# INTERACTIVE CALCULUS Q2



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# Interactive Calculus Q2

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This text was compiled on 01/17/2026

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# Licensing

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# 1.1: Approximating Areas

**Learning Objectives**

- Use sigma (summation) notation to calculate sums and powers of integers.
- Use the sum of rectangular areas to approximate the area under a curve.
- Use Riemann sums to approximate area.

**Archimedes** was fascinated with calculating the areas of various shapes—in other words, the amount of space enclosed by the shape. He used a process that has come to be known as the **method of exhaustion**, which used smaller and smaller shapes, the areas of which could be calculated exactly, to fill an irregular region and thereby obtain closer and closer approximations to the total area. In this process, an area bounded by curves is filled with rectangles, triangles, and shapes with exact area formulas. These areas are then summed to approximate the area of the curved region.

In this section, we develop techniques to approximate the area between a curve, defined by a function $f(x)$, and the x-axis on a closed interval $[a, b]$. Like Archimedes, we first approximate the area under the curve using shapes of known area (namely, rectangles). By using smaller and smaller rectangles, we get closer and closer approximations to the area. Taking a limit allows us to calculate the exact area under the curve.

Let's start by introducing some notation to make the calculations easier. We then consider the case when $f(x)$ is continuous and nonnegative. Later in the chapter, we relax some of these restrictions and develop techniques that apply in more general cases.

## Sigma (Summation) Notation

As mentioned, we will use shapes of known area to approximate the area of an irregular region bounded by curves. This process often requires adding up long strings of numbers. To make it easier to write down these lengthy sums, we look at some new notation here, called **sigma notation** (also known as **summation notation**). The Greek capital letter $\Sigma$, sigma, is used to express long sums of values in a compact form. For example, if we want to add all the integers from 1 to 20 without sigma notation, we have to write

$$1+2+3+4+5+6+7+8+9+10+11+12+13 \\ +14+15+16+17+18+19+20. \tag{1.1.1}$$

We could probably skip writing a couple of terms and write

$$1+2+3+4+\cdots+19+20, \tag{1.1.2}$$

which is better, but still cumbersome. With sigma notation, we write this sum as

$$\sum_{i=1}^{20} i \tag{1.1.3}$$

which is much more compact. Typically, sigma notation is presented in the form

$$\sum_{i=1}^{n} a_i \tag{1.1.4}$$

where $a_i$ describes the terms to be added, and the $i$ is called the *index*. Each term is evaluated, then we sum all the values, beginning with the value when $i = 1$ and ending with the value when $i = n$. For example, an expression like $\sum_{i=2}^{7} s_i$ is interpreted as $s_2 + s_3 + s_4 + s_5 + s_6 + s_7$. Note that the index is used only to keep track of the terms to be added; it does not factor into the calculation of the sum itself. The index is therefore called a **dummy variable**. We can use any letter we like for the index. Typically, mathematicians use $i$, $j$, $k$, $m$, and $n$ for indices.

Let's try a couple of examples of using sigma notation.

**Example 1.1.1: Using Sigma Notation**

a. Write in sigma notation and evaluate the sum of terms $3^i$ for $i = 1, 2, 3, 4, 5$.
b. Write the sum in sigma notation:

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25}.$$

**Solution**

a. Write

$$\sum_{i=1}^{5} 3^i = 3 + 3^2 + 3^3 + 3^4 + 3^5 = 363.$$

b. The denominator of each term is a perfect square. Using sigma notation, this sum can be written as $\sum_{i=1}^{5} \frac{1}{i^2}$.

**Exercise 1.1.1**

Write in sigma notation and evaluate the sum of terms $2^i$ for $i = 3, 4, 5, 6$.

**Hint**

Use the solving steps in Example 1.1.1 as a guide.

**Answer**

$$\sum_{i=3}^{6} 2^i = 2^3 + 2^4 + 2^5 + 2^6 = 120$$

The properties associated with the summation process are given in the following rule.

**Rule: Properties of Sigma Notation**

Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ represent two sequences of terms and let $c$ be a constant. The following properties hold for all positive integers $n$ and for integers $m$, with $1 \leq m \leq n$.

i. $\sum_{i=1}^{n} c = nc$
ii. $\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$
iii. $\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$
iv. $\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$
v. $\sum_{i=1}^{n} a_i = \sum_{i=1}^{m} a_i + \sum_{i=m+1}^{n} a_i$

**Proof**

We prove properties 2. and 3. here, and leave proof of the other properties to the Exercises.

2. We have

$$\sum_{i=1}^{n} ca_i = ca_1 + ca_2 + ca_3 + \cdots + ca_n = c(a_1 + a_2 + a_3 + \cdots + a_n) = c\sum_{i=1}^{n} a_i. \tag{1.1.5}$$

3. We have

$$\sum_{i=1}^{n}(a_i+b_i)=(a_1+b_1)+(a_2+b_2)+(a_3+b_3)+\cdots+(a_n+b_n) \tag{1.1.6}$$

$$=(a_1+a_2+a_3+\cdots+a_n)+(b_1+b_2+b_3+\cdots+b_n) \tag{1.1.7}$$

$$=\sum_{i=1}^{n}a_i+\sum_{i=1}^{n}b_i. \tag{1.1.8}$$

□

A few more formulas for frequently found functions simplify the summation process further. These are shown in the next rule, for **sums and powers of integers**, and we use them in the next set of examples.

**Rule: Sums and Powers of Integers**

1. The sum of $n$ integers is given by

$$\sum_{i=1}^{n}i=1+2+\cdots+n=\frac{n(n+1)}{2}. \tag{1.1.9}$$

2. The sum of consecutive integers squared is given by

$$\sum_{i=1}^{n}i^2=1^2+2^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}. \tag{1.1.10}$$

3. The sum of consecutive integers cubed is given by

$$\sum_{i=1}^{n}i^3=1^3+2^3+\cdots+n^3=\frac{n^2(n+1)^2}{4}. \tag{1.1.11}$$

**Example 1.1.2: Evaluation Using Sigma Notation**

Write using sigma notation and evaluate:

a. The sum of the terms $(i-3)^2$ for $i=1,2,\ldots,200$.
b. The sum of the terms $(i^3-i^2)$ for $i=1,2,3,4,5,6$

**Solution**

a. Multiplying out $(i-3)^2$, we can break the expression into three terms.

$$\begin{aligned}\sum_{i=1}^{200}(i-3)^2&=\sum_{i=1}^{200}(i^2-6i+9)\\&=\sum_{i=1}^{200}i^2-\sum_{i=1}^{200}6i+\sum_{i=1}^{200}9\\&=\sum_{i=1}^{200}i^2-6\sum_{i=1}^{200}i+\sum_{i=1}^{200}9\\&=\frac{200(200+1)(400+1)}{6}-6\left[\frac{200(200+1)}{2}\right]+9(200)\\&=2,686,700-120,600+1800\\&=2,567,900\end{aligned}$$

b. Use sigma notation property iv. and the rules for the sum of squared terms and the sum of cubed terms.

$$\begin{aligned}\sum_{i=1}^{6}(i^3-i^2) &= \sum_{i=1}^{6} i^3 - \sum_{i=1}^{6} i^2 \\ &= \frac{6^2(6+1)^2}{4} - \frac{6(6+1)(2(6)+1)}{6} \\ &= \frac{1764}{4} - \frac{546}{6} \\ &= 350\end{aligned}$$

**Exercise 1.1.2**

Find the sum of the values of $4+3i$ for $i = 1, 2, \ldots, 100$.

**Hint**

Use the properties of sigma notation to solve the problem.

**Answer**

$15,550$

**Example 1.1.3: Finding the Sum of the Function Values**

Find the sum of the values of $f(x) = x^3$ over the integers $1, 2, 3, \ldots, 10$.

**Solution**

Using Equation 1.1.11, we have

$$\sum_{i=0}^{10} i^3 = \frac{(10)^2(10+1)^2}{4} = \frac{100(121)}{4} = 3025$$

**Exercise 1.1.3**

Evaluate the sum indicated by the notation $\sum_{k=1}^{20}(2k+1)$.

**Hint**

Use the rule on sum and powers of integers (Equations 1.1.9-1.1.11).

**Answer**

$440$

## Approximating Area

Now that we have the necessary notation, we return to the problem at hand: approximating the area under a curve. Let $f(x)$ be a continuous, nonnegative function defined on the closed interval $[a, b]$. We want to approximate the area $A$ bounded by $f(x)$ above, the $x$-axis below, the line $x = a$ on the left, and the line $x = b$ on the right (Figure 1.1.1).

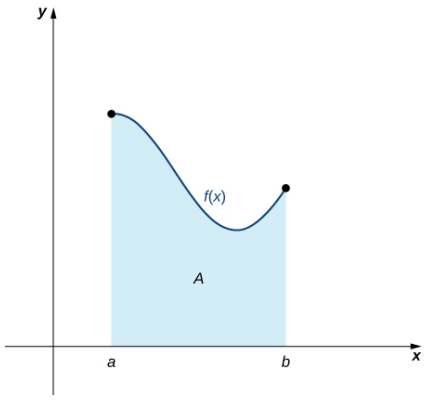



Figure 1.1.1: An area (shaded region) bounded by the curve $f(x)$ at top, the $x$-axis at bottom, the line $x = a$ to the left, and the line $x = b$ at right.

How do we approximate the area under this curve? The approach is a geometric one. By dividing a region into many small shapes that have known area formulas, we can sum these areas and obtain a reasonable estimate of the true area. We begin by dividing the interval $[a, b]$ into $n$ subintervals of equal width, $\frac{b-a}{n}$. We do this by selecting equally spaced points $x_0, x_1, x_2, \ldots, x_n$ with $x_0 = a, x_n = b$, and

$$x_i - x_{i-1} = \frac{b-a}{n} \tag{1.1.12}$$

for $i = 1, 2, 3, \ldots, n$.

We denote the width of each subinterval with the notation $\Delta x$, so $\Delta x = \frac{b-a}{n}$ and

$$x_i = x_0 + i\Delta x \tag{1.1.13}$$

for $i = 1, 2, 3, \ldots, n$. This notion of dividing an interval $[a, b]$ into subintervals by selecting points from within the interval is used quite often in approximating the area under a curve, so let's define some relevant terminology.

**Definition: Partitions**

A set of points $P = x_i$ for $i = 0, 1, 2, \ldots, n$ with $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, which divides the interval $[a, b]$ into subintervals of the form $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$ is called a **partition** of $[a, b]$. If the subintervals all have the same width, the set of points forms a **regular partition** (or uniform partition) of the interval $[a, b]$.

We can use this regular partition as the basis of a method for estimating the area under the curve. We next examine two methods: the left-endpoint approximation and the right-endpoint approximation.

**Rule: Left-Endpoint Approximation**

On each subinterval $[x_{i-1}, x_i]$ (for $i = 1, 2, 3, \ldots, n$), construct a rectangle with width $\Delta x$ and height equal to $f(x_{i-1})$, which is the function value at the left endpoint of the subinterval. Then the area of this rectangle is $f(x_{i-1})\Delta x$. Adding the areas of all these rectangles, we get an approximate value for $A$ (Figure 1.1.2). We use the notation $L_n$ to denote that this is a **left-endpoint approximation** of $A$ using $n$ subintervals.

$$A \approx L_n = f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(xn-1)\Delta x = \sum_{i=1}^{n} f(x_{i-1})\Delta x \tag{1.1.14}$$

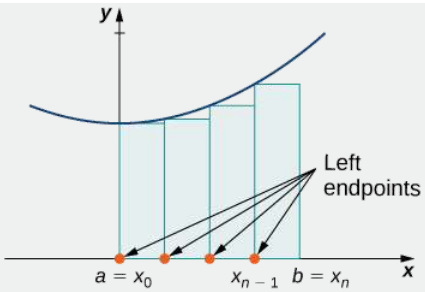



Figure 1.1.2: In the left-endpoint approximation of area under a curve, the height of each rectangle is determined by the function value at the left of each subinterval.

The second method for approximating area under a curve is the right-endpoint approximation. It is almost the same as the left-endpoint approximation, but now the heights of the rectangles are determined by the function values at the right of each subinterval.'



**Rule: Right-Endpoint Approximation**

Construct a rectangle on each subinterval $[x_{i-1}, x_i]$, only this time the height of the rectangle is determined by the function value $f(x_i)$ at the right endpoint of the subinterval. Then, the area of each rectangle is $f(x_i)\,\Delta x$ and the approximation for $A$ is given by

$$A \approx R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x = \sum_{i=1}^{n} f(x_i)\Delta x. \tag{1.1.15}$$

The notation $R_n$ indicates this is a **right-endpoint approximation** for $A$ (Figure 1.1.3).

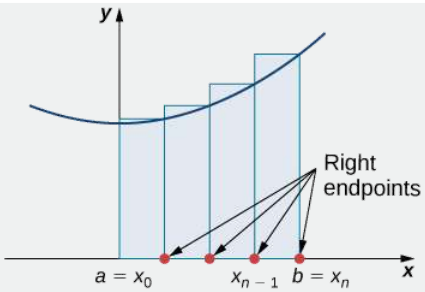
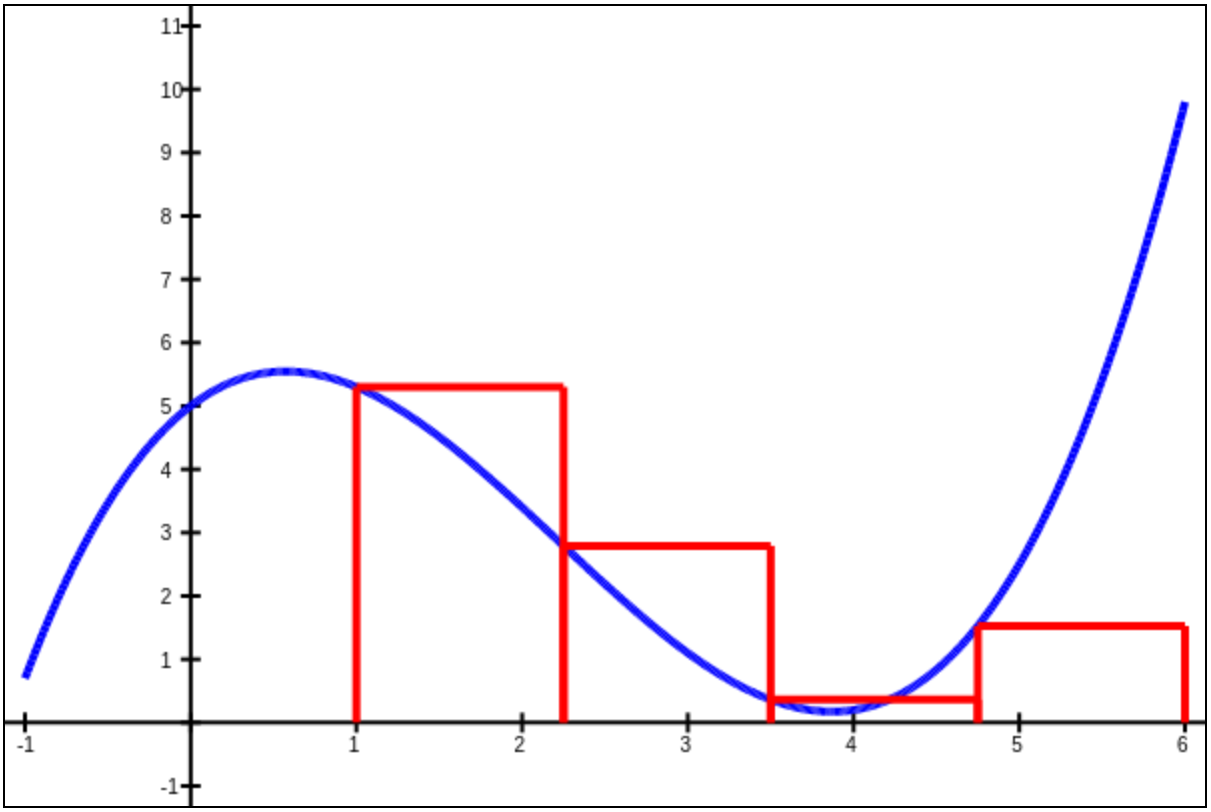



Figure 1.1.3: In the right-endpoint approximation of area under a curve, the height of each rectangle is determined by the function value at the right of each subinterval. Note that the right-endpoint approximation differs from the left-endpoint approximation in Figure 1.1.2.

## Interactive Left and Right Endpoint Approximation Activity

Choose Left or Right Endpoint Approximation and move the slider to see the approximations with different numbers of rectangles. You will then see the approximation of the area under the curve $y = .3x^3 - 2x^2 + 2x + 5$ for $1 < x < 6$.




n = 4

◉ Left-Endpoint Approximation ○ Right-Endpoint Approximation

Area ≈ 6.63 + 3.49 + 0.45 + 1.91 = 12.48

The graphs in Figure 1.1.4 represent the curve $f(x) = \frac{x^2}{2}$. In Figure 1.1.4b we divide the region represented by the interval $[0, 3]$ into six subintervals, each of width $0.5$. Thus, $\Delta x = 0.5$. We then form six rectangles by drawing vertical lines perpendicular to $x_{i-1}$, the

left endpoint of each subinterval. We determine the height of each rectangle by calculating $f(x_{i-1})$ for $i = 1, 2, 3, 4, 5, 6$. The intervals are $[0, 0.5], [0.5, 1], [1, 1.5], [1.5, 2], [2, 2.5], [2.5, 3]$. We find the area of each rectangle by multiplying the height by the width. Then, the sum of the rectangular areas approximates the area between $f(x)$ and the $x$-axis. When the left endpoints are used to calculate height, we have a left-endpoint approximation. Thus,

$$\begin{aligned} A \approx L_6 &= \sum_{i=1}^{6} f(x_{i-1})\Delta x = f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x \\ &= f(0)0.5 + f(0.5)0.5 + f(1)0.5 + f(1.5)0.5 + f(2)0.5 + f(2.5)0.5 \\ &= (0)0.5 + (0.125)0.5 + (0.5)0.5 + (1.125)0.5 + (2)0.5 + (3.125)0.5 \\ &= 0 + 0.0625 + 0.25 + 0.5625 + 1 + 1.5625 \\ &= 3.4375 \text{ units}^2 \end{aligned}$$

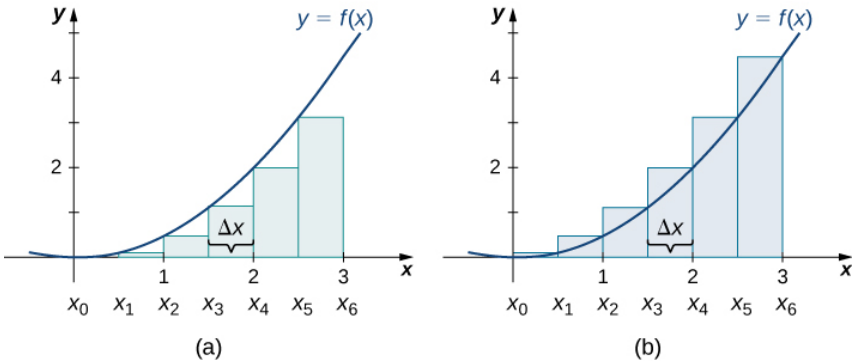



Figure 1.1.4: Methods of approximating the area under a curve by using (a) the left endpoints and (b) the right endpoints.

In Figure 1.1.4*b*, we draw vertical lines perpendicular to $x_i$ such that $x_i$ is the right endpoint of each subinterval, and calculate $f(x_i)$ for $i = 1, 2, 3, 4, 5, 6$. We multiply each $f(x_i)$ by $\Delta x$ to find the rectangular areas, and then add them. This is a right-endpoint approximation of the area under $f(x)$. Thus,

$$\begin{aligned} A \approx R_6 &= \sum_{i=1}^{6} f(x_i)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x + f(x_6)\Delta x \\ &= f(0.5)0.5 + f(1)0.5 + f(1.5)0.5 + f(2)0.5 + f(2.5)0.5 + f(3)0.5 \\ &= (0.125)0.5 + (0.5)0.5 + (1.125)0.5 + (2)0.5 + (3.125)0.5 + (4.5)0.5 \\ &= 0.0625 + 0.25 + 0.5625 + 1 + 1.5625 + 2.25 \\ &= 5.6875 \text{ units}^2. \end{aligned}$$


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**Example 1.1.4: Approximating the Area Under a Curve**

Use both left-endpoint and right-endpoint approximations to approximate the area under the curve of $f(x) = x^2$ on the interval $[0, 2]$; use $n = 4$.

**Solution**

First, divide the interval $[0, 2]$ into $n$ equal subintervals. Using $n = 4$, $\Delta x = \dfrac{(2-0)}{4} = 0.5$. This is the width of each rectangle. The intervals $[0, 0.5], [0.5, 1], [1, 1.5], [1.5, 2]$ are shown in Figure 1.1.5. Using a left-endpoint approximation, the heights are $f(0) = 0$, $f(0.5) = 0.25$, $f(1) = 1$, and $f(1.5) = 2.25$. Then,

$$\begin{aligned} L_4 &= f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x \\ &= 0(0.5) + 0.25(0.5) + 1(0.5) + 2.25(0.5) \\ &= 1.75 \text{ units}^2 \end{aligned}$$

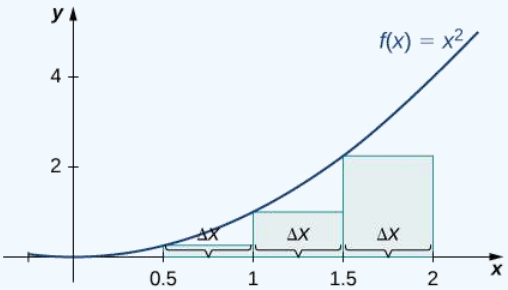



Figure 1.1.5: The graph shows the left-endpoint approximation of the area under $f(x) = x^2$ from 0 to 2.

The right-endpoint approximation is shown in Figure 1.1.6. The intervals are the same, $\Delta x = 0.5$, but now use the right endpoint to calculate the height of the rectangles. We have

$$\begin{aligned} R_4 &= f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x \\ &= 0.25(0.5) + 1(0.5) + 2.25(0.5) + 4(0.5) \\ &= 3.75 \text{ units}^2 \end{aligned}$$


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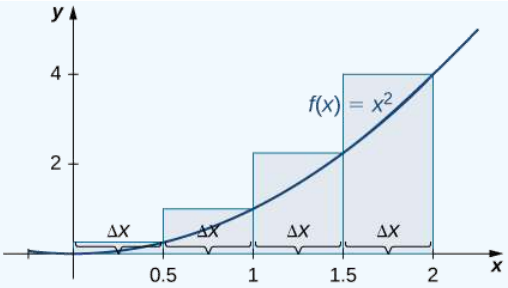



Figure 1.1.6: The graph shows the right-endpoint approximation of the area under $f(x) = x^2$ from 0 to 2.

The left-endpoint approximation is $1.75\ \text{units}^2$; the right-endpoint approximation is $3.75\ \text{units}^2$.

**Exercise 1.1.4**

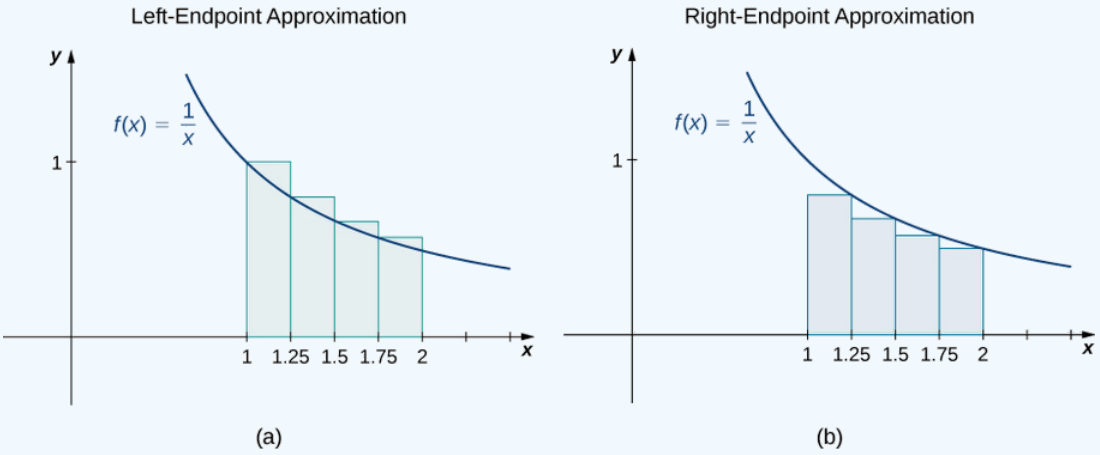
Sketch left-endpoint and right-endpoint approximations for $f(x) = \frac{1}{x}$ on $[1, 2]$; use $n = 4$. Approximate the area using both methods.

**Hint**

Follow the solving strategy in Example 1.1.4 step-by-step.

**Answer**

The left-endpoint approximation is $0.7595\ \text{units}^2$. The right-endpoint approximation is $0.6345\ \text{units}^2$. See the below Media.




Looking at Figure 1.1.4 and the graphs in Example 1.1.4, we can see that when we use a small number of intervals, neither the left-endpoint approximation nor the right-endpoint approximation is a particularly accurate estimate of the area under the curve. However, it seems logical that if we increase the number of points in our partition, our estimate of $A$ will improve. We will have more rectangles, but each rectangle will be thinner, so we will be able to fit the rectangles to the curve more precisely.

We can demonstrate the improved approximation obtained through smaller intervals with an example. Let's explore the idea of increasing $n$, first in a left-endpoint approximation with four rectangles, then eight rectangles, and finally 32 rectangles. Then, let's do the same thing in a right-endpoint approximation, using the same sets of intervals, of the same curved region. Figure 1.1.7 shows the area of the region under the curve $f(x) = (x-1)^3 + 4$ on the interval $[0, 2]$ using a left-endpoint approximation where $n = 4$. The width of each rectangle is

$$\Delta x = \frac{2-0}{4} = \frac{1}{2}.$$

The area is approximated by the summed areas of the rectangles, or

$$L_4 = f(0)(0.5) + f(0.5)(0.5) + f(1)(0.5) + f(1.5)0.5 = 7.5\ \text{units}^2$$


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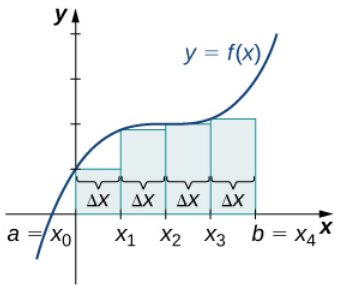



Figure 1.1.7: With a left-endpoint approximation and dividing the region from $a$ to $b$ into four equal intervals, the area under the curve is approximately equal to the sum of the areas of the rectangles.

Figure 1.1.8 shows the same curve divided into eight subintervals. Comparing the graph with four rectangles in Figure 1.1.7 with this graph with eight rectangles, we can see there appears to be less white space under the curve when $n = 8$. This white space is area under the curve we are unable to include using our approximation. The area of the rectangles is

$$L_8 = f(0)(0.25) + f(0.25)(0.25) + f(0.5)(0.25) + f(0.75)(0.25) + f(1)(0.25) + f(1.25)(0.25) + f(1.5)(0.25) + f(1.75)(0.25) = 7.75 \text{ units}^2$$

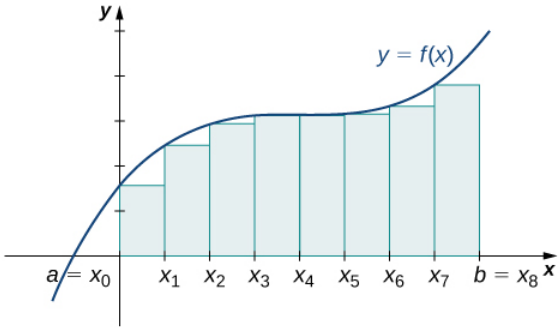



Figure 1.1.8: The region under the curve is divided into $n = 8$ rectangular areas of equal width for a left-endpoint approximation.

The graph in Figure 1.1.9 shows the same function with 32 rectangles inscribed under the curve. There appears to be little white space left. The area occupied by the rectangles is

$$L_{32} = f(0)(0.0625) + f(0.0625)(0.0625) + f(0.125)(0.0625) + \cdots + f(1.9375)(0.0625) = 7.9375 \text{ units}^2.$$

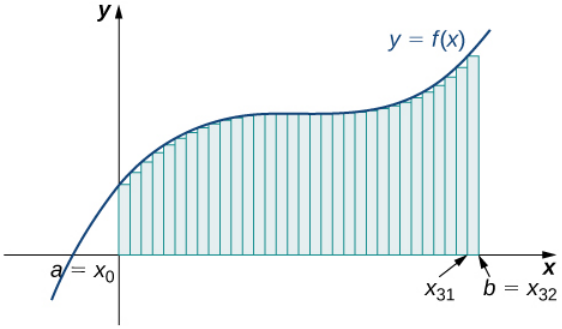



Figure 1.1.9: Here, 32 rectangles are inscribed under the curve for a left-endpoint approximation.

We can carry out a similar process for the right-endpoint approximation method. A right-endpoint approximation of the same curve, using four rectangles (Figure 1.1.10), yields an area

$$R_4 = f(0.5)(0.5) + f(1)(0.5) + f(1.5)(0.5) + f(2)(0.5) = 8.5 \text{ units}^2.$$

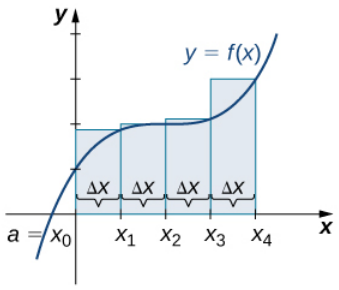



Figure 1.1.10: Now we divide the area under the curve into four equal subintervals for a right-endpoint approximation.

Dividing the region over the interval $[0, 2]$ into eight rectangles results in $\Delta x = \frac{2-0}{8} = 0.25$. The graph is shown in Figure 1.1.11. The area is

$$R_8 = f(0.25)(0.25) + f(0.5)(0.25) + f(0.75)(0.25) + f(1)(0.25) + f(1.25)(0.25) + f(1.5)(0.25) + f(1.75)(0.25) + f(2)(0.25) = 8.25 \text{ units}^2$$

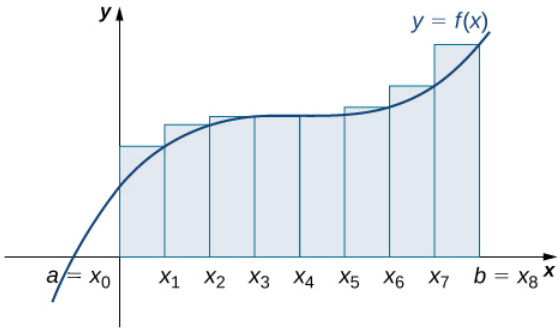



Figure 1.1.11: Here we use right-endpoint approximation for a region divided into eight equal subintervals.

Last, the right-endpoint approximation with $n = 32$ is close to the actual area (Figure 1.1.12). The area is approximately

$$R_{32} = f(0.0625)(0.0625) + f(0.125)(0.0625) + f(0.1875)(0.0625) + \cdots + f(2)(0.0625) = 8.0625 \text{ units}^2$$

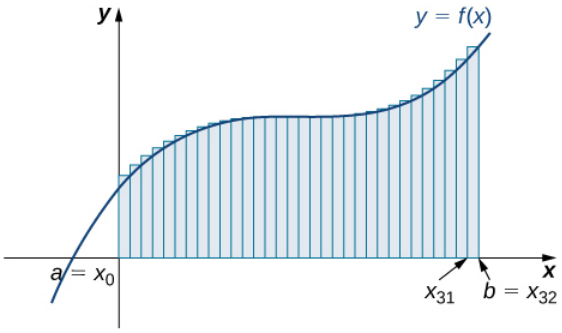



Figure 1.1.12: The region is divided into 32 equal subintervals for a right-endpoint approximation.

Based on these figures and calculations, it appears we are on the right track; the rectangles appear to approximate the area under the curve better as $n$ gets larger. Furthermore, as $n$ increases, both the left-endpoint and right-endpoint approximations appear to approach an area of 8 square units. Table 1.1.15 shows a numerical comparison of the left- and right-endpoint methods. The idea that the approximations of the area under the curve get better and better as $n$ gets larger and larger is very important, and we now explore this idea in more detail.

Table 1.1.15: Converging Values of Left- and Right-Endpoint Approximations as $n$ Increases

| Value of $n$ | Approximate Area $L_n$ | Approximate Area $R_n$ |
|---|---|---|
| $n = 4$ | 7.5 | 8.5 |
| $n = 8$ | 7.75 | 8.25 |
| $n = 32$ | 7.94 | 8.06 |

## Forming Riemann Sums

So far we have been using rectangles to approximate the area under a curve. The heights of these rectangles have been determined by evaluating the function at either the right or left endpoints of the subinterval $[x_{i-1}, x_i]$. In reality, there is no reason to restrict evaluation of the function to one of these two points only. We could evaluate the function at any point $x_i^*$ in the subinterval $[x_{i-1}, x_i]$, and use $f(x_i^*)$ as the height of our rectangle. This gives us an estimate for the area of the form

$$A \approx \sum_{i=1}^{n} f(x_i^*)\,\Delta x. \tag{1.1.16}$$

A sum of this form is called a Riemann sum, named for the 19th-century mathematician Bernhard Riemann, who developed the idea.

**Definition: Riemann sum**

Let $f(x)$ be defined on a closed interval $[a, b]$ and let $P$ be any partition of $[a, b]$. Let $\Delta x_i$ be the width of each subinterval $[x_{i-1}, x_i]$ and for each $i$, let $x_i^*$ be any point in $[x_{i-1}, x_i]$. A Riemann sum is defined for $f(x)$ as

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x_i. \tag{1.1.17}$$

At this point, we'll choose a regular partition $P$, as we have in our examples above. This forces all $\Delta x_i$ to be equal to $\Delta x = \dfrac{b-a}{n}$ for any natural number of intervals $n$.

Recall that with the left- and right-endpoint approximations, the estimates seem to get better and better as $n$ get larger and larger. The same thing happens with Riemann sums. Riemann sums give better approximations for larger values of $n$. We are now ready to define the area under a curve in terms of Riemann sums.

**Definition: Area Under the Curve**

Let $f(x)$ be a continuous, nonnegative function on an interval $[a, b]$, and let $\sum_{i=1}^{n} f(x_i^*)\,\Delta x$ be a Riemann sum for $f(x)$ with a regular partition $P$. Then, the **area under the curve** $y = f(x)$ on $[a, b]$ is given by

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x. \tag{1.1.18}$$

See a graphical demonstration of the construction of a Riemann sum.

Some subtleties here are worth discussing. First, note that taking the limit of a sum is a little different from taking the limit of a function $f(x)$ as $x$ goes to infinity. Limits of sums are discussed in detail in the chapter on Sequences and Series; however, for now we can assume that the computational techniques we used to compute limits of functions can also be used to calculate limits of sums.

Second, we must consider what to do if the expression converges to different limits for different choices of $x_i^*$. Fortunately, this does not happen. Although the proof is beyond the scope of this text, it can be shown that if $f(x)$ is continuous on the closed interval $[a, b]$, then $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$ exists and is unique (in other words, it does not depend on the choice of $x_i^*$).

We look at some examples shortly. But, before we do, let's take a moment and talk about some specific choices for $x_i^*$. Although any choice for $x_i^*$ gives us an estimate of the area under the curve, we don't necessarily know whether that estimate is too high (overestimate) or too low (underestimate). If it is important to know whether our estimate is high or low, we can select our value for $x_i^*$ to guarantee one result or the other.

If we want an overestimate, for example, we can choose $x_i^*$ such that for $i = 1, 2, 3, \ldots, n$, $f(x_i^*) \geq f(x)$ for all $x \in [x_i - 1, x_i]$. In other words, we choose $x_i^*$ so that for $i = 1, 2, 3, \ldots, n$, $f(x_i^*)$ is the maximum function value on the interval $[x_{i-1}, x_i]$. If we select $x_i^*$ in this way, then the Riemann sum $\sum_{i=1}^{n} f(x_i^*)\Delta x$ is called an **upper sum**. Similarly, if we want an underestimate, we can choose $x * i$ so that for $i = 1, 2, 3, \ldots, n$, $f(x_i^*)$ is the minimum function value on the interval $[x_{i-1}, x_i]$. In this case, the associated Riemann sum is called a **lower sum**. Note that if $f(x)$ is either increasing or decreasing throughout the interval $[a, b]$, then the maximum and minimum values of the function occur at the endpoints of the subintervals, so the upper and lower sums are just the same as the left- and right-endpoint approximations.

### Example 1.1.5: Finding Lower and Upper Sums

Find a lower sum for $f(x) = 10 - x^2$ on $[1, 2]$; let $n = 4$ subintervals.

**Solution**

With $n = 4$ over the interval $[1, 2]$, $\Delta x = \frac{1}{4}$. We can list the intervals as $[1, 1.25]$, $[1.25, 1.5]$, $[1.5, 1.75]$, and $[1.75, 2]$. Because the function is decreasing over the interval $[1, 2]$, Figure shows that a lower sum is obtained by using the right endpoints.

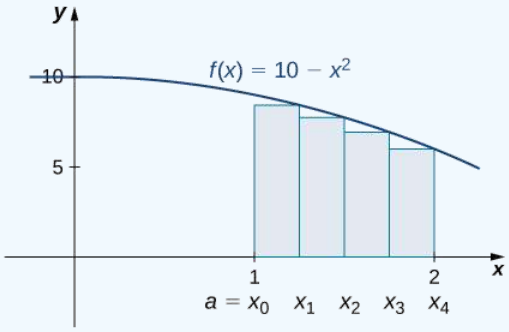



Figure 1.1.13: The graph of $f(x) = 10 - x^2$ is set up for a right-endpoint approximation of the area bounded by the curve and the $x$-axis on $[1, 2]$, and it shows a lower sum.

The Riemann sum is

$$\begin{aligned}\sum_{k=1}^{4}(10 - x^2)(0.25) &= 0.25[10 - (1.25)^2 + 10 - (1.5)^2 + 10 - (1.75)^2 + 10 - (2)^2] \\ &= 0.25[8.4375 + 7.75 + 6.9375 + 6] \\ &= 7.28 \text{ units}^2.\end{aligned}$$

The area of $7.28$ units$^2$ is a lower sum and an underestimate.

### Exercise 1.1.5

a. Find an upper sum for $f(x) = 10 - x^2$ on $[1, 2]$; let $n = 4$.
b. Sketch the approximation.

**Hint**

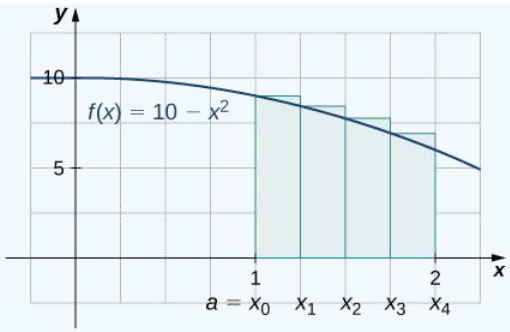
$f(x)$ is decreasing on $[1, 2]$, so the maximum function values occur at the left endpoints of the subintervals.

**Answer**

a. Upper sum=$8.0313$ units$^2$.

b.


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**Example 1.1.6: Finding Lower and Upper Sums for $f(x) = \sin x$**

Find a lower sum for $f(x) = \sin x$ over the interval $[a, b] = [0, \frac{\pi}{2}]$; let $n = 6$.

**Solution**

Let's first look at the graph in Figure 1.1.14 to get a better idea of the area of interest.

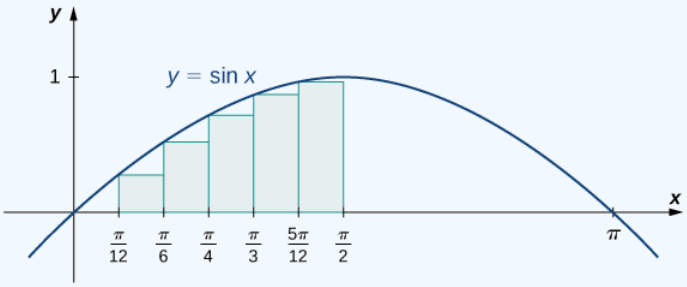



Figure 1.1.14: The graph of $y = \sin x$ is divided into six regions: $\Delta x = \frac{\pi/2}{6} = \frac{\pi}{12}$.

The intervals are $[0, \frac{\pi}{12}]$, $[\frac{\pi}{12}, \frac{\pi}{6}]$, $[\frac{\pi}{6}, \frac{\pi}{4}]$, $[\frac{\pi}{4}, \frac{\pi}{3}]$, $[\frac{\pi}{3}, \frac{5\pi}{12}]$, and $[\frac{5\pi}{12}, \frac{\pi}{2}]$. Note that $f(x) = \sin x$ is increasing on the interval $[0, \frac{\pi}{2}]$, so a left-endpoint approximation gives us the lower sum. A left-endpoint approximation is the Riemann sum $\sum_{i=0}^{5} \sin x_i \left(\frac{\pi}{12}\right)$.We have

$$A \approx \sin(0)\left(\tfrac{\pi}{12}\right) + \sin\left(\tfrac{\pi}{12}\right)\left(\tfrac{\pi}{12}\right) + \sin\left(\tfrac{\pi}{6}\right)\left(\tfrac{\pi}{12}\right) + \sin\left(\tfrac{\pi}{4}\right)\left(\tfrac{\pi}{12}\right) + \sin\left(\tfrac{\pi}{3}\right)\left(\tfrac{\pi}{12}\right) + \sin\left(\tfrac{5\pi}{12}\right)\left(\tfrac{\pi}{12}\right) \approx 0.863 \text{ units}^2.$$

**Exercise 1.1.6**

Using the function $f(x) = \sin x$ over the interval $[0, \frac{\pi}{2}]$, find an upper sum; let $n = 6$.

**Hint**

Follow the steps from Example 1.1.6.

**Answer**

$A \approx 1.125 \text{ units}^2$

## Key Concepts

- The use of sigma (summation) notation of the form $\sum_{i=1}^{n} a_i$ is useful for expressing long sums of values in compact form.
- For a continuous function defined over an interval $[a, b]$, the process of dividing the interval into $n$ equal parts, extending a rectangle to the graph of the function, calculating the areas of the series of rectangles, and then summing the areas yields an approximation of the area of that region.
- When using a regular partition, the width of each rectangle is $\Delta x = \frac{b-a}{n}$.

- Riemann sums are expressions of the form $\sum_{i=1}^{n} f(x_i^*)\Delta x$, and can be used to estimate the area under the curve $y = f(x)$. Left- and right-endpoint approximations are special kinds of Riemann sums where the values of $x_i^*$ are chosen to be the left or right endpoints of the subintervals, respectively.
- Riemann sums allow for much flexibility in choosing the set of points $x_i^*$ at which the function is evaluated, often with an eye to obtaining a lower sum or an upper sum.

## Key Equations

- **Properties of Sigma Notation**

$$\sum_{i=1}^{n} c = nc$$

$$\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$$

$$\sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

$$\sum_{i=1}^{n}(a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$$

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{m} a_i + \sum_{i=m+1}^{n} a_i$$

- **Sums and Powers of Integers**

$$\sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=0}^{n} i^3 = 1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

- **Left-Endpoint Approximation**

$$A \approx L_n = f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x = \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

- **Right-Endpoint Approximation**

$$A \approx R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x = \sum_{i=1}^{n} f(x_i)\Delta x$$

## Glossary

**left-endpoint approximation**

an approximation of the area under a curve computed by using the left endpoint of each subinterval to calculate the height of the vertical sides of each rectangle

**lower sum**

a sum obtained by using the minimum value of $f(x)$ on each subinterval

**partition**

a set of points that divides an interval into subintervals

**regular partition**

a partition in which the subintervals all have the same width

**riemann sum**

an estimate of the area under the curve of the form $A \approx \sum_{i=1}^{n} f(x_i^*)\Delta x$

**right-endpoint approximation**

the right-endpoint approximation is an approximation of the area of the rectangles under a curve using the right endpoint of each subinterval to construct the vertical sides of each rectangle

**sigma notation**

(also, **summation notation**) the Greek letter sigma ($\Sigma$) indicates addition of the values; the values of the index above and below the sigma indicate where to begin the summation and where to end it

**upper sum**

a sum obtained by using the maximum value of $f(x)$ on each subinterval

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# 1.1E: Exercises for Section 1.1

1.) State whether the given sums are equal or unequal.

a. $\sum_{i=1}^{10} i$ and $\sum_{k=1}^{10} k$

b. $\sum_{i=1}^{10} i$ and $\sum_{i=6}^{15} (i-5)$

c. $\sum_{i=1}^{10} i(i-1)$ and $\sum_{j=0}^{9} (j+1)j$

d. $\sum_{i=1}^{10} i(i-1)$ and $\sum_{k=1}^{10} (k^2-k)$

**Answer**

a. They are equal; both represent the sum of the first 10 whole numbers.

b. They are equal; both represent the sum of the first 10 whole numbers.

c. They are equal by substituting $j = i-1$.

d. They are equal; the first sum factors the terms of the second.

**In exercises 2 - 3, use the rules for sums of powers of integers to compute the sums.**

2) $\sum_{i=5}^{10} i$

3) $\sum_{i=5}^{10} i^2$

**Answer**

$$\sum_{i=5}^{10} i^2 = 385 - 30 = 355$$

**Suppose that $\sum_{i=1}^{100} a_i = 15$ and $\sum_{i=1}^{100} b_i = -12$.**

**In exercises 4 - 7, compute the sums.**

4) $\sum_{i=1}^{100} (a_i + b_i)$

5) $\sum_{i=1}^{100} (a_i - b_i)$

**Answer**

$$\sum_{i=1}^{100} (a_i - b_i) = 15 - (-12) = 27$$

6) $\sum_{i=1}^{100} (3a_i - 4b_i)$

7) $\sum_{i=1}^{100} (5a_i + 4b_i)$

**Answer**

$$\sum_{i=1}^{100} (5a_i + 4b_i) = 5(15) + 4(-12) = 27$$

**In exercises 8 - 11, use summation properties and formulas to rewrite and evaluate the sums.**

8) $\sum_{k=1}^{20} 100(k^2 - 5k + 1)$

9) $\sum_{j=1}^{50} (j^2 - 2j)$

**Answer**

$$\sum_{j=1}^{50} j^2 - 2\sum_{j=1}^{50} j = \frac{(50)(51)(101)}{6} - \frac{2(50)(51)}{2} = 40,375$$

10) $\sum_{j=11}^{20} (j^2 - 10j)$

11) $\sum_{k=1}^{25} [(2k)^2 - 100k]$

**Answer**

$$4\sum_{k=1}^{25} k^2 - 100\sum_{k=1}^{25} k = \frac{4(25)(26)(51)}{9} - 50(25)(26) = -10,400$$

**Let $L_n$ denote the left-endpoint sum using $n$ subintervals and let $R_n$ denote the corresponding right-endpoint sum.**

**In exercises 12 - 19, compute the indicated left and right sums for the given functions on the indicated interval.**

12) $L_4$ for $f(x) = \frac{1}{x-1}$ on $[2, 3]$

13) $R_4$ for $g(x) = \cos(\pi x)$ on $[0, 1]$

**Answer**

$R_4 = 0.25$

14) $L_6$ for $f(x) = \frac{1}{x(x-1)}$ on $[2, 5]$

15) $R_6$ for $f(x) = \frac{1}{x(x-1)}$ on $[2, 5]$

**Answer**

$R_6 = 0.372$

16) $R_4$ for $\frac{1}{x^2+1}$ on $[-2, 2]$

17) $L_4$ for $\frac{1}{x^2+1}$ on $[-2, 2]$

**Answer**

$L_4 = 2.20$

18) $R_4$ for $x^2 - 2x + 1$ on $[0, 2]$

19) $L_8$ for $x^2 - 2x + 1$ on $[0, 2]$

**Answer**

$L_8 = 0.6875$

20) Compute the left and right Riemann sums— $L_4$ and $R_4$, respectively—for $f(x) = (2 - |x|)$ on $[-2, 2]$. Compute their average value and compare it with the area under the graph of $f$.

21) Compute the left and right Riemann sums— $L_6$ and $R_6$, respectively—for $f(x) = (3 - |3 - x|)$ on $[0, 6]$. Compute their average value and compare it with the area under the graph of $f$.

**Answer**

$L_6 = 9.000 = R_6$. The graph of $f$ is a triangle with area $9\,\text{units}^2$.

22) Compute the left and right Riemann sums— $L_4$ and $R_4$, respectively—for $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$ and compare their values.

23) Compute the left and right Riemann sums— $L_6$ and $R_6$, respectively—for $f(x) = \sqrt{9 - (x-3)^2}$ on $[0, 6]$ and compare their values.

**Answer**

$L_6 = 13.12899 = R_6$. They are equal.

**For exercises 24 - 27, express the following endpoint sums in sigma notation but do not evaluate them.**

24) $L_{30}$ for $f(x) = x^2$ on $[1, 2]$

25) $L_{10}$ for $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$

**Answer**

$$L_{10} = \frac{4}{10} \sum_{i=1}^{10} \sqrt{4 - (-2 + 4\frac{(i-1)}{10})}$$

26) $R_{20}$ for $f(x) = \sin x$ on $[0, \pi]$

27) $R_{100}$ for $\ln x$ on $[1, e]$

**Answer**

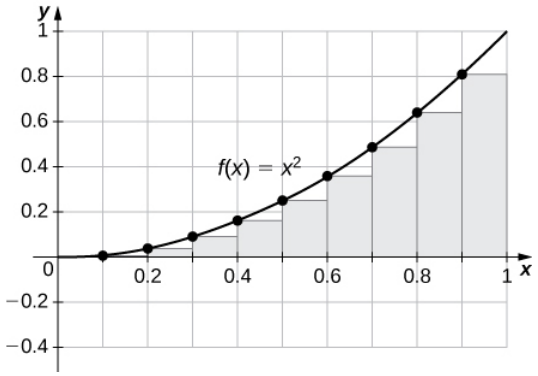
$$R_{100} = \frac{e-1}{100} \sum_{i=1}^{100} ln(1 + (e-1)\frac{i}{100})$$

**In exercises 28 - 33, graph the function then use a calculator or a computer program to evaluate the following left and right endpoint sums. Is the area under the curve between the left and right endpoint sums?**

28) [T] $L_{100}$ and $R_{100}$ for $y = x^2 - 3x + 1$ on the interval $[-1, 1]$

29) [T] $L_{100}$ and $R_{100}$ for $y = x^2$ on the interval $[0, 1]$

**Answer**



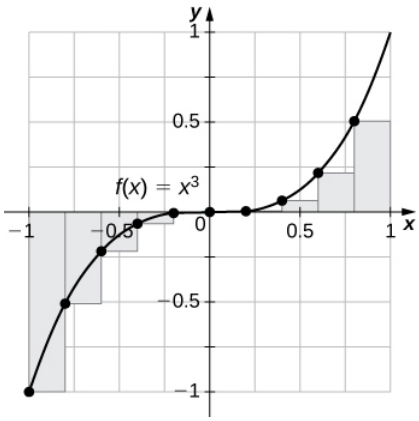

$R_{100} = 0.33835, \quad L_{100} = 0.32835.$

The plot shows that the left Riemann sum is an underestimate because the function is increasing. Similarly, the right Riemann sum is an overestimate. The area lies between the left and right Riemann sums. Ten rectangles are shown for visual clarity. This behavior persists for more rectangles.

30) [T] $L_{50}$ and $R_{50}$ for $y = \dfrac{x+1}{x^2-1}$ on the interval $[2, 4]$

31) [T] $L_{100}$ and $R_{100}$ for $y = x^3$ on the interval $[-1, 1]$

**Answer**



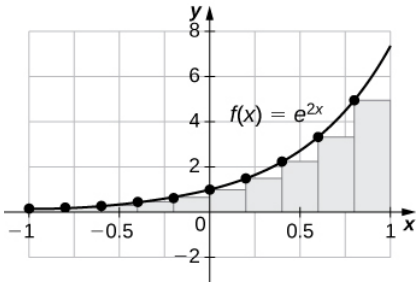

$L_{100} = -0.02, \quad R_{100} = 0.02.$

The left endpoint sum is an underestimate because the function is increasing. Similarly, a right endpoint approximation is an overestimate. The area lies between the left and right endpoint estimates.

32) [T] $L_{50}$ and $R_{50}$ for $y = \tan(x)$ on the interval $[0, \frac{\pi}{4}]$

33) [T] $L_{100}$ and $R_{100}$ for $y = e^{2x}$ on the interval $[-1, 1]$

**Answer**




$L_{100} = 3.555, \quad R_{100} = 3.670.$

The plot shows that the left Riemann sum is an underestimate because the function is increasing. Ten rectangles are shown for visual clarity. This behavior persists for more rectangles.

34) Let $t_j$ denote the time that it took Tejay van Garteren to ride the $j^{\text{th}}$ stage of the Tour de France in 2014. If there were a total of 21 stages, interpret $\sum_{j=1}^{21} t_j$.

35) Let $r_j$ denote the total rainfall in Portland on the $j^{\text{th}}$ day of the year in 2009. Interpret $\sum_{j=1}^{31} r_j$.

**Answer**

The sum represents the cumulative rainfall in January 2009.

36) Let $d_j$ denote the hours of daylight and $\delta_j$ denote the increase in the hours of daylight from day $j-1$ to day $j$ in Fargo, North Dakota, on the $j^{\text{th}}$ day of the year. Interpret $d1 + \sum_{j=2}^{365} \delta_j$.

37) To help get in shape, Joe gets a new pair of running shoes. If Joe runs 1 mi each day in week 1 and adds $\frac{1}{10}$ mi to his daily routine each week, what is the total mileage on Joe's shoes after 25 weeks?

**Answer**

The total mileage is $7 \times \sum_{i=1}^{25}(1 + \frac{(i-1)}{10}) = 7 \times 25 + \frac{7}{10} \times 12 \times 25 = 385$ mi.

38) The following table gives approximate values of the average annual atmospheric rate of increase in carbon dioxide ($CO_2$) each decade since 1960, in parts per million (ppm). Estimate the total increase in atmospheric $CO_2$ between 1964 and 2013.

| Decade | Ppm/y |
|---|---|
| 1964-1973 | 1.07 |
| 1976-1983 | 1.34 |
| 1984-1993 | 1.40 |
| 1994-2003 | 1.87 |
| 2004-2013 | 2.07 |

Average Annual Atmospheric CO2 Increase, 1964–2013 Source: http://www.esrl.noaa.gov/gmd/ccgg/trends/.

39) The following table gives the approximate increase in sea level in inches over 20 years starting in the given year. Estimate the net change in mean sea level from 1870 to 2010.

| Starting Year | 20- Year Change |
|---|---|
| 1870 | 0.3 |
| 1890 | 1.5 |
| 1910 | 0.2 |
| 1930 | 2.8 |
| 1950 | 0.7 |
| 1970 | 1.1 |
| 1990 | 1.5 |

Approximate 20-Year Sea Level Increases, 1870–1990

Source: http://link.springer.com/article/10....712-011-9119-1

**Answer**

Add the numbers to get 8.1-in. net increase.

40) The following table gives the approximate increase in dollars in the average price of a gallon of gas per decade since 1950. If the average price of a gallon of gas in 2010 was $2.60, what was the average price of a gallon of gas in 1950?

| Starting Year | 10- Year Change |
|---|---|
| 1950 | 0.03 |
| 1960 | 0.05 |
| 1970 | 0.86 |
| 1980 | −0.03 |
| 1990 | 0.29 |
| 2000 | 1.12 |

Approximate 10-Year Gas Price Increases, 1950–2000

Source: epb.lbl.gov/homepages/Rick_Di...011-trends.pdf.

41) The following table gives the percent growth of the U.S. population beginning in July of the year indicated. If the U.S. population was 281,421,906 in July 2000, estimate the U.S. population in July 2010.

| Year | % Change/Year |
|---|---|
| 2000 | 1.12 |
| 2001 | 0.99 |
| 2002 | 0.93 |
| 2003 | 0.86 |
| 2004 | 0.93 |
| 2005 | 0.93 |
| 2006 | 0.97 |
| 2007 | 0.96 |
| 2008 | 0.95 |
| 2009 | 0.88 |

Annual Percentage Growth of U.S. Population, 2000–2009

Source: www.census.gov/popest/data.

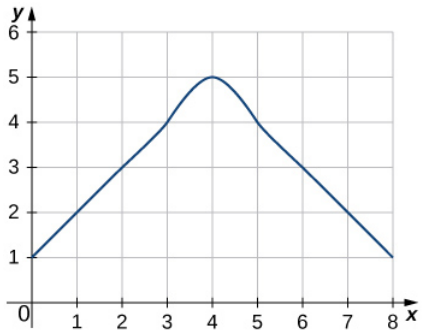
(Hint: To obtain the population in July 2001, multiply the population in July 2000 by 1.0112 to get 284,573,831.)

**Answer**

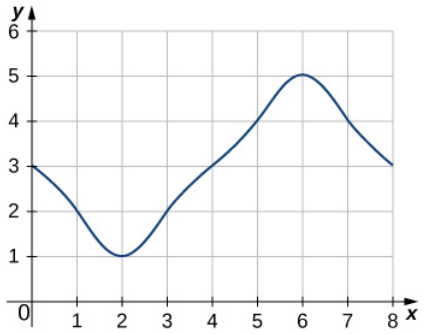
309,389,957

**In exercises 42 - 45, estimate the areas under the curves by computing the left Riemann sums, $L_8$.**

42)



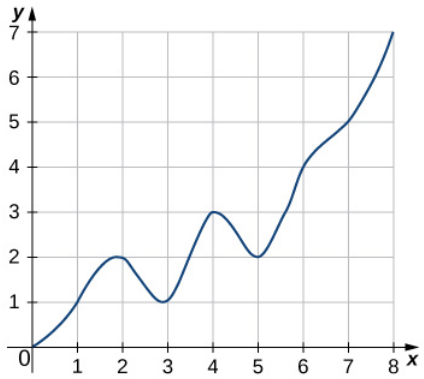

43)



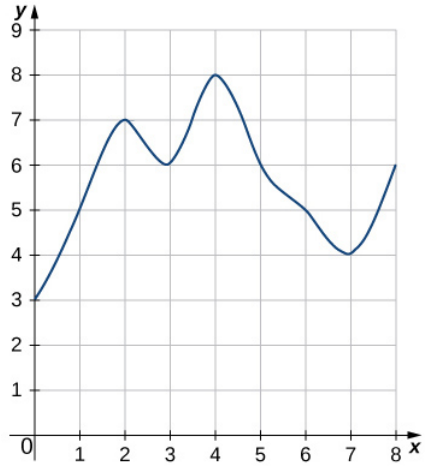

**Answer**

$L_8 = 3+2+1+2+3+4+5+4 = 24$

44)




45)




**Answer**

$L_8 = 3+5+7+6+8+6+5+4 = 44$

46) [T] Use a computer algebra system to compute the Riemann sum, $L_N$, for $N = 10, 30, 50$ for $f(x) = \sqrt{1-x^2}$ on $[-1, 1]$.

47) [T] Use a computer algebra system to compute the Riemann sum, $L_N$, for $N = 10, 30, 50$ for $f(x) = \dfrac{1}{\sqrt{1+x^2}}$ on $[-1, 1]$.

**Answer**

$L_{10} \approx 1.7604, \quad L_{30} \approx 1.7625, \quad L_{50} \approx 1.76265$


https://math.libretexts.org/@go/page/83590

48) [T] Use a computer algebra system to compute the Riemann sum, $L_N$, for $N = 10, 30, 50$ for $f(x) = \sin^2 x$ on $[0, 2\pi]$. Compare these estimates with $\pi$.

**In exercises 49-50, use a calculator or a computer program to evaluate the endpoint sums $R_N$ and $L_N$ for $N = 1, 10, 100$. How do these estimates compare with the exact answers, which you can find via geometry?**

49) [T] $y = \cos(\pi x)$ on the interval $[0, 1]$

**Answer**

$R_1 = -1, \quad L_1 = 1,$
$R_{10} = -0.1, \quad L_{10} = 0.1,$
$L_{100} = 0.01, \quad R_{100} = -0.01.$
By symmetry of the graph, the exact area is zero.

50) [T] $y = 3x + 2$ on the interval $[3, 5]$

**In exercises 51 - 52, use a calculator or a computer program to evaluate the endpoint sums $R_N$ and $L_N$ for $N = 1, 10, 100$.**

51) [T] $y = x^4 - 5x^2 + 4$ on the interval $[-2, 2]$, which has an exact area of $\frac{32}{15}$

**Answer**

$R_1 = 0, \quad L_1 = 0,$
$R_{10} = 2.4499, \quad L_{10} = 2.4499,$
$R_{100} = 2.1365, \quad L_{100} = 2.1365$

52) [T] $y = \ln x$ on the interval $[1, 2]$, which has an exact area of $2\ln(2) - 1$

53) Explain why, if $f(a) \geq 0$ and $f$ is increasing on $[a, b]$, that the left endpoint estimate is a lower bound for the area below the graph of $f$ on $[a, b]$.

**Answer**

If $[c, d]$is a subinterval of $[a, b]$ under one of the left-endpoint sum rectangles, then the area of the rectangle contributing to the left-endpoint estimate is $f(c)(d - c)$. But, $f(c) \leq f(x)$ for $c \leq x \leq d$, so the area under the graph of $f$ between $c$ and $d$ is $f(c)(d - c)$ plus the area below the graph of $f$ but above the horizontal line segment at height $f(c)$, which is positive. As this is true for each left-endpoint sum interval, it follows that the left Riemann sum is less than or equal to the area below the graph of $f$ on $[a, b]$.

54) Explain why, if $f(b) \geq 0$ and f is decreasing on $[a, b]$, that the left endpoint estimate is an upper bound for the area below the graph of $f$ on $[a, b]$.

55) Show that, in general, $R_N - L_N = (b - a) \times \dfrac{f(b) - f(a)}{N}$.

**Answer**

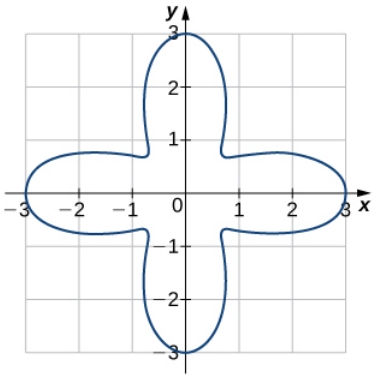
$L_N = \dfrac{b-a}{N} \displaystyle\sum_{i=1}^{N} f(a + (b-a)\dfrac{i-1}{N}) = \dfrac{b-a}{N} \displaystyle\sum_{i=0}^{N-1} f(a + (b-a)\dfrac{i}{N})$ and $R_N = \dfrac{b-a}{N} \displaystyle\sum_{i=1}^{N} f(a + (b-a)\dfrac{i}{N})$. The left sum has a term corresponding to $i = 0$ and the right sum has a term corresponding to $i = N$. In $R_N - L_N$, any term corresponding to $i = 1, 2, \ldots, N-1$ occurs once with a plus sign and once with a minus sign, so each such term cancels and one is left with $R_N - L_N = \dfrac{b-a}{N}(f(a + (b-a))\dfrac{N}{N}) - (f(a) + (b-a)\dfrac{0}{N}) = \dfrac{b-a}{N}(f(b) - f(a))$.

56) Explain why, if $f$ is increasing on $[a, b]$, the error between either $L_N$ or $R_N$ and the area $A$ below the graph of $f$ is at most $(b - a)\dfrac{f(b) - f(a)}{N}$.

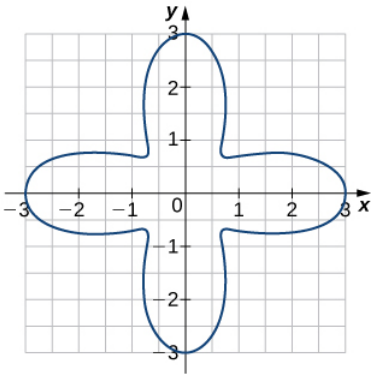
57) For each of the three graphs:

a. Obtain a lower bound $L(A)$ for the area enclosed by the curve by adding the areas of the squares enclosed completely by the curve.

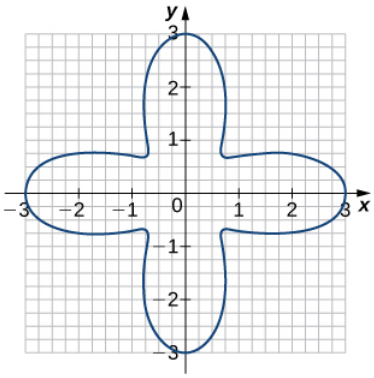
b. Obtain an upper bound $U(A)$ for the area by adding to $L(A)$ the areas $B(A)$ of the squares enclosed partially by the curve.




Graph 1




Graph 2




Graph 3

**Answer**

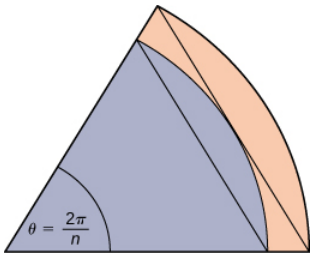
Graph 1: a. $L(A) = 0, \quad B(A) = 20; \quad b. U(A) = 20.$

Graph 2: $a. L(A) = 9; \quad b. B(A) = 11, \quad U(A) = 20.$

Graph 3: a. $L(A) = 11.0; \quad b. B(A) = 4.5, \quad U(A) = 15.5.$

58) In the previous exercise, explain why $L(A)$ gets no smaller while $U(A)$ gets no larger as the squares are subdivided into four boxes of equal area.

59) A unit circle is made up of n wedges equivalent to the inner wedge in the figure. The base of the inner triangle is 1 unit and its height is $\sin(\frac{\pi}{n})$. The base of the outer triangle is $B = \cos(\frac{\pi}{n}) + \sin(\frac{\pi}{n})\tan(\frac{\pi}{n})$ and the height is $H = B\sin(\frac{2\pi}{n})$. Use this information to argue that the area of a unit circle is equal to $\pi$.




**Answer**

Let $A$ be the area of the unit circle. The circle encloses $n$ congruent triangles each of area $\frac{\sin(\frac{2\pi}{n})}{2}$, so $\frac{n}{2}\sin(\frac{2\pi}{n}) \leq A$. Similarly, the circle is contained inside $n$ congruent triangles each of area $\frac{BH}{2} = \frac{1}{2}(\cos(\frac{\pi}{n}) + \sin(\frac{\pi}{n})\tan(\frac{\pi}{n}))\sin(\frac{2\pi}{n})$, so $A \leq \frac{n}{2}\sin(\frac{2\pi}{n})(\cos(\frac{\pi}{n})) + \sin(\frac{\pi}{n})\tan(\frac{\pi}{n})$. As $n \to \infty$, $\frac{n}{2}\sin(\frac{2\pi}{n}) = \frac{\pi\sin(\frac{2\pi}{n})}{(\frac{2\pi}{n})} \to \pi$, so we conclude $\pi \leq A$. Also, as $n \to \infty$, $\cos(\frac{\pi}{n}) + \sin(\frac{\pi}{n})\tan(\frac{\pi}{n}) \to 1$, so we also have $A \leq \pi$. By the squeeze theorem for limits, we conclude that $A = \pi$.

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# 1.2: The Definite Integral

**Learning Objectives**

- State the definition of the definite integral.
- Explain the terms integrand, limits of integration, and variable of integration.
- Explain when a function is integrable.
- Describe the relationship between the definite integral and net area.
- Use geometry and the properties of definite integrals to evaluate them.
- Calculate the average value of a function.

In the preceding section we defined the area under a curve in terms of Riemann sums:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x. \tag{1.2.2}$$

However, this definition came with restrictions. We required $f(x)$ to be continuous and nonnegative. Unfortunately, real-world problems don't always meet these restrictions. In this section, we look at how to apply the concept of the area under the curve to a broader set of functions through the use of the definite integral.

## Definition and Notation

The definite integral generalizes the concept of the area under a curve. We lift the requirements that $f(x)$ be continuous and nonnegative, and define the definite integral as follows.

**Definition: Definite Integral**

If $f(x)$ is a function defined on an interval $[a, b]$, the ***definite integral*** of $f$ from $a$ to $b$ is given by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x, \tag{1.2.3}$$

provided the limit exists. If this limit exists, the function $f(x)$ is said to be integrable on $[a, b]$, or is an integrable function.

The integral symbol in the previous definition should look familiar. We have seen similar notation in the chapter on Applications of Derivatives, where we used the indefinite integral symbol (without the $a$ and $b$ above and below) to represent an antiderivative. Although the notation for indefinite integrals may look similar to the notation for a definite integral, they are not the same. A definite integral is a number. An indefinite integral is a family of functions. Later in this chapter we examine how these concepts are related. However, close attention should always be paid to notation so we know whether we're working with a definite integral or an indefinite integral.

Integral notation goes back to the late seventeenth century and is one of the contributions of Gottfried Wilhelm **Leibniz**, who is often considered to be the codiscoverer of calculus, along with Isaac Newton. The integration symbol $\int$ is an elongated $S$, suggesting sigma or summation. On a definite integral, above and below the summation symbol are the boundaries of the interval, $[a, b]$. The numbers $a$ and $b$ are $x$-values and are called the **limits of integration**; specifically, $a$ is the lower limit and $b$ is the upper limit. To clarify, we are using the word limit in two different ways in the context of the definite integral. First, we talk about the limit of a sum as $n \to \infty$. Second, the boundaries of the region are called the *limits of integration.*

We call the function $f(x)$ the **integrand**, and the $dx$ indicates that $f(x)$ is a function with respect to $x$, called the **variable of integration**. Note that, like the index in a sum, the variable of integration is a **dummy variable**, and has no impact on the computation of the integral. We could use any variable we like as the variable of integration:

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(u)\,du \tag{1.2.4}$$

Previously, we discussed the fact that if $f(x)$ is continuous on $[a,b]$, then the limit $\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$ exists and is unique. This leads to the following theorem, which we state without proof.

**Continuous Functions Are Integrable**

If $f(x)$ is continuous on $[a,b]$, then $f$ is integrable on $[a,b]$.

Functions that are not continuous on $[a,b]$ may still be integrable, depending on the nature of the discontinuities. For example, functions with a finite number of jump discontinuities or removable discontinuities on a closed interval are integrable.

It is also worth noting here that we have retained the use of a regular partition in the Riemann sums. This restriction is not strictly necessary. Any partition can be used to form a Riemann sum. However, if a nonregular partition is used to define the definite integral, it is not sufficient to take the limit as the number of subintervals goes to infinity. Instead, we must take the limit as the width of the largest subinterval goes to zero. This introduces a little more complex notation in our limits and makes the calculations more difficult without really gaining much additional insight, so we stick with regular partitions for the Riemann sums.



**Example 1.2.1: Evaluating an Integral Using the Definition**

Use the definition of the definite integral to evaluate $\int_0^2 x^2\,dx$. Use a right-endpoint approximation to generate the Riemann sum.

**Solution**

We first want to set up a Riemann sum. Based on the limits of integration, we have $a=0$ and $b=2$. For $i=0,1,2,\ldots,n$, let $P=x_i$ be a regular partition of $[0,2]$. Then

$$\Delta x=\frac{b-a}{n}=\frac{2}{n}.$$

Since we are using a right-endpoint approximation to generate Riemann sums, for each $i$, we need to calculate the function value at the right endpoint of the interval $[x_{i-1},x_i]$. The right endpoint of the interval is $x_i$, and since $P$ is a regular partition,

$$x_i=x_0+i\Delta x=0+i\left[\frac{2}{n}\right]=\frac{2i}{n}.$$

Thus, the function value at the right endpoint of the interval is

$$f(x_i)=x_i^2=\left(\frac{2i}{n}\right)^2=\frac{4i^2}{n^2}.$$

Then the Riemann sum takes the form

$$\sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} \left(\frac{4i^2}{n^2}\right)\frac{2}{n} = \sum_{i=1}^{n} \frac{8i^2}{n^3} = \frac{8}{n^3}\sum_{i=1}^{n} i^2.$$

Using the summation formula for $\sum_{i=1}^{n} i^2$, we have

$$\begin{aligned}\sum_{i=1}^{n} f(x_i)\Delta x &= \frac{8}{n^3}\sum_{i=1}^{n} i^2 \\ &= \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] \\ &= \frac{8}{n^3}\left[\frac{2n^3+3n^2+n}{6}\right] \\ &= \frac{16n^3+24n^2+n}{6n^3} \\ &= \frac{8}{3}+\frac{4}{n}+\frac{1}{6n^2}.\end{aligned}$$

Now, to calculate the definite integral, we need to take the limit as $n \to \infty$. We get

$$\begin{aligned}\int_0^2 x^2\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x \\ &= \lim_{n\to\infty}\left(\frac{8}{3}+\frac{4}{n}+\frac{1}{6n^2}\right) \\ &= \lim_{n\to\infty}\left(\frac{8}{3}\right)+\lim_{n\to\infty}\left(\frac{4}{n}\right)+\lim_{n\to\infty}\left(\frac{1}{6n^2}\right) \\ &= \frac{8}{3}+0+0=\frac{8}{3}.\end{aligned}$$

**Exercise 1.2.1**

Use the definition of the definite integral to evaluate $\int_0^3 (2x-1)\,dx$.

Use a right-endpoint approximation to generate the Riemann sum.

**Hint**

Use the solving strategy from Example **1.2.1**.

**Answer**

6

## Evaluating Definite Integrals

Evaluating definite integrals this way can be quite tedious because of the complexity of the calculations. Later in this chapter we develop techniques for evaluating definite integrals *without* taking limits of Riemann sums. However, for now, we can rely on the fact that definite integrals represent the area under the curve, and we can evaluate definite integrals by using geometric formulas to calculate that area. We do this to confirm that definite integrals do, indeed, represent areas, so we can then discuss what to do in the case of a curve of a function dropping below the $x$-axis.

**Example 1.2.2: Using Geometric Formulas to Calculate Definite Integrals**

Use the formula for the area of a circle to evaluate $\int_3^6 \sqrt{9-(x-3)^2}\,dx$.

