kz_1 \rangle \tag{4.3.7}$$

- **Vector addition**:

$$\vec{\mathbf{v}} + \vec{\mathbf{w}} = \langle x_1, y_1, z_1 \rangle + \langle x_2, y_2, z_2 \rangle = \langle x_1 + x_2, y_1 + y_2, z_1 + z_2 \rangle \tag{4.3.8}$$

- **Vector subtraction**:

$$\vec{\mathbf{v}} - \vec{\mathbf{w}} = \langle x_1, y_1, z_1 \rangle - \langle x_2, y_2, z_2 \rangle = \langle x_1 - x_2, y_1 - y_2, z_1 - z_2 \rangle \tag{4.3.9}$$

- **Vector magnitude**:

$$\|\vec{\mathbf{v}}\| = \sqrt{x_1^2 + y_1^2 + z_1^2} \tag{4.3.10}$$

- **Unit vector in the direction of $\vec{\mathbf{v}}$**:

$$\frac{1}{\|\vec{\mathbf{v}}\|}\vec{\mathbf{v}} = \frac{1}{\|\vec{\mathbf{v}}\|}\langle x_1, y_1, z_1 \rangle = \langle \frac{x_1}{\|\vec{\mathbf{v}}\|}, \frac{y_1}{\|\vec{\mathbf{v}}\|}, \frac{z_1}{\|\vec{\mathbf{v}}\|} \rangle, \quad \text{if } \vec{\mathbf{v}} \neq \vec{\mathbf{0}} \tag{4.3.11}$$

We have seen that vector addition in two dimensions satisfies the commutative, associative, and additive inverse properties. These properties of vector operations are valid for three-dimensional vectors as well. Scalar multiplication of vectors satisfies the distributive property, and the zero vector acts as an additive identity. The proofs to verify these properties in three dimensions are straightforward extensions of the proofs in two dimensions.

Example 4.3.9: Vector Operations in Three Dimensions

Let $\vec{\mathbf{v}} = \langle -2, 9, 5 \rangle$ and $\vec{\mathbf{w}} = \langle 1, -1, 0 \rangle$ (Figure 4.3.19). Find the following vectors.

a. $3\vec{\mathbf{v}} - 2\vec{\mathbf{w}}$
b. $5\|\vec{\mathbf{w}}\|$
c. $\|5\vec{\mathbf{w}}\|$
d. A unit vector in the direction of $\vec{\mathbf{v}}$

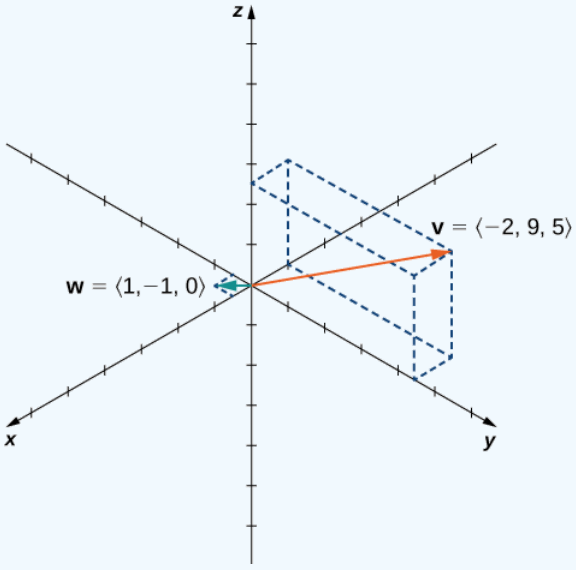


Figure 4.3.19: The vectors $\vec{\mathbf{v}} = \langle -2, 9, 5 \rangle$ *and* $\vec{\mathbf{w}} = \langle 1, -1, 0 \rangle$.

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Solution

a. First, use scalar multiplication of each vector, then subtract:

$$\begin{aligned} 3\vec{\mathbf{v}} - 2\vec{\mathbf{w}} &= 3\langle -2, 9, 5\rangle - 2\langle 1, -1, 0\rangle \\ &= \langle -6, 27, 15\rangle - \langle 2, -2, 0\rangle \\ &= \langle -6-2, 27-(-2), 15-0\rangle \\ &= \langle -8, 29, 15\rangle. \end{aligned}$$

b. Write the equation for the magnitude of the vector, then use scalar multiplication:

$$5\|\vec{\mathbf{w}}\| = 5\sqrt{1^2+(-1)^2+0^2} = 5\sqrt{2}.$$

c. First, use scalar multiplication, then find the magnitude of the new vector. Note that the result is the same as for part b.:

$$\|5\vec{\mathbf{w}}\| = \| \langle 5, -5, 0\rangle \| = \sqrt{5^2+(-5)^2+0^2} = \sqrt{50} = 5\sqrt{2}$$

d. Recall that to find a unit vector in two dimensions, we divide a vector by its magnitude. The procedure is the same in three dimensions:

$$\begin{aligned} \frac{\vec{\mathbf{v}}}{\|\vec{\mathbf{v}}\|} &= \frac{1}{\|\vec{\mathbf{v}}\|}\langle -2, 9, 5\rangle \\ &= \frac{1}{\sqrt{(-2)^2+9^2+5^2}}\langle -2, 9, 5\rangle \\ &= \frac{1}{\sqrt{110}}\langle -2, 9, 5\rangle \\ &= \left\langle \frac{-2}{\sqrt{110}}, \frac{9}{\sqrt{110}}, \frac{5}{\sqrt{110}}\right\rangle. \end{aligned}$$



Exercise 4.3.9:

Let $\vec{\mathbf{v}} = \langle -1, -1, 1\rangle$ and $\vec{\mathbf{w}} = \langle 2, 0, 1\rangle$. Find a unit vector in the direction of $5\vec{\mathbf{v}} + 3\vec{\mathbf{w}}$.

Hint

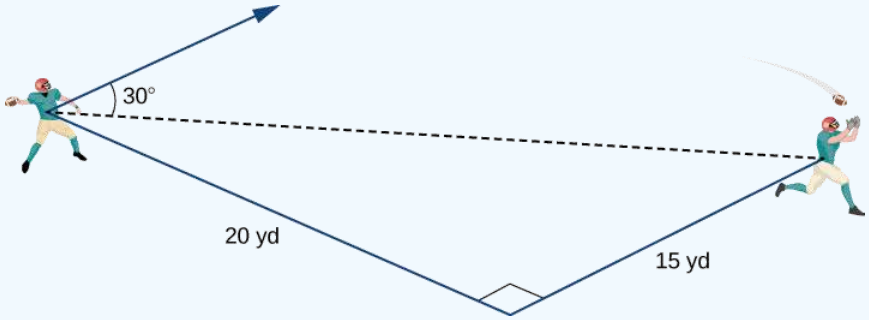
Start by writing $5\vec{\mathbf{v}}+3\vec{\mathbf{w}}$ in component form.

Answer

$$\left\langle \frac{1}{3\sqrt{10}}, -\frac{5}{3\sqrt{10}}, \frac{8}{3\sqrt{10}} \right\rangle$$

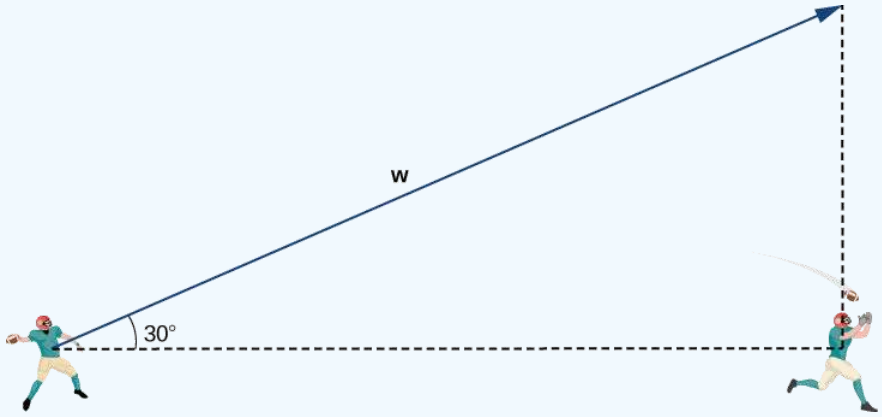
Example 4.3.10: Throwing a Forward Pass

A quarterback is standing on the football field preparing to throw a pass. His receiver is standing 20 yd down the field and 15 yd to the quarterback's left. The quarterback throws the ball at a velocity of 60 mph toward the receiver at an upward angle of 30° (see the following figure). Write the initial velocity vector of the ball, $\vec{\mathbf{v}}$, in component form.



Solution

The first thing we want to do is find a vector in the same direction as the velocity vector of the ball. We then scale the vector appropriately so that it has the right magnitude. Consider the vector $\vec{\mathbf{w}}$ extending from the quarterback's arm to a point directly above the receiver's head at an angle of 30° (see the following figure). This vector would have the same direction as $\vec{\mathbf{v}}$, but it may not have the right magnitude.



The receiver is 20 yd down the field and 15 yd to the quarterback's left. Therefore, the straight-line distance from the quarterback to the receiver is

$$\textbf{Dist from QB to receiver} = \sqrt{15^2+20^2} = \sqrt{225+400} = \sqrt{625} = 25 \text{ yd.}$$

We have $\dfrac{25}{\|\vec{\mathbf{w}}\|} = \cos 30^\circ$. Then the magnitude of $\vec{\mathbf{w}}$ is given by

$$\|\vec{\mathbf{w}}\| = \frac{25}{\cos 30^\circ} = \frac{25 \cdot 2}{\sqrt{3}} = \frac{50}{\sqrt{3}} \text{ yd}$$

and the vertical distance from the receiver to the terminal point of $\vec{\mathbf{w}}$ is

$$\textbf{Vert dist from receiver to terminal point of } \vec{\mathbf{w}} = \|\vec{\mathbf{w}}\| \sin 30^\circ = \frac{50}{\sqrt{3}} \cdot \frac{1}{2} = \frac{25}{\sqrt{3}} \text{ yd.}$$

Then $\vec{\mathbf{w}} = \langle 20, 15, \frac{25}{\sqrt{3}} \rangle$, and has the same direction as $\vec{\mathbf{v}}$.

Recall, though, that we calculated the magnitude of $\vec{\mathbf{w}}$ to be $\|\vec{\mathbf{w}}\| = \frac{50}{\sqrt{3}}$, and $\vec{\mathbf{v}}$ has magnitude 60 mph. So, we need to multiply vector $\vec{\mathbf{w}}$ by an appropriate constant, k. We want to find a value of k so that $\|k\vec{\mathbf{w}}\| = 60$ mph. We have

$$\|k\vec{\mathbf{w}}\| = k\|\vec{\mathbf{w}}\| = k\frac{50}{\sqrt{3}} \text{ mph},$$

so we want

$$k\frac{50}{\sqrt{3}} = 60$$
$$k = \frac{60\sqrt{3}}{50}$$
$$k = \frac{6\sqrt{3}}{5}.$$

Then

$$\vec{\mathbf{v}} = k\vec{\mathbf{w}} = k\langle 20, 15, \frac{25}{\sqrt{3}} \rangle = \frac{6\sqrt{3}}{5}\langle 20, 15, \frac{25}{\sqrt{3}} \rangle = \langle 24\sqrt{3}, 18\sqrt{3}, 30 \rangle.$$

Let's double-check that $\|\vec{\mathbf{v}}\| = 60$. We have

$$\|\vec{\mathbf{v}}\| = \sqrt{(24\sqrt{3})^2 + (18\sqrt{3})^2 + (30)^2} = \sqrt{1728 + 972 + 900} = \sqrt{3600} = 60 \text{ mph}.$$

So, we have found the correct components for $\vec{\mathbf{v}}$.

Exercise 4.3.10

Assume the quarterback and the receiver are in the same place as in the previous example. This time, however, the quarterback throws the ball at velocity of 40 mph and an angle of 45°. Write the initial velocity vector of the ball, $\vec{\mathbf{v}}$, in component form.

Hint

Follow the process used in the previous example.

Answer

$v = \langle 16\sqrt{2}, 12\sqrt{2}, 20\sqrt{2} \rangle$

Key Concepts

- The three-dimensional coordinate system is built around a set of three axes that intersect at right angles at a single point, the origin. Ordered triples (x, y, z) are used to describe the location of a point in space.
- The distance d between points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by the formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

- In three dimensions, the equations $x = a$, $y = b$, and $z = c$ describe planes that are parallel to the coordinate planes.
- The standard equation of a sphere with center (a, b, c) and radius r is

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2.$$

- In three dimensions, as in two, vectors are commonly expressed in component form, $\vec{\mathbf{v}} = \langle x, y, z \rangle$, or in terms of the standard unit vectors, $\vec{\mathbf{v}} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\hat{\mathbf{k}}$.
- Properties of vectors in space are a natural extension of the properties for vectors in a plane. Let $\vec{\mathbf{v}} = \langle x_1, y_1, z_1 \rangle$ and $\vec{\mathbf{w}} = \langle x_2, y_2, z_2 \rangle$ be vectors, and let k be a scalar.

  Scalar multiplication:

$$k\vec{\mathbf{v}} = \langle kx_1, ky_1, kz_1 \rangle$$

Vector addition:

$$\vec{\mathbf{v}} + \vec{\mathbf{w}} = \langle x_1, y_1, z_1 \rangle + \langle x_2, y_2, z_2 \rangle = \langle x_1 + x_2, y_1 + y_2, z_1 + z_2 \rangle$$

Vector subtraction:

$$\vec{\mathbf{v}} - \vec{\mathbf{w}} = \langle x_1, y_1, z_1 \rangle - \langle x_2, y_2, z_2 \rangle = \langle x_1 - x_2, y_1 - y_2, z_1 - z_2 \rangle$$

Vector magnitude:

$$\|\vec{\mathbf{v}}\| = \sqrt{x_1^2 + y_1^2 + z_1^2}$$

Unit vector in the direction of $\vec{\mathbf{v}}$:

$$\frac{\vec{\mathbf{v}}}{\|\vec{\mathbf{v}}\|} = \frac{1}{\|\vec{\mathbf{v}}\|}\langle x_1, y_1, z_1 \rangle = \langle \frac{x_1}{\|\vec{\mathbf{v}}\|}, \frac{y_1}{\|\vec{\mathbf{v}}\|}, \frac{z_1}{\|\vec{\mathbf{v}}\|} \rangle, \ \vec{\mathbf{v}} \neq \vec{\mathbf{0}}$$

Key Equations

Distance between two points in space:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Sphere with center (a, b, c) and radius r:

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$$

Glossary

coordinate plane
a plane containing two of the three coordinate axes in the three-dimensional coordinate system, named by the axes it contains: the xy-plane, xz-plane, or the yz-plane

right-hand rule
a common way to define the orientation of the three-dimensional coordinate system; when the right hand is curved around the z-axis in such a way that the fingers curl from the positive x-axis to the positive y-axis, the thumb points in the direction of the positive z-axis

octants
the eight regions of space created by the coordinate planes

sphere
the set of all points equidistant from a given point known as the *center*

standard equation of a sphere
$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$ describes a sphere with center (a, b, c) and radius r

three-dimensional rectangular coordinate system
a coordinate system defined by three lines that intersect at right angles; every point in space is described by an ordered triple (x, y, z) that plots its location relative to the defining axes

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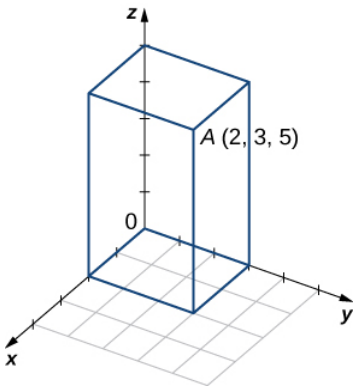
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4.3E: Exercises for Section 4.3

1) Consider a rectangular box with one of the vertices at the origin, as shown in the following figure. If point $A(2,3,5)$ is the opposite vertex to the origin, then find

a. the coordinates of the other six vertices of the box and

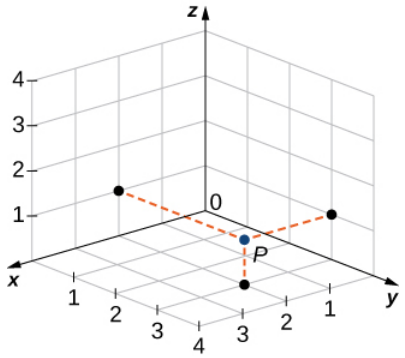
b. the length of the diagonal of the box determined by the vertices O and A.



Answer

a. $(2,0,5),(2,0,0),(2,3,0),(0,3,0),(0,3,5),(0,0,5)$ b. $\sqrt{38}$

2) Find the coordinates of point P and determine its distance to the origin.

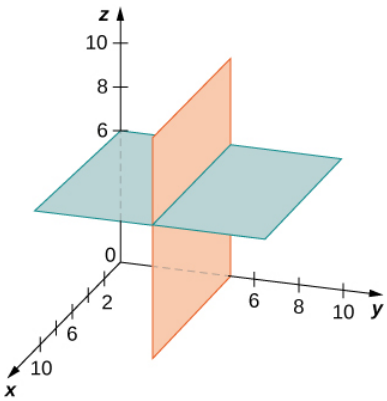


For exercises 3-6, describe and graph the set of points that satisfies the given equation.

3) $(y-5)(z-6)=0$

Answer

A union of two planes: $y=5$ (a plane parallel to the xz-plane) and $z=6$ (a plane parallel to the xy-plane)

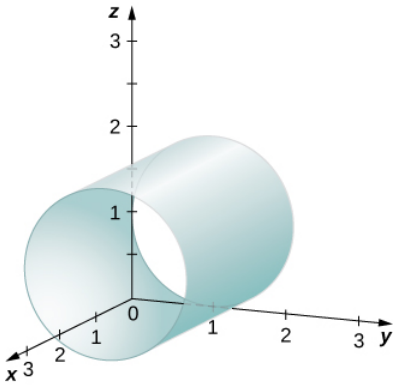


4) $(z-2)(z-5)=0$

5) $(y-1)^2+(z-1)^2=1$

Answer

A cylinder of radius 1 centered on the line $y=1, z=1$



6) $(x-2)^2+(z-5)^2=4$

7) Write the equation of the plane passing through point $(1,1,1)$ that is parallel to the xy-plane.

Answer

$z=1$

8) Write the equation of the plane passing through point $(1,-3,2)$ that is parallel to the xz-plane.

9) Find an equation of the plane passing through points $(1,-3,-2)$, $(0,3,-2)$,and $(1,0,-2)$.

Answer

$z=-2$

10) Find an equation of the plane passing through points $(1,9,2)$, $(1,3,6)$,and $(1,-7,8)$.

For exercises 11-14, find the equation of the sphere in standard form that satisfies the given conditions.

11) Center $C(-1,7,4)$ and radius 4

Answer

$(x+1)^2+(y-7)^2+(z-4)^2=16$

12) Center $C(-4,7,2)$ and radius 6

13) Diameter PQ, where $P(-1,5,7)$ and $Q(-5,2,9)$

Answer

$x+3)^2+(y-3.5)^2+(z-8)^2=\dfrac{29}{4}$

14) Diameter PQ, where $P(-16,-3,9)$ and $Q(-2,3,5)$

For exercises 15 and 16, find the center and radius of the sphere with an equation in general form that is given.

15) $x^2+y^2+z^2-4z+3=0$

Answer

Center $C(0,0,2)$ and radius 1

16) $x^2+y^2+z^2-6x+8y-10z+25=0$

For exercises 17-20, express vector $\overrightarrow{PQ}$ with the initial point at P and the terminal point at Q

a. in component form and

b. by using standard unit vectors.

17) $P(3,0,2)$ and $Q(-1,-1,4)$

Answer

$a.\ \overrightarrow{PQ} = \langle -4,-1,2\rangle$
$b.\ \overrightarrow{PQ} = -4\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$

18) $P(0,10,5)$ and $Q(1,1,-3)$

19) $P(-2,5,-8)$ and $M(1,-7,4)$, where M is the midpoint of the line segment $\overline{PQ}$

Answer

$a.\ \overrightarrow{PQ} = \langle 6,-24,24\rangle$
$b.\ \overrightarrow{PQ} = 6\hat{\mathbf{i}} - 24\hat{\mathbf{j}} + 24\hat{\mathbf{k}}$

20) $Q(0,7,-6)$ and $M(-1,3,2)$, where M is the midpoint of the line segment $\overline{PQ}$

21) Find terminal point Q of vector $\overrightarrow{PQ} = \langle 7,-1,3\rangle$ with the initial point at $P(-2,3,5)$.

Answer

$Q(5,2,8)$

22) Find initial point P of vector $\overrightarrow{PQ} = \langle -9,1,2\rangle$ with the terminal point at $Q(10,0,-1)$.

For exercises 23-26, use the given vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ to find and express the vectors $\vec{\mathbf{a}} + \vec{\mathbf{b}}$, $4\vec{\mathbf{a}}$, and $-5\vec{\mathbf{a}} + 3\vec{\mathbf{b}}$ in component form.

23) $\vec{\mathbf{a}} = \langle -1,-2,4\rangle, \quad \vec{\mathbf{b}} = \langle -5,6,-7\rangle$

Answer

$\vec{\mathbf{a}} + \vec{\mathbf{b}} = \langle -6,4,-3\rangle, 4\vec{\mathbf{a}} = \langle -4,-8,16\rangle, -5\vec{\mathbf{a}} + 3\vec{\mathbf{b}} = \langle -10,28,-41\rangle$

24) $\vec{\mathbf{a}} = \langle 3,-2,4\rangle, \quad \vec{\mathbf{b}} = \langle -5,6,-9\rangle$

25) $\vec{\mathbf{a}} = -\hat{\mathbf{k}}, \quad \vec{\mathbf{b}} = -\hat{\mathbf{i}}$

Answer

$\vec{\mathbf{a}} + \vec{\mathbf{b}} = \langle -1,0,-1\rangle, 4\vec{\mathbf{a}} = \langle 0,0,-4\rangle, -5\vec{\mathbf{a}} + 3\vec{\mathbf{b}} = \langle -3,0,5\rangle$

26) $\vec{\mathbf{a}} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}, \quad \vec{\mathbf{b}} = 2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$

For exercises 27-30, vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are given. Find the magnitudes of vectors $\vec{\mathbf{u}} - \vec{\mathbf{v}}$ and $-2\vec{\mathbf{u}}$.

27) $\vec{\mathbf{u}} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}, \quad \vec{\mathbf{v}} = -\hat{\mathbf{i}} + 5\hat{\mathbf{j}} - \hat{\mathbf{k}}$

Answer

$\|\vec{\mathbf{u}} - \vec{\mathbf{v}}\| = \sqrt{38}, \quad \|-2\vec{\mathbf{u}}\| = 2\sqrt{29}$

28) $\vec{\mathbf{u}} = \hat{\mathbf{i}} + \hat{\mathbf{j}}, \quad \vec{\mathbf{v}} = \hat{\mathbf{j}} - \hat{\mathbf{k}}$

29) $\vec{\mathbf{u}} = \langle 2\cos t, -2\sin t, 3\rangle, \quad \vec{\mathbf{v}} = \langle 0,0,3\rangle, \quad$ where t is a real number.

Answer

$\|\vec{\mathbf{u}} - \vec{\mathbf{v}}\| = 2, \quad \|-2\vec{\mathbf{u}}\| = 2\sqrt{13}$

30) $\vec{\mathbf{u}} = \langle 0, 1, \sinh t \rangle, \quad \vec{\mathbf{v}} = \langle 1, 1, 0 \rangle, \quad$ where t is a real number.

For exercises 31-36, find the unit vector in the direction of the given vector $\vec{\mathbf{a}}$ and express it using standard unit vectors.

31) $\vec{\mathbf{a}} = 3\hat{\mathbf{i}} - 4\hat{\mathbf{j}}$

Answer

$\frac{3}{5}\hat{\mathbf{i}} - \frac{4}{5}\hat{\mathbf{j}}$

32) $\vec{\mathbf{a}} = \langle 4, -3, 6 \rangle$

33) $\vec{\mathbf{a}} = \overrightarrow{PQ}$, where $P(-2, 3, 1)$ and $Q(0, -4, 4)$

Answer

$\frac{\sqrt{62}}{31}\hat{\mathbf{i}} - \frac{7\sqrt{62}}{62}\hat{\mathbf{j}} + \frac{3\sqrt{62}}{62}\hat{\mathbf{k}}$

34) $\vec{\mathbf{a}} = \overrightarrow{OP}$, where $P(-1, -1, 1)$

35) $\vec{\mathbf{a}} = \vec{\mathbf{u}} - \vec{\mathbf{v}} + \vec{\mathbf{w}}$, where $\vec{\mathbf{u}} = \hat{\mathbf{i}} - \hat{\mathbf{j}} - \hat{\mathbf{k}}, \quad \vec{\mathbf{v}} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}, \quad$ and $\vec{\mathbf{w}} = -\hat{\mathbf{i}} + \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$

Answer

$-\frac{\sqrt{6}}{3}\hat{\mathbf{i}} + \frac{\sqrt{6}}{6}\hat{\mathbf{j}} + \frac{\sqrt{6}}{6}\hat{\mathbf{k}}$

36) $\vec{\mathbf{a}} = 2\vec{\mathbf{u}} + \vec{\mathbf{v}} - \vec{\mathbf{w}}, \quad$ where $\vec{\mathbf{u}} = \hat{\mathbf{i}} - \hat{\mathbf{k}}, \quad \vec{\mathbf{v}} = 2\hat{\mathbf{j}}$, and $\vec{\mathbf{w}} = \hat{\mathbf{i}} - \hat{\mathbf{j}}$

37) Determine whether $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ are equivalent vectors, where $A(1, 1, 1)$, $B(3, 3, 3)$, $P(1, 4, 5)$, and $Q(3, 6, 7)$.

Answer

Equivalent vectors

38) Determine whether the vectors $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ are equivalent, where $A(1, 4, 1)$, $B(-2, 2, 0)$, $P(2, 5, 7)$, and $Q(-3, 2, 1)$.

For exercises 39-42, find vector $\vec{\mathbf{u}}$ with a magnitude that is given and satisfies the given conditions.

39) $\vec{\mathbf{v}} = \langle 7, -1, 3 \rangle$, $\|\vec{\mathbf{u}}\| = 10$, and $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ have the same direction

Answer

$\vec{\mathbf{u}} = \langle \frac{70\sqrt{59}}{59}, -\frac{10\sqrt{59}}{59}, \frac{30\sqrt{59}}{59} \rangle$

40) $\vec{\mathbf{v}} = \langle 2, 4, 1 \rangle$, $\|\vec{\mathbf{u}}\| = 15$, and $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ have the same direction

41) $\vec{\mathbf{v}} = \langle 2\sin t, \ 2\cos t, 1 \rangle$, $\|\vec{\mathbf{u}}\| = 2$, $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ have opposite directions for any t, where t is a real number

Answer

$\vec{\mathbf{u}} = \langle -\frac{4\sqrt{5}}{5}\sin t, -\frac{4\sqrt{5}}{5}\cos t, -\frac{2\sqrt{5}}{5} \rangle$

42) $\vec{\mathbf{v}} = \langle 3\sinh t, 0, 3 \rangle$, $\|\vec{\mathbf{u}}\| = 5$, and $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ have opposite directions for any t, where t is a real number

43) Determine a vector of magnitude 5 in the direction of vector $\overrightarrow{AB}$, where $A(2, 1, 5)$ and $B(3, 4, -7)$.

Answer

$\langle \frac{5\sqrt{154}}{154}, \frac{15\sqrt{154}}{154}, -\frac{30\sqrt{154}}{77} \rangle$

44) Find a vector of magnitude 2 that points in the opposite direction than vector $\overrightarrow{AB}$, where $A(-1, -1, 1)$ and $B(0, 1, 1)$. Express the answer in component form.

45) Consider the points $A(2, \alpha, 0)$, $B(0, 1, \beta)$, and $C(1, 1, \beta)$, where α and β are negative real numbers. Find α and β such that $\|\overrightarrow{OA} - \overrightarrow{OB} + \overrightarrow{OC}\| = \|\overrightarrow{OB}\| = 4$.

Answer

$\alpha = -\sqrt{7},\ \beta = -\sqrt{15}$

46) Consider points $A(\alpha, 0, 0)$, $B(0, \beta, 0)$, and $C(\alpha, \beta, \beta)$, where α and β are positive real numbers. Find α and β such that $\|\overrightarrow{OA} + \overrightarrow{OB}\| = \sqrt{2}$ and $\|\overrightarrow{OC}\| = \sqrt{3}$.

47) Let $P(x, y, z)$ be a point situated at an equal distance from points $A(1, -1, 0)$ and $B(-1, 2, 1)$. Show that point P lies on the plane of equation $-2x + 3y + z = 2$.

48) Let $P(x, y, z)$ be a point situated at an equal distance from the origin and point $A(4, 1, 2)$. Show that the coordinates of point P satisfy the equation $8x + 2y + 4z = 21$.

49) The points A, B, and C are collinear (in this order) if the relation $\|\overrightarrow{AB}\| + \|\overrightarrow{BC}\| = \|\overrightarrow{AC}\|$ is satisfied. Show that $A(5, 3, -1)$, $B(-5, -3, 1)$, and $C(-15, -9, 3)$ are collinear points.

50) Show that points $A(1, 0, 1)$, $B(0, 1, 1)$, and $C(1, 1, 1)$ are not collinear.

51) [T] A force $\vec{\mathbf{F}}$ of 50 N acts on a particle in the direction of the vector $\overrightarrow{OP}$, where $P(3, 4, 0)$.

a. Express the force as a vector in component form.

b. Find the angle between force $\vec{\mathbf{F}}$ and the positive direction of the x-axis. Express the answer in degrees rounded to the nearest integer.

Answer

$a.\ \vec{\mathbf{F}} = \langle 30, 40, 0 \rangle; \quad b.\ 53^\circ$

52) [T] A force $\vec{\mathbf{F}}$ of 40 N acts on a box in the direction of the vector $\overrightarrow{OP}$, where $P(1, 0, 2)$.

a. Express the force as a vector by using standard unit vectors.

b. Find the angle between force $\vec{\mathbf{F}}$ and the positive direction of the x-axis.

53) If $\vec{\mathbf{F}}$ is a force that moves an object from point $P_1(x_1, y_1, z_1)$ to another point $P_2(x_2, y_2, z_2)$, then the displacement vector is defined as $\vec{\mathbf{D}} = (x_2 - x_1)\hat{\mathbf{i}} + (y_2 - y_1)\hat{\mathbf{j}} + (z_2 - z_1)\hat{\mathbf{k}}$. A metal container is lifted 10 m vertically by a constant force $\vec{\mathbf{F}}$. Express the displacement vector $\vec{\mathbf{D}}$ by using standard unit vectors.

Answer

$\vec{\mathbf{D}} = 10\hat{\mathbf{k}}$

54) A box is pulled 4 yd horizontally in the x-direction by a constant force $\vec{\mathbf{F}}$. Find the displacement vector in component form.

55) The sum of the forces acting on an object is called the resultant or net force. An object is said to be in static equilibrium if the resultant force of the forces that act on it is zero. Let $\vec{\mathbf{F}}_1 = \langle 10, 6, 3 \rangle$, $\vec{\mathbf{F}}_2 = \langle 0, 4, 9 \rangle$, and $\vec{\mathbf{F}}_3 = \langle 10, -3, -9 \rangle$ be three forces acting on a box. Find the force $\vec{\mathbf{F}}_4$ acting on the box such that the box is in static equilibrium. Express the answer in component form.

Answer

$\vec{\mathbf{F}}_4 = \langle -20, -7, -3 \rangle$

56) [T] Let $\vec{\mathbf{F}}_k = \langle 1, k, k^2 \rangle, k = 1, \ldots, n$ be n forces acting on a particle, with $n \geq 2$.

a. Find the net force $\vec{\mathbf{F}} = \sum_{k=1}^{n} \vec{\mathbf{F}}_k$. Express the answer using standard unit vectors.

b. Use a computer algebra system (CAS) to find n such that $\|\vec{\mathbf{F}}\| < 100$.

57) The force of gravity $\vec{\mathbf{F}}$ acting on an object is given by $\vec{\mathbf{F}} = m\vec{\mathbf{g}}$, where m is the mass of the object (expressed in kilograms) and $\vec{\mathbf{g}}$ is acceleration resulting from gravity, with $\|\vec{\mathbf{g}}\| = 9.8\ N/kg$. A 2-kg disco ball hangs by a chain from the ceiling of a room.

a. Find the force of gravity $\vec{\mathbf{F}}$ acting on the disco ball and find its magnitude.

b. Find the force of tension $\vec{\mathbf{T}}$ in the chain and its magnitude.

Express the answers using standard unit vectors.



Figure 18: (credit: modification of work by Kenneth Lu, Flickr)

Answer

$a.\ \vec{\mathbf{F}} = -19.6\hat{\mathbf{k}}, \quad \|\vec{\mathbf{F}}\| = 19.6\ N$
$b.\ \vec{\mathbf{T}} = 19.6\hat{\mathbf{k}}, \quad \|\vec{\mathbf{T}}\| = 19.6\ N$

58) A 5-kg pendant chandelier is designed such that the alabaster bowl is held by four chains of equal length, as shown in the following figure.

a. Find the magnitude of the force of gravity acting on the chandelier.

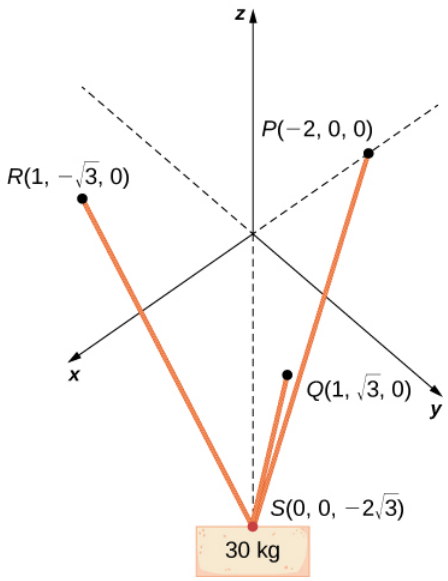
b. Find the magnitudes of the forces of tension for each of the four chains (assume chains are essentially vertical).



59) [T] A 30-kg block of cement is suspended by three cables of equal length that are anchored at points $P(-2, 0, 0), Q(1, \sqrt{3}, 0)$, and $R(1, -\sqrt{3}, 0)$. The load is located at $S(0, 0, -2\sqrt{3})$, as shown in the following figure. Let $\vec{\mathbf{F}}_1, \vec{\mathbf{F}}_2$, and $\vec{\mathbf{F}}_3$ be the forces of tension resulting from the load in cables RS, QS, and PS, respectively.

a. Find the gravitational force $\vec{\mathbf{F}}$ acting on the block of cement that counterbalances the sum $\vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 + \vec{\mathbf{F}}_3$ of the forces of tension in the cables.

b. Find forces $\vec{\mathbf{F}}_1, \vec{\mathbf{F}}_2$, and $\vec{\mathbf{F}}_3$. Express the answer in component form.

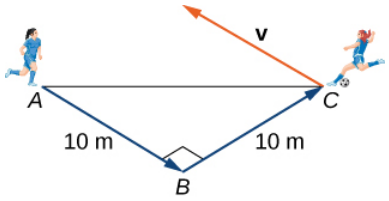


Answer

a. $\vec{\mathbf{F}} = -294\hat{\mathbf{k}}$ N;

b. $\vec{\mathbf{F}}_1 = \langle -\frac{49\sqrt{3}}{3}, 49, -98\rangle, \vec{\mathbf{F}}_2 = \langle -\frac{49\sqrt{3}}{3}, -49, -98\rangle$, and $\vec{\mathbf{F}}_3 = \langle \frac{98\sqrt{3}}{3}, 0, -98\rangle$ (each component is expressed in newtons)

60) Two soccer players are practicing for an upcoming game. One of them runs 10 m from point A to point B. She then turns left at 90° and runs 10 m until she reaches point C. Then she kicks the ball with a speed of 10 m/sec at an upward angle of 45° to her teammate, who is located at point A. Write the velocity of the ball in component form.



61) Let $\vec{\mathbf{r}}(t) = \langle x(t), y(t), z(t)\rangle$ be the position vector of a particle at the time $t \in [0, T]$, where x, y, and z are smooth functions on $[0, T]$. The instantaneous velocity of the particle at time t is defined by vector $\vec{\mathbf{v}}(t) = \langle x'(t), y'(t), z'(t)\rangle$, with components that are the derivatives with respect to t, of the functions x, y, and z, respectively. The magnitude $\|\vec{\mathbf{v}}(t)\|$ of the instantaneous velocity vector is called the speed of the particle at time t. Vector $\vec{\mathbf{a}}(t) = \langle x''(t), y''(t), z''(t)\rangle$, with components that are the second derivatives with respect to t, of the functions x, y, and z, respectively, gives the acceleration of the particle at time t. Consider $\vec{\mathbf{r}}(t) = \langle \cos t, \sin t, 2t\rangle$ the position vector of a particle at time $t \in [0, 30]$, where the components of $\vec{\mathbf{r}}$ are expressed in centimeters and time is expressed in seconds.

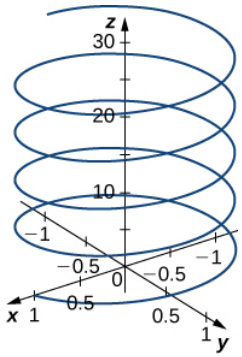
a. Find the instantaneous velocity, speed, and acceleration of the particle after the first second. Round your answer to two decimal places.

b. Use a CAS to visualize the path of the particle—that is, the set of all points of coordinates $(\cos t, \sin t, 2t)$, where $t \in [0, 30]$.

Answer

a. $\vec{\mathbf{v}}(1) = \langle -0.84, 0.54, 2\rangle$(each component is expressed in centimeters per second); $\|\vec{\mathbf{v}}(1)\| = 2.24$(expressed in centimeters per second); $\vec{\mathbf{a}}(1) = \langle -0.54, -0.84, 0\rangle$(each component expressed in centimeters per second squared);

b.



62) [T] Let $\vec{\mathbf{r}}(t) = \langle t, 2t^2, 4t^2 \rangle$ be the position vector of a particle at time t (in seconds), where $t \in [0, 10]$ (here the components of $\vec{\mathbf{r}}$ are expressed in centimeters).

a. Find the instantaneous velocity, speed, and acceleration of the particle after the first two seconds. Round your answer to two decimal places.

b. Use a CAS to visualize the path of the particle defined by the points $(t,\ 2t^2,\ 4t^2)$, where $t \in [0,\ 60]$.

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4.4: The Dot Product

Learning Objectives

- Calculate the dot product of two given vectors.
- Determine whether two given vectors are perpendicular.
- Find the direction cosines of a given vector.
- Explain what is meant by the vector projection of one vector onto another vector, and describe how to compute it.
- Calculate the work done by a given force.

If we apply a force to an object so that the object moves, we say that work is done by the force. Previously, we looked at a constant force and we assumed the force was applied in the direction of motion of the object. Under those conditions, work can be expressed as the product of the force acting on an object and the distance the object moves. In this chapter, however, we have seen that both force and the motion of an object can be represented by vectors.

In this section, we develop an operation called the dot product, which allows us to calculate work in the case when the force vector and the motion vector have different directions. The dot product essentially tells us how much of the force vector is applied in the direction of the motion vector. The dot product can also help us measure the angle formed by a pair of vectors and the position of a vector relative to the coordinate axes. It even provides a simple test to determine whether two vectors meet at a right angle.

The Dot Product and Its Properties

We have already learned how to add and subtract vectors. In this chapter, we investigate two types of vector multiplication. The first type of vector multiplication is called the dot product, based on the notation we use for it, and it is defined as follows:

Definition: dot product

The *dot product* of vectors $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$ is given by the sum of the products of the components

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = u_1 v_1 + u_2 v_2 + u_3 v_3. \tag{4.4.1}$$

Note that if u and v are two-dimensional vectors, we calculate the dot product in a similar fashion. Thus, if $\vec{\mathbf{u}} = \langle u_1, u_2 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2 \rangle$, then

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = u_1 v_1 + u_2 v_2. \tag{4.4.2}$$

When two vectors are combined under addition or subtraction, the result is a vector. When two vectors are combined using the dot product, the result is a scalar. For this reason, the dot product is often called the *scalar product.* It may also be called the *inner product.*

Example 4.4.1: Calculating Dot Products

a. Find the dot product of $\vec{\mathbf{u}} = \langle 3, 5, 2 \rangle$ and $\vec{\mathbf{v}} = \langle -1, 3, 0 \rangle$.

b. Find the scalar product of $\vec{\mathbf{p}} = 10\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 7\hat{\mathbf{k}}$ and $\vec{\mathbf{q}} = -2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 6\hat{\mathbf{k}}$.

Solution:

a. Substitute the vector components into the formula for the dot product:

$$\begin{aligned} \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} &= u_1 v_1 + u_2 v_2 + u_3 v_3 \\ &= 3(-1) + 5(3) + 2(0) \\ &= -3 + 15 + 0 \\ &= 12. \end{aligned}$$

b. The calculation is the same if the vectors are written using standard unit vectors. We still have three components for each vector to substitute into the formula for the dot product:

$$\begin{aligned}\vec{\mathbf{p}} \cdot \vec{\mathbf{q}} &= u_1 v_1 + u_2 v_2 + u_3 v_3 \\ &= 10(-2) + (-4)(1) + (7)(6) \\ &= -20 - 4 + 42 \\ &= 18.\end{aligned}$$

Exercise 4.4.1

Find $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}$, where $\vec{\mathbf{u}} = \langle 2, 9, -1 \rangle$ and $\vec{\mathbf{v}} = \langle -3, 1, -4 \rangle$.

Hint

Multiply corresponding components and then add their products.

Answer

7

Like vector addition and subtraction, the dot product has several algebraic properties. We prove three of these properties and leave the rest as exercises.

Properties of the Dot Product

Let $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$ be vectors, and let c be a scalar.

i. *Commutative property*

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = \vec{\mathbf{v}} \cdot \vec{\mathbf{u}} \tag{4.4.3}$$

ii. *Distributive property*

$$\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} + \vec{\mathbf{w}}) = \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} + \vec{\mathbf{u}} \cdot \vec{\mathbf{w}} \tag{4.4.4}$$

iii. *Associative property*

$$c(\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}) = (c\vec{\mathbf{u}}) \cdot \vec{\mathbf{v}} = \vec{\mathbf{u}} \cdot (c\vec{\mathbf{v}}) \tag{4.4.5}$$

iv. *Property of magnitude*

$$\vec{\mathbf{v}} \cdot \vec{\mathbf{v}} = \|\vec{\mathbf{v}}\|^2 \tag{4.4.6}$$

Proof

Let $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$. Then

$$\begin{aligned}\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} &= \langle u_1, u_2, u_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle \\ &= u_1 v_1 + u_2 v_2 + u_3 v_3 \\ &= v_1 u_1 + v_2 u_2 + v_3 u_3 \\ &= \langle v_1, v_2, v_3 \rangle \cdot \langle u_1, u_2, u_3 \rangle \\ &= \vec{\mathbf{v}} \cdot \vec{\mathbf{u}}.\end{aligned}$$

The associative property looks like the associative property for real-number multiplication, but pay close attention to the difference between scalar and vector objects:

$$\begin{aligned} c(\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}) &= c(u_1v_1 + u_2v_2 + u_3v_3) \\ &= c(u_1v_1) + c(u_2v_2) + c(u_3v_3) \\ &= (cu_1)v_1 + (cu_2)v_2 + (cu_3)v_3 \\ &= \langle cu_1, cu_2, cu_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle \\ &= c\langle u_1, u_2, u_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle \\ &= (c\vec{\mathbf{u}}) \cdot \vec{\mathbf{v}}. \end{aligned}$$

The proof that $c(\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}) = \vec{\mathbf{u}} \cdot (c\vec{\mathbf{v}})$ is similar.

The fourth property shows the relationship between the magnitude of a vector and its dot product with itself:

$$\begin{aligned} \vec{\mathbf{v}} \cdot \vec{\mathbf{v}} &= \langle v_1, v_2, v_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle \\ &= (v_1)^2 + (v_2)^2 + (v_3)^2 \\ &= \left[\sqrt{(v_1)^2 + (v_2)^2 + (v_3)^2} \right]^2 \\ &= \|\vec{\mathbf{v}}\|^2. \end{aligned}$$

□

Note that by property iv. we have $\vec{\mathbf{0}} \cdot \vec{\mathbf{v}} = 0$. Also by property iv. if $\vec{\mathbf{v}} \cdot \vec{\mathbf{v}} = 0$, then $\vec{\mathbf{v}} = \vec{\mathbf{0}}$.

Example 4.4.2: Using Properties of the Dot Product

Let $\vec{\mathbf{a}} = \langle 1, 2, -3 \rangle$, $\vec{\mathbf{b}} = \langle 0, 2, 4 \rangle$, and $\vec{\mathbf{c}} = \langle 5, -1, 3 \rangle$.

Find each of the following products.

a. $(\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})\vec{\mathbf{c}}$
b. $\vec{\mathbf{a}} \cdot (2\vec{\mathbf{c}})$
c. $\|\vec{\mathbf{b}}\|^2$

Solution

a. Note that this expression asks for the scalar multiple of $\vec{\mathbf{c}}$ by $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}$:

$$\begin{aligned} (\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})\vec{\mathbf{c}} &= (\langle 1, 2, -3 \rangle \cdot \langle 0, 2, 4 \rangle)\langle 5, -1, 3 \rangle \\ &= (1(0) + 2(2) + (-3)(4))\langle 5, -1, 3 \rangle \\ &= -8\langle 5, -1, 3 \rangle \\ &= \langle -40, 8, -24 \rangle. \end{aligned}$$

b. This expression is a dot product of vector $\vec{\mathbf{a}}$ and scalar multiple $2\vec{\mathbf{c}}$:

$$\begin{aligned} \vec{\mathbf{a}} \cdot (2\vec{\mathbf{c}}) &= 2(\vec{\mathbf{a}} \cdot \vec{\mathbf{c}}) \\ &= 2(\langle 1, 2, -3 \rangle \cdot \langle 5, -1, 3 \rangle) \\ &= 2(1(5) + 2(-1) + (-3)(3)) \\ &= 2(-6) = -12. \end{aligned}$$

c. Simplifying this expression is a straightforward application of the dot product:

$$\begin{aligned}\|\vec{\mathbf{b}}\|^2 &= \vec{\mathbf{b}} \cdot \vec{\mathbf{b}} \\ &= \langle 0,2,4\rangle \cdot \langle 0,2,4\rangle \\ &= 0^2 + 2^2 + 4^2 \\ &= 0 + 4 + 16 \\ &= 20.\end{aligned}$$

Exercise 4.4.2

Find the following products for $\vec{\mathbf{p}} = \langle 7,0,2\rangle$, $\vec{\mathbf{q}} = \langle -2,2,-2\rangle$, and $\vec{\mathbf{r}} = \langle 0,2,-3\rangle$.

a. $(\vec{\mathbf{r}} \cdot \vec{\mathbf{p}})\vec{\mathbf{q}}$
b. $\|\vec{\mathbf{p}}\|^2$

Hint

$\vec{\mathbf{r}} \cdot \vec{\mathbf{p}}$ is a scalar.

Answer

a. $(\vec{\mathbf{r}} \cdot \vec{\mathbf{p}})\vec{\mathbf{q}} = \langle 12,-12,12\rangle$; *b.* $\|\vec{\mathbf{p}}\|^2 = 53$

Using the Dot Product to Find the Angle between Two Vectors

When two nonzero vectors are placed in standard position, whether in two dimensions or three dimensions, they form an angle between them (Figure 4.4.1). The dot product provides a way to find the measure of this angle. This property is a result of the fact that we can express the dot product in terms of the cosine of the angle formed by two vectors.

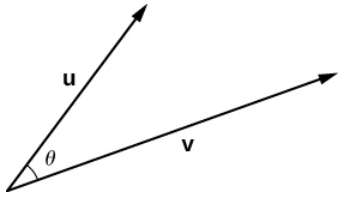


Figure 4.4.1: Let θ be the angle between two nonzero vectors $\vec{\mathbf{u}}$ *and* $\vec{\mathbf{v}}$ *such that* $0 \le \theta \le \pi$.

Evaluating a Dot Product

The dot product of two vectors is the product of the magnitude of each vector and the cosine of the angle between them:

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = \|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\| \cos\theta. \tag{4.4.7}$$

Proof

Place vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ in standard position and consider the vector $\vec{\mathbf{v}} - \vec{\mathbf{u}}$ (Figure 4.4.2). These three vectors form a triangle with side lengths $\|\vec{\mathbf{u}}\|$, $\|\vec{\mathbf{v}}\|$, and $\|\vec{\mathbf{v}} - \vec{\mathbf{u}}\|$.

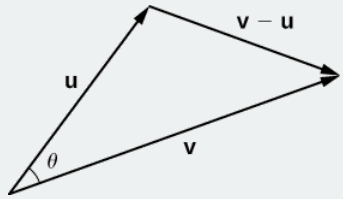


Figure 4.4.2: The lengths of the sides of the triangle are given by the magnitudes of the vectors that form the triangle.

Recall from trigonometry that the law of cosines describes the relationship among the side lengths of the triangle and the angle θ. Applying the law of cosines here gives

$$\|\vec{\mathbf{v}} - \vec{\mathbf{u}}\|^2 = \|\vec{\mathbf{u}}\|^2 + \|\vec{\mathbf{v}}\|^2 - 2\|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\| \cos\theta. \tag{4.4.8}$$

The dot product provides a way to rewrite the left side of Equation 4.4.8:

$$\begin{aligned}\|\vec{\mathbf{v}}-\vec{\mathbf{u}}\|^2 &= (\vec{\mathbf{v}}-\vec{\mathbf{u}})\cdot(\vec{\mathbf{v}}-\vec{\mathbf{u}})\\ &= (\vec{\mathbf{v}}-\vec{\mathbf{u}})\cdot\vec{\mathbf{v}}-(\vec{\mathbf{v}}-\vec{\mathbf{u}})\cdot\vec{\mathbf{u}}\\ &= \vec{\mathbf{v}}\cdot\vec{\mathbf{v}}-\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}-\vec{\mathbf{v}}\cdot\vec{\mathbf{u}}+\vec{\mathbf{u}}\cdot\vec{\mathbf{u}}\\ &= \vec{\mathbf{v}}\cdot\vec{\mathbf{v}}-\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}-\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}+\vec{\mathbf{u}}\cdot\vec{\mathbf{u}}\\ &= \|\vec{\mathbf{v}}\|^2-2\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}+\|\vec{\mathbf{u}}\|^2.\end{aligned}$$

Substituting into the law of cosines yields

$$\begin{aligned}\|\vec{\mathbf{v}}-\vec{\mathbf{u}}\|^2 &= \|\vec{\mathbf{u}}\|^2+\|\vec{\mathbf{v}}\|^2-2\|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\|\cos\theta\\ \|\vec{\mathbf{v}}\|^2-2\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}+\|\vec{\mathbf{u}}\|^2 &= \|\vec{\mathbf{u}}\|^2+\|\vec{\mathbf{v}}\|^2-2\|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\|\cos\theta\\ -2\vec{\mathbf{u}}\cdot\vec{\mathbf{v}} &= -2\|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\|\cos\theta\\ \vec{\mathbf{u}}\cdot\vec{\mathbf{v}} &= \|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\|\cos\theta.\end{aligned}$$

□

We can use the form of the dot product in Equation 4.4.7 to find the measure of the angle between two nonzero vectors by rearranging Equation 4.4.7 to solve for the cosine of the angle:

$$\cos\theta=\frac{\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\|}. \tag{4.4.9}$$

Using this equation, we can find the cosine of the angle between two nonzero vectors. Since we are considering the smallest angle between the vectors, we assume $0^\circ\le\theta\le180^\circ$ (or $0\le\theta\le\pi$ if we are working in radians). The inverse cosine is unique over this range, so we are then able to determine the measure of the angle θ.

Example 4.4.3: Finding the Angle between Two Vectors

Find the measure of the angle between each pair of vectors.

a. $\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$ and $2\hat{\mathbf{i}}-\hat{\mathbf{j}}-3\hat{\mathbf{k}}$
b. $\langle 2,5,6\rangle$ and $\langle -2,-4,4\rangle$

Solution

a. To find the cosine of the angle formed by the two vectors, substitute the components of the vectors into Equation 4.4.9:

$$\begin{aligned}\cos\theta &= \frac{(\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}})\cdot(2\hat{\mathbf{i}}-\hat{\mathbf{j}}-3\hat{\mathbf{k}})}{\left\|\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}\right\|\cdot\left\|2\hat{\mathbf{i}}-\hat{\mathbf{j}}-3\hat{\mathbf{k}}\right\|}\\ &= \frac{1(2)+(1)(-1)+(1)(-3)}{\sqrt{1^2+1^2+1^2}\sqrt{2^2+(-1)^2+(-3)^2}}\\ &= \frac{-2}{\sqrt{3}\sqrt{14}}=\frac{-2}{\sqrt{42}}.\end{aligned}$$

Therefore, $\theta=\arccos\dfrac{-2}{\sqrt{42}}$ rad.

b. Start by finding the value of the cosine of the angle between the vectors:

$$\begin{aligned}\cos\theta &= \frac{\langle 2,5,6\rangle \cdot \langle -2,-4,4\rangle}{\|\langle 2,5,6\rangle\| \cdot \|\langle -2,-4,4\rangle\|} \\ &= \frac{2(-2)+(5)(-4)+(6)(4)}{\sqrt{2^2+5^2+6^2}\sqrt{(-2)^2+(-4)^2+4^2}} \\ &= \frac{0}{\sqrt{65}\sqrt{36}} = 0.\end{aligned}$$

Now, $\cos\theta = 0$ and $0 \le \theta \le \pi$, so $\theta = \pi/2$.



Exercise 4.4.3

Find the measure of the angle, in radians, formed by vectors $\vec{\mathbf{a}} = \langle 1,2,0\rangle$ and $\vec{\mathbf{b}} = \langle 2,4,1\rangle$. Round to the nearest hundredth.

Hint

Use the Equation 4.4.9

Answer

$\theta \approx 0.22$ rad

The angle between two vectors can be acute $(0 < \cos\theta < 1)$, obtuse $(-1 < \cos\theta < 0)$, or straight $(\cos\theta = -1)$. If $\cos\theta = 1$, then both vectors have the same direction. If $\cos\theta = 0$, then the vectors, when placed in standard position, form a right angle (Figure 4.4.3). We can formalize this result into a theorem regarding orthogonal (perpendicular) vectors.

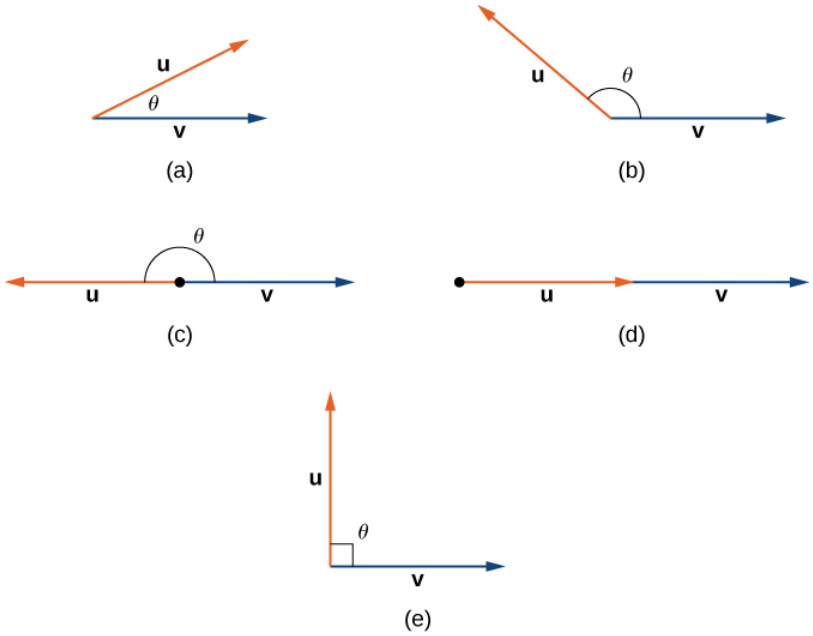


Figure 4.4.3: (a) An acute angle has $0 < \cos\theta < 1$. (b) An obtuse angle has $-1 < \cos\theta < 0$. (c) A straight line has $\cos\theta = -1$. (d) If the vectors have the same direction, $\cos\theta = 1$. (e) If the vectors are orthogonal (perpendicular), $\cos\theta = 0$.



Orthogonal Vectors

The nonzero vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are **orthogonal vectors** if and only if $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = 0$.

Proof

Let $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ be nonzero vectors, and let θ denote the angle between them. First, assume $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = 0$. Then

$$\|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\|\cos\theta = 0. \tag{4.4.10}$$

However, $\|\vec{\mathbf{u}}\| \neq 0$ and $\|\vec{\mathbf{v}}\| \neq 0$, so we must have $\cos\theta = 0$. Hence, $\theta = 90^\circ$, and the vectors are orthogonal.

Now assume $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are orthogonal. Then $\theta = 90^\circ$ and we have

$$\begin{aligned}\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} &= \|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\| \cos\theta \\ &= \|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\| \cos 90^\circ \\ &= \|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\|(0) \\ &= 0.\end{aligned}$$

□

The terms **orthogonal, perpendicular,** and **normal** each indicate that mathematical objects are intersecting at right angles. The use of each term is determined mainly by its context. We say that vectors are orthogonal and lines are perpendicular. The term *normal* is used most often when measuring the angle made with a plane or other surface.

Example 4.4.4: Identifying Orthogonal Vectors

Determine whether $\vec{\mathbf{p}} = \langle 1, 0, 5 \rangle$ and $\vec{\mathbf{q}} = \langle 10, 3, -2 \rangle$ are orthogonal vectors.

Solution

Using the definition, we need only check the dot product of the vectors:

$$\vec{\mathbf{p}} \cdot \vec{\mathbf{q}} = 1(10) + (0)(3) + (5)(-2) = 10 + 0 - 10 = 0.$$

Because $\vec{\mathbf{p}} \cdot \vec{\mathbf{q}} = 0,$ the vectors are orthogonal (Figure 4.4.4).

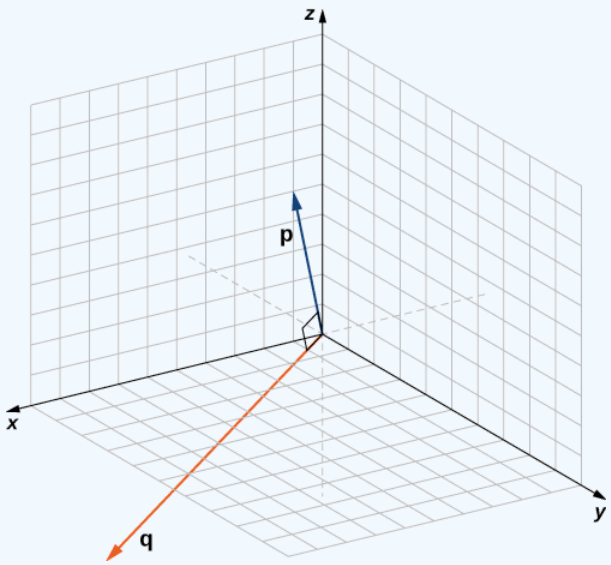


Figure 4.4.4: Vectors $\vec{\mathbf{p}}$ and $\vec{\mathbf{q}}$ form a right angle when their initial points are aligned.

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Exercise 4.4.4

For which value of x is $\vec{\mathbf{p}} = \langle 2, 8, -1 \rangle$ orthogonal to $\vec{\mathbf{q}} = \langle x, -1, 2 \rangle$?

Hint

Vectors $\vec{\mathbf{p}}$ and $\vec{\mathbf{q}}$ are orthogonal if and only if $\vec{\mathbf{p}} \cdot \vec{\mathbf{q}} = 0$.

Answer

$x = 5$

Example 4.4.5: Measuring the Angle Formed by Two Vectors

Let $\vec{\mathbf{v}} = \langle 2, 3, 3 \rangle$. Find the measures of the angles formed by the following vectors.

a. $\vec{\mathbf{v}}$ and $\hat{\mathbf{i}}$
b. $\vec{\mathbf{v}}$ and $\hat{\mathbf{j}}$
c. $\vec{\mathbf{v}}$ and $\hat{\mathbf{k}}$

Solution

a.Let α be the angle formed by $\vec{\mathbf{v}}$ and $\hat{\mathbf{i}}$:

$$\cos\alpha = \frac{\vec{\mathbf{v}} \cdot \hat{\mathbf{i}}}{\|\vec{\mathbf{v}}\| \cdot \|\hat{\mathbf{i}}\|} = \frac{\langle 2, 3, 3 \rangle \cdot \langle 1, 0, 0 \rangle}{\sqrt{2^2 + 3^2 + 3^2}\sqrt{1}} = \frac{2}{\sqrt{22}}$$

$$\alpha = \arccos \frac{2}{\sqrt{22}} \approx 1.130 \text{ rad.}$$

b. Let β represent the angle formed by $\vec{\mathbf{v}}$ and $\hat{\mathbf{j}}$:

$$\cos\beta = \frac{\vec{\mathbf{v}} \cdot \hat{\mathbf{j}}}{\|\vec{\mathbf{v}}\| \cdot \|\hat{\mathbf{j}}\|} = \frac{\langle 2, 3, 3 \rangle \cdot \langle 0, 1, 0 \rangle}{\sqrt{2^2 + 3^2 + 3^2}\sqrt{1}} = \frac{3}{\sqrt{22}}$$

$$\beta = \arccos \frac{3}{\sqrt{22}} \approx 0.877 \text{ rad.}$$

c. Let γ represent the angle formed by $\vec{\mathbf{v}}$ and $\hat{\mathbf{k}}$:

$$\cos\gamma = \frac{\vec{\mathbf{v}} \cdot \hat{\mathbf{k}}}{\|\vec{\mathbf{v}}\| \cdot \|\hat{\mathbf{k}}\|} = \frac{\langle 2,3,3\rangle \cdot \langle 0,0,1\rangle}{\sqrt{2^2+3^2+3^2}\sqrt{1}} = \frac{3}{\sqrt{22}}$$

$$\gamma = \arccos\frac{3}{\sqrt{22}} \approx 0.877 \text{ rad.}$$

Exercise 4.4.5

Let $\vec{\mathbf{v}} = \langle 3, -5, 1\rangle$. Find the measure of the angles formed by each pair of vectors.

a. $\vec{\mathbf{v}}$ and $\hat{\mathbf{i}}$
b. $\vec{\mathbf{v}}$ and $\hat{\mathbf{j}}$
c. $\vec{\mathbf{v}}$ and $\hat{\mathbf{k}}$

Hint

$\hat{\mathbf{i}} = \langle 1,0,0\rangle$, $\hat{\mathbf{j}} = \langle 0,1,0\rangle$, and $\hat{\mathbf{k}} = \langle 0,0,1\rangle$

Answer

a. $\alpha \approx 1.04$ rad; b. $\beta \approx 2.58$ rad; c. $\gamma \approx 1.40$ rad



The angle a vector makes with each of the coordinate axes, called a direction angle, is very important in practical computations, especially in a field such as engineering. For example, in astronautical engineering, the angle at which a rocket is launched must be determined very precisely. A very small error in the angle can lead to the rocket going hundreds of miles off course. Direction angles are often calculated by using the dot product and the cosines of the angles, called the direction cosines. Therefore, we define both these angles and their cosines.

Definition: direction angles

The angles formed by a nonzero vector and the coordinate axes are called the **direction angles** for the vector (Figure 4.4.5). The cosines for these angles are called the **direction cosines.**

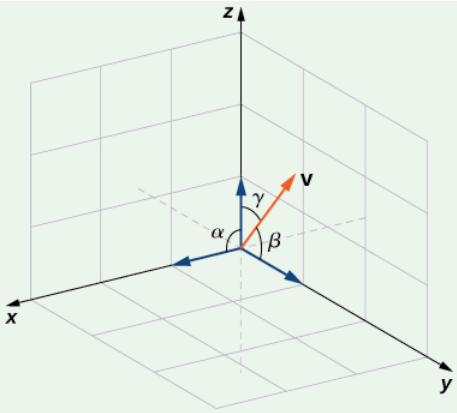


Figure 4.4.5: Angle α is formed by vector $\vec{\mathbf{v}}$ and unit vector $\hat{\mathbf{i}}$. Angle β is formed by vector $\vec{\mathbf{v}}$ and unit vector $\hat{\mathbf{j}}$. Angle γ is formed by vector $\vec{\mathbf{v}}$ and unit vector $\hat{\mathbf{k}}$.

In Example, the direction cosines of $\vec{\mathbf{v}} = \langle 2, 3, 3 \rangle$ are $\cos\alpha = \dfrac{2}{\sqrt{22}}, \cos\beta = \dfrac{3}{\sqrt{22}}$, and $\cos\gamma = \dfrac{3}{\sqrt{22}}$. The direction angles of $\vec{\mathbf{v}}$ are $\alpha = 1.130$ rad, $\beta = 0.877$ rad, and $\gamma = 0.877$ rad.

So far, we have focused mainly on vectors related to force, movement, and position in three-dimensional physical space. However, vectors are often used in more abstract ways. For example, suppose a fruit vendor sells apples, bananas, and oranges. On a given day, he sells 30 apples, 12 bananas, and 18 oranges. He might use a quantity vector, $\vec{\mathbf{q}} = \langle 30, 12, 18 \rangle$, to represent the quantity of fruit he sold that day. Similarly, he might want to use a price vector, $\vec{\mathbf{p}} = \langle 0.50, 0.25, 1 \rangle$,to indicate that he sells his apples for 50¢ each, bananas for 25¢ each, and oranges for $1 apiece. In this example, although we could still graph these vectors, we do not interpret them as literal representations of position in the physical world. We are simply using vectors to keep track of particular pieces of information about apples, bananas, and oranges.

This idea might seem a little strange, but if we simply regard vectors as a way to order and store data, we find they can be quite a powerful tool. Going back to the fruit vendor, let's think about the dot product, $\vec{\mathbf{q}} \cdot \vec{\mathbf{p}}$. We compute it by multiplying the number of apples sold (30) by the price per apple (50¢), the number of bananas sold by the price per banana, and the number of oranges sold by the price per orange. We then add all these values together. So, in this example, the dot product tells us how much money the fruit vendor had in sales on that particular day.

When we use vectors in this more general way, there is no reason to limit the number of components to three. What if the fruit vendor decides to start selling grapefruit? In that case, he would want to use four-dimensional quantity and price vectors to represent the number of apples, bananas, oranges, and grapefruit sold, and their unit prices. As you might expect, to calculate the dot product of four-dimensional vectors, we simply add the products of the components as before, but the sum has four terms instead of three.

Example 4.4.6: Using Vectors in an Economic Context

AAA Party Supply Store sells invitations, party favors, decorations, and food service items such as paper plates and napkins. When AAA buys its inventory, it pays 25¢ per package for invitations and party favors. Decorations cost AAA 50¢ each, and food service items cost 20¢ per package. AAA sells invitations for $2.50 per package and party favors for $1.50 per package. Decorations sell for $4.50 each and food service items for $1.25 per package.

During the month of May, AAA Party Supply Store sells 1258 invitations, 342 party favors, 2426 decorations, and 1354 food service items. Use vectors and dot products to calculate how much money AAA made in sales during the month of May. How much did the store make in profit?

Solution

The cost, price, and quantity vectors are

$$\vec{\mathbf{c}} = \langle 0.25, 0.25, 0.50, 0.20 \rangle$$

$$\vec{\mathbf{p}} = \langle 2.50, 1.50, 4.50, 1.25 \rangle$$

$$\vec{\mathbf{q}} = \langle 1258, 342, 2426, 1354 \rangle.$$

AAA sales for the month of May can be calculated using the dot product $\vec{\mathbf{p}} \cdot \vec{\mathbf{q}}$. We have

$$\begin{aligned} \vec{\mathbf{p}} \cdot \vec{\mathbf{q}} &= \langle 2.50, 1.50, 4.50, 1.25 \rangle \cdot \langle 1258, 342, 2426, 1354 \rangle \\ &= 3145 + 513 + 10917 + 1692.5 \\ &= 16267.5. \end{aligned}$$

So, AAA took in \$16,267.50 during the month of May. To calculate the profit, we must first calculate how much AAA paid for the items sold. We use the dot product $c \cdot q$ to get

$$\begin{aligned} \vec{\mathbf{c}} \cdot \vec{\mathbf{q}} &= \langle 0.25, 0.25, 0.50, 0.20 \rangle \cdot \langle 1258, 342, 2426, 1354 \rangle \\ &= 314.5 + 85.5 + 1213 + 270.8 \\ &= 1883.8. \end{aligned}$$

So, AAA paid \$1,883.30 for the items they sold. Their profit, then, is given by

$$\vec{\mathbf{p}} \cdot \vec{\mathbf{q}} - \vec{\mathbf{c}} \cdot \vec{\mathbf{q}} = 16267.5 - 1883.8 = 14383.7.$$

Therefore, AAA Party Supply Store made \$14,383.70 in May.

Exercise 4.4.6

On June 1, AAA Party Supply Store decided to increase the price they charge for party favors to \$2 per package. They also changed suppliers for their invitations, and are now able to purchase invitations for only 10¢ per package. All their other costs and prices remain the same. If AAA sells 1408 invitations, 147 party favors, 2112 decorations, and 1894 food service items in the month of June, use vectors and dot products to calculate their total sales and profit for June.

Hint

Use four-dimensional vectors for cost, price, and quantity sold.

Answer

Sales = \$15,685.50; profit = \$14,073.15

Projections

As we have seen, addition combines two vectors to create a resultant vector. But what if we are given a vector and we need to find its component parts? We use vector projections to perform the opposite process; they can break down a vector into its components. The magnitude of a vector projection is a scalar projection. For example, if a child is pulling the handle of a wagon at a 55° angle, we can use projections to determine how much of the force on the handle is actually moving the wagon forward (4.4.6). We return to this example and learn how to solve it after we see how to calculate projections.

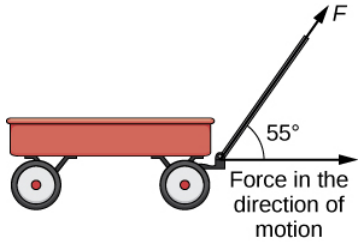


Figure 4.4.6: When a child pulls a wagon, only the horizontal component of the force propels the wagon forward.

Definition: Vector and Projection

The **vector projection** of $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$ is the vector labeled $\text{proj}_{\vec{\mathbf{u}}} \vec{\mathbf{v}}$ in Figure 4.4.7. It has the same initial point as $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ and the same direction as $\vec{\mathbf{u}}$, and represents the component of $\vec{\mathbf{v}}$ that acts in the direction of $\vec{\mathbf{u}}$. If θ represents the angle between $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$, then, by properties of triangles, we know the length of $\text{proj}_{\vec{\mathbf{u}}} \vec{\mathbf{v}}$ is $\|\text{proj}_{\vec{\mathbf{u}}} \vec{\mathbf{v}}\| = \|\vec{\mathbf{v}}\| \cos\theta$. When expressing $\cos\theta$ in terms of the dot product, this becomes

$$\|\text{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}\| = \|\vec{\mathbf{v}}\|\cos\theta = \|\vec{\mathbf{v}}\|\left(\frac{\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\|}\right) = \frac{\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|}. \tag{4.4.11}$$

We now multiply by a unit vector in the direction of $\vec{\mathbf{u}}$ to get $\text{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}$:

$$\text{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}} = \frac{\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|}\left(\frac{1}{\|\vec{\mathbf{u}}\|}\vec{\mathbf{u}}\right) = \frac{\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|^2}\vec{\mathbf{u}}. \tag{4.4.12}$$

The length of this vector is also known as the **scalar projection** of $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$ and is denoted by

$$\|\text{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}\| = \text{comp}_{\vec{\mathbf{u}}}\vec{\mathbf{v}} = \frac{\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|}. \tag{4.4.13}$$

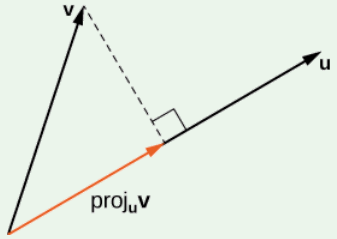


Figure 4.4.7: The projection of $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$ shows the component of vector $\vec{\mathbf{v}}$ in the direction of $\vec{\mathbf{u}}$.

Example 4.4.7: Finding Projections

Find the projection of $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$.

a. $\vec{\mathbf{v}} = \langle 3, 5, 1\rangle$ and $\vec{\mathbf{u}} = \langle -1, 4, 3\rangle$

b. $\vec{\mathbf{v}} = 3\hat{\mathbf{i}} - 2\hat{\mathbf{j}}$ and $\vec{\mathbf{u}} = \hat{\mathbf{i}} + 6\hat{\mathbf{j}}$

Solution

a. Substitute the components of $\vec{\mathbf{v}}$ and $\vec{\mathbf{u}}$ into the formula for the projection:

$$\begin{aligned}\text{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}} &= \frac{\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|^2}\vec{\mathbf{u}} \\ &= \frac{\langle -1, 4, 3\rangle\cdot\langle 3, 5, 1\rangle}{\|\langle -1, 4, 3\rangle\|^2}\langle -1, 4, 3\rangle \\ &= \frac{-3 + 20 + 3}{(-1)^2 + 4^2 + 3^2}\langle -1, 4, 3\rangle \\ &= \frac{20}{26}\langle -1, 4, 3\rangle \\ &= \langle -\frac{10}{13}, \frac{40}{13}, \frac{30}{13}\rangle.\end{aligned}$$

b. To find the two-dimensional projection, simply adapt the formula to the two-dimensional case:

$$\begin{aligned}\text{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}} &= \frac{\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|^2}\vec{\mathbf{u}} \\ &= \frac{(\hat{\mathbf{i}}+6\hat{\mathbf{j}})\cdot(3i-2j)}{\left\|\hat{\mathbf{i}}+6\hat{\mathbf{j}}\right\|^2}(i+6j) \\ &= \frac{1(3)+6(-2)}{1^2+6^2}(\hat{\mathbf{i}}+6\hat{\mathbf{j}}) \\ &= -\frac{9}{37}(\hat{\mathbf{i}}+6\hat{\mathbf{j}}) \\ &= -\frac{9}{37}\hat{\mathbf{i}}-\frac{54}{37}\hat{\mathbf{j}}.\end{aligned}$$



Sometimes it is useful to decompose vectors—that is, to break a vector apart into a sum. This process is called the **resolution of a vector into components.** Projections allow us to identify two orthogonal vectors having a desired sum. For example, let $\vec{\mathbf{v}} = \langle 6, -4\rangle$ and let $\vec{\mathbf{u}} = \langle 3, 1\rangle$. We want to decompose the vector $\vec{\mathbf{v}}$ into orthogonal components such that one of the component vectors has the same direction as $\vec{\mathbf{u}}$.

We first find the component that has the same direction as $\vec{\mathbf{u}}$ by projecting $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$. Let $\vec{\mathbf{p}} = \text{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}$. Then, we have

$$\begin{aligned}\vec{\mathbf{p}} &= \frac{\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|^2}\vec{\mathbf{u}} \\ &= \frac{18-4}{9+1}\vec{\mathbf{u}} \\ &= \frac{7}{5}\vec{\mathbf{u}} = \frac{7}{5}\langle 3,1\rangle = \langle\frac{21}{5},\frac{7}{5}\rangle.\end{aligned}$$

Now consider the vector $\vec{\mathbf{q}} = \vec{\mathbf{v}} - \vec{\mathbf{p}}$. We have

$$\begin{aligned}\vec{\mathbf{q}} &= \vec{\mathbf{v}} - \vec{\mathbf{p}} \\ &= \langle 6,-4\rangle - \langle\frac{21}{5},\frac{7}{5}\rangle \\ &= \langle\frac{9}{5},-\frac{27}{5}\rangle.\end{aligned}$$

Clearly, by the way we defined $\vec{\mathbf{q}}$, we have $\vec{\mathbf{v}} = \vec{\mathbf{q}} + \vec{\mathbf{p}}$, and

$$\begin{aligned}\vec{\mathbf{q}}\cdot\vec{\mathbf{p}} &= \langle\frac{9}{5},-\frac{27}{5}\rangle\cdot\langle\frac{21}{5},\frac{7}{5}\rangle \\ &= \frac{9(21)}{25}+-\frac{27(7)}{25} \\ &= \frac{189}{25}-\frac{189}{25}=0.\end{aligned}$$

Therefore, $\vec{\mathbf{q}}$ and $\vec{\mathbf{p}}$ are orthogonal.

Example 4.4.8: Resolving Vectors into Components

Express $\vec{\mathbf{v}}=\langle 8,-3,-3\rangle$ as a sum of orthogonal vectors such that one of the vectors has the same direction as $\vec{\mathbf{u}}=\langle 2,3,2\rangle$.

Solution

Let $\vec{\mathbf{p}}$ represent the projection of $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$:

$$\begin{aligned}\vec{\mathbf{p}} &= \text{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}} \\ &= \frac{\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|^2}\vec{\mathbf{u}} \\ &= \frac{\langle 2,3,2\rangle\cdot\langle 8,-3,-3\rangle}{\|\langle 2,3,2\rangle\|^2}\langle 2,3,2\rangle \\ &= \frac{16-9-6}{2^2+3^2+2^2}\langle 2,3,2\rangle \\ &= \frac{1}{17}\langle 2,3,2\rangle \\ &= \langle\frac{2}{17},\frac{3}{17},\frac{2}{17}\rangle.\end{aligned}$$

Then,

$$\begin{aligned}\vec{\mathbf{q}} &= \vec{\mathbf{v}}-\vec{\mathbf{p}}=\langle 8,-3,-3\rangle-\langle\frac{2}{17},\frac{3}{17},\frac{2}{17}\rangle \\ &= \langle\frac{134}{17},-\frac{54}{17},-\frac{53}{17}\rangle.\end{aligned}$$

To check our work, we can use the dot product to verify that $\vec{\mathbf{p}}$ and $\vec{\mathbf{q}}$ are orthogonal vectors:

$$\begin{aligned}\vec{\mathbf{p}}\cdot\vec{\mathbf{q}} &= \langle\frac{2}{17},\frac{3}{17},\frac{2}{17}\rangle\cdot\langle\frac{134}{17},-\frac{54}{17},-\frac{53}{17}\rangle \\ &= \frac{268}{17}-\frac{162}{17}-\frac{106}{17}=0.\end{aligned}$$

Then,

$$\vec{\mathbf{v}}=\vec{\mathbf{p}}+\vec{\mathbf{q}}=\langle\frac{2}{17},\frac{3}{17},\frac{2}{17}\rangle+\langle\frac{134}{17},-\frac{54}{17},-\frac{53}{17}\rangle.$$

Exercise 4.4.7

Express $\vec{\mathbf{v}}=5\,\hat{\mathbf{i}}-\hat{\mathbf{j}}$ as a sum of orthogonal vectors such that one of the vectors has the same direction as $\vec{\mathbf{u}}=4\,\hat{\mathbf{i}}+2\hat{\mathbf{j}}$.

Hint

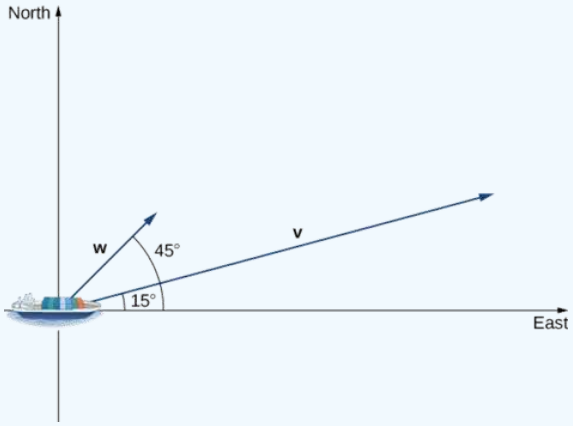
Start by finding the projection of $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$.

Answer

$\vec{\mathbf{v}} = \vec{\mathbf{p}} + \vec{\mathbf{q}}$, where $\vec{\mathbf{p}} = \frac{18}{5}\hat{\mathbf{i}} + \frac{9}{5}\hat{\mathbf{j}}$ and $\vec{\mathbf{q}} = \frac{7}{5}\hat{\mathbf{i}} - \frac{14}{5}\hat{\mathbf{j}}$

Example 4.4.9: Scalar Projection of Velocity

A container ship leaves port traveling 15° north of east. Its engine generates a speed of 20 knots along that path (see the following figure). In addition, the ocean current moves the ship northeast at a speed of 2 knots. Considering both the engine and the current, how fast is the ship moving in the direction 15° north of east? Round the answer to two decimal places.



Solution

Let $\vec{\mathbf{v}}$ be the velocity vector generated by the engine, and let w be the velocity vector of the current. We already know $\|\vec{\mathbf{v}}\| = 20$ along the desired route. We just need to add in the scalar projection of $\vec{\mathbf{w}}$ onto $\vec{\mathbf{v}}$. We get

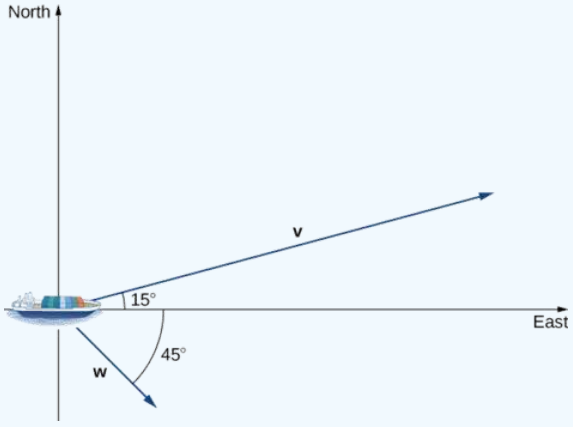
$$\text{comp}_{\vec{\mathbf{v}}}\vec{\mathbf{w}} = \frac{\vec{\mathbf{v}} \cdot \vec{\mathbf{w}}}{\|\vec{\mathbf{v}}\|}$$

$$= \frac{\|\vec{\mathbf{v}}\|\|\vec{\mathbf{w}}\|\cos(30^\circ)}{\|\vec{\mathbf{v}}\|} = \|\vec{\mathbf{w}}\|\cos(30^\circ) = 2\frac{\sqrt{3}}{2} = \sqrt{3} \approx 1.73 \text{ knots.}$$

The ship is moving at 21.73 knots in the direction 15° north of east.

Exercise 4.4.8

Repeat the previous example, but assume the ocean current is moving southeast instead of northeast, as shown in the following figure.



Hint

Compute the scalar projection of $\vec{\mathbf{w}}$ onto $\vec{\mathbf{v}}$.

https://math.libretexts.org/@go/page/84012

Answer

21 knots

Work

Now that we understand dot products, we can see how to apply them to real-life situations. The most common application of the dot product of two vectors is in the calculation of work.

From physics, we know that work is done when an object is moved by a force. When the force is constant and applied in the same direction the object moves, then we define the work done as the product of the force and the distance the object travels: $W = Fd$. We saw several examples of this type in earlier chapters. Now imagine the direction of the force is different from the direction of motion, as with the example of a child pulling a wagon. To find the work done, we need to multiply the component of the force that acts in the direction of the motion by the magnitude of the displacement. The dot product allows us to do just that. If we represent an applied force by a vector $\vec{\mathbf{F}}$ and the displacement of an object by a vector $\vec{\mathbf{s}}$, then the **work done by the force** is the dot product of $\vec{\mathbf{F}}$ and $\vec{\mathbf{s}}$.

Definition: Constant Force

When a constant force is applied to an object so the object moves in a straight line from point P to point Q, the work W done by the force $\vec{\mathbf{F}}$, acting at an angle θ from the line of motion, is given by

$$W = \vec{\mathbf{F}} \cdot \overrightarrow{PQ} = \| \vec{\mathbf{F}} \| \| \overrightarrow{PQ} \| \cos\theta. \tag{4.4.14}$$

Let's revisit the problem of the child's wagon introduced earlier. Suppose a child is pulling a wagon with a force having a magnitude of 8 lb on the handle at an angle of 55°. If the child pulls the wagon 50 ft, find the work done by the force (Figure 4.4.8).

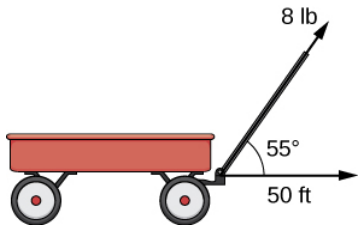


Figure 4.4.8: The horizontal component of the force is the projection of $\vec{\mathbf{F}}$ *onto the positive* x*-axis.*

We have

$$W = \| \vec{\mathbf{F}} \| \| \overrightarrow{PQ} \| \cos\theta = 8(50)(\cos(55^\circ)) \approx 229 \text{ ft·lb}.$$

In U.S. standard units, we measure the magnitude of force $\left\| \vec{\mathbf{F}} \right\|$ in pounds. The magnitude of the displacement vector $\left\| \overrightarrow{PQ} \right\|$ tells us how far the object moved, and it is measured in feet. The customary unit of measure for work, then, is the foot-pound. One foot-pound is the amount of work required to move an object weighing 1 lb a distance of 1 ft straight up. In the metric system, the unit of measure for force is the newton (N), and the unit of measure of magnitude for work is a newton-meter (N·m), or a joule (J).

Example 4.4.10: Calculating Work

A conveyor belt generates a force $\vec{\mathbf{F}} = 5\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + \hat{\mathbf{k}}$ that moves a suitcase from point $(1, 1, 1)$ to point $(9, 4, 7)$ along a straight line. Find the work done by the conveyor belt. The distance is measured in meters and the force is measured in newtons.

Solution

The displacement vector $\overrightarrow{PQ}$ has initial point $(1, 1, 1)$ and terminal point $(9, 4, 7)$:

$$\overrightarrow{PQ} = \langle 9-1, 4-1, 7-1 \rangle = \langle 8, 3, 6 \rangle = 8\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}}.$$

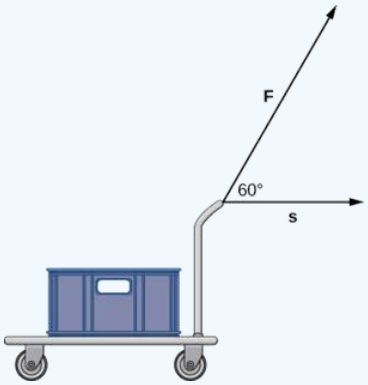
Work is the dot product of force and displacement:

$$\begin{aligned} W &= \vec{\mathbf{F}} \cdot \overrightarrow{PQ} \\ &= (5\,\hat{\mathbf{i}} - 3\,\hat{\mathbf{j}} + \hat{\mathbf{k}}) \cdot (8\,\hat{\mathbf{i}} + 3\,\hat{\mathbf{j}} + 6\hat{\mathbf{k}}) \\ &= 5(8) + (-3)(3) + 1(6) \\ &= 37\,\text{N}\cdot\text{m} \\ &= 37\,\text{J} \end{aligned}$$



Exercise 4.4.9

A constant force of 30 lb is applied at an angle of 60° to pull a handcart 10 ft across the ground. What is the work done by this force?



Hint

Use the definition of work as the dot product of force and distance.

Answer

150 ft-lb

Key Concepts

- The dot product, or scalar product, of two vectors $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$ is $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = u_1 v_1 + u_2 v_2 + u_3 v_3$.
- The dot product satisfies the following properties:
  - $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = \vec{\mathbf{v}} \cdot \vec{\mathbf{u}}$
  - $\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} + \vec{\mathbf{w}}) = \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} + \vec{\mathbf{u}} \cdot \vec{\mathbf{w}}$
  - $c(\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}) = (c\vec{\mathbf{u}}) \cdot \vec{\mathbf{v}} = \vec{\mathbf{u}} \cdot (c\vec{\mathbf{v}})$
  - $\vec{\mathbf{v}} \cdot \vec{\mathbf{v}} = \|\vec{\mathbf{v}}\|^2$
- The dot product of two vectors can be expressed, alternatively, as $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = \|\vec{\mathbf{u}}\| \|\vec{\mathbf{v}}\| \cos\theta$. This form of the dot product is useful for finding the measure of the angle formed by two vectors.
- Vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are orthogonal if $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = 0$.
- The angles formed by a nonzero vector and the coordinate axes are called the *direction angles* for the vector. The cosines of these angles are known as the *direction cosines.*
- The vector projection of $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$ is the vector $\text{proj}_{\vec{\mathbf{u}}} \vec{\mathbf{v}} = \dfrac{\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|^2} \vec{\mathbf{u}}$. The magnitude of this vector is known as the *scalar projection* of $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$, given by $\text{comp}_{\vec{\mathbf{u}}} \vec{\mathbf{v}} = \dfrac{\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|}$.
- Work is done when a force is applied to an object, causing displacement. When the force is represented by the vector $\vec{\mathbf{F}}$ and the displacement is represented by the vector $\vec{\mathbf{s}}$, then the work done W is given by the formula $W = \vec{\mathbf{F}} \cdot \vec{\mathbf{s}} = \| \vec{\mathbf{F}} \| \, \| \vec{\mathbf{s}} \| \cos\theta$.

Key Equations

- **Dot product of $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$**

  $$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = u_1 v_1 + u_2 v_2 + u_3 v_3 = \|\vec{\mathbf{u}}\| \|\vec{\mathbf{v}}\| \cos\theta$$

- **Cosine of the angle formed by $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$**

  $$\cos\theta = \frac{\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\| \|\vec{\mathbf{v}}\|}$$

- **Vector projection of $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$**

  $$\text{proj}_{\vec{\mathbf{u}}} \vec{\mathbf{v}} = \frac{\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|^2} \vec{\mathbf{u}}$$

- **Scalar projection of $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$**

  $$\text{comp}_{\vec{\mathbf{u}}} \vec{\mathbf{v}} = \frac{\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|}$$

- **Work done by a force $\vec{\mathbf{F}}$ to move an object through displacement vector $\overrightarrow{PQ}$**

  $$W = \vec{\mathbf{F}} \cdot \overrightarrow{PQ} = \| \vec{\mathbf{F}} \| \| \overrightarrow{PQ} \| \cos\theta$$

Glossary

direction angles
the angles formed by a nonzero vector and the coordinate axes

direction cosines
the cosines of the angles formed by a nonzero vector and the coordinate axes

dot product or scalar product
$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = u_1 v_1 + u_2 v_2 + u_3 v_3$ where $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$

scalar projection
the magnitude of the vector projection of a vector

orthogonal vectors

vectors that form a right angle when placed in standard position

vector projection

the component of a vector that follows a given direction

work done by a force

work is generally thought of as the amount of energy it takes to move an object; if we represent an applied force by a vector $\vec{\mathbf{F}}$ and the displacement of an object by a vector $\vec{\mathbf{s}}$, then the work done by the force is the dot product of $\vec{\mathbf{F}}$ and $\vec{\mathbf{s}}$.

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- edited for vector notation by Paul Seeburger

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4.4E: Exercises for Section 4.4

For exercises 1-4, the vectors $\vec{u}$ and $\vec{v}$ are given. Calculate the dot product $\vec{u} \cdot \vec{v}$.

1) $\vec{u} = \langle 3, 0 \rangle, \quad \vec{v} = \langle 2, 2 \rangle$

Answer

$\vec{u} \cdot \vec{v} = 6$

2) $\vec{u} = \langle 3, -4 \rangle, \quad \vec{v} = \langle 4, 3 \rangle$

3) $\vec{u} = \langle 2, 2, -1 \rangle, \quad \vec{v} = \langle -1, 2, 2 \rangle$

Answer

$\vec{u} \cdot \vec{v} = 0$

4) $\vec{u} = \langle 4, 5, -6 \rangle, \quad \vec{v} = \langle 0, -2, -3 \rangle$

For exercises 5-8, the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are given. Determine the vectors $(\vec{a} \cdot \vec{b})\vec{c}$ and $(\vec{a} \cdot \vec{c})\vec{b}$. Express the vectors in component form.

5) $\vec{a} = \langle 2, 0, -3 \rangle, \quad \vec{b} = \langle -4, -7, 1 \rangle, \quad \vec{c} = \langle 1, 1, -1 \rangle$

Answer

$(\vec{a} \cdot \vec{b})\vec{c} = \langle -11, -11, 11 \rangle; (\vec{a} \cdot \vec{c})\vec{b} = \langle -20, -35, 5 \rangle$

6) $\vec{a} = \langle 0, 1, 2 \rangle, \quad \vec{b} = \langle -1, 0, 1 \rangle, \quad \vec{c} = \langle 1, 0, -1 \rangle$

7) $\vec{a} = \hat{i} + \hat{j}, \quad \vec{b} = \hat{i} - \hat{k}, \quad \vec{c} = \hat{i} - 2\hat{k}$

Answer

$(\vec{a} \cdot \vec{b})\vec{c} = \langle 1, 0, -2 \rangle; (\vec{a} \cdot \vec{c})\vec{b} = \langle 1, 0, -1 \rangle$

8) $\vec{a} = \hat{i} - \hat{j} + \hat{k}, \quad \vec{b} = \hat{j} + 3\hat{k}, \quad \vec{c} = -\hat{i} + 2\hat{j} - 4\hat{k}$

For exercises 9-12, two vectors are given.

a. Find the measure of the angle θ between these two vectors. Express the answer in radians rounded to two decimal places, if it is not possible to express it exactly.

b. Is θ an acute angle?

9) [T] $\vec{a} = \langle 3, -1 \rangle, \quad \vec{b} = \langle -4, 0 \rangle$

Answer

$a.\ \theta = 2.82$ rad; $b.\ \theta$ is not acute.

10) [T] $\vec{a} = \langle 2, 1 \rangle, \quad \vec{b} = \langle -1, 3 \rangle$

11) $\vec{u} = 3\hat{i}, \quad \vec{v} = 4\hat{i} + 4\hat{j}$

Answer

$a.\ \theta = \frac{\pi}{4}$ rad; $b.\ \theta$ is acute.

12) $\vec{u} = 5\hat{i}, \quad \vec{v} = -6\hat{i} + 6\hat{j}$

For exercises 13-18, find the measure of the angle between the three-dimensional vectors $\vec{a}$ and $\vec{b}$. Express the answer in radians rounded to two decimal places, if it is not possible to express it exactly.

13) $\vec{a} = \langle 3, -1, 2 \rangle, \quad \vec{b} = \langle 1, -1, -2 \rangle$

Answer

$\theta = \frac{\pi}{2}$

14) $\vec{\mathbf{a}} = \langle 0, -1, -3 \rangle, \quad \vec{\mathbf{b}} = \langle 2, 3, -1 \rangle$

15) $\vec{\mathbf{a}} = \hat{\mathbf{i}} + \hat{\mathbf{j}}, \quad \vec{\mathbf{b}} = \hat{\mathbf{j}} - \hat{\mathbf{k}}$

Answer

$\theta = \frac{\pi}{3}$

16) $\vec{\mathbf{a}} = \hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}, \quad \vec{\mathbf{b}} = \hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}$

17) [T] $\vec{\mathbf{a}} = 3\hat{\mathbf{i}} - \hat{\mathbf{j}} - 2\hat{\mathbf{k}}, \quad \vec{\mathbf{b}} = \vec{\mathbf{v}} + \vec{\mathbf{w}}$, where $\vec{\mathbf{v}} = -2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ and $\vec{\mathbf{w}} = \hat{\mathbf{i}} + 2\hat{\mathbf{k}}$

Answer

$\theta = 2$ rad

18) [T] $\vec{\mathbf{a}} = 3\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}, \quad \vec{\mathbf{b}} = \vec{\mathbf{v}} - \vec{\mathbf{w}}$, where $\vec{\mathbf{v}} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 4\hat{\mathbf{k}}$ and $\vec{\mathbf{w}} = 6\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$

For exercises 19-22, determine whether the given vectors are orthogonal.

19) $\vec{\mathbf{a}} = \langle x, y \rangle, \quad \vec{\mathbf{b}} = \langle -y, x \rangle$, where x and y are nonzero real numbers

Answer

Orthogonal

20) $\vec{\mathbf{a}} = \langle x, x \rangle, \quad \vec{\mathbf{b}} = \langle -y, y \rangle$, where x and y are nonzero real numbers

21) $\vec{\mathbf{a}} = 3\hat{\mathbf{i}} - \hat{\mathbf{j}} - 2\hat{\mathbf{k}}, \quad \vec{\mathbf{b}} = -2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + \hat{\mathbf{k}}$

Answer

Not orthogonal

22) $\vec{\mathbf{a}} = \hat{\mathbf{i}} - \hat{\mathbf{j}}, \quad \vec{\mathbf{b}} = 7\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$

23) Find all two-dimensional vectors $\vec{\mathbf{a}}$ orthogonal to vector $\vec{\mathbf{b}} = \langle 3, 4 \rangle$. Express the answer in component form.

Answer

$\vec{\mathbf{a}} = \langle -\frac{4\alpha}{3}, \alpha \rangle$, where $\alpha \neq 0$ is a real number

24) Find all two-dimensional vectors $\vec{\mathbf{a}}$ orthogonal to vector $\vec{\mathbf{b}} = \langle 5, -6 \rangle$. Express the answer by using standard unit vectors.

25) Determine all three-dimensional vectors $\vec{\mathbf{u}}$ orthogonal to vector $\vec{\mathbf{v}} = \langle 1, 1, 0 \rangle$. Express the answer by using standard unit vectors.

Answer

$\vec{\mathbf{u}} = -\alpha\hat{\mathbf{i}} + \alpha\hat{\mathbf{j}} + \beta\hat{\mathbf{k}}$, where α and β are real numbers such that $\alpha^2 + \beta^2 \neq 0$

26) Determine all three-dimensional vectors $\vec{\mathbf{u}}$ orthogonal to vector $\vec{\mathbf{v}} = \hat{\mathbf{i}} - \hat{\mathbf{j}} - \hat{\mathbf{k}}$. Express the answer in component form.

27) Determine the real number α such that vectors $\vec{\mathbf{a}} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$ and $\vec{\mathbf{b}} = 9\hat{\mathbf{i}} + \alpha\hat{\mathbf{j}}$ are orthogonal.

Answer

$\alpha = -6$

28) Determine the real number α such that vectors $\vec{\mathbf{a}} = -3\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$ and $\vec{\mathbf{b}} = 2\hat{\mathbf{i}} + \alpha\hat{\mathbf{j}}$ are orthogonal.

29) [T] Consider the points $P(4, 5)$ and $Q(5, -7)$, and note that O represents the origin.

a. Determine vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$. Express the answer by using standard unit vectors.

b. Determine the measure of angle O in triangle OPQ. Express the answer in degrees rounded to two decimal places.

Answer

$a.\ \overrightarrow{OP} = 4\hat{\mathbf{i}} + 5\hat{\mathbf{j}}, \quad \overrightarrow{OQ} = 5\hat{\mathbf{i}} - 7\hat{\mathbf{j}}; \quad b.\ 105.8^\circ$

30) [T] Consider points $A(1, 1), B(2, -7)$, and $C(6, 3)$.

a. Determine vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$. Express the answer in component form.

b. Determine the measure of angle B in triangle ABC. Express the answer in degrees rounded to two decimal places.

31) Determine the measure of angle A in triangle ABC, where $A(1, 1, 8), B(4, -3, -4)$, and $C(-3, 1, 5)$. Express your answer in degrees rounded to two decimal places.

Answer

68.33°

32) Consider points $P(3, 7, -2)$ and $Q(1, 1, -3)$. Determine the angle between vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$. [Remember that O represents the origin.] Express the answer in degrees rounded to two decimal places.

For exercises 33-34, determine which (if any) pairs of the following vectors are orthogonal.

33) $\vec{\mathbf{u}} = \langle 3, 7, -2 \rangle, \quad \vec{\mathbf{v}} = \langle 5, -3, -3 \rangle, \quad \vec{\mathbf{w}} = \langle 0, 1, -1 \rangle$

Answer

$\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are orthogonal; $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ are orthogonal, but $\vec{\mathbf{u}}$ and $\vec{\mathbf{w}}$ are **not** orthogonal.

34) $\vec{\mathbf{u}} = \hat{\mathbf{i}} - \hat{\mathbf{k}}, \quad \vec{\mathbf{v}} = 5\hat{\mathbf{j}} - 5\hat{\mathbf{k}}, \quad \vec{\mathbf{w}} = 10\hat{\mathbf{j}}$

35) Use vectors to show that a parallelogram with equal diagonals is a **rectangle**.

36) Use vectors to show that the diagonals of a **rhombus** are perpendicular.

37) Show that $\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} + \vec{\mathbf{w}}) = \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} + \vec{\mathbf{u}} \cdot \vec{\mathbf{w}}$ is true for any vectors $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$.

38) Verify the identity $\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} + \vec{\mathbf{w}}) = \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} + \vec{\mathbf{u}} \cdot \vec{\mathbf{w}}$ for vectors $\vec{\mathbf{u}} = \langle 1, 0, 4 \rangle$, $\vec{\mathbf{v}} = \langle -2, 3, 5 \rangle$, and $\vec{\mathbf{w}} = \langle 4, -2, 6 \rangle$.

For exercises 39-41, determine $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}$ using the given information.

39) $\|\vec{\mathbf{u}}\| = 5, \quad \|\vec{\mathbf{v}}\| = 3$, and the angle between $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ is $\pi/6$ rad.

Answer

$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = (5)(3)\cos\frac{\pi}{6} = 15 \cdot \frac{\sqrt{3}}{2} = \frac{15\sqrt{3}}{2}$

40) $\|\vec{\mathbf{u}}\| = 20, \quad \|\vec{\mathbf{v}}\| = 15$, and the angle between $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ is $5\pi/4$ rad.

41) $\|\vec{\mathbf{u}}\| = 7, \quad \|\vec{\mathbf{v}}\| = 12$, and the angle between $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ is $\pi/2$ rad.

Answer

$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = (7)(12)\cos\frac{\pi}{2} = 84 \cdot 0 = 0$

42) Considering the definition of the dot product, $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = \|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\|\cos\theta$, where θ is the angle between $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$, what can you say about the angle θ between two nonzero vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ if:

a. $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} > 0$?

b. $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} < 0$?

c. $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = 0$?

In exercises 43-45, you are given the vertices of a triangle. Use dot products to determine whether each triangle is *acute*, *obtuse*, or *right*.

43) $(2,3,0),\ (3,1,-2),\ (-1,4,5)$

Answer

Label the points P, Q and R. Then determine the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$.

Here we see that, if $P=(2,3,0),\ Q=(3,1,-2)$ and $R=(-1,4,5)$, so we have $\overrightarrow{PQ}=<1,-2,-2>$ and $\overrightarrow{PR}=<-3,1,5>$.

Then $\overrightarrow{PQ}\cdot\overrightarrow{PR}=-15<0$, so angle QPR is obtuse. Therefore this triangle must be an ***obtuse triangle***.

44) $(5,1,0),\ (7,1,1),\ (6,3,2)$

45) $(6,1,4),\ (3,2,-1),\ (2,3,1)$

Solution:

Label the points P, Q and R. Then determine the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$.

Here we see that, if $P=(6,1,4),\ Q=(3,2,-1)$ and $R=(2,3,1)$, so we have $\overrightarrow{PQ}=<-3,1,-5>$ and $\overrightarrow{PR}=<-4,2,-3>$.

Then $\overrightarrow{PQ}\cdot\overrightarrow{PR}=29>0$, so angle QPR is **acute**. This triangle could still be any of the three types, so we need to check another angle.

Considering angle PQR we find $\overrightarrow{QP}=<3,-1,5>$ and $\overrightarrow{QR}=<-1,1,2>$, so $\overrightarrow{QP}\cdot\overrightarrow{QR}=6>0$, showing angle PQR is also **acute**.

We then must check the third angle to determine whether this is a right triangle or an acute triangle.

To check angle PRQ we find $\overrightarrow{RP}=<4,-2,3>$ and $\overrightarrow{RQ}=<1,-1,-2>$, giving us $\overrightarrow{RP}\cdot\overrightarrow{RQ}=0$. Angle PRQ is thus a **right** angle.

Therefore this triangle is a ***right triangle***.

46) Which of the following operations are allowed for nonzero vectors $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$, and which are not. Explain your reasoning.

a. $\vec{\mathbf{u}}+(\vec{\mathbf{v}}\cdot\vec{\mathbf{w}})$ b. $(\vec{\mathbf{u}}\cdot\vec{\mathbf{v}})\cdot\vec{\mathbf{w}}$ c. $(\vec{\mathbf{u}}\cdot\vec{\mathbf{v}})\,\vec{\mathbf{w}}$
d. $(\vec{\mathbf{u}}+\vec{\mathbf{v}})\cdot\vec{\mathbf{w}}$ e. $(\vec{\mathbf{u}}+\vec{\mathbf{v}})\cdot\|\vec{\mathbf{w}}\|$ f. $\|\vec{\mathbf{u}}+\vec{\mathbf{v}}\|\|\vec{\mathbf{w}}\|$

Projections

For the exercises 47-50, given the vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$:

a. Find the vector projection $\text{Proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}$ of vector $\vec{\mathbf{v}}$ onto vector $\vec{\mathbf{u}}$ and the component of $\vec{\mathbf{v}}$ that is orthogonal to $\vec{\mathbf{u}}$, i.e., $\vec{\mathbf{v}}_{\text{perp}}$. Express your answers in component form.

b. Find the scalar projection $\text{comp}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}=\|\text{Proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}\|$ of vector $\vec{\mathbf{v}}$ onto vector $\vec{\mathbf{u}}$.

c. Find the vector projection $\text{Proj}_{\vec{\mathbf{v}}}\vec{\mathbf{u}}$ of vector $\vec{\mathbf{u}}$ onto vector $\vec{\mathbf{v}}$ and the component of $\vec{\mathbf{u}}$ that is orthogonal to $\vec{\mathbf{v}}$, i.e., $\vec{\mathbf{u}}_{\text{perp}}$. Express your answers in unit vector form.

d. Find the scalar projection $\text{comp}_{\vec{\mathbf{v}}}\vec{\mathbf{u}}$ of vector $\vec{\mathbf{u}}$ onto vector $\vec{\mathbf{v}}$.

47) $\vec{\mathbf{u}}=5\,\hat{\mathbf{i}}+2\hat{\mathbf{j}},\quad \vec{\mathbf{v}}=2\,\hat{\mathbf{i}}+3\hat{\mathbf{j}}$

Answer

a. $\text{Proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}=\langle\frac{80}{29},\frac{32}{29}\rangle$ and $\vec{\mathbf{v}}_{\text{perp}}=<-\frac{22}{29},\frac{55}{29}>$; b. $\text{comp}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}=\frac{16}{\sqrt{29}}=\frac{16\sqrt{29}}{29}$;
c. $\text{Proj}_{\vec{\mathbf{v}}}\vec{\mathbf{u}}=\frac{32}{13}\,\hat{\mathbf{i}}+\frac{48}{13}\,\hat{\mathbf{j}}$ and $\vec{\mathbf{u}}_{\text{perp}}=\frac{33}{13}\,\hat{\mathbf{i}}-\frac{22}{13}\,\hat{\mathbf{j}}$; d. $\text{comp}_{\vec{\mathbf{v}}}\vec{\mathbf{u}}=\frac{16}{\sqrt{13}}=\frac{16\sqrt{13}}{13}$

48) $\vec{\mathbf{u}}=\langle-4,7\rangle,\quad \vec{\mathbf{v}}=\langle3,5\rangle$

49) $\vec{\mathbf{u}}=3\,\hat{\mathbf{i}}+2\hat{\mathbf{k}},\quad \vec{\mathbf{v}}=2\,\hat{\mathbf{j}}+4\hat{\mathbf{k}}$

Answer

a. $\text{Proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}} = \langle \frac{24}{13}, 0, \frac{16}{13} \rangle$ and $\vec{\mathbf{v}}_{\text{perp}} = < -\frac{24}{13}, 2, \frac{36}{13} >$; b. $\text{comp}_{\vec{\mathbf{u}}}\vec{\mathbf{v}} = \frac{8}{\sqrt{13}} = \frac{8\sqrt{13}}{13}$

c. $\text{Proj}_{\vec{\mathbf{v}}}\vec{\mathbf{u}} = \frac{4}{5}\hat{\mathbf{j}} + \frac{8}{5}\hat{\mathbf{k}}$ and $\vec{\mathbf{u}}_{\text{perp}} = 3\hat{\mathbf{i}} - \frac{4}{5}\hat{\mathbf{j}} + \frac{2}{5}\hat{\mathbf{k}}$; d. $\text{comp}_{\vec{\mathbf{v}}}\vec{\mathbf{u}} = \frac{\sqrt{80}}{5} = \frac{4\sqrt{5}}{5}$;

50) $\vec{\mathbf{u}} = \langle 4, 4, 0 \rangle, \quad \vec{\mathbf{v}} = \langle 0, 4, 1 \rangle$

51) Consider the vectors $\vec{\mathbf{u}} = 4\hat{\mathbf{i}} - 3\hat{\mathbf{j}}$ and $\vec{\mathbf{v}} = 3\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$.

a. Find the component form of vector $\text{Proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}$ that represents the projection of $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$.

b. Write the decomposition $\vec{\mathbf{v}} = \vec{\mathbf{w}} + \vec{\mathbf{q}}$ of vector $\vec{\mathbf{v}}$ into the orthogonal components $\vec{\mathbf{w}}$ and $\vec{\mathbf{q}}$, where $\vec{\mathbf{w}}$ is the projection of $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$ and $\vec{\mathbf{q}}$ is the vector component of $\vec{\mathbf{v}}$ orthogonal to the direction of $\vec{\mathbf{u}}$. That is, $\vec{\mathbf{q}} = \vec{\mathbf{v}}_{\text{perp}}$.

Answer

a. $\text{Proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}} = \langle \frac{24}{25}, -\frac{18}{25} \rangle$; b. $\vec{\mathbf{q}} = \vec{\mathbf{v}}_{\text{perp}} = \langle \frac{51}{25}, \frac{68}{25} \rangle$,
$\vec{\mathbf{v}} = \vec{\mathbf{w}} + \vec{\mathbf{q}} = \text{Proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}} + \vec{\mathbf{v}}_{\text{perp}} = \langle \frac{24}{25}, -\frac{18}{25} \rangle + \langle \frac{51}{25}, \frac{68}{25} \rangle$
So we have that, $\vec{\mathbf{v}} = \langle \frac{24}{25}, -\frac{18}{25} \rangle + \langle \frac{51}{25}, \frac{68}{25} \rangle$.

52) Consider vectors $\vec{\mathbf{u}} = 2\hat{\mathbf{i}} + 4\hat{\mathbf{j}}$ and $\vec{\mathbf{v}} = 4\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$.

a. Find the component form of vector $\vec{\mathbf{w}} = \text{Proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}$ that represents the projection of $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$.

b. Write the decomposition $\vec{\mathbf{v}} = \vec{\mathbf{w}} + \vec{\mathbf{q}}$ of vector $\vec{\mathbf{v}}$ into the orthogonal components $\vec{\mathbf{w}}$ and $\vec{\mathbf{q}}$, where $\vec{\mathbf{w}}$ is the projection of $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$ and $\vec{\mathbf{q}}$ is a vector orthogonal to the direction of $\vec{\mathbf{u}}$.

53) A $50,000$-pound truck is parked on a hill with a 5° slope above the horizontal (in the positive x-direction). Considering only the force due to gravity, find

a. the component of the weight of the truck that is pulling the truck down the hill (aligned with the road)

b. the component of the weight of the truck that is perpendicular to the road.

c. the magnitude of the force needed to keep the truck from rolling down the hill

d. the magnitude of the force that is perpendicular to the road.