**Solution**

The function describes a semicircle with radius 3. To find

$$\int_3^6 \sqrt{9-(x-3)^2}\,dx$$

we want to find the area under the curve over the interval $[3,6]$. The formula for the area of a circle is $A=\pi r^2$. The area of a semicircle is just one-half the area of a circle, or $A=\frac{1}{2}\pi r^2$. The shaded area in Figure covers one-half of the semicircle, or $A=\frac{1}{4}\pi r^2$. Thus,

$$\int_3^6 \sqrt{9-(x-3)^2}\,dx=\frac{1}{4}\pi(3)^2=\frac{9}{4}\pi\approx 7.069.$$

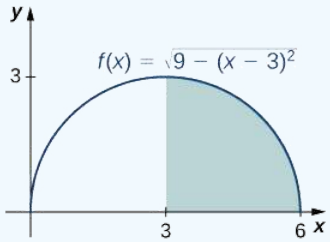



Figure 1.2.1: The value of the integral of the function $f(x)$ over the interval $[3,6]$ is the area of the shaded region.

## Exercise 1.2.2

Use the formula for the area of a trapezoid to evaluate $\int_2^4 (2x+3)\,dx$.

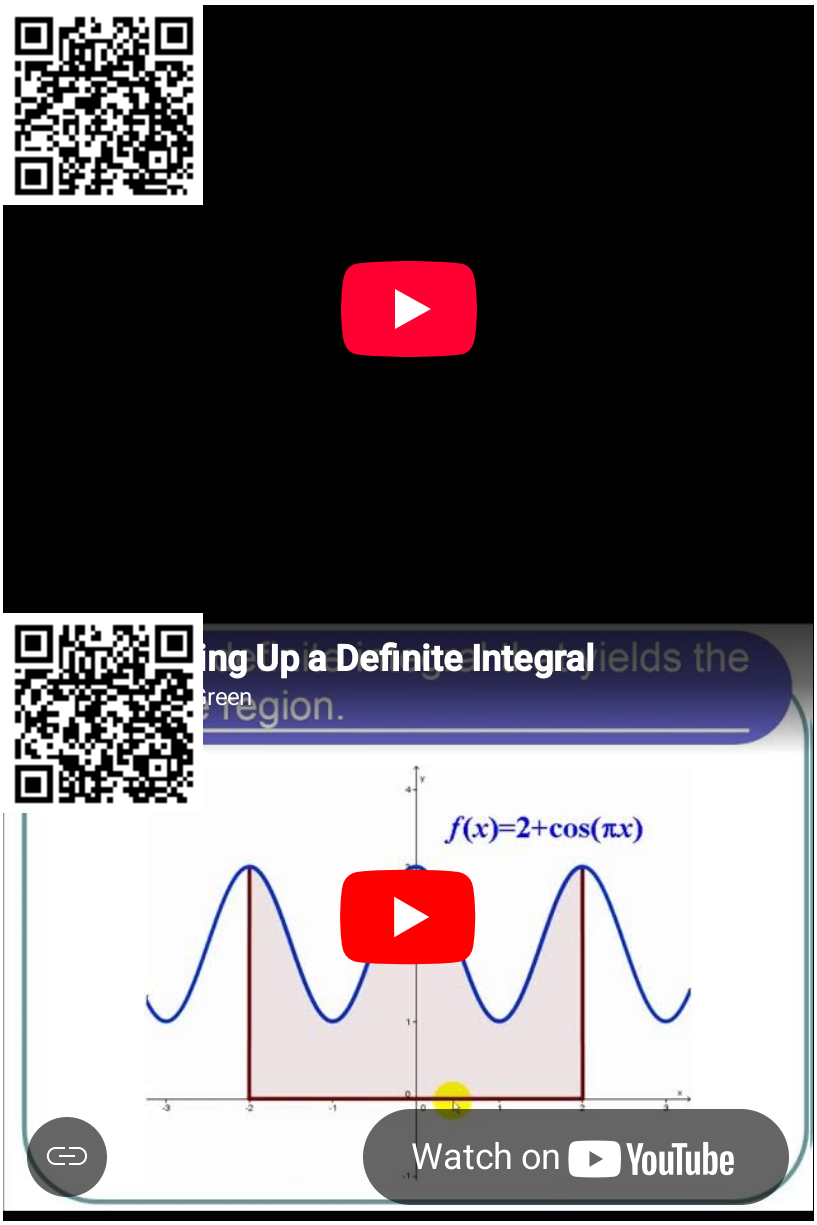
**Hint**

Graph the function $f(x)$ and calculate the area under the function on the interval $[2,4]$.

**Answer**

18 square units






## Area and the Definite Integral

When we defined the definite integral, we lifted the requirement that $f(x)$ be nonnegative. But how do we interpret "the area under the curve" when $f(x)$ is negative?

### Net Signed Area

Let us return to the Riemann sum. Consider, for example, the function $f(x) = 2 - 2x^2$ (shown in Figure 1.2.2) on the interval $[0, 2]$. Use $n = 8$ and choose $\{x_i^*\}$ as the left endpoint of each interval. Construct a rectangle on each subinterval of height $f(x_i^*)$ and width $\Delta x$. When $f(x_i^*)$ is positive, the product $f(x_i^*)\Delta x$ represents the area of the rectangle, as before. When $f(x_i^*)$ is negative, however, the product $f(x_i^*)\Delta x$ represents the negative of the area of the rectangle. The Riemann sum then becomes

$$\sum_{i=1}^{8} f(x_i^*)\Delta x = (\text{Area of rectangles above the } x\text{-axis}) - (\text{Area of rectangles below the } x\text{-axis})$$

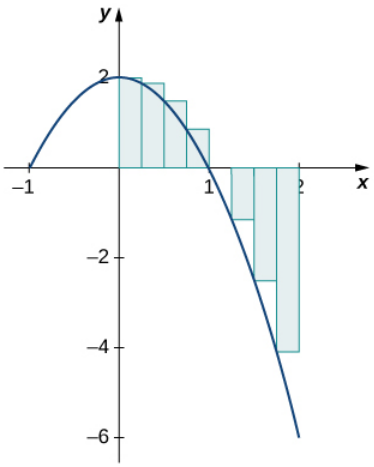



Figure 1.2.2: For a function that is partly negative, the Riemann sum is the area of the rectangles above the $x$-axis less the area of the rectangles below the $x$-axis.

Taking the limit as $n \to \infty$, the Riemann sum approaches the area between the curve above the $x$-axis and the $x$-axis, less the area between the curve below the $x$-axis and the $x$-axis, as shown in Figure 1.2.3. Then,

$$\int_0^2 f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x = A_1 - A_2. \tag{1.2.5}$$

The quantity $A_1 - A_2$ is called the **net signed area.**

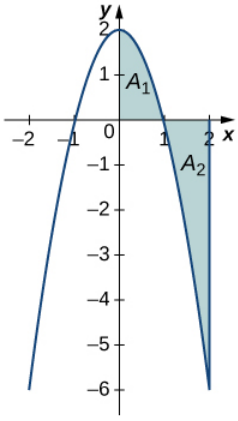



Figure 1.2.3: In the limit, the definite integral equals area $A_1$ less area $A_2$, or the net signed area.

Notice that net signed area can be positive, negative, or zero. If the area above the $x$-axis is larger, the net signed area is positive. If the area below the $x$-axis is larger, the net signed area is negative. If the areas above and below the $x$-axis are equal, the net signed area is zero.

**Example 1.2.3: Finding the Net Signed Area**

Find the net signed area between the curve of the function $f(x) = 2x$ and the $x$-axis over the interval $[-3, 3]$.

**Solution**

The function produces a straight line that forms two triangles: one from $x = -3$ to $x = 0$ and the other from $x = 0$ to $x = 3$ (Figure 1.2.4). Using the geometric formula for the area of a triangle, $A = \frac{1}{2}bh$, the area of triangle $A_1$, above the axis, is

$$A_1 = \frac{1}{2}3(6) = 9,$$

where $3$ is the base and $2(3) = 6$ is the height. The area of triangle $A_2$, below the axis, is

$$A_2 = \frac{1}{2}(3)(6) = 9,$$

where $3$ is the base and $6$ is the height. Thus, the net area is

$$\int_{-3}^{3} 2x\,dx = A_1 - A_2 = 9 - 9 = 0.$$

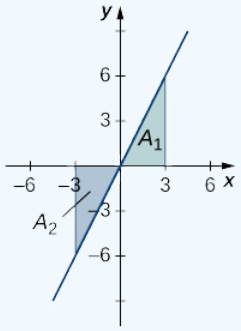


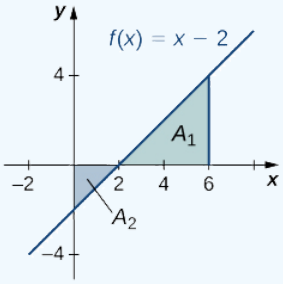

Figure 1.2.4: The area above the curve and below the $x$-axis equals the area below the curve and above the $x$-axis.

**Analysis**

If $A_1$ is the area above the $x$-axis and $A_2$ is the area below the $x$-axis, then the net area is $A_1 - A_2$. Since the areas of the two triangles are equal, the net area is zero.

**Exercise 1.2.3**

Find the net signed area of $f(x) = x - 2$ over the interval $[0, 6]$, illustrated in the following image.




**Hint**

Use the solving method described in Example 1.2.3.

**Answer**

6

## Total Area

One application of the definite integral is finding **displacement** when given a velocity function. If $v(t)$ represents the velocity of an object as a function of time, then the area under the curve tells us how far the object is from its original position. This is a very important application of the definite integral, and we examine it in more detail later in the chapter. For now, we're just going to look at some basics to get a feel for how this works by studying constant velocities.

When velocity is a constant, the area under the curve is just velocity times time. This idea is already very familiar. If a car travels away from its starting position in a straight line at a speed of $70$ mph for $2$ hours, then it is $140$ mi away from its original position (Figure 1.2.5). Using integral notation, we have

$$\int_0^2 70\,dt = 140 \text{ miles}.$$

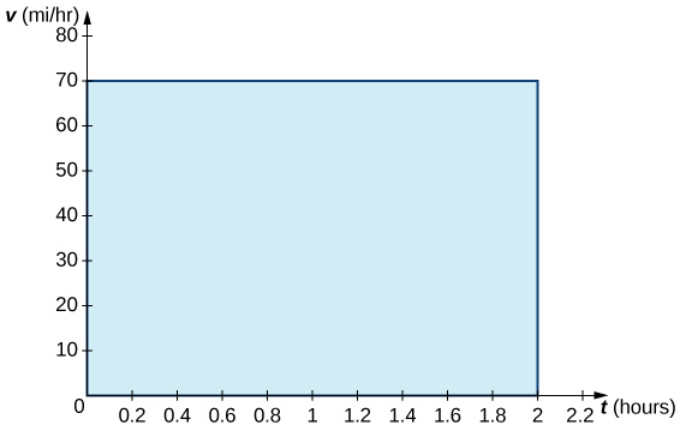



Figure **1.2.5**: The area under the curve $v(t) = 70$ tells us how far the car is from its starting point at a given time.

In the context of displacement, net signed area allows us to take direction into account. If a car travels straight north at a speed of 60 mph for 2 hours, it is 120 mi north of its starting position. If the car then turns around and travels south at a speed of 40 mph for 3 hours, it will be back at it starting position (Figure **1.2.6**). Again, using integral notation, we have

$$\int_0^2 60\,dt + \int_2^5 -40\,dt = 120 - 120 = 0.$$

In this case the displacement is zero.

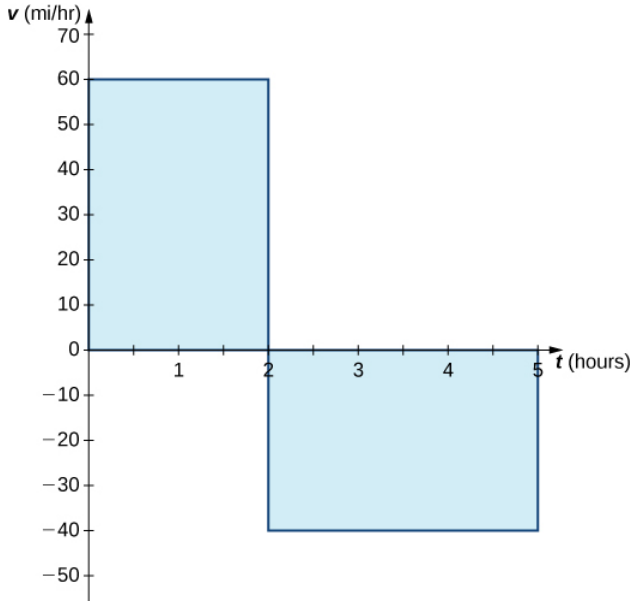



Figure **1.2.6**: The area above the axis and the area below the axis are equal, so the net signed area is zero.

Suppose we want to know how far the car travels overall, regardless of direction. In this case, we want to know the area between the curve and the $t$-axis, regardless of whether that area is above or below the axis. This is called the **total area.**

Graphically, it is easiest to think of calculating total area by adding the areas above the axis and the areas below the axis (rather than subtracting the areas below the axis, as we did with net signed area). To accomplish this mathematically, we use the absolute value function. Thus, the total distance traveled by the car is

$$\int_0^2 |60|\,dt + \int_2^5 |-40|\,dt = \int_0^2 60\,dt + \int_2^5 40\,dt = 120 + 120 = 240.$$

Bringing these ideas together formally, we state the following definitions.

**Definition: Net Signed Area**

Let $f(x)$ be an integrable function defined on an interval $[a, b]$. Let $A_1$ represent the area between $f(x)$ and the $x$-axis that lies above the axis and let $A_2$ represent the area between $f(x)$ and the $x$-axis that lies below the axis. Then, the *net signed area* between $f(x)$ and the $x$-axis is given by

$$\int_a^b f(x)\,dx = A_1 - A_2. \tag{1.2.6}$$

The total area between $f(x)$ and the $x$-axis is given by

$$\int_a^b |f(x)|\,dx = A_1 + A_2. \tag{1.2.7}$$

**Example 1.2.4: Finding the Total Area**

Find the total area between $f(x) = x - 2$ and the $x$-axis over the interval $[0, 6]$.

**Solution**

Calculate the $x$-intercept as $(2, 0)$ (set $y = 0$, solve for $x$). To find the total area, take the area below the $x$-axis over the subinterval $[0, 2]$ and add it to the area above the $x$-axis on the subinterval $[2, 6]$ (Figure 1.2.7).

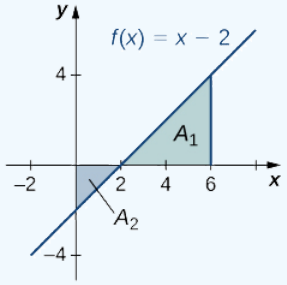



Figure 1.2.7: The total area between the line and the $x$-axis over $[0, 6]$ is $A_2$ plus $A_1$.

We have

$$\int_0^6 |(x-2)|\,dx = A_2 + A_1.$$

Then, using the formula for the area of a triangle, we obtain

$$A_2 = \frac{1}{2}bh = \frac{1}{2}\cdot 2 \cdot 2 = 2$$

$$A_1 = \frac{1}{2}bh = \frac{1}{2}\cdot 4 \cdot 4 = 8.$$

The total area, then, is

$$A_1 + A_2 = 8 + 2 = 10 \text{ units}^2.$$

**Exercise 1.2.4**

Find the total area between the function $f(x) = 2x$ and the $x$-axis over the interval $[-3, 3]$.

**Hint**

Review the solving strategy in Example 1.2.4.

**Answer**

$18 \text{ units}^2$

## Properties of the Definite Integral

The properties of indefinite integrals apply to definite integrals as well. Definite integrals also have properties that relate to the limits of integration. These properties, along with the rules of integration that we examine later in this chapter, help us manipulate expressions to evaluate definite integrals.

### Rule: Properties of the Definite Integral

1.

$$\int_a^a f(x)\,dx = 0 \tag{1.2.8}$$

If the limits of integration are the same, the integral is just a line and contains no area.

2.

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx \tag{1.2.9}$$

If the limits are reversed, then place a negative sign in front of the integral.

3.

$$\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx \tag{1.2.10}$$

The integral of a sum is the sum of the integrals.

4.

$$\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx \tag{1.2.11}$$

The integral of a difference is the difference of the integrals

5.

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx \tag{1.2.12}$$

for constant $c$. The integral of the product of a constant and a function is equal to the constant multiplied by the integral of the function.

6.

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \tag{1.2.13}$$

Although this formula normally applies when $c$ is between $a$ and $b$, the formula holds for all values of $a$, $b$, and $c$, provided $f(x)$ is integrable on the largest interval.

### Example 1.2.5: Using the Properties of the Definite Integral

Use the properties of the definite integral to express the definite integral of $f(x) = -3x^3 + 2x + 2$ over the interval $[-2, 1]$ as the sum of three definite integrals.

**Solution**

Using integral notation, we have $\int_{-2}^{1} (-3x^3 + 2x + 2)\,dx$. We apply properties 3. and 5. to get

$$\int_{-2}^{1}(-3x^3+2x+2)\,dx=\int_{-2}^{1}-3x^3\,dx+\int_{-2}^{1}2x\,dx+\int_{-2}^{1}2\,dx$$
$$=-3\int_{-2}^{1}x^3\,dx+2\int_{-2}^{1}x\,dx+\int_{-2}^{1}2\,dx.$$

## Exercise 1.2.5

Use the properties of the definite integral to express the definite integral of $f(x)=6x^3-4x^2+2x-3$ over the interval $[1,3]$ as the sum of four definite integrals.

**Hint**

Use the solving strategy from Example **1.2.5** and the properties of definite integrals.

**Answer**

$$6\int_1^3 x^3\,dx-4\int_1^3 x^2\,dx+2\int_1^3 x\,dx-\int_1^3 3\,dx$$

## Example 1.2.6: Using the Properties of the Definite Integral

If it is known that $\int_0^8 f(x)\,dx=10$ and $\int_0^5 f(x)\,dx=5$, find the value of $\int_5^8 f(x)\,dx$.

**Solution**

By property 6,

$$\int_a^b f(x)\,dx=\int_a^c f(x)\,dx+\int_c^b f(x)\,dx.$$

Thus,

$$\int_0^8 f(x)\,dx=\int_0^5 f(x)\,dx+\int_5^8 f(x)\,dx$$
$$10=5+\int_5^8 f(x)\,dx$$
$$5=\int_5^8 f(x)\,dx.$$




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**Exercise 1.2.6**

If it is known that $\int_1^5 f(x)\,dx = -3$ and $\int_2^5 f(x)\,dx = 4$, find the value of $\int_1^2 f(x)\,dx$.

**Hint**

Use the solving strategy from Example 1.2.6 and the rule on properties of definite integrals.

**Answer**

$-7$

## Comparison Properties of Integrals

A picture can sometimes tell us more about a function than the results of computations. Comparing functions by their graphs as well as by their algebraic expressions can often give new insight into the process of integration. Intuitively, we might say that if a function $f(x)$ is above another function $g(x)$, then the area between $f(x)$ and the $x$-axis is greater than the area between $g(x)$ and the $x$-axis. This is true depending on the interval over which the comparison is made. The properties of definite integrals are valid whether $a < b$, $a = b$, or $a > b$. The following properties, however, concern only the case $a \le b$, and are used when we want to compare the sizes of integrals.

**Comparison Theorem**

i. If $f(x) \ge 0$ for $a \le x \le b$, then

$$\int_a^b f(x)\,dx \ge 0. \tag{1.2.14}$$

ii. If $f(x) \ge g(x)$ for $a \le x \le b$, then

$$\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx. \tag{1.2.15}$$

iii. If $m$ and $M$ are constants such that $m \le f(x) \le M$ for $a \le x \le b$, then

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a). \tag{1.2.16}$$

**Example 1.2.7: Comparing Two Functions over a Given Interval**

Compare $f(x) = \sqrt{1+x^2}$ and $g(x) = \sqrt{1+x}$ over the interval $[0,1]$.

**Solution**

Graphing these functions is necessary to understand how they compare over the interval $[0,1]$. Initially, when graphed on a graphing calculator, $f(x)$ appears to be above $g(x)$ everywhere. However, on the interval $[0,1]$, the graphs appear to be on top of each other. We need to zoom in to see that, on the interval $[0,1]$, $g(x)$ is above $f(x)$. The two functions intersect at $x = 0$ and $x = 1$ (Figure 1.2.8).

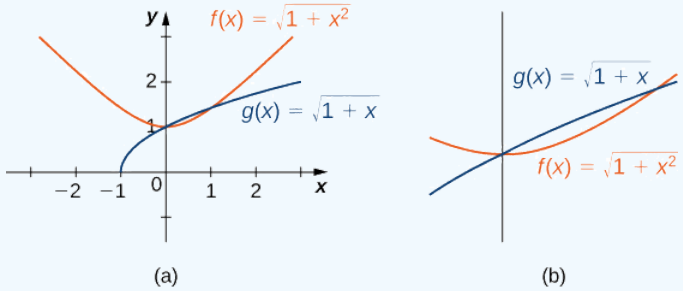
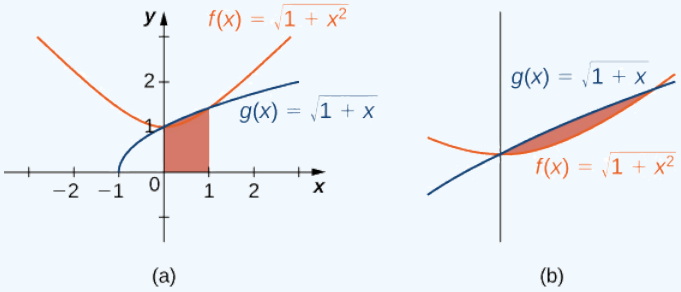



Figure 1.2.8: (a) The function $f(x)$ appears above the function $g(x)$ except over the interval $[0,1]$ (b) Viewing the same graph with a greater zoom shows this more clearly.

We can see from the graph that over the interval $[0,1]$, $g(x) \geq f(x)$. Comparing the integrals over the specified interval $[0,1]$, we also see that $\int_0^1 g(x)\,dx \geq \int_0^1 f(x)\,dx$ (Figure 1.2.9). The thin, red-shaded area shows just how much difference there is between these two integrals over the interval $[0,1]$.




(a) (b)

Figure 1.2.9: (a) The graph shows that over the interval $[0,1], g(x) \geq f(x)$, where equality holds only at the endpoints of the interval. (b) Viewing the same graph with a greater zoom shows this more clearly.

## Average Value of a Function

We often need to find the average of a set of numbers, such as an average test grade. Suppose you received the following test scores in your algebra class: 89, 90, 56, 78, 100, and 69. Your semester grade is your average of test scores and you want to know what grade to expect. We can find the average by adding all the scores and dividing by the number of scores. In this case, there are six test scores. Thus,

$$\frac{89+90+56+78+100+69}{6} = \frac{482}{6} \approx 80.33.$$

Therefore, your average test grade is approximately 80.33, which translates to a B− at most schools.

Suppose, however, that we have a function $v(t)$ that gives us the speed of an object at any time $t$, and we want to find the object's average speed. The function $v(t)$ takes on an infinite number of values, so we can't use the process just described. Fortunately, we can use a definite integral to find the average value of a function such as this.

Let $f(x)$ be continuous over the interval $[a,b]$ and let $[a,b]$ be divided into n subintervals of width $\Delta x = (b-a)/n$. Choose a representative $x_i^*$ in each subinterval and calculate $f(x_i^*)$ for $i = 1, 2, \ldots, n$. In other words, consider each $f(x_i^*)$ as a sampling of the function over each subinterval. The average value of the function may then be approximated as

$$f_{ave} \approx \frac{f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)}{n}, \tag{1.2.17}$$

which is basically the same expression used to calculate the average of discrete values.

But we know $\Delta x = \frac{b-a}{n}$, so $n = \frac{b-a}{\Delta x}$, and we get

$$f_{ave} \approx \frac{f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)}{n} = \frac{f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)}{\left(\frac{b-a}{\Delta x}\right)}. \tag{1.2.18}$$

Following through with the algebra, the numerator is a sum that is represented as $\sum_{i=1}^{n} f(x * i)$, and we are dividing by a fraction. To divide by a fraction, invert the denominator and multiply. Thus, an approximate value for the average value of the function is given by

$$\frac{\sum_{i=1}^{n} f(x_i^*)}{\left(\frac{b-a}{\Delta x}\right)} = \left(\frac{\Delta x}{b-a}\right)\sum_{i=1}^{n} f(x_i^*) = \left(\frac{1}{b-a}\right)\sum_{i=1}^{n} f(x_i^*)\Delta x. \tag{1.2.19}$$

This is a Riemann sum. Then, to get the exact average value, take the limit as $n$ goes to infinity. Thus, the average value of a function is given by

$$\frac{1}{b-a}\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \frac{1}{b-a}\int_a^b f(x)dx. \tag{1.2.20}$$

**Definition: Average Value of a Function**

Let $f(x)$ be continuous over the interval $[a, b]$. Then, the *average value of the function* $f(x)$ (or $f_{ave}$) on $[a, b]$ is given by

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx. \tag{1.2.21}$$

**Example 1.2.8: Finding the Average Value of a Linear Function**

Find the average value of $f(x) = x + 1$ over the interval $[0, 5]$.

**Solution**

First, graph the function on the stated interval, as shown in Figure 1.2.10.

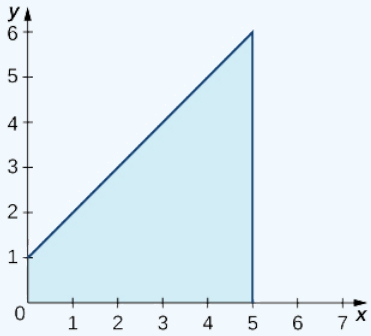



Figure 1.2.10:The graph shows the area under the function $(x) = x + 1$ over $[0, 5]$.

The region is a trapezoid lying on its side, so we can use the area formula for a trapezoid $A = \frac{1}{2}h(a + b)$, where $h$ represents height, and $a$ and $b$ represent the two parallel sides. Then,

$$\int_0^5 x + 1\,dx = \frac{1}{2}h(a + b) = \frac{1}{2}\cdot 5\cdot (1 + 6) = \frac{35}{2}.$$

Thus the average value of the function is

$$\frac{1}{5-0}\int_0^5 x + 1\,dx = \frac{1}{5}\cdot\frac{35}{2} = \frac{7}{2}.$$

**Exercise 1.2.7**

Find the average value of $f(x) = 6 - 2x$ over the interval $[0, 3]$.

**Hint**

Use the average value formula (Equation 1.2.21), and use geometry to evaluate the integral.

**Answer**

3

## Key Concepts

- The definite integral can be used to calculate net signed area, which is the area above the $x$-axis less the area below the $x$-axis. Net signed area can be positive, negative, or zero.
- The component parts of the definite integral are the integrand, the variable of integration, and the limits of integration.
- Continuous functions on a closed interval are integrable. Functions that are not continuous may still be integrable, depending on the nature of the discontinuities.
- The properties of definite integrals can be used to evaluate integrals.
- The area under the curve of many functions can be calculated using geometric formulas.
- The average value of a function can be calculated using definite integrals.

## Key Equations

- **Definite Integral**

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$$

- **Properties of the Definite Integral**

$$\int_a^a f(x)\,dx = 0$$

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

$$\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

$$\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx, \text{ for constant } c$$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

## Glossary

**average value of a function**
(or $f_{ave}$) the average value of a function on an interval can be found by calculating the definite integral of the function and dividing that value by the length of the interval

**definite integral**
a primary operation of calculus; the area between the curve and the $x$-axis over a given interval is a definite integral

**integrable function**
a function is integrable if the limit defining the integral exists; in other words, if the limit of the Riemann sums as $n$ goes to infinity exists

**integrand**
the function to the right of the integration symbol; the integrand includes the function being integrated

**limits of integration**
these values appear near the top and bottom of the integral sign and define the interval over which the function should be integrated

**net signed area**
the area between a function and the $x$-axis such that the area below the $x$-axis is subtracted from the area above the $x$-axis; the result is the same as the definite integral of the function

**total area**
total area between a function and the $x$-axis is calculated by adding the area above the $x$-axis and the area below the $x$-axis; the result is the same as the definite integral of the absolute value of the function

**variable of integration**
indicates which variable you are integrating with respect to; if it is $x$, then the function in the integrand is followed by $dx$

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# 1.2E: Exercises for Section 1.2

**In exercises 1 - 4, express the limits as integrals.**

1) $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}(x_i^*)\Delta x$ over $[1,3]$

2) $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}(5(x_i^*)^2-3(x_i^*)^3)\Delta x$ over $[0,2]$

**Answer**

$$\int_0^2 (5x^2-3x^3)\,dx$$

3) $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\sin^2(2\pi x_i^*)\Delta x$ over $[0,1]$

4) $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\cos^2(2\pi x_i^*)\Delta x$ over $[0,1]$

**Answer**

$$\int_0^1 \cos^2(2\pi x)\,dx$$

**In exercises 5 - 10, given $L_n$ or $R_n$ as indicated, express their limits as $n\to\infty$ as definite integrals, identifying the correct intervals.**

5) $L_n=\frac{1}{n}\sum\limits_{i=1}^{n}\frac{i-1}{n}$

6) $R_n=\frac{1}{n}\sum\limits_{i=1}^{n}\frac{i}{n}$

**Answer**

$$\int_0^1 x\,dx$$

7) $Ln=\frac{2}{n}\sum\limits_{i=1}^{n}(1+2\frac{i-1}{n})$

8) $R_n=\frac{3}{n}\sum\limits_{i=1}^{n}(3+3\frac{i}{n})$

**Answer**

$$\int_3^6 x\,dx$$

9) $L_n=\frac{2\pi}{n}\sum\limits_{i=1}^{n}2\pi\frac{i-1}{n}\cos(2\pi\frac{i-1}{n})$

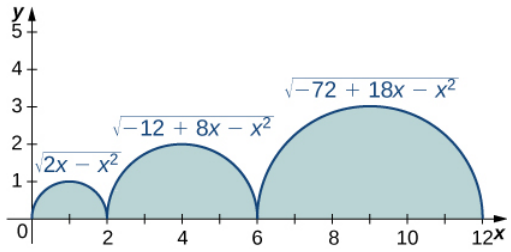
10 $R_n=\frac{1}{n}\sum\limits_{i=1}^{n}(1+\frac{i}{n})\log((1+\frac{i}{n})^2)$

**Answer**

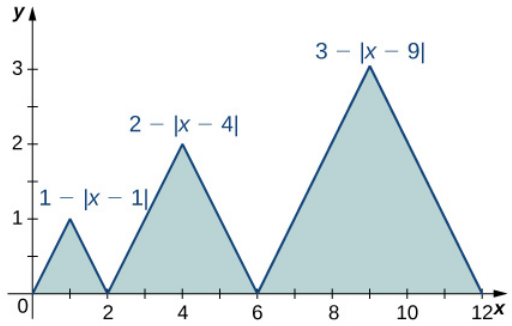
$$\int_1^2 x\log(x^2)\,dx$$

**In exercises 11 - 16, evaluate the integrals of the functions graphed using the formulas for areas of triangles and circles, and subtracting the areas below the $x$-axis.**

11)



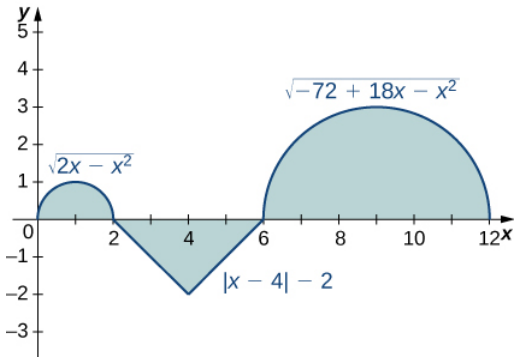

12)



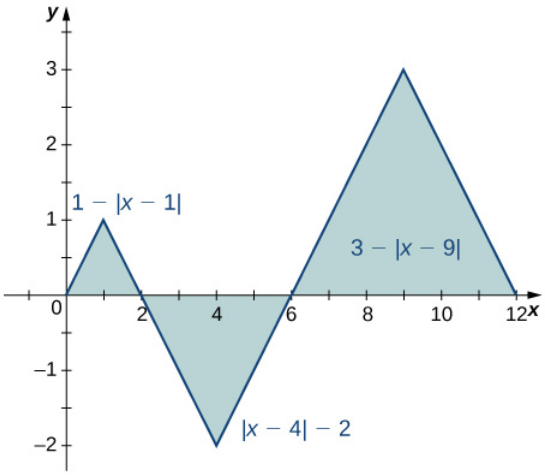

**Answer**

$1 + 2 \cdot 2 + 3 \cdot 3 = 14$

13)




14)



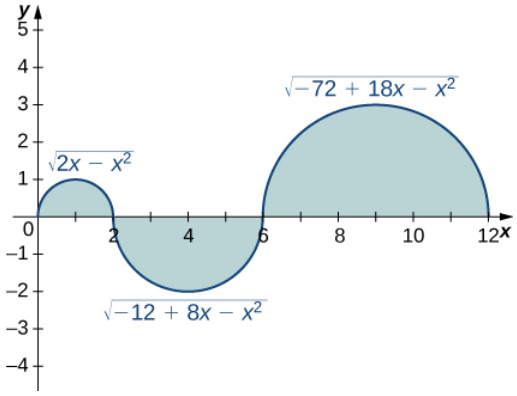


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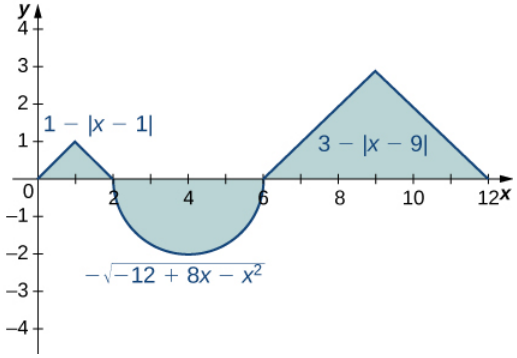
**Answer**

$1-4+9=6$

15)




16)




**Answer**

$1-2\pi+9=10-2\pi$

**In exercises 17 - 24, evaluate the integral using area formulas.**

17) $\int_0^3 (3-x)\,dx$

18) $\int_2^3 (3-x)\,dx$

**Answer**

The integral is the area of the triangle, $\frac{1}{2}$.

19) $\int_{-3}^3 (3-|x|)\,dx$

20) $\int_0^6 (3-|x-3|)\,dx$

**Answer**

The integral is the area of the triangle, $9$.

21) $\int_{-2}^2 \sqrt{4-x^2}\,dx$

22) $\int_1^5 \sqrt{4-(x-3)^2}\,dx$

**Answer**

The integral is the area $\frac{1}{2}\pi r^2 = 2\pi$.

23) $\int_0^{12} \sqrt{36-(x-6)^2}\,dx$

24) $\int_{-2}^{3} (3-|x|)\,dx$

**Answer**

The integral is the area of the "big" triangle less the "missing" triangle, $9-\frac{1}{2}$.

**In exercises 25 - 28, use averages of values at the left (L) and right (R) endpoints to compute the integrals of the piecewise linear functions with graphs that pass through the given list of points over the indicated intervals.**

25) $(0,0),(2,1),(4,3),(5,0),(6,0),(8,3)$ over $[0,8]$

26) $(0,2),(1,0),(3,5),(5,5),(6,2),(8,0)$ over $[0,8]$

**Answer**

$L = 2+0+10+5+4 = 21,\ R = 0+10+10+2+0 = 22,\ \dfrac{L+R}{2} = 21.5$

27) $(-4,-4),(-2,0),(0,-2),(3,3),(4,3)$ over $[-4,4]$

28) $(-4,0),(-2,2),(0,0),(1,2),(3,2),(4,0)$ over $[-4,4]$

**Answer**

$L = 0+4+0+4+2 = 10,\ R = 4+0+2+4+0 = 10,\ \dfrac{L+R}{2} = 10$

**Suppose that $\int_0^4 f(x)\,dx = 5$ and $\int_0^2 f(x)\,dx = -3$, and $\int_0^4 g(x)\,dx = -1$ and $\int_0^2 g(x)\,dx = 2$. In exercises 29 - 34, compute the integrals.**

29) $\int_0^4 (f(x)+g(x))\,dx$

30) $\int_2^4 (f(x)+g(x))\,dx$

**Answer**

$\int_2^4 f(x)\,dx + \int_2^4 g(x)\,dx = 8-3 = 5$

31) $\int_0^2 (f(x)-g(x))\,dx$

32) $\int_2^4 (f(x)-g(x))\,dx$

**Answer**

$\int_2^4 f(x)\,dx - \int_2^4 g(x)\,dx = 8+3 = 11$

33) $\int_0^2 (3f(x)-4g(x))\,dx$

34) $\int_2^4 (4f(x)-3g(x))\,dx$

**Answer**

$4\int_2^4 f(x)\,dx - 3\int_2^4 g(x)\,dx = 32+9 = 41$

**In exercises 35 - 38, use the identity $\int_{-A}^{A} f(x)\,dx = \int_{-A}^{0} f(x)\,dx + \int_{0}^{A} f(x)\,dx$ to compute the integrals.**

35) $\int_{-\pi}^{\pi} \frac{\sin t}{1+t^2}\,dt$ (Hint: $\sin(-t) = -\sin(t)$)

36) $\int_{\sqrt{-\pi}}^{\sqrt{\pi}} \frac{t}{1+\cos t}\,dt$

**Answer**

The integrand is odd; the integral is zero.

37) $\int_{1}^{3} (2-x)\,dx$ (Hint: Look at the graph of $f$.)

38) $\int_{2}^{4} (x-3)^3\,dx$ (Hint: Look at the graph of $f$.)

**Answer**

The integrand is antisymmetric with respect to $x = 3$. The integral is zero.

**In exercises 39 - 44, given that $\int_0^1 x\,dx = \frac{1}{2}$, $\int_0^1 x^2\,dx = \frac{1}{3}$, and $\int_0^1 x^3\,dx = \frac{1}{4}$, compute the integrals.**

39) $\int_{0}^{1} (1+x+x^2+x^3)\,dx$

40) $\int_{0}^{1} (1-x+x^2-x^3)\,dx$

**Answer**

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = \frac{7}{12}$$

41) $\int_{0}^{1} (1-x)^2\,dx$

42) $\int_{0}^{1} (1-2x)^3\,dx$

**Answer**

$$\int_0^1 (1 - 6x + 12x^2 - 8x^3)\,dx = 1 - 6\left(\frac{1}{2}\right) + 12\left(\frac{1}{3}\right) - 8\left(\frac{1}{4}\right) = 1 - 3 + 4 - 2 = 0$$

43) $\int_{0}^{1} \left(6x - \frac{4}{3}x^2\right)\,dx$

44) $\int_{0}^{1} (7 - 5x^3)\,dx$

**Answer**

$7 - \frac{5}{4} = \frac{23}{4}$

**In exercises 45 - 50, use the comparison theorem.**

45) Show that $\int_{0}^{3} (x^2 - 6x + 9)\,dx \geq 0$.

46) Show that $\int_{-2}^{3} (x-3)(x+2)\,dx \leq 0$.

**Answer**

The integrand is negative over $[-2, 3]$.

47) Show that $\int_{0}^{1} \sqrt{1+x^3}\,dx \leq \int_{0}^{1} \sqrt{1+x^2}\,dx$.

48) Show that $\int_1^2 \sqrt{1+x}\,dx \le \int_1^2 \sqrt{1+x^2}\,dx$.

**Answer**

$x \le x^2$ over $[1,2]$, so $\sqrt{1+x} \le \sqrt{1+x^2}$ over $[1,2]$.

49) Show that $\int_0^{\pi/2} \sin t dt \ge \frac{\pi}{4}$ (Hint: $\sin t \ge \frac{2t}{\pi}$ over $[0, \frac{\pi}{2}]$)

50) Show that $\int_{-\pi/4}^{\pi/4} \cos t\,dt \ge \pi\sqrt{2}/4$.

**Answer**

$\cos(t) \ge \frac{\sqrt{2}}{2}$. Multiply by the length of the interval to get the inequality.

**In exercises 51 - 56, find the average value $f_{ave}$ of $f$ between $a$ and $b$, and find a point $c$, where $f(c) = f_{ave}$**

51) $f(x) = x^2,\ a = -1,\ b = 1$

52) $f(x) = x^5,\ a = -1,\ b = 1$

**Answer**

$f_{ave} = 0;\ c = 0$

53) $f(x) = \sqrt{4 - x^2},\ a = 0,\ b = 2$

54) $f(x) = 3 - |x|,\ a = -3,\ b = 3$

**Answer**

$\frac{3}{2}$ when $c = \pm\frac{3}{2}$

55) $f(x) = \sin x,\ a = 0,\ b = 2\pi$

56) $f(x) = \cos x,\ a = 0,\ b = 2\pi$

**Answer**

$f_{ave} = 0;\ c = \frac{\pi}{2},\ \frac{3\pi}{2}$

**In exercises 57 - 60, approximate the average value using Riemann sums $L_{100}$ and $R_{100}$. How does your answer compare with the exact given answer?**

57) [T] $y = \ln(x)$ over the interval $[1,4]$; the exact solution is $\frac{\ln(256)}{3} - 1$.

58) [T] $y = e^{x/2}$ over the interval $[0,1]$; the exact solution is $2(\sqrt{e} - 1)$.

**Answer**

$L_{100} = 1.294,\ R_{100} = 1.301$; the exact average is between these values.

59) [T] $y = \tan x$ over the interval $[0, \frac{\pi}{4}]$; the exact solution is $\frac{2\ln(2)}{\pi}$.

60) [T] $y = \frac{x+1}{\sqrt{4-x^2}}$ over the interval $[-1,1]$; the exact solution is $\frac{\pi}{6}$.

**Answer**

$L_{100} \times (\frac{1}{2}) = 0.5178,\ R_{100} \times (\frac{1}{2}) = 0.5294$

**In exercises 61 - 64, compute the average value using the left Riemann sums $L_N$ for $N = 1, 10, 100$. How does the accuracy compare with the given exact value?**

61) [T] $y = x^2 - 4$ over the interval $[0, 2]$; the exact solution is $-\frac{8}{3}$.

62) [T] $y = xe^{x^2}$ over the interval $[0, 2]$; the exact solution is $\frac{1}{4}(e^4 - 1)$.

**Answer**

$L_1 = 0,\ L_{10} \times (\frac{1}{2}) = 8.743493,\ L_{100} \times (\frac{1}{2}) = 12.861728.$ The exact answer $\approx 26.799$, so $L_{100}$ is not accurate.

63) [T] $y = \left(\frac{1}{2}\right)^x$ over the interval $[0, 4]$; the exact solution is $\frac{15}{64 \ln(2)}$.

64) [T] $y = x \sin(x^2)$ over the interval $[-\pi, 0]$; the exact solution is $\frac{\cos(\pi^2) - 1}{2\pi.}$

**Answer**

$L_1 \times (\frac{1}{\pi}) = 1.352, L_{10} \times (\frac{1}{\pi}) = -0.1837, L_{100} \times (1\pi) = -0.2956.$ The exact answer $\approx -0.303$, so $L_{100}$ is not accurate to the first decimal.

65) Suppose that $A = \int_0^{2\pi} \sin^2 t\, dt$ and $B = \int_0^{2\pi} \cos^2 t\, dt$. Show that $A + B = 2\pi$ and $A = B$.

66) Suppose that $A = \int_{-\pi/4}^{\pi/4} \sec^2 t\, dt = \pi$ and $B = \int_{-\pi/}^{\pi/4} 4 \tan^2 t\, dt$. Show that $A - B = \frac{\pi}{2}$.

**Answer**

Use $\tan^2 \theta + 1 = \sec^2 \theta$. Then, $B - A = \int_{-\pi/4}^{\pi/4} 1\, dx = \frac{\pi}{2}$.

67) Show that the average value of $\sin^2 t$ over $[0, 2\pi]$ is equal to $1/2$. Without further calculation, determine whether the average value of $\sin^2 t$ over $[0, \pi]$ is also equal to $1/2$.

68) Show that the average value of $\cos^2 t$ over $[0, 2\pi]$ is equal to $1/2$. Without further calculation, determine whether the average value of $\cos^2(t)$ over $[0, \pi]$ is also equal to $1/2$.

**Answer**

$\int_0^{2\pi} \cos^2 t\, dt = \pi$, so divide by the length $2\pi$ of the interval. $\cos^2 t$ has period $\pi$, so yes, it is true.

69) Explain why the graphs of a quadratic function (parabola) $p(x)$ and a linear function $\ell(x)$ can intersect in at most two points. Suppose that $p(a) = \ell(a)$ and $p(b) = \ell(b)$, and that $\int_a^b p(t)\, dt > \int_a^b \ell(t)\, dt$. Explain why $\int_c^d p(t)\, dt > \int_c^d \ell(t)\, dt$ whenever $a \le c < d \le b$.

70) Suppose that parabola $p(x) = ax^2 + bx + c$ opens downward ($a < 0$) and has a vertex of $y = \frac{-b}{2a} > 0$. For which interval $[A, B]$ is $\int_A^B (ax^2 + bx + c)\, dx$ as large as possible?

**Answer**

The integral is maximized when one uses the largest interval on which $p$ is nonnegative. Thus, $A = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$ and $B = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$.

71) Suppose $[a, b]$ can be subdivided into subintervals $a = a_0 < a_1 < a_2 < \cdots < a_N = b$ such that either $f \ge 0$ over $[a_{i-1}, a_i]$ or $f \le 0$ over $[a_{i-1}, a_i]$. Set $A_i = \int_{a_{i-1}}^{a_i} f(t)\, dt$.

a. Explain why $\int_a^b f(t)\, dt = A_1 + A_2 + \cdots + A_N$.

b. Then, explain why $\int_a^b f(t)\, dt \le \int_a^b |f(t)|\, dt$.

72) Suppose $f$ and $g$ are continuous functions such that $\int_c^d f(t)\,dt \le \int_c^d g(t)\,dt$ for every subinterval $[c,d]$ of $[a,b]$. Explain why $f(x) \le g(x)$ for all values of $x$.

**Answer**

If $f(t_0) > g(t_0)$ for some $t_0 \in [a,b]$, then since $f-g$ is continuous, there is an interval containing $t_0$ such that $f(t) > g(t)$ over the interval $[c,d]$, and then $\int_d^d f(t)\,dt > \int_c^d g(t)\,dt$over this interval.

73) Suppose the average value of $f$ over $[a,b]$ is 1 and the average value of $f$ over $[b,c]$ is 1 where $a < c < b$. Show that the average value of $f$ over $[a,c]$ is also 1.

74) Suppose that $[a,b]$ can be partitioned. taking $a = a_0 < a_1 < \cdots < a_N = b$ such that the average value of $f$ over each subinterval $[a_{i-1}, a_i] = 1$ is equal to 1 for each $i = 1, \ldots, N$. Explain why the average value of f over $[a,b]$ is also equal to 1.

**Answer**

The integral of f over an interval is the same as the integral of the average of f over that interval. Thus, $\int_a^b f(t)\,dt$
$= \int_{a_0}^{a_1} f(t)\,dt + \int_a^{a_2} 1_f(t)\,dt + \cdots + \int_{a_{N+1}}^{a_N} f(t)\,dt = \int_{a_0}^{a_1} 1\,dt + \int_{a_1}^{a_2} 1\,dt + \cdots + \int_{a_{N+1}}^{a_N} 1\,dt$
$= (a_1 - a_0) + (a_2 - a_1) + \cdots + (a_N - a_{N-1}) = a_N - a_0 = b - a.$
Dividing through by $b-a$ gives the desired identity.

75) Suppose that for each $i$ such that $1 \le i \le N$ one has $\int_{i-1}^{i} f(t)\,dt = i$. Show that $\int_0^N f(t)\,dt = \frac{N(N+1)}{2}$.

76) Suppose that for each $i$ such that $1 \le i \le N$ one has $\int_{i-1}^{i} f(t)\,dt = i^2$. Show that $\int_0^N f(t)\,dt = \frac{N(N+1)(2N+1)}{6}$.

**Answer**

$$\int_0^N f(t)\,dt = \sum_{i=1}^{N} \int_{i-1}^{i} f(t)\,dt = \sum_{i=1}^{N} i^2 = \frac{N(N+1)(2N+1)}{6}$$

77) [T] Compute the left and right Riemann sums $L_{10}$ and $R_{10}$ and their average $\frac{L_{10}+R_{10}}{2}$ for $f(t) = t^2$over $[0,1]$. Given that $\int_0^1 t^2\,dt = 1/3$, to how many decimal places is $\frac{L_{10}+R_{10}}{2}$ accurate?

78) [T] Compute the left and right Riemann sums, $L_10$ and $R_{10}$, and their average $\frac{L_{10}+R_{10}}{2}$ for $f(t) = (4-t^2)$ over $[1,2]$. Given that $\int_1^2 (4-t^2)\,dt = 1.66$, to how many decimal places is $\frac{L_{10}+R_{10}}{2}$ accurate?

**Answer**

$L_{10} = 1.815,\ R_{10} = 1.515,\ \frac{L_{10}+R_{10}}{2} = 1.665,$ so the estimate is accurate to two decimal places.

79) If $\int_1^5 \sqrt{1+t^4}\,dt = 41.7133\ldots$, what is $\int_1^5 \sqrt{1+u^4}\,du$?

80) Estimate $\int_0^1 t\,dt$ using the left and right endpoint sums, each with a single rectangle. How does the average of these left and right endpoint sums compare with the actual value $\int_0^1 t\,dt$?

**Answer**

The average is $1/2$, which is equal to the integral in this case.

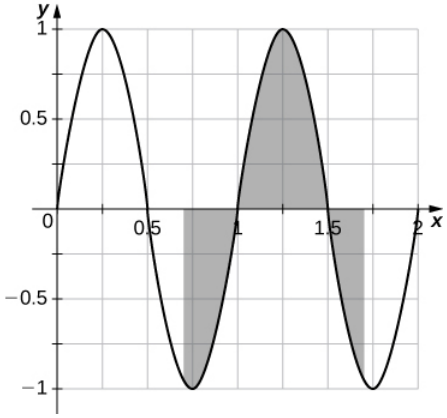
81) Estimate $\int_0^1 t\,dt$ by comparison with the area of a single rectangle with height equal to the value of $t$ at the midpoint $t = \frac{1}{2}$. How does this midpoint estimate compare with the actual value $\int_0^1 t\,dt$?

82) From the graph of $\sin(2\pi x)$ shown:

a. Explain why $\int_0^1 \sin(2\pi t)\,dt = 0$.

b. Explain why, in general, $\int_a^{a+1} \sin(2\pi t)\,dt = 0$ for any value of $a$.

**Answer**




a. The graph is antisymmetric with respect to $t = \frac{1}{2}$ over $[0, 1]$, so the average value is zero.

b. For any value of $a$, the graph between $[a, a + 1]$ is a shift of the graph over $[0, 1]$, so the net areas above and below the axis do not change and the average remains zero.

83) If f is 1-periodic ($f(t + 1) = f(t)$), odd, and integrable over $[0, 1]$, is it always true that $\int_0^1 f(t)\,dt = 0$?

84) If f is 1-periodic and $\int 10f(t)\,dt = A$, is it necessarily true that $\int_a^{1+a} f(t)\,dt = A$ for all $A$?

**Answer**

Yes, the integral over any interval of length 1 is the same.

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# 1.3: The Fundamental Theorem of Calculus

**Learning Objectives**

- Describe the meaning of the Mean Value Theorem for Integrals.
- State the meaning of the Fundamental Theorem of Calculus, Part 1.
- Use the Fundamental Theorem of Calculus, Part 1, to evaluate derivatives of integrals.
- State the meaning of the Fundamental Theorem of Calculus, Part 2.
- Use the Fundamental Theorem of Calculus, Part 2, to evaluate definite integrals.
- Explain the relationship between differentiation and integration.

In the previous two sections, we looked at the definite integral and its relationship to the area under the curve of a function. Unfortunately, so far, the only tools we have available to calculate the value of a definite integral are geometric area formulas and limits of Riemann sums, and both approaches are extremely cumbersome. In this section we look at some more powerful and useful techniques for evaluating definite integrals.

These new techniques rely on the relationship between differentiation and integration. This relationship was discovered and explored by both Sir Isaac Newton and Gottfried Wilhelm Leibniz (among others) during the late 1600s and early 1700s, and it is codified in what we now call the **Fundamental Theorem of Calculus**, which has two parts that we examine in this section. Its very name indicates how central this theorem is to the entire development of calculus.

Isaac Newton's contributions to mathematics and physics changed the way we look at the world. The relationships he discovered, codified as Newton's laws and the law of universal gravitation, are still taught as foundational material in physics today, and his calculus has spawned entire fields of mathematics.

Before we get to this crucial theorem, however, let's examine another important theorem, the Mean Value Theorem for Integrals, which is needed to prove the Fundamental Theorem of Calculus.

## The Mean Value Theorem for Integrals

The Mean Value Theorem for Integrals states that a continuous function on a closed interval takes on its average value at the same point in that interval. The theorem guarantees that if $f(x)$ is continuous, a point $c$ exists in an interval $[a, b]$ such that the value of the function at $c$ is equal to the average value of $f(x)$ over $[a, b]$. We state this theorem mathematically with the help of the formula for the average value of a function that we presented at the end of the preceding section.

**The Mean Value Theorem for Integrals, Part 1**

If $f(x)$ is continuous over an interval $[a, b]$, then there is at least one point $c \in [a, b]$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx. \tag{1.3.1}$$

This formula can also be stated as

$$\int_a^b f(x)\,dx = f(c)(b-a). \tag{1.3.2}$$

**Proof**

Since $f(x)$ is continuous on $[a, b]$, by the extreme value theorem (see section on Maxima and Minima), it assumes minimum and maximum values—$m$ and $M$, respectively—on $[a, b]$. Then, for all $x$ in $[a, b]$, we have $m \le f(x) \le M$. Therefore, by the comparison theorem (see Section on The Definite Integral), we have

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a). \tag{1.3.3}$$

Dividing by $b-a$ gives us

$$m \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq M. \tag{1.3.4}$$

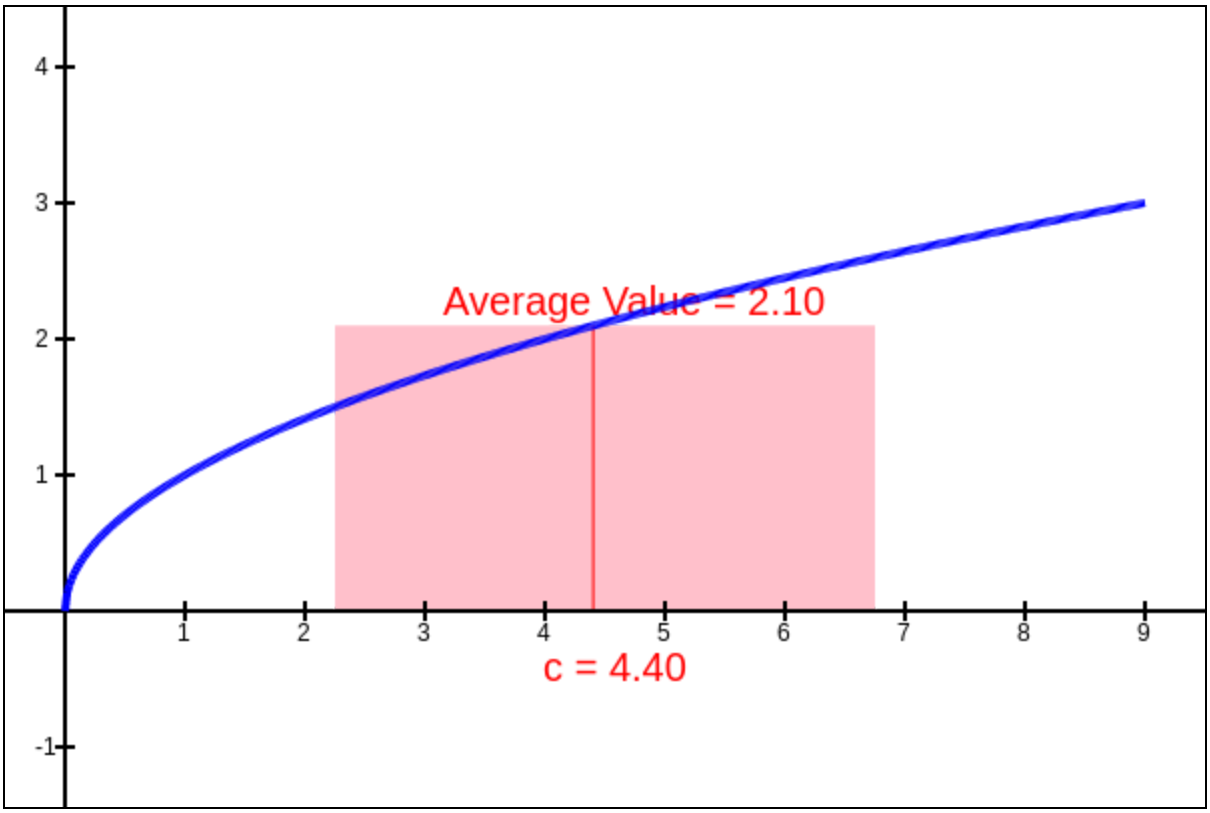
Since $\frac{1}{b-a}\int_a^b f(x)\,dx$ is a number between $m$ and $M$, and since $f(x)$ is continuous and assumes the values $m$ and $M$ over $[a,b]$, by the Intermediate Value Theorem, there is a number $c$ over $[a,b]$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx, \tag{1.3.5}$$

and the proof is complete.□

## Average Value Visualization

The graph of a function will be given. You can change $a$ and $b$ and see the average value for that function between $a$ and $b$.




a = 2.25

b = 6.75

$f(x) = \sqrt{x}$

New Function

## Example 1.3.1: Finding the Average Value of a Function

Find the average value of the function $f(x) = 8 - 2x$ over the interval $[0, 4]$ and find $c$ such that $f(c)$ equals the average value of the function over $[0, 4]$.

**Solution**

The formula states the mean value of $f(x)$ is given by

$$\frac{1}{4-0}\int_0^4 (8 - 2x)\, dx.$$

We can see in Figure 1.3.1 that the function represents a straight line and forms a right triangle bounded by the $x$- and $y$-axes. The area of the triangle is $A = \frac{1}{2}(base)(height)$. We have

$$A = \frac{1}{2}(4)(8) = 16.$$

The average value is found by multiplying the area by $1/(4-0)$. Thus, the average value of the function is

$$\frac{1}{4}(16) = 4$$

Set the average value equal to $f(c)$ and solve for $c$.

$$8 - 2c = 4$$
$$c = 2$$

At $c = 2, f(2) = 4$.

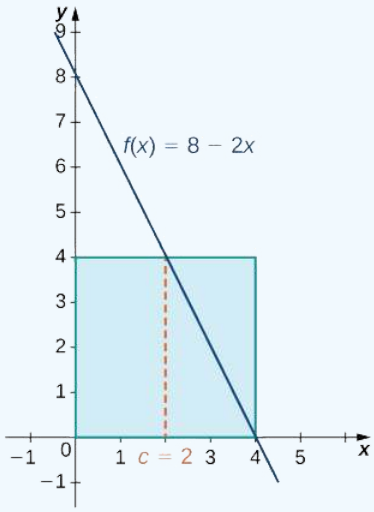



Figure 1.3.1: By the Mean Value Theorem, the continuous function $f(x)$ takes on its average value at $c$ at least once over a closed interval.

## Exercise 1.3.1

Find the average value of the function $f(x) = \frac{x}{2}$ over the interval $[0, 6]$ and find c such that $f(c)$ equals the average value of the function over $[0, 6]$.

**Hint**

Use the procedures from Example 1.3.1 to solve the problem

**Answer**

The average value is 1.5 and $c = 3$.

**Example 1.3.2: Finding the Point Where a Function Takes on Its Average Value**

Given $\int_0^3 x^2\,dx = 9$, find $c$ such that $f(c)$ equals the average value of $f(x) = x^2$ over $[0, 3]$.

**Solution**

We are looking for the value of $c$ such that

$$f(c) = \frac{1}{3-0}\int_0^3 x^2\,dx = \frac{1}{3}(9) = 3.$$

Replacing $f(c)$ with $c^2$, we have

$$c^2 = 3$$
$$c = \pm\sqrt{3}.$$

Since $-\sqrt{3}$ is outside the interval, take only the positive value. Thus, $c = \sqrt{3}$ (Figure 1.3.2).

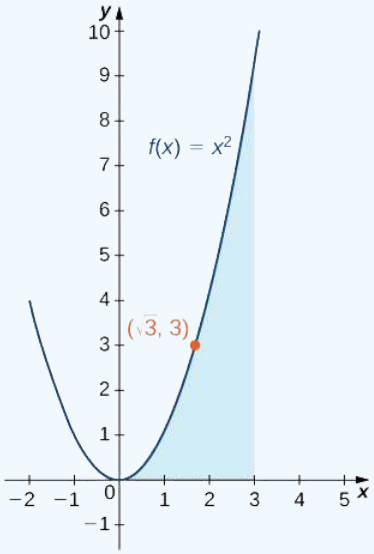



Figure 1.3.2: Over the interval $[0, 3]$, the function $f(x) = x^2$ takes on its average value at $c = \sqrt{3}$.

**Exercise 1.3.2**

Given $\int_0^3 (2x^2 - 1)\,dx = 15$, find $c$ such that $f(c)$ equals the average value of $f(x) = 2x^2 - 1$ over $[0, 3]$.

**Hint**

Use the procedures from Example 1.3.2 to solve the problem.

**Answer**

$c = \sqrt{3}$

## Fundamental Theorem of Calculus Part 1: Integrals and Antiderivatives

As mentioned earlier, the Fundamental Theorem of Calculus is an extremely powerful theorem that establishes the relationship between differentiation and integration, and gives us a way to evaluate definite integrals without using Riemann sums or calculating areas. The theorem is comprised of two parts, the first of which, the **Fundamental Theorem of Calculus, Part 1,** is stated here. Part 1 establishes the relationship between differentiation and integration.

**Fundamental Theorem of Calculus, part 1**

If $f(x)$ is continuous over an interval $[a, b]$, and the function $F(x)$ is defined by

$$F(x) = \int_a^x f(t)\,dt, \tag{1.3.6}$$

then $F'(x) = f(x)$ over $[a, b]$.

Before we delve into the proof, a couple of subtleties are worth mentioning here. First, a comment on the notation. Note that we have defined a function, $F(x)$, as the definite integral of another function, $f(t)$, from the point a to the point $x$. At first glance, this is confusing, because we have said several times that a definite integral is a number, and here it looks like it's a function. The key here is to notice that for any particular value of $x$, the definite integral is a number. So the function $F(x)$ returns a number (the value of the definite integral) for each value of $x$.

Second, it is worth commenting on some of the key implications of this theorem. There is a reason it is called the *Fundamental Theorem of Calculus*. Not only does it establish a relationship between integration and differentiation, but also it guarantees that any integrable function has an antiderivative. Specifically, it guarantees that any continuous function has an antiderivative.

**Proof: Fundamental Theorem of Calculus, Part 1**

Applying the definition of the derivative, we have

$$\begin{aligned} F'(x) &= \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left[\int_a^{x+h} f(t)dt - \int_a^x f(t)\,dt\right] \\ &= \lim_{h\to 0} \frac{1}{h}\left[\int_a^{x+h} f(t)\,dt + \int_x^a f(t)\,dt\right] \\ &= \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt. \end{aligned}$$

Looking carefully at this last expression, we see $\frac{1}{h}\int_x^{x+h} f(t)\,dt$ is just the average value of the function $f(x)$ over the interval $[x, x+h]$. Therefore, by Equation 1.3.2, there is some number $c$ in $[x, x+h]$ such that

$$\frac{1}{h}\int_x^{x+h} f(t)\,dt = f(c).$$

In addition, since $c$ is between $x$ and $h$, $c$ approaches $x$ as $h$ approaches zero. Also, since $f(x)$ is continuous, we have

$$\lim_{h\to 0} f(c) = \lim_{c\to x} f(c) = f(x)$$

Putting all these pieces together, we have

$$F'(x) = \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt = \lim_{h\to 0} f(c) = f(x),$$

and the proof is complete.

□



**Example 1.3.3: Finding a Derivative with the Fundamental Theorem of Calculus**

Use the Fundamental Theorem of Calculus, Part 1 to find the derivative of

$$g(x) = \int_1^x \frac{1}{t^3+1}\,dt.$$

**Solution**

According to the Fundamental Theorem of Calculus, the derivative is given by

$$g'(x) = \frac{1}{x^3+1}.$$

**Exercise 1.3.3**

Use the Fundamental Theorem of Calculus, Part 1 to find the derivative of $g(r) = \int_0^r \sqrt{x^2+4}\,dx$.

**Hint**

Follow the procedures from Example **1.3.3** to solve the problem.

**Answer**

$g'(r) = \sqrt{r^2+4}$



**Example 1.3.4: Using the Fundamental Theorem and the Chain Rule to Calculate Derivatives**

Let $F(x) = \int_1^{\sqrt{x}} \sin t \, dt$. Find $F'(x)$.

**Solution**

Letting $u(x) = \sqrt{x}$, we have $F(x) = \int_1^{u(x)} \sin t \, dt$.

Thus, by the Fundamental Theorem of Calculus and the chain rule,

$$F'(x) = \sin(u(x))\frac{du}{dx} = \sin(u(x)) \cdot \left(\frac{1}{2}x^{-1/2}\right) = \frac{\sin\sqrt{x}}{2\sqrt{x}}.$$

**Exercise 1.3.4**

Let $F(x) = \int_1^{x^3} \cos t \, dt$. Find $F'(x)$.

**Hint**

Use the chain rule to solve the problem.

**Answer**

$F'(x) = 3x^2 \cos x^3$



**Example 1.3.5: Using the Fundamental Theorem of Calculus with Two Variable Limits of Integration**

Let $F(x) = \int_x^{2x} t^3\,dt$. Find $F'(x)$.

**Solution**

We have $F(x) = \int_x^{2x} t^3\,dt$. Both limits of integration are variable, so we need to split this into two integrals. We get

$$\begin{aligned} F(x) &= \int_x^{2x} t^3\,dt = \int_x^0 t^3\,dt + \int_0^{2x} t^3\,dt \\ &= -\int_0^x t^3\,dt + \int_0^{2x} t^3\,dt. \end{aligned}$$

Differentiating the first term, we obtain

$$\frac{d}{dx}\left[-\int_0^x t^3\,dt\right] = -x^3.$$

Differentiating the second term, we first let $(x) = 2x$. Then,

$$\begin{aligned} \frac{d}{dx}\left[\int_0^{2x} t^3\,dt\right] &= \frac{d}{dx}\left[\int_0^{u(x)} t^3\,dt\right] \\ &= (u(x))^3\,du\,dx \\ &= (2x)^3 \cdot 2 = 16x^3. \end{aligned}$$

Thus,

$$\begin{aligned} F'(x) &= \frac{d}{dx}\left[-\int_0^x t^3\,dt\right] + \frac{d}{dx}\left[\int_0^{2x} t^3\,dt\right] \\ &= -x^3 + 16x^3 = 15x^3 \end{aligned}$$

**Exercise 1.3.5**

Let $F(x) = \int_x^{x^2} \cos t\,dt$. Find $F'(x)$.

**Hint**

Use the procedures from Example **1.3.5** to solve the problem

**Answer**

$F'(x) = 2x \cos x^2 - \cos x$



## Fundamental Theorem of Calculus, Part 2: The Evaluation Theorem

The Fundamental Theorem of Calculus, Part 2, is perhaps the most important theorem in calculus. After tireless efforts by mathematicians for approximately 500 years, new techniques emerged that provided scientists with the necessary tools to explain many phenomena. Using calculus, astronomers could finally determine distances in space and map planetary orbits. Everyday financial problems such as calculating marginal costs or predicting total profit could now be handled with simplicity and accuracy. Engineers could calculate the bending strength of materials or the three-dimensional motion of objects. Our view of the world was forever changed with calculus.

After finding approximate areas by adding the areas of n rectangles, the application of this theorem is straightforward by comparison. It almost seems too simple that the area of an entire curved region can be calculated by just evaluating an antiderivative at the first and last endpoints of an interval.

### The Fundamental Theorem of Calculus, Part 2

If $f(x)$ is continuous over the interval $[a, b]$ and $F(x)$ is any antiderivative of $f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a). \tag{1.3.7}$$

We often see the notation $F(x)|_a^b$ to denote the expression $F(b) - F(a)$. We use this vertical bar and associated limits $a$ and $b$ to indicate that we should evaluate the function $F(x)$ at the upper limit (in this case, $b$), and subtract the value of the function $F(x)$ evaluated at the lower limit (in this case, $a$).

The **Fundamental Theorem of Calculus, Part 2** (also known as the **evaluation theorem**) states that if we can find an antiderivative for the integrand, then we can evaluate the definite integral by evaluating the antiderivative at the endpoints of the interval and subtracting.

**Proof**

Let $P = x_i, i = 0, 1, \ldots, n$ be a regular partition of $[a, b]$. Then, we can write

$$\begin{aligned} F(b) - F(a) &= F(x_n) - F(x_0) \\ &= [F(x_n) - F(x_{n-1})] + [F(x_{n-1}) - F(x_{n-2})] + \ldots + [F(x_1) - F(x_0)] \\ &= \sum_{i=1}^{n} [F(x_i) - F(x_{i-1})]. \end{aligned}$$

Now, we know $F$ is an antiderivative of $f$ over $[a, b]$, so by the Mean Value Theorem (see The Mean Value Theorem) for $i = 0, 1, \ldots, n$ we can find $c_i$ in $[x_{i-1}, x_i]$ such that

$$F(x_i) - F(x_{i-1}) = F'(c_i)(x_i - x_{i-1}) = f(c_i)\,\Delta x. \tag{1.3.8}$$

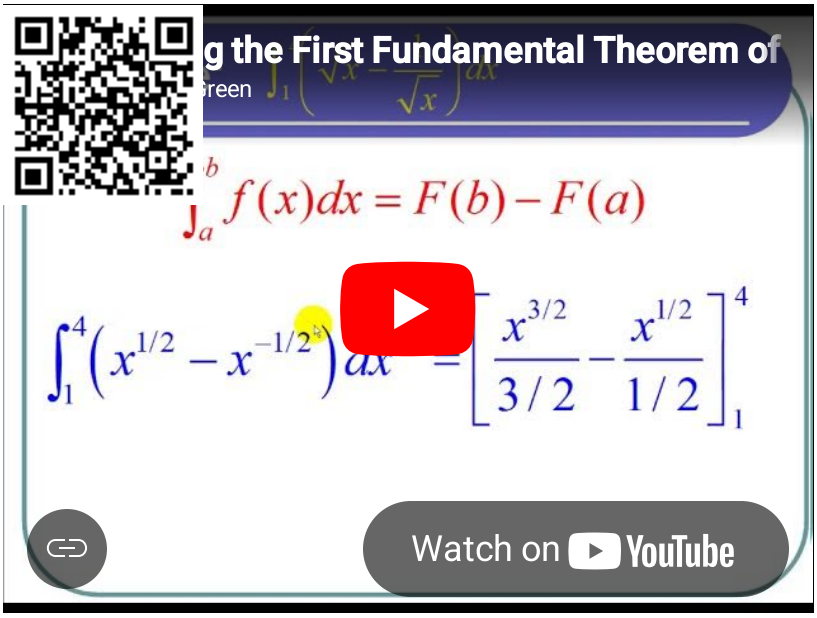
Then, substituting into the previous equation, we have

$$F(b) - F(a) = \sum_{i=1}^{n} f(c_i)\,\Delta x. \tag{1.3.9}$$

Taking the limit of both sides as $n \to \infty$, we obtain

$$F(b) - F(a) = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x = \int_a^b f(x)\,dx. \tag{1.3.10}$$

□




**Example 1.3.6: Evaluating an Integral with the Fundamental Theorem of Calculus**

Use Equation 1.3.7 to evaluate

$$\int_{-2}^{2} (t^2 - 4)\,dt.$$

**Solution**

Recall the power rule for Antiderivatives:

If $y = x^n$,

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C.$$

Use this rule to find the antiderivative of the function and then apply the theorem. We have

$$\begin{aligned}\int_{-2}^{2}(t^2-4)dt &= \left(\frac{t^3}{3}-4t\right)\Big|_{-2}^{2} \\ &= \left[\frac{(2)^3}{3}-4(2)\right]-\left[\frac{(-2)^3}{3}-4(-2)\right] \\ &= \left[\frac{8}{3}-8\right]-\left[-\frac{8}{3}+8\right] \\ &= \frac{8}{3}-8+\frac{8}{3}-8 \\ &= \frac{16}{3}-16=-\frac{32}{3}.\end{aligned}$$

**Analysis**

Notice that we did not include the "$+C$" term when we wrote the antiderivative. The reason is that, according to the Fundamental Theorem of Calculus, Part 2 (Equation 1.3.7), any antiderivative works. So, for convenience, we chose the antiderivative with $C = 0$. If we had chosen another antiderivative, the constant term would have canceled out. This always happens when evaluating a definite integral.

The region of the area we just calculated is depicted in Figure **1.3.3**. Note that the region between the curve and the $x$-axis is all below the $x$-axis. Area is always positive, but a definite integral can still produce a negative number (a net signed area). For example, if this were a profit function, a negative number indicates the company is operating at a loss over the given interval.

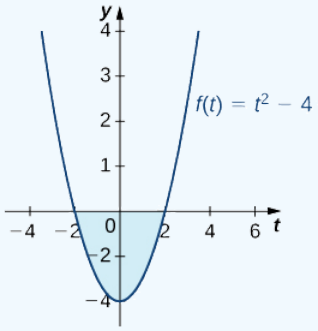



Figure **1.3.3**: The evaluation of a definite integral can produce a negative value, even though area is always positive.



## Example 1.3.7: Evaluating a Definite Integral Using the Fundamental Theorem of Calculus, Part 2

Evaluate the following integral using the Fundamental Theorem of Calculus, Part 2 (Equation 1.3.7):

$$\int_1^9 \frac{x-1}{\sqrt{x}}\,dx.$$

**Solution**

First, eliminate the radical by rewriting the integral using rational exponents. Then, separate the numerator terms by writing each one over the denominator:

$$\int_1^9 \frac{x-1}{x^{1/2}}\,dx = \int_1^9 \left(\frac{x}{x^{1/2}} - \frac{1}{x^{1/2}}\right)\,dx.$$

Use the properties of exponents to simplify:

$$\int_1^9 \left(\frac{x}{x^{1/2}} - \frac{1}{x^{1/2}}\right)\,dx = \int_1^9 (x^{1/2} - x^{-1/2})\,dx.$$

Now, integrate using the power rule:

$$\begin{aligned}\int_1^9 (x^{1/2} - x^{-1/2})\,dx &= \left(\frac{x^{3/2}}{\frac{3}{2}} - \frac{x^{1/2}}{\frac{1}{2}}\right)\Big|_1^9 \\ &= \left[\frac{(9)^{3/2}}{\frac{3}{2}} - \frac{(9)^{1/2}}{\frac{1}{2}}\right] - \left[\frac{(1)^{3/2}}{\frac{3}{2}} - \frac{(1)^{1/2}}{\frac{1}{2}}\right] \\ &= \left[\frac{2}{3}(27) - 2(3)\right] - \left[\frac{2}{3}(1) - 2(1)\right] \\ &= 18 - 6 - \frac{2}{3} + 2 = \frac{40}{3}.\end{aligned}$$

See Figure 1.3.4.

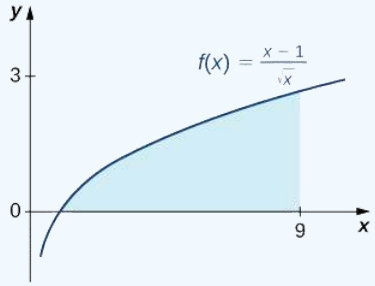



Figure 1.3.4: The area under the curve from $x = 1$ to $x = 9$ can be calculated by evaluating a definite integral.

## Exercise 1.3.6

Use Note to evaluate $\int_1^2 x^{-4}\,dx$.

**Hint**

Use the power rule.

**Answer**

$\frac{7}{24}$

## Using Calculus to Make a Basket

You are a coach for a basketball team and need to advise your player how fast, in meters per second, the player needs to throw the basketball so that it lands in the basket when the player is a given number of meters away. The vertical acceleration is 9.8 meters per second squared and the player will release the ball at an angle of $\frac{\pi}{4}$ radians, so that the initial upward velocity will be $\frac{\sqrt{2}}{2}s_0$ meters per second and the horizontal velocity will be a constant $\frac{\sqrt{2}}{2}s_0$ meters per second where $s_0$ is the initial speed of the ball in meters per second. The basketball hoop is 3 meters high and the player will release the ball when it is 1.8 meters from the ground. You will be given the horizontal distance $x$ meters from the basket the player will be releasing the ball, and you must instruct the player at what initial speed the ball should be released. Then you will watch the player shoot the ball and see if the player makes the basket.

The horizontal distance from the basket = 6.0 meters.

Initial speed: s =

Shoot

Basketball players and Cinematographers: Alize Reyes Hernandez and Alberto Morales Cesar

**Example 1.3.8: A Roller-Skating Race**

James and Kathy are racing on roller skates. They race along a long, straight track, and whoever has gone the farthest after 5 sec wins a prize. If James can skate at a velocity of $f(t) = 5 + 2t$ ft/sec and Kathy can skate at a velocity of $g(t) = 10 + \cos\left(\frac{\pi}{2}t\right)$ ft/sec, who is going to win the race?

**Solution**

We need to integrate both functions over the interval $[0, 5]$ and see which value is bigger. For James, we want to calculate

$$\int_0^5 (5 + 2t)\, dt.$$

Using the power rule, we have

$$\begin{aligned}\int_0^5 (5 + 2t)\, dt &= \left(5t + t^2\right)\Big|_0^5 \\ &= (25 + 25) \\ &= 50.\end{aligned}$$

Thus, James has skated 50 ft after 5 sec. Turning now to Kathy, we want to calculate

$$\int_0^5 10 + \cos\left(\frac{\pi}{2}t\right)\, dt.$$

We know $\sin t$ is an antiderivative of $\cos t$, so it is reasonable to expect that an antiderivative of $\cos\left(\frac{\pi}{2}t\right)$ would involve $\sin\left(\frac{\pi}{2}t\right)$. However, when we differentiate $\sin\left(\pi^2 t\right)$, we get $\pi^2 \cos\left(\pi^2 t\right)$ as a result of the chain rule, so we have to account for this additional coefficient when we integrate. We obtain

$$\begin{aligned}\int_0^5 10 + \cos\left(\frac{\pi}{2}t\right)\, dt &= \left(10t + \frac{2}{\pi}\sin\left(\frac{\pi}{2}t\right)\right)\Big|_0^5 \\ &= \left(50 + \frac{2}{\pi}\right) - \left(0 - \frac{2}{\pi}\sin 0\right) \approx 50.6.\end{aligned}$$

Kathy has skated approximately 50.6 ft after 5 sec. Kathy wins, but not by much!

Exercise 1.3.7

Suppose James and Kathy have a rematch, but this time the official stops the contest after only 3 sec. Does this change the outcome?

**Hint**

Change the limits of integration from those in Example 1.3.7.

**Answer**

Kathy still wins, but by a much larger margin: James skates 24 ft in 3 sec, but Kathy skates 29.3634 ft in 3 sec.

## A Parachutist in Free Fall

Julie is an avid skydiver with more than 300 jumps under her belt and has mastered the art of making adjustments to her body position in the air to control how fast she falls. If she arches her back and points her belly toward the ground, she reaches a terminal velocity of approximately 120 mph (176 ft/sec). If, instead, she orients her body with her head straight down, she falls faster, reaching a terminal velocity of 150 mph (220 ft/sec).



Figure 1.3.5: Skydivers can adjust the velocity of their dive by changing the position of their body during the free fall. (credit: Jeremy T. Lock)

Since Julie will be moving (falling) in a downward direction, we assume the downward direction is positive to simplify our calculations. Julie executes her jumps from an altitude of 12,500 ft. After she exits the aircraft, she immediately starts falling at a velocity given by $v(t) = 32t$.

She continues to accelerate according to this velocity function until she reaches terminal velocity. After she reaches terminal velocity, her speed remains constant until she pulls her ripcord and slows down to land.

On her first jump of the day, Julie orients herself in the slower “belly down” position (terminal velocity is 176 ft/sec). Using this information, answer the following questions.

1. How long after she exits the aircraft does Julie reach terminal velocity?
2. Based on your answer to question 1, set up an expression involving one or more integrals that represents the distance Julie falls after 30 sec.
3. If Julie pulls her ripcord at an altitude of 3000 ft, how long does she spend in a free fall?
4. Julie pulls her ripcord at 3000 ft. It takes 5 sec for her parachute to open completely and for her to slow down, during which time she falls another 400 ft. After her canopy is fully open, her speed is reduced to 16 ft/sec. Find the total time Julie spends in the air, from the time she leaves the airplane until the time her feet touch the ground. On Julie’s second jump of the day, she decides she wants to fall a little faster and orients herself in the “head down” position. Her terminal velocity in this position is 220 ft/sec. Answer these questions based on this velocity:

5. How long does it take Julie to reach terminal velocity in this case?
6. Before pulling her ripcord, Julie reorients her body in the “belly down” position so she is not moving quite as fast when her parachute opens. If she begins this maneuver at an altitude of 4000 ft, how long does she spend in a free fall before beginning the reorientation?

Some jumpers wear “**wingsuits**” (Figure **1.3.6**). These suits have fabric panels between the arms and legs and allow the wearer to glide around in a free fall, much like a flying squirrel. (Indeed, the suits are sometimes called “flying squirrel suits.”) When wearing these suits, terminal velocity can be reduced to about 30 mph (44 ft/sec), allowing the wearers a much longer time in the air. Wingsuit flyers still use parachutes to land; although the vertical velocities are within the margin of safety, horizontal velocities can exceed 70 mph, much too fast to land safely.



Figure **1.3.6**: The fabric panels on the arms and legs of a wingsuit work to reduce the vertical velocity of a skydiver’s fall. (credit: Richard Schneider)

Answer the following question based on the velocity in a wingsuit.

7. If Julie dons a wingsuit before her third jump of the day, and she pulls her ripcord at an altitude of 3000 ft, how long does she get to spend gliding around in the air

## Key Concepts

- The Mean Value Theorem for Integrals states that for a continuous function over a closed interval, there is a value c such that $f(c)$ equals the average value of the function.
- The Fundamental Theorem of Calculus, Part 1 shows the relationship between the derivative and the integral.
- The Fundamental Theorem of Calculus, Part 2 is a formula for evaluating a definite integral in terms of an antiderivative of its integrand. The total area under a curve can be found using this formula.

## Key Equations

- **Mean Value Theorem for Integrals**

  If $f(x)$ is continuous over an interval $[a, b]$, then there is at least one point $c \in [a, b]$ such that

  $$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

- **Fundamental Theorem of Calculus, Part 1**

  If $f(x)$ is continuous over an interval $[a, b]$, and the function $F(x)$ is defined by

  $$F(x) = \int_a^x f(t)\,dt,$$

  then

  $$F'(x) = f(x).$$

- **Fundamental Theorem of Calculus, Part 2**

  If $f$ is continuous over the interval $[a, b]$ and $F(x)$ is any antiderivative of $f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

## Glossary

**fundamental theorem of calculus**

the theorem, central to the entire development of calculus, that establishes the relationship between differentiation and integration

**fundamental theorem of calculus, part 1**

uses a definite integral to define an antiderivative of a function

**fundamental theorem of calculus, part 2**

(also, **evaluation theorem**) we can evaluate a definite integral by evaluating the antiderivative of the integrand at the endpoints of the interval and subtracting

**mean value theorem for integrals**

guarantees that a point $c$ exists such that $f(c)$ is equal to the average value of the function

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# 1.3E: Exercises for Section 1.3

1) Consider two athletes running at variable speeds $v_1(t)$ and $v_2(t)$. The runners start and finish a race at exactly the same time. Explain why the two runners must be going the same speed at some point.

2) Two mountain climbers start their climb at base camp, taking two different routes, one steeper than the other, and arrive at the peak at exactly the same time. Is it necessarily true that, at some point, both climbers increased in altitude at the same rate?

**Answer**

Yes. It is implied by the Mean Value Theorem for Integrals.

3) To get on a certain toll road a driver has to take a card that lists the mile entrance point. The card also has a timestamp. When going to pay the toll at the exit, the driver is surprised to receive a speeding ticket along with the toll. Explain how this can happen.

4) Set $F(x) = \int_1^x (1-t)\,dt$. Find $F'(2)$ and the average value of $F'$ over $[1,2]$.

**Answer**

$F'(2) = -1$; average value of $F'$ over $[1,2]$ is $-1/2$.

**In exercises 5 - 16, use the Fundamental Theorem of Calculus, Part 1, to find each derivative.**

5) $\frac{d}{dx}\left[\int_1^x e^{-t^2}\,dt\right]$

6) $\frac{d}{dx}\left[\int_1^x e^{\cos t}\,dt\right]$

**Answer**

$\frac{d}{dx}\left[\int_1^x e^{\cos t}\,dt\right] = e^{\cos x}$

7) $\frac{d}{dx}\left[\int_3^x \sqrt{9-y^2}\,dy\right]$

8) $\frac{d}{dx}\left[\int_3^x \frac{ds}{\sqrt{16-s^2}}\right]$

**Answer**

$\frac{d}{dx}\left[\int_3^x \frac{ds}{\sqrt{16-s^2}}\right] = \frac{1}{\sqrt{16-x^2}}$

9) $\frac{d}{dx}\left[\int^{2x} x_t\,dt\right]$

10) $\frac{d}{dx}\left[\int_0^{\sqrt{x}} t\,dt\right]$

**Answer**

$\frac{d}{dx}\left[\int_0^{\sqrt{x}} t\,dt\right] = \sqrt{x}\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2}$

11) $\frac{d}{dx}\left[\int_0^{\sin x} \sqrt{1-t^2}\,dt\right]$

12) $\frac{d}{dx}\left[\int_{\cos x}^1 \sqrt{1-t^2}\,dt\right]$

**Answer**

$\frac{d}{dx}\left[\int_{\cos x}^1 \sqrt{1-t^2}\,dt\right] = -\sqrt{1-\cos^2 x}\frac{d}{dx}\left(\cos x\right) = |\sin x|\sin x$

13) $\frac{d}{dx}\left[\int_1^{\sqrt{x}} \frac{t^2}{1+t^4}\,dt\right]$

14) $\frac{d}{dx}\left[\int_1^{x^2} \frac{\sqrt{t}}{1+t}\,dt\right]$

**Answer**

$$\frac{d}{dx}\left[\int_1^{x^2} \frac{\sqrt{t}}{1+t}\,dt\right] = 2x\frac{|x|}{1+x^2}$$

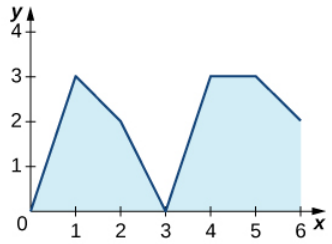
15) $\frac{d}{dx}\left[\int_0^{\ln x} e^t\,dt\right]$

16) $\frac{d}{dx}\left[\int_1^{e^x} \ln u^2\,du\right]$

**Answer**

$$\frac{d}{dx}\left[\int_1^{e^x} \ln u^2\,du\right] = \ln(e^{2x})\frac{d}{dx}\left(e^x\right) = 2xe^x$$

17) The graph of $y = \int_0^x f(t)\,dt$, where $f$ is a piecewise constant function, is shown here.

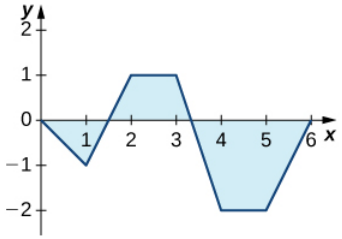



a. Over which intervals is $f$ positive? Over which intervals is it negative? Over which intervals, if any, is it equal to zero?

b. What are the maximum and minimum values of $f$?

c. What is the average value of $f$?

18) The graph of $y = \int_0^x f(t)\,dt$, where $f$ is a piecewise constant function, is shown here.




a. Over which intervals is $f$ positive? Over which intervals is it negative? Over which intervals, if any, is it equal to zero?

b. What are the maximum and minimum values of $f$?

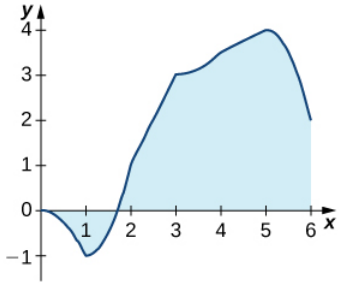
c. What is the average value of $f$?

**Answer**

a. $f$ is positive over $[1,2]$ and $[5,6]$, negative over $[0,1]$ and $[3,4]$, and zero over $[2,3]$ and $[4,5]$.
b. The maximum value is $2$ and the minimum is $-3$.
c. The average value is $0$.

19) The graph of $y = \int_0^x \ell(t)\,dt$, where $\ell$ is a piecewise linear function, is shown here.


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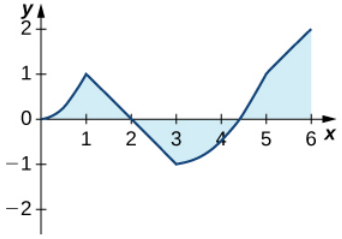



a. Over which intervals is $\ell$ positive? Over which intervals is it negative? Over which, if any, is it zero?

b. Over which intervals is $\ell$ increasing? Over which is it decreasing? Over which, if any, is it constant?

c. What is the average value of $\ell$?

20) The graph of $y = \int_0^x \ell(t)\,dt$, where $\ell$ is a piecewise linear function, is shown here.




a. Over which intervals is $\ell$ positive? Over which intervals is it negative? Over which, if any, is it zero?

b. Over which intervals is $\ell$ increasing? Over which is it decreasing? Over which intervals, if any, is it constant?

c. What is the average value of $\ell$?

**Answer**

a. $\ell$ is positive over $[0, 1]$ and $[3, 6]$, and negative over $[1, 3]$.

b. It is increasing over $[0, 1]$ and $[3, 5]$, and it is constant over $[1, 3]$ and $[5, 6]$.

c. Its average value is $\frac{1}{3}$.

**In exercises 21 - 26, use a calculator to estimate the area under the curve by computing $T_{10}$, the average of the left- and right-endpoint Riemann sums using $N = 10$ rectangles. Then, using the Fundamental Theorem of Calculus, Part 2, determine the exact area.**

21) [T] $y = x^2$ over $[0, 4]$

22) [T] $y = x^3 + 6x^2 + x - 5$ over $[-4, 2]$

**Answer**

$$T_{10} = 49.08, \quad \int_{-2}^{3} \left(x^3 + 6x^2 + x - 5\right)\,dx = 48$$

23) [T] $y = \sqrt{x^3}$ over $[0, 6]$

24) [T] $y = \sqrt{x} + x^2$ over $[1, 9]$

**Answer**

$$T_{10} = 260.836, \quad \int_{1}^{9} \left(\sqrt{x} + x^2\right)\,dx = 260$$

25) [T] $\int (\cos x - \sin x)\,dx$ over $[0, \pi]$

26) [T] $\int \frac{4}{x^2}\,dx$ over $[1,4]$

**Answer**

$T_{10} = 3.058, \quad \int_1^4 \frac{4}{x^2}\,dx = 3$

**In exercises 27 - 46, evaluate each definite integral using the Fundamental Theorem of Calculus, Part 2.**

27) $\int_{-1}^{2} (x^2 - 3x)\,dx$

28) $\int_{-2}^{3} (x^2 + 3x - 5)\,dx$

**Answer**

$F(x) = \frac{x^3}{3} + \frac{3x^2}{2} - 5x, \quad F(3) - F(-2) = -\frac{35}{6}$

29) $\int_{-2}^{3} (t+2)(t-3)dt$

30) $\int_{2}^{3} (t^2 - 9)(4 - t^2)dt$

**Answer**

$F(x) = -\frac{t^5}{5} + \frac{13t^3}{3} - 36t, \quad F(3) - F(2) = \frac{62}{15}$

31) $\int_1^2 x^9\,dx$

32) $\int_0^1 x^{99}\,dx$

**Answer**

$F(x) = \frac{x^{100}}{100}, \quad F(1) - F(0) = \frac{1}{100}$

33) $\int_4^8 (4t^{5/2} - 3t^{3/2})dt$

34) $\int_{1/4}^{4} \left(x^2 - \frac{1}{x^2}\right)\,dx$

**Answer**

$F(x) = \frac{x^3}{3} + \frac{1}{x}, \quad F(4) - F\left(\frac{1}{4}\right) = \frac{1125}{64}$

35) $\int_1^2 \frac{2}{x^3}\,dx$

36) $\int_1^4 \frac{1}{2\sqrt{x}}\,dx$

**Answer**

$F(x) = \sqrt{x}, \quad F(4) - F(1) = 1$

37) $\int_1^4 \frac{2 - \sqrt{t}}{t^2}\,dt$

38) $\int_1^{16} \frac{dt}{t^{1/4}}$

**Answer**

$F(t) = \frac{4}{3}t^{3/4}, \quad F(16) - F(1) = \frac{28}{3}$

39) $\int_0^{2\pi} \cos\theta \, d\theta$

40) $\int_0^{\pi/2} \sin\theta \, d\theta$

**Answer**

$F(\theta) = -\cos\theta, \quad F(\frac{\pi}{2}) - F(0) = 1$

41) $\int_0^{\pi/4} \sec^2\theta \, d\theta$

42) $\int_0^{\pi/4} \sec\theta\tan\theta \, d\theta$

**Answer**

$F(\theta) = \sec\theta, \quad F(\frac{\pi}{4}) - F(0) = \sqrt{2} - 1$

43) $\int_{\pi/3}^{\pi/4} \csc\theta\cot\theta \, d\theta$

44) $\int_{\pi/4}^{\pi/2} \csc^2\theta \, d\theta$

**Answer**

$F(\theta) = -\cot\theta, \quad F(\frac{\pi}{2}) - F(\frac{\pi}{4}) = 1$

45) $\int_1^2 \left(\frac{1}{t^2} - \frac{1}{t^3}\right) dt$

46) $\int_{-2}^{-1} \left(\frac{1}{t^2} - \frac{1}{t^3}\right) dt$

**Answer**

$F(t) = -\frac{1}{t} + \frac{1}{2t^2}, \quad F(-1) - F(-2) = \frac{7}{8}$

**In exercises 47 - 50, use the evaluation theorem to express the integral as a function $F(x)$.**

47) $\int_a^x t^2 \, dt$

48) $\int_1^x e^t \, dt$

**Answer**

$F(x) = e^x - e$

49) $\int_0^x \cos t \, dt$

50) $\int_{-x}^x \sin t \, dt$

**Answer**

$F(x) = 0$

**In exercises 51 - 54, identify the roots of the integrand to remove absolute values, then evaluate using the Fundamental Theorem of Calculus, Part 2.**

51) $\int_{-2}^3 |x| \, dx$

52) $\int_{-2}^4 |\, t^2 - 2t - 3\,| \, dt$

**Answer**

$$\int_{-2}^{-1}(t^2-2t-3)\,dt-\int_{-1}^{3}(t^2-2t-3)\,dt+\int_{3}^{4}(t^2-2t-3)\,dt=\frac{46}{3}$$

53) $\int_0^{\pi}|\cos t|\,dt$

54) $\int_{-\pi/2}^{\pi/2}|\sin t|\,dt$

**Answer**

$$-\int_{-\pi/2}^{0}\sin t\,dt+\int_{0}^{\pi/2}\sin t\,dt=2$$

55) Suppose that the number of hours of daylight on a given day in Seattle is modeled by the function $-3.75\cos(\frac{\pi t}{6})+12.25$, with $t$ given in months and $t=0$ corresponding to the winter solstice.

a. What is the average number of daylight hours in a year?

b. At which times $t_1$ and $t_2$, where $0\le t_1<t_2<12$, do the number of daylight hours equal the average number?

c. Write an integral that expresses the total number of daylight hours in Seattle between $t_1$ and $t_2$

d. Compute the mean hours of daylight in Seattle between $t_1$ and $t_2$, where $0\le t_1<t_2<12$, and then between $t_2$ and $t_1$, and show that the average of the two is equal to the average day length.

56) Suppose the rate of gasoline consumption in the United States can be modeled by a sinusoidal function of the form $(11.21-\cos(\frac{\pi t}{6}))\times 10^9$ gal/mo.

a. What is the average monthly consumption, and for which values of $t$ is the rate at time $t$ equal to the average rate?

b. What is the number of gallons of gasoline consumed in the United States in a year?

c. Write an integral that expresses the average monthly U.S. gas consumption during the part of the year between the beginning of April $(t=3)$ and the end of September $(t=9)$.

**Answer**

a. The average is $11.21\times 10^9$ since $\cos(\frac{\pi t}{6})$ has period 12 and integral 0 over any period. Consumption is equal to the average when $\cos(\frac{\pi t}{6})=0$, when $t=3$, and when $t=9$.

b. Total consumption is the average rate times duration: $11.21\times 12\times 10^9=1.35\times 10^{11}$

c. $10^9(11.21-\frac{1}{6}\int_3^9\cos(\frac{\pi t}{6})\,dt)=10^9(11.21+2\pi)=11.84x10^9$

57) Explain why, if $f$ is continuous over $[a,b]$, there is at least one point $c\in[a,b]$ such that $f(c)=\frac{1}{b-a}\int_a^b f(t)\,dt$.

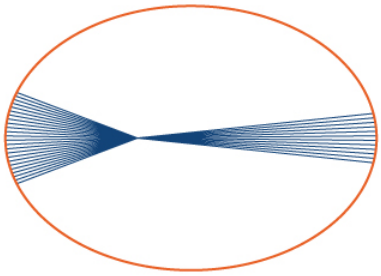
58) Explain why, if $f$ is continuous over $[a,b]$ and is not equal to a constant, there is at least one point $M\in[a,b]$ such that $f(M)=\frac{1}{b-a}\int_a^b f(t)\,dt$ and at least one point $m\in[a,b]$ such that $f(m)<\frac{1}{b-a}\int_a^b f(t)\,dt$.

**Answer**

If $f$ is not constant, then its average is strictly smaller than the maximum and larger than the minimum, which are attained over $[a,b]$ by the extreme value theorem.

59) Kepler's first law states that the planets move in elliptical orbits with the Sun at one focus. The closest point of a planetary orbit to the Sun is called the **perihelion** (for Earth, it currently occurs around January 3) and the farthest point is called the **aphelion** (for Earth, it currently occurs around July 4). Kepler's second law states that planets sweep out equal areas of their elliptical orbits in equal times. Thus, the two arcs indicated in the following figure are swept out in equal times. At what time of year is Earth moving fastest in its orbit? When is it moving slowest? Kepler's first law states that the planets move in elliptical orbits with the Sun at one focus. The closest point of a planetary orbit to the Sun is called the perihelion (for Earth, it currently occurs around January 3) and the farthest point is called the aphelion (for Earth, it currently occurs around July 4). Kepler's second law states that planets sweep

out equal areas of their elliptical orbits in equal times. Thus, the two arcs indicated in the following figure are swept out in equal times. At what time of year is Earth moving fastest in its orbit? When is it moving slowest?



60) A point on an ellipse with major axis length $2a$ and minor axis length $2b$ has the coordinates $(a\cos\theta, b\sin\theta), \quad 0 \le \theta \le 2\pi.$

a. Show that the distance from this point to the focus at $(-c, 0)$ is $d(\theta) = a + c\cos\theta$, where $c = \sqrt{a^2 - b^2}$.

Use these coordinates to show that the average distance $\bar{d}$ from a point on the ellipse to the focus at $(-c, 0)$, with respect to angle $\theta$, is $a$.

**Answer**

a. $d^2\theta = (a\cos\theta + c)^2 + b^2\sin^2\theta = a^2 + c^2\cos^2\theta + 2ac\cos\theta = (a + c\cos\theta)^2$;

b. $\bar{d} = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}(a + 2c\cos\theta)\,d\theta = a$

61) As implied earlier, according to Kepler's laws, Earth's orbit is an ellipse with the Sun at one focus. The perihelion for Earth's orbit around the Sun is 147,098,290 km and the aphelion is 152,098,232 km.

a. By placing the major axis along the $x$-axis, find the average distance from Earth to the Sun.

b. The classic definition of an astronomical unit (AU) is the distance from Earth to the Sun, and its value was computed as the average of the perihelion and aphelion distances. Is this definition justified?

62) The force of gravitational attraction between the Sun and a planet is $F(\theta) = \dfrac{GmM}{r^2(\theta)}$, where $m$ is the mass of the planet, $M$ is the mass of the Sun, $G$ is a universal constant, and $r(\theta)$ is the distance between the Sun and the planet when the planet is at an angle $\theta$ with the major axis of its orbit. Assuming that $M$, $m$, and the ellipse parameters $a$ and $b$ (half-lengths of the major and minor axes) are given, set up—but do not evaluate—an integral that expresses in terms of $G$, $m$, $M$, $a$, $b$ the average gravitational force between the Sun and the planet.

**Answer**

Mean gravitational force $= \dfrac{GmM}{2}\displaystyle\int_0^{2\pi}\frac{1}{(a + 2\sqrt{a^2 - b^2}\cos\theta)^2}\,d\theta.$

The displacement from rest of a mass attached to a spring satisfies the simple harmonic motion equation $x(t) = A\cos(\omega t - \phi)$, where $\phi$ is a phase constant, $\omega$ is the angular frequency, and $A$ is the amplitude. Find the average velocity, the average speed (magnitude of velocity), the average displacement, and the average distance from rest (magnitude of displacement) of the mass.

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# 1.4: Integration Formulas and the Net Change Theorem

**Learning Objectives**

- Apply the basic integration formulas.
- Explain the significance of the net change theorem.
- Use the net change theorem to solve applied problems.
- Apply the integrals of odd and even functions.

In this section, we use some basic integration formulas studied previously to solve some key applied problems. It is important to note that these formulas are presented in terms of indefinite integrals. Although definite and indefinite integrals are closely related, there are some key differences to keep in mind. A definite integral is either a number (when the limits of integration are constants) or a single function (when one or both of the limits of integration are variables). An indefinite integral represents a family of functions, all of which differ by a constant. As you become more familiar with integration, you will get a feel for when to use definite integrals and when to use indefinite integrals. You will naturally select the correct approach for a given problem without thinking too much about it. However, until these concepts are cemented in your mind, think carefully about whether you need a definite integral or an indefinite integral and make sure you are using the proper notation based on your choice.

## Basic Integration Formulas

Recall the integration formulas given in the section on Antiderivatives and the properties of definite integrals. Let's look at a few examples of how to apply these formulas and properties.

**Example 1.4.1: Integrating a Function Using the Power Rule**

Use the power rule to integrate the function $\int_1^4 \sqrt{t}(1+t)\,dt$.

**Solution**

The first step is to rewrite the function and simplify it so we can apply the power rule:

$$\begin{aligned}\int_1^4 \sqrt{t}(1+t)\,dt &= \int_1^4 t^{1/2}(1+t)\,dt \\ &= \int_1^4 (t^{1/2}+t^{3/2})\,dt.\end{aligned}$$

Now apply the power rule:

$$\begin{aligned}\int_1^4 (t^{1/2}+t^{3/2})\,dt &= \left(\frac{2}{3}t^{3/2}+\frac{2}{5}t^{5/2}\right)\Big|_1^4 \\ &= \left[\frac{2}{3}(4)^{3/2}+\frac{2}{5}(4)^{5/2}\right]-\left[\frac{2}{3}(1)^{3/2}+\frac{2}{5}(1)^{5/2}\right] \\ &= \frac{256}{15}.\end{aligned}$$

**Exercise 1.4.1**

Find the definite integral of $f(x)=x^2-3x$ over the interval $[1,3]$.

**Hint**

Follow the process from Example 1.4.1 to solve the problem.

**Answer**

$$\int_1^3 (x^2-3x)\,dx=-\frac{10}{3}$$

## The Net Change Theorem

The **net change theorem** considers the integral of a **rate of change**. It says that when a quantity changes, the new value equals the initial value plus the integral of the rate of change of that quantity. The formula can be expressed in two ways. The second is more familiar; it is simply the definite integral.

**Net Change Theorem**

The new value of a changing quantity equals the initial value plus the integral of the rate of change:

$$F(b) = F(a) + \int_a^b F'(x)dx \tag{1.4.1}$$

or

$$\int_a^b F'(x)dx = F(b) - F(a). \tag{1.4.2}$$

Subtracting $F(a)$ from both sides of the Equation 1.4.1 yields Equation 1.4.2. Since they are equivalent formulas, which one we use depends on the application.

The significance of the net change theorem lies in the results. Net change can be applied to area, distance, and volume, to name only a few applications. Net change accounts for negative quantities automatically without having to write more than one integral. To illustrate, let's apply the net change theorem to a **velocity** function in which the result is **displacement.**

We looked at a simple example of this in The Definite Integral section. Suppose a car is moving due north (the positive direction) at 40 mph between 2 p.m. and 4 p.m., then the car moves south at 30 mph between 4 p.m. and 5 p.m. We can graph this motion as shown in Figure 1.4.1.

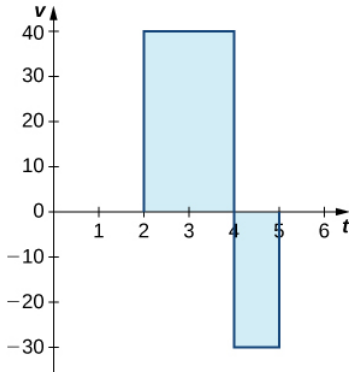



Figure 1.4.1: The graph shows speed versus time for the given motion of a car.

Just as we did before, we can use definite integrals to calculate the net displacement as well as the total distance traveled. The **net displacement** is given by

$$\int_2^5 v(t)\,dt = \int_2^4 40\,dt + \int_4^5 -30\,dt = 80 - 30 = 50.$$

Thus, at 5 p.m. the car is 50 mi north of its starting position. The **total distance** traveled is given by

$$\int_2^5 |v(t)|\,dt = \int_2^4 40\,dt + \int_4^5 30\,dt = 80 + 30 = 110.$$

Therefore, between 2 p.m. and 5 p.m., the car traveled a total of 110 mi.

To summarize, net displacement may include both positive and negative values. In other words, the velocity function accounts for both forward distance and backward distance. To find net displacement, integrate the velocity function over the interval. Total distance traveled, on the other hand, is always positive. To find the total distance traveled by an object, regardless of direction, we need to integrate the absolute value of the velocity function.



**Example 1.4.2: Finding Net Displacement**

Given a velocity function $v(t) = 3t - 5$ (in meters per second) for a particle in motion from time $t = 0$ to time $t = 3$, find the net displacement of the particle.

**Solution**

Applying the net change theorem, we have

$$\int_0^3 (3t-5)\,dt = \left(\frac{3t^2}{2} - 5t\right)\Big|_0^3 = \left[\frac{3(3)^2}{2} - 5(3)\right] - 0 = \frac{27}{2} - 15 = \frac{27}{2} - \frac{30}{2} = -\frac{3}{2}.$$

The net displacement is $-\frac{3}{2}$ m (Figure 1.4.2).

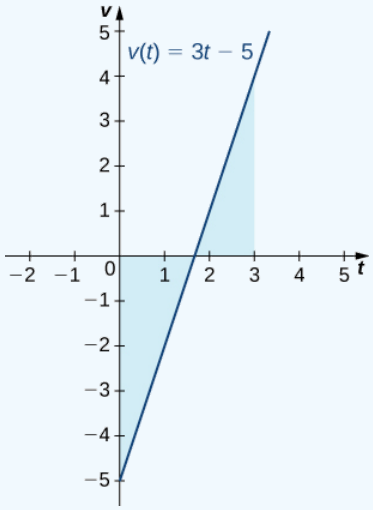



Figure 1.4.2: The graph shows velocity versus time for a particle moving with a linear velocity function.


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**Example 1.4.3: Finding the Total Distance Traveled**

Use Example 1.4.2 to find the total distance traveled by a particle according to the velocity function $v(t) = 3t - 5$ m/sec over a time interval $[0, 3]$.

**Solution**

The total distance traveled includes both the positive and the negative values. Therefore, we must integrate the absolute value of the velocity function to find the total distance traveled.

To continue with the example, use two integrals to find the total distance. First, find the $t$-intercept of the function, since that is where the division of the interval occurs. Set the equation equal to zero and solve for $t$. Thus,

$$\begin{aligned} 3t - 5 &= 0 \\ 3t &= 5 \\ t &= \frac{5}{3}. \end{aligned}$$

The two subintervals are $[0, \frac{5}{3}]$ and $[\frac{5}{3}, 3]$. To find the total distance traveled, integrate the absolute value of the function. Since the function is negative over the interval $[0, \frac{5}{3}]$, we have $|v(t)| = -v(t)$ over that interval. Over $[\frac{5}{3}, 3]$, the function is positive, so $|v(t)| = v(t)$. Thus, we have

$$\begin{aligned} \int_0^3 |v(t)|\, dt &= \int_0^{5/3} -v(t)\, dt + \int_{5/3}^3 v(t)\, dt \\ &= \int_0^{5/3} 5 - 3t\, dt + \int_{5/3}^3 3t - 5\, dt \\ &= \left(5t - \frac{3t^2}{2}\right)\Big|_0^{5/3} + \left(\frac{3t^2}{2} - 5t\right)\Big|_{5/3}^3 \\ &= \left[5(\frac{5}{3}) - \frac{3(5/3)^2}{2}\right] - 0 + \left[\frac{27}{2} - 15\right] - \left[\frac{3(5/3)^2}{2} - \frac{25}{3}\right] \\ &= \frac{25}{3} - \frac{25}{6} + \frac{27}{2} - 15 - \frac{25}{6} + \frac{25}{3} \\ &= \frac{41}{6} \end{aligned}$$

So, the total distance traveled is $\frac{14}{6}$ m.

**Exercise 1.4.2**

Find the net displacement and total distance traveled in meters given the velocity function $f(t) = \frac{1}{2}e^t - 2$ over the interval $[0, 2]$.

**Hint**

Follow the procedures from Examples 1.4.2 and 1.4.3. Note that $f(t) \leq 0$ for $t \leq \ln 4$ and $f(t) \geq 0$ for $t \geq \ln 4$.

**Answer**

Net displacement: $\frac{e^2-9}{2} \approx -0.8055$ m; total distance traveled: $4 \ln 4 - 7.5 + \frac{e^2}{2} \approx 1.740$ m.

## Applying the Net Change Theorem

The net change theorem can be applied to the flow and consumption of fluids, as shown in Example 1.4.4.

**Example 1.4.4: How Many Gallons of Gasoline Are Consumed?**

If the motor on a motorboat is started at $t = 0$ and the boat consumes gasoline at the rate of $5 - t^3$ gal/hr, how much gasoline is used in the first 2 hours?

**Solution**

Express the problem as a definite integral, integrate, and evaluate using the Fundamental Theorem of Calculus. The limits of integration are the endpoints of the interval [0,2]. We have

$$\begin{aligned} \int_0^2 (5 - t^3)\, dt &= \left(5t - \frac{t^4}{4}\right) \Big|_0^2 \\ &= \left[5(2) - \frac{(2)^4}{4}\right] - 0 \\ &= 10 - \frac{16}{4} \\ &= 6. \end{aligned}$$

Thus, the motorboat uses 6 gal of gas in 2 hours.

## Choose the Car With the Best Velocity Game

You will be competing in a car race where you get to choose the car. You will be given the velocity functions of each car and based on this information, pick the car that will win the race, i.e. travel the farthest during the given time frame. Click on "Next Race" to get started.

Next Race

**Example 1.4.5: Chapter Opener: Iceboats**

As we saw at the beginning of the chapter, top **iceboat** racers can attain speeds of up to five times the wind speed. Andrew is an intermediate iceboater, though, so he attains speeds equal to only twice the wind speed.




Figure **1.4.3**: (credit: modification of work by Carter Brown, Flickr)

Suppose Andrew takes his iceboat out one morning when a light 5-mph breeze has been blowing all morning. As Andrew gets his iceboat set up, though, the wind begins to pick up. During his first half hour of iceboating, the wind speed increases according to the function $v(t) = 20t + 5$. For the second half hour of Andrew's outing, the wind remains steady at 15 mph. In other words, the wind speed is given by

$$v(t) = \begin{cases} 20t + 5, & \text{for } 0 \le t \le \frac{1}{2} \\ 15, & \text{for } \frac{1}{2} \le t \le 1 \end{cases}$$

Recalling that Andrew's iceboat travels at twice the wind speed, and assuming he moves in a straight line away from his starting point, how far is Andrew from his starting point after 1 hour?

**Solution**

To figure out how far Andrew has traveled, we need to integrate his velocity, which is twice the wind speed. Then

$$\text{Distance} = \int_0^1 2v(t)\,dt.$$

Substituting the expressions we were given for $v(t)$, we get

$$\begin{aligned} \int_0^1 2v(t)\,dt &= \int_0^{1/2} 2v(t)\,dt + \int_{1/2}^1 2v(t)\,dt \\ &= \int_0^{1/2} 2(20t+5)\,dt + \int_{1/3}^1 2(15)\,dt \\ &= \int_0^{1/2} (40t+10)\,dt + \int_{1/2}^1 30\,dt \\ &= [20t^2 + 10t]\Big|_0^{1/2} + [30t]\Big|_{1/2}^1 \\ &= \left(\frac{20}{4} + 5\right) - 0 + (30 - 15) \\ &= 25. \end{aligned}$$

Andrew is 25 mi from his starting point after 1 hour.

**Exercise 1.4.3**

Suppose that, instead of remaining steady during the second half hour of Andrew's outing, the wind starts to die down according to the function $v(t) = -10t + 15$. In other words, the wind speed is given by

$$v(t) = \begin{cases} 20t + 5, & \text{for } 0 \le t \le \frac{1}{2} \\ -10t + 15, & \text{for } \frac{1}{2} \le t \le 1 \end{cases}.$$

Under these conditions, how far from his starting point is Andrew after 1 hour?

**Hint**

Don't forget that Andrew's iceboat moves twice as fast as the wind.

**Answer**

17.5 mi

## Integrating Even and Odd Functions

We saw in Functions and Graphs that an **even function** is a function in which $f(-x) = f(x)$ for all $x$ in the domain—that is, the graph of the curve is unchanged when $x$ is replaced with $-x$. The graphs of even functions are symmetric about the $y$-axis. An **odd function** is one in which $f(-x) = -f(x)$ for all $x$ in the domain, and the graph of the function is symmetric about the origin.

Integrals of even functions, when the limits of integration are from $-a$ to $a$, involve two equal areas, because they are symmetric about the $y$-axis. Integrals of odd functions, when the limits of integration are similarly $[-a, a]$, evaluate to zero because the areas above and below the $x$-axis are equal.

### Integrals of Even and Odd Functions

For continuous even functions such that $f(-x) = f(x)$,

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx. \tag{1.4.3}$$

For continuous odd functions such that $f(-x) = -f(x)$,

$$\int_{-a}^{a} f(x)\,dx = 0. \tag{1.4.4}$$

### Example 1.4.6: Integrating an Even Function

Integrate the even function $\int_{-2}^{2}(3x^8 - 2)\,dx$ and verify that the integration formula for even functions holds.

**Solution**

The symmetry appears in the graphs in Figure 1.4.4. Graph (a) shows the region below the curve and above the $x$-axis. We have to zoom in to this graph by a huge amount to see the region. Graph (b) shows the region above the curve and below the $x$-axis. The signed area of this region is negative. Both views illustrate the symmetry about the $y$-axis of an even function. We have

$$\begin{aligned}\int_{-2}^{2}(3x^8 - 2)\,dx &= \left(\frac{x^9}{3} - 2x\right)\Big|_{-2}^{2} \\ &= \left[\frac{(2)^9}{3} - 2(2)\right] - \left[\frac{(-2)^9}{3} - 2(-2)\right] \\ &= \left(\frac{512}{3} - 4\right) - \left(-\frac{512}{3} + 4\right) \\ &= \frac{1000}{3}.\end{aligned}$$

To verify the integration formula for even functions, we can calculate the integral from 0 to 2 and double it, then check to make sure we get the same answer.

$$\int_{0}^{2}(3x^8 - 2)\,dx = \left(\frac{x^9}{3} - 2x\right)\Big|_{0}^{2} = \frac{512}{3} - 4 = \frac{500}{3}$$

Since $2 \cdot \frac{500}{3} = \frac{1000}{3}$, we have verified the formula for even functions in this particular example.

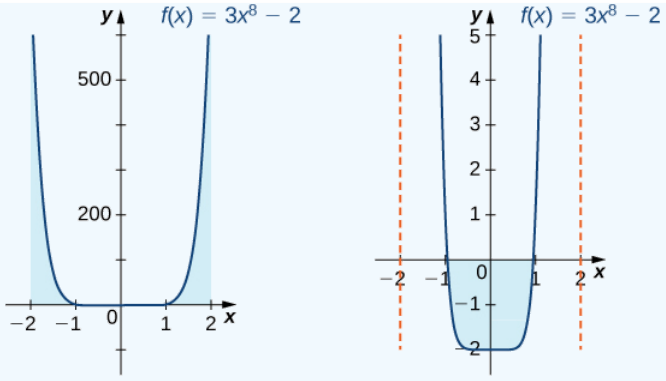



Figure 1.4.4: Graph (a) shows the positive area between the curve and the $x$-axis, whereas graph (b) shows the negative area between the curve and the $x$-axis. Both views show the symmetry about the $y$-axis.

## Example 1.4.7: Integrating an Odd Function

Evaluate the definite integral of the odd function $-5\sin x$ over the interval $[-\pi, \pi]$.

**Solution**

The graph is shown in Figure 1.4.5. We can see the symmetry about the origin by the positive area above the $x$-axis over $[-\pi, 0]$, and the negative area below the $x$-axis over $[0, \pi]$. we have

$$\begin{aligned}\int_{-\pi}^{\pi} -5\sin x\,dx &= -5(-\cos x)\Big|_{-\pi}^{\pi} \\ &= 5\cos x\Big|_{-\pi}^{\pi} \\ &= [5\cos\pi] - [5\cos(-\pi)] \\ &= -5 - (-5) = 0.\end{aligned}$$

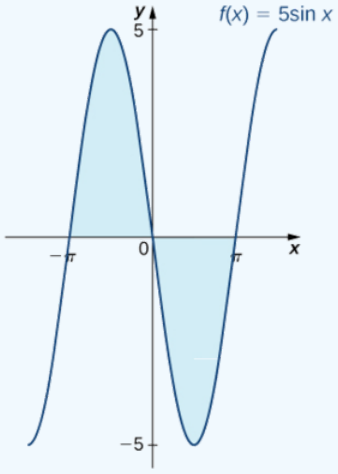



Figure 1.4.5:The graph shows areas between a curve and the $x$-axis for an odd function.

## Exercise 1.4.4

Integrate the function $\int_{-2}^{2} x^4\,dx$.

**Hint**

Integrate an even function.

**Answer**


https://math.libretexts.org/@go/page/83595

$\frac{64}{5}$

## Key Concepts

- The net change theorem states that when a quantity changes, the final value equals the initial value plus the integral of the rate of change. Net change can be a positive number, a negative number, or zero.
- The area under an even function over a symmetric interval can be calculated by doubling the area over the positive $x$-axis. For an odd function, the integral over a symmetric interval equals zero, because half the area is negative.

## Key Equations

- Net Change Theorem

$$F(b) = F(a) + \int_a^b F'(x)\,dx$$

or

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

## Glossary

**net change theorem**

if we know the rate of change of a quantity, the net change theorem says the future quantity is equal to the initial quantity plus the integral of the rate of change of the quantity

## Contributors and Attributions

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# 1.4E: Exercises for Section 1.4

**In exercises 1 - 6, use basic integration formulas to compute the following antiderivatives.**

1) $\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right) dx$

**Answer**

$$\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right) dx = \int x^{1/2}\, dx - \int x^{-1/2}\, dx = \frac{2}{3}x^{3/2} + C_1 - 2x^{1/2} + C_2 = \frac{2}{3}x^{3/2} - 2x^{1/2} + C$$

2) $\int \left(e^{2x} - \frac{1}{2}e^{x/2}\right) dx$

3) $\int \frac{dx}{2x}$

**Answer**

$$\int \frac{dx}{2x} = \frac{1}{2}\ln|x| + C$$

4) $\int \frac{x-1}{x^2}\, dx$

5) $\int_0^{\pi} (\sin x - \cos x)\, dx$

**Answer**

$$\int_0^{\pi} \sin x\, dx - \int_0^{\pi} \cos x\, dx = -\cos x\big|_0^{\pi} - (\sin x)\big|_0^{\pi} = (-(-1) + 1) - (0 - 0) = 2$$

6) $\int_0^{\pi/2} (x - \sin x)\, dx$

7) Write an integral that expresses the increase in the perimeter $P(s)$ of a square when its side length $s$ increases from 2 units to 4 units and evaluate the integral.

**Answer**

$P(s) = 4s$, so $\frac{dP}{ds} = 4$ and $\int_2^4 4\, ds = 8$.

8) Write an integral that quantifies the change in the area $A(s) = s^2$ of a square when the side length doubles from $S$ units to $2S$ units and evaluate the integral.

9) A regular $N$-gon (an $N$-sided polygon with sides that have equal length $s$, such as a pentagon or hexagon) has perimeter $Ns$. Write an integral that expresses the increase in perimeter of a regular $N$-gon when the length of each side increases from 1 unit to 2 units and evaluate the integral.

**Answer**

$$\int_1^2 N\, ds = N$$

10) The area of a regular pentagon with side length $a > 0$ is $pa^2$ with $p = \frac{1}{4}\sqrt{5 + \sqrt{5 + 2\sqrt{5}}}$. The Pentagon in Washington, DC, has inner sides of length 360 ft and outer sides of length 920 ft. Write an integral to express the area of the roof of the Pentagon according to these dimensions and evaluate this area.

11) A dodecahedron is a Platonic solid with a surface that consists of 12 pentagons, each of equal area. By how much does the surface area of a dodecahedron increase as the side length of each pentagon doubles from 1 unit to 2 units?

**Answer**

With $p$ as in the previous exercise, each of the 12 pentagons increases in area from $2p$ to $4p$ units so the net increase in the area of the dodecahedron is $36p$ units.

12) An icosahedron is a Platonic solid with a surface that consists of 20 equilateral triangles. By how much does the surface area of an icosahedron increase as the side length of each triangle doubles from a unit to $2a$ units?

13) Write an integral that quantifies the change in the area of the surface of a cube when its side length doubles from $s$ unit to $2s$ units and evaluate the integral.

**Answer**

$18s^2 = 6\int_s^{2s} 2x\,dx$

14) Write an integral that quantifies the increase in the volume of a cube when the side length doubles from $s$ unit to $2s$ units and evaluate the integral.

15) Write an integral that quantifies the increase in the surface area of a sphere as its radius doubles from $R$ unit to $2R$ units and evaluate the integral.

**Answer**

$12\pi R^2 = 8\pi\int_R^{2R} r\,dr$

16) Write an integral that quantifies the increase in the volume of a sphere as its radius doubles from $R$ unit to $2R$ units and evaluate the integral.

17) Suppose that a particle moves along a straight line with velocity $v(t) = 4 - 2t$, where $0 \le t \le 2$ (in meters per second). Find the displacement at time $t$ and the total distance traveled up to $t = 2$.

**Answer**

$d(t) = \int_0^t v(s)\,ds = 4t - t^2$. The total distance is $d(2) = 4$ m.

18) Suppose that a particle moves along a straight line with velocity defined by $v(t) = t^2 - 3t - 18$, where $0 \le t \le 6$ (in meters per second). Find the displacement at time $t$ and the total distance traveled up to $t = 6$.

19) Suppose that a particle moves along a straight line with velocity defined by $v(t) = |2t - 6|$, where $0 \le t \le 6$ (in meters per second). Find the displacement at time $t$ and the total distance traveled up to $t = 6$.

**Answer**

$d(t) = \int_0^t v(s)\,ds$. For $t < 3$, $d(t) = \int_0^t (6 - 2t)\,dt = 6t - t^2$. For $t > 3$, $d(t) = d(3) + \int_3^t (2t - 6)\,dt = 9 + (t^2 - 6t)$. The total distance is $d(6) = 9$ m.

20) Suppose that a particle moves along a straight line with acceleration defined by $a(t) = t - 3$, where $0 \le t \le 6$ (in meters per second). Find the velocity and displacement at time $t$ and the total distance traveled up to $t = 6$ if $v(0) = 3$ and $d(0) = 0$.

21) A ball is thrown upward from a height of 1.5 m at an initial speed of 40 m/sec. Acceleration resulting from gravity is $-9.8$ $\text{m/sec}^2$. Neglecting air resistance, solve for the velocity $v(t)$ and the height $h(t)$ of the ball $t$ seconds after it is thrown and before it returns to the ground.

**Answer**

$v(t) = 40 - 9.8t;\ h(t) = 1.5 + 40t - 4.9t^2$ m/s

22) A ball is thrown upward from a height of 3 m at an initial speed of 60 m/sec. Acceleration resulting from gravity is $-9.8\,\text{m/sec}^2$. Neglecting air resistance, solve for the velocity $v(t)$ and the height $h(t)$ of the ball $t$ seconds after it is thrown and before it returns to the ground.

23) The area $A(t)$ of a circular shape is growing at a constant rate. If the area increases from $4\pi$ units to $9\pi$ units between times $t = 2$ and $t = 3$, find the net change in the radius during that time.

**Answer**

The net increase is 1 unit.

24) A spherical balloon is being inflated at a constant rate. If the volume of the balloon changes from $36\pi\ \text{in}^3$ to $288\pi\ \text{in}^3$ between time $t = 30$ and $t = 60$ seconds, find the net change in the radius of the balloon during that time.

25) Water flows into a conical tank with cross-sectional area $\pi x^2$ at height $x$ and volume $\frac{\pi x^3}{3}$ up to height $x$. If water flows into the tank at a rate of 1 $\text{m}^3/\text{min}$, find the height of water in the tank after 5 min. Find the change in height between 5 min and 10 min.

**Answer**

At $t = 5$, the height of water is $x = \left(\frac{15}{\pi}\right)^{1/3}$ m. The net change in height from $t = 5$ to $t = 10$ is $\left(\left(\frac{30}{\pi}\right)^{1/3} - \left(\frac{15}{\pi}\right)^{1/3}\right)$ m.

26) A horizontal cylindrical tank has cross-sectional area $A(x) = 4(6x - x^2)\ \text{m}^2$ at height $x$ meters above the bottom when $x \leq 3$.

a. The volume $V$ between heights $a$ and $b$ is $\int_a^b A(x)\,dx$. Find the volume at heights between 2 m and 3 m.

b. Suppose that oil is being pumped into the tank at a rate of 50 L/min. Using the chain rule, $\frac{dx}{dt} = \frac{dx}{dV}\frac{dV}{dt}$, at how many meters per minute is the height of oil in the tank changing, expressed in terms of $x$, when the height is at $x$ meters?

c. How long does it take to fill the tank to 3 m starting from a fill level of 2 m?

27) The following table lists the electrical power in gigawatts—the rate at which energy is consumed—used in a certain city for different hours of the day, in a typical 24-hour period, with hour 1 corresponding to midnight to 1 a.m.

| Hour | Power | Hour | Power |
|---|---|---|---|
| 1 | 28 | 13 | 48 |
| 2 | 25 | 14 | 49 |
| 3 | 24 | 15 | 49 |
| 4 | 23 | 16 | 50 |
| 5 | 24 | 17 | 50 |
| 6 | 27 | 18 | 50 |
| 7 | 29 | 19 | 46 |
| 8 | 32 | 20 | 43 |
| 9 | 34 | 21 | 42 |
| 10 | 39 | 22 | 40 |
| 11 | 42 | 23 | 37 |
| 12 | 46 | 24 | 34 |

Find the total amount of power in gigawatt-hours (gW-h) consumed by the city in a typical 24-hour period.

**Answer**

The total daily power consumption is estimated as the sum of the hourly power rates, or 911 gW-h.

28) The average residential electrical power use (in hundreds of watts) per hour is given in the following table.

| Hour | Power | Hour | Power |
|---|---|---|---|
| 1 | 8 | 13 | 12 |
| 2 | 6 | 14 | 13 |
| 3 | 5 | 15 | 14 |
| 4 | 4 | 16 | 15 |
| 5 | 5 | 17 | 17 |
| 6 | 6 | 18 | 19 |
| 7 | 7 | 19 | 18 |
| 8 | 8 | 20 | 17 |
| 9 | 9 | 21 | 16 |
| 10 | 10 | 22 | 16 |
| 11 | 10 | 23 | 13 |
| 12 | 11 | 24 | 11 |

a. Compute the average total energy used in a day in kilowatt-hours (kWh).

b. If a ton of coal generates 1842 kWh, how long does it take for an average residence to burn a ton of coal?

c. Explain why the data might fit a plot of the form $p(t) = 11.5 - 7.5 \sin\left(\frac{\pi t}{12}\right)$.

29) The data in the following table are used to estimate the average power output produced by Peter Sagan for each of the last 18 sec of Stage 1 of the 2012 **Tour de France.**

| Second | Watts | Second | Watts |
|---|---|---|---|
| 1 | 600 | 10 | 1200 |
| 2 | 500 | 11 | 1170 |
| 3 | 575 | 12 | 1125 |
| 4 | 1050 | 13 | 1100 |
| 5 | 925 | 14 | 1075 |
| 6 | 950 | 15 | 1000 |
| 7 | 1050 | 16 | 950 |
| 8 | 950 | 17 | 900 |
| 9 | 1100 | 18 | 780 |

Average Power OutputSource: sportsexercisengineering.com

Estimate the net energy used in kilojoules (kJ), noting that 1W = 1 J/s, and the average power output by Sagan during this time interval.

**Answer**

**17** kJ

30) The data in the following table are used to estimate the average power output produced by Peter Sagan for each 15-min interval of Stage 1 of the 2012 Tour de France.

| Minutes | Watts | Minutes | Watts |
|---|---|---|---|
| 15 | 200 | 165 | 170 |
| 30 | 180 | 180 | 220 |
| 45 | 190 | 195 | 140 |
| 60 | 230 | 210 | 225 |
| 75 | 240 | 225 | 170 |
| 90 | 210 | 240 | 10 |
| 105 | 210 | 255 | 200 |
| 1120 | 220 | 270 | 220 |
| 135 | 210 | 285 | 250 |
| 150 | 150 | 300 | 400 |

Average Power Output*Source*: sportsexercisengineering.com

Estimate the net energy used in kilojoules, noting that 1W = 1 J/s.


https://math.libretexts.org/@go/page/83596

31) The distribution of incomes as of 2012 in the United States in $5000 increments is given in the following table. The kth row denotes the percentage of households with incomes between $\$5000xk$ and $5000xk + 4999$. The row $k = 40$ contains all households with income between $200,000 and $250,000 and $k = 41$ accounts for all households with income exceeding $250,000.

| | | | |
|---|---|---|---|
| 0 | 3.5 | 21 | 1.5 |
| 1 | 4.1 | 22 | 1.4 |
| 2 | 5.9 | 23 | 1.3 |
| 3 | 5.7 | 24 | 1.3 |
| 4 | 5.9 | 25 | 1.1 |
| 5 | 5.4 | 26 | 1.0 |
| 6 | 5.5 | 27 | 0.75 |
| 7 | 5.1 | 28 | 0.8 |
| 8 | 4.8 | 29 | 1.0 |
| 9 | 4.1 | 30 | 0.6 |
| 10 | 4.3 | 31 | 0.6 |
| 11 | 3.5 | 32 | .5 |
| 12 | 3.7 | 33 | 0.5 |
| 13 | 3.2 | 34 | 0.4 |
| 14 | 3.0 | 35 | 0.3 |
| 15 | 2.8 | 36 | 0.3 |
| 16 | 2.5 | 37 | 0.3 |
| 17 | 2.2 | 38 | 0.2 |
| 18 | 2.2 | 39 | 1.8 |
| 19 | 1.8 | 40 | 2.3 |
| 20 | 2.1 | 41 | |

Income DistributionsSource: www.census.gov/prod/2013pubs/p60-245.pdf

a. Estimate the percentage of U.S. households in 2012 with incomes less than $55,000.

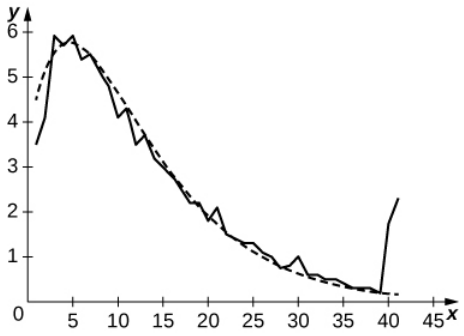
b. What percentage of households had incomes exceeding $85,000?

c. Plot the data and try to fit its shape to that of a graph of the form $a(x + c)e^{-b(x+e)}$ for suitable $a, b, c$.

**Answer**

a. 54.3%;
b. 27.00%;
c. The curve in the following plot is $2.35(t + 3)e^{-0.15(t+3)}$.


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32) Newton's law of gravity states that the gravitational force exerted by an object of mass $M$ and one of mass $m$ with centers that are separated by a distance r is $F = G\frac{mM}{r^2}$, with $G$ an empirical constant $G = 6.67x10^{-11}\ \text{m}^3/(\text{kg}\cdot\text{s}^2)$. The work done by a variable force over an interval $[a, b]$ is defined as $W = \int_a^b F(x)\,dx$. If Earth has mass $5.97219 \times 10^{24}$ and radius 6371 km, compute the amount of work to elevate a polar weather satellite of mass 1400 kg to its orbiting altitude of 850 km above Earth.

33) For a given motor vehicle, the maximum achievable **deceleration** from braking is approximately $7\ \text{m/sec}^2$ on dry concrete. On wet asphalt, it is approximately $2.5\ \text{m/sec}^2$. Given that 1 mph corresponds to 0.447 m/sec, find the total distance that a car travels in meters on dry concrete after the brakes are applied until it comes to a complete stop if the initial velocity is 67 mph (30 m/sec) or if the initial braking velocity is 56 mph (25 m/sec). Find the corresponding distances if the surface is slippery wet asphalt.

**Answer**

In dry conditions, with initial velocity $v_0 = 30$ m/s, $D = 64.3$ and, if $v_0 = 25,\ D = 44.64$. In wet conditions, if $v_0 = 30$, and $D = 180$ and if $v_0 = 25,\ D = 125$.

34) John is a 25-year old man who weighs 160 lb. He burns $500 - 50t$ calories/hr while riding his bike for $t$ hours. If an oatmeal cookie has 55 cal and John eats $4t$ cookies during the $t^{\text{th}}$ hour, how many net calories has he lost after 3 hours riding his bike?

35) Sandra is a 25-year old woman who weighs 120 lb. She burns $300 - 50t$ cal/hr while walking on her treadmill for $t$ hours. Her caloric intake from drinking Gatorade is $100t$ calories during the $t^{\text{th}}$ hour. What is her net decrease in calories after walking for 3 hours?

**Answer**

225 cal

36) A motor vehicle has a maximum efficiency of 33 mpg at a cruising speed of 40 mph. The efficiency drops at a rate of 0.1 mpg/mph between 40 mph and 50 mph, and at a rate of 0.4 mpg/mph between 50 mph and 80 mph. What is the efficiency in miles per gallon if the car is cruising at 50 mph? What is the efficiency in miles per gallon if the car is cruising at 80 mph? If gasoline costs $3.50/gal, what is the cost of fuel to drive 50 mi at 40 mph, at 50 mph, and at 80 mph?

37) Although some engines are more efficient at given a horsepower than others, on average, fuel efficiency decreases with horsepower at a rate of $1/25$ mpg/horsepower. If a typical 50-horsepower engine has an average fuel efficiency of 32 mpg, what is the average fuel efficiency of an engine with the following horsepower: 150, 300, 450?

**Answer**

$E(150) = 28,\ E(300) = 22,\ E(450) = 16$

38) [T] The following table lists the 2013 schedule of **federal income tax** versus taxable income.

| Taxable Income Range | The Tax Is ... | ... Of the Amount Over |
|---|---|---|
| $0–$8925 | 10% | $0 |
| $8925–$36,250 | $892.50 + 15% | $8925 |
| $36,250–$87,850 | $4,991.25 + 25% | $36,250 |
| $87,850–$183,250 | $17,891.25 + 28% | $87,850 |
| $183,250–$398,350 | $44,603.25 + 33% | $183,250 |
| $398,350–$400,000 | $115,586.25 + 35% | $398,350 |
| > $400,000 | $116,163.75 + 39.6% | $400,000 |

Federal Income Tax Versus Taxable IncomeSource: http://www.irs.gov/pub/irs-prior/i1040tt--2013.pdf.

Suppose that Steve just received a $10,000 raise. How much of this raise is left after federal taxes if Steve's salary before receiving the raise was $40,000? If it was $90,000? If it was $385,000?

39) [T] The following table provides hypothetical data regarding the level of service for a certain highway.

| Highway Speed Range (mph) | Vehicles per Hour per Lane | Density Range (vehicles/mi) |
|---|---|---|
| >60 | <600 | <10 |
| 60-57 | 300-1000 | 10-20 |
| 57-54 | 1000-1500 | 20-30 |
| 57-54 | 1500-1900 | 30-45 |
| 46-30 | 1900-2100 | 48-70 |
| <30 | Unstable | 70-200 |

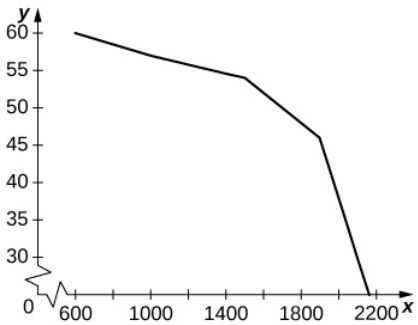
a. Plot vehicles per hour per lane on the $x$-axis and highway speed on the $y$-axis.

b. Compute the average decrease in speed (in miles per hour) per unit increase in congestion (vehicles per hour per lane) as the latter increases from 600 to 1000, from 1000 to 1500, and from 1500 to 2100. Does the decrease in miles per hour depend linearly on the increase in vehicles per hour per lane?

c. Plot minutes per mile (60 times the reciprocal of miles per hour) as a function of vehicles per hour per lane. Is this function linear?

**Answer**

a.

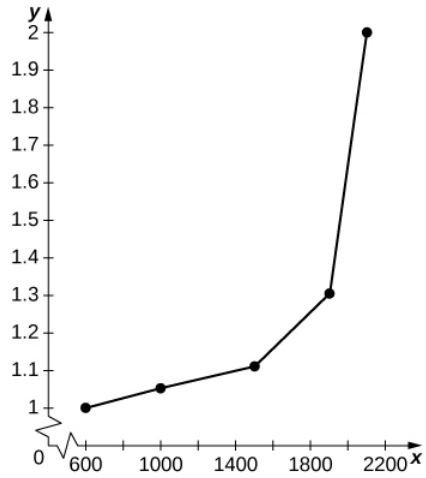




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b. Between 600 and 1000 the average decrease in vehicles per hour per lane is −0.0075. Between 1000 and 1500 it is −0.006 per vehicles per hour per lane, and between 1500 and 2100 it is −0.04 vehicles per hour per lane.

c.



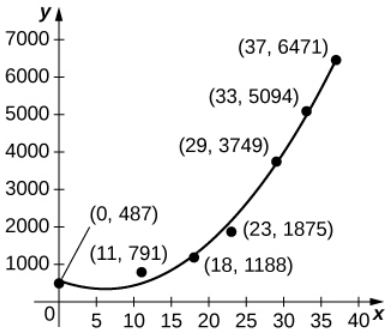

The graph is nonlinear, with minutes per mile increasing dramatically as vehicles per hour per lane reach 2000.

**For the next two exercises use the data in the following table, which displays bald eagle populations from 1963 to 2000 in the continental United States.**

| Year | Population of Breeding Pairs of Bald Eagles |
|---|---|
| 1963 | 487 |
| 1974 | 791 |
| 1981 | 1188 |
| 1986 | 1875 |
| 1992 | 3749 |
| 1996 | 5094 |
| 2000 | 6471 |

Population of Breeding Bald Eagle Pairs*Source*: www.fws.gov/Midwest/eagle/pop.../chtofprs.html.

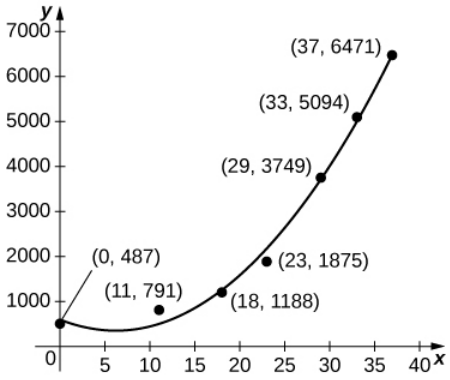
40) [T] The graph below plots the quadratic $p(t) = 6.48t^2 - 80.31t + 585.69$ against the data in preceding table, normalized so that $t = 0$ corresponds to 1963. Estimate the average number of bald eagles per year present for the 37 years by computing the average value of $p$ over $[0, 37]$.





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41) [T] The graph below plots the cubic $p(t) = 0.07t^3 + 2.42t^2 - 25.63t + 521.23$ against the data in the preceding table, normalized so that $t = 0$ corresponds to 1963. Estimate the average number of bald eagles per year present for the 37 years by computing the average value of $p$ over $[0, 37]$.



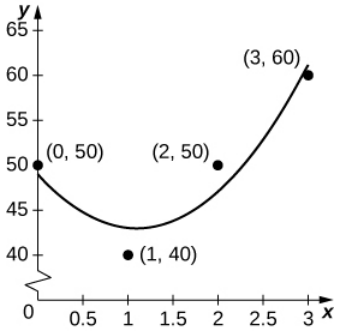

**Answer**

$$\frac{1}{37}\int_0^{37} p(t)\,dt = \frac{0.07(37)^3}{4} + \frac{2.42(37)^2}{3} - \frac{25.63(37)}{2} + 521.23 \approx 2037$$

Solution:

42) [T] Suppose you go on a road trip and record your speed at every half hour, as compiled in the following table. The best quadratic fit to the data is $q(t) = 5x^2 - 11x + 49$, shown in the accompanying graph. Integrate $q$ to estimate the total distance driven over the 3 hours.

| Time (hr) | Speed (m[h) |
|---|---|
| 0 (start) | 50 |
| 1 | 40 |
| 2 | 50 |
| 3 | 60 |



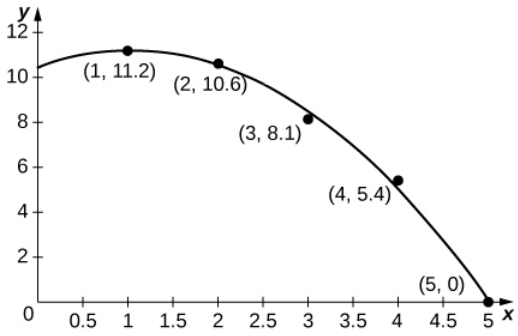


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**As a car accelerates, it does not accelerate at a constant rate; rather, the acceleration is variable. For the following exercises, use the following table, which contains the acceleration measured at every second as a driver merges onto a freeway.**

| Time (sec) | Acceleration (mph/sec) |
|---|---|
| 1 | 11.2 |
| 2 | 10.6 |
| 3 | 8.1 |
| 4 | 5.4 |
| 5 | 0 |

43) [T] The accompanying graph plots the best quadratic fit, $a(t) = -0.70t^2 + 1.44t + 10.44$, to the data from the preceding table. Compute the average value of $a(t)$ to estimate the average acceleration between $t = 0$ and $t = 5$.




**Answer**

Average acceleration is $A = \frac{1}{5}\int_0^5 a(t)\,dt = -\frac{0.7(5^2)}{3} + \frac{1.44(5)}{2} + 10.44 \approx 8.2$ mph/s

44) [T] Using your acceleration equation from the previous exercise, find the corresponding velocity equation. Assuming the final velocity is 0 mph, find the velocity at time $t = 0$.

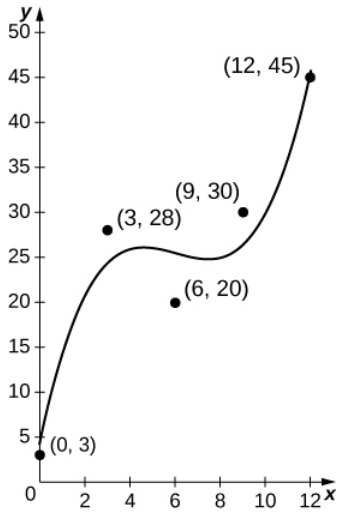
45) [T] Using your velocity equation from the previous exercise, find the corresponding distance equation, assuming your initial distance is 0 mi. How far did you travel while you accelerated your car? (Hint: You will need to convert time units.)

**Answer**

$d(t) = \int_0^1 |v(t)|\,dt = \int_0^t \left(\frac{7}{30}t^3 - 0.72t^2 - 10.44t + 41.033\right)dt = \frac{7}{120}t^4 - 0.24t^3 - 5.22t^3 + 41.033t$. Then, $d(5) \approx 81.12\,\text{mph} \times \text{sec} \approx 119$ feet.

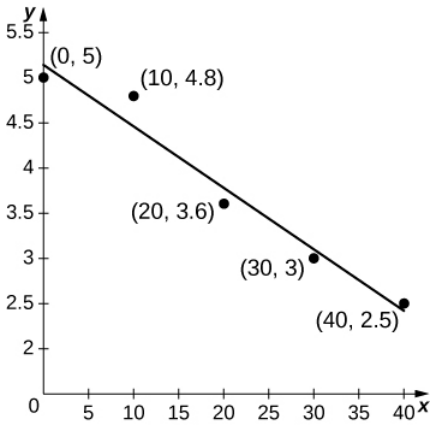
46) [T] The number of hamburgers sold at a restaurant throughout the day is given in the following table, with the accompanying graph plotting the best cubic fit to the data, $b(t) = 0.12t^3 - 2.13t^3 + 12.13t + 3.91$, with $t = 0$ corresponding to 9 a.m. and $t = 12$ corresponding to 9 p.m. Compute the average value of $b(t)$ to estimate the average number of hamburgers sold per hour.

| **Hours Past Midnight** | **No. of Burgers Sold** |
|---|---|
| 9 | 3 |
| 12 | 28 |
| 15 | 20 |
| 18 | 30 |
| 21 | 45 |




47) [T] An athlete runs by a motion detector, which records her speed, as displayed in the following table. The best linear fit to this data, $\ell(t) = -0.068t + 5.14$, is shown in the accompanying graph. Use the average value of $\ell(t)$ between $t = 0$ and $t = 40$ to estimate the runner's average speed.

| Minutes | Speed (m/sec) |
|---|---|
| 0 | 5 |
| 10 | 4.8 |
| 20 | 3.6 |
| 30 | 3.0 |
| 40 | 2.5 |




**Answer**

$$\frac{1}{40}\int_0^{40}(-0.068t + 5.14)\,dt = -\frac{0.068(40)}{2} + 5.14 = 3.78 \text{ m/sec}$$

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# 1.5: Substitution

**Learning Objectives**

- Use substitution to evaluate indefinite integrals.
- Use substitution to evaluate definite integrals.

The Fundamental Theorem of Calculus gave us a method to evaluate integrals without using Riemann sums. The drawback of this method, though, is that we must be able to find an antiderivative, and this is not always easy. In this section we examine a technique, called *integration by substitution*, to help us find antiderivatives. Specifically, this method helps us find antiderivatives when the integrand is the result of a chain-rule derivative.

At first, the approach to the substitution procedure may not appear very obvious. However, it is primarily a visual task—that is, the integrand shows you what to do; it is a matter of recognizing the form of the function. So, what are we supposed to see? We are looking for an integrand of the form $f\big[g(x)\big]g'(x)\,dx$. For example, in the integral

$$\int (x^2-3)^3\,2x\,dx. \tag{1.5.1}$$

we have

$$f(x)=x^3$$

and

$$g(x)=x^2-3.$$

Then

$$g'(x)=2x.$$

and

$$f[g(x)]g'(x)=(x^2-3)^3(2x),$$

and we see that our integrand is in the correct form. The method is called substitution because we substitute part of the integrand with the variable $u$ and part of the integrand with $du$. It is also referred to as *change of variables* because we are changing variables to obtain an expression that is easier to work with for applying the integration rules.