Solution:

Let $\vec{\mathbf{F}}$ be the force due to gravity on the truck (its weight) and let $\vec{\mathbf{r}}$ be the unit vector pointing up the hill to the right.

Then

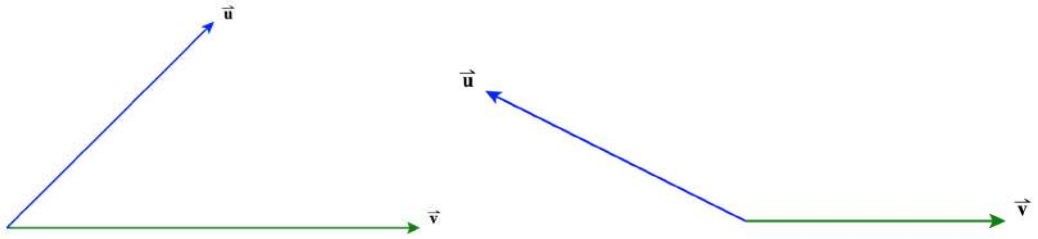
a. the component of the weight of the truck that is pulling the truck down the hill is $\text{Proj}_{\vec{\mathbf{r}}}\vec{\mathbf{F}}$.
Now $\vec{\mathbf{F}} = -50,000\hat{\mathbf{j}}$ and $\vec{\mathbf{r}} = \cos 5^\circ\ \hat{\mathbf{i}} + \sin 5^\circ\ \hat{\mathbf{j}}$ and $\vec{\mathbf{F}} \cdot \vec{\mathbf{r}} = -50,000 \sin 5^\circ \approx -4357.78714$
Note that $\|\vec{\mathbf{r}}\| = 1$.
Then: $\text{Proj}_{\vec{\mathbf{r}}}\vec{\mathbf{F}} = \left(\frac{\vec{\mathbf{F}} \cdot \vec{\mathbf{r}}}{\|\vec{\mathbf{r}}\|^2} \right) \vec{\mathbf{r}} = (-50,000 \sin 5^\circ)\, \vec{\mathbf{r}} = -50000 \sin 5^\circ \cos 5^\circ\ \hat{\mathbf{i}} - 50000 \sin^2 5^\circ\ \hat{\mathbf{j}}$
Using the identity, $\sin 2\theta = 2\cos\theta \sin\theta$, we can simplify this as
$\text{Proj}_{\vec{\mathbf{r}}}\vec{\mathbf{F}} = -25000 \sin 10^\circ\ \hat{\mathbf{i}} - 50000 \sin^2 5^\circ\ \hat{\mathbf{j}} \approx -4341.2\ \hat{\mathbf{i}} - 379.8\ \hat{\mathbf{j}}$

b. the component of the weight of the truck that is perpendicular to the road is $\vec{\mathbf{F}}_{\text{perp}}$.
Since we know that $\vec{\mathbf{F}} = \text{Proj}_{\vec{\mathbf{r}}}\vec{\mathbf{F}} + \vec{\mathbf{F}}_{\text{perp}}$, we have that $\vec{\mathbf{F}}_{\text{perp}} = \vec{\mathbf{F}} - \text{Proj}_{\vec{\mathbf{r}}}\vec{\mathbf{F}}$.
So $\vec{\mathbf{F}}_{\text{perp}} = -50000\hat{\mathbf{j}} - \left(-25000 \sin 10^\circ\ \hat{\mathbf{i}} - 50000 \sin^2 5^\circ\ \hat{\mathbf{j}} \right)$
$= 25000 \sin 10^\circ\ \hat{\mathbf{i}} + \left(50000 \sin^2 5^\circ - 50000 \right) \hat{\mathbf{j}} = 25000 \sin 10^\circ\ \hat{\mathbf{i}} - 50000 \cos^2 5^\circ\ \hat{\mathbf{j}} \approx 4341.2\ \hat{\mathbf{i}} - 49620.2\ \hat{\mathbf{j}}$

c. the magnitude of the force needed to keep the truck from rolling down the hill is $\|\text{Proj}_{\vec{\mathbf{r}}}\vec{\mathbf{F}}\| = 50,000 \sin 5^\circ$lbs
≈ 4357.8lbs. Note that this force is actually the negative of the projection vector, but its magnitude is the same.

d. the magnitude of the truck's weight that is perpendicular to the road is $\|\vec{\mathbf{F}}_{\text{perp}}\| \approx 49809.7$lbs.

54) Given the vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ shown in each diagram below, draw in $\text{Proj}_{\vec{\mathbf{v}}}\vec{\mathbf{u}}$ and $\vec{\mathbf{u}}_{\text{perp}}$.



Work

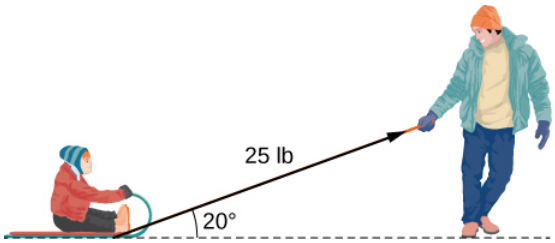
55) Find the work done by force $\vec{\mathbf{F}} = \langle 5, 6, -2 \rangle$ (measured in Newtons) that moves a particle from point $P(3, -1, 0)$ to point $Q(2, 3, 1)$ along a straight line (the distance is measured in meters).

Answer

$17\,\text{N}\cdot\text{m}$

56) [T] A sled is pulled by exerting a force of 100 N on a rope that makes an angle of 25° with the horizontal. Find the work done in pulling the sled 40 m. (Round the answer to one decimal place.)

57) [T] A father is pulling his son on a sled at an angle of 20° with the horizontal with a force of 25 lb (see the following image). He pulls the sled in a straight path of 50 ft. How much work was done by the man pulling the sled? (Round the answer to the nearest integer.)



Answer

1175 ft·lb

58) [T] A car is towed using a force of 1600 N. The rope used to pull the car makes an angle of 25° with the horizontal. Find the work done in towing the car 2 km. Express the answer in joules (1 J= 1 N·m) rounded to the nearest integer.

59) [T] A boat sails north aided by a wind blowing in a direction of $N30^\circ E$ with a magnitude of 500 lb. How much work is performed by the wind as the boat moves 100 ft? (Round the answer to two decimal places.)

Answer

$25000\sqrt{3}$ft-lbs $\approx 43,301.27$ft-lbs

Solution:

Vector representing the wind: $\vec{\mathbf{w}} = 500\cos 60^\circ\,\hat{\mathbf{i}} + 500\sin 60^\circ\,\hat{\mathbf{j}}$

Vector representing the displacement to the north: $\vec{\mathbf{d}} = 100\hat{\mathbf{j}}$

Work done by the wind: $W = \vec{\mathbf{w}} \cdot \vec{\mathbf{d}} = 25000\sqrt{3}$ ft-lbs $\approx 43,301.27$ft-lbs

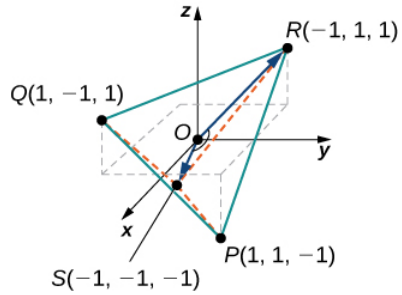
Other Applications of the Dot Product

60) [T] Find the vectors that join the center of a clock to the hours 1:00, 2:00, and 3:00. Assume the clock is circular with a radius of 1 unit.

61) A methane molecule has a carbon atom situated at the origin and four hydrogen atoms located at points $P(1, 1, -1), Q(1, -1, 1), R(-1, 1, 1)$, and $S(-1, -1, -1)$ (see figure).

a. Find the distance between the hydrogen atoms located at $P$and R.

b. Find the angle between vectors $\overrightarrow{OS}$ and $\overrightarrow{OR}$ that connect the carbon atom with the hydrogen atoms located at S and R, which is also called the *bond angle*. Express the answer in degrees rounded to two decimal places.



Answer

$a.\ 2\sqrt{2};\quad b.\ 109.47^\circ$

62) Vector $\vec{\mathbf{p}} = \langle 150, 225, 375 \rangle$ represents the price of certain models of bicycles sold by a bicycle shop. Vector $\vec{\mathbf{n}} = \langle 10, 7, 9 \rangle$ represents the number of bicycles sold of each model, respectively. Compute the dot product $\vec{\mathbf{p}} \cdot \vec{\mathbf{n}}$ and state its meaning.

63) [T] Two forces $\vec{\mathbf{F}}_1$ and $\vec{\mathbf{F}}_2$ are represented by vectors with initial points that are at the origin. The first force has a magnitude of 20 lb and passes through the point $P(1, 1, 0)$. The second force has a magnitude of 40 lb and passes through the point $Q(0, 1, 1)$. Let $\vec{\mathbf{F}}$ be the resultant force of forces $\vec{\mathbf{F}}_1$ and $\vec{\mathbf{F}}_2$.

a. Find the magnitude of $\vec{\mathbf{F}}$. (Round the answer to one decimal place.)

b. Find the direction angles of $\vec{\mathbf{F}}$. (Express the answer in degrees rounded to one decimal place.)

Answer

$a.\ \| \vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 \| = 52.9$ lb; b. The direction angles are $\alpha = 74.5^\circ$, $\beta = 36.7^\circ$,and $\gamma = 57.7^\circ$.

64) [T] Consider $\vec{\mathbf{r}}(t) = \langle \cos t, \sin t, 2t \rangle$ the position vector of a particle at time $t \in [0, 30]$, where the components of $\vec{\mathbf{r}}$ are expressed in centimeters and time in seconds. Let $\overrightarrow{OP}$ be the position vector of the particle after 1 sec.

a. Show that all vectors $\overrightarrow{PQ}$, where $Q(x, y, z)$ is an arbitrary point, orthogonal to the instantaneous velocity vector $\vec{\mathbf{v}}(1)$ of the particle after 1 sec, can be expressed as $\overrightarrow{PQ} = \langle x - \cos 1, y - \sin 1, z - 2 \rangle$, where $x \sin 1 - y \cos 1 - 2z + 4 = 0$. The set of point Q describes a plane called the *normal plane* to the path of the particle at point P.

b. Use a CAS to visualize the instantaneous velocity vector and the normal plane at point P along with the path of the particle.

Contributors

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Problems 39-46 and 53 & 54 were created by Paul Seeburger (Monroe Community College).
Solution to Problem 59 also was added by Paul Seeburger.

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4.5: The Cross Product

Learning Objectives

- Calculate the cross product of two given vectors.
- Use determinants to calculate a cross product.
- Find a vector orthogonal to two given vectors.
- Determine areas and volumes by using the cross product.
- Calculate the torque of a given force and position vector.

Imagine a mechanic turning a wrench to tighten a bolt. The mechanic applies a force at the end of the wrench. This creates rotation, or torque, which tightens the bolt. We can use vectors to represent the force applied by the mechanic, and the distance (radius) from the bolt to the end of the wrench. Then, we can represent torque by a vector oriented along the axis of rotation. Note that the torque vector is orthogonal to both the force vector and the radius vector.

In this section, we develop an operation called the **cross product**, which allows us to find a vector orthogonal to two given vectors. Calculating torque is an important application of cross products, and we examine torque in more detail later in the section.

The Cross Product and Its Properties

The dot product is a multiplication of two vectors that results in a scalar. In this section, we introduce a product of two vectors that generates a third vector orthogonal to the first two. Consider how we might find such a vector. Let $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$ be nonzero vectors. We want to find a vector $\vec{\mathbf{w}} = \langle w_1, w_2, w_3 \rangle$ orthogonal to both $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$—that is, we want to find $\vec{\mathbf{w}}$ such that $\vec{\mathbf{u}} \cdot \vec{\mathbf{w}} = 0$ and $\vec{\mathbf{v}} \cdot \vec{\mathbf{w}} = 0$. Therefore, w_1, w_2, and w_3 must satisfy

$$u_1 w_1 + u_2 w_2 + u_3 w_3 = 0 \tag{4.5.1}$$

$$v_1 w_1 + v_2 w_2 + v_3 w_3 = 0. \tag{4.5.2}$$

If we multiply the top equation by v_3 and the bottom equation by u_3 and subtract, we can eliminate the variable w_3, which gives

$$(u_1 v_3 - v_1 u_3) w_1 + (u_2 v_3 - v_2 u_3) w_2 = 0.$$

If we select

$$\begin{aligned} w_1 &= u_2 v_3 - u_3 v_2 \\ w_2 &= -(u_1 v_3 - u_3 v_1), \end{aligned}$$

we get a possible solution vector. Substituting these values back into the original equations (Equations 4.5.1 and 4.5.2) gives

$$w_3 = u_1 v_2 - u_2 v_1.$$

That is, vector

$$\vec{\mathbf{w}} = \langle u_2 v_3 - u_3 v_2, -(u_1 v_3 - u_3 v_1), u_1 v_2 - u_2 v_1 \rangle$$

is orthogonal to both $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$, which leads us to define the following operation, called the *cross product*.

Definition: Cross Product

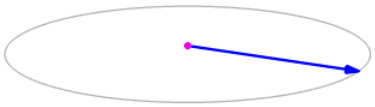
Let $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$. Then, the *cross product* $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ is vector

$$\begin{aligned} \vec{\mathbf{u}} \times \vec{\mathbf{v}} &= (u_2 v_3 - u_3 v_2)\hat{\mathbf{i}} - (u_1 v_3 - u_3 v_1)\hat{\mathbf{j}} + (u_1 v_2 - u_2 v_1)\hat{\mathbf{k}} \\ &= \langle u_2 v_3 - u_3 v_2, -(u_1 v_3 - u_3 v_1), u_1 v_2 - u_2 v_1 \rangle. \end{aligned} \tag{4.5.3}$$

From the way we have developed $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$, it should be clear that the cross product is orthogonal to both $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$. However, it never hurts to check. To show that $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ is orthogonal to $\vec{\mathbf{u}}$, we calculate the dot product of $\vec{\mathbf{u}}$ and $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$.

$$\begin{aligned}\vec{\mathbf{u}}\cdot(\vec{\mathbf{u}}\times\vec{\mathbf{v}}) &= \langle u_1,u_2,u_3\rangle\cdot\langle u_2v_3-u_3v_2,-u_1v_3+u_3v_1,u_1v_2-u_2v_1\rangle\\ &= u_1(u_2v_3-u_3v_2)+u_2(-u_1v_3+u_3v_1)+u_3(u_1v_2-u_2v_1)\\ &= u_1u_2v_3-u_1u_3v_2-u_1u_2v_3+u_2u_3v_1+u_1u_3v_2-u_2u_3v_1\\ &= (u_1u_2v_3-u_1u_2v_3)+(-u_1u_3v_2+u_1u_3v_2)+(u_2u_3v_1-u_2u_3v_1)\\ &= 0\end{aligned}$$

In a similar manner, we can show that the cross product is also orthogonal to $\vec{\mathbf{v}}$.



The cross product $\vec{\mathbf{a}}\times\vec{\mathbf{b}}$ (vertical, in pink) changes as the angle between the vectors $\vec{\mathbf{a}}$ (blue) and $\vec{\mathbf{b}}$ (red) changes. The cross product (purple) is always perpendicular to both vectors, and has magnitude zero when the vectors are parallel and maximum magnitude $\|\vec{\mathbf{a}}\|\|\vec{\mathbf{b}}\|$ when they are perpendicular. (Public Domain; LucasVB).

Example 4.5.1: Finding a Cross Product

Let $\vec{\mathbf{p}}=\langle -1,2,5\rangle$ and $\vec{\mathbf{q}}=\langle 4,0,-3\rangle$ (Figure 4.5.1). Find $\vec{\mathbf{p}}\times\vec{\mathbf{q}}$.

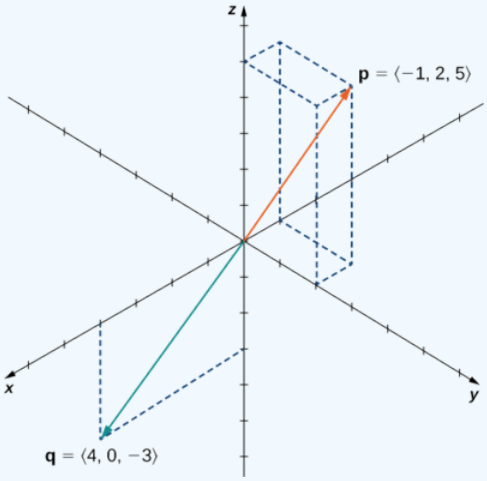


Figure 4.5.1: Finding a cross product to two given vectors.

Solution

Substitute the components of the vectors into Equation 4.5.3:

$$\begin{aligned}\vec{\mathbf{p}}\times\vec{\mathbf{q}} &= \langle -1,2,5\rangle\times\langle 4,0,-3\rangle\\ &= \langle p_2q_3-p_3q_2,-(p_1q_3-p_3q_1),p_1q_2-p_2q_1\rangle\\ &= \langle 2(-3)-5(0),-(-1)(-3)+5(4),(-1)(0)-2(4)\rangle\\ &= \langle -6,17,-8\rangle.\end{aligned}$$

Exercise 4.5.1

Find $\vec{\mathbf{p}} \times \vec{\mathbf{q}}$ for $\vec{\mathbf{p}} = \langle 5, 1, 2 \rangle$ and $\vec{\mathbf{q}} = \langle -2, 0, 1 \rangle$. Express the answer using standard unit vectors.

Hint

Use the formula $\vec{\mathbf{u}} \times \vec{\mathbf{v}} = (u_2 v_3 - u_3 v_2)\hat{\mathbf{i}} - (u_1 v_3 - u_3 v_1)\hat{\mathbf{j}} + (u_1 v_2 - u_2 v_1)\hat{\mathbf{k}}$.

Answer

$\vec{\mathbf{p}} \times \vec{\mathbf{q}} = \hat{\mathbf{i}} - 9\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$

Although it may not be obvious from Equation 4.5.3, the direction of $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ is given by the right-hand rule. If we hold the right hand out with the fingers pointing in the direction of $\vec{\mathbf{u}}$, then curl the fingers toward vector $\vec{\mathbf{v}}$, the thumb points in the direction of the cross product, as shown in Figure 4.5.2.

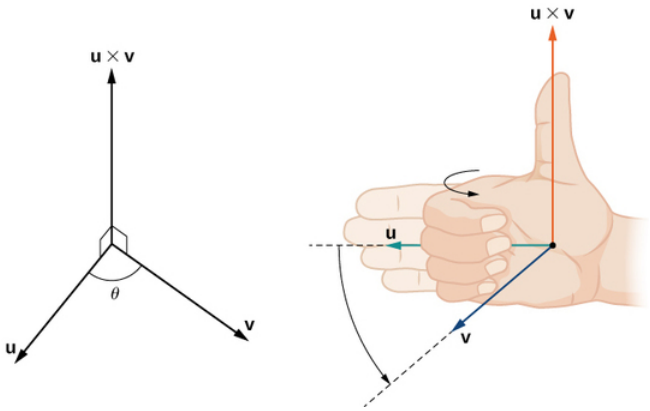


Figure 4.5.2: The direction of $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ *is determined by the right-hand rule.*

Notice what this means for the direction of $\vec{\mathbf{v}} \times \vec{\mathbf{u}}$. If we apply the right-hand rule to $\vec{\mathbf{v}} \times \vec{\mathbf{u}}$, we start with our fingers pointed in the direction of $\vec{\mathbf{v}}$, then curl our fingers toward the vector $\vec{\mathbf{u}}$. In this case, the thumb points in the opposite direction of $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$. (Try it!)

Example 4.5.2: Anticommutativity of the Cross Product

Let $\vec{\mathbf{u}} = \langle 0, 2, 1 \rangle$ and $\vec{\mathbf{v}} = \langle 3, -1, 0 \rangle$. Calculate $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ and $\vec{\mathbf{v}} \times \vec{\mathbf{u}}$ and graph them.

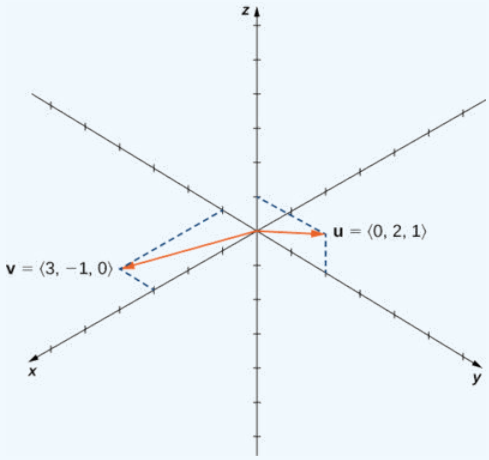


Figure 4.5.3: Are the cross products $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ and $\vec{\mathbf{v}} \times \vec{\mathbf{u}}$ in the same direction?

Solution

We have

$$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = \langle (0+1), -(0-3), (0-6) \rangle = \langle 1, 3, -6 \rangle$$

$$\vec{\mathbf{v}} \times \vec{\mathbf{u}} = \langle (-1-0), -(3-0), (6-0) \rangle = \langle -1, -3, 6 \rangle.$$

We see that, in this case, $\vec{\mathbf{u}} \times \vec{\mathbf{v}} = -(\vec{\mathbf{v}} \times \vec{\mathbf{u}})$ (Figure 4.5.4). We prove this in general later in this section.

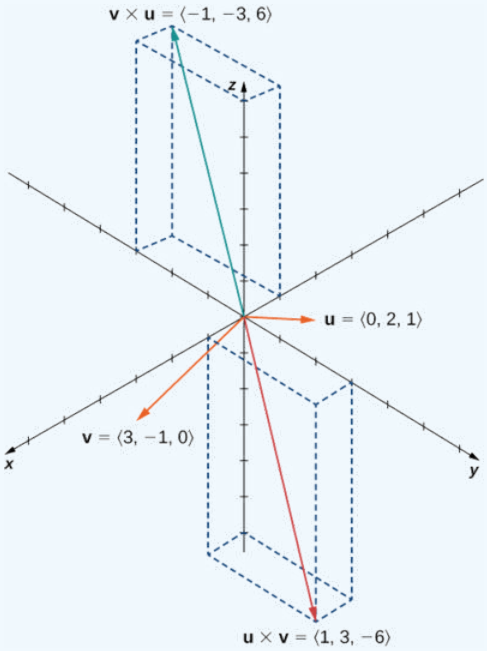


Figure 4.5.4: The cross products $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ *and* $\vec{\mathbf{v}} \times \vec{\mathbf{u}}$ *are both orthogonal to* $\vec{\mathbf{u}}$ *and* $\vec{\mathbf{v}}$*, but in opposite directions.*

Exercise 4.5.2

Suppose vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ lie in the xy-plane (the z-component of each vector is zero). Now suppose the x- and y-components of $\vec{\mathbf{u}}$ and the y-component of $\vec{\mathbf{v}}$ are all positive, whereas the x-component of $\vec{\mathbf{v}}$ is negative. Assuming the coordinate axes are oriented in the usual positions, in which direction does $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ point?

Hint

Remember the right-hand rule (Figure 4.5.2).

Answer

Up (the positive z-direction)

The cross products of the standard unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ can be useful for simplifying some calculations, so let's consider these cross products. A straightforward application of the definition shows that

$$\hat{\mathbf{i}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}} \times \hat{\mathbf{k}} = \vec{\mathbf{0}}. \tag{4.5.4}$$

(The cross product of two vectors is a vector, so each of these products results in the zero vector, not the scalar 0.) It's up to you to verify the calculations on your own.

Furthermore, because the cross product of two vectors is orthogonal to each of these vectors, we know that the cross product of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ is parallel to $\hat{\mathbf{k}}$. Similarly, the vector product of $\hat{\mathbf{i}}$ and $\hat{\mathbf{k}}$ is parallel to $\hat{\mathbf{j}}$, and the vector product of $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ is parallel to $\hat{\mathbf{i}}$.

We can use the right-hand rule to determine the direction of each product. Then we have

$$\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}$$
$$\hat{\mathbf{j}} \times \hat{\mathbf{i}} = -\hat{\mathbf{k}}$$

$$\hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}}$$
$$\hat{\mathbf{k}} \times \hat{\mathbf{j}} = -\hat{\mathbf{i}}$$

$$\hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}}$$
$$\hat{\mathbf{i}} \times \hat{\mathbf{k}} = -\hat{\mathbf{j}}.$$

These formulas come in handy later.

Example 4.5.3: Cross Product of Standard Unit Vectors

Find $\hat{\mathbf{i}} \times (\hat{\mathbf{j}} \times \hat{\mathbf{k}})$.

Solution

We know that $\hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}}$. Therefore, $\hat{\mathbf{i}} \times (\hat{\mathbf{j}} \times \hat{\mathbf{k}}) = \hat{\mathbf{i}} \times \hat{\mathbf{i}} = \vec{\mathbf{0}}$.

Exercise 4.5.3

Find $(\hat{\mathbf{i}} \times \hat{\mathbf{j}}) \times (\hat{\mathbf{k}} \times \hat{\mathbf{i}})$.

Hint

Remember the right-hand rule (Figure 4.5.2).

Answer

$-\hat{\mathbf{i}}$

As we have seen, the dot product is often called the *scalar product* because it results in a scalar. The cross product results in a vector, so it is sometimes called the **vector product.** These operations are both versions of vector multiplication, but they have very different properties and applications. Let's explore some properties of the cross product. We prove only a few of them. Proofs of the other properties are left as exercises.

Properties of the Cross Product

Let $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$ be vectors in space, and let c be a scalar.

i. Anticommutative property:

$$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = -(\vec{\mathbf{v}} \times \vec{\mathbf{u}}) \tag{4.5.5}$$

ii. Distributive property:

$$\vec{\mathbf{u}} \times (\vec{\mathbf{v}} + \vec{\mathbf{w}}) = \vec{\mathbf{u}} \times \vec{\mathbf{v}} + \vec{\mathbf{u}} \times \vec{\mathbf{w}} \tag{4.5.6}$$

iii. Multiplication by a constant:

$$c(\vec{\mathbf{u}} \times \vec{\mathbf{v}}) = (c\vec{\mathbf{u}}) \times \vec{\mathbf{v}} = \vec{\mathbf{u}} \times (c\vec{\mathbf{v}}) \tag{4.5.7}$$

iv. Cross product of the zero vector:

$$\vec{\mathbf{u}} \times \vec{\mathbf{0}} = \vec{\mathbf{0}} \times \vec{\mathbf{u}} = \vec{\mathbf{0}} \tag{4.5.8}$$

v. Cross product of a vector with itself:

$$\vec{\mathbf{v}} \times \vec{\mathbf{v}} = \vec{\mathbf{0}} \tag{4.5.9}$$

vi. Scalar triple product:

$$\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) = (\vec{\mathbf{u}} \times \vec{\mathbf{v}}) \cdot \vec{\mathbf{w}} \tag{4.5.10}$$

Proof

For property i, we want to show $\vec{\mathbf{u}} \times \vec{\mathbf{v}} = -(\vec{\mathbf{v}} \times \vec{\mathbf{u}})$. We have

$$\begin{aligned} \vec{\mathbf{u}} \times \vec{\mathbf{v}} &= \langle u_1, u_2, u_3 \rangle \times \langle v_1, v_2, v_3 \rangle \\ &= \langle u_2 v_3 - u_3 v_2, -u_1 v_3 + u_3 v_1, u_1 v_2 - u_2 v_1 \rangle \\ &= -\langle u_3 v_2 - u_2 v_3, -u_3 v_1 + u_1 v_3, u_2 v_1 - u_1 v_2 \rangle \\ &= -\langle v_1, v_2, v_3 \rangle \times \langle u_1, u_2, u_3 \rangle \\ &= -(\vec{\mathbf{v}} \times \vec{\mathbf{u}}). \end{aligned}$$

Unlike most operations we've seen, the cross product is not commutative. This makes sense if we think about the right-hand rule.

For property $iv.$, this follows directly from the definition of the cross product. We have

$$\vec{\mathbf{u}} \times \vec{\mathbf{0}} = \langle u_2(0) - u_3(0), -(u_2(0) - u_3(0)), u1(0) - u_2(0) \rangle = \langle 0, 0, 0 \rangle = \vec{\mathbf{0}}. \tag{4.5.11}$$

Then, by property i., $\vec{\mathbf{0}} \times \vec{\mathbf{u}} = \vec{\mathbf{0}}$ as well. Remember that the dot product of a vector and the zero vector is the *scalar* 0, whereas the cross product of a vector with the zero vector is the *vector* $\vec{\mathbf{0}}$.

Property $vi.$ looks like the associative property, but note the change in operations:

$$\begin{aligned} \vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) &= u \cdot \langle v_2 w_3 - v_3 w_2, -v_1 w_3 + v_3 w_1, v_1 w_2 - v_2 w_1 \rangle \\ &= u_1(v_2 w_3 - v_3 w_2) + u_2(-v_1 w_3 + v_3 w_1) + u_3(v_1 w_2 - v_2 w_1) \\ &= u_1 v_2 w_3 - u_1 v_3 w_2 - u_2 v_1 w_3 + u_2 v_3 w_1 + u_3 v_1 w_2 - u_3 v_2 w_1 \\ &= (u_2 v_3 - u_3 v_2) w_1 + (u_3 v_1 - u_1 v_3) w_2 + (u_1 v_2 - u_2 v_1) w_3 \\ &= \langle u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1 \rangle \cdot \langle w_1, w_2, w_3 \rangle = (\vec{\mathbf{u}} \times \vec{\mathbf{v}}) \cdot \vec{\mathbf{w}}. \end{aligned}$$

□

Example 4.5.4: Using the Properties of the Cross Product

Use the cross product properties to calculate $(2\hat{\mathbf{i}} \times 3\hat{\mathbf{j}}) \times \hat{\mathbf{j}}$.

Solution

$$\begin{aligned} (2\hat{\mathbf{i}} \times 3\hat{\mathbf{j}}) \times \hat{\mathbf{j}} &= 2(\hat{\mathbf{i}} \times 3\hat{\mathbf{j}}) \times \hat{\mathbf{j}} \\ &= 2(3)(\hat{\mathbf{i}} \times \hat{\mathbf{j}}) \times \hat{\mathbf{j}} \\ &= (6\hat{\mathbf{k}}) \times \hat{\mathbf{j}} \\ &= 6(\hat{\mathbf{k}} \times \hat{\mathbf{j}}) \\ &= 6(-\hat{\mathbf{i}}) = -6\hat{\mathbf{i}}. \end{aligned}$$

Exercise 4.5.4

Use the properties of the cross product to calculate $(\hat{\mathbf{i}} \times \hat{\mathbf{k}}) \times (\hat{\mathbf{k}} \times \hat{\mathbf{j}})$.

Hint

$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = -(\vec{\mathbf{v}} \times \vec{\mathbf{u}})$

Answer

$-\hat{\mathbf{k}}$

So far in this section, we have been concerned with the direction of the vector $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$, but we have not discussed its magnitude. It turns out there is a simple expression for the magnitude of $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ involving the magnitudes of $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$, and the sine of the angle between them.

Magnitude of the Cross Product

Let $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ be vectors, and let θ be the angle between them. Then, $\|\vec{\mathbf{u}} \times \vec{\mathbf{v}}\| = \|\vec{\mathbf{u}}\| \cdot \|\vec{\mathbf{v}}\| \cdot \sin\theta$.

Proof

Let $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$ be vectors, and let θ denote the angle between them. Then

$$\begin{aligned}
\|\vec{\mathbf{u}} \times \vec{\mathbf{v}}\|^2 &= (u_2v_3 - u_3v_2)^2 + (u_3v_1 - u_1v_3)^2 + (u_1v_2 - u_2v_1)^2 \\
&= u_2^2v_3^2 - 2u_2u_3v_2v_3 + u_3^2v_2^2 + u_3^2v_1^2 - 2u_1u_3v_1v_3 + u_1^2v_3^2 + u_1^2v_2^2 - 2u_1u_2v_1v_2 + u_2^2v_1^2 \\
&= u_1^2v_1^2 + u_1^2v_2^2 + u_1^2v_3^2 + u_2^2v_1^2 + u_2^2v_2^2 + u_2^2v_3^2 + u_3^2v_1^2 + u_3^2v_2^2 + u_3^2v_3^2 \\
&\quad - (u_1^2v_1^2 + u_2^2v_2^2 + u_3^2v_3^2 + 2u_1u_2v_1v_2 + 2u_1u_3v_1v_3 + 2u_2u_3v_2v_3) \\
&= (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1v_1 + u_2v_2 + u_3v_3)^2 \\
&= \|\vec{\mathbf{u}}\|^2\|\vec{\mathbf{v}}\|^2 - (\vec{\mathbf{u}} \cdot \vec{\mathbf{v}})^2 \\
&= \|\vec{\mathbf{u}}\|^2\|\vec{\mathbf{v}}\|^2 - \|\vec{\mathbf{u}}\|^2\|\vec{\mathbf{v}}\|^2 \cos^2\theta \\
&= \|\vec{\mathbf{u}}\|^2\|\vec{\mathbf{v}}\|^2(1 - \cos^2\theta) \\
&= \|\vec{\mathbf{u}}\|^2\|\vec{\mathbf{v}}\|^2(\sin^2\theta).
\end{aligned}$$

Taking square roots and noting that $\sqrt{\sin^2\theta} = \sin\theta$ for $0 \le \theta \le 180^\circ$, we have the desired result:

$$\|\vec{\mathbf{u}} \times \vec{\mathbf{v}}\| = \|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\| \sin\theta. \tag{4.5.12}$$

□

This definition of the cross product allows us to visualize or interpret the product geometrically. It is clear, for example, that the cross product is defined only for vectors in three dimensions, not for vectors in two dimensions. In two dimensions, it is impossible to generate a vector simultaneously orthogonal to two nonparallel vectors.

Example 4.5.5: Calculating the Cross Product

Use Note to find the magnitude of the cross product of $\vec{\mathbf{u}} = \langle 0, 4, 0 \rangle$ and $\vec{\mathbf{v}} = \langle 0, 0, -3 \rangle$.

Solution

We have

$$\begin{aligned}
\|\vec{\mathbf{u}} \times \vec{\mathbf{v}}\| &= \|\vec{\mathbf{u}}\| \cdot \|\vec{\mathbf{v}}\| \cdot \sin\theta \\
&= \sqrt{0^2 + 4^2 + 0^2} \cdot \sqrt{0^2 + 0^2 + (-3)^2} \cdot \sin\frac{\pi}{2} \\
&= 4(3)(1) = 12
\end{aligned}$$

Exercise 4.5.5

Use Note to find the magnitude of $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$, where $\vec{\mathbf{u}} = \langle -8, 0, 0 \rangle$ and $\vec{\mathbf{v}} = \langle 0, 2, 0 \rangle$.

Hint

Vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are orthogonal.

Answer

Determinants and the Cross Product

Using Equation 4.5.3 to find the cross product of two vectors is straightforward, and it presents the cross product in the useful component form. The formula, however, is complicated and difficult to remember. Fortunately, we have an alternative. We can calculate the cross product of two vectors using determinant notation.

A 2×2 determinant is defined by

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - b_1 a_2. \tag{4.5.13}$$

For example,

$$\begin{vmatrix} 3 & -2 \\ 5 & 1 \end{vmatrix} = 3(1) - 5(-2) = 3 + 10 = 13. \tag{4.5.14}$$

A 3×3 determinant is defined in terms of 2×2 determinants as follows:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}. \tag{4.5.15}$$

Equation 4.5.15 is referred to as the expansion of the determinant along the first row. Notice that the multipliers of each of the 2×2 determinants on the right side of this expression are the entries in the first row of the 3×3 determinant. Furthermore, each of the 2×2 determinants contains the entries from the 3×3 determinant that would remain if you crossed out the row and column containing the multiplier. Thus, for the first term on the right, a_1 is the multiplier, and the 2×2 determinant contains the entries that remain if you cross out the first row and first column of the 3×3 determinant. Similarly, for the second term, the multiplier is a_2, and the 2×2 determinant contains the entries that remain if you cross out the first row and second column of the 3×3 determinant. Notice, however, that the coefficient of the second term is negative. The third term can be calculated in similar fashion.

Example 4.5.6: Using Expansion Along the First Row to Compute a 3×3 Determinant

Evaluate the determinant $\begin{vmatrix} 2 & 5 & -1 \\ -1 & 1 & 3 \\ -2 & 3 & 4 \end{vmatrix}$.

Solution

We have

$$\begin{aligned} \begin{vmatrix} 2 & 5 & -1 \\ -1 & 1 & 3 \\ -2 & 3 & 4 \end{vmatrix} &= 2 \begin{vmatrix} 1 & 3 \\ 3 & 4 \end{vmatrix} - 5 \begin{vmatrix} -1 & 3 \\ -2 & 4 \end{vmatrix} - 1 \begin{vmatrix} -1 & 1 \\ -2 & 3 \end{vmatrix} \\ &= 2(4-9) - 5(-4+6) - 1(-3+2) \\ &= 2(-5) - 5(2) - 1(-1) = -10 - 10 + 1 \\ &= -19 \end{aligned}$$

Exercise 4.5.6

Evaluate the determinant $\begin{vmatrix} 1 & -2 & -1 \\ 3 & 2 & -3 \\ 1 & 5 & 4 \end{vmatrix}$.

Hint

Expand along the first row. Don't forget the second term is negative!

Answer

40

Technically, determinants are defined only in terms of arrays of real numbers. However, the determinant notation provides a useful mnemonic device for the cross product formula.

Rule: Cross Product Calculated by a Determinant

Let $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$ be vectors. Then the cross product $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ is given by

$$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \hat{\mathbf{i}} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \hat{\mathbf{j}} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \hat{\mathbf{k}}. \tag{4.5.16}$$

Example 4.5.7: Using Determinant Notation to find $\vec{\mathbf{p}} \times \vec{\mathbf{q}}$

Let $\vec{\mathbf{p}} = \langle -1, 2, 5 \rangle$ and $\vec{\mathbf{q}} = \langle 4, 0, -3 \rangle$. Find $\vec{\mathbf{p}} \times \vec{\mathbf{q}}$.

Solution

We set up our determinant by putting the standard unit vectors across the first row, the components of $\vec{\mathbf{u}}$ in the second row, and the components of $\vec{\mathbf{v}}$ in the third row. Then, we have

$$\begin{aligned} \vec{\mathbf{p}} \times \vec{\mathbf{q}} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -1 & 2 & 5 \\ 4 & 0 & -3 \end{vmatrix} = \begin{vmatrix} 2 & 5 \\ 0 & -3 \end{vmatrix} \hat{\mathbf{i}} - \begin{vmatrix} -1 & 5 \\ 4 & -3 \end{vmatrix} \hat{\mathbf{j}} + \begin{vmatrix} -1 & 2 \\ 4 & 0 \end{vmatrix} \hat{\mathbf{k}} \\ &= (-6 - 0)\hat{\mathbf{i}} - (3 - 20)\hat{\mathbf{j}} + (0 - 8)\hat{\mathbf{k}} \\ &= -6\hat{\mathbf{i}} + 17\hat{\mathbf{j}} - 8\hat{\mathbf{k}}. \end{aligned}$$

Notice that this answer confirms the calculation of the cross product in Example 4.5.1.

Exercise 4.5.7

Use determinant notation to find $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$, where $\vec{\mathbf{a}} = \langle 8, 2, 3 \rangle$ and $\vec{\mathbf{b}} = \langle -1, 0, 4 \rangle$.

Hint

Calculate the determinant $\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 8 & 2 & 3 \\ -1 & 0 & 4 \end{vmatrix}$.

Answer

$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = 8\hat{\mathbf{i}} - 35\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$

Using the Cross Product

The cross product is very useful for several types of calculations, including finding a vector orthogonal to two given vectors, computing areas of triangles and parallelograms, and even determining the volume of the three-dimensional geometric shape made of parallelograms known as a *parallelepiped*. The following examples illustrate these calculations.

Example 4.5.8: Finding a Unit Vector Orthogonal to Two Given Vectors

Let $\vec{\mathbf{a}} = \langle 5, 2, -1 \rangle$ and $\vec{\mathbf{b}} = \langle 0, -1, 4 \rangle$. Find a unit vector orthogonal to both $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$.

Solution

The cross product $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$ is orthogonal to both vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$. We can calculate it with a determinant:

$$\begin{aligned} \vec{\mathbf{a}} \times \vec{\mathbf{b}} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 5 & 2 & -1 \\ 0 & -1 & 4 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ -1 & 4 \end{vmatrix} \hat{\mathbf{i}} - \begin{vmatrix} 5 & -1 \\ 0 & 4 \end{vmatrix} \hat{\mathbf{j}} + \begin{vmatrix} 5 & 2 \\ 0 & -1 \end{vmatrix} \hat{\mathbf{k}} \\ &= (8-1)\hat{\mathbf{i}} - (20-0)\hat{\mathbf{j}} + (-5-0)\hat{\mathbf{k}} \\ &= 7\hat{\mathbf{i}} - 20\hat{\mathbf{j}} - 5\hat{\mathbf{k}}. \end{aligned}$$

Normalize this vector to find a unit vector in the same direction:

$$\|\vec{\mathbf{a}} \times \vec{\mathbf{b}}\| = \sqrt{(7)^2 + (-20)^2 + (-5)^2} = \sqrt{474}\,.$$

Thus, $\left\langle \frac{7}{\sqrt{474}}, \frac{-20}{\sqrt{474}}, \frac{-5}{\sqrt{474}} \right\rangle$ is a unit vector orthogonal to $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$.

Simplified, this vector becomes $\left\langle \frac{7\sqrt{474}}{474}, \frac{-10\sqrt{474}}{237}, \frac{-5\sqrt{474}}{474} \right\rangle$.

Exercise 4.5.8

Find a unit vector orthogonal to both $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$, where $\vec{\mathbf{a}} = \langle 4, 0, 3 \rangle$ and $\vec{\mathbf{b}} = \langle 1, 1, 4 \rangle$.

Hint

Normalize the cross product.

Answer

$\left\langle \frac{-3}{\sqrt{194}}, \frac{-13}{\sqrt{194}}, \frac{4}{\sqrt{194}} \right\rangle$ or, simplified as $\left\langle \frac{-3\sqrt{194}}{194}, \frac{-13\sqrt{194}}{194}, \frac{2\sqrt{194}}{97} \right\rangle$

To use the cross product for calculating areas, we state and prove the following theorem.

Area of a Parallelogram

If we locate vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ such that they form adjacent sides of a parallelogram, then the area of the parallelogram is given by $\|\vec{\mathbf{u}} \times \vec{\mathbf{v}}\|$ (Figure 4.5.5).

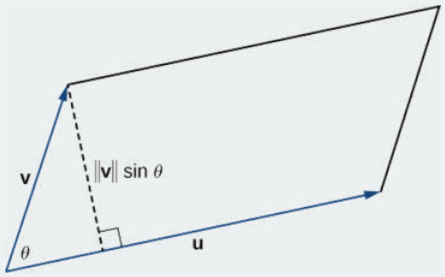


Figure 4.5.5: The parallelogram with adjacent sides $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ has base $\|\vec{\mathbf{u}}\|$ and height $\|\vec{\mathbf{v}}\| \sin\theta$.

Proof

We show that the magnitude of the cross product is equal to the base times height of the parallelogram.

$$\begin{aligned}\text{Area of a parallelogram} &= \text{base} \times \text{height} \\ &= \|\vec{\mathbf{u}}\|(\|\vec{\mathbf{v}}\| \sin\theta) \\ &= \|\vec{\mathbf{u}} \times \vec{\mathbf{v}}\|\end{aligned}$$

□

Example 4.5.9: Finding the Area of a Triangle

Let $P = (1, 0, 0), Q = (0, 1, 0)$, and $R = (0, 0, 1)$ be the vertices of a triangle (Figure 4.5.6). Find its area.

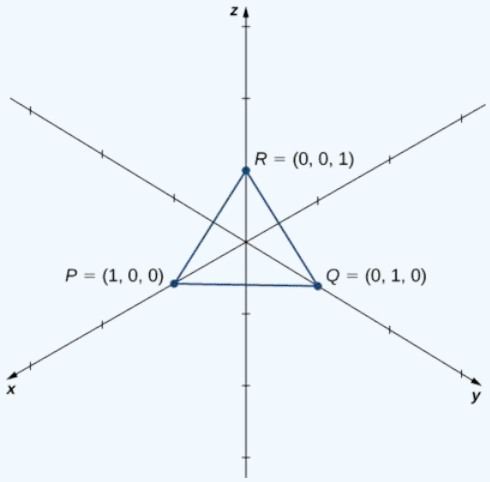


Figure 4.5.6: Finding the area of a triangle by using the cross product.

Solution

We have $\overrightarrow{PQ} = \langle 0-1, 1-0, 0-0\rangle = \langle -1, 1, 0\rangle$ and $\overrightarrow{PR} = \langle 0-1, 0-0, 1-0\rangle = \langle -1, 0, 1\rangle$. The area of the parallelogram with adjacent sides $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ is given by $\left\|\overrightarrow{PQ} \times \overrightarrow{PR}\right\|$:

$$\begin{aligned}\overrightarrow{PQ} \times \overrightarrow{PR} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} \\ &= (1-0)\hat{\mathbf{i}} - (-1-0)\hat{\mathbf{j}} + (0-(-1))\hat{\mathbf{k}} \\ &= \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}\end{aligned}$$

$$\begin{aligned}\left\|\overrightarrow{PQ} \times \overrightarrow{PR}\right\| &= \|\langle 1, 1, 1\rangle\| \\ &= \sqrt{1^2 + 1^2 + 1^2} \\ &= \sqrt{3}.\end{aligned}$$

The area of ΔPQR is half the area of the parallelogram or $\sqrt{3}/2$ units^2.



Exercise 4.5.9

Find the area of the parallelogram $PQRS$ with vertices $P(1,1,0)$, $Q(7,1,0)$, $R(9,4,2)$, and $S(3,4,2)$.

Hint

Sketch the parallelogram and identify two vectors that form adjacent sides of the parallelogram.

Answer

$6\sqrt{13}\,\text{units}^2$

The Triple Scalar Product

Because the cross product of two vectors is a vector, it is possible to combine the dot product and the cross product. The dot product of a vector with the cross product of two other vectors is called the **triple scalar product** because the result is a scalar.

Definition: Triple Scalar Product

The triple scalar product of vectors $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$ is

$$\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}). \tag{4.5.17}$$

Calculating a Triple Scalar Product

The triple scalar product of vectors

$$\vec{\mathbf{u}} = u_1\,\hat{\mathbf{i}} + u_2\,\hat{\mathbf{j}} + u_3\hat{\mathbf{k}} \tag{4.5.18}$$

$$\vec{\mathbf{v}} = v_1\,\hat{\mathbf{i}} + v_2\,\hat{\mathbf{j}} + v_3\hat{\mathbf{k}} \tag{4.5.19}$$

and

$$\vec{\mathbf{w}} = w_1\,\hat{\mathbf{i}} + w_2\,\hat{\mathbf{j}} + w_3\hat{\mathbf{k}} \tag{4.5.20}$$

is the determinant of the 3×3 matrix formed by the components of the vectors:

$$\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}. \tag{4.5.21}$$

Proof

The calculation is straightforward.

$$\begin{aligned}\vec{\mathbf{u}}\cdot(\vec{\mathbf{v}}\times\vec{\mathbf{w}}) &= \langle u_1,u_2,u_3\rangle\cdot\langle v_2w_3-v_3w_2,-v_1w_3+v_3w_1,v_1w_2-v_2w_1\rangle\\ &= u_1(v_2w_3-v_3w_2)+u_2(-v_1w_3+v_3w_1)+u_3(v_1w_2-v_2w_1)\\ &= u_1(v_2w_3-v_3w_2)-u_2(v_1w_3-v_3w_1)+u_3(v_1w_2-v_2w_1)\\ &= \begin{vmatrix} u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\\ w_1 & w_2 & w_3\end{vmatrix}.\end{aligned}$$

□

Example 4.5.10: Calculating the Triple Scalar Product

Let $\vec{\mathbf{u}}=\langle 1,3,5\rangle$, $\vec{\mathbf{v}}=\langle 2,-1,0\rangle$ and $\vec{\mathbf{w}}=\langle -3,0,-1\rangle$. Calculate the triple scalar product $\vec{\mathbf{u}}\cdot(\vec{\mathbf{v}}\times\vec{\mathbf{w}})$.

Solution

Apply Equation 4.5.21 directly:

$$\begin{aligned}\vec{\mathbf{u}}\cdot(\vec{\mathbf{v}}\times\vec{\mathbf{w}}) &= \begin{vmatrix} 1 & 3 & 5\\ 2 & -1 & 0\\ -3 & 0 & -1\end{vmatrix}\\ &= 1\begin{vmatrix}-1 & 0\\ 0 & -1\end{vmatrix}-3\begin{vmatrix}2 & 0\\ -3 & -1\end{vmatrix}+5\begin{vmatrix}2 & -1\\ -3 & 0\end{vmatrix}\\ &= (1-0)-3(-2-0)+5(0-3)\\ &= 1+6-15=-8.\end{aligned}$$

Exercise 4.5.10

Calculate the triple scalar product $\vec{\mathbf{a}}\cdot(\vec{\mathbf{b}}\times\vec{\mathbf{c}})$, where $\vec{\mathbf{a}}=\langle 2,-4,1\rangle$, $\vec{\mathbf{b}}=\langle 0,3,-1\rangle$, and $\vec{\mathbf{c}}=\langle 5,-3,3\rangle$.

Hint

Place the vectors as the rows of a 3×3 matrix, then calculate the determinant.

Answer

17

When we create a matrix from three vectors, we must be careful about the order in which we list the vectors. If we list them in a matrix in one order and then rearrange the rows, the absolute value of the determinant remains unchanged. However, each time two rows switch places, the determinant changes sign:

$$\begin{vmatrix} a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}=d \qquad \begin{vmatrix} b_1 & b_2 & b_3\\ a_1 & a_2 & a_3\\ c_1 & c_2 & c_3\end{vmatrix}=-d \qquad \begin{vmatrix} b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\\ a_1 & a_2 & a_3\end{vmatrix}=d \qquad \begin{vmatrix} c_1 & c_2 & c_3\\ b_1 & b_2 & b_3\\ a_1 & a_2 & a_3\end{vmatrix}=-d$$

Verifying this fact is straightforward, but rather messy. Let's take a look at this with an example:

$$\begin{aligned}\begin{vmatrix} 1 & 2 & 1\\ -2 & 0 & 3\\ 4 & 1 & -1\end{vmatrix} &= \begin{vmatrix}0 & 3\\ 1 & -1\end{vmatrix}-2\begin{vmatrix}-2 & 3\\ 4 & -1\end{vmatrix}+\begin{vmatrix}-2 & 0\\ 4 & 1\end{vmatrix}\\ &= (0-3)-2(2-12)+(-2-0)\\ &= -3+20-2=15.\end{aligned}$$

Switching the top two rows we have

$$\begin{vmatrix} -2 & 0 & 3 \\ 1 & 2 & 1 \\ 4 & 1 & -1 \end{vmatrix} = -2\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} + 3\begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix}$$
$$= -2(-2-1) + 3(1-8)$$
$$= 6 - 21 = -15.$$

Rearranging vectors in the triple products is equivalent to reordering the rows in the matrix of the determinant. Let $\vec{\mathbf{u}} = u_1\,\hat{\mathbf{i}} + u_2\hat{\mathbf{j}} + u_3\hat{\mathbf{k}}$, $\vec{\mathbf{v}} = v_1\,\hat{\mathbf{i}} + v_2\hat{\mathbf{j}} + v_3\hat{\mathbf{k}}$, and $\vec{\mathbf{w}} = w_1\,\hat{\mathbf{i}} + w_2\hat{\mathbf{j}} + w_3\hat{\mathbf{k}}$. Applying Note, we have

$$\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \tag{4.5.22}$$

and

$$\vec{\mathbf{u}} \cdot (\vec{\mathbf{w}} \times \vec{\mathbf{v}}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \\ v_1 & v_2 & v_3 \end{vmatrix}. \tag{4.5.23}$$

We can obtain the determinant for calculating $\vec{\mathbf{u}} \cdot (\vec{\mathbf{w}} \times \vec{\mathbf{v}})$ by switching the bottom two rows of $\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}})$. Therefore, $\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) = -\vec{\mathbf{u}} \cdot (\vec{\mathbf{w}} \times \vec{\mathbf{v}})$.

Following this reasoning and exploring the different ways we can interchange variables in the triple scalar product lead to the following identities:

$$\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) = -\vec{\mathbf{u}} \cdot (\vec{\mathbf{w}} \times \vec{\mathbf{v}}) \tag{4.5.24}$$

$$\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) = \vec{\mathbf{v}} \cdot (\vec{\mathbf{w}} \times \vec{\mathbf{u}}) = \vec{\mathbf{w}} \cdot (\vec{\mathbf{u}} \times \vec{\mathbf{v}}). \tag{4.5.25}$$

Let $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ be two vectors in standard position. If $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are not scalar multiples of each other, then these vectors form adjacent sides of a parallelogram. We saw in Note that the area of this parallelogram is $||\vec{\mathbf{u}} \times \vec{\mathbf{v}}||$. Now suppose we add a third vector $\vec{\mathbf{w}}$ that does not lie in the same plane as $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ but still shares the same initial point. Then these vectors form three edges of a **parallelepiped**, a three-dimensional prism with six faces that are each parallelograms, as shown in Figure 4.5.7. The volume of this prism is the product of the figure's height and the area of its base. The triple scalar product of $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$ provides a simple method for calculating the volume of the parallelepiped defined by these vectors.

Volume of a Parallelepiped

The volume of a parallelepiped with adjacent edges given by the vectors $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$ is the absolute value of the triple scalar product (Figure 4.5.7):

$$V = ||\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}})||. \tag{4.5.26}$$

Note that, as the name indicates, the triple scalar product produces a scalar. The volume formula just presented uses the absolute value of a scalar quantity.

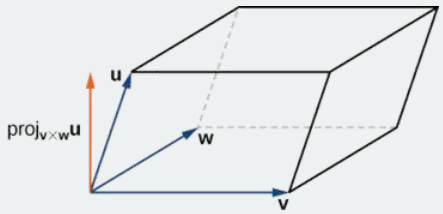


Figure 4.5.7: The height of the parallelepiped is given by $\|\text{proj}_{\vec{\mathbf{v}} \times \vec{\mathbf{w}}}\,\vec{\mathbf{u}}\|$.

Proof

The area of the base of the parallelepiped is given by $\|\vec{\mathbf{v}} \times \vec{\mathbf{w}}\|$. The height of the figure is given by $\|\text{proj}_{\vec{\mathbf{v}} \times \vec{\mathbf{w}}} \vec{\mathbf{u}}\|$. The volume of the parallelepiped is the product of the height and the area of the base, so we have

$$\begin{aligned} V &= \| \text{proj}_{\vec{\mathbf{v}} \times \vec{\mathbf{w}}} \vec{\mathbf{u}} \| \, \|\vec{\mathbf{v}} \times \vec{\mathbf{w}}\| \\ &= \| \frac{\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}})}{\|\vec{\mathbf{v}} \times \vec{\mathbf{w}}\|} \| \, \|\vec{\mathbf{v}} \times \vec{\mathbf{w}}\| \\ &= \|\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}})\|. \end{aligned}$$

□

Example 4.5.11: Calculating the Volume of a Parallelepiped

Let $\vec{\mathbf{u}} = \langle -1, -2, 1 \rangle$, $\vec{\mathbf{v}} = \langle 4, 3, 2 \rangle$, and $\vec{\mathbf{w}} = \langle 0, -5, -2 \rangle$. Find the volume of the parallelepiped with adjacent edges $\vec{\mathbf{u}}, \vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$ (Figure 4.5.8).

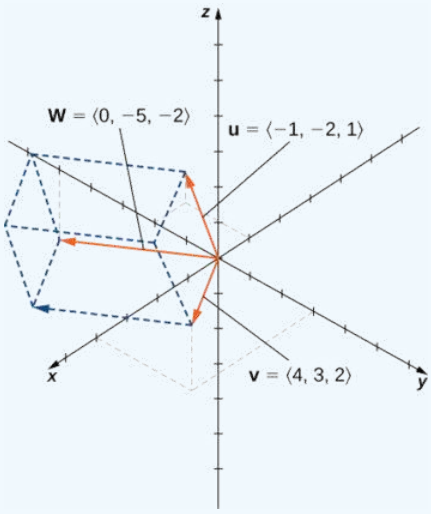


Figure 4.5.8

Solution

We have

$$\begin{aligned} \vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) &= \begin{vmatrix} -1 & -2 & 1 \\ 4 & 3 & 2 \\ 0 & -5 & -2 \end{vmatrix} \\ &= (-1) \begin{vmatrix} 3 & 2 \\ -5 & -2 \end{vmatrix} + 2 \begin{vmatrix} 4 & 2 \\ 0 & -2 \end{vmatrix} + \begin{vmatrix} 4 & 3 \\ 0 & -5 \end{vmatrix} \\ &= (-1)(-6+10) + 2(-8-0) + (-20-0) \\ &= -4 - 16 - 20 \\ &= -40. \end{aligned}$$

Thus, the volume of the parallelepiped is $|-40| = 40$ units^3



Exercise 4.5.11

Find the volume of the parallelepiped formed by the vectors $\vec{\mathbf{a}} = 3\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - \hat{\mathbf{k}}, \vec{\mathbf{b}} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} - \hat{\mathbf{k}},$ and $\vec{\mathbf{c}} = 3\hat{\mathbf{j}} + \hat{\mathbf{k}}$.

Hint

Calculate the triple scalar product by finding a determinant.

Answer

8 units^3

Applications of the Cross Product

The cross product appears in many practical applications in mathematics, physics, and engineering. Let's examine some of these applications here, including the idea of torque, with which we began this section. Other applications show up in later chapters, particularly in our study of vector fields such as gravitational and electromagnetic fields (Introduction to Vector Calculus).

Example 4.5.12: Using the Triple Scalar Product

Use the triple scalar product to show that vectors $\vec{\mathbf{u}} = \langle 2, 0, 5\rangle, \vec{\mathbf{v}} = \langle 2, 2, 4\rangle$, and $\vec{\mathbf{w}} = \langle 1, -1, 3\rangle$ are coplanar—that is, show that these vectors lie in the same plane.

Solution

Start by calculating the triple scalar product to find the volume of the parallelepiped defined by $\vec{\mathbf{u}}, \vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$:

$$\begin{aligned}\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) &= \begin{vmatrix} 2 & 0 & 5 \\ 2 & 2 & 4 \\ 1 & -1 & 3 \end{vmatrix} \\ &= [2(2)(3) + (0)(4)(1) + 5(2)(-1)] - [5(2)(1) + (2)(4)(-1) + (0)(2)(3)] \\ &= 2 - 2 = 0.\end{aligned}$$

The volume of the parallelepiped is 0 units^3, so one of the dimensions must be zero. Therefore, the three vectors all lie in the same plane.

Exercise 4.5.12

Are the vectors $\vec{\mathbf{a}} = \hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}, \vec{\mathbf{b}} = \hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}},$ and $\vec{\mathbf{c}} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ coplanar?

Hint

Calculate the triple scalar product.

Answer

No, the triple scalar product is $-4 \neq 0$, so the three vectors form the adjacent edges of a parallelepiped. They are not coplanar.

Example 4.5.13: Finding an Orthogonal Vector

Only a single plane can pass through any set of three noncolinear points. Find a vector orthogonal to the plane containing points $P = (9, -3, -2), Q = (1, 3, 0)$, and $R = (-2, 5, 0)$.

Solution

The plane must contain vectors $\overrightarrow{PQ}$ and $\overrightarrow{QR}$:

$$\overrightarrow{PQ} = \langle 1 - 9, 3 - (-3), 0 - (-2) \rangle = \langle -8, 6, 2 \rangle$$

$$\overrightarrow{QR} = \langle -2 - 1, 5 - 3, 0 - 0 \rangle = \langle -3, 2, 0 \rangle.$$

The cross product $\overrightarrow{PQ} \times \overrightarrow{QR}$ produces a vector orthogonal to both $\overrightarrow{PQ}$ and $\overrightarrow{QR}$. Therefore, the cross product is orthogonal to the plane that contains these two vectors:

$$\begin{aligned} \overrightarrow{PQ} \times \overrightarrow{QR} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -8 & 6 & 2 \\ -3 & 2 & 0 \end{vmatrix} \\ &= 0\hat{\mathbf{i}} - 6\hat{\mathbf{j}} - 16\hat{\mathbf{k}} - (-18\hat{\mathbf{k}} + 4\hat{\mathbf{i}} + 0\hat{\mathbf{j}}) \\ &= -4\hat{\mathbf{i}} - 6\hat{\mathbf{j}} + 2\hat{\mathbf{k}}. \end{aligned}$$

We have seen how to use the triple scalar product and how to find a vector orthogonal to a plane. Now we apply the cross product to real-world situations.

Sometimes a force causes an object to rotate. For example, turning a screwdriver or a wrench creates this kind of rotational effect, called torque.

Definition: Torque

Torque, $\vec{\tau}$ (the Greek letter tau), measures the tendency of a force to produce rotation about an axis of rotation. Let $\vec{\mathbf{r}}$ be a vector with an initial point located on the axis of rotation and with a terminal point located at the point where the force is applied, and let vector $\vec{\mathbf{F}}$ represent the force. Then torque is equal to the cross product of r and F:

$$\vec{\tau} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}. \tag{4.5.27}$$

See Figure 4.5.9.

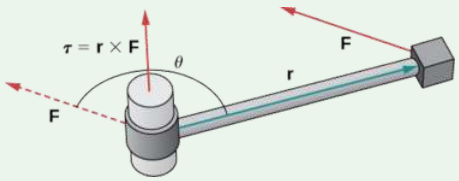


Figure 4.5.9: Torque measures how a force causes an object to rotate.

Think about using a wrench to tighten a bolt. The torque τ applied to the bolt depends on how hard we push the wrench (force) and how far up the handle we apply the force (distance). The torque increases with a greater force on the wrench at a greater distance from the bolt. Common units of torque are the newton-meter or foot-pound. Although torque is dimensionally equivalent to work (it has the same units), the two concepts are distinct. Torque is used specifically in the context of rotation, whereas work typically involves motion along a line.

Example 4.5.14: Evaluating Torque

A bolt is tightened by applying a force of 6 N to a 0.15-m wrench (Figure 4.5.10). The angle between the wrench and the force vector is 40°. Find the magnitude of the torque about the center of the bolt. Round the answer to two decimal places.

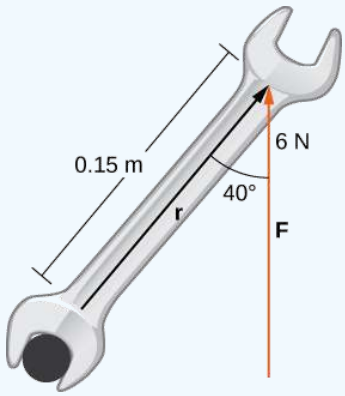


Figure 4.5.10: Torque describes the twisting action of the wrench.

Solution:

Substitute the given information into the equation defining torque:

$$\begin{aligned} \|\vec{\boldsymbol{\tau}}\| &= \|\vec{\mathbf{r}} \times \vec{\mathbf{F}}\| \\ &= \|\vec{\mathbf{r}}\| \, \|\vec{\mathbf{F}}\| \sin\theta \\ &= (0.15 \text{ m})(6 \text{ N}) \sin 40^\circ \\ &\approx 0.58 \text{ N·m}. \end{aligned}$$



Exercise 4.5.14

Calculate the force required to produce 15 N·m torque at an angle of 30° from a 150-cm rod.

Hint

$\|\vec{\boldsymbol{\tau}}\| = 15$ N·m and $\|\vec{\mathbf{r}}\| = 1.5$ m

Answer

20 N

Deciding to Use the Dot Product, Cross Product, or Other Vector Arithmetic

You are working as a consultant and will be given a question that involves the use of the dot product, the cross product, or other vector arithmetic such as vector addition, subtraction or scalar multiplication. You must guide the client towards the best of the three choices so that the client can most efficiently get the job done.

◯ Dot ◯ Cross ◯ Other

Click on the Start button to bring in your first client.

Start

Key Concepts

- The cross product $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ of two vectors $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$ is a vector orthogonal to both $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$. Its length is given by $\|\vec{\mathbf{u}} \times \vec{\mathbf{v}}\| = \|\vec{\mathbf{u}}\| \cdot \|\vec{\mathbf{v}}\| \cdot \sin\theta$, where θ is the angle between $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$. Its direction is given by the right-hand rule.
- The algebraic formula for calculating the cross product of two vectors,

  $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$, is

  $\vec{\mathbf{u}} \times \vec{\mathbf{v}} = (u_2 v_3 - u_3 v_2)\hat{\mathbf{i}} - (u_1 v_3 - u_3 v_1)\hat{\mathbf{j}} + (u_1 v_2 - u_2 v_1)\hat{\mathbf{k}}$.
- The cross product satisfies the following properties for vectors $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$, and scalar c:

$$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = -(\vec{\mathbf{v}} \times \vec{\mathbf{u}})$$
$$\vec{\mathbf{u}} \times (\vec{\mathbf{v}} + \vec{\mathbf{w}}) = \vec{\mathbf{u}} \times \vec{\mathbf{v}} + \vec{\mathbf{u}} \times \vec{\mathbf{w}}$$
$$c(\vec{\mathbf{u}} \times \vec{\mathbf{v}}) = (c\vec{\mathbf{u}}) \times \vec{\mathbf{v}} = \vec{\mathbf{u}} \times (c\vec{\mathbf{v}})$$
$$\vec{\mathbf{u}} \times \vec{\mathbf{0}} = \vec{\mathbf{0}} \times \vec{\mathbf{u}} = \vec{\mathbf{0}}$$
$$\vec{\mathbf{v}} \times \vec{\mathbf{v}} = \vec{\mathbf{0}}$$
$$\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) = (\vec{\mathbf{u}} \times \vec{\mathbf{v}}) \cdot \vec{\mathbf{w}}$$

- The cross product of vectors $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$ is the determinant $\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$
- If vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ form adjacent sides of a parallelogram, then the area of the parallelogram is given by $\|\vec{\mathbf{u}} \times \vec{\mathbf{v}}\|$.
- The triple scalar product of vectors $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$ is $\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}})$.
- The volume of a parallelepiped with adjacent edges given by vectors $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$ is $V = |\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}})|$.
- If the triple scalar product of vectors $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$ is zero, then the vectors are coplanar. The converse is also true: If the vectors are coplanar, then their triple scalar product is zero.
- The cross product can be used to identify a vector orthogonal to two given vectors or to a plane.
- Torque $\vec{\boldsymbol{\tau}}$ measures the tendency of a force to produce rotation about an axis of rotation. If force $\vec{\mathbf{F}}$ is acting at a distance (displacement) $\vec{\mathbf{r}}$ from the axis, then torque is equal to the cross product of $\vec{\mathbf{r}}$ and $\vec{\mathbf{F}}$: $\vec{\boldsymbol{\tau}} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}$.

Key Equations

- **The cross product of two vectors in terms of the unit vectors**

$$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = (u_2v_3 - u_3v_2)\hat{\mathbf{i}} - (u_1v_3 - u_3v_1)\hat{\mathbf{j}} + (u_1v_2 - u_2v_1)\hat{\mathbf{k}} \tag{4.5.28}$$

Glossary

cross product

$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = (u_2v_3 - u_3v_2)\hat{\mathbf{i}} - (u_1v_3 - u_3v_1)\hat{\mathbf{j}} + (u_1v_2 - u_2v_1)\hat{\mathbf{k}}$, where $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$

determinant

a real number associated with a square matrix

parallelepiped

a three-dimensional prism with six faces that are parallelograms

torque

the effect of a force that causes an object to rotate

triple scalar product

the dot product of a vector with the cross product of two other vectors: $\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}})$

vector product

the cross product of two vectors

Contributors and Attributions

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4.5E: Exercises for Section 4.5

For exercises 1-4, the vectors $\vec{u}$ and $\vec{v}$ are given.

a. Find the cross product $\vec{u} \times \vec{v}$ of the vectors $\vec{u}$ and $\vec{v}$. Express the answer in component form.

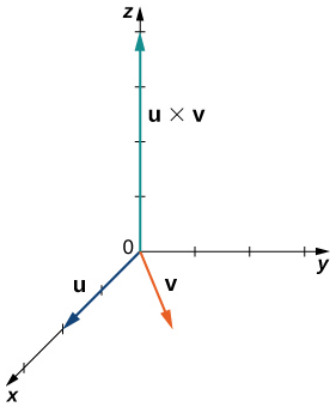
b. Sketch the vectors $\vec{u}$, $\vec{v}$, and $\vec{u} \times \vec{v}$.

1) $\vec{u} = \langle 2, 0, 0 \rangle, \quad \vec{v} = \langle 2, 2, 0 \rangle$

Answer

$a.\ \vec{u} \times \vec{v} = \langle 0, 0, 4 \rangle;$

$b.$



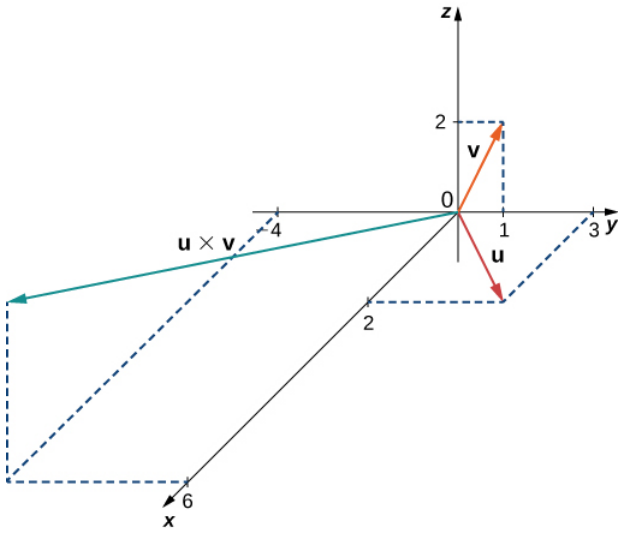
2) $\vec{u} = \langle 3, 2, -1 \rangle, \quad \vec{v} = \langle 1, 1, 0 \rangle$

3) $\vec{u} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}}, \quad \vec{v} = \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$

Answer

$a.\ \vec{u} \times \vec{v} = \langle 6, -4, 2 \rangle;$

$b.$



4) $\vec{u} = 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}, \quad \vec{v} = 3\hat{\mathbf{i}} + \hat{\mathbf{k}}$

5) Simplify $(\hat{\mathbf{i}} \times \hat{\mathbf{i}} - 2\hat{\mathbf{i}} \times \hat{\mathbf{j}} - 4\hat{\mathbf{i}} \times \hat{\mathbf{k}} + 3\hat{\mathbf{j}} \times \hat{\mathbf{k}}) \times \hat{\mathbf{i}}.$

Answer

$-2\hat{\mathbf{j}} - 4\hat{\mathbf{k}}$

6) Simplify $\hat{\mathbf{j}} \times (\hat{\mathbf{k}} \times \hat{\mathbf{j}} + 2\hat{\mathbf{j}} \times \hat{\mathbf{i}} - 3\hat{\mathbf{j}} \times \hat{\mathbf{j}} + 5\hat{\mathbf{i}} \times \hat{\mathbf{k}})$.

In exercises 7-10, vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are given. Find unit vector $\vec{\mathbf{w}}$ in the direction of the cross product vector $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$. Express your answer using standard unit vectors.

7) $\vec{\mathbf{u}} = \langle 3, -1, 2 \rangle, \quad \vec{\mathbf{v}} = \langle -2, 0, 1 \rangle$

Answer

$\vec{\mathbf{w}} = -\frac{\sqrt{6}}{18}\hat{\mathbf{i}} - \frac{7\sqrt{6}}{18}\hat{\mathbf{j}} - \frac{\sqrt{6}}{9}\hat{\mathbf{k}}$

8) $\vec{\mathbf{u}} = \langle 2, 6, 1 \rangle, \quad \vec{\mathbf{v}} = \langle 3, 0, 1 \rangle$

9) $\vec{\mathbf{u}} = \overrightarrow{AB}, \quad \vec{\mathbf{v}} = \overrightarrow{AC},$ where $A(1, 0, 1)$, $B(1, -1, 3)$, and $C(0, 0, 5)$

Answer

$\vec{\mathbf{w}} = -\frac{4\sqrt{21}}{21}\hat{\mathbf{i}} - \frac{2\sqrt{21}}{21}\hat{\mathbf{j}} - \frac{\sqrt{21}}{21}\hat{\mathbf{k}}$

10) $\vec{\mathbf{u}} = \overrightarrow{OP}, \quad \vec{\mathbf{v}} = \overrightarrow{PQ},$ where $P(-1, 1, 0)$ and $Q(0, 2, 1)$

11) Determine the real number α such that $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ and $\hat{\mathbf{i}}$ are orthogonal, where $\vec{\mathbf{u}} = 3\hat{\mathbf{i}} + \hat{\mathbf{j}} - 5\hat{\mathbf{k}}$ and $\vec{\mathbf{v}} = 4\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \alpha\hat{\mathbf{k}}$.

Answer

$\alpha = 10$

12) Show that $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ and $2\hat{\mathbf{i}} - 14\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ cannot be orthogonal for any α real number, where $\vec{\mathbf{u}} = \hat{\mathbf{i}} + 7\hat{\mathbf{j}} - \hat{\mathbf{k}}$ and $\vec{\mathbf{v}} = \alpha\hat{\mathbf{i}} + 5\hat{\mathbf{j}} + \hat{\mathbf{k}}$.

13) Show that $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ is orthogonal to $\vec{\mathbf{u}} + \vec{\mathbf{v}}$ and $\vec{\mathbf{u}} - \vec{\mathbf{v}}$, where $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are nonzero vectors.

14) Show that $\vec{\mathbf{v}} \times \vec{\mathbf{u}}$ is orthogonal to $(\vec{\mathbf{u}} \cdot \vec{\mathbf{v}})(\vec{\mathbf{u}} + \vec{\mathbf{v}}) + \vec{\mathbf{u}}$, where $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are nonzero vectors.

15) Calculate the determinant $\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -1 & 7 \\ 2 & 0 & 3 \end{vmatrix}$.

Answer

$-3\hat{\mathbf{i}} + 11\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$

16) Calculate the determinant $\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 3 & -4 \\ 1 & 6 & -1 \end{vmatrix}$.

For exercises 17-18, the vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are given. Use determinant notation to find vector $\vec{\mathbf{w}}$ orthogonal to vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$.

17) $\vec{\mathbf{u}} = \langle -1, 0, e^t \rangle, \quad \vec{\mathbf{v}} = \langle 1, e^{-t}, 0 \rangle,$ where t is a real number

Answer

$\vec{\mathbf{w}} = \langle -1, e^t, -e^{-t} \rangle$

18) $\vec{\mathbf{u}} = \langle 1, 0, x \rangle, \quad \vec{\mathbf{v}} = \langle \frac{2}{x}, 1, 0 \rangle,$ where x is a nonzero real number

19) Find vector $(\vec{\mathbf{a}} - 2\vec{\mathbf{b}}) \times \vec{\mathbf{c}},$ where $\vec{\mathbf{a}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & -1 & 5 \\ 0 & 1 & 8 \end{vmatrix}, \vec{\mathbf{b}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 1 & 1 \\ 2 & -1 & -2 \end{vmatrix},$ and $\vec{\mathbf{c}} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$.

Answer

$-26\hat{\mathbf{i}}+17\hat{\mathbf{j}}+9\hat{\mathbf{k}}$

20) Find vector $\vec{\mathbf{c}}\times(\vec{\mathbf{a}}+3\vec{\mathbf{b}})$, where $\vec{\mathbf{a}}=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 5 & 0 & 9\\ 0 & 1 & 0\end{vmatrix}$, $\vec{\mathbf{b}}=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 0 & -1 & 1\\ 7 & 1 & -1\end{vmatrix}$, and $\vec{\mathbf{c}}=\hat{\mathbf{i}}-\hat{\mathbf{k}}$.

21) [T] Use the cross product $\vec{\mathbf{u}}\times\vec{\mathbf{v}}$ to find the acute angle between vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$, where $\vec{\mathbf{u}}=\hat{\mathbf{i}}+2\hat{\mathbf{j}}$ and $\vec{\mathbf{v}}=\hat{\mathbf{i}}+\hat{\mathbf{k}}$. Express the answer in degrees rounded to the nearest integer.

Answer

72°

22) [T] Use the cross product $\vec{\mathbf{u}}\times\vec{\mathbf{v}}$ to find the obtuse angle between vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$, where $\vec{\mathbf{u}}=-\hat{\mathbf{i}}+3\hat{\mathbf{j}}+\hat{\mathbf{k}}$ and $\vec{\mathbf{v}}=\hat{\mathbf{i}}-2\hat{\mathbf{j}}$. Express the answer in degrees rounded to the nearest integer.

23) Use the sine and cosine of the angle between two nonzero vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ to prove Lagrange's identity: $\|\vec{\mathbf{u}}\times\vec{\mathbf{v}}\|^2=\|\vec{\mathbf{u}}\|^2\|\vec{\mathbf{v}}\|^2-(\vec{\mathbf{u}}\cdot\vec{\mathbf{v}})^2$.

24) Verify Lagrange's identity $\|\vec{\mathbf{u}}\times\vec{\mathbf{v}}\|^2=\|\vec{\mathbf{u}}\|^2\|\vec{\mathbf{v}}\|^2-(\vec{\mathbf{u}}\cdot\vec{\mathbf{v}})^2$ for vectors $\vec{\mathbf{u}}=-\hat{\mathbf{i}}+\hat{\mathbf{j}}-2\hat{\mathbf{k}}$ and $\vec{\mathbf{v}}=2\hat{\mathbf{i}}-\hat{\mathbf{j}}$.

25) Nonzero vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are called collinear if there exists a nonzero scalar α such that $\vec{\mathbf{v}}=\alpha\vec{\mathbf{u}}$. Show that $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are collinear if and only if $\vec{\mathbf{u}}\times\vec{\mathbf{v}}=0$.

26) Nonzero vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are called *collinear* if there exists a nonzero scalar α such that $\vec{\mathbf{v}}=\alpha\vec{\mathbf{u}}$. Show that vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are collinear, where $A(4,1,0)$, $B(6,5,-2)$,and $C(5,3,-1)$.

27) Find the area of the parallelogram with adjacent sides $\vec{\mathbf{u}}=\langle 3,2,0\rangle$ and $\vec{\mathbf{v}}=\langle 0,2,1\rangle$.

Answer

7

28) Find the area of the parallelogram with adjacent sides $\vec{\mathbf{u}}=\hat{\mathbf{i}}+\hat{\mathbf{j}}$ and $\vec{\mathbf{v}}=\hat{\mathbf{i}}+\hat{\mathbf{k}}$.

29) Consider points $A(3,-1,2)$, $B(2,1,5)$,and $C(1,-2,-2)$.

a. Find the area of parallelogram $ABCD$ with adjacent sides $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

b. Find the area of triangle ABC.

c. Find the distance from point A to line BC.

Answer

a. $5\sqrt{6}$; b. $\frac{5\sqrt{6}}{2}$; c. $\frac{5\sqrt{6}}{\sqrt{59}}=\frac{5\sqrt{354}}{59}$

30) Consider points $A(2,-3,4)$, $B(0,1,2)$,and $C(-1,2,0)$.

a. Find the area of parallelogram $ABCD$ with adjacent sides $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

b. Find the area of triangle ABC.

c. Find the distance from point B to line AC.

In exercises 31-32, vectors $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$ are given.

a. Find the triple scalar product $\vec{\mathbf{u}}\cdot(\vec{\mathbf{v}}\times\vec{\mathbf{w}})$.

b. Find the volume of the parallelepiped with the adjacent edges $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$.

31) $\vec{\mathbf{u}}=\hat{\mathbf{i}}+\hat{\mathbf{j}}$, $\vec{\mathbf{v}}=\hat{\mathbf{j}}+\hat{\mathbf{k}}$, and $\vec{\mathbf{w}}=\hat{\mathbf{i}}+\hat{\mathbf{k}}$

Answer

$a.\ 2; \quad b.\ 2\text{units}^3$

32) $\vec{\mathbf{u}} = \langle -3, 5, -1\rangle, \quad \vec{\mathbf{v}} = \langle 0, 2, -2\rangle$, and $\quad \vec{\mathbf{w}} = \langle 3, 1, 1\rangle$

33) Calculate the triple scalar products $\vec{\mathbf{v}} \cdot (\vec{\mathbf{u}} \times \vec{\mathbf{w}})$ and $\vec{\mathbf{w}} \cdot (\vec{\mathbf{u}} \times \vec{\mathbf{v}})$, where $\vec{\mathbf{u}} = \langle 1, 1, 1\rangle$, $\vec{\mathbf{v}} = \langle 7, 6, 9\rangle$, and $\vec{\mathbf{w}} = \langle 4, 2, 7\rangle$.

Answer

$\vec{\mathbf{v}} \cdot (\vec{\mathbf{u}} \times \vec{\mathbf{w}}) = -1, \quad \vec{\mathbf{w}} \cdot (\vec{\mathbf{u}} \times \vec{\mathbf{v}}) = 1$

34) Calculate the triple scalar products $\vec{\mathbf{w}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{u}})$ and $\vec{\mathbf{u}} \cdot (\vec{\mathbf{w}} \times \vec{\mathbf{v}})$, where $\vec{\mathbf{u}} = \langle 4, 2, -1\rangle, \vec{\mathbf{v}} = \langle 2, 5, -3\rangle$, and $\vec{\mathbf{w}} = \langle 9, 5, -10\rangle$.

35) Find vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, and $\vec{\mathbf{c}}$ with a triple scalar product given by the determinant $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 5 \\ 8 & 9 & 2 \end{vmatrix}$. Determine their triple scalar product.

Answer

$\vec{\mathbf{a}} = \langle 1, 2, 3\rangle, \quad \vec{\mathbf{b}} = \langle 0, 2, 5\rangle, \quad \vec{\mathbf{c}} = \langle 8, 9, 2\rangle; \quad \vec{\mathbf{a}} \cdot (\vec{\mathbf{b}} \times \vec{\mathbf{c}}) = -9$

36) The triple scalar product of vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, and $\vec{\mathbf{c}}$ is given by the determinant $\begin{vmatrix} 0 & -2 & 1 \\ 0 & 1 & 4 \\ 1 & -3 & 7 \end{vmatrix}$. Find vector $\vec{\mathbf{a}} - \vec{\mathbf{b}} + \vec{\mathbf{c}}$.

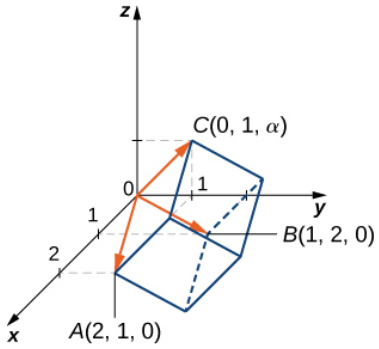
37) Consider the parallelepiped with edges OA, OB, and OC, where $A(2, 1, 0)$, $B(1, 2, 0)$,and $C(0, 1, \alpha)$.

a. Find the real number $\alpha > 0$ such that the volume of the parallelepiped is 3 units^3.

b. For $\alpha = 1$, find the height h from vertex C of the parallelepiped. Sketch the parallelepiped.

Answer

$a.\ \alpha = 1; \quad b.\ h = 1$ unit,



38) Consider points $A(\alpha, 0, 0)$, $B(0, \beta, 0)$,and $C(0, 0, \gamma)$, with α, β, and γ positive real numbers.

a. Determine the volume of the parallelepiped with adjacent sides $\overrightarrow{OA}$, $\overrightarrow{OB}$, and $\overrightarrow{OC}$.

b. Find the volume of the tetrahedron with vertices O, A, B, and C. (Hint: The volume of the tetrahedron is $1/6$ of the volume of the parallelepiped.)

c. Find the distance from the origin to the plane determined by A, B, and C. Sketch the parallelepiped and tetrahedron.

39) Let u, v, and w be three-dimensional vectors and c be a real number. Prove the following properties of the cross product.

a. $\vec{\mathbf{u}} \times \vec{\mathbf{u}} = \vec{\mathbf{0}}$

b. $\vec{\mathbf{u}} \times (\vec{\mathbf{v}} + \vec{\mathbf{w}}) = (\vec{\mathbf{u}} \times \vec{\mathbf{v}}) + (\vec{\mathbf{u}} \times \vec{\mathbf{w}})$

c. $c(\vec{\mathbf{u}} \times \vec{\mathbf{v}}) = (c\vec{\mathbf{u}}) \times \vec{\mathbf{v}} = \vec{\mathbf{u}} \times (c\vec{\mathbf{v}})$

d. $\vec{\mathbf{u}} \cdot (\vec{\mathbf{u}} \times \vec{\mathbf{v}}) = \vec{\mathbf{0}}$

40) Show that vectors $\vec{\mathbf{u}} = \langle 1, 0, -8 \rangle$, $\vec{\mathbf{v}} = \langle 0, 1, 6 \rangle$, and $\vec{\mathbf{w}} = \langle -1, 9, 3 \rangle$ satisfy the following properties of the cross product.

a. $\vec{\mathbf{u}} \times \vec{\mathbf{u}} = \vec{\mathbf{0}}$

b. $\vec{\mathbf{u}} \times (\vec{\mathbf{v}} + \vec{\mathbf{w}}) = (\vec{\mathbf{u}} \times \vec{\mathbf{v}}) + (\vec{\mathbf{u}} \times \vec{\mathbf{w}})$

c. $c(\vec{\mathbf{u}} \times \vec{\mathbf{v}}) = (c\vec{\mathbf{u}}) \times \vec{\mathbf{v}} = \vec{\mathbf{u}} \times (c\vec{\mathbf{v}})$

d. $\vec{\mathbf{u}} \cdot (\vec{\mathbf{u}} \times \vec{\mathbf{v}}) = \vec{\mathbf{0}}$

41) Nonzero vectors $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$ are said to be linearly dependent if one of the vectors is a linear combination of the other two. For instance, there exist two nonzero real numbers α and β such that $\vec{\mathbf{w}} = \alpha\vec{\mathbf{u}} + \beta\vec{\mathbf{v}}$. Otherwise, the vectors are called linearly independent. Show that $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$ could be placed on the same plane if and only if they are linear dependent.

42) Consider vectors $\vec{\mathbf{u}} = \langle 1, 4, -7 \rangle$, $\vec{\mathbf{v}} = \langle 2, -1, 4 \rangle$, $\vec{\mathbf{w}} = \langle 0, -9, 18 \rangle$, and $\vec{\mathbf{p}} = \langle 0, -9, 17 \rangle$.

a. Show that $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$ can be placed on the same plane by using their triple scalar product

b. Show that $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$ can be placed on the same plane by using the definition that there exist two nonzero real numbers α and β such that $w = \alpha u + \beta v$.

c. Show that $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, and $\vec{\mathbf{p}}$ are linearly independent—that is, none of the vectors is a linear combination of the other two.

43) Consider points $A(0, 0, 2), B(1, 0, 2), C(1, 1, 2)$,and $D(0, 1, 2)$. Are vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$, and $\overrightarrow{AD}$ linearly dependent (that is, one of the vectors is a linear combination of the other two)?

Answer

Yes, $\overrightarrow{AD} = \alpha\overrightarrow{AB} + \beta\overrightarrow{AC}$, where $\alpha = -1$ and $\beta = 1$.

44) Show that vectors $\hat{\mathbf{i}} + \hat{\mathbf{j}}$, $\hat{\mathbf{i}} - \hat{\mathbf{j}}$, and $\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ are linearly independent—that is, there exist two nonzero real numbers α and β such that $\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}} = \alpha(\hat{\mathbf{i}} + \hat{\mathbf{j}}) + \beta(\hat{\mathbf{i}} - \hat{\mathbf{j}})$.

45) Let $\vec{\mathbf{u}} = \langle u_1, u_2 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2 \rangle$ be two-dimensional vectors. The cross product of vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ is not defined. However, if the vectors are regarded as the three-dimensional vectors $\tilde{\vec{\mathbf{u}}} = \langle u_1, u_2, 0 \rangle$ and $\tilde{\vec{\mathbf{v}}} = \langle v_1, v_2, 0 \rangle$, respectively, then, in this case, we can define the cross product of $\tilde{\vec{\mathbf{u}}}$ and $\tilde{\vec{\mathbf{v}}}$. In particular, in determinant notation, the cross product of $\tilde{\vec{\mathbf{u}}}$ and $\tilde{\vec{\mathbf{v}}}$ is given by

$$\tilde{\vec{\mathbf{u}}} \times \tilde{\vec{\mathbf{v}}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ u_1 & u_2 & 0 \\ v_1 & v_2 & 0 \end{vmatrix}.$$

Use this result to compute $(\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}) \times (\sin\theta\,\hat{\mathbf{i}} - \cos\theta\,\hat{\mathbf{j}})$, where θ is a real number.

Answer

$-\hat{\mathbf{k}}$

46) Consider points $P(2, 1), Q(4, 2)$,and $R(1, 2)$.

a. Find the area of triangle PQR.

b. Determine the distance from point R to the line passing through P and Q.

47) Determine a vector of magnitude 10 perpendicular to the plane passing through the x-axis and point $P(1, 2, 4)$.

Answer

$\langle 0, \pm 4\sqrt{5}, 2\sqrt{5} \rangle$

48) Determine a unit vector perpendicular to the plane passing through the z-axis and point $A(3, 1, -2)$.

49) Consider $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ two three-dimensional vectors. If the magnitude of the cross product vector $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ is k times larger than the magnitude of vector $\vec{\mathbf{u}}$, show that the magnitude of $\vec{\mathbf{v}}$ is greater than or equal to k, where k is a natural number.

50) [T] Assume that the magnitudes of two nonzero vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are known. The function $f(\theta) = \|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\| \sin\theta$ defines the magnitude of the cross product vector $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$, where $\theta \in [0, \pi]$ is the angle between $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$.

a. Graph the function f.

b. Find the absolute minimum and maximum of function f. Interpret the results.

c. If $\|\vec{\mathbf{u}}\| = 5$ and $\|\vec{\mathbf{v}}\| = 2$, find the angle between $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ if the magnitude of their cross product vector is equal to 9.

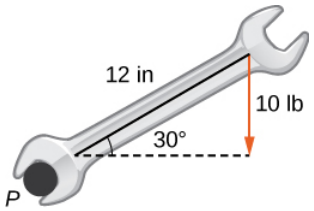
51) Find all vectors $\vec{\mathbf{w}} = \langle w_1, w_2, w_3 \rangle$ that satisfy the equation $\langle 1, 1, 1 \rangle \times \vec{\mathbf{w}} = \langle -1, -1, 2 \rangle$. Hint: You should be able to write all components of $\vec{\mathbf{w}}$ in terms of one of the constants w_1, w_2, or w_3.

Answer

Writing all components in terms of the constant w_3, one way to represent these vectors is: $\vec{\mathbf{w}} = \langle w_3 - 1, w_3 + 1, w_3 \rangle$, where w_3 is any real number.
Note that we could use any parameter we wish here. We could set $w_3 = a$. Then $\vec{\mathbf{w}} = \langle a - 1, a + 1, a \rangle$ would also represent these vectors.

52) Solve the equation $\vec{\mathbf{w}} \times \langle 1, 0, -1 \rangle = \langle 3, 0, 3 \rangle$, where $\vec{\mathbf{w}} = \langle w_1, w_2, w_3 \rangle$ is a nonzero vector with a magnitude of 3.

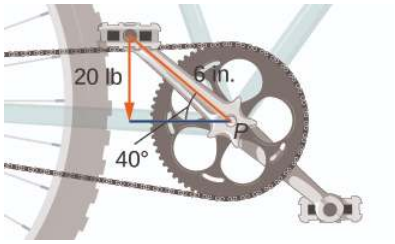
53) [T] A mechanic uses a 12-in. wrench to turn a bolt. The wrench makes a 30° angle with the horizontal. If the mechanic applies a vertical force of 10 lb on the wrench handle, what is the magnitude of the torque at point P (see the following figure)? Express the answer in foot-pounds rounded to two decimal places.



Answer

8.66 ft-lb

54) [T] A boy applies the brakes on a bicycle by applying a downward force of 20 lb on the pedal when the 6-in. crank makes a 40° angle with the horizontal (see the following figure). Find the torque at point P. Express your answer in foot-pounds rounded to two decimal places.



55) [T] Find the magnitude of the force that needs to be applied to the end of a 20-cm wrench located on the positive direction of the y-axis if the force is applied in the direction $\langle 0, 1, -2 \rangle$ and it produces a $100\,\text{N}\cdot\text{m}$ torque to the bolt located at the origin.

Answer

$250\sqrt{5}\,\text{N} \approx 559\,\text{N}$

56) [T] What is the magnitude of the force required to be applied to the end of a 1-ft wrench at an angle of 35° to produce a torque of $20\,\text{N}\cdot\text{m}$?

57) [T] The force vector $\vec{\mathbf{F}}$ acting on a proton with an electric charge of $1.6 \times 10^{-19}\,C$ (in coulombs) moving in a magnetic field $\vec{\mathbf{B}}$ where the velocity vector $\vec{\mathbf{v}}$ is given by $\vec{\mathbf{F}} = 1.6 \times 10^{-19}(\vec{\mathbf{v}} \times \vec{\mathbf{B}})$ (here, $\vec{\mathbf{v}}$ is expressed in meters per second, $\vec{\mathbf{B}}$ is in tesla [T],

and $\vec{\mathbf{F}}$ is in newtons [N]). Find the force that acts on a proton that moves in the xy-plane at velocity $\vec{\mathbf{v}} = 10^5\,\hat{\mathbf{i}} + 10^5\,\hat{\mathbf{j}}$ (in meters per second) in a magnetic field given by $\vec{\mathbf{B}} = 0.3\,\hat{\mathbf{j}}$.

Answer

$\vec{\mathbf{F}} = 4.8 \times 10^{-15}\ kN$

58) [T] The force vector $\vec{\mathbf{F}}$ acting on a proton with an electric charge of $1.6 \times 10^{-19}\ C$ moving in a magnetic field $\vec{\mathbf{B}}$ where the velocity vector $\vec{\mathbf{v}}$ is given by $\vec{\mathbf{F}} = 1.6 \times 10^{-19}(\vec{\mathbf{v}} \times \vec{\mathbf{B}})$ (here, $\vec{\mathbf{v}}$ is expressed in meters per second, $\vec{\mathbf{B}}$ in T, and $\vec{\mathbf{F}}$ in N). If the magnitude of force $\vec{\mathbf{F}}$ acting on a proton is $5.9 \times 10^{-17}\ N$ and the proton is moving at the speed of 300 m/sec in magnetic field $\vec{\mathbf{B}}$ of magnitude 2.4 T, find the angle between velocity vector $\vec{\mathbf{v}}$ of the proton and magnetic field $\vec{\mathbf{B}}$. Express the answer in degrees rounded to the nearest integer.

59) [T] Consider $\vec{\mathbf{r}}(t) = \langle \cos t,\ \sin t,\ 2t \rangle$ the position vector of a particle at time $t \in [0, 30]$, where the components of $\vec{\mathbf{r}}$ are expressed in centimeters and time in seconds. Let $\overrightarrow{OP}$ be the position vector of the particle after 1 sec.

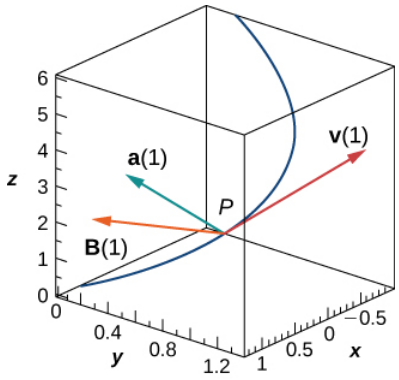
a. Determine unit vector $\vec{\mathbf{B}}(t)$ (called the binormal unit vector) that has the direction of cross product vector $\vec{\mathbf{v}}(t) \times \vec{\mathbf{a}}(t)$, where $\vec{\mathbf{v}}(t)$ and $\vec{\mathbf{a}}(t)$ are the instantaneous velocity vector and, respectively, the acceleration vector of the particle after t seconds.

b. Use a CAS to visualize vectors $\vec{\mathbf{v}}(1)$, $\vec{\mathbf{a}}(1)$, and $\vec{\mathbf{B}}(1)$ as vectors starting at point P along with the path of the particle.

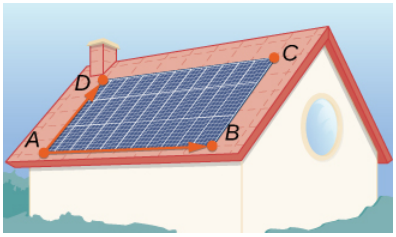
Answer

a. $\vec{\mathbf{B}}(t) = \langle \frac{2\sqrt{5}\sin t}{5}, -\frac{2\sqrt{5}\cos t}{5}, \frac{\sqrt{5}}{5} \rangle$;

b.



60) A solar panel is mounted on the roof of a house. The panel may be regarded as positioned at the points of coordinates (in meters) $A(8, 0, 0), B(8, 18, 0), C(0, 18, 8)$, and $D(0, 0, 8)$ (see the following figure).



a. Find vector $\vec{\mathbf{n}} = \overrightarrow{AB} \times \overrightarrow{AD}$ perpendicular to the surface of the solar panels. Express the answer using standard unit vectors. Note that the magnitude of this vector should give us the area of rectangle $ABCD$.

b. Assume unit vector $\vec{\mathbf{s}} = \frac{1}{\sqrt{3}}\,\hat{\mathbf{i}} + \frac{1}{\sqrt{3}}\,\hat{\mathbf{j}} + \frac{1}{\sqrt{3}}\,\hat{\mathbf{k}}$ points toward the Sun at a particular time of the day and the flow of solar energy is $\vec{\mathbf{F}} = 900\,\vec{\mathbf{s}}$ (in watts per square meter $[W/m^2]$). Find the predicted amount of electrical power the panel can produce, which is given by the dot product of vectors $\vec{\mathbf{F}}$ and $\vec{\mathbf{n}}$ (expressed in watts).

c. Determine the angle of elevation of the Sun above the solar panel. Express the answer in degrees rounded to the nearest whole number. (Hint: The angle between vectors $\vec{\mathbf{n}}$ and $\vec{\mathbf{s}}$ and the angle of elevation are complementary.)

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4.6: Equations of Lines and Planes in Space

Learning Objectives

- Write the vector, parametric, and symmetric of a line through a given point in a given direction, and a line through two given points.
- Find the distance from a point to a given line.
- Write the vector and scalar equations of a plane through a given point with a given normal.
- Find the distance from a point to a given plane.
- Find the angle between two planes.

By now, we are familiar with writing equations that describe a line in two dimensions. To write an equation for a line, we must know two points on the line, or we must know the direction of the line and at least one point through which the line passes. In two dimensions, we use the concept of slope to describe the orientation, or direction, of a line. In three dimensions, we describe the direction of a line using a vector parallel to the line. In this section, we examine how to use equations to describe lines and planes in space.

Equations for a Line in Space

Let's first explore what it means for two vectors to be parallel. Recall that parallel vectors must have the same or opposite directions. If two nonzero vectors, $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$, are parallel, we claim there must be a scalar, k, such that $\vec{\mathbf{u}} = k\vec{\mathbf{v}}$. If $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ have the same direction, simply choose

$$k = \frac{\|\vec{\mathbf{u}}\|}{\|\vec{\mathbf{v}}\|}. \tag{4.6.1}$$

If $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ have opposite directions, choose

$$k = -\frac{\|\vec{\mathbf{u}}\|}{\|\vec{\mathbf{v}}\|}. \tag{4.6.2}$$

Note that the converse holds as well. If $\vec{\mathbf{u}} = k\vec{\mathbf{v}}$ for some scalar k, then either $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ have the same direction $(k > 0)$ or opposite directions $(k < 0)$, so $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are parallel. Therefore, two nonzero vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are parallel if and only if $\vec{\mathbf{u}} = k\vec{\mathbf{v}}$ for some scalar k. By convention, the zero vector $\vec{\mathbf{0}}$ is considered to be parallel to all vectors.

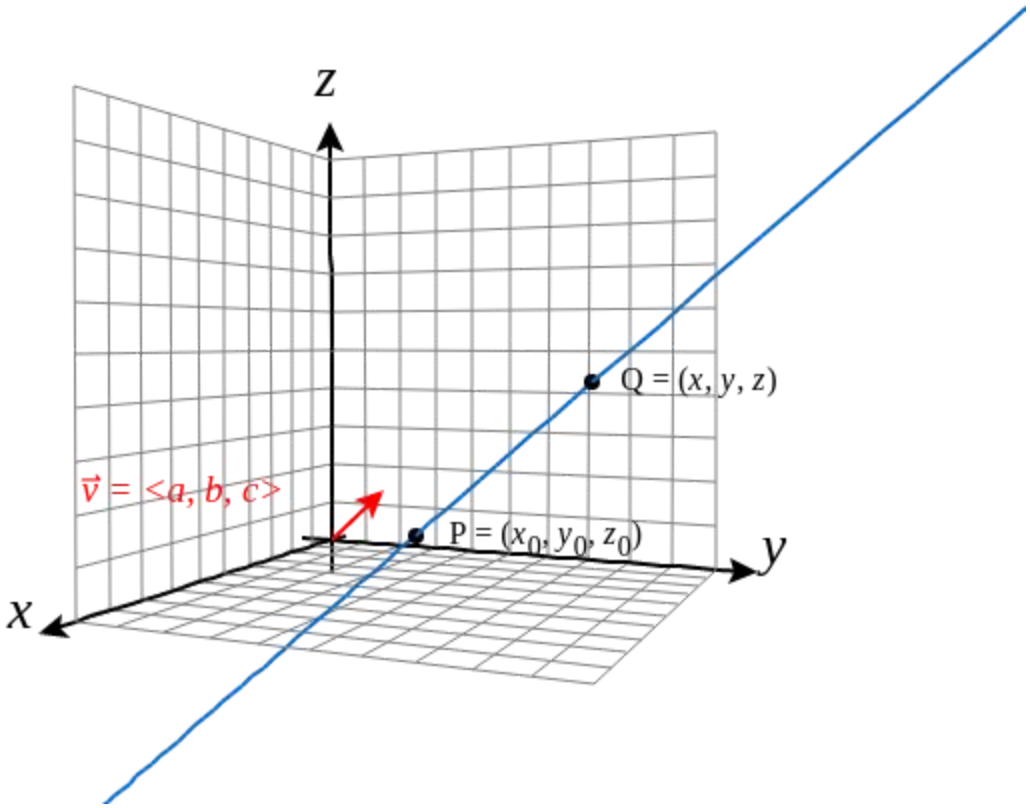


Figure* 4.6.1*: *Vector $\vec{\mathbf{v}}$ is the direction vector for $\overrightarrow{PQ}$.*

As in two dimensions, we can describe a line in space using a point on the line and the direction of the line, or a parallel vector, which we call the **direction vector** (Figure 4.6.1). Let L be a line in space passing through point $P(x_0, y_0, z_0)$. Let $\vec{\mathbf{v}} = \langle a, b, c \rangle$ be a vector parallel to L. Then, for any point on line $Q(x, y, z)$, we know that $\overrightarrow{PQ}$ is parallel to $\vec{\mathbf{v}}$. Thus, as we just discussed, there is a scalar, t, such that $\overrightarrow{PQ} = t\vec{\mathbf{v}}$, which gives

$$\begin{aligned} \overrightarrow{PQ} &= t\vec{\mathbf{v}} \\ \langle x - x_0, y - y_0, z - z_0 \rangle &= t\langle a, b, c \rangle \\ \langle x - x_0, y - y_0, z - z_0 \rangle &= \langle ta, tb, tc \rangle. \end{aligned} \tag{4.6.3}$$

Using vector operations, we can rewrite Equation 4.6.3

$$\begin{aligned} \langle x - x_0, y - y_0, z - z_0 \rangle &= \langle ta, tb, tc \rangle \\ \langle x, y, z \rangle - \langle x_0, y_0, z_0 \rangle &= t\langle a, b, c \rangle \\ \underbrace{\langle x, y, z \rangle}_{\vec{\mathbf{r}}} &= \underbrace{\langle x_0, y_0, z_0 \rangle}_{\vec{\mathbf{r}}_0} + t\underbrace{\langle a, b, c \rangle}_{\vec{\mathbf{v}}}. \end{aligned}$$

Setting $\vec{\mathbf{r}} = \langle x, y, z \rangle$ and $\vec{\mathbf{r}}_0 = \langle x_0, y_0, z_0 \rangle$, we now have the **vector equation of a line:**

$$\vec{\mathbf{r}} = \vec{\mathbf{r}}_0 + t\vec{\mathbf{v}}. \tag{4.6.4}$$

Equating components, Equation 4.6.4 shows that the following equations are simultaneously true: $x - x_0 = ta, y - y_0 = tb,$ and $z - z_0 = tc$. If we solve each of these equations for the component variables $x, y,$ and z, we get a set of equations in which each

variable is defined in terms of the parameter t and that, together, describe the line. This set of three equations forms a set of **parametric equations of a line:**

$$x = x_0 + ta$$
$$y = y_0 + tb$$
$$z = z_0 + tc.$$

If we solve each of the equations for t assuming a, b, and c are nonzero, we get a different description of the same line:

$$\frac{x - x_0}{a} = t$$
$$\frac{y - y_0}{b} = t$$
$$\frac{z - z_0}{c} = t.$$

Because each expression equals t, they all have the same value. We can set them equal to each other to create **symmetric equations of a line:**

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

We summarize the results in the following theorem.

Theorem: Parametric and Symmetric Equations of a Line

A line L parallel to vector $\vec{\mathbf{v}} = \langle a, b, c \rangle$ and passing through point $P(x_0, y_0, z_0)$ can be described by the following parametric equations:

$$x = x_0 + ta, y = y_0 + tb, \tag{4.6.5}$$

and

$$z = z_0 + tc. \tag{4.6.6}$$

If the constants a, b, and c are all nonzero, then L can be described by the symmetric equation of the line:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}. \tag{4.6.7}$$

The parametric equations of a line are not unique. Using a different parallel vector or a different point on the line leads to a different, equivalent representation. Each set of parametric equations leads to a related set of symmetric equations, so it follows that a symmetric equation of a line is not unique either.

Example 4.6.1: Equations of a Line in Space

Find parametric and symmetric equations of the line passing through points $(1, 4, -2)$ and $(-3, 5, 0)$.

Solution

First, identify a vector parallel to the line:

$$\vec{\mathbf{v}} = \langle -3 - 1, 5 - 4, 0 - (-2) \rangle = \langle -4, 1, 2 \rangle.$$

Use either of the given points on the line to complete the parametric equations:

$$x = 1 - 4t$$
$$y = 4 + t,$$

and

$$z = -2 + 2t.$$

Solve each equation for t to create the symmetric equation of the line:

$$\frac{x-1}{-4} = y-4 = \frac{z+2}{2}.$$

Exercise 4.6.1

Find parametric and symmetric equations of the line passing through points $(1,-3,2)$ and $(5,-2,8)$.

Hint:

Start by finding a vector parallel to the line.

Answer

Possible set of parametric equations: $x = 1+4t, y = -3+t, z = 2+6t$; related set of symmetric equations:

$$\frac{x-1}{4} = y+3 = \frac{z-2}{6}$$

Sometimes we don't want the equation of a whole line, just a line segment. In this case, we limit the values of our parameter t. For example, let $P(x_0,y_0,z_0)$ and $Q(x_1,y_1,z_1)$ be points on a line, and let $\vec{\mathbf{p}} = \langle x_0,y_0,z_0\rangle$ and $\vec{\mathbf{q}} = \langle x_1,y_1,z_1\rangle$ be the associated position vectors. In addition, let $\vec{\mathbf{r}} = \langle x,y,z\rangle$. We want to find a vector equation for the line segment between P and Q. Using P as our known point on the line, and $\overrightarrow{PQ} = \langle x_1-x_0, y_1-y_0, z_1-z_0\rangle$ as the direction vector equation, Equation 4.6.4 gives

$$\vec{\mathbf{r}} = \vec{\mathbf{p}} + t(\overrightarrow{PQ}). \tag{4.6.8}$$

Equation 4.6.8 can be expanded using properties of vectors:

$$\begin{aligned}\vec{\mathbf{r}} &= \vec{\mathbf{p}} + t(\overrightarrow{PQ})\\ &= \langle x_0,y_0,z_0\rangle + t\langle x_1-x_0, y_1-y_0, z_1-z_0\rangle\\ &= \langle x_0,y_0,z_0\rangle + t(\langle x_1,y_1,z_1\rangle - \langle x_0,y_0,z_0\rangle)\\ &= \langle x_0,y_0,z_0\rangle + t\langle x_1,y_1,z_1\rangle - t\langle x_0,y_0,z_0\rangle\\ &= (1-t)\langle x_0,y_0,z_0\rangle + t\langle x_1,y_1,z_1\rangle\\ &= (1-t)\vec{\mathbf{p}} + t\vec{\mathbf{q}}.\end{aligned}$$

Thus, the vector equation of the line passing through P and Q is

$$\vec{\mathbf{r}} = (1-t)\vec{\mathbf{p}} + t\vec{\mathbf{q}}. \tag{4.6.9}$$

Remember that we did not want the equation of the whole line, just the line segment between P and Q. Notice that when $t=0$, we have $r=p$, and when $t=1$, we have $\vec{\mathbf{r}} = \vec{\mathbf{q}}$. Therefore, the vector equation of the line segment between P and Q is

$$\vec{\mathbf{r}} = (1-t)\vec{\mathbf{p}} + t\vec{\mathbf{q}}, 0 \le t \le 1. \tag{4.6.10}$$

Going back to Equation 4.6.4, we can also find parametric equations for this line segment. We have

$$\begin{aligned}\vec{\mathbf{r}} &= \vec{\mathbf{p}} + t(\overrightarrow{PQ})\\ \langle x,y,z\rangle &= \langle x_0,y_0,z_0\rangle + t\langle x_1-x_0, y_1-y_0, z_1-z_0\rangle\\ &= \langle x_0+t(x_1-x_0), y_0+t(y_1-y_0), z_0+t(z_1-z_0)\rangle.\end{aligned}$$

Then, the parametric equations are

$$\begin{aligned}x &= x_0 + t(x_1-x_0)\\ y &= y_0 + t(y_1-y_0)\\ z &= z_0 + t(z_1-z_0),\ 0 \le t \le 1.\end{aligned} \tag{4.6.11}$$

Example 4.6.2: Parametric Equations of a Line Segment

Find parametric equations of the line segment between the points $P(2,1,4)$ and $Q(3,-1,3)$.

Solution

Start with the parametric equations for a line (Equations 4.6.1) and work with each component separately:

$$\begin{aligned} x &= x_0 + t(x_1 - x_0) \\ &= 2 + t(3-2) \\ &= 2 + t, \\ y &= y_0 + t(y_1 - y_0) \\ &= 1 + t(-1-1) \\ &= 1 - 2t, \end{aligned}$$

and

$$\begin{aligned} z &= z_0 + t(z_1 - z_0) \\ &= 4 + t(3-4) \\ &= 4 - t. \end{aligned}$$

Therefore, the parametric equations for the line segment are

$$\begin{aligned} x &= 2 + t \\ y &= 1 - 2t \\ z &= 4 - t,\ 0 \le t \le 1. \end{aligned}$$

Exercise 4.6.2

Find parametric equations of the line segment between points $P(-1,3,6)$ and $Q(-8,2,4)$.