**Substitution with Indefinite Integrals**

Let $u=g(x)$,, where $g'(x)$ is continuous over an interval, let $f(x)$ be continuous over the corresponding range of $g$, and let $F(x)$ be an antiderivative of $f(x)$. Then,

$$\begin{aligned}\int f[g(x)]g'(x)\,dx &= \int f(u)\,du\\ &= F(u)+C\\ &= F(g(x))+C\end{aligned}$$

**Proof**

Let $f$, $g$, $u$, and $F$ be as specified in the theorem. Then

$$\frac{d}{dx}\big[F(g(x))\big]=F'(g(x))g'(x)=f[g(x)]g'(x). \tag{1.5.2}$$

Integrating both sides with respect to $x$, we see that

$$\int f[g(x)]g'(x)\,dx=F(g(x))+C. \tag{1.5.3}$$

If we now substitute $u = g(x)$, and $du = g'(x)\,dx$, we get

$$\int f[g(x)]g'(x)\,dx = \int f(u)\,du = F(u) + C = F(g(x)) + C. \tag{1.5.4}$$

□

Returning to the problem we looked at originally, we let $u = x^2 - 3$ and then $du = 2x\,dx$.

Rewrite the integral (Equation 1.5.1) in terms of $u$:

$$\int (x^2 - 3)^3 (2x\,dx) = \int u^3\,du.$$

Using the power rule for integrals, we have

$$\int u^3\,du = \frac{u^4}{4} + C.$$

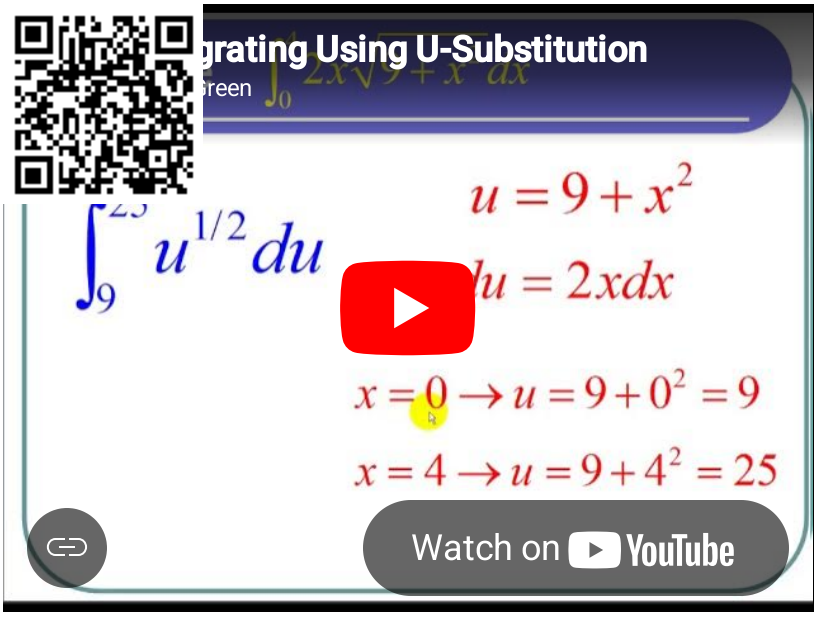
Substitute the original expression for $x$ back into the solution:

$$\frac{u^4}{4} + C = \frac{(x^2 - 3)^4}{4} + C.$$

We can generalize the procedure in the following Problem-Solving Strategy.

### Problem-Solving Strategy: Integration by Substitution

1. Look carefully at the integrand and select an expression $g(x)$ within the integrand to set equal to u. Let's select $g(x)$. such that $g'(x)$ is also part of the integrand.
2. Substitute $u = g(x)$ and $du = g'(x)dx$. into the integral.
3. We should now be able to evaluate the integral with respect to $u$. If the integral can't be evaluated we need to go back and select a different expression to use as $u$.
4. Evaluate the integral in terms of $u$.
5. Write the result in terms of $x$ and the expression $g(x)$.



### Example 1.5.1: Using Substitution to Find an Antiderivative

Use substitution to find the antiderivative of $\int 6x(3x^2+4)^4\,dx$.

**Solution**

The first step is to choose an expression for $u$. We choose $u=3x^2+4$ because then $du=6x\,dx$ and we already have $du$ in the integrand. Write the integral in terms of $u$:

$$\int 6x(3x^2+4)^4\,dx=\int u^4\,du.$$

Remember that $du$ is the derivative of the expression chosen for $u$, regardless of what is inside the integrand. Now we can evaluate the integral with respect to $u$:

$$\int u^4\,du=\frac{u^5}{5}+C=\frac{(3x^2+4)^5}{5}+C.$$

**Analysis**

We can check our answer by taking the derivative of the result of integration. We should obtain the integrand. Picking a value for $C$ of 1, we let $y=\frac{1}{5}(3x^2+4)^5+1$. We have

$$y=\frac{1}{5}(3x^2+4)^5+1,$$

so

$$\begin{aligned} y' &= \left(\frac{1}{5}\right)5(3x^2+4)^4 6x \\ &= 6x(3x^2+4)^4. \end{aligned}$$

This is exactly the expression we started with inside the integrand.

### Exercise 1.5.1

Use substitution to find the antiderivative of $\int 3x^2(x^3-3)^2\,dx$.

**Hint**

Let $u=x^3-3$.

**Answer**

$$\int 3x^2(x^3-3)^2\,dx=\frac{1}{3}(x^3-3)^3+C$$

Sometimes we need to adjust the constants in our integral if they don't match up exactly with the expressions we are substituting.



**Example 1.5.2**: Using Substitution with Alteration

Use substitution to find the antiderivative of

$$\int z\sqrt{z^2-5}\,dz.$$

**Solution**

Rewrite the integral as $\int z(z^2-5)^{1/2}\,dz$. Let $u=z^2-5$ and $du=2z\,dz$. Now we have a problem because $du=2z\,dz$ and the original expression has only $z\,dz$. We have to alter our expression for $du$ or the integral in $u$ will be twice as large as it should be. If we multiply both sides of the $du$ equation by $\frac{1}{2}$. we can solve this problem. Thus,

$$u=z^2-5$$

$$du=2z\,dz$$

$$\frac{1}{2}du=\frac{1}{2}(2z)\,dz=z\,dz.$$

Write the integral in terms of $u$, but pull the $\frac{1}{2}$ outside the integration symbol:

$$\int z(z^2-5)^{1/2}\,dz=\frac{1}{2}\int u^{1/2}\,du.$$

Integrate the expression in $u$:

$$\begin{aligned}\frac{1}{2}\int u^{1/2}\,du &= \left(\frac{1}{2}\right)\frac{u^{3/2}}{\frac{3}{2}}+C\\ &=\left(\frac{1}{2}\right)\left(\frac{2}{3}\right)u^{3/2}+C\\ &=\frac{1}{3}u^{3/2}+C\\ &=\frac{1}{3}(z^2-5)^{3/2}+C\end{aligned}$$

**Exercise 1.5.2**

Use substitution to find the antiderivative of $\int x^2(x^3+5)^9\,dx$.

**Hint**

Multiply the du equation by $\frac{1}{3}$.

**Answer**

$$\int x^2(x^3+5)^9\,dx = \frac{(x^3+5)^{10}}{30} + C$$

**Example 1.5.3: Using Substitution with Integrals of Trigonometric Functions**

Use substitution to evaluate the integral $\int \frac{\sin t}{\cos^3 t}\,dt$.

**Solution**

We know the derivative of $\cos t$ is $-\sin t$, so we set $u = \cos t$. Then $du = -\sin t\,dt$.

Substituting into the integral, we have

$$\int \frac{\sin t}{\cos^3 t}\,dt = -\int \frac{du}{u^3}.$$

Evaluating the integral, we get

$$-\int \frac{du}{u^3} = -\int u^{-3}\,du = -\left(-\frac{1}{2}\right)u^{-2} + C.$$

Putting the answer back in terms of t, we get

$$\int \frac{\sin t}{\cos^3 t}\,dt = \frac{1}{2u^2} + C = \frac{1}{2\cos^2 t} + C.$$

**Exercise 1.5.3**

Use substitution to evaluate the integral $\int \frac{\cos t}{\sin^2 t}\,dt$.

**Hint**

Use the process from Example **1.5.3** to solve the problem.

**Answer**

$$\int \frac{\cos t}{\sin^2 t}\,dt = -\frac{1}{\sin t} + C$$

**Exercise 1.5.4**

Use substitution to evaluate the indefinite integral $\int \cos^3 t \sin t\,dt$.

**Hint**

Use the process from Example **1.5.3** to solve the problem.

**Answer**

$$\int \cos^3 t \sin t\,dt = -\frac{\cos^4 t}{4} + C$$

Sometimes we need to manipulate an integral in ways that are more complicated than just multiplying or dividing by a constant. We need to eliminate all the expressions within the integrand that are in terms of the original variable. When we are done, $u$ should be the only variable in the integrand. In some cases, this means solving for the original variable in terms of $u$. This technique should become clear in the next example.

### Example 1.5.4: Finding an Antiderivative Using $u$-Substitution

Use substitution to find the antiderivative of

$$\int \frac{x}{\sqrt{x-1}}\,dx.$$

**Solution**

If we let $u = x - 1$, then $du = dx$. But this does not account for the $x$ in the numerator of the integrand. We need to express $x$ in terms of $u$. If $u = x - 1$, then $x = u + 1$. Now we can rewrite the integral in terms of $u$:

$$\int \frac{x}{\sqrt{x-1}}\,dx = \int \frac{u+1}{\sqrt{u}}\,du = \int \left(\sqrt{u} + \frac{1}{\sqrt{u}}\right)\,du = \int \left(u^{1/2} + u^{-1/2}\right)\,du.$$

Then we integrate in the usual way, replace $u$ with the original expression, and factor and simplify the result. Thus,

$$\begin{aligned}
\int (u^{1/2} + u^{-1/2})\,du &= \frac{2}{3}u^{3/2} + 2u^{1/2} + C \\
&= \frac{2}{3}(x-1)^{3/2} + 2(x-1)^{1/2} + C \\
&= (x-1)^{1/2}\left[\frac{2}{3}(x-1) + 2\right] + C \\
&= (x-1)^{1/2}\left(\frac{2}{3}x - \frac{2}{3} + \frac{6}{3}\right) \\
&= (x-1)^{1/2}\left(\frac{2}{3}x + \frac{4}{3}\right) \\
&= \frac{2}{3}(x-1)^{1/2}(x+2) + C.
\end{aligned}$$



## Substitution for Definite Integrals

Substitution can be used with definite integrals, too. However, using substitution to evaluate a definite integral requires a change to the limits of integration. If we change variables in the integrand, the limits of integration change as well.

### Substitution with Definite Integrals

Let $u = g(x)$ and let $g'$ be continuous over an interval $[a, b]$, and let $f$ be continuous over the range of $u = g(x)$. Then,

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du. \tag{1.5.5}$$

Although we will not formally prove this theorem, we justify it with some calculations here. From the substitution rule for indefinite integrals, if $F(x)$ is an antiderivative of $f(x)$, we have

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C. \tag{1.5.6}$$

Then

$$\begin{aligned}\int_a^b f[g(x)]g'(x)\,dx &= F(g(x))\Big|_{x=a}^{x=b} \\ &= F(g(b)) - F(g(a)) \\ &= F(u)\Big|_{u=g(a)}^{u=g(b)} \\ &= \int_{g(a)}^{g(b)} f(u)\,du\end{aligned}$$

and we have the desired result.

### Example 1.5.5: Using Substitution to Evaluate a Definite Integral

Use substitution to evaluate

$$\int_0^1 x^2(1+2x^3)^5\,dx.$$

**Solution**

Let $u = 1 + 2x^3$, so $du = 6x^2\,dx$. Since the original function includes one factor of $x^2$ and $du = 6x^2\,dx$, multiply both sides of the $du$ equation by $1/6$. Then,

$$du = 6x^2\,dx$$

$$\text{becomes} \quad \frac{1}{6}du = x^2\,dx.$$

To adjust the limits of integration, note that when $x = 0, u = 1 + 2(0) = 1$, and when $x = 1,\ u = 1 + 2(1) = 3$.

Then

$$\int_0^1 x^2(1+2x^3)^5 dx = \frac{1}{6}\int_1^3 u^5\,du.$$

Evaluating this expression, we get

$$\begin{aligned}\frac{1}{6}\int_1^3 u^5\,du &= \left(\frac{1}{6}\right)\left(\frac{u^6}{6}\right)\Big|_1^3 \\ &= \frac{1}{36}\left[(3)^6 - (1)^6\right] \\ &= \frac{182}{9}.\end{aligned}$$

**Exercise 1.5.5**

Use substitution to evaluate the definite integral $\int_{-1}^{0} y(2y^2-3)^5\,dy$.

**Hint**

Use the steps from Example 1.5.5 to solve the problem.

**Answer**

$$\int_{-1}^{0} y(2y^2-3)^5\,dy = \frac{91}{3}$$

**Exercise 1.5.6**

Use substitution to evaluate $\int_0^1 x^2\cos\left(\frac{\pi}{2}x^3\right)\,dx$.

**Hint**

Use the process from Example 1.5.5 to solve the problem.

**Answer**

$$\int_0^1 x^2\cos\left(\frac{\pi}{2}x^3\right)\,dx = \frac{2}{3\pi} \approx 0.2122$$

**Example 1.5.6: Using Substitution with an Exponential Function**

Use substitution to evaluate

$$\int_0^1 xe^{4x^2+3}\,dx.$$

**Solution**

Let $u = 4x^3+3$. Then, $du = 8x\,dx$. To adjust the limits of integration, we note that when $x=0$, $u=3$, and when $x=1$, $u=7$. So our substitution gives

$$\begin{aligned}\int_0^1 xe^{4x^2+3}\,dx &= \frac{1}{8}\int_3^7 e^u\,du \\ &= \frac{1}{8}e^u\Big|_3^7 \\ &= \frac{e^7-e^3}{8} \\ &\approx 134.568\end{aligned}$$

Substitution may be only one of the techniques needed to evaluate a definite integral. All of the properties and rules of integration apply independently, and trigonometric functions may need to be rewritten using a trigonometric identity before we can apply substitution. Also, we have the option of replacing the original expression for $u$ after we find the antiderivative, which means that we do not have to change the limits of integration. These two approaches are shown in Example 1.5.7.

**Example 1.5.7: Using Substitution to Evaluate a Trigonometric Integral**

Use substitution to evaluate

$$\int_0^{\pi/2} \cos^2\theta\,d\theta.$$

**Solution**

Let us first use a trigonometric identity to rewrite the integral. The trig identity $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$ allows us to rewrite the integral as

$$\int_0^{\pi/2} \cos^2\theta\, d\theta = \int_0^{\pi/2} \frac{1+\cos 2\theta}{2}\, d\theta.$$

Then,

$$\begin{aligned}\int_0^{\pi/2} \left(\frac{1+\cos 2\theta}{2}\right) d\theta &= \int_0^{\pi/2} \left(\frac{1}{2}+\frac{1}{2}\cos 2\theta\right) d\theta \\ &= \frac{1}{2}\int_0^{\pi/2} d\theta + \int_0^{\pi/2} \cos 2\theta\, d\theta.\end{aligned}$$

We can evaluate the first integral as it is, but we need to make a substitution to evaluate the second integral. Let $u = 2\theta$. Then, $du = 2\, d\theta$, or $\dfrac{1}{2}\, du = d\theta$. Also, when $\theta = 0$, $u = 0$, and when $\theta = \pi/2$, $u = \pi$. Expressing the second integral in terms of $u$, we have

$$\begin{aligned}\frac{1}{2}\int_0^{\pi/2} d\theta + \frac{1}{2}\int_0^{\pi/2} \cos 2\theta\, d\theta &= \frac{1}{2}\int_0^{\pi/2} d\theta + \frac{1}{2}\left(\frac{1}{2}\right)\int_0^{\pi} \cos u\, du \\ &= \frac{\theta}{2}\Big|_{\theta=0}^{\theta=\pi/2} + \frac{1}{4}\sin u\Big|_{u=0}^{u=\theta} \\ &= \left(\frac{\pi}{4}-0\right) + (0-0) = \frac{\pi}{4}\end{aligned}$$

## Key Concepts

- Substitution is a technique that simplifies the integration of functions that are the result of a chain-rule derivative. The term 'substitution' refers to changing variables or substituting the variable $u$ and $du$ for appropriate expressions in the integrand.
- When using substitution for a definite integral, we also have to change the limits of integration.

## Key Equations

- **Substitution with Indefinite Integrals**

$$\int f[g(x)]g'(x)\, dx = \int f(u)\, du = F(u) + C = F(g(x)) + C$$

- **Substitution with Definite Integrals**

$$\int_a^b f(g(x))g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$$

## Glossary

**change of variables**
the substitution of a variable, such as $u$, for an expression in the integrand

**integration by substitution**
a technique for integration that allows integration of functions that are the result of a chain-rule derivative

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# 1.5E: Exercises for Section 1.5

1) Why is $u$-substitution referred to as a *change of variable*?

2) If $f = g \circ h$, when reversing the chain rule, $\frac{d}{dx}(g \circ h)(x) = g'(h(x))h'(x)$, should you take $u = g(x)$ or $u = h(x)$?

**Answer**

$u = h(x)$

**In exercises 3 - 7, verify each identity using differentiation. Then, using the indicated $u$-substitution, identify $f$ such that the integral takes the form $\int f(u)\,du$.**

3) $\int x\sqrt{x+1}\,dx = \frac{2}{15}(x+1)^{3/2}(3x-2)+C; \quad u = x+1$

4) $\int \frac{x^2}{\sqrt{x-1}}\,dx = \frac{2}{15}\sqrt{x-1}(3x^2+4x+8)+C, \quad (x>1); \quad u = x-1$

**Answer**

$f(u) = \frac{(u+1)^2}{\sqrt{u}}$

5) $\int x\sqrt{4x^2+9}\,dx = \frac{1}{12}(4x^2+9)^{3/2}+C; \quad u = 4x^2+9$

6) $\int \frac{x}{\sqrt{4x^2+9}}\,dx = \frac{1}{4}\sqrt{4x^2+9}+C; \quad u = 4x^2+9$

**Answer**

$du = 8x\,dx; \quad f(u) = \frac{1}{8\sqrt{u}}$

7) $\int \frac{x}{(4x^2+9)^2}\,dx = -\frac{1}{8(4x^2+9)}+C; \quad u = 4x^2+9$

**In exercises 8 - 17, find the antiderivative using the indicated substitution.**

8) $\int (x+1)^4\,dx; \quad u = x+1$

**Answer**

$\int (x+1)^4\,dx = \frac{1}{5}(x+1)^5+C$

9) $\int (x-1)^5\,dx; \quad u = x-1$

10) $\int (2x-3)^{-7}\,dx; \quad u = 2x-3$

**Answer**

$\int (2x-3)^{-7}\,dx = -\frac{1}{12(2x-3)^6}+C$

11) $\int (3x-2)^{-11}\,dx; \quad u = 3x-2$

12) $\int \frac{x}{\sqrt{x^2+1}}\,dx; \quad u = x^2+1$

**Answer**

$\int \frac{x}{\sqrt{x^2+1}}\,dx = \sqrt{x^2+1}+C$

13) $\int \frac{x}{\sqrt{1-x^2}}\,dx; \quad u = 1-x^2$

14) $\int (x-1)(x^2-2x)^3\,dx; \quad u=x^2-2x$

**Answer**

$$\int (x-1)(x^2-2x)^3\,dx = \frac{1}{8}(x^2-2x)^4 + C$$

15) $\int (x^2-2x)(x^3-3x^2)^2\,dx; \quad u=x^3=3x^2$

16) $\int \cos^3\theta\,d\theta; \quad u=\sin\theta$ (Hint: $\cos^2\theta = 1-\sin^2\theta$)

**Answer**

$$\int \cos^3\theta\,d\theta = \sin\theta - \frac{\sin^3\theta}{3} + C$$

17) $\int \sin^3\theta\,d\theta; \quad u=\cos\theta$ (Hint: $\sin^2\theta = 1-\cos^2\theta$)

**In exercises 18 - 34, use a suitable change of variables to determine the indefinite integral.**

18) $\int x(1-x)^{99}\,dx$

**Answer**

$$\int x(1-x)^{99}\,dx = \frac{(1-x)^{101}}{101} - \frac{(1-x)^{100}}{100} + C$$
$$= -\frac{(1-x)^{100}}{10100}\left[100x+1\right] + C$$

19) $\int t(1-t^2)^{10}dt$

20) $\int (11x-7)^{-3}\,dx$

**Answer**

$$\int (11x-7)^{-3}\,dx = -\frac{1}{22(11x-7)^2} + C$$

21) $\int (7x-11)^4\,dx$

22) $\int \cos^3\theta\sin\theta\,d\theta$

**Answer**

$$\int \cos^3\theta\sin\theta\,d\theta = -\frac{\cos^4\theta}{4} + C$$

23) $\int \sin^7\theta\cos\theta\,d\theta$

24) $\int \cos^2(\pi t)\sin(\pi t)\,dt$

**Answer**

$$\int \cos^2(\pi t)\sin(\pi t)\,dt = -\frac{cos^3(\pi t)}{3\pi} + C$$

25) $\int \sin^2 x\cos^3 x\,dx$ (Hint: $\sin^2 x + \cos^2 x = 1$)

26) $\int t\sin(t^2)\cos(t^2)\,dt$

**Answer**

$$\int t\sin(t^2)\cos(t^2)\,dt = -\frac{1}{4}\cos^2(t^2) + C$$

27) $\int t^2\cos^2(t^3)\sin(t^3)\,dt$

28) $\int \frac{x^2}{(x^3-3)^2}\,dx$

**Answer**

$$\int \frac{x^2}{(x^3-3)^2}\,dx = -\frac{1}{3(x^3-3)} + C$$

29) $\int \frac{x^3}{\sqrt{1-x^2}}\,dx$

30) $\int \frac{y^5}{(1-y^3)^{3/2}}\,dy$

**Answer**

$$\int \frac{y^5}{(1-y^3)^{3/2}}\,dy = -\frac{2(y^3-2)}{3\sqrt{1-y^3}} + C$$

31) $\int \cos\theta(1-\cos\theta)^{99}\sin\theta\,d\theta$

32) $\int (1-\cos^3\theta)^{10}\cos^2\theta\sin\theta\,d\theta$

**Answer**

$$\int (1-\cos^3\theta)^{10}\cos^2\theta\sin\theta\,d\theta = \frac{1}{33}(1-\cos^3\theta)^{11} + C$$

33) $\int (\cos\theta - 1)(\cos^2\theta - 2\cos\theta)^3\sin\theta\,d\theta$

34) $\int (\sin^2\theta - 2\sin\theta)(\sin^3\theta - 3\sin^2\theta)^3\cos\theta\,d\theta$

**Answer**

$$\int (\sin^2\theta - 2\sin\theta)(\sin^3\theta - 3\sin^2\theta)^3\cos\theta\,d\theta = \frac{1}{12}(\sin^3\theta - 3\sin^2\theta)^4 + C$$

**In exercises 35 - 38, use a calculator to estimate the area under the curve using left Riemann sums with 50 terms, then use substitution to solve for the exact answer.**

35) [T] $y = 3(1-x)^2$ over $[0, 2]$

36) [T] $y = x(1-x^2)^3$ over $[-1, 2]$

**Answer**

$L_{50} = -8.5779$. The exact area is $\frac{-81}{8}$ units$^2$.

37) [T] $y = \sin x(1-\cos x)^2$ over $[0, \pi]$

38) [T] $y = \frac{x}{(x^2+1)^2}$ over $[-1, 1]$

**Answer**

$L_{50} = -0.006399$. The exact area is 0.

**In exercises 39 - 44, use a change of variables to evaluate the definite integral.**

39) $\int_0^1 x\sqrt{1-x^2}\,dx$

40) $\int_0^1 \frac{x}{\sqrt{1+x^2}}\,dx$

**Answer**

$$u = 1+x^2, \quad du = 2x\,dx, \quad \int_0^1 \frac{x}{\sqrt{1+x^2}}\,dx = \frac{1}{2}\int_1^2 u^{-1/2}du = \sqrt{2}-1$$

41) $\int_0^2 \frac{t}{\sqrt{5+t^2}}\,dt$

42) $\int_0^1 \frac{t^2}{\sqrt{1+t^3}}\,dt$

**Answer**

$$u = 1+t^3, \quad du = 3t^2, \quad \int_0^1 \frac{t^2}{\sqrt{1+t^3}}\,dt = \frac{1}{3}\int_1^2 u^{-1/2}du = \frac{2}{3}(\sqrt{2}-1)$$

43) $\int_0^{\pi/4} \sec^2\theta\tan\theta\,d\theta$

44) $\int_0^{\pi/4} \frac{\sin\theta}{\cos^4\theta}\,d\theta$

**Answer**

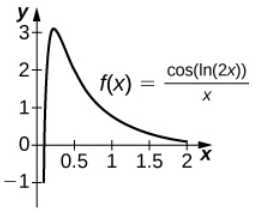
$$u = \cos\theta, \quad du = -\sin\theta\,d\theta, \quad \int_0^{\pi/4} \frac{\sin\theta}{\cos^4\theta}\,d\theta = -\int_1^{\sqrt{2}/2} u^{-4}\,du = \int_{\sqrt{2}/2}^1 u^{-4}\,du = \frac{1}{3}(2\sqrt{2}-1)$$

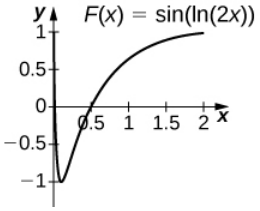
**In exercises 45 - 50, evaluate the indefinite integral $\int f(x)\,dx$ with constant $C = 0$ using $u$-substitution. Then, graph the function and the antiderivative over the indicated interval. If possible, estimate a value of $C$ that would need to be added to the antiderivative to make it equal to the definite integral $F(x) = \int_a^x f(t)\,dt$, with a the left endpoint of the given interval.**

45) [T] $\int (2x+1)e^{x^2+x-6}\,dx$ over $[-3,2]$

46) [T] $\int \frac{\cos(\ln(2x))}{x}\,dx$ on $[0,2]$

**Answer**



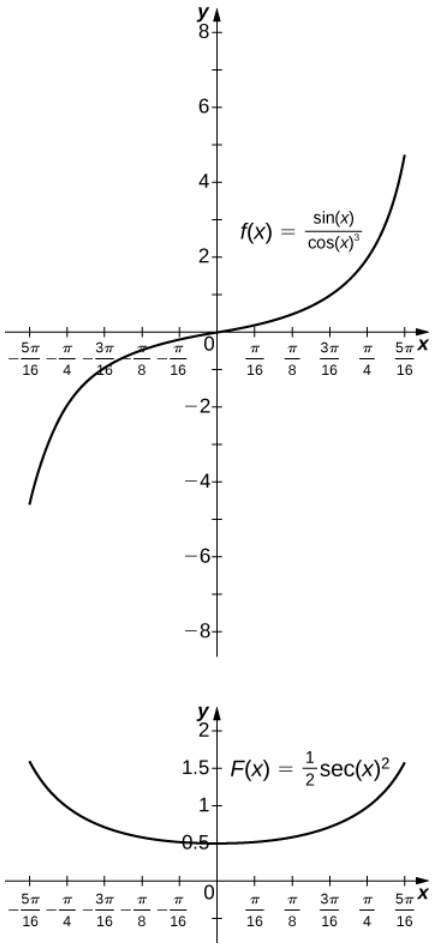




The antiderivative is $y = \sin(\ln(2x))$. Since the antiderivative is not continuous at $x = 0$, one cannot find a value of C that would make $y = \sin(\ln(2x)) - C$ work as a definite integral.

47) [T] $\int \frac{3x^2+2x+1}{\sqrt{x^3+x^2+x+4}}\,dx$ over $[-1,2]$

48) [T] $\int \frac{\sin x}{\cos^3 x}\,dx$ over $[-\frac{\pi}{3}, \frac{\pi}{3}]$

**Answer**

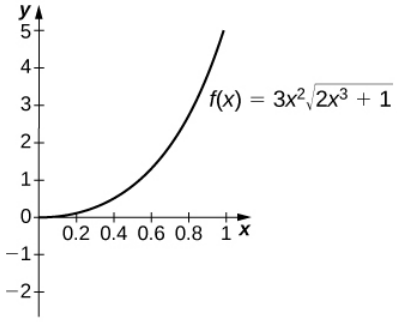


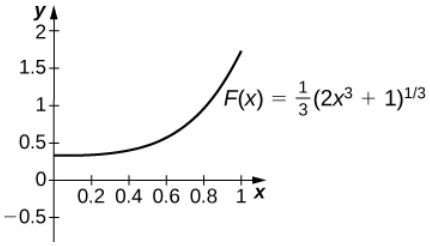

The antiderivative is $y = \frac{1}{2}\sec^2 x$. You should take $C = -2$ so that $F(-\frac{\pi}{3}) = 0$.

49) [T] $\int (x+2)e^{-x^2-4x+3}\,dx$ over $[-5, 1]$

50) [T] $\int 3x^2\sqrt{2x^3+1}\,dx$ over $[0, 1]$

**Answer**




The antiderivative is $y = \frac{1}{3}(2x^3 + 1)^{3/2}$. One should take $C = -\frac{1}{3}$.

51) If $h(a) = h(b)$ in $\int_a^b g'(h(x))h(x)\,dx$, what can you say about the value of the integral?

52) Is the substitution $u = 1 - x^2$ in the definite integral $\int_0^2 \frac{x}{1-x^2}\,dx$ okay? If not, why not?

**Answer**

No, because the integrand is discontinuous at $x = 1$.

**In exercises 53 - 59, use a change of variables to show that each definite integral is equal to zero.**

53) $\int_0^{\pi} \cos^2(2\theta)\sin(2\theta)\,d\theta$

54) $\int_0^{\sqrt{\pi}} t\cos(t^2)\sin(t^2)\,dt$

**Answer**

$u = \sin(t^2)$; the integral becomes $\frac{1}{2}\int_0^0 u\,du$.

55) $\int_0^1 (1-2t)\,dt$

56) $\int_0^1 \frac{1-2t}{1+(t-\frac{1}{2})^2}\,dt$

**Answer**

$u = 1 + (t - \frac{1}{2})^2$; the integral becomes $-\int_{5/4}^{5/4} \frac{1}{u}\,du$.

57) $\int_0^{\pi} \sin\left((t-\frac{\pi}{2})^3\right)\cos\left(t-\frac{\pi}{2}\right)\,dt$

58) $\int_0^2 (1-t)\cos(\pi t)\,dt$

**Answer**

$u = 1 - t$; Since the integrand is odd, the integral becomes

$$\int_1^{-1} u\cos\big(\pi(1-u)\big)\,du = \int_1^{-1} u[\cos\pi\cos u - \sin\pi\sin u]\,du = -\int_1^{-1} u\cos u\,du = \int_{-1}^{1} u\cos u\,du = 0$$

59) $\int_{\pi/4}^{3\pi/4} \sin^2 t\cos t\,dt$

60) Show that the average value of $f(x)$ over an interval $[a,b]$ is the same as the average value of $f(cx)$ over the interval $[\frac{a}{c},\frac{b}{c}]$ for $c>0$.

**Answer**

Setting $u=cx$ and $du=c\,dx$ gets you $\frac{1}{\frac{b}{c}-\frac{a}{c}}\int_{a/c}^{b/c} f(cx)\,dx = \frac{c}{b-a}\int_{u=a}^{u=b} f(u)\frac{du}{c} = \frac{1}{b-a}\int_a^b f(u)\,du.$

61) Find the area under the graph of $f(t)=\frac{t}{(1+t^2)^a}$ between $t=0$ and $t=x$ where $a>0$ and $a\neq 1$ is fixed, and evaluate the limit as $x\to\infty$.

62) Find the area under the graph of $g(t)=\frac{t}{(1-t^2)^a}$ between $t=0$ and $t=x$, where $0<x<1$ and $a>0$ is fixed. Evaluate the limit as $x\to 1$.

**Answer**

$\int_0^x g(t)\,dt = \frac{1}{2}\int_{u=1-x^2}^{1}\frac{du}{u^a} = \frac{1}{2(1-a)}u^{1-a}\,|\,1u = \frac{1}{2(1-a)}(1-(1-x^2)^{1-a})$ As $x\to 1$ the limit is $\frac{1}{2(1-a)}$ if $a<1$, and the limit diverges to $+\infty$ if $a>1$.

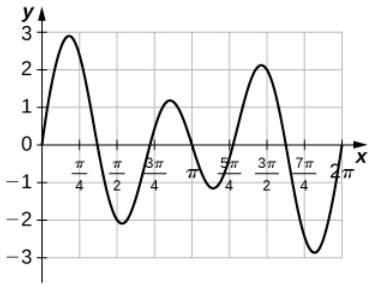
63) The area of a semicircle of radius $1$ can be expressed as $\int_{-1}^{1}\sqrt{1-x^2}\,dx$. Use the substitution $x=\cos t$ to express the area of a semicircle as the integral of a trigonometric function. You do not need to compute the integral.

64) The area of the top half of an ellipse with a major axis that is the $x$-axis from $x=-1$ to a and with a minor axis that is the $y$-axis from $y=-b$ to $y=b$ can be written as $\int_{-a}^{a} b\sqrt{1-\frac{x^2}{a^2}}\,dx$. Use the substitution $x=a\cos t$ to express this area in terms of an integral of a trigonometric function. You do not need to compute the integral.

**Answer**

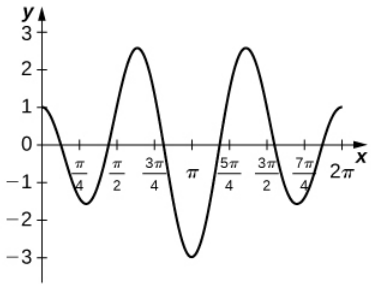
$\int_{t=\pi}^{t=0} b\sqrt{1-\cos^2 t}\times(-a\sin t)\,dt = \int_{t=0}^{t=\pi} ab\sin^2 t\,dt$

65) [T] The following graph is of a function of the form $f(t)=a\sin(nt)+b\sin(mt)$. Estimate the coefficients $a$ and $b$ and the frequency parameters $n$ and $m$. Use these estimates to approximate $\int_0^{\pi} f(t)\,dt$.




66) [T] The following graph is of a function of the form $f(x)=a\cos(nt)+b\cos(mt)$. Estimate the coefficients $a$ and $b$ and the frequency parameters $n$ and $m$. Use these estimates to approximate $\int_0^{\pi} f(t)\,dt$.


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**Answer**

$$f(t) = 2\cos(3t) - \cos(2t); \quad \int_0^{\pi/2} (2\cos(3t) - \cos(2t))\, dt = -\frac{2}{3}$$

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# 1.6: Integrals Involving Exponential and Logarithmic Functions

**Learning Objectives**

- Integrate functions involving exponential functions.
- Integrate functions involving logarithmic functions.

Exponential and logarithmic functions are used to model population growth, cell growth, and financial growth, as well as depreciation, radioactive decay, and resource consumption, to name only a few applications. In this section, we explore integration involving exponential and logarithmic functions.

## Integrals of Exponential Functions

The exponential function is perhaps the most efficient function in terms of the operations of calculus. The exponential function, $y = e^x$, is its own derivative and its own integral.

**Rule: Integrals of Exponential Functions**

Exponential functions can be integrated using the following formulas.

$$\int e^x \, dx = e^x + C \tag{1.6.1}$$

$$\int a^x \, dx = \frac{a^x}{\ln a} + C \tag{1.6.2}$$

**Example 1.6.1: Finding an Antiderivative of an Exponential Function**

Find the antiderivative of the exponential function $e^{-x}$.

**Solution**

Use substitution, setting $u = -x$, and then $du = -1\,dx$. Multiply the $du$ equation by $-1$, so you now have $-du = dx$. Then,

$$\int e^{-x} \, dx = -\int e^u \, du = -e^u + C = -e^{-x} + C.$$

**Exercise 1.6.1**

Find the antiderivative of the function using substitution: $x^2 e^{-2x^3}$.

**Hint**

Let $u$ equal the exponent on $e$.

**Answer**

$$\int x^2 e^{-2x^3} \, dx = -\frac{1}{6} e^{-2x^3} + C$$

A common mistake when dealing with exponential expressions is treating the exponent on $e$ the same way we treat exponents in polynomial expressions. We cannot use the power rule for the exponent on $e$. This can be especially confusing when we have both exponentials and polynomials in the same expression, as in the previous checkpoint. In these cases, we should always double-check to make sure we're using the right rules for the functions we're integrating.

**Example 1.6.2: Square Root of an Exponential Function**

Find the antiderivative of the exponential function $e^x \sqrt{1 + e^x}$.

**Solution**

First rewrite the problem using a rational exponent:

$$\int e^x\sqrt{1+e^x}\,dx = \int e^x(1+e^x)^{1/2}\,dx.$$

Using substitution, choose $u = 1 + e^x$. Then, $du = e^x\,dx$. We have

$$\int e^x(1+e^x)^{1/2}\,dx = \int u^{1/2}\,du.$$

Then

$$\int u^{1/2}\,du = \frac{u^{3/2}}{3/2} + C = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(1+e^x)^{3/2} + C$$

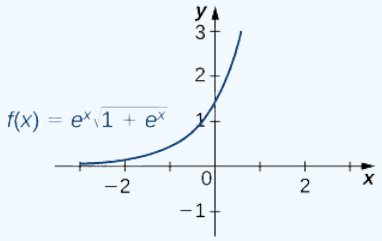



Figure 1.6.1: The graph shows an exponential function times the square root of an exponential function.

## Exercise 1.6.2

Find the antiderivative of $e^x(3e^x - 2)^2$.

**Hint**

Let $u = 3e^x - 2$.

**Answer**

$$\int e^x(3e^x-2)^2\,dx = \frac{1}{9}(3e^x-2)^3 + C$$

## Example 1.6.3: Using Substitution with an Exponential Function

Use substitution to evaluate the indefinite integral $\int 3x^2e^{2x^3}\,dx$.

**Solution**

Here we choose to let $u$ equal the expression in the exponent on $e$. Let $u = 2x^3$ and $du = 6x^2\,dx$. Again, $du$ is off by a constant multiplier; the original function contains a factor of $3x^2$, not $6x^2$. Multiply both sides of the equation by $\frac{1}{2}$ so that the integrand in $u$ equals the integrand in $x$. Thus,

$$\int 3x^2e^{2x^3}\,dx = \frac{1}{2}\int e^u\,du.$$

Integrate the expression in $u$ and then substitute the original expression in $x$ back into the $u$-integral:

$$\frac{1}{2}\int e^u\,du = \frac{1}{2}e^u + C = \frac{1}{2}e^{x^3} + C.$$

## Exercise 1.6.3

Evaluate the indefinite integral $\int 2x^3e^{x^4}\,dx$.

**Hint**

Let $u = x^4$.

**Answer**

$$\int 2x^3 e^{x^4}\,dx = \frac{1}{2}e^{x^4} + C$$

As mentioned at the beginning of this section, exponential functions are used in many real-life applications. The number $e$ is often associated with compounded or accelerating growth, as we have seen in earlier sections about the derivative. Although the derivative represents a rate of change or a growth rate, the integral represents the total change or the total growth. Let's look at an example in which integration of an exponential function solves a common business application.

A price–demand function tells us the relationship between the quantity of a product demanded and the price of the product. In general, price decreases as quantity demanded increases. The marginal price–demand function is the derivative of the price–demand function and it tells us how fast the price changes at a given level of production. These functions are used in business to determine the price–elasticity of demand, and to help companies determine whether changing production levels would be profitable.

**Example 1.6.4: Finding a Price–Demand Equation**

Find the price–demand equation for a particular brand of toothpaste at a supermarket chain when the demand is $50$ tubes per week at \$2.35 per tube, given that the marginal price—demand function, $p'(x)$, for $x$ number of tubes per week, is given as

$$p'(x) = -0.015e^{-0.01x}.$$

If the supermarket chain sells $100$ tubes per week, what price should it set?

**Solution**

To find the price–demand equation, integrate the marginal price–demand function. First find the antiderivative, then look at the particulars. Thus,

$$p(x) = \int -0.015e^{-0.01x}\,dx = -0.015\int e^{-0.01x}\,dx.$$

Using substitution, let $u = -0.01x$ and $du = -0.01\,dx$. Then, divide both sides of the $du$ equation by $-0.01$. This gives

$$\frac{-0.015}{-0.01}\int e^u\,du = 1.5\int e^u\,du = 1.5e^u + C = 1.5e^{-0.01}x + C.$$

The next step is to solve for $C$. We know that when the price is \$2.35 per tube, the demand is $50$ tubes per week. This means

$$p(50) = 1.5e^{-0.01(50)} + C = 2.35.$$

Now, just solve for $C$:

$$C = 2.35 - 1.5e^{-0.5} = 2.35 - 0.91 = 1.44.$$

Thus,

$$p(x) = 1.5e^{-0.01x} + 1.44.$$

If the supermarket sells $100$ tubes of toothpaste per week, the price would be

$$p(100) = 1.5e - 0.01(100) + 1.44 = 1.5e - 1 + 1.44 \approx 1.99.$$

The supermarket should charge \$1.99 per tube if it is selling $100$ tubes per week.

**Example 1.6.5: Evaluating a Definite Integral Involving an Exponential Function**

Evaluate the definite integral $\int_1^2 e^{1-x}\,dx$.

**Solution**

Again, substitution is the method to use. Let $u = 1 - x$, so $du = -1\,dx$ or $-du = dx$. Then $\int e^{1-x}\,dx = -\int e^u\,du$.

Next, change the limits of integration. Using the equation $u = 1 - x$, we have:

$$\text{When } x = 1, \quad u = 1 - (1) = 0,$$

$$\text{and when } x = 2, \quad u = 1 - (2) = -1.$$

The integral then becomes

$$\begin{aligned} \int_1^2 e^{1-x}\, dx &= -\int_0^{-1} e^u\, du \\ &= \int_{-1}^0 e^u\, du \\ &= e^u\Big|_{-1}^0 = e^0 - (e^{-1}) \\ &= -e^{-1} + 1. \end{aligned}$$

See Figure 1.6.2.

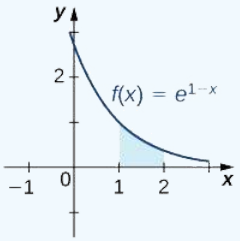



Figure 1.6.2: The indicated area can be calculated by evaluating a definite integral using substitution.

## Exercise 1.6.4

Evaluate $\int_0^2 e^{2x}\, dx$.

**Hint**

Let $u = 2x$.

**Answer**

$$\frac{1}{2}\int_0^4 e^u\, du = \frac{1}{2}(e^4 - 1)$$



**Example 1.6.6: Growth of Bacteria in a Culture**

Suppose the rate of growth of bacteria in a Petri dish is given by $q(t) = 3^t$, where $t$ is given in hours and $q(t)$ is given in thousands of bacteria per hour. If a culture starts with $10,000$ bacteria, find a function $Q(t)$ that gives the number of bacteria in the Petri dish at any time $t$. How many bacteria are in the dish after $2$ hours?

**Solution**

We have

$$Q(t) = \int 3^t dt = \frac{3^t}{\ln 3} + C.$$

Then, at $t = 0$ we have $Q(0) = 10 = \frac{1}{\ln 3} + C$, so $C \approx 9.090$ and we get

$$Q(t) = \frac{3^t}{\ln 3} + 9.090.$$

At time $t = 2$, we have

$$\begin{aligned} Q(2) &= \frac{3^2}{\ln 3} + 9.090 \\ &\approx 17.282. \end{aligned}$$

After 2 hours, there are 17,282 bacteria in the dish.

**Exercise 1.6.5**

From Example, suppose the bacteria grow at a rate of $q(t) = 2^t$. Assume the culture still starts with $10,000$ bacteria. Find $Q(t)$. How many bacteria are in the dish after $3$ hours?

**Hint**

Use the procedure from Example $1.6.6$ to solve the problem

**Answer**

$$Q(t) = \frac{2^t}{\ln 2} + 8.557.$$

$$Q(3) \approx 20,099$$

So there are $20,099$ bacteria in the dish after $3$ hours.

## Determine the Best Time to Remove a Tumor Based on Its Growth

You are a cardiologist and after you have reviewed the initial examination and discovered a small tumor, it is your job to decide when it is best to perform surgery. You will see the current size of the tumor, its growth rate, and the optimal size to surgically remove it for this particular patient. Then fill in the box with the number of days accurate to two decimal places and click on Schedule Appointment to watch it grow and see if you scheduled the appointment at the optimal time to save the patient's life or if you scheduled it at the wrong time severely risking the patient's life.

Next Patient

### Example 1.6.7: Fruit Fly Population Growth

Suppose a population of **fruit flies** increases at a rate of $g(t) = 2e^{0.02t}$, in flies per day. If the initial population of fruit flies is $100$ flies, how many flies are in the population after $10$ days?

**Solution**

Let $G(t)$ represent the number of flies in the population at time $t$. Applying the net change theorem, we have

$$\begin{aligned} G(10) = G(0) + \int_0^{10} 2e^{0.02t}\,dt \\ &= 100 + \left[\frac{2}{0.02}e^{0.02t}\right]\Big|_0^{10} \\ &= 100 + \left[100e^{0.02t}\right]\Big|_0^{10} \\ &= 100 + 100e^{0.2} - 100 \\ &\approx 122. \end{aligned}$$

There are $122$ flies in the population after $10$ days.

### Exercise 1.6.6

Suppose the rate of growth of the fly population is given by $g(t) = e^{0.01t}$, and the initial fly population is $100$ flies. How many flies are in the population after $15$ days?

**Hint**

Use the process from Example $1.6.7$ to solve the problem.

**Answer**

There are $116$ flies.

### Example 1.6.8: Evaluating a Definite Integral Using Substitution

Evaluate the definite integral using substitution:

$$\int_1^2 \frac{e^{1/x}}{x^2}\,dx.$$

**Solution**

This problem requires some rewriting to simplify applying the properties. First, rewrite the exponent on e as a power of $x$, then bring the $x^2$ in the denominator up to the numerator using a negative exponent. We have

$$\int_1^2 \frac{e^{1/x}}{x^2}\,dx = \int_1^2 e^{x^{-1}}x^{-2}\,dx.$$

Let $u = x^{-1}$, the exponent on $e$. Then

$$du = -x^{-2}\,dx$$

$$-du = x^{-2}\,dx.$$

Bringing the negative sign outside the integral sign, the problem now reads

$$-\int e^u\,du.$$

Next, change the limits of integration:

$$u = (1)^{-1} = 1$$

$$u = (2)^{-1} = \frac{1}{2}.$$

Notice that now the limits begin with the larger number, meaning we can multiply by $-1$ and interchange the limits. Thus,

$$-\int_1^{1/2} e^u\,du = \int_{1/2}^1 e^u\,du = e^u\big|_{1/2}^1 = e - e^{1/2} = e - \sqrt{e}.$$

**Exercise 1.6.7**

Evaluate the definite integral using substitution:

$$\int_1^2 \frac{1}{x^3}e^{4x^{-2}}\,dx.$$

**Hint**

Let $u = 4x^{-2}$.

**Answer**

$$\int_1^2 \frac{1}{x^3}e^{4x^{-2}}\,dx = \frac{1}{8}[e^4 - e].$$

## Integrals Involving Logarithmic Functions

Integrating functions of the form $f(x) = x^{-1}$ result in the absolute value of the natural log function, as shown in the following rule. Integral formulas for other logarithmic functions, such as $f(x) = \ln x$ and $f(x) = \log_a x$, are also included in the rule.

**Rule: Integration Formulas Involving Logarithmic Functions**

The following formulas can be used to evaluate integrals involving logarithmic functions.

$$\int x^{-1}\,dx = \ln|x| + C$$

$$\int \ln x\,dx = x\ln x - x + C = x(\ln x - 1) + C$$

$$\int \log_a x\,dx = \frac{x}{\ln a}(\ln x - 1) + C$$



**Example 1.6.9: Finding an Antiderivative Involving $\ln x$**

Find the antiderivative of the function $\dfrac{3}{x-10}$.

**Solution**

First factor the $3$ outside the integral symbol. Then use the $u^{-1}$ rule. Thus,

$$\int \frac{3}{x-10}\,dx = 3\int \frac{1}{x-10}\,dx = 3\int \frac{du}{u} = 3\ln|u| + C = 3\ln|x-10| + C, \quad x \neq 10.$$

See Figure 1.6.3.

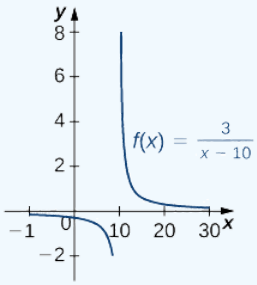



Figure 1.6.3: The domain of this function is $x \neq 10$.

**Exercise 1.6.8**

Find the antiderivative of $\dfrac{1}{x+2}$.

**Hint**

Follow the pattern from Example 1.6.9 to solve the problem.

**Answer**

$$\int \frac{1}{x+2}\,dx = \ln|x+2| + C$$

### Example 1.6.10: Finding an Antiderivative of a Rational Function

Find the antiderivative of $\dfrac{2x^3+3x}{x^4+3x^2}$.

**Solution**

This can be rewritten as $\int (2x^3+3x)(x^4+3x^2)^{-1}\,dx$. Use substitution.

Let $u=x^4+3x^2$, then $du=(4x^3+6x)\,dx$. Alter $du$ by factoring out the $2$. Thus,

$$du=(4x^3+6x)\,dx=2(2x^3+3x)\,dx$$

$$\frac{1}{2}\,du=(2x^3+3x)\,dx.$$

Rewrite the integrand in $u$:

$$\int (2x^3+3x)(x^4+3x^2)^{-1}\,dx=\frac{1}{2}\int u^{-1}\,du.$$

Then we have

$$\frac{1}{2}\int u^{-1}\,du=\frac{1}{2}\ln|u|+C=\frac{1}{2}\ln\,|\,x^4+3x^2\,|\,+C.$$

### Example 1.6.11: Finding an Antiderivative of a Logarithmic Function

Find the antiderivative of the log function $\log_2 x$.

**Solution**

Follow the format in the formula listed in the rule on integration formulas involving logarithmic functions. Based on this format, we have

$$\int \log_2 x\,dx=\frac{x}{\ln 2}(\ln x-1)+C.$$

### Exercise 1.6.9

Find the antiderivative of $\log_3 x$.

**Hint**

Follow Example 1.6.11 and refer to the rule on integration formulas involving logarithmic functions.

**Answer**

$$\int \log_3 x\,dx=\frac{x}{\ln 3}(\ln x-1)+C$$



Example **1.6.12** is a definite integral of a trigonometric function. With trigonometric functions, we often have to apply a trigonometric property or an identity before we can move forward. Finding the right form of the integrand is usually the key to a smooth integration.

### Example 1.6.12: Evaluating a Definite Integral

Evaluate the definite integral

$$\int_0^{\pi/2} \frac{\sin x}{1+\cos x}\, dx.$$

**Solution**

We need substitution to evaluate this problem. Let $u = 1 + \cos x$ so $du = -\sin x\ dx$.

Rewrite the integral in terms of $u$, changing the limits of integration as well. Thus,

$$u = 1 + \cos(0) = 2$$
$$u = 1 + \cos\left(\frac{\pi}{2}\right) = 1.$$

Then

$$\begin{aligned}\int_0^{\pi/2} \frac{\sin x}{1+\cos x} &= -\int_2^1 u^{-1}\, du \\ &= \int_1^2 u^{-1}\, du \\ &= \ln|u| \Big|_1^2 \\ &= [\ln 2 - \ln 1] = \ln 2\end{aligned}$$

## Key Concepts

- Exponential and logarithmic functions arise in many real-world applications, especially those involving growth and decay.
- Substitution is often used to evaluate integrals involving exponential functions or logarithms.

## Key Equations

- **Integrals of Exponential Functions**

$$\int e^x\,dx = e^x + C$$

$$\int a^x\,dx = \frac{a^x}{\ln a} + C$$

- **Integration Formulas Involving Logarithmic Functions**

$$\int x^{-1}\,dx = \ln|x| + C$$

$$\int \ln x\,dx = x\ln x - x + C = x(\ln x - 1) + C$$

$$\int \log_a x\,dx = \frac{x}{\ln a}(\ln x - 1) + C$$

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# 1.6E: Exercises for Section 1.6

**For exercises 1 - 8, compute each indefinite integral.**

1) $\int e^{2x}\,dx$

2) $\int e^{-3x}\,dx$

**Answer**

$$\int e^{-3x}\,dx \quad = \quad \frac{-1}{3}e^{-3x}+C$$

3) $\int 2^{x}\,dx$

4) $\int 3^{-x}\,dx$

**Answer**

$$\int 3^{-x}\,dx \quad = \quad -\frac{3^{-x}}{\ln 3}+C$$

5) $\int \frac{1}{2x}\,dx$

6) $\int \frac{2}{x}\,dx$

**Answer**

$$\int \frac{2}{x}\,dx \quad = \quad 2\ln|x|+C \quad = \quad \ln(x^2)+C$$

7) $\int \frac{1}{x^2}\,dx$

8) $\int \frac{1}{\sqrt{x}}\,dx$

**Answer**

$$\int \frac{1}{\sqrt{x}}\,dx \quad = \quad 2\sqrt{x}+C$$

**In exercises 9 - 16, find each indefinite integral by using appropriate substitutions.**

9) $\int \frac{\ln x}{x}\,dx$

10) $\int \frac{dx}{x(\ln x)^2}$

**Answer**

$$\int \frac{dx}{x(\ln x)^2} \quad = \quad -\frac{1}{\ln x}+C$$

11) $\int \frac{dx}{x\ln x} \quad (x>1)$

12) $\int \frac{dx}{x\ln x\ln(\ln x)}$

**Answer**

$$\int \frac{dx}{x\ln x\ln(\ln x)} \quad = \quad \ln(\ln(\ln x))+C$$

13) $\int \tan\theta\,d\theta$

14) $\int \frac{\cos x - x \sin x}{x \cos x}\, dx$

**Answer**

$$\int \frac{\cos x - x \sin x}{x \cos x}\, dx \quad = \quad \ln(x \cos x) + C$$

15) $\int \frac{\ln(\sin x)}{\tan x}\, dx$

16) $\int \ln(\cos x) \tan x\, dx$

**Answer**

$$\int \ln(\cos x) \tan x\, dx \quad = \quad -\frac{1}{2}(\ln(\cos(x)))^2 + C$$

17) $\int x e^{-x^2}\, dx$

18) $\int x^2 e^{-x^3}\, dx$

**Answer**

$$\int x^2 e^{-x^3}\, dx \quad = \quad \frac{-e^{-x^3}}{3} + C$$

19) $\int e^{\sin x} \cos x\, dx$

20) $\int e^{\tan x} \sec^2 x\, dx$

**Answer**

$$\int e^{\tan x} \sec^2 x\, dx \quad = \quad e^{\tan x} + C$$

21) $\int \frac{e^{\ln x}}{x}\, dx$

22) $\int \frac{e^{\ln(1-t)}}{1-t}\, dt$

**Answer**

$$\int \frac{e^{\ln(1-t)}}{1-t}\, dt = \int \frac{1-t}{1-t}\, dt = \int 1\, dt \quad = \quad t + C$$

**In exercises 23 - 28, verify by differentiation that $\int \ln x\, dx = x(\ln x - 1) + C$, then use appropriate changes of variables to compute the integral.**

23) $\int \ln x\, dx$ (Hint: $\int \ln x\, dx = \frac{1}{2} \int x \ln(x^2)\, dx$)

24) $\int x^2 \ln^2 x\, dx$

**Answer**

$$\int x^2 \ln^2 x\, dx \quad = \quad \frac{1}{9} x^3 (\ln(x^3) - 1) + C$$

25) $\int \frac{\ln x}{x^2}\, dx$ (Hint: Set $u = \frac{1}{x}$.)

26) $\int \frac{\ln x}{\sqrt{x}}\, dx$ (Hint: Set $u = \sqrt{x}$.)

**Answer**

$$\int \frac{\ln x}{\sqrt{x}}\,dx \quad = \quad 2\sqrt{x}(\ln x - 2) + C$$

27) Write an integral to express the area under the graph of $y = \frac{1}{t}$ from $t = 1$ to $e^x$ and evaluate the integral.

28) Write an integral to express the area under the graph of $y = e^t$ between $t = 0$ and $t = \ln x$, and evaluate the integral.

**Answer**

$$\int_0^{\ln x} e^t\,dt = e^t\Big|_0^{\ln x} = e^{\ln x} - e^0 = x - 1$$

**In exercises 29 - 35, use appropriate substitutions to express the trigonometric integrals in terms of compositions with logarithms.**

29) $\int \tan(2x)\,dx$

30) $\int \frac{\sin(3x) - \cos(3x)}{\sin(3x) + \cos(3x)}\,dx$

**Answer**

$$\int \frac{\sin(3x) - \cos(3x)}{\sin(3x) + \cos(3x)}\,dx \quad = \quad -\frac{1}{3}\ln|\sin(3x) + \cos(3x)| + C$$

31) $\int \frac{x\sin(x^2)}{\cos(x^2)}\,dx$

32) $\int x\csc(x^2)\,dx$

**Answer**

$$\int x\csc(x^2)\,dx \quad = \quad -\frac{1}{2}\ln|\csc(x^2) + \cot(x^2)| + C$$

33) $\int \ln(\cos x)\tan x\,dx$

34) $\int \ln(\csc x)\cot x\,dx$

**Answer**

$$\int \ln(\csc x)\cot x\,dx \quad = \quad -\frac{1}{2}(\ln(\csc x))^2 + C$$

35) $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$

**In exercises 36 - 40, evaluate the definite integral.**

36) $\int_1^2 \frac{1 + 2x + x^2}{3x + 3x^2 + x^3}\,dx$

**Answer**

$$\int_1^2 \frac{1 + 2x + x^2}{3x + 3x^2 + x^3}\,dx \quad = \quad \frac{1}{3}\ln\left(\tfrac{26}{7}\right)$$

37) $\int_0^{\pi/4} \tan x\,dx$

38) $\int_0^{\pi/3} \frac{\sin x - \cos x}{\sin x + \cos x}\,dx$

**Answer**

$$\int_0^{\pi/3} \frac{\sin x - \cos x}{\sin x + \cos x}\,dx \quad = \quad \ln(\sqrt{3} - 1)$$

39) $\int_{\pi/6}^{\pi/2} \csc x \, dx$

40) $\int_{\pi/4}^{\pi/3} \cot x \, dx$

**Answer**

$$\int_{\pi/4}^{\pi/3} \cot x \, dx \quad = \quad \frac{1}{2}\ln\frac{3}{2}$$

**In exercises 41 - 46, integrate using the indicated substitution.**

41) $\int \frac{x}{x-100}\, dx; \quad u = x - 100$

42) $\int \frac{y-1}{y+1}\, dy; \quad u = y + 1$

**Answer**

$$\int \frac{y-1}{y+1}\, dy \quad = \quad y - 2\ln|y+1| + C$$

43) $\int \frac{1-x^2}{3x-x^3}\, dx; \quad u = 3x - x^3$

44) $\int \frac{\sin x + \cos x}{\sin x - \cos x}\, dx; \quad u = \sin x - \cos x$

**Answer**

$$\int \frac{\sin x + \cos x}{\sin x - \cos x}\, dx \quad = \quad \ln|\sin x - \cos x| + C$$

45) $\int e^{2x}\sqrt{1-e^{2x}}\, dx; \quad u = 1 - e^{2x}$

46) $\int \ln(x)\frac{\sqrt{1-(\ln x)^2}}{x}\, dx; \quad u = 1 - (\ln x)^2$

**Answer**

$$\int \ln(x)\frac{\sqrt{1-(\ln x)^2}}{x}\, dx \quad = \quad -\frac{1}{3}(1-(\ln x)^2)^{3/2} + C$$

47) $\int \frac{\sqrt{x}}{\sqrt{x}+2}\, dx; \quad u = \sqrt{x} + 2$

**Answer**

$$\int \frac{\sqrt{x}}{\sqrt{x}+2}\, dx \quad = \quad \left(\sqrt{x}+2\right)^2 - 8\left(\sqrt{x}+2\right) + 8\ln\left(\sqrt{x}+2\right) + C$$

48) $\int e^x \sec(e^x+1)\tan(e^x+1)\, dx; \quad u = e^x + 1$

**Answer**

$$\int e^x \sec(e^x+1)\tan(e^x+1)\, dx \quad = \quad \sec(e^x+1) + C$$

**In exercises 49 - 54, state whether the right-endpoint approximation overestimates or underestimates the exact area. Then calculate the right endpoint estimate $R_{50}$ and solve for the exact area.**

49) [T] $y = e^x$ over $[0, 1]$

50) [T] $y = e^{-x}$ over $[0, 1]$

**Answer**

Since $f$ is decreasing, the right endpoint estimate underestimates the area.

Exact solution: $\frac{e-1}{e}$, $R_{50} = 0.6258$.

51) [T] $y = \ln(x)$ over $[1, 2]$

52) [T] $y = \frac{x+1}{x^2+2x+6}$ over $[0, 1]$

**Answer**

Since $f$ is increasing, the right endpoint estimate overestimates the area.

Exact solution: $\frac{2\ln(3) - \ln(6)}{2}$, $R_{50} = 0.2033$.

53) [T] $y = 2^x$ over $[-1, 0]$

54) [T] $y = -2^{-x}$ over $[0, 1]$

**Answer**

Since $f$ is increasing, the right endpoint estimate overestimates the area (the actual area is a larger negative number).

Exact solution: $-\frac{1}{\ln(4)}$, $R_{50} = -0.7164$.

**In exercises 55 - 58, $f(x) \geq 0$ for $a \leq x \leq b$. Find the area under the graph of $f(x)$ between the given values $a$ and $b$ by integrating.**

55) $f(x) = \frac{\log_{10}(x)}{x}$; $a = 10, b = 100$

56) $f(x) = \frac{\log_2(x)}{x}$; $a = 32, b = 64$

**Answer**

$\frac{11}{2}\ln 2$

57) $f(x) = 2^{-x}$; $a = 1, b = 2$

58) $f(x) = 2^{-x}$; $a = 3, b = 4$

**Answer**

$\frac{1}{\ln(65,536)}$

59) Find the area under the graph of the function $f(x) = xe^{-x^2}$ between $x = 0$ and $x = 5$.

60) Compute the integral of $f(x) = xe^{-x^2}$ and find the smallest value of $N$ such that the area under the graph $f(x) = xe^{-x^2}$ between $x = N$ and $x = N + 1$ is, at most, $0.01$.

**Answer**

$\int_N^{N+1} xe^{-x^2}\,dx = \frac{1}{2}(e^{-N^2} - e^{-(N+1)^2})$. The quantity is less than $0.01$ when $N = 2$.

61) Find the limit, as $N$ tends to infinity, of the area under the graph of $f(x) = xe^{-x^2}$ between $x = 0$ and $x = 5$.

62) Show that $\int_a^b \frac{dt}{t} = \int_{1/b}^{1/a} \frac{dt}{t}$ when $0 < a \leq b$.

**Answer**

$\int_a^b \frac{dx}{x} = \ln(b) - \ln(a) = \ln(\frac{1}{a}) - \ln(\frac{1}{b}) = \int_{1/b}^{1/a} \frac{dx}{x}$

63) Suppose that $f(x) > 0$ for all $x$ and that $f$ and $g$ are differentiable. Use the identity $f^g = e^{g\ln f}$ and the chain rule to find the derivative of $f^g$.

64) Use the previous exercise to find the antiderivative of $h(x) = x^x(1 + \ln x)$ and evaluate $\int_2^3 x^x(1 + \ln x)\, dx$.

**Answer**

23

65) Show that if $c > 0$, then the integral of $\frac{1}{x}$ from $ac$ to $bc$ $(\text{for } 0 < a < b)$ is the same as the integral of $\frac{1}{x}$ from $a$ to $b$.

**The following exercises are intended to derive the fundamental properties of the natural log starting from the definition $\ln(x) = \int_1^x \frac{dt}{t}$, using properties of the definite integral and making no further assumptions.**

66) Use the identity $\ln(x) = \int_1^x \frac{dt}{t}$ to derive the identity $\ln\left(\frac{1}{x}\right) = -\ln x$.

**Answer**

We may assume that $x > 1$,so $\frac{1}{x} < 1$. Then, $\int_1^{1/x} \frac{dt}{t}$. Now make the substitution $u = \frac{1}{t}$, so $du = -\frac{dt}{t^2}$ and $\frac{du}{u} = -\frac{dt}{t}$, and change endpoints: $\int_1^{1/x} \frac{dt}{t} = -\int_1^x \frac{du}{u} = -\ln x$.

67) Use a change of variable in the integral $\int_1^{xy} \frac{1}{t}\, dt$ to show that $\ln xy = \ln x + \ln y$ for $x, y > 0$.

68) Use the identity $\ln x = \int_1^x \frac{dt}{t}$ to show that $\ln(x)$ is an increasing function of $x$ on $[0, \infty)$, and use the previous exercises to show that the range of $\ln(x)$ is $(-\infty, \infty)$. Without any further assumptions, conclude that $\ln(x)$ has an inverse function defined on $(-\infty, \infty)$.

69) Pretend, for the moment, that we do not know that $e^x$ is the inverse function of $\ln(x)$, but keep in mind that $\ln(x)$ has an inverse function defined on $(-\infty, \infty)$. Call it $E$. Use the identity $\ln xy = \ln x + \ln y$ to deduce that $E(a + b) = E(a)E(b)$ for any real numbers $a, b$.

70) Pretend, for the moment, that we do not know that $e^x$ is the inverse function of $\ln x$, but keep in mind that $\ln x$ has an inverse function defined on $(-\infty, \infty)$. Call it $E$. Show that $E'(t) = E(t)$.

**Answer**

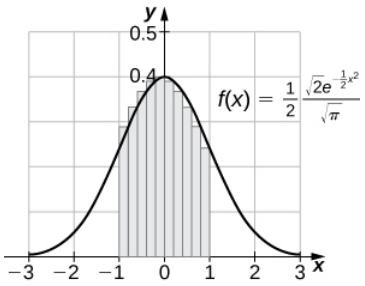
$x = E(\ln(x))$. Then, $1 = \frac{E'(\ln x)}{x}$ or $x = E'(\ln x)$. Since any number $t$ can be written $t = \ln x$ for some $x$, and for such $t$ we have $x = E(t)$, it follows that for any $t$, $E'(t) = E(t)$.

71) The sine integral, defined as $S(x) = \int_0^x \frac{\sin t}{t}\, dt$ is an important quantity in engineering. Although it does not have a simple closed formula, it is possible to estimate its behavior for large $x$. Show that for $k \geq 1$, $\quad |S(2\pi k) - S(2\pi(k+1))| \leq \frac{1}{k(2k+1)\pi}$. (Hint: $\sin(t + \pi) = -\sin t$)

72) [T] The normal distribution in probability is given by $p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$, where $\sigma$ is the standard deviation and $\mu$ is the average. The standard normal distribution in probability, $p_s$, corresponds to $\mu = 0$ and $\sigma = 1$. Compute the left endpoint estimates $R_{10}$ and $R_{100}$ of $\int_{-1}^1 \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$.

**Answer**

$R_{10} = 0.6811, \quad R_{100} = 0.6827$




73) [T] Compute the right endpoint estimates $R_{50}$ and $R_{100}$ of $\int_{-3}^{5} \frac{1}{2\sqrt{2\pi}} e^{-(x-1)^2/8}$.

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- Paul Seeburger (Monroe Community College) added problems 47-48 to Section 5.6 exercises.

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# 1.7: Integrals Resulting in Inverse Trigonometric Functions

**Learning Objectives**

- Integrate functions resulting in inverse trigonometric functions

In this section we focus on integrals that result in inverse trigonometric functions. We have worked with these functions before. Recall, that trigonometric functions are not one-to-one unless the domains are restricted. When working with inverses of trigonometric functions, we always need to be careful to take these restrictions into account. Also, we previously developed formulas for derivatives of inverse trigonometric functions. The formulas developed there give rise directly to integration formulas involving inverse trigonometric functions.

## Integrals that Result in Inverse Trigonometric Functions

Let us begin this last section of the chapter with the three formulas. Along with these formulas, we use substitution to evaluate the integrals. We prove the formula for the inverse sine integral.

**Rule: Integration Formulas Resulting in Inverse Trigonometric Functions**

The following integration formulas yield inverse trigonometric functions:

$$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C \tag{1.7.1}$$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C \tag{1.7.2}$$

$$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{|u|}{a}\right) + C \tag{1.7.3}$$

**Proof of the first formula**

Let $y = \sin^{-1}\frac{x}{a}$. Then $a\sin y = x$. Now using implicit differentiation, we obtain

$$\frac{d}{dx}(a\sin y) = \frac{d}{dx}(x) \tag{1.7.4}$$

$$a\cos y\frac{dy}{dx} = 1 \tag{1.7.5}$$

$$\frac{dy}{dx} = \frac{1}{a\cos y}. \tag{1.7.6}$$

For $-\frac{\pi}{2} \le y \le \frac{\pi}{2}, \cos y \ge 0$. Thus, applying the Pythagorean identity $\sin^2 y + \cos^2 y = 1$, we have $\cos y = \sqrt{1-\sin^2 y}$. This gives

$$\frac{1}{a\cos y} = \frac{1}{a\sqrt{1-\sin^2 y}} \tag{1.7.7}$$

$$= \frac{1}{\sqrt{a^2-a^2\sin^2 y}} \tag{1.7.8}$$

$$= \frac{1}{\sqrt{a^2-x^2}}. \tag{1.7.9}$$

Then for $-a \le x \le a$, we have

$$\int \frac{1}{\sqrt{a^2-u^2}}\,du = \sin^{-1}\left(\frac{u}{a}\right) + C. \tag{1.7.10}$$

□

**Example 1.7.1: Evaluating a Definite Integral Using Inverse Trigonometric Functions**

Evaluate the definite integral

$$\int_0^{1/2} \frac{dx}{\sqrt{1-x^2}}.$$

**Solution**

We can go directly to the formula for the antiderivative in the rule on integration formulas resulting in inverse trigonometric functions, and then evaluate the definite integral. We have

$$\int_0^{1/2} \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x \Big|_0^{1/2} = \sin^{-1} \tfrac{1}{2} - \sin^{-1} 0 = \frac{\pi}{6} - 0 = \frac{\pi}{6}.$$

Note that since the integrand is simply the derivative of $\sin^{-1} x$, we are really just using this fact to find the antiderivative here.

**Exercise 1.7.1**

Find the indefinite integral using an inverse trigonometric function and substitution for $\int \frac{dx}{\sqrt{9-x^2}}$.

**Hint**

Use the formula in the rule on integration formulas resulting in inverse trigonometric functions.

**Answer**

$$\int \frac{dx}{\sqrt{9-x^2}} = \sin^{-1}\left(\frac{x}{3}\right) + C$$

In many integrals that result in inverse trigonometric functions in the antiderivative, we may need to use substitution to see how to use the integration formulas provided above.



**Example 1.7.2: Finding an Antiderivative Involving an Inverse Trigonometric Function using substitution**

Evaluate the integral

$$\int \frac{dx}{\sqrt{4-9x^2}}.$$

**Solution**

Substitute $u = 3x$. Then $du = 3\,dx$ and we have

$$\int \frac{dx}{\sqrt{4-9x^2}} = \frac{1}{3}\int \frac{du}{\sqrt{4-u^2}}.$$

Applying the formula with $a = 2$, we obtain

$$\int \frac{dx}{\sqrt{4-9x^2}} = \frac{1}{3}\int \frac{du}{\sqrt{4-u^2}} = \frac{1}{3}\sin^{-1}\left(\frac{u}{2}\right) + C = \frac{1}{3}\sin^{-1}\left(\frac{3x}{2}\right) + C.$$

**Exercise 1.7.2**

Find the antiderivative of $\int \frac{dx}{\sqrt{1-16x^2}}$.

**Hint**

Substitute $u = 4x$.

**Answer**

$$\int \frac{dx}{\sqrt{1-16x^2}} = \frac{1}{4}\sin^{-1}(4x) + C$$

**Example 1.7.3: Evaluating a Definite Integral**

Evaluate the definite integral

$$\int_0^{\sqrt{3}/2} \frac{du}{\sqrt{1-u^2}}.$$

**Solution**

The format of the problem matches the inverse sine formula. Thus,

$$\int_0^{\sqrt{3}/2} \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u\Big|_0^{\sqrt{3}/2} = [\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)] - [\sin^{-1}(0)] = \frac{\pi}{3}.$$

## Integrals Resulting in Other Inverse Trigonometric Functions

There are six inverse trigonometric functions. However, only three integration formulas are noted in the rule on integration formulas resulting in inverse trigonometric functions because the remaining three are negative versions of the ones we use. The only difference is whether the integrand is positive or negative. Rather than memorizing three more formulas, if the integrand is negative, simply factor out −1 and evaluate the integral using one of the formulas already provided. To close this section, we examine one more formula: the integral resulting in the inverse tangent function.

**Example 1.7.4: Finding an Antiderivative Involving the Inverse Tangent Function**

Find the antiderivative of $\int \frac{1}{9+x^2}\,dx$.

**Solution**

Apply the formula with $a = 3$. Then,

$$\int \frac{dx}{9+x^2} = \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C.$$

**Exercise 1.7.3**

Find the antiderivative of $\int \frac{dx}{16+x^2}$.

**Hint**

Follow the steps in Example **1.7.4**.

**Answer**

$$\int \frac{dx}{16+x^2} = \frac{1}{4}\tan^{-1}\left(\frac{x}{4}\right)+C$$



**Example 1.7.5: Applying the Integration Formulas WITH SUBSTITUTION**

Find an antiderivative of $\int \frac{1}{1+4x^2}\,dx$.

**Solution**

Comparing this problem with the formulas stated in the rule on integration formulas resulting in inverse trigonometric functions, the integrand looks similar to the formula for $\tan^{-1}u+C$. So we use substitution, letting $u=2x$, then $du=2\,dx$ and $\frac{1}{2}\,du=dx$.Then, we have

$$\frac{1}{2}\int \frac{1}{1+u^2}\,du = \frac{1}{2}\tan^{-1}u+C = \frac{1}{2}\tan^{-1}(2x)+C.$$

**Exercise 1.7.4**

Use substitution to find the antiderivative of $\int \frac{dx}{25+4x^2}$.

**Hint**

Use the solving strategy from Example **1.7.5** and the rule on integration formulas resulting in inverse trigonometric functions.

**Answer**

$$\int \frac{dx}{25+4x^2} = \frac{1}{10}\tan^{-1}\left(\frac{2x}{5}\right)+C$$

**Example 1.7.6: Evaluating a Definite Integral**

Evaluate the definite integral $\int_{\sqrt{3}/3}^{\sqrt{3}} \frac{dx}{1+x^2}$.

**Solution**

Use the formula for the inverse tangent. We have

$$\int_{\sqrt{3}/3}^{\sqrt{3}} \frac{dx}{1+x^2} = \tan^{-1} x \Big|_{\sqrt{3}/3}^{\sqrt{3}} = [\tan^{-1}(\sqrt{3})] - [\tan^{-1}\left(\frac{\sqrt{3}}{3}\right)] = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}.$$

**Exercise 1.7.5**

Evaluate the definite integral $\int_0^2 \frac{dx}{4+x^2}$.

**Hint**

Follow the procedures from Example 1.7.6 to solve the problem.

**Answer**

$$\int_0^2 \frac{dx}{4+x^2} = \frac{\pi}{8}$$



## Key Concepts

- Formulas for derivatives of inverse trigonometric functions developed in Derivatives of Exponential and Logarithmic Functions lead directly to integration formulas involving inverse trigonometric functions.
- Use the formulas listed in the rule on integration formulas resulting in inverse trigonometric functions to match up the correct format and make alterations as necessary to solve the problem.
- Substitution is often required to put the integrand in the correct form.

## Key Equations

- **Integrals That Produce Inverse Trigonometric Functions**

$$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right)+C$$

$$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{|u|}{a}\right)+C$$

## Contributors and Attributions

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- Includes some added textual clarifications and edits by Paul Seeburger (Monroe Community College)

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# 1.7E: Exercises for Section 1.7

**In exercises 1 - 6, evaluate each integral in terms of an inverse trigonometric function.**

1) $\int_0^{\sqrt{3}/2} \frac{dx}{\sqrt{1-x^2}}$

**Answer**

$$\int_0^{\sqrt{3}/2} \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x \Big|_0^{\sqrt{3}/2} = \frac{\pi}{3}$$

2) $\int_{-1/2}^{1/2} \frac{dx}{\sqrt{1-x^2}}$

3) $\int_{\sqrt{3}}^{1} \frac{dx}{\sqrt{1+x^2}}$

**Answer**

$$\int_{\sqrt{3}}^{1} \frac{dx}{\sqrt{1+x^2}} = \tan^{-1} x \Big|_{\sqrt{3}}^{1} = -\frac{\pi}{12}$$

4) $\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{dx}{1+x^2}$

5) $\int_1^{\sqrt{2}} \frac{dx}{|x|\sqrt{x^2-1}}$

**Answer**

$$\int_1^{\sqrt{2}} \frac{dx}{|x|\sqrt{x^2-1}} = \sec^{-1} x \Big|_1^{\sqrt{2}} = \frac{\pi}{4}$$

6) $\int_1^{\frac{2}{\sqrt{3}}} \frac{dx}{|x|\sqrt{x^2-1}}$

**In exercises 7 - 12, find each indefinite integral, using appropriate substitutions.**

7) $\int \frac{dx}{\sqrt{9-x^2}}$

**Answer**

$$\int \frac{dx}{\sqrt{9-x^2}} = \sin^{-1}\left(\frac{x}{3}\right) + C$$

8) $\int \frac{dx}{\sqrt{1-16x^2}}$

9) $\int \frac{dx}{9+x^2}$

**Answer**

$$\int \frac{dx}{9+x^2} = \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C$$

10) $\int \frac{dx}{25+16x^2}$

11) $\int \frac{dx}{x\sqrt{x^2-9}}$

**Answer**

$$\int \frac{dx}{x\sqrt{x^2-9}} = \frac{1}{3}\sec^{-1}\left(\frac{|x|}{3}\right) + C$$

12) $\int \frac{dx}{x\sqrt{4x^2-16}}$

13) Explain the relationship $-\cos^{-1} t + C = \int \frac{dt}{\sqrt{1-t^2}} = \sin^{-1} t + C$. Is it true, in general, that $\cos^{-1} t = -\sin^{-1} t$?

**Answer**

$\cos(\frac{\pi}{2} - \theta) = \sin\theta$. So, $\sin^{-1} t = \frac{\pi}{2} - \cos^{-1} t$. They differ by a constant.

14) Explain the relationship $\sec^{-1} t + C = \int \frac{dt}{|t|\sqrt{t^2-1}} = -\csc^{-1} t + C$. Is it true, in general, that $\sec^{-1} t = -\csc^{-1} t$?

15) Explain what is wrong with the following integral: $\int_1^2 \frac{dt}{\sqrt{1-t^2}}$.

**Answer**

$\sqrt{1-t^2}$ is not defined as a real number when $t > 1$.

16) Explain what is wrong with the following integral: $\int_{-1}^1 \frac{dt}{|t|\sqrt{t^2-1}}$.

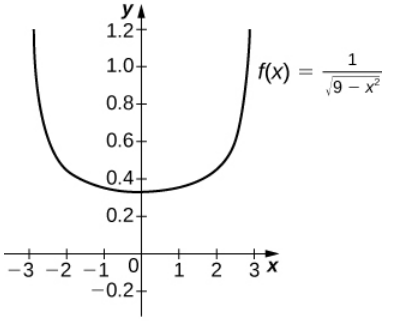
**Answer**

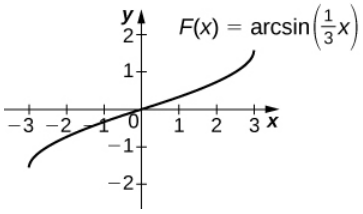
$\sqrt{t^2-1}$ is not defined as a real number when $-1 < t < 1$, and the integrand is undefined when $t = -1$ or $t = 1$.

**In exercises 17 - 20, solve for the antiderivative of $f$ with $C = 0$, then use a calculator to graph $f$ and the antiderivative over the given interval $[a, b]$. Identify a value of $C$ such that adding $C$ to the antiderivative recovers the definite integral $F(x) = \int_a^x f(t)\,dt$.**

17) [T] $\int \frac{1}{\sqrt{9-x^2}}\,dx$ over $[-3, 3]$

**Answer**



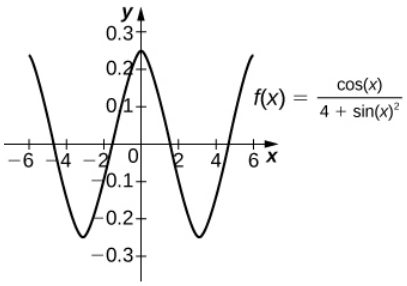



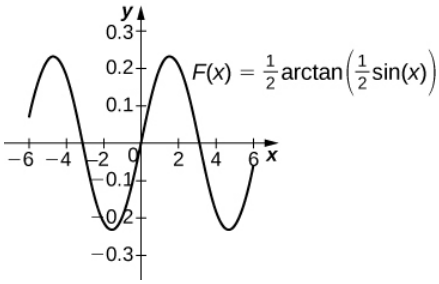

The antiderivative is $\sin^{-1}(\frac{x}{3}) + C$. Taking $C = \frac{\pi}{2}$ recovers the definite integral.

18) [T] $\int \frac{9}{9+x^2}\,dx$ over $[-6, 6]$

19) [T] $\int \frac{\cos x}{4+\sin^2 x}\,dx$ over $[-6, 6]$

**Answer**




The antiderivative is $\frac{1}{2}\tan^{-1}(\frac{\sin x}{2})+C$. Taking $C=\frac{1}{2}\tan^{-1}(\frac{\sin(6)}{2})$ recovers the definite integral.

20) [T] $\int \frac{e^x}{1+e^{2x}}\,dx$ over $[-6,6]$

**In exercises 21 - 26, compute the antiderivative using appropriate substitutions.**

21) $\int \frac{\sin^{-1} t}{\sqrt{1-t^2}}\,dt$

**Answer**

$$\int \frac{\sin^{-1} t\,dt}{\sqrt{1-t^2}} \quad = \quad \tfrac{1}{2}(\sin^{-1} t)^2+C$$

22) $\int \frac{dt}{\sin^{-1} t\sqrt{1-t^2}}$

23) $\int \frac{\tan^{-1}(2t)}{1+4t^2}\,dt$

**Answer**

$$\int \frac{\tan^{-1}(2t)}{1+4t^2}\,dt \quad = \quad \frac{1}{4}(\tan^{-1}(2t))^2+C$$

24) $\int \frac{t\tan^{-1}(t^2)}{1+t^4}\,dt$

25) $\int \frac{\sec^{-1}(\frac{t}{2})}{|t|\sqrt{t^2-4}}\,dt$

**Answer**

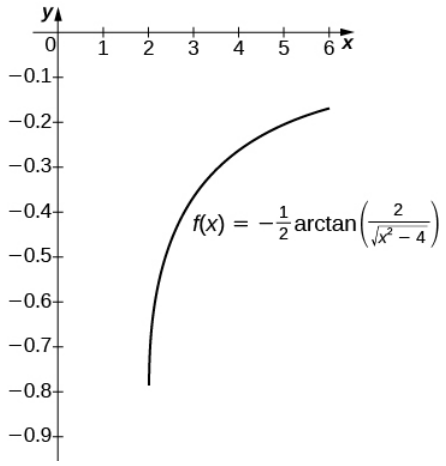
$$\int \frac{\sec^{-1}(\frac{t}{2})}{|t|\sqrt{t^2-4}}\,dt \quad = \quad \tfrac{1}{4}(\sec^{-1}(\tfrac{t}{2}))^2+C$$

26) $\int \frac{t\sec^{-1}(t^2)}{t^2\sqrt{t^4-1}}\,dt$

**In exercises 27 - 32, use a calculator to graph the antiderivative of $f$ with $C=0$ over the given interval $[a,b]$. Approximate a value of $C$, if possible, such that adding $C$ to the antiderivative gives the same value as the definite integral $F(x)=\int_a^x f(t)\,dt$.**

27) [T] $\int \frac{1}{x\sqrt{x^2-4}}\,dx$ over $[2,6]$

**Answer**

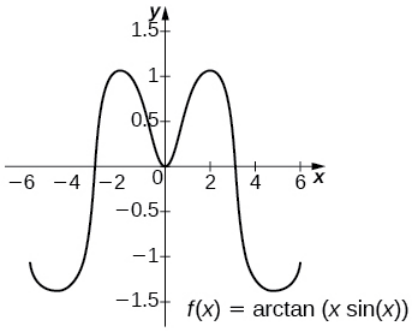



The antiderivative is $\frac{1}{2}\sec^{-1}(\frac{x}{2}) + C$. Taking $C = 0$ recovers the definite integral over $[2, 6]$.

28) [T] $\int \frac{1}{(2x+2)\sqrt{x}}\,dx$ over $[0, 6]$

29) [T] $\int \frac{(\sin x + x\cos x)}{1 + x^2\sin^2 x\,dx}$ over $[-6, 6]$

**Answer**

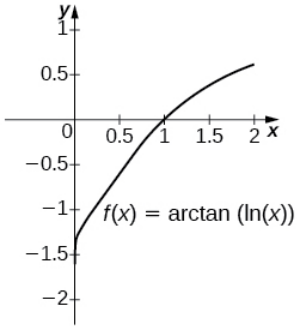



The general antiderivative is $\tan^{-1}(x\sin x) + C$. Taking $C = -\tan^{-1}(6\sin(6))$ recovers the definite integral.

30) [T] $\int \frac{2e^{-2x}}{\sqrt{1-e^{-4x}}}\,dx$ over $[0, 2]$

31) [T] $\int \frac{1}{x + x\ln 2x}$ over $[0, 2]$

**Answer**




The general antiderivative is $\tan^{-1}(\ln x) + C$. Taking $C = \frac{\pi}{2} = \lim_{t\to\infty} \tan^{-1} t$ recovers the definite integral.


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32) [T] $\int \frac{\sin^{-1} x}{\sqrt{1-x^2}}$ over $[-1,1]$

**In exercises 33 - 38, compute each integral using appropriate substitutions.**

33) $\int \frac{e^t}{\sqrt{1-e^{2t}}}\,dt$

**Answer**

$$\int \frac{e^t}{\sqrt{1-e^{2t}}}\,dt \quad = \quad \sin^{-1}(e^t)+C$$

34) $\int \frac{e^t}{1+e^{2t}}\,dt$

35) $\int \frac{dt}{t\sqrt{1-\ln^2 t}}$

**Answer**

$$\int \frac{dt}{t\sqrt{1-\ln^2 t}} \quad = \quad \sin^{-1}(\ln t)+C$$

36) $\int \frac{dt}{t(1+\ln^2 t)}$

37) $\int \frac{\cos^{-1}(2t)}{\sqrt{1-4t^2}}\,dt$

**Answer**

$$\int \frac{\cos^{-1}(2t)}{\sqrt{1-4t^2}}\,dt \quad = \quad -\frac{1}{2}(\cos^{-1}(2t))^2+C$$

38) $\int \frac{e^t \cos^{-1}(e^t)}{\sqrt{1-e^{2t}}}\,dt$

**In exercises 39 - 42, compute each definite integral.**

39) $\int_0^{1/2} \frac{\tan(\sin^{-1} t)}{\sqrt{1-t^2}}\,dt$

**Answer**

$$\int_0^{1/2} \frac{\tan(\sin^{-1} t)}{\sqrt{1-t^2}}\,dt \quad = \quad \frac{1}{2}\ln\left(\frac{4}{3}\right)$$

40) $\int_{1/4}^{1/2} \frac{\tan(\cos^{-1} t)}{\sqrt{1-t^2}}\,dt$

41) $\int_0^{1/2} \frac{\sin(\tan^{-1} t)}{1+t^2}\,dt$

**Answer**

$$\int_0^{1/2} \frac{\sin(\tan^{-1} t)}{1+t^2}\,dt \quad = \quad 1-\frac{2}{\sqrt{5}}$$

42) $\int_0^{1/2} \frac{\cos(\tan^{-1} t)}{1+t^2}\,dt$

43) For $A>0$, compute $I(A)=\int_{-A}^{A} \frac{dt}{1+t^2}$ and evaluate $\lim_{a\to\infty} I(A)$, the area under the graph of $\frac{1}{1+t^2}$ on $[-\infty,\infty]$.

**Answer**

$2\tan^{-1}(A)\to\pi$ as $A\to\infty$

44) For $1<B<\infty$, compute $I(B)=\int_1^B \frac{dt}{t\sqrt{t^2-1}}$ and evaluate $\lim_{B\to\infty} I(B)$, the area under the graph of $\frac{1}{t\sqrt{t^2-1}}$ over $[1,\infty)$.

45) Use the substitution $u = \sqrt{2}\cot x$ and the identity $1 + \cot^2 x = \csc^2 x$ to evaluate $\int \frac{dx}{1+\cos^2 x}$. (Hint: Multiply the top and bottom of the integrand by $\csc^2 x$.)

**Answer**

Using the hint, one has $\int \frac{\csc^2 x}{\csc^2 x + \cot^2 x}\,dx = \int \frac{\csc^2 x}{1+2\cot^2 x}\,dx$. Set $u = \sqrt{2}\cot x$. Then, $du = -\sqrt{2}\csc^2 x$ and the integral is $-\frac{1}{\sqrt{2}}\int \frac{du}{1+u^2} = -\frac{\sqrt{2}}{2}\tan^{-1} u + C = \frac{\sqrt{2}}{2}\tan^{-1}(\sqrt{2}\cot x) + C$. If one uses the identity $\tan^{-1} s + \tan^{-1}(\frac{1}{s}) = \frac{\pi}{2}$, then this can also be written $\frac{\sqrt{2}}{2}\tan^{-1}(\frac{\tan x}{\sqrt{2}}) + C$.

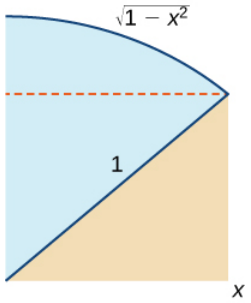
46) [T] Approximate the points at which the graphs of $f(x) = 2x^2 - 1$ and $g(x) = (1+4x^2)^{-3/2}$ intersect, and approximate the area between their graphs accurate to three decimal places.

47) [T] Approximate the points at which the graphs of $f(x) = x^2 - 1$ and $f(x) = x^2 - 1$ intersect, and approximate the area between their graphs accurate to three decimal places.

**Answer**

$x \approx \pm 1.13525$. The left endpoint estimate with $N = 100$ is 2.796 and these decimals persist for $N = 500$.

48) Use the following graph to prove that $\int_0^x \sqrt{1-t^2}\,dt = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x$.




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# 1.8: Chapter 1 Review Exercises

**In exercises 1 - 4, answer *True or False.* Justify your answer with a proof or a counterexample. Assume all functions $f$ and $g$ are continuous over their domains.**

1) If $f(x) > 0,\ f'(x) > 0$ for all $x$, then the right-hand rule underestimates the integral $\int_a^b f(x)\,dx$. Use a graph to justify your answer.

**Answer**

False

2) $\int_a^b f(x)^2\,dx = \int_a^b f(x)\,dx$

3) If $f(x) \le g(x)$ for all $x \in [a,b]$, then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$.

**Answer**

True

4) All continuous functions have an antiderivative.

**In exercises 5 - 8, evaluate the Riemann sums $L_4$ and $R_4$ for the given functions over the specified interval. Compare your answer with the exact answer, when possible, or use a calculator to determine the answer.**

5) $y = 3x^2 - 2x + 1)$ over $[-1,1]$

**Answer**

$L_4 = 5.25,\ R_4 = 3.25,$ exact answer: 4

6) $y = \ln(x^2+1)$ over $[0,e]$

7) $y = x^2 \sin x$ over $[0,\pi]$

**Answer**

$L_4 = 5.364,\ R_4 = 5.364,$ exact answer: $5.870$

8) $y = \sqrt{x} + \frac{1}{x}$ over $[1,4]$

**In exercises 9 - 12, evaluate the integrals.**

9) $\int_{-1}^{1} (x^3 - 2x^2 + 4x)\,dx$

**Answer**

$-\frac{4}{3}$

10) $\int_0^4 \frac{3t}{\sqrt{1+6t^2}}\,dt$

11) $\int_{\pi/3}^{\pi/2} 2\sec(2\theta)\tan(2\theta)\,d\theta$

**Answer**

1

12) $\int_0^{\pi/4} e^{\cos^2 x} \sin x \cos x\,dx$

**In exercises 13 - 16, find the antiderivative.**

13) $\int \frac{dx}{(x+4)^3}$

**Answer**

$$-\frac{1}{2(x+4)^2}+C$$

14) $\int x\ln(x^2)\,dx$

15) $\int \frac{4x^2}{\sqrt{1-x^6}}\,dx$

**Answer**

$$\frac{4}{3}\sin^{-1}(x^3)+C$$

16) $\int \frac{e^{2x}}{1+e^{4x}}\,dx$

**In exercises 17 - 20, find the derivative.**

17) $\frac{d}{dt}\int_0^t \frac{\sin x}{\sqrt{1+x^2}}\,dx$

**Answer**

$$\frac{\sin t}{\sqrt{1+t^2}}$$

18) $\frac{d}{dx}\int_1^{x^3} \sqrt{4-t^2}\,dt$

19) $\frac{d}{dx}\int_1^{\ln(x)} (4t+e^t)\,dt$

**Answer**

$$4\frac{\ln x}{x}+1$$

20) $\frac{d}{dx}\int_0^{\cos x} e^{t^2}\,dt$

**In exercises 21 - 23, consider the historic average cost per gigabyte of RAM on a computer.**

| Year | 5-Year Change ($) |
|---|---|
| 1980 | 0 |
| 1985 | −5,468,750 |
| 1990 | −755,495 |
| 1995 | −73,005 |
| 2000 | −29,768 |
| 2005 | −918 |
| 2010 | −177 |

21) If the average cost per gigabyte of RAM in 2010 is $12, find the average cost per gigabyte of RAM in 1980.

**Answer**

$6,328,113

Solution: $6,328,113

22) The average cost per gigabyte of RAM can be approximated by the function $C(t) = 8,500,000(0.65)^t$, where $t$ is measured in years since 1980, and $C$ is cost in US dollars. Find the average cost per gigabyte of RAM for the period from 1980 to 2010.

23) Find the average cost of 1 GB RAM from 2005 to 2010.

**Answer**

$73.36

24) The velocity of a bullet from a rifle can be approximated by $v(t) = 6400t^2 - 6505t + 2686$, where $t$ is seconds after the shot and v is the velocity measured in feet per second. This equation only models the velocity for the first half-second after the shot: $0 \leq t \leq 0.5$. What is the total distance the bullet travels in 0.5 sec?

25) What is the average velocity of the bullet for the first half-second?

**Answer**

$\frac{19117}{12}$ ft/sec, or about 1593 ft/sec

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# CHAPTER OVERVIEW

## 2: Applications of Integration

In this chapter, we use definite integrals to calculate the force exerted on the dam when the reservoir is full and we examine how changing water levels affect that force. Hydrostatic force is only one of the many applications of definite integrals we explore in this chapter. From geometric applications such as surface area and volume, to physical applications such as mass and work, to growth and decay models, definite integrals are a powerful tool to help us understand and model the world around us.


2.1: Prelude to Applications of Integration
2.2: Areas between Curves
2.2E: Exercises for Section 2.2
2.3: Determining Volumes by Slicing
2.3E: Exercises for Section 2.3
2.4: Volumes of Revolution - Cylindrical Shells
2.4E: Exercises for Section 2.4
2.5: Arc Length of a Curve and Surface Area
2.5E: Exercises for Section 2.5
2.6: Physical Applications of Integration
2.6E: Exercises for Section 2.6
2.7: Moments and Centers of Mass
2.7E: Exercises for Section 2.7
2.8: Integrals, Exponential Functions, and Logarithms
2.8E: Exercises for Section 2.8
2.9: Exponential Growth and Decay
2.9E: Exercises for Section 2.9
2.10: Calculus of the Hyperbolic Functions
2.10E: Exercises for Section 2.10
2.11: Chapter 2 Review Exercises


*Thumbnail: A region between two functions.*

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# 2.1: Prelude to Applications of Integration

The Hoover Dam is an engineering marvel. When Lake Mead, the reservoir behind the dam, is full, the dam withstands a great deal of force. However, water levels in the lake vary considerably as a result of droughts and varying water demands. Later in this chapter, we use definite integrals to calculate the force exerted on the dam when the reservoir is full and we examine how changing water levels affect that force. Hydrostatic force is only one of the many applications of definite integrals we explore in this chapter. From geometric applications such as surface area and volume, to physical applications such as mass and work, to growth and decay models, definite integrals are a powerful tool to help us understand and model the world around us.



Figure **2.1.1**: Hoover Dam is one of the United States' iconic landmarks, and provides irrigation and hydroelectric power for millions of people in the southwest United States. (credit: modification of work by Lynn Betts, Wikimedia).

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# 2.2: Areas between Curves

**Learning Objectives**

- Determine the area of a region between two curves by integrating with respect to the independent variable.
- Find the area of a compound region.
- Determine the area of a region between two curves by integrating with respect to the dependent variable.

In Introduction to Integration, we developed the concept of the definite integral to calculate the area below a curve on a given interval. In this section, we expand that idea to calculate the area of more complex regions. We start by finding the area between two curves that are functions of $x$, beginning with the simple case in which one function value is always greater than the other. We then look at cases when the graphs of the functions cross. Last, we consider how to calculate the area between two curves that are functions of $y$.

## Area of a Region between Two Curves

Let $f(x)$ and $g(x)$ be continuous functions over an interval $[a, b]$ such that $f(x) \geq g(x)$ on $[a, b]$. We want to find the area between the graphs of the functions, as shown in Figure 2.2.1.

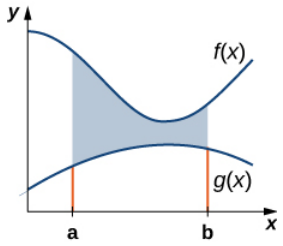



Figure 2.2.1: The area between the graphs of two functions, $f(x)$ and $g(x)$, on the interval $[a, b]$

As we did before, we are going to partition the interval on the **x-axis** and approximate the area between the graphs of the functions with rectangles. So, for $i = 0, 1, 2, \ldots, n$, let $P = x_i$ be a regular partition of $[a, b]$. Then, for $i = 1, 2, \ldots, n$, choose a point $x_i^* \in [x_{i-1}, x_i]$, and on each interval $[x_{i-1}, x_i]$ construct a rectangle that extends vertically from $g(x_i^*)$ to $f(x_i^*)$. Figure 2.2.2a shows the rectangles when $x_i^*$ is selected to be the left endpoint of the interval and $n = 10$. Figure 2.2.2b shows a representative rectangle in detail.

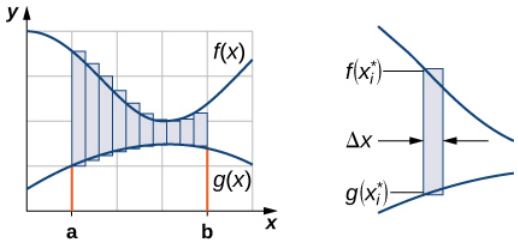



Figure 2.2.2: (a)We can approximate the area between the graphs of two functions, $f(x)$ and $g(x)$, with rectangles. (b) The area of a typical rectangle goes from one curve to the other.

The height of each individual rectangle is $f(x_i^*) - g(x_i^*)$ and the width of each rectangle is $\Delta x$. Adding the areas of all the rectangles, we see that the area between the curves is approximated by

$$A \approx \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)] \Delta x. \tag{2.2.1}$$

This is a Riemann sum, so we take the limit as $n \to \infty$ and we get

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)] \Delta x = \int_a^b [f(x) - g(x)] dx. \tag{2.2.2}$$

These findings are summarized in the following theorem.

### Finding the Area between Two Curves

Let $f(x)$ and $g(x)$ be continuous functions such that $f(x) \geq g(x)$ over an interval $[a, b]$. Let R denote the region bounded above by the graph of $f(x)$, below by the graph of $g(x)$, and on the left and right by the lines $x = a$ and $x = b$, respectively. Then, the area of $\mathbf{R}$ is given by

$$A = \int_a^b [f(x) - g(x)]dx. \tag{2.2.3}$$

We apply this theorem in the following example.

### Example 2.2.1: Finding the Area of a Region between Two Curves I

If $\mathbf{R}$ is the region bounded above by the graph of the function $f(x) = x + 4$ and below by the graph of the function $g(x) = 3 - \frac{x}{2}$ over the interval $[1, 4]$, find the area of region $\mathbf{R}$.

**Solution**

The region is depicted in the following figure.

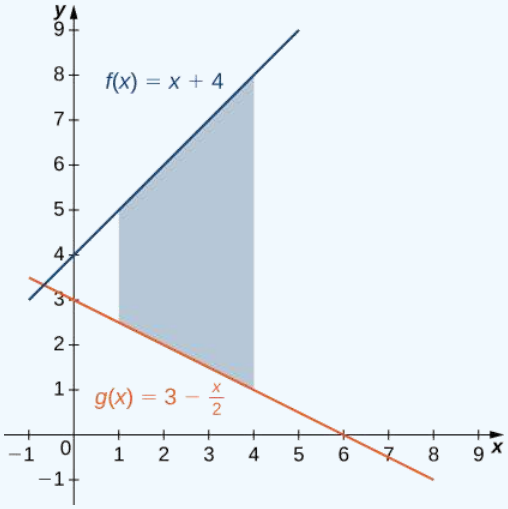



Figure 2.2.3: A region between two curves is shown where one curve is always greater than the other.

We have

$$A = \int_a^b [f(x) - g(x)]\,dx$$
$$= \int_1^4 [(x+4) - (3 - \frac{x}{2})]\,dx = \int_1^4 \left[\frac{3x}{2} + 1\right]\,dx$$
$$= [\frac{3x^2}{4} + x]\Big|_1^4 = (16 - \frac{7}{4}) = \frac{57}{4}.$$

The area of the region is $\frac{57}{4} units^2$.

### Exercise 2.2.1

If $\mathbf{R}$ is the region bounded by the graphs of the functions $f(x) = \frac{x}{2} + 5$ and $g(x) = x + \frac{1}{2}$ over the interval $[1, 5]$, find the area of region $\mathbf{R}$.

**Hint**

Graph the functions to determine which function's graph forms the upper bound and which forms the lower bound, then follow the process used in Example.

**Answer**

**12** units$^2$

In Example **2.2.1**, we defined the interval of interest as part of the problem statement. Quite often, though, we want to define our interval of interest based on where the graphs of the two functions intersect. This is illustrated in the following example.

## Example 2.2.2: Finding the Area of a Region between Two Curves II

If $\mathbf{R}$ is the region bounded above by the graph of the function $f(x) = 9 - (x/2)^2$ and below by the graph of the function $g(x) = 6 - x$, find the area of region $\mathbf{R}$.

**Solution**

The region is depicted in the following figure.

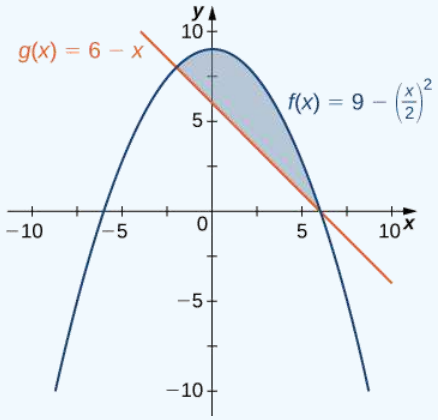



Figure **2.2.4**: This graph shows the region below the graph of $f(x)$ and above the graph of $g(x)$.

We first need to compute where the graphs of the functions intersect. Setting $f(x) = g(x)$, we get

$$\begin{aligned} f(x) &= g(x) \\ 9 - (\frac{x}{2})^2 &= 6 - x \\ 9 - \frac{x^2}{4} &= 6 - x \\ 36 - x^2 &= 24 - 4x \\ x^2 - 4x - 12 &= 0 \\ (x-6)(x+2) &= 0. \end{aligned}$$

The graphs of the functions intersect when $x = 6$ or $x = -2$, so we want to integrate from $-2$ to $6$. Since $f(x) \geq g(x)$ for $-2 \leq x \leq 6$, we obtain

$$\begin{aligned} A &= \int_a^b [f(x) - g(x)]\, dx \\ &= \int_{-2}^{6} \left[9 - (\frac{x}{2})^2 - (6 - x)\right] dx \\ &= \int_{-2}^{6} \left[3 - \frac{x^2}{4} + x\right] dx \\ &= \left[3x - \frac{x^3}{12} + \frac{x^2}{2}\right]\Big|_{-2}^{6} = \frac{64}{3}. \end{aligned}$$

The area of the region is **64/3** units$^2$.

**Exercise 2.2.2**

If $\mathbf{R}$ is the region bounded above by the graph of the function $f(x) = x$ and below by the graph of the function $g(x) = x^4$, find the area of region $\mathbf{R}$.

**Hint**

Use the process from Example 2.2.2.

**Answer**

$\frac{3}{10}$ unit$^2$

## Areas of Compound Regions

So far, we have required $f(x) \geq g(x)$ over the entire interval of interest, but what if we want to look at regions bounded by the graphs of functions that cross one another? In that case, we modify the process we just developed by using the absolute value function.

**Finding the Area of a Region between Curves That Cross**

Let $f(x)$ and $g(x)$ be continuous functions over an interval $[a, b]$. Let $\mathbf{R}$ denote the region between the graphs of $f(x)$ and $g(x)$, and be bounded on the left and right by the lines $x = a$ and $x = b$, respectively. Then, the area of $\mathbf{R}$ is given by

$$A = \int_a^b |f(x) - g(x)|\, dx. \tag{2.2.4}$$

In practice, applying this theorem requires us to break up the interval $[a, b]$ and evaluate several integrals, depending on which of the function values is greater over a given part of the interval. We study this process in the following example.

**Example 2.2.3: Finding the Area of a Region Bounded by Functions That Cross**

If $\mathbf{R}$ is the region between the graphs of the functions $f(x) = \sin x$ and $g(x) = \cos x$ over the interval $[0, \pi]$, find the area of region $\mathbf{R}$.

**Solution**

The region is depicted in the following figure.

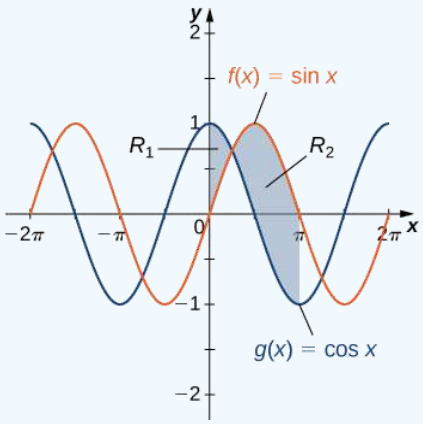



Figure 2.2.5: The region between two curves can be broken into two sub-regions.

The graphs of the functions intersect at $x = \pi/4$. For $x \in [0, \pi/4], \cos x \geq \sin x$, so

$$|f(x) - g(x)| = |\sin x - \cos x| = \cos x - \sin x.$$

On the other hand, for $x \in [\pi/4, \pi], \sin x \geq \cos x$, so

$$|f(x) - g(x)| = |\sin x - \cos x| = \sin x - \cos x.$$

Then

$$A = \int_a^b |f(x) - g(x)|\, dx$$

$$= \int_0^{\pi} |\sin x - \cos x|\, dx = \int_0^{\pi/4} (\cos x - \sin x) dx + \int_{\pi/4}^{\pi} (\sin x - \cos x) dx$$

$$= [\sin x + \cos x]|_0^{\pi/4} + [-\cos x - \sin x]|_{\pi/4}^{\pi}$$

$$= (\sqrt{2} - 1) + (1 + \sqrt{2}) = 2\sqrt{2}.$$

The area of the region is $2\sqrt{2}$ units$^2$.



**Exercise 2.2.3**

If $\mathbf{R}$ is the region between the graphs of the functions $f(x) = \sin x$ and $g(x) = \cos x$ over the interval $[\pi/2, 2\pi]$, find the area of region $\mathbf{R}$.

**Hint**

The two curves intersect at $x = (5\pi)/4$.

**Answer**

$2 + 2\sqrt{2}$ units$^2$

**Example 2.2.4: Finding the Area of a Complex Region**

Consider the region depicted in Figure 2.2.6. Find the area of $\mathbf{R}$.

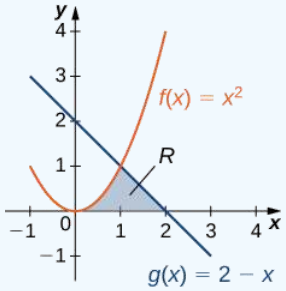



Figure **2.2.6**: Two integrals are required to calculate the area of this region.

**Solution**

As with Example **2.2.3**, we need to divide the interval into two pieces. The graphs of the functions intersect at $x = 1$ (set $f(x) = g(x)$ and solve for x), so we evaluate two separate integrals: one over the interval $[0, 1]$ and one over the interval $[1, 2]$.

Over the interval $[0, 1]$, the region is bounded above by $f(x) = x^2$ and below by the x-axis, so we have

$$A_1 = \int_0^1 x^2 dx = \frac{x^3}{3} \Big|_0^1 = \frac{1}{3}.$$

Over the interval $[1, 2]$, the region is bounded above by $g(x) = 2 - x$ and below by the **x-axis**, so we have

$$A_2 = \int_1^2 (2 - x) dx = \left[2x - \frac{x^2}{2}\right] \Big|_1^2 = \frac{1}{2}.$$

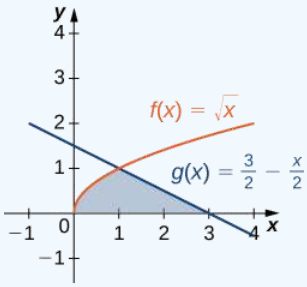
Adding these areas together, we obtain

$$A = A_1 + A_2 = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}.$$

The area of the region is $5/6$ units$^2$.

**Exercise 2.2.4**

Consider the region depicted in the following figure. Find the area of **R**.




**Hint**

The two curves intersect at **x=1**

**Answer**

$\frac{5}{3}$ units$^2$

## Regions Defined with Respect to y

In Example **2.2.4**, we had to evaluate two separate integrals to calculate the area of the region. However, there is another approach that requires only one integral. What if we treat the curves as functions of $y$, instead of as functions of $x$? Review Figure. Note that the left graph, shown in red, is represented by the function $y = f(x) = x^2$. We could just as easily solve this for x and represent the curve by the function $x = v(y) = \sqrt{y}$. (Note that $x = -\sqrt{y}$ is also a valid representation of the function $y = f(x) = x^2$ as a function of $y$. However, based on the graph, it is clear we are interested in the positive square root.) Similarly, the right graph is represented by

the function $y = g(x) = 2 - x$, but could just as easily be represented by the function $x = u(y) = 2 - y$. When the graphs are represented as functions of $y$, we see the region is bounded on the left by the graph of one function and on the right by the graph of the other function. Therefore, if we integrate with respect to $y$, we need to evaluate one integral only. Let's develop a formula for this type of integration.

Let $u(y)$ and $v(y)$ be continuous functions over an interval $[c, d]$ such that $u(y) \geq v(y)$ for all $y \in [c, d]$. We want to find the area between the graphs of the functions, as shown in Figure 2.2.7.

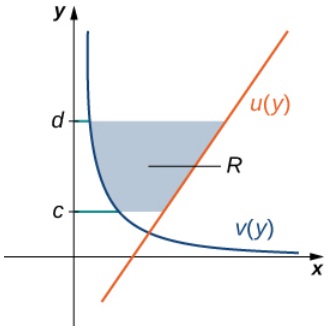



Figure 2.2.7: We can find the area between the graphs of two functions, $u(y)$ and $v(y)$.

This time, we are going to partition the interval on the **y-axis** and use horizontal rectangles to approximate the area between the functions. So, for $i = 0, 1, 2, \ldots, n$, let $Q = y_i$ be a regular partition of $[c, d]$. Then, for $i = 1, 2, \ldots, n$, choose a point $y_i^* \in [y_{i-1}, y_i]$, then over each interval $[y_{i-1}, y_i]$ construct a rectangle that extends horizontally from $v(y^0 *_i)$ to $u(y_i^*)$. Figure 2.2.8*a* shows the rectangles when $y_i^*$ is selected to be the lower endpoint of the interval and $n = 10$. Figure 2.2.8*b* shows a representative rectangle in detail.

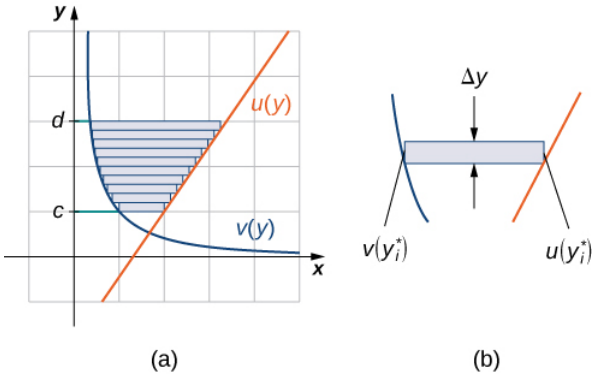



Figure 2.2.8: (a) Approximating the area between the graphs of two functions, $u(y)$ and $v(y)$, with rectangles. (b) The area of a typical rectangle.

The height of each individual rectangle is $\Delta y$ and the width of each rectangle is $u(y_i^*) - v(y_i^*)$. Therefore, the area between the curves is approximately

$$A \approx \sum_{i=1}^{n} [u(y_i^*) - v(y_i^*)] \Delta y.$$

This is a Riemann sum, so we take the limit as $n \to \infty$, obtaining

$$\begin{aligned} A &= \lim_{n \to \infty} \sum_{i=1}^{n} [u(y_i^*) - v(y_i^*)] \Delta y \\ &= \int_c^d [u(y) - v(y)] dy. \end{aligned}$$

These findings are summarized in the following theorem.

### Finding the Area between Two Curves, Integrating along the y-axis

Let $u(y)$ and $v(y)$ be continuous functions such that $u(y) \geq v(y)$for all $y \in [c, d]$. Let $\mathbf{R}$ denote the region bounded on the right by the graph of $u(y)$, on the left by the graph of $v(y)$, and above and below by the lines $y = d$ and $y = c$, respectively. Then, the area of $\mathbf{R}$ is given by

$$A = \int_c^d [u(y) - v(y)]dy. \tag{2.2.5}$$

### Example 2.2.5: Integrating with Respect to y

Let's revisit Example 2.2.4, only this time let's integrate with respect to $y$. Let $\mathbf{R}$ be the region depicted in Figure 2.2.9. Find the area of $\mathbf{R}$ by integrating with respect to $y$.

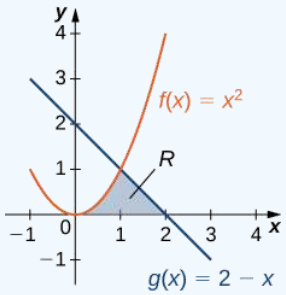



Figure 2.2.9: The area of region $\mathbf{R}$ can be calculated using one integral only when the curves are treated as functions of $y$.

**Solution**

We must first express the graphs as functions of $y$. As we saw at the beginning of this section, the curve on the left can be represented by the function $x = v(y) = \sqrt{y}$, and the curve on the right can be represented by the function $x = u(y) = 2 - y$.

Now we have to determine the limits of integration. The region is bounded below by the x-axis, so the lower limit of integration is $y = 0$. The upper limit of integration is determined by the point where the two graphs intersect, which is the point $(1, 1)$, so the upper limit of integration is $y = 1$. Thus, we have $[c, d] = [0, 1]$.

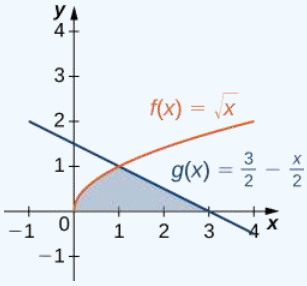
Calculating the area of the region, we get

$$\begin{aligned} A &= \int_c^d [u(y) - v(y)]dy \\ &= \int_0^1 [(2 - y) - \sqrt{y}]dy \\ &= [2y - \frac{y^2}{2} - \frac{2}{3}y^{3/2}] \,|_0^1 \\ &= \frac{5}{6}. \end{aligned}$$

The area of the region is $5/6$ units$^2$.

### Exercise 2.2.5

Let's revisit the checkpoint associated with Example 2.2.4, only this time, let's integrate with respect to $y$. Let $\mathbf{R}$ be the region depicted in the following figure. Find the area of $\mathbf{R}$ by integrating with respect to $y$.




**Hint**

Follow the process from the previous example.

**Answer**

$\frac{5}{3}$ units$^2$

## Key Concepts

- Just as definite integrals can be used to find the area under a curve, they can also be used to find the area between two curves.
- To find the area between two curves defined by functions, integrate the difference of the functions.
- If the graphs of the functions cross, or if the region is complex, use the absolute value of the difference of the functions. In this case, it may be necessary to evaluate two or more integrals and add the results to find the area of the region.
- Sometimes it can be easier to integrate with respect to y to find the area. The principles are the same regardless of which variable is used as the variable of integration.

## Key Equations

- **Area between two curves, integrating on the x-axis**

$$A = \int_a^b [f(x) - g(x)]dx$$

- **Area between two curves, integrating on the y-axis**

$$A = \int_c^d [u(y) - v(y)]dy$$

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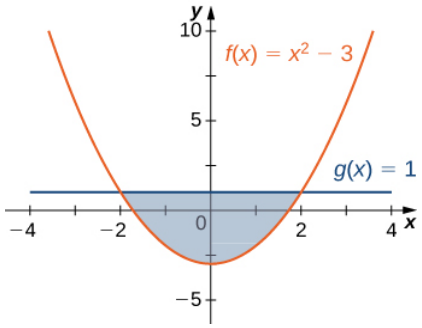
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# 2.2E: Exercises for Section 2.2

**For exercises 1 - 2, determine the area of the region between the two curves in the given figure by integrating over the $x$-axis.**

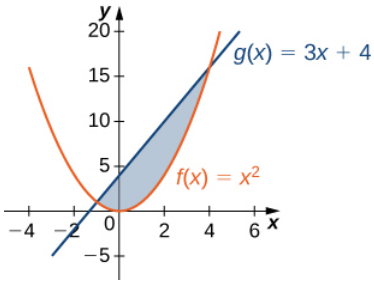
1) $y = x^2 - 3$ and $y = 1$




**Answer**

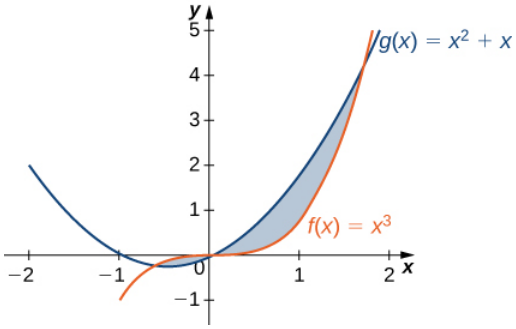
$\frac{32}{3}$ units$^2$

2) $y = x^2$ and $y = 3x + 4$




**For exercises 3 - 4, split the region between the two curves into two smaller regions, then determine the area by integrating over the $x$-axis. Note that you will have two integrals to solve.**

3) $y = x^3$ and $y = x^2 + x$

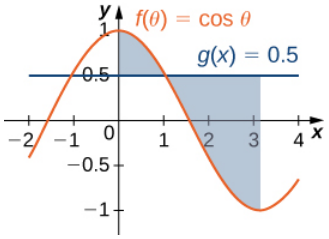



**Answer**

$\frac{13}{12}$ units$^2$

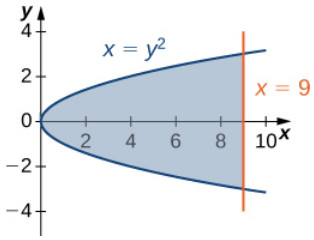
4) $y = \cos\theta$ and $y = 0.5$, for $0 \le \theta \le \pi$


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**For exercises 5-6, determine the area of the region between the two curves by integrating over the $y$-axis.**

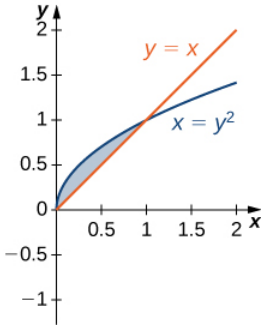
5) $x = y^2$ and $x = 9$




**Answer**

$36 \text{ units}^2$

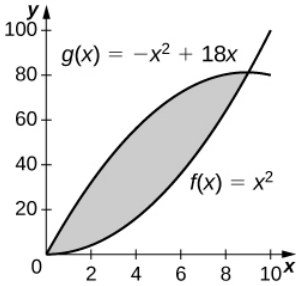
6) $y = x$ and $x = y^2$




**For exercises 7 - 13, graph the equations and shade the area of the region between the curves. Determine its area by integrating over the $x$-axis.**

7) $y = x^2$ and $y = -x^2 + 18x$

**Answer**



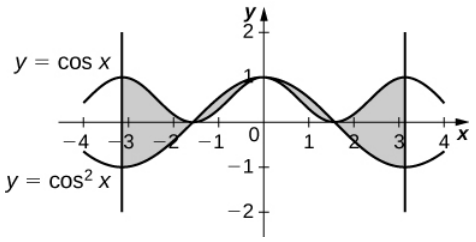

243 square units

8) $y = \dfrac{1}{x}$, $y = \dfrac{1}{x^2}$, and $x = 3$

9) $y = \cos x$ and $y = \cos^2 x$ on $x \in [-\pi, \pi]$


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**Answer**

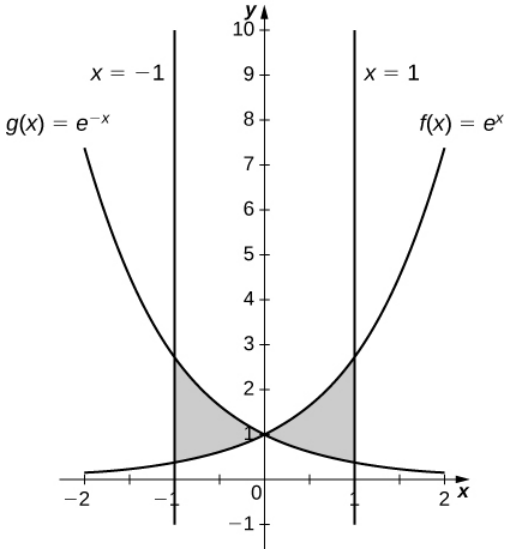



4 square units

10) $y = e^x$, $y = e^{2x-1}$, and $x = 0$

11) $y = e^x$, $y = e^{-x}$, $x = -1$ and $x = 1$

**Answer**

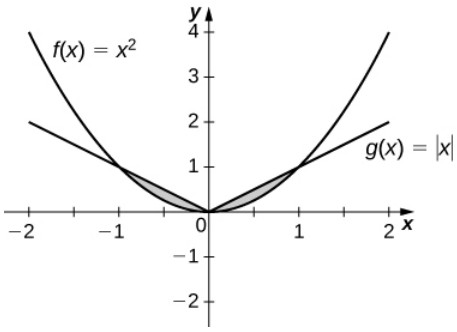



$\frac{2(e-1)^2}{e}$ units$^2$

12) $y = e$, $y = e^x$, and $y = e^{-x}$

13) $y = |x|$ and $y = x^2$

**Answer**




$\frac{1}{3}$ units$^2$

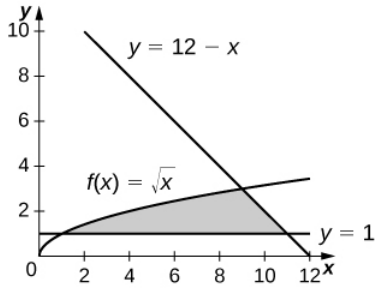

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**For exercises 14 - 19, graph the equations and shade the area of the region between the curves. If necessary, break the region into sub-regions to determine its entire area.**

14) $y = \sin(\pi x)$, $y = 2x$, and $x > 0$

15) $y = 12 - x$, $y = \sqrt{x}$, and $y = 1$

**Answer**

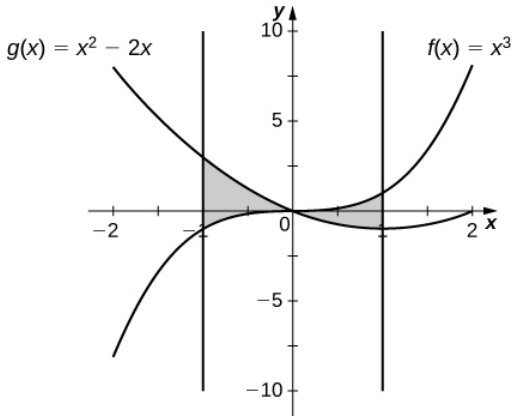



$\frac{34}{3}$ units$^2$

16) $y = \sin x$ and $y = \cos x$ over $x \in [-\pi, \pi]$

17) $y = x^3$ and $y = x^2 - 2x$ over $x \in [-1, 1]$

**Answer**



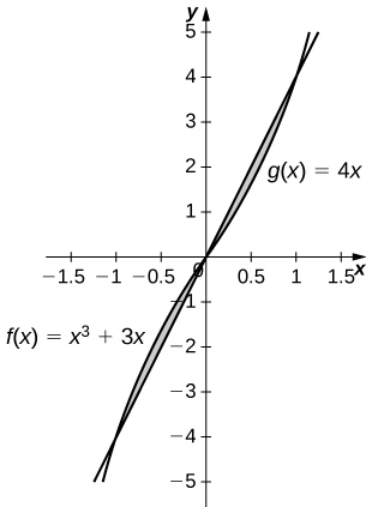

$\frac{5}{2}$ units$^2$

18) $y = x^2 + 9$ and $y = 10 + 2x$ over $x \in [-1, 3]$

19) $y = x^3 + 3x$ and $y = 4x$

**Answer**


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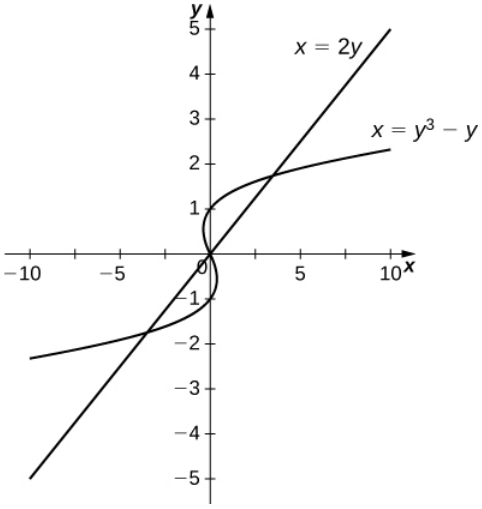

$\frac{1}{2}$ units$^2$

**For exercises 20 -25, graph the equations and shade the area of the region between the curves. Determine its area by integrating over the $y$-axis.**

20) $x = y^3$ and $x = 3y - 2$

21) $x = 2y$ and $x = y^3 - y$

**Answer**



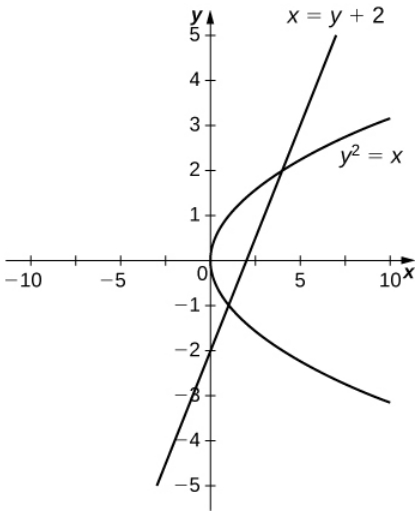

$\frac{9}{2}$ units$^2$

22) $x = -3 + y^2$ and $x = y - y^2$

23) $y^2 = x$ and $x = y + 2$

**Answer**


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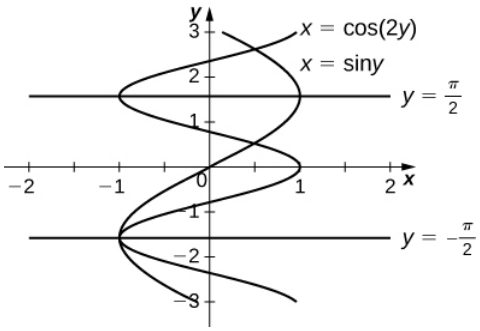



$\frac{9}{2}$ units$^2$

24) $x = |y|$ and $2x = -y^2 + 2$

25) $x = \sin y, \quad x = \cos(2y), \quad y = \pi/2$, and $y = -\pi/2$

**Answer**



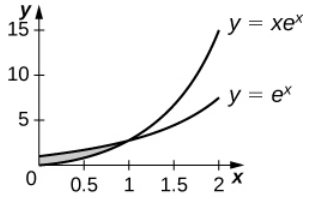

$\frac{3\sqrt{3}}{2}$ units$^2$

**For exercises 26 - 37, graph the equations and shade the area of the region between the curves. Determine its area by integrating over the $x$-axis or $y$-axis, whichever seems more convenient.**

26) $x = y^4$ and $x = y^5$

27) $y = xe^x, \quad y = e^x, \quad x = 0$, and $x = 1$.

**Answer**

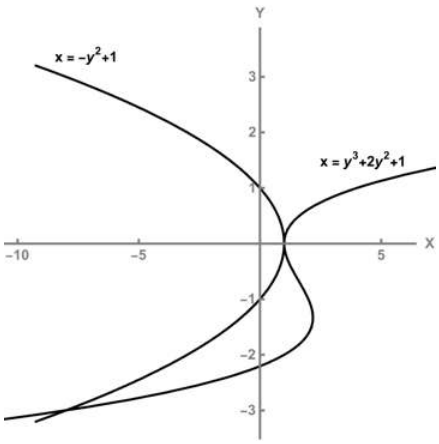



$(e - 2)$ units$^2$

28) $y = x^6$ and $y = x^4$

29) $x = y^3 + 2y^2 + 1$ and $x = -y^2 + 1$

**Answer**

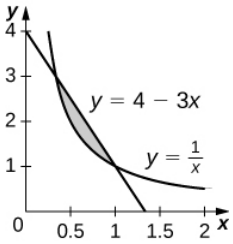



$\frac{27}{4}$ units$^2$

30) $y = |x|$ and $y = x^2 - 1$

31) $y = 4 - 3x$ and $y = \frac{1}{x}$

**Answer**

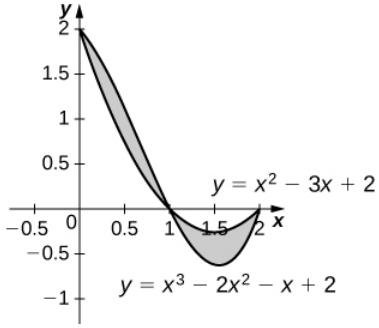



$\left(\frac{4}{3} - \ln(3)\right)$ units$^2$

32) $y = \sin x, \quad x = -\pi/6, \quad x = \pi/6$, and $y = \cos^3 x$

33) $y = x^2 - 3x + 2$ and $y = x^3 - 2x^2 - x + 2$

**Answer**

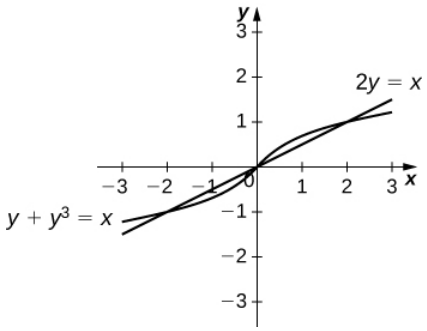



$\frac{1}{2}$ square units

34) $y = 2\cos^3(3x), \quad y = -1, \quad x = \frac{\pi}{4}$, and $x = -\frac{\pi}{4}$

35) $y + y^3 = x$ and $2y = x$

**Answer**

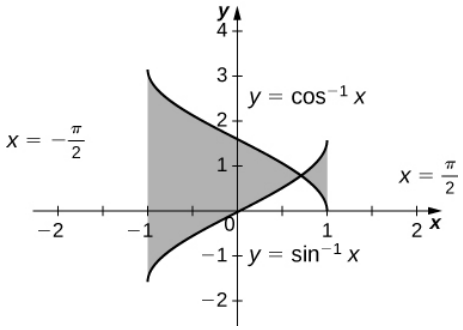



$\frac{1}{2}$ square units

36) $y = \sqrt{1 - x^2}$ and $y = x^2 - 1$

37) $y = \cos^{-1} x$, $y = \sin^{-1} x$, $x = -1$, and $x = 1$

**Answer**




$-2(\sqrt{2} - \pi)$ square units

**For exercises 38 - 47, find the exact area of the region bounded by the given equations if possible. If you are unable to determine the intersection points analytically, use a calculator to approximate the intersection points with three decimal places and determine the approximate area of the region.**

38) [T] $x = e^y$ and $y = x - 2$

39) [T] $y = x^2$ and $y = \sqrt{1 - x^2}$

**Answer**

1.067 square units

40) [T] $y = 3x^2 + 8x + 9$ and $3y = x + 24$

41) [T] $x = \sqrt{4 - y^2}$ and $y^2 = 1 + x^2$

**Answer**

0.852 square units

42) [T] $x^2 = y^3$ and $x = 3y$

43) [T] $y = \sin^3 x + 2$, $y = \tan x$, $x = -1.5$, and $x = 1.5$

**Answer**

7.523 square units

44) [T] $y = \sqrt{1 - x^2}$ and $y^2 = x^2$


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45) [T] $y = \sqrt{1 - x^2}$ and $y = x^2 + 2x + 1$

**Answer**

$\frac{3\pi - 4}{12}$ square units

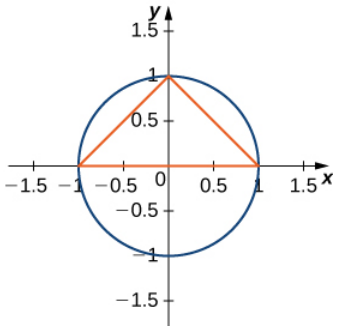
46) [T] $x = 4 - y^2$ and $x = 1 + 3y + y^2$

47) [T] $y = \cos x, \quad y = e^x, \quad x = -\pi, \quad$ and $\quad x = 0$

**Answer**

$1.429$ square units

48) The largest triangle with a base on the $x$-axis that fits inside the upper half of the unit circle $y^2 + x^2 = 1$ is given by $y = 1 + x$ and $y = 1 - x$. See the following figure. What is the area inside the semicircle but outside the triangle?




49) A factory selling cell phones has a marginal cost function $C(x) = 0.01x^2 - 3x + 229$, where $x$ represents the number of cell phones, and a marginal revenue function given by $R(x) = 429 - 2x$. Find the area between the graphs of these curves and $x = 0$. What does this area represent?

**Answer**

$33,333.33$ total profit for $200$ cell phones sold

50) An amusement park has a marginal cost function $C(x) = 1000e - x + 5$, where $x$ represents the number of tickets sold, and a marginal revenue function given by $R(x) = 60 - 0.1x$. Find the total profit generated when selling $550$ tickets. Use a calculator to determine intersection points, if necessary, to two decimal places.

51) The tortoise versus the hare: The speed of the hare is given by the sinusoidal function $H(t) = 1 - \cos((\pi t)/2)$ whereas the speed of the tortoise is $T(t) = (1/2)\tan^{-1}(t/4)$, where $t$ is time measured in hours and the speed is measured in miles per hour. Find the area between the curves from time $t = 0$ to the first time after one hour when the tortoise and hare are traveling at the same speed. What does it represent? Use a calculator to determine the intersection points, if necessary, accurate to three decimal places.

**Answer**

$3.263$ mi represents how far ahead the hare is from the tortoise

52) The tortoise versus the hare: The speed of the hare is given by the sinusoidal function $H(t) = (1/2) - (1/2)\cos(2\pi t)$ whereas the speed of the tortoise is $T(t) = \sqrt{t}$, where $t$ is time measured in hours and speed is measured in kilometers per hour. If the race is over in 1 hour, who won the race and by how much? Use a calculator to determine the intersection points, if necessary, accurate to three decimal places.

**For exercises 53 - 55, find the area between the curves by integrating with respect to $x$ and then with respect to $y$. Is one method easier than the other? Do you obtain the same answer?**

53) $y = x^2 + 2x + 1$ and $y = -x^2 - 3x + 4$

**Answer**


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$\frac{343}{24}$ square units

54) $y = x^4$ and $x = y^5$

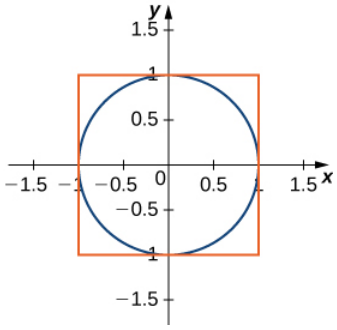
55) $x = y^2 - 2$ and $x = 2y$

**Answer**

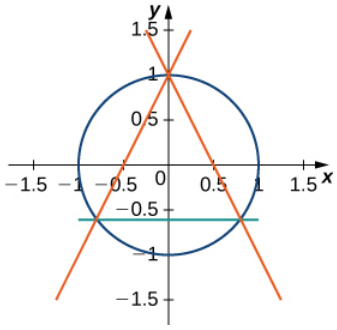
$4\sqrt{3}$ square units

**For exercises 56 - 57, solve using calculus, then check your answer with geometry.**

56) Determine the equations for the sides of the square that touches the unit circle on all four sides, as seen in the following figure. Find the area between the perimeter of this square and the unit circle. Is there another way to solve this without using calculus?




57) Find the area between the perimeter of the unit circle and the triangle created from $y = 2x + 1$, $y = 1 - 2x$ and $y = -\frac{3}{5}$, as seen in the following figure. Is there a way to solve this without using calculus?




**Answer**

$\left(\pi - \frac{32}{25}\right)$ square units

## Contributors

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# 2.3: Determining Volumes by Slicing

**Learning Objectives**

- Determine the volume of a solid by integrating a cross-section (the slicing method).
- Find the volume of a solid of revolution using the disk method.
- Find the volume of a solid of revolution with a cavity using the washer method.

In the preceding section, we used definite integrals to find the area between two curves. In this section, we use definite integrals to find volumes of three-dimensional solids. We consider three approaches—slicing, disks, and washers—for finding these volumes, depending on the characteristics of the solid.

## Volume and the Slicing Method

Just as area is the numerical measure of a two-dimensional region, volume is the numerical measure of a three-dimensional solid. Most of us have computed volumes of solids by using basic geometric formulas. The volume of a rectangular solid, for example, can be computed by multiplying length, width, and height: $V = lwh$. The formulas for the volumes of:

- a sphere

$$V_{sphere} = \frac{4}{3}\pi r^3, \tag{2.3.1}$$

- a cone

$$V_{cone} = \frac{1}{3}\pi r^2 h \tag{2.3.2}$$

- and a pyramid

$$V_{pyramid} = \frac{1}{3}Ah \tag{2.3.3}$$

have also been introduced. Although some of these formulas were derived using geometry alone, all these formulas can be obtained by using integration.

We can also calculate the volume of a cylinder. Although most of us think of a cylinder as having a circular base, such as a soup can or a metal rod, in mathematics the word cylinder has a more general meaning. To discuss cylinders in this more general context, we first need to define some vocabulary.

We define the *cross-section* of a solid to be the intersection of a plane with the solid. A cylinder is defined as any solid that can be generated by translating a plane region along a line perpendicular to the region, called the axis of the cylinder. Thus, all cross-sections perpendicular to the axis of a cylinder are identical. The solid shown in Figure **2.3.1** is an example of a cylinder with a noncircular base. To calculate the volume of a cylinder, then, we simply multiply the area of the cross-section by the height of the cylinder: $V = A \cdot h$. In the case of a right circular cylinder (soup can), this becomes $V = \pi r^2 h$.

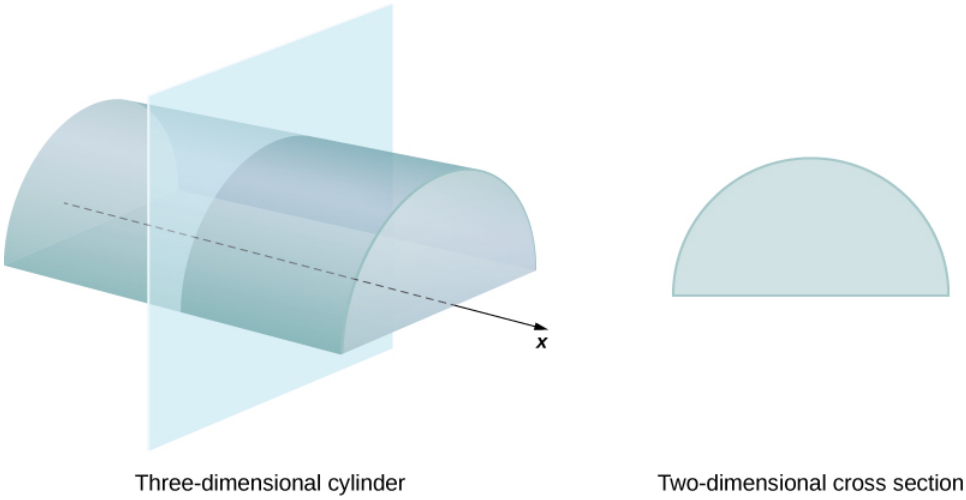



Figure **2.3.1**: *Each cross-section of a particular cylinder is identical to the others.*

If a solid does not have a constant cross-section (and it is not one of the other basic solids), we may not have a formula for its volume. In this case, we can use a definite integral to calculate the volume of the solid. We do this by slicing the solid into pieces, estimating the volume of each slice, and then adding those estimated volumes together. The slices should all be parallel to one another, and when we put all the slices together, we should get the whole solid. Consider, for example, the solid S shown in Figure **2.3.2**, extending along the $x$**-axis.**

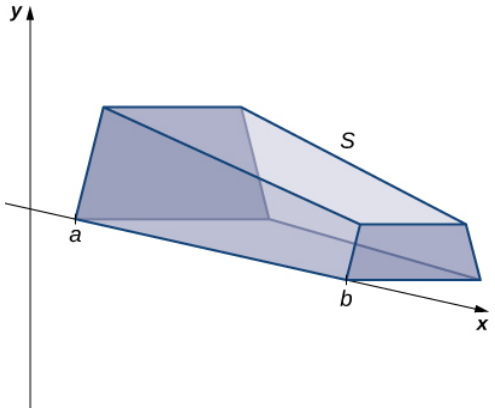



Figure **2.3.2**: A solid with a varying cross-section.

We want to divide $S$ into slices perpendicular to the $x$**-axis**. As we see later in the chapter, there may be times when we want to slice the solid in some other direction—say, with slices perpendicular to the $y$-axis. The decision of which way to slice the solid is very important. If we make the wrong choice, the computations can get quite messy. Later in the chapter, we examine some of these situations in detail and look at how to decide which way to slice the solid. For the purposes of this section, however, we use slices perpendicular to the $x$**-axis.**

Because the cross-sectional area is not constant, we let $A(x)$ represent the area of the cross-section at point x. Now let $P = x_0, x_1 \ldots, X_n$ be a regular partition of $[a, b]$, and for $i = 1, 2, \ldots n$, let $S_i$ represent the slice of $S$ stretching from $x_{i-1}$ to $x_i$. The following figure shows the sliced solid with $n = 3$.

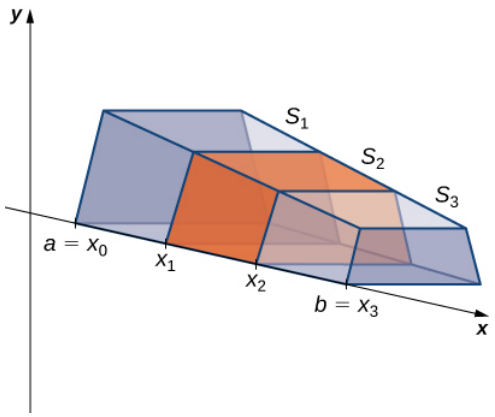



Figure **2.3.3**: The solid $S$ has been divided into three slices perpendicular to the $x$-**axis.**

Finally, for $i = 1, 2, \ldots n$, let $x_i^*$ be an arbitrary point in $[x_{i-1}, x_i]$. Then the volume of slice $S_i$ can be estimated by $V(S_i) \approx A(x_i^*)\,\Delta x$. Adding these approximations together, we see the volume of the entire solid $S$ can be approximated by

$$V(S) \approx \sum_{i=1}^{n} A(x_i^*)\,\Delta x. \tag{2.3.4}$$

By now, we can recognize this as a Riemann sum, and our next step is to take the limit as $n \to \infty$. Then we have

$$V(S) = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\,\Delta x = \int_a^b A(x)\,dx. \tag{2.3.5}$$

The technique we have just described is called the slicing method. To apply it, we use the following strategy.

**Problem-Solving Strategy: Finding Volumes by the Slicing Method**

1. Examine the solid and determine the shape of a cross-section of the solid. It is often helpful to draw a picture if one is not provided.
2. Determine a formula for the area of the cross-section.
3. Integrate the area formula over the appropriate interval to get the volume.

Recall that in this section, we assume the slices are perpendicular to the $x$-**axis**. Therefore, the area formula is in terms of x and the limits of integration lie on the $x$-**axis.** However, the problem-solving strategy shown here is valid regardless of how we choose to slice the solid.

**Example 2.3.1: Deriving the Formula for the Volume of a Pyramid**

We know from geometry that the formula for the volume of a pyramid is $V = \frac{1}{3}Ah$. If the pyramid has a square base, this becomes $V = \frac{1}{3}a^2h$, where a denotes the length of one side of the base. We are going to use the slicing method to derive this formula.

**Solution**

We want to apply the slicing method to a pyramid with a square base. To set up the integral, consider the pyramid shown in Figure **2.3.4**, oriented along the $x$-**axis.**

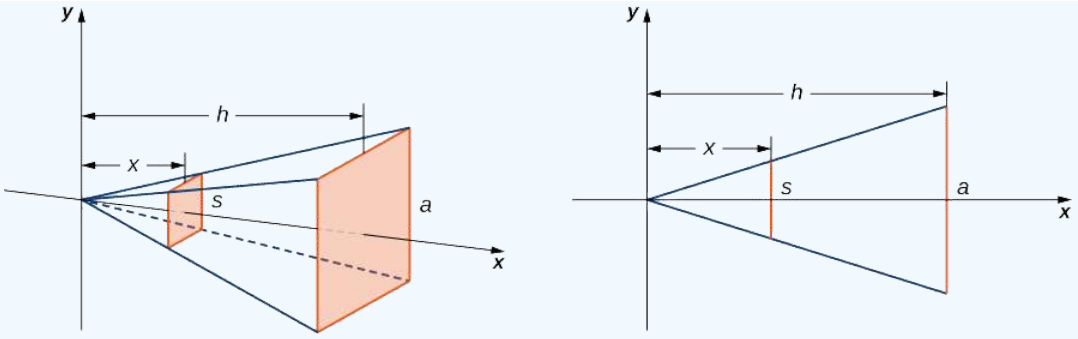



Figure 2.3.4: (a) A pyramid with a square base is oriented along the $x$-axis. (b) A two-dimensional view of the pyramid is seen from the side.