Answer

$$x = -1 - 7t, y = 3 - t, z = 6 - 2t, 0 \le t \le 1$$

Distance between a Point and a Line

We already know how to calculate the distance between two points in space. We now expand this definition to describe the distance between a point and a line in space. Several real-world contexts exist when it is important to be able to calculate these distances. When building a home, for example, builders must consider "setback" requirements, when structures or fixtures have to be a certain distance from the property line. Air travel offers another example. Airlines are concerned about the distances between populated areas and proposed flight paths.

Let L be a line in the plane and let M be any point not on the line. Then, we define distance d from M to L as the length of line segment $\overline{MP}$, where P is a point on L such that $\overline{MP}$ is perpendicular to L (Figure 4.6.2).

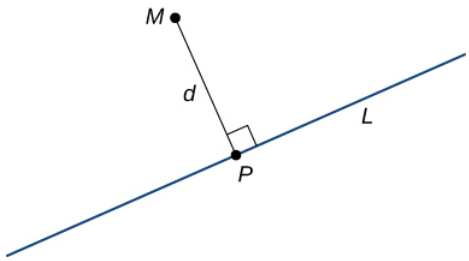


Figure 4.6.2: The distance from point M to line L is the length of $\overline{MP}$.

When we're looking for the distance between a line and a point in space, Figure 4.6.2 still applies. We still define the distance as the length of the perpendicular line segment connecting the point to the line. In space, however, there is no clear way to know which point on the line creates such a perpendicular line segment, so we select an arbitrary point on the line and use properties of vectors to calculate the distance. Therefore, let P be an arbitrary point on line L and let $\vec{\mathbf{v}}$ be a direction vector for L (Figure 4.6.3).

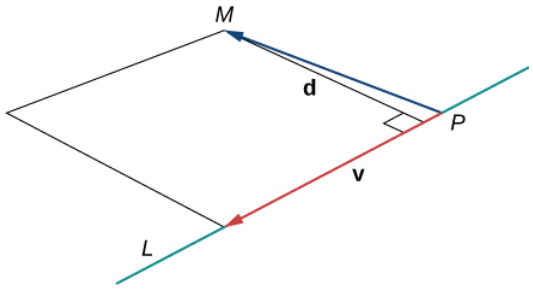


Figure 4.6.3: Vectors $\overrightarrow{PM}$ and $\vec{\mathbf{v}}$ *form two sides of a parallelogram with base* $\|\vec{\mathbf{v}}\|$ *and height* d*, which is the distance between a line and a point in space.*

Vectors $\overrightarrow{PM}$ and $\vec{\mathbf{v}}$ form two sides of a parallelogram with area $\|\overrightarrow{PM}\times\vec{\mathbf{v}}\|$. Using a formula from geometry, the area of this parallelogram can also be calculated as the product of its base and height:

$$\|\overrightarrow{PM}\times\vec{\mathbf{v}}\| = \|\vec{\mathbf{v}}\|d. \tag{4.6.12}$$

We can use this formula to find a general formula for the distance between a line in space and any point not on the line.

Distance from a Point to a Line

Let L be a line in space passing through point P with direction vector $\vec{\mathbf{v}}$. If M is any point not on L, then the distance from M to L is

$$d = \frac{\|\overrightarrow{PM}\times\vec{\mathbf{v}}\|}{\|\vec{\mathbf{v}}\|}. \tag{4.6.13}$$

Example 4.6.3: Calculating the Distance from a Point to a Line

Find the distance between the point $M=(1,1,3)$ and line $\dfrac{x-3}{4}=\dfrac{y+1}{2}=z-3$.

Solution:

From the symmetric equations of the line, we know that vector $\vec{\mathbf{v}}=\langle 4,2,1\rangle$ is a direction vector for the line. Setting the symmetric equations of the line equal to zero, we see that point $P(3,-1,3)$ lies on the line. Then,

$$\begin{aligned}\overrightarrow{PM} &= \langle 1-3, 1-(-1), 3-3\rangle \\ &= \langle -2,2,0\rangle.\end{aligned}$$

To calculate the distance, we need to find $\overrightarrow{PM}\times\vec{\mathbf{v}}$:

$$\begin{aligned}\overrightarrow{PM}\times\vec{\mathbf{v}} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -2 & 2 & 0 \\ 4 & 2 & 1 \end{vmatrix} \\ &= (2-0)\hat{\mathbf{i}} - (-2-0)\hat{\mathbf{j}} + (-4-8)\hat{\mathbf{k}} \\ &= 2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 12\hat{\mathbf{k}}.\end{aligned}$$

Therefore, the distance between the point and the line is (Figure 4.6.4)

$$d = \frac{\|\overrightarrow{PM} \times \vec{\mathbf{v}}\|}{\|\vec{\mathbf{v}}\|}$$
$$= \frac{\sqrt{2^2 + 2^2 + 12^2}}{\sqrt{4^2 + 2^2 + 1^2}}$$
$$= \frac{2\sqrt{38}}{\sqrt{21}}$$
$$= \frac{2\sqrt{798}}{21} \text{ units}$$

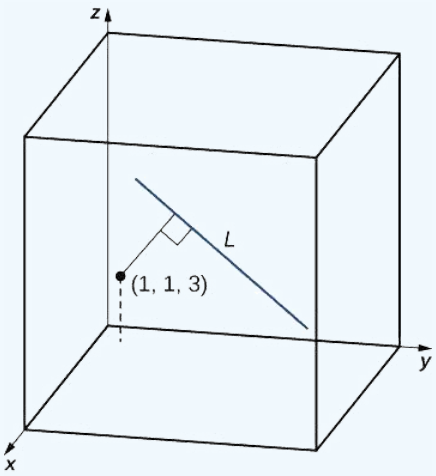


Figure 4.6.4: Point $(1, 1, 3)$ is approximately 2.7 units from the line with symmetric equations $\frac{x-3}{4} = \frac{y+1}{2} = z - 3$.

Exercise 4.6.3

Find the distance between point $(0, 3, 6)$ and the line with parametric equations $x = 1 - t, y = 1 + 2t, z = 5 + 3t$.

Hint

Find a vector with initial point $(0, 3, 6)$ and a terminal point on the line, and then find a direction vector for the line.

Answer

$$\sqrt{\frac{10}{7}} = \frac{\sqrt{70}}{7} \text{ units}$$

Relationships between Lines

Given two lines in the two-dimensional plane, the lines are equal, they are parallel but not equal, or they intersect in a single point. In three dimensions, a fourth case is possible. If two lines in space are not parallel, but do not intersect, then the lines are said to be **skew lines** (Figure 4.6.5).

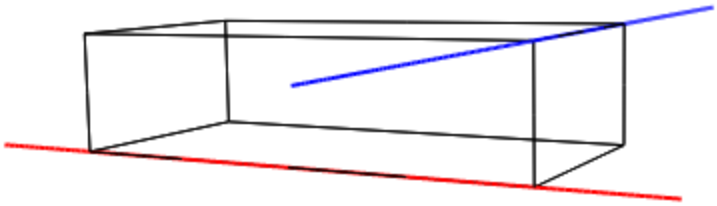


Figure* 4.6.5*: *In three dimensions, it is possible that two lines do not cross, even when they have different directions.*

To classify lines as parallel but not equal, equal, intersecting, or skew, we need to know two things: whether the direction vectors are parallel and whether the lines share a point (Figure 4.6.6).

Direction Vectors Are Parallel?	Lines Share A Common Point? Yes	Lines Share A Common Point? No
Yes	Equal	Parallel but not equal
No	Intersecting	Skew

Figure 4.6.6: Determine the relationship between two lines based on whether their direction vectors are parallel and whether they share a point.

Example 4.6.4: Classifying Lines in Space

For each pair of lines, determine whether the lines are equal, parallel but not equal, skew, or intersecting.

a.

- $L_1 : x = 2s - 1, y = s - 1, z = s - 4$

- $L_2 : x = t - 3, y = 3t + 8, z = 5 - 2t$

b.

- $L_1 : x = -y = z$
- $L_2 : \dfrac{x-3}{2} = y = z - 2$

c.

- $L_1 : x = 6s - 1, y = -2s, z = 3s + 1$
- $L_2 : \dfrac{x-4}{6} = \dfrac{y+3}{-2} = \dfrac{z-1}{3}$

Solution

a. Line L_1 has direction vector $\vec{\mathbf{v}}_1 = \langle 2, 1, 1 \rangle$; line L_2 has direction vector $\vec{\mathbf{v}}_2 = \langle 1, 3, -2 \rangle$. Because the direction vectors are not parallel vectors, the lines are either intersecting or skew. To determine whether the lines intersect, we see if there is a point, (x, y, z), that lies on both lines. To find this point, we use the parametric equations to create a system of equalities:

$$2s - 1 = t - 3; \tag{4.6.14}$$

$$s - 1 = 3t + 8; \tag{4.6.15}$$

$$s - 4 = 5 - 2t. \tag{4.6.16}$$

By the first equation, $t = 2s + 2$. Substituting into the second equation yields

$$s - 1 = 3(2s + 2) + 8$$

$$s - 1 = 6s + 6 + 8$$

$$5s = -15$$

$$s = -3.$$

Substitution into the third equation, however, yields a contradiction:

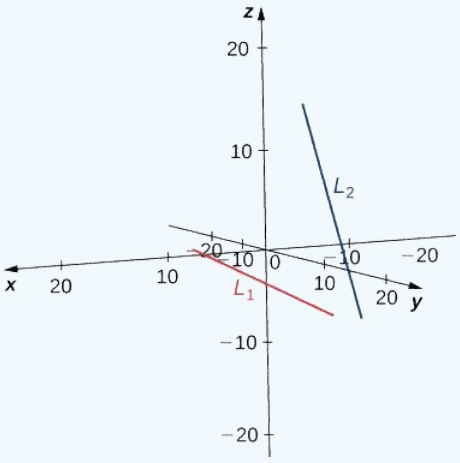
$$s - 4 = 5 - 2(2s + 2)$$

$$s - 4 = 5 - 4s - 4$$

$$5s = 5$$

$$s = 1.$$

There is no single point that satisfies the parametric equations for L_1 and L_2 simultaneously. These lines do not intersect, so they are skew (see the following figure).



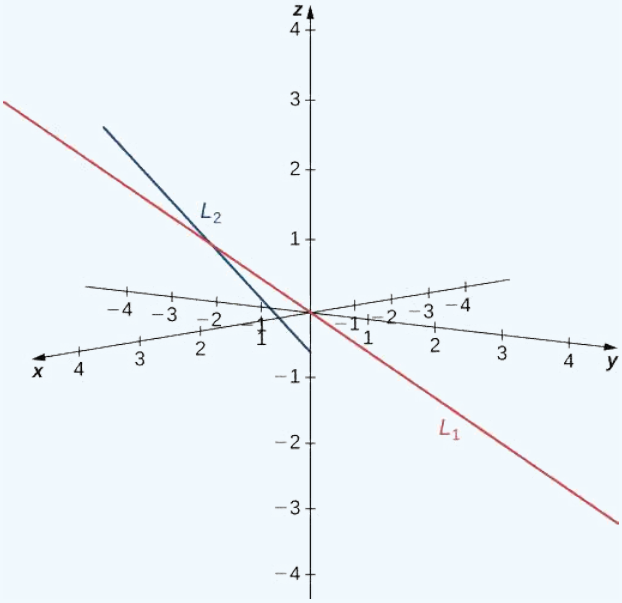
b. Line L_1 has direction vector $\vec{\mathbf{v}}_1 = \langle 1, -1, 1 \rangle$ and passes through the origin, $(0, 0, 0)$. Line L_2 has a different direction vector, $\vec{\mathbf{v}}_2 = \langle 2, 1, 1 \rangle$, so these lines are not parallel or equal. Let r represent the parameter for line L_1 and let s represent the parameter for L_2:

$$x = r \; x = 2s + 3$$

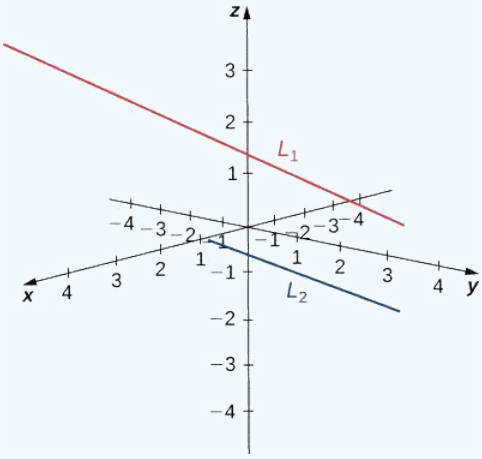
$$y = -r \; y = s$$

$$z = r \; z = s + 2.$$

Solve the system of equations to find $r = 1$ and $s = -1$. If we need to find the point of intersection, we can substitute these parameters into the original equations to get $(1, -1, 1)$ (see the following figure).



c. Lines L_1 and L_2 have equivalent direction vectors: $\vec{\mathbf{v}} = \langle 6, -2, 3 \rangle$. These two lines are parallel (see the following figure).



Exercise 4.6.4

Describe the relationship between the lines with the following parametric equations:

$$x = 1 - 4t, y = 3 + t, z = 8 - 6t$$

$$x = 2 + 3s, y = 2s, z = -1 - 3s.$$

Hint

Start by identifying direction vectors for each line. Is one a multiple of the other?

Answer

These lines are skew because their direction vectors are not parallel and there is no point (x, y, z) that lies on both lines.

Equations for a Plane

We know that a line is determined by two points. In other words, for any two distinct points, there is exactly one line that passes through those points, whether in two dimensions or three. Similarly, given any three points that do not all lie on the same line, there is a unique plane that passes through these points. Just as a line is determined by two points, a plane is determined by three.

This may be the simplest way to characterize a plane, but we can use other descriptions as well. For example, given two distinct, intersecting lines, there is exactly one plane containing both lines. A plane is also determined by a line and any point that does not lie on the line. These characterizations arise naturally from the idea that a plane is determined by three points. Perhaps the most surprising characterization of a plane is actually the most useful.

Imagine a pair of orthogonal vectors that share an initial point. Visualize grabbing one of the vectors and twisting it. As you twist, the other vector spins around and sweeps out a plane. Here, we describe that concept mathematically. Let $\vec{\mathbf{n}} = \langle a, b, c \rangle$ be a vector and $P = (x_0, y_0, z_0)$ be a point. Then the set of all points $Q = (x, y, z)$ such that $\overrightarrow{PQ}$ is orthogonal to $\vec{\mathbf{n}}$ forms a plane (Figure 4.6.7). We say that $\vec{\mathbf{n}}$ is a **normal vector**, or perpendicular to the plane. Remember, the dot product of orthogonal vectors is zero. This fact generates the **vector equation of a plane**:

$$\vec{\mathbf{n}} \cdot \overrightarrow{PQ} = 0. \tag{4.6.17}$$

Rewriting this equation provides additional ways to describe the plane:

$$\begin{aligned} \vec{\mathbf{n}} \cdot \overrightarrow{PQ} &= 0 \\ \langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle &= 0 \\ a(x - x_0) + b(y - y_0) + c(z - z_0) &= 0. \end{aligned}$$

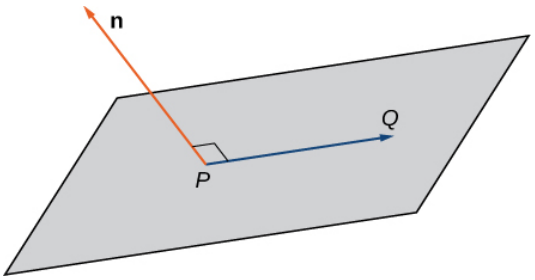


Figure 4.6.7: Given a point P and vector $\vec{\mathbf{n}}$, *the set of all points Q with $\overrightarrow{PQ}$ orthogonal to $\vec{\mathbf{n}}$ forms a plane.*

Definition: scalar equation of a plane

Given a point P and vector $\vec{\mathbf{n}}$, the set of all points Q satisfying the equation $\vec{\mathbf{n}} \cdot \overrightarrow{PQ} = 0$ forms a plane. The equation

$$\vec{\mathbf{n}} \cdot \overrightarrow{PQ} = 0$$

is known as the *vector equation of a plane*.

The *scalar equation of a plane* containing point $P = (x_0, y_0, z_0)$ with normal vector $\vec{n} = \langle a, b, c \rangle$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

This equation can be expressed as $ax + by + cz + d = 0$, where $d = -ax_0 - by_0 - cz_0$. This form of the equation is sometimes called the *general form of the equation of a plane.*

As described earlier in this section, any three points that do not all lie on the same line determine a plane. Given three such points, we can find an equation for the plane containing these points.

Example 4.6.5: Writing an Equation of a Plane Given Three Points in the Plane

Write an equation for the plane containing points $P=(1,1,-2), Q=(0,2,1)$, and $R=(-1,-1,0)$ in both standard and general forms.

Solution

To write an equation for a plane, we must find a normal vector for the plane. We start by identifying two vectors in the plane:

$$\begin{aligned}\overrightarrow{PQ} &= \langle 0-1, 2-1, 1-(-2)\rangle \\ &= \langle -1, 1, 3\rangle \\ \overrightarrow{QR} &= \langle -1-0, -1-2, 0-1\rangle \\ &= \langle -1, -3, -1\rangle.\end{aligned}$$

The cross product $\overrightarrow{PQ}\times\overrightarrow{QR}$ is orthogonal to both $\overrightarrow{PQ}$ and $\overrightarrow{QR}$, so it is normal to the plane that contains these two vectors:

$$\begin{aligned}\vec{\mathbf{n}} &= \overrightarrow{PQ}\times\overrightarrow{QR} \\ &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -1 & 1 & 3 \\ -1 & -3 & -1 \end{vmatrix} \\ &= (-1+9)\hat{\mathbf{i}} - (1+3)\hat{\mathbf{j}} + (3+1)\hat{\mathbf{k}} \\ &= 8\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 4\hat{\mathbf{k}}.\end{aligned}$$

Thus, $n=\langle 8,-4,4\rangle$, and we can choose any of the three given points to write an equation of the plane:

$$\begin{aligned}8(x-1)-4(y-1)+4(z+2) &= 0 \\ 8x-4y+4z+4 &= 0.\end{aligned}$$



The scalar equations of a plane vary depending on the normal vector and point chosen.

Example 4.6.6: Writing an Equation for a Plane Given a Point and a Line

Find an equation of the plane that passes through point $(1,4,3)$ and contains the line given by $x=\dfrac{y-1}{2}=z+1$.

Solution

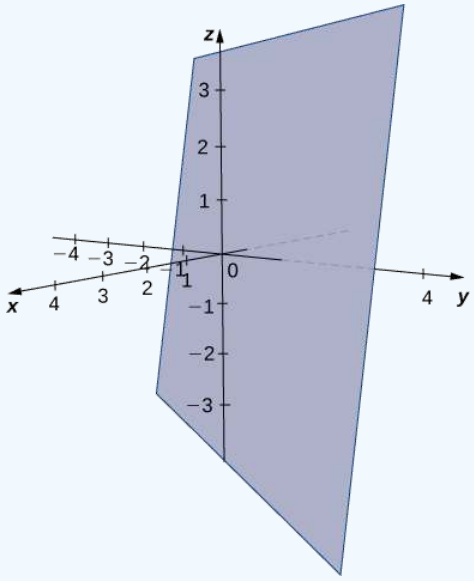
Symmetric equations describe the line that passes through point $(0,1,-1)$ parallel to vector $\vec{\mathbf{v}}_1=\langle 1,2,1\rangle$ (see the following figure). Use this point and the given point, $(1,4,3)$, to identify a second vector parallel to the plane:

$$\vec{\mathbf{v}}_2=\langle 1-0,4-1,3-(-1)\rangle=\langle 1,3,4\rangle.$$

Use the cross product of these vectors to identify a normal vector for the plane:

$$\begin{aligned}\vec{\mathbf{n}}&=\vec{\mathbf{v}}_1\times\vec{\mathbf{v}}_2\\&=\begin{vmatrix}\hat{\mathbf{i}}&\hat{\mathbf{j}}&\hat{\mathbf{k}}\\1&2&1\\1&3&4\end{vmatrix}\\&=(8-3)\hat{\mathbf{i}}-(4-1)\hat{\mathbf{j}}+(3-2)\hat{\mathbf{k}}\\&=5\hat{\mathbf{i}}-3\hat{\mathbf{j}}+\hat{\mathbf{k}}.\end{aligned}$$

The scalar equations for the plane are $5x-3(y-1)+(z+1)=0$ and $5x-3y+z+4=0$.



Exercise 4.6.6

Find an equation of the plane containing the lines L_1 and L_2:

$$L_1: x=-y=z$$

$$L_2: \frac{x-3}{2}=y=z-2.$$

Hint

Hint: The cross product of the lines' direction vectors gives a normal vector for the plane.

Answer

$$-2(x-1)+(y+1)+3(z-1)=0$$

or

$$-2x+y+3z=0$$

Now that we can write an equation for a plane, we can use the equation to find the distance d between a point P and the plane. It is defined as the shortest possible distance from P to a point on the plane.

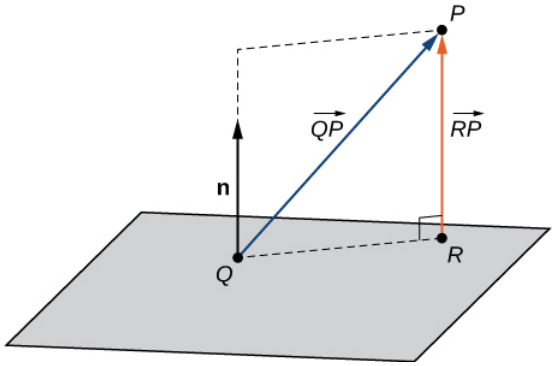


Figure 4.6.8: We want to find the shortest distance from point P to the plane. Let point R be the point in the plane such that, for any other point in the plane Q, $\|\overrightarrow{RP}\| < \|\overrightarrow{QP}\|$.

Just as we find the two-dimensional distance between a point and a line by calculating the length of a line segment perpendicular to the line, we find the three-dimensional distance between a point and a plane by calculating the length of a line segment perpendicular to the plane. Let R bet the point in the plane such that $\overrightarrow{RP}$ is orthogonal to the plane, and let Q be an arbitrary point in the plane. Then the projection of vector $\overrightarrow{QP}$ onto the normal vector describes vector $\overrightarrow{RP}$, as shown in Figure.

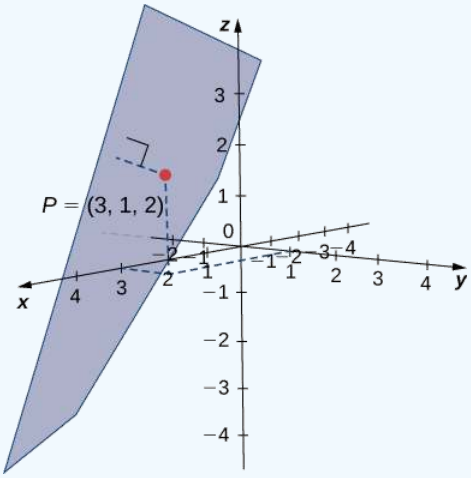
The Distance between a Plane and a Point

Suppose a plane with normal vector $\vec{\mathbf{n}}$ passes through point Q. The distance d from the plane to a point P not in the plane is given by

$$d = \|\text{proj}_{\vec{\mathbf{n}}}\,\overrightarrow{QP}\| = |\,\text{comp}_{\vec{\mathbf{n}}}\,\overrightarrow{QP}\,| = \frac{\left|\overrightarrow{QP}\cdot\vec{\mathbf{n}}\right|}{\|\vec{\mathbf{n}}\|}. \tag{4.6.18}$$

Example 4.6.7: Distance between a Point and a Plane

Find the distance between point $P=(3,1,2)$ and the plane given by $x-2y+z=5$ (see the following figure).



Solution

The coefficients of the plane's equation provide a normal vector for the plane: $\vec{\mathbf{n}}=\langle 1,-2,1\rangle$. To find vector $\overrightarrow{QP}$, we need a point in the plane. Any point will work, so set $y=z=0$ to see that point $Q=(5,0,0)$ lies in the plane. Find the component

form of the vector from Q to P:

$$\overrightarrow{QP} = \langle 3-5, 1-0, 2-0\rangle = \langle -2, 1, 2\rangle.$$

Apply the distance formula from Equation:

$$\begin{aligned} d &= \frac{\left|\overrightarrow{QP}\cdot\vec{\mathbf{n}}\right|}{\|\vec{\mathbf{n}}\|} \\ &= \frac{|\langle -2,1,2\rangle \cdot \langle 1,-2,1\rangle|}{\sqrt{1^2+(-2)^2+1^2}} \\ &= \frac{|-2-2+2|}{\sqrt{6}} \\ &= \frac{2}{\sqrt{6}} = \frac{\sqrt{6}}{3} \text{ units.} \end{aligned}$$

Exercise 4.6.7

Find the distance between point $P = (5, -1, 0)$ and the plane given by $4x + 2y - z = 3$.

Hint

Point $(0, 0, -3)$ lies on the plane.

Answer

$$\frac{15}{\sqrt{21}} = \frac{5\sqrt{21}}{7} \text{ units} \tag{4.6.19}$$



Parallel and Intersecting Planes

We have discussed the various possible relationships between two lines in two dimensions and three dimensions. When we describe the relationship between two planes in space, we have only two possibilities: the two distinct planes are parallel or they intersect. When two planes are parallel, their normal vectors are parallel. When two planes intersect, the intersection is a line (Figure 4.6.9).

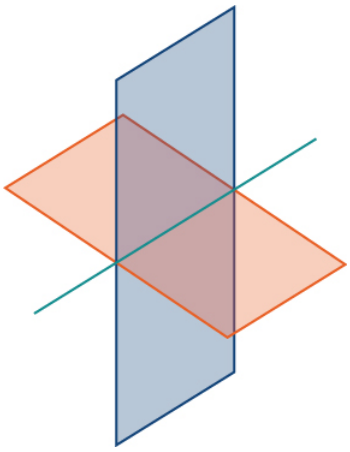
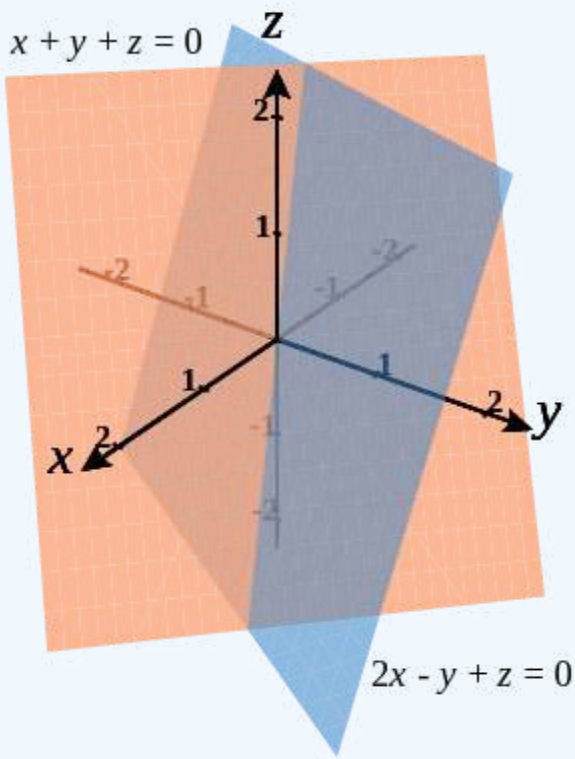


Figure 4.6.9: The intersection of two nonparallel planes is always a line.

We can use the equations of the two planes to find parametric equations for the line of intersection.

Example 4.6.8: Finding the Line of Intersection for Two Planes

Find parametric and symmetric equations for the line formed by the intersection of the planes given by $x+y+z=0$ and $2x-y+z=0$ (see the following figure).



Solution

Note that the two planes have nonparallel normals, so the planes intersect. Further, the origin satisfies each equation, so we know the line of intersection passes through the origin. Add the plane equations so we can eliminate one of the variables, in

this case, y:

$$x+y+z=0$$
$$2x-y+z=0$$
$$\overline{\qquad\qquad\qquad}$$
$$3x+2z=0\,.$$

This gives us $x=-\frac{2}{3}z$. We substitute this value into the first equation to express y in terms of z:

$$x+y+z=0$$
$$-\frac{2}{3}z+y+z=0$$
$$y+\frac{1}{3}z=0$$
$$y=-\frac{1}{3}z$$

We now have the first two variables, x and y, in terms of the third variable, z. Now we define z in terms of t. To eliminate the need for fractions, we choose to define the parameter t as $t=-\frac{1}{3}z$. Then, $z=-3t$. Substituting the parametric representation of z back into the other two equations, we see that the parametric equations for the line of intersection are $x=2t, y=t, z=-3t$. The symmetric equations for the line are $\frac{x}{2}=y=\frac{z}{-3}$.

Exercise 4.6.8

Find parametric equations for the line formed by the intersection of planes $x+y-z=3$ and $3x-y+3z=5$.

Hint

Add the two equations, then express z in terms of x. Then, express y in terms of x.

Answer

$$x=t, y=7-3t, z=4-2t$$

In addition to finding the equation of the line of intersection between two planes, we may need to find the angle formed by the intersection of two planes. For example, builders constructing a house need to know the angle where different sections of the roof meet to know whether the roof will look good and drain properly. We can use normal vectors to calculate the angle between the two planes. We can do this because the angle between the normal vectors is the same as the angle between the planes. Figure 4.6.10 shows why this is true.

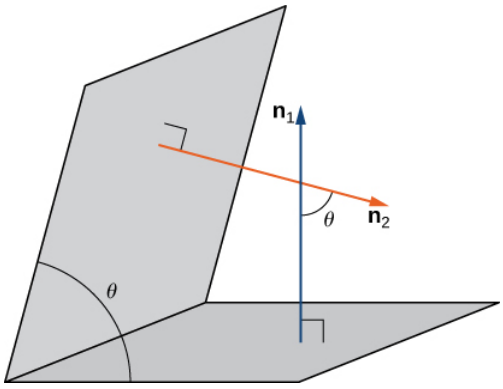


Figure 4.6.10: The angle between two planes has the same measure as the angle between the normal vectors for the planes.

We can find the measure of the angle θ between two intersecting planes by first finding the cosine of the angle, using the following equation:

$$\cos\theta = \frac{|\vec{\mathbf{n}}_1 \cdot \vec{\mathbf{n}}_2|}{\|\vec{\mathbf{n}}_1\|\|\vec{\mathbf{n}}_2\|}. \tag{4.6.20}$$

We can then use the angle to determine whether two planes are parallel or orthogonal or if they intersect at some other angle.

Example 4.6.9: Finding the Angle between Two Planes

Determine whether each pair of planes is parallel, orthogonal, or neither. If the planes are intersecting, but not orthogonal, find the measure of the angle between them. Give the answer in radians and round to two decimal places.

a. $x+2y-z=8$ and $2x+4y-2z=10$
b. $2x-3y+2z=3$ and $6x+2y-3z=1$
c. $x+y+z=4$ and $x-3y+5z=1$

Solution:

a. The normal vectors for these planes are $\vec{\mathbf{n}}_1 = \langle 1, 2, -1\rangle$ and $\vec{\mathbf{n}}_2 = \langle 2, 4, -2\rangle$. These two vectors are scalar multiples of each other. The normal vectors are parallel, so the planes are parallel.
b. The normal vectors for these planes are $\vec{\mathbf{n}}_1 = \langle 2, -3, 2\rangle$ and $\vec{\mathbf{n}}_2 = \langle 6, 2, -3\rangle$. Taking the dot product of these vectors, we have

$$\begin{aligned}\vec{\mathbf{n}}_1 \cdot \vec{\mathbf{n}}_2 &= \langle 2, -3, 2\rangle \cdot \langle 6, 2, -3\rangle \\ &= 2(6) - 3(2) + 2(-3) = 0.\end{aligned}$$

The normal vectors are orthogonal, so the corresponding planes are orthogonal as well.
c. The normal vectors for these planes are $\vec{\mathbf{n}}_1 = \langle 1, 1, 1\rangle$ and $\vec{\mathbf{n}}_2 = \langle 1, -3, 5\rangle$:

$$\begin{aligned}\cos\theta &= \frac{|\vec{\mathbf{n}}_1 \cdot \vec{\mathbf{n}}_2|}{\|\vec{\mathbf{n}}_1\|\|\vec{\mathbf{n}}_2\|} \\ &= \frac{|\langle 1, 1, 1\rangle \cdot \langle 1, -3, 5\rangle|}{\sqrt{1^2+1^2+1^2}\sqrt{1^2+(-3)^2+5^2}} \\ &= \frac{3}{\sqrt{105}}\end{aligned}$$

Then $\theta = \arccos\frac{3}{\sqrt{105}} \approx 1.27$ rad.
Thus the angle between the two planes is about 1.27 rad, or approximately 73°.

Exercise 4.6.9

Find the measure of the angle between planes $x+y-z=3$ and $3x-y+3z=5$. Give the answer in radians and round to two decimal places.

Hint

Use the coefficients of the variables in each equation to find a normal vector for each plane.

Answer

$1.44\,\text{rad}$

When we find that two planes are parallel, we may need to find the distance between them. To find this distance, we simply select a point in one of the planes. The distance from this point to the other plane is the distance between the planes.

Previously, we introduced the formula for calculating this distance in Equation 4.6.18:

$$d = \frac{\overrightarrow{QP} \cdot \vec{\mathbf{n}}}{\|\vec{\mathbf{n}}\|}, \tag{4.6.21}$$

where Q is a point on the plane, P is a point not on the plane, and $\vec{n}$ is the normal vector that passes through point Q. Consider the distance from point (x_0, y_0, z_0) to plane $ax + by + cz + k = 0$. Let (x_1, y_1, z_1) be any point in the plane. Substituting into the formula yields

$$\begin{aligned} d &= \frac{|a(x_0 - x_1) + b(y_0 - y_1) + c(z_0 - z_1)|}{\sqrt{a^2 + b^2 + c^2}} \\ &= \frac{|ax_0 + by_0 + cz_0 + k|}{\sqrt{a^2 + b^2 + c^2}}. \end{aligned}$$

We state this result formally in the following theorem.

Distance from a Point to a Plane

Let $P(x_0, y_0, z_0)$ be a point. The distance from P to plane $a_x + b_y + c_z + k = 0$ is given by

$$d = \frac{|ax_0 + by_0 + cz_0 + k|}{\sqrt{a^2 + b^2 + c^2}}. \tag{4.6.22}$$

Example 4.6.10: Finding the Distance between Parallel Planes

Find the distance between the two parallel planes given by $2x + y - z = 2$ and $2x + y - z = 8$.

Solution

Point $(1, 0, 0)$ lies in the first plane. The desired distance, then, is

$$\begin{aligned} d &= \frac{|ax_0 + by_0 + cz_0 + k|}{\sqrt{a^2 + b^2 + c^2}} \\ &= \frac{|2(1) + 1(0) + (-1)(0) + (-8)|}{\sqrt{2^2 + 1^2 + (-1)^2}} \\ &= \frac{6}{\sqrt{6}} = \sqrt{6} \text{ units} \end{aligned}$$

Exercise 4.6.10:

Find the distance between parallel planes $5x - 2y + z = 6$ and $5x - 2y + z = -3$.

Hint

Set $x = y = 0$ to find a point on the first plane.

Answer

$$\frac{9}{\sqrt{30}} = \frac{3\sqrt{30}}{10} \text{ units}$$

Distance between Two Skew Lines

Finding the distance from a point to a line or from a line to a plane seems like a pretty abstract procedure. But, if the lines represent pipes in a chemical plant or tubes in an oil refinery or roads at an intersection of highways, confirming that the distance between them meets specifications can be both important and awkward to measure. One way is to model the two pipes as lines, using the techniques in this chapter, and then calculate the distance between them. The calculation involves forming vectors along the directions of the lines and using both the cross product and the dot product.

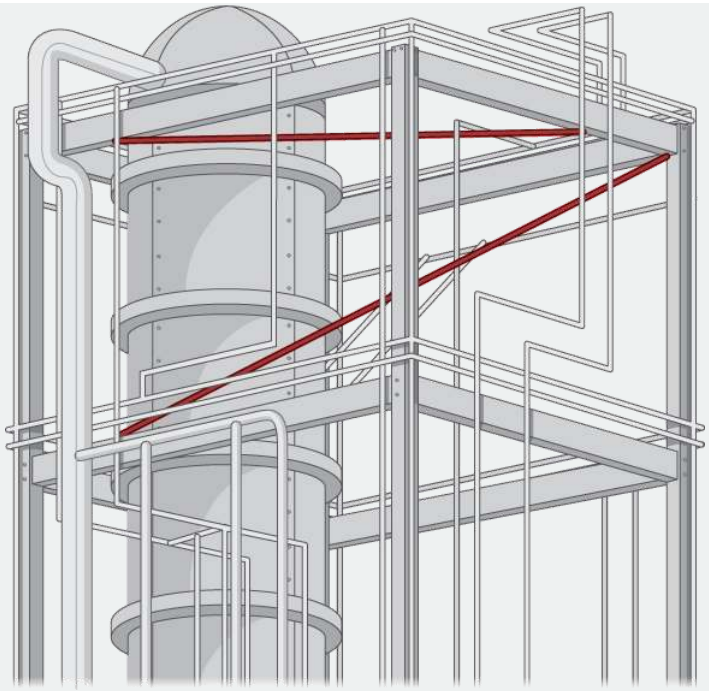


Figure 4.6.11: Industrial pipe installations often feature pipes running in different directions. How can we find the distance between two skew pipes?

The symmetric forms of two lines, L_1 and L_2, are

$$L_1 : \frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1} \tag{4.6.23}$$

$$L_2 : \frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}. \tag{4.6.24}$$

You are to develop a formula for the distance d between these two lines, in terms of the values $a_1, b_1, c_1; a_2, b_2, c_2; x_1, y_1, z_1;$ and x_2, y_2, z_2. The distance between two lines is usually taken to mean the minimum distance, so this is the length of a line segment or the length of a vector that is perpendicular to both lines and intersects both lines.

1. First, write down two vectors, $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$, that lie along L_1 and L_2, respectively.

2. Find the cross product of these two vectors and call it $\vec{\mathbf{N}}$. This vector is perpendicular to $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$, and hence is perpendicular to both lines.

3. From vector $\vec{\mathbf{N}}$, form a unit vector $\vec{\mathbf{n}}$ in the same direction.

4. Use symmetric equations to find a convenient vector $\vec{\mathbf{v}}_{12}$ that lies between any two points, one on each line. Again, this can be done directly from the symmetric equations.

5. The dot product of two vectors is the magnitude of the projection of one vector onto the other—that is, $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = \|\vec{\mathbf{A}}\| \|\vec{\mathbf{B}}\| \cos\theta$, where θ is the angle between the vectors. Using the dot product, find the projection of vector $\vec{\mathbf{v}}_{12}$ found in step 4 onto unit vector $\vec{\mathbf{n}}$ found in step 3. This projection is perpendicular to both lines, and hence its length must be the perpendicular distance d between them. Note that the value of d may be negative, depending on your choice of vector $\vec{\mathbf{v}}_{12}$ or the order of the cross product, so use absolute value signs around the numerator.

6. Check that your formula gives the correct distance of $|-25|/\sqrt{198} \approx 1.78$ between the following two lines:

$$L_1 : \frac{x-5}{2} = \frac{y-3}{4} = \frac{z-1}{3} \tag{4.6.25}$$

$$L_2 : \frac{x-6}{3} = \frac{y-1}{5} = \frac{z}{7}. \tag{4.6.26}$$

7. Is your general expression valid when the lines are parallel? If not, why not? (Hint: What do you know about the value of the cross product of two parallel vectors? Where would that result show up in your expression for d?)

8. Demonstrate that your expression for the distance is zero when the lines intersect. Recall that two lines intersect if they are not parallel and they are in the same plane. Hence, consider the direction of $\vec{\mathbf{n}}$ and $\vec{\mathbf{v}}_{12}$. What is the result of their dot product?

9. Consider the following application. Engineers at a refinery have determined they need to install support struts between many of the gas pipes to reduce damaging vibrations. To minimize cost, they plan to install these struts at the closest points between adjacent skewed pipes. Because they have detailed schematics of the structure, they are able to determine the correct lengths of the struts needed, and hence manufacture and distribute them to the installation crews without spending valuable time making measurements.

The rectangular frame structure has the dimensions $4.0 \times 15.0 \times 10.0\,\text{m}$ (height, width, and depth). One sector has a pipe entering the lower corner of the standard frame unit and exiting at the diametrically opposed corner (the one farthest away at the top); call this L_1. A second pipe enters and exits at the two different opposite lower corners; call this L_2 (Figure 4.6.12).

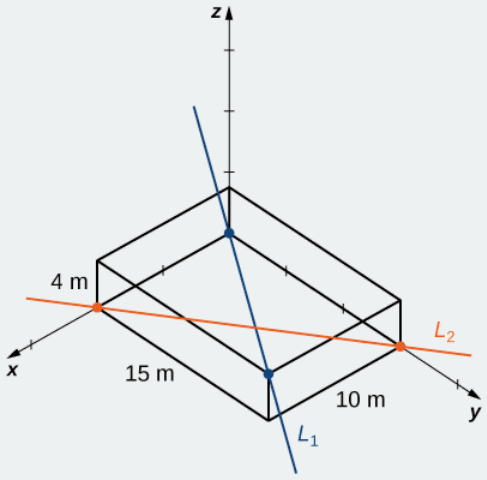


Figure 4.6.12: Two pipes cross through a standard frame unit.

Write down the vectors along the lines representing those pipes, find the cross product between them from which to create the unit vector $\vec{\mathbf{n}}$, define a vector that spans two points on each line, and finally determine the minimum distance between the lines. (Take the origin to be at the lower corner of the first pipe.) Similarly, you may also develop the symmetric equations for each line and substitute directly into your formula.

Key Concepts

- In three dimensions, the direction of a line is described by a direction vector. The vector equation of a line with direction vector $\vec{\mathbf{v}} = \langle a, b, c \rangle$ passing through point $P = (x_0, y_0, z_0)$ is $\vec{\mathbf{r}} = \vec{\mathbf{r}}_0 + t\vec{\mathbf{v}}$, where $\vec{\mathbf{r}}_0 = \langle x_0, y_0, z_0 \rangle$ is the position vector of point P. This equation can be rewritten to form the parametric equations of the line: $x = x_0 + ta, y = y_0 + tb$, and $z = z_0 + tc$. The line can also be described with the symmetric equations $\dfrac{x - x_0}{a} = \dfrac{y - y_0}{b} = \dfrac{z - z_0}{c}$.
- Let L be a line in space passing through point P with direction vector $\vec{\mathbf{v}}$. If Q is any point not on L, then the distance from Q to L is $d = \dfrac{\|\overrightarrow{PQ} \times \vec{\mathbf{v}}\|}{\|\vec{\mathbf{v}}\|}$.
- In three dimensions, two lines may be parallel but not equal, equal, intersecting, or skew.
- Given a point P and vector $\vec{\mathbf{n}}$, the set of all points Q satisfying equation $\vec{\mathbf{n}} \cdot \overrightarrow{PQ} = 0$ forms a plane. Equation $\vec{\mathbf{n}} \cdot \overrightarrow{PQ} = 0$ is known as the *vector equation of a plane.*
- The scalar equation of a plane containing point $P = (x_0, y_0, z_0)$ with normal vector $\vec{\mathbf{n}} = \langle a, b, c \rangle$ is $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$. This equation can be expressed as $ax + by + cz + d = 0$, where $d = -ax_0 - by_0 - cz_0$. This form of the equation is sometimes called the general *form of the equation of a plane.*

- Suppose a plane with normal vector n passes through point Q. The distance D from the plane to point P not in the plane is given by

$$D = \|\text{proj}_{\vec{\mathbf{n}}}\overrightarrow{QP}\| = |\text{ comp}_{\vec{\mathbf{n}}}\overrightarrow{QP}| = \frac{\left|\overrightarrow{QP}\cdot\vec{\mathbf{n}}\right|}{\|\vec{\mathbf{n}}\|}. \tag{4.6.27}$$

- The normal vectors of parallel planes are parallel. When two planes intersect, they form a line.
- The measure of the angle θ between two intersecting planes can be found using the equation: $\cos\theta = \frac{|\vec{\mathbf{n}}_1 \cdot \vec{\mathbf{n}}_2|}{\|\vec{\mathbf{n}}_1\|\|\vec{\mathbf{n}}_2\|}$, where $\vec{\mathbf{n}}_1$ and $\vec{\mathbf{n}}_2$ are normal vectors to the planes.
- The distance D from point (x_0, y_0, z_0) to plane $ax + by + cz + d = 0$ is given by

$$D = \frac{|a(x_0 - x_1) + b(y_0 - y_1) + c(z_0 - z_1)|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}} \tag{4.6.28}$$

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Key Equations

- **Vector Equation of a Line**

  $\vec{\mathbf{r}} = \vec{\mathbf{r}}_0 + t\vec{\mathbf{v}}$

- **Parametric Equations of a Line**

  $x = x_0 + ta, y = y_0 + tb,$ and $z = z_0 + tc$

- **Symmetric Equations of a Line**

  $\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$

- **Vector Equation of a Plane**

  $\vec{\mathbf{n}} \cdot \overrightarrow{PQ} = 0$

- **Scalar Equation of a Plane**

  $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

- **Distance between a Plane and a Point**

  $d = \|\text{proj}_{\vec{\mathbf{n}}}\overrightarrow{QP}\| = |\text{ comp}_{\vec{\mathbf{n}}}\overrightarrow{QP}| = \frac{\left|\overrightarrow{QP}\cdot\vec{\mathbf{n}}\right|}{\|\vec{\mathbf{n}}\|}$

Glossary

direction vector

a vector parallel to a line that is used to describe the direction, or orientation, of the line in space

general form of the equation of a plane

an equation in the form $ax + by + cz + d = 0,$ where $\vec{\mathbf{n}} = \langle a, b, c\rangle$ is a normal vector of the plane, $P = (x_0, y_0, z_0)$ is a point on the plane, and $d = -ax_0 - by_0 - cz_0$

normal vector

a vector perpendicular to a plane

parametric equations of a line

the set of equations $x = x_0 + ta, y = y_0 + tb,$ and $z = z_0 + tc$ describing the line with direction vector $v = \langle a, b, c\rangle$ passing through point (x_0, y_0, z_0)

scalar equation of a plane

the equation $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$ used to describe a plane containing point $P=(x_0,y_0,z_0)$ with normal vector $n=\langle a,b,c\rangle$ or its alternate form $ax+by+cz+d=0$, where $d=-ax_0-by_0-cz_0$

skew lines

two lines that are not parallel but do not intersect

symmetric equations of a line

the equations $\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}$ describing the line with direction vector $v=\langle a,b,c\rangle$ passing through point (x_0,y_0,z_0)

vector equation of a line

the equation $\vec{\mathbf{r}}=\vec{\mathbf{r}}_0+t\vec{\mathbf{v}}$ used to describe a line with direction vector $\vec{\mathbf{v}}=\langle a,b,c\rangle$ passing through point $P=(x_0,y_0,z_0)$, where $\vec{\mathbf{r}}_0=\langle x_0,y_0,z_0\rangle$, is the position vector of point P

vector equation of a plane

the equation $\vec{\mathbf{n}}\cdot\overrightarrow{PQ}=0,$ where P is a given point in the plane, Q is any point in the plane, and $\vec{\mathbf{n}}$ is a normal vector of the plane

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4.6E: Exercises for Section 4.6

In exercises 1 - 4, points P and Q are given. Let L be the line passing through points P and Q.

a. Find the vector equation of line L.

b. Find parametric equations of line L.

c. Find symmetric equations of line L.

d. Find parametric equations of the line segment determined by P and Q.

1) $P(-3,5,9), \quad Q(4,-7,2)$

Answer

a. $\vec{\mathbf{r}} = \langle -3,5,9\rangle + t\langle 7,-12,-7\rangle, \quad t \in R;$
b. $x = -3+7t, \quad y = 5-12t, \quad z = 9-7t, \quad t \in R;$
c. $\dfrac{x+3}{7} = \dfrac{y-5}{-12} = \dfrac{z-9}{-7};$
d. $x = -3+7t, \quad y = 5-12t, \quad z = 9-7t, \quad 0 \le t \le 1$

2) $P(4,0,5), \quad Q(2,3,1)$

3) $P(-1,0,5), \quad Q(4,0,3)$

Answer

a. $\vec{\mathbf{r}} = \langle -1,0,5\rangle + t\langle 5,0,-2\rangle, \quad t \in R;$
b. $x = -1+5t, y = 0, z = 5-2t, \quad t \in R;$
c. $\dfrac{x+1}{5} = \dfrac{z-5}{-2}, \quad y = 0;$
d. $x = -1+5t, \quad y = 0, \quad z = 5-2t, \quad t \in [0,1]$

4) $P(7,-2,6), \quad Q(-3,0,6)$

For exercises 5 - 8, point P and vector $\vec{\mathbf{v}}$ are given. Let L be the line passing through point P with direction $\vec{\mathbf{v}}$.

a. Find parametric equations of line L.

b. Find symmetric equations of line L.

c. Find the intersection of the line with the xy-plane.

5) $P(1,-2,3), \ \vec{\mathbf{v}} = \langle 1,2,3\rangle$

Answer

a. $x = 1+t, \quad y = -2+2t, \quad z = 3+3t, \quad t \in R;$
b. $\dfrac{x-1}{1} = \dfrac{y+2}{2} = \dfrac{z-3}{3};$
c. $(0,-4,0)$

6) $P(3,1,5), \ \vec{\mathbf{v}} = \langle 1,1,1\rangle$

7) $P(3,1,5), \ \vec{\mathbf{v}} = \overrightarrow{QR}$, where $Q(2,2,3)$ and $R(3,2,3)$

Answer

a. $x = 3+t, \quad y = 1, \quad z = 5, \quad t \in R;$
b. $y = 1, \quad z = 5;$
c. The line does not intersect the xy-plane.

8) $P(2,3,0), \ \vec{\mathbf{v}} = \overrightarrow{QR}$, where $Q(0,4,5)$ and $R(0,4,6)$

For exercises 9 and 10, line L is given.

a. Find a point P that belongs to the line and a direction vector $\vec{\mathbf{v}}$ of the line. Express $\vec{\mathbf{v}}$ in component form.

b. Find the distance from the origin to line L.

9) $x = 1 + t, \quad y = 3 + t, \quad z = 5 + 4t, \quad t \in R$

Answer

a. A possible point and direction vector are $P(1, 3, 5)$ and $\vec{\mathbf{v}} = \langle 1, 1, 4 \rangle$, but these answers are not unique.
b. $\sqrt{3}$ units

10) $-x = y + 1, \quad z = 2$

11) Find the distance between point $A(-3, 1, 1)$ and the line of symmetric equations

$x = -y = -z.$

Answer

$\frac{2\sqrt{2}}{\sqrt{3}} = \frac{2\sqrt{6}}{3}$ units

12) Find the distance between point $A(4, 2, 5)$ and the line of parametric equations

$x = -1 - t, \; y = -t, \; z = 2, \; t \in R.$

For exercises 13 - 14, lines L_1 and L_2 are given.

a. Verify whether lines L_1 and L_2 are parallel.

b. If the lines L_1 and L_2 are parallel, then find the distance between them.

13) $L_1 : x = 1 + t, \quad y = t, \quad z = 2 + t, \quad t \in R \quad$ and $L_2 : x - 3 = y - 1 = z - 3$

Answer

a. Parallel;
b. $\frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{6}}{3}$ units

14) $L_1 : x = 2, \quad y = 1, \quad z = t, \quad t \in R \;$ and $L_2 : x = 1, \quad y = 1, \quad z = 2 - 3t, \quad t \in R$

15) Show that the line passing through points $P(3, 1, 0)$ and $Q(1, 4, -3)$ is perpendicular to the line with equation $x = 3t, \quad y = -32 + 8t, \quad z = -9 + 6t, \quad t \in R.$

Answer

$\overrightarrow{PQ} = \langle -2, 3, -3 \rangle$ is the direction vector of the line through points P and Q, and the direction vector of the line defined by the parametric equations above is $\vec{\mathbf{v}} = \langle 3, 8, 6 \rangle$.

Since $\vec{\mathbf{v}} \cdot \overrightarrow{PQ} = -6 + 24 - 18 = 0$, the two direction vectors are orthogonal.
Now all we need to show is that the two lines intersect.
The line through points $P(3, 1, 0)$ and $Q(1, 4, -3)$ has parametric equations: $x = 3 - 2u$, $y = 1 + 3u$, and $z = -3u$.
Setting the x- and z-coordinates of the two lines equal, we obtain the system of equations:

$$3t = 3 - 2u \quad \text{and} \quad -9 + 6t = -3u$$

Solving this system using substitution gives us, $u = -3$ and $t = 3$. Plugging these values of t and u back into the parametric equations of these two lines gives us the intersection point with coordinates $(9, -8, 9)$ on both lines.

Therefore the lines intersect and the line through points P and Q with direction vector $\overrightarrow{PQ}$ is perpendicular to the other line.

16) Are the lines of equations $x = -2 + 2t, \quad y = -6, \quad z = 2 + 6t, \quad t \in R \;$ and $x = -1 + t, \quad y = 1 + t, \quad z = t, \quad t \in R,$ perpendicular to each other?

17) Find the point of intersection of the lines of equations $x=-2y=3z$ and $x=-5-t, \quad y=-1+t, \quad z=t-11, \quad t\in R.$

Answer

$(-12,6,-4)$

18) Find the intersection point of the x-axis with the line of parametric equations $x=10+t, \quad y=2-2t, \quad z=-3+3t, \quad t\in R.$

For exercises 19 - 22, lines L_1 and L_2 are given. Determine whether the lines are *equal, parallel but not equal, skew,* or *intersecting*.

19) $L_1: x=y-1=-z$ and $L_2: x-2=-y=\dfrac{z}{2}$

Answer

The lines are skew.

20) $L_1: x=2t, \quad y=0, \quad z=3, \quad t\in R$ and $L_2: x=0, \quad y=8+s, \quad z=7+s, \quad s\in R$

21) $L_1: x=-1+2t, \quad y=1+3t, \quad z=7t, \quad t\in R$ and $L_2: x-1=\frac{2}{3}(y-4)=\frac{2}{7}z-2$

Answer

The lines are equal.

22) $L_1: 3x=y+1=2z$ and $L_2: x=6+2t, \quad y=17+6t, \quad z=9+3t, \quad t\in R$

23) Consider line L of symmetric equations $x-2=-y=\dfrac{z}{2}$ and point $A(1,1,1)$.

a. Find parametric equations for a line parallel to L that passes through point A.

b. Find symmetric equations of a line skew to L and that passes through point A.

c. Find symmetric equations of a line that intersects L and passes through point A.

Answer

a. $x=1+t, \quad y=1-t, \quad z=1+2t, \quad t\in R$
b. For instance, the line passing through A with direction vector j: $x=1, z=1$
c. For instance, the line passing through A and point $(2,0,0)$ that belongs to L is a line that intersects;
$L: \frac{x-1}{-1}=y-1=z-1$

24) Consider line L of parametric equations $x=t, \quad y=2t, \quad z=3, \quad t\in R.$

a. Find parametric equations for a line parallel to L that passes through the origin.

b. Find parametric equations of a line skew to L that passes through the origin.

c. Find symmetric equations of a line that intersects L and passes through the origin.

For exercises 25 - 28, point P and vector $\vec{\mathbf{n}}$ are given.

a. Find the scalar equation of the plane that passes through P and has normal vector $\vec{\mathbf{n}}$.

b. Find the general form of the equation of the plane that passes through P and has normal vector $\vec{\mathbf{n}}$.

25) $P(0,0,0), \quad \vec{\mathbf{n}}=3\hat{\mathbf{i}}-2\hat{\mathbf{j}}+4\hat{\mathbf{k}}$

Answer

a. $3x-2y+4z=0$
b. $3x-2y+4z=0$

26) $P(3,2,2), \quad \vec{\mathbf{n}}=2\hat{\mathbf{i}}+3\hat{\mathbf{j}}-\hat{\mathbf{k}}$

27) $P(1,2,3), \quad \vec{\mathbf{n}} = \langle 1,2,3 \rangle$

Answer

a. $(x-1)+2(y-2)+3(z-3)=0$
b. $x+2y+3z-14=0$

28) $P(0,0,0), \quad \vec{\mathbf{n}} = \langle -3,2,-1 \rangle$

For exercises 29 - 32, the equation of a plane is given.

a. Find normal vector $\vec{\mathbf{n}}$ to the plane. Express $\vec{\mathbf{n}}$ using standard unit vectors.

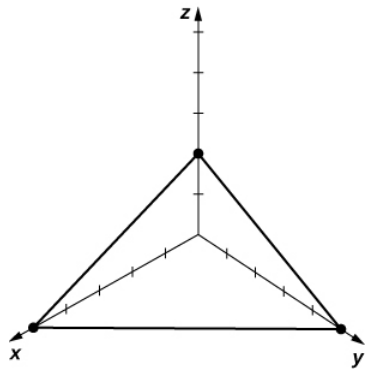
b. Find the intersections of the plane with each of the coordinate axes (its intercepts).

c. Sketch the plane.

29) [T] $4x+5y+10z-20=0$

Answer

a. $\vec{\mathbf{n}} = 4\hat{\mathbf{i}}+5\hat{\mathbf{j}}+10\hat{\mathbf{k}}$
b. $(5,0,0), \ (0,4,0),$ and $(0,0,2)$
c.

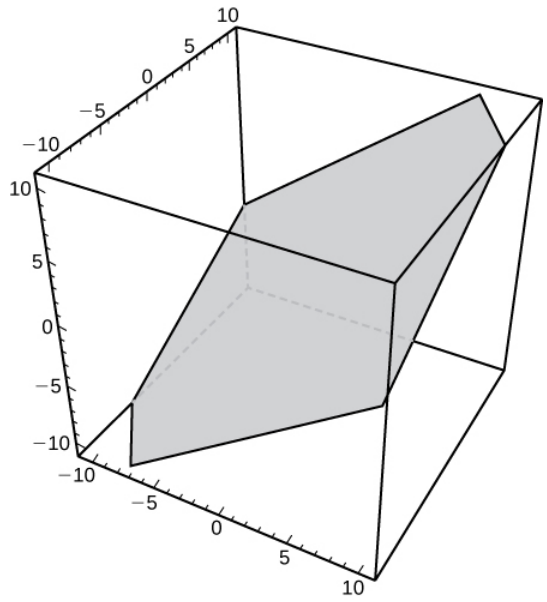


30) $3x+4y-12=0$

31) $3x-2y+4z=0$

Answer

a. $\vec{\mathbf{n}} = 3\hat{\mathbf{i}}-2\hat{\mathbf{j}}+4\hat{\mathbf{k}}$
b. $(0,0,0)$
c.



32) $x+z=0$

33) Given point $P(1,2,3)$ and vector $\vec{\mathbf{n}}=\hat{\mathbf{i}}+\hat{\mathbf{j}}$, find point Q on the x-axis such that $\overrightarrow{PQ}$ and $\vec{\mathbf{n}}$ are orthogonal.

Answer

$(3,0,0)$

34) Show there is no plane perpendicular to $\vec{\mathbf{n}}=\hat{\mathbf{i}}+\hat{\mathbf{j}}$ that passes through points $P(1,2,3)$ and $Q(2,3,4)$.

35) Find parametric equations of the line passing through point $P(-2,1,3)$ that is perpendicular to the plane of equation $2x-3y+z=7$.

Answer

$x=-2+2t, \quad y=1-3t, \quad z=3+t, \quad t\in R$

36) Find symmetric equations of the line passing through point $P(2,5,4)$ that is perpendicular to the plane of equation $2x+3y-5z=0$.

37) Show that line $\dfrac{x-1}{2}=\dfrac{y+1}{3}=\dfrac{z-2}{4}$ is parallel to plane $x-2y+z=6$.

38) Find the real number α such that the line of parametric equations $x=t, \quad y=2-t, \quad z=3+t, \quad t\in R$ is parallel to the plane of equation $\alpha x+5y+z-10=0$.

For exercises 39 - 42, the equations of two planes are given.

a. Determine whether the planes are *parallel, orthogonal,* or *neither*.

b. If the planes are neither parallel nor orthogonal, then find the measure of the angle between the planes. Express the answer in degrees rounded to the nearest integer.

c. If the planes intersect, find the line of intersection of the planes, providing the parametric equations of this line.

39) [T] $x+y+z=0, \quad 2x-y+z-7=0$

Answer

a. The planes are neither parallel nor orthogonal.

b. 62°

c. $x=-1+2t$

$y = -4 + t$
$z = 5 - 3t$

40) $5x - 3y + z = 4, \quad x + 4y + 7z = 1$

41) $x - 5y - z = 1, \quad 5x - 25y - 5z = -3$

Answer

a. The planes are parallel.

42) [T] $x - 3y + 6z = 4, \quad 5x + y - z = 4$

For exercises 43 - 46, determine whether the given line intersects with the given plane. If they do intersect, state the point of intersection.

43) Plane: $2x + y - z = 11$ Line: $x = 1 + t, \quad y = 3 - 2t, \quad z = 2 + 4t$

Answer

They intersect at point $(-1, 7, -6)$.

44) Plane: $-x + 2y + z = 2$ Line: $x = 1 + 2t, \quad y = -2 + t, \quad z = 5 - 3t$

Answer

They intersect at point $\left(-\frac{1}{3}, -\frac{8}{3}, 7\right)$.

45) Plane: $x - 3y + 2z = 4$ Line: $x = 2 - t, \quad y = t, \quad z = 4 + 2t$

Answer

The line does not intersect with this plane.

46) Plane: $x - 3y + 2z = 10$ Line: $x = 2 - t, \quad y = t, \quad z = 4 + 2t$

Answer

The line is actually fully contained in this plane, so every point on the line is on the plane. For example, when $t = 0$ we have the point, $(2, 0, 4)$.

47) Show that the lines of equations $x = t, \quad y = 1 + t, \quad z = 2 + t, \quad t \in R,$ and $\frac{x}{2} = \frac{y - 1}{3} = z - 3$ are skew, and find the distance between them.

Answer

$\frac{1}{\sqrt{6}} = \frac{\sqrt{6}}{6}$ units

48) Show that the lines of equations $x = -1 + t, \quad y = -2 + t, \quad z = 3t, \quad t \in R,$ and $x = 5 + s, \quad y = -8 + 2s, \quad z = 7s, \quad s \in R$ are skew, and find the distance between them.

49) Consider point $C(-3, 2, 4)$ and the plane of equation $2x + 4y - 3z = 8$.

a. Find the radius of the sphere with center C tangent to the given plane.

b. Find point P of tangency.

Answer

a. $r = \frac{18}{\sqrt{29}} = \frac{18\sqrt{29}}{29}$

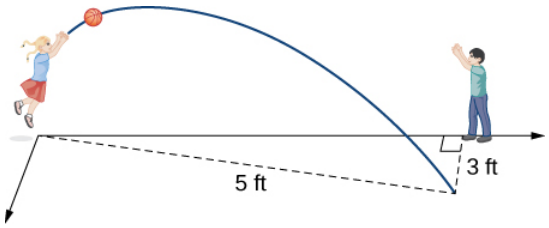
b. $P\left(-\frac{51}{29}, \frac{130}{29}, \frac{62}{29}\right)$

50) Consider the plane of equation $x-y-z-8=0$.

a. Find the equation of the sphere with center C at the origin that is tangent to the given plane.

b. Find parametric equations of the line passing through the origin and the point of tangency.

51) Two children are playing with a ball. The girl throws the ball to the boy. The ball travels in the air, curves 3 ft to the right, and falls 5 ft away from the girl (see the following figure). If the plane that contains the trajectory of the ball is perpendicular to the ground, find its equation.



Answer

$4x-3y=0$

52) [T] John allocates d dollars to consume monthly three goods of prices a, b, and c. In this context, the budget equation is defined as $ax+by+cz=d$, where $x\geq 0,\ y\geq 0$, and $z\geq 0$ represent the number of items bought from each of the goods. The budget set is given by $\{(x,y,z)\,|\,ax+by+cz\leq d,\ x\geq 0,\ y\geq 0,\ z\geq 0\}$, and the budget plane is the part of the plane of equation $ax+by+cz=d$ for which $x\geq 0,\ y\geq 0$, and $z\geq 0$. Consider $a=\$8,\ b=\$5,\ c=\$10$, and $d=\$500$.

a. Use a CAS to graph the budget set and budget plane.

b. For $z=25$, find the new budget equation and graph the budget set in the same system of coordinates.

53) [T] Consider $\vec{\mathbf{r}}(t)=\langle \sin t, \cos t, 2t\rangle$ the position vector of a particle at time $t\in[0,3]$, where the components of $\vec{\mathbf{r}}$ are expressed in centimeters and time is measured in seconds. Let $\overrightarrow{OP}$ be the position vector of the particle after 1 sec.

a. Determine the velocity vector $\vec{\mathbf{v}}(1)$ of the particle after 1 sec.

b. Find the scalar equation of the plane that is perpendicular to $v(1)$ and passes through point P. This plane is called the normal plane to the path of the particle at point P.

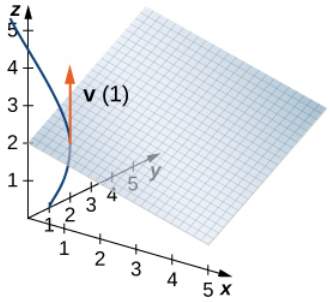
c. Use a CAS to visualize the path of the particle along with the velocity vector and normal plane at point P.

Answer

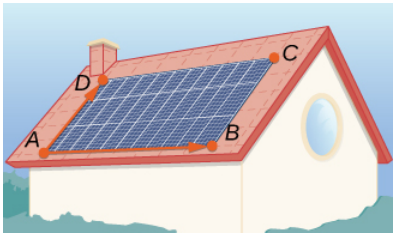
a. $\vec{\mathbf{v}}(1)=\langle \cos 1, -\sin 1, 2\rangle$

b. $(\cos 1)(x-\sin 1)-(\sin 1)(y-\cos 1)+2(z-2)=0$

c.



54) [T] A solar panel is mounted on the roof of a house. The panel may be regarded as positioned at the points of coordinates (in meters) $A(8,0,0),\ B(8,18,0),\ C(0,18,8)$, and $D(0,0,8)$ (see the following figure).



a. Find the general form of the equation of the plane that contains the solar panel by using points A, B, and C, and show that its normal vector is equivalent to $\overrightarrow{AB} \times \overrightarrow{AD}$.

b. Find parametric equations of line L_1 that passes through the center of the solar panel and has direction vector $\vec{\mathbf{s}} = \frac{1}{\sqrt{3}}\hat{\mathbf{i}} + \frac{1}{\sqrt{3}}\hat{\mathbf{j}} + \frac{1}{\sqrt{3}}\hat{\mathbf{k}}$, which points toward the position of the Sun at a particular time of day.

c. Find symmetric equations of line L_2 that passes through the center of the solar panel and is perpendicular to it.

d. Determine the angle of elevation of the Sun above the solar panel by using the angle between lines L_1 and L_2.

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Problems 15 and 43 - 46 created by Paul Seeburger

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4.7: Quadric Surfaces

Learning Objectives

- Identify a cylinder as a type of three-dimensional surface.
- Recognize the main features of ellipsoids, paraboloids, and hyperboloids.
- Use traces to draw the intersections of quadric surfaces with the coordinate planes.

We have been exploring vectors and vector operations in three-dimensional space, and we have developed equations to describe lines, planes, and spheres. In this section, we use our knowledge of planes and spheres, which are examples of three-dimensional figures called *surfaces*, to explore a variety of other surfaces that can be graphed in a three-dimensional coordinate system.

Identifying Cylinders

The first surface we'll examine is the cylinder. Although most people immediately think of a hollow pipe or a soda straw when they hear the word cylinder, here we use the broad mathematical meaning of the term. As we have seen, cylindrical surfaces don't have to be circular. A rectangular heating duct is a cylinder, as is a rolled-up yoga mat, the cross-section of which is a spiral shape.

In the two-dimensional coordinate plane, the equation $x^2+y^2=9$ describes a circle centered at the origin with radius 3. In three-dimensional space, this same equation represents a surface. Imagine copies of a circle stacked on top of each other centered on the z-axis (Figure $4.7.1$), forming a hollow tube. We can then construct a cylinder from the set of lines parallel to the z-axis passing through the circle $x^2+y^2=9$ in the xy-plane, as shown in the figure. In this way, any curve in one of the coordinate planes can be extended to become a surface.

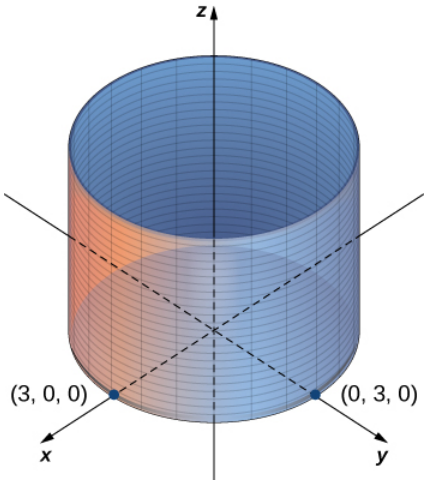


Figure $4.7.1$: In three-dimensional space, the graph of equation $x^2+y^2=9$ is a cylinder with radius 3 centered on the z-axis. It continues indefinitely in the positive and negative directions.

Definition: cylinders and rulings

A set of lines parallel to a given line passing through a given curve is known as a cylindrical surface, or *cylinder*. The parallel lines are called rulings.

From this definition, we can see that we still have a cylinder in three-dimensional space, even if the curve is not a circle. Any curve can form a cylinder, and the rulings that compose the cylinder may be parallel to any given line (Figure $4.7.2$).

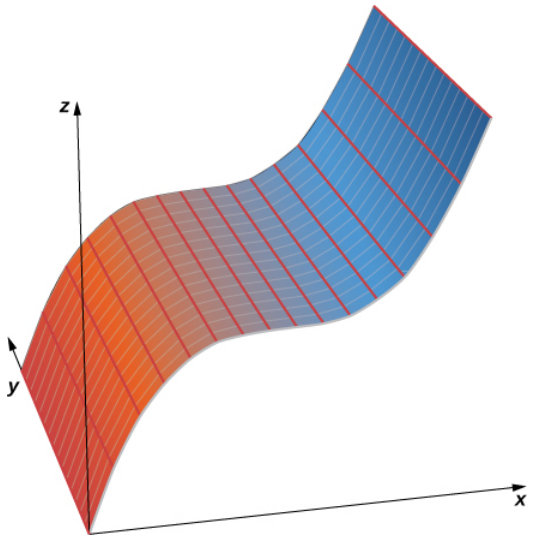
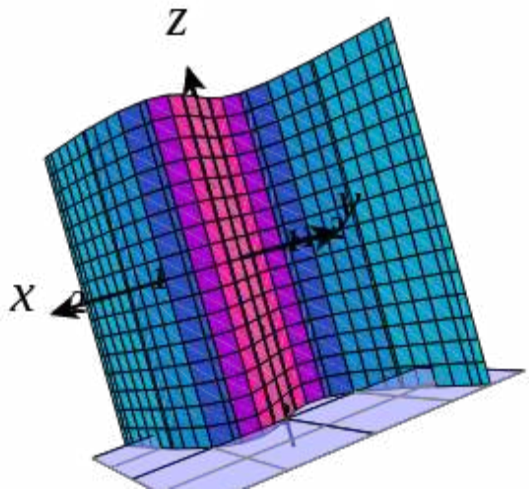


Figure 4.7.2: In three-dimensional space, the graph of equation $z = x^3$ is a cylinder, or a cylindrical surface with rulings parallel to the y-axis.

Interactive View of a Cylindrical Surface

Below is the graph of the cylindrical surface $y + x^3 - x = 0$. You can move it around and also chane its equation to see the effect on the graph.

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Example 4.7.1: Graphing Cylindrical Surfaces

Sketch the graphs of the following cylindrical surfaces.

a. $x^2 + z^2 = 25$
b. $z = 2x^2 - y$
c. $y = \sin x$

Solution

a. The variable y can take on any value without limit. Therefore, the lines ruling this surface are parallel to the y-axis. The intersection of this surface with the xz-plane forms a circle centered at the origin with radius 5 (see Figure $4.7.3$).

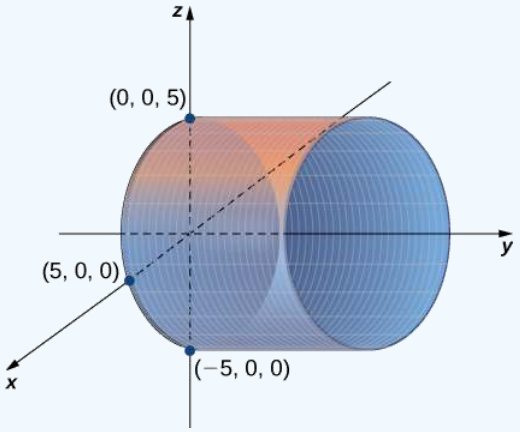


Figure $4.7.3$: The graph of equation $x^2 + z^2 = 25$ is a cylinder with radius 5 centered on the y-axis.

b. In this case, the equation contains all three variables —x, y, and z— so none of the variables can vary arbitrarily. The easiest way to visualize this surface is to use a computer graphing utility (Figure $4.7.4$).

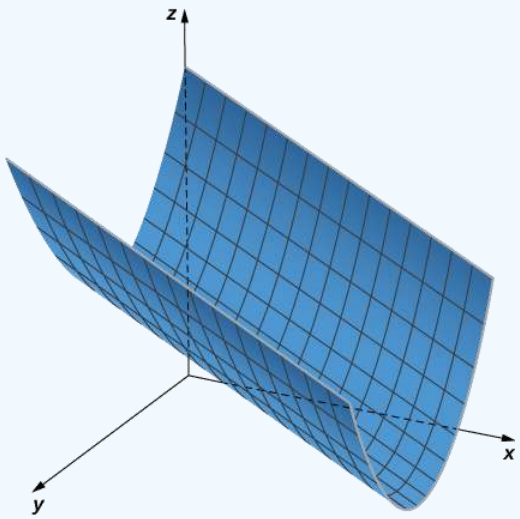


Figure 4.7.4

c. In this equation, the variable z can take on any value without limit. Therefore, the lines composing this surface are parallel to the z-axis. The intersection of this surface with the xy-plane outlines curve $y = \sin x$ (Figure $4.7.5$).

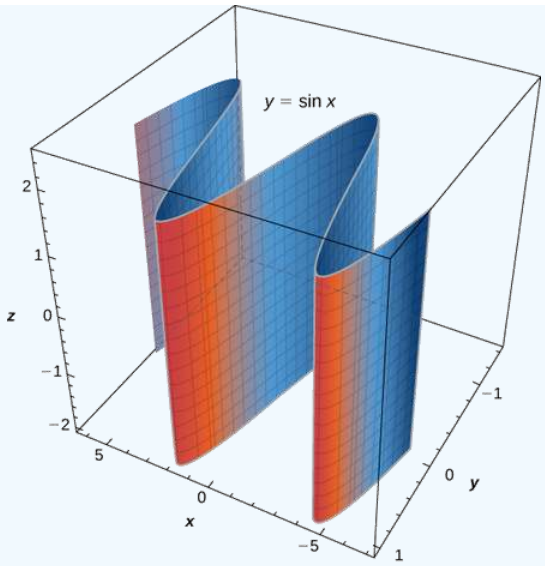


Figure 4.7.5: The graph of equation $y = \sin x$ is formed by a set of lines parallel to the z-axis passing through curve $y = \sin x$ in the xy-plane.



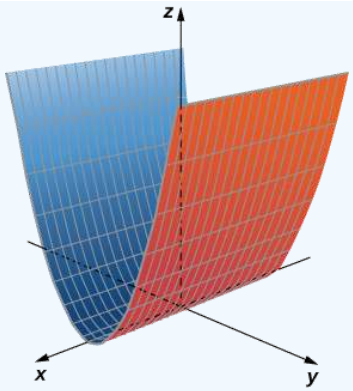
Exercise 4.7.1:

Sketch or use a graphing tool to view the graph of the cylindrical surface defined by equation $z = y^2$.

Hint

The variable x can take on any value without limit.

Answer

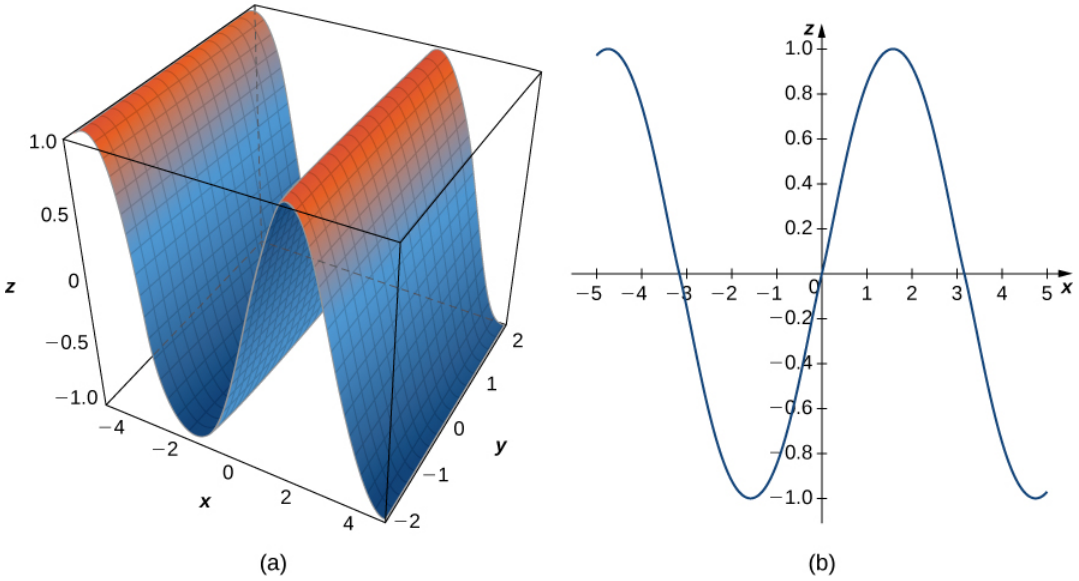


When sketching surfaces, we have seen that it is useful to sketch the intersection of the surface with a plane parallel to one of the coordinate planes. These curves are called traces. We can see them in the plot of the cylinder in Figure 4.7.6.

Definition: traces

The *traces* of a surface are the cross-sections created when the surface intersects a plane parallel to one of the coordinate planes.

Traces are useful in sketching cylindrical surfaces. For a cylinder in three dimensions, though, only one set of traces is useful. Notice, in Figure 4.7.6, that the trace of the graph of $z = \sin x$ in the *xz*-plane is useful in constructing the graph. The trace in the *xy*-plane, though, is just a series of parallel lines, and the trace in the *yz*-plane is simply one line.



(a) (b)

Figure 4.7.6: (a) This is one view of the graph of equation $z = \sin x$. (b) To find the trace of the graph in the xz-plane, set $y = 0$. The trace is simply a two-dimensional sine wave.

Cylindrical surfaces are formed by a set of parallel lines. Not all surfaces in three dimensions are constructed so simply, however. We now explore more complex surfaces, and traces are an important tool in this investigation.



Quadric Surfaces

We have learned about surfaces in three dimensions described by first-order equations; these are planes. Some other common types of surfaces can be described by second-order equations. We can view these surfaces as three-dimensional extensions of the conic sections we discussed earlier: the ellipse, the parabola, and the hyperbola. We call these graphs quadric surfaces

Definition: Quadric surfaces and conic sections

Quadric surfaces are the graphs of equations that can be expressed in the form

$$Ax^2+By^2+Cz^2+Dxy+Exz+Fyz+Gx+Hy+Jz+K=0. \tag{4.7.1}$$

When a quadric surface intersects a coordinate plane, the trace is a conic section.

An *ellipsoid* is a surface described by an equation of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. Set $x=0$ to see the trace of the ellipsoid in the yz-plane. To see the traces in the xy- and xz-planes, set $z=0$ and $y=0$, respectively. Notice that, if $a=b$, the trace in the xy-plane is a circle. Similarly, if $a=c$, the trace in the xz-plane is a circle and, if $b=c$, then the trace in the yz-plane is a circle. A sphere, then, is an ellipsoid with $a=b=c$.

Example 4.7.2: Sketching an Ellipsoid

Sketch the ellipsoid

$$\frac{x^2}{2^2}+\frac{y^2}{3^2}+\frac{z^2}{5^2}=1. \tag{4.7.2}$$

Solution

Start by sketching the traces. To find the trace in the xy-plane, set $z=0$: $\frac{x^2}{2^2}+\frac{y^2}{3^2}=1$ (Figure 4.7.7). To find the other traces, first set $y=0$ and then set $x=0$.

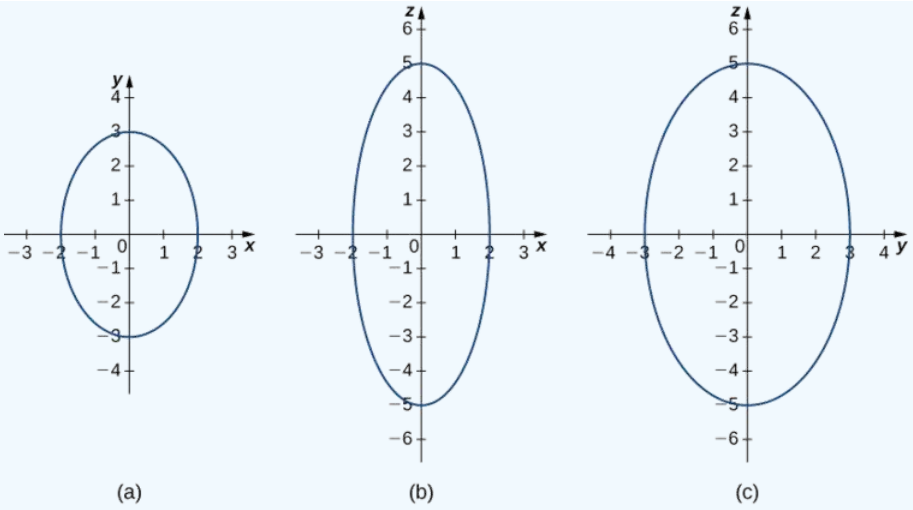


Figure 4.7.7: (a) This graph represents the trace of equation $\frac{x^2}{2^2}+\frac{y^2}{3^2}+\frac{z^2}{5^2}=1$ in the xy-plane, when we set $z=0$. (b) When we set $y=0$, we get the trace of the ellipsoid in the xz-plane, which is an ellipse. (c) When we set $x=0$, we get the trace of the ellipsoid in the yz-plane, which is also an ellipse.

Now that we know what traces of this solid look like, we can sketch the surface in three dimensions (Figure 4.7.8).

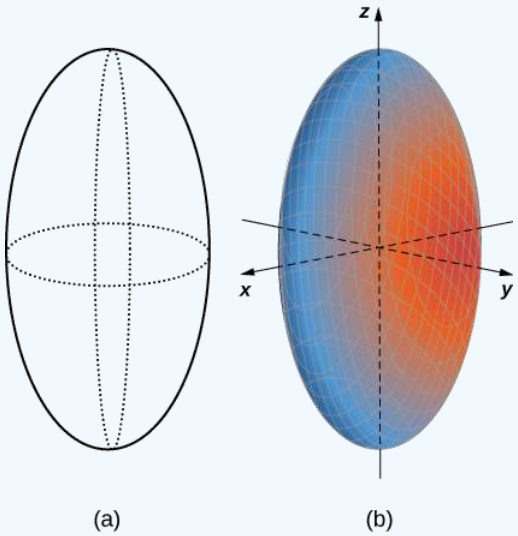


Figure 4.7.8: (a) The traces provide a framework for the surface. (b) The center of this ellipsoid is the origin.

The trace of an ellipsoid is an ellipse in each of the coordinate planes. However, this does not have to be the case for all quadric surfaces. Many quadric surfaces have traces that are different kinds of conic sections, and this is usually indicated by the name of the surface. For example, if a surface can be described by an equation of the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z}{c} \tag{4.7.3}$$

then we call that surface an **elliptic paraboloid**. The trace in the *xy*-plane is an ellipse, but the traces in the *xz*-plane and *yz*-plane are parabolas (Figure 4.7.9). Other elliptic paraboloids can have other orientations simply by interchanging the variables to give us a different variable in the linear term of the equation $\frac{x^2}{a^2}+\frac{z^2}{c^2}=\frac{y}{b}$ or $\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{x}{a}$.

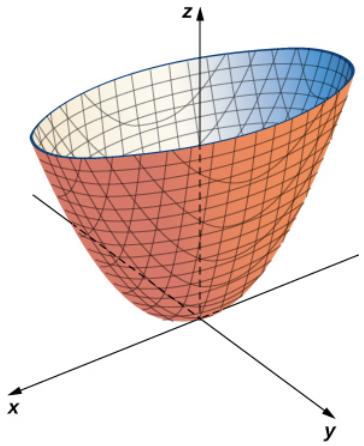
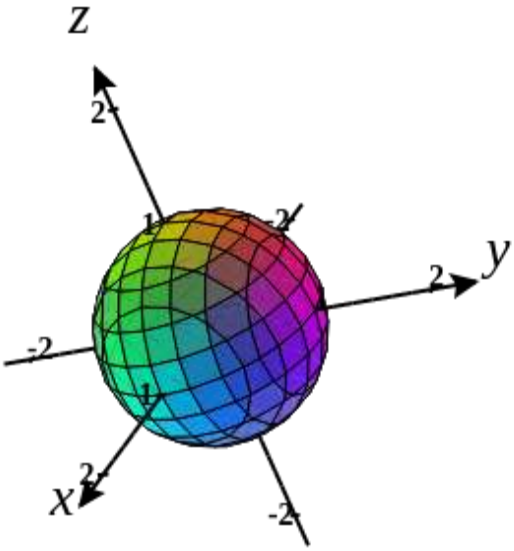


Figure 4.7.9: This quadric surface is called an elliptic paraboloid.

Quadric Surface Interactive Exploration

Move the sliders to change the coefficients of $ax^2+by^2+cz^2=d$. You can also drag and rotate the axes to change your perspective.

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Example 4.7.3: Identifying Traces of Quadric Surfaces

Describe the traces of the elliptic paraboloid $x^2+\frac{y^2}{2^2}=\frac{z}{5}$.

Solution

To find the trace in the xy-plane, set $z=0: x^2+\frac{y^2}{2^2}=0$. The trace in the plane $z=0$ is simply one point, the origin. Since a single point does not tell us what the shape is, we can move up the z-axis to an arbitrary plane to find the shape of other traces of the figure.

The trace in plane $z = 5$ is the graph of equation $x^2 + \frac{y^2}{2^2} = 1$, which is an ellipse. In the xz-plane, the equation becomes $z = 5x^2$. The trace is a parabola in this plane and in any plane with the equation $y = b$.

In planes parallel to the yz-plane, the traces are also parabolas, as we can see in Figure 4.7.10

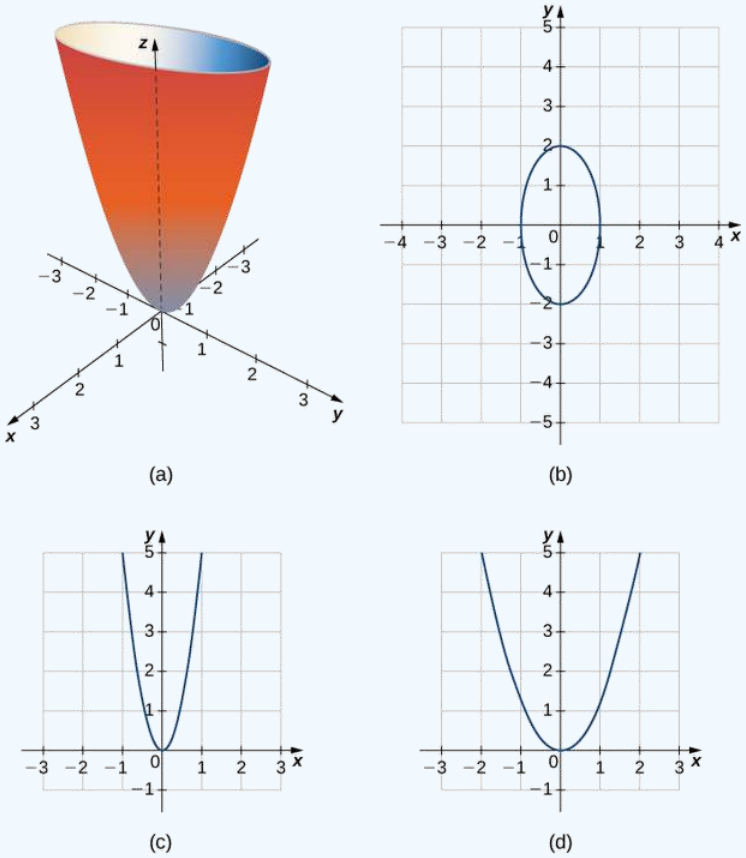


Figure 4.7.10: (a) The paraboloid $x^2 + \frac{y^2}{2^2} = \frac{z}{5}$. (b) The trace in plane $z = 5$. (c) The trace in the xz-plane. (d) The trace in the yz-plane.

Exercise 4.7.2:

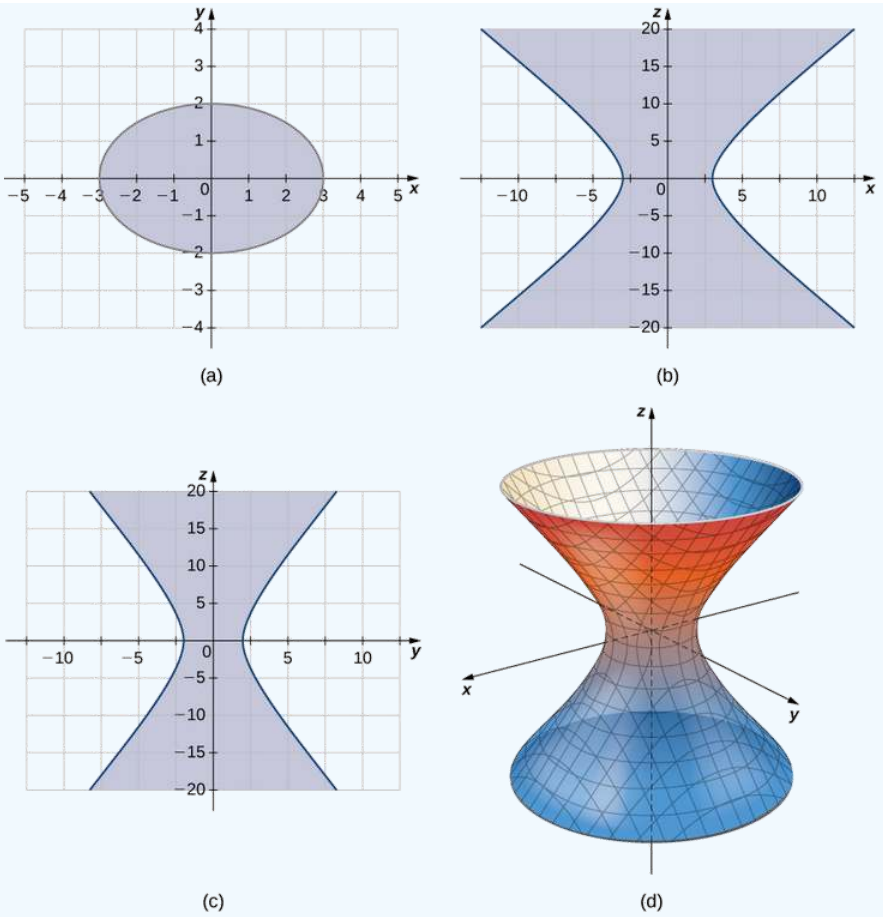
A hyperboloid of one sheet is any surface that can be described with an equation of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$. Describe the traces of the hyperboloid of one sheet given by equation $\frac{x^2}{3^2} + \frac{y^2}{2^2} - \frac{z^2}{5^2} = 1$.

Hint

To find the traces in the coordinate planes, set each variable to zero individually.

Answer

The traces parallel to the xy-plane are ellipses and the traces parallel to the xz- and yz-planes are hyperbolas. Specifically, the trace in the xy-plane is ellipse $\frac{x^2}{3^2} + \frac{y^2}{2^2} = 1$, the trace in the xz-plane is hyperbola $\frac{x^2}{3^2} - \frac{z^2}{5^2} = 1$, and the trace in the yz-plane is hyperbola $\frac{y^2}{2^2} - \frac{z^2}{5^2} = 1$ (see the following figure).



Hyperboloids of one sheet have some fascinating properties. For example, they can be constructed using straight lines, such as in the sculpture in Figure 4.7.1*a*. In fact, cooling towers for nuclear power plants are often constructed in the shape of a hyperboloid. The builders are able to use straight steel beams in the construction, which makes the towers very strong while using relatively little material (Figure 4.7.1*b*).

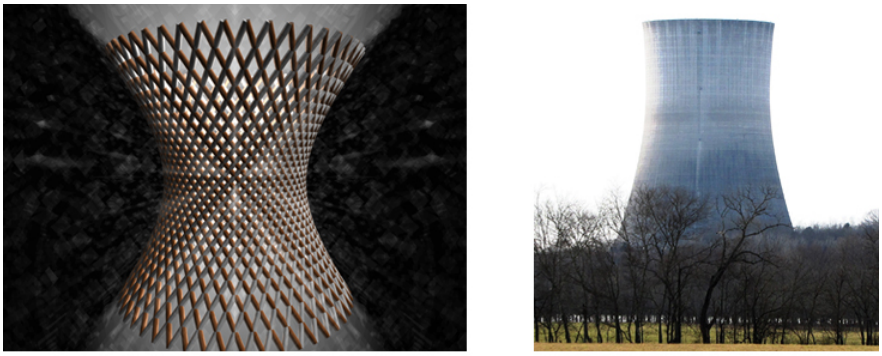


Figure 4.7.11: (a) A sculpture in the shape of a hyperboloid can be constructed of straight lines. (b) Cooling towers for nuclear power plants are often built in the shape of a hyperboloid.

Example 4.7.4: Chapter Opener: Finding the Focus of a Parabolic Reflector

Energy hitting the surface of a parabolic reflector is concentrated at the focal point of the reflector (Figure 4.7.12). If the surface of a parabolic reflector is described by equation $\frac{x^2}{100}+\frac{y^2}{100}=\frac{z}{4}$, where is the focal point of the reflector?

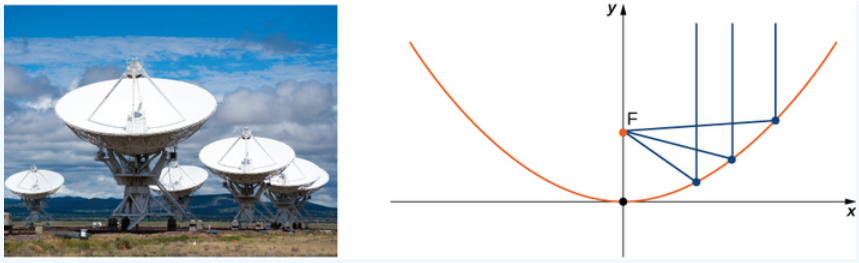


Figure 4.7.12: Energy reflects off of the parabolic reflector and is collected at the focal point. (credit: modification of CGP Grey, Wikimedia Commons)

Solution

Since z is the first-power variable, the axis of the reflector corresponds to the z-axis. The coefficients of x^2 and y^2 are equal, so the cross-section of the paraboloid perpendicular to the z-axis is a circle. We can consider a trace in the xz-plane or the yz-plane; the result is the same. Setting $y = 0$, the trace is a parabola opening up along the z-axis, with standard equation $x^2 = 4pz$, where p is the focal length of the parabola. In this case, this equation becomes $x^2 = 100 \cdot \frac{z}{4} = 4pz$ or $25 = 4p$. So p is 6.25 m, which tells us that the focus of the paraboloid is 6.25 m up the axis from the vertex. Because the vertex of this surface is the origin, the focal point is $(0, 0, 6.25)$.

Seventeen standard quadric surfaces can be derived from the general equation

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Jz + K = 0. \tag{4.7.4}$$

The following figures summarizes the most important ones.

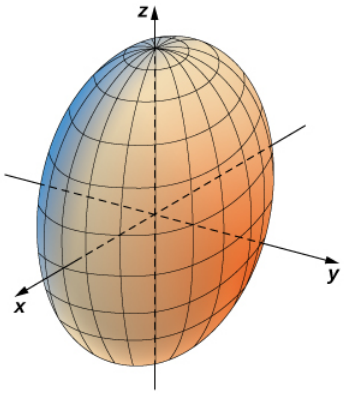
Characteristics of Common Quadric Surfaces

Ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Traces
In plane $z = p$: an ellipse
In plane $y = q$: an ellipse
In plane $x = r$: an ellipse

If $a = b = c$, then this surface is a sphere.

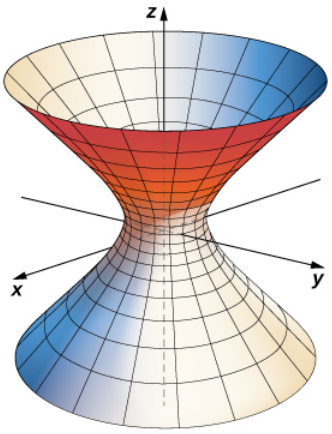


Hyperboloid of One Sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

Traces
In plane $z = p$: an ellipse
In plane $y = q$: a hyperbola
In plane $x = r$: a hyperbola

In the equation for this surface, two of the variables have positive coefficients and one has a negative coefficient. The axis of the surface corresponds to the variable with the negative coefficient.



Hyperboloid of Two Sheets

$$\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Traces
In plane $z = p$: an ellipse or the empty set (no trace)
In plane $y = q$: a hyperbola
In plane $x = r$: a hyperbola

In the equation for this surface, two of the variables have negative coefficients and one has a positive coefficient. The axis of the surface corresponds to the variable with a positive coefficient. The surface does not intersect the coordinate plane perpendicular to the axis.

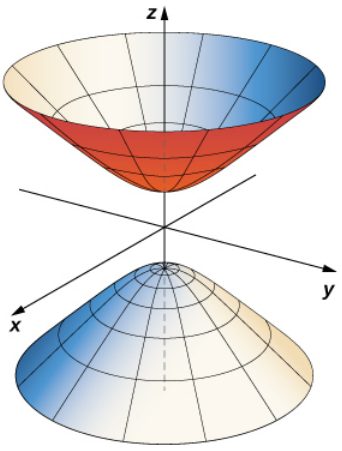


Figure 4.7.13: Characteristics of Common Quadratic Surfaces: Ellipsoid, **Hyperboloid of One Sheet, Hyperboloid of Two Sheets.**

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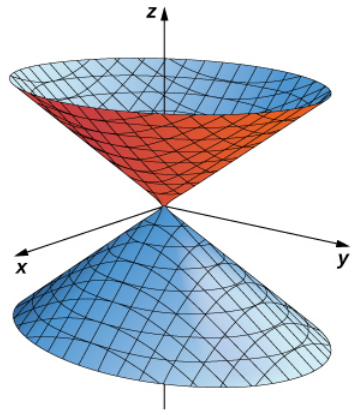
Characteristics of Common Quadric Surfaces

Elliptic Cone

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$$

Traces
In plane $z = p$: an ellipse
In plane $y = q$: a hyperbola
In plane $x = r$: a hyperbola
In the xz – plane: a pair of lines that intersect at the origin
In the yz – plane: a pair of lines that intersect at the origin

The axis of the surface corresponds to the variable with a negative coefficient. The traces in the coordinate planes parallel to the axis are intersecting lines.

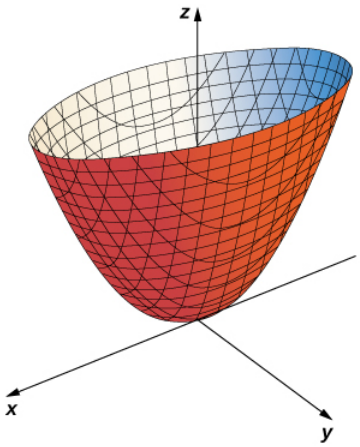


Elliptic Paraboloid

$$z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$$

Traces
In plane $z = p$: an ellipse
In plane $y = q$: a parabola
In plane $x = r$: a parabola

The axis of the surface corresponds to the linear variable.



Hyperbolic Paraboloid

$$z=\frac{x^2}{a^2}-\frac{y^2}{b^2}$$

Traces
In plane $z = p$: a hyperbola
In plane $y = q$: a parabola
In plane $x = r$: a parabola

The axis of the surface corresponds to the linear variable.

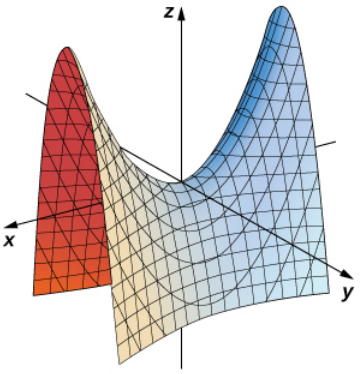


Figure 4.7.14: Characteristics of Common Quadratic Surfaces: **Elliptic Cone, Elliptic Paraboloid,** Hyperbolic Paraboloid.

Example 4.7.5: Identifying Equations of Quadric Surfaces

Identify the surfaces represented by the given equations.

a. $16x^2+9y^2+16z^2=144$
b. $9x^2-18x+4y^2+16y-36z+25=0$

Solution

a. The x, y, and z terms are all squared, and are all positive, so this is probably an ellipsoid. However, let's put the equation into the standard form for an ellipsoid just to be sure. We have

$$16x^2+9y^2+16z^2=144.$$

Dividing through by 144 gives

$$\frac{x^2}{9}+\frac{y^2}{16}+\frac{z^2}{9}=1.$$

So, this is, in fact, an ellipsoid, centered at the origin.

b. We first notice that the z term is raised only to the first power, so this is either an elliptic paraboloid or a hyperbolic paraboloid. We also note there are x terms and y terms that are not squared, so this quadric surface is not centered at the origin. We need to complete the square to put this equation in one of the standard forms. We have

$$\begin{aligned}
9x^2-18x+4y^2+16y-36z+25&=0\\
9x^2-18x+4y^2+16y+25&=36z\\
9(x^2-2x)+4(y^2+4y)+25&=36z\\
9(x^2-2x+1-1)+4(y^2+4y+4-4)+25&=36z\\
9(x-1)^2-9+4(y+2)^2-16+25&=36z\\
9(x-1)^2+4(y+2)^2&=36z\\
\frac{(x-1)^2}{4}+\frac{(y-2)^2}{9}&=z.
\end{aligned}$$

This is an elliptic paraboloid centered at $(1,2,0)$.





Exercise 4.7.3

Identify the surface represented by equation $9x^2 + y^2 - z^2 + 2z - 10 = 0$.

Hint

Look at the signs and powers of the x, y, and z terms

Answer

Hyperboloid of one sheet, centered at $(0, 0, 1)$.

Key Concepts

- A set of lines parallel to a given line passing through a given curve is called a *cylinder,* or a cylindrical surface. The parallel lines are called *rulings.*
- The intersection of a three-dimensional surface and a plane is called a trace. To find the trace in the xy-, yz-, or xz-planes, set $z = 0, x = 0,$ or $y = 0$, respectively.
- Quadric surfaces are three-dimensional surfaces with traces composed of conic sections. Every quadric surface can be expressed with an equation of the form

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Jz + K = 0.$$

- To sketch the graph of a quadric surface, start by sketching the traces to understand the framework of the surface.
- Important quadric surfaces are summarized in Figures 4.7.13and 4.7.14

Glossary

cylinder

a set of lines parallel to a given line passing through a given curve

ellipsoid

a three-dimensional surface described by an equation of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$; all traces of this surface are ellipses

elliptic cone

a three-dimensional surface described by an equation of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$; traces of this surface include ellipses and intersecting lines

elliptic paraboloid

a three-dimensional surface described by an equation of the form $z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$; traces of this surface include ellipses and parabolas

hyperboloid of one sheet

a three-dimensional surface described by an equation of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1;$ traces of this surface include ellipses and hyperbolas

hyperboloid of two sheets

a three-dimensional surface described by an equation of the form $\frac{z^2}{c^2}-\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$; traces of this surface include ellipses and hyperbolas

quadric surfaces

surfaces in three dimensions having the property that the traces of the surface are conic sections (ellipses, hyperbolas, and parabolas)

rulings

parallel lines that make up a cylindrical surface

trace

the intersection of a three-dimensional surface with a coordinate plane

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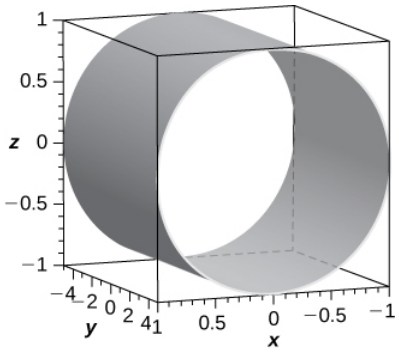
4.7E: Exercises for Section 4.7

For exercises 1 - 6, sketch and describe the cylindrical surface of the given equation.

1) [T] $x^2 + z^2 = 1$

Answer

The surface is a cylinder with the rulings parallel to the y-axis.

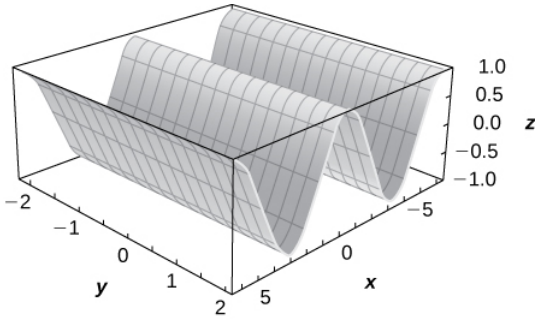


2) [T] $x^2 + y^2 = 9$

3) [T] $z = \cos\left(\frac{\pi}{2} + x\right)$

Answer

The surface is a cylinder with rulings parallel to the y-axis.



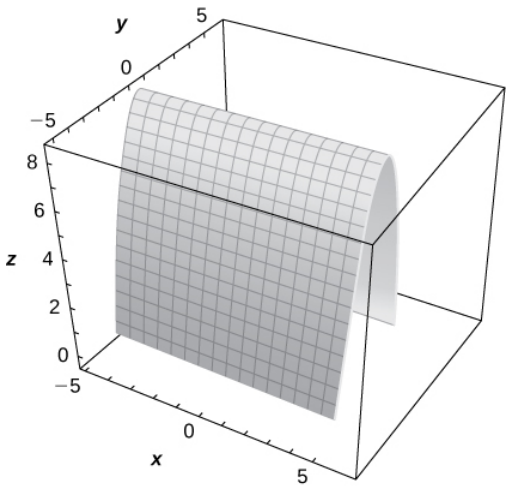
4) [T] $z = e^x$

5) [T] $z = 9 - y^2$

Answer

The surface is a cylinder with rulings parallel to the x-axis.

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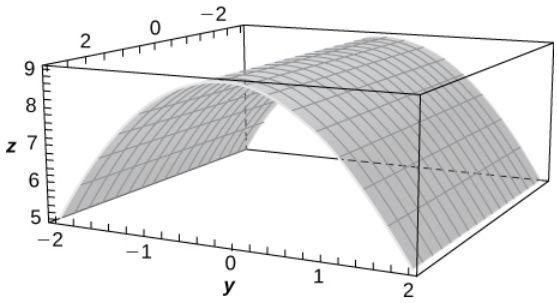
6) [T] $z = \ln x$

For exercises 7 - 10, the graph of a quadric surface is given.

a. Specify the name of the quadric surface.

b. Determine the axis of symmetry of the quadric surface.

7)

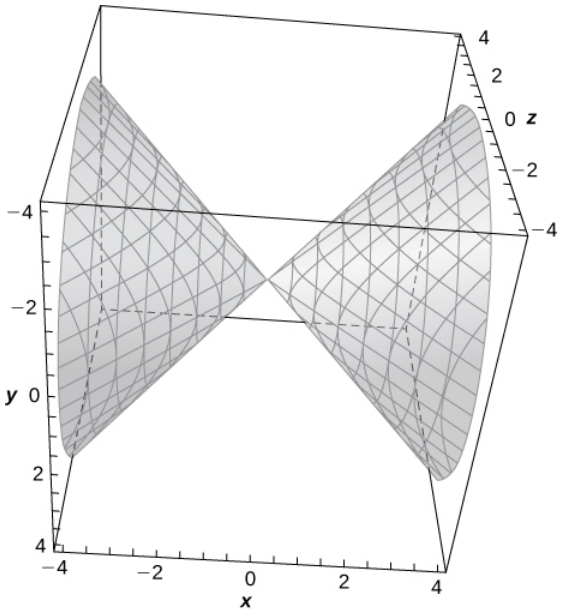


Answer

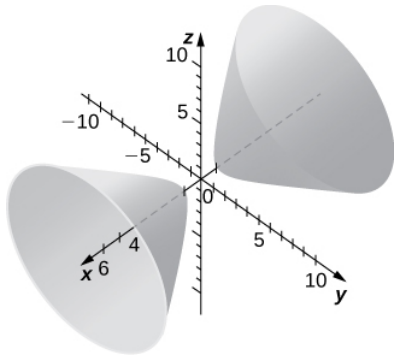
a. Cylinder; b. The x-axis

8)

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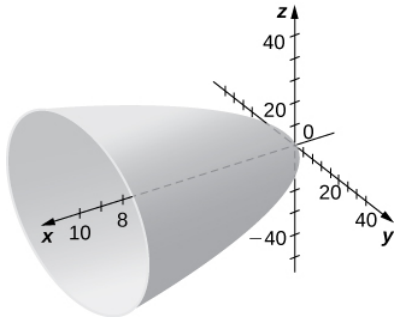
9)



Answer

a. Hyperboloid of two sheets; b. The x-axis

10)



For exercises 11 - 16, match the given quadric surface with its corresponding equation in standard form.

a. $\frac{x^2}{4}+\frac{y^2}{9}-\frac{z^2}{12}=1$

b. $\frac{x^2}{4}-\frac{y^2}{9}-\frac{z^2}{12}=1$

c. $\frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{12}=1$

d. $z=4x^2+3y^2$

e. $z=4x^2-y^2$

f. $4x^2+y^2-z^2=0$

11) Hyperboloid of two sheets

Answer

b.

12) Ellipsoid

13) Elliptic paraboloid

Answer

d.

14) Hyperbolic paraboloid

15) Hyperboloid of one sheet

Answer

a.

16) Elliptic cone

For exercises 17 - 28, rewrite the given equation of the quadric surface in standard form. Identify the surface.

17) $-x^2+36y^2+36z^2=9$

Answer

$-\frac{x^2}{9}+\frac{y^2}{\frac{1}{4}}+\frac{z^2}{\frac{1}{4}}=1,$ hyperboloid of one sheet with the x-axis as its axis of symmetry

18) $-4x^2+25y^2+z^2=100$

19) $-3x^2+5y^2-z^2=10$

Answer

$-\frac{x^2}{\frac{10}{3}}+\frac{y^2}{2}-\frac{z^2}{10}=1,$ hyperboloid of two sheets with the y-axis as its axis of symmetry

20) $3x^2-y^2-6z^2=18$

21) $5y=x^2-z^2$

Answer

$y=-\frac{z^2}{5}+\frac{x^2}{5},$ hyperbolic paraboloid with the y-axis as its axis of symmetry

22) $8x^2-5y^2-10z=0$

23) $x^2+5y^2+3z^2-15=0$

Answer

$\frac{x^2}{15}+\frac{y^2}{3}+\frac{z^2}{5}=1,$ ellipsoid

24) $63x^2+7y^2+9z^2-63=0$

25) $x^2+5y^2-8z^2=0$

Answer

$\frac{x^2}{40}+\frac{y^2}{8}-\frac{z^2}{5}=0,$ elliptic cone with the z-axis as its axis of symmetry

26) $5x^2-4y^2+20z^2=0$

27) $6x=3y^2+2z^2$

Answer

$x=\frac{y^2}{2}+\frac{z^2}{3},$ elliptic paraboloid with the x-axis as its axis of symmetry

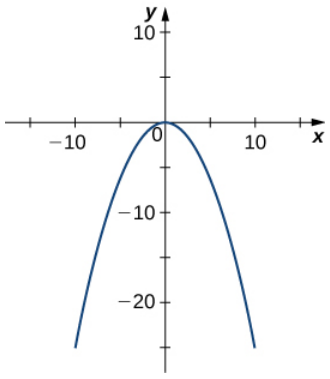
28) $49y=x^2+7z^2$

For exercises 29 - 34, find the trace of the given quadric surface in the specified plane of coordinates and sketch it.

29) [T] $x^2+z^2+4y=0, \quad z=0$

Answer

Parabola $y=-\frac{x^2}{4},$

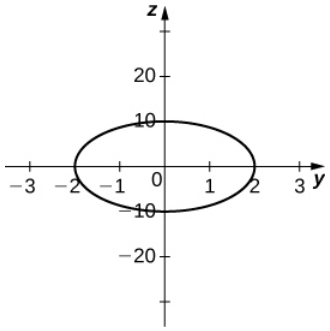


30) [T] $x^2+z^2+4y=0, \quad x=0$

31) [T] $-4x^2+25y^2+z^2=100, \quad x=0$

Answer

Ellipse $\frac{y^2}{4}+\frac{z^2}{100}=1,$



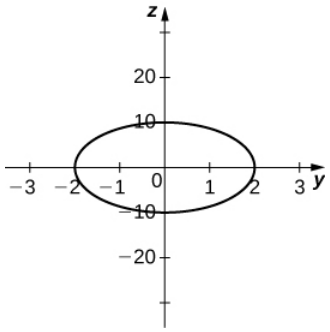
32) [T] $-4x^2+25y^2+z^2=100, \quad y=0$

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33) [T] $x^2+\frac{y^2}{4}+\frac{z^2}{100}=1, \quad x=0$

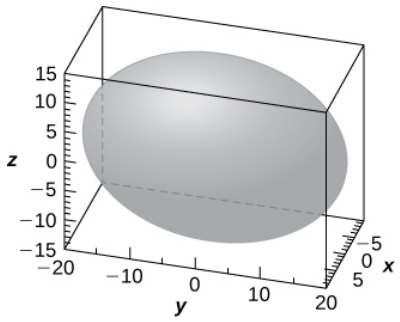
Answer

Ellipse $\frac{y^2}{4}+\frac{z^2}{100}=1,$



34) [T] $x^2-y-z^2=1, \quad y=0$

35) Use the graph of the given quadric surface to answer the questions.



a. Specify the name of the quadric surface.

b. Which of the equations—$16x^2+9y^2+36z^2=3600,\ 9x^2+36y^2+16z^2=3600,$ or $36x^2+9y^2+16z^2=3600$ — corresponds to the graph?

c. Use b. to write the equation of the quadric surface in standard form.

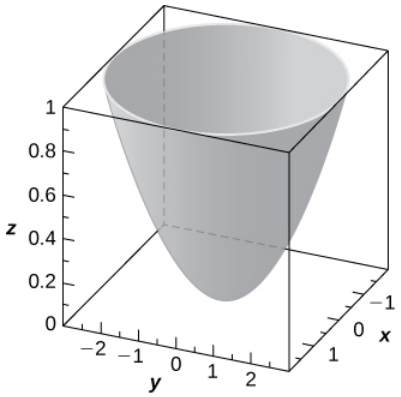
Answer

a. Ellipsoid

b. The third equation

c. $\frac{x^2}{100}+\frac{y^2}{400}+\frac{z^2}{225}=1$

36) Use the graph of the given quadric surface to answer the questions.



a. Specify the name of the quadric surface.

b. Which of the equations—$36z = 9x^2 + y^2$, $9x^2 + 4y^2 = 36z$, or $-36z = -81x^2 + 4y^2$ —corresponds to the graph above?

c. Use b. to write the equation of the quadric surface in standard form.

For exercises 37 - 42, the equation of a quadric surface is given.

a. Use the method of completing the square to write the equation in standard form.

b. Identify the surface.

37) $x^2 + 2z^2 + 6x - 8z + 1 = 0$

Answer

a. $\dfrac{(x+3)^2}{16} + \dfrac{(z-2)^2}{8} = 1$

b. Cylinder centered at $(-3, 2)$ with rulings parallel to the y-axis

38) $4x^2 - y^2 + z^2 - 8x + 2y + 2z + 3 = 0$

39) $x^2 + 4y^2 - 4z^2 - 6x - 16y - 16z + 5 = 0$

Answer

a. $\dfrac{(x-3)^2}{4} + (y-2)^2 - (z+2)^2 = 1$

b. Hyperboloid of one sheet centered at $(3, 2, -2)$, with the z-axis as its axis of symmetry

40) $x^2 + z^2 - 4y + 4 = 0$

41) $x^2 + \dfrac{y^2}{4} - \dfrac{z^2}{3} + 6x + 9 = 0$

Answer

a. $(x+3)^2 + \dfrac{y^2}{4} - \dfrac{z^2}{3} = 0$

b. Elliptic cone centered at $(-3, 0, 0)$, with the z-axis as its axis of symmetry

42) $x^2 - y^2 + z^2 - 12z + 2x + 37 = 0$

43) Write the standard form of the equation of the ellipsoid centered at the origin that passes through points $A(2, 0, 0)$, $B(0, 0, 1)$, and $C\left(12, \sqrt{11}, \frac{1}{2}\right)$.

Answer

$\dfrac{x^2}{4} + \dfrac{y^2}{16} + z^2 = 1$

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44) Write the standard form of the equation of the ellipsoid centered at point $P(1,1,0)$ that passes through points $A(6,1,0)$, $B(4,2,0)$and $C(1,2,1)$.

45) Determine the intersection points of elliptic cone $x^2-y^2-z^2=0$ with the line of symmetric equations $\frac{x-1}{2}=\frac{y+1}{3}=z$.

Answer

$(1,-1,0)$ and $\left(\frac{13}{3},4,\frac{5}{3}\right)$

46) Determine the intersection points of parabolic hyperboloid $z=3x^2-2y^2$ with the line of parametric equations $x=3t,\ y=2t,\ z=19t$, where $t\in R$.

47) Find the equation of the quadric surface with points $P(x,y,z)$ that are equidistant from point $Q(0,-1,0)$ and plane of equation $y=1$. Identify the surface.

Answer

$x^2+z^2+4y=0,$ elliptic paraboloid

48) Find the equation of the quadric surface with points $P(x,y,z)$ that are equidistant from point $Q(0,2,0)$and plane of equation $y=-2$. Identify the surface.

49) If the surface of a parabolic reflector is described by equation $400z=x^2+y^2,$ find the focal point of the reflector.

Answer

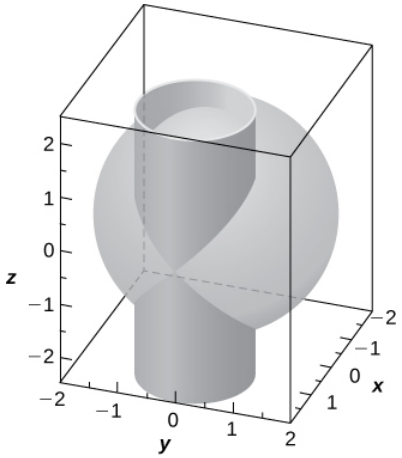
$(0,0,100)$

50) Consider the parabolic reflector described by equation $z=20x^2+20y^2$. Find its focal point.

51) Show that quadric surface $x^2+y^2+z^2+2xy+2xz+2yz+x+y+z=0$ reduces to two parallel planes.

52) Show that quadric surface $x^2+y^2+z^2-2xy-2xz+2yz-1=0$ reduces to two parallel planes passing.

53) [T] The intersection between cylinder $(x-1)^2+y^2=1$ and sphere $x^2+y^2+z^2=4$ is called a *Viviani curve.*



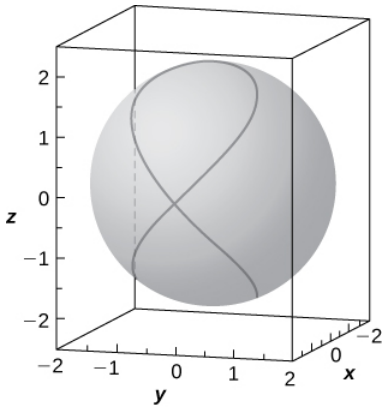
a. Solve the system consisting of the equations of the surfaces to find the equation of the intersection curve. (Hint: Find x and y in terms of z.)

b. Use a computer algebra system (CAS) or CalcPlot3D to visualize the intersection curve on sphere $x^2+y^2+z^2=4$.

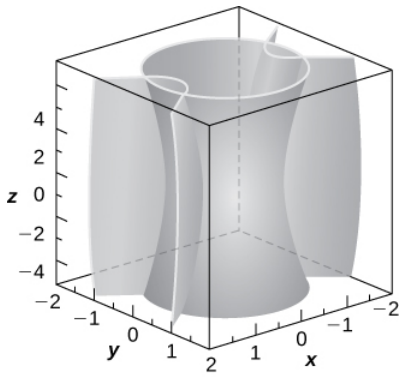
Answer

a. $x=2-\frac{z^2}{2},\quad y=\pm\frac{z}{2}\sqrt{4-z^2},$ where $z\in[-2,2]$;

b.



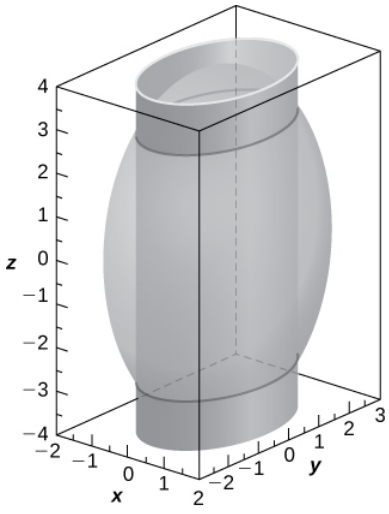
54) Hyperboloid of one sheet $25x^2+25y^2-z^2=25$ and elliptic cone $-25x^2+75y^2+z^2=0$ are represented in the following figure along with their intersection curves. Identify the intersection curves and find their equations (Hint: Find y from the system consisting of the equations of the surfaces.)



55) [T] Use a CAS or CalcPlot3D to create the intersection between cylinder $9x^2+4y^2=18$ and ellipsoid $36x^2+16y^2+9z^2=144$, and find the equations of the intersection curves.

Answer

two ellipses of equations $\dfrac{x^2}{2}+\dfrac{y^2}{\frac{9}{2}}=1$ in planes $z=\pm 2\sqrt{2}$



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56) [T] A spheroid is an ellipsoid with two equal semiaxes. For instance, the equation of a spheroid with the z-axis as its axis of symmetry is given by $\frac{x^2}{a^2}+\frac{y^2}{a^2}+\frac{z^2}{c^2}=1$, where a and c are positive real numbers. The spheroid is called oblate if $c<a$, and prolate for $c>a$.

a. The eye cornea is approximated as a prolate spheroid with an axis that is the eye, where $a=8.7mm$ and $c=9.6mm$.Write the equation of the spheroid that models the cornea and sketch the surface.

b. Give two examples of objects with prolate spheroid shapes.

57) [T] In cartography, Earth is approximated by an oblate spheroid rather than a sphere. The radii at the equator and poles are approximately 3963mi and 3950mi, respectively.

a. Write the equation in standard form of the ellipsoid that represents the shape of Earth. Assume the center of Earth is at the origin and that the trace formed by plane $z=0$ corresponds to the equator.

b. Sketch the graph.

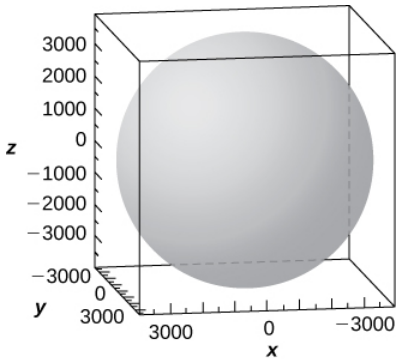
c. Find the equation of the intersection curve of the surface with plane $z=1000$ that is parallel to the xy-plane. The intersection curve is called a *parallel.*

d. Find the equation of the intersection curve of the surface with plane $x+y=0$ that passes through the z-axis. The intersection curve is called a meridian.

Answer

a. $\frac{x^2}{3963^2}+\frac{y^2}{3963^2}+\frac{z^2}{3950^2}=1$

b.



c. The intersection curve is the ellipse of equation $\frac{x^2}{3963^2}+\frac{y^2}{3963^2}=\frac{(2950)(4950)}{3950^2}$, and the intersection is an ellipse.

d. The intersection curve is the ellipse of equation $\frac{2y^2}{3963^2}+\frac{z^2}{3950^2}=1$.

58) [T] A set of buzzing stunt magnets (or “rattlesnake eggs”) includes two sparkling, polished, superstrong spheroid-shaped magnets well-known for children’s entertainment. Each magnet is 1.625 in. long and 0.5 in. wide at the middle. While tossing them into the air, they create a buzzing sound as they attract each other.

a. Write the equation of the prolate spheroid centered at the origin that describes the shape of one of the magnets.

b. Write the equations of the prolate spheroids that model the shape of the buzzing stunt magnets. Use a CAS or CalcPlot3D to create the graphs.

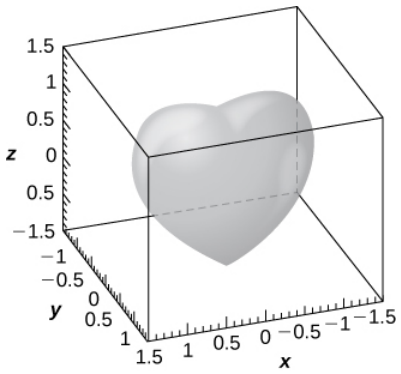
59) [T] A heart-shaped surface is given by equation $(x^2+\frac{9}{4}y^2+z^2-1)^3-x^2z^3-\frac{9}{80}y^2z^3=0$.

a. Use a CAS or CalcPlot3D to graph the surface that models this shape.

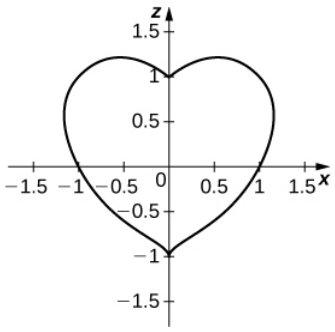
b. Determine and sketch the trace of the heart-shaped surface on the xz-plane.

Answer

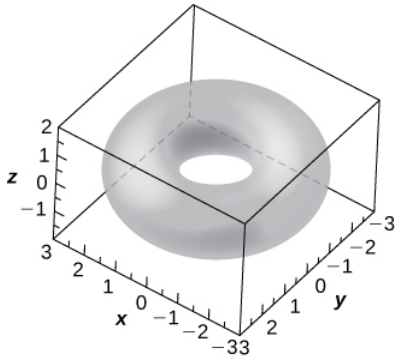
a.



b. The intersection curve is $(x^2+z^2-1)^3-x^2z^3=0$.



60) [T] The ring torus symmetric about the z-axis is a special type of surface in topology and its equation is given by $(x^2+y^2+z^2+R^2-r^2)^2=4R^2(x^2+y^2)$, where $R>r>0$. The numbers R and r are called are the major and minor radii, respectively, of the surface. The following figure shows a ring torus for which $R=2$ and $r=1$.



a. Write the equation of the ring torus with $R=2$ and $r=1$, and use a CAS or CalcPlot3D to graph the surface. Compare the graph with the figure given.

b. Determine the equation and sketch the trace of the ring torus from a. on the xy-plane.

c. Give two examples of objects with ring torus shapes.

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4.8: Cylindrical and Spherical Coordinates

Learning Objectives

- Convert from cylindrical to rectangular coordinates.
- Convert from rectangular to cylindrical coordinates.
- Convert from spherical to rectangular coordinates.
- Convert from rectangular to spherical coordinates.

The Cartesian coordinate system provides a straightforward way to describe the location of points in space. Some surfaces, however, can be difficult to model with equations based on the Cartesian system. This is a familiar problem; recall that in two dimensions, polar coordinates often provide a useful alternative system for describing the location of a point in the plane, particularly in cases involving circles. In this section, we look at two different ways of describing the location of points in space, both of them based on extensions of polar coordinates. As the name suggests, cylindrical coordinates are useful for dealing with problems involving cylinders, such as calculating the volume of a round water tank or the amount of oil flowing through a pipe. Similarly, spherical coordinates are useful for dealing with problems involving spheres, such as finding the volume of domed structures.

Cylindrical Coordinates

When we expanded the traditional Cartesian coordinate system from two dimensions to three, we simply added a new axis to model the third dimension. Starting with polar coordinates, we can follow this same process to create a new three-dimensional coordinate system, called the cylindrical coordinate system. In this way, cylindrical coordinates provide a natural extension of polar coordinates to three dimensions.

Definition: The Cylindrical Coordinate System

In the cylindrical coordinate system**,** a point in space (Figure $4.8.1$) is represented by the ordered triple (r, θ, z), where

- (r, θ) are the polar coordinates of the point's projection in the xy-plane
- z is the usual z-**coordinate** in the Cartesian coordinate system

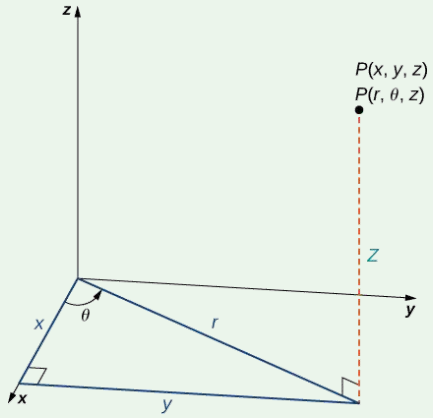


Figure $4.8.1$: The right triangle lies in the xy-plane. The length of the hypotenuse is r and θ is the measure of the angle formed by the positive x-axis and the hypotenuse. The z-coordinate describes the location of the point above or below the xy-plane.

In the xy-plane, the right triangle shown in Figure $4.8.1$ provides the key to transformation between cylindrical and Cartesian, or rectangular, coordinates.

Conversion between Cylindrical and Cartesian Coordinates

The rectangular coordinates (x, y, z) and the cylindrical coordinates (r, θ, z) of a point are related as follows:

These equations are used to convert from cylindrical coordinates to rectangular coordinates.

- $x = r\cos\theta$
- $y = r\sin\theta$

- $z = z$

These equations are used to convert from rectangular coordinates to cylindrical coordinates

1. $r^2 = x^2 + y^2$
2. $\tan\theta = \dfrac{y}{x}$
3. $z = z$

As when we discussed conversion from rectangular coordinates to polar coordinates in two dimensions, it should be noted that the equation $\tan\theta = \dfrac{y}{x}$ has an infinite number of solutions. However, if we restrict θ to values between 0 and 2π, then we can find a unique solution based on the quadrant of the xy-plane in which original point (x, y, z) is located. Note that if $x = 0$, then the value of θ is either $\dfrac{\pi}{2}, \dfrac{3\pi}{2}$, or 0, depending on the value of y.

Notice that these equations are derived from properties of right triangles. To make this easy to see, consider point P in the xy-plane with rectangular coordinates $(x, y, 0)$ and with cylindrical coordinates $(r, \theta, 0)$, as shown in Figure 4.8.2.

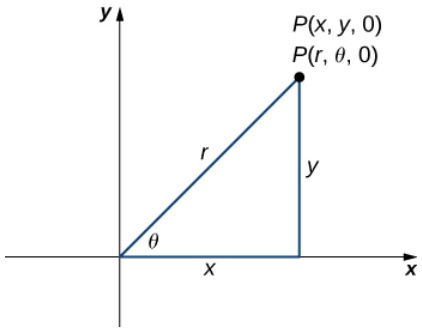


Figure 4.8.2: The Pythagorean theorem provides equation $r^2 = x^2 + y^2$. Right-triangle relationships tell us that $x = r\cos\theta, y = r\sin\theta,$ and $\tan\theta = y/x$.

Let's consider the differences between rectangular and cylindrical coordinates by looking at the surfaces generated when each of the coordinates is held constant. If c is a constant, then in rectangular coordinates, surfaces of the form $x = c, y = c$, or $z = c$ are all planes. Planes of these forms are parallel to the yz-plane, the xz-plane, and the xy-plane, respectively. When we convert to cylindrical coordinates, the z-coordinate does not change. Therefore, in cylindrical coordinates, surfaces of the form $z = c$ are planes parallel to the xy-plane. Now, let's think about surfaces of the form $r = c$. The points on these surfaces are at a fixed distance from the z-axis. In other words, these surfaces are vertical circular cylinders. Last, what about $\theta = c$? The points on a surface of the form $\theta = c$ are at a fixed angle from the x-axis, which gives us a half-plane that starts at the z-axis (Figures 4.8.3 and 4.8.4).

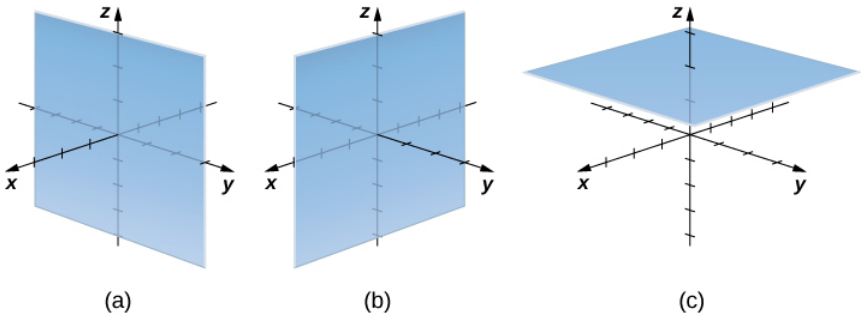


Figure 4.8.3: In rectangular coordinates, (a) surfaces of the form $x = c$ are planes parallel to the yz-plane, (b) surfaces of the form $y = c$ are planes parallel to the xz-plane, and (c) surfaces of the form $z = c$ are planes parallel to the xy-plane.

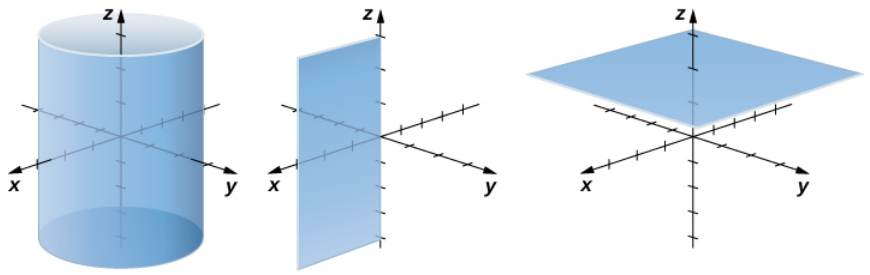


Figure 4.8.4: In cylindrical coordinates, (a) surfaces of the form $r = c$ are vertical cylinders of radius r, (b) surfaces of the form $\theta = c$ are half-planes at angle θ from the x-axis, and (c) surfaces of the form $z = c$ are planes parallel to the xy-plane.

Example 4.8.1: Converting from Cylindrical to Rectangular Coordinates

Plot the point with cylindrical coordinates $(4, \frac{2\pi}{3}, -2)$ and express its location in rectangular coordinates.

Solution

Conversion from cylindrical to rectangular coordinates requires a simple application of the equations listed in Note:

$$\begin{aligned} x &= r\cos\theta = 4\cos\frac{2\pi}{3} = -2 \\ y &= r\sin\theta = 4\sin\frac{2\pi}{3} = 2\sqrt{3} \\ z &= -2 \end{aligned}.$$

The point with cylindrical coordinates $(4, \frac{2\pi}{3}, -2)$ has rectangular coordinates $(-2, 2\sqrt{3}, -2)$ (Figure 4.8.5).

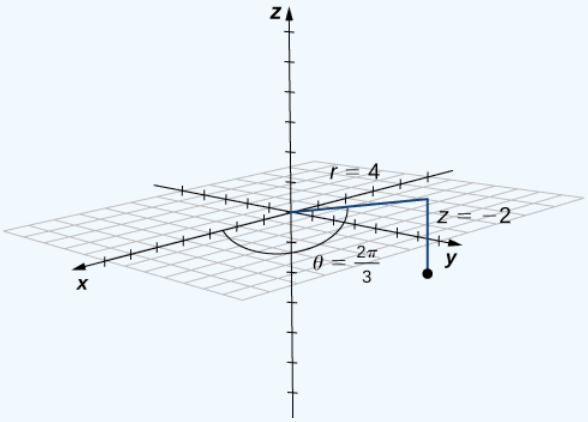


Figure 4.8.5: The projection of the point in the xy-plane is 4 units from the origin. The line from the origin to the point's projection forms an angle of $\frac{2\pi}{3}$ with the positive x-axis. The point lies 2 units below the xy-plane.

Exercise 4.8.1

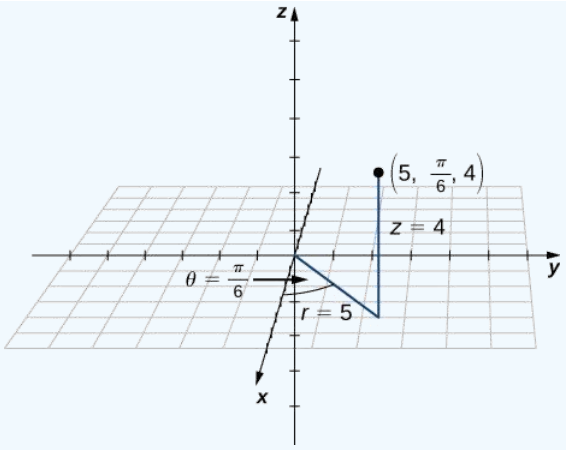
Point R has cylindrical coordinates $(5, \frac{\pi}{6}, 4)$. Plot R and describe its location in space using rectangular, or Cartesian, coordinates.

Hint

The first two components match the polar coordinates of the point in the xy-plane.

Answer

The rectangular coordinates of the point are $(\frac{5\sqrt{3}}{2}, \frac{5}{2}, 4)$.



If this process seems familiar, it is with good reason. This is exactly the same process that we followed in Introduction to Parametric Equations and Polar Coordinates to convert from polar coordinates to two-dimensional rectangular coordinates.

Example 4.8.2: Converting from Rectangular to Cylindrical Coordinates

Convert the rectangular coordinates $(1, -3, 5)$ to cylindrical coordinates.

Solution

Use the second set of equations from Note to translate from rectangular to cylindrical coordinates:

$$\begin{aligned} r^2 &= x^2 + y^2 \\ r &= \pm\sqrt{1^2 + (-3)^2} \\ &= \pm\sqrt{10}. \end{aligned}$$

We choose the positive square root, so $r = \sqrt{10}$.Now, we apply the formula to find θ. In this case, y is negative and x is positive, which means we must select the value of θ between $\dfrac{3\pi}{2}$ and 2π:

$$\begin{aligned} \tan\theta = \frac{y}{x} &= \frac{-3}{1} \\ \theta = \arctan(-3) &\approx 5.03 \text{ rad}. \end{aligned}$$

In this case, the z-coordinates are the same in both rectangular and cylindrical coordinates:

$$z = 5.$$

The point with rectangular coordinates $(1, -3, 5)$ has cylindrical coordinates approximately equal to $(\sqrt{10}, 5.03, 5)$.

Exercise 4.8.2

Convert point $(-8, 8, -7)$ from Cartesian coordinates to cylindrical coordinates.

Hint

$r^2 = x^2 + y^2$ and $\tan\theta = \frac{y}{x}$

Answer

$(8\sqrt{2}, \frac{3\pi}{4}, -7)$

The use of cylindrical coordinates is common in fields such as physics. Physicists studying electrical charges and the capacitors used to store these charges have discovered that these systems sometimes have a cylindrical symmetry. These systems have

complicated modeling equations in the Cartesian coordinate system, which make them difficult to describe and analyze. The equations can often be expressed in more simple terms using cylindrical coordinates. For example, the cylinder described by equation $x^2+y^2=25$ in the Cartesian system can be represented by cylindrical equation $r=5$.

Example 4.8.3: Identifying Surfaces in the Cylindrical Coordinate System

Describe the surfaces with the given cylindrical equations.

a. $\theta=\dfrac{\pi}{4}$
b. $r^2+z^2=9$
c. $z=r$

Solution

a. When the angle θ is held constant while r and z are allowed to vary, the result is a half-plane (Figure 4.8.6).

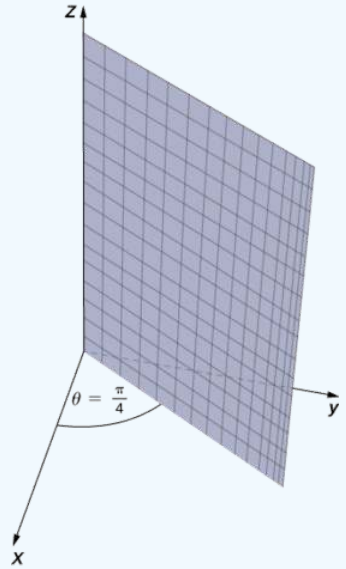


Figure 4.8.6: In polar coordinates, the equation $\theta=\pi/4$ describes the ray extending diagonally through the first quadrant. In three dimensions, this same equation describes a half-plane.

b. Substitute $r^2=x^2+y^2$ into equation $r^2+z^2=9$ to express the rectangular form of the equation: $x^2+y^2+z^2=9$. This equation describes a sphere centered at the origin with radius **3** (Figure 4.8.7).

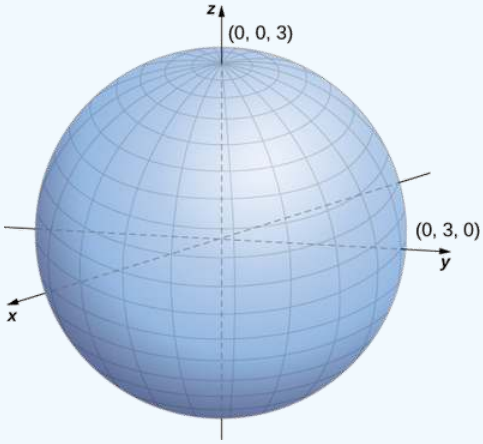


Figure 4.8.7: The sphere centered at the origin with radius **3** can be described by the cylindrical equation $r^2+z^2=9$.

c. To describe the surface defined by equation $z=r$, is it useful to examine traces parallel to the xy-plane. For example, the trace in plane $z=1$ is circle $r=1$, the trace in plane $z=3$ is circle $r=3$, and so on. Each trace is a circle. As the value of z increases, the radius of the circle also increases. The resulting surface is a cone (Figure 4.8.8).

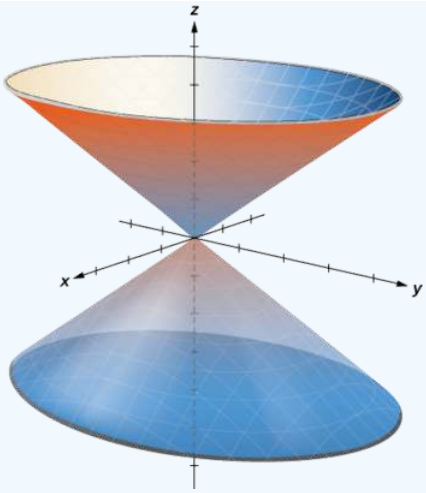


Figure 4.8.8: The traces in planes parallel to the xy-plane are circles. The radius of the circles increases as z increases.



Exercise 4.8.3

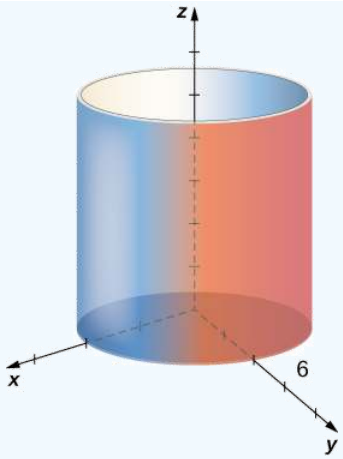
Describe the surface with cylindrical equation $r = 6$.

Hint

The θ and z components of points on the surface can take any value.

Answer

This surface is a cylinder with radius 6.



Spherical Coordinates

In the Cartesian coordinate system, the location of a point in space is described using an ordered triple in which each coordinate represents a distance. In the cylindrical coordinate system, location of a point in space is described using two distances $(r$ and $z)$ and an angle measure (θ). In the spherical coordinate system, we again use an ordered triple to describe the location of a point in space. In this case, the triple describes one distance and two angles. Spherical coordinates make it simple to describe a sphere, just as cylindrical coordinates make it easy to describe a cylinder. Grid lines for spherical coordinates are based on angle measures, like those for polar coordinates.

Definition: spherical coordinate system

In the *spherical coordinate system*, a point P in space (Figure $4.8.9$) is represented by the ordered triple (ρ, θ, φ) where

- ρ (the Greek letter rho) is the distance between P and the origin $(\rho \neq 0)$;
- θ is the same angle used to describe the location in cylindrical coordinates;
- φ (the Greek letter phi) is the angle formed by the positive z-axis and line segment $\bar{OP}$, where O is the origin and $0 \leq \varphi \leq \pi$.

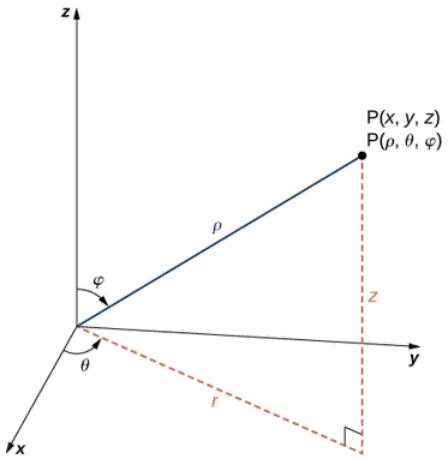


Figure 4.8.9: The relationship among spherical, rectangular, and cylindrical coordinates.

By convention, the origin is represented as $(0, 0, 0)$ in spherical coordinates.

HOWTO: Converting among Spherical, Cylindrical, and Rectangular Coordinates

Rectangular coordinates (x, y, z), cylindrical coordinates (r, θ, z), and spherical coordinates (ρ, θ, φ) of a point are related as follows:

Convert from spherical coordinates to rectangular coordinates

These equations are used to convert from spherical coordinates to rectangular coordinates.

- $x = \rho \sin\varphi \cos\theta$
- $y = \rho \sin\varphi \sin\theta$
- $z = \rho \cos\varphi$

Convert from rectangular coordinates to spherical coordinates

These equations are used to convert from rectangular coordinates to spherical coordinates.

- $\rho^2 = x^2 + y^2 + z^2$
- $\tan\theta = \dfrac{y}{x}$
- $\varphi = \arccos\left(\dfrac{z}{\sqrt{x^2+y^2+z^2}}\right)$.

Convert from spherical coordinates to cylindrical coordinates

These equations are used to convert from spherical coordinates to cylindrical coordinates.

- $r = \rho \sin\varphi$
- $\theta = \theta$
- $z = \rho \cos\varphi$

Convert from cylindrical coordinates to spherical coordinates

These equations are used to convert from cylindrical coordinates to spherical coordinates.

- $\rho = \sqrt{r^2+z^2}$
- $\theta = \theta$
- $\varphi = \arccos\left(\dfrac{z}{\sqrt{r^2+z^2}}\right)$

The formulas to convert from spherical coordinates to rectangular coordinates may seem complex, but they are straightforward applications of trigonometry. Looking at Figure, it is easy to see that $r = \rho \sin\varphi$. Then, looking at the triangle in the xy-plane with r as its hypotenuse, we have $x = r\cos\theta = \rho \sin\varphi \cos\theta$. The derivation of the formula for y is similar. Figure also shows that $\rho^2 = r^2 + z^2 = x^2 + y^2 + z^2$ and $z = \rho\cos\varphi$. Solving this last equation for φ and then substituting $\rho = \sqrt{r^2+z^2}$ (from the first equation) yields $\varphi = \arccos\left(\dfrac{z}{\sqrt{r^2+z^2}}\right)$. Also, note that, as before, we must be careful when using the formula $\tan\theta = \dfrac{y}{x}$ to choose the correct value of θ.

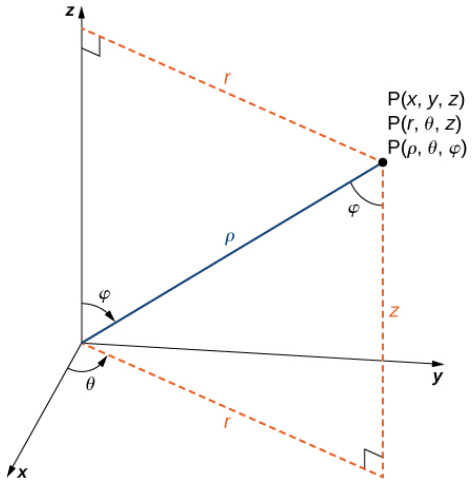


Figure 4.8.10: The equations that convert from one system to another are derived from right-triangle relationships.

As we did with cylindrical coordinates, let's consider the surfaces that are generated when each of the coordinates is held constant. Let c be a constant, and consider surfaces of the form $\rho = c$. Points on these surfaces are at a fixed distance from the origin and form a sphere. The coordinate θ in the spherical coordinate system is the same as in the cylindrical coordinate system, so surfaces

of the form $\theta = c$ are half-planes, as before. Last, consider surfaces of the form $\varphi = 0$. The points on these surfaces are at a fixed angle from the z-axis and form a half-cone (Figure 4.8.11).

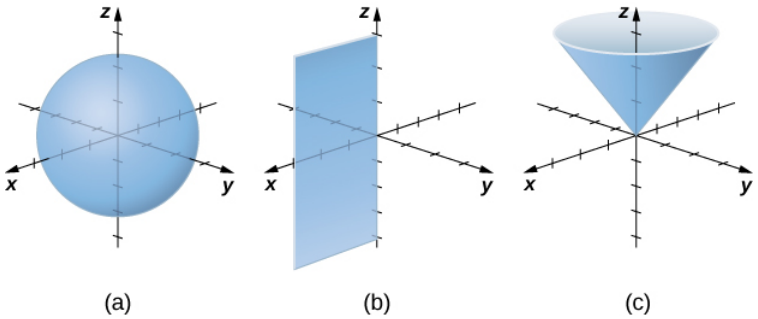


Figure 4.8.11: In spherical coordinates, surfaces of the form $\rho = c$ are spheres of radius ρ (a), surfaces of the form $\theta = c$ are half-planes at an angle θ from the x-axis (b), and surfaces of the form $\phi = c$ are half-cones at an angle ϕ from the z-axis (c).

Example 4.8.4: Converting from Spherical Coordinates

Plot the point with spherical coordinates $(8, \frac{\pi}{3}, \frac{\pi}{6})$ and express its location in both rectangular and cylindrical coordinates.

Solution

Use the equations in Note to translate between spherical and cylindrical coordinates (Figure 4.8.12):

$$\begin{aligned} x &= \rho \sin\varphi \cos\theta \\ &= 8 \sin(\frac{\pi}{6}) \cos(\frac{\pi}{3}) \\ &= 8(\frac{1}{2})\frac{1}{2} \\ &= 2 \\ y &= \rho \sin\varphi \sin\theta \\ &= 8 \sin(\frac{\pi}{6}) \sin(\frac{\pi}{3}) \\ &= 8(\frac{1}{2})\frac{\sqrt{3}}{2} \\ &= 2\sqrt{3} \\ z &= \rho \cos\varphi \\ &= 8 \cos(\frac{\pi}{6}) \\ &= 8(\frac{\sqrt{3}}{2}) \\ &= 4\sqrt{3} \end{aligned}$$

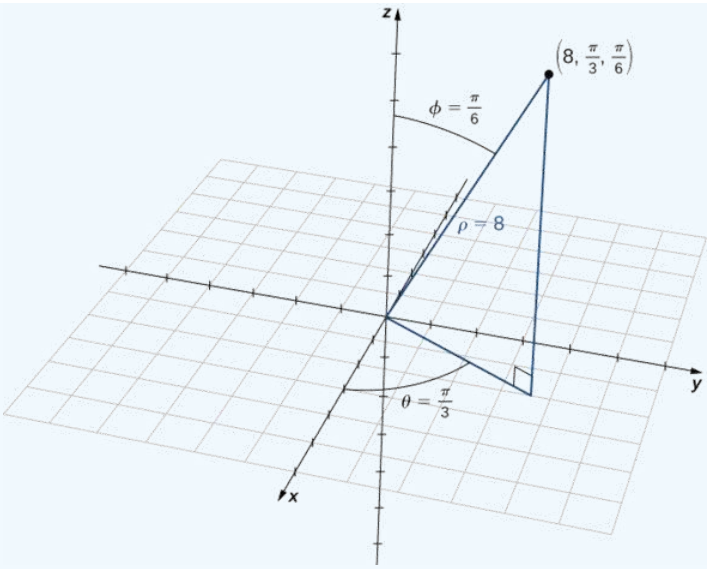


Figure 4.8.12: The projection of the point in the xy-plane is 4 units from the origin. The line from the origin to the point's projection forms an angle of $\pi/3$ with the positive x-axis. The point lies $4\sqrt{3}$ units above the xy-plane.

The point with spherical coordinates $(8, \frac{\pi}{3}, \frac{\pi}{6})$ has rectangular coordinates $(2, 2\sqrt{3}, 4\sqrt{3})$.

Finding the values in cylindrical coordinates is equally straightforward:

$$\begin{aligned} r &= \rho \sin \varphi \\ &= 8 \sin \frac{\pi}{6} \qquad = 4 \\ \theta &= \theta \\ z &= \rho \cos \varphi \\ &= 8 \cos \frac{\pi}{6} \\ &= 4\sqrt{3}. \end{aligned}$$

Thus, cylindrical coordinates for the point are $(4, \frac{\pi}{3}, 4\sqrt{3})$.

Exercise 4.8.4

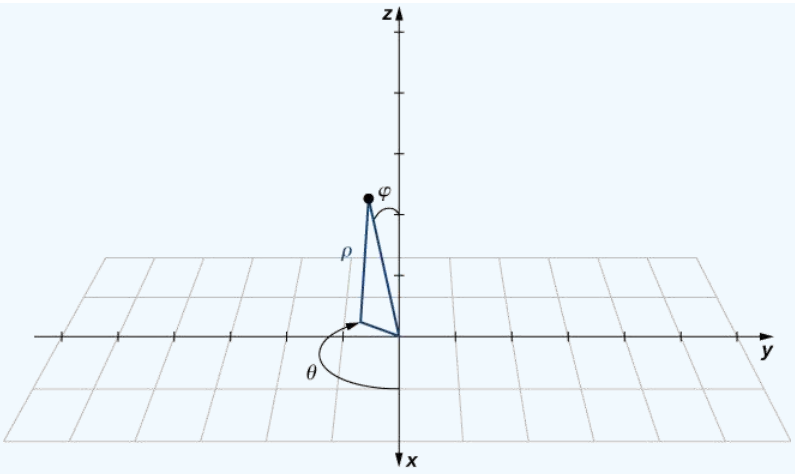
Plot the point with spherical coordinates $(2, -\frac{5\pi}{6}, \frac{\pi}{6})$ and describe its location in both rectangular and cylindrical coordinates.

Hint

Converting the coordinates first may help to find the location of the point in space more easily.

Answer

Cartesian: $(-\frac{\sqrt{3}}{2}, -\frac{1}{2}, \sqrt{3})$, cylindrical: $(1, -\frac{5\pi}{6}, \sqrt{3})$



Example 4.8.5: Converting from Rectangular Coordinates

Convert the rectangular coordinates $(-1, 1, \sqrt{6})$ to both spherical and cylindrical coordinates.

Solution

Start by converting from rectangular to spherical coordinates:

$$\rho^2 = x^2 + y^2 + z^2 = (-1)^2 + 1^2 + (\sqrt{6})^2 = 8$$

$$\tan\theta = \frac{1}{-1}$$

$$\rho = 2\sqrt{2} \text{ and } \theta = \arctan(-1) = \frac{3\pi}{4}.$$

Because $(x, y) = (-1, 1)$, then the correct choice for θ is $\frac{3\pi}{4}$.

There are actually two ways to identify φ. We can use the equation $\varphi = \arccos(\frac{z}{\sqrt{x^2+y^2+z^2}})$. A more simple approach, however, is to use equation $z = \rho\cos\varphi$. We know that $z = \sqrt{6}$ and $\rho = 2\sqrt{2}$, so

$$\sqrt{6} = 2\sqrt{2}\cos\varphi, \text{ so } \cos\varphi = \frac{\sqrt{6}}{2\sqrt{2}} = \frac{\sqrt{3}}{2}$$

and therefore $\varphi = \frac{\pi}{6}$. The spherical coordinates of the point are $(2\sqrt{2}, \frac{3\pi}{4}, \frac{\pi}{6})$.

To find the cylindrical coordinates for the point, we need only find r:

$$r = \rho\sin\varphi = 2\sqrt{2}\sin(\frac{\pi}{6}) = \sqrt{2}.$$

The cylindrical coordinates for the point are $(\sqrt{2}, \frac{3\pi}{4}, \sqrt{6})$.



Example 4.8.6: Identifying Surfaces in the Spherical Coordinate System

Describe the surfaces with the given spherical equations.

a. $\theta = \dfrac{\pi}{3}$
b. $\varphi = \dfrac{5\pi}{6}$
c. $\rho = 6$
d. $\rho = \sin\theta \sin\varphi$

Solution

a. The variable θ represents the measure of the same angle in both the cylindrical and spherical coordinate systems. Points with coordinates $(\rho, \dfrac{\pi}{3}, \varphi)$ lie on the plane that forms angle $\theta = \dfrac{\pi}{3}$ with the positive x-axis. Because $\rho > 0$, the surface described by equation $\theta = \dfrac{\pi}{3}$ is the half-plane shown in Figure 4.8.13.

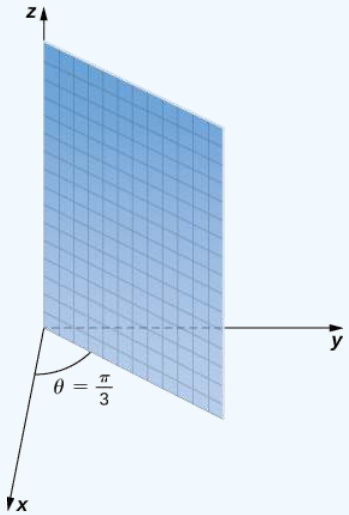


Figure 4.8.13: The surface described by equation $\theta = \dfrac{\pi}{3}$ is a half-plane.

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b. Equation $\varphi = \dfrac{5\pi}{6}$ describes all points in the spherical coordinate system that lie on a line from the origin forming an angle measuring $\dfrac{5\pi}{6}$ rad with the positive z-axis. These points form a half-cone (Figure). Because there is only one value for φ that is measured from the positive z-axis, we do not get the full cone (with two pieces).

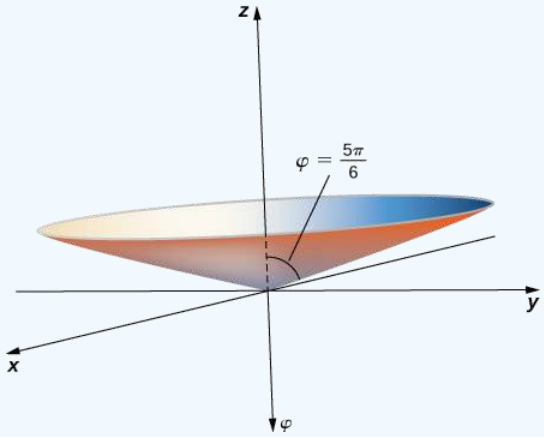


Figure 4.8.14: The equation $\varphi = \dfrac{5\pi}{6}$ describes a cone.

To find the equation in rectangular coordinates, use equation $\varphi = \arccos\left(\dfrac{z}{\sqrt{x^2+y^2+z^2}}\right)$.

$$\begin{aligned} \frac{5\pi}{6} &= \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) \\ \cos\frac{5\pi}{6} &= \frac{z}{\sqrt{x^2+y^2+z^2}} \\ -\frac{\sqrt{3}}{2} &= \frac{z}{\sqrt{x^2+y^2+z^2}} \\ \frac{3}{4} &= \frac{z^2}{x^2+y^2+z^2} \\ \frac{3x^2}{4}+\frac{3y^2}{4}+\frac{3z^2}{4} &= z^2 \\ \frac{3x^2}{4}+\frac{3y^2}{4}-\frac{z^2}{4} &= 0. \end{aligned}$$

This is the equation of a cone centered on the z-axis.

c. Equation $\rho = 6$ describes the set of all points 6 units away from the origin—a sphere with radius 6 (Figure 4.8.15).

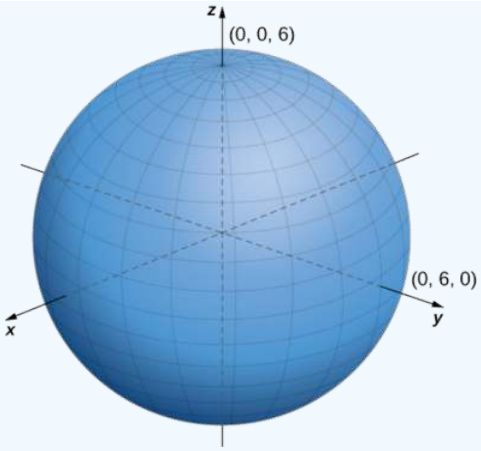


Figure 4.8.15: Equation $\rho = 6$ describes a sphere with radius 6.

d. To identify this surface, convert the equation from spherical to rectangular coordinates, using equations $y = \rho sin\varphi \sin\theta$ and $\rho^2 = x^2 + y^2 + z^2$:

$$\rho = \sin\theta \sin\varphi$$

$\rho^2 = \rho \sin\theta \sin\varphi$ Multiply both sides of the equation by ρ.

$x^2 + y^2 + z^2 = y$ Substitute rectangular variables using the equations above.

$x^2 + y^2 - y + z^2 = 0$ Subtract y from both sides of the equation.

$x^2 + y^2 - y + \frac{1}{4} + z^2 = \frac{1}{4}$ Complete the square.

$x^2 + (y - \frac{1}{2})^2 + z^2 = \frac{1}{4}$. Rewrite the middle terms as a perfect square.

The equation describes a sphere centered at point $(0, \frac{1}{2}, 0)$ with radius $\frac{1}{2}$.

Exercise 4.8.5

Describe the surfaces defined by the following equations.

a. $\rho = 13$
b. $\theta = \frac{2\pi}{3}$
c. $\varphi = \frac{\pi}{4}$

Hint

Think about what each component represents and what it means to hold that component constant.

Answer a

This is the set of all points 13 units from the origin. This set forms a sphere with radius 13.

Answer b

This set of points forms a half plane. The angle between the half plane and the positive x-axis is $\theta = \frac{2\pi}{3}$.

Answer c

Let P be a point on this surface. The position vector of this point forms an angle of $\varphi = \frac{\pi}{4}$ with the positive z-axis, which means that points closer to the origin are closer to the axis. These points form a half-cone.

Spherical coordinates are useful in analyzing systems that have some degree of symmetry about a point, such as the volume of the space inside a domed stadium or wind speeds in a planet's atmosphere. A sphere that has Cartesian equation $x^2+y^2+z^2=c^2$ has the simple equation $\rho=c$ in spherical coordinates.

In geography, latitude and longitude are used to describe locations on Earth's surface, as shown in Figure. Although the shape of Earth is not a perfect sphere, we use spherical coordinates to communicate the locations of points on Earth. Let's assume Earth has the shape of a sphere with radius 4000 mi. We express angle measures in degrees rather than radians because latitude and longitude are measured in degrees.

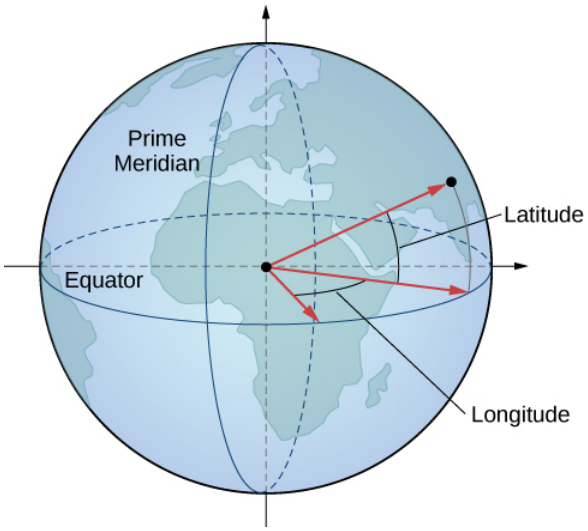


Figure 4.8.16: In the latitude–longitude system, angles describe the location of a point on Earth relative to the equator and the prime meridian.

Let the center of Earth be the center of the sphere, with the ray from the center through the North Pole representing the positive z-axis. The prime meridian represents the trace of the surface as it intersects the xz-plane. The equator is the trace of the sphere intersecting the xy-plane.

Example 4.8.7: Converting Latitude and Longitude to Spherical Coordinates

The latitude of Columbus, Ohio, is 40° N and the longitude is 83° W, which means that Columbus is 40° north of the equator. Imagine a ray from the center of Earth through Columbus and a ray from the center of Earth through the equator directly south of Columbus. The measure of the angle formed by the rays is 40°. In the same way, measuring from the prime meridian, Columbus lies 83° to the west. Express the location of Columbus in spherical coordinates.

Solution

The radius of Earth is 4000mi, so $\rho=4000$. The intersection of the prime meridian and the equator lies on the positive x-axis. Movement to the west is then described with negative angle measures, which shows that $\theta=-83^\circ$, Because Columbus lies 40° north of the equator, it lies 50° south of the North Pole, so $\varphi=50^\circ$. In spherical coordinates, Columbus lies at point $(4000,-83^\circ,50^\circ)$.

Exercise 4.8.6

Sydney, Australia is at $34^\circ S$ and $151^\circ E$. Express Sydney's location in spherical coordinates.

Hint

Because Sydney lies south of the equator, we need to add 90° to find the angle measured from the positive z-axis.

Answer

$(4000,151^\circ,124^\circ)$

Cylindrical and spherical coordinates give us the flexibility to select a coordinate system appropriate to the problem at hand. A thoughtful choice of coordinate system can make a problem much easier to solve, whereas a poor choice can lead to unnecessarily complex calculations. In the following example, we examine several different problems and discuss how to select the best coordinate system for each one.

Example 4.8.8: Choosing the Best Coordinate System

In each of the following situations, we determine which coordinate system is most appropriate and describe how we would orient the coordinate axes. There could be more than one right answer for how the axes should be oriented, but we select an orientation that makes sense in the context of the problem. Note: There is not enough information to set up or solve these problems; we simply select the coordinate system (Figure 4.8.17).

a. Find the center of gravity of a bowling ball.
b. Determine the velocity of a submarine subjected to an ocean current.
c. Calculate the pressure in a conical water tank.
d. Find the volume of oil flowing through a pipeline.
e. Determine the amount of leather required to make a football.

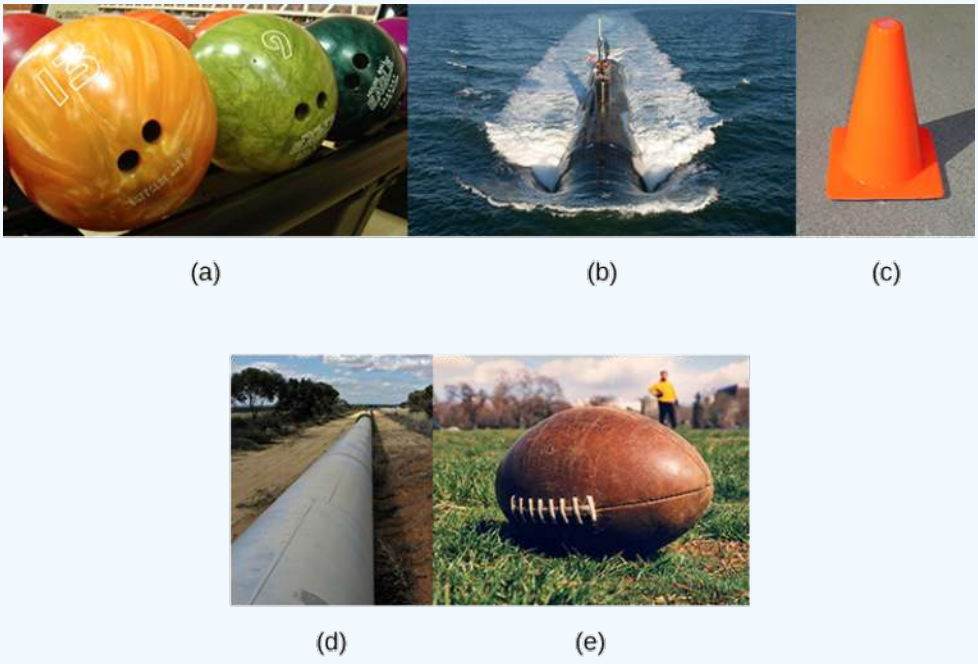


Figure 4.8.17: (credit: (a) modification of work by scl hua, Wikimedia, (b) modification of work by DVIDSHUB, Flickr, (c) modification of work by Michael Malak, Wikimedia, (d) modification of work by Sean Mack, Wikimedia, (e) modification of work by Elvert Barnes, Flickr)

Solution

a. Clearly, a bowling ball is a sphere, so spherical coordinates would probably work best here. The origin should be located at the physical center of the ball. There is no obvious choice for how the x-, y- and z-axes should be oriented. Bowling balls normally have a weight block in the center. One possible choice is to align the z-axis with the axis of symmetry of the weight block.
b. A submarine generally moves in a straight line. There is no rotational or spherical symmetry that applies in this situation, so rectangular coordinates are a good choice. The z-axis should probably point upward. The x- and y-axes could be aligned to point east and north, respectively. The origin should be some convenient physical location, such as the starting position of the submarine or the location of a particular port.
c. A cone has several kinds of symmetry. In cylindrical coordinates, a cone can be represented by equation $z = kr$, where k is a constant. In spherical coordinates, we have seen that surfaces of the form $\varphi = c$ are half-cones. Last, in rectangular coordinates, elliptic cones are quadric surfaces and can be represented by equations of the form $z^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$. In this case, we could choose any of the three. However, the equation for the surface is more complicated in rectangular

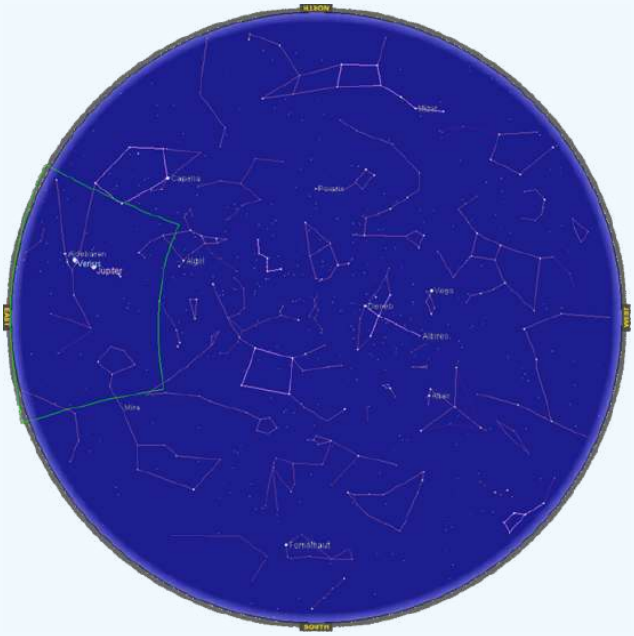
coordinates than in the other two systems, so we might want to avoid that choice. In addition, we are talking about a water tank, and the depth of the water might come into play at some point in our calculations, so it might be nice to have a component that represents height and depth directly. Based on this reasoning, cylindrical coordinates might be the best choice. Choose the z-axis to align with the axis of the cone. The orientation of the other two axes is arbitrary. The origin should be the bottom point of the cone.

d. A pipeline is a cylinder, so cylindrical coordinates would be best the best choice. In this case, however, we would likely choose to orient our z-axis with the center axis of the pipeline. The x-axis could be chosen to point straight downward or to some other logical direction. The origin should be chosen based on the problem statement. Note that this puts the z-axis in a horizontal orientation, which is a little different from what we usually do. It may make sense to choose an unusual orientation for the axes if it makes sense for the problem.

e. A football has rotational symmetry about a central axis, so cylindrical coordinates would work best. The z-axis should align with the axis of the ball. The origin could be the center of the ball or perhaps one of the ends. The position of the x-axis is arbitrary.

Exercise 4.8.7

Which coordinate system is most appropriate for creating a star map, as viewed from Earth (see the following figure)?



How should we orient the coordinate axes?

Hint

What kinds of symmetry are present in this situation?

Answer

Spherical coordinates with the origin located at the center of the earth, the z-axis aligned with the North Pole, and the x-axis aligned with the prime meridian

Key Concepts

- In the cylindrical coordinate system, a point in space is represented by the ordered triple (r, θ, z), where (r, θ) represents the polar coordinates of the point's projection in the xy-plane and z represents the point's projection onto the z-axis.
- To convert a point from cylindrical coordinates to Cartesian coordinates, use equations $x = r\cos\theta, y = r\sin\theta,$ and $z = z$.
- To convert a point from Cartesian coordinates to cylindrical coordinates, use equations $r^2 = x^2 + y^2, \tan\theta = \dfrac{y}{x},$ and $z = z$.

- In the spherical coordinate system, a point P in space is represented by the ordered triple (ρ, θ, φ), where ρ is the distance between P and the origin $(\rho \neq 0)$, θ is the same angle used to describe the location in cylindrical coordinates, and φ is the angle formed by the positive z-axis and line segment $\bar{OP}$, where O is the origin and $0 \leq \varphi \leq \pi$.
- To convert a point from spherical coordinates to Cartesian coordinates, use equations $x = \rho \sin \varphi \cos \theta, y = \rho \sin \varphi \sin \theta,$ and $z = \rho \cos \varphi$.
- To convert a point from Cartesian coordinates to spherical coordinates, use equations $\rho^2 = x^2 + y^2 + z^2, \tan \theta = \frac{y}{x},$ and $\varphi = \arccos\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$.
- To convert a point from spherical coordinates to cylindrical coordinates, use equations $r = \rho \sin \varphi, \theta = \theta,$ and $z = \rho \cos \varphi$.
- To convert a point from cylindrical coordinates to spherical coordinates, use equations $\rho = \sqrt{r^2 + z^2}, \theta = \theta,$ and $\varphi = \arccos\left(\frac{z}{\sqrt{r^2 + z^2}}\right)$.

Glossary

cylindrical coordinate system

a way to describe a location in space with an ordered triple (r, θ, z), where (r, θ) represents the polar coordinates of the point's projection in the xy-plane, and z represents the point's projection onto the z-axis

spherical coordinate system

a way to describe a location in space with an ordered triple (ρ, θ, φ), where ρ is the distance between P and the origin $(\rho \neq 0)$, θ is the same angle used to describe the location in cylindrical coordinates, and φ is the angle formed by the positive z-axis and line segment $\bar{OP}$, where O is the origin and $0 \leq \varphi \leq \pi$

Contributors and Attributions

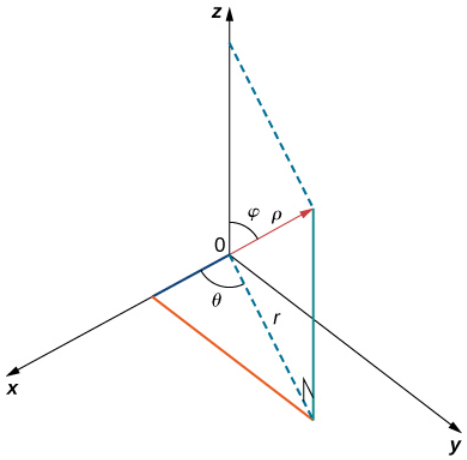
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4.8E: Exercises for Section 4.8

Use the following figure as an aid in identifying the relationship between the rectangular, cylindrical, and spherical coordinate systems.



For exercises 1 - 4, the cylindrical coordinates (r, θ, z) of a point are given. Find the rectangular coordinates (x, y, z) of the point.

1) $\left(4, \frac{\pi}{6}, 3\right)$

Answer

$(2\sqrt{3}, 2, 3)$

2) $\left(3, \frac{\pi}{3}, 5\right)$

3) $\left(4, \frac{7\pi}{6}, 3\right)$

Answer

$(-2\sqrt{3}, -2, 3)$

4) $(2, \pi, -4)$

For exercises 5 - 8, the rectangular coordinates (x, y, z) of a point are given. Find the cylindrical coordinates (r, θ, z) of the point.

5) $(1, \sqrt{3}, 2)$

Answer

$\left(2, \frac{\pi}{3}, 2\right)$

6) $(1, 1, 5)$

7) $(3, -3, 7)$

Answer

$\left(3\sqrt{2}, -\frac{\pi}{4}, 7\right)$

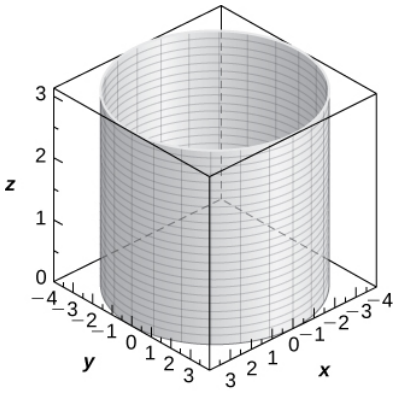
8) $(-2\sqrt{2}, 2\sqrt{2}, 4)$

For exercises 9 - 16, the equation of a surface in cylindrical coordinates is given. Find the equation of the surface in rectangular coordinates. Identify and graph the surface.

9) [T] $r = 4$

Answer

A cylinder of equation $x^2+y^2=16$, with its center at the origin and rulings parallel to the z-axis,

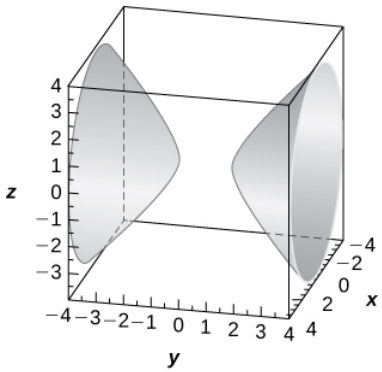


10) [T] $z=r^2\cos^2\theta$

11) [T] $r^2\cos(2\theta)+z^2+1=0$

Answer

Hyperboloid of two sheets of equation $-x^2+y^2-z^2=1$, with the y-axis as the axis of symmetry,

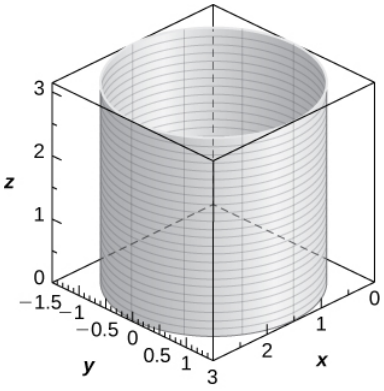


12) [T] $r=3\sin\theta$

13) [T] $r=2\cos\theta$

Answer

Cylinder of equation $x^2-2x+y^2=0$, with a center at $(1,0,0)$ and radius 1, with rulings parallel to the z-axis,

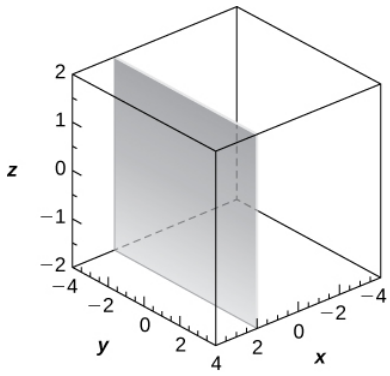


14) [T] $r^2+z^2=5$

15) [T] $r=2\sec\theta$

Answer

Plane of equation $x = 2$,



16) [T] $r = 3\csc\theta$

For exercises 17 - 22, the equation of a surface in rectangular coordinates is given. Find the equation of the surface in cylindrical coordinates.

17) $z = 3$

Answer

$z = 3$

18) $x = 6$

19) $x^2 + y^2 + z^2 = 9$

Answer

$r^2 + z^2 = 9$

20) $y = 2x^2$

21) $x^2 + y^2 - 16x = 0$

Answer

$r = 16\cos\theta, \quad r = 0$

22) $x^2 + y^2 - 3\sqrt{x^2 + y^2} + 2 = 0$

For exercises 23 - 26, the spherical coordinates (ρ, θ, φ) of a point are given. Find the rectangular coordinates (x, y, z) of the point.

23) $(3, 0, \pi)$

Answer

$(0, 0, -3)$

24) $\left(1, \frac{\pi}{6}, \frac{\pi}{6}\right)$

25) $\left(12, -\frac{\pi}{4}, \frac{\pi}{4}\right)$

Answer

$(6, -6, \sqrt{2})$

26) $\left(3, \frac{\pi}{4}, \frac{\pi}{6}\right)$

For exercises 27 - 30, the rectangular coordinates (x, y, z) of a point are given. Find the spherical coordinates (ρ, θ, φ) of the point. Express the measure of the angles in degrees rounded to the nearest integer.

27) $(4, 0, 0)$

Answer

$(4, 0, 90^\circ)$

28) $(-1, 2, 1)$

29) $(0, 3, 0)$

Answer

$(3, 90^\circ, 90^\circ)$

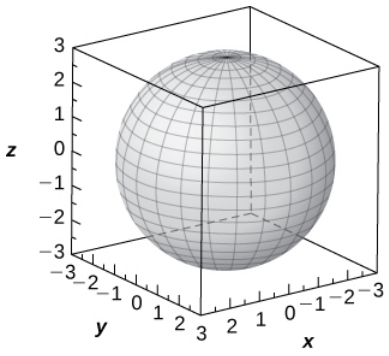
30) $(-2, 2\sqrt{3}, 4)$

For exercises 31 - 36, the equation of a surface in spherical coordinates is given. Find the equation of the surface in rectangular coordinates. Identify and graph the surface.

31) [T] $\rho = 3$

Answer

Sphere of equation $x^2 + y^2 + z^2 = 9$ centered at the origin with radius 3,

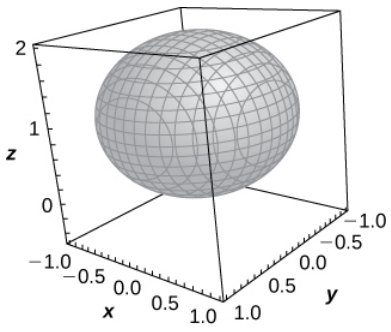


32) [T] $\varphi = \frac{\pi}{3}$

33) [T] $\rho = 2\cos\varphi$

Answer

Sphere of equation $x^2 + y^2 + (z-1)^2 = 1$ centered at $(0, 0, 1)$ with radius 1,

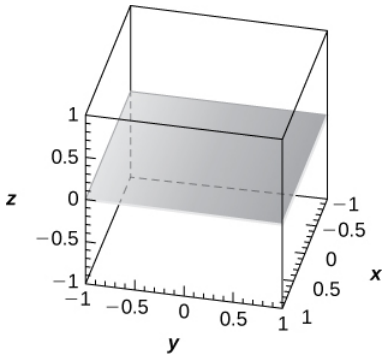


34) [T] $\rho = 4\csc\varphi$

35) [T] $\varphi = \frac{\pi}{2}$

Answer

The xy-plane of equation $z = 0$,



36) [T] $\rho = 6 \csc \varphi \sec \theta$

For exercises 37 - 40, the equation of a surface in rectangular coordinates is given. Find the equation of the surface in spherical coordinates. Identify the surface.

37) $x^2 + y^2 - 3z^2 = 0, \quad z \neq 0$

Answer

$\varphi = \frac{\pi}{3}$ or $\varphi = \frac{2\pi}{3}$; Elliptic cone

38) $x^2 + y^2 + z^2 - 4z = 0$

39) $z = 6$

Answer

$\rho \cos \varphi = 6$; Plane at $z = 6$

40) $x^2 + y^2 = 9$

For exercises 41 - 44, the cylindrical coordinates of a point are given. Find its associated spherical coordinates, with the measure of the angle φ in radians rounded to four decimal places.

41) [T] $\left(1, \frac{\pi}{4}, 3\right)$

Answer

$\left(\sqrt{10}, \frac{\pi}{4}, 0.3218\right)$

42) [T] $(5, \pi, 12)$

43) $\left(3, \frac{\pi}{2}, 3\right)$

Answer

$(3\sqrt{2}, \frac{\pi}{2}, \frac{\pi}{4})$

44) $\left(3, -\frac{\pi}{6}, 3\right)$

For exercises 45 - 48, the spherical coordinates of a point are given. Find its associated cylindrical coordinates.

45) $\left(2, -\frac{\pi}{4}, \frac{\pi}{2}\right)$

Answer

$\left(2, -\frac{\pi}{4}, 0\right)$

46) $\left(4, \frac{\pi}{4}, \frac{\pi}{6}\right)$

47) $\left(8, \frac{\pi}{3}, \frac{\pi}{2}\right)$

Answer

$\left(8, \frac{\pi}{3}, 0\right)$

48) $\left(9, -\frac{\pi}{6}, \frac{\pi}{3}\right)$

For exercises 49 - 52, find the most suitable system of coordinates to describe the solids.

49) The solid situated in the first octant with a vertex at the origin and enclosed by a cube of edge length a, where $a > 0$

Answer

Cartesian system, $\{(x, y, z) \mid 0 \le x \le a,\ 0 \le y \le a,\ 0 \le z \le a\}$

50) A spherical shell determined by the region between two concentric spheres centered at the origin, of radii of a and b, respectively, where $b > a > 0$

51) A solid inside sphere $x^2 + y^2 + z^2 = 9$ and outside cylinder $\left(x - \frac{3}{2}\right)^2 + y^2 = \frac{9}{4}$

Answer

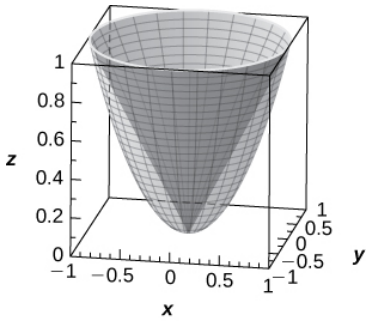
Cylindrical system, $\{(r, \theta, z) \mid r^2 + z^2 \le 9,\ r \ge 3\cos\theta,\ 0 \le \theta \le 2\pi\}$

52) A cylindrical shell of height 10 determined by the region between two cylinders with the same center, parallel rulings, and radii of 2 and 5, respectively

53) [T] Use a CAS or CalcPlot3D to graph in cylindrical coordinates the region between elliptic paraboloid $z = x^2 + y^2$ and cone $x^2 + y^2 - z^2 = 0$.

Answer

The region is described by the set of points $\{(r, \theta, z) \mid 0 \le r \le 1,\ 0 \le \theta \le 2\pi,\ r^2 \le z \le r\}$.



54) [T] Use a CAS or CalcPlot3D to graph in spherical coordinates the "ice cream-cone region" situated above the xy-plane between sphere $x^2 + y^2 + z^2 = 4$ and elliptical cone $x^2 + y^2 - z^2 = 0$.

55) Washington, DC, is located at 39° N and 77° W (see the following figure). Assume the radius of Earth is 4000 mi. Express the location of Washington, DC, in spherical coordinates.



Answer

$(4000, -77^\circ, 51^\circ)$

https://math.libretexts.org/@go/page/84021

56) San Francisco is located at $37.78^\circ N$ and $122.42^\circ W$. Assume the radius of Earth is 4000mi. Express the location of San Francisco in spherical coordinates.

57) Find the latitude and longitude of Rio de Janeiro if its spherical coordinates are $(4000, -43.17^\circ, 102.91^\circ)$.

Answer

$43.17^\circ W, 22.91^\circ S$

58) Find the latitude and longitude of Berlin if its spherical coordinates are $(4000, 13.38^\circ, 37.48^\circ)$.

59) [T] Consider the torus of equation $\left(x^2+y^2+z^2+R^2-r^2\right)^2 = 4R^2(x^2+y^2), \quad$ where $R \geq r > 0$.

a. Write the equation of the torus in spherical coordinates.

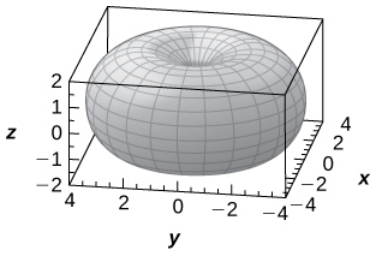
b. If $R = r$, the surface is called a horn torus. Show that the equation of a horn torus in spherical coordinates is $\rho = 2R\sin\varphi$.

c. Use a CAS or CalcPlot3D to graph the horn torus with $R = r = 2$ in spherical coordinates.

Answer

a. $\rho = 0, \quad \rho + R^2 - r^2 - 2R\sin\varphi = 0$

c.



60) [T] The "bumpy sphere" with an equation in spherical coordinates is $\rho = a + b\cos(m\theta)\sin(n\varphi)$, with $\theta \in [0, 2\pi]$ and $\varphi \in [0, \pi]$, where a and b are positive numbers and m and n are positive integers, may be used in applied mathematics to model tumor growth.

a. Show that the "bumpy sphere" is contained inside a sphere of equation $\rho = a + b$. Find the values of θ and φ at which the two surfaces intersect.

b. Use a CAS or CalcPlot3D to graph the surface for $a = 14, b = 2, m = 4,$ and $n = 6$ along with sphere $\rho = a + b$.

c. Find the equation of the intersection curve of the surface at b. with the cone $\varphi = \frac{\pi}{12}$. Graph the intersection curve in the plane of intersection.

Contributors

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Links to CalcPlot3D added by Paul Seeburger (Monroe Community College).

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4.9: Chapter 4 Review Exercises

For exercises 1 - 4, determine whether the statement is *true* or *false*. Justify the answer with a proof or a counterexample.

1) For vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ and any given scalar c, $c(\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}) = (c\vec{\mathbf{a}}) \cdot \vec{\mathbf{b}}$.

Answer

True; See proof in Section 11.3

2) For vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ and any given scalar c, $c(\vec{\mathbf{a}} \times \vec{\mathbf{b}}) = (c\vec{\mathbf{a}}) \times \vec{\mathbf{b}}$.

3) The symmetric equation for the line of intersection between two planes $x+y+z=2$ and $x+2y-4z=5$ is given by $-\frac{x-1}{6} = \frac{y-1}{5} = z$.

Answer

False, Converting the symmetric equations above to the parametric equations of the line, we get:
$x = 1-6t$
$y = 1+5t$
$z = t$

If this line lies on each plane, we should get an identity (like 5 = 5) when we substitute each expression of t in the each plane's equation.

Substituting into the first plane equation, we get: $(1-6t)+(1+5t)+t = 2$ ✓

Thus we know that this line does lie on the first plane.

But when we substitute into the second plane equation, we obtain:
$(1-6t)+2(1+5t)-4(t) = 1-6t+2+10t-4t = 3 \neq 5$

Since we do not get an identity, we know this line is **not** on the second plane and can therefore not be the line of intersection of the two planes.

4) If $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = 0$, then $\vec{\mathbf{a}}$ is perpendicular to $\vec{\mathbf{b}}$.

Answer

False, since $\vec{\mathbf{a}}$ or $\vec{\mathbf{b}}$ could also be the zero vector.

For exercises 5 and 6, use the given vectors to find the quantities.

5) $\vec{\mathbf{a}} = 9\hat{\mathbf{i}} - 2\hat{\mathbf{j}}, \quad \vec{\mathbf{b}} = -3\hat{\mathbf{i}} + \hat{\mathbf{k}}$

a. $3\vec{\mathbf{a}} + \vec{\mathbf{b}}$

b. $\|\vec{\mathbf{a}}\|$

c. $\vec{\mathbf{a}} \times \|\vec{\mathbf{b}} \times \vec{\mathbf{a}}\|$

d. $\|\vec{\mathbf{b}} \times \vec{\mathbf{a}}\|$

Answer

a. $\langle 24, -6, 1\rangle$
b. $\sqrt{85}$
c. Can't cross a vector with a scalar
d. 11

6) $\vec{\mathbf{a}} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} - 9\hat{\mathbf{k}}, \quad \vec{\mathbf{b}} = -\hat{\mathbf{i}} + 2\hat{\mathbf{k}}, \quad \vec{\mathbf{c}} = 4\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$

a. $2\vec{\mathbf{a}} - \vec{\mathbf{b}}$

b. $\|\vec{\mathbf{b}} \times \vec{\mathbf{c}}\|$

c. $\vec{\mathbf{b}} \times \left(\vec{\mathbf{b}} \times \vec{\mathbf{c}}\right)$

d. $\vec{\mathbf{c}} \times \|\vec{\mathbf{b}} \times \vec{\mathbf{a}}\|$

e. $\text{Proj}_{\vec{\mathbf{a}}} \vec{\mathbf{b}}$

7) Find the values of a such that vectors $\langle 2, 4, a \rangle$ and $\langle 0, -1, a \rangle$ are orthogonal.

Answer

$a = \pm 2$

For exercises 8 and 9, find the unit vectors.

8) Find the unit vector that has the same direction as vector $\vec{\mathbf{v}}$ that begins at $(0, -3)$ and ends at $(4, 10)$.

9) Find the unit vector that has the same direction as vector $\vec{\mathbf{v}}$ that begins at $(1, 4, 10)$ and ends at $(3, 0, 4)$.

Answer

$\langle \frac{1}{\sqrt{14}}, -\frac{2}{\sqrt{14}}, -\frac{3}{\sqrt{14}} \rangle = \langle \frac{\sqrt{14}}{14}, -\frac{\sqrt{14}}{7}, -\frac{3\sqrt{14}}{14} \rangle$

For exercises 10 and 11, find the area or volume of the given shapes.

10) The parallelogram spanned by vectors $\vec{\mathbf{a}} = \langle 1, 13 \rangle$ and $\vec{\mathbf{b}} = \langle 3, 21 \rangle$

11) The parallelepiped formed by $\vec{\mathbf{a}} = \langle 1, 4, 1 \rangle$ and $\vec{\mathbf{b}} = \langle 3, 6, 2 \rangle$, and $\vec{\mathbf{c}} = \langle -2, 1, -5 \rangle$

Answer

27 units^2

For exercises 12 and 13, find the parametric equations and the vector equation of the line with the given properties.

12) The line that passes through point $(2, -3, 7)$ that is parallel to vector $\langle 1, 3, -2 \rangle$

13) The line that passes through points $(1, 3, 5)$ and $(-2, 6, -3)$

Answer

$x = 1 - 3t, y = 3 + 3t, z = 5 - 8t, \quad \vec{\mathbf{r}}(t) = (1 - 3t)\hat{\mathbf{i}} + 3(1 + t)\hat{\mathbf{j}} + (5 - 8t)\hat{\mathbf{k}}$

For exercises 14 and 15, find the equation of the plane with the given properties.

14) The plane that passes through point $(4, 7, -1)$ and has normal vector $\vec{\mathbf{n}} = \langle 3, 4, 2 \rangle$

15) The plane that passes through points $(0, 1, 5)$, $(2, -1, 6)$,and $(3, 2, 5)$.

Answer

$-x + 3y + 8z = 43$

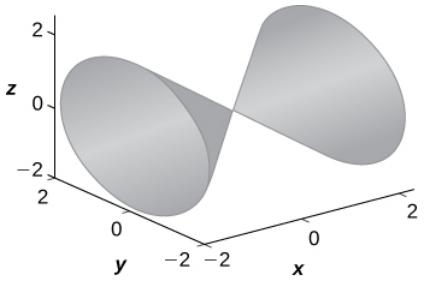
For exercises 16 and 17, find the traces for the surfaces in planes $x = k$, $y = k$, and $z = k$. Then, describe and draw the surfaces.

16) $9x^2 + 4y^2 - 16y + 36z^2 = 20$

17) $x^2 = y^2 + z^2$

Answer

$x = k$ trace: $k^2 = y^2 + z^2$ is a circle, $y = k$ trace: $x^2 - z^2 = k^2$ is a hyperbola (or a pair of lines if $k = 0$), $z = k$ trace: $x^2 - y^2 = k^2$ is a hyperbola (or a pair of lines if $k = 0$). The surface is a cone.



For exercises 18 and 19, write the given equation in cylindrical coordinates and spherical coordinates.

18) $x^2+y^2+z^2=144$

19) $z=x^2+y^2-1$

Answer

Cylindrical: $z=r^2-1$, spherical: $\cos\varphi=\rho\sin^2\varphi-\frac{1}{\rho}$

For exercises 20 and 21, convert the given equations from cylindrical or spherical coordinates to rectangular coordinates. Identify the given surface.

20) $\rho^2(\sin^2(\varphi)-\cos^2(\varphi))=1$

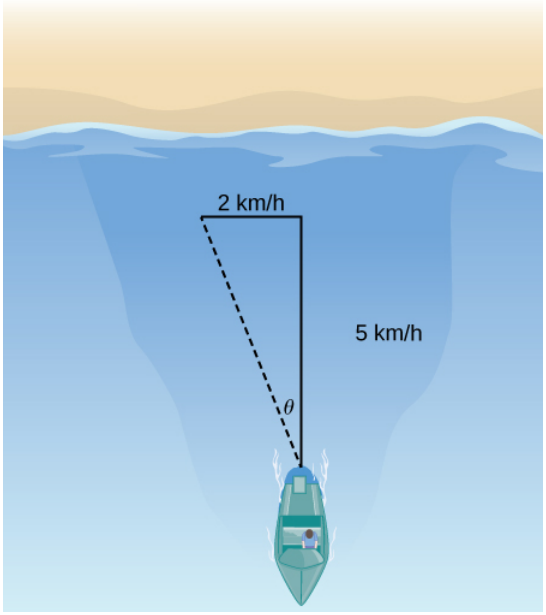
21) $r^2-2r\cos(\theta)+z^2=1$

Answer

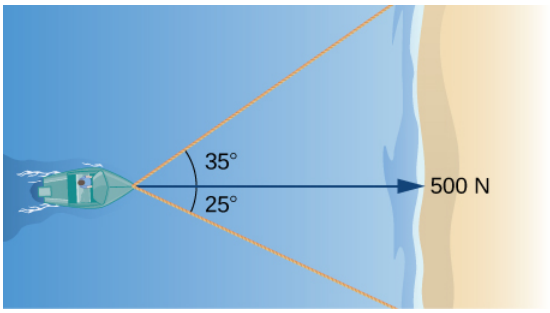
$x^2-2x+y^2+z^2=1$, sphere

For exercises 22 and 23, consider a small boat crossing a river.

22) If the boat velocity is 5 km/h due north in still water and the water has a current of 2 km/h due west (see the following figure), what is the velocity of the boat relative to shore? What is the angle θ that the boat is actually traveling?



23) When the boat reaches the shore, two ropes are thrown to people to help pull the boat ashore. One rope is at an angle of 25° and the other is at 35°. If the boat must be pulled straight and at a force of 500 N, find the magnitude of force for each rope (see the following figure).



Answer

331 N and 244 N

24) An airplane is flying in the direction of 52° east of north with a speed of 450 mph. A strong wind has a bearing 33° east of north with a speed of 50 mph. What is the resultant ground speed and bearing of the airplane?

25) Calculate the work done by moving a particle from position $(1, 2, 0)$ to $(8, 4, 5)$ along a straight line with a force $\vec{\mathbf{F}} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}}$.

Answer

15 J

In problems 26 and 27, consider your unsuccessful attempt to take the tire off your car using a wrench to loosen the bolts. Assume the wrench is 0.3 m long and you are able to apply a 200-N force.

26) Because your tire is flat, you are only able to apply your force at a 60° angle. What is the torque at the center of the bolt? Assume this force is not enough to loosen the bolt.

27) Someone lends you a tire jack and you are now able to apply a 200-N force at an 80° angle. Is your resulting torque going to be more or less? What is the new resulting torque at the center of the bolt? Assume this force is not enough to loosen the bolt.

Answer

More, 59.09J

Contributors

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CHAPTER OVERVIEW

5: Differentiation of Functions of Several Variables

When dealing with a function of more than one independent variable, several questions naturally arise. For example, how do we calculate limits of functions of more than one variable? The definition of derivative we used before involved a limit. Does the new definition of derivative involve limits as well? Do the rules of differentiation apply in this context? Can we find relative extrema of functions using derivatives? All these questions are answered in this chapter.

5.1: Prelude to Differentiation of Functions of Several Variables
5.2: Functions of Several Variables
5.2E: Exercises for Section 5.2
5.3: Limits and Continuity
5.3E: Exercises for Section 5.3
5.4: Partial Derivatives
5.4E: Exercises for Section 5.4
5.5: Tangent Planes and Linear Approximations
5.5E: Exercises for Section 5.5
5.6: The Chain Rule for Multivariable Functions
5.6E: Exercises for Section 5.6
5.7: Directional Derivatives and the Gradient
5.7E: Exercises for Section 5.7
5.8: Maxima/Minima Problems
5.8E: Exercises for Section 5.8
5.9: Lagrange Multipliers
5.9E: Exercises for Section 5.9
5.10: Chapter 5 Review Exercises

Thumbnail: Real function of two real variables. (Public Domain; Maschen).

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5.1: Prelude to Differentiation of Functions of Several Variables

In Introduction to Applications of Derivatives, we studied how to determine the maximum and minimum of a function of one variable over a closed interval. This function might represent the temperature over a given time interval, the position of a car as a function of time, or the altitude of a jet plane as it travels from New York to San Francisco. In each of these examples, the function has one independent variable.



Figure 5.1.1: Americans use (and lose) millions of golf balls a year, which keeps golf ball manufacturers in business. In this chapter, we study a profit model and learn methods for calculating optimal production levels for a typical golf ball manufacturing company. (credit: modification of work by oatsy40, Flickr)

Suppose, however, that we have a quantity that depends on more than one variable. For example, temperature can depend on location and the time of day, or a company's profit model might depend on the number of units sold and the amount of money spent on advertising. In this chapter, we look at a company that produces golf balls. We develop a profit model and, under various restrictions, we find that the optimal level of production and advertising dollars spent determines the maximum possible profit. Depending on the nature of the restrictions, both the method of solution and the solution itself changes.

When dealing with a function of more than one independent variable, several questions naturally arise. For example, how do we calculate limits of functions of more than one variable? The definition of derivative we used before involved a limit. Does the new definition of derivative involve limits as well? Do the rules of differentiation apply in this context? Can we find relative extrema of functions using derivatives? All these questions are answered in this chapter.

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5.2: Functions of Several Variables

Learning Objectives

- Recognize a function of two variables and identify its domain and range.
- Sketch a graph of a function of two variables.
- Sketch several traces or level curves of a function of two variables.
- Recognize a function of three or more variables and identify its level surfaces.

Our first step is to explain what a function of more than one variable is, starting with functions of two independent variables. This step includes identifying the domain and range of such functions and learning how to graph them. We also examine ways to relate the graphs of functions in three dimensions to graphs of more familiar planar functions.

Functions of Two Variables

The definition of a function of two variables is very similar to the definition for a function of one variable. The main difference is that, instead of mapping values of one variable to values of another variable, we map ordered pairs of variables to another variable.

Definition: function of two variables

A *function of two variables* $z = (x, y)$ maps each ordered pair (x, y) in a subset D of the real plane R^2 to a unique real number z. The set D is called the domain of the function. The range of f is the set of all real numbers z that has at least one ordered pair $(x, y) \in D$ such that $f(x, y) = z$ as shown in Figure 5.2.1.

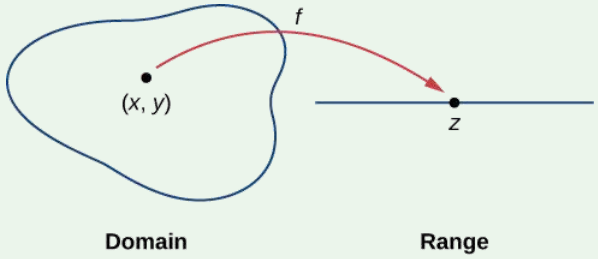


Figure 5.2.1: The domain of a function of two variables consists of ordered pairs (x, y).

Determining the domain of a function of two variables involves taking into account any domain restrictions that may exist. Let's take a look.

Example 5.2.1: Domains and Ranges for Functions of Two Variables

Find the domain and range of each of the following functions:

a. $f(x, y) = 3x + 5y + 2$
b. $g(x, y) = \sqrt{9 - x^2 - y^2}$

Solution

a. This is an example of a linear function in two variables. There are no values or combinations of x and y that cause $f(x, y)$ to be undefined, so the domain of f is R^2. To determine the range, first pick a value for z. We need to find a solution to the equation $f(x, y) = z$, or $3x - 5y + 2 = z$. One such solution can be obtained by first setting $y = 0$, which yields the equation $3x + 2 = z$. The solution to this equation is $x = \dfrac{z-2}{3}$, which gives the ordered pair $\left(\dfrac{z-2}{3}, 0\right)$ as a solution to the equation $f(x, y) = z$ for any value of z. Therefore, the range of the function is all real numbers, or R.

b. For the function $g(x, y)$ to have a real value, the quantity under the square root must be nonnegative:

$$9 - x^2 - y^2 \geq 0.$$

This inequality can be written in the form

$$x^2 + y^2 \leq 9.$$

Therefore, the domain of $g(x,y)$ is $\{(x,y) \in R^2 \mid x^2+y^2 \leq 9\}$. The graph of this set of points can be described as a disk of radius 3 centered at the origin. The domain includes the boundary circle as shown in the following graph.

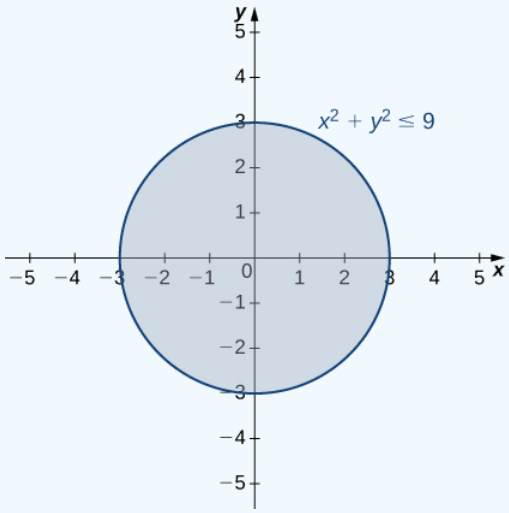


Figure 5.2.2: The domain of the function $g(x,y) = \sqrt{9-x^2-y^2}$ is a closed disk of radius 3.

To determine the range of $g(x,y) = \sqrt{9-x^2-y^2}$ we start with a point (x_0, y_0) on the boundary of the domain, which is defined by the relation $x^2+y^2=9$. It follows that $x_0^2+y_0^2=9$ and

$$\begin{aligned} g(x_0,y_0) &= \sqrt{9-x_0^2-y_0^2} \\ &= \sqrt{9-(x_0^2+y_0^2)} \\ &= \sqrt{9-9} \\ &= 0. \end{aligned}$$

If $x_0^2+y_0^2=0$ (in other words, $x_0=y_0=0$) , then

$$\begin{aligned} g(x_0,y_0) &= \sqrt{9-x_0^2-y_0^2} \\ &= \sqrt{9-(x_0^2+y_0^2)} \\ &= \sqrt{9-0} = 3. \end{aligned}$$

This is the maximum value of the function. Given any value c between 0 and 3, we can find an entire set of points inside the domain of g such that $g(x,y)=c$:

$$\begin{aligned} \sqrt{9-x^2-y^2} &= c \\ 9-x^2-y^2 &= c^2 \\ x^2+y^2 &= 9-c^2. \end{aligned}$$

Since $9-c^2>0$, this describes a circle of radius $\sqrt{9-c^2}$ centered at the origin. Any point on this circle satisfies the equation $g(x,y)=c$. Therefore, the range of this function can be written in interval notation as $[0,3]$.



Exercise 5.2.1

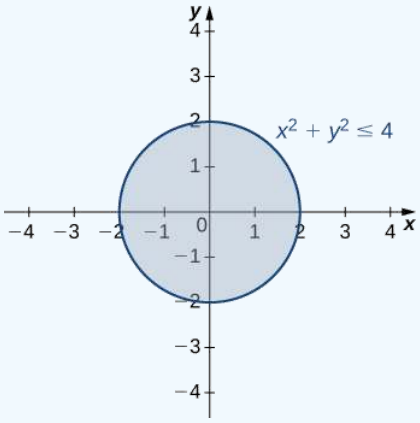
Find the domain and range of the function $f(x,y)=\sqrt{36-9x^2-9y^2}$.

Hint

Determine the set of ordered pairs that do not make the radicand negative.

Solution

The domain is $\{(x,y)|x^2+y^2\leq 4\}$ the shaded circle defined by the inequality $x^2+y^2\leq 4$, which has a circle of radius 2 as its boundary. The range is $[0,6]$.



Graphing Functions of Two Variables

Suppose we wish to graph the function $z=(x,y)$. This function has two independent variables (x and y) and one dependent variable (z). When graphing a function $y=f(x)$ of one variable, we use the Cartesian plane. We are able to graph any ordered pair (x,y) in the plane, and every point in the plane has an ordered pair (x,y) associated with it. With a function of two variables, each ordered pair (x,y) in the domain of the function is mapped to a real number z. Therefore, the graph of the function f consists of ordered triples (x,y,z). The graph of a function $z=(x,y)$ of two variables is called a surface.

To understand more completely the concept of plotting a set of ordered triples to obtain a surface in three-dimensional space, imagine the (x,y) coordinate system laying flat. Then, every point in the domain of the function f has a unique z-value associated

with it. If z is positive, then the graphed point is located above the xy-plane, if z is negative, then the graphed point is located below the xy-plane. The set of all the graphed points becomes the two-dimensional surface that is the graph of the function f.

Example 5.2.2: Graphing Functions of Two Variables

Create a graph of each of the following functions:

a. $g(x,y)=\sqrt{9-x^2-y^2}$
b. $f(x,y)=x^2+y^2$

Solution

a. In Example 5.2.2, we determined that the domain of $g(x,y)=\sqrt{9-x^2-y^2}$ is $\{(x,y)\in R^2 \mid x^2+y^2\leq 9\}$ and the range is $\{z\in R^2 \mid 0\leq z\leq 3\}$. When $x^2+y^2=9$ we have $g(x,y)=0$. Therefore any point on the circle of radius 3 centered at the origin in the xy-plane maps to $z=0$ in R^3. If $x^2+y^2=8$, then $g(x,y)=1$, so any point on the circle of radius $2\sqrt{2}$ centered at the origin in the xy-plane maps to $z=1$ in R^3. As x^2+y^2 gets closer to zero, the value of z approaches 3. When $x^2+y^2=0$, then $g(x,y)=3$. This is the origin in the xy-plane If x^2+y^2 is equal to any other value between 0 and 9, then $g(x,y)$ equals some other constant between 0 and 3. The surface described by this function is a hemisphere centered at the origin with radius 3 as shown in the following graph.

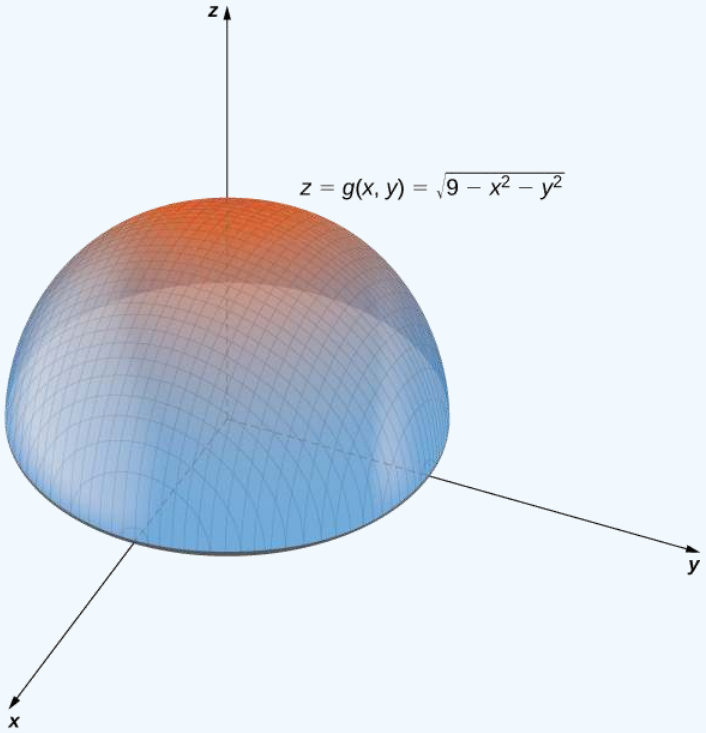


Figure 5.2.3: Graph of the hemisphere represented by the given function of two variables.

b. This function also contains the expression x^2+y^2. Setting this expression equal to various values starting at zero, we obtain circles of increasing radius. The minimum value of $f(x,y)=x^2+y^2$ is zero (attained when $x=y=0$.. When $x=0$, the function becomes $z=y^2$, and when $y=0$, then the function becomes $z=x^2$. These are cross-sections of the graph, and are parabolas. Recall from Introduction to Vectors in Space that the name of the graph of $f(x,y)=x^2+y^2$ is a **paraboloid**. The graph of f appears in the following graph.

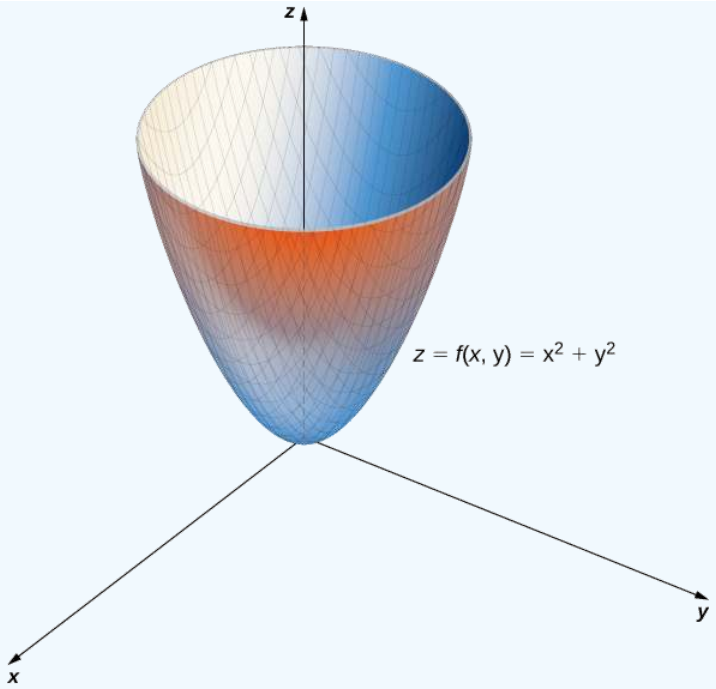


Figure 5.2.4: A paraboloid is the graph of the given function of two variables.

Example 5.2.3: Nuts and Bolts

A profit function for a hardware manufacturer is given by

$$f(x,y) = 16 - (x-3)^2 - (y-2)^2,$$

where x is the number of nuts sold per month (measured in thousands) and y represents the number of bolts sold per month (measured in thousands). Profit is measured in thousands of dollars. Sketch a graph of this function.

Solution

This function is a polynomial function in two variables. The domain of f consists of (x,y) coordinate pairs that yield a nonnegative profit:

$$16 - (x-3)^2 - (y-2)^2 \geq 0$$
$$(x-3)^2 + (y-2)^2 \leq 16.$$

This is a disk of radius 4 centered at $(3,2)$. A further restriction is that both x and y must be nonnegative. When $x = 3$ and $y = 2$, $f(x,y) = 16$. Note that it is possible for either value to be a noninteger; for example, it is possible to sell 2.5 thousand nuts in a month. The domain, therefore, contains thousands of points, so we can consider all points within the disk. For any $z < 16$, we can solve the equation $f(x,y) = 16$:

$$16 - (x-3)^2 - (y-2)^2 = z$$
$$(x-3)^2 + (y-2)^2 = 16 - z.$$

Since $z < 16,$ we know that $16 - z > 0,$ so the previous equation describes a circle with radius $\sqrt{16-z}$ centered at the point $(3,2)$. Therefore. the range of $f(x,y)$ is $\{z \in \mathbb{R} | z \leq 16\}$. The graph of $f(x,y)$ is also a paraboloid, and this paraboloid points downward as shown.

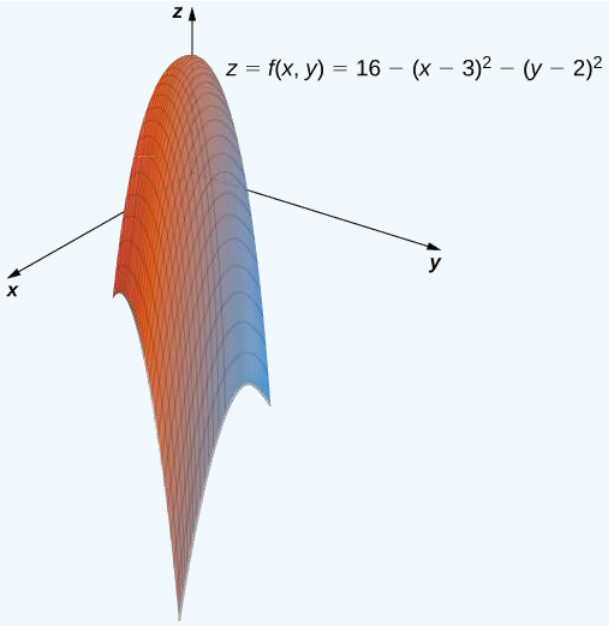
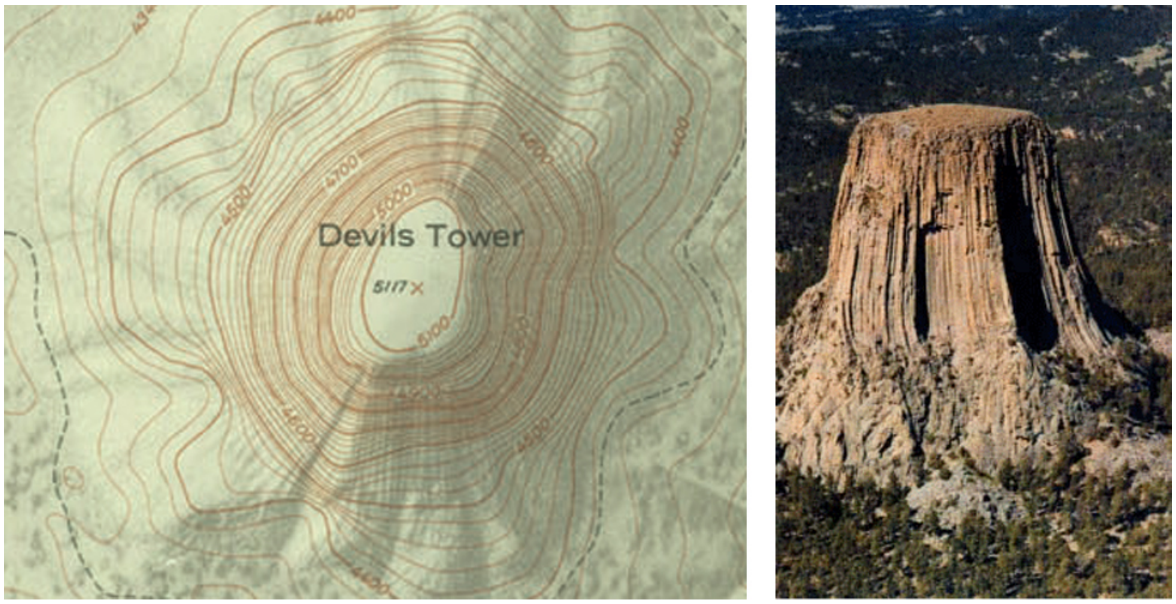


Figure 5.2.5: The graph of the given function of two variables is also a paraboloid.



Level Curves

If hikers walk along rugged trails, they might use a topographical map that shows how steeply the trails change. A *topographical map* contains curved lines called contour lines. Each contour line corresponds to the points on the map that have equal elevation (Figure 5.2.6). A level curve of a function of two variables $f(x, y)$ is completely analogous to a contour line on a topographical map.



(a) (b)

Figure 5.2.6: (a) A topographical map of Devil's Tower, Wyoming. Lines that are close together indicate very steep terrain. (b) A perspective photo of Devil's Tower shows just how steep its sides are. Notice the top of the tower has the same shape as the center of the topographical map.

Definition: level curves

Given a function $f(x,y)$ and a number c in the range of f, a level curve of a function of two variables for the value c is defined to be the set of points satisfying the equation $f(x,y)=c$.

Returning to the function $g(x,y)=\sqrt{9-x^2-y^2}$, we can determine the level curves of this function. The range of g is the closed interval $[0,3]$. First, we choose any number in this closed interval—say, $c=2$. The level curve corresponding to $c=2$ is described by the equation

$$\sqrt{9-x^2-y^2}=2. \tag{5.2.1}$$

To simplify, square both sides of this equation:

$$9-x^2-y^2=4. \tag{5.2.2}$$

Now, multiply both sides of the equation by -1 and add 9to each side:

$$x^2+y^2=5. \tag{5.2.3}$$

This equation describes a circle centered at the origin with radius $\sqrt{5}$. Using values of c between 0 and 3 yields other circles also centered at the origin. If $c=3$, then the circle has radius 0, so it consists solely of the origin. Figure 5.2.7 is a graph of the level curves of this function corresponding to $c=0,1,2$, and 3. Note that in the previous derivation it may be possible that we introduced extra solutions by squaring both sides. This is not the case here because the range of the square root function is nonnegative.

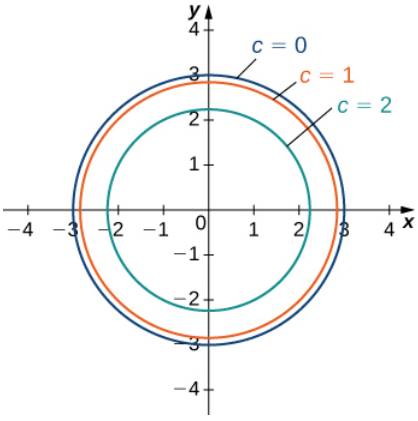


Figure 5.2.7: Level curves of the function $g(x,y)=\sqrt{9-x^2-y^2}$, using $c=0,1,2,$ and $3 (c=3$ corresponds to the origin).

A graph of the various level curves of a function is called a *contour map*.

Example 5.2.4: Making a Contour Map

Given the function $f(x,y)=\sqrt{8+8x-4y-4x^2-y^2}$, find the level curve corresponding to $c=0$. Then create a contour map for this function. What are the domain and range of f?