We first want to determine the shape of a cross-section of the pyramid. We know the base is a square, so the cross-sections are squares as well (step 1). Now we want to determine a formula for the area of one of these cross-sectional squares. Looking at Figure 2.3.4 (b), and using a proportion, since these are similar triangles, we have

$$\frac{s}{a} = \frac{x}{h} \tag{2.3.6}$$

or

$$s = \frac{ax}{h}. \tag{2.3.7}$$

Therefore, the area of one of the cross-sectional squares is

$$A(x) = s^2 = \left(\frac{ax}{h}\right)^2 \qquad \text{(step 2)} \tag{2.3.8}$$

Then we find the volume of the pyramid by integrating from $0$ to $h$ (step 3):

$$V = \int_0^h A(x)\,dx = \int_0^h \left(\frac{ax}{h}\right)^2 dx = \frac{a^2}{h^2}\int_0^h x^2\,dx = \left[\frac{a^2}{h^2}\left(\frac{1}{3}x^3\right)\right]\Big|_0^h = \frac{1}{3}a^2h. \tag{2.3.9}$$

This is the formula we were looking for.



**Exercise 2.3.1**

Use the slicing method to derive the formula

$$V = \frac{1}{3}\pi r^2 h \tag{2.3.10}$$

for the volume of a circular cone.

**Hint**

Use similar triangles, as in Example **2.3.1**.

## Solids of Revolution

If a region in a plane is revolved around a line in that plane, the resulting solid is called a **solid of revolution**, as shown in the following figure.

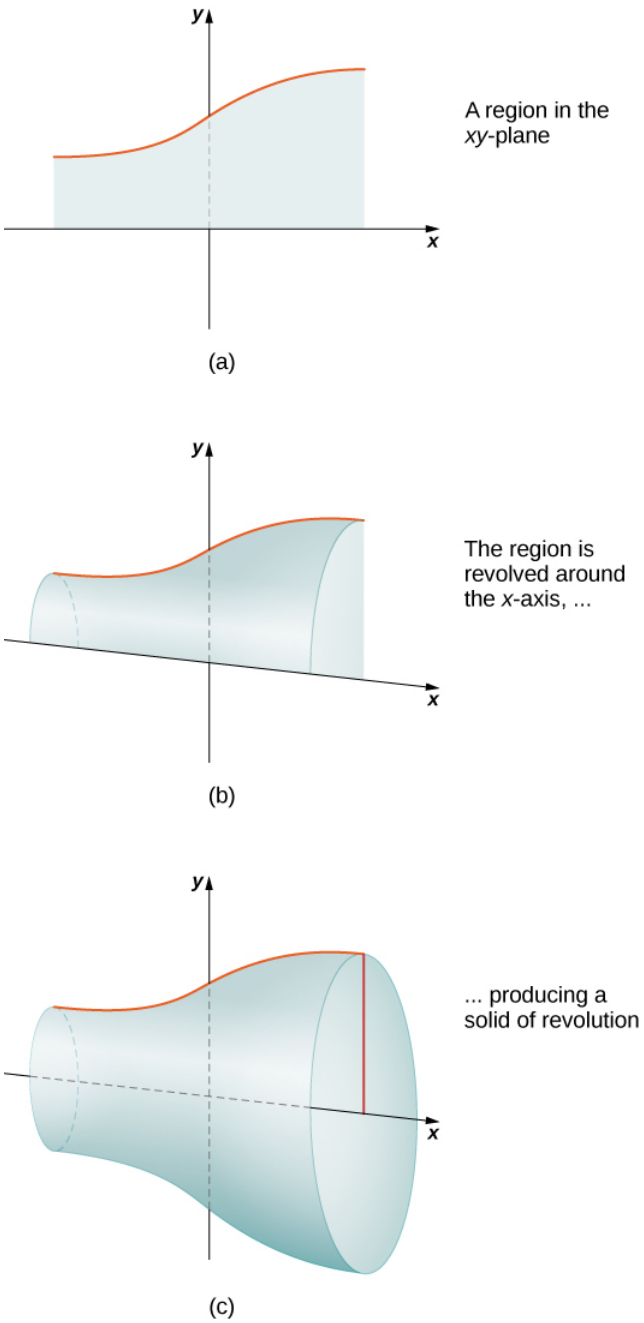



Figure **2.3.5**: (a) This is the region that is revolved around the $x$-axis. (b) As the region begins to revolve around the axis, it sweeps out a solid of revolution. (c) This is the solid that results when the revolution is complete.

Solids of revolution are common in mechanical applications, such as machine parts produced by a lathe. We spend the rest of this section looking at solids of this type. The next example uses the slicing method to calculate the volume of a solid of revolution.

**Example 2.3.2: Using the Slicing Method to find the Volume of a Solid of Revolution**

Use the slicing method to find the volume of the solid of revolution bounded by the graphs of $f(x) = x^2 - 4x + 5, x = 1$,and $x = 4$, and rotated about the $x$-axis.

**Solution**

Using the problem-solving strategy, we first sketch the graph of the quadratic function over the interval $[1, 4]$ as shown in the following figure.

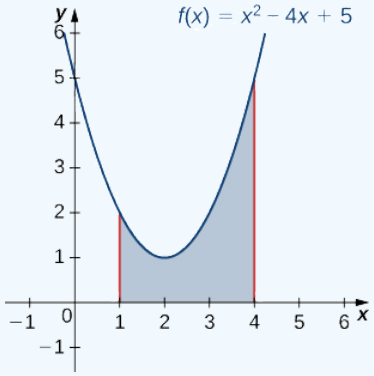
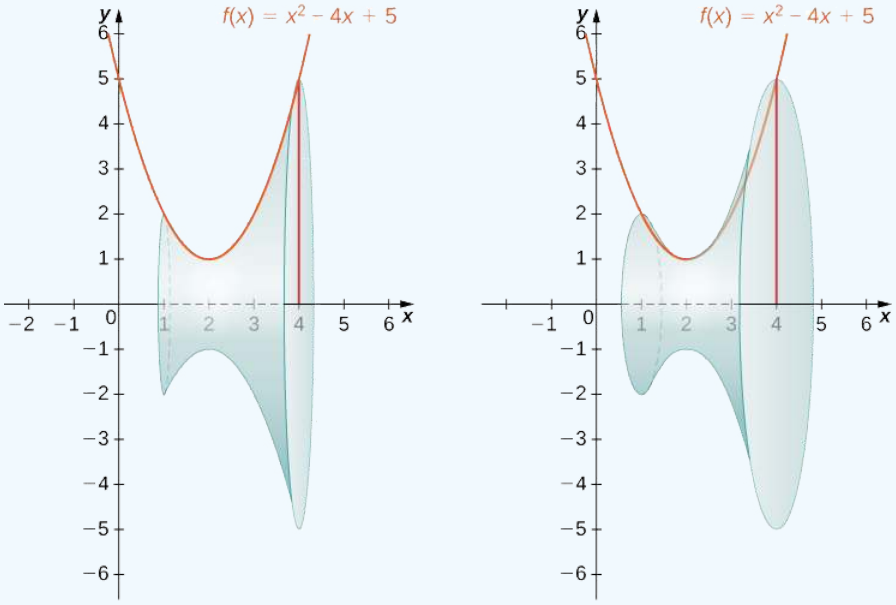



Figure **2.3.6**: A region used to produce a solid of revolution.

Next, revolve the region around the $x$-axis, as shown in the following figure.




(a) (b)

Figure **2.3.7**: Two views, (a) and (b), of the solid of revolution produced by revolving the region in Figure **2.3.6** about the $x$-**axis.**

Since the solid was formed by revolving the region around the $x$-**axis,** the cross-sections are circles (step 1). The area of the cross-section, then, is the area of a circle, and the radius of the circle is given by $f(x)$. Use the formula for the area of the circle:

$$A(x) = \pi r^2 = \pi[f(x)]^2 = \pi(x^2 - 4x + 5)^2 \qquad \text{(step 2)}. \tag{2.3.11}$$

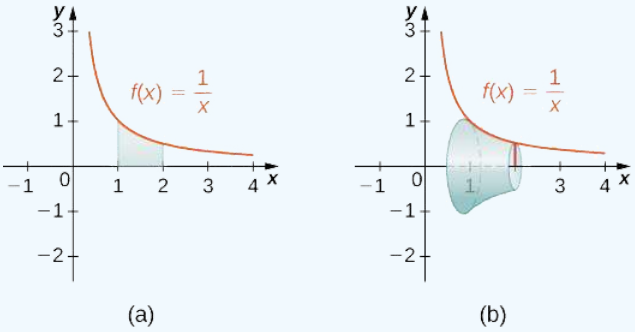
The volume, then, is (step 3)

$$\begin{aligned} V &= \int_a^b A(x)\,dx \\ &= \int_1^4 \pi(x^2 - 4x + 5)^2\,dx \\ &= \pi \int_1^4 (x^4 - 8x^3 + 26x^2 - 40x + 25)\,dx \\ &= \pi\left(\frac{x^5}{5} - 2x^4 + \frac{26x^3}{3} - 20x^2 + 25x\right)\Big|_1^4 \\ &= \frac{78}{5}\pi \end{aligned}$$

The volume is $78\pi/5$ **units**$^3$.

**Exercise 2.3.2**

Use the method of slicing to find the volume of the solid of revolution formed by revolving the region between the graph of the function $f(x) = 1/x$ and the $x$-**axis** over the interval $[1, 2]$ around the $x$-**axis.** See the following figure.




(a) (b)

**Hint**

Use the problem-solving strategy presented earlier and follow Example 2.3.2 to help with step 2.

**Answer**

$\frac{\pi}{2}$ units$^3$

## The Disk Method

When we use the slicing method with solids of revolution, it is often called the disk method because, for solids of revolution, the slices used to over approximate the volume of the solid are disks. To see this, consider the solid of revolution generated by revolving the region between the graph of the function $f(x) = (x-1)^2 + 1$ and the $x$-**axis** over the interval $[-1, 3]$ around the $x$-**axis**. The graph of the function and a representative disk are shown in Figure 2.3.8 (a) and (b). The region of revolution and the resulting solid are shown in Figure 2.3.8 (c) and (d).

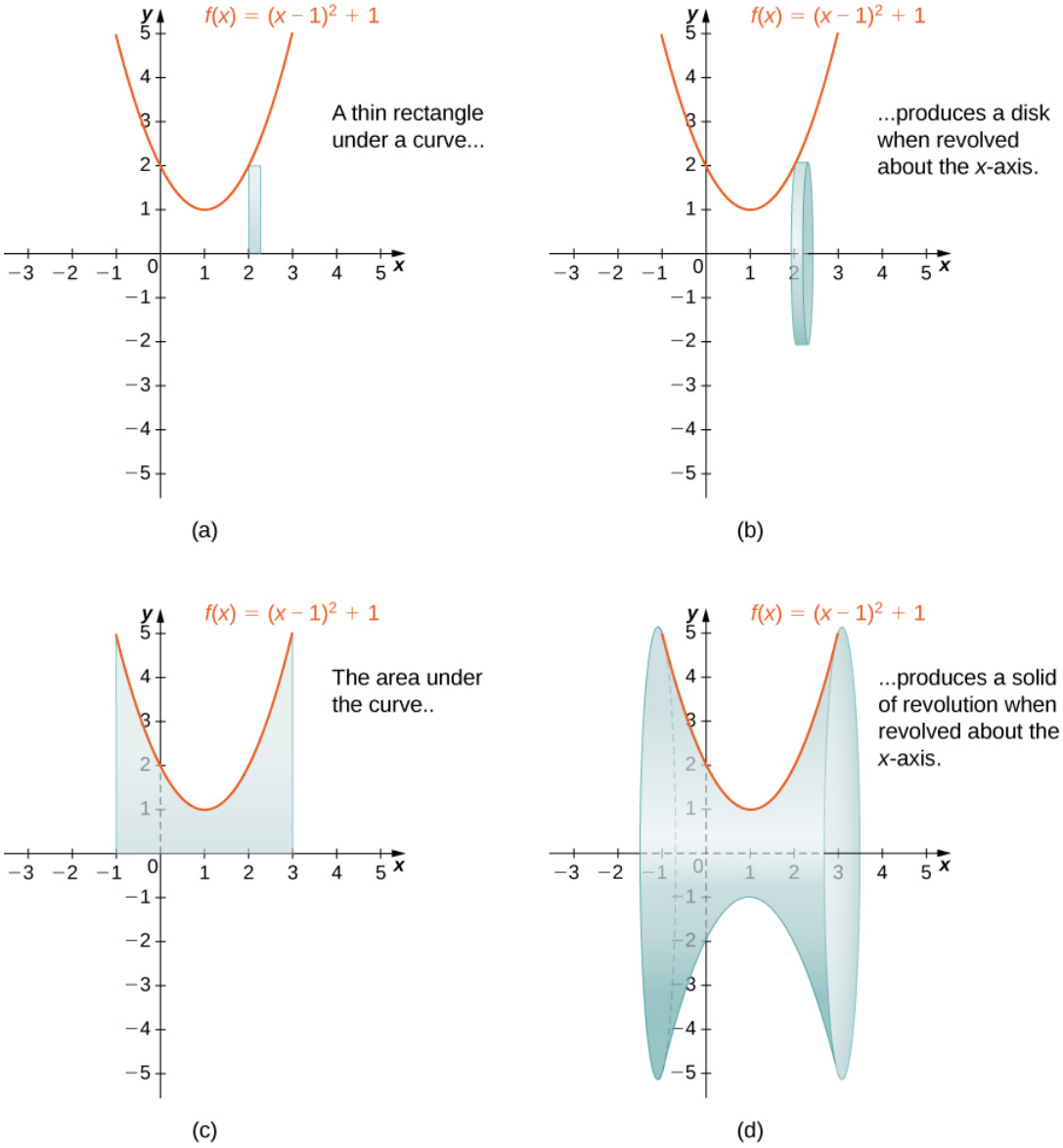



Figure **2.3.8**: (a) A thin rectangle for approximating the area under a curve. (b) A representative disk formed by revolving the rectangle about the $x$**-axis.** (c) The region under the curve is revolved about the $x$**-axis,** resulting in (d) the solid of revolution.


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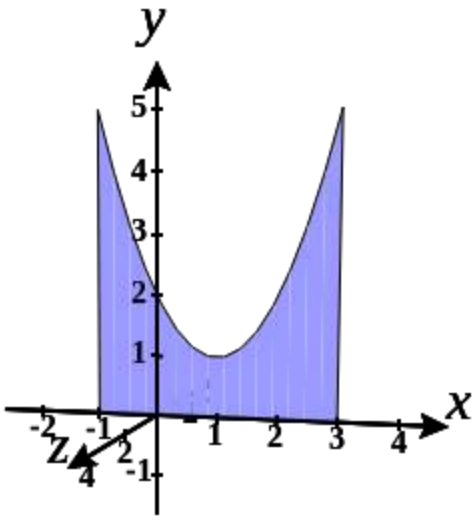



Figure 2.3.8: (e) A dynamic version of this solid of revolution generated using CalcPlot3D.

We already used the formal Riemann sum development of the volume formula when we developed the slicing method. We know that

$$\int_a^b A(x)\,dx.$$

The only difference with the disk method is that we know the formula for the cross-sectional area ahead of time; it is the area of a circle. This gives the following rule.

**The Disk Method**

Let $f(x)$ be continuous and nonnegative. Define $R$ as the region bounded above by the graph of $f(x)$, below by the $x$-**axis**, on the left by the line $x = a$, and on the right by the line $x = b$. Then, the volume of the solid of revolution formed by revolving $R$ around the $x$-**axis** is given by

$$V = \int_a^b \pi[f(x)]^2\,dx. \tag{2.3.12}$$

The volume of the solid we have been studying (Figure 2.3.8) is given by

$$
\begin{aligned}
V &= \int_a^b \pi [f(x)]^2 \, dx \\
&= \int_{-1}^{3} \pi \left[(x-1)^2+1\right]^2 dx = \pi \int_{-1}^{3} \left[(x-1)^4+2(x-1)^2+1\right]^2 dx \\
&= \pi \left[\frac{1}{5}(x-1)^5+\frac{2}{3}(x-1)^3+x\right]\Big|_{-1}^{3} \\
&= \pi \left[\left(\frac{32}{5}+\frac{16}{3}+3\right)-\left(-\frac{32}{5}-\frac{16}{3}-1\right)\right] \\
&= \frac{412\pi}{15} \text{ units}^3.
\end{aligned}
$$

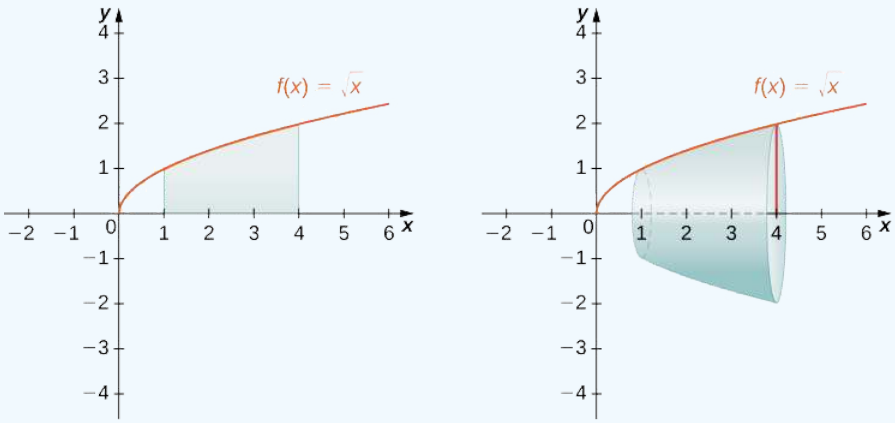
Let's look at some examples.

### Example 2.3.3: Using the Disk Method to Find the Volume of a Solid of Revolution 1

Use the disk method to find the volume of the solid of revolution generated by rotating the region between the graph of $f(x) = \sqrt{x}$ and the $x$-axis over the interval $[1, 4]$ around the $x$-axis.

**Solution**

The graphs of the function and the solid of revolution are shown in the following figure.




(a) (b)

Figure 2.3.9: (a) The function $f(x) = \sqrt{x}$ over the interval $[1, 4]$. (b) The solid of revolution obtained by revolving the region under the graph of $f(x)$ about the $x$-axis.

We have

$$
\begin{aligned}
V &= \int_a^b \pi \left[f(x)\right]^2 dx \\
&= \int_1^4 \pi \left[\sqrt{x}\right]^2 dx = \pi \int_1^4 x \, dx \\
&= \frac{\pi}{2} x^2 \Big|_1^4 = \frac{15\pi}{2}
\end{aligned}
$$

The volume is $(15\pi)/2 \text{ units}^3$.

### Exercise 2.3.3

Use the disk method to find the volume of the solid of revolution generated by rotating the region between the graph of $f(x) = \sqrt{4-x}$ and the $x$-axis over the interval $[0, 4]$ around the $x$-axis.

**Hint**

Use the procedure from Example 2.3.3.

**Answer**

$8\pi$ units$^3$

So far, our examples have all concerned regions revolved around the $x$-**axis**, but we can generate a solid of revolution by revolving a plane region around any horizontal or vertical line. In the next example, we look at a solid of revolution that has been generated by revolving a region around the $y$-**axis**. The mechanics of the disk method are nearly the same as when the $x$-**axis** is the axis of revolution, but we express the function in terms of $y$ and we integrate with respect to y as well. This is summarized in the following rule.

### Rule: The Disk Method for Solids of Revolution around the $y$-axis

Let $g(y)$ be continuous and nonnegative. Define $Q$ as the region bounded on the right by the graph of $g(y)$, on the left by the $y$-**axis**, below by the line $y = c$, and above by the line $y = d$. Then, the volume of the solid of revolution formed by revolving $Q$ around the $y$-**axis** is given by

$$V = \int_c^d \pi [g(y)]^2 \, dy. \tag{2.3.13}$$

The next example shows how this rule works in practice.

### Example 2.3.4: Using the Disk Method to Find the Volume of a Solid of Revolution 2

Let $R$ be the region bounded by the graph of $g(y) = \sqrt{4 - y}$ and the $y$-**axis** over the $y$-**axis** interval $[0, 4]$. Use the disk method to find the volume of the solid of revolution generated by rotating $R$ around the $y$-**axis.**

**Solution**

Figure 2.3.10 shows the function and a representative disk that can be used to estimate the volume. Notice that since we are revolving the function around the $y$-**axis,** the disks are horizontal, rather than vertical.

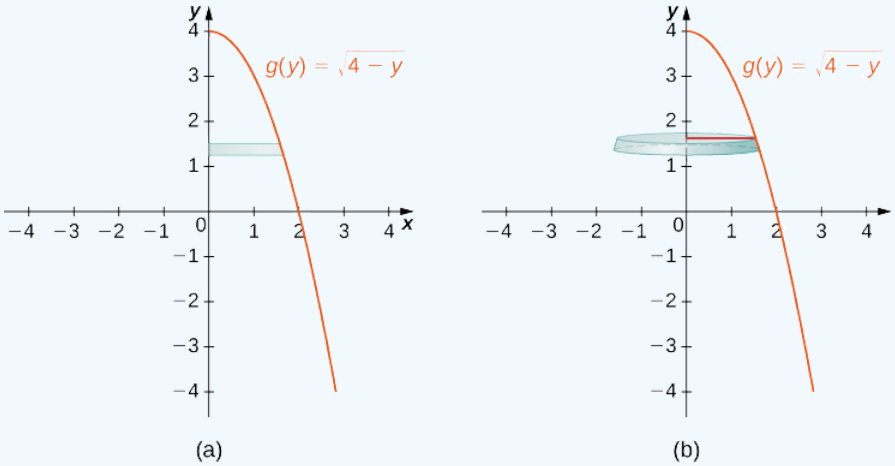



Figure 2.3.10: (a) Shown is a thin rectangle between the curve of the function $g(y) = \sqrt{4 - y}$ and the $y$-**axis.** (b) The rectangle forms a representative disk after revolution around the $y$-**axis.**

The region to be revolved and the full solid of revolution are depicted in the following figure.


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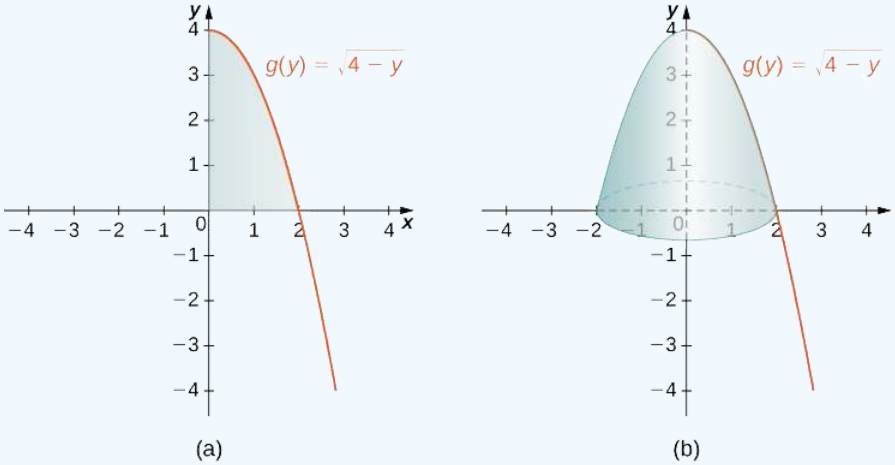



Figure 2.3.11: (a) The region to the left of the function $g(y) = \sqrt{4-y}$ over the $y$-axis interval $[0, 4]$. (b) The solid of revolution formed by revolving the region about the $y$-axis.

To find the volume, we integrate with respect to $y$. We obtain

$$V = \int_c^d \pi [g(y)]^2 \, dy = \int_0^4 \pi \left[\sqrt{4-y}\right]^2 \, dy \qquad (2.3.14)$$
$$= \pi \int_0^4 (4-y) \, dy = \pi \left[4y - \frac{y^2}{2}\right]\Big|_0^4 = 8\pi.$$

The volume is $8\pi$ units$^3$.

**Exercise 2.3.4**

Use the disk method to find the volume of the solid of revolution generated by rotating the region between the graph of $g(y) = y$ and the $y$-axis over the interval $[1, 4]$ around the $y$-axis.

**Hint**

Use the procedure from Example 2.3.4.

**Answer**

$21\pi$ units$^3$

## The Washer Method

Some solids of revolution have cavities in the middle; they are not solid all the way to the axis of revolution. Sometimes, this is just a result of the way the region of revolution is shaped with respect to the axis of revolution. In other cases, cavities arise when the region of revolution is defined as the region between the graphs of two functions. A third way this can happen is when an axis of revolution other than the $x$-axis or $y$-axis is selected.

When the solid of revolution has a cavity in the middle, the slices used to approximate the volume are not disks, but washers (disks with holes in the center). For example, consider the region bounded above by the graph of the function $f(x) = \sqrt{x}$ and below by the graph of the function $g(x) = 1$ over the interval $[1, 4]$. When this region is revolved around the $x$-axis, the result is a solid with a cavity in the middle, and the slices are washers. The graph of the function and a representative washer are shown in Figure 2.3.12 (a) and (b). The region of revolution and the resulting solid are shown in Figure 2.3.12 (c) and (d).

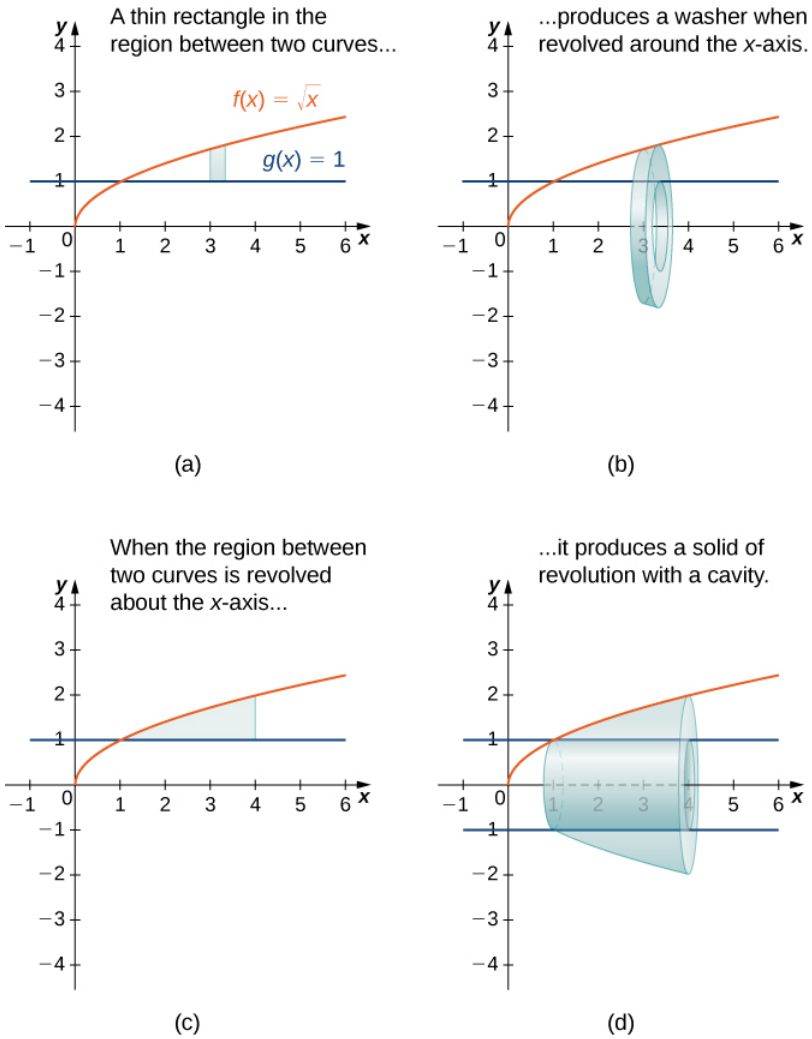



Figure **2.3.12**: (a) A thin rectangle in the region between two curves. (b) A representative disk formed by revolving the rectangle about the $x$**-axis.** (c) The region between the curves over the given interval. (d) The resulting solid of revolution.


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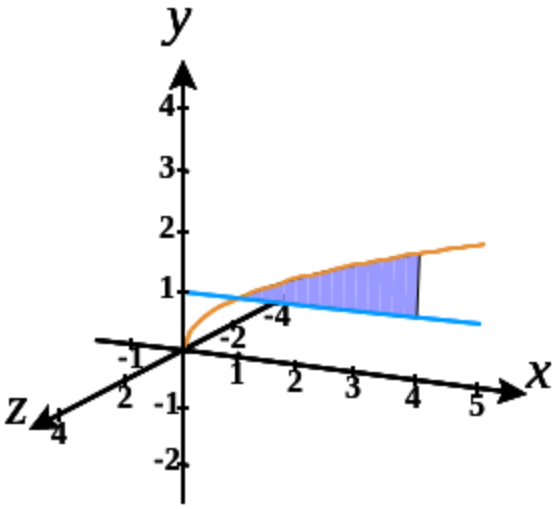



Figure 2.3.12: (e) A dynamic version of this solid of revolution generated using CalcPlot3D.

The cross-sectional area, then, is the area of the outer circle less the area of the inner circle. In this case,

$$A(x) = \pi\left(\sqrt{x}\right)^2 - \pi(1)^2 = \pi(x-1).$$

Then the volume of the solid is

$$V = \int_a^b A(x)\,dx = \int_1^4 \pi(x-1)\,dx = \pi\left[\frac{x^2}{2} - x\right]\Big|_1^4 = \frac{9}{2}\pi \text{ units}^3. \tag{2.3.15}$$

Generalizing this process gives the **washer method.**

**Rule: The Washer Method**

Suppose $f(x)$ and $g(x)$ are continuous, nonnegative functions such that $f(x) \geq g(x)$ over $[a,b]$. Let $R$ denote the region bounded above by the graph of $f(x)$, below by the graph of $g(x)$, on the left by the line $x = a$, and on the right by the line $x = b$. Then, the volume of the solid of revolution formed by revolving $R$ around the $x$-**axis** is given by

$$V = \int_a^b \pi\left[(f(x))^2 - (g(x))^2\right]\,dx. \tag{2.3.16}$$

## Example 2.3.5: Using the Washer Method

Find the volume of a solid of revolution formed by revolving the region bounded above by the graph of $f(x) = x$ and below by the graph of $g(x) = 1/x$ over the interval $[1, 4]$ around the $x$-axis.

**Solution**

The graphs of the functions and the solid of revolution are shown in the following figure.

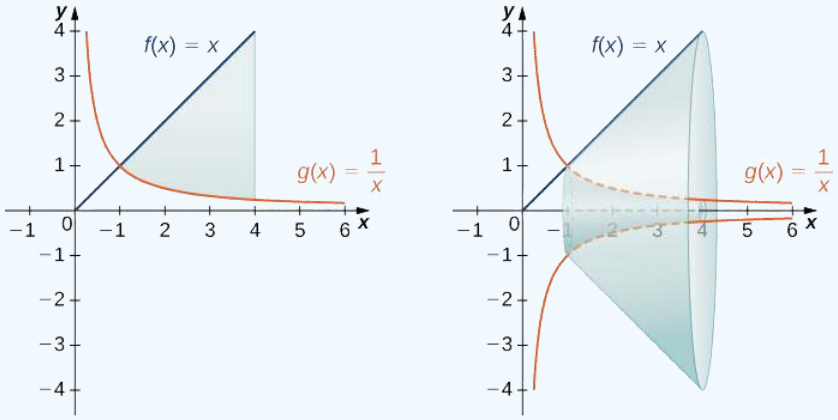



Figure 2.3.13: (a) The region between the graphs of the functions $f(x) = x$ and $g(x) = 1/x$ over the interval $[1, 4]$. (b) Revolving the region about the $x$-axis generates a solid of revolution with a cavity in the middle.

We have

$$V = \int_a^b \pi[(f(x))^2 - (g(x))^2]\,dx = \pi \int_1^4 \left[x^2 - \left(\frac{1}{x}\right)^2\right] dx$$
$$= \pi \left[\frac{x^3}{3} + \frac{1}{x}\right]\Bigg|_1^4$$
$$= \frac{81\pi}{4} \text{ units}^3.$$

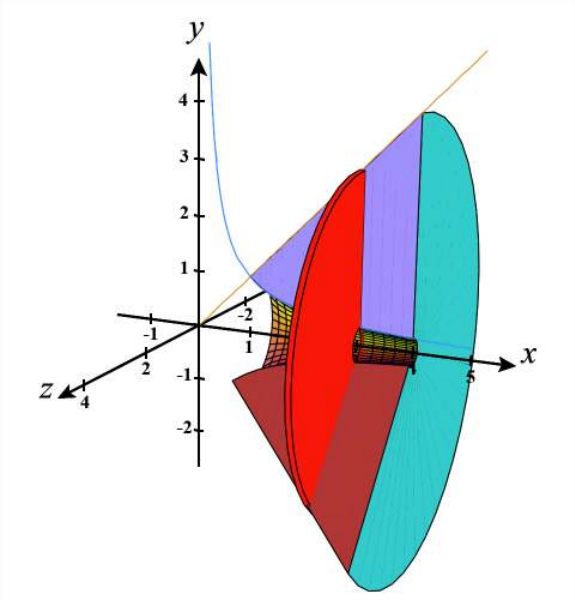



Figure 2.3.13: (c) A dynamic version of this solid of revolution generated using CalcPlot3D.



**Exercise 2.3.5**

Find the volume of a solid of revolution formed by revolving the region bounded by the graphs of $f(x) = \sqrt{x}$ and $g(x) = 1/x$ over the interval $[1, 3]$ around the $x$-**axis.**

**Hint**

Graph the functions to determine which graph forms the upper bound and which graph forms the lower bound, then use the procedure from Example **2.3.5**.

**Answer**

$\frac{10\pi}{3}$ **units**$^3$

As with the disk method, we can also apply the washer method to solids of revolution that result from revolving a region around the $y$-axis. In this case, the following rule applies.

**Rule: The Washer Method for Solids of Revolution around the $y$-axis**

Suppose $u(y)$ and $v(y)$ are continuous, nonnegative functions such that $v(y) \leq u(y)$ for $y \in [c, d]$. Let $Q$ denote the region bounded on the right by the graph of $u(y)$, on the left by the graph of $v(y)$, below by the line $y = c$, and above by the line $y = d$. Then, the volume of the solid of revolution formed by revolving $Q$ around the $y$-**axis** is given by

$$V = \int_c^d \pi \left[ (u(y))^2 - (v(y))^2 \right] dy. \tag{2.3.17}$$

Rather than looking at an example of the washer method with the $y$-**axis** as the axis of revolution, we now consider an example in which the axis of revolution is a line other than one of the two coordinate axes. The same general method applies, but you may have to visualize just how to describe the cross-sectional area of the volume.

**Example 2.3.6**:

Find the volume of a solid of revolution formed by revolving the region bounded above by $f(x) = 4 - x$ and below by the $x$-**axis** over the interval $[0, 4]$ around the line $y = -2$.

**Solution**

The graph of the region and the solid of revolution are shown in the following figure.

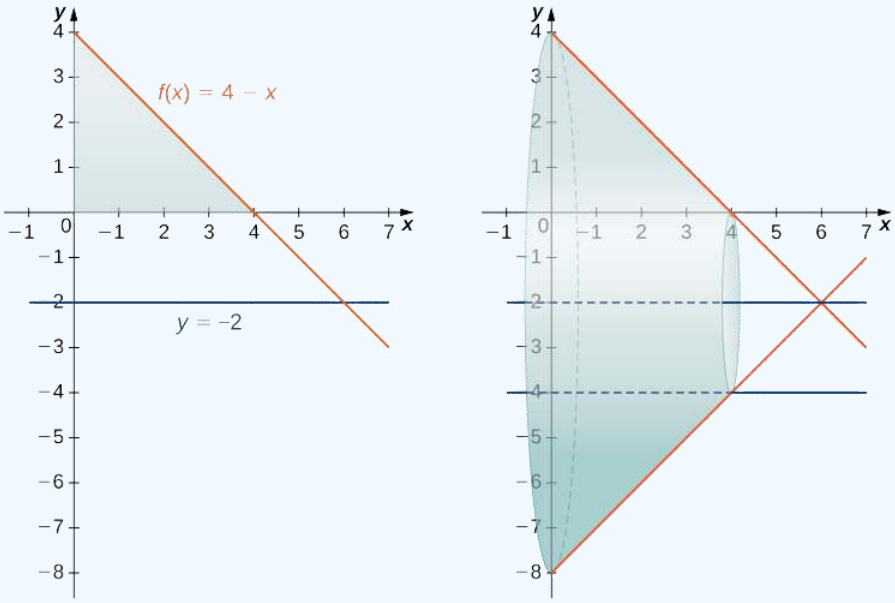



Figure **2.3.14**: (a) The region between the graph of the function $f(x) = 4 - x$ and the $x$-**axis** over the interval $[0, 4]$. (b) Revolving the region about the line $y = -2$ generates a solid of revolution with a cylindrical hole through its middle.

We can't apply the volume formula to this problem directly because the axis of revolution is not one of the coordinate axes. However, we still know that the area of the cross-section is the area of the outer circle less the area of the inner circle. Looking at the graph of the function, we see the radius of the outer circle is given by $f(x) + 2$, which simplifies to

$$f(x) + 2 = (4 - x) + 2 = 6 - x.$$

The radius of the inner circle is $g(x) = 2$. Therefore, we have

$$\begin{aligned} V &= \int_0^4 \pi \left[(6 - x)^2 - (2)^2\right] \, dx \\ &= \pi \int_0^4 (x^2 - 12x + 32) \, dx = \pi \left[\frac{x^3}{3} - 6x^2 + 32x\right]\Big|_0^4 \\ &= \frac{160\pi}{3} \text{ units}^3. \end{aligned}$$

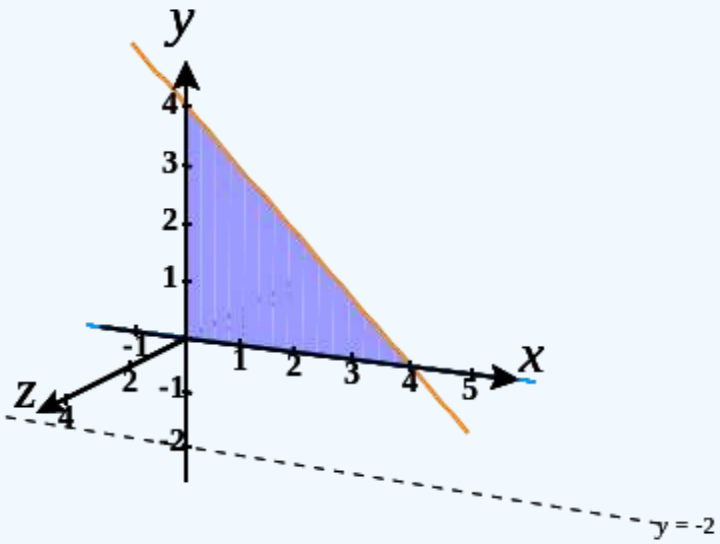
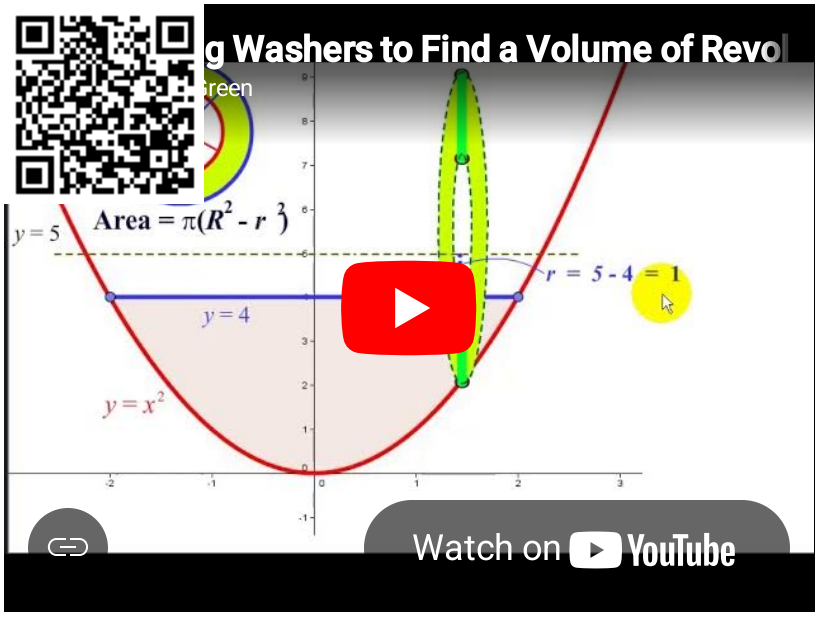



Figure 2.3.14: (c) A dynamic version of this solid of revolution generated using CalcPlot3D.





**Exercise 2.3.6**

Find the volume of a solid of revolution formed by revolving the region bounded above by the graph of $f(x) = x + 2$ and below by the $x$-axis over the interval $[0, 3]$ around the line $y = -1$.

**Hint**

Use the procedure from Example 2.3.6.

**Answer**

$60\pi$ units$^3$

## Key Concepts

- Definite integrals can be used to find the volumes of solids. Using the slicing method, we can find a volume by integrating the cross-sectional area.
- For solids of revolution, the volume slices are often disks and the cross-sections are circles. The method of disks involves applying the method of slicing in the particular case in which the cross-sections are circles, and using the formula for the area of a circle.
- If a solid of revolution has a cavity in the center, the volume slices are washers. With the method of washers, the area of the inner circle is subtracted from the area of the outer circle before integrating.

## Key Equations

- **Disk Method along the $x$-axis**

$$V = \int_a^b \pi [f(x)]^2 \, dx$$

- **Disk Method along the $y$-axis**

$$V = \int_c^d \pi [g(y)]^2 \, dy$$

- **Washer Method**

$$V = \int_a^b \pi \left[ (f(x))^2 - (g(x))^2 \right] \, dx$$

## Glossary

**cross-section**

the intersection of a plane and a solid object

**disk method**

a special case of the slicing method used with solids of revolution when the slices are disks

**slicing method**

a method of calculating the volume of a solid that involves cutting the solid into pieces, estimating the volume of each piece, then adding these estimates to arrive at an estimate of the total volume; as the number of slices goes to infinity, this estimate becomes an integral that gives the exact value of the volume

**solid of revolution**

a solid generated by revolving a region in a plane around a line in that plane

**washer method**

a special case of the slicing method used with solids of revolution when the slices are washers

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# 2.3E: Exercises for Section 2.3

1) Derive the formula for the volume of a sphere using the slicing method.

2) Use the slicing method to derive the formula for the volume of a cone.

3) Use the slicing method to derive the formula for the volume of a tetrahedron with side length $a$.

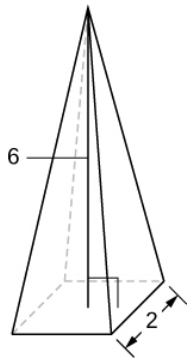
4) Use the disk method to derive the formula for the volume of a trapezoidal cylinder.

5) Explain when you would use the disk method versus the washer method. When are they interchangeable?

## Volumes by Slicing

**For exercises 6 - 10, draw a typical slice and find the volume using the slicing method for the given volume.**

6) A pyramid with height 6 units and square base of side 2 units, as pictured here.



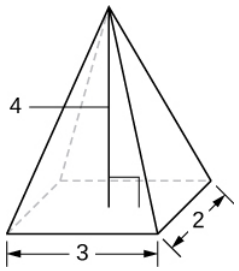

**Solution:**

Here the cross-sections are squares taken perpendicular to the $y$-axis.
We use the vertical cross-section of the pyramid through its center to obtain an equation relating $x$ and $y$.
Here this would be the equation, $y = 6 - 6x$. Since we need the dimensions of the square at each $y$-level, we solve this equation for $x$ to get, $x = 1 - \frac{y}{6}$.
This is half the distance across the square cross-section at the $y$-level, so the side length of the square cross-section is, $s = 2\left(1 - \frac{y}{6}\right)$.
Thus, we have the area of a cross-section is,

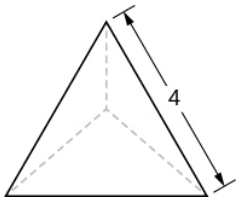
$$A(y) = \left[2\left(1-\tfrac{y}{6}\right)\right]^2 = 4\left(1-\tfrac{y}{6}\right)^2.$$

$$\begin{aligned}
\text{Then,}\quad V &= \int_0^6 4\left(1-\tfrac{y}{6}\right)^2\,dy \\
&= -24\int_1^0 u^2\,du, \quad \text{where } u = 1-\tfrac{y}{6},\ \text{so } du = -\tfrac{1}{6}\,dy, \quad \Longrightarrow \quad -6\,du = dy \\
&= 24\int_0^1 u^2\,du = 24\frac{u^3}{3}\Big|_0^1 \\
&= 8u^3\Big|_0^1 \\
&= 8\left(1^3 - 0^3\right) \quad = \quad 8\,\text{units}^3
\end{aligned}$$

7) A pyramid with height 4 units and a rectangular base with length 2 units and width 3 units, as pictured here.



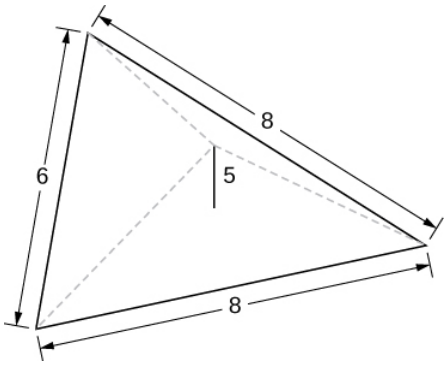

8) A tetrahedron with a base side of 4 units,as seen here.



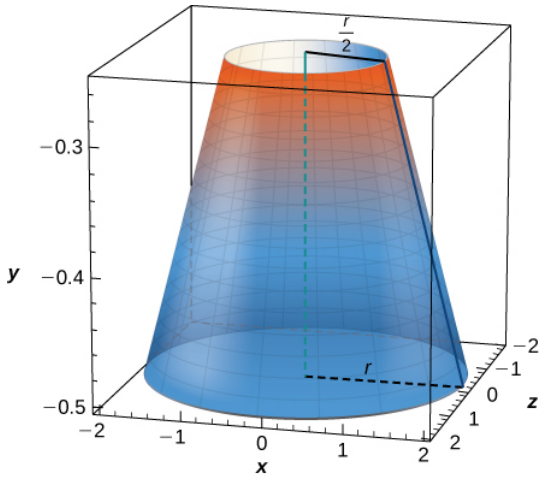

**Answer**

$V = \frac{32}{3\sqrt{2}} = \frac{16\sqrt{2}}{3}$ units$^3$

9) A pyramid with height 5 units, and an isosceles triangular base with lengths of 6 units and 8 units, as seen here.




10) A cone of radius $r$ and height $h$ has a smaller cone of radius $r/2$ and height $h/2$ removed from the top, as seen here. The resulting solid is called a frustum.




**Answer**

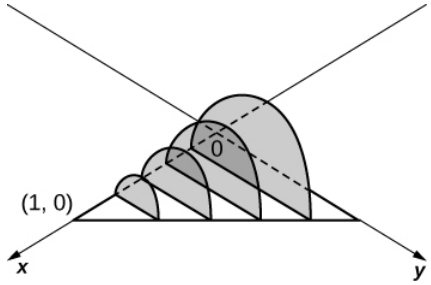
$V = \frac{7\pi}{12} hr^2$ units$^3$

**For exercises 11 - 16, draw an outline of the solid and find the volume using the slicing method.**

11) The base is a circle of radius $a$. The slices perpendicular to the base are squares.

12) The base is a triangle with vertices $(0,0), (1,0)$, and $(0,1)$. Slices perpendicular to the $xy$-plane are semicircles.

**Answer**

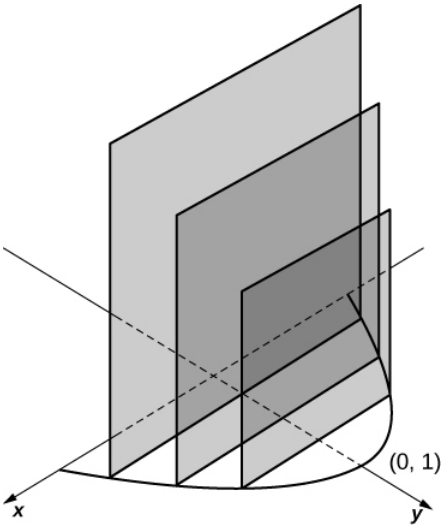



$$V = \int_0^1 \frac{\pi(1-x)^2}{8}\,dx \quad = \quad \frac{\pi}{24} \text{ units}^3$$

13) The base is the region under the parabola $y = 1 - x^2$ in the first quadrant. Slices perpendicular to the $xy$-plane are squares.

14) The base is the region under the parabola $y = 1 - x^2$ and above the **$x$-axis**. Slices perpendicular to the **$y$-axis** are squares.

**Answer**

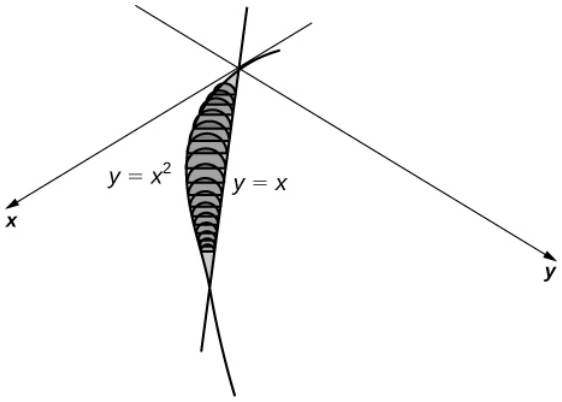



$$V = \int_0^1 4(1-y)\,dy \quad = \quad 2 \text{ units}^3$$

15) The base is the region enclosed by $y = x^2$ and $y = 9$. Slices perpendicular to the $x$-axis are right isosceles triangles.

16) The base is the area between $y = x$ and $y = x^2$. Slices perpendicular to the $x$-axis are semicircles.

**Answer**




$$V = \int_0^1 \frac{\pi}{8}(x - x^2)^2 \, dx \quad = \quad \frac{\pi}{240} \text{ units}^3$$

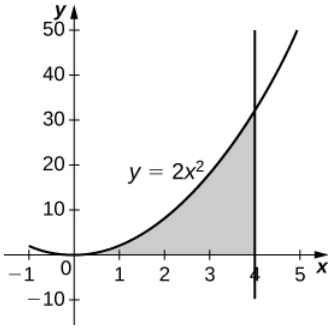
## Disk and Washer Method

**For exercises 17 - 24, draw the region bounded by the curves. Then, use the disk or washer method to find the volume when the region is rotated around the $x$-axis.**

17) $x + y = 8$, $x = 0$, and $y = 0$

18) $y = 2x^2$, $x = 0$, $x = 4$, and $y = 0$

**Answer**

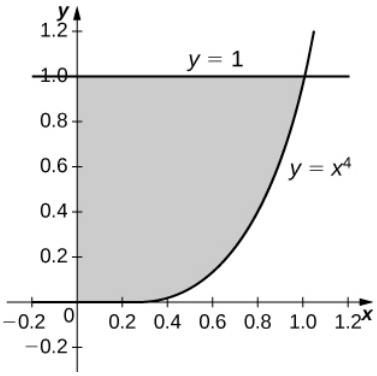



$$V = \int_0^4 4\pi x^4 \, dx \quad = \quad \frac{4096\pi}{5} \text{ units}^3$$

19) $y = e^x + 1$, $x = 0$, $x = 1$, and $y = 0$

20) $y = x^4$, $x = 0$, and $y = 1$

**Answer**





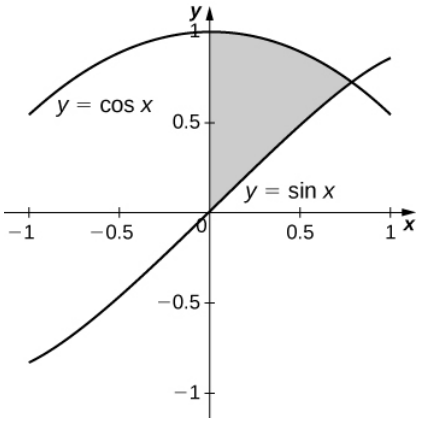
https://math.libretexts.org/@go/page/83609

$$V = \int_0^1 \pi\left(1^2 - \left(x^4\right)^2\right)\,dx = \int_0^1 \pi\left(1 - x^8\right)\,dx \quad = \quad \frac{8\pi}{9} \text{ units}^3$$

21) $y = \sqrt{x}, \quad x = 0, \quad x = 4$, and $y = 0$

22) $y = \sin x, \quad y = \cos x$, and $x = 0$

**Answer**

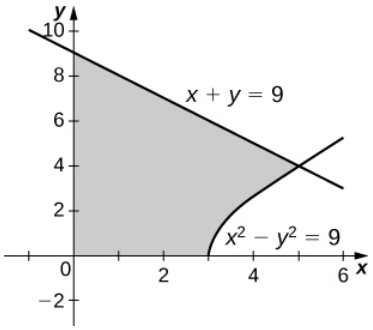



$$V = \int_0^{\pi/4} \pi\left(\cos^2 x - \sin^2 x\right)\,dx = \int_0^{\pi/4} \pi \cos 2x\,dx \quad = \quad \frac{\pi}{2} \text{ units}^3$$

23) $y = \dfrac{1}{x}, \quad x = 2$, and $y = 3$

24) $x^2 - y^2 = 9$ and $x + y = 9, \quad y = 0$ and $x = 0$

**Answer**



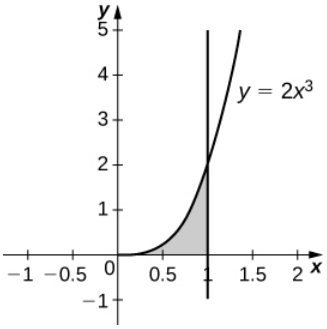

$V = 207\pi$ units$^3$

**For exercises 25 - 32, draw the region bounded by the curves. Then, find the volume when the region is rotated around the $y$ -axis.**

25) $y = 4 - \dfrac{1}{2}x, \quad x = 0$, and $y = 0$

26) $y = 2x^3, \quad x = 0, \quad x = 1$, and $y = 0$

**Answer**

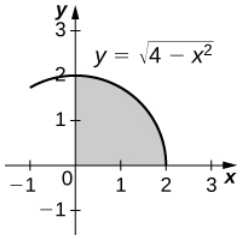



$V = \frac{4\pi}{5}$ units$^3$

27) $y = 3x^2$, $x = 0$, and $y = 3$

28) $y = \sqrt{4 - x^2}$, $y = 0$, and $x = 0$

**Answer**

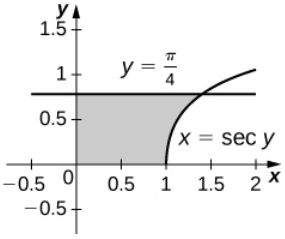



$V = \frac{16\pi}{3}$ units$^3$

29) $y = \dfrac{1}{\sqrt{x+1}}$, $x = 0$, and $x = 3$

30) $x = \sec(y)$ and $y = \dfrac{\pi}{4}$, $y = 0$ and $x = 0$

**Answer**

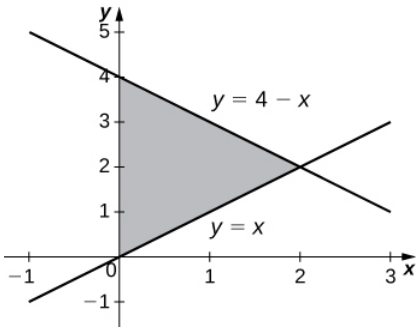



$V = \pi$ units$^3$

31) $y = \dfrac{1}{x+1}$, $x = 0$, and $x = 2$

32) $y = 4 - x$, $y = x$, and $x = 0$

**Answer**



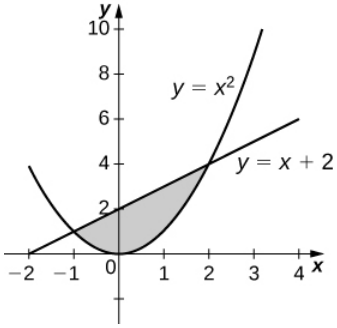

$V = \frac{16\pi}{3}$ units$^3$

**For exercises 33 - 40, draw the region bounded by the curves. Then, find the volume when the region is rotated around the $x$ -axis.**

33) $y = x + 2, \quad y = x + 6, \quad x = 0$, and $x = 5$

34) $y = x^2$ and $y = x + 2$

**Answer**

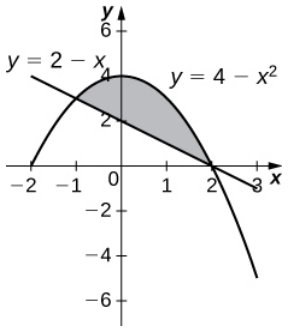



$V = \frac{72\pi}{5}$ units$^3$

35) $x^2 = y^3$ and $x^3 = y^2$

36) $y = 4 - x^2$ and $y = 2 - x$

**Answer**

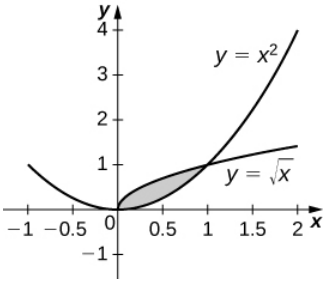



$V = \frac{108\pi}{5}$ units$^3$

37) [T] $y = \cos x, \quad y = e^{-x}, \quad x = 0$, and $x = 1.2927$

38) $y = \sqrt{x}$ and $y = x^2$

**Answer**

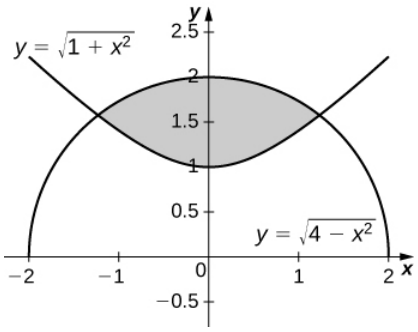



$V = \frac{3\pi}{10}$ units$^3$

39) $y = \sin x, \quad y = 5\sin x, \quad x = 0$ and $x = \pi$

40) $y = \sqrt{1 + x^2}$ and $y = \sqrt{4 - x^2}$

**Answer**



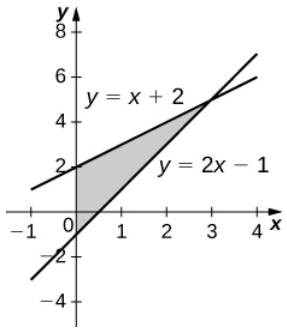

$V = 2\sqrt{6}\pi$ units$^3$

**For exercises 41 - 45, draw the region bounded by the curves. Then, use the washer method to find the volume when the region is revolved around the $y$-axis.**

41) $y = \sqrt{x}, \quad x = 4$, and $y = 0$

42) $y = x + 2, \quad y = 2x - 1$, and $x = 0$

**Answer**

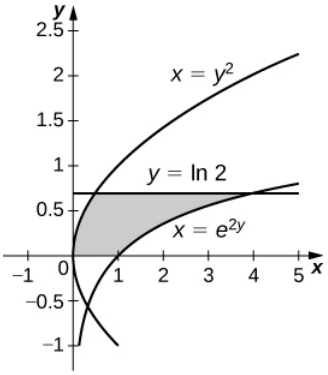



$V = 9\pi$ units$^3$

43) $y = \frac{3}{x}$ and $y = x^3$

44) $x = e^{2y}, \quad x = y^2, \quad y = 0$, and $y = \ln(2)$

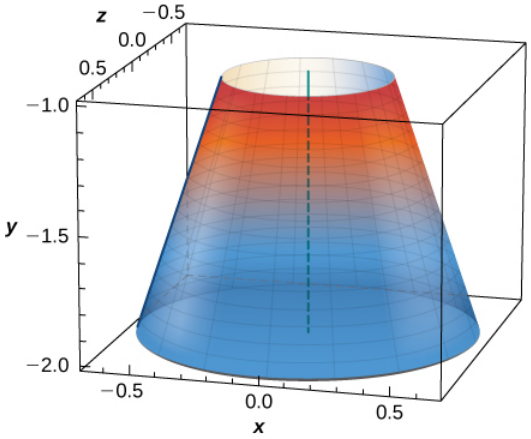
**Answer**




$V = \frac{\pi}{20}(75 - 4\ln^5(2))$ units$^3$

45) $x = \sqrt{9 - y^2}$, $x = e^{-y}$, $y = 0$, and $y = 3$

46) Yogurt containers can be shaped like frustums. Rotate the line $y = \left(\frac{1}{m}\right)x$ around the $y$-axis to find the volume between $y = a$ and $y = b$.



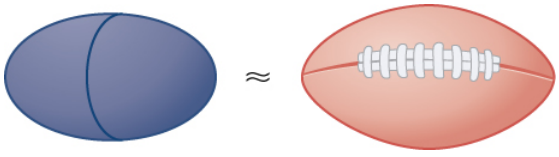




**Answer**

$V = \frac{m^2\pi}{3}(b^3 - a^3)$ units$^3$

47) Rotate the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ around the $x$-axis to approximate the volume of a football, as seen here.

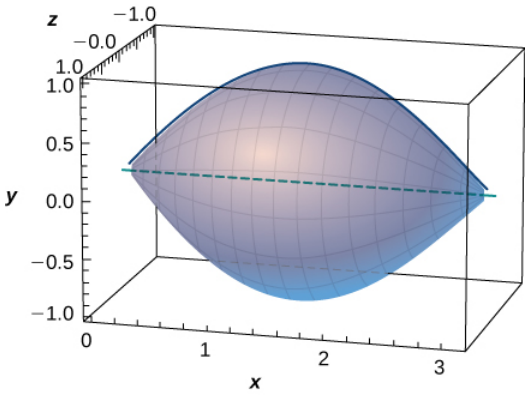


48) Rotate the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ around the $y$-axis to approximate the volume of a football.

**Answer**

$V=\frac{4a^2b\pi}{3}$ units$^3$

49) A better approximation of the volume of a football is given by the solid that comes from rotating $y=\sin x$ around the $x$-axis from $x=0$ to $x=\pi$. What is the volume of this football approximation, as seen here?




**For exercises 51 - 56, find the volume of the solid described.**

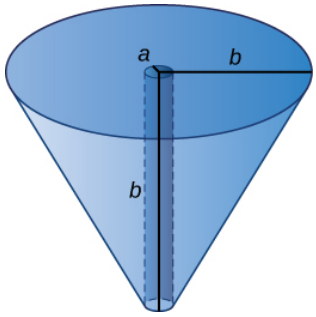
51) The base is the region between $y=x$ and $y=x^2$. Slices perpendicular to the $x$-axis are semicircles.

52) The base is the region enclosed by the generic ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Slices perpendicular to the $x$-axis are semicircles.

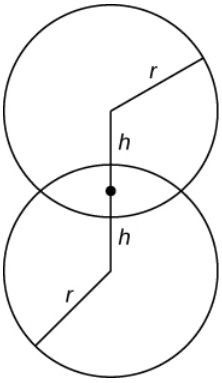
**Answer**

$V=\frac{2ab^2\pi}{3}$ units$^3$

53) Bore a hole of radius a down the axis of $a$ right cone and through the base of radius $b$, as seen here.




54) Find the volume common to two spheres of radius $r$ with centers that are $2h$ apart, as shown here.



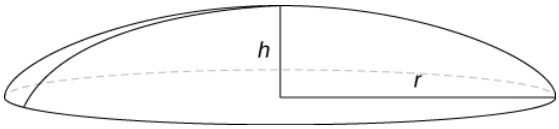


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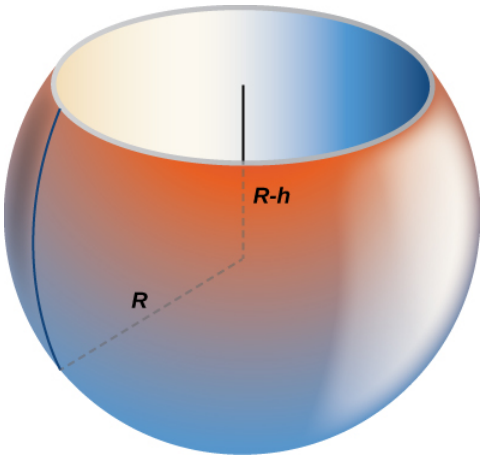
**Answer**

$V = \frac{\pi}{12}(r+h)^2(6r-h)$ units$^3$

55) Find the volume of a spherical cap of height $h$ and radius $r$ where $h < r$, as seen here.




56) Find the volume of a sphere of radius $R$ with a cap of height $h$ removed from the top, as seen here.




**Answer**

$V = \frac{\pi}{3}(h+R)(h-2R)^2$ units$^3$

## Contributors

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# 2.4: Volumes of Revolution - Cylindrical Shells

**Learning Objectives**

- Calculate the volume of a solid of revolution by using the method of cylindrical shells.
- Compare the different methods for calculating a volume of revolution.

In this section, we examine the method of cylindrical shells, the final method for finding the volume of a solid of revolution. We can use this method on the same kinds of solids as the disk method or the washer method; however, with the disk and washer methods, we integrate along the coordinate axis parallel to the axis of revolution. With the method of cylindrical shells, we integrate along the coordinate *axis perpendicular* to the axis of revolution. The ability to choose which variable of integration we want to use can be a significant advantage with more complicated functions. Also, the specific geometry of the solid sometimes makes the method of using cylindrical shells more appealing than using the washer method. In the last part of this section, we review all the methods for finding volume that we have studied and lay out some guidelines to help you determine which method to use in a given situation.

## The Method of Cylindrical Shells

Again, we are working with a solid of revolution. As before, we define a region $R$, bounded above by the graph of a function $y = f(x)$, below by the $x$-**axis**, and on the left and right by the lines $x = a$ and $x = b$, respectively, as shown in Figure **2.4.1***a*. We then revolve this region around the $y$-axis, as shown in Figure **2.4.1***b*. Note that this is different from what we have done before. Previously, regions defined in terms of functions of $x$ were revolved around the $x$-**axis** or a line parallel to it.

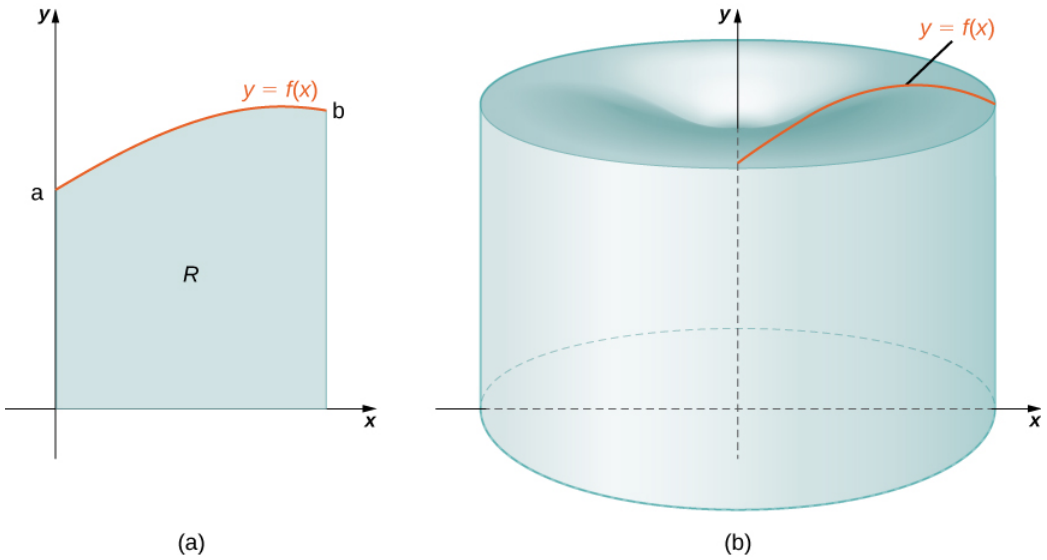



Figure **2.4.1**: (a) A region bounded by the graph of a function of $x$. (b) The solid of revolution formed when the region is revolved around the $y$-**axis.**

As we have done many times before, partition the interval $[a, b]$ using a regular partition, $P = x_0, x_1, \ldots, x_n$ and, for $i = 1, 2, \ldots, n$, choose a point $x_i^* \in [x_{i-1}, x_i]$. Then, construct a rectangle over the interval $[x_{i-1}, x_i]$ of height $f(x_i^*)$ and width $\Delta x$. A representative rectangle is shown in Figure **2.4.2***a*. When that rectangle is revolved around the $y$-axis, instead of a disk or a washer, we get a cylindrical shell, as shown in Figure **2.4.2**.

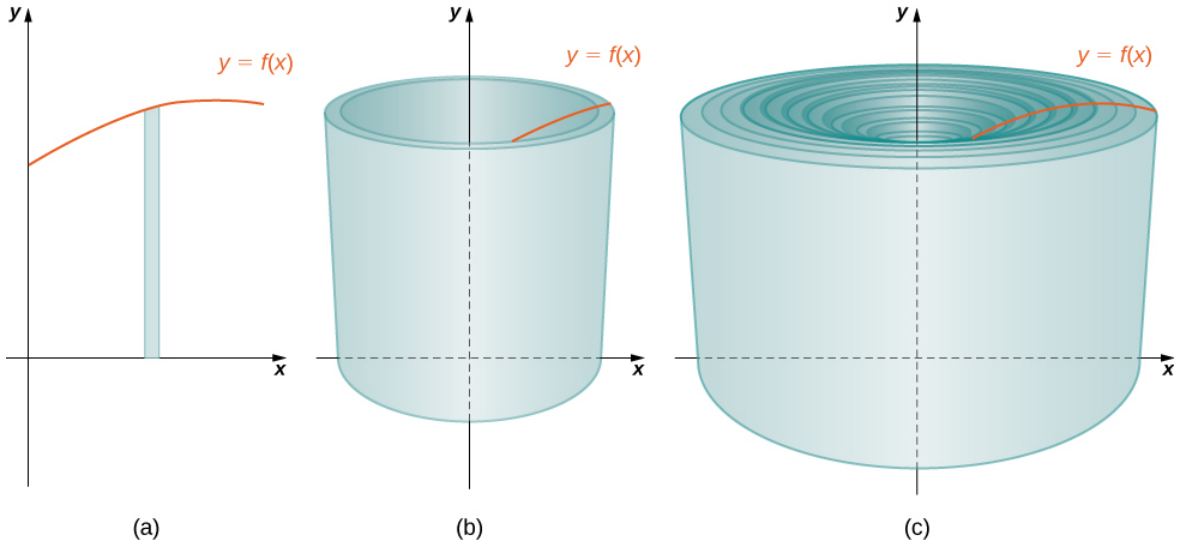



Figure 2.4.2: (a) A representative rectangle. (b) When this rectangle is revolved around the $y$-**axis**, the result is a cylindrical shell. (c) When we put all the shells together, we get an approximation of the original solid.

To calculate the volume of this shell, consider Figure 2.4.3.

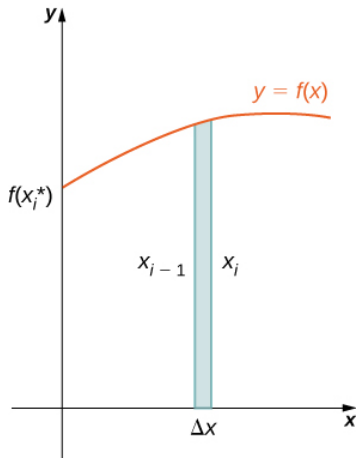



Figure 2.4.3: Calculating the volume of the shell.

The shell is a cylinder, so its volume is the cross-sectional area multiplied by the height of the cylinder. The cross-sections are annuli (ring-shaped regions—essentially, circles with a hole in the center), with outer radius $x_i$ and inner radius $x_{i-1}$. Thus, the cross-sectional area is $\pi x_i^2 - \pi x_{i-1}^2$. The height of the cylinder is $f(x_i^*)$. Then the volume of the shell is

$$\begin{aligned} V_{shell} &= f(x_i^*)(\pi x_i^2 - \pi x_{i-1}^2) \\ &= \pi f(x_i^*)(x_i^2 - x_{i-1}^2) \\ &= \pi f(x_i^*)(x_i + x_{i-1})(x_i - x_{i-1}) \\ &= 2\pi f(x_i^*)\left(\frac{x_i + x_{i-1}}{2}\right)(x_i - x_{i-1}). \end{aligned}$$

Note that $x_i - x_{i-1} = \Delta x$, so we have

$$V_{shell} = 2\pi f(x_i^*)\left(\frac{x_i + x_{i-1}}{2}\right)\Delta x. \tag{2.4.1}$$

Furthermore, $\frac{x_i + x_{i-1}}{2}$ is both the midpoint of the interval $[x_{i-1}, x_i]$ and the average radius of the shell, and we can approximate this by $x_i^*$. We then have

$$V_{shell} \approx 2\pi f(x_i^*)x_i^* \Delta x. \tag{2.4.2}$$

Another way to think of this is to think of making a vertical cut in the shell and then opening it up to form a flat plate (Figure 2.4.4).

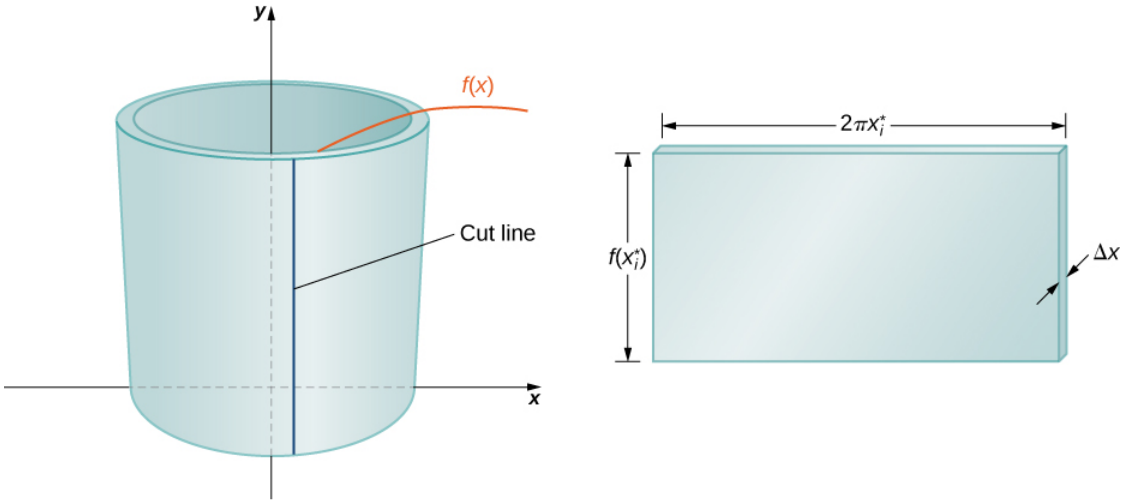



Figure 2.4.4: (a) Make a vertical cut in a representative shell. (b) Open the shell up to form a flat plate.