Solution

To find the level curve for $c=0$, we set $f(x,y)=0$ and solve. This gives

$$0=\sqrt{8+8x-4y-4x^2-y^2}.$$

We then square both sides and multiply both sides of the equation by -1:

$$4x^2+y^2-8x+4y-8=0.$$

Now, we rearrange the terms, putting the x terms together and the y terms together, and add 8 to each side:

$$4x^2-8x+y^2+4y=8.$$

Next, we group the pairs of terms containing the same variable in parentheses, and factor 4 from the first pair:

$$4(x^2-2x)+(y^2+4y)=8.$$

Then we complete the square in each pair of parentheses and add the correct value to the right-hand side:

$$4(x^2-2x+1)+(y^2+4y+4)=8+4(1)+4.$$

Next, we factor the left-hand side and simplify the right-hand side:

$$4(x-1)^2+(y+2)^2=16.$$

Last, we divide both sides by 16 :

$$\frac{(x-1)^2}{4}+\frac{(y+2)^2}{16}=1.$$

This equation describes an ellipse centered at $(1,-2)$. The graph of this ellipse appears in the following graph.

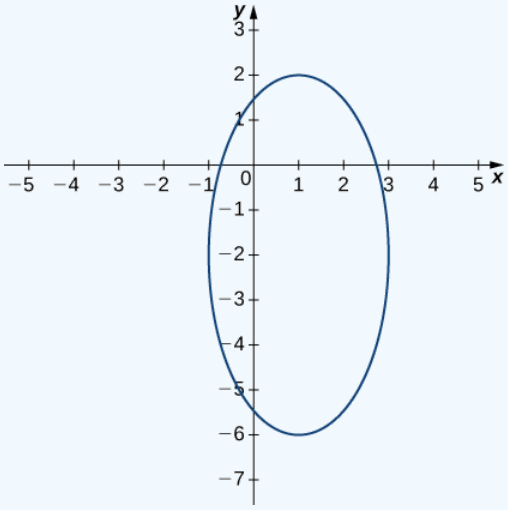


Figure 5.2.8: Level curve of the function $f(x,y)=\sqrt{8+8x-4y-4x^2-y^2}$ corresponding to $c=0$

We can repeat the same derivation for values of c less than 4. Then, Equation becomes

$$\frac{4(x-1)^2}{16-c^2}+\frac{(y+2)^2}{16-c^2}=1$$

for an arbitrary value of c. Figure $5.2.9$ shows a contour map for $f(x,y)$ using the values $c=0,1,2$, and 3. When $c=4$, the level curve is the point $(-1,2)$.

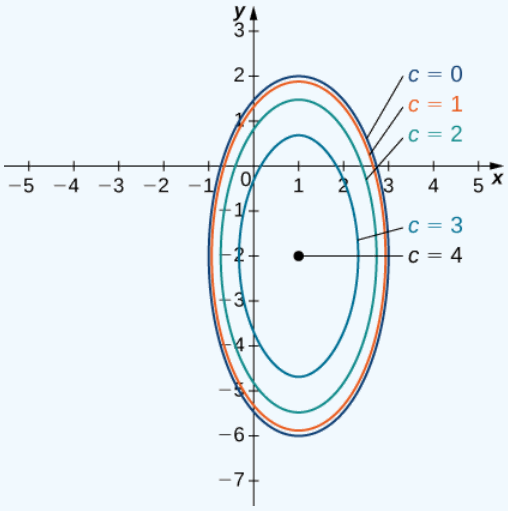


Figure 5.2.9: Contour map for the function $f(x,y)=\sqrt{8+8x-4y-4x^2-y^2}$ using the values $c=0,1,2,3,$ and 4.

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Exercise 5.2.4

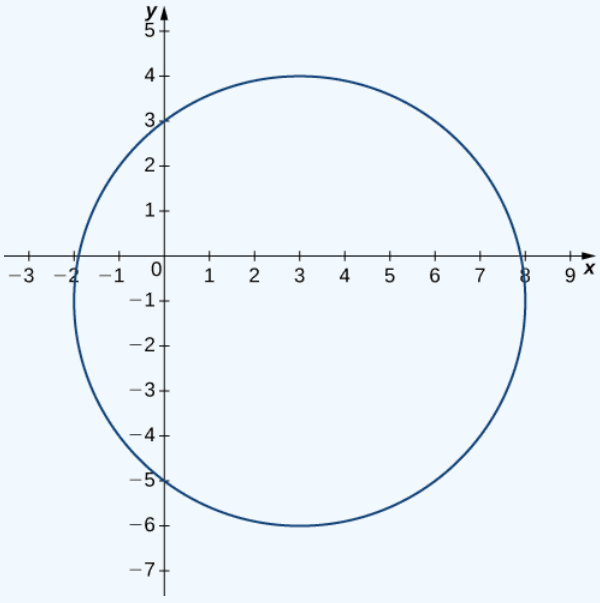
Find and graph the level curve of the function $g(x,y)=x^2+y^2-6x+2y$ corresponding to $c=15$.

Hint

First, set $g(x,y)=15$ and then complete the square.

Solution

The equation of the level curve can be written as $(x-3)^2+(y+1)^2=25,$ which is a circle with radius 5 centered at $(3,-1)$.



Another useful tool for understanding the **graph of a function of two variables** is called a vertical trace. Level curves are always graphed in the $xy-plane$, but as their name implies, vertical traces are graphed in the $xz-$ or $yz-$ planes.

Definition: vertical traces

Consider a function $z = f(x, y)$ with domain $D \subseteq \mathbb{R}^2$. A *vertical trace* of the function can be either the set of points that solves the equation $f(a, y) = z$ for a given constant $x = a$ or $f(x, b) = z$ for a given constant $y = b$.

Example 5.2.5: Finding Vertical Traces

Find vertical traces for the function $f(x, y) = \sin x \cos y$ corresponding to $x = -\frac{\pi}{4}, 0$, and $\frac{\pi}{4}$, and $y = -\frac{\pi}{4}, 0$, and $\frac{\pi}{4}$.

Solution

First set $x = -\frac{\pi}{4}$ in the equation $z = \sin x \cos y$:

$$z = \sin(-\frac{\pi}{4}) \cos y = -\frac{\sqrt{2} \cos y}{2} \approx -0.7071 \cos y.$$

This describes a cosine graph in the plane $x = -\frac{\pi}{4}$. The other values of z appear in the following table.

Vertical Traces Parallel to the $xz - Plane$ for the Function $f(x, y) = \sin x \cos y$

c	Vertical Trace for $x = c$
$-\frac{\pi}{4}$	$z = -\frac{\sqrt{2} \cos y}{2}$
0	$z = 0$
$\frac{\pi}{4}$	$z = \frac{\sqrt{2} \cos y}{2}$

In a similar fashion, we can substitute the $y - values$ in the equation $f(x, y)$ to obtain the traces in the $yz - plane$, as listed in the following table.

Vertical Traces Parallel to the $yz - Plane$ for the Function $f(x, y) = \sin x \cos y$

d	Vertical Trace for $y = d$
$\frac{\pi}{4}$	$z = \frac{\sqrt{2} \sin x}{2}$
0	$z = \sin x$
$-\frac{\pi}{4}$	$z = \frac{\sqrt{2} \sin x}{2}$

The three traces in the $xz - plane$ are cosine functions; the three traces in the $yz - plane$ are sine functions. These curves appear in the intersections of the surface with the planes $x = -\frac{\pi}{4}, x = 0, x = \frac{\pi}{4}$ and $y = -\frac{\pi}{4}, y = 0, y = \frac{\pi}{4}$ as shown in the following figure.

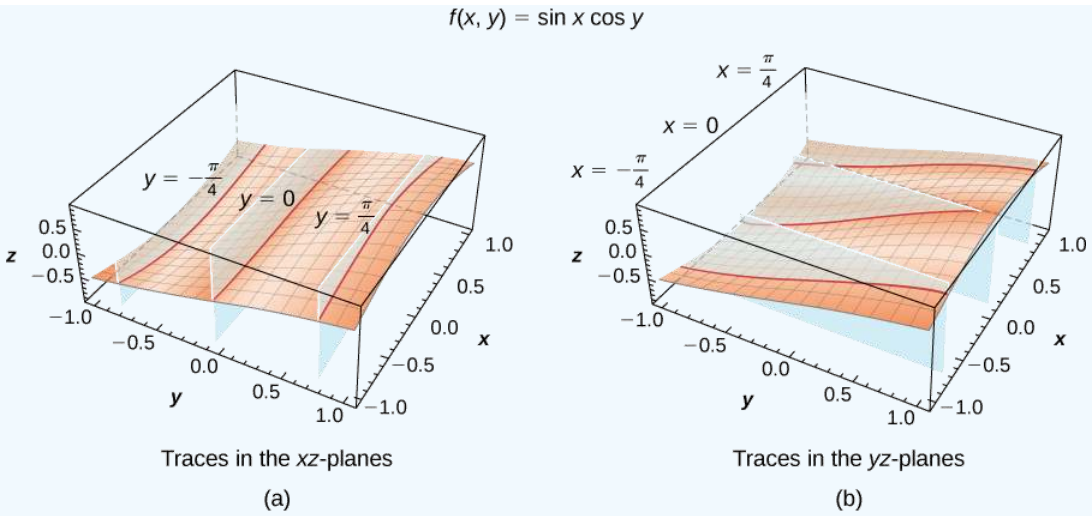


Figure 5.2.10: Vertical traces of the function $f(x,y)$ are cosine curves in the $xz-planes$ (a) and sine curves in the $yz-planes$ (b).

Exercise 5.2.5

Determine the equation of the vertical trace of the function $g(x,y)=-x^2-y^2+2x+4y-1$ corresponding to $y=3$, and describe its graph.

Hint

Set $y=3$ in the equation $z=-x^2-y^2+2x+4y-1$ and complete the square.

Solution

$z=3-(x-1)^2$. This function describes a parabola opening downward in the plane $y=3$.

Functions of two variables can produce some striking-looking surfaces. Figure 5.2.11 shows two examples.

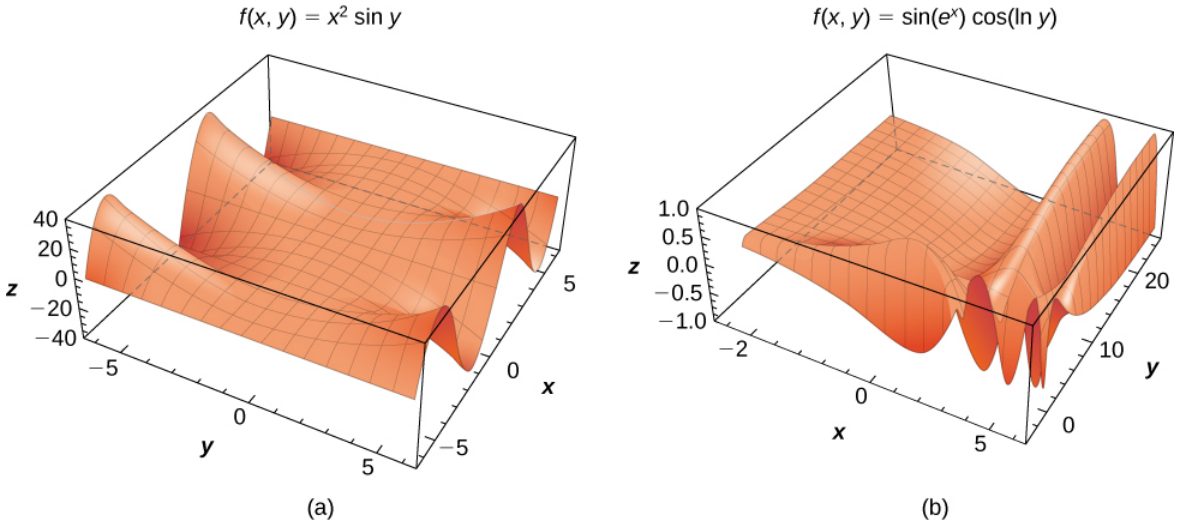


Figure 5.2.11: Examples of surfaces representing functions of two variables: (a) a combination of a power function and a sine function and (b) a combination of trigonometric, exponential, and logarithmic functions.

Functions of More Than Two Variables

So far, we have examined only functions of two variables. However, it is useful to take a brief look at functions of more than two variables. Two such examples are

$$\underbrace{f(x,y,z)=x^2-2xy+y^2+3yz-z^2+4x-2y+3x-6}_{\text{a polynomial in three variables}} \tag{5.2.4}$$

and

$$g(x,y,t)=(x^2-4xy+y^2)\sin t-(3x+5y)\cos t. \tag{5.2.5}$$

In the first function, (x,y,z) represents a point in space, and the function f maps each point in space to a fourth quantity, such as temperature or wind speed. In the second function, (x,y) can represent a point in the plane, and t can represent time. The function might map a point in the plane to a third quantity (for example, pressure) at a given time t. The method for finding the domain of a function of more than two variables is analogous to the method for functions of one or two variables.

Example 5.2.6: Domains for Functions of Three Variables

Find the domain of each of the following functions:

a. $f(x,y,z)=\dfrac{3x-4y+2z}{\sqrt{9-x^2-y^2-z^2}}$

b. $g(x,y,t)=\dfrac{\sqrt{2t-4}}{x^2-y^2}$

Solution:

a. For the function $f(x,y,z)=\dfrac{3x-4y+2z}{\sqrt{9-x^2-y^2-z^2}}$ to be defined (and be a real value), two conditions must hold:

1. The denominator **cannot** be zero.
2. The radicand **cannot** be negative.

Combining these conditions leads to the inequality

$$9-x^2-y^2-z^2>0.$$

Moving the variables to the other side and reversing the inequality gives the domain as

$$domain(f)=\{(x,y,z)\in R^3 \mid x^2+y^2+z^2<9\},$$

which describes a ball of radius 3 centered at the origin. (Note: The surface of the ball is not included in this domain.)

b. For the function $g(x,y,t)=\dfrac{\sqrt{2t-4}}{x^2-y^2}$ to be defined (and be a real value), two conditions must hold:

1. The radicand cannot be negative.
2. The denominator cannot be zero.

Since the radicand cannot be negative, this implies $2t-4\geq 0$, and therefore that $t\geq 2$. Since the denominator cannot be zero, $x^2-y^2\neq 0$, or $x^2\neq y^2$, Which can be rewritten as $y=\pm x$, which are the equations of two lines passing through the origin. Therefore, the domain of g is

$$domain(g)=\{(x,y,t)|y\neq\pm x, t\geq 2\}.$$

Exercise 5.2.6

Find the domain of the function $h(x,y,t)=(3t-6)\sqrt{y-4x^2+4}$.

Hint

Check for values that make radicands negative or denominators equal to zero.

Solution

$$domain(h)=\{(x,y,t)\in\mathbb{R}^3 \mid y\geq 4x^2-4\}$$

Functions of two variables have level curves, which are shown as curves in the $xy-plane$. However, when the function has three variables, the curves become surfaces, so we can define level surfaces for functions of three variables.

Definition: level surface of a function of three variables

Given a function $f(x,y,z)$ and a number c in the range of f, a *level surface of a function of three variables* is defined to be the set of points satisfying the equation $f(x,y,z)=c$.

Example 5.2.7: Finding a Level Surface

Find the level surface for the function $f(x,y,z)=4x^2+9y^2-z^2$ corresponding to $c=1$.

Solution

The level surface is defined by the equation $4x^2+9y^2-z^2=1$. This equation describes a hyperboloid of one sheet as shown in Figure 5.2.12

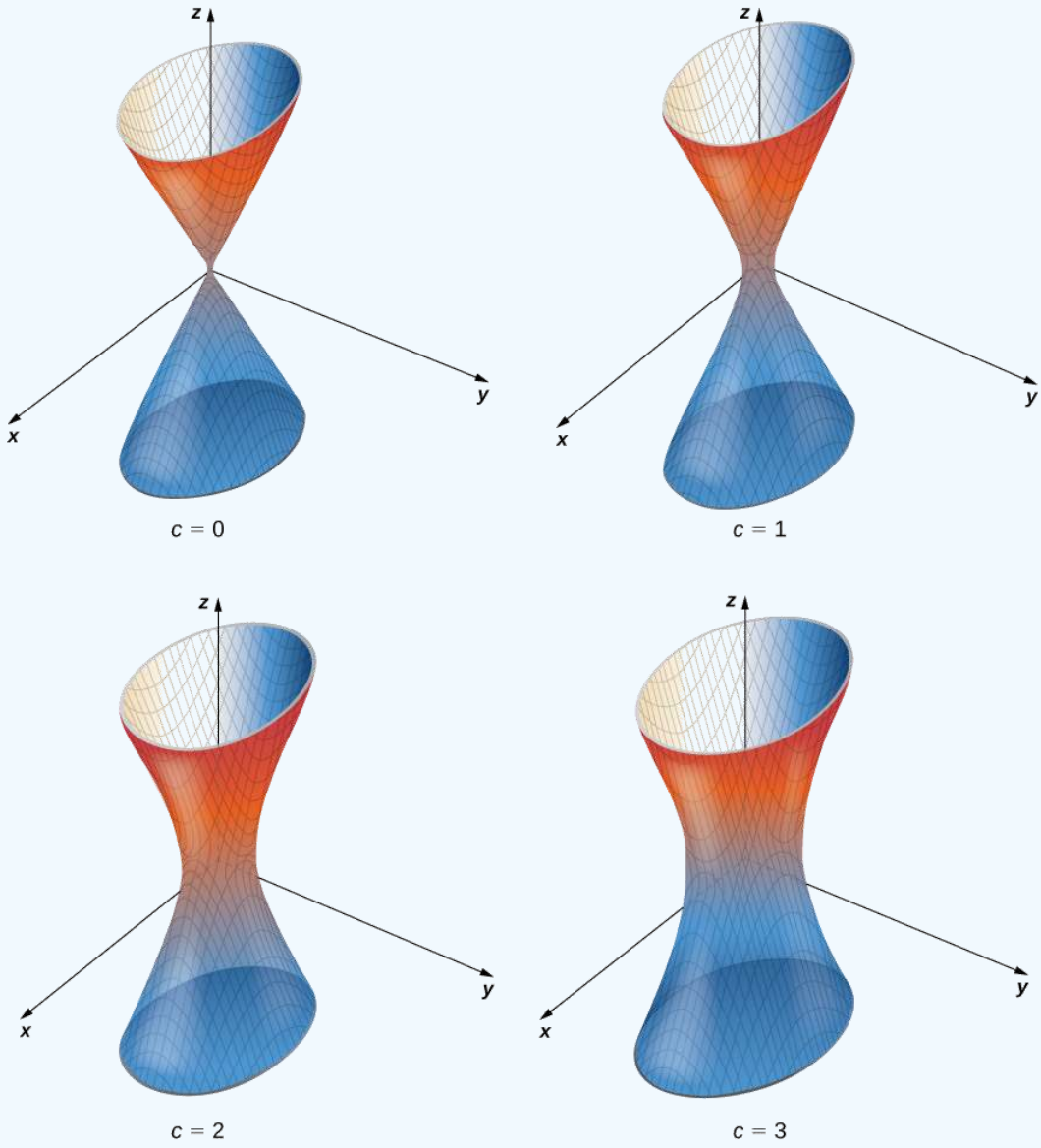


Figure 5.2.12: A hyperboloid of one sheet with some of its level surfaces.

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Exercise 5.2.5

Find the equation of the level surface of the function

$$g(x,y,z)=x^2+y^2+z^2-2x+4y-6z$$

corresponding to $c=2$, and describe the surface, if possible.

Hint

Set $g(x,y,z)=c$ and complete the square.

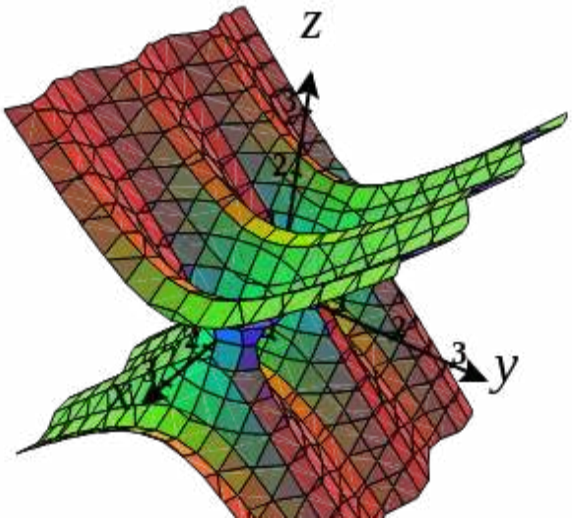
Solution

((x−1)^2+(y+2)^2+(z−3)^2=16\) describes a sphere of radius 4 centered at the point $(1,-2,3)$.

Using CalcPlot3D to Visualize a Level Surface

Often the level surfaces are too complex to graph enough of them, so using a computer program such as CalcPlot3D is the way to go. If you just set the function equal to a, a slider will automatically be created so you can just dynamically observe the level surfaces. Below is the dynamic computer plot of the level surface for $f(x,y,z)=\sin(3x)+y^2-z^2$. You can also explore further by changing the equation.

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Summary

- The graph of a function of two variables is a surface in $\mathbb{R}^3$ and can be studied using level curves and vertical traces.
- A set of level curves is called a contour map.

Key Equations

- **Vertical trace**

  $f(a,y)=z$ for $x=a$ or $f(x,b)=z$ for $y=b$

- **Level surface of a function of three variables**

$f(x, y, z) = c$

Glossary

contour map
a plot of the various level curves of a given function $f(x, y)$

function of two variables
a function $z = f(x, y)$ that maps each ordered pair (x, y) in a subset D of R^2 to a unique real number z

graph of a function of two variables
a set of ordered triples (x, y, z) that satisfies the equation $z = f(x, y)$ plotted in three-dimensional Cartesian space

level curve of a function of two variables
the set of points satisfying the equation $f(x, y) = c$ for some real number c in the range of f

level surface of a function of three variables
the set of points satisfying the equation $f(x, y, z) = c$ for some real number c in the range of f

surface
the graph of a function of two variables, $z = f(x, y)$

vertical trace
the set of ordered triples (c, y, z) that solves the equation $f(c, y) = z$ for a given constant $x = c$ or the set of ordered triples (x, d, z) that solves the equation $f(x, d) = z$ for a given constant $y = d$

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5.2E: Exercises for Section 5.2

For the following exercises, evaluate each function at the indicated values.

1) $W(x,y)=4x^2+y^2$. Find $W(2,-1), W(-3,6)$

Answer

$W(2,-1)=17, \quad W(-3,6)=72$

2) $W(x,y)=4x^2+y^2$. Find $W(2+h,3+h)$.

3) The volume of a right circular cylinder is calculated by a function of two variables, $V(x,y)=\pi x^2 y$, where x is the radius of the right circular cylinder and y represents the height of the cylinder. Evaluate $V(2,5)$ and explain what this means.

Answer

$V(2,5)=20\pi \text{ units}^3$ This is the volume when the radius is 2 and the height is 5.

4) An oxygen tank is constructed of a right cylinder of height y and radius x with two hemispheres of radius x mounted on the top and bottom of the cylinder. Express the volume of the cylinder as a function of two variables, x and y, find $V(10,2)$, and explain what this means.

For exercises 5 - 10, find the domain and range of the given function. State the domain in set-builder notation and the range in interval notation.

5) $V(x,y)=4x^2+y^2$

Answer

Domain: $\{(x,y)\,|\,x\in\mathbb{R}, \text{y}\in\mathbb{R}\}$ That is, all points in the xy-plane
Range: $[0,\infty)$

6) $f(x,y)=\sqrt{x^2+y^2-4}$

Answer

Domain: $\{(x,y)\,|\,x^2+y^2\geq 4\}$
Range: $[0,\infty)$

7) $f(x,y)=4\ln(y^2-x)$

Answer

Domain: $\{(x,y)\,|\,x<y^2\}$
Range: $(-\infty,\infty)$

8) $g(x,y)=\sqrt{16-4x^2-y^2}$

Answer

Domain: $\{(x,y)\,|\,\dfrac{x^2}{4}+\dfrac{y^2}{16}\leq 1\}$
Range: $[0,4]$

9) $z=\arccos(y-x)$

Answer

Domain: $\{(x,y)\,|\,x-1\leq y\leq x+1\}$ That is, all points between the graphs of $y=x-1$ and $y=x+1$.
Range: $[0,\pi]$

10) $f(x,y)=\dfrac{y+2}{x^2}$

Answer

Domain: $\{(x,y)\,|\,x\neq 0\}$
Range: $(-\infty,\infty)$

Find the range of the functions.

11) $g(x,y)=\sqrt{16-4x^2-y^2}$

Answer

$\{z\,|\,0\leq z\leq 4\}$ or in interval notation: $[0,4]$

12) $V(x,y)=4x^2+y^2$

13) $z=y^2-x^2$

Answer

The set $\mathbb{R}$

In exercises 14 - 29, find the level curves of each function at the indicated values of c to visualize the given function. Sketch a contour plot for those exercises where you are asked for more than 3 values of c.

14) $z(x,y)=y^2-x^2,\quad c=1$

15) $z(x,y)=y^2-x^2,\quad c=4$

Answer

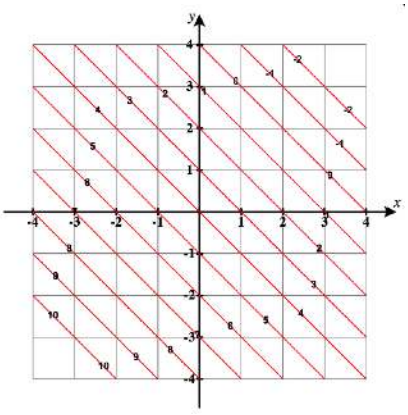
$y^2-x^2=4,$ a hyperbola

16) $g(x,y)=x^2+y^2;\quad c=0,1,2,3,4,9$

17) $g(x,y)=4-x-y;\quad c=0,1,2,3,4$

Answer

Level curves are lines with $y=-x+(4-c)$.
For each value of c these are:
$c=0:\ y=-x+4$,
$c=1:\ y=-x+3$,
$c=2:\ y=-x+2$,
$c=3:\ y=-x+1$,
$c=4:\ y=-x$.
The contour plot consists of a series of parallel lines.



18) $f(x,y)=xy;c=1;\quad c=-1$

19) $h(x,y)=2x-y;\quad c=-2,0,2$

Answer

$2x-y=0, 2x-y=-2, 2x-y=2$; three lines

20) $f(x,y)=x^2-y; \quad c=1,2$

21) $g(x,y)=\dfrac{x}{x+y}; c=-1,0,1,2$

Answer

Level curves are lines with the form $y=x\left(\dfrac{1-c}{c}\right)$. At $c=0$, we solve it directly from the equation $\dfrac{x}{x+y}=0$ to get $x=0$.

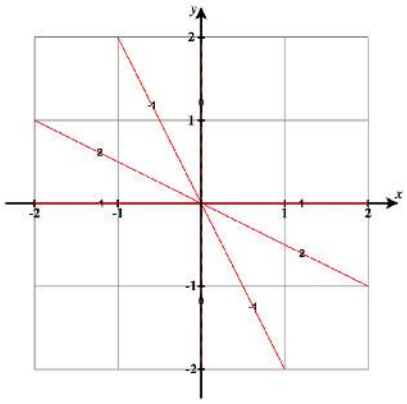
For each value of c these are:

$c=-1: \ y=-2x$,

$c=0: \ x=0, \ \text{with} \ y\neq 0$,

$c=1: \ y=0, \ \text{with} \ x\neq 0$,

$c=2: \ y=-\frac{1}{2}x$.



22) $g(x,y)=x^3-y; \quad c=-1,0,2$

23) $g(x,y)=e^{xy}; \quad c=\frac{1}{2},3$

Answer

The level curves have the form, $y=\dfrac{\ln c}{x}$.

For each value of c these are:

$c=\frac{1}{2}: \ y=\dfrac{\ln\frac{1}{2}}{x}$ that can be rewritten as, $y=-\dfrac{\ln 2}{x}$

$c=3: \ y=\dfrac{\ln 3}{x}$.

24) $f(x,y)=x^2; \quad c=4,9$

25) $f(x,y)=xy-x; \quad c=-2,0,2$

Answer

Level curves have the form: $y=\dfrac{c}{x}+1$.

Here $y=\dfrac{-2}{x}+1, \quad y=1, \quad y=\dfrac{2}{x}+1$ or $xy-x=-2, \ xy-x=0, \ xy-x=2$

26) $h(x,y)=\ln(x^2+y^2); \quad c=-1,0,1$

27) $g(x,y)=\ln\left(\dfrac{y}{x^2}\right); \quad c=-2,0,2$

Answer

The level curves have the form, $y = e^c x^2$.

For each value of c these are:

$c = -2: \; y = e^{-2} x^2$,

$c = 0: \; y = x^2$,

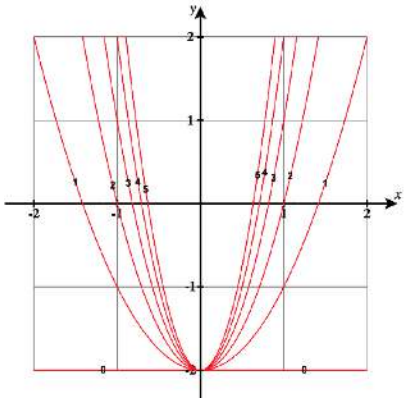
$c = 2: \; y = e^2 x^2$.

28) $z = f(x, y) = \sqrt{x^2 + y^2}, \quad c = 3$

29) $f(x, y) = \dfrac{y+2}{x^2}, \quad c =$ any constant

Answer

The level curves are parabolas of the form $y = cx^2 - 2, \text{ with } x \neq 0$.



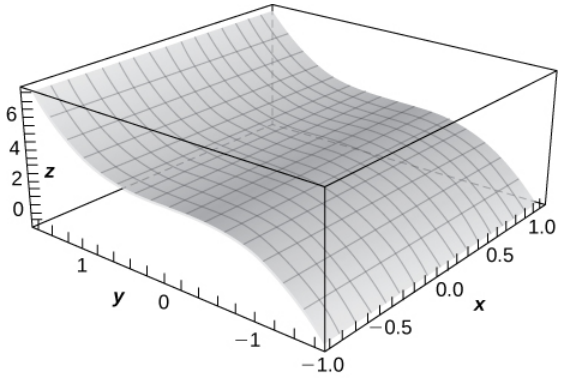
In exercises 30-32, find the vertical traces of the functions at the indicated values of x and y, and plot the traces.

30) $z = 4 - x - y, \quad x = 2$

31) $f(x, y) = 3x + y^3, \quad x = 1$

Answer

$z = 3 + y^3$, a curve in the zy-**plane** with rulings parallel to the x-axis



32) $z = \cos\sqrt{x^2 + y^2}, \quad x = 1$

In exercises 33 - 38, find the domain and range of each function.

33) $z = \sqrt{100 - 4x^2 - 25y^2}$

Answer

Domain: $\left\{(x,y)\,|\,\dfrac{x^2}{25}+\dfrac{y^2}{4}\le 1\right\}$

Range: $[0,10]$

34) $z=\ln(x-y^2)$

35) $f(x,y,z)=\dfrac{1}{\sqrt{36-4x^2-9y^2-z^2}}$

Answer

Domain: $\left\{(x,y,z)\,|\,\dfrac{x^2}{9}+\dfrac{y^2}{4}+\dfrac{z^2}{36}<1\right\}$

Range: $\left[\frac{1}{6},\infty\right)$

36) $f(x,y,z)=\sqrt{49-x^2-y^2-z^2}$

37) $f(x,y,z)=\sqrt[3]{16-x^2-y^2-z^2}$

Answer

Domain: All points in xyz-space

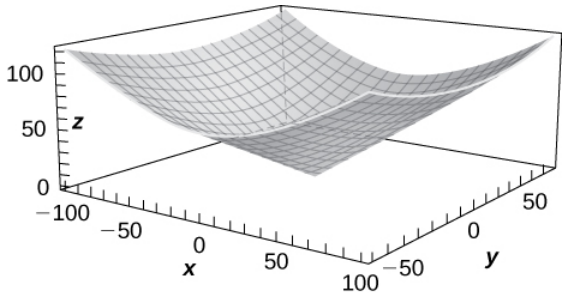
Range: $\left(-\infty,\sqrt[3]{16}\right]$

38) $f(x,y)=\cos\sqrt{x^2+y^2}$

In exercises 39 - 40, plot a graph of the function.

39) $z=f(x,y)=\sqrt{x^2+y^2}$

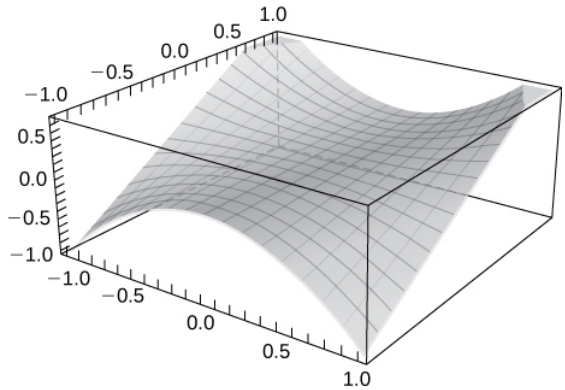
Answer



40) $z=x^2+y^2$

41) Use technology to graph $z=x^2y$.

Answer

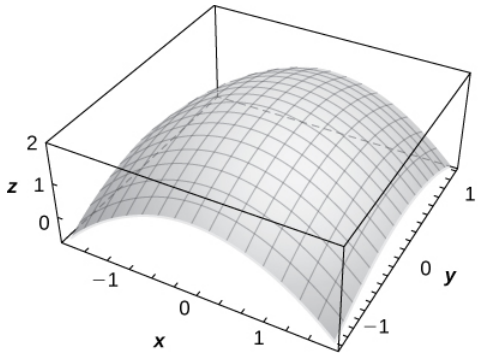


In exercises 42 - 46, sketch the function by finding its level curves. Verify the graph using technology, such as CalcPlot3D.

42) $f(x,y)=\sqrt{4-x^2-y^2}$

43) $f(x,y)=2-\sqrt{x^2+y^2}$

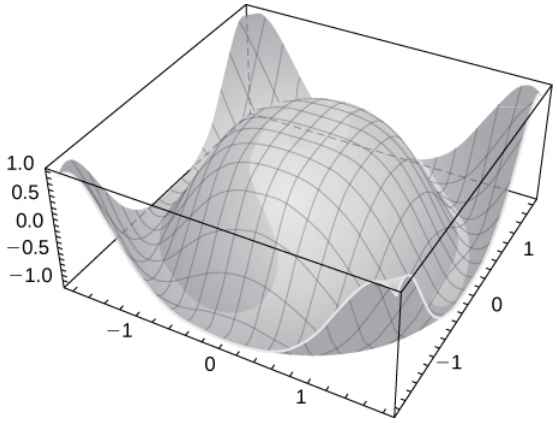
Answer



44) $z=1+e^{-x^2-y^2}$

45) $z=\cos\sqrt{x^2+y^2}$

Answer



46) $z=y^2-x^2$

47) Describe the contour lines for several values of c for $z=x^2+y^2-2x-2y$.

Answer

The contour lines are concentric circles centered at the point, $(1,1)$.
You can see this by completing the square after setting this function equal to c.
That is, we write $x^2-2x+1+y^2-2y+1=c+2$ which can be rewritten as, $(x-1)^2+(y-1)^2=c+2$.
This gives us circles centered at the point, $(1,1)$, each with a radius of $\sqrt{c+2}$.

In exercises, 48 - 52, find the level surface for the given value of c for each function of three variables and describe it.

48) $w(x,y,z)=x-2y+z, \quad c=4$

49) $w(x,y,z)=x^2+y^2+z^2, \quad c=9$

Answer

$x^2+y^2+z^2=9$, a sphere of radius 3

50) $w(x,y,z)=x^2+y^2-z^2, \quad c=-4$

51) $w(x,y,z)=x^2+y^2-z^2, \quad c=4$

Answer

$x^2+y^2-z^2=4,$ a hyperboloid of one sheet

52) $w(x,y,z)=9x^2-4y^2+36z^2, \quad c=0$

In exercises 53 - 55, find an equation of the level curve of f that contains the point P.

53) $f(x,y)=1-4x^2-y^2, \quad P(0,1)$

Answer

$4x^2+y^2=1,$

54) $g(x,y)=y^2\arctan x, \quad P(1,2)$

55) $g(x,y)=e^{xy}(x^2+y^2), \quad P(1,0)$

Answer

$1=e^{xy}(x^2+y^2)$

56) The strength E of an electric field at point (x,y,z) resulting from an infinitely long charged wire lying along the y-axis is given by $E(x,y,z)=k/\sqrt{x^2+y^2}$, where k is a positive constant. For simplicity, let $k=1$ and find the equations of the level surfaces for $E=10$ and $E=100$.

57) A thin plate made of iron is located in the xy-plane The temperature T in degrees Celsius at a point $P(x,y)$ is inversely proportional to the square of its distance from the origin. Express T as a function of x and y.

Answer

$$T(x,y)=\frac{k}{x^2+y^2}$$

58) Refer to the preceding problem. Using the temperature function found there, determine the proportionality constant if the temperature at point $P(1,2)$ is $50^\circ C$. Use this constant to determine the temperature at point $Q(3,4)$.

59) Refer to the preceding problem. Find the level curves for $T=40^\circ C$ and $T=100^\circ C$, and describe what the level curves represent.

Answer

$x^2+y^2=\frac{k}{40}, \quad x^2+y^2=\frac{k}{100}$. The level curves represent circles of radii $\sqrt{10k}/20$ and $\sqrt{k}/10$

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- Paul Seeburger (Monroe Community College) added the contour plots to answers for problems 17, 21 and 29.

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5.3: Limits and Continuity

Learning Objectives

- Calculate the limit of a function of two variables.
- Learn how a function of two variables can approach different values at a boundary point, depending on the path of approach.
- State the conditions for continuity of a function of two variables.
- Verify the continuity of a function of two variables at a point.
- Calculate the limit of a function of three or more variables and verify the continuity of the function at a point.

We have now examined functions of more than one variable and seen how to graph them. In this section, we see how to take the limit of a function of more than one variable, and what it means for a function of more than one variable to be continuous at a point in its domain. It turns out these concepts have aspects that just don't occur with functions of one variable.

Limit of a Function of Two Variables

Recall from Section 2.5 that the definition of a limit of a function of one variable:

Let $f(x)$ be defined for all $x \neq a$ in an open interval containing a. Let L be a real number. Then

$$\lim_{x \to a} f(x) = L \tag{5.3.1}$$

if for every $\varepsilon > 0$, there exists a $\delta > 0$, such that if $0 < |x - a| < \delta$ for all x in the domain of f, then

$$|f(x) - L| < \varepsilon. \tag{5.3.2}$$

Before we can adapt this definition to define a limit of a function of two variables, we first need to see how to extend the idea of an open interval in one variable to an open interval in two variables.

Definition: δ Disks

Consider a point $(a, b) \in \mathbb{R}^2$. A δ **disk** centered at point (a, b) is defined to be an open disk of radius δ centered at point (a, b) —that is,

$$\{(x, y) \in \mathbb{R}^2 \mid (x - a)^2 + (y - b)^2 < \delta^2\} \tag{5.3.3}$$

as shown in Figure 5.3.1.

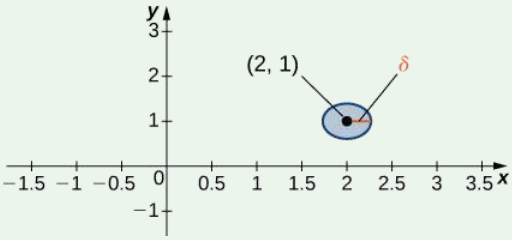


Figure 5.3.1: A δ disk centered around the point $(2, 1)$.

The idea of a δ disk appears in the definition of the limit of a function of two variables. If δ is small, then all the points (x, y) in the δ disk are close to (a, b). This is completely analogous to x being close to a in the definition of a limit of a function of one variable. In one dimension, we express this restriction as

$$a - \delta < x < a + \delta. \tag{5.3.4}$$

In more than one dimension, we use a δ disk.

Definition: Limit of a Function of Two Variables

Let f be a function of two variables, x and y. The limit of $f(x,y)$ as (x,y) approaches (a,b) is L, written

$$\lim_{(x,y)\to(a,b)} f(x,y) = L \tag{5.3.5}$$

if for each $\varepsilon > 0$ there exists a small enough $\delta > 0$ such that for all points (x,y) in a δ disk around (a,b), except possibly for (a,b) itself, the value of $f(x,y)$ is no more than ε away from L (Figure 5.3.2).

Using symbols, we write the following: For any $\varepsilon > 0$, there exists a number $\delta > 0$ such that

$$|f(x,y) - L| < \varepsilon \tag{5.3.6}$$

whenever

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta. \tag{5.3.7}$$

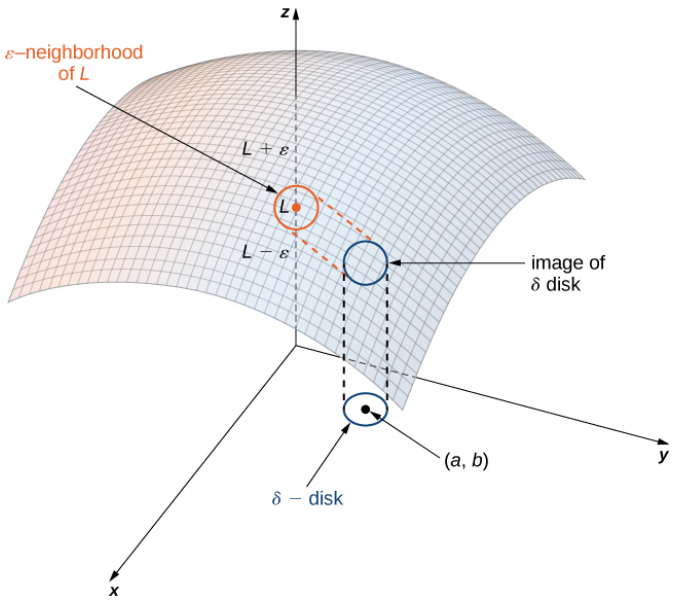


Figure 5.3.2: The limit of a function involving two variables requires that $f(x,y)$ be within ε of L whenever (x,y) is within δ of (a,b). The smaller the value of ε, the smaller the value of δ.

Proving that a limit exists using the definition of a **limit of a function of two variables** can be challenging. Instead, we use the following theorem, which gives us shortcuts to finding limits. The formulas in this theorem are an extension of the formulas in the limit laws theorem in The Limit Laws.

Limit Laws for Functions of Two Variables

Let $f(x,y)$ and $g(x,y)$ be defined for all $(x,y) \neq (a,b)$ in a neighborhood around (a,b), and assume the neighborhood is contained completely inside the domain of f. Assume that L and M are real numbers such that

$$\lim_{(x,y)\to(a,b)} f(x,y) = L \tag{5.3.8}$$

and

$$\lim_{(x,y)\to(a,b)} g(x,y) = M, \tag{5.3.9}$$

and let c be a constant. Then each of the following statements holds:

Constant Law:

$$\lim_{(x,y)\to(a,b)} c = c \tag{5.3.10}$$

Identity Laws:

$$\lim_{(x,y)\to(a,b)} x = a \tag{5.3.11}$$

$$\lim_{(x,y)\to(a,b)} y = b \tag{5.3.12}$$

Sum Law:

$$\lim_{(x,y)\to(a,b)} (f(x,y)+g(x,y)) = L+M \tag{5.3.13}$$

Difference Law:

$$\lim_{(x,y)\to(a,b)} (f(x,y)-g(x,y)) = L-M \tag{5.3.14}$$

Constant Multiple Law:

$$\lim_{(x,y)\to(a,b)} (cf(x,y)) = cL \tag{5.3.15}$$

Product Law:

$$\lim_{(x,y)\to(a,b)} (f(x,y)g(x,y)) = LM \tag{5.3.16}$$

Quotient Law:

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M} \text{ for } M \neq 0 \tag{5.3.17}$$

Power Law:

$$\lim_{(x,y)\to(a,b)} (f(x,y))^n = L^n \tag{5.3.18}$$

for any positive integer n.

Root Law:

$$\lim_{(x,y)\to(a,b)} \sqrt[n]{f(x,y)} = \sqrt[n]{L} \tag{5.3.19}$$

for all L if n is odd and positive, and for $L \geq 0$ if n is even and positive.

The proofs of these properties are similar to those for the limits of functions of one variable. We can apply these laws to finding limits of various functions.

Example 5.3.1: Finding the Limit of a Function of Two Variables

Find each of the following limits:

a. $\lim_{(x,y)\to(2,-1)} (x^2-2xy+3y^2-4x+3y-6)$

b. $\lim_{(x,y)\to(2,-1)} \frac{2x+3y}{4x-3y}$

Solution

a. First use the sum and difference laws to separate the terms:

$$\begin{aligned}\lim_{(x,y)\to(2,-1)} (x^2-2xy+3y^2-4x+3y-6) \\ = \left(\lim_{(x,y)\to(2,-1)} x^2\right) - \left(\lim_{(x,y)\to(2,-1)} 2xy\right) + \left(\lim_{(x,y)\to(2,-1)} 3y^2\right) - \left(\lim_{(x,y)\to(2,-1)} 4x\right) \\ + \left(\lim_{(x,y)\to(2,-1)} 3y\right) - \left(\lim_{(x,y)\to(2,-1)} 6\right).\end{aligned}$$

Next, use the constant multiple law on the second, third, fourth, and fifth limits:

$$= (\lim_{(x,y)\to(2,-1)} x^2) - 2(\lim_{(x,y)\to(2,-1)} xy) + 3(\lim_{(x,y)\to(2,-1)} y^2) - 4(\lim_{(x,y)\to(2,-1)} x) + 3(\lim_{(x,y)\to(2,-1)} y) - \lim_{(x,y)\to(2,-1)} 6.$$

Now, use the power law on the first and third limits, and the product law on the second limit:

$$\left(\lim_{(x,y)\to(2,-1)} x\right)^2 - 2\left(\lim_{(x,y)\to(2,-1)} x\right)\left(\lim_{(x,y)\to(2,-1)} y\right) + 3\left(\lim_{(x,y)\to(2,-1)} y\right)^2 - 4\left(\lim_{(x,y)\to(2,-1)} x\right) + 3\left(\lim_{(x,y)\to(2,-1)} y\right) - \lim_{(x,y)\to(2,-1)} 6.$$

Last, use the identity laws on the first six limits and the constant law on the last limit:

$$\lim_{(x,y)\to(2,-1)} (x^2 - 2xy + 3y^2 - 4x + 3y - 6) = (2)^2 - 2(2)(-1) + 3(-1)^2 - 4(2) + 3(-1) - 6 = -6.$$

b. Before applying the quotient law, we need to verify that the limit of the denominator is nonzero. Using the difference law, constant multiple law, and identity law,

$$\lim_{(x,y)\to(2,-1)} (4x - 3y) = \lim_{(x,y)\to(2,-1)} 4x - \lim_{(x,y)\to(2,-1)} 3y = 4(\lim_{(x,y)\to(2,-1)} x) - 3(\lim_{(x,y)\to(2,-1)} y) = 4(2) - 3(-1) = 11.$$

Since the limit of the denominator is nonzero, the quotient law applies. We now calculate the limit of the numerator using the difference law, constant multiple law, and identity law:

$$\lim_{(x,y)\to(2,-1)} (2x + 3y) = \lim_{(x,y)\to(2,-1)} 2x + \lim_{(x,y)\to(2,-1)} 3y = 2(\lim_{(x,y)\to(2,-1)} x) + 3(\lim_{(x,y)\to(2,-1)} y) = 2(2) + 3(-1) = 1.$$

Therefore, according to the quotient law we have

$$\lim_{(x,y)\to(2,-1)} \frac{2x+3y}{4x-3y} = \frac{\lim_{(x,y)\to(2,-1)} (2x+3y)}{\lim_{(x,y)\to(2,-1)} (4x-3y)} = \frac{1}{11}.$$

Exercise 5.3.1:

Evaluate the following limit:

$$\lim_{(x,y)\to(5,-2)} \sqrt[3]{\frac{x^2 - y}{y^2 + x - 1}}.$$

Hint

Use the limit laws.

Answer

$$\lim_{(x,y)\to(5,-2)} \sqrt[3]{\frac{x^2-y}{y^2+x-1}} = \frac{3}{2}$$



Since we are taking the limit of a function of two variables, the point (a,b) is in $\mathbb{R}^2$, and it is possible to approach this point from an infinite number of directions. Sometimes when calculating a limit, the answer varies depending on the path taken toward (a,b). If this is the case, then the limit fails to exist. In other words, the limit must be unique, regardless of path taken.

Example 5.3.2: Limits That Fail to Exist

Show that neither of the following limits exist:

a. $\lim\limits_{(x,y)\to(0,0)} \dfrac{2xy}{3x^2+y^2}$

b. $\lim\limits_{(x,y)\to(0,0)} \dfrac{4xy^2}{x^2+3y^4}$

Solution

a. The domain of the function $f(x,y) = \dfrac{2xy}{3x^2+y^2}$ consists of all points in the xy-plane except for the point $(0,0)$ (Figure 5.3.3). To show that the limit does not exist as (x,y) approaches $(0,0)$, we note that it is impossible to satisfy the definition of a limit of a function of two variables because of the fact that the function takes different values along different lines passing through point $(0,0)$. First, consider the line $y=0$ in the xy-plane. Substituting $y=0$ into $f(x,y)$ gives

$$f(x,0) = \frac{2x(0)}{3x^2+0^2} = 0$$

for any value of x. Therefore the value of f remains constant for any point on the x-axis, and as y approaches zero, the function remains fixed at zero.

Next, consider the line $y=x$. Substituting $y=x$ into $f(x,y)$ gives

$$f(x,x) = \frac{2x(x)}{3x^2+x^2} = \frac{2x^2}{4x^2} = \tfrac{1}{2}.$$

This is true for any point on the line $y=x$. If we let x approach zero while staying on this line, the value of the function remains fixed at $\frac{1}{2}$, regardless of how small x is.

Choose a value for ε that is less than $1/2$—say, $1/4$. Then, no matter how small a δ disk we draw around $(0,0)$, the values of $f(x,y)$ for points inside that δ disk will include both 0 and $\frac{1}{2}$. Therefore, the definition of limit at a point is never satisfied and the limit fails to exist.

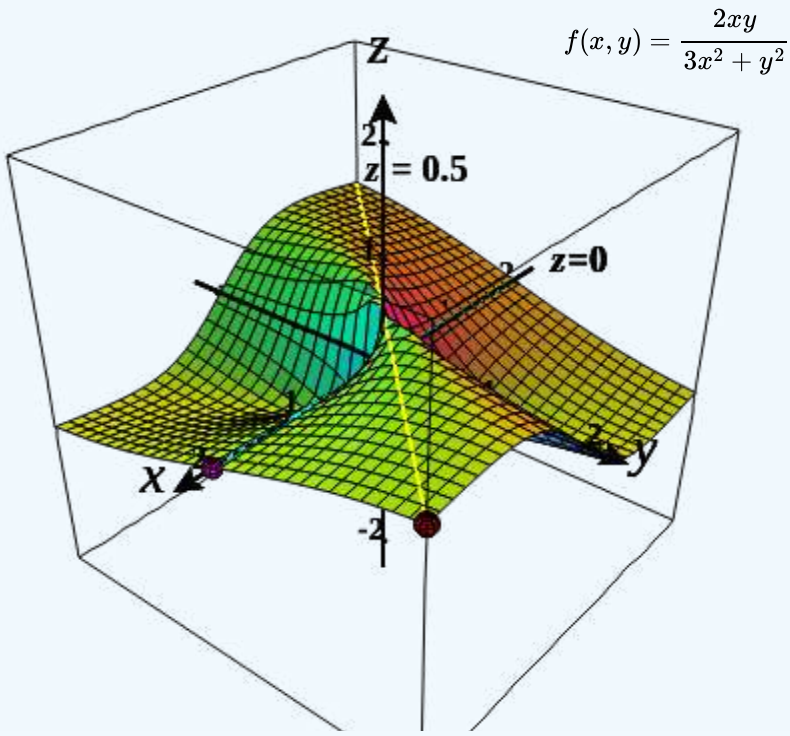


Figure 5.3.3: Graph of the function $f(x,y) = \dfrac{2xy}{3x^2+y^2}$. Along the line $y=0$, the function is equal to zero; along the line $y=x$, the function is equal to $\frac{1}{2}$.

b. In a similar fashion to a., we can approach the origin along any straight line passing through the origin. If we try the x-axis (i.e., $y=0$), then the function remains fixed at zero. The same is true for the y-axis. Suppose we approach the origin along a straight line of slope k. The equation of this line is $y=kx$. Then the limit becomes

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+3y^4} &= \lim_{(x,y)\to(0,0)} \frac{4x(kx)^2}{x^2+3(kx)^4} \\ &= \lim_{(x,y)\to(0,0)} \frac{4k^2x^3}{x^2+3k^4x^4} \\ &= \lim_{(x,y)\to(0,0)} \frac{4k^2x}{1+3k^4x^2} \\ &= \frac{\lim\limits_{(x,y)\to(0,0)}(4k^2x)}{\lim\limits_{(x,y)\to(0,0)}(1+3k^4x^2)} \\ &= 0.\end{aligned}$$

regardless of the value of k. It would seem that the limit is equal to zero. What if we chose a curve passing through the origin instead? For example, we can consider the parabola given by the equation $x=y^2$. Substituting y^2 in place of x in $f(x,y)$ gives

$$\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+3y^4} = \lim_{(x,y)\to(0,0)} \frac{4(y^2)y^2}{(y^2)^2+3y^4}$$
$$= \lim_{(x,y)\to(0,0)} \frac{4y^4}{y^4+3y^4}$$
$$= \lim_{(x,y)\to(0,0)} 1$$
$$= 1.$$

By the same logic in part a, it is impossible to find a δ disk around the origin that satisfies the definition of the limit for any value of $\varepsilon < 1$. Therefore,

$$\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+3y^4}$$

does **not** exist.



Exercise 5.3.2:

Show that

$$\lim_{(x,y)\to(2,1)} \frac{(x-2)(y-1)}{(x-2)^2+(y-1)^2}$$

does not exist.

Hint

Pick a line with slope k passing through point $(2,1)$.

Answer

If $y = k(x-2)+1$, then $\lim_{(x,y)\to(2,1)} \frac{(x-2)(y-1)}{(x-2)^2+(y-1)^2} = \frac{k}{1+k^2}$. Since the answer depends on k, the limit fails to exist.

Interior Points and Boundary Points

To study continuity and differentiability of a function of two or more variables, we first need to learn some new terminology.

Definition: Interior and Boundary Points

Let S be a subset of $\mathbb{R}^2$ (Figure 5.3.4).

1. A point P_0 is called an *interior point* of S if there is a δ disk centered around P_0 contained completely in S.
2. A point P_0 is called a *boundary point* of S if every δ disk centered around P_0 contains points both inside and outside S.

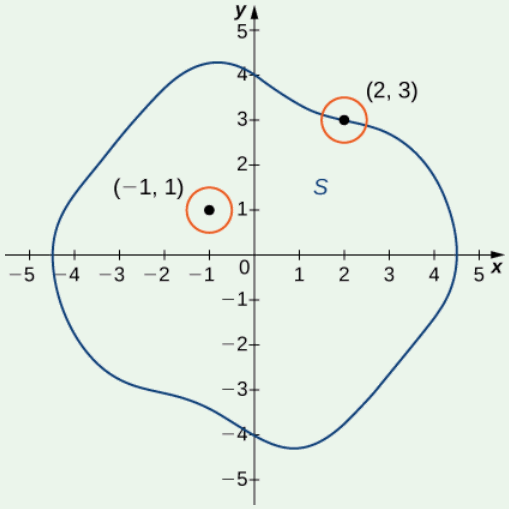


Figure 5.3.4: In the set S shown, $(-1,1)$ is an interior point and $(2,3)$ is a boundary point.

Definition: Open and Closed Sets

Let S be a subset of $\mathbb{R}^2$ (Figure 5.3.4).

1. S is called an *open set* if every point of S is an interior point.
2. S is called a *closed set* if it contains all its boundary points.

An example of an open set is a δ disk. If we include the boundary of the disk, then it becomes a closed set. A set that contains some, but not all, of its boundary points is neither open nor closed. For example if we include half the boundary of a δ disk but not the other half, then the set is neither open nor closed.

Definition: Connected Sets and Regions

Let S be a subset of $\mathbb{R}^2$ (Figure 5.3.4).

1. An open set S is a *connected set* if it cannot be represented as the union of two or more disjoint, nonempty open subsets.
2. A set S is a *region* if it is open, connected, and nonempty.

The definition of a limit of a function of two variables requires the δ disk to be contained inside the domain of the function. However, if we wish to find the limit of a function at a boundary point of the domain, the δ **disk** is not contained inside the domain. By definition, some of the points of the δ **disk** are inside the domain and some are outside. Therefore, we need only consider points that are inside both the δ disk and the domain of the function. This leads to the definition of the limit of a function at a boundary point.

Definition

Let f be a function of two variables, x and y, and suppose (a,b) is on the boundary of the domain of f. Then, the limit of $f(x,y)$ as (x,y) approaches (a,b) is L, written

$$\lim_{(x,y)\to(a,b)} f(x,y) = L, \tag{5.3.20}$$

if for any $\varepsilon > 0$, there exists a number $\delta > 0$ such that for any point (x,y) inside the domain of f and within a suitably small distance positive δ of (a,b), the value of $f(x,y)$ is no more than ε away from L (Figure 5.3.2). Using symbols, we can write: For any $\varepsilon > 0$, there exists a number $\delta > 0$ such that

$$|f(x,y)-L|<\varepsilon \text{ whenever } 0<\sqrt{(x-a)^2+(y-b)^2}<\delta. \tag{5.3.21}$$

Example 5.3.3: Limit of a Function at a Boundary Point

Prove

$$\lim_{(x,y)\to(4,3)}\sqrt{25-x^2-y^2}=0.$$

Solution

The domain of the function $f(x,y)=\sqrt{25-x^2-y^2}$ is $\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\le 25\}$, which is a circle of radius 5 centered at the origin, along with its interior as shown in Figure 5.3.5.

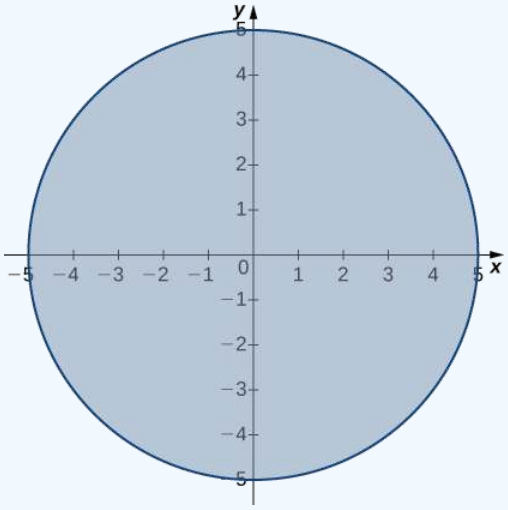


Figure 5.3.5: Domain of the function $f(x,y)=\sqrt{25-x^2-y^2}$.

We can use the limit laws, which apply to limits at the boundary of domains as well as interior points:

$$\begin{aligned}\lim_{(x,y)\to(4,3)}\sqrt{25-x^2-y^2} &= \sqrt{\lim_{(x,y)\to(4,3)}(25-x^2-y^2)}\\ &= \sqrt{\lim_{(x,y)\to(4,3)}25-\lim_{(x,y)\to(4,3)}x^2-\lim_{(x,y)\to(4,3)}y^2}\\ &= \sqrt{25-4^2-3^2}\\ &= 0\end{aligned}$$

See the following graph.

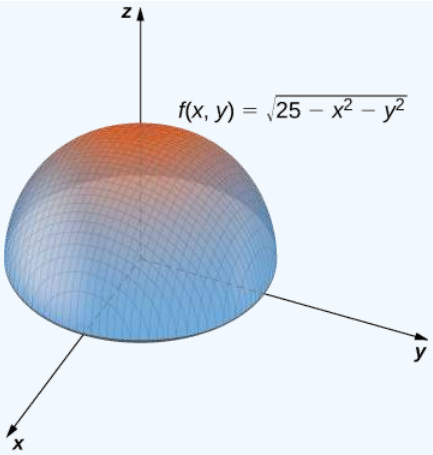


Figure 5.3.6: Graph of the function $f(x,y)=\sqrt{25-x^2-y^2}$.

Exercise 5.3.3

Evaluate the following limit:

$$\lim_{(x,y)\to(5,-2)}\sqrt{29-x^2-y^2}.$$

Hint

Determine the domain of $f(x,y)=\sqrt{29-x^2-y^2}$.

Answer

$$\lim_{(x,y)\to(5,-2)}\sqrt{29-x^2-y^2}$$

Continuity of Functions of Two Variables

In Continuity, we defined the continuity of a function of one variable and saw how it relied on the limit of a function of one variable. In particular, three conditions are necessary for $f(x)$ to be continuous at point $x=a$

1. $f(a)$ exists.
2. $\lim\limits_{x\to a} f(x)$ exists.
3. $\lim\limits_{x\to a} f(x)=f(a)$.

These three conditions are necessary for continuity of a function of two variables as well.

Definition: Continuous Functions

A function $f(x,y)$ is continuous at a point (a,b) in its domain if the following conditions are satisfied:

1. $f(a,b)$ exists.
2. $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ exists.
3. $\lim\limits_{(x,y)\to(a,b)} f(x,y)=f(a,b)$.

Example 5.3.4: Demonstrating Continuity for a Function of Two Variables

Show that the function

$$f(x,y)=\frac{3x+2y}{x+y+1}$$

is continuous at point $(5,-3)$.

Solution

There are three conditions to be satisfied, per the definition of continuity. In this example, $a = 5$ and $b = -3$.

1. $f(a,b)$ exists. This is true because the domain of the function f consists of those ordered pairs for which the denominator is nonzero (i.e., $x + y + 1 \neq 0$). Point $(5, -3)$ satisfies this condition. Furthermore,

$$f(a,b) = f(5,-3) = \frac{3(5)+2(-3)}{5+(-3)+1} = \frac{15-6}{2+1} = 3.$$

2. $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ exists. This is also true:

$$\begin{aligned} \lim_{(x,y)\to(a,b)} f(x,y) &= \lim_{(x,y)\to(5,-3)} \frac{3x+2y}{x+y+1} \\ &= \frac{\lim\limits_{(x,y)\to(5,-3)} (3x+2y)}{\lim\limits_{(x,y)\to(5,-3)} (x+y+1)} \\ &= \frac{15-6}{5-3+1} \\ &= 3. \end{aligned}$$

3. $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$. This is true because we have just shown that both sides of this equation equal three.



Exercise 5.3.4

Show that the function

$$f(x,y) = \sqrt{26 - 2x^2 - y^2}$$

is continuous at point $(2, -3)$.

Hint

Use the three-part definition of continuity.

Answer

1. The domain of f contains the ordered pair $(2, -3)$ because $f(a,b) = f(2,-3) = \sqrt{16 - 2(2)^2 - (-3)^2} = 3$

2. $\lim\limits_{(x,y)\to(a,b)} f(x,y) = 3$
3. $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b) = 3$

Continuity of a function of any number of variables can also be defined in terms of delta and epsilon. A function of two variables is continuous at a point (x_0, y_0) in its domain if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that, whenever $\sqrt{(x-x_0)^2+(y-y_0)^2} < \delta$ it is true, $|f(x,y) - f(a,b)| < \varepsilon$. This definition can be combined with the formal definition (that is, the *epsilon–delta definition*) of continuity of a function of one variable to prove the following theorems:

The Sum of Continuous Functions Is Continuous

If $f(x,y)$ is continuous at (x_0, y_0), and $g(x,y)$ is continuous at (x_0, y_0), then $f(x,y) + g(x,y)$ is continuous at (x_0, y_0).

The Product of Continuous Functions Is Continuous

If $g(x)$ is continuous at x_0 and $h(y)$ is continuous at y_0, then $f(x,y) = g(x)h(y)$ is continuous at (x_0, y_0).

The Composition of Continuous Functions Is Continuous

Let g be a function of two variables from a domain $D \subseteq \mathbb{R}^2$ to a range $R \subseteq R$. Suppose g is continuous at some point $(x_0, y_0) \in D$ and define $z_0 = g(x_0, y_0)$. Let f be a function that maps R to R such that z_0 is in the domain of f. Last, assume f is continuous at z_0. Then $f \circ g$ is continuous at (x_0, y_0) as shown in Figure 5.3.7.

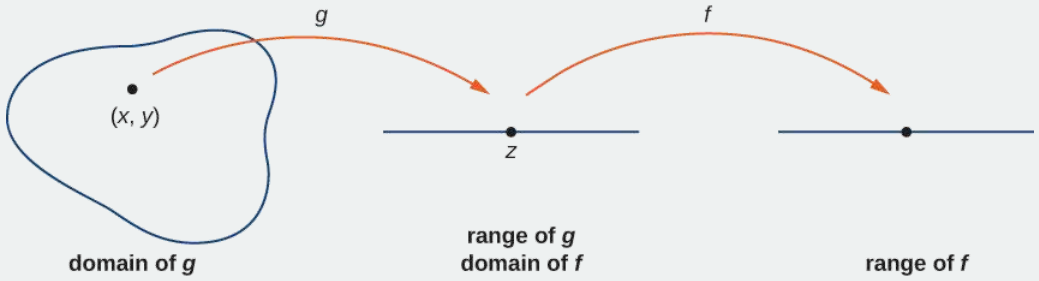


Figure 5.3.7: The composition of two continuous functions is continuous.

Let's now use the previous theorems to show continuity of functions in the following examples.

Example 5.3.5: More Examples of Continuity of a Function of Two Variables

Show that the functions $f(x,y) = 4x^3y^2$ and $g(x,y) = \cos(4x^3y^2)$ are continuous everywhere.

Solution

The polynomials $g(x) = 4x^3$ and $h(y) = y^2$ are continuous at every real number, and therefore by the product of continuous functions theorem, $f(x,y) = 4x^3y^2$ is continuous at every point (x,y) in the xy-plane. Since $f(x,y) = 4x^3y^2$ is continuous at every point (x,y) in the xy-plane and $g(x) = \cos x$ is continuous at every real number x, the continuity of the composition of functions tells us that $g(x,y) = \cos(4x^3y^2)$ is continuous at every point (x,y) in the xy-plane.

Exercise 5.3.5

Show that the functions $f(x,y) = 2x^2y^3 + 3$ and $g(x,y) = (2x^2y^3 + 3)^4$ are continuous everywhere.

Hint

Use the continuity of the sum, product, and composition of two functions.

Answer

The polynomials $g(x) = 2x^2$ and $h(y) = y^3$ are continuous at every real number; therefore, by the product of continuous functions theorem, $f(x,y) = 2x^2y^3$ is continuous at every point (x,y) in the xy-plane. Furthermore, any constant function is continuous everywhere, so $g(x,y) = 3$ is continuous at every point (x,y) in the xy-plane. Therefore,

$f(x,y)=2x^2y^3+3$ is continuous at every point (x,y) in the xy-plane. Last, $h(x)=x^4$ is continuous at every real number x, so by the continuity of composite functions theorem $g(x,y)=(2x^2y^3+3)^4$ is continuous at every point (x,y) in the xy-plane.

Functions of Three or More Variables

The limit of a function of three or more variables occurs readily in applications. For example, suppose we have a function $f(x,y,z)$ that gives the temperature at a physical location (x,y,z) in three dimensions. Or perhaps a function $g(x,y,z,t)$ can indicate air pressure at a location (x,y,z) at time t. How can we take a limit at a point in $\mathbb{R}^3$? What does it mean to be continuous at a point in four dimensions?

The answers to these questions rely on extending the concept of a δ disk into more than two dimensions. Then, the ideas of the limit of a function of three or more variables and the continuity of a function of three or more variables are very similar to the definitions given earlier for a function of two variables.

Definition: δ-balls

Let (x_0,y_0,z_0) be a point in $\mathbb{R}^3$. Then, a δ-ball in three dimensions consists of all points in $\mathbb{R}^3$ lying at a distance of less than δ from (x_0,y_0,z_0) —that is,

$$\{(x,y,z)\in\mathbb{R}^3 \mid \sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}<\delta\}. \tag{5.3.22}$$

To define a δ-ball in higher dimensions, add additional terms under the radical to correspond to each additional dimension. For example, given a point $P=(w_0,x_0,y_0,z_0)$ in $\mathbb{R}^4$, a δ ball around P can be described by

$$\{(w,x,y,z)\in\mathbb{R}^4 \mid \sqrt{(w-w_0)^2+(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}<\delta\}. \tag{5.3.23}$$

To show that a limit of a function of three variables exists at a point (x_0,y_0,z_0), it suffices to show that for any point in a δ ball centered at (x_0,y_0,z_0), the value of the function at that point is arbitrarily close to a fixed value (the limit value). All the limit laws for functions of two variables hold for functions of more than two variables as well.

Example 5.3.6: Finding the Limit of a Function of Three Variables

Find

$$\lim_{(x,y,z)\to(4,1,-3)}\frac{x^2y-3z}{2x+5y-z}.$$

Solution

Before we can apply the quotient law, we need to verify that the limit of the denominator is nonzero. Using the difference law, the identity law, and the constant law,

$$\begin{aligned}\lim_{(x,y,z)\to(4,1,-3)}(2x+5y-z) &= 2\left(\lim_{(x,y,z)\to(4,1,-3)}x\right)+5\left(\lim_{(x,y,z)\to(4,1,-3)}y\right)-\left(\lim_{(x,y,z)\to(4,1,-3)}z\right)\\ &=2(4)+5(1)-(-3)\\ &=16.\end{aligned}$$

Since this is nonzero, we next find the limit of the numerator. Using the product law, power law, difference law, constant multiple law, and identity law,

$$\begin{aligned}\lim_{(x,y,z)\to(4,1,-3)}(x^2y-3z) &= \left(\lim_{(x,y,z)\to(4,1,-3)}x\right)^2\left(\lim_{(x,y,z)\to(4,1,-3)}y\right)-3\lim_{(x,y,z)\to(4,1,-3)}z\\ &=(4^2)(1)-3(-3)\\ &=16+9\\ &=25\end{aligned}$$

Last, applying the quotient law:

$$\lim_{(x,y,z)\to(4,1,-3)} \frac{x^2y-3z}{2x+5y-z} = \frac{\lim\limits_{(x,y,z)\to(4,1,-3)}(x^2y-3z)}{\lim\limits_{(x,y,z)\to(4,1,-3)}(2x+5y-z)} = \frac{25}{16}$$

Exercise 5.3.6

Find

$$\lim_{(x,y,z)\to(4,-1,3)} \sqrt{13-x^2-2y^2+z^2}$$

Hint

Use the limit laws and the continuity of the composition of functions.

Answer

$$\lim_{(x,y,z)\to(4,-1,3)} \sqrt{13-x^2-2y^2+z^2} = 2$$

Key Concepts

- To study limits and continuity for functions of two variables, we use a δ disk centered around a given point.
- A function of several variables has a limit if for any point in a δ ball centered at a point P, the value of the function at that point is arbitrarily close to a fixed value (the limit value).
- The limit laws established for a function of one variable have natural extensions to functions of more than one variable.
- A function of two variables is continuous at a point if the limit exists at that point, the function exists at that point, and the limit and function are equal at that point.

Glossary

boundary point

a point P_0 of R is a boundary point if every δ disk centered around P_0 contains points both inside and outside R

closed set

a set S that contains all its boundary points

connected set

an open set S that cannot be represented as the union of two or more disjoint, nonempty open subsets

δ disk

an open disk of radius δ centered at point (a,b)

δ ball

all points in $\mathbb{R}^3$ lying at a distance of less than δ from (x_0,y_0,z_0)

interior point

a point P_0 of $\mathbb{R}$ is a boundary point if there is a δ disk centered around P_0 contained completely in $\mathbb{R}$

open set

a set S that contains none of its boundary points

region

an open, connected, nonempty subset of $\mathbb{R}^2$

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5.3E: Exercises for Section 5.3

1) Use the limit laws for functions of two variables to evaluate each limit below, given that $\lim\limits_{(x,y)\to(a,b)} f(x,y)=5$ and $\lim\limits_{(x,y)\to(a,b)} g(x,y)=2$.

a. $\lim\limits_{(x,y)\to(a,b)} [f(x,y)+g(x,y)]$

b. $\lim\limits_{(x,y)\to(a,b)} [f(x,y)g(x,y)]$

c. $\lim\limits_{(x,y)\to(a,b)} \left[\dfrac{7f(x,y)}{g(x,y)}\right]$

d. $\lim\limits_{(x,y)\to(a,b)} \left[\dfrac{2f(x,y)-4g(x,y)}{f(x,y)-g(x,y)}\right]$

Answer

a. $\lim\limits_{(x,y)\to(a,b)} [f(x,y)+g(x,y)] = \lim\limits_{(x,y)\to(a,b)} f(x,y) + \lim\limits_{(x,y)\to(a,b)} g(x,y) = 5+2=7$

b. $\lim\limits_{(x,y)\to(a,b)} [f(x,y)g(x,y)] = \left(\lim\limits_{(x,y)\to(a,b)} f(x,y)\right)\left(\lim\limits_{(x,y)\to(a,b)} g(x,y)\right) = 5(2)=10$

c. $\lim\limits_{(x,y)\to(a,b)} \left[\dfrac{7f(x,y)}{g(x,y)}\right] = \dfrac{7\left(\lim\limits_{(x,y)\to(a,b)} f(x,y)\right)}{\lim\limits_{(x,y)\to(a,b)} g(x,y)} = \dfrac{7(5)}{2} = 17.5$

d. $\lim\limits_{(x,y)\to(a,b)} \left[\dfrac{2f(x,y)-4g(x,y)}{f(x,y)-g(x,y)}\right] = \dfrac{2\left(\lim\limits_{(x,y)\to(a,b)} f(x,y)\right)-4\left(\lim\limits_{(x,y)\to(a,b)} g(x,y)\right)}{\lim\limits_{(x,y)\to(a,b)} f(x,y)-\lim\limits_{(x,y)\to(a,b)} g(x,y)} = \dfrac{2(5)-4(2)}{5-2} = \dfrac{2}{3}$

In exercises 2 - 4, find the limit of the function.

2) $\lim\limits_{(x,y)\to(1,2)} x$

3) $\lim\limits_{(x,y)\to(1,2)} \dfrac{5x^2y}{x^2+y^2}$

Answer

$\lim\limits_{(x,y)\to(1,2)} \dfrac{5x^2y}{x^2+y^2} = 2$

4) Show that the limit $\lim\limits_{(x,y)\to(0,0)} \dfrac{5x^2y}{x^2+y^2}$ exists and is the same along the paths: y-axis and x-axis, and along $y=x$.

In exercises 5 - 19, evaluate the limits at the indicated values of x and y. If the limit does not exist, state this and explain why the limit does not exist.

5) $\lim\limits_{(x,y)\to(0,0)} \dfrac{4x^2+10y^2+4}{4x^2-10y^2+6}$

Answer

$\lim\limits_{(x,y)\to(0,0)} \dfrac{4x^2+10y^2+4}{4x^2-10y^2+6} = \dfrac{2}{3}$

6) $\lim\limits_{(x,y)\to(11,13)} \sqrt{\dfrac{1}{xy}}$

7) $\lim\limits_{(x,y)\to(0,1)} \dfrac{y^2\sin x}{x}$

Answer

$$\lim_{(x,y)\to(0,1)} \frac{y^2 \sin x}{x} = 1$$

8) $\lim_{(x,y)\to(0,0)} \sin\left(\frac{x^8+y^7}{x-y+10}\right)$

9) $\lim_{(x,y)\to(\pi/4,1)} \frac{y\tan x}{y+1}$

Answer

$$\lim_{(x,y)\to(\pi/4,1)} \frac{y\tan x}{y+1} = \frac{1}{2}$$

10) $\lim_{(x,y)\to(0,\pi/4)} \frac{\sec x+2}{3x-\tan y}$

11) $\lim_{(x,y)\to(2,5)} \left(\frac{1}{x}-\frac{5}{y}\right)$

Answer

$$\lim_{(x,y)\to(2,5)} \left(\frac{1}{x}-\frac{5}{y}\right) = -\frac{1}{2}$$

12) $\lim_{(x,y)\to(4,4)} x\ln y$

13) $\lim_{(x,y)\to(4,4)} e^{-x^2-y^2}$

Answer

$$\lim_{(x,y)\to(4,4)} e^{-x^2-y^2} = e^{-32}$$

14) $\lim_{(x,y)\to(0,0)} \sqrt{9-x^2-y^2}$

15) $\lim_{(x,y)\to(1,2)} (x^2y^3-x^3y^2+3x+2y)$

Answer

$$\lim_{(x,y)\to(1,2)} (x^2y^3-x^3y^2+3x+2y) = 11$$

16) $\lim_{(x,y)\to(\pi,\pi)} x\sin\left(\frac{x+y}{4}\right)$

17) $\lim_{(x,y)\to(0,0)} \frac{xy+1}{x^2+y^2+1}$

Answer

$$\lim_{(x,y)\to(0,0)} \frac{xy+1}{x^2+y^2+1} = 1$$

18) $\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}$

19) $\lim_{(x,y)\to(0,0)} \ln(x^2+y^2)$

Answer

The limit does not exist because when x and y both approach zero, the function approaches $\ln 0$, which is undefined (approaches negative infinity).

In exercises 20 - 21, complete the statement.

20) A point (x_0, y_0) in a plane region R is an interior point of R if _____.

21) A point (x_0, y_0) in a plane region R is called a boundary point of R if _____.

Answer

Every open disk centered at (x_0, y_0) contains points inside R and outside R.

In exercises 22 - 25, use algebraic techniques to evaluate the limit.

22) $\lim\limits_{(x,y)\to(2,1)} \dfrac{x-y-1}{\sqrt{x-y}-1}$

23) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^4-4y^4}{x^2+2y^2}$

Answer

$$\lim_{(x,y)\to(0,0)} \frac{x^4-4y^4}{x^2+2y^2} = 0$$

24) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3-y^3}{x-y}$

25) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-xy}{\sqrt{x}-\sqrt{y}}$

Answer

$$\lim_{(x,y)\to(0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}} = 0$$

In exercises 26 - 27, evaluate the limits of the functions of three variables.

26) $\lim\limits_{(x,y,z)\to(1,2,3)} \dfrac{xz^2-y^2z}{xyz-1}$

27) $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{x^2-y^2-z^2}{x^2+y^2-z^2}$

Answer

The limit does not exist.

In exercises 28 - 31, evaluate the limit of the function by determining the value the function approaches along the indicated paths. If the limit does not exist, explain why not.

28) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy+y^3}{x^2+y^2}$

a. Along the x-axis $(y=0)$

b. Along the y-axis $(x=0)$

c. Along the path $y=2x$

29) Evaluate $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy+y^3}{x^2+y^2}$ using the results of previous problem.

Answer

The limit does not exist. The function approaches two different values along different paths.

30) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^4+y^2}$

a. Along the x-axis $(y=0)$

b. Along the y-axis $(x=0)$

c. Along the path $y=x^2$

31) Evaluate $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^4+y^2}$ using the results of previous problem.

Answer

The limit does not exist because the function approaches two different values along the paths.

In exercises 32 - 35, discuss the continuity of each function. Find the largest region in the xy-plane in which each function is continuous.

32) $f(x,y)=\sin(xy)$

33) $f(x,y)=\ln(x+y)$

Answer

The function f is continuous in the region $y>-x$.

34) $f(x,y)=e^{3xy}$

35) $f(x,y)=\dfrac{1}{xy}$

Answer

The function f is continuous at all points in the xy-plane except at points on the x- and y-axes.

In exercises 36 - 38, determine the region in which the function is continuous. Explain your answer.

36) $f(x,y)=\dfrac{x^2y}{x^2+y^2}$

37) $f(x,y)=\begin{cases} \dfrac{x^2y}{x^2+y^2} & if(x,y)\neq(0,0) \\ 0 & if(x,y)=(0,0) \end{cases}$

Hint:

Show that the function approaches different values along two different paths.

Answer

The function is continuous at $(0,0)$ since the limit of the function at $(0,0)$ is 0, the same value of $f(0,0)$.

38) $f(x,y)=\dfrac{\sin(x^2+y^2)}{x^2+y^2}$

39) Determine whether $g(x,y)=\dfrac{x^2-y^2}{x^2+y^2}$ is continuous at $(0,0)$.

Answer

The function is discontinuous at $(0,0)$. The limit at $(0,0)$ fails to exist and $g(0,0)$ does not exist.

40) Create a plot using graphing software to determine where the limit does not exist. Determine the region of the coordinate plane in which $f(x,y)=\dfrac{1}{x^2-y}$ is continuous.

41) Determine the region of the xy-plane in which the composite function $g(x,y)=\arctan(\frac{xy^2}{x+y})$ is continuous. Use technology to support your conclusion.

Answer

Since the function $\arctan x$ is continuous over $(-\infty,\infty)$, $g(x,y)=\arctan(\frac{xy^2}{x+y})$ is continuous where $z=\dfrac{xy^2}{x+y}$ is continuous. The inner function z is continuous on all points of the xy-plane except where $y=-x$. Thus, $g(x,y)=\arctan(\frac{xy^2}{x+y})$ is continuous on all points of the coordinate plane except at points at which $y=-x$.

42) Determine the region of the xy-plane in which $f(x,y)=\ln(x^2+y^2-1)$ is continuous. Use technology to support your conclusion. (Hint: Choose the range of values for x and y carefully!)

43) At what points in space is $g(x,y,z)=x^2+y^2-2z^2$ continuous?

Answer

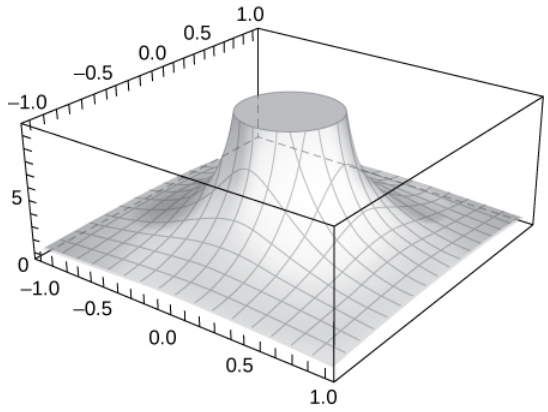
All points $P(x,y,z)$ in space

44) At what points in space is $g(x,y,z)=\dfrac{1}{x^2+z^2-1}$ continuous?

45) Show that $\lim\limits_{(x,y)\to(0,0)}\dfrac{1}{x^2+y^2}$ does not exist at $(0,0)$ by plotting the graph of the function.

Answer

The graph increases without bound as x and y both approach zero.



46) [T] Evaluate $\lim\limits_{(x,y)\to(0,0)}\dfrac{-xy^2}{x^2+y^4}$ by plotting the function using a CAS. Determine analytically the limit along the path $x=y^2$.

47) [T]

a. Use a CAS to draw a contour map of $z=\sqrt{9-x^2-y^2}$.

b. What is the name of the geometric shape of the level curves?

c. Give the general equation of the level curves.

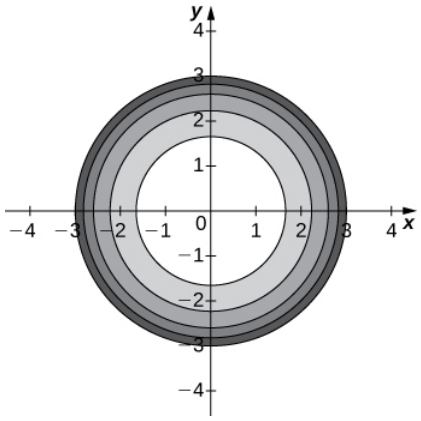
d. What is the maximum value of z?

e. What is the domain of the function?

f. What is the range of the function?

Answer

a.



b. The level curves are circles centered at $(0,0)$ with radius $9-c$.
c. $x^2+y^2=9-c$
d. $z=3$
e. $\{(x,y)\in R^2 \mid x^2+y^2\leq 9\}$
f. $\{z|0\leq z\leq 3\}$

48) True or False: If we evaluate $\lim\limits_{(x,y)\to(0,0)} f(x)$ along several paths and each time the limit is 1, we can conclude that $\lim\limits_{(x,y)\to(0,0)} f(x)=1$.

49) Use polar coordinates to find $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}$. You can also find the limit using L'Hôpital's rule.

Answer

$$\lim_{(x,y)\to(0,0)} \frac{\sin\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}=1$$

50) Use polar coordinates to find $\lim\limits_{(x,y)\to(0,0)} \cos(x^2+y^2)$.

51) Discuss the continuity of $f(g(x,y))$ where $f(t)=1/t$ and $g(x,y)=2x-5y$.

Answer

$f(g(x,y))$ is continuous at all points (x,y) that are not on the line $2x-5y=0$.

52) Given $f(x,y)=x^2-4y$, find $\lim\limits_{h\to 0} \dfrac{f(x+h,y)-f(x,y)}{h}$.

53) Given $f(x,y)=x^2-4y$, find $\lim\limits_{h\to 0} \dfrac{f(1+h,y)-f(1,y)}{h}$.

Answer

$$\lim_{h\to 0} \frac{f(1+h,y)-f(1,y)}{h}=2$$

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- Paul Seeburger (Monroe Community College) created problem 1.

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5.4: Partial Derivatives

Learning Objectives

- Calculate the partial derivatives of a function of two variables.
- Calculate the partial derivatives of a function of more than two variables.
- Determine the higher-order derivatives of a function of two variables.
- Explain the meaning of a partial differential equation and give an example.

Now that we have examined limits and continuity of functions of two variables, we can proceed to study derivatives. Finding derivatives of functions of two variables is the key concept in this chapter, with as many applications in mathematics, science, and engineering as differentiation of single-variable functions. However, we have already seen that limits and continuity of multivariable functions have new issues and require new terminology and ideas to deal with them. This carries over into differentiation as well.

Derivatives of a Function of Two Variables

When studying derivatives of functions of one variable, we found that one interpretation of the derivative is an instantaneous rate of change of y as a function of x. Leibniz notation for the derivative is dy/dx, which implies that y is the dependent variable and x is the independent variable. For a function $z = f(x, y)$ of two variables, x and y are the independent variables and z is the dependent variable. This raises two questions right away: How do we adapt Leibniz notation for functions of two variables? Also, what is an interpretation of the derivative? The answer lies in partial derivatives.

Definition: Partial Derivatives

Let $f(x, y)$ be a function of two variables. Then the *partial derivative* of f with respect to x, written as $\partial f/\partial x$, or f_x, is defined as

$$\frac{\partial f}{\partial x} = f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h} \tag{5.4.1}$$

The partial derivative of f with respect to y, written as $\partial f/\partial y$, or f_y, is defined as

$$\frac{\partial f}{\partial y} = f_y(x, y) = \lim_{k \to 0} \frac{f(x, y + k) - f(x, y)}{k}. \tag{5.4.2}$$

This definition shows two differences already. First, the notation changes, in the sense that we still use a version of Leibniz notation, but the d in the original notation is replaced with the symbol ∂. (This rounded "d" is usually called "partial," so $\partial f/\partial x$ is spoken as the "partial of f with respect to x.") This is the first hint that we are dealing with partial derivatives. Second, we now have two different derivatives we can take, since there are two different independent variables. Depending on which variable we choose, we can come up with different partial derivatives altogether, and often do.

Example 5.4.1: Calculating Partial Derivatives from the Definition

Use the definition of the partial derivative as a limit to calculate $\partial f/\partial x$ and $\partial f/\partial y$ for the function

$$f(x, y) = x^2 - 3xy + 2y^2 - 4x + 5y - 12.$$

Solution

First, calculate $f(x + h, y)$.

$$\begin{aligned} f(x + h, y) &= (x + h)^2 - 3(x + h)y + 2y^2 - 4(x + h) + 5y - 12 \\ &= x^2 + 2xh + h^2 - 3xy - 3hy + 2y^2 - 4x - 4h + 5y - 12. \end{aligned}$$

Next, substitute this into Equation 5.4.1 and simplify:

$$\begin{aligned}\frac{\partial f}{\partial x} &= \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}\\ &= \lim_{h\to 0}\frac{(x^2+2xh+h^2-3xy-3hy+2y^2-4x-4h+5y-12)-(x^2-3xy+2y^2-4x+5y-12)}{h}\\ &= \lim_{h\to 0}\frac{x^2+2xh+h^2-3xy-3hy+2y^2-4x-4h+5y-12-x^2+3xy-2y^2+4x-5y+12}{h}\\ &= \lim_{h\to 0}\frac{2xh+h^2-3hy-4h}{h}\\ &= \lim_{h\to 0}\frac{h(2x+h-3y-4)}{h}\\ &= \lim_{h\to 0}(2x+h-3y-4)\\ &= 2x-3y-4.\end{aligned}$$

To calculate $\frac{\partial f}{\partial y}$, first calculate $f(x,y+h)$:

$$\begin{aligned}f(x+h,y) &= x^2-3x(y+h)+2(y+h)^2-4x+5(y+h)-12\\ &= x^2-3xy-3xh+2y^2+4yh+2h^2-4x+5y+5h-12.\end{aligned}$$

Next, substitute this into Equation 5.4.2 and simplify:

$$\begin{aligned}\frac{\partial f}{\partial y} &= \lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}\\ &= \lim_{h\to 0}\frac{(x^2-3xy-3xh+2y^2+4yh+2h^2-4x+5y+5h-12)-(x^2-3xy+2y^2-4x+5y-12)}{h}\\ &= \lim_{h\to 0}\frac{x^2-3xy-3xh+2y^2+4yh+2h^2-4x+5y+5h-12-x^2+3xy-2y^2+4x-5y+12}{h}\\ &= \lim_{h\to 0}\frac{-3xh+4yh+2h^2+5h}{h}\\ &= \lim_{h\to 0}\frac{h(-3x+4y+2h+5)}{h}\\ &= \lim_{h\to 0}(-3x+4y+2h+5)\\ &= -3x+4y+5\end{aligned}$$



Exercise 5.4.1

Use the definition of the partial derivative as a limit to calculate $\partial f/\partial x$ and $\partial f/\partial y$ for the function

$$f(x,y) = 4x^2 + 2xy - y^2 + 3x - 2y + 5.$$

Hint

Use Equations 5.4.1and 5.4.2from the definition of partial derivatives.

Answer

$$\frac{\partial f}{\partial x} = 8x + 2y + 3$$
$$\frac{\partial f}{\partial y} = 2x - 2y - 2$$

The idea to keep in mind when calculating partial derivatives is to treat all independent variables, other than the variable with respect to which we are differentiating, as constants. Then proceed to differentiate as with a function of a single variable. To see why this is true, first fix y and define $g(x) = f(x,y)$ as a function of x. Then

$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h} \\ &= \frac{\partial f}{\partial x}. \end{aligned}$$

The same is true for calculating the partial derivative of f with respect to y. This time, fix x and define $h(y) = f(x,y)$ as a function of y. Then

$$\begin{aligned} h'(x) &= \lim_{k\to 0} \frac{h(x+k) - h(x)}{k} \\ &= \lim_{k\to 0} \frac{f(x,y+k) - f(x,y)}{k} \\ &= \frac{\partial f}{\partial y}. \end{aligned}$$

All differentiation rules apply.

Example 5.4.2: Calculating Partial Derivatives

Calculate $\partial f/\partial x$ and $\partial f/\partial y$ for the following functions by holding the opposite variable constant then differentiating:

a. $f(x,y) = x^2 - 3xy + 2y^2 - 4x + 5y - 12$
b. $g(x,y) = \sin(x^2 y - 2x + 4)$

Solution:

a. To calculate $\partial f/\partial x$, treat the variable y as a constant. Then differentiate $f(x,y)$ with respect to x using the sum, difference, and power rules:

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\left[x^2 - 3xy + 2y^2 - 4x + 5y - 12\right] \\ &= \frac{\partial}{\partial x}[x^2] - \frac{\partial}{\partial x}[3xy] + \frac{\partial}{\partial x}[2y^2] - \frac{\partial}{\partial x}[4x] + \frac{\partial}{\partial x}[5y] - \frac{\partial}{\partial x}[12] \\ &= 2x - 3y + 0 - 4 + 0 - 0 \\ &= 2x - 3y - 4. \end{aligned}$$

The derivatives of the third, fifth, and sixth terms are all zero because they do not contain the variable x, so they are treated as constant terms. The derivative of the second term is equal to the coefficient of x, which is $-3y$. Calculating $\partial f/\partial y$:

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left[x^2 - 3xy + 2y^2 - 4x + 5y - 12\right] \\ &= \frac{\partial}{\partial y}[x^2] - \frac{\partial}{\partial y}[3xy] + \frac{\partial}{\partial y}[2y^2] - \frac{\partial}{\partial y}[4x] + \frac{\partial}{\partial y}[5y] - \frac{\partial}{\partial y}[12] \\ &= -3x + 4y - 0 + 5 - 0 \\ &= -3x + 4y + 5.\end{aligned}$$

These are the same answers obtained in Example 5.4.1.

b. To calculate $\partial g/\partial x$, treat the variable y as a constant. Then differentiate $g(x, y)$ with respect to x using the chain rule and power rule:

$$\begin{aligned}\frac{\partial g}{\partial x} &= \frac{\partial}{\partial x}\left[\sin(x^2 y - 2x + 4)\right] \\ &= \cos(x^2 y - 2x + 4)\frac{\partial}{\partial x}[x^2 y - 2x + 4] \\ &= (2xy - 2)\cos(x^2 y - 2x + 4).\end{aligned}$$

To calculate $\partial g/\partial y$, treat the variable x as a constant. Then differentiate $g(x, y)$ with respect to y using the chain rule and power rule:

$$\begin{aligned}\frac{\partial g}{\partial y} &= \frac{\partial}{\partial y}\left[\sin(x^2 y - 2x + 4)\right] \\ &= \cos(x^2 y - 2x + 4)\frac{\partial}{\partial y}[x^2 y - 2x + 4] \\ &= x^2\cos(x^2 y - 2x + 4).\end{aligned}$$



Exercise 5.4.2

Calculate $\partial f/\partial x$ and $\partial f/\partial y$ for the function

$$f(x, y) = \tan(x^3 - 3x^2 y^2 + 2y^4)$$

by holding the opposite variable constant, then differentiating.

Hint

Use Equations 5.4.1and 5.4.1from the definition of partial derivatives.

Answer

$$\frac{\partial f}{\partial x} = (3x^2 - 6xy^2)\sec^2(x^3 - 3x^2y^2 + 2y^4)$$
$$\frac{\partial f}{\partial y} = (-6x^2y + 8y^3)\sec^2(x^3 - 3x^2y^2 + 2y^4)$$

How can we interpret these partial derivatives? Recall that the graph of a function of two variables is a surface in R^3. If we remove the limit from the definition of the partial derivative with respect to x, the difference quotient remains:

$$\frac{f(x+h,y) - f(x,y)}{h}. \tag{5.4.3}$$

This resembles the difference quotient for the derivative of a function of one variable, except for the presence of the y variable. Figure 5.4.1 illustrates a surface described by an arbitrary function $z = f(x,y)$.

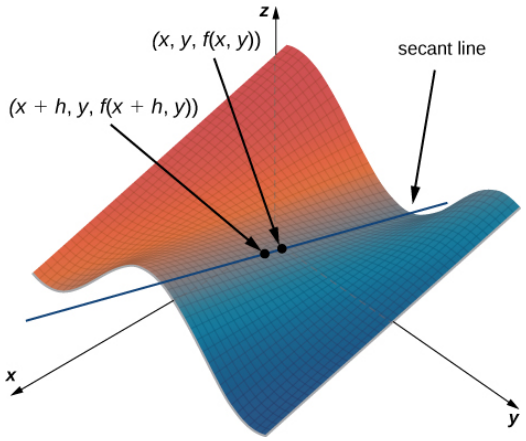


Figure 5.4.1: Secant line passing through the points $(x, y, f(x,y))$ and $(x+h, y, f(x+h,y))$.

In Figure 5.4.1, the value of h is positive. If we graph $f(x,y)$ and $f(x+h,y)$ for an arbitrary point (x,y), then the slope of the secant line passing through these two points is given by

$$\frac{f(x+h,y) - f(x,y)}{h}. \tag{5.4.4}$$

This line is parallel to the x-axis. Therefore, the slope of the secant line represents an average rate of change of the function f as we travel parallel to the x-axis. As h approaches zero, the slope of the secant line approaches the slope of the tangent line.

If we choose to change y instead of x by the same incremental value h, then the secant line is parallel to the y-axis and so is the tangent line. Therefore, $\partial f/\partial x$ represents the slope of the tangent line passing through the point $(x, y, f(x,y))$parallel to the x-axis and $\partial f/\partial y$ represents the slope of the tangent line passing through the point $(x, y, f(x,y))$parallel to the y-axis. If we wish to find the slope of a tangent line passing through the same point in any other direction, then we need what are called directional derivatives.

We now return to the idea of contour maps, which we introduced in Functions of Several Variables. We can use a **contour map** to estimate partial derivatives of a function $g(x,y)$.

Example 5.4.3: Partial Derivatives from a Contour Map

Use a contour map to estimate $\partial g/\partial x$ at the point $(\sqrt{5}, 0)$ for the function

$$g(x,y) = \sqrt{9 - x^2 - y^2}.$$

Solution

Figure 5.4.2 represents a contour map for the function $g(x,y)$.

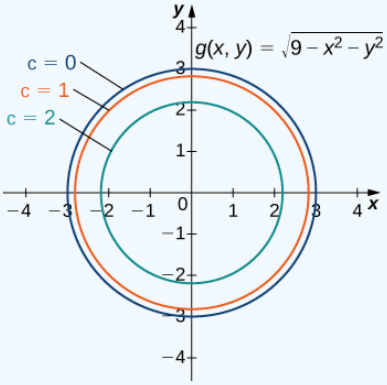


Figure 5.4.2: Contour map for the function $g(x,y)=\sqrt{9-x^2-y^2}$, using $c=0,1,2,$ and $3(c=3$ corresponds to the origin).

The inner circle on the contour map corresponds to $c=2$ and the next circle out corresponds to $c=1$. The first circle is given by the equation $2=\sqrt{9-x^2-y^2}$; the second circle is given by the equation $1=\sqrt{9-x^2-y^2}$. The first equation simplifies to $x^2+y^2=5$ and the second equation simplifies to $x^2+y^2=8$. The x-intercept of the first circle is $(\sqrt{5},0)$ and the x-intercept of the second circle is $(2\sqrt{2},0)$. We can estimate the value of $\partial g/\partial x$ evaluated at the point $(\sqrt{5},0)$ using the slope formula:

$$\begin{aligned}\left.\frac{\partial g}{\partial x}\right|_{(x,y)=(\sqrt{5},0)} &\approx \frac{g(\sqrt{5},0)-g(2\sqrt{2},0)}{\sqrt{5}-2\sqrt{2}}\\ &=\frac{2-1}{\sqrt{5}-2\sqrt{2}}\\ &=\frac{1}{\sqrt{5}-2\sqrt{2}}\approx -1.688.\end{aligned}$$

To calculate the exact value of $\partial g/\partial x$ evaluated at the point $(\sqrt{5},0)$, we start by finding $\partial g/\partial x$ using the chain rule. First, we rewrite the function as

$$g(x,y)=\sqrt{9-x^2-y^2}=(9-x^2-y^2)^{1/2}$$

and then differentiate with respect to x while holding y constant:

$$\begin{aligned}\frac{\partial g}{\partial x} &=\frac{1}{2}(9-x^2-y^2)^{-1/2}(-2x)\\ &=-\frac{x}{\sqrt{9-x^2-y^2}}.\end{aligned}$$

Next, we evaluate this expression using $x=\sqrt{5}$ and $y=0$:

$$\begin{aligned}\left.\frac{\partial g}{\partial x}\right|_{(x,y)=(\sqrt{5},0)} &=-\frac{\sqrt{5}}{\sqrt{9-(\sqrt{5})^2-(0)^2}}\\ &=-\frac{\sqrt{5}}{\sqrt{4}}\\ &=-\frac{\sqrt{5}}{2}\approx -1.118.\end{aligned}$$

The estimate for the partial derivative corresponds to the slope of the secant line passing through the points $(\sqrt{5},0,g(\sqrt{5},0))$ and $(2\sqrt{2},0,g(2\sqrt{2},0))$. It represents an approximation to the slope of the tangent line to the surface through the point $(\sqrt{5},0,g(\sqrt{5},0))$, which is parallel to the x-axis.

Exercise 5.4.3

Use a contour map to estimate $\partial f/\partial y$ at point $(0,\sqrt{2})$ for the function

$$f(x,y)=x^2-y^2.$$

Compare this with the exact answer.

Hint

Create a contour map for f using values of c from -3 to 3. Which of these curves passes through point $(0, \sqrt{2})$?

Answer

Using the curves corresponding to $c = -2$ and $c = -3$, we obtain

$$\begin{aligned} \left.\frac{\partial f}{\partial y}\right|_{(x,y)=(0,\sqrt{2})} &\approx \frac{f(0,\sqrt{3}) - f(0,\sqrt{2})}{\sqrt{3}-\sqrt{2}} \\ &= \frac{-3-(-2)}{\sqrt{3}-\sqrt{2}} \cdot \frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}} \\ &= -\sqrt{3}-\sqrt{2} \approx -3.146. \end{aligned}$$

The exact answer is

$$\left.\frac{\partial f}{\partial y}\right|_{(x,y)=(0,\sqrt{2})} = (-2y|_{(x,y)=(0,\sqrt{2})} = -2\sqrt{2} \approx -2.828.$$

Functions of More Than Two Variables

Suppose we have a function of three variables, such as $w = f(x, y, z)$. We can calculate partial derivatives of w with respect to any of the independent variables, simply as extensions of the definitions for partial derivatives of functions of two variables.

Definition: Partial Derivatives

Let $f(x, y, z)$ be a function of three variables. Then, the partial derivative of f with respect to x, written as $\partial f/\partial x$, or f_x, is defined to be

$$\frac{\partial f}{\partial x} = f_x(x, y, z) = \lim_{h\to 0} \frac{f(x+h, y, z) - f(x, y, z)}{h}. \tag{5.4.5}$$

The *partial derivative* of f with *respect* to y, written as $\partial f/\partial y$, or f_y, is defined to be

$$\frac{\partial f}{\partial y} = f_y(x, y, z) = \lim_{k\to 0} \frac{f(x, y+k, z) - f(x, y, z)}{k.} \tag{5.4.6}$$

The *partial derivative* of f with respect to z, written as $\partial f/\partial z$, or f_z, is defined to be

$$\frac{\partial f}{\partial z} = f_z(x, y, z) = \lim_{m\to 0} \frac{f(x, y, z+m) - f(x, y, z)}{m}. \tag{5.4.7}$$

We can calculate a partial derivative of a function of three variables using the same idea we used for a function of two variables. For example, if we have a function f of x, y, and z, and we wish to calculate $\partial f/\partial x$, then we treat the other two independent variables as if they are constants, then differentiate with respect to x.

Example 5.4.4: Calculating Partial Derivatives for a Function of Three Variables

Use the limit definition of partial derivatives to calculate $\partial f/\partial x$ for the function

$$f(x, y, z) = x^2 - 3xy + 2y^2 - 4xz + 5yz^2 - 12x + 4y - 3z.$$

Then, find $\partial f/\partial y$ and $\partial f/\partial z$ by setting the other two variables constant and differentiating accordingly.

Solution:

We first calculate $\partial f/\partial x$ using Equation 5.4.5, then we calculate the other two partial derivatives by holding the remaining variables constant. To use the equation to find $\partial f/\partial x$, we first need to calculate $f(x+h, y, z)$:

$$\begin{aligned} f(x+h, y, z) &= (x+h)^2 - 3(x+h)y + 2y^2 - 4(x+h)z + 5yz^2 - 12(x+h) + 4y - 3z \\ &= x^2 + 2xh + h^2 - 3xy - 3xh + 2y^2 - 4xz - 4hz + 5yz^2 - 12x - 12h + 4y - 3z \end{aligned}$$

and recall that $f(x, y, z) = x^2 - 3xy + 2y^2 - 4zx + 5yz^2 - 12x + 4y - 3z$. Next, we substitute these two expressions into the equation:

$$\frac{\partial f}{\partial x} = \lim_{h\to 0}\left[\frac{\begin{array}{l} x^2+2xh+h^2-3xy-3hy+2y^2-4xz-4hz+5yz^2-12x-12h+4y-3zh-x^2-3xy+2y^2-4xz+5yz^2 \\ -12x+4y-3z \end{array}}{h}\right]$$

$$= \lim_{h\to 0}\left[\frac{2xh+h^2-3hy-4hz-12h}{h}\right]$$

$$= \lim_{h\to 0}\left[\frac{h(2x+h-3y-4z-12)}{h}\right]$$

$$= \lim_{h\to 0}(2x+h-3y-4z-12)$$

$$= 2x-3y-4z-12.$$

Then we find $\partial f/\partial y$ by holding x and z constant. Therefore, any term that does not include the variable y is constant, and its derivative is zero. We can apply the sum, difference, and power rules for functions of one variable:

$$\frac{\partial}{\partial y}\left[x^2-3xy+2y^2-4xz+5yz^2-12x+4y-3z\right]$$

$$=\frac{\partial}{\partial y}[x^2]-\frac{\partial}{\partial y}[3xy]+\frac{\partial}{\partial y}[2y^2]-\frac{\partial}{\partial y}[4xz]+\frac{\partial}{\partial y}[5yz^2]-\frac{\partial}{\partial y}[12x]+\frac{\partial}{\partial y}[4y]-\frac{\partial}{\partial z}[3z]$$

$$=0-3x+4y-0+5z^2-0+4-0$$

$$=-3x+4y+5z^2+4.$$

To calculate $\partial f/\partial z$, we hold x and y constant and apply the sum, difference, and power rules for functions of one variable:

$$\frac{\partial}{\partial z}[x^2-3xy+2y^2-4xz+5yz^2-12x+4y-3z]$$

$$=\frac{\partial}{\partial z}[x^2]-\frac{\partial}{\partial z}[3xy]+\frac{\partial}{\partial z}[2y^2]-\frac{\partial}{\partial z}[4xz]+\frac{\partial}{\partial z}[5yz^2]-\frac{\partial}{\partial z}[12x]+\frac{\partial}{\partial z}[4y]-\frac{\partial}{\partial z}[3z]$$

$$=0-0+0-4x+10yz-0+0-3$$

$$=-4x+10yz-3$$

Exercise 5.4.4

Use the limit definition of partial derivatives to calculate $\partial f/\partial x$ for the function

$$f(x,y,z)=2x^2-4x^2y+2y^2+5xz^2-6x+3z-8.$$

Then find $\partial f/\partial y$ and $\partial f/\partial z$ by setting the other two variables constant and differentiating accordingly.

Hint

Use the strategy in the preceding example.

Answer

$\frac{\partial f}{\partial x}=4x-8xy+5z^2-6, \frac{\partial f}{\partial y}=-4x^2+4y, \frac{\partial f}{\partial z}=10xz+3$

Example 5.4.5: Calculating Partial Derivatives for a Function of Three Variables

Calculate the three partial derivatives of the following functions.

a. $f(x,y,z)=x^2y-4xz+y^2x-3yz$
b. $g(x,y,z)=\sin(x^2y-z)+\cos(x^2-yz)$

Solution

In each case, treat all variables as constants except the one whose partial derivative you are calculating.

a.

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[\frac{x^2y-4xz+y^2}{x-3yz}\right]$$

$$= \frac{\frac{\partial}{\partial x}(x^2y-4xz+y^2)(x-3yz)-(x^2y-4xz+y^2)\frac{\partial}{\partial x}(x-3yz)}{(x-3yz)^2}$$

$$= \frac{(2xy-4z)(x-3yz)-(x^2y-4xz+y^2)(1)}{(x-3yz)^2}$$

$$= \frac{2x^2y-6xy^2z-4xz+12yz^2-x^2y+4xz-y^2}{(x-3yz)^2}$$

$$= \frac{x^2y-6xy^2z-4xz+12yz^2+4xz-y^2}{(x-3yz)^2}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[\frac{x^2y-4xz+y^2}{x-3yz}\right]$$

$$= \frac{\frac{\partial}{\partial y}(x^2y-4xz+y^2)(x-3yz)-(x^2y-4xz+y^2)\frac{\partial}{\partial y}(x-3yz)}{(x-3yz)^2}$$

$$= \frac{(x^2+2y)(x-3yz)-(x^2y-4xz+y^2)(-3z)}{(x-3yz)^2}$$

$$= \frac{x^3-3x^2yz+2xy-6y^2z+3x^2yz-12xz^2+3y^2z}{(x-3yz)^2}$$

$$= \frac{x^3+2xy-3y^2z-12xz^2}{(x-3yz)^2}$$

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left[\frac{x^2y-4xz+y^2}{x-3yz}\right]$$

$$= \frac{\frac{\partial}{\partial z}(x^2y-4xz+y^2)(x-3yz)-(x^2y-4xz+y^2)\frac{\partial}{\partial z}(x-3yz)}{(x-3yz)^2}$$

$$= \frac{(-4x)(x-3yz)-(x^2y-4xz+y^2)(-3y)}{(x-3yz)^2}$$

$$= \frac{-4x^2+12xyz+3x^2y^2-12xyz+3y^3}{(x-3yz)^2}$$

$$= \frac{-4x^2+3x^2y^2+3y^3}{(x-3yz)^2}$$

b.

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[\sin(x^2y-z)+\cos(x^2-yz)\right]$$

$$= (\cos(x^2y-z))\frac{\partial}{\partial x}(x^2y-z)-(\sin(x^2-yz))\frac{\partial}{\partial x}(x^2-yz)$$

$$= 2xy\cos(x^2y-z)-2x\sin(x^2-yz)$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[\sin(x^2y-z)+\cos(x^2-yz)]$$

$$= (\cos(x^2y-z))\frac{\partial}{\partial y}(x^2y-z)-(\sin(x^2-yz))\frac{\partial}{\partial y}(x^2-yz)$$

$$= x^2\cos(x^2y-z)+z\sin(x^2-yz)$$

$$\begin{aligned}\frac{\partial f}{\partial z} &= \frac{\partial}{\partial z}[\sin(x^2y-z)+\cos(x^2-yz)] \\ &= (\cos(x^2y-z))\frac{\partial}{\partial z}(x^2y-z)-(\sin(x^2-yz))\frac{\partial}{\partial z}(x^2-yz) \\ &= -\cos(x^2y-z)+y\sin(x^2-yz)\end{aligned}$$

Exercise 5.4.5

Calculate $\partial f/\partial x, \partial f/\partial y,$ and $\partial f/\partial z$ for the function

$$f(x,y,z)=\sec(x^2y)-\tan(x^3yz^2).$$

Hint

Use the strategy in the preceding example.

Answer

$$\frac{\partial f}{\partial x}=2xy\sec(x^2y)\tan(x^2y)-3x^2yz^2\sec^2(x^3yz^2)$$

$$\frac{\partial f}{\partial y}=x^2\sec(x^2y)\tan(x^2y)-x^3z^2\sec^2(x^3yz^2)$$

$$\frac{\partial f}{\partial z}=-2x^3yz\sec^2(x^3yz^2)$$

Higher-Order Partial Derivatives

Consider the function

$$f(x,y)=2x^3-4xy^2+5y^3-6xy+5x-4y+12. \tag{5.4.8}$$

Its partial derivatives are

$$\frac{\partial f}{\partial x}=6x^2-4y^2-6y+5 \tag{5.4.9}$$

and

$$\frac{\partial f}{\partial y}=-8xy+15y^2-6x-4. \tag{5.4.10}$$

Each of these partial derivatives is a function of two variables, so we can calculate partial derivatives of these functions. Just as with derivatives of single-variable functions, we can call these *second-order derivatives*, third-order derivatives, and so on. In general, they are referred to as **higher-order partial derivatives.** There are four second-order partial derivatives for any function (provided they all exist):

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right] \\ \frac{\partial^2 f}{\partial y\partial x} &= \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] \\ \frac{\partial^2 f}{\partial x\partial y} &= \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] \\ \frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right].\end{aligned}$$

An alternative notation for each is $f_{xx}, f_{xy}, f_{yx},$ and $f_{yy},$ respectively. Higher-order partial derivatives calculated with respect to different variables, such as f_{xy} and $f_{yx},$ are commonly called **mixed partial derivatives.**

Example 5.4.6: Calculating Second Partial Derivatives

Calculate all four second partial derivatives for the function

$$f(x,y)=xe^{-3y}+\sin(2x-5y). \tag{5.4.11}$$

Solution:

To calculate $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial y \partial x}$, we first calculate $\partial f/\partial x$:

$$\frac{\partial f}{\partial x} = e^{-3y} + 2\cos(2x - 5y). \tag{5.4.12}$$

To calculate $\frac{\partial^2 f}{\partial x^2}$, differentiate $\partial f/\partial x$ (Equation 5.4.12) with respect to x:

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right] \\ &= \frac{\partial}{\partial x}[e^{-3y} + 2\cos(2x - 5y)] \\ &= -4\sin(2x - 5y).\end{aligned}$$

To calculate $\frac{\partial^2 f}{\partial y \partial x}$, differentiate $\partial f/\partial x$ (Equation 5.4.12) with respect to y:

$$\begin{aligned}\frac{\partial^2 f}{\partial y\, \partial x} &= \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] \\ &= \frac{\partial}{\partial y}[e^{-3y} + 2\cos(2x - 5y)] \\ &= -3e^{-3y} + 10\sin(2x - 5y).\end{aligned}$$

To calculate $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y^2}$, first calculate $\partial f/\partial y$:

$$\frac{\partial f}{\partial y} = -3xe^{-3y} - 5\cos(2x - 5y). \tag{5.4.13}$$

To calculate $\frac{\partial^2 f}{\partial x \partial y}$, differentiate $\partial f/\partial y$ (Equation 5.4.13) with respect to x:

$$\begin{aligned}\frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] \\ &= \frac{\partial}{\partial x}[-3xe^{-3y} - 5\cos(2x - 5y)] \\ &= -3e^{-3y} + 10\sin(2x - 5y).\end{aligned}$$

To calculate $\frac{\partial^2 f}{\partial y^2}$, differentiate $\partial f/\partial y$ (Equation 5.4.13) with respect to y:

$$\begin{aligned}\frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right] \\ &= \frac{\partial}{\partial y}[-3xe^{-3y} - 5\cos(2x - 5y)] \\ &= 9xe^{-3y} - 25\sin(2x - 5y).\end{aligned}$$

Exercise 5.4.6

Calculate all four second partial derivatives for the function

$$f(x, y) = \sin(3x - 2y) + \cos(x + 4y).$$

Hint

Follow the same steps as in the previous example.

Answer

$$\frac{\partial^2 f}{\partial x^2} = -9\sin(3x-2y) - \cos(x+4y)$$
$$\frac{\partial^2 f}{\partial y \partial x} = 6\sin(3x-2y) - 4\cos(x+4y)$$
$$\frac{\partial^2 f}{\partial x \partial y} = 6\sin(3x-2y) - 4\cos(x+4y)$$
$$\frac{\partial^2 f}{\partial y^2} = -4\sin(3x-2y) - 16\cos(x+4y)$$

At this point we should notice that, in both Example 5.4.6 and the checkpoint, it was true that $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$. Under certain conditions, this is always true. In fact, it is a direct consequence of the following theorem.

Equality of Mixed Partial Derivatives (Clairaut's Theorem)

Suppose that $f(x,y)$ is defined on an open disk D that contains the point (a,b). If the functions f_{xy} and f_{yx} are continuous on D, then $f_{xy} = f_{yx}$.

Clairaut's theorem guarantees that as long as mixed second-order derivatives are continuous, the order in which we choose to differentiate the functions (i.e., which variable goes first, then second, and so on) does not matter. It can be extended to higher-order derivatives as well. The proof of Clairaut's theorem can be found in most advanced calculus books.

Two other second-order partial derivatives can be calculated for any function $f(x,y)$. The partial derivative f_{xx} is equal to the partial derivative of f_x with respect to x, and f_{yy} is equal to the partial derivative of f_y with respect to y.

Partial Differential Equations

Previously, we studied differential equations in which the unknown function had one independent variable. A partial differential equation is an equation that involves an unknown function of more than one independent variable and one or more of its partial derivatives. Examples of partial differential equations are

$$u_t = c^2(u_{xx} + u_{yy}) \tag{5.4.14}$$
heat equation in two dimensions

$$u_{tt} = c^2(u_{xx} + u_{yy}) \tag{5.4.15}$$
wave equation in two dimensions

$$u_{xx} + u_{yy} = 0 \tag{5.4.16}$$
Laplace's equation in two dimensions

In the heat and wave equations, the unknown function u has three independent variables: t, x, and y with c is an arbitrary constant. The independent variables x and y are considered to be spatial variables, and the variable t represents time. In Laplace's equation, the unknown function u has two independent variables x and y.

Example 5.4.7: A Solution to the Wave Equation

Verify that

$$u(x,y,t) = 5\sin(3\pi x)\sin(4\pi y)\cos(10\pi t) \tag{5.4.17}$$

is a solution to the wave equation

$$u_{tt} = 4(u_{xx} + u_{yy}). \tag{5.4.18}$$

Solution

First, we calculate u_{tt}, u_{xx}, and u_{yy}:

$$
\begin{aligned}
u_{tt}(x,y,t) &= \frac{\partial}{\partial t}\left[\frac{\partial u}{\partial t}\right] \\
&= \frac{\partial}{\partial t}[5\sin(3\pi x)\sin(4\pi y)(-10\pi\sin(10\pi t))] \\
&= \frac{\partial}{\partial t}\left[-50\pi\sin(3\pi x)\sin(4\pi y)\sin(10\pi t)\right] \\
&= -500\pi^2\sin(3\pi x)\sin(4\pi y)\cos(10\pi t) \\
u_{xx}(x,y,t) &= \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x}\right] \\
&= \frac{\partial}{\partial x}\left[15\pi\cos(3\pi x)\sin(4\pi y)\cos(10\pi t)\right] \\
&= -45\pi^2\sin(3\pi x)\sin(4\pi y)\cos(10\pi t) \\
u_{yy}(x,y,t) &= \frac{\partial}{\partial y}\left[\frac{\partial u}{\partial y}\right] \\
&= \frac{\partial}{\partial y}\left[5\sin(3\pi x)(4\pi\cos(4\pi y))\cos(10\pi t)\right] \\
&= \frac{\partial}{\partial y}\left[20\pi\sin(3\pi x)\cos(4\pi y)\cos(10\pi t)\right] \\
&= -80\pi^2\sin(3\pi x)\sin(4\pi y)\cos(10\pi t).
\end{aligned}
$$

Next, we substitute each of these into the right-hand side of Equation 5.4.18 and simplify:

$$
\begin{aligned}
4(u_{xx}+u_{yy}) &= 4(-45\pi^2\sin(3\pi x)\sin(4\pi y)\cos(10\pi t) + -80\pi^2\sin(3\pi x)\sin(4\pi y)\cos(10\pi t)) \\
&= 4(-125\pi^2\sin(3\pi x)\sin(4\pi y)\cos(10\pi t)) \\
&= -500\pi^2\sin(3\pi x)\sin(4\pi y)\cos(10\pi t) \\
&= u_{tt}.
\end{aligned}
$$

This verifies the solution.



Exercise 5.4.7: A Solution to the Heat Equation

Verify that

$$u(x,y,t) = 2\sin\left(\frac{x}{3}\right)\sin\left(\frac{y}{4}\right)e^{-25t/16}$$

is a solution to the heat equation

$$u_t = 9(u_{xx} + u_{yy}).$$

Hint

Calculate the partial derivatives and substitute into the right-hand side.

Answer

TBA

Since the solution to the two-dimensional heat equation is a function of three variables, it is not easy to create a visual representation of the solution. We can graph the solution for fixed values of t, which amounts to snapshots of the heat distributions at fixed times. These snapshots show how the heat is distributed over a two-dimensional surface as time progresses. The graph of the preceding solution at time $t = 0$ appears in Figure 5.4.3. As time progresses, the extremes level out, approaching zero as t approaches infinity.

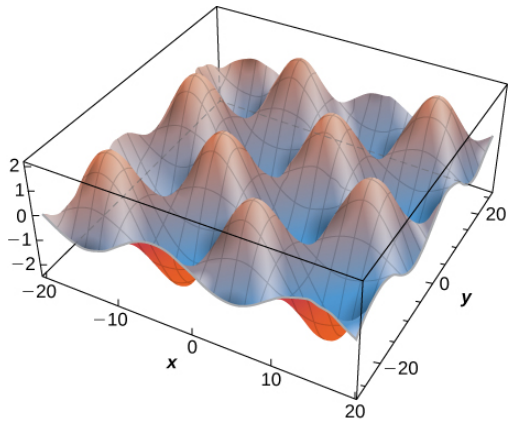


Figure 5.4.3

If we consider the heat equation in one dimension, then it is possible to graph the solution over time. The heat equation in one dimension becomes

$$u_t = c^2 u_{xx},$$

where c^2 represents the thermal diffusivity of the material in question. A solution of this differential equation can be written in the form

$$u_m(x,t) = e^{-\pi^2 m^2 c^2 t}\sin(m\pi x)$$

where m is any positive integer. A graph of this solution using $m = 1$ appears in Figure 5.4.4, where the initial temperature distribution over a wire of length 1 is given by $u(x,0) = \sin \pi x$. Notice that as time progresses, the wire cools off. This is seen because, from left to right, the highest temperature (which occurs in the middle of the wire) decreases and changes color from red to blue.

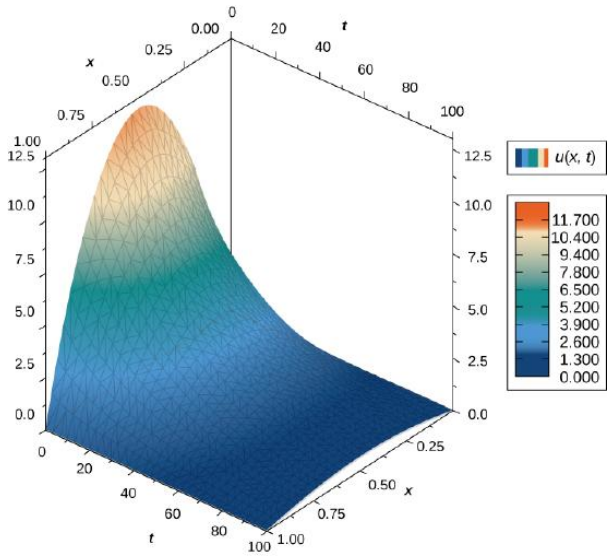
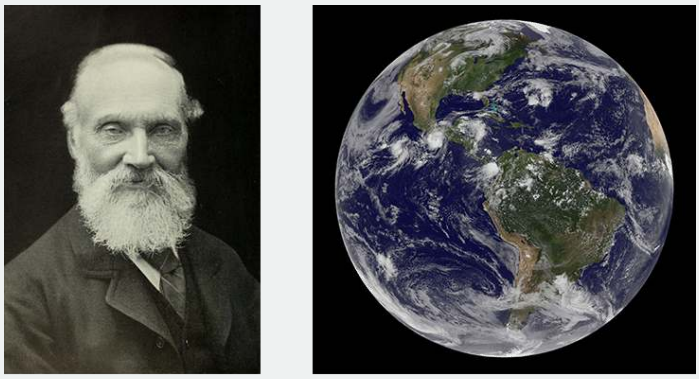


Figure 5.4.4: Graph of a solution of the heat equation in one dimension over time.

Lord Kelvin and the Age of Earth

During the late 1800s, the scientists of the new field of geology were coming to the conclusion that Earth must be “millions and millions” of years old. At about the same time, Charles Darwin had published his treatise on evolution. Darwin’s view was that evolution needed many millions of years to take place, and he made a bold claim that the Weald chalk fields, where important fossils were found, were the result of 300 million years of erosion.



(a) (b)

Figure 5.4.5: (a) William Thomson (Lord Kelvin), 1824-1907, was a British physicist and electrical engineer; (b) Kelvin used the heat diffusion equation to estimate the age of Earth (credit: modification of work by NASA).

At that time, eminent physicist William Thomson (Lord Kelvin) used an important partial differential equation, known as the heat diffusion equation, to estimate the age of Earth by determining how long it would take Earth to cool from molten rock to what we had at that time. His conclusion was a range of **20** to **400** million years, but most likely about **50** million years. For many decades, the proclamations of this irrefutable icon of science did not sit well with geologists or with Darwin.

- Read Kelvin’s paper on estimating the age of the Earth.

Kelvin made reasonable assumptions based on what was known in his time, but he also made several assumptions that turned out to be wrong. One incorrect assumption was that Earth is solid and that the cooling was therefore via conduction only, hence justifying the use of the diffusion equation. But the most serious error was a forgivable one—omission of the fact that Earth contains radioactive elements that continually supply heat beneath Earth’s mantle. The discovery of radioactivity came near the end of Kelvin’s life and he acknowledged that his calculation would have to be modified.

Kelvin used the simple one-dimensional model applied only to Earth’s outer shell, and derived the age from graphs and the roughly known temperature gradient near Earth’s surface. Let’s take a look at a more appropriate version of the diffusion equation in radial coordinates, which has the form

$$\frac{\partial T}{\partial t} = K\left[\frac{\partial^2 T}{\partial^2 r} + \frac{2}{r}\frac{\partial T}{\partial r}\right] \tag{5.4.19}$$

.

Here, $T(r,t)$ is temperature as a function of r (measured from the center of Earth) and time t. K is the heat conductivity—for molten rock, in this case. The standard method of solving such a partial differential equation is by separation of variables, where we express the solution as the product of functions containing each variable separately. In this case, we would write the temperature as

$$T(r,t) = R(r)f(t). \tag{5.4.20}$$

1. Substitute this form into Equation 5.4.19 and, noting that $f(t)$ is constant with respect to distance (r) and $R(r)$ is constant with respect to time (t), show that

$$\frac{1}{f}\frac{\partial f}{\partial t} = \frac{K}{R}\left[\frac{\partial^2 R}{\partial r^2} + \frac{2}{r}\frac{\partial R}{\partial r}\right]. \tag{5.4.21}$$

2. This equation represents the separation of variables we want. The left-hand side is only a function of t and the right-hand side is only a function of r, and they must be equal for all values of r and t. Therefore, they both must be equal to a constant. Let's call that constant $-\lambda^2$. (The convenience of this choice is seen on substitution.) So, we have

$$\frac{1}{f}\frac{\partial f}{\partial t} = -\lambda^2 \text{and} \frac{K}{R}\left[\frac{\partial^2 R}{\partial r^2} + \frac{2}{r}\frac{\partial R}{\partial r}\right] = -\lambda^2. \tag{5.4.22}$$

3. Now, we can verify through direct substitution for each equation that the solutions are $f(t) = Ae^{-\lambda^2 t}$ and $R(r) = B\left(\frac{\sin \alpha r}{r}\right) + C\left(\frac{\cos \alpha r}{r}\right)$, where $\alpha = \lambda/\sqrt{K}$. Note that $f(t) = Ae^{+\lambda n^2 t}$ is also a valid solution, so we could have chosen $+\lambda^2$ for our constant. Can you see why it would not be valid for this case as time increases?
4. Let's now apply boundary conditions.
   a. The temperature must be finite at the center of Earth, $r = 0$. Which of the two constants, B or C, must therefore be zero to keep R finite at $r = 0$? (Recall that $\sin(\alpha r)/r \to \alpha =$ as $r \to 0$, but $\cos(\alpha r)/r$ behaves very differently.)
   b. Kelvin argued that when magma reaches Earth's surface, it cools very rapidly. A person can often touch the surface within weeks of the flow. Therefore, the surface reached a moderate temperature very early and remained nearly constant at a surface temperature T_s. For simplicity, let's set $T = 0$ at $r = R_E$ and find α such that this is the temperature there for all time t. (Kelvin took the value to be $300K \approx 80^\circ F$. We can add this $300K$ constant to our solution later.) For this to be true, the sine argument must be zero at $r = R_E$. Note that α has an infinite series of values that satisfies this condition. Each value of α represents a valid solution (each with its own value for A). The total or general solution is the sum of all these solutions.
   c. At $t = 0$, we assume that all of Earth was at an initial hot temperature T_0 (Kelvin took this to be about $7000K$.) The application of this boundary condition involves the more advanced application of Fourier coefficients. As noted in part b. each value of α_n represents a valid solution, and the general solution is a sum of all these solutions. This results in a series solution:

$$T(r,t) = \left(\frac{T_0 R_E}{\pi}\right)\sum_n \frac{(-1)^{n-1}}{n} e^{-\lambda n^2 t}\frac{\sin(\alpha_n r)}{r}$$

   where $\alpha_n = n\pi/R_E$.

Note how the values of α_n come from the boundary condition applied in part b. The term $\frac{-1^{n-1}}{n}$ is the constant A_n for each term in the series, determined from applying the Fourier method. Letting $\beta = \frac{\pi}{R_E}$, examine the first few terms of this solution shown here and note how λ^2 in the exponential causes the higher terms to decrease quickly as time progresses:

$$T(r,t) \tag{5.4.23}$$
$$= \frac{T_0 R_E}{\pi r}\left(e^{-K\beta^2 t}(\sin \beta r) - \frac{1}{2}e^{-4K\beta^2 t}(\sin 2\beta r) + \frac{1}{3}e^{-9K\beta^2 t}(\sin 3\beta r) - \frac{1}{4}e^{-16K\beta^2 t}(\sin 4\beta r) + \frac{1}{5}e^{-25K\beta^2 t}(\sin 5\beta r)\ldots\right).$$

Near time $t = 0$, many terms of the solution are needed for accuracy. Inserting values for the conductivity K and $\beta = \pi/R_E$ for time approaching merely thousands of years, only the first few terms make a significant contribution. Kelvin only needed to look at the solution near Earth's surface (Figure 5.4.6) and, after a long time, determine what time best yielded the estimated temperature gradient known during his era ($1^\circ F$ increase per $50ft$). He simply chose a range of times with a gradient close to this value. In Figure 5.4.6, the solutions are plotted and scaled, with the $300 - K$ surface temperature added. Note that the center of Earth would be relatively cool. At the time, it was thought Earth must be solid.

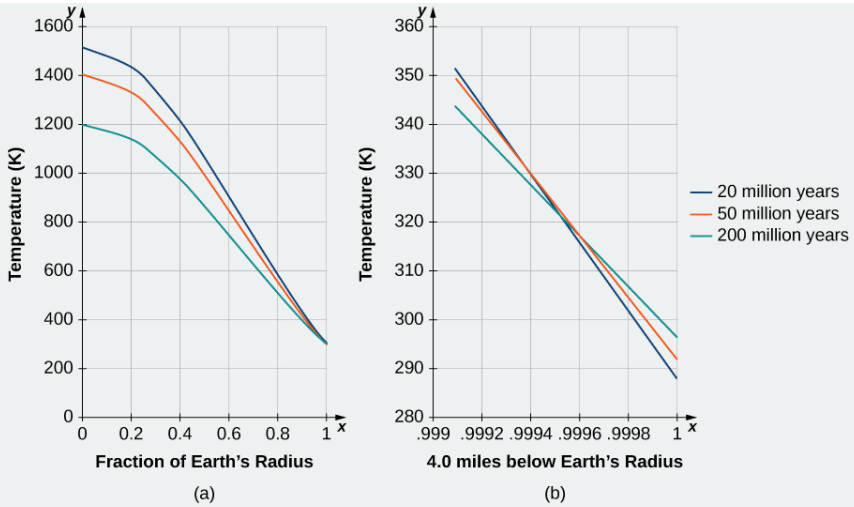


Figure 5.4.6: Temperature versus radial distance from the center of Earth. (a) Kelvin's results, plotted to scale. (b) A close-up of the results at a depth of 4.0 miles below Earth's surface.

Epilog

On May **20, 1904,** physicist **Ernest Rutherford** spoke at the Royal Institution to announce a revised calculation that included the contribution of radioactivity as a source of Earth's heat. In Rutherford's own words:

> "I came into the room, which was half-dark, and presently spotted Lord Kelvin in the audience, and realized that I was in for trouble at the last part of my speech dealing with the age of the Earth, where my views conflicted with his. To my relief, Kelvin fell fast asleep, but as I came to the important point, I saw the old bird sit up, open an eye and cock a baleful glance at me.
>
> Then a sudden inspiration came, and I said Lord Kelvin had limited the age of the Earth, provided no new source [of heat] was discovered. That prophetic utterance referred to what we are now considering tonight, radium! Behold! The old boy beamed upon me."

Rutherford calculated an age for Earth of about **500** million years. Today's accepted value of Earth's age is about **4.6** billion years.

Key Concepts

- A partial derivative is a derivative involving a function of more than one independent variable.
- To calculate a partial derivative with respect to a given variable, treat all the other variables as constants and use the usual differentiation rules.
- Higher-order partial derivatives can be calculated in the same way as higher-order derivatives.

Key Equations

Partial derivative of f with respect to x

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

Partial derivative of f with respect to y

$$\frac{\partial f}{\partial y} = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}$$

Glossary

higher-order partial derivatives
second-order or higher partial derivatives, regardless of whether they are mixed partial derivatives

mixed partial derivatives
second-order or higher partial derivatives, in which at least two of the differentiations are with respect to different variables

partial derivative
a derivative of a function of more than one independent variable in which all the variables but one are held constant

partial differential equation

an equation that involves an unknown function of more than one independent variable and one or more of its partial derivatives

Contributors and Attributions

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5.4E: Exercises for Section 5.4

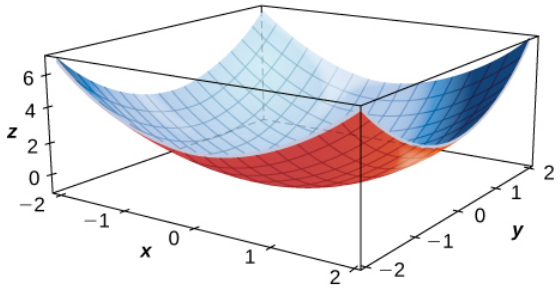
In the following exercise, calculate the partial derivative using the limit definitions only.

1) $\dfrac{\partial z}{\partial y}$ for $z = x^2 - 3xy + y^2$

Answer

$\dfrac{\partial z}{\partial y} = -3x + 2y$

For exercises 2 - 5, calculate the sign of the partial derivative using the graph of the surface.



2) $f_x(1,1)$

3) $f_x(-1,1)$

Answer

The sign is negative.

4) $f_y(1,1)$

5) $f_x(0,0)$

Answer

The partial derivative is zero at the origin.

In exercises 6 - 16, calculate the requested partial derivatives.

6) $\dfrac{\partial z}{\partial x}$ for $z = \sin(3x)\cos(3y)$

7) $\dfrac{\partial z}{\partial y}$ for $z = \sin(3x)\cos(3y)$

Answer

$\dfrac{\partial z}{\partial y} = -3\sin(3x)\sin(3y)$

8) $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ for $z = x^8 e^3 y$

9) $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ for $z = \ln(x^6 + y^4)$

Answer

$\dfrac{\partial z}{\partial x} = \frac{6x^5}{x^6+y^4}$; $\dfrac{\partial z}{\partial y} = \frac{4y^3}{x^6+y^4}$

10) Find $f_y(x,y)$ for $f(x,y) = e^{xy}\cos(x)\sin(y)$.

11) Let $z = e^{xy}$. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

Answer

$$\frac{\partial z}{\partial x} = ye^{xy}; \quad \frac{\partial z}{\partial y} = xe^{xy}$$

12) Let $z = \ln(\frac{x}{y})$. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

13) Let $z = \tan(2x - y)$. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

Answer

$$\frac{\partial z}{\partial x} = 2\sec^2(2x - y), \quad \frac{\partial z}{\partial y} = -\sec^2(2x - y)$$

14) Let $z = \sinh(2x + 3y)$. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

15) Let $f(x, y) = \arctan(\frac{y}{x})$. Evaluate $f_x(2, -2)$ and $f_y(2, -2)$.

Answer

$f_x(2, -2) = \frac{1}{4} = f_y(2, -2)$

16) Let $f(x, y) = \dfrac{xy}{x - y}$. Find $f_x(2, -2)$ and $f_y(2, -2)$.

17) Find $\dfrac{\partial z}{\partial x}$ at $(0, 1)$ for $z = e^{-x}cos(y)$.

Answer

$$\frac{\partial z}{\partial x} = -\cos(1)$$

18) Given $f(x, y, z) = x^3yz^2$, find $\dfrac{\partial^2 f}{\partial x \partial y}$ and $f_z(1, 1, 1)$.

19) Given $f(x, y, z) = 2\sin(x + y)$, find $f_x(0, \frac{\pi}{2}, -4)$, $f_y(0, \frac{\pi}{2}, -4)$, and $f_z(0, \frac{\pi}{2}, -4)$.

Answer

$$f_x(x, y, z) = 0, \quad f_y(x, y, z) = 0, \quad f_z(x, y, z) = 0$$

20) The area of a parallelogram with adjacent side lengths that are a and b, and in which the angle between these two sides is θ, is given by the function $A(a, b, \theta) = ba\sin(\theta)$. Find the rate of change of the area of the parallelogram with respect to the following:

a. Side a

b. Side b

c. Angle θ

21) Express the volume of a right circular cylinder as a function of two variables:

a. its radius r and its height h.

b. Show that the rate of change of the volume of the cylinder with respect to its radius is the product of its circumference multiplied by its height.

c. Show that the rate of change of the volume of the cylinder with respect to its height is equal to the area of the circular base.

Answer

a. $V(r,h)=\pi r^2 h$
b. $\dfrac{\partial V}{\partial r}=2\pi rh$
c. $\dfrac{\partial V}{\partial h}=\pi r^2$

22) Calculate $\dfrac{\partial w}{\partial z}$ for $w=z\sin(xy^2+2z)$.

In exercises 23 - 39, find the indicated higher-order partial derivatives.

23) $f_{xy}(x,y)$ for $f(x,y)=\ln(x-y)$

Answer

$f_{xy}(x,y)=\frac{1}{(x-y)^2}$

24) $f_{yx}(x,y)$ for $f(x,y)=\ln(x-y)$

25) Let $z=x^2+3xy+2y^2$. Find $\dfrac{\partial^2 z}{\partial x^2}$ and $\dfrac{\partial^2 z}{\partial y^2}$.

Answer

$\dfrac{\partial^2 z}{\partial x^2}=2, \quad \dfrac{\partial^2 z}{\partial y^2}=4$

26) Given $z=e^x\tan y$, find $\dfrac{\partial^2 z}{\partial x\partial y}$ and $\dfrac{\partial^2 z}{\partial y\partial x}$.

27) Given $f(x,y,z)=xyz$, find $f_{xyy}(x,y,z)$, $f_{yxy}(x,y,z)$, and $f_{yyx}(x,y,z)$.

Answer

$f_{xyy}(x,y,z)=f_{yxy}(x,y,z)=f_{yyx}(x,y,z)=0$

28) Given $f(x,y,z)=e^{-2x}\sin(z^2y)$, show that $f_{xyy}(x,y,z)=f_{yxy}(x,y,z)$.

29) Show that $z=\frac{1}{2}(e^y-e^{-y})\sin x$ is a solution of the differential equation $\dfrac{\partial^2 z}{\partial x^2}+\dfrac{\partial^2 z}{\partial y^2}=0$.

Answer

$\dfrac{d^2 z}{dx^2}=-\frac{1}{2}(e^y-e^{-y})\sin x$
$\dfrac{d^2 z}{dy^2}=\frac{1}{2}(e^y-e^{-y})\sin x$
$\dfrac{d^2 z}{dx^2}+\dfrac{d^2 z}{dy^2}=0$

30) Find $f_{xx}(x,y)$ for $f(x,y)=\frac{4x^2}{y}+\frac{y^2}{2x}$.

31) Let $f(x,y,z)=x^2y^3z-3xy^2z^3+5x^2z-y^3z$. Find f_{xyz}.

Answer

$f_{xyz}(x,y,z)=6y^2x-18yz^2$

32) Let $F(x,y,z)=x^3yz^2-2x^2yz+3xz-2y^3z$. Find $F_{xyz}(x,y,z)$.

33) Given $f(x,y)=x^2+x-3xy+y^3-5$, find all points at which $f_x(x,y)=f_y(x,y)=0$ simultaneously.

Answer

$(\frac{1}{4},\frac{1}{2}), \quad (1,1)$

34) Given $f(x,y)=2x^2+2xy+y^2+2x-3$, find all points at which $\dfrac{\partial f}{\partial x}=0$ and $\dfrac{\partial f}{\partial y}=0$ simultaneously.

35) Given $f(x,y)=y^3-3yx^2-3y^2-3x^2+1$, find all points on f at which $f_x(x,y)=f_y(x,y)=0$ simultaneously.

Answer

$(0,0), \quad (0,2), \quad (\sqrt{3},-1), \quad (-\sqrt{3},-1)$

36) Given $f(x,y)=15x^3-3xy+15y^3$, find all points at which $f_x(x,y)=f_y(x,y)=0$ simultaneously.

37) Show that $z=e^x\sin y$ satisfies the equation $\dfrac{\partial^2 z}{\partial x^2}+\dfrac{\partial^2 z}{\partial y^2}=0$.

Answer

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=e^x\sin y-e^x\sin y=0$$

38) Show that $f(x,y)=\ln(x^2+y^2)$ solves Laplace's equation $\dfrac{\partial^2 z}{\partial x^2}+\dfrac{\partial^2 z}{\partial y^2}=0$.

39) Show that $z=e^{-t}\cos(\frac{x}{c})$ satisfies the heat equation $\dfrac{\partial z}{\partial t}=-e^{-t}\cos(\frac{x}{c})$.

Answer

$$c^2\frac{\partial^2 z}{\partial x^2}=e^{-t}\cos(\tfrac{x}{c})$$

40) Find $\lim\limits_{\Delta x\to 0}\dfrac{f(x+\Delta x)-f(x,y)}{\Delta x}$ for $f(x,y)=-7x-2xy+7y$.

41) Find $\lim\limits_{\Delta y\to 0}\dfrac{f(x,y+\Delta y)-f(x,y)}{\Delta y}$ for $f(x,y)=-7x-2xy+7y$.

Answer

$$\frac{\partial f}{\partial y}=-2x+7$$

42) Find $\lim\limits_{\Delta x\to 0}\dfrac{\Delta f}{\Delta x}=\lim\limits_{\Delta x\to 0}\dfrac{f(x+\Delta x,y)-f(x,y)}{\Delta x}$ for $f(x,y)=x^2y^2+xy+y$.

43) Find $\lim\limits_{\Delta x\to 0}\dfrac{\Delta f}{\Delta x}=\lim\limits_{\Delta x\to 0}\dfrac{f(x+\Delta x,y)-f(x,y)}{\Delta x}$ for $f(x,y)=\sin(xy)$.

Answer

$$\frac{\partial f}{\partial x}=y\cos xy$$

44) The function $P(T,V)=\dfrac{nRT}{V}$ gives the pressure at a point in a gas as a function of temperature T and volume V. The letters n and R are constants. Find $\dfrac{\partial P}{\partial V}$ and $\dfrac{\partial P}{\partial T}$, and explain what these quantities represent.

45) The equation for heat flow in the xy-plane is $\dfrac{\partial f}{\partial t}=\dfrac{\partial^2 f}{\partial x^2}+\dfrac{\partial^2 f}{\partial y^2}$. Show that $f(x,y,t)=e^{-2t}\sin x\sin y$ is a solution.

46) The basic wave equation is $f_{tt}=f_{xx}$. Verify that $f(x,t)=\sin(x+t)$ and $f(x,t)=\sin(x-t)$ are solutions.

47) The law of cosines can be thought of as a function of three variables. Let x, y, and θ be two sides of any triangle where the angle θ is the included angle between the two sides. Then, $F(x,y,\theta)=x^2+y^2-2xy\cos\theta$ gives the square of the third side of the triangle. Find $\dfrac{\partial F}{\partial \theta}$ and $\dfrac{\partial F}{\partial x}$ when $x=2$, $y=3$, and $\theta=\frac{\pi}{6}$.

Answer

$$\frac{\partial F}{\partial \theta} = 6, \quad \frac{\partial F}{\partial x} = 4 - 3\sqrt{3}$$

48) Suppose the sides of a rectangle are changing with respect to time. The first side is changing at a rate of 2in./sec whereas the second side is changing at the rate of 4 in/sec. How fast is the diagonal of the rectangle changing when the first side measures 16 in. and the second side measures 20 in.? (Round answer to three decimal places.)

49) A **Cobb-Douglas production function** is $f(x, y) = 200x^{0.7}y^{0.3}$, where x and y represent the amount of labor and capital available. Let $x = 500$ and $y = 1000$. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at these values, which represent the marginal productivity of labor and capital, respectively.

Answer

$$\frac{\partial f}{\partial x} \text{ at } (500, 1000) = 172.36, \quad \frac{\partial f}{\partial y} \text{at } (500, 1000) = 36.93$$

50) The apparent temperature index is a measure of how the temperature feels, and it is based on two variables: h, which is relative humidity, and t, which is the air temperature.

$A = 0.885t - 22.4h + 1.20th - 0.544$. Find $\frac{\partial A}{\partial t}$ and $\frac{\partial A}{\partial h}$ when $t = 20^\circ F$ and $h = 0.90$.

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5.5: Tangent Planes and Linear Approximations

Learning Objectives

- Determine the equation of a plane tangent to a given surface at a point.
- Use the tangent plane to approximate a function of two variables at a point.
- Explain when a function of two variables is differentiable.
- Use the total differential to approximate the change in a function of two variables.

In this section, we consider the problem of finding the tangent plane to a surface, which is analogous to finding the equation of a tangent line to a curve when the curve is defined by the graph of a function of one variable, $y = f(x)$. The slope of the tangent line at the point $x = a$ is given by $m = f'(a)$; what is the slope of a tangent plane? We learned about the equation of a plane in Equations of Lines and Planes in Space; in this section, we see how it can be applied to the problem at hand.

Tangent Planes

Intuitively, it seems clear that, in a plane, only one line can be tangent to a curve at a point. However, in three-dimensional space, many lines can be tangent to a given point. If these lines lie in the same plane, they determine the tangent plane at that point. A more intuitive way to think of a tangent plane is to assume the surface is smooth at that point (no corners). Then, a tangent line to the surface at that point in any direction does not have any abrupt changes in slope because the direction changes smoothly. Therefore, in a small-enough neighborhood around the point, a tangent plane touches the surface at that point only.

Definition: tangent lines

Let $P_0 = (x_0, y_0, z_0)$ be a point on a surface S, and let C be any curve passing through P_0 and lying entirely in S. If the tangent lines to all such curves C at P_0 lie in the same plane, then this plane is called the *tangent plane* to S at P_0 (Figure 5.5.1).

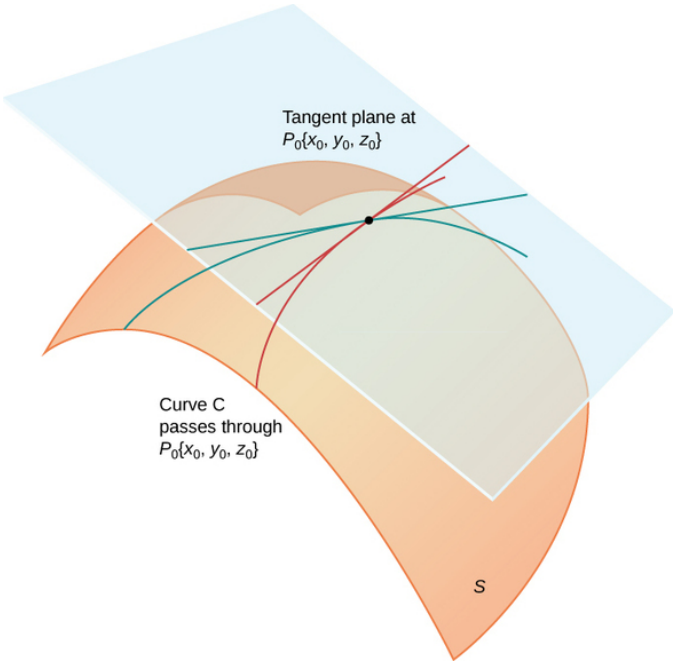
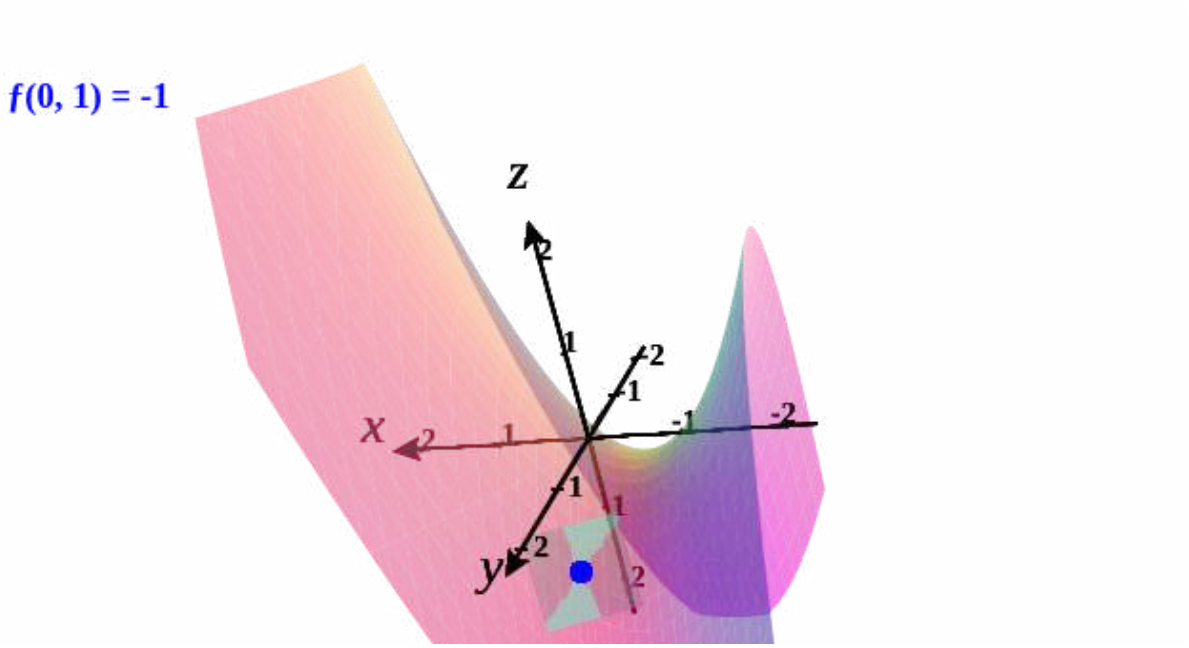


Figure 5.5.1: The tangent plane to a surface S at a point P_0 contains all the tangent lines to curves in S *that pass through* P_0.

Interactive Tangent Plane to a Surface

Below is an interactive tangent plane viewer. You can move the point's xy-coordinates and see the tangent plane drawn. You can also change the function.



For a tangent plane to a surface to exist at a point on that surface, it is sufficient for the function that defines the surface to be differentiable at that point. We define the term tangent plane here and then explore the idea intuitively.

Definition: tangent planes

Let S be a surface defined by a differentiable function $z = f(x, y)$, and let $P_0 = (x_0, y_0)$ be a point in the domain of f. Then, the equation of the tangent plane to S at P_0 is given by

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0). \tag{5.5.1}$$

To see why this formula is correct, let's first find two tangent lines to the surface S. The equation of the tangent line to the curve that is represented by the intersection of S with the vertical trace given by $x = x_0$ is $z = f(x_0, y_0) + f_y(x_0, y_0)(y - y_0)$. Similarly, the equation of the tangent line to the curve that is represented by the intersection of S with the vertical trace given by $y = y_0$ is $z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0)$. A parallel vector to the first tangent line is $\vec{\mathbf{a}} = \hat{\mathbf{j}} + f_y(x_0, y_0)\,\hat{\mathbf{k}}$; a parallel vector to the second tangent line is $\vec{\mathbf{b}} = \hat{\mathbf{i}} + f_x(x_0, y_0)\,\hat{\mathbf{k}}$. We can take the cross product of these two vectors:

$$\begin{aligned}
\vec{\mathbf{a}} \times \vec{\mathbf{b}} &= (\hat{\mathbf{j}} + f_y(x_0, y_0)\,\hat{\mathbf{k}}) \times (\hat{\mathbf{i}} + f_x(x_0, y_0)\,\hat{\mathbf{k}}) \\
&= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 1 & f_y(x_0, y_0) \\ 1 & 0 & f_x(x_0, y_0) \end{vmatrix} \\
&= f_x(x_0, y_0)\,\hat{\mathbf{i}} + f_y(x_0, y_0)\,\hat{\mathbf{j}} - \hat{\mathbf{k}}.
\end{aligned}$$

This vector is perpendicular to both lines and is therefore perpendicular to the tangent plane. We can use this vector as a normal vector to the tangent plane, along with the point $P_0 = (x_0, y_0, f(x_0, y_0))$ in the equation for a plane:

$$\begin{aligned}
\vec{\mathbf{n}} \cdot ((x - x_0)\,\hat{\mathbf{i}} + (y - y_0)\,\hat{\mathbf{j}} + (z - f(x_0, y_0))\,\hat{\mathbf{k}}) &= 0 \\
(f_x(x_0, y_0)\,\hat{\mathbf{i}} + f_y(x_0, y_0)\,\hat{\mathbf{j}} - \hat{\mathbf{k}}) \cdot ((x - x_0)\,\hat{\mathbf{i}} + (y - y_0)\,\hat{\mathbf{j}} + (z - f(x_0, y_0))\,\hat{\mathbf{k}}) &= 0 \\
f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - f(x_0, y_0)) &= 0.
\end{aligned}$$

Solving this equation for z gives Equation 5.5.1.

Example 5.5.1: Finding a Tangent Plane

Find the equation of the tangent plane to the surface defined by the function $f(x,y)=2x^2-3xy+8y^2+2x-4y+4$ at point $(2,-1)$.

Solution

First, we must calculate $f_x(x,y)$ and $f_y(x,y)$, then use Equation with $x_0=2$ and $y_0=-1$:

$$\begin{aligned} f_x(x,y) &= 4x-3y+2 \\ f_y(x,y) &= -3x+16y-4 \\ f(2,-1) &= 2(2)^2-3(2)(-1)+8(-1)^2+2(2)-4(-1)+4=34 \\ f_x(2,-1) &= 4(2)-3(-1)+2=13 \\ f_y(2,-1) &= -3(2)+16(-1)-4=-26. \end{aligned}$$

Then Equation 5.5.1 becomes

$$\begin{aligned} z &= f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0) \\ z &= 34+13(x-2)-26(y-(-1)) \\ z &= 34+13x-26-26y-26 \\ z &= 13x-26y-18. \end{aligned}$$

(See the following figure).

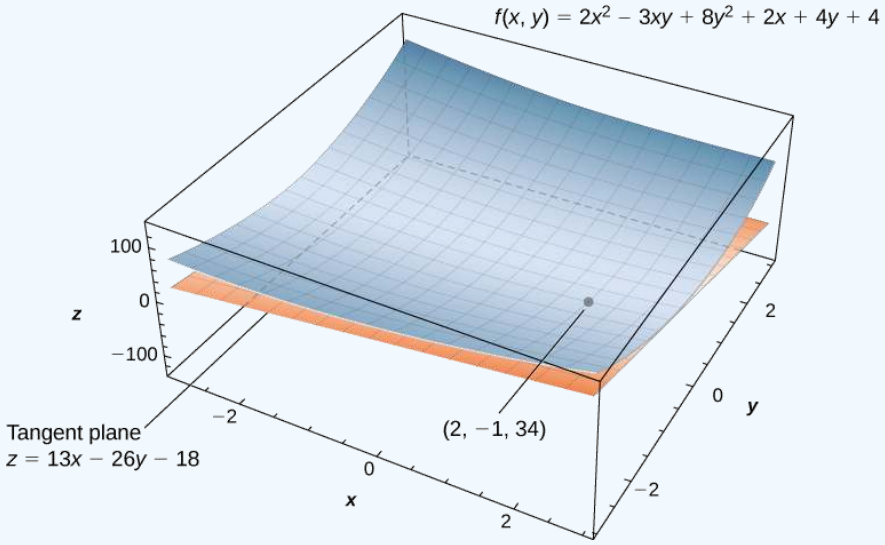


Figure 5.5.2: Calculating the equation of a tangent plane to a given surface at a given point.



Exercise 5.5.1

Find the equation of the tangent plane to the surface defined by the function $f(x,y)=x^3-x^2y+y^2-2x+3y-2$ at point $(-1,3)$.

Hint

First, calculate $f_x(x,y)$ and $f_y(x,y)$, then use Equation 5.5.1.

Answer

$z=7x+8y-3$

Example 5.5.2: Finding Another Tangent Plane

Find the equation of the tangent plane to the surface defined by the function $f(x,y)=\sin(2x)\cos(3y)$ at the point $(\pi/3,\pi/4)$.

Solution

First, calculate $f_x(x,y)$ and $f_y(x,y)$, then use Equation 5.5.1 with $x_0=\pi/3$ and $y_0=\pi/4$:

$$\begin{aligned} f_x(x,y) &= 2\cos(2x)\cos(3y) \\ f_y(x,y) &= -3\sin(2x)\sin(3y) \\ f\left(\frac{\pi}{3},\frac{\pi}{4}\right) &= \sin\left(2\left(\frac{\pi}{3}\right)\right)\cos\left(3\left(\frac{\pi}{4}\right)\right)=\left(\frac{\sqrt{3}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right)=-\frac{\sqrt{6}}{4} \\ f_x\left(\frac{\pi}{3},\frac{\pi}{4}\right) &= 2\cos\left(2\left(\frac{\pi}{3}\right)\right)\cos\left(3\left(\frac{\pi}{4}\right)\right)=2\left(-\frac{1}{2}\right)\left(-\frac{\sqrt{2}}{2}\right)=\frac{\sqrt{2}}{2} \\ f_y\left(\frac{\pi}{3},\frac{\pi}{4}\right) &= -3\sin\left(2\left(\frac{\pi}{3}\right)\right)\sin\left(3\left(\frac{\pi}{4}\right)\right)=-3\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)=-\frac{3\sqrt{6}}{4}. \end{aligned}$$

Then Equation 5.5.1 becomes

$$\begin{aligned} z &= f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0) \\ &= -\frac{\sqrt{6}}{4}+\frac{\sqrt{2}}{2}\left(x-\frac{\pi}{3}\right)-\frac{3\sqrt{6}}{4}\left(y-\frac{\pi}{4}\right) \\ &= \frac{\sqrt{2}}{2}x-\frac{3\sqrt{6}}{4}y-\frac{\sqrt{6}}{4}-\frac{\pi\sqrt{2}}{6}+\frac{3\pi\sqrt{6}}{16} \end{aligned}$$

A tangent plane to a surface does not always exist at every point on the surface. Consider the piecewise function

$$f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}. \tag{5.5.2}$$

The graph of this function follows.

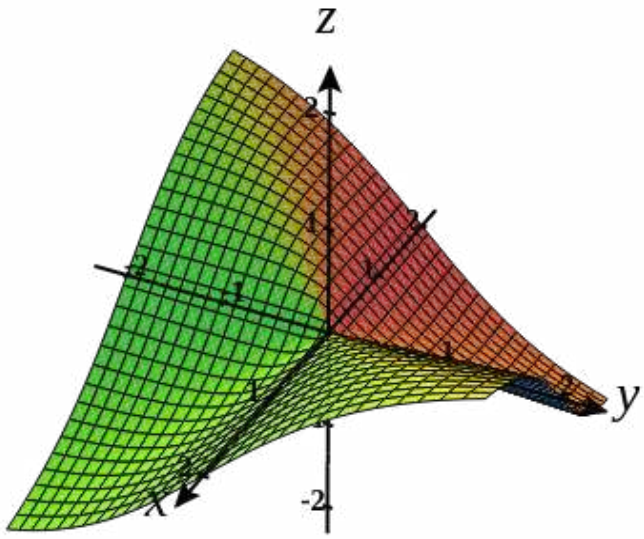


Figure* 5.5.3*: *Graph of a function that does not have a tangent plane at the origin. Dynamic figure powered by CalcPlot3D.*

If either $x=0$ or $y=0$, then $f(x,y)=0$, so the value of the function does not change on either the x- or y-axis. Therefore, $f_x(x,0)=f_y(0,y)=0$, so as either x or y approach zero, these partial derivatives stay equal to zero. Substituting them into Equation gives $z=0$ as the equation of the tangent line. However, if we approach the origin from a different direction, we get a different story. For example, suppose we approach the origin along the line $y=x$. If we put $y=x$ into the original function, it becomes

$$f(x,x)=\frac{x(x)}{\sqrt{x^2+(x)^2}}=\frac{x^2}{\sqrt{2x^2}}=\frac{|x|}{\sqrt{2}}. \tag{5.5.3}$$

When $x>0$, the slope of this curve is equal to $\sqrt{2}/2$; when $x<0$, the slope of this curve is equal to $-(\sqrt{2}/2)$. This presents a problem. In the definition of tangent plane, we presumed that all tangent lines through point P (in this case, the origin) lay in the same plane. This is clearly not the case here. When we study differentiable functions, we will see that this function is not differentiable at the origin.

Linear Approximations

Recall from Linear Approximations and Differentials that the formula for the linear approximation of a function $f(x)$ at the point $x=a$ is given by

$$y\approx f(a)+f'(a)(x-a). \tag{5.5.4}$$

The diagram for the linear approximation of a function of one variable appears in the following graph.

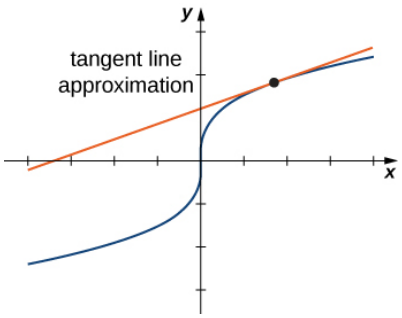


Figure 5.5.4: Linear approximation of a function in one variable.

The tangent line can be used as an approximation to the function $f(x)$ for values of x reasonably close to $x = a$. When working with a function of two variables, the tangent line is replaced by a tangent plane, but the approximation idea is much the same.

Definition: Linear Approximation

Given a function $z = f(x, y)$ with continuous partial derivatives that exist at the point (x_0, y_0), the linear approximation of f at the point (x_0, y_0) is given by the equation

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0). \tag{5.5.5}$$

Notice that this equation also represents the tangent plane to the surface defined by $z = f(x, y)$ at the point (x_0, y_0). The idea behind using a linear approximation is that, if there is a point (x_0, y_0) at which the precise value of $f(x, y)$ is known, then for values of (x, y) reasonably close to (x_0, y_0), the linear approximation (i.e., tangent plane) yields a value that is also reasonably close to the exact value of $f(x, y)$ (Figure). Furthermore the plane that is used to find the linear approximation is also the tangent plane to the surface at the point (x_0, y_0).

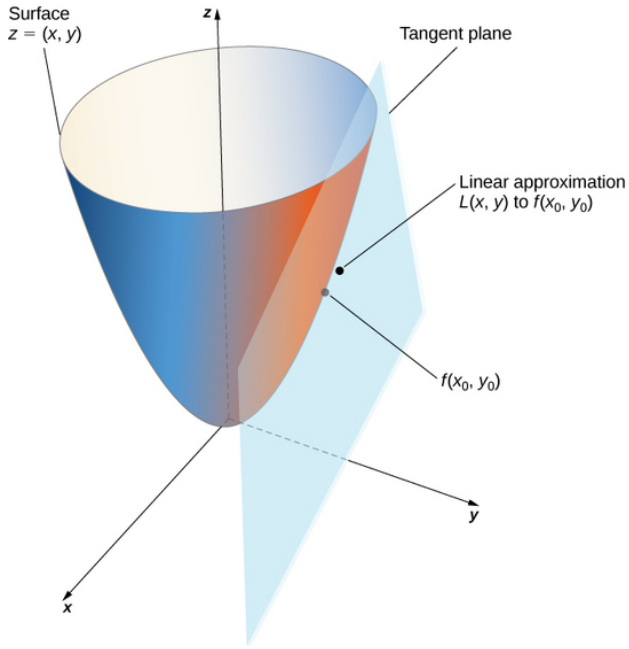


Figure 5.5.5: Using a tangent plane for linear approximation at a point.

Example 5.5.3: Using a Tangent Plane Approximation

Given the function $f(x, y) = \sqrt{41 - 4x^2 - y^2}$, approximate $f(2.1, 2.9)$ using point $(2, 3)$ for (x_0, y_0). What is the approximate value of $f(2.1, 2.9)$ to four decimal places?

Solution

To apply Equation 5.5.5, we first must calculate $f(x_0, y_0)$, $f_x(x_0, y_0)$, and $f_y(x_0, y_0)$ using $x_0 = 2$ and $y_0 = 3$:

$$f(x_0,y_0) = f(2,3) = \sqrt{41-4(2)^2-(3)^2} = \sqrt{41-16-9} = \sqrt{16} = 4$$

$$f_x(x,y) = -\frac{4x}{\sqrt{41-4x^2-y^2}} \text{ so } f_x(x_0,y_0) = -\frac{4(2)}{\sqrt{41-4(2)^2-(3)^2}} = -2$$

$$f_y(x,y) = -\frac{y}{\sqrt{41-4x^2-y^2}} \text{ so } f_y(x_0,y_0) = -\frac{3}{\sqrt{41-4(2)^2-(3)^2}} = -\frac{3}{4}.$$

Now we substitute these values into Equation 5.5.5:

$$\begin{aligned} L(x,y) &= f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0) \\ &= 4-2(x-2)-\frac{3}{4}(y-3) \\ &= \frac{41}{4}-2x-\frac{3}{4}y. \end{aligned}$$

Last, we substitute $x = 2.1$ and $y = 2.9$ into $L(x,y)$:

$$L(2.1,2.9) = \frac{41}{4}-2(2.1)-\frac{3}{4}(2.9) = 10.25-4.2-2.175 = 3.875.$$

The approximate value of $f(2.1,2.9)$to four decimal places is

$$f(2.1,2.9) = \sqrt{41-4(2.1)^2-(2.9)^2} = \sqrt{14.95} \approx 3.8665,$$

which corresponds to a 0.2 error in approximation.



Exercise 5.5.2

Given the function $f(x,y) = e^{5-2x+3y}$, approximate $f(4.1,0.9)$using point $(4,1)$ for (x_0,y_0). What is the approximate value of $f(4.1,0.9)$to four decimal places?

Hint

First calculate $f(x_0,y_0)$, $f_x(x_0,y_0)$, and $f_y(x_0,y_0)$ using $x_0 = 4$ and $y_0 = 1$, then use Equation 5.5.5

Answer

$L(x,y) = 6-2x+3y$, so $L(4.1,0.9) = 6-2(4.1)+3(0.9) = 0.5$ $f(4.1,0.9) = e^{5-2(4.1)+3(0.9)} = e^{-0.5} \approx 0.6065$.

Differentiability

When working with a function $y = f(x)$ of one variable, the function is said to be differentiable at a point $x = a$ if $f'(a)$ exists. Furthermore, if a function of one variable is differentiable at a point, the graph is "smooth" at that point (i.e., no corners exist) and a tangent line is well-defined at that point.

The idea behind differentiability of a function of two variables is connected to the idea of smoothness at that point. In this case, a surface is considered to be smooth at point P if a tangent plane to the surface exists at that point. If a function is differentiable at a point, then a tangent plane to the surface exists at that point. Recall the formula (Equation 5.5.1) for a tangent plane at a point (x_0, y_0) is given by

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

For a tangent plane to exist at the point (x_0, y_0), the partial derivatives must therefore exist at that point. However, this is not a sufficient condition for smoothness, as was illustrated in Figure. In that case, the partial derivatives existed at the origin, but the function also had a corner on the graph at the origin.

Definition: differentiable Functions

A function $f(x, y)$ is *differentiable* at a point $P(x_0, y_0)$ if, for all points (x, y) in a δ disk around P, we can write

$$f(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + E(x, y), \tag{5.5.6}$$

where the error term E satisfies

$$\lim_{(x,y)\to(x_0,y_0)} \frac{E(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0. \tag{5.5.7}$$

The last term in Equation 5.5.6 is to as the **error term** and it represents how closely the tangent plane comes to the surface in a small neighborhood (δ disk) of point P. For the function f to be differentiable at P, the function must be smooth—that is, the graph of f must be close to the tangent plane for points near P.

Example 5.5.4: Demonstrating Differentiability

Show that the function $f(x, y) = 2x^2 - 4y$ is differentiable at point $(2, -3)$.

Solution

First, we calculate $f(x_0, y_0)$, $f_x(x_0, y_0)$, and $f_y(x_0, y_0)$ using $x_0 = 2$ and $y_0 = -3$, then we use Equation 5.5.6:

$$\begin{aligned} f(2, -3) &= 2(2)^2 - 4(-3) = 8 + 12 = 20 \\ f_x(2, -3) &= 4(2) = 8 \\ f_y(2, -3) &= -4. \end{aligned}$$

Therefore $m_1 = 8$ and $m_2 = -4$, and Equation 5.5.6 becomes

$$\begin{aligned} f(x, y) &= f(2, -3) + f_x(2, -3)(x - 2) + f_y(2, -3)(y + 3) + E(x, y) \\ 2x^2 - 4y &= 20 + 8(x - 2) - 4(y + 3) + E(x, y) \\ 2x^2 - 4y &= 20 + 8x - 16 - 4y - 12 + E(x, y) \\ 2x^2 - 4y &= 8x - 4y - 8 + E(x, y) \\ E(x, y) &= 2x^2 - 8x + 8. \end{aligned}$$

Next, we calculate the limit in Equation 5.5.7:

$$\begin{aligned}\lim_{(x,y)\to(x_0,y_0)}\frac{E(x,y)}{\sqrt{(x-x+0)^2+(y-y_0)^2}} &= \lim_{(x,y)\to(2,-3)}\frac{2x^2-8x+8}{\sqrt{(x-2)^2+(y+3)^2}}\\ &= \lim_{(x,y)\to(2,-3)}\frac{2(x^2-4x+4)}{\sqrt{(x-2)^2+(y+3)^2}}\\ &= \lim_{(x,y)\to(2,-3)}\frac{2(x-2)^2}{\sqrt{(x-2)^2+(y+3)^2}}\\ &= \lim_{(x,y)\to(2,-3)}\frac{2((x-2)^2+(y+3)^2)}{\sqrt{(x-2)^2+(y+3)^2}}\\ &= \lim_{(x,y)\to(2,-3)}2\sqrt{(x-2)^2+(y+3)^2}\\ &= 0.\end{aligned}$$

Since $E(x,y)\geq 0$ for any value of x or y, the original limit must be equal to zero. Therefore, $f(x,y)=2x^2-4y$ is differentiable at point $(2,-3)$.

Exercise 5.5.3

Show that the function $f(x,y)=3x-4y^2$ is differentiable at point $(-1,2)$.

Hint

First, calculate $f(x_0,y_0)$, $f_x(x_0,y_0)$, and $f_y(x_0,y_0)$ using $x_0=-1$ and $y_0=2$, then use Equation 5.5.7 to find $E(x,y)$. Last, calculate the limit.

Answer

$$f(-1,2)=-19,\quad f_x(-1,2)=3,\quad f_y(-1,2)=-16,\quad E(x,y)=-4(y-2)^2.$$

$$\begin{aligned}\lim_{(x,y)\to(x_0,y_0)}\frac{E(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} &= \lim_{(x,y)\to(-1,2)}\frac{-4(y-2)^2}{\sqrt{(x+1)^2+(y-2)^2}}\\ &\leq \lim_{(x,y)\to(-1,2)}\frac{-4((x+1)^2+(y-2)^2)}{\sqrt{(x+1)^2+(y-2)^2}}\\ &= \lim_{(x,y)\to(-1,2)}-4\sqrt{(x+1)^2+(y-2)^2}\\ &= 0.\end{aligned}$$

This function from (Equation 5.5.2)

$$f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

is not differentiable at the origin (Figure 5.5.3). We can see this by calculating the partial derivatives. This function appeared earlier in the section, where we showed that $f_x(0,0)=f_y(0,0)=0$. Substituting this information into Equations 5.5.6 and 5.5.7 using $x_0=0$ and $y_0=0$, we get

$$\begin{aligned}f(x,y) &= f(0,0)+f_x(0,0)(x-0)+f_y(0,0)(y-0)+E(x,y)\\ E(x,y) &= \frac{xy}{\sqrt{x^2+y^2}}.\end{aligned}$$

Calculating

$$\lim_{(x,y)\to(x_0,y_0)}\frac{E(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} \tag{5.5.8}$$

gives

$$\lim_{(x,y)\to(x_0,y_0)} \frac{E(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = \lim_{(x,y)\to(0,0)} \frac{\frac{xy}{\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}}$$
$$= \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}.$$

Depending on the path taken toward the origin, this limit takes different values. Therefore, the limit does not exist and the function f is not differentiable at the origin as shown in the following figure.

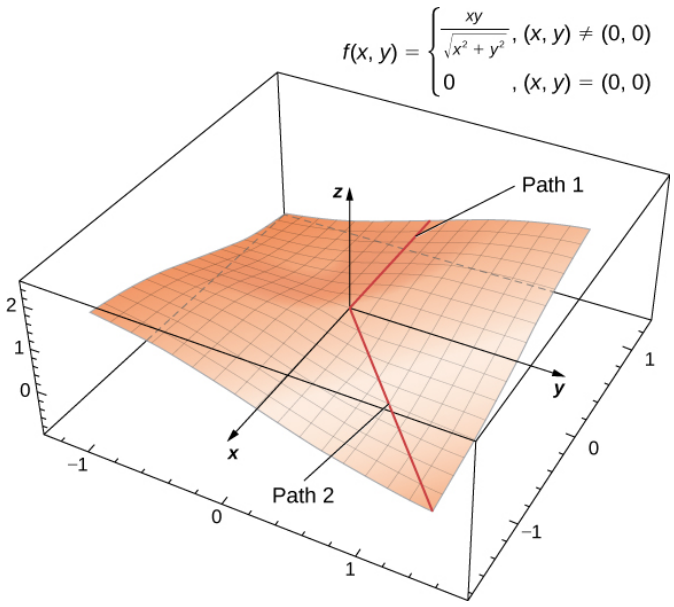


Figure 5.5.6: This function $f(x,y)$ (Equation 5.5.2) is not differentiable at the origin.

Differentiability and continuity for functions of two or more variables are connected, the same as for functions of one variable. In fact, with some adjustments of notation, the basic theorem is the same.

THEOREM: Differentiability Implies Continuity

Let $z=f(x,y)$ be a function of two variables with (x_0,y_0) in the domain of f. If $f(x,y)$ is differentiable at (x_0,y_0), then $f(x,y)$ is continuous at (x_0,y_0).

Note shows that if a function is differentiable at a point, then it is continuous there. However, if a function is continuous at a point, then it is not necessarily differentiable at that point. For example, the function discussed above (Equation 5.5.2)

$$f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

is *continuous* at the origin, but it is *not differentiable* at the origin. This observation is also similar to the situation in single-variable calculus.

We can further explores the connection between continuity and differentiability at a point. This next theorem says that if the function and its partial derivatives are continuous at a point, the function is differentiable.

Theorem: Continuity of First Partials Implies Differentiability

Let $z=f(x,y)$ be a function of two variables with (x_0,y_0) in the domain of f. If $f(x,y)$, $f_x(x,y)$, and $f_y(x,y)$ all exist in a neighborhood of (x_0,y_0) and are continuous at (x_0,y_0), then $f(x,y)$ is differentiable there.

Recall that earlier we showed that the function in Equation 5.5.2 was not differentiable at the origin. Let's calculate the partial derivatives f_x and f_y:

$$\frac{\partial f}{\partial x} = \frac{y^3}{(x^2+y^2)^{3/2}} \tag{5.5.9}$$

and

$$\frac{\partial f}{\partial y} = \frac{x^3}{(x^2+y^2)^{3/2}}. \tag{5.5.10}$$

The contrapositive of the preceding theorem states that if a function is not differentiable, then at least one of the hypotheses must be false. Let's explore the condition that $f_x(0,0)$ must be continuous. For this to be true, it must be true that

$$\lim_{(x,y)\to(0,0)} f_x(x,y) = f_x(0,0) \tag{5.5.11}$$

therefor

$$\lim_{(x,y)\to(0,0)} f_x(x,y) = \lim_{(x,y)\to(0,0)} \frac{y^3}{(x^2+y^2)^{3/2}}. \tag{5.5.12}$$

Let $x = ky$. Then

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{y^3}{(x^2+y^2)^{3/2}} &= \lim_{y\to 0} \frac{y^3}{((ky)^2+y^2)^{3/2}} \\
&= \lim_{y\to 0} \frac{y^3}{(k^2y^2+y^2)^{3/2}} \\
&= \lim_{y\to 0} \frac{y^3}{|y|^3(k^2+1)^{3/2}} \\
&= \frac{1}{(k^2+1)^{3/2}} \lim_{y\to 0} \frac{|y|}{y}.
\end{aligned}$$

If $y > 0$, then this expression equals $1/(k^2+1)^{3/2}$; if $y < 0$, then it equals $-(1/(k^2+1)^{3/2})$. In either case, the value depends on k, so the limit fails to exist.

Differentials

In Linear Approximations and Differentials we first studied the concept of differentials. The differential of y, written dy, is defined as $f'(x)dx$. The differential is used to approximate $\Delta y = f(x+\Delta x) - f(x)$, where $\Delta x = dx$. Extending this idea to the linear approximation of a function of two variables at the point (x_0, y_0) yields the formula for the total differential for a function of two variables.

Definition: Total Differential

Let $z = f(x,y)$ be a function of two variables with (x_0, y_0) in the domain of f, and let Δx and Δy be chosen so that $(x_0+\Delta x, y_0+\Delta y)$ is also in the domain of f. If f is differentiable at the point (x_0, y_0), then the differentials dx and dy are defined as

$$dx = \Delta x \tag{5.5.13}$$

and

$$dy = \Delta y. \tag{5.5.14}$$

The differential dz, also called the ***total differential*** of $z = f(x,y)$ at (x_0, y_0), is defined as

$$dz = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy. \tag{5.5.15}$$

Notice that the symbol ∂ is not used to denote the total differential; rather, d appears in front of z. Now, let's define $\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y)$. We use dz to approximate Δz, so

$$\Delta z \approx dz = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy. \tag{5.5.16}$$

Therefore, the differential is used to approximate the change in the function $z = f(x_0, y_0)$ at the point (x_0, y_0) for given values of Δx and Δy. Since $\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y)$, this can be used further to approximate $f(x+\Delta x, y+\Delta y)$:

$$f(x+\Delta x, y+\Delta y) = f(x,y) + \Delta z \approx f(x,y) + fx(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y. \tag{5.5.17}$$

See the following figure.

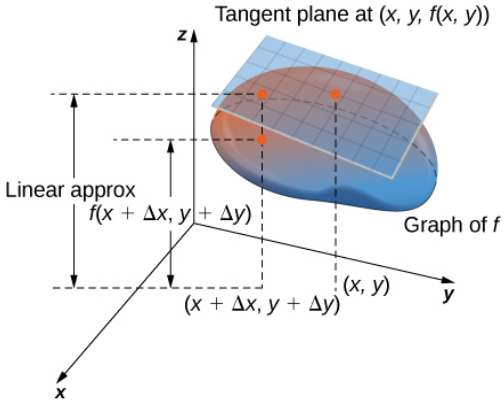


Figure 5.5.7: The linear approximation is calculated via the formula $f(x+\Delta x, y+\Delta y) \approx f(x,y) + f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y$.

One such application of this idea is to determine error propagation. For example, if we are manufacturing a gadget and are off by a certain amount in measuring a given quantity, the differential can be used to estimate the error in the total volume of the gadget.

Example 5.5.5: Approximation by Differentials

Find the differential dz of the function $f(x,y) = 3x^2 - 2xy + y^2$ and use it to approximate Δz at point $(2,-3)$. Use $\Delta x = 0.1$ and $\Delta y = -0.05$. What is the exact value of Δz?

Solution

First, we must calculate $f(x_0,y_0)$, $f_x(x_0,y_0)$, and $f_y(x_0,y_0)$ using $x_0 = 2$ and $y_0 = -3$:

$$\begin{aligned} f(x_0,y_0) &= f(2,-3) = 3(2)^2 - 2(2)(-3) + (-3)^2 = 12+12+9 = 33 \\ f_x(x,y) &= 6x - 2y \end{aligned}$$

$$\begin{aligned} f_y(x,y) &= -2x + 2y \\ f_x(x_0,y_0) &= fx(2,-3) \\ &= 6(2) - 2(-3) = 12+6 = 18 \end{aligned}$$

$$\begin{aligned} f_y(x_0,y_0) &= f_y(2,-3) \\ &= -2(2) + 2(-3) \\ &= -4 - 6 = -10. \end{aligned}$$

Then, we substitute these quantities into Equation 5.5.15:

$$\begin{aligned} dz &= f_x(x_0,y_0)dx + f_y(x_0,y_0)dy \\ dz &= 18(0.1) - 10(-0.05) = 1.8 + 0.5 = 2.3. \end{aligned}$$

This is the approximation to $\Delta z = f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0)$. The exact value of Δz is given by

$$\begin{aligned} \Delta z &= f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0) \\ &= f(2+0.1, -3-0.05) - f(2,-3) \\ &= f(2.1,-3.05) - f(2,-3) \\ &= 2.3425. \end{aligned}$$



Exercise 5.5.4

Find the differential dz of the function $f(x,y) = 4y^2 + x^2y - 2xy$ and use it to approximate Δz at point $(1,-1)$. Use $\Delta x = 0.03$ and $\Delta y = -0.02$. What is the exact value of Δz?

Hint

First, calculate $f_x(x_0,y_0)$ and $f_y(x_0,y_0)$ using $x_0 = 1$ and $y_0 = -1$, then use Equation 5.5.15

Answer

$dz = 0.18$

$\Delta z = f(1.03,-1.02) - f(1,-1) = 0.180682$

Differentiability of a Function of Three Variables

All of the preceding results for differentiability of functions of two variables can be generalized to functions of three variables. First, the definition:

Definition: Differentiability at a point

A function $f(x,y,z)$ is differentiable at a point $P(x_0,y_0,z_0)$ if for all points (x,y,z) in a δ disk around P we can write

$$f(x,y) = f(x_0,y_0,z_0) + f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0) + E(x,y,z), \quad (5.5.18)$$

where the error term E satisfies

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} \frac{E(x,y,z)}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}} = 0. \quad (5.5.19)$$

If a function of three variables is differentiable at a point (x_0,y_0,z_0), then it is continuous there. Furthermore, continuity of first partial derivatives at that point guarantees differentiability.

Key Concepts

- The analog of a tangent line to a curve is a tangent plane to a surface for functions of two variables.
- Tangent planes can be used to approximate values of functions near known values.
- A function is differentiable at a point if it is ”smooth” at that point (i.e., no corners or discontinuities exist at that point).
- The total differential can be used to approximate the change in a function $z = f(x_0,y_0)$ at the point (x_0,y_0) for given values of Δx and Δy.

Key Equations

- **Tangent plane**

  $z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$

- **Linear approximation**

  $L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$

- **Total differential**

  $dz = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy$.

- **Differentiability (two variables)**

  $f(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + E(x, y),$

  where the error term E satisfies

  $$\lim_{(x,y)\to(x_0,y_0)} \frac{E(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0\,.$$

- **Differentiability (three variables)**

  $f(x, y) = f(x_0, y_0, z_0) + f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) + E(x, y, z),$

  where the error term E satisfies

  $$\lim_{(x,y,z)\to(x_0,y_0,z_0)} \frac{E(x, y, z)}{\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}} = 0\,.$$

Glossary

differentiable

a function $f(x, y)$ is differentiable at (x_0, y_0) if $f(x, y)$ can be expressed in the form $f(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + E(x, y),$

where the error term $E(x, y)$ satisfies $\lim_{(x,y)\to(x_0,y_0)} \frac{E(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0$

linear approximation

given a function $f(x, y)$ and a tangent plane to the function at a point (x_0, y_0), we can approximate $f(x, y)$ for points near (x_0, y_0) using the tangent plane formula

tangent plane

given a function $f(x, y)$ that is differentiable at a point (x_0, y_0), the equation of the tangent plane to the surface $z = f(x, y)$ is given by $z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$

total differential

the total differential of the function $f(x, y)$ at (x_0, y_0) is given by the formula $dz = f_x(x_0, y_0)dx + fy(x_0, y_0)dy$

Contributors and Attributions

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5.5E: Exercises for Section 5.5

In exercises 1 - 2, find a unit normal vector to the surface at the indicated point.

1) $f(x,y)=x^3, \quad (2,-1,8)$

Answer

$(\frac{\sqrt{145}}{145})(12\hat{\mathbf{i}}-\hat{\mathbf{k}})$

2) $\ln\left(\dfrac{x}{y-z}\right)=0$ when $x=y=1$

In exercises 3 - 7, find a normal vector and a tangent vector at point P.

3) $x^2+xy+y^2=3, \quad P(-1,-1)$

Answer

Normal vector: $\hat{\mathbf{i}}+\hat{\mathbf{j}}$, tangent vector: $\hat{\mathbf{i}}-\hat{\mathbf{j}}$

4) $(x^2+y^2)^2=9(x^2-y^2), \quad P(\sqrt{2},1)$

5) $xy^2-2x^2+y+5x=6, \quad P(4,2)$

Answer

Normal vector: $7\hat{\mathbf{i}}-17\hat{\mathbf{j}}$, tangent vector: $17\hat{\mathbf{i}}+7\hat{\mathbf{j}}$

6) $2x^3-x^2y^2=3x-y-7, \quad P(1,-2)$

7) $ze^{x^2-y^2}-3=0, \quad P(2,2,3)$

Answer

Normal vector: $-12\hat{\mathbf{i}}+12\hat{\mathbf{j}}-\hat{\mathbf{k}}$ or $12\hat{\mathbf{i}}-12\hat{\mathbf{j}}+\hat{\mathbf{k}}$,
Tangent vector: $0\hat{\mathbf{i}}+1\hat{\mathbf{j}}+12\hat{\mathbf{k}}$ or $1\hat{\mathbf{i}}+0\hat{\mathbf{j}}-12\hat{\mathbf{k}}$

In exercises 8 - 19, find the equation for the tangent plane to the surface at the indicated point. (Hint: If the given function is not already solved for z, start by solving it for z in terms of x and y.)

8) $-8x-3y-7z=-19, \quad P(1,-1,2)$

9) $z=-9x^2-3y^2, \quad P(2,1,-39)$

Answer

$-36x-6y-z=-39$

10) $x^2+10xyz+y^2+8z^2=0, \quad P(-1,-1,-1)$

11) $z=\ln(10x^2+2y^2+1), \quad P(0,0,0)$

Answer

$z=0$

12) $z=e^{7x^2+4y^2}, \quad P(0,0,1)$

13) $xy+yz+zx=11, \quad P(1,2,3)$

Answer

$5x+4y+3z-22=0$

14) $x^2+4y^2=z^2, \quad P(3,2,5)$

15) $x^3+y^3=3xyz, \quad P(1,2,\frac{3}{2})$

Answer

$4x-5y+4z=0$

16) $z=axy, \quad P(1,\frac{1}{a},1)$

17) $z=\sin x+\sin y+\sin(x+y), \quad P(0,0,0)$

Answer

$2x+2y-z=0$

18) $h(x,y)=\ln\sqrt{x^2+y^2}, \quad P(3,4)$

19) $z=x^2-2xy+y^2, \quad P(1,2,1)$

Answer

$-2(x-1)+2(y-2)-(z-1)=0$

In exercises 20 - 25, find parametric equations for the normal line to the surface at the indicated point. (Recall that to find the equation of a line in space, you need a point on the line, $P_0(x_0,y_0,z_0)$, and a vector $\vec{\mathbf{v}}=\langle a,b,c\rangle$ that is parallel to the line. Then the equations of the line are: $x=x_0+at, \quad y=y_0+bt, \quad z=z_0+ct.$)

20) $-3x+9y+4z=-4, \quad P(1,-1,2)$

21) $z=5x^2-2y^2, \quad P(2,1,18)$

Answer

$x=20t+2, y=-4t+1, z=-t+18$

22) $x^2-8xyz+y^2+6z^2=0, \quad P(1,1,1)$

23) $z=\ln(3x^2+7y^2+1), \quad P(0,0,0)$

Answer

$x=0, y=0, z=t$

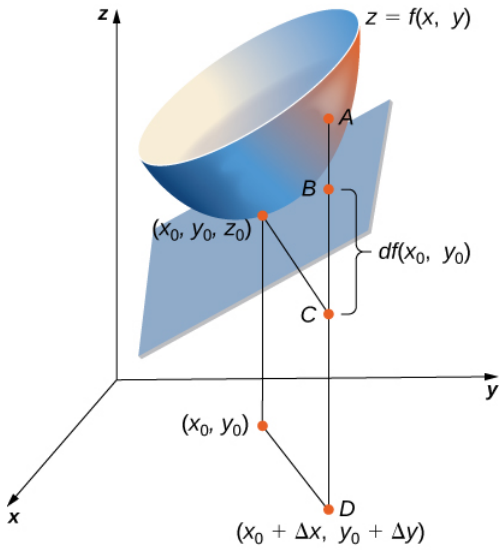
24) $z=e^{4x^2+6y^2}, \quad P(0,0,1)$

25) $z=x^2-2xy+y^2$ at point $P(1,2,1)$

Answer

$x-1=2t; y-2=-2t; z-1=t$

In exercises 26 - 28, use the figure shown here.



26) The length of line segment AC is equal to what mathematical expression?

27) The length of line segment BC is equal to what mathematical expression?

Answer

The differential of the function $z(x,y) = dz = f_x dx + f_y dy$

28) Using the figure, explain what the length of line segment AB represents.

29) Show that $f(x,y) = e^{xy}x$ is differentiable at point $(1,0)$.

Answer

Using the definition of differentiability, we have $e^{xy}x \approx x + y$.

30) Show that $f(x,y) = x^2 + 3y$ is differentiable at every point. In other words, show that $\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y) = f_x \Delta x + f_y \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$, where both ε_1 and ε_2 approach zero as $(\Delta x, \Delta y)$ approaches $(0,0)$.

Answer

$\Delta z = 2x\Delta x + 3\Delta y + (\Delta x)^2 . (\Delta x)^2 \to 0$ for small Δx and z satisfies the definition of differentiability.

31) Find the total differential of each function:

a. $z = x^3 + y^3 - 5$
b. $z = e^{xy}$
c. $z = y\cos x + \sin y$
d. $P = t^2 + 3t + tu^3$
e. $w = e^y \cos(x) + z^2$

Answers:

a. $dz = 3x^2\, dx + 3y^2\, dy$
b. $dz = ye^{xy}\, dx + xe^{xy}\, dy$
c. $dz = -y\sin x\, dx + (\cos x + \cos y)\, dy$
d. $dP = (2t + 3 + u^3)\, dt + 3tu^2\, du$
e. $dw = -e^y \sin(x)\, dx + e^y \cos(x)\, dy + 2z\, dz$

32) a. Find the total differential dz of the function $z = \dfrac{xy}{y+x}$ and then

b. State its value where x changes from 10 to 10.5 and y changes from 15 to 13.

Answer

a. $dz = \dfrac{y^2}{(x+y)^2}\,dx + \dfrac{x^2}{(x+y)^2}\,dy$
b. $dx = 0.5$ and $dy = -2$ so

$$\begin{aligned} dz &= f_x(10,15)\,dx + f_y(10,15)\,dy \\ &= \frac{15^2}{25^2}\,dx + \frac{10^2}{25^2}\,dy \\ &= \frac{225}{625}\,(0.5) + \frac{100}{625}(-2) \\ &= \frac{9}{25}\left(\frac{1}{2}\right) + \frac{4}{25}(-2) \\ &= \frac{18}{100} - \frac{32}{100} \\ &= .18 - .32 = -0.14 \end{aligned}$$

33) Let $z = f(x,y) = xe^y$. State its total differential. Then compute Δz from $P(1,2)$ to $Q(1.05, 2.1)$ and then find the approximate change in z, dz, from point P to point Q. Recall $\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y)$, and dz and Δz should be approximately equal, if dx and dy are both reasonably small.

Answer

Total Differential: $dz = e^y\,dx + xe^y\,dy$
$\Delta z \approx 1.185422$and $dz \approx 1.108$. Note that they are relatively close.

34) The volume of a right circular cylinder is given by $V(r,h) = \pi r^2 h$. Find the differential dV. Interpret the formula geometrically.

Answer

$dV = 2\pi rh\,dr + \pi r^2\,dh$

35) See the preceding problem. Use differentials to estimate the amount of aluminum in an enclosed aluminum can with diameter $8.0cm$ and height $12cm$ if the aluminum is 0.04 cm thick.

Answer

16 cm^3

36) Use the differential dz to approximate the change in $z = \sqrt{4 - x^2 - y^2}$ as (x,y) moves from point $(1,1)$ to point $(1.01, 0.97)$.Compare this approximation with the actual change in the function.

37) Let $z = f(x,y) = x^2 + 3xy - y^2$. Find the exact change in the function and the approximate change in the function as x changes from 2.00 to 2.05 and y changes from 3.00 to 2.96.

Answer

$\Delta z =$ exact change $= 0.6449$, approximate change is $dz = 0.65$. The two values are close.

38) The centripetal acceleration of a particle moving in a circle is given by $a(r,v) = \frac{v^2}{r}$, where v is the velocity and r is the radius of the circle. Approximate the maximum percent error in measuring the acceleration resulting from errors of 3% in v and 2% in r. (Recall that the percentage error is the ratio of the amount of error over the original amount. So, in this case, the percentage error in a is given by $\frac{da}{a}$.)

39) The radius r and height h of a right circular cylinder are measured with possible errors of 4% and 5%, respectively. Approximate the maximum possible percentage error in measuring the volume (Recall that the percentage error is the ratio of the amount of error over the original amount. So, in this case, the percentage error in V is given by $\frac{dV}{V}$.)

Answer

13% or 0.13

40) The base radius and height of a right circular cone are measured as 10 in. and 25 in., respectively, with a possible error in measurement of as much as 0.1 in. each. Use differentials to estimate the maximum error in the calculated volume of the cone.

41) The **electrical resistance** R produced by wiring resistors R_1 and R_2 in parallel can be calculated from the formula $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$. If R_1 and R_2 are measured to be 7Ω and 6Ω, respectively, and if these measurements are accurate to within 0.05Ω, estimate the maximum possible error in computing R. (The symbol Ω represents an ohm, the unit of electrical resistance.)

Answer

0.025

42) The area of an ellipse with axes of length $2a$ and $2b$ is given by the formula $A = \pi ab$. Approximate the percent change in the area when a increases by 2% and b increases by 1.5%.

43) The period T of a **simple pendulum** with small oscillations is calculated from the formula $T = 2\pi\sqrt{\frac{L}{g}}$, where L is the length of the pendulum and g is the acceleration resulting from gravity. Suppose that L and g have errors of, at most, 0.5% and 0.1%, respectively. Use differentials to approximate the maximum percentage error in the calculated value of T.

Answer

0.3%

44) **Electrical power** P is given by $P = \frac{V^2}{R}$, where V is the voltage and R is the resistance. Approximate the maximum percentage error in calculating power if $120V$ is applied to a $2000 - \Omega$ resistor and the possible percent errors in measuring V and R are 3% and 4%, respectively.

For exercises 45 - 49, find the linear approximation of each function at the indicated point.

45) $f(x,y) = x\sqrt{y}, \quad P(1,4)$

Answer

$L(x,y) = 2x + \frac{1}{4}y - 1$

46) $f(x,y) = e^x \cos y; \quad P(0,0)$

47) $f(x,y) = \arctan(x+2y), \quad P(1,0)$

Answer

$L(x,y) = \frac{1}{2}x + y + \frac{1}{4}\pi - \frac{1}{2}$

48) $f(x,y) = \sqrt{20 - x^2 - 7y^2}, \quad P(2,1)$

49) $f(x,y,z) = \sqrt{x^2 + y^2 + z^2}, \quad P(3,2,6)$

Answer

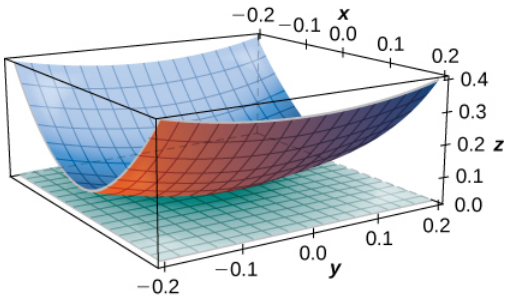
$L(x,y,z) = \frac{3}{7}x + \frac{2}{7}y + \frac{6}{7}z$

50) [T] Find the equation of the tangent plane to the surface $f(x,y) = x^2 + y^2$ at point $(1,2,5)$, and graph the surface and the tangent plane at the point.

51) [T] Find the equation for the tangent plane to the surface at the indicated point, and graph the surface and the tangent plane: $z = \ln(10x^2 + 2y^2 + 1), \quad P(0,0,0)$.

Answer

$z = 0$



52) [T] Find the equation of the tangent plane to the surface $z = f(x, y) = \sin(x + y^2)$ at point $\left(\frac{\pi}{4}, 0, \frac{\sqrt{2}}{2}\right)$, and graph the surface and the tangent plane.

Contributors

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- Paul Seeburger (Monroe Community College) created all but part e of exercise 31.

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5.6: The Chain Rule for Multivariable Functions

Learning Objectives

- State the chain rules for one or two independent variables.
- Use tree diagrams as an aid to understanding the chain rule for several independent and intermediate variables.
- Perform implicit differentiation of a function of two or more variables.

In single-variable calculus, we found that one of the most useful differentiation rules is the chain rule, which allows us to find the derivative of the composition of two functions. The same thing is true for multivariable calculus, but this time we have to deal with more than one form of the chain rule. In this section, we study extensions of the chain rule and learn how to take derivatives of compositions of functions of more than one variable.

Chain Rules for One or Two Independent Variables

Recall that the chain rule for the derivative of a composite of two functions can be written in the form

$$\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x). \tag{5.6.1}$$

In this equation, both $f(x)$ and $g(x)$ are functions of one variable. Now suppose that f is a function of two variables and g is a function of one variable. Or perhaps they are both functions of two variables, or even more. How would we calculate the derivative in these cases? The following theorem gives us the answer for the case of one independent variable.

Chain Rule for One Independent Variable

Suppose that $x = g(t)$ and $y = h(t)$ are differentiable functions of t and $z = f(x, y)$ is a differentiable function of x and y. Then $z = f(x(t), y(t))$ is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}, \tag{5.6.2}$$

where the ordinary derivatives are evaluated at t and the partial derivatives are evaluated at (x, y).

Proof

The proof of this theorem uses the definition of differentiability of a function of two variables. Suppose that f is differentiable at the point $P(x_0, y_0)$, where $x_0 = g(t_0)$ and $y_0 = h(t_0)$ for a fixed value of t_0. We wish to prove that $z = f(x(t), y(t))$ is differentiable at $t = t_0$ and that Equation 5.6.2 holds at that point as well.

Since f is differentiable at P, we know that

$$z(t) = f(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + E(x, y),$$

where

$$\lim_{(x,y)\to(x_0,y_0)} \frac{E(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0.$$

We then subtract $z_0 = f(x_0, y_0)$ from both sides of this equation:

$$z(t) - z(t_0) = f(x(t), y(t)) - f(x(t_0), y(t_0))$$
$$= f_x(x_0, y_0)(x(t) - x(t_0)) + f_y(x_0, y_0)(y(t) - y(t_0)) + E(x(t), y(t)).$$

Next, we divide both sides by $t - t_0$:

$$z(t) - z(t_0)t - t_0 = fx(x_0, y_0)(x(t) - x(t_0)t - t_0) + f_y(x_0, y_0)(y(t) - y(t_0)t - t_0) + E(x(t), y(t))t - t_0.$$

Then we take the limit as t approaches t_0:

$$\lim_{t\to t_0}\frac{z(t)-z(t_0)}{t-t_0}=f_x(x_0,y_0)\lim_{t\to t_0}\left(\frac{x(t)-x(t_0)}{t-t_0}\right)$$
$$+f_y(x_0,y_0)\lim_{t\to t_0}\left(\frac{y(t)-y(t_0)}{t-t_0}\right)$$
$$+\lim_{t\to t_0}\frac{E(x(t),y(t))}{t-t_0}.$$

The left-hand side of this equation is equal to dz/dt, which leads to

$$\frac{dz}{dt}=f_x(x_0,y_0)\frac{dx}{dt}+f_y(x_0,y_0)\frac{dy}{dt}+\lim_{t\to t_0}\frac{E(x(t),y(t))}{t-t_0}.$$

The last term can be rewritten as

$$\lim_{t\to t_0}\frac{E(x(t),y(t))}{t-t_0}=\lim_{t\to t_0}\frac{(E(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}}\frac{\sqrt{(x-x_0)^2+(y-y_0)^2}}{t-t_0})$$
$$=\lim_{t\to t_0}\left(\frac{E(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}}\right)\lim_{t\to t_0}\left(\frac{\sqrt{(x-x_0)^2+(y-y_0)^2}}{t-t_0}\right).$$

As t approaches t_0, $(x(t),y(t))$ approaches $(x(t_0),y(t_0))$, so we can rewrite the last product as

$$\lim_{(x,y)\to(x_0,y_0)}\frac{(E(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}}\lim_{(x,y)\to(x_0,y_0)}(\frac{\sqrt{(x-x_0)^2+(y-y_0)^2}}{t-t_0}).$$

Since the first limit is equal to zero, we need only show that the second limit is finite:

$$\lim_{(x,y)\to(x_0,y_0)}\frac{\sqrt{(x-x_0)^2+(y-y_0)^2}}{t-t+0}=\lim_{(x,y)\to(x_0,y_0)}\sqrt{\frac{(x-x_0)^2+(y-y_0)^2}{(t-t_0)^2}}$$
$$=\lim_{(x,y)\to(x_0,y_0)}\sqrt{\left(\frac{x-x_0}{t-t_0}\right)^2+\left(\frac{y-y_0}{t-t_0}\right)^2}$$
$$=\sqrt{\left[\lim_{(x,y)\to(x_0,y_0)}\left(\frac{x-x_0}{t-t_0}\right)\right]^2+\left[\lim_{(x,y)\to(x_0,y_0)}\left(\frac{y-y_0}{t-t_0}\right)\right]^2}.$$

Since $x(t)$ and $y(t)$ are both differentiable functions of t, both limits inside the last radical exist. Therefore, this value is finite. This proves the chain rule at $t=t_0$; the rest of the theorem follows from the assumption that all functions are differentiable over their entire domains.

□

Closer examination of Equation 5.6.2 reveals an interesting pattern. The first term in the equation is $\frac{\partial f}{\partial x}\cdot\frac{dx}{dt}$ and the second term is $\frac{\partial f}{\partial y}\cdot\frac{dy}{dt}$. Recall that when multiplying fractions, cancelation can be used. If we treat these derivatives as fractions, then each product "simplifies" to something resembling $\partial f/dt$. The variables x and y that disappear in this simplification are often called **intermediate variables**: they are independent variables for the function f, but are dependent variables for the variable t. Two terms appear on the right-hand side of the formula, and f is a function of two variables. This pattern works with functions of more than two variables as well, as we see later in this section.

Example 5.6.1: Using the Chain Rule

Calculate dz/dt for each of the following functions:

a. $z=f(x,y)=4x^2+3y^2, x=x(t)=\sin t, y=y(t)=\cos t$

b. $z=f(x,y)=\sqrt{x^2-y^2}, x=x(t)=e^{2t}, y=y(t)=e^{-t}$

Solution

a. To use the chain rule, we need four quantities—$\partial z/\partial x$, $\partial z/\partial y$, dx/dt, and dy/dt:

- $\dfrac{\partial z}{\partial x} = 8x$
- $\dfrac{dx}{dt} = \cos t$
- $\dfrac{\partial z}{\partial y} = 6y$
- $\dfrac{dy}{dt} = -\sin t$

Now, we substitute each of these into Equation 5.6.2:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt} = (8x)(\cos t) + (6y)(-\sin t) = 8x\cos t - 6y\sin t.$$

This answer has three variables in it. To reduce it to one variable, use the fact that $x(t) = \sin t$ and $y(t) = \cos t$. We obtain

$$\frac{dz}{dt} = 8x\cos t - 6y\sin t = 8(\sin t)\cos t - 6(\cos t)\sin t = 2\sin t\cos t.$$

This derivative can also be calculated by first substituting $x(t)$ and $y(t)$ into $f(x, y)$, then differentiating with respect to t:

$$z = f(x, y) = f(x(t), y(t)) = 4(x(t))^2 + 3(y(t))^2 = 4\sin^2 t + 3\cos^2 t.$$

Then

$$\frac{dz}{dt} = 2(4\sin t)(\cos t) + 2(3\cos t)(-\sin t) = 8\sin t\cos t - 6\sin t\cos t = 2\sin t\cos t,$$

which is the same solution. However, it may not always be this easy to differentiate in this form.

b. To use the chain rule, we again need four quantities—$\partial z/\partial x$, $\partial z/dy$, dx/dt, and dy/dt :

- $\dfrac{\partial z}{\partial x} = \dfrac{x}{\sqrt{x^2 - y^2}}$
- $\dfrac{dx}{dt} = 2e^{2t}$
- $\dfrac{\partial z}{\partial y} = \dfrac{-y}{\sqrt{x^2 - y^2}}$
- $\dfrac{dx}{dt} = -e^{-t}$.

We substitute each of these into Equation 5.6.2:

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt} \\ &= \left(\frac{x}{\sqrt{x^2 - y^2}}\right)(2e^{2t}) + \left(\frac{-y}{\sqrt{x^2 - y^2}}\right)(-e^{-t}) \\ &= \frac{2xe^{2t} - ye^{-t}}{\sqrt{x^2 - y^2}}.\end{aligned}$$

To reduce this to one variable, we use the fact that $x(t) = e^{2t}$ and $y(t) = e^{-t}$. Therefore,

$$\frac{dz}{dt} = \frac{2xe^2t + ye^{-t}}{\sqrt{x^2 - y^2}}$$
$$= \frac{2(e^{2t})e^{2t} + (e^{-t})e^{-t}}{\sqrt{e^{4t} - e^{-2t}}}$$
$$= \frac{2e^{4t} + e^{-2t}}{\sqrt{e^{4t} - e^{-2t}}}.$$

To eliminate negative exponents, we multiply the top by e^{2t} and the bottom by $\sqrt{e^{4t}}$:

$$\frac{dz}{dt} = \frac{2e^{4t} + e^{-2t}}{\sqrt{e^{4t} - e^{-2t}}} \cdot \frac{e^{2t}}{\sqrt{e^{4t}}}$$
$$= \frac{2e^{6t} + 1}{\sqrt{e^{8t} - e^{2t}}}$$
$$= \frac{2e^{6t} + 1}{\sqrt{e^{2t}(e^{6t} - 1)}}$$
$$= \frac{2e^{6t} + 1}{e^t\sqrt{e^{6t} - 1}}.$$

Again, this derivative can also be calculated by first substituting $x(t)$ and $y(t)$ into $f(x, y)$, then differentiating with respect to t:

$$z = f(x, y)$$
$$= f(x(t), y(t))$$
$$= \sqrt{(x(t))^2 - (y(t))^2}$$
$$= \sqrt{e^{4t} - e^{-2t}}$$
$$= (e^{4t} - e^{-2t})^{1/2}.$$

Then

$$\frac{dz}{dt} = \frac{1}{2}(e^{4t} - e^{-2t})^{-1/2}\left(4e^{4t} + 2e^{-2t}\right)$$
$$= \frac{2e^{4t} + e^{-2t}}{\sqrt{e^{4t} - e^{-2t}}}.$$

This is the same solution.

Exercise 5.6.1

Calculate dz/dt given the following functions. Express the final answer in terms of t.

$$z = f(x, y) = x^2 - 3xy + 2y^2$$
$$x = x(t) = 3\sin 2t, y = y(t) = 4\cos 2t$$

Hint

Calculate $\partial z/\partial x, \partial z/dy, dx/dt$,and dy/dt, then use Equation 5.6.2

Answer

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$
$$= (2x - 3y)(6\cos 2t) + (-3x + 4y)(-8\sin 2t)$$

$$= -92 \sin 2t \cos 2t - 72(\cos^2 2t - \sin^2 2t)$$
$$= -46 \sin 4t - 72 \cos 4t.$$

It is often useful to create a visual representation of Equation for the chain rule. This is called a **tree diagram** for the chain rule for functions of one variable and it provides a way to remember the formula (Figure 5.6.1). This diagram can be expanded for functions of more than one variable, as we shall see very shortly.

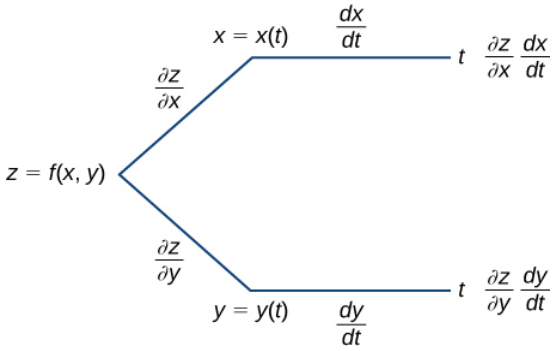


Figure 5.6.1: Tree diagram for the case $\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$.

In this diagram, the leftmost corner corresponds to $z = f(x, y)$. Since f has two **independent variables**, there are two lines coming from this corner. The upper branch corresponds to the variable x and the lower branch corresponds to the variable y. Since each of these variables is then dependent on one variable t, one branch then comes from x and one branch comes from y. Last, each of the branches on the far right has a label that represents the path traveled to reach that branch. The top branch is reached by following the x branch, then the t branch; therefore, it is labeled $(\partial z/\partial x) \times (dx/dt)$. The bottom branch is similar: first the y branch, then the t branch. This branch is labeled $(\partial z/\partial y) \times (dy/dt)$. To get the formula for dz/dt, add all the terms that appear on the rightmost side of the diagram. This gives us Equation.



In Note, $z = f(x, y)$ is a function of x and y, and both $x = g(u, v)$ and $y = h(u, v)$ are functions of the independent variables u and v.

Chain Rule for Two Independent Variables

Suppose $x = g(u, v)$ and $y = h(u, v)$ are differentiable functions of u and v, and $z = f(x, y)$ is a differentiable function of x and y. Then, $z = f(g(u, v), h(u, v))$ is a differentiable function of u and v, and

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \tag{5.6.3}$$

and

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}. \tag{5.6.4}$$

We can draw a tree diagram for each of these formulas as well as follows.

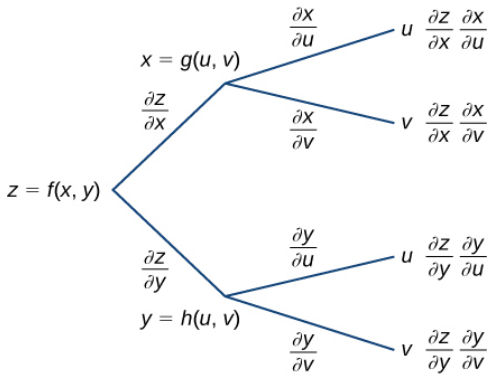


Figure 5.6.2: Tree diagram for $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$ and $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$.

To derive the formula for $\partial z/\partial u$, start from the left side of the diagram, then follow only the branches that end with u and add the terms that appear at the end of those branches. For the formula for $\partial z/\partial v$, follow only the branches that end with v and add the terms that appear at the end of those branches.

There is an important difference between these two chain rule theorems. In Note, the left-hand side of the formula for the derivative is not a partial derivative, but in Note it is. The reason is that, in Note, z is ultimately a function of t alone, whereas in Note, z is a function of both u and v.

Example 5.6.2: Using the Chain Rule for Two Variables

Calculate $\partial z/\partial u$ and $\partial z/\partial v$ using the following functions:

$$z = f(x,y) = 3x^2 - 2xy + y^2,\ x = x(u,v) = 3u + 2v,\ y = y(u,v) = 4u - v.$$

Solution

To implement the chain rule for two variables, we need six partial derivatives—$\partial z/\partial x,\ \partial z/\partial y,\ \partial x/\partial u,\ \partial x/\partial v,\ \partial y/\partial u$,and $\partial y/\partial v$:

$$\frac{\partial z}{\partial x} = 6x - 2y \quad \frac{\partial z}{\partial y} = -2x + 2y$$
$$\frac{\partial x}{\partial u} = 3 \quad \frac{\partial x}{\partial v} = 2$$
$$\frac{\partial y}{\partial u} = 4 \quad \frac{\partial y}{\partial v} = -1.$$

To find $\partial z/\partial u$, we use Equation 5.6.3:

$$\begin{aligned}\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}\\ &= 3(6x - 2y) + 4(-2x + 2y)\\ &= 10x + 2y.\end{aligned}$$

Next, we substitute $x(u,v) = 3u + 2v$ and $y(u,v) = 4u - v$:

$$\frac{\partial z}{\partial u} = 10x + 2y$$
$$= 10(3u + 2v) + 2(4u - v)$$
$$= 38u + 18v.$$

To find $\partial z/\partial v$, we use Equation ???:

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$
$$= 2(6x - 2y) + (-1)(-2x + 2y)$$
$$= 14x - 6y.$$

Then we substitute $x(u, v) = 3u + 2v$ and $y(u, v) = 4u - v$:

$$\frac{\partial z}{\partial v} = 14x - 6y$$
$$= 14(3u + 2v) - 6(4u - v)$$
$$= 18u + 34v$$



Exercise 5.6.2

Calculate $\partial z/\partial u$ and $\partial z/\partial v$ given the following functions:

$$z = f(x, y) = \frac{2x - y}{x + 3y},\ x(u, v) = e^{2u}\cos 3v,\ y(u, v) = e^{2u}\sin 3v.$$

Hint

Calculate $\partial z/\partial x,\ \partial z/\partial y,\ \partial x/\partial u,\ \partial x/\partial v,\ \partial y/\partial u$,and $\partial y/\partial v$, then use Equation 5.6.3and Equation ???.

Answer

$$\frac{\partial z}{\partial u} = 0, \frac{\partial z}{\partial v} = \frac{-21}{(3\sin 3v + \cos 3v)^2}$$

The Generalized Chain Rule

Now that we've see how to extend the original chain rule to functions of two variables, it is natural to ask: Can we extend the rule to more than two variables? The answer is yes, as the *generalized chain rule* states.

Generalized Chain Rule

Let $w = f(x_1, x_2, \ldots, x_m)$ be a differentiable function of m independent variables, and for each $i \in 1, \ldots, m$, let $x_i = x_i(t_1, t_2, \ldots, t_n)$ be a differentiable function of n independent variables. Then

$$\frac{\partial w}{\partial t_j} = \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_j} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_j} + \cdots + \frac{\partial w}{\partial x_m}\frac{\partial x_m}{\partial t_j} \tag{5.6.5}$$

for any $j \in 1, 2, \ldots, n$.

In the next example we calculate the derivative of a function of three independent variables in which each of the three variables is dependent on two other variables.

Example 5.6.3: Using the Generalized Chain Rule

Calculate $\partial w/\partial u$ and $\partial w/\partial v$ using the following functions:

$$\begin{aligned} w = f(x, y, z) &= 3x^2 - 2xy + 4z^2 \\ x = x(u, v) &= e^u \sin v \\ y = y(u, v) &= e^u \cos v \\ z = z(u, v) &= e^u. \end{aligned}$$

Solution

The formulas for $\partial w/\partial u$ and $\partial w/\partial v$ are

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial u}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial v}.$$

Therefore, there are nine different partial derivatives that need to be calculated and substituted. We need to calculate each of them:

$$\frac{\partial w}{\partial x} = 6x - 2y \quad \frac{\partial w}{\partial y} = -2x \quad \frac{\partial w}{\partial z} = 8z$$

$$\frac{\partial x}{\partial u} = e^u \sin v \quad \frac{\partial y}{\partial u} = e^u \cos v \quad \frac{\partial z}{\partial u} = e^u$$

$$dfrac\partial x\partial v = e^u \cos v \quad \frac{\partial y}{\partial v} = -e^u \sin v \quad \frac{\partial z}{\partial v} = 0.$$

Now, we substitute each of them into the first formula to calculate $\partial w/\partial u$:

$$\begin{aligned} \frac{\partial w}{\partial u} &= \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial u} \\ &= (6x - 2y)e^u \sin v - 2xe^u \cos v + 8ze^u, \end{aligned}$$

then substitute $x(u, v) = e^u \sin v, y(u, v) = e^u \cos v,$ and $z(u, v) = e^u$ into this equation:

$$\frac{\partial w}{\partial u} = (6x-2y)e^u \sin v - 2xe^u \cos v + 8ze^u$$
$$= (6e^u \sin v - 2eu \cos v)e^u \sin v - 2(e^u \sin v)e^u \cos v + 8e^{2u}$$
$$= 6e^{2u}\sin^2 v - 4e^{2u}\sin v \cos v + 8e^{2u}$$
$$= 2e^{2u}(3\sin^2 v - 2\sin v \cos v + 4).$$

Next, we calculate $\partial w/\partial v$:

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial v}$$
$$= (6x-2y)e^u \cos v - 2x(-e^u \sin v) + 8z(0),$$

then we substitute $x(u,v) = e^u \sin v, y(u,v) = e^u \cos v,$ and $z(u,v) = e^u$ into this equation:

$$\frac{\partial w}{\partial v} = (6x-2y)e^u \cos v - 2x(-e^u \sin v)$$
$$= (6e^u \sin v - 2e^u \cos v)e^u \cos v + 2(e^u \sin v)(e^u \sin v)$$
$$= 2e^{2u}\sin^2 v + 6e^{2u}\sin v \cos v - 2e^{2u}\cos^2 v$$
$$= 2e^{2u}(\sin^2 v + \sin v \cos v - \cos^2 v).$$

Exercise 5.6.3

Calculate $\partial w/\partial u$ and $\partial w/\partial v$ given the following functions:

$$w = f(x,y,z) = \frac{x+2y-4z}{2x-y+3z}$$
$$x = x(u,v) = e^{2u}\cos 3v$$
$$y = y(u,v) = e^{2u}\sin 3v$$
$$z = z(u,v) = e^{2u}.$$

Hint

Calculate nine partial derivatives, then use the same formulas from Example 5.6.3

Answer

$$\frac{\partial w}{\partial u} = 0$$
$$\frac{\partial w}{\partial v} = \frac{15 - 33\sin 3v + 6\cos 3v}{(3 + 2\cos 3v - \sin 3v)^2}$$

Example 5.6.4: Drawing a Tree Diagram

Create a tree diagram for the case when

$$w = f(x,y,z), x = x(t,u,v), y = y(t,u,v), z = z(t,u,v)$$

and write out the formulas for the three partial derivatives of w.

Solution

Starting from the left, the function f has three independent variables: $x, y,$ and z. Therefore, three branches must be emanating from the first node. Each of these three branches also has three branches, for each of the variables $t, u,$ and v.

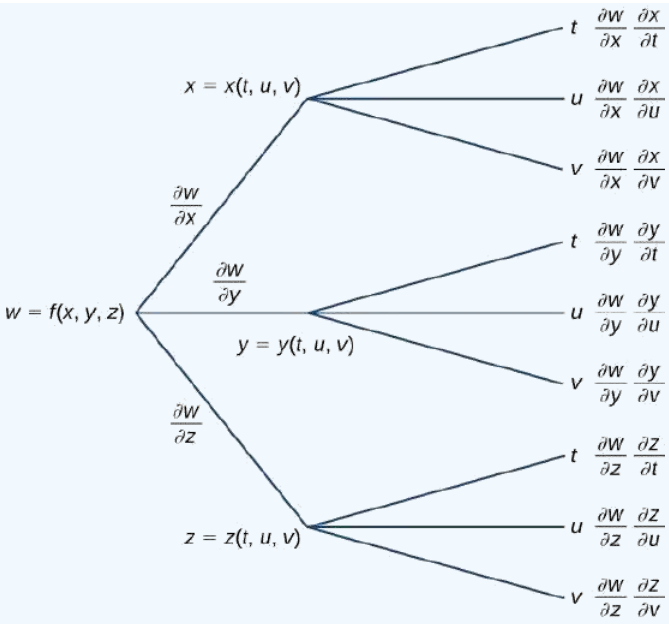


Figure 5.6.3: Tree diagram for a function of three variables, each of which is a function of three independent variables.

The three formulas are

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$$

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}.$$

Exercise 5.6.4

Create a tree diagram for the case when

$$w = f(x, y), x = x(t, u, v), y = y(t, u, v)$$

and write out the formulas for the three partial derivatives of w.

Hint

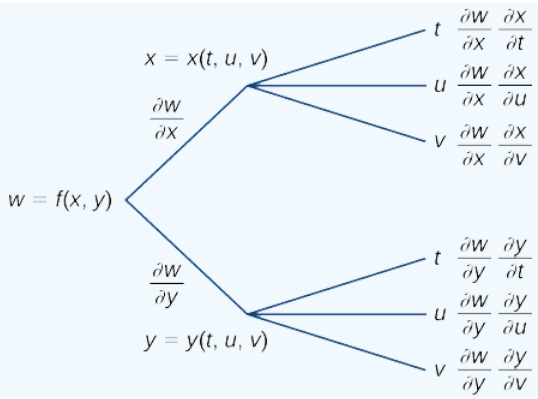
Determine the number of branches that emanate from each node in the tree.

Answer

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t}$$

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v}$$



Implicit Differentiation

Recall from implicit differentiation provides a method for finding dy/dx when y is defined implicitly as a function of x. The method involves differentiating both sides of the equation defining the function with respect to x, then solving for dy/dx. Partial derivatives provide an alternative to this method.

Consider the ellipse defined by the equation $x^2+3y^2+4y-4=0$ as follows.

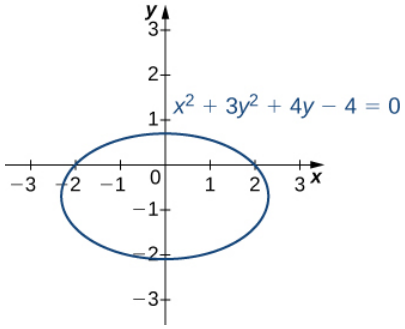


Figure 5.6.4: Graph of the ellipse defined by $x^2+3y^2+4y-4=0$.

This equation implicitly defines y as a function of x. As such, we can find the derivative dy/dx using the method of implicit differentiation:

$$\begin{aligned}\frac{d}{dx}(x^2+3y^2+4y-4) &= \frac{d}{dx}(0)\\ 2x+6y\frac{dy}{dx}+4\frac{dy}{dx} &= 0\\ (6y+4)\frac{dy}{dx} &= -2x\\ \frac{dy}{dx} &= -\frac{x}{3y+2}\end{aligned}$$

We can also define a function $z=f(x,y)$ by using the left-hand side of the equation defining the ellipse. Then $f(x,y)=x^2+3y^2+4y-4$. The ellipse $x^2+3y^2+4y-4=0$ can then be described by the equation $f(x,y)=0$. Using this function and the following theorem gives us an alternative approach to calculating dy/dx.