In reality, the outer radius of the shell is greater than the inner radius, and hence the back edge of the plate would be slightly longer than the front edge of the plate. However, we can approximate the flattened shell by a flat plate of height $f(x_i^*)$, width $2\pi x_i^*$, and thickness $\Delta x$ (Figure). The volume of the shell, then, is approximately the volume of the flat plate. Multiplying the height, width, and depth of the plate, we get

$$V_{shell} \approx f(x_i^*)(2\pi x_i^*)\,\Delta x, \tag{2.4.3}$$

which is the same formula we had before.

To calculate the volume of the entire solid, we then add the volumes of all the shells and obtain

$$V \approx \sum_{i=1}^{n} (2\pi x_i^* f(x_i^*)\,\Delta x). \tag{2.4.4}$$

Here we have another Riemann sum, this time for the function $2\pi x\, f(x)$. Taking the limit as $n \to \infty$ gives us

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} (2\pi x_i^* f(x_i^*)\,\Delta x) = \int_a^b (2\pi x\, f(x))\,dx. \tag{2.4.5}$$

This leads to the following rule for the *method of cylindrical shells*.

**Rule: The Method of Cylindrical Shells**

Let $f(x)$ be continuous and nonnegative. Define $R$ as the region bounded above by the graph of $f(x)$, below by the **$x$-axis,** on the left by the line $x = a$, and on the right by the line $x = b$. Then the volume of the solid of revolution formed by revolving $R$ around the $y$-axis is given by

$$V = \int_a^b (2\pi x\, f(x))\,dx. \tag{2.4.6}$$

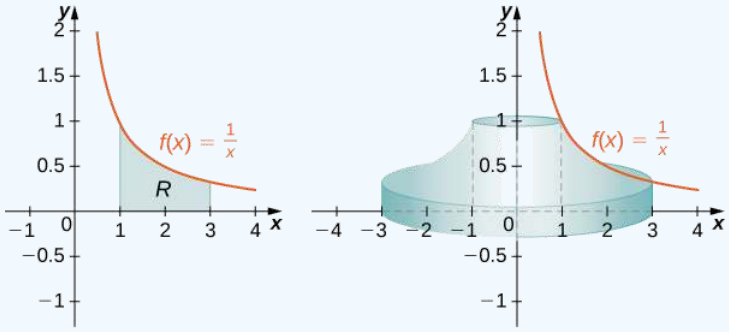
Now let's consider an example.

**Example 2.4.1: The Method of Cylindrical Shells I**

Define $R$ as the region bounded above by the graph of $f(x) = 1/x$ and below by the **$x$-axis** over the interval $[1, 3]$. Find the volume of the solid of revolution formed by revolving $R$ around the $y$-axis.

**Solution**

First we must graph the region $R$ and the associated solid of revolution, as shown in Figure 2.4.5.




(a) (b)

Figure 2.4.5: (a) The region $R$ under the graph of $f(x) = 1/x$ over the interval $[1, 3]$. (b) The solid of revolution generated by revolving $R$ about the $y$-axis.

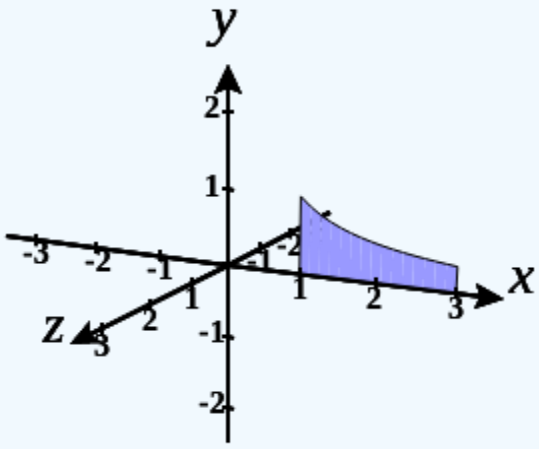



Figure 2.4.5 (c) Visualizing the solid of revolution with CalcPlot3D.

Then the volume of the solid is given by

$$V = \int_a^b (2\pi x f(x))\,dx$$
$$= \int_1^3 \left(2\pi x \left(\frac{1}{x}\right)\right) dx$$
$$= \int_1^3 2\pi\,dx$$
$$= 2\pi x\Big|_1^3 = 4\pi \text{ units}^3.$$



**Exercise 2.4.1**

Define R as the region bounded above by the graph of $f(x) = x^2$ and below by the $x$-axis over the interval $[1, 2]$. Find the volume of the solid of revolution formed by revolving $R$ around the $y$-axis.

**Hint**

Use the procedure from Example 2.4.1.

**Answer**

$\frac{15\pi}{2}$ units$^3$

**Example 2.4.2: The Method of Cylindrical Shells II**

Define $R$ as the region bounded above by the graph of $f(x) = 2x - x^2$ and below by the $x$-axis over the interval $[0, 2]$. Find the volume of the solid of revolution formed by revolving $R$ around the $y$-axis.

**Solution**

First graph the region $R$ and the associated solid of revolution, as shown in Figure 2.4.6.

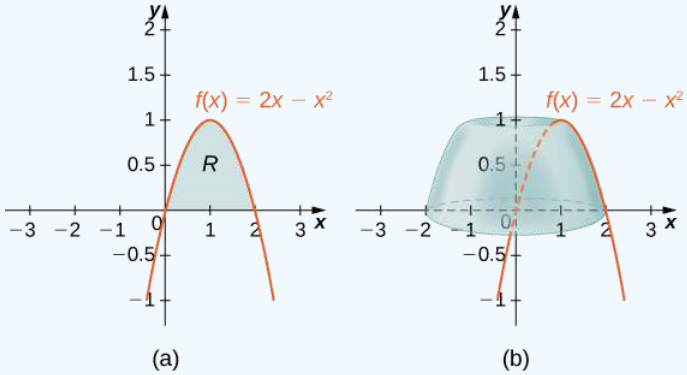



Figure 2.4.6: (a) The region $R$ *under the graph of* $f(x) = 2x - x^2$ *over the interval* $[0, 2]$. *(b) The volume of revolution obtained by revolving* $R$ *about the* $y$*-axis.*

Then the volume of the solid is given by

$$\begin{aligned} V &= \int_a^b (2\pi x f(x))\, dx \\ &= \int_0^2 (2\pi x(2x - x^2))\, dx \\ &= 2\pi \int_0^2 (2x^2 - x^3)\, dx \\ &= 2\pi \left[\frac{2x^3}{3} - \frac{x^4}{4}\right]\Big|_0^2 \\ &= \frac{8\pi}{3} \text{ units}^3 \end{aligned}$$

**Exercise 2.4.2**

Define $R$ as the region bounded above by the graph of $f(x) = 3x - x^2$ and below by the $x$-axis over the interval $[0, 2]$. Find the volume of the solid of revolution formed by revolving $R$ around the $y$-axis.

**Hint**

Use the process from Example 2.4.2.

**Answer**

$8\pi$ units$^3$

As with the disk method and the washer method, we can use the method of cylindrical shells with solids of revolution, revolved around the $x$-axis, when we want to integrate with respect to $y$. The analogous rule for this type of solid is given here.

**Rule: The Method of Cylindrical Shells for Solids of Revolution around the $x$-axis**

Let $g(y)$ be continuous and nonnegative. Define $Q$ as the region bounded on the right by the graph of $g(y)$, on the left by the $y$-axis, below by the line $y = c$, and above by the line $y = d$. Then, the volume of the solid of revolution formed by revolving $Q$ around the $x$-axis is given by

$$V = \int_c^d (2\pi y\, g(y))\, dy. \tag{2.4.7}$$

**Example 2.4.3: The Method of Cylindrical Shells for a Solid Revolved around the $x$-axis**

Define $Q$ as the region bounded on the right by the graph of $g(y) = 2\sqrt{y}$ and on the left by the $y$-axis for $y \in [0, 4]$. Find the volume of the solid of revolution formed by revolving $Q$ around the $x$-axis.

**Solution**

First, we need to graph the region $Q$ and the associated solid of revolution, as shown in Figure 2.4.7.

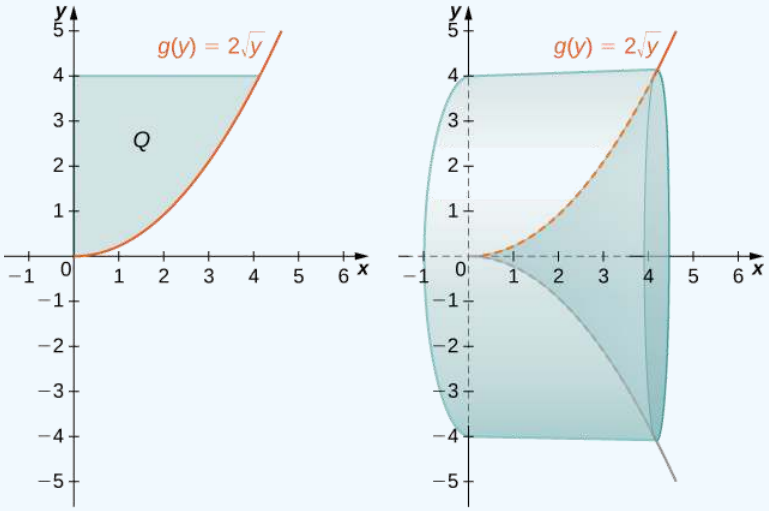



Figure 2.4.7: (a) The region $Q$ to the left of the function $g(y)$ over the interval $[0, 4]$. (b) The solid of revolution generated by revolving $Q$ around the $x$-axis.

Label the shaded region $Q$. Then the volume of the solid is given by

$$\begin{aligned} V &= \int_c^d (2\pi\, y\, g(y))\, dy \\ &= \int_0^4 (2\pi\, y(2\sqrt{y}))\, dy \\ &= 4\pi \int_0^4 y^{3/2}\, dy \\ &= 4\pi \left[\frac{2y^{5/2}}{5}\right] \Big|_0^4 \\ &= \frac{256\pi}{5} \text{ units}^3 \end{aligned}$$

**Exercise 2.4.3**

Define $Q$ as the region bounded on the right by the graph of $g(y) = 3/y$ and on the left by the $y$-axis for $y \in [1, 3]$. Find the volume of the solid of revolution formed by revolving $Q$ around the $x$-axis.

**Hint**

Use the process from Example 2.4.3.

**Answer**

$12\pi$ units$^3$

For the next example, we look at a solid of revolution for which the graph of a function is revolved around a line other than one of the two coordinate axes. To set this up, we need to revisit the development of the method of cylindrical shells. Recall that we found the volume of one of the shells to be given by

$$\begin{aligned} V_{shell} &= f(x_i^*)(\pi x_i^2 - \pi x_{i-1}^2) \\ &= \pi f(x_i^*)(x_i^2 - x_{i-1}^2) \\ &= \pi f(x_i^*)(x_i + x_{i-1})(x_i - x_{i-1}) \\ &= 2\pi f(x_i^*)\left(\frac{x_i + x_{i-1}}{2}\right)(x_i - x_{i-1}). \end{aligned}$$

This was based on a shell with an outer radius of $x_i$ and an inner radius of $x_{i-1}$. If, however, we rotate the region around a line other than the $y$-axis, we have a different outer and inner radius. Suppose, for example, that we rotate the region around the line $x = -k$, where $k$ is some positive constant. Then, the outer radius of the shell is $x_i + k$ and the inner radius of the shell is $x_{i-1} + k$. Substituting these terms into the expression for volume, we see that when a plane region is rotated around the line $x = -k$, the volume of a shell is given by

$$\begin{aligned} V_{shell} &= 2\pi f(x_i^*)\left(\frac{(x_i + k) + (x_{i-1} + k)}{2}\right)((x_i + k) - (x_{i-1} + k)) \\ &= 2\pi f(x_i^*)\left(\left(\frac{x_i + x_{i-2}}{2}\right) + k\right)\Delta x. \end{aligned}$$

As before, we notice that $\frac{x_i + x_{i-1}}{2}$ is the midpoint of the interval $[x_{i-1}, x_i]$ and can be approximated by $x_i^*$. Then, the approximate volume of the shell is

$$V_{shell} \approx 2\pi(x_i^* + k)f(x_i^*)\Delta x. \tag{2.4.8}$$

The remainder of the development proceeds as before, and we see that

$$V = \int_a^b (2\pi(x + k)f(x))dx. \tag{2.4.9}$$

We could also rotate the region around other horizontal or vertical lines, such as a vertical line in the right half plane. In each case, the volume formula must be adjusted accordingly. Specifically, the $x$-term in the integral must be replaced with an expression representing the radius of a shell. To see how this works, consider the following example.

**Example 2.4.4: A Region of Revolution Revolved around a Line**

Define $R$ as the region bounded above by the graph of $f(x) = x$ and below by the $x$-axis over the interval $[1, 2]$. Find the volume of the solid of revolution formed by revolving $R$ around the line $x = -1$.

**Solution**

First, graph the region $R$ and the associated solid of revolution, as shown in Figure 2.4.8.

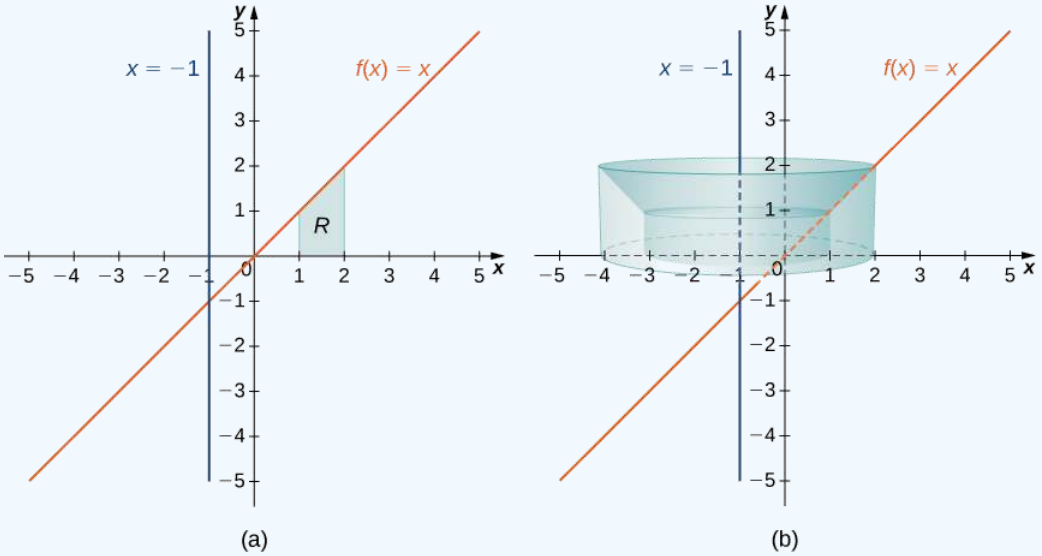
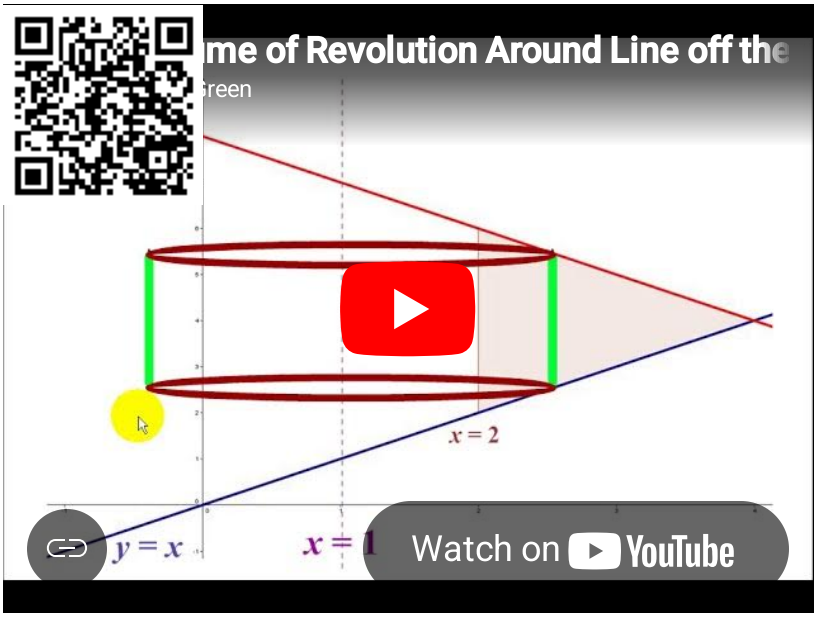



Figure 2.4.8: (a) The region $R$ between the graph of $f(x)$ and the $x$-axis over the interval $[1, 2]$. (b) The solid of revolution generated by revolving $R$ around the line $x = -1$.

Note that the radius of a shell is given by $x + 1$. Then the volume of the solid is given by

$$V = \int_1^2 2\pi(x+1)f(x)\,dx$$
$$= \int_1^2 2\pi(x+1)x\,dx = 2\pi\int_1^2 x^2 + x\,dx$$
$$= 2\pi\left[\frac{x^3}{3} + \frac{x^2}{2}\right]\Big|_1^2$$
$$= \frac{23\pi}{3}\ \text{units}^3$$




**Exercise 2.4.4**

Define $R$ as the region bounded above by the graph of $f(x) = x^2$ and below by the $x$-axis over the interval $[0, 1]$. Find the volume of the solid of revolution formed by revolving $R$ around the line $x = -2$.

**Hint**

Use the process from Example 2.4.4.

**Answer**

$\frac{11\pi}{6}$ units$^3$

For our final example in this section, let's look at the volume of a solid of revolution for which the region of revolution is bounded by the graphs of two functions.

**Example 2.4.5: A Region of Revolution Bounded by the Graphs of Two Functions**

Define $R$ as the region bounded above by the graph of the function $f(x) = \sqrt{x}$ and below by the graph of the function $g(x) = 1/x$ over the interval $[1, 4]$. Find the volume of the solid of revolution generated by revolving $R$ around the $y$-axis.

**Solution**

First, graph the region $R$ and the associated solid of revolution, as shown in Figure 2.4.9.

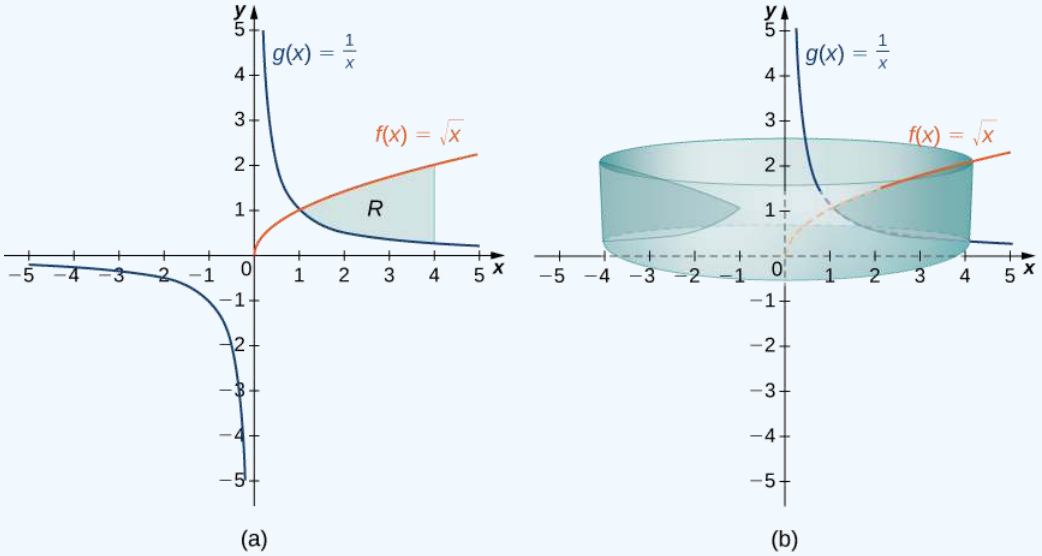



Figure 2.4.9: (a) The region $R$ between the graph of $f(x)$ and the graph of $g(x)$ over the interval $[1,4]$. (b) The solid of revolution generated by revolving $R$ around the $y$-axis.

Note that the axis of revolution is the $y$-axis, so the radius of a shell is given simply by $x$. We don't need to make any adjustments to the x-term of our integrand. The height of a shell, though, is given by $f(x) - g(x)$, so in this case we need to adjust the $f(x)$ term of the integrand. Then the volume of the solid is given by

$$V = \int_1^4 (2\pi x(f(x) - g(x)))\, dx$$

$$= \int_1^4 (2\pi x(\sqrt{x} - \frac{1}{x}))\, dx = 2\pi \int_1^4 (x^{3/2} - 1) dx$$

$$= 2\pi \left[\frac{2x^{5/2}}{5} - x\right]\Big|_1^4 = \frac{94\pi}{5} \text{ units}^3.$$

**Exercise 2.4.5**

Define $R$ as the region bounded above by the graph of $f(x) = x$ and below by the graph of $g(x) = x^2$ over the interval $[0,1]$. Find the volume of the solid of revolution formed by revolving $R$ around the $y$-axis.

**Hint**

Hint: Use the process from Example 2.4.5.

**Answer**

$\frac{\pi}{6}$ units$^3$

## Which Method Should We Use?

We have studied several methods for finding the volume of a solid of revolution, but how do we know which method to use? It often comes down to a choice of which integral is easiest to evaluate. Figure 2.4.10 describes the different approaches for solids of revolution around the $x$-axis. It's up to you to develop the analogous table for solids of revolution around the $y$-axis.

Comparing the Methods for Finding the Volume of a Solid Revolution around the $x$-axis

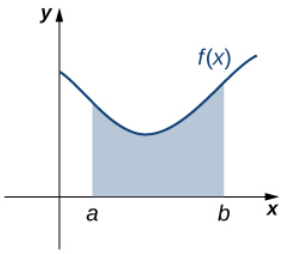
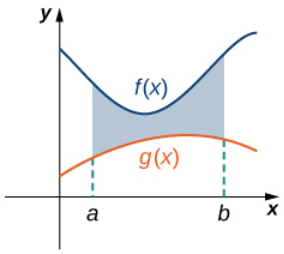
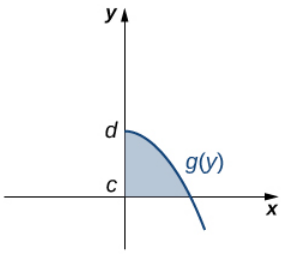
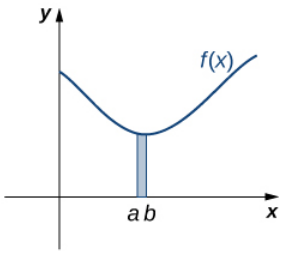
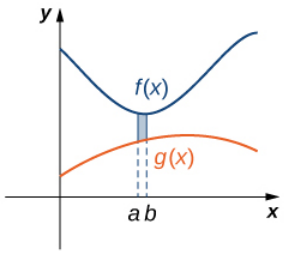
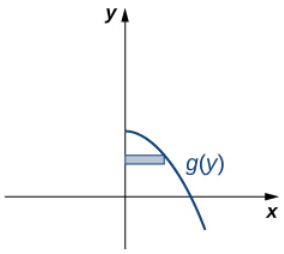
| Compare | Disk Method | Washer Method | Shell Method |
|---|---|---|---|
| Volume formula | $V = \int_a^b \pi[f(x)]^2\,dx$ | $V = \int_a^b \pi\left[(f(x))^2 - (g(x))^2\right]dx$ | $V = \int_c^d 2\pi y\, g(y)\,dy$ |
| Solid | No cavity in the center | Cavity in the center | With or without a cavity in the center |
| Interval to partition | $[a, b]$ on $x$-axis | $[a, b]$ on $x$-axis | $[c, d]$ on $y$-axis |
| Rectangle | Vertical | Vertical | Horizontal |
| Typical region |   |   |   |
| Typical element |   |   |   |

Figure 2.4.10

Let's take a look at a couple of additional problems and decide on the best approach to take for solving them.

**Example 2.4.6: Selecting the Best Method**

For each of the following problems, select the best method to find the volume of a solid of revolution generated by revolving the given region around the $x$-axis, and set up the integral to find the volume (do not evaluate the integral).

a. The region bounded by the graphs of $y = x, y = 2 - x$, and the $x$-axis.
b. The region bounded by the graphs of $y = 4x - x^2$ and the $x$-axis.

**Solution**

a.

First, sketch the region and the solid of revolution as shown.

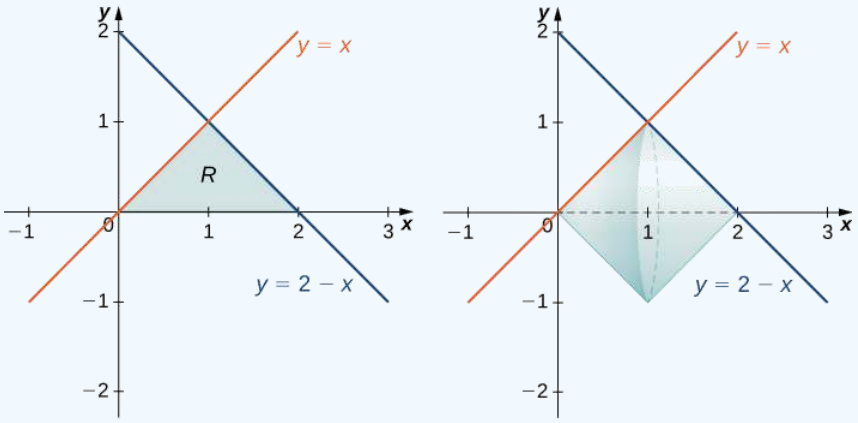



Figure 2.4.11: (a) The region $R$ bounded by two lines and the $x$-axis. (b) The solid of revolution generated by revolving $R$ about the $x$-axis.

Looking at the region, if we want to integrate with respect to $x$, we would have to break the integral into two pieces, because we have different functions bounding the region over $[0, 1]$ and $[1, 2]$. In this case, using the disk method, we would have

$$V = \int_0^1 \pi x^2 \, dx + \int_1^2 \pi (2 - x)^2 \, dx.$$

If we used the shell method instead, we would use functions of y to represent the curves, producing

$$V = \int_0^1 2\pi y[(2 - y) - y] \, dy = \int_0^1 2\pi y[2 - 2y] \, dy.$$

Neither of these integrals is particularly onerous, but since the shell method requires only one integral, and the integrand requires less simplification, we should probably go with the shell method in this case.

b.

First, sketch the region and the solid of revolution as shown.

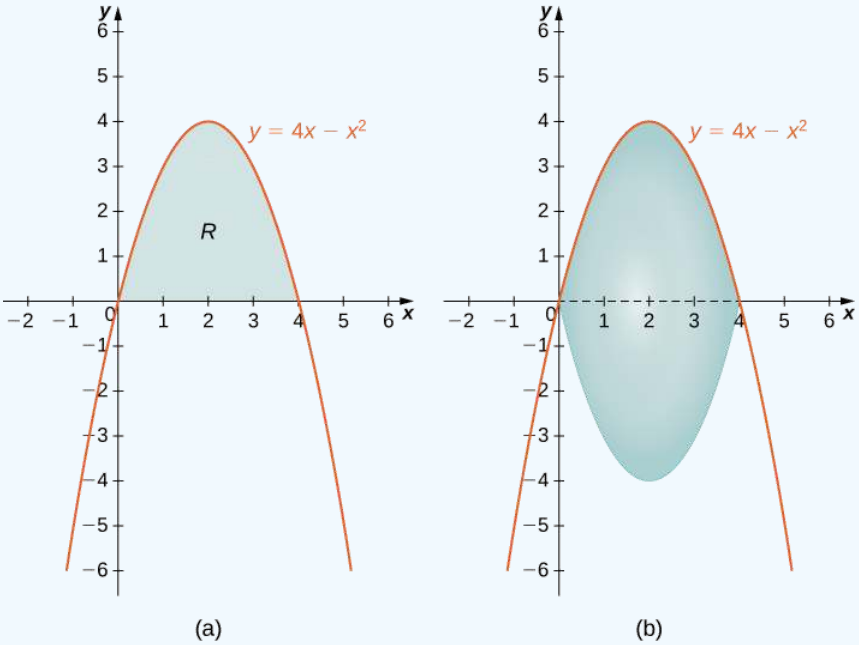



Figure 2.4.12: (a) The region $R$ between the curve and the $x$-axis. (b) The solid of revolution generated by revolving $R$ about the $x$-axis.

Looking at the region, it would be problematic to define a horizontal rectangle; the region is bounded on the left and right by the same function. Therefore, we can dismiss the method of shells. The solid has no cavity in the middle, so we can use the method of disks. Then

$$V = \int_0^4 \pi \left(4x - x^2\right)^2 \, dx$$

**Exercise 2.4.6**

Select the best method to find the volume of a solid of revolution generated by revolving the given region around the $x$-axis, and set up the integral to find the volume (do not evaluate the integral): the region bounded by the graphs of $y = 2 - x^2$ and $y = x^2$.

**Hint**

Sketch the region and use Figure **2.4.12** to decide which integral is easiest to evaluate.

**Answer**

Use the method of washers;

$$V = \int_{-1}^1 \pi \left[\left(2 - x^2\right)^2 - \left(x^2\right)^2\right] \, dx$$

## Key Concepts

- The method of cylindrical shells is another method for using a definite integral to calculate the volume of a solid of revolution. This method is sometimes preferable to either the method of disks or the method of washers because we integrate with respect to the other variable. In some cases, one integral is substantially more complicated than the other.
- The geometry of the functions and the difficulty of the integration are the main factors in deciding which integration method to use.

### Key Equations

- **Method of Cylindrical Shells**

$$V = \int_a^b \left(2\pi x \, f(x)\right) \, dx$$

## Glossary

**method of cylindrical shells**

a method of calculating the volume of a solid of revolution by dividing the solid into nested cylindrical shells; this method is different from the methods of disks or washers in that we integrate with respect to the opposite variable

## Contributors and Attributions

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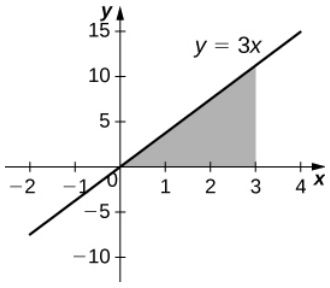
# 2.4E: Exercises for Section 2.4

**For exercises 1 - 6, find the volume generated when the region between the two curves is rotated around the given axis. Use both the shell method and the washer method. Use technology to graph the functions and draw a typical slice by hand.**

1) [T] Over the curve of $y = 3x$, $x = 0$, and $y = 3$ rotated around the $y$-axis.

2) [T] Under the curve of $y = 3x$, $x = 0$, and $x = 3$ rotated around the $y$-axis.

**Answer**

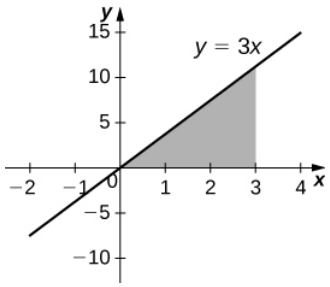



$V = 54\pi$ units$^3$

3) [T] Over the curve of $y = 3x$, $x = 0$, and $y = 3$ rotated around the $x$-axis.

4) [T] Under the curve of $y = 3x$, $x = 0$, and $x = 3$ rotated around the $x$-axis.

**Answer**

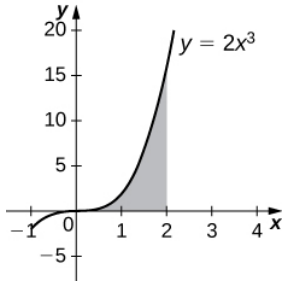



$V = 81\pi$ units$^3$

5) [T] Under the curve of $y = 2x^3$, $x = 0$, and $x = 2$ rotated around the $y$-axis.

6) [T] Under the curve of $y = 2x^3$, $x = 0$, and $x = 2$ rotated around the $x$-axis.

**Answer**




$V = \frac{512\pi}{7}$ units$^3$

**For exercises 7 - 16, use shells to find the volumes of the given solids. Note that the rotated regions lie between the curve and the $x$-axis and are rotated around the $y$-axis.**

7) $y = 1 - x^2$, $x = 0$, and $x = 1$

8) $y = 5x^3$, $x = 0$, and $x = 1$

**Answer**

$V = 2\pi$ units$^3$

9) $y = \dfrac{1}{x}$, $x = 1$, and $x = 100$

10) $y = \sqrt{1 - x^2}$, $x = 0$, and $x = 1$

**Answer**

$V = \frac{2\pi}{3}$ units$^3$

11) $y = \dfrac{1}{1 + x^2}$, $x = 0$,and $x = 3$

12) $y = \sin x^2, x = 0$, and $x = \sqrt{\pi}$

**Answer**

$V = 2\pi$ units$^3$

13) $y = \dfrac{1}{\sqrt{1 - x^2}}$, $x = 0$, and $x = \frac{1}{2}$

14) $y = \sqrt{x}$, $x = 0$, and $x = 1$

**Answer**

$V = \frac{4\pi}{5}$ units$^3$

15) $y = (1 + x^2)^3$, $x = 0$, and $x = 1$

16) $y = 5x^3 - 2x^4$, $x = 0$, and $x = 2$

**Answer**

$V = \frac{64\pi}{3}$ units$^3$

**For exercises 17 - 26, use shells to find the volume generated by rotating the regions between the given curve and $y = 0$ around the $x$-axis.**

17) $y = \sqrt{1 - x^2}$, $x = 0$, and $x = 1$

18) $y = x^2$, $x = 0$, and $x = 2$

**Answer**

$V = \frac{32\pi}{5}$ units$^3$

19) $y = e^x$, $x = 0$, and $x = 1$

20) $y = \ln(x)$, $x = 1$, and $x = e$

**Answer**

$V = \pi(e - 2)$ units$^3$

21) $x = \dfrac{1}{1 + y^2}$, $y = 1$, and $y = 4$

22) $x = \dfrac{1 + y^2}{y}$, $y = 0$, and $y = 2$

**Answer**

$V = \frac{28\pi}{3}$ units$^3$

23) $x = \cos y$, $y = 0$, and $y = \pi$

24) $x = y^3 - 4y^2$, $x = -1$, and $x = 2$

**Answer**

$V = \frac{84\pi}{5}$ units$^3$

25) $x = ye^y$, $x = -1$, and $x = 2$

26) $x = e^y \cos y$, $x = 0$, and $x = \pi$

**Answer**

$V = e^{\pi}\pi^2$ units$^3$

**For exercises 27 - 36, find the volume generated when the region between the curves is rotated around the given axis.**

27) $y = 3 - x$, $y = 0$, $x = 0$, and $x = 2$ rotated around the $y$-axis.

28) $y = x^3$, $y = 0$, $x = 0$, and $y = 8$ rotated around the $y$-axis.

**Answer**

$V = \frac{64\pi}{5}$ units$^3$

29) $y = x^2$, $y = x$, rotated around the $y$-axis.

30) $y = \sqrt{x}$, $x = 0$, and $x = 1$ rotated around the line $x = 2$.

**Answer**

$V = \frac{28\pi}{15}$ units$^3$

31) $y = \dfrac{1}{4 - x}$, $x = 1$, and $x = 2$ rotated around the line $x = 4$.

32) $y = \sqrt{x}$ and $y = x^2$ rotated around the $y$-axis.

**Answer**

$V = \frac{3\pi}{10}$ units$^3$

33) $y = \sqrt{x}$ and $y = x^2$ rotated around the line $x = 2$.

34) $x = y^3$, $y = \dfrac{1}{x}$, $x = 1$, and $y = 2$ rotated around the $x$-axis.

**Answer**

$\frac{52\pi}{5}$ units$^3$

35) $x = y^2$ and $y = x$ rotated around the line $y = 2$.

36) [T] Left of $x = \sin(\pi y)$, right of $y = x$, around the $y$-axis.

**Answer**

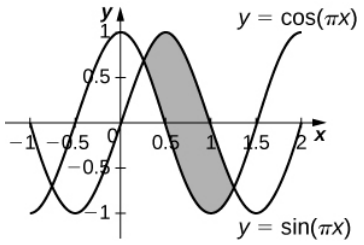
$V \approx 0.9876$ units$^3$

**For exercises 37 - 44, use technology to graph the region. Determine which method you think would be easiest to use to calculate the volume generated when the function is rotated around the specified axis. Then, use your chosen method to find the volume.**

37) [T] $y = x^2$ and $y = 4x$ rotated around the $y$-axis.

38) [T] $y = \cos(\pi x), y = \sin(\pi x), x = \frac{1}{4}$, and $x = \frac{5}{4}$ rotated around the $y$-axis.

**Answer**

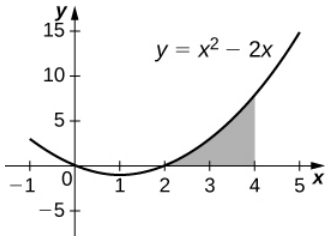



$V = 3\sqrt{2}$ units$^3$

39) [T] $y = x^2 - 2x,\ x = 2$, and $x = 4$ rotated around the $y$-axis.

40) [T] $y = x^2 - 2x,\ x = 2$, and $x = 4$ rotated around the $x$-axis.

**Answer**

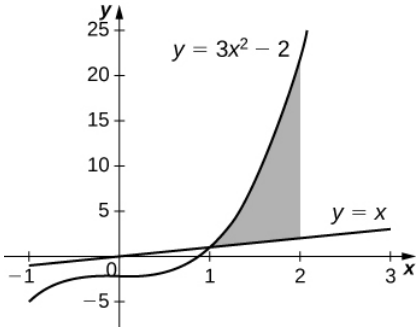



$V = \frac{496\pi}{15}$ units$^3$

41) [T] $y = 3x^3 - 2,\ y = x$, and $x = 2$ rotated around the $x$-axis.

42) [T] $y = 3x^3 - 2,\ y = x$, and $x = 2$ rotated around the $y$-axis.

**Answer**

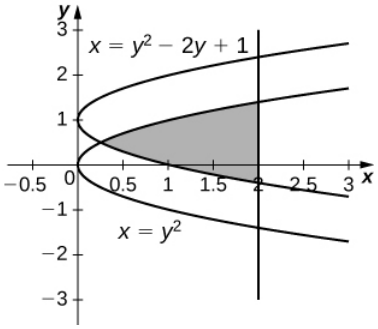



$V = \frac{398\pi}{15}$ units$^3$

43) [T] $x = \sin(\pi y^2)$ and $x = \sqrt{2}y$ rotated around the $x$-axis.

44) [T] $x = y^2,\ x = y^2 - 2y + 1$, and $x = 2$ rotated around the $y$-axis.

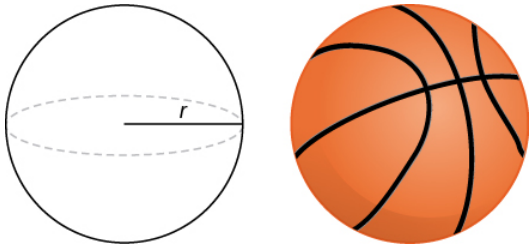
**Answer**



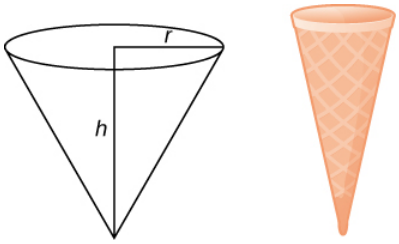

$V = 15.9074$ units$^3$

**For exercises 45 - 51, use the method of shells to approximate the volumes of some common objects, which are pictured in accompanying figures.**

45) Use the method of shells to find the volume of a sphere of radius $r$.



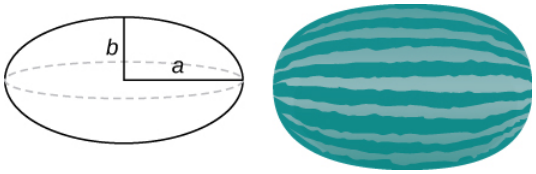

46) Use the method of shells to find the volume of a cone with radius $r$ and height $h$.



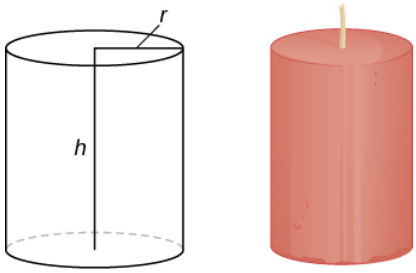

**Answer**

$V = \frac{1}{3}\pi r^2 h$ units$^3$

47) Use the method of shells to find the volume of an ellipse $(x^2/a^2) + (y^2/b^2) = 1$ rotated around the $x$-axis.



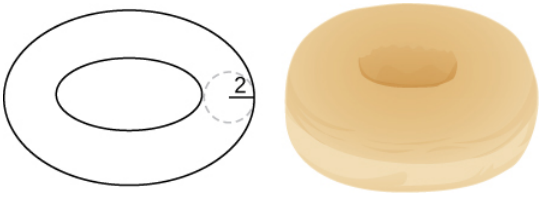

48) Use the method of shells to find the volume of a cylinder with radius $r$ and height $h$.




**Answer**

$V = \pi r^2 h$ units$^3$

49) Use the method of shells to find the volume of the donut created when the circle $x^2 + y^2 = 4$ is rotated around the line $x = 4$.




50) Consider the region enclosed by the graphs of $y = f(x)$, $y = 1 + f(x)$, $x = 0$, $y = 0$, and $x = a > 0$. What is the volume of the solid generated when this region is rotated around the $y$-axis? Assume that the function is defined over the interval $[0, a]$.

**Answer**

$V = \pi a^2$ units$^3$

51) Consider the function $y = f(x)$, which decreases from $f(0) = b$ to $f(1) = 0$. Set up the integrals for determining the volume, using both the shell method and the disk method, of the solid generated when this region, with $x = 0$ and $y = 0$, is rotated around the $y$-axis. Prove that both methods approximate the same volume. Which method is easier to apply? (Hint: Since $f(x)$ is one-to-one, there exists an inverse $f^{-1}(y)$.)

## Contributors

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# 2.5: Arc Length of a Curve and Surface Area

**Learning Objectives**

- Determine the length of a curve, $y = f(x)$, between two points.
- Determine the length of a curve, $x = g(y)$, between two points.
- Find the surface area of a solid of revolution.

In this section, we use definite integrals to find the arc length of a curve. We can think of **arc length** as the distance you would travel if you were walking along the path of the curve. Many real-world applications involve arc length. If a rocket is launched along a parabolic path, we might want to know how far the rocket travels. Or, if a curve on a map represents a road, we might want to know how far we have to drive to reach our destination.

We begin by calculating the arc length of curves defined as functions of $x$, then we examine the same process for curves defined as functions of $y$. (The process is identical, with the roles of $x$ and $y$ reversed.) The techniques we use to find arc length can be extended to find the surface area of a surface of revolution, and we close the section with an examination of this concept.

## Arc Length of the Curve y = f(x)

In previous applications of integration, we required the function $f(x)$ to be integrable, or at most continuous. However, for calculating arc length we have a more stringent requirement for $f(x)$. Here, we require $f(x)$ to be differentiable, and furthermore we require its derivative, $f'(x)$, to be continuous. Functions like this, which have continuous derivatives, are called **smooth.** (This property comes up again in later chapters.)

Let $f(x)$ be a smooth function defined over $[a, b]$. We want to calculate the length of the curve from the point $(a, f(a))$ to the point $(b, f(b))$. We start by using line segments to approximate the length of the curve. For $i = 0, 1, 2, \ldots, n$, let $P = x_i$ be a regular partition of $[a, b]$. Then, for $i = 1, 2, \ldots, n$, construct a line segment from the point $(x_{i-1}, f(x_{i-1}))$ to the point $(x_i, f(x_i))$. Although it might seem logical to use either horizontal or vertical line segments, we want our line segments to approximate the curve as closely as possible. Figure 2.5.1 depicts this construct for $n = 5$.

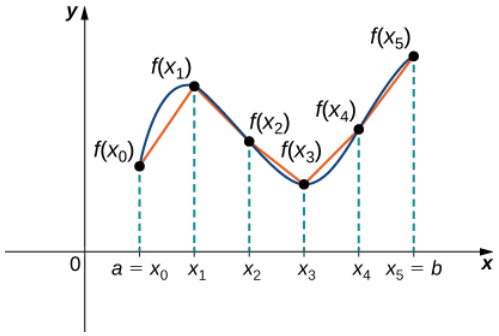



Figure 2.5.1: We can approximate the length of a curve by adding line segments.

To help us find the length of each line segment, we look at the change in vertical distance as well as the change in horizontal distance over each interval. Because we have used a regular partition, the change in horizontal distance over each interval is given by $\Delta x$. The change in vertical distance varies from interval to interval, though, so we use $\Delta y_i = f(x_i) - f(x_{i-1})$ to represent the change in vertical distance over the interval $[x_{i-1}, x_i]$, as shown in Figure 2.5.2. Note that some (or all) $\Delta y_i$ may be negative.

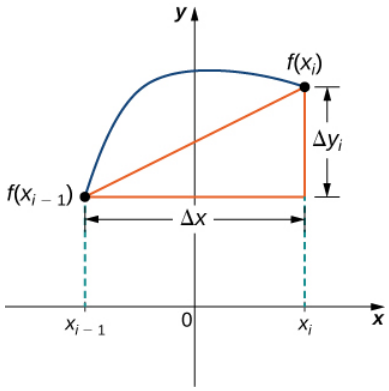



Figure **2.5.2**: A representative line segment approximates the curve over the interval $[x_{i-1}, x_i]$.

By the Pythagorean theorem, the length of the line segment is

$$\sqrt{(\Delta x)^2 + (\Delta y_i)^2}.$$

We can also write this as

$$\Delta x\sqrt{1 + ((\Delta y_i)/(\Delta x))^2}.$$

Now, by the Mean Value Theorem, there is a point $x_i^* \in [x_{i-1}, x_i]$ such that $f'(x_i^*) = (\Delta y_i)/(\Delta x)$. Then the length of the line segment is given by

$$\Delta x\sqrt{1 + [f'(x_i^*)]^2}.$$

Adding up the lengths of all the line segments, we get

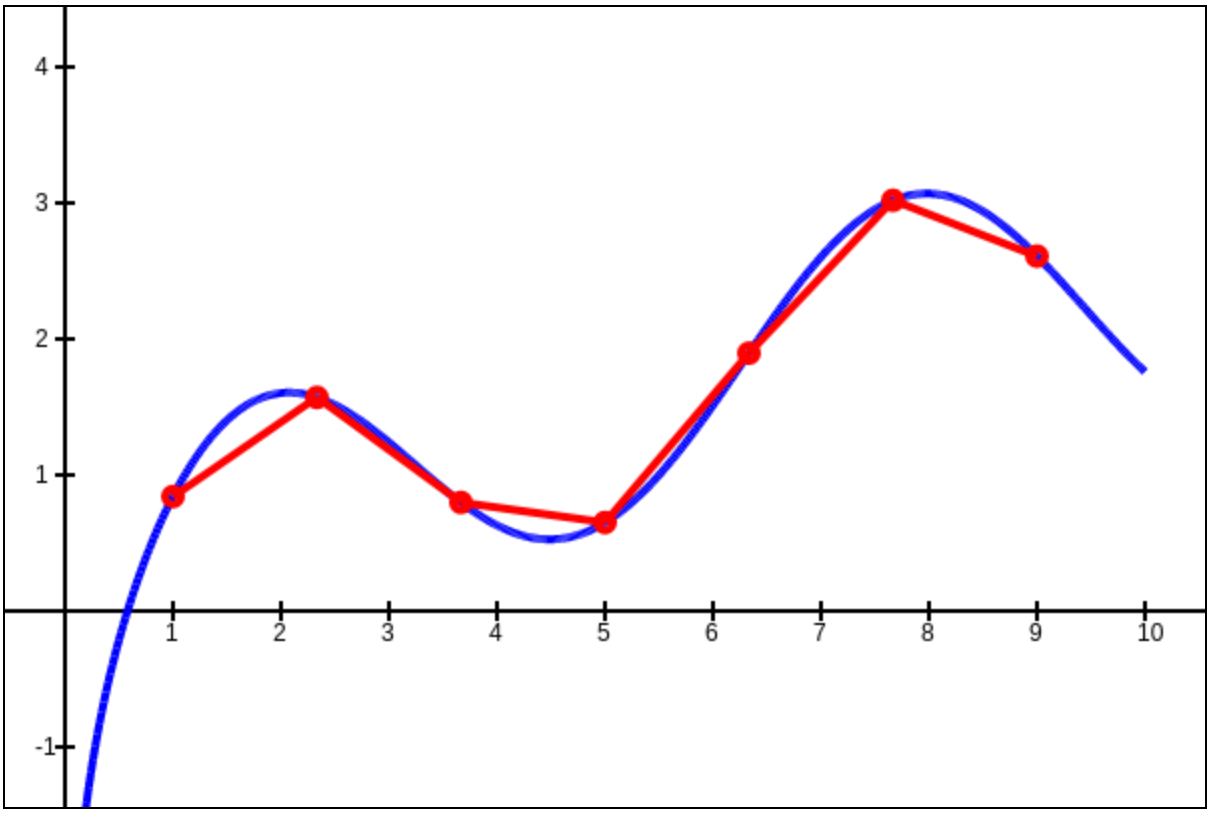
$$\text{Arc Length} \approx \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]^2}\Delta x.$$

This is a Riemann sum. Taking the limit as $n \to \infty$, we have

$$\begin{aligned}\text{Arc Length} &= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]^2}\Delta x \\ &= \int_a^b \sqrt{1 + [f'(x)]^2}dx.\end{aligned}$$

## Interactive Visualization of the Arc Length Approximation

This diagram shows how the arc length of the function $f(x) = \sin(x) + \ln(x)$ from $x = 1$ to $x = 9$ is approximated. Move the slider to change the number of points used in the approximation.




n = 6

Arc Length ≈ 1.52 + 1.541 + 1.341 + 1.825 + 1.744 + 1.395 = 9.366

We summarize these findings in the following theorem.

**Arc Length for $y = f(x)$**

Let $f(x)$ be a smooth function over the interval $[a, b]$. Then the arc length of the portion of the graph of $f(x)$ from the point $(a, f(a))$ to the point $(b, f(b))$ is given by

$$\text{Arc Length} = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx. \tag{2.5.1}$$

Note that we are integrating an expression involving $f'(x)$, so we need to be sure $f'(x)$ is integrable. This is why we require $f(x)$ to be smooth. The following example shows how to apply the theorem.

**Example 2.5.1: Calculating the Arc Length of a Function of x**

Let $f(x) = 2x^{3/2}$. Calculate the arc length of the graph of $f(x)$ over the interval $[0, 1]$. Round the answer to three decimal places.

**Solution**

We have $f'(x) = 3x^{1/2}$, so $[f'(x)]^2 = 9x$. Then, the arc length is

$$\text{Arc Length} = \int_a^b \sqrt{1+[f'(x)]^2}dx$$
$$= \int_0^1 \sqrt{1+9x}dx.$$

Substitute $u = 1+9x$. Then, $du = 9dx$. When $x = 0$, then $u = 1$, and when $x = 1$, then $u = 10$. Thus,

$$\text{Arc Length} = \int_0^1 \sqrt{1+9x}dx$$
$$= \frac{1}{9}\int_0^1 \sqrt{1+9x}9dx$$
$$= \frac{1}{9}\int_1^{10} \sqrt{u}du$$
$$= \frac{1}{9}\cdot\frac{2}{3}u^{3/2}\,|_1^{10} = \frac{2}{27}[10\sqrt{10}-1]$$
$$\approx 2.268 units.$$



**Exercise 2.5.1**

Let $f(x) = (4/3)x^{3/2}$. Calculate the arc length of the graph of $f(x)$ over the interval $[0,1]$. Round the answer to three decimal places.

**Hint**

Use the process from the previous example. Don't forget to change the limits of integration.

**Answer**

$$\frac{1}{6}(5\sqrt{5}-1) \approx 1.697$$

Although it is nice to have a formula for calculating arc length, this particular theorem can generate expressions that are difficult to integrate. We study some techniques for integration in Introduction to Techniques of Integration. In some cases, we may have to use a computer or calculator to approximate the value of the integral.

**Example 2.5.2: Using a Computer or Calculator to Determine the Arc Length of a Function of x**

Let $f(x) = x^2$. Calculate the arc length of the graph of $f(x)$ over the interval $[1, 3]$.

**Solution**

We have $f'(x) = 2x$, so $[f'(x)]^2 = 4x^2$. Then the arc length is given by

$$\begin{aligned} \text{Arc Length} &= \int_a^b \sqrt{1 + [f'(x)]^2}\, dx \\ &= \int_1^3 \sqrt{1 + 4x^2}\, dx. \end{aligned}$$

Using a computer to approximate the value of this integral, we get

$$\int_1^3 \sqrt{1 + 4x^2}\, dx \approx 8.26815.$$

**Exercise 2.5.2**

Let $f(x) = \sin x$. Calculate the arc length of the graph of $f(x)$ over the interval $[0, \pi]$. Use a computer or calculator to approximate the value of the integral.

**Hint**

Use the process from the previous example.

**Answer**

$$\text{Arc Length} \approx 3.8202 \tag{2.5.2}$$

## Arc Length of the Curve $x = g(y)$

We have just seen how to approximate the length of a curve with line segments. If we want to find the arc length of the graph of a function of $y$, we can repeat the same process, except we partition the y-axis instead of the x-axis. Figure 2.5.3 shows a representative line segment.

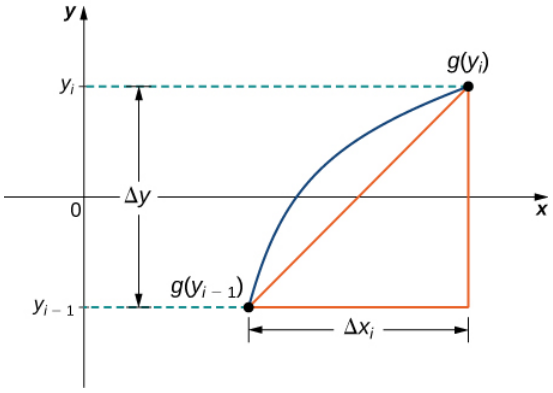



Figure 2.5.3: A representative line segment over the interval $[y_{i-1}, y_i]$.

Then the length of the line segment is

$$\sqrt{(\Delta y)^2 + (\Delta x_i)^2}, \tag{2.5.3}$$

which can also be written as

$$\Delta y \sqrt{1 + \left(\frac{\Delta x_i}{\Delta y}\right)^2}. \tag{2.5.4}$$

If we now follow the same development we did earlier, we get a formula for arc length of a function $x = g(y)$.

**Arc Length for x = g(y)**

Let $g(y)$ be a smooth function over an interval $[c, d]$. Then, the arc length of the graph of $g(y)$ from the point $(c, g(c))$ to the point $(d, g(d))$ is given by

$$\text{Arc Length} = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy. \tag{2.5.5}$$

**Example 2.5.3: Calculating the Arc Length of a Function of $y$**

Let $g(y) = 3y^3$. Calculate the arc length of the graph of $g(y)$ over the interval $[1, 2]$.

**Solution**

We have $g'(y) = 9y^2$, so $[g'(y)]^2 = 81y^4$. Then the arc length is

$$\begin{aligned} \text{Arc Length} &= \int_c^d \sqrt{1 + [g'(y)]^2}\, dy \\ &= \int_1^2 \sqrt{1 + 81y^4}\, dy. \end{aligned}$$

Using a computer to approximate the value of this integral, we obtain

$$\int_1^2 \sqrt{1 + 81y^4}\, dy \approx 21.0277.$$

**Exercise 2.5.3**

Let $g(y) = 1/y$. Calculate the arc length of the graph of $g(y)$ over the interval $[1, 4]$. Use a computer or calculator to approximate the value of the integral.

**Hint**

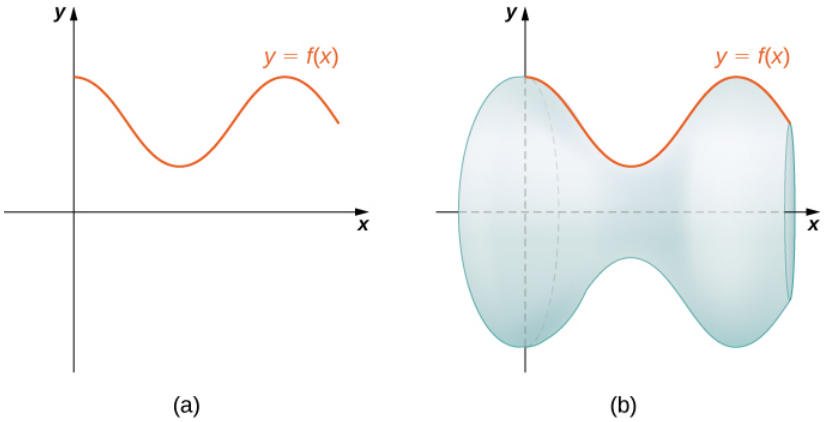
Use the process from the previous example.

**Answer**

$$\text{Arc Length} = 3.15018$$

## Area of a Surface of Revolution

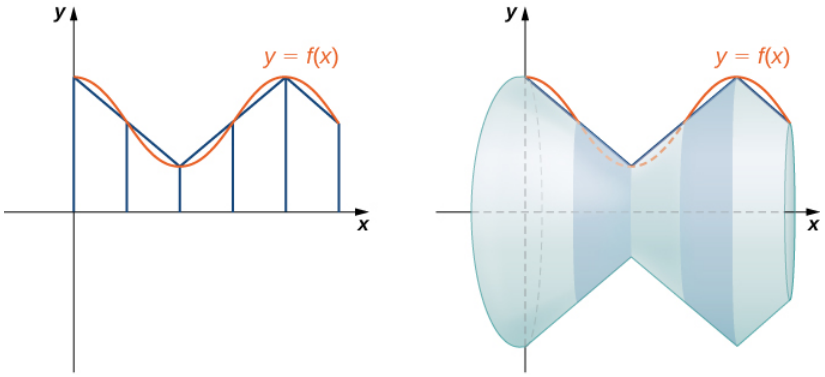
The concepts we used to find the arc length of a curve can be extended to find the surface area of a surface of revolution. **Surface area** is the total area of the outer layer of an object. For objects such as cubes or bricks, the surface area of the object is the sum of the areas of all of its faces. For curved surfaces, the situation is a little more complex. Let $f(x)$ be a nonnegative smooth function over the interval $[a, b]$. We wish to find the surface area of the surface of revolution created by revolving the graph of $y = f(x)$ around the $x$-axis as shown in the following figure.




(a) (b)

Figure 2.5.4: (a) A curve representing the function $f(x)$. (b) The surface of revolution formed by revolving the graph of $f(x)$ around the $x-axis$.

As we have done many times before, we are going to partition the interval $[a, b]$ and approximate the surface area by calculating the surface area of simpler shapes. We start by using line segments to approximate the curve, as we did earlier in this section. For $i = 0, 1, 2, \ldots, n$, let $P = x_i$ be a regular partition of $[a, b]$. Then, for $i = 1, 2, \ldots, n$, construct a line segment from the point $(x_{i-1}, f(x_{i-1}))$ to the point $(x_i, f(x_i))$. Now, revolve these line segments around the $x$-axis to generate an approximation of the surface of revolution as shown in the following figure.




(a) (b)

Figure 2.5.5: (a) Approximating $f(x)$ with line segments. (b) The surface of revolution formed by revolving the line segments around the $x-axis$.

Notice that when each line segment is revolved around the axis, it produces a band. These bands are actually pieces of cones (think of an ice cream cone with the pointy end cut off). A piece of a cone like this is called a **frustum** of a cone.

To find the surface area of the band, we need to find the lateral surface area, $S$, of the frustum (the area of just the slanted outside surface of the frustum, not including the areas of the top or bottom faces). Let $r_1$ and $r_2$ be the radii of the wide end and the narrow end of the frustum, respectively, and let $l$ be the slant height of the frustum as shown in the following figure.

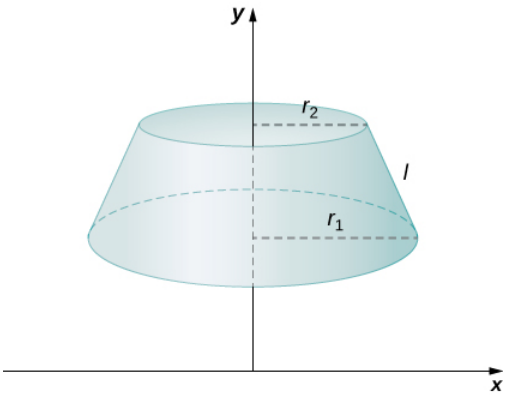



Figure 2.5.6: A frustum of a cone can approximate a small part of surface area.

We know the lateral surface area of a cone is given by

$$\text{Lateral Surface Area } = \pi rs, \tag{2.5.6}$$

where $r$ is the radius of the base of the cone and $s$ is the **slant height** (Figure 2.5.7).

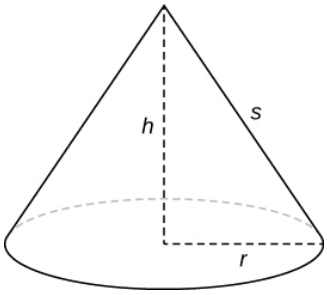



Figure 2.5.7: The lateral surface area of the cone is given by $\pi rs$.

Since a frustum can be thought of as a piece of a cone, the lateral surface area of the frustum is given by the lateral surface area of the whole cone less the lateral surface area of the smaller cone (the pointy tip) that was cut off (Figure 2.5.8).

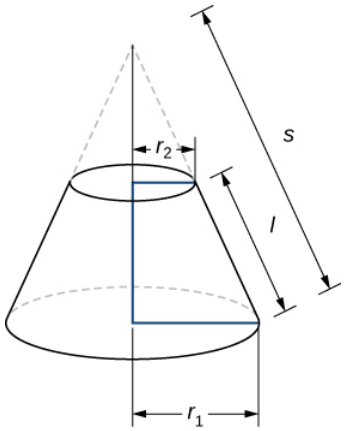



Figure 2.5.8: Calculating the lateral surface area of a frustum of a cone.

The cross-sections of the small cone and the large cone are similar triangles, so we see that

$$\frac{r_2}{r_1} = \frac{s-l}{s} \tag{2.5.7}$$

Solving for $s$, we get =s−ls

$$\frac{r_2}{r_1} = \frac{s-l}{s}$$
$$r_2 s = r_1(s-l)$$
$$r_2 s = r_1 s - r_1 l$$
$$r_1 l = r_1 s - r_2 s$$
$$r_1 l = (r_1 - r_2)s$$
$$\frac{r_1 l}{r_1 - r_2} = s$$

Then the lateral surface area (SA) of the frustum is

$$\begin{aligned} S &= \text{(Lateral SA of large cone)} - \text{(Lateral SA of small cone)} \\ &= \pi r_1 s - \pi r_2 (s-l) \\ &= \pi r_1 \left(\frac{r_1 l}{r_1 - r_2}\right) - \pi r_2 \left(\frac{r_1 l}{r_1 - r_2 - l}\right) \\ &= \frac{\pi r_1^2 l}{r^1 - r^2} - \frac{\pi r_1 r_2 l}{r_1 - r_2} + \pi r_2 l \\ &= \frac{\pi r_1^2 l}{r_1 - r_2} - \frac{\pi r_1 r 2_l}{r_1 - r_2} + \frac{\pi r_2 l (r_1 - r_2)}{r_1 - r_2} \\ &= \frac{\pi r_1^2}{l r_1 - r_2} - \frac{\pi r_1 r_2 l}{r_1 - r_2} + \frac{\pi r_1 r_2 l}{r_1 - r_2} - \frac{\pi r_2^2 l}{r_1 - r_3} \\ &= \frac{\pi(r_1^2 - r_2^2)l}{r_1 - r_2} = \frac{\pi(r_1 - r + 2)(r1 + r2)l}{r_1 - r_2} \\ &= \pi(r_1 + r_2)l. \end{aligned}$$

Let's now use this formula to calculate the surface area of each of the bands formed by revolving the line segments around the $x$ $-$ $axis$. A representative band is shown in the following figure.

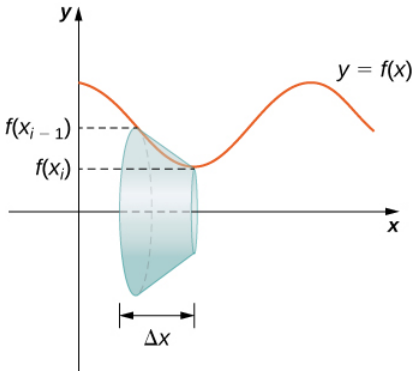



Figure 2.5.9: A representative band used for determining surface area.

Note that the slant height of this frustum is just the length of the line segment used to generate it. So, applying the surface area formula, we have

$$\begin{aligned} S &= \pi(r_1 + r_2)l \\ &= \pi(f(x_{i-1}) + f(x_i))\sqrt{\Delta x^2 + (\Delta yi)^2} \\ &= \pi(f(x_{i-1}) + f(x_i))\Delta x\sqrt{1 + \left(\frac{\Delta y_i}{\Delta x}\right)^2} \end{aligned}$$

Now, as we did in the development of the arc length formula, we apply the Mean Value Theorem to select $x_i^* \in [x_{i-1}, x_i]$ such that $f'(x_i^*) = (\Delta y_i)/\Delta x$. This gives us

$$S = \pi(f(x_{i-1}) + f(x_i))\Delta x\sqrt{1 + (f'(x_i^*))^2}$$

Furthermore, since $f(x)$ is continuous, by the *Intermediate Value Theorem*, there is a point $x_i^{**} \in [x_{i-1}, x[i]$ such that \(f(x^{**}_i)= (1/2)[f(xi−1)+f(xi)],

so we get

$$S = 2\pi f(x_i^{**})\Delta x\sqrt{1+(f'(x_i^*))^2}.$$

Then the approximate surface area of the whole surface of revolution is given by

$$\text{Surface Area} \approx \sum_{i=1}^{n} 2\pi f(x_i^{**})\Delta x\sqrt{1+(f'(x_i^*))^2}.$$

This almost looks like a Riemann sum, except we have functions evaluated at two different points, $x_i^*$ and $x_i^{**}$, over the interval $[x_{i-1}, x_i]$. Although we do not examine the details here, it turns out that because $f(x)$ is smooth, if we let n$\to\infty$, the limit works the same as a Riemann sum even with the two different evaluation points. This makes sense intuitively. Both $x_i^*$ and x^{**}_i\) are in the interval $[x_{i-1}, x_i]$, so it makes sense that as $n\to\infty$, both $x_i^*$ and $x_i^{**}$ approach $x$ Those of you who are interested in the details should consult an advanced calculus text.

Taking the limit as $n\to\infty$, we get

$$\begin{aligned}\text{Surface Area} &= \lim_{n\to\infty}\sum_{i=1} n^2\pi f(x_i^{**})\Delta x\sqrt{1+(f'(x_i^*))^2} \\ &= \int_a^b (2\pi f(x)\sqrt{1+(f'(x))^2})\end{aligned}$$

As with arc length, we can conduct a similar development for functions of $y$ to get a formula for the surface area of surfaces of revolution about the $y-axis$. These findings are summarized in the following theorem.

### Surface Area of a Surface of Revolution

Let $f(x)$ be a nonnegative smooth function over the interval $[a,b]$. Then, the surface area of the surface of revolution formed by revolving the graph of $f(x)$ around the x-axis is given by

$$\text{Surface Area} = \int_a^b (2\pi f(x)\sqrt{1+(f'(x))^2})dx \tag{2.5.8}$$

Similarly, let $g(y)$ be a nonnegative smooth function over the interval $[c,d]$. Then, the surface area of the surface of revolution formed by revolving the graph of $g(y)$ around the $y-axis$ is given by

$$\text{Surface Area} = \int_c^d (2\pi g(y)\sqrt{1+(g'(y))^2}dy \tag{2.5.9}$$

### Example 2.5.4: Calculating the Surface Area of a Surface of Revolution 1.

Let $f(x)=\sqrt{x}$ over the interval $[1,4]$. Find the surface area of the surface generated by revolving the graph of $f(x)$ around the $x$ -axis. Round the answer to three decimal places.

**Solution**

The graph of $f(x)$ and the surface of rotation are shown in Figure 2.5.10.

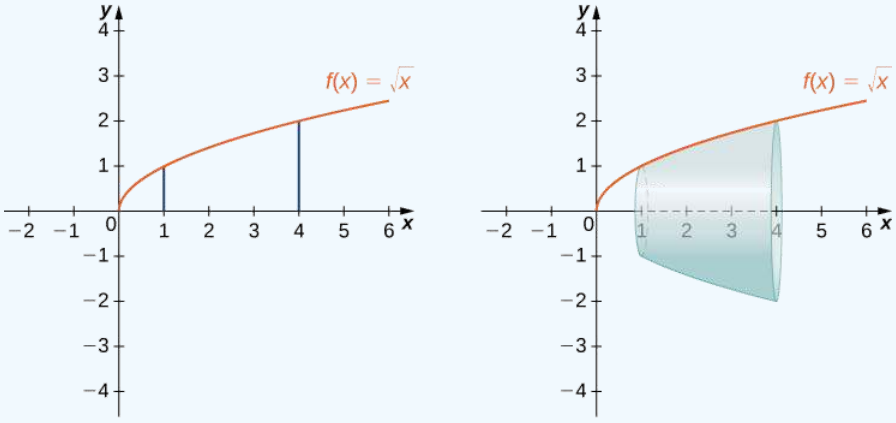



Figure **2.5.10**: (a) The graph of $f(x)$. (b) The surface of revolution.

We have $f(x) = \sqrt{x}$. Then, $f'(x) = 1/(2\sqrt{x})$ and $(f'(x))^2 = 1/(4x)$. Then,

$$\begin{aligned}\text{Surface Area} &= \int_a^b (2\pi f(x)\sqrt{1+(f'(x))^2}dx \\ &= \int_1^4 (\sqrt{2\pi\sqrt{x}1 + \frac{1}{4x}})dx \\ &= \int_1^4 (2\pi\sqrt{x+14}dx.\end{aligned}$$

Let $u = x + 1/4$. Then, $du = dx$. When $x = 1, u = 5/4$, and when $x = 4, u = 17/4$. This gives us

$$\begin{aligned}\int_0^1 (2\pi\sqrt{x+\frac{1}{4}})dx &= \int_{5/4}^{17/4} 2\pi\sqrt{u}du \\ &= 2\pi\left[\frac{2}{3}u^{3/2}\right]\Big|_{5/4}^{17/4} \\ &= \frac{\pi}{6}[17\sqrt{17} - 5\sqrt{5}] \approx 30.846\end{aligned}$$



**Exercise 2.5.4**

Let $f(x) = \sqrt{1-x}$ over the interval $[0, 1/2]$. Find the surface area of the surface generated by revolving the graph of $f(x)$ around the $x$-axis. Round the answer to three decimal places.

**Hint**

Use the process from the previous example.

**Answer**

$$\frac{\pi}{6}(5\sqrt{5} - 3\sqrt{3}) \approx 3.133 \tag{2.5.10}$$

**Example 2.5.5: Calculating the Surface Area of a Surface of Revolution 2**

Let $f(x) = y = \sqrt[3]{3x}$. Consider the portion of the curve where $0 \le y \le 2$. Find the surface area of the surface generated by revolving the graph of $f(x)$ around the $y$-axis.

**Solution**

Notice that we are revolving the curve around the $y$-axis, and the interval is in terms of $y$, so we want to rewrite the function as a function of $y$. We get $x = g(y) = (1/3)y^3$. The graph of $g(y)$ and the surface of rotation are shown in the following figure.

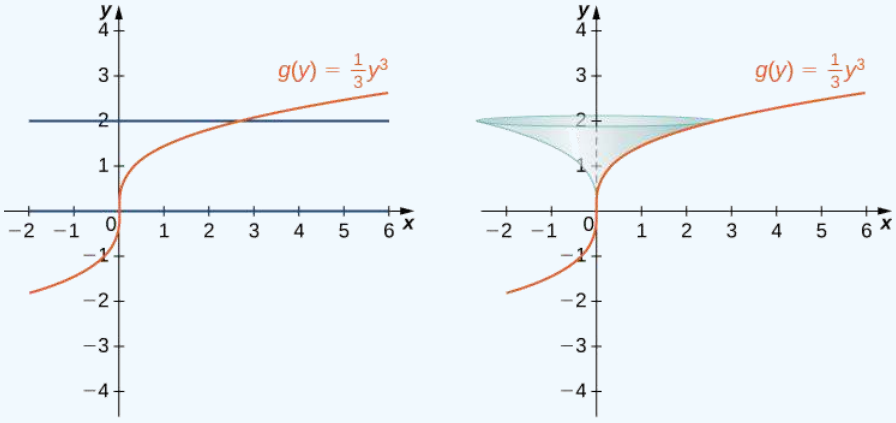



Figure 2.5.11: (a) The graph of $g(y)$. (b) The surface of revolution.

We have $g(y) = (1/3)y^3$, so $g'(y) = y^2$ and $(g'(y))^2 = y^4$. Then

$$\begin{aligned} \text{Surface Area} &= \int_c^d (2\pi g(y)\sqrt{1 + (g'(y))^2})dy \\ &= \int_0^2 (2\pi(\frac{1}{3}y^3)\sqrt{1+y^4})dy \\ &= \frac{2\pi}{3}\int_0^2 (y^3\sqrt{1+y^4})dy. \end{aligned}$$

Let $u = y^4 + 1$. Then $du = 4y^3dy$. When $y = 0, u = 1$, and when $y = 2, u = 17$. Then

$$\begin{aligned} \frac{2\pi}{3}\int_0^2 (y^3\sqrt{1+y^4})dy &= \frac{2\pi}{3}\int_1^{17} \frac{1}{4}\sqrt{u}du \\ &= \frac{\pi}{6}[\frac{2}{3}u^{3/2}]\,|_1^{17} = \frac{\pi}{9}[(17)^{3/2} - 1] \approx 24.118. \end{aligned}$$



**Exercise 2.5.5**

Let $g(y) = \sqrt{9 - y^2}$ over the interval $y \in [0, 2]$. Find the surface area of the surface generated by revolving the graph of $g(y)$ around the $y$-axis.