Theorem: Implicit Differentiation of a Function of Two or More Variables

Suppose the function $z=f(x,y)$ defines y implicitly as a function $y=g(x)$ of x via the equation $f(x,y)=0$. Then

$$\frac{dy}{dx}=-\frac{\partial f/\partial x}{\partial f/\partial y} \tag{5.6.6}$$

provided $f_y(x,y)\neq 0$.

If the equation $f(x,y,z)=0$ defines z implicitly as a differentiable function of x and y, then

$$\frac{dz}{dx}=-\frac{\partial f/\partial x}{\partial f/\partial z} \text{ and } \frac{dz}{dy}=-\frac{\partial f/\partial y}{\partial f/\partial z} \tag{5.6.7}$$

as long as $f_z(x,y,z)\neq 0$.

Equation 5.6.6 is a direct consequence of Equation 5.6.3. In particular, if we assume that y is defined implicitly as a function of x via the equation $f(x,y)=0$, we can apply the chain rule to find dy/dx :

$$\begin{aligned}\frac{d}{dx}f(x,y)&=\frac{d}{dx}(0)\\ \frac{\partial f}{\partial x}\cdot\frac{dx}{dx}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dx}&=0\\ \frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dx}&=0.\end{aligned}$$

Solving this equation for dy/dx gives Equation 5.6.6. Equation 5.6.6 can be derived in a similar fashion.

Let's now return to the problem that we started before the previous theorem. Using Note and the function $f(x,y)=x^2+3y^2+4y-4$, we obtain

$$\begin{aligned}\frac{\partial f}{\partial x}&=2x\\ \frac{\partial f}{\partial y}&=6y+4.\end{aligned}$$

Then Equation 5.6.6 gives

$$\frac{dy}{dx}=-\frac{\partial f/\partial x}{\partial f/\partial y}=-\frac{2x}{6y+4}=-\frac{x}{3y+2}, \tag{5.6.8}$$

which is the same result obtained by the earlier use of implicit differentiation.

Example 5.6.5: Implicit Differentiation by Partial Derivatives

a. Calculate dy/dx if y is defined implicitly as a function of x via the equation $3x^2-2xy+y^2+4x-6y-11=0$. What is the equation of the tangent line to the graph of this curve at point $(2,1)$?
b. Calculate $\partial z/\partial x$ and $\partial z/\partial y$, given $x^2e^y-yze^x=0$.

Solution

a. Set $f(x,y)=3x^2-2xy+y^2+4x-6y-11=0$, then calculate f_x and f_y : $f_x=6x-2y+4$ $f_y=-2x+2y-6$.

The derivative is given by

$$\frac{dy}{dx}=-\frac{\partial f/\partial x}{\partial f/\partial y}=\frac{6x-2y+4}{-2x+2y-6}=\frac{3x-y+2}{x-y+3}.$$

The slope of the tangent line at point $(2,1)$ is given by

$$\frac{dy}{dx}\Big|_{(x,y)=(2,1)}=\frac{3(2)-1+2}{2-1+3}=\frac{7}{4}$$

To find the equation of the tangent line, we use the point-slope form (Figure 5.6.5):

$$y - y_0 = m(x - x_0)$$

$$y - 1 = \frac{7}{4}(x - 2)$$

$$y = \frac{7}{4}x - \frac{7}{2} + 1$$

$$y = \frac{7}{4}x - \frac{5}{2}.$$

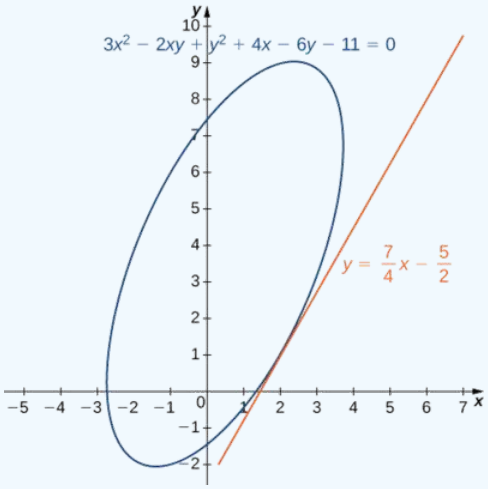


Figure 5.6.5: Graph of the rotated ellipse defined by $3x^2 - 2xy + y^2 + 4x - 6y - 11 = 0$.

b. We have $f(x, y, z) = x^2 e^y - yze^x$. Therefore,

$$\frac{\partial f}{\partial x} = 2xe^y - yze^x$$

$$\frac{\partial f}{\partial y} = x^2 e^y - ze^x$$

$$\frac{\partial f}{\partial z} = -ye^x$$

Using Equation 5.6.7,

$$\frac{\partial z}{\partial x} = -\frac{\partial f/\partial x}{\partial f/\partial y}\,\frac{\partial z}{\partial y} = -\frac{\partial f/\partial y}{\partial f/\partial z}$$

$$= -\frac{2xe^y - yze^x}{-ye^x} \text{ and } = -\frac{x^2 e^y - ze^x}{-ye^x}$$

$$= \frac{2xe^y - yze^x}{ye^x} = \frac{x^2 e^y - ze^x}{ye^x}$$



Exercise 5.6.5

Find dy/dx if y is defined implicitly as a function of x by the equation $x^2+xy-y^2+7x-3y-26=0$. What is the equation of the tangent line to the graph of this curve at point $(3,-2)$?

Hint

Calculate $\partial f/dx$ and $\partial f/dy$, then use Equation 5.6.6

Solution

$$\frac{dy}{dx}=\frac{2x+y+7}{2y-x+3}\Big|_{(3,-2)}=\frac{2(3)+(-2)+7}{2(-2)-(3)+3}=-\frac{11}{4}$$

Equation of the tangent line: $y=-\frac{11}{4}x+\frac{25}{4}$

Key Concepts

- The chain rule for functions of more than one variable involves the partial derivatives with respect to all the independent variables.
- Tree diagrams are useful for deriving formulas for the chain rule for functions of more than one variable, where each independent variable also depends on other variables.

Key Equations

- **Chain rule, one independent variable**

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

- **Chain rule, two independent variables**

$$\frac{dz}{du}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}\frac{dz}{dv}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

- **Generalized chain rule**

$$\frac{\partial w}{\partial t_j}=\frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_j}+\frac{\partial w}{\partial x_2}\frac{\partial x_1}{\partial t_j}+\cdots+\frac{\partial w}{\partial x_m}\frac{\partial x_m}{\partial t_j}$$

Glossary

generalized chain rule

the chain rule extended to functions of more than one independent variable, in which each independent variable may depend on one or more other variables

intermediate variable

given a composition of functions (e.g., $f(x(t), y(t))$), the intermediate variables are the variables that are independent in the outer function but dependent on other variables as well; in the function $f(x(t), y(t))$, the variables x and y are examples of intermediate variables

tree diagram

illustrates and derives formulas for the generalized chain rule, in which each independent variable is accounted for

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5.6E: Exercises for Section 5.6

In exercises 1 - 6, use the information provided to solve the problem.

1) Let $w(x,y,z) = xy\cos z$, where $x = t, y = t^2,$ and $z = \arcsin t$. Find $\dfrac{dw}{dt}$.

Answer

$$\frac{dw}{dt} = y\cos z + x\cos z(2t) - \frac{xy\sin z}{\sqrt{1-t^2}}$$

2) Let $w(t,v) = e^{tv}$ where $t = r+s$ and $v = rs$. Find $\dfrac{\partial w}{\partial r}$ and $\dfrac{\partial w}{\partial s}$.

3) If $w = 5x^2 + 2y^2, \quad x = -3u + v,$ and $y = u - 4v,$ find $\dfrac{\partial w}{\partial u}$ and $\dfrac{\partial w}{\partial v}$.

Answer

$$\frac{\partial w}{\partial u} = -30x + 4y \quad = \quad -30(-3u+v) + 4(u-4v) \quad = \quad 90u - 30v + 4u - 16v \quad = \quad 94u - 46v \quad ,$$
$$\frac{\partial w}{\partial v} = 10x - 16y \quad = \quad 10(-3u+v) - 16(u-4v) \quad = \quad -30u + 10v - 16u + 64v \quad = \quad -46u + 74v$$

4) If $w = xy^2, x = 5\cos(2t)$, and $y = 5\sin(2t)$, find $\dfrac{\partial w}{\partial t}$.

5) If $f(x,y) = xy, x = r\cos\theta$, and $y = r\sin\theta$, find $\dfrac{\partial f}{\partial r}$ and express the answer in terms of r and θ.

Answer

$$\frac{\partial f}{\partial r} = r\sin(2\theta)$$

6) Suppose $f(x,y) = x + y, u = e^x \sin y, \quad x = t^2$ and $y = \pi t$, where $x = r\cos\theta$ and $y = r\sin\theta$. Find $\dfrac{\partial f}{\partial \theta}$.

In exercises 7 - 12, find $\dfrac{dz}{dt}$ in two ways, first using the chain rule and then by direct substitution.

7) $z = x^2 + y^2, \quad x = t, y = t^2$

Answer

$$\frac{dz}{dt} = 2t + 4t^3$$

8) $z = \sqrt{x^2+y^2}, \quad y = t^2, x = t$

9) $z = xy, \quad x = 1 - \sqrt{t}, y = 1 + \sqrt{t}$

Answer

$$\frac{dz}{dt} = -1$$

10) $z = \frac{x}{y}, \quad x = e^t, y = 2e^t$

11) $z = \ln(x+y), \quad x = e^t, y = e^t$

Answer

$$\frac{dz}{dt} = 1$$

12) $z = x^4, \quad x = t, y = t$

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13) Let $w(x,y,z)=x^2+y^2+z^2, \quad x=cost, y=sint, \quad$ and $z=e^t$. Express w as a function of t and find $\frac{dw}{dt}$ directly. Then, find $\frac{dw}{dt}$ using the chain rule.

Answer

$\frac{dw}{dt}=2e^{2t}$ in both cases

14) Let $z=x^2y,$ where $x=t^2$ and $y=t^3$. Find $\frac{dz}{dt}$.

15) Let $u=e^x \sin y,$ where $x=-\ln 2t$ and $y=\pi t$. Find $\frac{du}{dt}$ when $x=\ln 2$ and $y=\frac{\pi}{4}$.

Answer

$\frac{du}{dt}=\sqrt{2}(\pi-4)$

In exercises 16 - 33, find $\frac{dy}{dx}$ using partial derivatives.

16) $\sin(6x)+\tan(8y)+5=0$

17) $x^3+y^2x-3=0$

Answer

$\frac{dy}{dx}=-\frac{3x^2+y^2}{2xy}$

18) $\sin(x+y)+\cos(x-y)=4$

19) $x^2-2xy+y^4=4$

Answer

$\frac{dy}{dx}=\frac{y-x}{-x+2y^3}$

20) $xe^y+ye^x-2x^2y=0$

21) $x^{2/3}+y^{2/3}=a^{2/3}$

Answer

$\frac{dy}{dx}=-\sqrt[3]{\frac{y}{x}}$

22) $x\cos(xy)+y\cos x=2$

23) $e^{xy}+ye^y=1$

Answer

$\frac{dy}{dx}=-\frac{ye^{xy}}{xe^{xy}+e^y(1+y)}$

24) $x^2y^3+\cos y=0$

25) Find $\frac{dz}{dt}$ using the chain rule where $z=3x^2y^3, \ x=t^4, \$ and $y=t^2$.

Answer

$\frac{dz}{dt}=42t^{13}$

26) Let $z = 3\cos x - \sin(xy), x = \frac{1}{t},$ and $y = 3t$. Find $\dfrac{dz}{dt}$.

27) Let $z = e^{1-xy},\ x = t^{1/3},$ and $y = t^3$. Find $\dfrac{dz}{dt}$.

Answer

$$\frac{dz}{dt} = -\frac{10}{3}t^{7/3} \times e^{1-t^{10/3}}$$

28) Find $\dfrac{dz}{dt}$ by the chain rule where $z = \cosh^2(xy),\ x = \frac{1}{2}t,$ and $y = e^t$.

29) Let $z = \dfrac{x}{y},\ x = 2\cos u,$ and $y = 3\sin v$. Find $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$.

Answer

$$\frac{\partial z}{\partial u} = \frac{-2\sin u}{3\sin v} \text{ and } \frac{\partial z}{\partial v} = \frac{-2\cos u\cos v}{3\sin^2 v}$$

30) Let $z = e^{x^2y}$, where $x = \sqrt{uv}$ and $y = \frac{1}{v}$. Find $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$.

31) If $z = xye^{x/y},\ x = r\cos\theta,$ and $y = r\sin\theta$, find $\dfrac{\partial z}{\partial r}$ and $\dfrac{\partial z}{\partial \theta}$ when $r = 2$ and $\theta = \frac{\pi}{6}$.

Answer

$$\frac{\partial z}{\partial r} = \sqrt{3}e^{\sqrt{3}}, \frac{\partial z}{\partial \theta} = (2 - 4\sqrt{3})e^{\sqrt{3}}$$

32) Find $\dfrac{\partial w}{\partial s}$ if $w = 4x + y^2 + z^3,\ x = e^{rs^2},\ y = \ln(\frac{r+s}{t}),$ and $z = rst^2$.

33) If $w = \sin(xyz),\ x = 1 - 3t,\ y = e^{1-t},$ and $z = 4t$, find $\dfrac{\partial w}{\partial t}$.

Answer

$$\frac{\partial w}{\partial t} = -3yz\cos(xyz) - xze^{1-t}\cos(xyz) + 4xy\cos(xyz)$$

In exercises 34 - 36, use this information: A function $f(x,y)$ is said to be homogeneous of degree n if $f(tx,ty) = t^n f(x,y)$. For all homogeneous functions of degree n, the following equation is true: $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} = nf(x,y)$. Show that the given function is homogeneous and verify that $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} = nf(x,y)$.

34) $f(x,y) = 3x^2 + y^2$

35) $f(x,y) = \sqrt{x^2 + y^2}$

Answer

$$f(tx,ty) = \sqrt{t^2x^2 + t^2y^2} = t^1 f(x,y), \quad \frac{\partial f}{\partial y} = x\tfrac{1}{2}(x^2+y^2)^{-1/2} \times 2x + y\tfrac{1}{2}(x^2+y^2)^{-1/2} \times 2y = 1f(x,y)$$

36) $f(x,y) = x^2y - 2y^3$

37) The volume of a right circular cylinder is given by $V(x,y) = \pi x^2 y$, where x is the radius of the cylinder and y is the cylinder height. Suppose x and y are functions of t given by $x = \frac{1}{2}t$ and $y = \frac{1}{3}t$ so that x and y are both increasing with time. How fast is the volume increasing when $x = 2$ and $y = 5$? Assume time is measured in seconds.

Answer

$\ddfrac{dV}{dt} = \frac{34\pi}{3}$ units^3/s

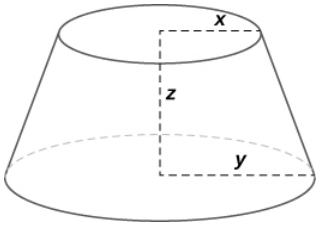
38) The pressure P of a gas is related to the volume and temperature by the formula $PV = kT$, where temperature is expressed in kelvins. Express the pressure of the gas as a function of both V and T. Find $\frac{dP}{dt}$ when $k = 1$, $\frac{dV}{dt} = 2$ cm^3/min, $\frac{dT}{dt} = 12$ K/min, $V = 20cm^3$, and $T = 20^\circ F$.

39) The radius of a right circular cone is increasing at 3 cm/min whereas the height of the cone is decreasing at 2 cm/min. Find the rate of change of the volume of the cone when the radius is 13 cm and the height is 18 cm.

Answer

$\frac{dV}{dt} = \frac{1066\pi}{3}\ \text{cm}^3/\text{min}$

40) The volume of a frustum of a cone is given by the formula $V = \frac{1}{3}\pi z(x^2 + y^2 + xy)$, where x is the radius of the smaller circle, y is the radius of the larger circle, and z is the height of the frustum (see figure). Find the rate of change of the volume of this frustum when $x = 10$ in., $y = 12$ in., and $z = 18$ in.



41) A closed box is in the shape of a rectangular solid with dimensions x, y, and z. (Dimensions are in inches.) Suppose each dimension is changing at the rate of 0.5 in./min. Find the rate of change of the total surface area of the box when $x = 2$ in., $y = 3$ in., and $z = 1$ in.

Answer

$\frac{dA}{dt} = 12\ \text{in.}^2/\text{min}$

42) The total resistance in a circuit that has three individual resistances represented by x, y, and z is given by the formula $R(x, y, z) = \frac{xyz}{yz + xz + xy}$. Suppose at a given time the x resistance is $100\,\Omega$, the y resistance is $200\,\Omega$, and the z resistance is $300\,\Omega$. Also, suppose the x resistance is changing at a rate of $2\,\Omega/\text{min}$, the y resistance is changing at the rate of $1\,\Omega/\text{min}$, and the z resistance has no change. Find the rate of change of the total resistance in this circuit at this time.

43) The temperature T at a point (x, y) is $T(x, y)$ and is measured using the Celsius scale. A fly crawls so that its position after t seconds is given by $x = \sqrt{1 + t}$ and $y = 2 + \frac{1}{3}t$, where x and y are measured in centimeters. The temperature function satisfies $T_x(2, 3) = 4$ and $T_y(2, 3) = 3$. How fast is the temperature increasing on the fly's path after 3 sec?

Answer

2°C/sec

44) The x and y components of a fluid moving in two dimensions are given by the following functions: $u(x, y) = 2y$ and $v(x, y) = -2x$ with $x \geq 0$ and $y \geq 0$. The speed of the fluid at the point (x, y) is $s(x, y) = \sqrt{u(x, y)^2 + v(x, y)^2}$. Find $\frac{\partial s}{\partial x}$ and $\frac{\partial s}{\partial y}$ using the chain rule.

45) Let $u = u(x, y, z)$, where $x = x(w, t)$, $y = y(w, t)$, $z = z(w, t)$, $w = w(r, s)$, and $t = t(r, s)$. Use a tree diagram and the chain rule to find an expression for $\frac{\partial u}{\partial r}$.

Answer

$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\left(\frac{\partial x}{\partial w}\frac{\partial w}{\partial r} + \frac{\partial x}{\partial t}\frac{\partial t}{\partial r}\right) + \frac{\partial u}{\partial y}\left(\frac{\partial y}{\partial w}\frac{\partial w}{\partial r} + \frac{\partial y}{\partial t}\frac{\partial t}{\partial r}\right) + \frac{\partial u}{\partial z}\left(\frac{\partial z}{\partial w}\frac{\partial w}{\partial r} + \frac{\partial z}{\partial t}\frac{\partial t}{\partial r}\right)$

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5.7: Directional Derivatives and the Gradient

Learning Objectives

- Determine the directional derivative in a given direction for a function of two variables.
- Determine the gradient vector of a given real-valued function.
- Explain the significance of the gradient vector with regard to direction of change along a surface.
- Use the gradient to find the tangent to a level curve of a given function.
- Calculate directional derivatives and gradients in three dimensions.

A function $z = f(x, y)$ has two partial derivatives: $\partial z/\partial x$ and $\partial z/\partial y$. These derivatives correspond to each of the independent variables and can be interpreted as instantaneous rates of change (that is, as slopes of a tangent line). For example, $\partial z/\partial x$ represents the slope of a tangent line passing through a given point on the surface defined by $z = f(x, y)$, assuming the tangent line is parallel to the x**-axis**. Similarly, $\partial z/\partial y$ represents the slope of the tangent line parallel to the y**-axis.** Now we consider the possibility of a tangent line parallel to neither axis.

Directional Derivatives

We start with the graph of a surface defined by the equation $z = f(x, y)$. Given a point (a, b) in the domain of f, we choose a direction to travel from that point. We measure the direction using an angle θ, which is measured counterclockwise in the xy-plane, starting at zero from the positive x-axis (Figure 5.7.1). The distance we travel is h and the direction we travel is given by the unit vector $\vec{\mathbf{u}} = (\cos\theta)\,\hat{\mathbf{i}} + (\sin\theta)\,\hat{\mathbf{j}}$. Therefore, the z-coordinate of the second point on the graph is given by $z = f(a + h\cos\theta, b + h\sin\theta)$.

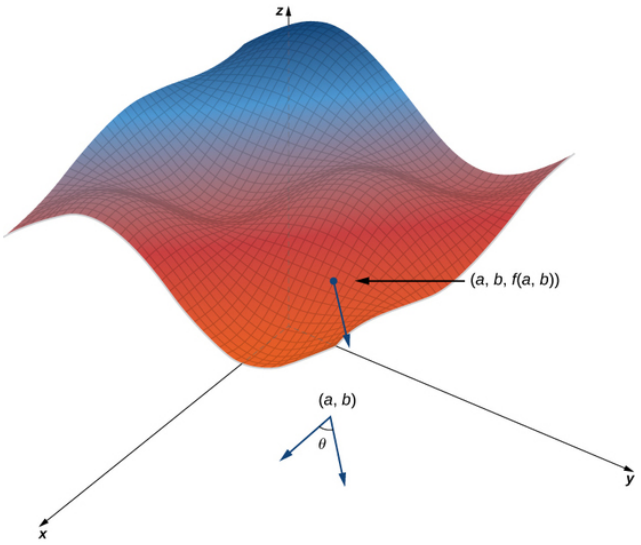


Figure 5.7.1: Finding the directional derivative at a point on the graph of $z = f(x, y)$. The slope of the blue arrow on the graph indicates the value of the directional derivative at that point.

We can calculate the slope of the secant line by dividing the difference in z-values by the length of the line segment connecting the two points in the domain. The length of the line segment is h. Therefore, the slope of the secant line is

$$m_{sec} = \frac{f(a + h\cos\theta, b + h\sin\theta) - f(a, b)}{h} \tag{5.7.1}$$

To find the slope of the tangent line in the same direction, we take the limit as h approaches zero.

Definition: Directional Derivatives

Suppose $z = f(x, y)$ is a function of two variables with a domain of D. Let $(a, b) \in D$ and define $\vec{\mathbf{u}} = (\cos\theta)\,\hat{\mathbf{i}} + (\sin\theta)\,\hat{\mathbf{j}}$. Then the directional derivative of f in the direction of $\vec{\mathbf{u}}$ is given by

$$D_{\vec{\mathbf{u}}} f(a, b) = \lim_{h \to 0} \frac{f(a + h\cos\theta, b + h\sin\theta) - f(a, b)}{h} \tag{5.7.2}$$

provided the limit exists.

Equation 5.7.2 provides a formal definition of the directional derivative that can be used in many cases to calculate a directional derivative.

Note that since the point (a,b) is chosen randomly from the domain D of the function f, we can use this definition to find the directional derivative as a function of x and y.

That is,

$$D_{\vec{\mathbf{u}}} f(x,y) = \lim_{h\to 0} \frac{f(x+h\cos\theta, y+h\sin\theta) - f(x,y)}{h} \tag{5.7.3}$$

Example 5.7.1: Finding a Directional Derivative from the Definition

Let $\theta = \arccos(3/5)$. Find the directional derivative $D_{\vec{\mathbf{u}}} f(x,y)$ of $f(x,y) = x^2 - xy + 3y^2$ in the direction of $\vec{\mathbf{u}} = (\cos\theta)\,\hat{\mathbf{i}} + (\sin\theta)\,\hat{\mathbf{j}}$.

Then determine $D_{\vec{\mathbf{u}}} f(-1,2)$.

Solution

First of all, since $\cos\theta = 3/5$ and θ is acute, this implies

$$\sin\theta = \sqrt{1-\left(\frac{3}{5}\right)^2} = \sqrt{\frac{16}{25}} = \frac{4}{5}.$$

Using $f(x,y) = x^2 - xy + 3y^2$, we first calculate $f(x+h\cos\theta, y+h\sin\theta)$:

$$\begin{aligned} f(x+h\cos\theta, y+h\sin\theta) &= (x+h\cos\theta)^2 - (x+h\cos\theta)(y+h\sin\theta) + 3(y+h\sin\theta)^2 \\ &= x^2 + 2xh\cos\theta + h^2\cos^2\theta - xy - xh\sin\theta - yh\cos\theta - h^2\sin\theta\cos\theta + 3y^2 + 6yh\sin\theta + 3h^2\sin^2\theta \\ &= x^2 + 2xh(\frac{3}{5}) + \frac{9h^2}{25} - xy - \frac{4xh}{5} - \frac{3yh}{5} - \frac{12h^2}{25} + 3y^2 + 6yh(\frac{4}{5}) + 3h^2(\frac{16}{25}) \\ &= x^2 - xy + 3y^2 + \frac{2xh}{5} + \frac{9h^2}{5} + \frac{21yh}{5}. \end{aligned}$$

We substitute this expression into Equation 5.7.2 with $a = x$ and $b = y$:

$$\begin{aligned} D_{\vec{\mathbf{u}}} f(x,y) &= \lim_{h\to 0} \frac{f(x+h\cos\theta, y+h\sin\theta) - f(x,y)}{h} \\ &= \lim_{h\to 0} \frac{(x^2 - xy + 3y^2 + \frac{2xh}{5} + \frac{9h^2}{5} + \frac{21yh}{5}) - (x^2 - xy + 3y^2)}{h} \\ &= \lim_{h\to 0} \frac{\frac{2xh}{5} + \frac{9h^2}{5} + \frac{21yh}{5}}{h} \\ &= \lim_{h\to 0} \frac{2x}{5} + \frac{9h}{5} + \frac{21y}{5} \\ &= \frac{2x+21y}{5}. \end{aligned}$$

To calculate $D_{\vec{\mathbf{u}}} f(-1,2)$, we substitute $x = -1$ and $y = 2$ into this answer (Figure 5.7.2):

$$D_{\vec{\mathbf{u}}} f(-1,2) = \frac{2(-1)+21(2)}{5} = \frac{-2+42}{5} = 8.$$

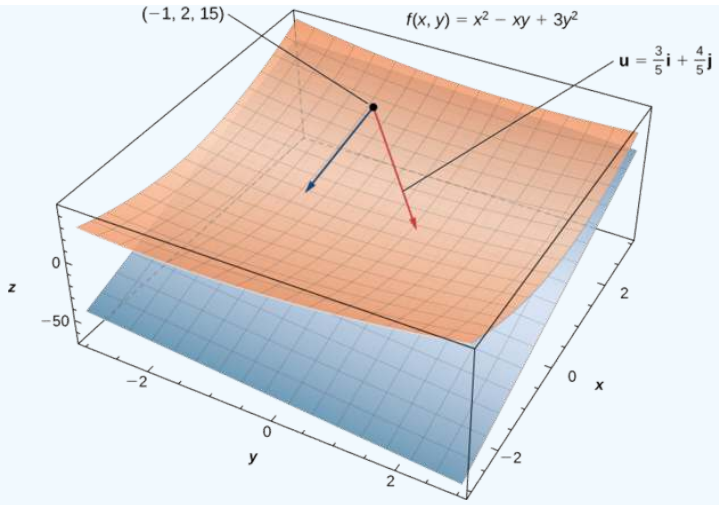


Figure 5.7.2: Finding the directional derivative in a given direction $\vec{\mathbf{u}}$ at a given point on a surface. The plane is tangent to the surface at the given point $(-1, 2, 15)$.

An easier approach to calculating directional derivatives that involves partial derivatives is outlined in the following theorem.

Directional Derivative of a Function of Two Variables

Let $z = f(x, y)$ be a function of two variables x and y, and assume that f_x and f_y exist. Then the directional derivative of f in the direction of $\vec{\mathbf{u}} = (\cos\theta)\,\hat{\mathbf{i}} + (\sin\theta)\,\hat{\mathbf{j}}$ is given by

$$D_{\vec{\mathbf{u}}} f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta. \tag{5.7.4}$$

Proof

Applying the definition of a directional derivative stated above in Equation 5.7.2, the directional derivative of f in the direction of $\vec{\mathbf{u}} = (\cos\theta)\,\hat{\mathbf{i}} + (\sin\theta)\,\hat{\mathbf{j}}$ at a point (x_0, y_0) in the domain of f can be written

$$D_{\vec{\mathbf{u}}} f((x_0, y_0)) = \lim_{t\to 0} \frac{f(x_0 + t\cos\theta, y_0 + t\sin\theta) - f(x_0, y_0)}{t}. \tag{5.7.5}$$

Let $x = x_0 + t\cos\theta$ and $y = y_0 + t\sin\theta$, and define $g(t) = f(x, y)$. Since f_x and f_y both exist, we can use the chain rule for functions of two variables to calculate $g'(t)$:

$$g'(t) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta. \tag{5.7.6}$$

If $t = 0$, then $x = x_0$ and $y = y_0$, so

$$g'(0) = f_x(x_0, y_0)\cos\theta + f_y(x_0, y_0)\sin\theta \tag{5.7.7}$$

By the definition of $g'(t)$, it is also true that

$$g'(0) = \lim_{t\to 0} \frac{g(t) - g(0)}{t} = \lim_{t\to 0} \frac{f(x_0 + t\cos\theta, y_0 + t\sin\theta) - f(x_0, y_0)}{t}. \tag{5.7.8}$$

Therefore, $D_{\vec{\mathbf{u}}} f(x_0, y_0) = f_x(x_0, y_0)\cos\theta + f_y(x_0, y_0)\sin\theta$.

Since the point (x_0, y_0) is an arbitrary point from the domain of f, this result holds for all points in the domain of f for which the partials f_x and f_y exist.

Therefore,

$$D_{\vec{\mathbf{u}}} f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta. \tag{5.7.9}$$

□

Example 5.7.2: Finding a Directional Derivative: Alternative Method

Let $\theta = \arccos(3/5)$. Find the directional derivative $D_{\vec{\mathbf{u}}} f(x,y)$ of $f(x,y) = x^2 - xy + 3y^2$ in the direction of $\vec{\mathbf{u}} = (\cos\theta)\,\hat{\mathbf{i}} + (\sin\theta)\,\hat{\mathbf{j}}$.

Then determine $D_{\vec{\mathbf{u}}} f(-1,2)$.

Solution

First, we must calculate the partial derivatives of f:

$$\begin{aligned} f_x(x,y) &= 2x - y \\ f_y(x,y) &= -x + 6y, \end{aligned}$$

Then we use Equation 5.7.4 with $\theta = \arccos(3/5)$:

$$\begin{aligned} D_{\vec{\mathbf{u}}} f(x,y) &= f_x(x,y)\cos\theta + f_y(x,y)\sin\theta \\ &= (2x-y)\frac{3}{5} + (-x+6y)\frac{4}{5} \\ &= \frac{6x}{5} - \frac{3y}{5} - \frac{4x}{5} + \frac{24y}{5} \\ &= \frac{2x+21y}{5}. \end{aligned}$$

To calculate $D_{\vec{\mathbf{u}}} f(-1,2)$, let $x = -1$ and $y = 2$:

$$D_{\vec{\mathbf{u}}} f(-1,2) = \frac{2(-1)+21(2)}{5} = \frac{-2+42}{5} = 8.$$

This is the same answer obtained in Example 5.7.1.



Exercise 5.7.1:

Find the directional derivative $D_{\vec{\mathbf{u}}} f(x,y)$ of $f(x,y) = 3x^2y - 4xy^3 + 3y^2 - 4x$ in the direction of $\vec{\mathbf{u}} = (\cos\frac{\pi}{3})\,\hat{\mathbf{i}} + (\sin\frac{\pi}{3})\,\hat{\mathbf{j}}$ using Equation 5.7.4.

What is $D_{\vec{\mathbf{u}}} f(3,4)$?

Hint

Calculate the partial derivatives and determine the value of θ.

Answer

$$D_{\vec{\mathbf{u}}} f(x,y) = \frac{(6xy - 4y^3 - 4)(1)}{2} + \frac{(3x^2 - 12xy^2 + 6y)\sqrt{3}}{2}$$

$$D_{\vec{\mathbf{u}}} f(3,4) = \frac{72 - 256 - 4}{2} + \frac{(27 - 576 + 24)\sqrt{3}}{2} = -94 - \frac{525\sqrt{3}}{2}$$

If the vector that is given for the direction of the derivative is not a unit vector, then it is only necessary to divide by the norm of the vector. For example, if we wished to find the directional derivative of the function in Example 5.7.2 in the direction of the vector $\langle -5, 12\rangle$, we would first divide by its magnitude to get $\vec{\mathbf{u}}$. This gives us $\vec{\mathbf{u}} = \langle -\frac{5}{13}, \frac{12}{13}\rangle$.

Then

$$\begin{aligned} D_{\vec{\mathbf{u}}} f(x,y) &= f_x(x,y)\cos\theta + f_y(x,y)\sin\theta \\ &= -\frac{5}{13}(2x-y) + \frac{12}{13}(-x+6y) \\ &= -\frac{22}{13}x + \frac{17}{13}y \end{aligned}$$

Gradient

The right-hand side of Equation 5.7.4 is equal to $f_x(x,y)\cos\theta + f_y(x,y)\sin\theta$, which can be written as the dot product of two vectors. Define the first vector as $\vec{\nabla} f(x,y) = f_x(x,y)\,\hat{\mathbf{i}} + f_y(x,y)\,\hat{\mathbf{j}}$ and the second vector as $\vec{\mathbf{u}} = (\cos\theta)\,\hat{\mathbf{i}} + (\sin\theta)\,\hat{\mathbf{j}}$. Then the right-hand side of the equation can be written as the dot product of these two vectors:

$$D_{\vec{\mathbf{u}}} f(x,y) = \vec{\nabla} f(x,y) \cdot \vec{\mathbf{u}}. \tag{5.7.10}$$

The first vector in Equation 5.7.10 has a special name: the gradient of the function f. The symbol ∇ is called **nabla** and the vector $\vec{\nabla} f$ is read "**del** f."

Definition: The Gradient

Let $z = f(x,y)$ be a function of x and y such that f_x and f_y exist. The vector $\vec{\nabla} f(x,y)$ is called the **gradient** of f and is defined as

$$\vec{\nabla} f(x,y) = f_x(x,y)\,\hat{\mathbf{i}} + f_y(x,y)\,\hat{\mathbf{j}}. \tag{5.7.11}$$

The vector $\vec{\nabla} f(x,y)$ is also written as "**grad** f."

Example 5.7.3: Finding Gradients

Find the gradient $\vec{\nabla} f(x,y)$ of each of the following functions:

a. $f(x,y) = x^2 - xy + 3y^2$
b. $f(x,y) = \sin 3x \cos 3y$

Solution

For both parts a. and b., we first calculate the partial derivatives f_x and f_y, then use Equation 5.7.11.

a. $f_x(x,y) = 2x - y$ and $f_y(x,y) = -x + 6y$, so

$$\begin{aligned} \vec{\nabla} f(x,y) &= f_x(x,y)\,\hat{\mathbf{i}} + f_y(x,y)\,\hat{\mathbf{j}} \\ &= (2x-y)\,\hat{\mathbf{i}} + (-x+6y)\,\hat{\mathbf{j}}. \end{aligned}$$

b. $f_x(x,y) = 3\cos 3x \cos 3y$ and $f_y(x,y) = -3\sin 3x \sin 3y$, so

$$\begin{aligned} \vec{\nabla} f(x,y) &= f_x(x,y)\,\hat{\mathbf{i}} + f_y(x,y)\,\hat{\mathbf{j}} \\ &= (3\cos 3x \cos 3y)\,\hat{\mathbf{i}} - (3\sin 3x \sin 3y)\,\hat{\mathbf{j}}. \end{aligned}$$

Exercise 5.7.2

Find the gradient $\vec{\nabla} f(x,y)$ of $f(x,y) = \dfrac{x^2 - 3y^2}{2x + y}$.

Hint

Calculate the partial derivatives, then use Equation 5.7.11.

Answer

$$\vec{\nabla} f(x,y) = \frac{2x^2 + 2xy + 6y^2}{(2x+y)^2}\,\hat{\mathbf{i}} - \frac{x^2 + 12xy + 3y^2}{(2x+y)^2}\,\hat{\mathbf{j}}$$

The gradient has some important properties. We have already seen one formula that uses the gradient: the formula for the directional derivative. Recall from The Dot Product that if the angle between two vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ is φ, then $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = \|\vec{\mathbf{a}}\| \|\vec{\mathbf{b}}\| \cos\varphi$. Therefore, if the angle between $\vec{\nabla} f(x_0, y_0)$ and $\vec{\mathbf{u}} = (cos\theta)\,\hat{\mathbf{i}} + (sin\theta)\,\hat{\mathbf{j}}$ is φ, we have

$$D_{\vec{\mathbf{u}}} f(x_0, y_0) = \vec{\nabla} f(x_0, y_0) \cdot \vec{\mathbf{u}} = \|\vec{\nabla} f(x_0, y_0)\| \|\vec{\mathbf{u}}\| \cos\varphi = \|\vec{\nabla} f(x_0, y_0)\| \cos\varphi. \tag{5.7.12}$$

The $\|\vec{\mathbf{u}}\|$ disappears because $\vec{\mathbf{u}}$ is a unit vector. Therefore, the directional derivative is equal to the magnitude of the gradient evaluated at (x_0, y_0) multiplied by $\cos\varphi$. Recall that $\cos\varphi$ ranges from -1 to 1.

If $\varphi = 0$, then $\cos\varphi = 1$ and $\vec{\nabla} f(x_0, y_0)$ and $\vec{\mathbf{u}}$ both point in the same direction.

If $\varphi = \pi$, then $\cos\varphi = -1$ and $\vec{\nabla} f(x_0, y_0)$ and $\vec{\mathbf{u}}$ point in opposite directions.

In the first case, the value of $D_{\vec{\mathbf{u}}} f(x_0, y_0)$ is maximized; in the second case, the value of $D_{\vec{\mathbf{u}}} f(x_0, y_0)$ is minimized.

We can also see that if $\vec{\nabla} f(x_0, y_0) = \vec{\mathbf{0}}$, then

$$D_{\vec{\mathbf{u}}} f(x_0, y_0) = \vec{\nabla} f(x_0, y_0) \cdot \vec{\mathbf{u}} = 0 \tag{5.7.13}$$

for any vector $\vec{\mathbf{u}}$. These three cases are outlined in the following theorem.

Properties of the Gradient

Suppose the function $z = f(x, y)$ is differentiable at (x_0, y_0) (Figure 5.7.3).

i. If $\vec{\nabla} f(x_0, y_0) = \vec{\mathbf{0}}$, then $D_{\vec{\mathbf{u}}} f(x_0, y_0) = 0$ for any unit vector $\vec{\mathbf{u}}$.
ii. If $\vec{\nabla} f(x_0, y_0) \neq \vec{\mathbf{0}}$, then $D_{\vec{\mathbf{u}}} f(x_0, y_0)$ is **maximized** when $\vec{\mathbf{u}}$ points in the same direction as $\vec{\nabla} f(x_0, y_0)$. The maximum value of $D_{\vec{\mathbf{u}}} f(x_0, y_0)$ is $\|\vec{\nabla} f(x_0, y_0)\|$.
iii. If $\vec{\nabla} f(x_0, y_0) \neq \vec{\mathbf{0}}$, then $D_{\vec{\mathbf{u}}} f(x_0, y_0)$ is **minimized** when $\vec{\mathbf{u}}$ points in the opposite direction from $\vec{\nabla} f(x_0, y_0)$. The minimum value of $D_{\vec{\mathbf{u}}} f(x_0, y_0)$ is $-\|\vec{\nabla} f(x_0, y_0)\|$.

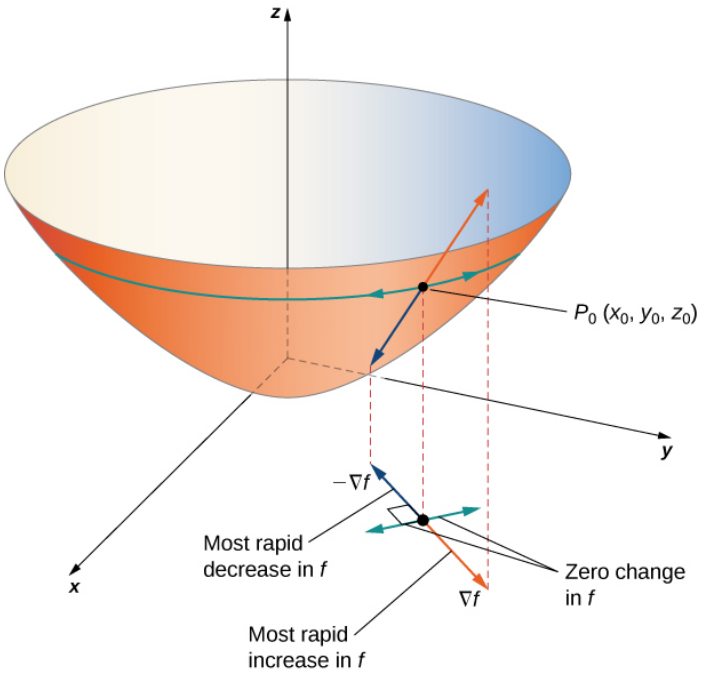


Figure 5.7.3: The gradient indicates the maximum and minimum values of the directional derivative at a point.

Example 5.7.4: Finding a Maximum Directional Derivative

Find the direction for which the directional derivative of $f(x, y) = 3x^2 - 4xy + 2y^2$ at $(-2, 3)$ is a maximum. What is the maximum value?

Solution

The maximum value of the directional derivative occurs when $\vec{\nabla} f$ and the unit vector point in the same direction. Therefore, we start by calculating $\vec{\nabla} f(x,y)$:

$$f_x(x,y) = 6x - 4y \text{ and } f_y(x,y) = -4x + 4y$$

so

$$\vec{\nabla} f(x,y) = f_x(x,y)\,\hat{\mathbf{i}} + f_y(x,y)\,\hat{\mathbf{j}} = (6x-4y)\,\hat{\mathbf{i}} + (-4x+4y)\,\hat{\mathbf{j}}.$$

Next, we evaluate the gradient at $(-2,3)$:

$$\vec{\nabla} f(-2,3) = (6(-2)-4(3))\,\hat{\mathbf{i}} + (-4(-2)+4(3))\,\hat{\mathbf{j}} = -24\,\hat{\mathbf{i}} + 20\,\hat{\mathbf{j}}.$$

We need to find a unit vector that points in the same direction as $\vec{\nabla} f(-2,3)$, so the next step is to divide $\vec{\nabla} f(-2,3)$ by its magnitude, which is $\sqrt{(-24)^2+(20)^2} = \sqrt{976} = 4\sqrt{61}$. Therefore,

$$\frac{\vec{\nabla} f(-2,3)}{\|\vec{\nabla} f(-2,3)\|} = \frac{-24}{4\sqrt{61}}i + \frac{20}{4\sqrt{61}}j = -\frac{6\sqrt{61}}{61}\,\hat{\mathbf{i}} + \frac{5\sqrt{61}}{61}\,\hat{\mathbf{j}}.$$

This is the unit vector that points in the same direction as $\vec{\nabla} f(-2,3)$. To find the angle corresponding to this unit vector, we solve the equations

$$\cos\theta = \frac{-6\sqrt{61}}{61} \text{ and } \sin\theta = \frac{5\sqrt{61}}{61}$$

for θ. Since cosine is negative and sine is positive, the angle must be in the second quadrant. Therefore, $\theta = \pi - \arcsin((5\sqrt{61})/61) \approx 2.45$ rad.

The maximum value of the directional derivative at $(-2,3)$ is $\|\vec{\nabla} f(-2,3)\| = 4\sqrt{61}$ (Figure 5.7.4).

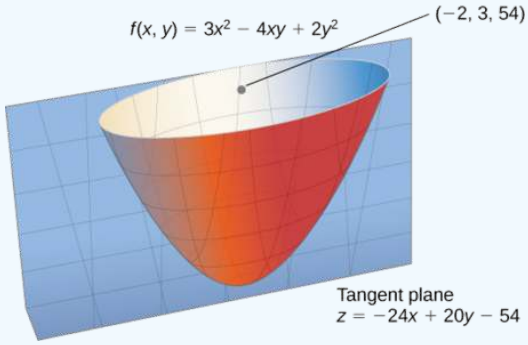


Figure 5.7.4: The maximum value of the directional derivative at $(-2,3)$ is in the direction of the gradient.

Exercise 5.7.3

Find the direction for which the directional derivative of $g(x,y) = 4x - xy + 2y^2$ at $(-2,3)$ is a maximum. What is the maximum value?

Hint

Evaluate the gradient of g at point $(-2,3)$.

Answer

The gradient of g at $(-2,3)$ is $\vec{\nabla} g(-2,3) = \hat{\mathbf{i}} + 14\,\hat{\mathbf{j}}$. The unit vector that points in the same direction as $\vec{\nabla} g(-2,3)$ is

$$\frac{\vec{\nabla} g(-2,3)}{\|\vec{\nabla} g(-2,3)\|} = \frac{1}{\sqrt{197}}\,\hat{\mathbf{i}} + \frac{14}{\sqrt{197}}\,\hat{\mathbf{j}} = \frac{\sqrt{197}}{197}\,\hat{\mathbf{i}} + \frac{14\sqrt{197}}{197}\,\hat{\mathbf{j}},$$

which gives an angle of $\theta = \arcsin((14\sqrt{197})/197) \approx 1.499$ rad.

The maximum value of the directional derivative is $\|\vec{\nabla} g(-2,3)\| = \sqrt{197}$.

Figure 5.7.5 shows a portion of the graph of the function $f(x, y) = 3 + \sin x \sin y$. Given a point (a, b) in the domain of f, the maximum value of the directional derivative at that point is given by $\|\vec{\nabla} f(a, b)\|$. This would equal the rate of greatest ascent if the surface represented a topographical map. If we went in the opposite direction, it would be the rate of greatest descent.

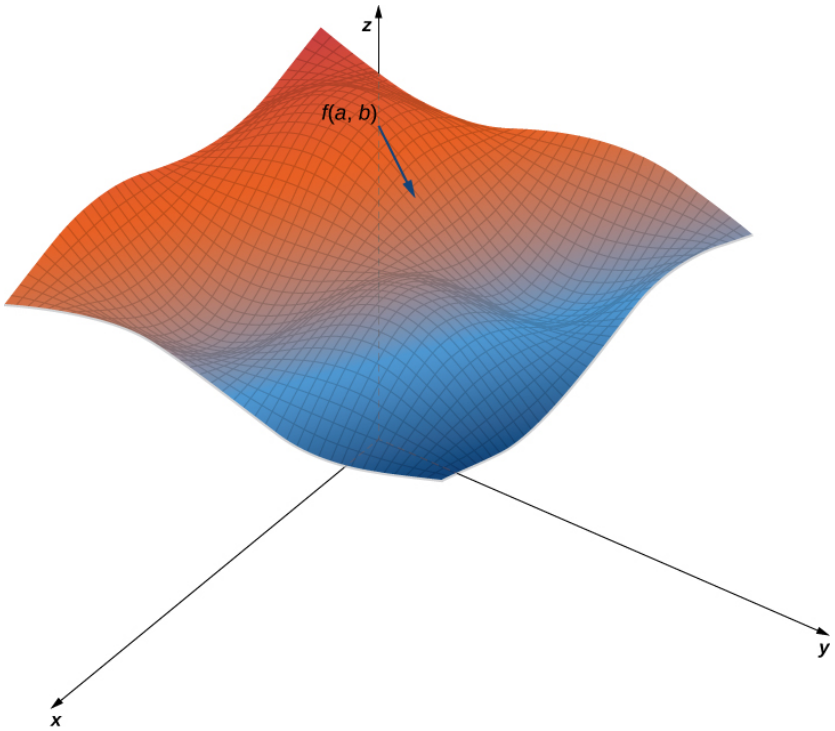
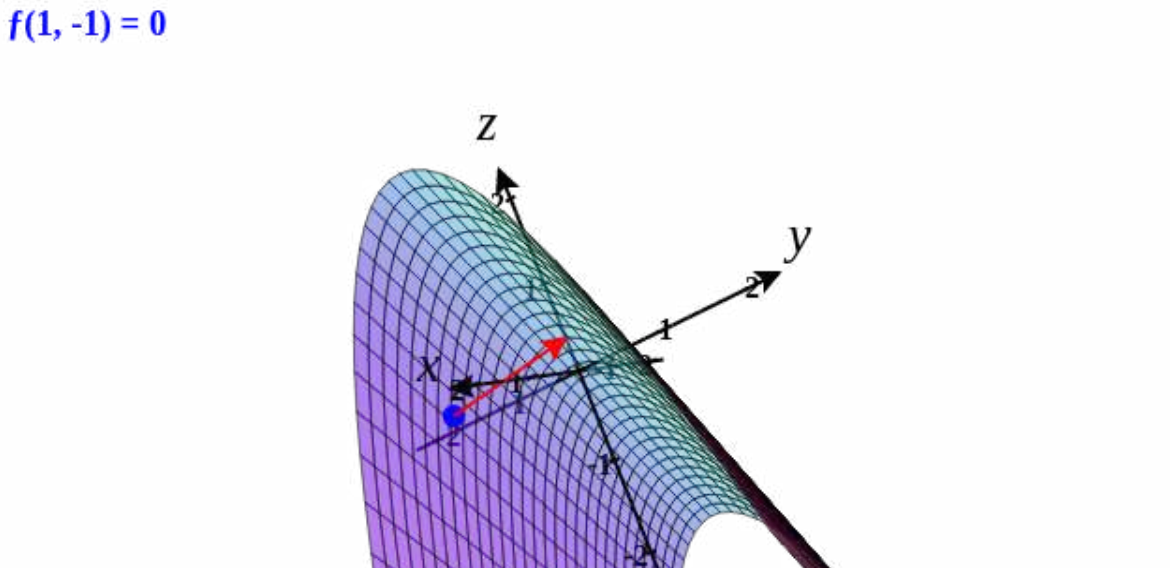


Figure 5.7.5: A typical surface in $\mathbb{R}^3$. Given a point on the surface, the directional derivative can be calculated using the gradient.

Visualizing the Gradient Vector on Its Surface

Below is the graph of the surface $f(x, y) = x - y^2$. You can move the point and see the associated gradient of the function at that point.

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When using a topographical map, the steepest slope is always in the direction where the contour lines are closest together (Figure 5.7.6). This is analogous to the contour map of a function, assuming the level curves are obtained for equally spaced values throughout the range of that function.

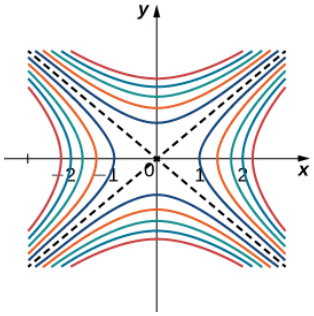


Figure 5.7.6: Contour map for the function $f(x,y)=x^2-y^2$ using level values between -5 and 5.

Gradients and Level Curves

Recall that if a curve is defined parametrically by the function pair $(x(t),y(t))$, then the vector $x'(t)\,\hat{\mathbf{i}}+y'(t)\,\hat{\mathbf{j}}$ is tangent to the curve for every value of t in the domain. Now let's assume $z=f(x,y)$ is a differentiable function of x and y, and (x_0,y_0) is in its domain. Let's suppose further that $x_0=x(t_0)$ and $y_0=y(t_0)$ for some value of t, and consider the level curve $f(x,y)=k$. Define $g(t)=f(x(t),y(t))$ and calculate $g'(t)$ on the level curve. By the chain Rule,

$$g'(t)=f_x(x(t),y(t))x'(t)+f_y(x(t),y(t))y'(t). \tag{5.7.14}$$

But $g'(t)=0$ because $g(t)=k$ for all t. Therefore, on the one hand,

$$f_x(x(t),y(t))x'(t)+f_y(x(t),y(t))y'(t)=0; \tag{5.7.15}$$

on the other hand,

$$f_x(x(t),y(t))x'(t)+f_y(x(t),y(t))y'(t)=\vec{\nabla}f(x,y)\cdot\langle x'(t),y'(t)\rangle. \tag{5.7.16}$$

Therefore,

$$\vec{\nabla}f(x,y)\cdot\langle x'(t),y'(t)\rangle=0. \tag{5.7.17}$$

Thus, the dot product of these vectors is equal to zero, which implies they are orthogonal. However, the second vector is tangent to the level curve, which implies the gradient must be normal to the level curve, which gives rise to the following theorem.

Gradient Is Normal to the Level Curve

Suppose the function $z=f(x,y)$ has continuous first-order partial derivatives in an open disk centered at a point (x_0,y_0). If $\vec{\nabla}f(x_0,y_0)\neq 0$, then $\vec{\nabla}f(x_0,y_0)$ is normal to the level curve of f at (x_0,y_0).

We can use this theorem to find tangent and normal vectors to level curves of a function.

Example 5.7.5: Finding Tangents to Level Curves

For the function $f(x,y)=2x^2-3xy+8y^2+2x-4y+4$, find a tangent vector to the level curve at point $(-2,1)$. Graph the level curve corresponding to $f(x,y)=18$ and draw in $\vec{\nabla}f(-2,1)$ and a tangent vector.

Solution

First, we must calculate $\vec{\nabla}f(x,y)$:

$$f_x(x,y)=4x-3y+2 \text{ and } f_y=-3x+16y-4 \text{ so } \vec{\nabla}f(x,y)=(4x-3y+2)\,\hat{\mathbf{i}}+(-3x+16y-4)\,\hat{\mathbf{j}}.$$

Next, we evaluate $\vec{\nabla}f(x,y)$ at $(-2,1)$:

$$\vec{\nabla}f(-2,1)=(4(-2)-3(1)+2)\,\hat{\mathbf{i}}+(-3(-2)+16(1)-4)\,\hat{\mathbf{j}}=-9\,\hat{\mathbf{i}}+18\,\hat{\mathbf{j}}.$$

This vector is orthogonal to the curve at point $(-2,1)$. We can obtain a tangent vector by reversing the components and multiplying either one by -1. Thus, for example, $-18\,\hat{\mathbf{i}}-9\,\hat{\mathbf{j}}$ is a tangent vector (Figure 5.7.7).

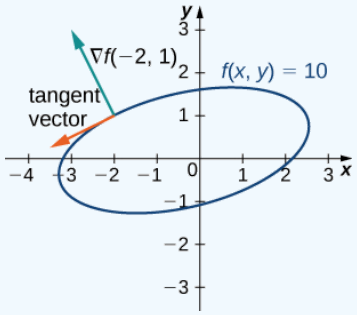


Figure 5.7.7: Tangent and normal vectors to $2x^2 - 3xy + 8y^2 + 2x - 4y + 4 = 18$ at point $(-2, 1)$.



Exercise 5.7.4

For the function $f(x, y) = x^2 - 2xy + 5y^2 + 3x - 2y + 3$, find the tangent to the level curve at point $(1, 1)$. Draw the graph of the level curve corresponding to $f(x, y) = 8$ and draw $\vec{\nabla} f(1, 1)$ and a tangent vector.

Hint

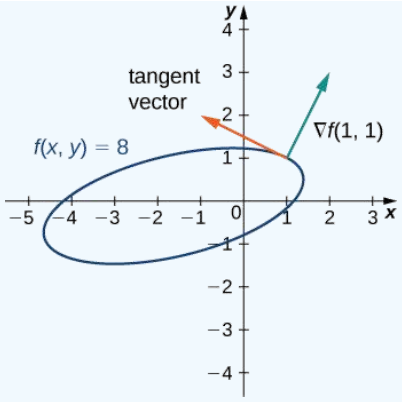
Calculate the gradient at point $(1, 1)$.

Answer

$\vec{\nabla} f(x, y) = (2x - 2y + 3)\,\hat{\mathbf{i}} + (-2x + 10y - 2)\,\hat{\mathbf{j}}$

$\vec{\nabla} f(1, 1) = 3\,\hat{\mathbf{i}} + 6\,\hat{\mathbf{j}}$

Tangent vector: $6\,\hat{\mathbf{i}} - 3\,\hat{\mathbf{j}}$ or $-6\,\hat{\mathbf{i}} + 3\,\hat{\mathbf{j}}$



Three-Dimensional Gradients and Directional Derivatives

The definition of a gradient can be extended to functions of more than two variables.

Definition: Gradients in 3D

Let $w = f(x, y, z)$ be a function of three variables such that f_x, f_y, and f_z exist. The vector $\vec{\nabla} f(x, y, z)$ is called the gradient of f and is defined as

$$\vec{\nabla} f(x, y, z) = f_x(x, y, z)\,\hat{\mathbf{i}} + f_y(x, y, z)\,\hat{\mathbf{j}} + f_z(x, y, z)\,\hat{\mathbf{k}}. \tag{5.7.18}$$

$\vec{\nabla} f(x, y, z)$ can also be written as **grad** $f(x, y, z)$.

Calculating the gradient of a function in three variables is very similar to calculating the gradient of a function in two variables. First, we calculate the partial derivatives f_x, f_y, and f_z, and then we use Equation 5.7.18.

Example 5.7.6: Finding Gradients in Three Dimensions

Find the gradient $\vec{\nabla} f(x, y, z)$ of each of the following functions:

a. $f(x, y, z) = 5x^2 - 2xy + y^2 - 4yz + z^2 + 3xz$
b. $f(x, y, z) = e^{-2z} \sin 2x \cos 2y$

Solution

For both parts a. and b., we first calculate the partial derivatives f_x, f_y, and f_z, then use Equation 5.7.18.

a. $f_x(x, y, z) = 10x - 2y + 3z$, $f_y(x, y, z) = -2x + 2y - 4z$, and $f_z(x, y, z) = 3x - 4y + 2z$, so

$$\begin{aligned}\vec{\nabla} f(x, y, z) &= f_x(x, y, z)\,\hat{\mathbf{i}} + f_y(x, y, z)\,\hat{\mathbf{j}} + f_z(x, y, z)\,\hat{\mathbf{k}} \\ &= (10x - 2y + 3z)\,\hat{\mathbf{i}} + (-2x + 2y - 4z)\,\hat{\mathbf{j}} + (3x - 4y + 2z)\,\hat{\mathbf{k}}.\end{aligned}$$

b. $f_x(x, y, z) = 2e^{-2z} \cos 2x \cos 2y$, $f_y(x, y, z) = -2e^{-2z} \sin 2x \sin 2y$, and $f_z(x, y, z) = -2e^{-2z} \sin 2x \cos 2y$, so

$$\begin{aligned}\vec{\nabla} f(x, y, z) &= f_x(x, y, z)\,\hat{\mathbf{i}} + f_y(x, y, z)\,\hat{\mathbf{j}} + f_z(x, y, z)\,\hat{\mathbf{k}} \\ &= (2e^{-2z} \cos 2x \cos 2y)\,\hat{\mathbf{i}} + (-2e^{-2z} \sin 2x \sin 2y)\,\hat{\mathbf{j}} + (-2e^{-2z} \sin 2x \cos 2y)\,\hat{\mathbf{k}} \\ &= 2e^{-2z}(\cos 2x \cos 2y\,\hat{\mathbf{i}} - \sin 2x \sin 2y\,\hat{\mathbf{j}} - \sin 2x \cos 2y\,\hat{\mathbf{k}}).\end{aligned}$$



Exercise 5.7.5:

Find the gradient $\vec{\nabla} f(x,y,z)$ of $f(x,y,z) = \dfrac{x^2 - 3y^2 + z^2}{2x + y - 4z.}$

Answer

$$\vec{\nabla} f(x,y,z) = \frac{2x^2 + 2xy + 6y^2 - 8xz - 2z^2}{(2x+y-4z)^2}\,\hat{\mathbf{i}} - \frac{x^2 + 12xy + 3y^2 - 24yz + z^2}{(2x+y-4z)^2}\,\hat{\mathbf{j}} + \frac{4x^2 - 12y^2 - 4z^2 + 4xz + 2yz}{(2x+y-4z)^2}\,\hat{\mathbf{k}}$$



The directional derivative can also be generalized to functions of three variables. To determine a direction in three dimensions, a vector with three components is needed. This vector is a unit vector, and the components of the unit vector are called *directional cosines***.** Given a three-dimensional unit vector $\vec{\mathbf{u}}$ in standard form (i.e., the initial point is at the origin), this vector forms three different angles with the positive x-, y-, and z-axes. Let's call these angles α, β, and γ. Then the directional cosines are given by $\cos\alpha, \cos\beta,$ and $\cos\gamma$. These are the components of the unit vector $\vec{\mathbf{u}}$; since $\vec{\mathbf{u}}$ is a unit vector, it is true that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

Definition: Directional Derivative of a Function of Three variables

Suppose $w = f(x, y, z)$ is a function of three variables with a domain of D. Let $(x_0, y_0, z_0) \in D$ and let $\vec{\mathbf{u}} = \cos\alpha\,\hat{\mathbf{i}} + \cos\beta\,\hat{\mathbf{j}} + \cos\gamma\,\hat{\mathbf{k}}$ be a unit vector. Then, the directional derivative of f in the direction of u is given by

$$D_{\vec{\mathbf{u}}} f(x_0, y_0, z_0) = \lim_{t\to 0} \frac{f(x_0 + t\cos\alpha, y_0 + t\cos\beta, z_0 + t\cos\gamma) - f(x_0, y_0, z_0)}{t} \tag{5.7.19}$$

provided the limit exists.

We can calculate the directional derivative of a function of three variables by using the gradient, leading to a formula that is analogous to Equation 5.7.4.

Directional Derivative of a Function of Three Variables

Let $f(x, y, z)$ be a differentiable function of three variables and let $\vec{\mathbf{u}} = \cos\alpha\,\hat{\mathbf{i}} + \cos\beta\,\hat{\mathbf{j}} + \cos\gamma\,\hat{\mathbf{k}}$ be a unit vector. Then, the directional derivative of f in the direction of $\vec{\mathbf{u}}$ is given by

$$D_{\vec{\mathbf{u}}} f(x, y, z) = \vec{\nabla} f(x, y, z) \cdot \vec{\mathbf{u}} = f_x(x, y, z)\cos\alpha + f_y(x, y, z)\cos\beta + f_z(x, y, z)\cos\gamma. \tag{5.7.20}$$

The three angles α, β, and γ determine the unit vector $\vec{\mathbf{u}}$. In practice, we can use an arbitrary (nonunit) vector, then divide by its magnitude to obtain a unit vector in the desired direction.

Example 5.7.7: Finding a Directional Derivative in Three Dimensions

Calculate $D_{\vec{\mathbf{v}}} f(1, -2, 3)$ in the direction of $\vec{\mathbf{v}} = -\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ for the function

$$f(x, y, z) = 5x^2 - 2xy + y^2 - 4yz + z^2 + 3xz.$$

Solution:

First, we find the magnitude of v:

$$\|\vec{\mathbf{v}}\| = \sqrt{(-1)^2 + (2)^2 + (2)^2} = \sqrt{9} = 3.$$

Therefore, $\dfrac{\vec{\mathbf{v}}}{\|\vec{\mathbf{v}}\|} = \dfrac{-\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}}{3} = -\dfrac{1}{3}\hat{\mathbf{i}} + \dfrac{2}{3}\hat{\mathbf{j}} + \dfrac{2}{3}\hat{\mathbf{k}}$ is a unit vector in the direction of $\vec{\mathbf{v}}$, so $\cos\alpha = -\dfrac{1}{3}, \cos\beta = \dfrac{2}{3}$, and $\cos\gamma = \dfrac{2}{3}$. Next, we calculate the partial derivatives of f:

$$\begin{aligned} f_x(x, y, z) &= 10x - 2y + 3z \\ f_y(x, y, z) &= -2x + 2y - 4z \\ f_z(x, y, z) &= -4y + 2z + 3x, \end{aligned}$$

then substitute them into Equation 5.7.20:

$$\begin{aligned} D_{\vec{\mathbf{v}}} f(x, y, z) &= f_x(x, y, z)\cos\alpha + f_y(x, y, z)\cos\beta + f_z(x, y, z)\cos\gamma \\ &= (10x - 2y + 3z)(-\tfrac{1}{3}) + (-2x + 2y - 4z)(\tfrac{2}{3}) + (-4y + 2z + 3x)(\tfrac{2}{3}) \\ &= -\frac{10x}{3} + \frac{2y}{3} - \frac{3z}{3} - \frac{4x}{3} + \frac{4y}{3} - \frac{8z}{3} - \frac{8y}{3} + \frac{4z}{3} + \frac{6x}{3} \\ &= -\frac{8x}{3} - \frac{2y}{3} - \frac{7z}{3}. \end{aligned}$$

Last, to find $D_{\vec{\mathbf{v}}} f(1, -2, 3)$, we substitute $x = 1$, $y = -2$, and $z = 3$:

$$\begin{aligned} D_{\vec{\mathbf{v}}} f(1, -2, 3) &= -\frac{8(1)}{3} - \frac{2(-2)}{3} - \frac{7(3)}{3} \\ &= -\frac{8}{3} + \frac{4}{3} - \frac{21}{3} \\ &= -\frac{25}{3}. \end{aligned}$$

Exercise 5.7.6:

Calculate $D_{\vec{\mathbf{v}}} f(x,y,z)$ and $D_{\vec{\mathbf{v}}} f(0,-2,5)$ in the direction of $\vec{\mathbf{v}} = -3\,\hat{\mathbf{i}} + 12\,\hat{\mathbf{j}} - 4\,\hat{\mathbf{k}}$ for the function

$$f(x,y,z) = 3x^2 + xy - 2y^2 + 4yz - z^2 + 2xz.$$

Hint

First, divide $\vec{\mathbf{v}}$ by its magnitude, calculate the partial derivatives of f, then use Equation 5.7.20

Answer

$$D_{\vec{\mathbf{v}}} f(x,y,z) = -\frac{3}{13}(6x+y+2z) + \frac{12}{13}(x-4y+4z) - \frac{4}{13}(2x+4y-2z)$$

$$D_{\vec{\mathbf{v}}} f(0,-2,5) = \frac{384}{13}$$



Summary

- A directional derivative represents a rate of change of a function in any given direction.
- The gradient can be used in a formula to calculate the directional derivative.
- The gradient indicates the direction of greatest change of a function of more than one variable.

Key Equations

- **directional derivative (two dimensions)**

$$D_{\vec{\mathbf{u}}} f(a,b) = \lim_{h \to 0} \frac{f(a + h\cos\theta, b + h\sin\theta) - f(a,b)}{h}$$

or

$$D_{\vec{\mathbf{u}}} f(x,y) = f_x(x,y)\cos\theta + f_y(x,y)\sin\theta$$

- **gradient (two dimensions)**

$$\vec{\nabla} f(x,y) = f_x(x,y)\,\hat{\mathbf{i}} + f_y(x,y)\,\hat{\mathbf{j}}$$

- **gradient (three dimensions)**

$$\vec{\nabla} f(x,y,z) = f_x(x,y,z)\,\hat{\mathbf{i}} + f_y(x,y,z)\,\hat{\mathbf{j}} + f_z(x,y,z)\,\hat{\mathbf{k}}$$

- **directional derivative (three dimensions)**

$$D_{\vec{\mathbf{u}}} f(x,y,z) = \vec{\nabla} f(x,y,z) \cdot \vec{\mathbf{u}} = f_x(x,y,z)\cos\alpha + f_y(x,y,z)\cos\beta + f_x(x,y,z)\cos\gamma$$

Glossary

directional derivative

the derivative of a function in the direction of a given unit vector

gradient

the gradient of the function $f(x,y)$ is defined to be $\vec{\nabla} f(x,y) = (\partial f/\partial x)\,\hat{\mathbf{i}} + (\partial f/\partial y)\,\hat{\mathbf{j}}$, which can be generalized to a function of any number of independent variables

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5.7E: Exercises for Section 5.7

In exercise 1, find the directional derivative using the limit definition only.

1) a. $f(x,y) = 5 - 2x^2 - \frac{1}{2}y^2$ at point $P(3,4)$ in the direction of $\vec{\mathbf{u}} = (\cos\frac{\pi}{4})\,\hat{\mathbf{i}} + (\sin\frac{\pi}{4})\,\hat{\mathbf{j}}$

Answer

a. $D_{\vec{\mathbf{u}}} f(3,4) = -8\sqrt{2}$

1) b. $f(x,y) = y^2\cos(2x)$ at point $P(\frac{\pi}{3},2)$ in the direction of $\vec{\mathbf{u}} = (\cos\frac{\pi}{4})\,\hat{\mathbf{i}} + (\sin\frac{\pi}{4})\,\hat{\mathbf{j}}$

Answer

b. $D_{\vec{\mathbf{u}}} f(\frac{\pi}{3},2) = -2\sqrt{6} - \sqrt{2}$

2) Find the directional derivative of $f(x,y) = y^2\sin(2x)$ at point $P(\frac{\pi}{4},2)$ in the direction of $\vec{\mathbf{u}} = 5\,\hat{\mathbf{i}} + 12\,\hat{\mathbf{j}}$.

In exercises 3 - 13, find the directional derivative of the function in the direction of $\vec{\mathbf{v}}$ as a function of x and y. Remember that you first need to find a unit vector in the direction of the direction vector. Then find the value of the directional derivative at point P.

3) $f(x,y) = xy, \quad P(-2,0), \quad \vec{\mathbf{v}} = \frac{1}{2}\,\hat{\mathbf{i}} + \frac{\sqrt{3}}{2}\,\hat{\mathbf{j}}$

Answer

$D_{\vec{\mathbf{v}}} f(x,y) = \frac{1}{2}y + \frac{\sqrt{3}}{2}x$
$D_{\vec{\mathbf{v}}} f(-2,0) = -\sqrt{3}$

4) $h(x,y) = e^x\sin y, \quad P(1,\frac{\pi}{2}), \quad \vec{\mathbf{v}} = -\,\hat{\mathbf{i}}$

5) $f(x,y) = x^2y, \quad P(-5,5), \quad \vec{\mathbf{v}} = 3\,\hat{\mathbf{i}} - 4\,\hat{\mathbf{j}}$

Answer

$D_{\vec{\mathbf{v}}} f(x,y) = \frac{6}{5}xy - \frac{4}{5}x^2$
$D_{\vec{\mathbf{v}}} f(-5,5) = -50$

6) $f(x,y) = xy, \quad P(1,1), \quad \vec{\mathbf{u}} = \langle\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\rangle$

7) $f(x,y) = x^2 - y^2, \quad P(1,0), \quad \vec{\mathbf{u}} = \langle\frac{\sqrt{3}}{2}, \frac{1}{2}\rangle$

Answer

$D_{\vec{\mathbf{u}}} f(x,y) = x\sqrt{3} - y$
$D_{\vec{\mathbf{u}}} f(1,0) = \sqrt{3}$

8) $f(x,y) = 3x + 4y + 7, \quad P(0,\frac{\pi}{2}), \quad \vec{\mathbf{u}} = \langle\frac{3}{5}, \frac{4}{5}\rangle$

9) $f(x,y) = e^x\cos y, \quad P = (0,\frac{\pi}{2}), \quad \vec{\mathbf{u}} = \langle 0,5\rangle$

Answer

$D_{\vec{\mathbf{u}}} f(x,y) = -e^x\sin y$
$D_{\vec{\mathbf{u}}} f(0,\frac{\pi}{2}) = -1$

10) $f(x,y) = y^{10}, \quad \vec{\mathbf{u}} = \langle 0,-3\rangle, \quad P = (1,-1)$

11) $f(x,y) = \ln(x^2 + y^2), \quad \vec{\mathbf{u}} = \langle 2,-5\rangle, \quad P(1,2)$

Answer

$$D_{\vec{\mathbf{u}}} f(x,y) = \frac{\sqrt{29}}{29}\left(\frac{4x-10y}{x^2+y^2}\right)$$
$$D_{\vec{\mathbf{u}}} f(1,2) = -\frac{16\sqrt{29}}{145}$$

12) $h(x,y,z) = xyz, \quad P(2,1,1), \quad \vec{\mathbf{v}} = 2\,\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$

Answer

$$D_{\vec{\mathbf{v}}} h(x,y,z) = \frac{\sqrt{6}}{6}(2yz + xz - xy)$$
$$D_{\vec{\mathbf{v}}} h(2,1,1) = \frac{2}{\sqrt{6}} = \frac{\sqrt{6}}{3}$$

13) $f(x,y,z) = y^2 + xz, \quad P(1,2,2), \quad \vec{\mathbf{v}} = \langle 2,-1,2\rangle$

Answer

$$D_{\vec{\mathbf{v}}} f(x,y,z) = \frac{2}{3}(z - y + x)$$
$$D_{\vec{\mathbf{v}}} f(1,2,2) = \frac{2}{3}$$

In exercises 14 - 19, find the directional derivative of the function in the direction of the unit vector $\vec{\mathbf{u}} = \cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}$.

14) $f(x,y) = x^2 + 2y^2, \quad \theta = \frac{\pi}{6}$

15) $f(x,y) = \dfrac{y}{x+2y}, \quad \theta = -\frac{\pi}{4}$

Answer

$$D_{\vec{\mathbf{u}}} f(x,y) = \frac{-\sqrt{2}(x+y)}{2(x+2y)^2}$$

16) $f(x,y) = \cos(3x+y), \quad \theta = \frac{\pi}{4}$

17) $w(x,y) = ye^x, \quad \theta = \frac{\pi}{3}$

Answer

$$D_{\vec{\mathbf{u}}} f(x,y) = \frac{e^x(y+\sqrt{3})}{2}$$

18) $f(x,y) = x\arctan(y), \quad \theta = \frac{\pi}{2}$

19) $f(x,y) = \ln(x+2y), \quad \theta = \frac{\pi}{3}$

Answer

$$D_{\vec{\mathbf{u}}} f(x,y) = \frac{1+2\sqrt{3}}{2(x+2y)}$$

In exercises 20 - 23, find the gradient $\vec{\nabla} f$.

20) Find the gradient of $f(x,y) = 3x^2 + y^3 - 3x + y$. Then find it's value at the point $P(2,3)$.

21) Find the gradient of $f(x,y) = \dfrac{14 - x^2 - y^2}{3}$. Then, find the gradient at point $P(1,2)$.

Answer

$$\vec{\nabla} f(x,y) = -\frac{2}{3}x\,\hat{\mathbf{i}} - \frac{2}{3}y\,\hat{\mathbf{j}}$$
$$\vec{\nabla} f(1,2) = -\frac{2}{3}\,\hat{\mathbf{i}} - \frac{4}{3}\,\hat{\mathbf{j}}$$

22) Find the gradient of $f(x,y) = \ln(4x^3 - 3y)$. Then, find the gradient at point $P(1,1)$.

23) Find the gradient of $f(x,y,z) = xy + yz + xz$. Then find the gradient at point $P(1,2,3)$.

Answer

$\vec{\nabla} f(x,y,z) = \langle y+z, x+z, y+x \rangle$

$\vec{\nabla} f(1,2,3) = \langle 5,4,3 \rangle$

In exercises 24 - 25, find the directional derivative of the function at point P in the direction of Q.

24) $f(x,y) = x^2 + 3y^2, \quad P(1,1), \quad Q(4,5)$

25) $f(x,y,z) = \dfrac{y}{x+z}, \quad P(2,1,-1), \quad Q(-1,2,0)$

Answer

$D_{\overrightarrow{PQ}} f(x,y) = \frac{3}{\sqrt{11}}$

26) Find the directional derivative of $f(x,y,z))$ at P and in the direction of $\vec{\mathbf{u}}$: $f(x,y,z) = \ln(x^2+2y^2+3z^2), \quad P(2,1,4), \quad \vec{\mathbf{u}} = \frac{-3}{13}\,\hat{\mathbf{i}} - \frac{4}{13}\,\hat{\mathbf{j}} - \frac{12}{13}\,\hat{\mathbf{k}}$.

In exercises 27 - 29, find the directional derivative of the function at P in the direction of $\vec{\mathbf{u}}$.

27) $f(x,y) = \ln(5x+4y), \quad P(3,9), \quad \vec{\mathbf{u}} = 6\,\hat{\mathbf{i}} + 8\,\hat{\mathbf{j}}$

Answer

$D_{\vec{\mathbf{u}}} f(3,9) = \frac{31}{255}$

28) $f(x,y) = -7x + 2y, \quad P(2,-4), \quad \vec{\mathbf{u}} = 4\,\hat{\mathbf{i}} - 3\,\hat{\mathbf{j}}$

29) $f(x,y,z) = 4x^5y^2z^3, \quad P(2,-1,1), \quad \vec{\mathbf{u}} = \frac{1}{3}\,\hat{\mathbf{i}} + \frac{2}{3}\,\hat{\mathbf{j}} - \frac{2}{3}\,\hat{\mathbf{k}}$

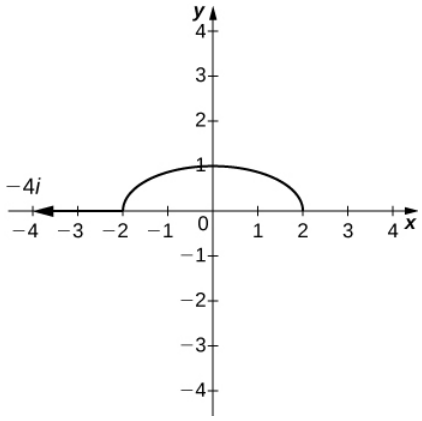
Answer

$D_{\vec{\mathbf{u}}} f(2,-1,1) = -320$

30) [T] Use technology to sketch the level curve of $f(x,y) = 4x - 2y + 3$ that passes through $P(1,2)$ and draw the gradient vector at P.

31) [T] Use technology to sketch the level curve of $f(x,y) = x^2 + 4y^2$ that passes through $P(-2,0)$ and draw the gradient vector at P.

Answer



In exercises 32 - 35, find the gradient vector at the indicated point.

32) $f(x,y) = xy^2 - yx^2, \quad P(-1,1)$

33) $f(x,y) = xe^y - \ln(x), \quad P(-3,0)$

Answer

$\vec{\nabla} f(-3,0) = \frac{4}{3}\,\hat{\mathbf{i}} - 3\,\hat{\mathbf{j}}$

34) $f(x,y,z) = xy - \ln(z), \quad P(2,-2,2)$

35) $f(x,y,z) = x\sqrt{y^2+z^2}, \quad P(-2,-1,-1)$

Answer

$\vec{\nabla} f(-2,-1,-1) = \sqrt{2}\,\hat{\mathbf{i}} + \sqrt{2}\,\hat{\mathbf{j}} + \sqrt{2}\,\hat{\mathbf{k}}$

In exercises 36 - 40, find the indicated directional derivative of the function.

36) $f(x,y) = x^2 + xy + y^2$ at point $(-5,-4)$ in the direction the function increases most rapidly.

37) $f(x,y) = e^{xy}$ at point $(6,7)$ in the direction the function increases most rapidly.

Answer

$1.6(10^{19})$

38) $f(x,y) = \arctan\left(\frac{y}{x}\right)$ at point $(-9,9)$ in the direction the function increases most rapidly.

39) $f(x,y,z) = \ln(xy + yz + zx)$ at point $(-9,-18,-27)$ in the direction the function increases most rapidly.

Answer

$\frac{5\sqrt{2}}{99}$

40) $f(x,y,z) = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$ at point $(5,-5,5)$ in the direction the function increases most rapidly.

In exercises 41 - 43, find the maximum rate of change of f at the given point and the direction in which it occurs.

41) $f(x,y) = xe^{-y}, \quad (-2,0)$

Answer

$\max\{D_{\vec{\mathbf{u}}} f(-2,0)\} = \sqrt{5}, \quad \langle 1,2\rangle$

Solution:

$\vec{\nabla} f(x,y) = e^{-y}\,\hat{\mathbf{i}} - xe^{-y}\,\hat{\mathbf{j}}$. So $\max\{D_{\vec{\mathbf{u}}} f(-2,0)\} = \|\vec{\nabla} f(-2,0)\| = \sqrt{5}$.

The direction in which it occurs will just be the direction of the gradient vector: $\vec{\nabla} f(-2,0) = \langle 1,2\rangle$.

42) $f(x,y) = \sqrt{x^2+2y}, \quad (4,10)$

43) $f(x,y) = \cos(3x+2y), \quad (\frac{\pi}{6}, -\frac{\pi}{8})$

Answer

$\max\{D_{\vec{\mathbf{u}}} f\left(\frac{\pi}{6}, -\frac{\pi}{8}\right)\} = \sqrt{\frac{13}{2}} = \frac{\sqrt{26}}{2}, \quad \langle -3,-2\rangle$

In exercises 44 - 47, find equations of

a. the tangent plane and

b. the normal line to the given surface at the given point.

44) The level curve $f(x,y,z) = 12$ for $f(x,y,z) = 4x^2 - 2y^2 + z^2$ at point $(2,2,2)$.

45) $f(x,y,z) = xy + yz + xz = 3$ at point $(1,1,1)$

Answer

a. tangent plane equation: $x + y + z = 3$,

b. normal line equations: $x - 1 = y - 1 = z - 1$

46) $f(x,y,z)=xyz=6$ at point $(1,2,3)$

47) $f(x,y,z)=xe^y\cos z-z=1$ at point $(1,0,0)$

Answer

a. tangent plane equation: $x+y-z=1$,
b. normal line equations: $x-1=y=-z$

In exercises 48 - 51, solve the stated problem.

48) The temperature T in a metal sphere is inversely proportional to the distance from the center of the sphere (the origin: $(0,0,0)$). The temperature at point $(1,2,2)$ is 120°C.

a. Find the rate of change of the temperature at point $(1,2,2)$ in the direction toward point $(2,1,3)$.

b. Show that, at any point in the sphere, the direction of greatest increase in temperature is given by a vector that points toward the origin.

49) The **electrical potential** (voltage) in a certain region of space is given by the function $V(x,y,z)=5x^2-3xy+xyz$.

a. Find the rate of change of the voltage at point $(3,4,5)$ in the direction of the vector $\langle 1,1,-1\rangle$.

b. In which direction does the voltage change most rapidly at point $(3,4,5)$?

c. What is the maximum rate of change of the voltage at point $(3,4,5)$?

Answer

a. $\frac{32}{\sqrt{3}}$,
b. $\langle 38,6,12\rangle$,
c. $2\sqrt{406}$

50) If the electric potential at a point (x,y) in the xy-plane is $V(x,y)=e^{-2x}\cos(2y)$, then the electric intensity vector at (x,y) is $E=-\vec{\nabla}V(x,y)$.

a. Find the electric intensity vector at $(\frac{\pi}{4},0)$.

b. Show that, at each point in the plane, the electric potential decreases most rapidly in the direction of the vector E.

51) In two dimensions, the motion of an ideal fluid is governed by a velocity potential φ. The velocity components of the fluid u in the x-direction and v in the y-direction, are given by $\langle u,v\rangle=\vec{\nabla}\varphi$. Find the velocity components associated with the velocity potential $\varphi(x,y)=\sin\pi x\sin 2\pi y$.

Answer

$\langle u,v\rangle=\langle \pi\cos(\pi x)\sin(2\pi y),2\pi\sin(\pi x)\cos(2\pi y)\rangle$

Contributors

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- Paul Seeburger (Monroe Community College) created problems 20 and 22 and added a solution to problem 41.

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5.8: Maxima/Minima Problems

Learning Objectives

- Use partial derivatives to locate critical points for a function of two variables.
- Apply a second derivative test to identify a critical point as a local maximum, local minimum, or saddle point for a function of two variables.
- Examine critical points and boundary points to find absolute maximum and minimum values for a function of two variables.

One of the most useful applications for derivatives of a function of one variable is the determination of maximum and/or minimum values. This application is also important for functions of two or more variables, but as we have seen in earlier sections of this chapter, the introduction of more independent variables leads to more possible outcomes for the calculations. The main ideas of finding critical points and using derivative tests are still valid, but new wrinkles appear when assessing the results.

Critical Points

For functions of a single variable, we defined critical points as the values of the variable at which the function's derivative equals zero or does not exist. For functions of two or more variables, the concept is essentially the same, except for the fact that we are now working with partial derivatives.

Definition: Critical Points

Let $z = f(x, y)$ be a function of two variables that is differentiable on an open set containing the point (x_0, y_0). The point (x_0, y_0) is called a *critical point* of a function of two variables f if one of the two following conditions holds:

1. $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$
2. Either $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ does not exist.

Example 5.8.1: Finding Critical Points

Find the critical points of each of the following functions:

a. $f(x, y) = \sqrt{4y^2 - 9x^2 + 24y + 36x + 36}$
b. $g(x, y) = x^2 + 2xy - 4y^2 + 4x - 6y + 4$

Solution

a. First, we calculate $f_x(x, y)$ and $f_y(x, y)$:

$$\begin{aligned} f_x(x, y) &= \frac{1}{2}(-18x + 36)(4y^2 - 9x^2 + 24y + 36x + 36)^{-1/2} \\ &= \frac{-9x + 18}{\sqrt{4y^2 - 9x^2 + 24y + 36x + 36}} \\ f_y(x, y) &= \frac{1}{2}(8y + 24)(4y^2 - 9x^2 + 24y + 36x + 36)^{-1/2} \\ &= \frac{4y + 12}{\sqrt{4y^2 - 9x^2 + 24y + 36x + 36}} \end{aligned}.$$

Next, we set each of these expressions equal to zero:

$$\begin{aligned} \frac{-9x + 18}{\sqrt{4y^2 - 9x^2 + 24y + 36x + 36}} &= 0 \\ \frac{4y + 12}{\sqrt{4y^2 - 9x^2 + 24y + 36x + 36}} &= 0. \end{aligned}$$

Then, multiply each equation by its common denominator:

$$-9x+18 = 0$$
$$4y+12 = 0.$$

Therefore, $x=2$ and $y=-3$, so $(2,-3)$ is a critical point of f.

We must also check for the possibility that the denominator of each partial derivative can equal zero, thus causing the partial derivative not to exist. Since the denominator is the same in each partial derivative, we need only do this once:

$$4y^2-9x^2+24y+36x+36=0. \tag{5.8.1}$$

Equation 5.8.1 represents a hyperbola. We should also note that the domain of f consists of points satisfying the inequality

$$4y^2-9x^2+24y+36x+36\geq 0. \tag{5.8.2}$$

Therefore, any points on the hyperbola are not only critical points, they are also on the boundary of the domain. To put the hyperbola in standard form, we use the method of completing the square:

$$\begin{aligned} 4y^2-9x^2+24y+36x+36 &= 0 \\ 4y^2-9x^2+24y+36x &= -36 \\ 4y^2+24y-9x^2+36x &= -36 \\ 4(y^2+6y)-9(x^2-4x) &= -36 \\ 4(y^2+6y+9)-9(x^2-4x+4) &= -36-36+36 \\ 4(y+3)^2-9(x-2)^2 &= -36. \end{aligned}$$

Dividing both sides by -36 puts the equation in standard form:

$$\begin{aligned} \frac{4(y+3)^2}{-36}-\frac{9(x-2)^2}{-36} &= 1 \\ \frac{(x-2)^2}{4}-\frac{(y+3)^2}{9} &= 1. \end{aligned}$$

Notice that point $(2,-3)$ is the center of the hyperbola.

Thus, the critical points of the function f are $(2,-3)$ and all points on the hyperbola, $\frac{(x-2)^2}{4}-\frac{(y+3)^2}{9}=1$.

b. First, we calculate $g_x(x,y)$ and $g_y(x,y)$:

$$\begin{aligned} g_x(x,y) &= 2x+2y+4 \\ g_y(x,y) &= 2x-8y-6. \end{aligned}$$

Next, we set each of these expressions equal to zero, which gives a system of equations in x and y:

$$\begin{aligned} 2x+2y+4 &= 0 \\ 2x-8y-6 &= 0. \end{aligned}$$

Subtracting the second equation from the first gives $10y+10=0$, so $y=-1$. Substituting this into the first equation gives $2x+2(-1)+4=0$, so $x=-1$.

Therefore $(-1,-1)$ is a critical point of g. There are no points in $\mathbb{R}^2$ that make either partial derivative not exist.

Figure 5.8.1 shows the behavior of the surface at the critical point.

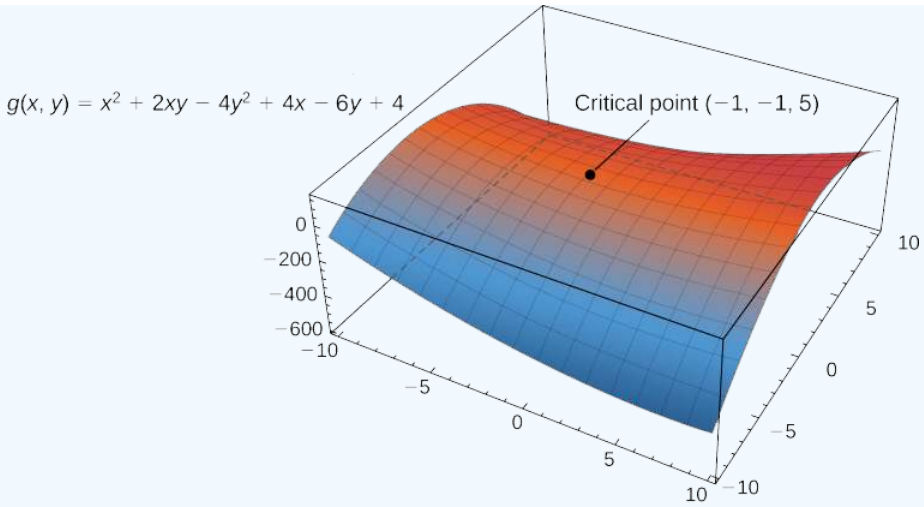


Figure 5.8.1: The function $g(x,y)$ has a critical point at $(-1,-1,5)$.

Exercise 5.8.1

Find the critical point of the function $f(x,y)=x^3+2xy-2x-4y$.

Hint

Calculate $f_x(x,y)$ and $f_y(x,y)$, then set them equal to zero.

Answer

The only critical point of f is $(2,-5)$.



The main purpose for determining critical points is to locate relative maxima and minima, as in single-variable calculus. When working with a function of one variable, the definition of a local extremum involves finding an interval around the critical point such that the function value is either greater than or less than all the other function values in that interval. When working with a function of two or more variables, we work with an open disk around the point.

Definition: Global and Local Extrema

Let $z = f(x, y)$ be a function of two variables that is defined and continuous on an open set containing the point (x_0, y_0). Then f has a **local maximum** at (x_0, y_0) if

$$f(x_0, y_0) \geq f(x, y) \tag{5.8.3}$$

for all points (x, y) within some disk centered at (x_0, y_0). The number $f(x_0, y_0)$ is called a local maximum value. If the preceding inequality holds for every point (x, y) in the domain of f, then f has a **global maximum** (also called an absolute maximum) at (x_0, y_0).

The function f has a local minimum at (x_0, y_0) if

$$f(x_0, y_0) \leq f(x, y) \tag{5.8.4}$$

for all points (x, y) within some disk centered at (x_0, y_0). The number $f(x_0, y_0)$ is called a local minimum value. If the preceding inequality holds for every point (x, y) in the domain of f, then f has a **global minimum** (also called an absolute minimum) at (x_0, y_0).

If $f(x_0, y_0)$ is either a local maximum or local minimum value, then it is called a **local extremum** (see the following figure).

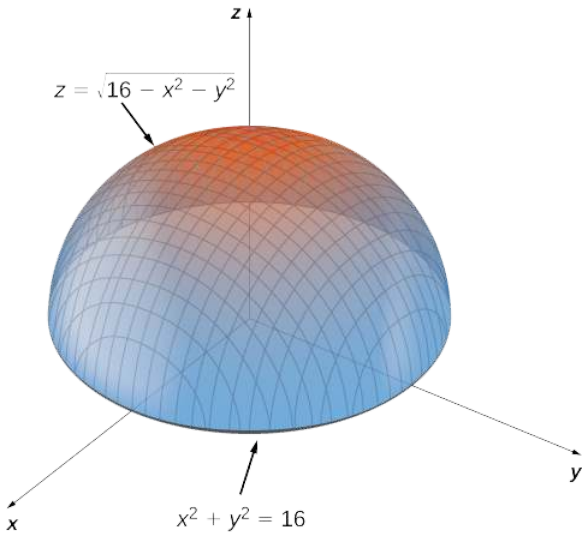


Figure 5.8.2: The graph of $z = \sqrt{16 - x^2 - y^2}$ has a maximum value when $(x, y) = (0, 0)$. It attains its minimum value at the boundary of its domain, which is the circle $x^2 + y^2 = 16$.

In Calculus 1, we showed that extrema of functions of one variable occur at critical points. The same is true for functions of more than one variable, as stated in the following theorem.

Fermat's Theorem for Functions of Two Variables

Let $z = f(x, y)$ be a function of two variables that is defined and continuous on an open set containing the point (x_0, y_0). Suppose f_x and f_y each exist at (x_0, y_0). If f has a local extremum at (x_0, y_0), then (x_0, y_0) is a critical point of f.



Second Derivative Test

Consider the function $f(x) = x^3$. This function has a critical point at $x = 0$, since $f'(0) = 3(0)^2 = 0$. However, f does not have an extreme value at $x = 0$. Therefore, the existence of a critical value at $x = x_0$ does not guarantee a local extremum at $x = x_0$. The same is true for a function of two or more variables. One way this can happen is at a **saddle point**. An example of a saddle point appears in the following figure.

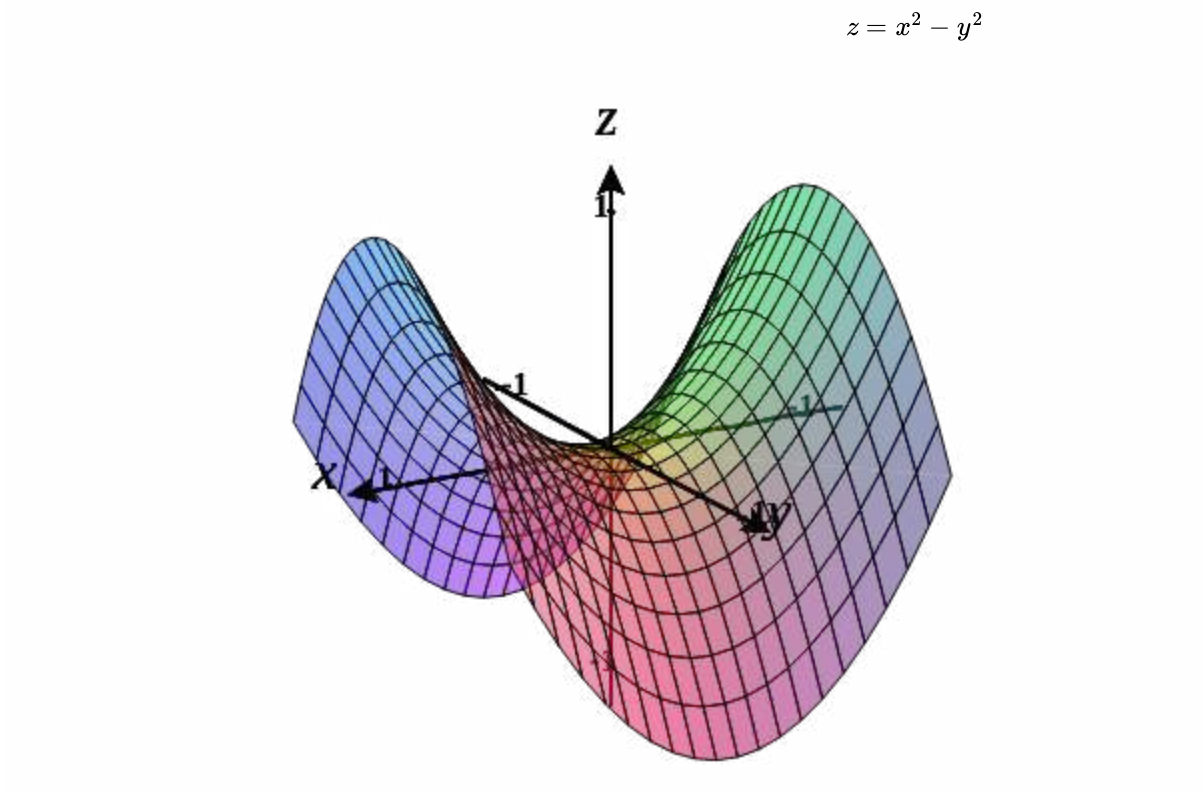


Figure 5.8.3: Graph of the function $z = x^2 - y^2$. This graph has a saddle point at the origin.

In this graph, the origin is a saddle point. This is because the first partial derivatives of $f(x, y) = x^2 - y^2$ are both equal to zero at this point, but it is neither a maximum nor a minimum for the function. Furthermore the vertical trace corresponding to $y = 0$ is

$z = x^2$ (a parabola opening upward), but the vertical trace corresponding to $x = 0$ is $z = -y^2$ (a parabola opening downward). Therefore, it is both a global maximum for one trace and a global minimum for another.

Definition: Saddle Point

Given the function $z = f(x, y)$, the point $\big(x_0, y_0, f(x_0, y_0)\big)$ is a saddle point if both $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$, but f does not have a local extremum at (x_0, y_0).

The second derivative test for a function of one variable provides a method for determining whether an extremum occurs at a critical point of a function. When extending this result to a function of two variables, an issue arises related to the fact that there are, in fact, four different second-order partial derivatives, although equality of mixed partials reduces this to three. The second derivative test for a function of two variables, stated in the following theorem, uses a **discriminant** D that replaces $f''(x_0)$ in the second derivative test for a function of one variable.

Second Derivative Test

Let $z = f(x, y)$ be a function of two variables for which the first- and second-order partial derivatives are continuous on some disk containing the point (x_0, y_0). Suppose $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$. Define the quantity

$$D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - \big(f_{xy}(x_0, y_0)\big)^2. \tag{5.8.5}$$

Then:

i. If $D > 0$ and $f_{xx}(x_0, y_0) > 0$, then f has a local minimum at (x_0, y_0).
ii. If $D > 0$ and $f_{xx}(x_0, y_0) < 0$, then f has a local maximum at (x_0, y_0).
iii. If $D < 0$, then f has a saddle point at (x_0, y_0).
iv. If $D = 0$, then the test is inconclusive.

See Figure 5.8.4.

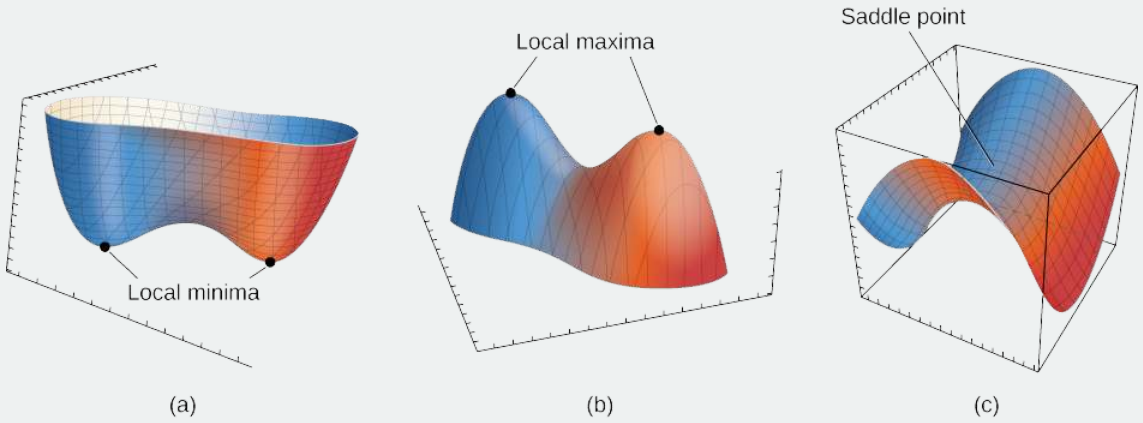


Figure 5.8.4: The second derivative test can often determine whether a function of two variables has local minima (a), local maxima (b), or a saddle point (c).

To apply the second derivative test, it is necessary that we first find the critical points of the function. There are several steps involved in the entire procedure, which are outlined in a problem-solving strategy.

Problem-Solving Strategy: Using the Second Derivative Test for Functions of Two Variables

Let $z = f(x, y)$ be a function of two variables for which the first- and second-order partial derivatives are continuous on some disk containing the point (x_0, y_0). To apply the second derivative test to find local extrema, use the following steps:

1. Determine the critical points (x_0, y_0) of the function f where $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$. Discard any points where at least one of the partial derivatives does not exist.
2. Calculate the discriminant $D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - \big(f_{xy}(x_0, y_0)\big)^2$ for each critical point of f.
3. Apply the four cases of the test to determine whether each critical point is a local maximum, local minimum, or saddle point, or whether the theorem is inconclusive.

Example 5.8.2: Using the Second Derivative Test

Find the critical points for each of the following functions, and use the second derivative test to find the local extrema:

a. $f(x,y)=4x^2+9y^2+8x-36y+24$

b. $g(x,y)=\frac{1}{3}x^3+y^2+2xy-6x-3y+4$

Solution

a. **Step 1** of the problem-solving strategy involves finding the critical points of f. To do this, we first calculate $f_x(x,y)$ and $f_y(x,y)$, then set each of them equal to zero:

$$\begin{aligned} f_x(x,y) &= 8x+8 \\ f_y(x,y) &= 18y-36. \end{aligned}$$

Setting them equal to zero yields the system of equations

$$\begin{aligned} 8x+8 &= 0 \\ 18y-36 &= 0. \end{aligned}$$

The solution to this system is $x=-1$ and $y=2$. Therefore $(-1,2)$ is a critical point of f.

Step 2 of the problem-solving strategy involves calculating D. To do this, we first calculate the second partial derivatives of f :

$$\begin{aligned} f_{xx}(x,y) &= 8 \\ f_{xy}(x,y) &= 0 \\ f_{yy}(x,y) &= 18. \end{aligned}$$

Therefore, $D=f_{xx}(-1,2)f_{yy}(-1,2)-\big(f_{xy}(-1,2)\big)^2=(8)(18)-(0)^2=144$.

Step 3 states to apply the four cases of the test to classify the function's behavior at this critical point.

Since $D>0$ and $f_{xx}(-1,2)>0$, this corresponds to case 1. Therefore, f has a local minimum at $(-1,2)$ as shown in the following figure.

$z = 4x^2 + 9y^2 + 8x - 36y + 24$

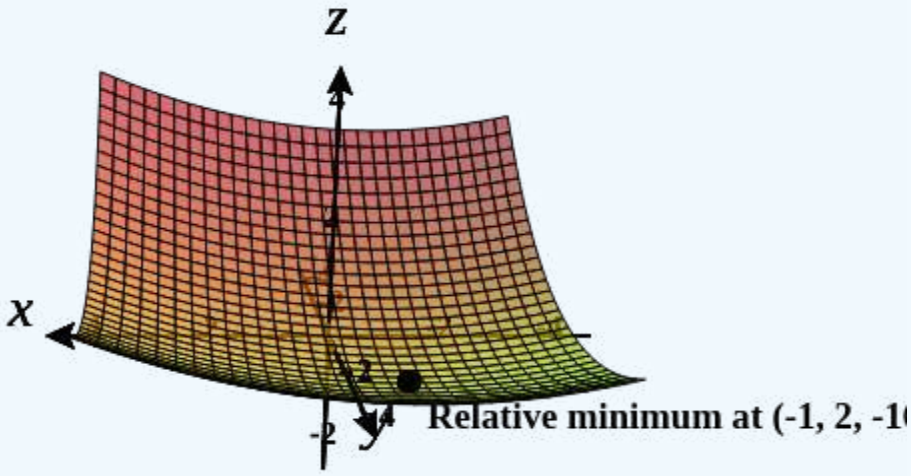


Figure 5.8.5: The function $f(x, y)$ has a local minimum at $(-1, 2, -16)$. Note the scale on the y-axis in this plot is in thousands.

b. For **step 1**, we first calculate $g_x(x, y)$ and $g_y(x, y)$, then set each of them equal to zero:

$$\begin{aligned} g_x(x, y) &= x^2 + 2y - 6 \\ g_y(x, y) &= 2y + 2x - 3. \end{aligned}$$

Setting them equal to zero yields the system of equations

$$\begin{aligned} x^2 + 2y - 6 &= 0 \\ 2y + 2x - 3 &= 0. \end{aligned}$$

To solve this system, first solve the second equation for y. This gives $y = \dfrac{3 - 2x}{2}$. Substituting this into the first equation gives

$$\begin{aligned} x^2 + 3 - 2x - 6 &= 0 \\ x^2 - 2x - 3 &= 0 \\ (x - 3)(x + 1) &= 0. \end{aligned}$$

Therefore, $x = -1$ or $x = 3$. Substituting these values into the equation $y = \dfrac{3 - 2x}{2}$ yields the critical points $\left(-1, \frac{5}{2}\right)$ and $\left(3, -\frac{3}{2}\right)$.

Step 2 involves calculating the second partial derivatives of g:

$$\begin{aligned} g_{xx}(x, y) &= 2x \\ g_{xy}(x, y) &= 2 \\ g_{yy}(x, y) &= 2. \end{aligned}$$

Then, we find a general formula for D:

$$\begin{aligned} D(x_0, y_0) &= g_{xx}(x_0, y_0) g_{yy}(x_0, y_0) - \left(g_{xy}(x_0, y_0)\right)^2 \\ &= (2x_0)(2) - 2^2 \\ &= 4x_0 - 4. \end{aligned}$$

Next, we substitute each critical point into this formula:

$$\begin{aligned} D\left(-1, \tfrac{5}{2}\right) &= (2(-1))(2) - (2)^2 = -4 - 4 = -8 \\ D\left(3, -\tfrac{3}{2}\right) &= (2(3))(2) - (2)^2 = 12 - 4 = 8. \end{aligned}$$

In step 3, we note that, applying Note to point $\left(-1, \frac{5}{2}\right)$ leads to case 3, which means that $\left(-1, \frac{5}{2}\right)$ is a saddle point. Applying the theorem to point $\left(3, -\frac{3}{2}\right)$ leads to case 1, which means that $\left(3, -\frac{3}{2}\right)$ corresponds to a local minimum as shown in the following figure.

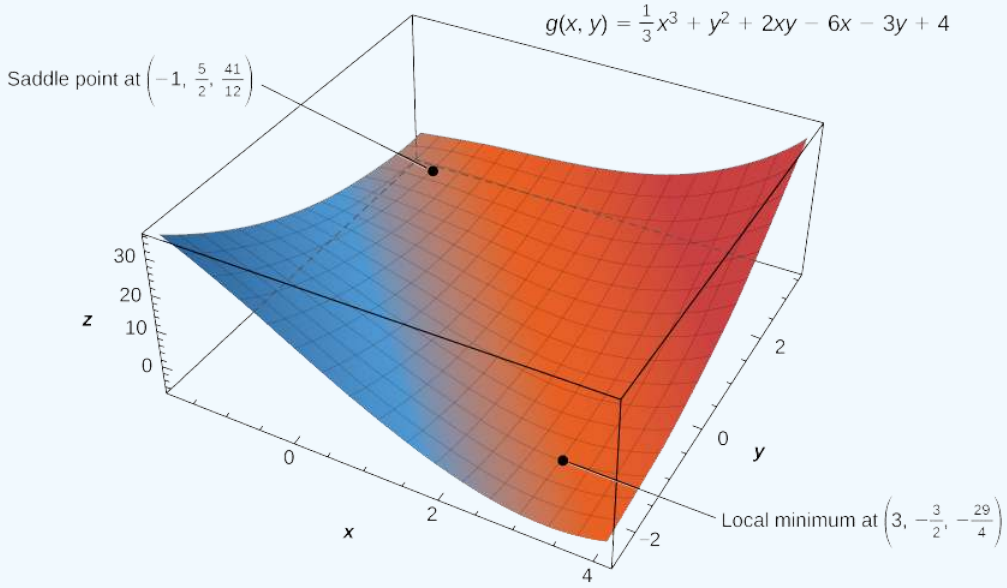


Figure 5.8.6: The function $g(x, y)$ has a local minimum and a saddle point.



https://math.libretexts.org/@go/page/84037

Exercise 5.8.2

Use the second derivative test to find the local extrema of the function

$$f(x,y)=x^3+2xy-6x-4y^2.$$

Hint

Follow the problem-solving strategy for applying the second derivative test.

Answer

$\left(\frac{4}{3},\frac{1}{3}\right)$ is a saddle point, $\left(-\frac{3}{2},-\frac{3}{8}\right)$ is a local maximum.

Absolute Maxima and Minima

When finding global extrema of functions of one variable on a closed interval, we start by checking the critical values over that interval and then evaluate the function at the endpoints of the interval. When working with a function of two variables, the closed interval is replaced by a closed, bounded set. A set is bounded if all the points in that set can be contained within a ball (or disk) of finite radius. First, we need to find the critical points inside the set and calculate the corresponding critical values. Then, it is necessary to find the maximum and minimum value of the function on the boundary of the set. When we have all these values, the largest function value corresponds to the global maximum and the smallest function value corresponds to the absolute minimum. First, however, we need to be assured that such values exist. The following theorem does this.

Extreme Value Theorem

A continuous function $f(x,y)$ on a closed and bounded set D in the plane attains an absolute maximum value at some point of D and an absolute minimum value at some point of D.

Now that we know any continuous function f defined on a closed, bounded set attains its extreme values, we need to know how to find them.

Finding Extreme Values of a Function of Two Variables

Assume $z=f(x,y)$ is a differentiable function of two variables defined on a closed, bounded set D. Then f will attain the absolute maximum value and the absolute minimum value, which are, respectively, the largest and smallest values found among the following:

1. The values of f at the critical points of f in D.
2. The values of f on the boundary of D.

The proof of this theorem is a direct consequence of the extreme value theorem and Fermat's theorem. In particular, if either extremum is not located on the boundary of D, then it is located at an interior point of D. But an interior point (x_0,y_0) of D that's an absolute extremum is also a local extremum; hence, (x_0,y_0) is a critical point of f by Fermat's theorem. Therefore the only possible values for the global extrema of f on D are the extreme values of f on the interior or boundary of D.

Problem-Solving Strategy: Finding Absolute Maximum and Minimum Values

Let $z=f(x,y)$ be a continuous function of two variables defined on a closed, bounded set D, and assume f is differentiable on D. To find the absolute maximum and minimum values of f on D, do the following:

1. Determine the critical points of f in D.
2. Calculate f at each of these critical points.
3. Determine the maximum and minimum values of f on the boundary of its domain.
4. The maximum and minimum values of f will occur at one of the values obtained in steps 2 and 3.

Finding the maximum and minimum values of f on the boundary of D can be challenging. If the boundary is a rectangle or set of straight lines, then it is possible to parameterize the line segments and determine the maxima on each of these segments, as seen in Example 5.8.3. The same approach can be used for other shapes such as circles and ellipses.

If the boundary of the set D is a more complicated curve defined by a function $g(x,y)=c$ for some constant c, and the first-order partial derivatives of g exist, then the method of Lagrange multipliers can prove useful for determining the extrema of f on the boundary which is introduced in Lagrange Multipliers.

Example 5.8.3: Finding Absolute Extrema

Use the problem-solving strategy for finding absolute extrema of a function to determine the absolute extrema of each of the following functions:

a. $f(x,y)=x^2-2xy+4y^2-4x-2y+24$ on the domain defined by $0\le x\le 4$ and $0\le y\le 2$
b. $g(x,y)=x^2+y^2+4x-6y$ on the domain defined by $x^2+y^2\le 16$

Solution

a. Using the problem-solving strategy, step 1 involves finding the critical points of f on its domain. Therefore, we first calculate $f_x(x,y)$ and $f_y(x,y)$, then set them each equal to zero:

$$\begin{aligned} f_x(x,y) &= 2x-2y-4 \\ f_y(x,y) &= -2x+8y-2. \end{aligned}$$

Setting them equal to zero yields the system of equations

$$\begin{aligned} 2x-2y-4 &= 0 \\ -2x+8y-2 &= 0. \end{aligned}$$

The solution to this system is $x=3$ and $y=1$. Therefore $(3,1)$ is a critical point of f. Calculating $f(3,1)$ gives $f(3,1)=17$.

The next step involves finding the extrema of f on the boundary of its domain. The boundary of its domain consists of four line segments as shown in the following graph:

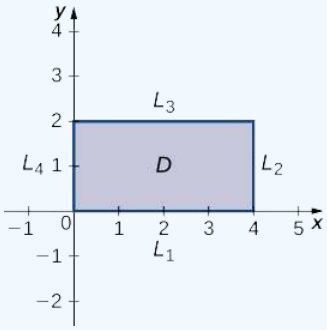


Figure 5.8.7: Graph of the domain of the function $f(x,y)=x^2-2xy+4y^2-4x-2y+24$.

L_1 is the line segment connecting $(0,0)$ and $(4,0)$, and it can be parameterized by the equations $x(t)=t, y(t)=0$ for $0\le t\le 4$. Define $g(t)=f\big(x(t),y(t)\big)$. This gives $g(t)=t^2-4t+24$. Differentiating g leads to $g'(t)=2t-4$. Therefore, g has a critical value at $t=2$, which corresponds to the point $(2,0)$. Calculating $f(2,0)$ gives the z-value 20.

L_2 is the line segment connecting $(4,0)$ and $(4,2)$, and it can be parameterized by the equations $x(t)=4, y(t)=t$ for $0\le t\le 2$. Again, define $g(t)=f\big(x(t),y(t)\big)$. This gives $g(t)=4t^2-10t+24$. Then, $g'(t)=8t-10$. g has a critical value at $t=\frac{5}{4}$, which corresponds to the point $\left(0,\frac{5}{4}\right)$. Calculating $f\left(0,\frac{5}{4}\right)$ gives the z-value 27.75.

L_3 is the line segment connecting $(0,2)$ and $(4,2)$, and it can be parameterized by the equations $x(t)=t, y(t)=2$ for $0\le t\le 4$. Again, define $g(t)=f\big(x(t),y(t)\big)$. This gives $g(t)=t^2-8t+36$. The critical value corresponds to the point $(4,2)$. So, calculating $f(4,2)$ gives the z-value 20.

L_4 is the line segment connecting $(0,0)$ and $(0,2)$, and it can be parameterized by the equations $x(t)=0, y(t)=t$ for $0\le t\le 2$. This time, $g(t)=4t^2-2t+24$ and the critical value $t=\frac{1}{4}$ correspond to the point $\left(0,\frac{1}{4}\right)$. Calculating $f\left(0,\frac{1}{4}\right)$ gives the z-value 23.75.

We also need to find the values of $f(x,y)$ at the corners of its domain. These corners are located at $(0,0),(4,0),(4,2)$ and $(0,2)$:

$$
\begin{aligned}
f(0,0) &= (0)^2 - 2(0)(0) + 4(0)^2 - 4(0) - 2(0) + 24 = 24 \\
f(4,0) &= (4)^2 - 2(4)(0) + 4(0)^2 - 4(4) - 2(0) + 24 = 24 \\
f(4,2) &= (4)^2 - 2(4)(2) + 4(2)^2 - 4(4) - 2(2) + 24 = 20 \\
f(0,2) &= (0)^2 - 2(0)(2) + 4(2)^2 - 4(0) - 2(2) + 24 = 36.
\end{aligned}
$$

The absolute maximum value is 36, which occurs at $(0,2)$, and the global minimum value is 20, which occurs at both $(4,2)$ and $(2,0)$ as shown in the following figure.

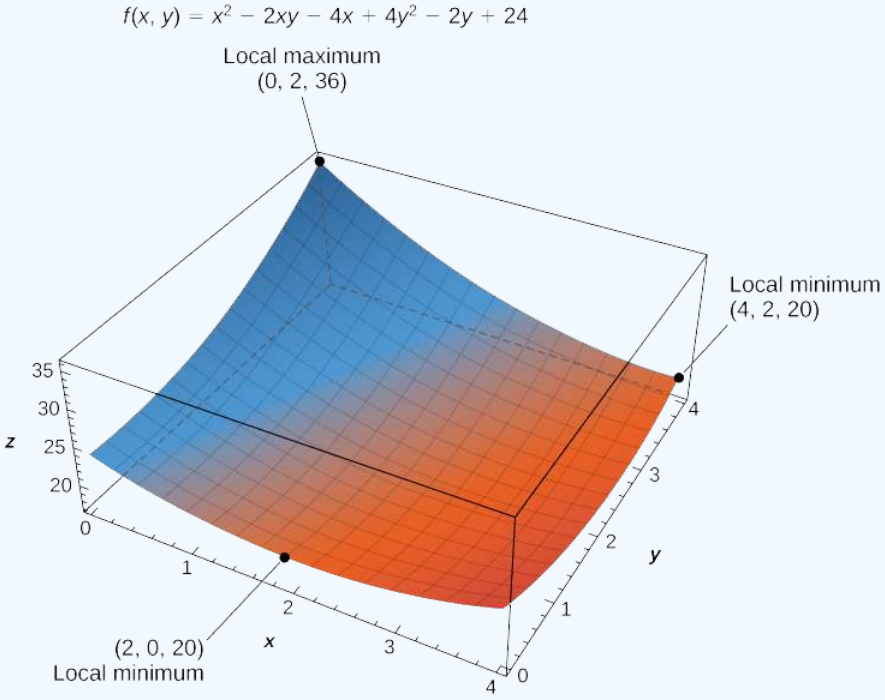


Figure 5.8.8: The function $f(x,y)$ has two global minima and one global maximum over its domain.

b. Using the problem-solving strategy, step 1 involves finding the critical points of g on its domain. Therefore, we first calculate $g_x(x,y)$ and $g_y(x,y)$, then set them each equal to zero:

$$
\begin{aligned}
g_x(x,y) &= 2x + 4 \\
g_y(x,y) &= 2y - 6.
\end{aligned}
$$

Setting them equal to zero yields the system of equations

$$
\begin{aligned}
2x + 4 &= 0 \\
2y - 6 &= 0.
\end{aligned}
$$

The solution to this system is $x = -2$ and $y = 3$. Therefore, $(-2,3)$ is a critical point of g. Calculating $g(-2,3)$, we get

$$
g(-2,3) = (-2)^2 + 3^2 + 4(-2) - 6(3) = 4 + 9 - 8 - 18 = -13. \tag{5.8.6}
$$

The next step involves finding the extrema of g on the boundary of its domain. The boundary of its domain consists of a circle of radius 4 centered at the origin as shown in the following graph.

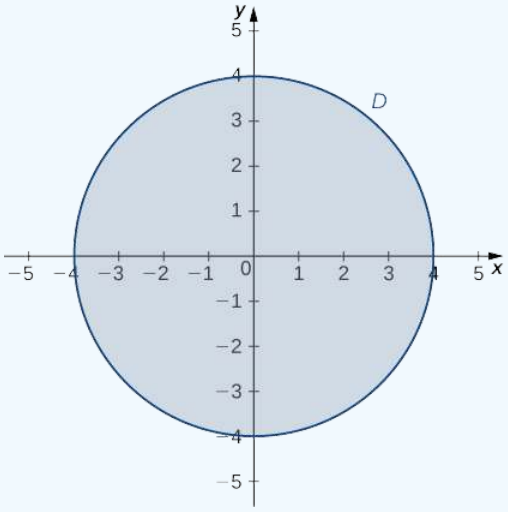


Figure 5.8.9: Graph of the restricted domain of the function $g(x,y)=x^2+y^2+4x-6y$.

The boundary of the domain of g can be parameterized using the functions $x(t)=4\cos t$, $y(t)=4\sin t$ for $0\le t\le 2\pi$. Define $h(t)=g\big(x(t),y(t)\big)$:

$$\begin{aligned} h(t) &= g\big(x(t),y(t)\big) \\ &= (4\cos t)^2+(4\sin t)^2+4(4\cos t)-6(4\sin t) \\ &= 16\cos^2 t+16\sin^2 t+16\cos t-24\sin t \\ &= 16+16\cos t-24\sin t. \end{aligned}$$

Setting $h'(t)=0$ leads to

$$\begin{aligned} -16\sin t-24\cos t &= 0 \\ -16\sin t &= 24\cos t \\ \frac{-16\sin t}{-16\cos t} &= \frac{24\cos t}{-16\cos t} \\ \tan t &= -\frac{3}{2}. \end{aligned}$$

This equation has two solutions over the interval $0\le t\le 2\pi$. One is $t=\pi-\arctan(\frac{3}{2})$ and the other is $t=2\pi-\arctan(\frac{3}{2})$. For the first angle,

$$\begin{aligned} \sin t &= \sin(\pi-\arctan(\tfrac{3}{2}))=\sin(\arctan(\tfrac{3}{2}))=\frac{3\sqrt{13}}{13} \\ \cos t &= \cos(\pi-\arctan(\tfrac{3}{2}))=-\cos(\arctan(\tfrac{3}{2}))=-\frac{2\sqrt{13}}{13}. \end{aligned}$$

Therefore, $x(t)=4\cos t=-\frac{8\sqrt{13}}{13}$ and $y(t)=4\sin t=\frac{12\sqrt{13}}{13}$, so $\left(-\frac{8\sqrt{13}}{13},\frac{12\sqrt{13}}{13}\right)$ is a critical point on the boundary and

$$\begin{aligned} g\left(-\tfrac{8\sqrt{13}}{13},\tfrac{12\sqrt{13}}{13}\right) &= \left(-\tfrac{8\sqrt{13}}{13}\right)^2+\left(\tfrac{12\sqrt{13}}{13}\right)^2+4\left(-\tfrac{8\sqrt{13}}{13}\right)-6\left(\tfrac{12\sqrt{13}}{13}\right) \\ &= \frac{144}{13}+\frac{64}{13}-\frac{32\sqrt{13}}{13}-\frac{72\sqrt{13}}{13} \\ &= \frac{208-104\sqrt{13}}{13}\approx -12.844. \end{aligned}$$

For the second angle,

$$\sin t = \sin(2\pi - \arctan(\tfrac{3}{2})) = -\sin(\arctan(\tfrac{3}{2})) = -\frac{3\sqrt{13}}{13}$$

$$\cos t = \cos(2\pi - \arctan(\tfrac{3}{2})) = \cos(\arctan(\tfrac{3}{2})) = \frac{2\sqrt{13}}{13}.$$

Therefore, $x(t) = 4\cos t = \frac{8\sqrt{13}}{13}$ and $y(t) = 4\sin t = -\frac{12\sqrt{13}}{13}$, so $\left(\frac{8\sqrt{13}}{13}, -\frac{12\sqrt{13}}{13}\right)$ is a critical point on the boundary and

$$\begin{aligned} g\left(\tfrac{8\sqrt{13}}{13}, -\tfrac{12\sqrt{13}}{13}\right) &= \left(\tfrac{8\sqrt{13}}{13}\right)^2 + \left(-\tfrac{12\sqrt{13}}{13}\right)^2 + 4\left(\tfrac{8\sqrt{13}}{13}\right) - 6\left(-\tfrac{12\sqrt{13}}{13}\right) \\ &= \frac{144}{13} + \frac{64}{13} + \frac{32\sqrt{13}}{13} + \frac{72\sqrt{13}}{13} \\ &= \frac{208 + 104\sqrt{13}}{13} \approx 44.844. \end{aligned}$$

The absolute minimum of g is -13, which is attained at the point $(-2, 3)$, which is an interior point of D. The absolute maximum of g is approximately equal to 44.844, which is attained at the boundary point $\left(\frac{8\sqrt{13}}{13}, -\frac{12\sqrt{13}}{13}\right)$. These are the absolute extrema of g on D as shown in the following figure.

$$f(x, y) = x^2 + 4x + y^2 - 6y$$

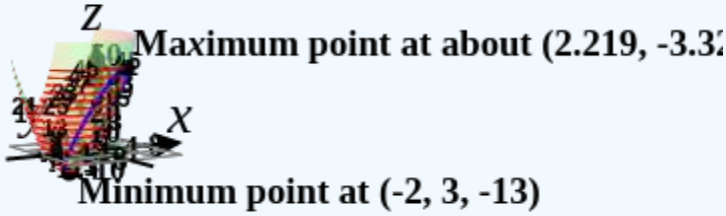


Figure 5.8.10: The function $f(x, y)$ has a local minimum and a local maximum.

Exercise 5.8.3:

Use the problem-solving strategy for finding absolute extrema of a function to find the absolute extrema of the function

$$f(x, y) = 4x^2 - 2xy + 6y^2 - 8x + 2y + 3$$

on the domain defined by $0 \le x \le 2$ and $-1 \le y \le 3$.

Hint

Calculate $f_x(x,y)$ and $f_y(x,y)$, and set them equal to zero. Then, calculate f for each critical point and find the extrema of f on the boundary of D.

Answer

The absolute minimum occurs at $(1,0) : f(1,0) = -1$.

The absolute maximum occurs at $(0,3) : f(0,3) = 63$.

Example 5.8.4: Profitable Golf Balls

Pro-T company has developed a profit model that depends on the number x of golf balls sold per month (measured in thousands), and the number of hours per month of advertising y, according to the function

$$z = f(x,y) = 48x + 96y - x^2 - 2xy - 9y^2, \tag{5.8.7}$$

where z is measured in thousands of dollars. The maximum number of golf balls that can be produced and sold is $50,000$, and the maximum number of hours of advertising that can be purchased is 25. Find the values of x and y that maximize profit, and find the maximum profit.



Figure 5.8.11: (credit: modification of work by oatsy40, Flickr)

Solution

Using the problem-solving strategy, step 1 involves finding the critical points of f on its domain. Therefore, we first calculate $f_x(x,y)$ and $f_y(x,y)$, then set them each equal to zero:

$$\begin{aligned} f_x(x,y) &= 48 - 2x - 2y \\ f_y(x,y) &= 96 - 2x - 18y. \end{aligned}$$

Setting them equal to zero yields the system of equations

$$\begin{aligned} 48 - 2x - 2y &= 0 \\ 96 - 2x - 18y &= 0. \end{aligned}$$

The solution to this system is $x = 21$ and $y = 3$. Therefore $(21,3)$ is a critical point of f. Calculating $f(21,3)$ gives $f(21,3) = 48(21) + 96(3) - 21^2 - 2(21)(3) - 9(3)^2 = 648$.

The domain of this function is $0 \le x \le 50$ and $0 \le y \le 25$ as shown in the following graph.

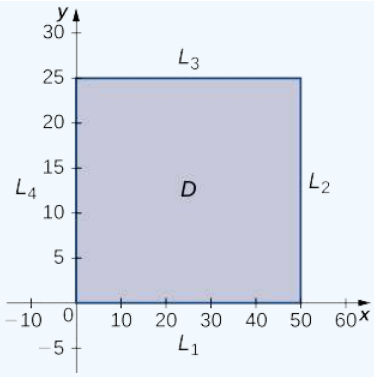


Figure 5.8.12: Graph of the domain of the function $f(x,y) = 48x + 96y - x^2 - 2xy - 9y^2$.

L_1 is the line segment connecting $(0,0)$ and $(50,0)$, and it can be parameterized by the equations $x(t) = t, y(t) = 0$ for $0 \le t \le 50$. We then define $g(t) = f(x(t), y(t))$:

$$\begin{aligned} g(t) &= f(x(t), y(t)) \\ &= f(t, 0) \\ &= 48t + 96(0) - y^2 - 2(t)(0) - 9(0)^2 \\ &= 48t - t^2. \end{aligned}$$

Setting $g'(t) = 0$ yields the critical point $t = 24$, which corresponds to the point $(24,0)$ in the domain of f. Calculating $f(24,0)$ gives 576.

L_2 is the line segment connecting $(50,0)$ and $(50,25)$, and it can be parameterized by the equations $x(t) = 50, y(t) = t$ for $0 \le t \le 25$. Once again, we define $g(t) = f\big(x(t), y(t)\big)$:

$$\begin{aligned} g(t) &= f\big(x(t), y(t)\big) \\ &= f(50, t) \\ &= 48(50) + 96t - 50^2 - 2(50)t - 9t^2 \\ &= -9t^2 - 4t - 100. \end{aligned}$$

This function has a critical point at $t = -\frac{2}{9}$, which corresponds to the point $(50, -29)$. This point is not in the domain of f.

L_3 is the line segment connecting $(0,25)$ and $(50,25)$, and it can be parameterized by the equations $x(t) = t, y(t) = 25$ for $0 \le t \le 50$. We define $g(t) = f\big(x(t), y(t)\big)$:

$$\begin{aligned} g(t) &= f\big(x(t), y(t)\big) \\ &= f(t, 25) \\ &= 48t + 96(25) - t^2 - 2t(25) - 9(25^2) \\ &= -t^2 - 2t - 3225. \end{aligned}$$

This function has a critical point at $t = -1$, which corresponds to the point $(-1, 25)$, which is not in the domain.

L_4 is the line segment connecting $(0,0)$ to $(0,25)$, and it can be parameterized by the equations $x(t) = 0, y(t) = t$ for $0 \le t \le 25$. We define $g(t) = f\big(x(t), y(t)\big)$:

$$\begin{aligned} g(t) &= f\big(x(t), y(t)\big) \\ &= f(0, t) \\ &= 48(0) + 96t - (0)^2 - 2(0)t - 9t^2 \\ &= 96t - 9t^2. \end{aligned}$$

This function has a critical point at $t = \frac{16}{3}$, which corresponds to the point $\left(0, \frac{16}{3}\right)$, which is on the boundary of the domain. Calculating $f\left(0, \frac{16}{3}\right)$ gives 256.

We also need to find the values of $f(x, y)$ at the corners of its domain. These corners are located at $(0, 0)$, $(50, 0)$, $(50, 25)$ and $(0, 25)$:

$$\begin{aligned} f(0,0) &= 48(0) + 96(0) - (0)^2 - 2(0)(0) - 9(0)^2 = 0 \\ f(50,0) &= 48(50) + 96(0) - (50)^2 - 2(50)(0) - 9(0)^2 = -100 \\ f(50,25) &= 48(50) + 96(25) - (50)^2 - 2(50)(25) - 9(25)^2 = -5825 \\ f(0,25) &= 48(0) + 96(25) - (0)^2 - 2(0)(25) - 9(25)^2 = -3225. \end{aligned}$$

The maximum value is 648, which occurs at $(21, 3)$. Therefore, a maximum profit of $\$648,000$ is realized when $21,000$ golf balls are sold and 3 hours of advertising are purchased per month as shown in the following figure.

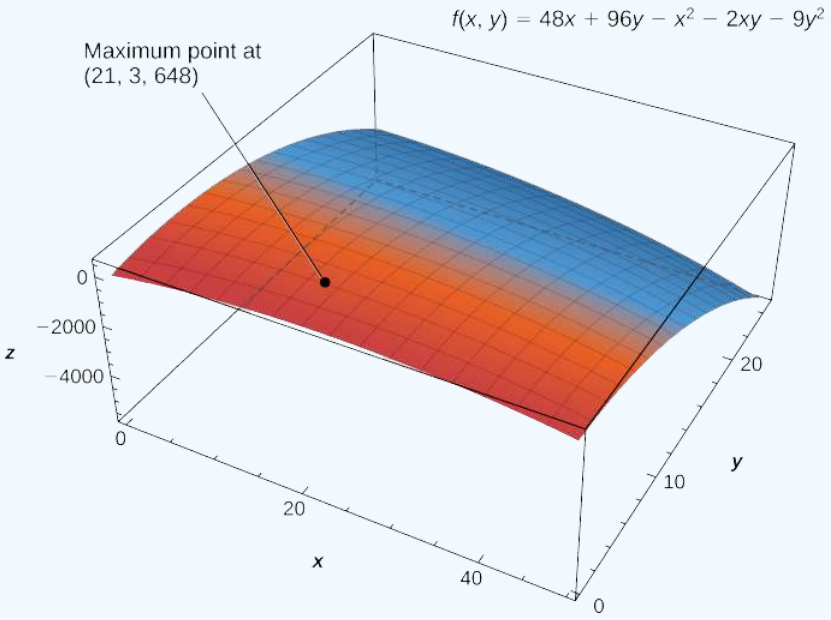


Figure 5.8.13: The profit function $f(x, y)$ has a maximum at $(21, 3, 648)$.

Key Concepts

- A critical point of the function $f(x, y)$ is any point (x_0, y_0) where either $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$, or at least one of $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ do not exist.
- A saddle point is a point (x_0, y_0) where $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$, but $f(x_0, y_0)$ is neither a maximum nor a minimum at that point.
- To find extrema of functions of two variables, first find the critical points, then calculate the discriminant and apply the second derivative test.

Key Equations

- **Discriminant**

$$D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2$$

Glossary

critical point of a function of two variables

the point (x_0, y_0) is called a critical point of $f(x, y)$ if one of the two following conditions holds:

1. $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$
2. At least one of $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ do not exist

discriminant

the discriminant of the function $f(x,y)$ is given by the formula $D = f_{xx}(x_0,y_0)f_{yy}(x_0,y_0) - (f_{xy}(x_0,y_0))^2$

saddle point

given the function $z = f(x,y)$, the point $(x_0, y_0, f(x_0,y_0))$ is a saddle point if both $f_x(x_0,y_0) = 0$ and $f_y(x_0,y_0) = 0$, but f does not have a local extremum at (x_0,y_0)

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5.8E: Exercises for Section 5.8

Finding Critical Points

In exercises 1 - 5, find all critical points.

1) $f(x,y) = 1 + x^2 + y^2$

Answer

$(0,0)$

2) $f(x,y) = 1 - (x-2)^2 + (y+3)^2$

3) $f(x,y) = (3x-2)^2 + (y-4)^2$

Answer

$\left(\frac{2}{3}, 4\right)$

4) $f(x,y) = x^4 + y^4 - 16xy$

Answer

$(0,0), \quad (-2,-2), \quad (2,2)$

5) $f(x,y) = 15x^3 - 3xy + 15y^3$

Answer

$(0,0), \quad \left(\frac{1}{15}, \frac{1}{15}\right)$

Finding Extrema & the Second Partials Test

In exercises 6 - 9, find the critical points of the function and test for extrema or saddle points by using algebraic techniques (completing the square) or by examining the form of the equation. Where possible, verify your results using the Second Partials Test.

6) $f(x,y) = -\sqrt{x^2 + y^2}$

Answer

Crit. pts.: $(0,0)$
Extrema: f has a relative maximum of 0 at $(0,0)$.
To justify this, consider the fact that the square root function cannot give a negative value, so this function cannot possibly return a positive value. Since it's value is 0 at the critical point $(0,0)$, we know it must be the function's **absolute maximum** value.

7) $f(x,y) = -x^2 - 5y^2 + 8x - 10y - 13$

Answer

Crit. pts.: $(4,-1)$
Extrema: f has a relative maximum of 8 at $(4,-1)$.
To justify this, we complete the square on this function, being careful to factor out the coefficient of the squared terms before completing the square.

$$\begin{aligned} f(x,y) &= -x^2 - 5y^2 + 8x - 10y - 13 \\ &= -(x^2 - 8x \quad\quad) - 5(y^2 + 2y \quad\quad) - 13 \\ &= -(x^2 - 8x + 16) - 5(y^2 + 2y + 1) - 13 + 16 + 5 \\ &= -(x-4)^2 - 5(y+1)^2 + 8 \end{aligned}$$

Note that this quadratic polynomial function takes the form $z=-(x^2+y^2)$, so we can see that it will have a **relative** (and, in fact, absolute) **maximum** at its vertex (the critical point $(4,-1)$). We can also argue that since we are subtracting squared terms from 8, we cannot possibly obtain a function value larger than 8, and since we do get a value of 8 at the critical point $(4,-1)$, we know it will be the absolute maximum value of this function.

8) $f(x,y)=x^2+y^2+2x-6y+6$

9) $f(x,y)=\sqrt{x^2+y^2}+1$

Answer

Crit. Pts.: $(0,0)$
Extrema: f has a relative minimum of 1 at $(0,0)$.
To justify this, consider the fact that the square root function cannot give a negative value, so this function cannot possibly return a value less than 1. Since it's value is 1 at the critical point $(0,0)$, we know 1 must be the function's **absolute minimum** value.

In exercises 10 - 34, identify any critical points and use the Second Partials Test to determine the behavior of the function at each critical point, whether there is a maximum, minimum, saddle point, or none of these. If the Second Partials Test fails, determine the function's behavior at the point using another method and justify your answer clearly.

10) $f(x,y)=-x^3+4xy-2y^2+1$

11) $f(x,y)=x^2y^2$

Answer

Crit. pts.: All points on the lines $x=0$ and $y=0$ are critical points of this function.
Exrema: The Second Partials Test fails.
Since $x^2y^2>0$ for all x and y different from zero, and $x^2y^2=0$ when either x or y equals zero (or both), then the absolute minimum of 0 occurs at all points on the x- or y-axes, that is, for all points on the lines $x=0$ and $y=0$.

12) $f(x,y)=x^2-6x+y^2+4y-8$

13) $f(x,y)=2xy+3x+4y$

Answer

Crit. pts.: $\left(-2,-\frac{3}{2}\right)$
Exrema: f has a saddle point at $\left(-2,-\frac{3}{2},-6\right)$.

14) $f(x,y)=8xy(x+y)+7$

15) $f(x,y)=x^2+4xy+y^2$

Answer

Crit. pts.: $(0,0)$
Exrema: f has a saddle point at $(0,0,0)$.

16) $f(x,y)=x^3+y^3-300x-75y-3$

17) $f(x,y)=9-x^4y^4$

Answer

Crit. pts.: All points on the lines $x=0$ and $y=0$ are critical points of this function.
Extrema: The Second Partials Test fails.
Since the term $-x^4y^4<0$ for all x and y different from zero, and $-x^4y^4=0$ when either x or y equals zero (or both), then this function cannot attain a value greater than 9 anywhere, but is 9 at the critical points. Thus f has an absolute maximum of 9 at all points on the x- or y-axes, that is, for all points on the lines $x=0$ and $y=0$.

18) $f(x,y)=x^2+10xy+y^2$

Answer

Crit. pts.: $(0,0)$
Extrema: f has a saddle point at $(0,0,0)$.

19) $f(x,y)=x^4+y^2+2xy+3$

Answer

Crit. pts.: $(0,0)$, $\left(-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$, $\left(\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)$
Extrema: f has a saddle point at $(0,0,3)$.
f has a local minimum of 2.75 at the point $\left(-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$.
f has a local minimum of 2.75 at the point $\left(\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)$.

20) $f(x,y)=7x^2y+9xy^2$

21) $f(x,y)=3x^2-2xy+y^2-8y$

Answer

Crit. pts.: $(2,6)$
Extrema: f has a relative minimum of -24 located at $(2,6)$.

22) $f(x,y)=3x^2+2xy+y^2$

23) $f(x,y)=y^2+xy+3y+2x+3$

Answer

Crit. pts.: $(1,-2)$
Extrema: f has a saddle point at $(1,-2,1)$.

24) $f(x,y)=x^2+xy+y^2-3x$

25) $f(x,y)=x^2+2y^2-x^2y$

Answer

Crit. pts.: $(0,0)$, $(-2,1)$, $(2,1)$
Extrema: f has a relative minimum of 0 at $(0,0)$ and saddle points at $(2,1,2)$ and $(-2,1,2)$.

26) $f(x,y)=x^2+y-e^y$

27) $f(x,y)=e^{-(x^2+y^2+2x)}$

Answer

Crit. pts.: $(-1,0)$
Extrema: f has a relative maximum of e located at $(-1,0)$.
See this problem illustrated in CalcPlot3D.

28) $f(x,y)=x^2+xy+y^2-x-y+1$

29) $f(x,y)=x^2y(9-x+y)$

Answer

Crit. pts.: $\left(\frac{9}{2},-\frac{9}{4}\right)$, $(9,0)$, and all points on the line $x=0$
Extrema: f has a saddle point at $(9,0,0)$ and a relative minimum of -102.515625 at $\left(\frac{9}{2},-\frac{9}{4}\right)$.
At the critical points on the line $x=0$, f has neither relative extrema nor saddle points, but they do represent a sort of trough on the surface.

30) $f(x,y) = -x^2 - 5y^2 + 10x - 30y - 62$

31) $f(x,y) = 120x + 120y - xy - x^2 - y^2$

Answer

Crit. pts.: $(40, 40)$

Extrema: f has a relative maximum of 4800located at $(40, 40)$.

32) $f(x,y) = 2x^2 + 2xy + y^2 + 2x - 3$

33) $f(x,y) = x^2 + x - 3xy + y^3 - 5$

Answer

Crit. pts.: $\left(\frac{1}{4}, \frac{1}{2}\right)$ and $(1, 1)$

Extrema: f has a saddle point at $\left(\frac{1}{4}, \frac{1}{2}, -\frac{79}{16}\right)$ and a relative minimum of -5 at $(1, 1)$.

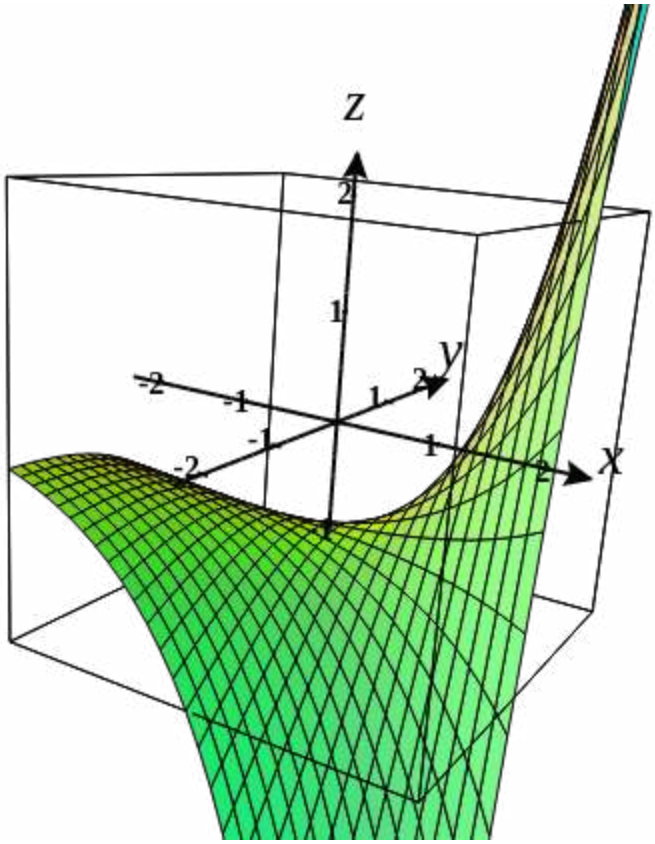
34) $f(x,y) = 2xye^{-x^2-y^2}$

In exercises 35 - 37, determine the extreme values and the saddle points. Use a CAS to graph the function.

35) [T] $f(x,y) = ye^x - e^y$

Answer

A saddle point is located at $(0, 0, -1)$.



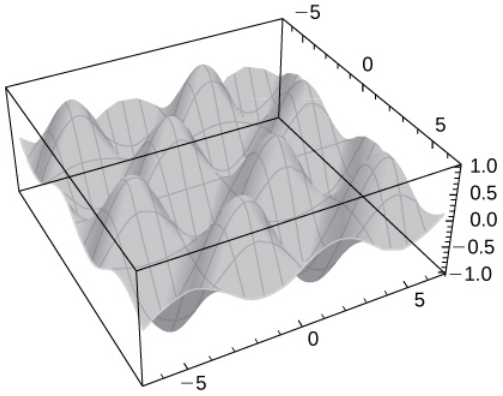
36) [T] $f(x,y) = x\sin(y)$

https://math.libretexts.org/@go/page/84038

37) [T] $f(x,y)=\sin(x)\sin(y), \quad x\in(0,2\pi), \quad y\in(0,2\pi)$

Answer

There is a saddle point at (π,π), local maxima at $\left(\frac{\pi}{2},\frac{\pi}{2}\right)$ and $\left(\frac{3\pi}{2},\frac{3\pi}{2}\right)$, and local minima at $\left(\frac{\pi}{2},\frac{3\pi}{2}\right)$ and $\left(\frac{3\pi}{2},\frac{\pi}{2}\right)$.



In exercises 38 - 41, find the absolute extrema of the given function on the indicated closed and bounded set R.

38) $f(x,y)=xy-x-3y$ where R is the triangular region with vertices $(0,0),\ (0,4),$ and $(5,0)$.

39) Find the absolute maximum and minimum values of $f(x,y)=x^2+y^2-2y+1$ on the region $R=\left\{(x,y)\,|\,x^2+y^2\le 4\right\}$.

Answer

$(0,1,0)$ is the absolute minimum and $(0,-2,9)$ is the absolute maximum.

40) $f(x,y)=x^3-3xy-y^3$ on $R=\left\{(x,y)\,|\,-2\le x\le 2,\ -2\le y\le 2\right\}$

41) $f(x,y)=\dfrac{-2y}{x^2+y^2+1}$ on $R=\left\{(x,y)\,|\,x^2+y^2\le 4\right\}$

Answer

There is an absolute minimum at $(0,1,-1)$ and an absolute maximum at $(0,-1,1)$.

42) Find three positive numbers the sum of which is 27, such that the sum of their squares is as small as possible.

43) Find the points on the surface $x^2-yz=5$ that are closest to the origin.

Answer

$\left(\sqrt{5},0,0\right),\ \left(-\sqrt{5},0,0\right)$

44) Find the maximum volume of a rectangular box with three faces in the coordinate planes and a vertex in the first octant on the line $x+y+z=1$.

45) The sum of the length and the girth (perimeter of a cross-section) of a package carried by a delivery service cannot exceed 108 in. Find the dimensions of the rectangular package of largest volume that can be sent.

Answer

18 by 36 by 18 in.

46) A cardboard box without a lid is to be made with a volume of 4 ft^3. Find the dimensions of the box that requires the least amount of cardboard.

47) Find the point on the surface $f(x,y)=x^2+y^2+10$ nearest the plane $x+2y-z=0$. Identify the point on the plane.

Answer

$\left(\frac{47}{24},\frac{47}{12},\frac{235}{24}\right)$

48) Find the point in the plane $2x - y + 2z = 16$ that is closest to the origin.

49) A company manufactures two types of athletic shoes: jogging shoes and cross-trainers. The total revenue from x units of jogging shoes and y units of cross-trainers is given by $R(x, y) = -5x^2 - 8y^2 - 2xy + 42x + 102y$, where x and y are in thousands of units. Find the values of x and y to maximize the total revenue.

Answer

$x = 3$ and $y = 6$

50) A shipping company handles rectangular boxes provided the sum of the length, width, and height of the box does not exceed 96in. Find the dimensions of the box that meets this condition and has the largest volume.

51) Find the maximum volume of a cylindrical soda can such that the sum of its height and circumference is 120 cm.

Answer

$V = \dfrac{64,000}{\pi} \approx 20,372 \text{ cm}^3$

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- Paul Seeburger (Monroe Community College) created problems 19 and 29, and added dynamic figures for problems 27 and 35.

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5.9: Lagrange Multipliers

Learning Objectives

- Use the method of Lagrange multipliers to solve optimization problems with one constraint.
- Use the method of Lagrange multipliers to solve optimization problems with two constraints.

Solving optimization problems for functions of two or more variables can be similar to solving such problems in single-variable calculus. However, techniques for dealing with multiple variables allow us to solve more varied optimization problems for which we need to deal with additional conditions or constraints. In this section, we examine one of the more common and useful methods for solving optimization problems with constraints.

Lagrange Multipliers

In the previous section, an applied situation was explored involving maximizing a profit function, subject to certain **constraints.** In that example, the constraints involved a maximum number of golf balls that could be produced and sold in 1 month (x), and a maximum number of advertising hours that could be purchased per month (y). Suppose these were combined into a single budgetary constraint, such as $20x+4y\le 216$, that took into account both the cost of producing the golf balls and the number of advertising hours purchased per month. The goal is still to maximize profit, but now there is a different type of constraint on the values of x and y. This constraint and the corresponding profit function

$$f(x,y)=48x+96y-x^2-2xy-9y^2$$

is an example of an **optimization problem**, and the function $f(x,y)$ is called the **objective function**. A graph of various level curves of the function $f(x,y)$ follows.

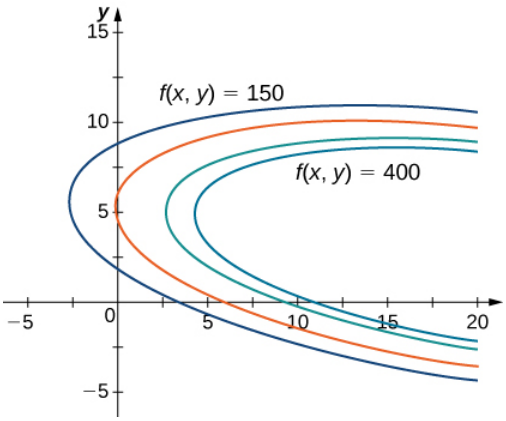


Figure 5.9.1: Graph showing level curves of the function $f(x,y)=48x+96y-x^2-2xy-9y^2$ corresponding to $c=150,250,350$, and 400.

In Figure 5.9.1, the value c represents different profit levels (i.e., values of the function f). As the value of c increases, the curve shifts to the right. Since our goal is to maximize profit, we want to choose a curve as far to the right as possible. If there were no restrictions on the number of golf balls the company could produce or the number of units of advertising available, then we could produce as many golf balls as we want, and advertise as much as we want, and there would be not be a maximum profit for the company. Unfortunately, we have a budgetary constraint that is modeled by the inequality $20x+4y\le 216$. To see how this constraint interacts with the profit function, Figure 5.9.2 shows the graph of the line $20x+4y=216$ superimposed on the previous graph.

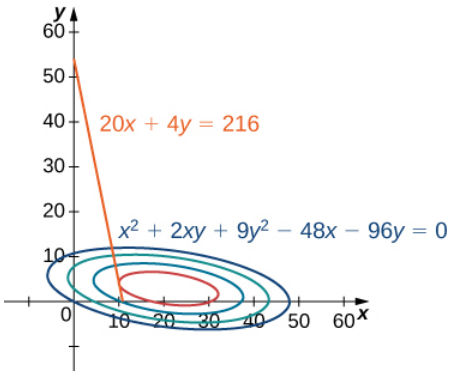


Figure 5.9.2: Graph of level curves of the function $f(x,y) = 48x + 96y - x^2 - 2xy - 9y^2$ corresponding to $c = 150, 250, 350,$ and 395. The red graph is the constraint function.

As mentioned previously, the maximum profit occurs when the level curve is as far to the right as possible. However, the level of production corresponding to this maximum profit must also satisfy the budgetary constraint, so the point at which this profit occurs must also lie on (or to the left of) the red line in Figure 5.9.2. Inspection of this graph reveals that this point exists where the line is tangent to the level curve of f. Trial and error reveals that this profit level seems to be around 395, when x and y are both just less than 5. We return to the solution of this problem later in this section. From a theoretical standpoint, at the point where the profit curve is tangent to the constraint line, the gradient of both of the functions evaluated at that point must point in the same (or opposite) direction. Recall that the gradient of a function of more than one variable is a vector. If two vectors point in the same (or opposite) directions, then one must be a constant multiple of the other. This idea is the basis of the **method of Lagrange multipliers.**

Method of Lagrange Multipliers: One Constraint

Theorem 5.9.1: Let f and g be functions of two variables with continuous partial derivatives at every point of some open set containing the smooth curve $g(x,y) = 0$. Suppose that f, when restricted to points on the curve $g(x,y) = 0$, has a local extremum at the point (x_0, y_0) and that $\vec{\nabla} g(x_0, y_0) \neq 0$. Then there is a number λ called a **Lagrange multiplier,** for which

$$\vec{\nabla} f(x_0, y_0) = \lambda \vec{\nabla} g(x_0, y_0). \tag{5.9.1}$$

Proof

Assume that a constrained extremum occurs at the point (x_0, y_0). Furthermore, we assume that the equation $g(x,y) = 0$ can be smoothly parameterized as

$$x = x(s) \text{ and } y = y(s)$$

where s is an arc length parameter with reference point (x_0, y_0) at $s = 0$. Therefore, the quantity $z = f(x(s), y(s))$ has a relative maximum or relative minimum at $s = 0$, and this implies that $\dfrac{dz}{ds} = 0$ at that point. From the chain rule,

$$\begin{aligned} \frac{dz}{ds} &= \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s} \\ &= \left(\frac{\partial f}{\partial x} \hat{\mathbf{i}} + \frac{\partial f}{\partial y} \hat{\mathbf{j}} \right) \cdot \left(\frac{\partial x}{\partial s} \hat{\mathbf{i}} + \frac{\partial y}{\partial s} \hat{\mathbf{j}} \right) \\ &= 0, \end{aligned}$$

where the derivatives are all evaluated at $s = 0$. However, the first factor in the dot product is the gradient of f, and the second factor is the unit tangent vector $\vec{\mathbf{T}}(0)$ to the constraint curve. Since the point (x_0, y_0) corresponds to $s = 0$, it follows from this equation that

$$\vec{\nabla} f(x_0, y_0) \cdot \vec{\mathbf{T}}(0) = 0,$$

which implies that the gradient is either the zero vector $\vec{\mathbf{0}}$ or it is normal to the constraint curve at a constrained relative extremum. However, the constraint curve $g(x,y) = 0$ is a level curve for the function $g(x,y)$ so that if $\vec{\nabla} g(x_0, y_0) \neq 0$ then

$\vec{\nabla} g(x_0, y_0)$ is normal to this curve at (x_0, y_0) It follows, then, that there is some scalar λ such that

$$\vec{\nabla} f(x_0, y_0) = \lambda \vec{\nabla} g(x_0, y_0)$$

□

To apply Theorem 5.9.1 to an optimization problem similar to that for the golf ball manufacturer, we need a problem-solving strategy.

Problem-Solving Strategy: Steps for Using Lagrange Multipliers

1. Determine the objective function $f(x, y)$ and the constraint function $g(x, y)$. Does the optimization problem involve maximizing or minimizing the objective function?
2. Set up a system of equations using the following template:

$$\vec{\nabla} f(x_0, y_0) = \lambda \vec{\nabla} g(x_0, y_0) \quad (5.9.2)$$
$$g(x_0, y_0) = 0 \quad (5.9.3)$$

3. Solve for x_0 and y_0.
4. The largest of the values of f at the solutions found in step 3 maximizes f; the smallest of those values minimizes f.

Example 5.9.1: Using Lagrange Multipliers

Use the method of Lagrange multipliers to find the minimum value of $f(x, y) = x^2 + 4y^2 - 2x + 8y$ subject to the constraint $x + 2y = 7$.

Solution

Let's follow the problem-solving strategy:

1. The objective function is $f(x, y) = x^2 + 4y^2 - 2x + 8y$. To determine the constraint function, we must first subtract 7 from both sides of the constraint. This gives $x + 2y - 7 = 0$. The constraint function is equal to the left-hand side, so $g(x, y) = x + 2y - 7$. The problem asks us to solve for the minimum value of f, subject to the constraint (Figure 5.9.3).

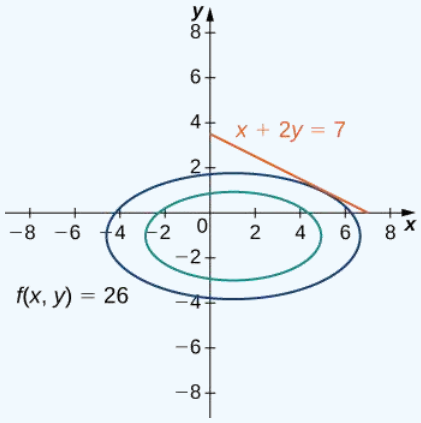


Figure 5.9.3: Graph of level curves of the function $f(x, y) = x^2 + 4y^2 - 2x + 8y$ corresponding to $c = 10$ and 26. The red graph is the constraint function.

2. We then must calculate the gradients of both f and g:

$$\vec{\nabla} f(x, y) = (2x - 2)\,\hat{\mathbf{i}} + (8y + 8)\,\hat{\mathbf{j}}$$
$$\vec{\nabla} g(x, y) = \hat{\mathbf{i}} + 2\hat{\mathbf{j}}.$$

The equation $\vec{\nabla} f(x_0, y_0) = \lambda \vec{\nabla} g(x_0, y_0)$ becomes

$$(2x_0 - 2)\,\hat{\mathbf{i}} + (8y_0 + 8)\,\hat{\mathbf{j}} = \lambda\left(\hat{\mathbf{i}} + 2\hat{\mathbf{j}}\right),$$

which can be rewritten as

$$(2x_0 - 2)\,\hat{\mathbf{i}} + (8y_0 + 8)\,\hat{\mathbf{j}} = \lambda\hat{\mathbf{i}} + 2\lambda\hat{\mathbf{j}}.$$

Next, we set the coefficients of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ equal to each other:

$$\begin{aligned} 2x_0 - 2 &= \lambda \\ 8y_0 + 8 &= 2\lambda. \end{aligned}$$

The equation $g(x_0, y_0) = 0$ becomes $x_0 + 2y_0 - 7 = 0$. Therefore, the system of equations that needs to be solved is

$$\begin{aligned} 2x_0 - 2 &= \lambda \\ 8y_0 + 8 &= 2\lambda \\ x_0 + 2y_0 - 7 &= 0. \end{aligned}$$

3. This is a linear system of three equations in three variables. We start by solving the second equation for λ and substituting it into the first equation. This gives $\lambda = 4y_0 + 4$, so substituting this into the first equation gives

$$2x_0 - 2 = 4y_0 + 4.$$

Solving this equation for x_0 gives $x_0 = 2y_0 + 3$. We then substitute this into the third equation:

$$\begin{aligned} (2y_0 + 3) + 2y_0 - 7 &= 0 \\ 4y_0 - 4 &= 0 \\ y_0 &= 1. \end{aligned}$$

Since $x_0 = 2y_0 + 3,$ this gives $x_0 = 5.$

4. Next, we evaluate $f(x, y) = x^2 + 4y^2 - 2x + 8y$ at the point $(5, 1)$,

$$f(5, 1) = 5^2 + 4(1)^2 - 2(5) + 8(1) = 27.$$

To ensure this corresponds to a minimum value on the constraint function, let's try some other points on the constraint from either side of the point $(5, 1)$, such as the intercepts of $g(x, y) = 0$, Which are $(7, 0)$ and $(0, 3.5)$.

We get $f(7, 0) = 35 > 27$ and $f(0, 3.5) = 77 > 27.$

So it appears that f has a relative minimum of 27 at $(5, 1)$, subject to the given constraint.



Exercise 5.9.1

Use the method of Lagrange multipliers to find the maximum value of

$$f(x,y)=9x^2+36xy-4y^2-18x-8y$$

subject to the constraint $3x+4y=32$.

Hint

Use the problem-solving strategy for the method of Lagrange multipliers.

Answer

Subject to the given constraint, f has a maximum value of 976 at the point $(8,2)$.

Let's now return to the problem posed at the beginning of the section.

Example 5.9.2: Golf Balls and Lagrange Multipliers

The golf ball manufacturer, Pro-T, has developed a profit model that depends on the number x of golf balls sold per month (measured in thousands), and the number of hours per month of advertising y, according to the function

$$z=f(x,y)=48x+96y-x^2-2xy-9y^2,$$

where z is measured in thousands of dollars. The budgetary constraint function relating the cost of the production of thousands golf balls and advertising units is given by $20x+4y=216$. Find the values of x and y that maximize profit, and find the maximum profit.

Solution:

Again, we follow the problem-solving strategy:

1. The objective function is $f(x,y)=48x+96y-x^2-2xy-9y^2$. To determine the constraint function, we first subtract 216 from both sides of the constraint, then divide both sides by 4, which gives $5x+y-54=0$. The constraint function is equal to the left-hand side, so $g(x,y)=5x+y-54$. The problem asks us to solve for the maximum value of f, subject to this constraint.
2. So, we calculate the gradients of both f and g:

$$\begin{aligned}\vec{\nabla}f(x,y) &= (48-2x-2y)\hat{\mathbf{i}}+(96-2x-18y)\hat{\mathbf{j}}\\ \vec{\nabla}g(x,y) &= 5\hat{\mathbf{i}}+\hat{\mathbf{j}}.\end{aligned}$$

The equation $\vec{\nabla}f(x_0,y_0)=\lambda\vec{\nabla}g(x_0,y_0)$ becomes

$$(48-2x_0-2y_0)\hat{\mathbf{i}}+(96-2x_0-18y_0)\hat{\mathbf{j}}=\lambda(5\hat{\mathbf{i}}+\hat{\mathbf{j}}),$$

which can be rewritten as

$$(48-2x_0-2y_0)\hat{\mathbf{i}}+(96-2x_0-18y_0)\hat{\mathbf{j}}=\lambda 5\hat{\mathbf{i}}+\lambda\hat{\mathbf{j}}.$$

We then set the coefficients of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ equal to each other:

$$\begin{aligned}48-2x_0-2y_0&=5\lambda\\ 96-2x_0-18y_0&=\lambda.\end{aligned}$$

The equation $g(x_0,y_0)=0$ becomes $5x_0+y_0-54=0$. Therefore, the system of equations that needs to be solved is

$$\begin{aligned}48-2x_0-2y_0&=5\lambda\\ 96-2x_0-18y_0&=\lambda\\ 5x_0+y_0-54&=0.\end{aligned}$$

3. We use the left-hand side of the second equation to replace λ in the first equation:

$$\begin{aligned} 48-2x_0-2y_0 &= 5(96-2x_0-18y_0) \\ 48-2x_0-2y_0 &= 480-10x_0-90y_0 \\ 8x_0 &= 432-88y_0 \\ x_0 &= 54-11y_0. \end{aligned}$$

Then we substitute this into the third equation:

$$\begin{aligned} 5(54-11y_0)+y_0-54 &= 0 \\ 270-55y_0+y_0-54 &= 0 \\ 216-54y_0 &= 0 \\ y_0 &= 4. \end{aligned}$$

Since $x_0 = 54-11y_0$, this gives $x_0 = 10$.

4. We then substitute $(10,4)$ into $f(x,y) = 48x+96y-x^2-2xy-9y^2$, which gives

$$\begin{aligned} f(10,4) &= 48(10)+96(4)-(10)^2-2(10)(4)-9(4)^2 \\ &= 480+384-100-80-144 \\ &= 540. \end{aligned}$$

Therefore the maximum profit that can be attained, subject to budgetary constraints, is $\$540,000$ with a production level of $10,000$ golf balls and 4 hours of advertising bought per month. Let's check to make sure this truly is a maximum. The endpoints of the line that defines the constraint are $(10.8,0)$ and $(0,54)$ Let's evaluate f at both of these points:

$$\begin{aligned} f(10.8,0) &= 48(10.8)+96(0)-10.8^2-2(10.8)(0)-9(0^2) \\ &= 401.76 \\ f(0,54) &= 48(0)+96(54)-0^2-2(0)(54)-9(54^2) \\ &= -21,060. \end{aligned}$$

The second value represents a loss, since no golf balls are produced. Neither of these values exceed 540, so it seems that our extremum is a maximum value of f, subject to the given constraint.

Exercise 5.9.2: Optimizing the Cobb-Douglas function

A company has determined that its production level is given by the Cobb-Douglas function $f(x,y) = 2.5x^{0.45}y^{0.55}$ where x represents the total number of labor hours in 1 year and y represents the total capital input for the company. Suppose 1 unit of labor costs $\$40$ and 1 unit of capital costs $\$50$. Use the method of Lagrange multipliers to find the maximum value of $f(x,y) = 2.5x^{0.45}y^{0.55}$ subject to a budgetary constraint of $\$500,000$ per year.

Hint

Use the problem-solving strategy for the method of Lagrange multipliers.

Answer

Subject to the given constraint, a maximum production level of 13890 occurs with 5625 labor hours and $\$5500$ of total capital input.

In the case of an objective function with three variables and a single constraint function, it is possible to use the method of Lagrange multipliers to solve an optimization problem as well. An example of an objective function with three variables could be the **Cobb-Douglas function** in Exercise 5.9.2: $f(x,y,z) = x^{0.2}y^{0.4}z^{0.4}$, where x represents the cost of labor, y represents capital input, and z represents the cost of advertising. The method is the same as for the method with a function of two variables; the equations to be solved are

$$\vec{\nabla} f(x,y,z) = \lambda \vec{\nabla} g(x,y,z)$$
$$g(x,y,z) = 0.$$

Example 5.9.3: Lagrange Multipliers with a Three-Variable objective function

Maximize the function $f(x,y,z) = x^2 + y^2 + z^2$ subject to the constraint $x + y + z = 1$.

Solution

1. The objective function is $f(x,y,z) = x^2 + y^2 + z^2$. To determine the constraint function, we subtract 1 from each side of the constraint: $x + y + z - 1 = 0$ which gives the constraint function as $g(x,y,z) = x + y + z - 1$.

2. Next, we calculate $\vec{\nabla} f(x,y,z)$ and $\vec{\nabla} g(x,y,z)$:

$$\vec{\nabla} f(x,y,z) = \langle 2x, 2y, 2z \rangle$$
$$\vec{\nabla} g(x,y,z) = \langle 1, 1, 1 \rangle.$$

This leads to the equations

$$\langle 2x_0, 2y_0, 2z_0 \rangle = \lambda \langle 1, 1, 1 \rangle$$
$$x_0 + y_0 + z_0 - 1 = 0$$

which can be rewritten in the following form:

$$2x_0 = \lambda$$
$$2y_0 = \lambda$$
$$2z_0 = \lambda$$
$$x_0 + y_0 + z_0 - 1 = 0.$$

3. Since each of the first three equations has λ on the right-hand side, we know that $2x_0 = 2y_0 = 2z_0$ and all three variables are equal to each other. Substituting $y_0 = x_0$ and $z_0 = x_0$ into the last equation yields $3x_0 - 1 = 0$, so $x_0 = \frac{1}{3}$ and $y_0 = \frac{1}{3}$ and $z_0 = \frac{1}{3}$ which corresponds to a critical point on the constraint curve.

4. Then, we evaluate f at the point $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$:

$$f\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 = \frac{3}{9} = \frac{1}{3}$$

Therefore, a possible extremum of the function is $\frac{1}{3}$. To verify it is a minimum, choose other points that satisfy the constraint from either side of the point we obtained above and calculate f at those points. For example,

$$f(1,0,0) = 1^2 + 0^2 + 0^2 = 1$$
$$f(0,-2,3) = 0^2 + +(-2)^2 + 3^2 = 13.$$

Both of these values are greater than $\frac{1}{3}$, leading us to believe the extremum is a minimum, subject to the given constraint.



Exercise 5.9.3

Use the method of Lagrange multipliers to find the minimum value of the function

$$f(x,y,z)=x+y+z$$

subject to the constraint $x^2+y^2+z^2=1$.

Hint

Use the problem-solving strategy for the method of Lagrange multipliers with an objective function of three variables.

Answer

Evaluating f at both points we obtained, gives us,

$$f\left(\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3}\right)=\frac{\sqrt{3}}{3}+\frac{\sqrt{3}}{3}+\frac{\sqrt{3}}{3}=\sqrt{3}$$

$$f\left(-\frac{\sqrt{3}}{3},-\frac{\sqrt{3}}{3},-\frac{\sqrt{3}}{3}\right)=-\frac{\sqrt{3}}{3}-\frac{\sqrt{3}}{3}-\frac{\sqrt{3}}{3}=-\sqrt{3}$$

Since the constraint is continuous, we compare these values and conclude that f has a relative minimum of $-\sqrt{3}$ at the point $\left(-\frac{\sqrt{3}}{3},-\frac{\sqrt{3}}{3},-\frac{\sqrt{3}}{3}\right)$, subject to the given constraint.

Problems with Two Constraints

The method of Lagrange multipliers can be applied to problems with more than one constraint. In this case the objective function, w is a function of three variables:

$$w=f(x,y,z) \tag{5.9.4}$$

and it is subject to two constraints:

$$g(x,y,z)=0 \text{ and } h(x,y,z)=0. \tag{5.9.5}$$

There are two Lagrange multipliers, λ_1 and λ_2, and the system of equations becomes

$$\begin{aligned}\vec{\nabla} f(x_0,y_0,z_0) &= \lambda_1 \vec{\nabla} g(x_0,y_0,z_0) + \lambda_2 \vec{\nabla} h(x_0,y_0,z_0)\\ g(x_0,y_0,z_0) &= 0\\ h(x_0,y_0,z_0) &= 0\end{aligned}$$

Example 5.9.4: Lagrange Multipliers with Two Constraints

Find the maximum and minimum values of the function

$$f(x,y,z) = x^2+y^2+z^2$$

subject to the constraints $z^2 = x^2+y^2$ and $x+y-z+1=0$.

Solution

Let's follow the problem-solving strategy:

1. The objective function is $f(x,y,z) = x^2+y^2+z^2$. To determine the constraint functions, we first subtract z^2 from both sides of the first constraint, which gives $x^2+y^2-z^2=0$, so $g(x,y,z) = x^2+y^2-z^2$. The second constraint function is $h(x,y,z) = x+y-z+1$.
2. We then calculate the gradients of f, g, and h:

$$\begin{aligned}\vec{\nabla} f(x,y,z) &= 2x\,\hat{\mathbf{i}} + 2y\hat{\mathbf{j}} + 2z\hat{\mathbf{k}}\\ \vec{\nabla} g(x,y,z) &= 2x\,\hat{\mathbf{i}} + 2y\hat{\mathbf{j}} - 2z\hat{\mathbf{k}}\\ \vec{\nabla} h(x,y,z) &= \hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}.\end{aligned}$$

   The equation $\vec{\nabla} f(x_0,y_0,z_0) = \lambda_1 \vec{\nabla} g(x_0,y_0,z_0) + \lambda_2 \vec{\nabla} h(x_0,y_0,z_0)$ becomes

$$2x_0\,\hat{\mathbf{i}} + 2y_0\hat{\mathbf{j}} + 2z_0\hat{\mathbf{k}} = \lambda_1(2x_0\,\hat{\mathbf{i}} + 2y_0\hat{\mathbf{j}} - 2z_0\hat{\mathbf{k}}) + \lambda_2(\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}),$$

   which can be rewritten as

$$2x_0\,\hat{\mathbf{i}} + 2y_0\hat{\mathbf{j}} + 2z_0\hat{\mathbf{k}} = (2\lambda_1 x_0 + \lambda_2)\hat{\mathbf{i}} + (2\lambda_1 y_0 + \lambda_2)\hat{\mathbf{j}} - (2\lambda_1 z_0 + \lambda_2)\hat{\mathbf{k}}.$$

   Next, we set the coefficients of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ equal to each other:

$$\begin{aligned}2x_0 &= 2\lambda_1 x_0 + \lambda_2\\ 2y_0 &= 2\lambda_1 y_0 + \lambda_2\\ 2z_0 &= -2\lambda_1 z_0 - \lambda_2.\end{aligned}$$

   The two equations that arise from the constraints are $z_0^2 = x_0^2+y_0^2$ and $x_0+y_0-z_0+1=0$. Combining these equations with the previous three equations gives

$$\begin{aligned}2x_0 &= 2\lambda_1 x_0 + \lambda_2\\ 2y_0 &= 2\lambda_1 y_0 + \lambda_2\\ 2z_0 &= -2\lambda_1 z_0 - \lambda_2\\ z_0^2 &= x_0^2+y_0^2\\ x_0+y_0-z_0+1 &= 0.\end{aligned}$$

3. The first three equations contain the variable λ_2. Solving the third equation for λ_2 and replacing into the first and second equations reduces the number of equations to four:

$$\begin{aligned}2x_0 &= 2\lambda_1 x_0 - 2\lambda_1 z_0 - 2z_0\\ 2y_0 &= 2\lambda_1 y_0 - 2\lambda_1 z_0 - 2z_0\\ z_0^2 &= x_0^2+y_0^2\\ x_0+y_0-z_0+1 &= 0.\end{aligned}$$

Next, we solve the first and second equation for λ_1. The first equation gives $\lambda_1 = \dfrac{x_0+z_0}{x_0-z_0}$, the second equation gives $\lambda_1 = \dfrac{y_0+z_0}{y_0-z_0}$. We set the right-hand side of each equation equal to each other and cross-multiply:

$$\begin{aligned}
\frac{x_0+z_0}{x_0-z_0} &= \frac{y_0+z_0}{y_0-z_0} \\
(x_0+z_0)(y_0-z_0) &= (x_0-z_0)(y_0+z_0) \\
x_0y_0 - x_0z_0 + y_0z_0 - z_0^2 &= x_0y_0 + x_0z_0 - y_0z_0 - z_0^2 \\
2y_0z_0 - 2x_0z_0 &= 0 \\
2z_0(y_0-x_0) &= 0.
\end{aligned}$$

Therefore, either $z_0 = 0$ or $y_0 = x_0$. If $z_0 = 0$, then the first constraint becomes $0 = x_0^2 + y_0^2$. The only real solution to this equation is $x_0 = 0$ and $y_0 = 0$, which gives the ordered triple $(0,0,0)$. This point does not satisfy the second constraint, so it is not a solution. Next, we consider $y_0 = x_0$, which reduces the number of equations to three:

$$\begin{aligned}
y_0 &= x_0 \\
z_0^2 &= x_0^2 + y_0^2 \\
x_0 + y_0 - z_0 + 1 &= 0.
\end{aligned}$$

We substitute the first equation into the second and third equations:

$$\begin{aligned}
z_0^2 &= x_0^2 + x_0^2 \\
&= x_0 + x_0 - z_0 + 1 \qquad = 0.
\end{aligned}$$

Then, we solve the second equation for z_0, which gives $z_0 = 2x_0 + 1$. We then substitute this into the first equation,

$$\begin{aligned}
z_0^2 &= 2x_0^2 \\
(2x_0^2+1)^2 &= 2x_0^2 \\
4x_0^2 + 4x_0 + 1 &= 2x_0^2 \\
2x_0^2 + 4x_0 + 1 &= 0,
\end{aligned}$$

and use the quadratic formula to solve for x_0:

$$x_0 = \frac{-4 \pm \sqrt{4^2 - 4(2)(1)}}{2(2)} = \frac{-4 \pm \sqrt{8}}{4} = \frac{-4 \pm 2\sqrt{2}}{4} = -1 \pm \frac{\sqrt{2}}{2}.$$

Recall $y_0 = x_0$, so this solves for y_0 as well. Then, $z_0 = 2x_0 + 1$, so

$$z_0 = 2x_0 + 1 = 2\left(-1 \pm \frac{\sqrt{2}}{2}\right) + 1 = -2 + 1 \pm \sqrt{2} = -1 \pm \sqrt{2}.$$

Therefore, there are two ordered triplet solutions:

$$\left(-1+\frac{\sqrt{2}}{2}, -1+\frac{\sqrt{2}}{2}, -1+\sqrt{2}\right) \text{ and } \left(-1-\frac{\sqrt{2}}{2}, -1-\frac{\sqrt{2}}{2}, -1-\sqrt{2}\right).$$

4. We substitute $\left(-1+\dfrac{\sqrt{2}}{2}, -1+\dfrac{\sqrt{2}}{2}, -1+\sqrt{2}\right)$ into $f(x,y,z) = x^2+y^2+z^2$, which gives

$$\begin{aligned}
f\left(-1+\frac{\sqrt{2}}{2}, -1+\frac{\sqrt{2}}{2}, -1+\sqrt{2}\right) &= \left(-1+\frac{\sqrt{2}}{2}\right)^2 + \left(-1+\frac{\sqrt{2}}{2}\right)^2 + (-1+\sqrt{2})^2 \\
&= \left(1-\sqrt{2}+\frac{1}{2}\right) + \left(1-\sqrt{2}+\frac{1}{2}\right) + (1-2\sqrt{2}+2) \\
&= 6 - 4\sqrt{2}.
\end{aligned}$$

Then, we substitute $\left(-1-\frac{\sqrt{2}}{2},-1+\frac{\sqrt{2}}{2},-1+\sqrt{2}\right)$ into $f(x,y,z)=x^2+y^2+z^2$, which gives

$$\begin{aligned} f\left(-1-\frac{\sqrt{2}}{2},-1+\frac{\sqrt{2}}{2},-1+\sqrt{2}\right) &= \left(-1-\frac{\sqrt{2}}{2}\right)^2+\left(-1-\frac{\sqrt{2}}{2}\right)^2+(-1-\sqrt{2})^2 \\ &= \left(1+\sqrt{2}+\frac{1}{2}\right)+\left(1+\sqrt{2}+\frac{1}{2}\right)+(1+2\sqrt{2}+2) \\ &= 6+4\sqrt{2}. \end{aligned}$$

$6+4\sqrt{2}$ is the maximum value and $6-4\sqrt{2}$ is the minimum value of $f(x,y,z)$, subject to the given constraints.



Exercise 5.9.4

Use the method of Lagrange multipliers to find the minimum value of the function

$$f(x,y,z)=x^2+y^2+z^2$$

subject to the constraints $2x+y+2z=9$ and $5x+5y+7z=29$.

Hint

Use the problem-solving strategy for the method of Lagrange multipliers with two constraints.

Answer

$f(2,1,2)=9$ is a minimum value of f, subject to the given constraints.

Key Concepts

- An objective function combined with one or more constraints is an example of an optimization problem.
- To solve optimization problems, we apply the method of Lagrange multipliers using a four-step problem-solving strategy.

Key Equations

- **Method of Lagrange multipliers, one constraint**

  $\vec{\nabla}f(x_0,y_0)=\lambda\vec{\nabla}g(x_0,y_0)$

  $g(x_0,y_0)=0$

- **Method of Lagrange multipliers, two constraints**

  $$\vec{\nabla} f(x_0, y_0, z_0) = \lambda_1 \vec{\nabla} g(x_0, y_0, z_0) + \lambda_2 \vec{\nabla} h(x_0, y_0, z_0)$$
  $$g(x_0, y_0, z_0) = 0$$
  $$h(x_0, y_0, z_0) = 0$$

Glossary

constraint
: an inequality or equation involving one or more variables that is used in an optimization problem; the constraint enforces a limit on the possible solutions for the problem

Lagrange multiplier
: the constant (or constants) used in the method of Lagrange multipliers; in the case of one constant, it is represented by the variable λ

method of Lagrange multipliers
: a method of solving an optimization problem subject to one or more constraints

objective function
: the function that is to be maximized or minimized in an optimization problem

optimization problem
: calculation of a maximum or minimum value of a function of several variables, often using Lagrange multipliers

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5.9E: Exercises for Section 5.9

In exercises 1-15, use the method of Lagrange multipliers to find the maximum and minimum values of the function subject to the given constraint.

1) Objective function: $f(x,y)=4xy$ Constraint: $\dfrac{x^2}{9}+\dfrac{y^2}{16}=1$

Answer

Subject to the given constraint, the function f has a relative minimum of -24 at both $\left(-\frac{3\sqrt{2}}{2},2\sqrt{2}\right)$ and $\left(\frac{3\sqrt{2}}{2},-2\sqrt{2}\right)$ and a relative maximum of 24 at both $\left(\frac{3\sqrt{2}}{2},2\sqrt{2}\right)$ and $\left(-\frac{3\sqrt{2}}{2},-2\sqrt{2}\right)$

2) Objective function: $f(x,y)=x^2y$ Constraint: $x^2+2y^2=6$

3) Objective function: $f(x,y)=x^2+y^2+2x-2y+1$ Constraint: $g(x,y)=x^2+y^2=2$

Answer

Subject to the given constraint, f has a relative minimum of -1 at $(-1,1)$ and a relative maximum of 7 at $(1,-1)$.

4) Objective function: $f(x,y)=xy$ Constraint: $4x^2+8y^2=16$

5) Objective function: $f(x,y)=x^2+y^2$ Constraint: $xy=1$

Answer

f has a relative minimum of 2 at both $(-1,-1)$ and $(1,1)$, subject to the given constraint.

6) Objective function: $f(x,y)=x^2-y^2$ Constraint: $x-2y+6=0$

7) Objective function: $f(x,y)=x^2+y^2$ Constraint: $x+2y-5=0$

Answer

Subject to the given constraint, f has a relative minimum of $f(1,2)=5$ at the point $(1,2)$.

8) Objective function: $f(x,y)=x^2+y^2$ Constraint: $(x-1)^2+4y^2=4$

9) Objective function: $f(x,y)=4x^3+y^2$ Constraint: $2x^2+y^2=1$

Answer

Subject to the given constraint, the function f has a relative minimum of $-\sqrt{2}$ at $\left(-\frac{\sqrt{2}}{2},0\right)$,
a relative minimum of $\frac{25}{27}$ at both points $\left(\frac{1}{3},-\frac{\sqrt{7}}{3}\right)$ and $\left(\frac{1}{3},\frac{\sqrt{7}}{3}\right)$,
a relative maximum of $\sqrt{2}$ at $\left(\frac{\sqrt{2}}{2},0\right)$, and a relative maximum of 1 at both points $(0,1)$ and $(0,-1)$.

Solution:

Let $g(x,y)=2x^2+y^2$ be the constraint function. Then:

$\vec{\nabla}f(x,y)=12x^2\,\hat{\mathbf{i}}+2y\,\hat{\mathbf{j}}$ and $\vec{\nabla}g(x,y)=4x\,\hat{\mathbf{i}}+2y\,\hat{\mathbf{j}}$

Using the Lagrange Multiplier equation,

$$\vec{\nabla}f(x,y)=\lambda\vec{\nabla}g(x,y),$$

we have:

$$12x^2\,\hat{\mathbf{i}}+2y\,\hat{\mathbf{j}}=4x\lambda\,\hat{\mathbf{i}}+2y\lambda\,\hat{\mathbf{j}}$$

giving us the system of equations:

$$12x^2 = 4x\lambda, \quad 2y = 2y\lambda, \quad \text{and the constraint} \quad 2x^2 + y^2 = 1$$

Rewriting the first two equations as zero-products (by moving to one-side and factoring), we get:

$$\begin{array}{ccc} 4x(3x-\lambda) = 0 & \text{and} & 2y(1-\lambda) = 0 \\ x = 0 \quad \text{or} \quad \lambda = 3x & & y = 0 \quad \text{or} \quad \lambda = 1 \end{array}$$

Now we consider the combinations of these solutions to the above two equations and plug each of these into the constraint equation to solve for the corresponding Lagrange points.

The combination $x = 0$ and $y = 0$ produces a contradiction when placed in the constraint equation, since this point is not on the ellipse.

Taking the combination $x = 0$ and $\lambda = 1$, we put 0 in for x in the constraint and solve for y, obtaining: $y = \pm 1$. This gives us two Lagrange points: $(0, 1)$ and $(0, -1)$.

Taking the combination $\lambda = 3x$ and $y = 0$, we put 0 in for y in the constraint and solve for x, obtaining: $x = \pm\frac{\sqrt{2}}{2}$. This gives us two Lagrange points: $\left(-\frac{\sqrt{2}}{2}, 0\right)$ and $\left(\frac{\sqrt{2}}{2}, 0\right)$.

Taking the combination $\lambda = 3x$ and $\lambda = 1$, we substitute 1 into the first equation for λ, giving us $1 = 3x$ so $x = \frac{1}{3}$. Plugging this value in for x in the constraint equation and solving for y, we obtain $y = \pm\frac{\sqrt{7}}{3}$ which gives us the two Lagrange points: $\left(\frac{1}{3}, -\frac{\sqrt{7}}{3}\right)$ and $\left(\frac{1}{3}, \frac{\sqrt{7}}{3}\right)$.

Evaluating the function f at these Lagrange points, we find:

$$\begin{array}{ccc} f(0,-1) = 1 & & f(0,1) = 1 \\ f\left(-\frac{\sqrt{2}}{2}, 0\right) = \frac{-4(\sqrt{2})^3}{8} = -\sqrt{2} & & f\left(\frac{\sqrt{2}}{2}, 0\right) = \frac{4(\sqrt{2})^3}{8} = \sqrt{2} \\ f\left(\frac{1}{3}, -\frac{\sqrt{7}}{3}\right) = \frac{25}{27} & & f\left(\frac{1}{3}, \frac{\sqrt{7}}{3}\right) = \frac{25}{27} \end{array}$$

Comparing these values with where the corresponding Lagrange points lie on the constraint curve, we conclude the results stated in the answer above.

10) Objective function: $f(x, y) = 2x^2 + y^2$ Constraint: $g(x, y) = x^2 + y^2 = 1$

11) Objective function: $f(x, y, z) = x + 3y - z$ Constraint: $x^2 + y^2 + z^2 = 4$

Answer

Subject to the given constraint, f has a relative minimum of $-2\sqrt{11}$ at the point $\left(-\frac{2\sqrt{11}}{11}, -\frac{6\sqrt{11}}{11}, \frac{2\sqrt{11}}{11}\right)$ and a relative maximum of $2\sqrt{11}$ at the point $\left(\frac{2\sqrt{11}}{11}, \frac{6\sqrt{11}}{11}, -\frac{2\sqrt{11}}{11}\right)$.

12) Objective function: $f(x, y, z) = x + y + z$ Constraint: $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = 1$

13) Objective function: $f(x, y, z) = xyz$ Constraint: $x^2 + 2y^2 + 3z^2 = 6$

Answer

Subject to the given constraint, f has a relative minimum of $-\frac{2\sqrt{3}}{3}$ at $\left(\sqrt{2},\ 1,\ -\frac{\sqrt{6}}{3}\right)$, $\left(\sqrt{2},\ -1,\ \frac{\sqrt{6}}{3}\right)$, $\left(-\sqrt{2},\ 1,\ \frac{\sqrt{6}}{3}\right)$, and $\left(-\sqrt{2},\ -1,\ -\frac{\sqrt{6}}{3}\right)$ and a relative maximum of $\frac{2\sqrt{3}}{3}$ at $\left(\sqrt{2},\ 1,\ \frac{\sqrt{6}}{3}\right)$, $\left(\sqrt{2},\ -1,\ -\frac{\sqrt{6}}{3}\right)$, $\left(-\sqrt{2},\ -1,\ \frac{\sqrt{6}}{3}\right)$, and $\left(-\sqrt{2},\ 1,\ -\frac{\sqrt{6}}{3}\right)$.

14) Objective function: $f(x,y,z)=x^2+y^2+z^2$ Constraint: $x^4+y^4+z^4=1$

15) Objective function: $f(x,y,z)=x^2+y^2+z^2$ Constraint: $xyz=4$

Answer

Subject to the given constraint, f has a relative minimum of $6\sqrt[3]{2}$ at the points $\left(\sqrt[3]{4},\ \sqrt[3]{4},\ \sqrt[3]{4}\right)$, $\left(\sqrt[3]{4},\ -\sqrt[3]{4},\ -\sqrt[3]{4}\right)$, $\left(-\sqrt[3]{4},\ \sqrt[3]{4},\ -\sqrt[3]{4}\right)$, and $\left(-\sqrt[3]{4},\ -\sqrt[3]{4},\ \sqrt[3]{4}\right)$.
To see a 3D visualization of this problem, see: CalcPlot3D for Problem 15.

In exercises 16-21, use the method of Lagrange multipliers to find the requested extremum of the given function subject to the given constraint.

16) Maximize $f(x,y)=\sqrt{6-x^2-y^2}$ subject to the constraint, $x+y-2=0$.

17) Maximize $f(x,y)=x^2-y^2$ subject to the constraints, $g(x,y)=y-x^2=0,\quad x>0,\quad y>0$.

Answer

Subject to the given constraints, f has a relative maximum of $f\left(\frac{\sqrt{2}}{2},\ \frac{1}{2}\right)=\frac{1}{4}$ at the point $\left(\frac{\sqrt{2}}{2},\ \frac{1}{2}\right)$. If $x>0$ were not a constraint, there would have been two other Lagrange points with relative extrema of f subject to the other two constraints. These would have been $(0,0)$ and $\left(-\frac{\sqrt{2}}{2},\ \frac{1}{2}\right)$.

To verify that f really has a relative **maximum** at the point $\left(\frac{\sqrt{2}}{2},\ \frac{1}{2}\right)$, we would need to check the value of f on either side of this point on the constraint curve, $y-x^2=0$.

If $x=0.5$ which is less than $\frac{\sqrt{2}}{2}$, y would be $y=(0.5)^2=0.25$.
If $x=1$ which is greater than $\frac{\sqrt{2}}{2}$, y would be $y=(1)^2=1$.

Then we compare the value of f at the Lagrange point, $\left(\frac{\sqrt{2}}{2},\ \frac{1}{2}\right)$, with the values of f at these other points on the constraint.
We have $f(0.5,0.25)=(0.5)^2-(0.25)^2=0.25-0.0625=0.1875<\frac{1}{4}$ and $f(1,1)=(1)^2-(1)^2=0<\frac{1}{4}$.

Therefore, we can conclude that f indeed has a relative maximum of $\frac{1}{4}$ at the point $\left(\frac{\sqrt{2}}{2},\ \frac{1}{2}\right)$.

18) Maximize $U(x,y)=8x^{4/5}y^{1/5}$ subject to the constraint, $4x+2y=12$.

19) Minimize $f(x,y,z)=x^2+y^2+z^2$ subject to the constraint, $x+y+z=1$.

Answer

Subject to the given constraint, f has a relative minimum of $f\left(\frac{1}{3},\ \frac{1}{3},\ \frac{1}{3}\right)=\frac{1}{3}$ at the point $\left(\frac{1}{3},\ \frac{1}{3},\ \frac{1}{3}\right)$.

20) Minimize $f(x,y)=xy$ on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

21) Maximize $f(x,y,z)=2x+3y+5z$ on the sphere $x^2+y^2+z^2=19$.

Answer

Subject to the given constraint, f has a relative maximum of $19\sqrt{2}$ at the point $\left(\sqrt{2},\ \frac{3\sqrt{2}}{2},\ \frac{5\sqrt{2}}{2}\right)$.

Note that, subject to this constraint, f also has a relative minimum of $-19\sqrt{2}$ at the point $\left(-\sqrt{2},\ -\frac{3\sqrt{2}}{2},\ -\frac{5\sqrt{2}}{2}\right)$.

To see a 3D visualization of this problem, see: CalcPlot3D for Problem 21.

In exercises 22-23, use the method of Lagrange multipliers with two constraints.

22) Optimize $f(x,y,z)=yz+xy$ subject to the constraints: $xy=1,\quad y^2+z^2=1$.

Answer

maximum: $\frac{3}{2}$, minimum: $\frac{1}{2}$

23) Minimize $f(x,y,z)=x^2+y^2+z^2$ when $x+y+z=9$ and $x+2y+3z=20$.

Answer

minimum: $f(2,3,4)=29$

Use the method of Lagrange multipliers to solve the following applied problems.

24) A large container in the shape of a rectangular solid must have a volume of 480 m^3. The bottom of the container costs \$5/m^2 to construct whereas the top and sides cost \$3/m^2 to construct. Use Lagrange multipliers to find the dimensions of the container of this size that has the minimum cost.

25) A rectangular box without a top (a topless box) is to be made from 12 ft^2 of cardboard. Find the maximum volume of such a box.

Answer

The maximum volume is 4 ft^3. The dimensions are $1\times 2\times 2$ ft.

26) Find the minimum distance from the parabola $y=x^2$ to point $(0,3)$.

27) Find the point on the line $y=2x+3$ that is closest to point $(4,2)$.

Answer

$(25,195)$

29) Find the minimum distance from point $(0,1)$ to the parabola $x^2=4y$.

Answer

1.0 unit

30) Find the minimum and maximum distances between the ellipse $x^2+xy+2y^2=1$ and the origin.

31) Find the minimum distance from the plane $x+y+z=1$ to point $(2,1,1)$.

Answer

$\sqrt{3}$ units

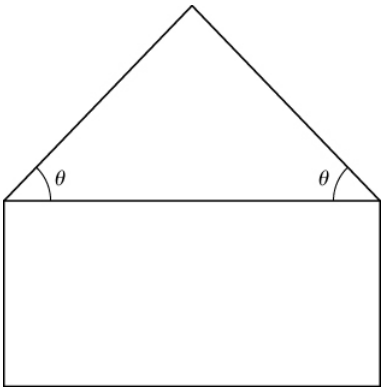
32) Find the point on the plane $4x+3y+z=2$ that is closest to the point $(1,-1,1)$.

33) Find the point on the surface $x^2-2xy+y^2-x+y=0$ closest to the point $(1,2,-3)$.

Answer

$\left(1,\ \frac{1}{2},\ -3\right)$

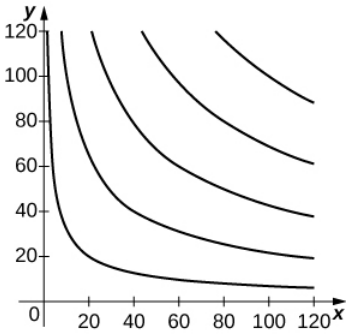
34) A pentagon is formed by placing an isosceles triangle on a rectangle, as shown in the diagram. If the perimeter of the pentagon is 10 in., find the lengths of the sides of the pentagon that will maximize the area of the pentagon.



35) **[T]** By investing x units of labor and y units of capital, a watch manufacturer can produce $P(x,y) = 50x^{0.4}y^{0.6}$ watches. Find the maximum number of watches that can be produced on a budget of $20,000 if labor costs $100/unit and capital costs $200/unit. Use a grapher like CalcPlot3D to sketch a contour plot of the function.

Answer

Roughly 3365 watches at the critical point $(80, 60)$.

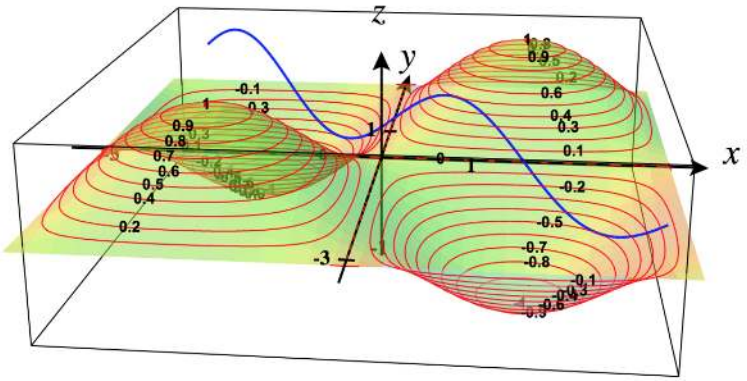


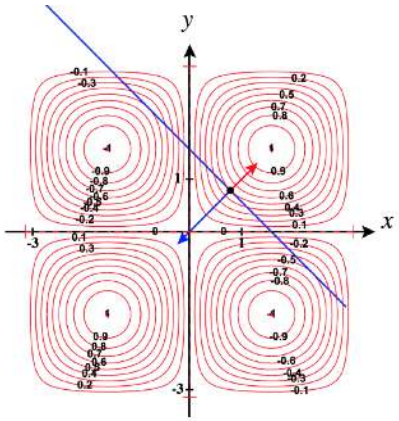
36) A rectangular solid is contained within a tetrahedron with vertices at $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, and the origin. The base of the box has dimensions x and y, and the height of the box is z. If the sum of x, y, and z is 1, find the dimensions that maximizes the volume of the rectangular solid.

37) Find the maximum value of $f(x,y) = \sin x \sin y$, where x and y denote the acute angles of a right triangle. Draw the surface plot and contour plot of the function using a CAS.

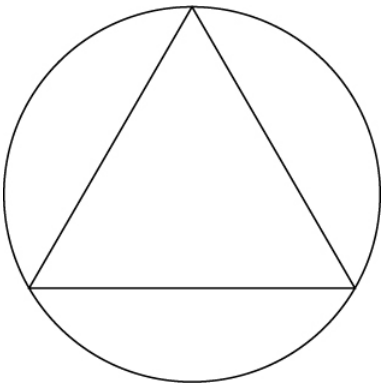
Answer

Subject to this constraint, f has a relative maximum of $\frac{1}{2}$ when $x = \frac{\pi}{4}$ and $y = \frac{\pi}{4}$.
Surface plot and contour plot for f:





38) Show that, of all the triangles inscribed in a circle of radius R (see diagram), the equilateral triangle has the largest perimeter.



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- Paul Seeburger (Monroe Community College) reordered these problems, adding problems 3 and 10 and answers for problems 15 and 17. He also added a full worked-out solution for problem 9 and a link to CalcPlot3D in problems 15, 21 and 35. He also created new images for Problem 37 and expanded the answers for many problems.

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5.10: Chapter 5 Review Exercises

For the following exercises, determine whether the statement is *true or false*. Justify your answer with a proof or a counterexample.

1. The domain of $f(x,y) = x^3 \arcsin(y)$ is $\{(x,y) \mid x \in \mathbb{R} \text{ and } -\pi \leq y \leq \pi\}$.

2. If the function $f(x,y)$ is continuous everywhere, then $f_{xy}(x,y) = f_{yx}(x,y)$.

Answer

True, by Clairaut's theorem

3. The linear approximation to the function of $f(x,y) = 5x^2 + x\tan y$ at the point $(2,\pi)$ is given by $L(x,y) = 22 + 21(x-2) + (y-\pi)$.

4. $(34, 916)$ is a critical point of $g(x,y) = 4x^3 - 2x^2y + y^2 - 2$.

Answer

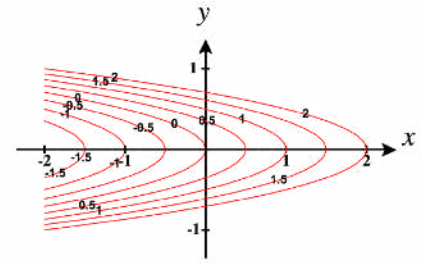
False

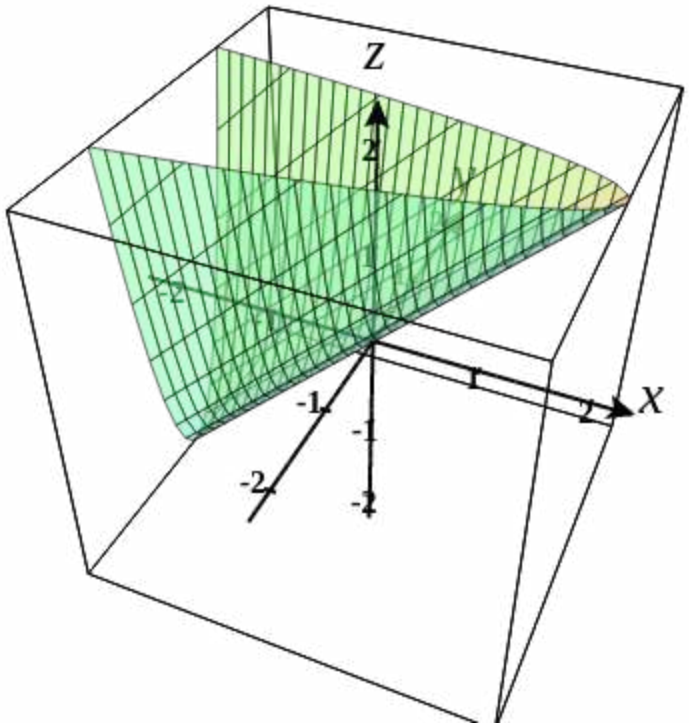
For the following exercises, sketch the function in one graph and, in a second, sketch several level curves.

5. $f(x,y) = e^{-(x^2+2y^2)}$

6. $f(x,y) = x + 4y^2$

Answer





For the following exercises, evaluate the following limits, if they exist. If they do not exist, prove it.

7. $\lim\limits_{(x,y)\to(1,1)} \dfrac{4xy}{x-2y^2}$

8. $\lim\limits_{(x,y)\to(0,0)} \dfrac{4xy}{x-2y^2}$

Answer

Does not exist.

For the following exercises, find the largest interval of continuity for the function.

9. $f(x,y)=x^3 \arcsin y$

10. $g(x,y)=\ln(4-x^2-y^2)$

Answer

Continuous at all points on the xy-plane, except where $x^2+y^2>4$.

For the following exercises, find all first partial derivatives.

11. $f(x,y)=x^2-y^2$

12. $u(x,y)=x^4-3xy+1,$ with $x=2t$ and $y=t^3$

Answer

$$\frac{\partial u}{\partial x}=4x^3-3y,$$

$$\frac{\partial u}{\partial y}=-3x,$$

$$\frac{dx}{dt} = 2 \text{ and } \frac{dy}{dt} = 3t^2$$

$$\begin{aligned}\frac{du}{dt} &= \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} \\ &= 8x^3 - 6y - 9xt^2 \\ &= 8(2t)^3 - 6(t^3) - 9(2t)t^2 \\ &= 64t^3 - 6t^3 - 18t^3 \\ &= 40t^3\end{aligned}$$

For the following exercises, find all second partial derivatives.

13. $g(t, x) = 3t^2 - \sin(x + t)$

14. $h(x, y, z) = \dfrac{x^3 e^{2y}}{z}$

Answer

$$\begin{aligned}h_{xx}(x, y, z) &= \frac{6xe^{2y}}{z}, \\ h_{xy}(x, y, z) &= \frac{6x^2 e^{2y}}{z}, \\ h_{xz}(x, y, z) &= -\frac{3x^2 e^{2y}}{z^2}, \\ h_{yx}(x, y, z) &= \frac{6x^2 e^{2y}}{z}, \\ h_{yy}(x, y, z) &= \frac{4x^3 e^{2y}}{z}, \\ h_{yz}(x, y, z) &= -\frac{2x^3 e^{2y}}{z^2}, \\ h_{zx}(x, y, z) &= -\frac{3x^2 e^{2y}}{z^2}, \\ h_{zy}(x, y, z) &= -\frac{2x^3 e^{2y}}{z^2}, \\ h_{zz}(x, y, z) &= \frac{2x^3 e^{2y}}{z^3}\end{aligned}$$

For the following exercises, find the equation of the tangent plane to the specified surface at the given point.

15. $z = x^3 - 2y^2 + y - 1$ at point $(1, 1, -1)$

16. $z = e^x + \dfrac{2}{y}$ at point $(0, 1, 3)$

Answer

$z = x - 2y + 5$

17. Approximate $f(x, y) = e^{x^2} + \sqrt{y}$ at $(0.1, 9.1)$. Write down your linear approximation function $L(x, y)$. How accurate is the approximation to the exact answer, rounded to four digits?

18. Find the differential dz of $h(x, y) = 4x^2 + 2xy - 3y$ and approximate Δz at the point $(1, -2)$. Let $\Delta x = 0.1$ and $\Delta y = 0.01$.

Answer

$dz = 4\,dx - dy,\ dz(0.1, 0.01) = 0.39,\ \Delta z = 0.432$

19. Find the directional derivative of $f(x, y) = x^2 + 6xy - y^2$ in the direction $\vec{\mathbf{v}} = \hat{\mathbf{i}} + 4\,\hat{\mathbf{j}}$.

20. Find the maximal directional derivative magnitude and direction for the function $f(x,y)=x^3+2xy-\cos(\pi y)$ at point $(3,0)$.

Answer

$3\sqrt{85}\langle 27,6\rangle$

For the following exercises, find the gradient.

21. $c(x,t)=e(t-x)^2+3\cos t$

22. $f(x,y)=\dfrac{\sqrt{x}+y^2}{xy}$

Answer

$$\vec{\nabla} f(x,y)=-\frac{\sqrt{x}+2y^2}{2x^2y}\,\hat{\mathbf{i}}+\left(\frac{1}{x}+\frac{1}{\sqrt{x}y^2}\right)\hat{\mathbf{j}}$$

For the following exercise, find and classify the critical points.

23. $z=x^3-xy+y^2-1$

For the following exercises, use Lagrange multipliers to find the maximum and minimum values for the functions with the given constraints.

24. $f(x,y)=x^2y$, subject to the constraint: $x^2+y^2=4$

Answer

maximum: $\dfrac{16}{3\sqrt{3}}$, minimum: $-\dfrac{16}{3\sqrt{3}}$,

25. $f(x,y)=x^2-y^2,$ subject to the constraint: $x+6y=4$

26. A machinist is constructing a right circular cone out of a block of aluminum. The machine gives an error of 5% in height and 2% in radius. Find the maximum error in the volume of the cone if the machinist creates a cone of height 6 cm and radius 2 cm.

Answer

2.3228cm^3

27. A trash compactor is in the shape of a cuboid. Assume the trash compactor is filled with incompressible liquid. The length and width are decreasing at rates of 2 ft/sec and 3 ft/sec, respectively. Find the rate at which the liquid level is rising when the length is 14 ft, the width is 10 ft, and the height is 4 ft.

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CHAPTER OVERVIEW

6: Appendices

6.1: A- Table of Derivatives
6.2: B- Table of Integrals

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6.1: A- Table of Derivatives

General Formulas

1. $\frac{d}{dx}(c) = 0$
2. $\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$
3. $\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$
4. $\frac{d}{dx}(x^n) = nx^{n-1}, \quad \text{for real numbers } n$
5. $\frac{d}{dx}(cf(x)) = cf'(x)$
6. $\frac{d}{dx}(f(x) - g(x)) = f'(x) - g'(x)$
7. $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$
8. $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$

Trigonometric Functions

9. $\frac{d}{dx}(\sin x) = \cos x$
10. $\frac{d}{dx}(\tan x) = \sec^2 x$
11. $\frac{d}{dx}(\sec x) = \sec x \tan x$
12. $\frac{d}{dx}(\cos x) = -\sin x$
13. $\frac{d}{dx}(\cot x) = -\csc^2 x$
14. $\frac{d}{dx}(\csc x) = -\csc x \cot x$

Inverse Trigonometric Functions

15. $\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$
16. $\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$
17. $\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{|x|\sqrt{x^2-1}}$
18. $\frac{d}{dx}\left(\cos^{-1} x\right) = \frac{-1}{\sqrt{1-x^2}}$
19. $\frac{d}{dx}\left(\cot^{-1} x\right) = \frac{-1}{1+x^2}$
20. $\frac{d}{dx}\left(\csc^{-1} x\right) = \frac{-1}{|x|\sqrt{x^2-1}}$

Exponential and Logarithmic Functions

21. $\frac{d}{dx}(e^x) = e^x$
22. $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$
23. $\frac{d}{dx}(b^x) = b^x \ln b$
24. $\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}$

Hyperbolic Functions

25. $\frac{d}{dx}(\sinh x) = \cosh x$
26. $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$
27. $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$
28. $\frac{d}{dx}(\cosh x) = \sinh x$
29. $\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$
30. $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$

Inverse Hyperbolic Functions

31. $\frac{d}{dx}\left(\sinh^{-1} x\right) = \frac{1}{\sqrt{x^2+1}}$
32. $\frac{d}{dx}\left(\tanh^{-1} x\right) = \frac{1}{1-x^2} \quad (|x| < 1)$
33. $\frac{d}{dx}\left(\operatorname{sech}^{-1} x\right) = \frac{-1}{x\sqrt{1-x^2}} \quad (0 < x < 1)$
34. $\frac{d}{dx}\left(\cosh^{-1} x\right) = \frac{1}{\sqrt{x^2-1}} \quad (x > 1)$
35. $\frac{d}{dx}\left(\coth^{-1} x\right) = \frac{1}{1-x^2} \quad (|x| > 1)$
36. $\frac{d}{dx}\left(\operatorname{csch}^{-1} x\right) = \frac{-1}{|x|\sqrt{1+x^2}} \quad (x \neq 0)$

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- **Appendix A: Table of Derivatives** by Edwin “Jed” Herman, Gilbert Strang is licensed CC BY-NC-SA 4.0. Original source: https://openstax.org/details/books/calculus-volume-1.

6.2: B- Table of Integrals

Basic Integrals

1. $\int u^n\,du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$
2. $\int \frac{du}{u} = \ln|u| + C$
3. $\int e^u\,du = e^u + C$
4. $\int a^u\,du = \frac{a^u}{\ln a} + C$
5. $\int \sin u\,du = -\cos u + C$
6. $\int \cos u\,du = \sin u + C$
7. $\int \sec^2 u\,du = \tan u + C$
8. $\int \csc^2 u\,du = -\cot u + C$
9. $\int \sec u \tan u\,du = \sec u + C$
10. $\int \csc u \cot u\,du = -\csc u + C$
11. $\int \tan u\,du = \ln|\sec u| + C$
12. $\int \cot u\,du = \ln|\sin u| + C$
13. $\int \sec u\,du = \ln|\sec u + \tan u| + C$
14. $\int \csc u\,du = \ln|\csc u - \cot u| + C$
15. $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$
16. $\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$
17. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{|u|}{a} + C$

Trigonometric Integrals

18. $\int \sin^2 u\,du = \frac{1}{2}u - \frac{1}{4}\sin 2u + C$
19. $\int \cos^2 u\,du = \frac{1}{2}u + \frac{1}{4}\sin 2u + C$
20. $\int \tan^2 u\,du = \tan u - u + C$
21. $\int \cot^2 u\,du = -\cot u - u + C$

22. $\int \sin^3 u\,du = -\frac{1}{3}(2+\sin^2 u)\cos u + C$

23. $\int \cos^3 u\,du = \frac{1}{3}(2+\cos^2 u)\sin u + C$

24. $\int \tan^3 u\,du = \frac{1}{2}\tan^2 u + \ln|\cos u| + C$

25. $\int \cot^3 u\,du = -\frac{1}{2}\cot^2 u - \ln|\sin u| + C$

26. $\int \sec^3 u\,du = \frac{1}{2}\sec u\tan u + \frac{1}{2}\ln|\sec u + \tan u| + C$

27. $\int \csc^3 u\,du = -\frac{1}{2}\csc u\cot u + \frac{1}{2}\ln|\csc u - \cot u| + C$

28. $\int \sin^n u\,du = \frac{-1}{n}\sin^{n-1} u\cos u + \frac{n-1}{n}\int \sin^{n-2} u\,du$

29. $\int \cos^n u\,du = \frac{1}{n}\cos^{n-1} u\sin u + \frac{n-1}{n}\int \cos^{n-2} u\,du$

30. $\int \tan^n u\,du = \frac{1}{n-1}\tan^{n-1} u - \int \tan^{n-2} u\,du$

31. $\int \cot^n u\,du = \frac{-1}{n-1}\cot^{n-1} u - \int \cot^{n-2} u\,du$

32. $\int \sec^n u\,du = \frac{1}{n-1}\tan u\sec^{n-2} u + \frac{n-2}{n-1}\int \sec^{n-2} u\,du$

33. $\int \csc^n u\,du = \frac{-1}{n-1}\cot u\csc^{n-2} u + \frac{n-2}{n-1}\int \csc^{n-2} u\,du$

34. $\int \sin au\sin bu\,du = \frac{\sin(a-b)u}{2(a-b)} - \frac{\sin(a+b)u}{2(a+b)} + C$

35. $\int \cos au\cos bu\,du = \frac{\sin(a-b)u}{2(a-b)} + \frac{\sin(a+b)u}{2(a+b)} + C$

36. $\int \sin au\cos bu\,du = -\frac{\cos(a-b)u}{2(a-b)} - \frac{\cos(a+b)u}{2(a+b)} + C$

37. $\int u\sin u\,du = \sin u - u\cos u + C$

38. $\int u\cos u\,du = \cos u + u\sin u + C$

39. $\int u^n \sin u\,du = -u^n\cos u + n\int u^{n-1}\cos u\,du$

40. $\int u^n \cos u\,du = u^n\sin u - n\int u^{n-1}\sin u\,du$

41. $$\int \sin^n u\cos^m u\,du = -\frac{\sin^{n-1} u\cos^{m+1} u}{n+m} + \frac{n-1}{n+m}\int \sin^{n-2} u\cos^m u\,du$$
$$= \frac{\sin^{n+1} u\cos^{m-1} u}{n+m} + \frac{m-1}{n+m}\int \sin^n u\cos^{m-2} u\,du$$

Exponential and Logarithmic Integrals

42. $\int ue^{au}\,du = \frac{1}{a^2}(au-1)e^{au} + C$

43. $\int u^n e^{au}\,du = \frac{1}{a}u^n e^{au} - \frac{n}{a}\int u^{n-1}e^{au}\,du$

44. $\int e^{au}\sin bu\,du = \frac{e^{au}}{a^2+b^2}(a\sin bu - b\cos bu) + C$

45. $\int e^{au}\cos bu\,du = \frac{e^{au}}{a^2+b^2}(a\cos bu + b\sin bu) + C$

46. $\int \ln u\,du = u\ln u - u + C$

47. $\int u^n \ln u\,du = \frac{u^{n+1}}{(n+1)^2}[(n+1)\ln u - 1] + C$

48. $\int \frac{1}{u\ln u}\,du = \ln|\ln u| + C$

Hyperbolic Integrals

49. $\int \sinh u\,du = \cosh u + C$

50. $\int \cosh u\,du = \sinh u + C$

51. $\int \tanh u\,du = \ln\cosh u + C$

52. $\int \coth u\,du = \ln|\sinh u| + C$

53. $\int \operatorname{sech} u\,du = \tan^{-1}|\sinh u| + C$

54. $\int \operatorname{csch} u\,du = \ln|\tanh\frac{1}{2}u| + C$

55. $\int \operatorname{sech}^2 u\,du = \tanh u + C$

56. $\int \operatorname{csch}^2 u\,du = -\coth u + C$

57. $\int \operatorname{sech} u\tanh u\,du = -\operatorname{sech} u + C$

58. $\int \operatorname{csch} u\coth u\,du = -\operatorname{csch} u + C$

Inverse Trigonometric Integrals

59. $\int \sin^{-1} u\,du = u\sin^{-1} u + \sqrt{1-u^2} + C$

60. $\int \cos^{-1} u\,du = u\cos^{-1} u - \sqrt{1-u^2} + C$

61. $\int \tan^{-1} u\,du = u\tan^{-1} u - \frac{1}{2}\ln(1+u^2) + C$

62. $\int u\sin^{-1} u\,du = \frac{2u^2-1}{4}\sin^{-1} u + \frac{u\sqrt{1-u^2}}{4} + C$

63. $\int u\cos^{-1} u\,du = \frac{2u^2-1}{4}\cos^{-1} u - \frac{u\sqrt{1-u^2}}{4} + C$

64. $\int u\tan^{-1} u\,du = \frac{u^2+1}{2}\tan^{-1} u - \frac{u}{2} + C$

65. $\int u^n \sin^{-1} u\, du = \frac{1}{n+1}\left[u^{n+1}\sin^{-1} u - \int \frac{u^{n+1}\, du}{\sqrt{1-u^2}}\right], \quad n \neq -1$

66. $\int u^n \cos^{-1} u\, du = \frac{1}{n+1}\left[u^{n+1}\cos^{-1} u + \int \frac{u^{n+1}\, du}{\sqrt{1-u^2}}\right], \quad n \neq -1$

67. $\int u^n \tan^{-1} u\, du = \frac{1}{n+1}\left[u^{n+1}\tan^{-1} u - \int \frac{u^{n+1}\, du}{1+u^2}\right], \quad n \neq -1$

Integrals Involving $a^2 + u^2$, $a > 0$

68. $\int \sqrt{a^2+u^2}\, du = \frac{u}{2}\sqrt{a^2+u^2} + \frac{a^2}{2}\ln\left(u+\sqrt{a^2+u^2}\right) + C$

69. $\int u^2\sqrt{a^2+u^2}\, du = \frac{u}{8}(a^2+2u^2)\sqrt{a^2+u^2} - \frac{a^4}{8}\ln\left(u+\sqrt{a^2+u^2}\right) + C$

70. $\int \frac{\sqrt{a^2+u^2}}{u}\, du = \sqrt{a^2+u^2} - a\ln\left|\frac{a+\sqrt{a^2+u^2}}{u}\right| + C$

71. $\int \frac{\sqrt{a^2+u^2}}{u^2}\, du = -\frac{\sqrt{a^2+u^2}}{u} + \ln\left(u+\sqrt{a^2+u^2}\right) + C$

72. $\int \frac{du}{\sqrt{a^2+u^2}} = \ln\left(u+\sqrt{a^2+u^2}\right) + C$

73. $\int \frac{u^2}{\sqrt{a^2+u^2}}\, du = \frac{u}{2}\left(\sqrt{a^2+u^2}\right) - \frac{a^2}{2}\ln\left(u+\sqrt{a^2+u^2}\right) + C$

74. $\int \frac{du}{u\sqrt{a^2+u^2}} = \frac{-1}{a}\ln\left|\frac{\sqrt{a^2+u^2}+a}{u}\right| + C$

75. $\int \frac{du}{u^2\sqrt{a^2+u^2}} = -\frac{\sqrt{a^2+u^2}}{a^2 u} + C$

76. $\int \frac{du}{(a^2+u^2)^{3/2}} = \frac{u}{a^2\sqrt{a^2+u^2}} + C$

Integrals Involving $u^2 - a^2$, $a > 0$

77. $\int \sqrt{u^2-a^2}\, du = \frac{u}{2}\sqrt{u^2-a^2} - \frac{a^2}{2}\ln\left|u+\sqrt{u^2-a^2}\right| + C$

78. $\int u^2\sqrt{u^2-a^2}\, du = \frac{u}{8}(2u^2-a^2)\sqrt{u^2-a^2} - \frac{a^4}{8}\ln\left|u+\sqrt{u^2-a^2}\right| + C$

79. $\int \frac{\sqrt{u^2-a^2}}{u}\, du = \sqrt{u^2-a^2} - a\cos^{-1}\frac{a}{|u|} + C$

80. $\int \frac{\sqrt{u^2-a^2}}{u^2}\, du = -\frac{\sqrt{u^2-a^2}}{u} + \ln\left|u+\sqrt{u^2-a^2}\right| + C$

81. $\int \frac{du}{\sqrt{u^2-a^2}} = \ln\left|u+\sqrt{u^2-a^2}\right| + C$

82. $\int \frac{u^2}{\sqrt{u^2-a^2}}\, du = \frac{u}{2}\sqrt{u^2-a^2} + \frac{a^2}{2}\ln\left|u+\sqrt{u^2-a^2}\right| + C$

83. $\int \frac{du}{u^2\sqrt{u^2-a^2}} = \frac{\sqrt{u^2-a^2}}{a^2 u} + C$

84. $\int \frac{du}{(u^2-a^2)^{3/2}} = -\frac{u}{a^2\sqrt{u^2-a^2}} + C$

Integrals Involving $a^2 - u^2$, $a > 0$

85. $$\int \sqrt{a^2-u^2}\,du = \frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$$

86. $$\int u^2\sqrt{a^2-u^2}\,du = \frac{u}{8}(2u^2-a^2)\sqrt{a^2-u^2} + \frac{a^4}{8}\sin^{-1}\frac{u}{a} + C$$

87. $$\int \frac{\sqrt{a^2-u^2}}{u}\,du = \sqrt{a^2-u^2} - a\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C$$

88. $$\int \frac{\sqrt{a^2-u^2}}{u^2}\,du = \frac{-1}{u}\sqrt{a^2-u^2} - \sin^{-1}\frac{u}{a} + C$$

89. $$\int \frac{u^2}{\sqrt{a^2-u^2}}\,du = \frac{1}{2}\left(-u\sqrt{a^2-u^2} + a^2\sin^{-1}\frac{u}{a}\right) + C$$

90. $$\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C$$

91. $$\int \frac{du}{u^2\sqrt{a^2-u^2}} = -\frac{1}{a^2u}\sqrt{a^2-u^2} + C$$

92. $$\int \left(a^2-u^2\right)^{3/2}\,du = -\frac{u}{8}\left(2u^2-5a^2\right)\sqrt{a^2-u^2} + \frac{3a^4}{8}\sin^{-1}\frac{u}{a} + C$$

93. $$\int \frac{du}{(a^2-u^2)^{3/2}} = -\frac{u}{a^2\sqrt{a^2-u^2}} + C$$

Integrals Involving $2au - u^2$, $a > 0$

94. $$\int \sqrt{2au-u^2}\,du = \frac{u-a}{2}\sqrt{2au-u^2} + \frac{a^2}{2}\cos^{-1}\left(\frac{a-u}{a}\right) + C$$

95. $$\int \frac{du}{\sqrt{2au-u^2}} = \cos^{-1}\left(\frac{a-u}{a}\right) + C$$

96. $$\int u\sqrt{2au-u^2}\,du = \frac{2u^2-au-3a^2}{6}\sqrt{2au-u^2} + \frac{a^3}{2}\cos^{-1}\left(\frac{a-u}{a}\right) + C$$

97. $$\int \frac{du}{u\sqrt{2au-u^2}} = -\frac{\sqrt{2au-u^2}}{au} + C$$

Integrals Involving $a + bu$, $a \neq 0$

98. $$\int \frac{u}{a+bu}\,du = \frac{1}{b^2}(a+bu-a\ln|a+bu|) + C$$

99. $$\int \frac{u^2}{a+bu}\,du = \frac{1}{2b^3}\left[(a+bu)^2 - 4a(a+bu) + 2a^2\ln|a+bu|\right] + C$$

100. $$\int \frac{du}{u(a+bu)} = \frac{1}{a}\ln\left|\frac{u}{a+bu}\right| + C$$

101. $$\int \frac{du}{u^2(a+bu)} = -\frac{1}{au} + \frac{b}{a^2}\ln\left|\frac{a+bu}{u}\right| + C$$

102. $$\int \frac{u}{(a+bu)^2}\,du = \frac{a}{b^2(a+bu)} + \frac{1}{b^2}\ln|a+bu| + C$$

103. $$\int \frac{u}{u(a+bu)^2}\,du = \frac{1}{a(a+bu)} - \frac{1}{a^2}\ln\left|\frac{a+bu}{u}\right| + C$$

104. $$\int \frac{u^2}{(a+bu)^2}\,du = \frac{1}{b^3}\left(a+bu - \frac{a^2}{a+bu} - 2a\ln|a+bu|\right) + C$$

105. $$\int u\sqrt{a+bu}\,du = \frac{2}{15b^2}(3bu-2a)(a+bu)^{3/2}+C$$

106. $$\int \frac{u}{\sqrt{a+bu}}\,du = \frac{2}{3b^2}(bu-2a)\sqrt{a+bu}+C$$

107. $$\int \frac{u^2}{\sqrt{a+bu}}\,du = \frac{2}{15b^3}(8a^2+3b^2u^2-4abu)\sqrt{a+bu}+C$$

108. $$\int \frac{du}{u\sqrt{a+bu}} = \begin{cases} \frac{1}{\sqrt{a}}\ln\left|\frac{\sqrt{a+bu}-\sqrt{a}}{\sqrt{a+bu}+\sqrt{a}}\right|+C, & \text{if } a>0 \\ \frac{\sqrt{2}}{\sqrt{-a}}\tan^{-1}\sqrt{\frac{a+bu}{-a}}+C, & \text{if } a<0 \end{cases}$$

109. $$\int \frac{\sqrt{a+bu}}{u}\,du = 2\sqrt{a+bu}+a\int \frac{du}{u\sqrt{a+bu}}$$

110. $$\int \frac{\sqrt{a+bu}}{u^2}\,du = -\frac{\sqrt{a+bu}}{u}+\frac{b}{2}\int \frac{du}{u\sqrt{a+bu}}$$

111. $$\int u^n\sqrt{a+bu}\,du = \frac{2}{b(2n+3)}\left[u^n(a+bu)^{3/2}-na\int u^{n-1}\sqrt{a+bu}\,du\right]$$

112. $$\int \frac{u^n}{\sqrt{a+bu}}\,du = \frac{2u^n\sqrt{a+bu}}{b(2n+1)}-\frac{2na}{b(2n+1)}\int \frac{u^{n-1}}{\sqrt{a+bu}}\,du$$

113. $$\int \frac{du}{u^n\sqrt{a+bu}} = -\frac{\sqrt{a+bu}}{a(n-1)u^{n-1}}-\frac{b(2n-3)}{2a(n-1)}\int \frac{du}{u^{n-1}\sqrt{a+bu}}$$

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