**Hint**

Use the process from the previous example.

**Answer**

$12\pi$

## Key Concepts

- The arc length of a curve can be calculated using a definite integral.
- The arc length is first approximated using line segments, which generates a Riemann sum. Taking a limit then gives us the definite integral formula. The same process can be applied to functions of $y$.
- The concepts used to calculate the arc length can be generalized to find the surface area of a surface of revolution.
- The integrals generated by both the arc length and surface area formulas are often difficult to evaluate. It may be necessary to use a computer or calculator to approximate the values of the integrals.

## Key Equations

- **Arc Length of a Function of x**

  **Arc Length** $= \int_a^b \sqrt{1 + [f'(x)]^2}dx$

- **Arc Length of a Function of y**

  **Arc Length** $= \int_c^d \sqrt{1 + [g'(y)]^2}dy$

- **Surface Area of a Function of x**

  **Surface Area** $= \int_a^b (2\pi f(x)\sqrt{1 + (f'(x))^2})dx$

## Glossary

**arc length**

the arc length of a curve can be thought of as the distance a person would travel along the path of the curve

**frustum**

a portion of a cone; a frustum is constructed by cutting the cone with a plane parallel to the base

**surface area**

the surface area of a solid is the total area of the outer layer of the object; for objects such as cubes or bricks, the surface area of the object is the sum of the areas of all of its faces

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# 2.5E: Exercises for Section 2.5

**For exercises 1 - 3, find the length of the functions over the given interval.**

1) $y = 5x$ from $x = 0$ to $x = 2$

**Answer**

$s = 2\sqrt{26}$ units

2) $y = -\frac{1}{2}x + 25$ from $x = 1$ to $x = 4$

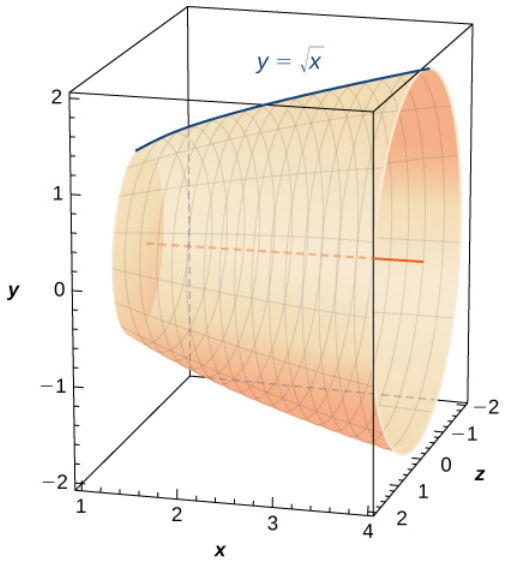
3) $x = 4y$ from $y = -1$ to $y = 1$

**Answer**

$s = 2\sqrt{17}$ units

4) Pick an arbitrary linear function $x = g(y)$ over any interval of your choice $(y_1, y_2)$. Determine the length of the function and then prove the length is correct by using geometry.

5) Find the surface area of the volume generated when the curve $y = \sqrt{x}$ revolves around the $x$-axis from $(1, 1)$ to $(4, 2)$, as seen here.




**Answer**

$A = \frac{\pi}{6}(17\sqrt{17} - 5\sqrt{5})$ units$^2$

6) Find the surface area of the volume generated when the curve $y = x^2$ revolves around the $y$-axis from $(1, 1)$ to $(3, 9)$.

**For exercises 7 - 16, find the lengths of the functions of $x$ over the given interval. If you cannot evaluate the integral exactly, use technology to approximate it.**

7) $y = x^{3/2}$ from $(0, 0)$ to $(1, 1)$

**Answer**

$s = \frac{13\sqrt{13} - 8}{27}$ units

8) $y = x^{2/3}$ from $(1, 1)$ to $(8, 4)$

9) $y = \frac{1}{3}(x^2 + 2)^{3/2}$ from $x = 0$ to $x = 1$

**Answer**

$s = \frac{4}{3}$ units

10) $y = \frac{1}{3}(x^2 - 2)^{3/2}$ from $x = 2$ to $x = 4$

11) [T] $y = e^x$ on $x = 0$ to $x = 1$

**Answer**

$s \approx 2.0035$ units

12) $y = \dfrac{x^3}{3} + \dfrac{1}{4x}$ from $x = 1$ to $x = 3$

13) $y = \dfrac{x^4}{4} + \dfrac{1}{8x^2}$ from $x = 1$ to $x = 2$

**Answer**

$s = \frac{123}{32}$ units

14) $y = \dfrac{2x^{3/2}}{3} - \dfrac{x^{1/2}}{2}$ from $x = 1$ to $x = 4$

15) $y = \frac{1}{27}(9x^2 + 6)^{3/2}$ from $x = 0$ to $x = 2$

**Answer**

$s = 10$ units

16) [T] $y = \sin x$ on $x = 0$ to $x = \pi$

**For exercises 17 - 26, find the lengths of the functions of $y$ over the given interval. If you cannot evaluate the integral exactly, use technology to approximate it.**

17) $y = \dfrac{5 - 3x}{4}$ from $y = 0$ to $y = 4$

**Answer**

$s = \frac{20}{3}$ units

18) $x = \frac{1}{2}(e^y + e^{-y})$ from $y = -1$ to $y = 1$

19) $x = 5y^{3/2}$ from $y = 0$ to $y = 1$

**Answer**

$s = \frac{1}{675}(229\sqrt{229} - 8)$ units

20) [T] $x = y^2$ from $y = 0$ to $y = 1$

21) $x = \sqrt{y}$ from $y = 0$ to $y = 1$

**Answer**

$s = \frac{1}{8}(4\sqrt{5} + \ln(9 + 4\sqrt{5}))$ units

22) $x = \frac{2}{3}(y^2 + 1)^{3/2}$ from $y = 1$ to $y = 3$

23) [T] $x = \tan y$ from $y = 0$ to $y = \frac{3}{4}$

**Answer**

$s \approx 1.201$ units

24) [T] $x = \cos^2 y$ from $y = -\frac{\pi}{2}$ to $y = \frac{\pi}{2}$

25) [T] $x = 4^y$ from $y = 0$ to $y = 2$

**Answer**

$s \approx 15.2341$ units

26) [T] $x = \ln(y)$ on $y = \dfrac{1}{e}$ to $y = e$

**For exercises 27 - 34, find the surface area of the volume generated when the following curves revolve around the $x$-axis. If you cannot evaluate the integral exactly, use your calculator to approximate it.**

27) $y = \sqrt{x}$ from $x = 2$ to $x = 6$

**Answer**

$A = \frac{49\pi}{3}$ units$^2$

28) $y = x^3$ from $x = 0$ to $x = 1$

29) $y = 7x$ from $x = -1$ to $x = 1$

**Answer**

$A = 70\pi\sqrt{2}$ units$^2$

30) [T] $y = \frac{1}{x^2}$ from $x = 1$ to $x = 3$

31) $y = \sqrt{4 - x^2}$ from $x = 0$ to $x = 2$

**Answer**

$A = 8\pi$ units$^2$

32) $y = \sqrt{4 - x^2}$ from $x = -1$ to $x = 1$

33) $y = 5x$ from $x = 1$ to $x = 5$

**Answer**

$A = 120\pi\sqrt{26}$ units$^2$

34) [T] $y = \tan x$ from $x = -\frac{\pi}{4}$ to $x = \frac{\pi}{4}$

**For exercises 35 - 42, find the surface area of the volume generated when the following curves revolve around the $y$-axis. If you cannot evaluate the integral exactly, use your calculator to approximate it.**

35) $y = x^2$ from $x = 0$ to $x = 2$

**Answer**

$A = \frac{\pi}{6}(17\sqrt{17} - 1)$ units$^2$

36) $y = \frac{1}{2}x^2 + \frac{1}{2}$ from $x = 0$ to $x = 1$

37) $y = x + 1$ from $x = 0$ to $x = 3$

**Answer**

$A = 9\sqrt{2}\pi$ units$^2$

38) [T] $y = \dfrac{1}{x}$ from $x = \dfrac{1}{2}$ to $x = 1$

39) $y = \sqrt[3]{x}$ from $x = 1$ to $x = 27$

**Answer**

$A = \frac{10\sqrt{10}\pi}{27}(73\sqrt{73} - 1)$ units$^2$

40) [T] $y = 3x^4$ from $x = 0$ to $x = 1$

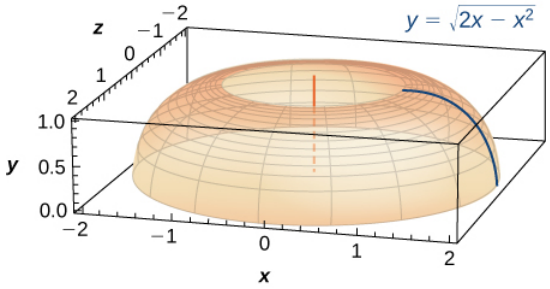
41) [T] $y = \dfrac{1}{\sqrt{x}}$ from $x = 1$ to $x = 3$

**Answer**

$A \approx 25.645$ units$^2$

42) [T] $y = \cos x$ from $x = 0$ to $x = \frac{\pi}{2}$

43) The base of a lamp is constructed by revolving a quarter circle $y = \sqrt{2x - x^2}$ around the $y$-axis from $x = 1$ to $x = 2$, as seen here. Create an integral for the surface area of this curve and compute it.



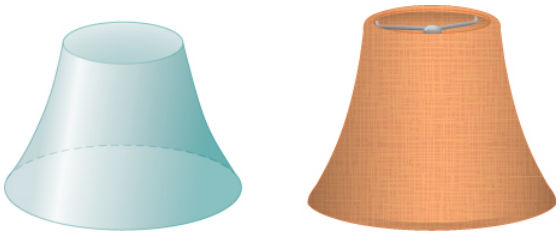

**Answer**

$A = 2\pi$ units$^2$

44) A light bulb is a sphere with radius $1/2$ in. with the bottom sliced off to fit exactly onto a cylinder of radius $1/4$ in. and length $1/3$ in., as seen here. The sphere is cut off at the bottom to fit exactly onto the cylinder, so the radius of the cut is $1/4$ in. Find the surface area (not including the top or bottom of the cylinder).



45) [T] A lampshade is constructed by rotating $y = 1/x$ around the $x$-axis from $y = 1$ to $y = 2$, as seen here. Determine how much material you would need to construct this lampshade—that is, the surface area—accurate to four decimal places.



**Answer**

$10.5017$ units$^2$

46) [T] An anchor drags behind a boat according to the function $y = 24e^{-x/2} - 24$, where $y$ represents the depth beneath the boat and $x$ is the horizontal distance of the anchor from the back of the boat. If the anchor is $23$ ft below the boat, how much rope do you have to pull to reach the anchor? Round your answer to three decimal places.

47) [T] You are building a bridge that will span $10$ ft. You intend to add decorative rope in the shape of $y = 5|\sin((x\pi)/5)|$, where $x$ is the distance in feet from one end of the bridge. Find out how much rope you need to buy, rounded to the nearest foot.

**Answer**

**23** ft

**For exercise 48, find the exact arc length for the following problems over the given interval.**

48) $y = \ln(\sin x)$ from $x = \frac{\pi}{4}$ to $x = \frac{3\pi}{4}$. (Hint: Recall trigonometric identities.)

49) Draw graphs of $y = x^2, y = x^6$, and $y = x^{10}$. For $y = x^n$, as $n$ increases, formulate a prediction on the arc length from $(0,0)$ to $(1,1)$. Now, compute the lengths of these three functions and determine whether your prediction is correct.

**Answer**

**2**

50) Compare the lengths of the parabola $x = y^2$ and the line $x = by$ from $(0,0)$ to $(b^2, b)$ as $b$ increases. What do you notice?

51) Solve for the length of $x = y^2$ from $(0,0)$ to $(1,1)$. Show that $x = \frac{y^2}{2}$ from $(0,0)$ to $(2,2)$ is twice as long. Graph both functions and explain why this is so.

**Answer**

Answers may vary

52) [T] Which is longer between $(1,1)$ and $(2, \frac{1}{2})$: the hyperbola $y = \frac{1}{x}$ or the graph of $x + 2y = 3$?

53) Explain why the surface area is infinite when $y = 1/x$ is rotated around the $x$-axis for $1 \leq x < \infty$, but the volume is finite.

**Answer**

For more information, look up Gabriel's Horn.

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# 2.6: Physical Applications of Integration

**Learning Objectives**

- Determine the mass of a one-dimensional object from its linear density function.
- Determine the mass of a two-dimensional circular object from its radial density function.
- Calculate the work done by a variable force acting along a line.
- Calculate the work done in pumping a liquid from one height to another.
- Find the hydrostatic force against a submerged vertical plate.

In this section, we examine some physical applications of integration. Let's begin with a look at calculating mass from a density function. We then turn our attention to work, and close the section with a study of hydrostatic force.

## Mass and Density

We can use integration to develop a formula for calculating mass based on a density function. First we consider a thin rod or wire. Orient the rod so it aligns with the $x$-axis, with the left end of the rod at $x = a$ and the right end of the rod at $x = b$ (Figure 2.6.1). Note that although we depict the rod with some thickness in the figures, for mathematical purposes we assume the rod is thin enough to be treated as a one-dimensional object.

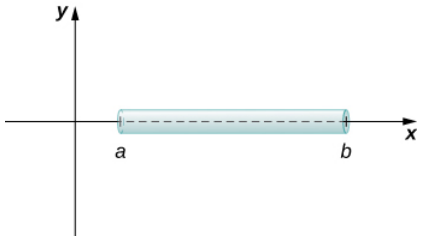



Figure 2.6.1: We can calculate the mass of a thin rod oriented along the $x$-axis by integrating its density function.

If the rod has constant density $\rho$, given in terms of mass per unit length, then the mass of the rod is just the product of the density and the length of the rod: $(b-a)\rho$. If the density of the rod is not constant, however, the problem becomes a little more challenging. When the density of the rod varies from point to point, we use a linear density function, $\rho(x)$, to denote the density of the rod at any point, $x$. Let $\rho(x)$ be an integrable linear density function. Now, for $i = 0, 1, 2, \ldots, n$ let $P = x_i$ be a regular partition of the interval $[a, b]$, and for $i = 1, 2, \ldots, n$ choose an arbitrary point $x_i^* \in [x_{i-1}, x_i]$. Figure 2.6.2 shows a representative segment of the rod.

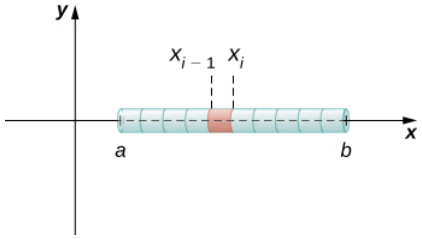



Figure 2.6.2: A representative segment of the rod.

The mass $m_i$ of the segment of the rod from $x_{i-1}$ to $x_i$ is approximated by

$$\begin{aligned} m_i &\approx \rho(x_i^*)(x_i - x_{i-1}) \\ &= \rho(x_i^*)\Delta x. \end{aligned}$$

Adding the masses of all the segments gives us an approximation for the mass of the entire rod:

$$\begin{aligned} m &= \sum_{i=1}^{n} m_i \\ &\approx \sum_{i=1}^{n} \rho(x_i^*)\Delta x. \end{aligned}$$

This is a Riemann sum. Taking the limit as $n \to \infty$, we get an expression for the exact mass of the rod:

$$\begin{aligned} m &= \lim_{n\to\infty} \sum_{i=1}^{n} \rho(x_i^*)\Delta x \\ &= \int_a^b \rho(x)dx. \end{aligned}$$

We state this result in the following theorem.

**Mass–Density Formula of a One-Dimensional Object**

Given a thin rod oriented along the $x$-axis over the interval $[a, b]$, let $\rho(x)$ denote a linear density function giving the density of the rod at a point $x$ in the interval. Then the mass of the rod is given by

$$m = \int_a^b \rho(x)dx. \tag{2.6.1}$$

We apply this theorem in the next example.

**Example 2.6.1: Calculating Mass from Linear Density**

Consider a thin rod oriented on the $x$-axis over the interval $[\pi/2, \pi]$. If the density of the rod is given by $\rho(x) = \sin x$, what is the mass of the rod?

**Solution**

Applying Equation 2.6.1 directly, we have

$$\begin{aligned} m &= \int_a^b \rho(x)dx \\ &= \int_{\pi/2}^{\pi} \sin x\, dx \\ &= -\cos x\Big|_{\pi/2}^{\pi} \\ &= 1. \end{aligned}$$

**Exercise 2.6.1**

Consider a thin rod oriented on the $x$-axis over the interval $[1, 3]$. If the density of the rod is given by $\rho(x) = 2x^2 + 3$, what is the mass of the rod?

**Hint**

Use the process from the previous example.

**Solution**

$70/3$

We now extend this concept to find the mass of a two-dimensional disk of radius $r$. As with the rod we looked at in the one-dimensional case, here we assume the disk is thin enough that, for mathematical purposes, we can treat it as a two-dimensional object. We assume the density is given in terms of mass per unit area (called **area density**), and further assume the density varies only along the disk's radius (called **radial density**). We orient the disk in the $xy-plane$, with the center at the origin. Then, the density of the disk can be treated as a function of $x$, denoted $\rho(x)$. We assume $\rho(x)$ is integrable. Because density is a function of $x$, we partition the interval from $[0, r]$ along the $x$-axis. For $i = 0, 1, 2, \ldots, n$, let $P = x_i$ be a regular partition of the interval $[0, r]$, and for $i = 1, 2, \ldots, n$, choose an arbitrary point $x_i^* \in [x_{i-1}, x_i]$. Now, use the partition to break up the disk into thin (two-dimensional) washers. A disk and a representative washer are depicted in the following figure.

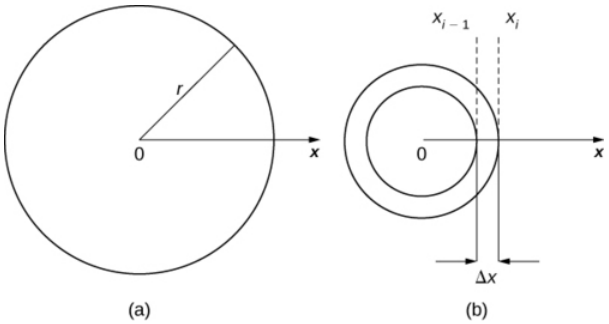



Figure **2.6.3**: (a) A thin disk in the *xy*-plane. (b) A representative washer.

We now approximate the density and area of the washer to calculate an approximate mass, $m_i$. Note that the area of the washer is given by

$$\begin{aligned} A_i &= \pi(x_i)^2 - \pi(x_{i-1})^2 \\ &= \pi[x_i^2 - x_{i-1}^2] \\ &= \pi(x_i + x_{i-1})(x_i - x_{i-1}) \\ &= \pi(x_i + x_{i-1})\Delta x. \end{aligned}$$

You may recall that we had an expression similar to this when we were computing volumes by shells. As we did there, we use $x_i^* \approx (x_i + x_{i-1})/2$ to approximate the average radius of the washer. We obtain

$$A_i = \pi(x_i + x_{i-1})\Delta x \approx 2\pi x_i^* \Delta x.$$

Using $\rho(x_i^*)$ to approximate the density of the washer, we approximate the mass of the washer by

$$m_i \approx 2\pi x_i^* \rho(x_i^*)\Delta x.$$

Adding up the masses of the washers, we see the mass $m$ of the entire disk is approximated by

$$m = \sum_{i=1}^{n} m_i \approx \sum_{i=1}^{n} 2\pi x_i^* \rho(x_i^*)\Delta x.$$

We again recognize this as a Riemann sum, and take the limit as $n \to \infty$. This gives us

$$\begin{aligned} m &= \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi x_i^* \rho(x_i^*)\Delta x \\ &= \int_0^r 2\pi x \rho(x) dx. \end{aligned}$$

We summarize these findings in the following theorem.

**Mass–Density Formula of a Circular Object**

Let $\rho(x)$ be an integrable function representing the radial density of a disk of radius $r$. Then the mass of the disk is given by

$$m = \int_0^r 2\pi x \rho(x) dx. \tag{2.6.2}$$

**Example 2.6.2: Calculating Mass from Radial Density**

Let $\rho(x) = \sqrt{x}$ represent the radial density of a disk. Calculate the mass of a disk of radius 4.

**Solution**

Applying Equation 2.6.2, we find

$$m = \int_0^r 2\pi x \rho(x)\,dx$$
$$= \int_0^4 2\pi x\sqrt{x}\,dx = 2\pi \int_0^4 x^{3/2}\,dx$$
$$= 2\pi \frac{2}{5} x^{5/2}\,|_0^4 = \frac{4\pi}{5}[32]$$
$$= \frac{128\pi}{5}.$$

**Exercise 2.6.2**

Let $\rho(x) = 3x + 2$ represent the radial density of a disk. Calculate the mass of a disk of radius 2.

**Hint**

Use the process from the previous example.

**Solution**

$24\pi$

## Work Done by a Force

We now consider work. In physics, work is related to force, which is often intuitively defined as a push or pull on an object. When a force moves an object, we say the force does work on the object. In other words, work can be thought of as the amount of energy it takes to move an object. According to physics, when we have a constant force, work can be expressed as the product of force and distance.

In the English system, the unit of force is the pound and the unit of distance is the foot, so work is given in foot-pounds. In the metric system, kilograms and meters are used. One newton is the force needed to accelerate 1 kilogram of mass at the rate of 1 m/sec$^2$. Thus, the most common unit of work is the newton-meter. This same unit is also called the joule. Both are defined as kilograms times meters squared over seconds squared ($kg \cdot m^2/s^2$).

When we have a constant force, things are pretty easy. It is rare, however, for a force to be constant. The work done to compress (or elongate) a spring, for example, varies depending on how far the spring has already been compressed (or stretched). We look at springs in more detail later in this section.

Suppose we have a variable force $F(x)$ that moves an object in a positive direction along the $x$-axis from point $a$ to point $b$. To calculate the work done, we partition the interval $[a, b]$ and estimate the work done over each subinterval. So, for $i = 0, 1, 2, \ldots, n$, let $P = x_i$ be a regular partition of the interval $[a, b]$, and for $i = 1, 2, \ldots, n$, choose an arbitrary point $x_i^* \in [x_{i-1}, x_i]$. To calculate the work done to move an object from point $x_{i-1}$ to point $x_i$, we assume the force is roughly constant over the interval, and use $F(x_i^*)$ to approximate the force. The work done over the interval $[x_{i-1}, x_i]$, then, is given by

$$W_i \approx F(x_i^*)(x_i - x_{i-1}) = F(x_i^*)\Delta x. \tag{2.6.3}$$

Therefore, the work done over the interval $[a, b]$ is approximately

$$W = \sum_{i=1}^{n} W_i \approx \sum_{i=1}^{n} F(x_i^*)\Delta x. \tag{2.6.4}$$

Taking the limit of this expression as $n \to \infty$ gives us the exact value for work:

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} F(x_i^*)\Delta x$$
$$= \int_a^b F(x)\,dx.$$

Thus, we can define work as follows.

**Definition: Work**

If a variable force $F(x)$ moves an object in a positive direction along the $x$-axis from point $a$ to point $b$, then the work done on the object is

$$W = \int_a^b F(x)dx. \tag{2.6.5}$$

Note that if $F$ is constant, the integral evaluates to $F \cdot (b - a) = F \cdot d$, which is the formula we stated at the beginning of this section.

Now let's look at the specific example of the work done to compress or elongate a spring. Consider a block attached to a horizontal spring. The block moves back and forth as the spring stretches and compresses. Although in the real world we would have to account for the force of friction between the block and the surface on which it is resting, we ignore friction here and assume the block is resting on a frictionless surface. When the spring is at its natural length (at rest), the system is said to be at equilibrium. In this state, the spring is neither elongated nor compressed, and in this equilibrium position the block does not move until some force is introduced. We orient the system such that $x = 0$ corresponds to the equilibrium position (Figure 2.6.4).

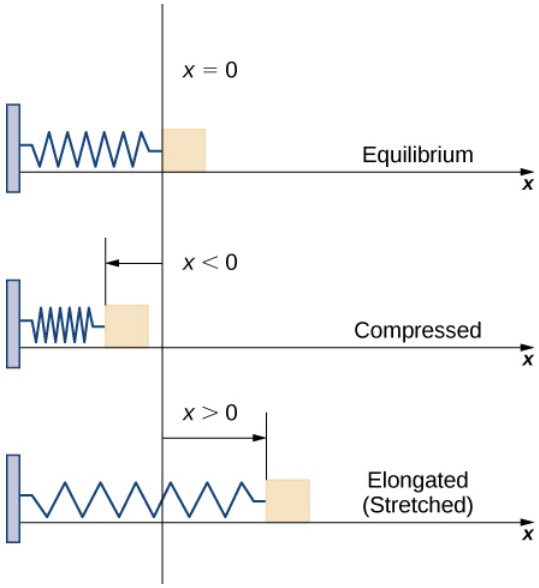



Figure 2.6.4: A block attached to a horizontal spring at equilibrium, compressed, and elongated.

According to Hooke's law, the force required to compress or stretch a spring from an equilibrium position is given by $F(x) = kx$, for some constant $k$. The value of k depends on the physical characteristics of the spring. The constant $k$ is called the spring constant and is always positive. We can use this information to calculate the work done to compress or elongate a spring, as shown in the following example.

**Example 2.6.3: The Work Required to Stretch or Compress a Spring**

Suppose it takes a force of 10 N (in the negative direction) to compress a spring 0.2 m from the equilibrium position. How much work is done to stretch the spring 0.5 m from the equilibrium position?

**Solution**

First find the spring constant, $k$. When $x = -0.2$, we know $F(x) = -10$, so

$$\begin{aligned} F(x) &= kx \\ -10 &= k(-0.2) \\ k &= 50 \end{aligned}$$

and $F(x) = 50x$. Then, to calculate work, we integrate the force function, obtaining

$$\begin{aligned} W &= \int_a^b F(x)dx \\ &= \int_0^{0.5} 50x\,dx \\ &= 25x^2\big|_0^{0.5} \\ &= 6.25. \end{aligned}$$

The work done to stretch the spring is $6.25$ J.



**Exercise 2.6.3**

Suppose it takes a force of $8$ lb to stretch a spring $6$ in. from the equilibrium position. How much work is done to stretch the spring $1$ ft from the equilibrium position?

**Hint**

Use the process from the previous example. Be careful with units.

**Solution**

$8$ ft-lb

## Work Done in Pumping

Consider the work done to pump water (or some other liquid) out of a tank. Pumping problems are a little more complicated than spring problems because many of the calculations depend on the shape and size of the tank. In addition, instead of being concerned about the work done to move a single mass, we are looking at the work done to move a volume of water, and it takes more work to move the water from the bottom of the tank than it does to move the water from the top of the tank.

We examine the process in the context of a cylindrical tank, then look at a couple of examples using tanks of different shapes. Assume a cylindrical tank of radius $4$ m and height $10$ m is filled to a depth of 8 m. How much work does it take to pump all the water over the top edge of the tank?

The first thing we need to do is define a frame of reference. We let $x$ represent the vertical distance below the top of the tank. That is, we orient the $x$-axis vertically, with the origin at the top of the tank and the downward direction being positive (Figure $2.6.5$).

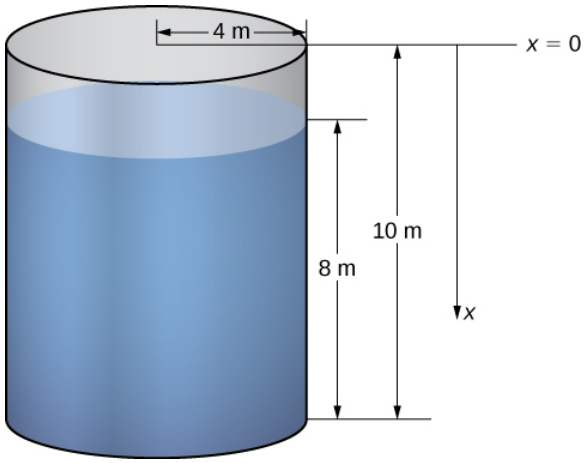



Figure 2.6.5: How much work is needed to empty a tank partially filled with water?

Using this coordinate system, the water extends from $x = 2$ to $x = 10$. Therefore, we partition the interval $[2, 10]$ and look at the work required to lift each individual "layer" of water. So, for $i = 0, 1, 2, \ldots, n$, let $P = x_i$ be a regular partition of the interval $[2, 10]$, and for $i = 1, 2, \ldots, n$, choose an arbitrary point $x_i^* \in [x_{i-1}, x_i]$. Figure 2.6.6 shows a representative layer.

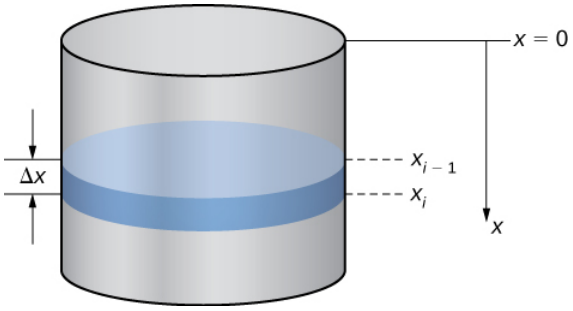



Figure 2.6.6: A representative layer of water.

In pumping problems, the force required to lift the water to the top of the tank is the force required to overcome gravity, so it is equal to the weight of the water. Given that the weight-density of water is $9800\ \text{N/m}^3$, or $62.4\ \text{lb/ft}^3$, calculating the volume of each layer gives us the weight. In this case, we have

$$V = \pi(4)^2 \Delta x = 16\pi\Delta x.$$

Then, the force needed to lift each layer is

$$F = 9800 \cdot 16\pi\Delta x = 156,800\pi\Delta x.$$

Note that this step becomes a little more difficult if we have a noncylindrical tank. We look at a noncylindrical tank in the next example.

We also need to know the distance the water must be lifted. Based on our choice of coordinate systems, we can use $x_i^*$ as an approximation of the distance the layer must be lifted. Then the work to lift the $i^{\text{th}}$ layer of water $W_i$ is approximately

$$W_i \approx 156,800\pi x_i^* \Delta x.$$

Adding the work for each layer, we see the approximate work to empty the tank is given by

$$W = \sum_{i=1}^{n} W_i$$

$$\approx \sum_{i=1}^{n} 156,800\pi x_i^* \Delta x.$$

This is a Riemann sum, so taking the limit as $n \to \infty$, we get

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} 156,800\pi x_i^* \Delta x$$

$$= 156,800\pi \int_2^{10} x dx$$

$$= 156,800\pi \left(\frac{x^2}{2}\right)\Big|_2^{10} = 7,526,400\pi \approx 23,644,883.$$

The work required to empty the tank is approximately 23,650,000 J.

For pumping problems, the calculations vary depending on the shape of the tank or container. The following problem-solving strategy lays out a step-by-step process for solving pumping problems.

**Problem-Solving Strategy: Solving Pumping Problems**

1. Sketch a picture of the tank and select an appropriate frame of reference.
2. Calculate the volume of a representative layer of water.
3. Multiply the volume by the weight-density of water to get the force.
4. Calculate the distance the layer of water must be lifted.
5. Multiply the force and distance to get an estimate of the work needed to lift the layer of water.
6. Sum the work required to lift all the layers. This expression is an estimate of the work required to pump out the desired amount of water, and it is in the form of a Riemann sum.
7. Take the limit as $n \to \infty$ and evaluate the resulting integral to get the exact work required to pump out the desired amount of water.

We now apply this problem-solving strategy in an example with a noncylindrical tank.

**Example 2.6.4: A Pumping Problem with a Noncylindrical Tank**

Assume a tank in the shape of an inverted cone, with height 12 ft and base radius 4 ft. The tank is full to start with, and water is pumped over the upper edge of the tank until the height of the water remaining in the tank is 4 ft. How much work is required to pump out that amount of water?

**Solution**

The tank is depicted in Figure 2.6.7. As we did in the example with the cylindrical tank, we orient the $x$-axis vertically, with the origin at the top of the tank and the downward direction being positive (step 1).

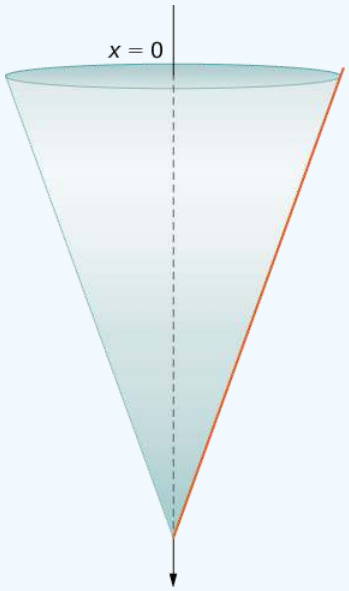



Figure 2.6.7: A water tank in the shape of an inverted cone.

The tank starts out full and ends with $4$ ft of water left, so, based on our chosen frame of reference, we need to partition the interval $[0,8]$. Then, for $i=0,1,2,\ldots,n$, let $P=x_i$ be a regular partition of the interval $[0,8]$, and for $i=1,2,\ldots,n$, choose an arbitrary point $x_i^* \in [x_{i-1},x_i]$. We can approximate the volume of a layer by using a disk, then use similar triangles to find the radius of the disk (Figure 2.6.8).

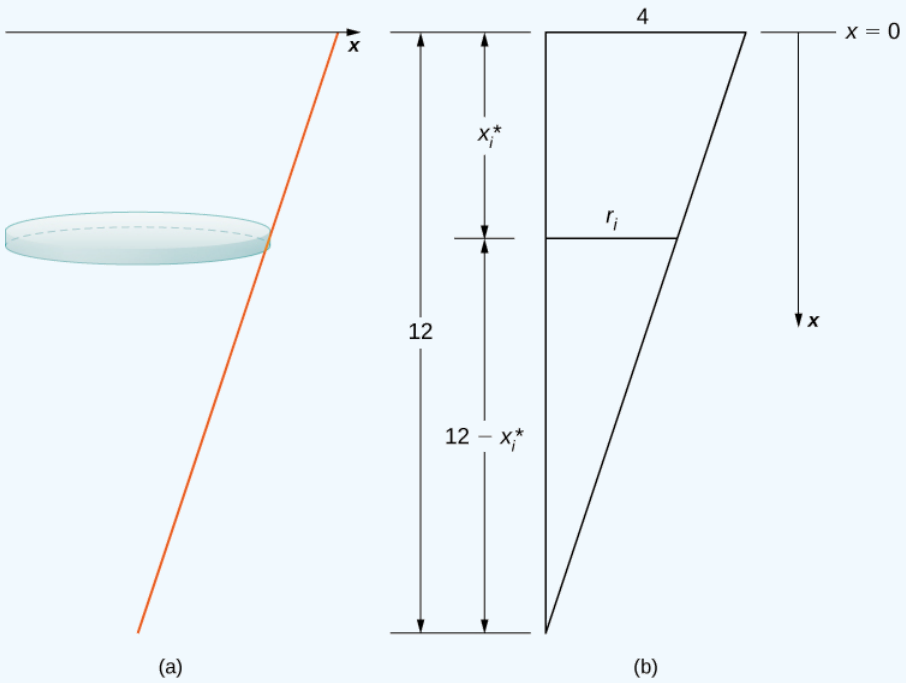



Figure 2.6.8: Using similar triangles to express the radius of a disk of water.

From properties of similar triangles, we have

$$\begin{aligned} \frac{r_i}{12-x_i^*} &= \frac{4}{12} = \frac{1}{3} && \text{(step 1)} \\ 3r_i &= 12-x_i^* \\ r_i &= \frac{12-x_i^*}{3} \\ &= 4-\frac{x_i^*}{3}. \end{aligned}$$

Then the volume of the disk is

$$V_i = \pi\left(4-\frac{x_i^*}{3}\right)^2 \Delta x. \qquad \text{(step 2)}$$

The weight-density of water is $62.4$lb/ft$^3$, so the force needed to lift each layer is approximately

$$F_i \approx 62.4\pi\left(4-\frac{x_i^*}{3}\right)^2 \Delta x \qquad \text{(step 3)}$$

Based on the diagram, the distance the water must be lifted is approximately $x_i^*$ feet (step 4), so the approximate work needed to lift the layer is

$$W_i \approx 62.4\pi x_i^*\left(4-\frac{x_i^*}{3}\right)^2 \Delta x. \qquad \text{(step 5)}$$

Summing the work required to lift all the layers, we get an approximate value of the total work:

$$W=\sum_{i=1}^{n} W_i \approx \sum_{i=1}^{n} 62.4\pi x_i^*\left(4-\frac{x_i^*}{3}\right)^2 \Delta x. \qquad \text{(step 6)}$$

Taking the limit as $n \to \infty$, we obtain

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} 62.4\pi x_i^*(4 - \frac{x_i^*}{3})^2 \Delta x$$

$$= \int_0^8 62.4\pi x \left(4 - \frac{x}{3}\right)^2 dx$$

$$= 62.4\pi \int_0^8 x\left(16 - \frac{8x}{3} + \frac{x^2}{9}\right) dx = 62.4\pi \int_0^8 \left(16x - \frac{8x^2}{3} + \frac{x^3}{9}\right) dx$$

$$= 62.4\pi \left[8x^2 - \frac{8x^3}{9} + \frac{x^4}{36}\right]\Big|_0^8 = 10,649.6\pi \approx 33,456.7.$$

It takes approximately $33,450$ ft-lb of work to empty the tank to the desired level.



**Exercise 2.6.4**

A tank is in the shape of an inverted cone, with height $10$ ft and base radius 6 ft. The tank is filled to a depth of 8 ft to start with, and water is pumped over the upper edge of the tank until 3 ft of water remain in the tank. How much work is required to pump out that amount of water?

**Hint**

Use the process from the previous example.

**Solution**

Approximately $43,255.2$ ft-lb





## Hydrostatic Force and Pressure

In this last section, we look at the force and pressure exerted on an object submerged in a liquid. In the English system, force is measured in pounds. In the metric system, it is measured in newtons. Pressure is force per unit area, so in the English system we have pounds per square foot (or, perhaps more commonly, pounds per square inch, denoted psi). In the metric system we have newtons per square meter, also called **pascals.**

Let's begin with the simple case of a plate of area $A$ submerged horizontally in water at a depth s (Figure 2.6.9). Then, the force exerted on the plate is simply the weight of the water above it, which is given by $F = \rho As$, where $\rho$ is the weight density of water (weight per unit volume). To find the **hydrostatic pressure**—that is, the pressure exerted by water on a submerged object—we divide the force by the area. So the pressure is $p = F/A = \rho s$.

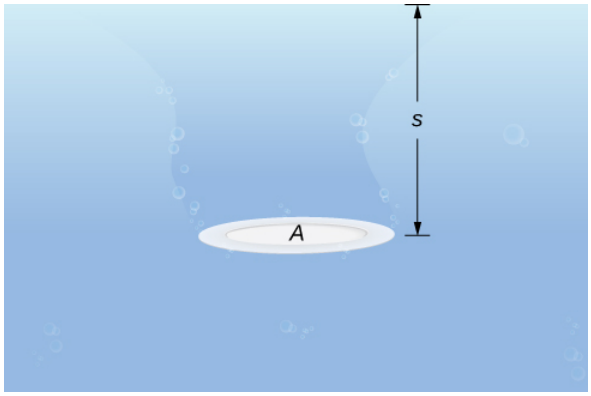



Figure 2.6.9: A plate submerged horizontally in water.

By Pascal's principle, the pressure at a given depth is the same in all directions, so it does not matter if the plate is submerged horizontally or vertically. So, as long as we know the depth, we know the pressure. We can apply Pascal's principle to find the force exerted on surfaces, such as dams, that are oriented vertically. We cannot apply the formula $F = \rho As$ directly, because the depth varies from point to point on a vertically oriented surface. So, as we have done many times before, we form a partition, a Riemann sum, and, ultimately, a definite integral to calculate the force.

Suppose a thin plate is submerged in water. We choose our frame of reference such that the $x$-axis is oriented vertically, with the downward direction being positive, and point $x = 0$ corresponding to a logical reference point. Let $s(x)$ denote the depth at point x. Note we often let $x = 0$ correspond to the surface of the water. In this case, depth at any point is simply given by $s(x) = x$. However, in some cases we may want to select a different reference point for $x = 0$, so we proceed with the development in the more general case. Last, let $w(x)$ denote the width of the plate at the point $x$.

Assume the top edge of the plate is at point $x = a$ and the bottom edge of the plate is at point $x = b$. Then, for $i = 0, 1, 2, \ldots, n$, let $P = x_i$ be a regular partition of the interval $[a, b]$, and for $i = 1, 2, \ldots, n$, choose an arbitrary point $x_i^* \in [x_{i-1}, x_i]$. The partition divides the plate into several thin, rectangular strips (Figure 2.6.10).

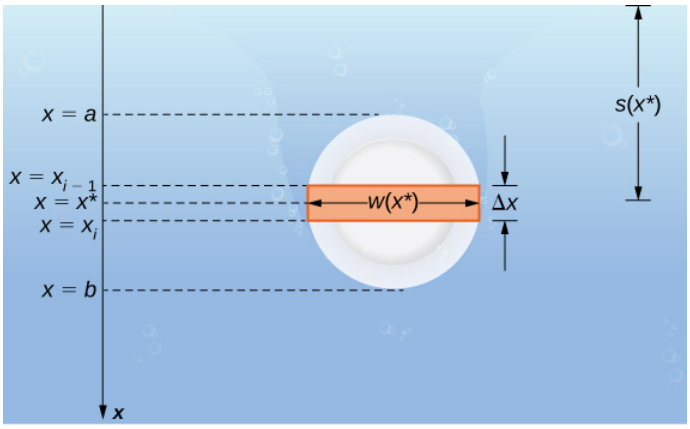



Figure 2.6.10: A thin plate submerged vertically in water.

Let's now estimate the force on a representative strip. If the strip is thin enough, we can treat it as if it is at a constant depth, $s(x_i^*)$. We then have

$$F_i = \rho As = \rho[w(x_i^*)\Delta x]s(x_i^*). \tag{2.6.6}$$

Adding the forces, we get an estimate for the force on the plate:

$$F \approx \sum_{i=1}^{n} F_i = \sum_{i=1}^{n} \rho[w(x_i^*)\Delta x]s(x_i^*). \tag{2.6.7}$$

This is a Riemann sum, so taking the limit gives us the exact force. We obtain

$$F = \lim_{n\to\infty} \sum_{i=1}^{n} \rho[w(x_i^*)\Delta x]s(x_i^*) = \int_a^b \rho w(x)s(x)dx. \tag{2.6.8}$$

Evaluating this integral gives us the force on the plate. We summarize this in the following problem-solving strategy.

**Problem-Solving Strategy: Finding Hydrostatic Force**

1. Sketch a picture and select an appropriate frame of reference. (Note that if we select a frame of reference other than the one used earlier, we may have to adjust Equation 2.6.8 accordingly.)
2. Determine the depth and width functions, $s(x)$ and $w(x)$.
3. Determine the weight-density of whatever liquid with which you are working. The weight-density of water is $62.4\,\text{lb/ft}^3$, or $9800\,\text{N/m}^3$.
4. Use the equation to calculate the total force.

**Example 2.6.5: Finding Hydrostatic Force**

A water trough 15 ft long has ends shaped like inverted isosceles triangles, with base 8 ft and height 3 ft. Find the force on one end of the trough if the trough is full of water.

**Solution**

Figure 2.6.11 shows the trough and a more detailed view of one end.

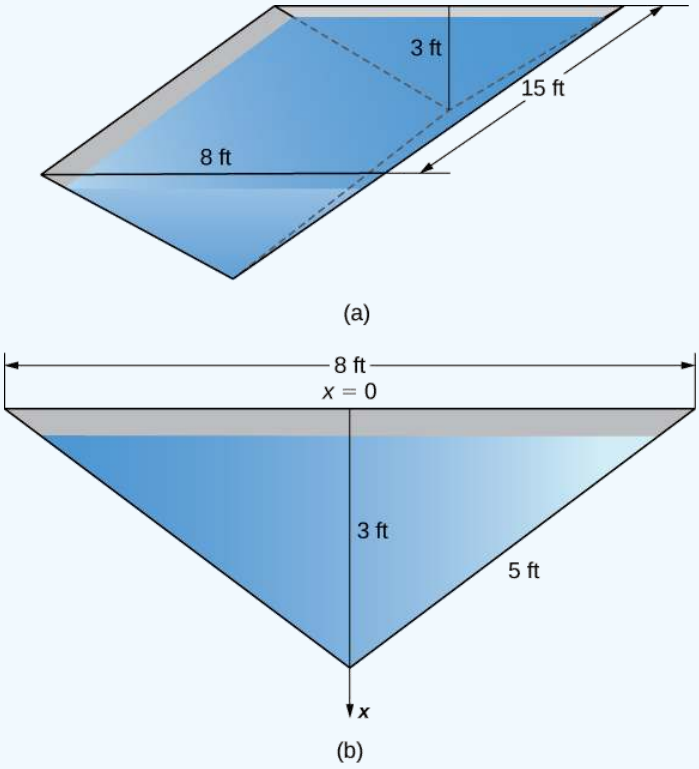



Figure 2.6.11: (a) A water trough with a triangular cross-section. (b) Dimensions of one end of the water trough.

Select a frame of reference with the $x$-axis oriented vertically and the downward direction being positive. Select the top of the trough as the point corresponding to $x = 0$ (step 1). The depth function, then, is $s(x) = x$. Using similar triangles, we see that $w(x) = 8 - (8/3)x$ (step 2). Now, the weight density of water is $62.4\,\text{lb/ft}^3$ (step 3), so applying Equation 2.6.8, we obtain

$$F = \int_a^b \rho w(x) s(x) dx$$

$$= \int_0^3 62.4 \left(8 - \frac{8}{3}x\right) x \, dx = 62.4 \int_0^3 \left(8x - \frac{8}{3}x^2\right) dx$$

$$= 62.4 \left[4x^2 - \frac{8}{9}x^3\right]\Big|_0^3 = 748.8.$$

The water exerts a force of 748.8 lb on the end of the trough (step 4).



**Exercise 2.6.5**

A water trough 12 m long has ends shaped like inverted isosceles triangles, with base 6 m and height 4 m. Find the force on one end of the trough if the trough is full of water.

**Hint**

Follow the problem-solving strategy and the process from the previous example.

**Solution**

156,800 N



## Engineering an Aquarium Window to Handle the Fluid Force

You are an engineer that is responsible for designing a window at an aquarium that will handle the fluid force on a circular aquarium window. You will be given how far the top of the window is from the surface of the sea water and will be given the radius of the window that is shown below. The weight density of the sea water in this aquarium is 10,094 Newtons per cubic meter. You must enter in the total fluid force on the window within an accuracy of 5% or your design will result in a disaster killing all the fish and injuring the visitors who are near the window when it breaks. Click on the Start button to see the first window.

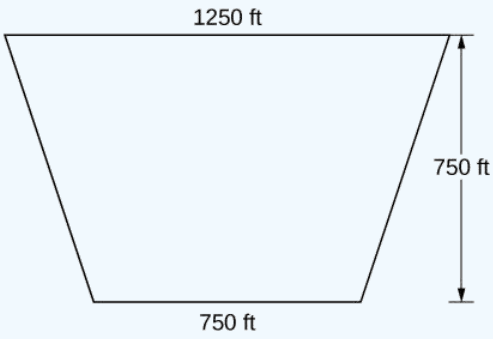
Click on the New Problem button to get started.

New Window

### Example 2.6.6: Finding Hydrostatic Force

We now return our attention to the Hoover Dam, mentioned at the beginning of this chapter. The actual dam is arched, rather than flat, but we are going to make some simplifying assumptions to help us with the calculations. Assume the face of the Hoover Dam is shaped like an isosceles trapezoid with lower base 750 ft, upper base 1250 ft, and height 750 ft (see the following figure).






When the reservoir is full, Lake Mead's maximum depth is about 530 ft, and the surface of the lake is about 10 ft below the top of the dam (see the following figure).

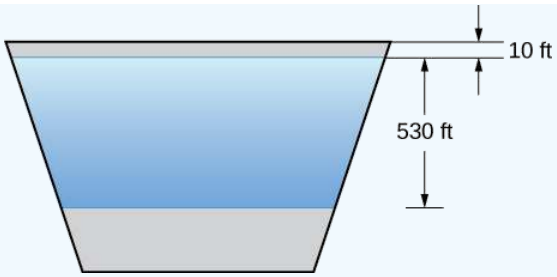



Figure **2.6.12**: A simplified model of the Hoover Dam with assumed dimensions.

a. Find the force on the face of the dam when the reservoir is full.
b. The southwest United States has been experiencing a drought, and the surface of Lake Mead is about 125 ft below where it would be if the reservoir were full. What is the force on the face of the dam under these circumstances?

**Solution**:

a.

We begin by establishing a frame of reference. As usual, we choose to orient the $x$-axis vertically, with the downward direction being positive. This time, however, we are going to let $x = 0$ represent the top of the dam, rather than the surface of the water. When the reservoir is full, the surface of the water is $10$ ft below the top of the dam, so $s(x) = x - 10$ (see the following figure).

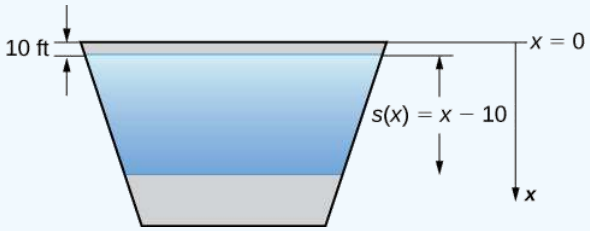



Figure **2.6.13**: We first choose a frame of reference.

To find the width function, we again turn to similar triangles as shown in the figure below.

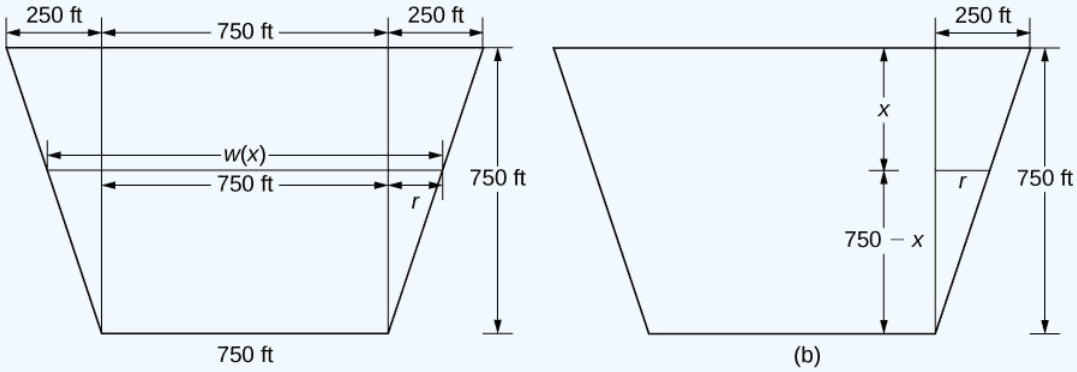



Figure **2.6.14**: We use similar triangles to determine a function for the width of the dam. (a) Assumed dimensions of the dam; (b) highlighting the similar triangles.

From the figure, we see that $w(x) = 750 + 2r$. Using properties of similar triangles, we get $r = 250 - (1/3)x$. Thus,

$$w(x) = 1250 - \frac{2}{3}x \qquad \text{(step 2)}$$

Using a weight-density of $62.4$lb/ft$^3$ (step 3) and applying Equation 2.6.8, we get

$$F = \int_a^b \rho w(x) s(x)\, dx$$
$$= \int_{10}^{540} 62.4 \left(1250 - \frac{2}{3}x\right)(x - 10)\, dx$$
$$= 62.4 \int_{10}^{540} -\frac{2}{3}[x^2 - 1885x + 18750]\, dx$$
$$= -62.4 \left(\frac{2}{3}\right)\left[\frac{x^3}{3} - \frac{1885x^2}{2} + 18750x\right]\Big|_{10}^{540} \approx 8,832,245,000\,\text{lb} = 4,416,122.5\,\text{t}.$$

Note the change from pounds to tons (2000lb = 1 ton) (step 4). This changes our depth function, $s(x)$, and our limits of integration. We have $s(x) = x - 135$. The lower limit of integration is 135. The upper limit remains 540. Evaluating the integral, we get

$$F = \int_a^b \rho w(x) s(x)\, dx$$
$$= \int_{135}^{540} 62.4 \left(1250 - \frac{2}{3}x\right)(x - 135)\, dx$$
$$= -62.4(\frac{2}{3}) \int_{135}^{540} (x - 1875)(x - 135)\, dx = -62.4 \left(\frac{2}{3}\right) \int_{135}^{540} (x^2 - 2010x + 253125)\, dx$$
$$= -62.4 \left(\frac{2}{3}\right)\left[\frac{x^3}{3} - 1005x^2 + 253125x\right]\Big|_{135}^{540} \approx 5,015,230,000\,\text{lb} = 2,507,615\,\text{t}.$$

**Exercise 2.6.6**

When the reservoir is at its average level, the surface of the water is about 50 ft below where it would be if the reservoir were full. What is the force on the face of the dam under these circumstances?

**Hint**

Change the depth function, $s(x)$, and the limits of integration.

**Solution**

Approximately 7,164,520,000 lb or 3,582,260 t

## Key Concepts

- Several physical applications of the definite integral are common in engineering and physics.
- Definite integrals can be used to determine the mass of an object if its density function is known.
- Work can also be calculated from integrating a force function, or when counteracting the force of gravity, as in a pumping problem.
- Definite integrals can also be used to calculate the force exerted on an object submerged in a liquid.

## Key Equations

- **Mass of a one-dimensional object**

  $$m = \int_a^b \rho(x) dx$$

- **Mass of a circular object**

  $$m = \int_0^r 2\pi x \rho(x) dx$$

- **Work done on an object**

  $$W = \int_a^b F(x) dx$$

- **Hydrostatic force on a plate**

$$F = \int_a^b \rho w(x) s(x) dx$$

## Glossary

**density function**

a density function describes how mass is distributed throughout an object; it can be a linear density, expressed in terms of mass per unit length; an area density, expressed in terms of mass per unit area; or a volume density, expressed in terms of mass per unit volume; weight-density is also used to describe weight (rather than mass) per unit volume

**Hooke's law**

this law states that the force required to compress (or elongate) a spring is proportional to the distance the spring has been compressed (or stretched) from equilibrium; in other words, $F = kx$, where $k$ is a constant

**hydrostatic pressure**

the pressure exerted by water on a submerged object

**work**

the amount of energy it takes to move an object; in physics, when a force is constant, work is expressed as the product of force and distance

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# 2.6E: Exercises for Section 2.6

## Basic Work Problems

**For exercises 1 - 6, find the work done.**

1) Find the work done when a constant force $F = 12$ lb moves a chair from $x = 0.9$ to $x = 1.1$ ft.

2) How much work is done when a person lifts a $50$ lb box of comics onto a truck that is $3$ ft off the ground?

**Answer**

$W = 150$ ft-lb

3) What is the work done lifting a $20$ kg child from the floor to a height of $2$ m? (Note that $1$ kg equates to $9.8$ N)

4) Find the work done when you push a box along the floor $2$ m, when you apply a constant force of $F = 100$ N.

**Answer**

$W = 200$ J

5) Compute the work done for a force $F = \frac{12}{x^2}$ N from $x = 1$ to $x = 2$ m.

6) What is the work done moving a particle from $x = 0$ to $x = 1$ m if the force acting on it is $F = 3x^2$ N?

**Answer**

$W = 1$ J

## Density Problems

**In exercises 7 - 11, find the mass of the one-dimensional object.**

7) A wire that is $2$ ft long (starting at $x = 0$) and has a density function of $\rho(x) = x^2 + 2x$ lb/ft

8) A car antenna that is $3$ ft long (starting at $x = 0$) and has a density function of $\rho(x) = 3x + 2$ lb/ft

**Answer**

$\frac{39}{2}$

9) A metal rod that is $8$ in. long (starting at $x = 0$) and has a density function of $\rho(x) = e^{1/2x}$ lb/in.

10) A pencil that is $4$ in. long (starting at $x = 2$) and has a density function of $\rho(x) = \frac{5}{x}$ oz/in.

**Answer**

$\ln(243)$

11) A ruler that is $12$ in. long (starting at $x = 5$) and has a density function of $\rho(x) = \ln(x) + (1/2)x^2$ oz/in.

**In exercises 12 - 16, find the mass of the two-dimensional object that is centered at the origin.**

12) An oversized hockey puck of radius $2$ in. with density function $\rho(x) = x^3 - 2x + 5$

**Answer**

$\frac{332\pi}{15}$

13) A frisbee of radius $6$ in. with density function $\rho(x) = e^{-x}$

14) A plate of radius $10$ in. with density function $\rho(x) = 1 + \cos(\pi x)$

**Answer**

$100\pi$

15) A jar lid of radius $3$ in. with density function $\rho(x) = \ln(x+1)$

16) A disk of radius $5$ cm with density function $\rho(x) = \sqrt{3x}$

**Answer**

$20\pi\sqrt{15}$

### Spring Work Problems

17) A $12$-in. spring is stretched to $15$ in. by a force of $75$ lb. What is the spring constant?

18) A spring has a natural length of $10$ cm. It takes $2$ J to stretch the spring to $15$ cm. How much work would it take to stretch the spring from $15$ cm to $20$ cm?

**Answer**

$W = 6$ J

19) A $1$-m spring requires $10$ J to stretch the spring to $1.1$ m. How much work would it take to stretch the spring from $1$ m to $1.2$ m?

20) A spring requires $5$ J to stretch the spring from $8$ cm to $12$ cm, and an additional $4$ J to stretch the spring from $12$ cm to $14$ cm. What is the natural length of the spring?

**Answer**

The natural length is$5$ cm.

21) A shock absorber is compressed 1 in. by a weight of 1 ton. What is the spring constant?

22) A force of $F = (20x - x^3)$ N stretches a nonlinear spring by $x$ meters. What work is required to stretch the spring from $x = 0$ to $x = 2$ m?

**Answer**

$W = 36$ J

### Cable and Chain Work Problems

23) Find the work done by winding up a hanging cable of length $100$ ft and weight-density $5$ lb/ft.

24) For the cable in the preceding exercise, how much work is done to lift the cable $50$ ft?

**Answer**

$W = 18,750$ ft-lb

25) For the cable in the preceding exercise, how much additional work is done by hanging a $200$ lb weight at the end of the cable?

### Pyramid & Satellite/Rocket Work Problems

26) [T] A pyramid of height $500$ ft has a square base $800$ ft by $800$ ft. Find the area $A$ at height $h$. If the rock used to build the pyramid weighs approximately $w = 100\,\text{lb/ft}^3$, how much work did it take to lift all the rock?

**Answer**

$W = \frac{32}{3} \times 10^9$ ft-lb

27) [T] For the pyramid in the preceding exercise, assume there were $1000$ workers each working $10$ hours a day, $5$ days a week, $50$ weeks a year. If each of the workers, on average, lifted ten 100-lb rocks $2$ ft/hr, how long did it take to build the pyramid?

28) [T] The force of gravity on a mass $m$ is $F = -((GMm)/x^2)$ newtons. For a rocket of mass $m = 1000$ kg, compute the work to lift the rocket from $x = 6400$ to $x = 6500$ km. (Note: $G = 6 \times 10^{-17}\,\text{N m}^2/\text{kg}^2$ and $M = 6 \times 10^{24}$ kg.)

**Answer**

$W = 8.65 \times 10^5$ J

29) [T] For the rocket in the preceding exercise, find the work to lift the rocket from $x = 6400$ to $x = \infty$.

### Hydrostatic Force and Pressure

30) [T] A rectangular dam is $40$ ft high and $60$ ft wide. Compute the total force $F$ on the dam when

a. the surface of the water is at the top of the dam and

b. the surface of the water is halfway down the dam.

**Answer**

a. $3,000,000$ lb,
b. $749,000$ lb

### Pumping Work Problems

31) [T] Find the work required to pump all the water out of a cylinder that has a circular base of radius $5$ft and height $200$ ft. Use the fact that the density of water is $62$ lb/ft$^3$.

32) [T] Find the work required to pump all the water out of the cylinder in the preceding exercise if the cylinder is only half full.

**Answer**

$W = 23.25\pi$ million ft-lb

33) [T] How much work is required to pump out a swimming pool if the area of the base is $800\,\text{ft}^2$, the water is $4$ ft deep, and the top is $1$ ft above the water level? Assume that the density of water is $62$ lb/ft$^3$.

34) A cylinder of depth $H$ and cross-sectional area $A$ stands full of water at density $\rho$. Compute the work to pump all the water to the top.

**Answer**

$$W = \frac{A\rho H^2}{2}$$

35) For the cylinder in the preceding exercise, compute the work to pump all the water to the top if the cylinder is only half full.

36) A cone-shaped tank has a cross-sectional area that increases with its depth: $A = \frac{\pi r^2 h^2}{H^3}$. Show that the work to empty it is half the work for a cylinder with the same height and base.

**Answer**

Answers may vary.

## Contributors

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# 2.7: Moments and Centers of Mass

**Learning Objectives**

- Find the center of mass of objects distributed along a line.
- Locate the center of mass of a thin plate.
- Use symmetry to help locate the centroid of a thin plate.
- Apply the theorem of Pappus for volume.

In this section, we consider centers of mass (also called *centroids*, under certain conditions) and moments. The basic idea of the center of mass is the notion of a balancing point. Many of us have seen performers who spin plates on the ends of sticks. The performers try to keep several of them spinning without allowing any of them to drop. If we look at a single plate (without spinning it), there is a sweet spot on the plate where it balances perfectly on the stick. If we put the stick anywhere other than that sweet spot, the plate does not balance and it falls to the ground. (That is why performers spin the plates; the spin helps keep the plates from falling even if the stick is not exactly in the right place.) Mathematically, that sweet spot is called the *center of mass of the plate.*

In this section, we first examine these concepts in a one-dimensional context, then expand our development to consider centers of mass of two-dimensional regions and symmetry. Last, we use centroids to find the volume of certain solids by applying the theorem of Pappus.

## Center of Mass and Moments

Let's begin by looking at the center of mass in a one-dimensional context. Consider a long, thin wire or rod of negligible mass resting on a fulcrum, as shown in Figure **2.7.1***a*. Now suppose we place objects having masses $m_1$ and $m_2$ at distances $d_1$ and $d_2$ from the fulcrum, respectively, as shown in Figure **2.7.1***b*.

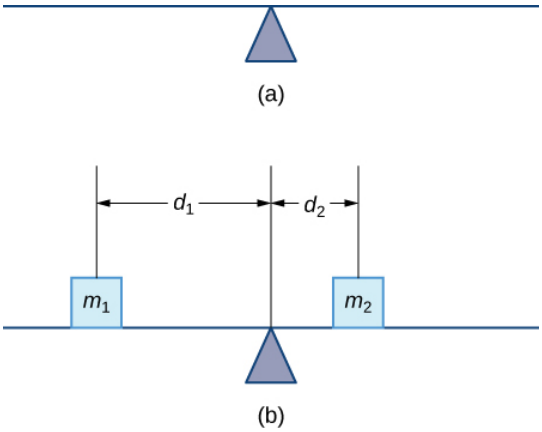



Figure **2.7.1**: (a) A thin rod rests on a fulcrum. (b) Masses are placed on the rod.

The most common real-life example of a system like this is a playground seesaw, or teeter-totter, with children of different weights sitting at different distances from the center. On a seesaw, if one child sits at each end, the heavier child sinks down and the lighter child is lifted into the air. If the heavier child slides in toward the center, though, the seesaw balances. Applying this concept to the masses on the rod, we note that the masses balance each other if and only if

$$m_1 d_1 = m_2 d_2.$$

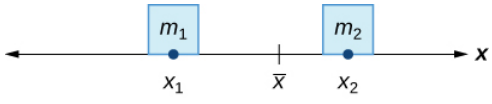



Figure **2.7.2**: The center of mass $\bar{x}$ is the balance point of the system.

In the seesaw example, we balanced the system by moving the masses (children) with respect to the fulcrum. However, we are really interested in systems in which the masses are not allowed to move, and instead we balance the system by moving the

fulcrum. Suppose we have two point masses, $m_1$ and $m_2$, located on a number line at points $x_1$ and $x_2$, respectively (Figure 2.7.2). The center of mass, $\bar{x}$, is the point where the fulcrum should be placed to make the system balance.

Thus, we have

$$\begin{aligned} m_1|x_1 - \bar{x}| &= m_2|x_2 - \bar{x}| \\ m_1(\bar{x} - x_1) &= m_2(x_2 - \bar{x}) \\ m_1\bar{x} - m_1x_1 &= m_2x_2 - m_2\bar{x} \\ \bar{x}(m_1 + m_2) &= m_1x_1 + m_2x_2 \end{aligned}$$

or

$$\bar{x} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} \tag{2.7.1}$$

The expression in the numerator of Equation 2.7.1, $m_1x_1 + m_2x_2$, is called the *first moment of the system* with respect to the origin. If the context is clear, we often drop the word first and just refer to this expression as the **moment** of the system. The expression in the denominator, $m_1 + m_2$, is the total mass of the system. Thus, the **center of mass** of the system is the point at which the total mass of the system could be concentrated without changing the moment.

This idea is not limited just to two point masses. In general, if $n$ masses, $m_1, m_2, \ldots, m_n$, are placed on a number line at points $x_1, x_2, \ldots, x_n$, respectively, then the center of mass of the system is given by

$$\bar{x} = \frac{\sum_{i=1}^{n} m_ix_i}{\sum_{i=1}^{n} m_i} \tag{2.7.2}$$

### Center of Mass of Objects on a Line

Let $m_1, m_2, \ldots, m_n$ be point masses placed on a number line at points $x_1, x_2, \ldots, x_n$, respectively, and let $m = \sum_{i=1}^{n} m_i$ denote the total mass of the system. Then, the *moment of the system* with respect to the origin is given by

$$M = \sum_{i=1}^{n} m_ix_i \tag{2.7.3}$$

and the *center of mass of the system* is given by

$$\bar{x} = \frac{M}{m}. \tag{2.7.4}$$

We apply this theorem in the following example.

### Example 2.7.1: Finding the Center of Mass of Objects along a Line

Suppose four point masses are placed on a number line as follows:

- $m_1 = 30\,kg$, placed at $x_1 = -2m$
- $m_2 = 5\,kg$, placed at $x_2 = 3m$
- $m_3 = 10\,kg$, placed at $x_3 = 6m$
- $m_4 = 15\,kg$, placed at $x_4 = -3m$.

**Solution**

Find the moment of the system with respect to the origin and find the center of mass of the system.

First, we need to calculate the moment of the system (Equation 2.7.3):

$$\begin{aligned} M &= \sum_{i=1}^{4} m_i x_i \\ &= -60 + 15 + 60 - 45 \\ &= -30. \end{aligned}$$

Now, to find the center of mass, we need the total mass of the system:

$$\begin{aligned} m &= \sum_{i=1}^{4} m_i \\ &= 30 + 5 + 10 + 15 \\ &= 60\,kg \end{aligned}$$

Then we have (from Equation 2.7.4)

$$\bar{x} = \frac{M}{m} = -\frac{30}{60} = -\frac{1}{2}.$$

The center of mass is located 1/2 m to the left of the origin.

## Find the Center of Mass of a Teeter-Totter



Right now the 4 meter long teeter-totter is placed so that the brothers are not balanced. After being given the masses of the two brothers, your job will be to decide where to place the center holder so that the brothers are balanced. You must place it within 5% of where it needs to be placed.

Begin

Picture: Brothers on the teeter-totter by Quinn Dombrowski

**Exercise 2.7.1**

Suppose four point masses are placed on a number line as follows:

- $m_1 = 12\,kg$ placed at $x_1 = -4m$
- $m_2 = 12\,kg$ placed at $x_2 = 4m$
- $m_3 = 30\,kg$ placed at $x_3 = 2m$
- $m_4 = 6\,kg$, placed at $x_4 = -6m$.

Find the moment of the system with respect to the origin and find the center of mass of the system.

**Hint**

Use the process from the previous example.

**Answer**

$M = 24, \bar{x} = \frac{2}{5}m$

We can generalize this concept to find the center of mass of a system of point masses in a plane. Let $m_1$ be a point mass located at point $(x_1, y_1)$ in the plane. Then the moment $M_x$ of the mass with respect to the $x$-axis is given by $M_x = m_1 y_1$. Similarly, the moment $M_y$ with respect to the $y$-axis is given by

$$M_y = m_1 x_1. \tag{2.7.5}$$

Notice that the $x$-coordinate of the point is used to calculate the moment with respect to the $y$-axis, and vice versa. The reason is that the $x$-coordinate gives the distance from the point mass to the $y$-axis, and the $y$-coordinate gives the distance to the $x$-axis (see the following figure).

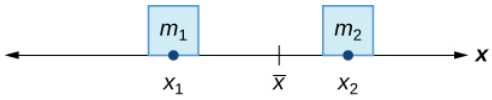



Figure **2.7.3**: Point mass $m_1$ is located at point $(x_1, y_1)$ in the plane.

If we have several point masses in the $xy$-plane, we can use the moments with respect to the $x$- and $y$-axes to calculate the $x$- and $y$-coordinates of the center of mass of the system.

### Center of Mass of Objects in a Plane

Let $m_1$, $m_2$, …, $m_n$ be point masses located in the $xy$-plane at points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, respectively, and let $m = \sum_{i=1}^{n} m_i$ denote the total mass of the system. Then the moments $M_x$ and $M_y$ of the system with respect to the $x$- and $y$-axes, respectively, are given by

$$M_x = \sum_{i=1}^{n} m_i y_i \tag{2.7.6}$$

and

$$M_y = \sum_{i=1}^{n} m_i x_i. \tag{2.7.7}$$

Also, the coordinates of the center of mass $(\bar{x}, \bar{y})$ of the system are

$$\bar{x} = \frac{M_y}{m} \tag{2.7.8}$$

and

$$\bar{y} = \frac{M_x}{m}. \tag{2.7.9}$$

The next example demonstrates how to the center of mass formulas (Equations 2.7.6 - 2.7.9) may be applied.

### Example 2.7.2: Finding the Center of Mass of Objects in a Plane

Suppose three point masses are placed in the $xy$-plane as follows (assume coordinates are given in meters):

- $m_1 = 2\,kg$ placed at $(-1, 3)$,
- $m_2 = 6\,kg$ placed at $(1, 1)$,
- $m_3 = 4\,kg$ placed at $(2, -2)$.

Find the center of mass of the system.

**Solution**

First we calculate the total mass of the system:

$$m = \sum_{i=1}^{3} m_i = 2 + 6 + 4 = 12\,kg.$$

Next we find the moments with respect to the $x$- and $y$-axes:

$$M_y = \sum_{i=1}^{3} m_i x_i = -2 + 6 + 8 = 12,$$

$$M_x = \sum_{i=1}^{3} m_i y_i = 6 + 6 - 8 = 4.$$

Then we have

$$\bar{x} = \frac{M_y}{m} = \frac{12}{12} = 1$$

and

$$\bar{y} = \frac{M_x}{m} = \frac{4}{12} = \frac{1}{3}.$$

The center of mass of the system is $(1, 1/3)$, in meters.

**Exercise 2.7.2**

Suppose three point masses are placed on a number line as follows (assume coordinates are given in meters):

- $m_1 = 5\,kg$, placed at $(-2, -3)$,
- $m_2 = 3\,kg$, placed at $(2, 3)$,
- $m_3 = 2\,kg$, placed at $(-3, -2)$.

Find the center of mass of the system.

**Hint**

Use the process from the previous example.

**Answer**

$(-1, -1)$ m

## Center of Mass of Thin Plates

So far we have looked at systems of point masses on a line and in a plane. Now, instead of having the mass of a system concentrated at discrete points, we want to look at systems in which the mass of the system is distributed continuously across a thin sheet of material. For our purposes, we assume the sheet is thin enough that it can be treated as if it is two-dimensional. Such a sheet is called a **lamina**. Next we develop techniques to find the center of mass of a lamina. In this section, we also assume the density of the lamina is constant.

Laminas are often represented by a two-dimensional region in a plane. The geometric center of such a region is called its **centroid**. Since we have assumed the density of the lamina is constant, the center of mass of the lamina depends only on the shape of the corresponding region in the plane; it does not depend on the density. In this case, the center of mass of the lamina corresponds to the centroid of the delineated region in the plane. As with systems of point masses, we need to find the total mass of the lamina, as well as the moments of the lamina with respect to the $x$- and $y$-axes.

We first consider a lamina in the shape of a rectangle. Recall that the center of mass of a lamina is the point where the lamina balances. For a rectangle, that point is both the horizontal and vertical center of the rectangle. Based on this understanding, it is

clear that the center of mass of a rectangular lamina is the point where the diagonals intersect, which is a result of the **symmetry principle**, and it is stated here without proof.

**The Symmetry Principle**

If a region $R$ is symmetric about a line $l$, then the centroid of $R$ lies on $l$.

Let's turn to more general laminas. Suppose we have a lamina bounded above by the graph of a continuous function $f(x)$, below by the $x$-axis, and on the left and right by the lines $x = a$ and $x = b$, respectively, as shown in the following figure.

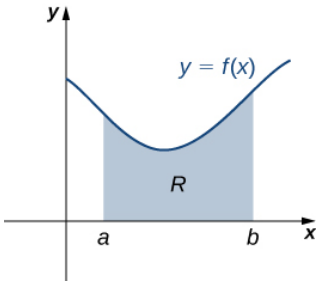



Figure 2.7.4: A region in the plane representing a lamina.

As with systems of point masses, to find the center of mass of the lamina, we need to find the total mass of the lamina, as well as the moments of the lamina with respect to the $x$- and $y$-axes. As we have done many times before, we approximate these quantities by partitioning the interval $[a, b]$ and constructing rectangles.

For $i = 0, 1, 2, \ldots, n$, let $P = x_i$ be a regular partition of $[a, b]$. Recall that we can choose any point within the interval $[x_{i-1}, x_i]$ as our $x_i^*$. In this case, we want $x_i^*$ to be the $x$-coordinate of the centroid of our rectangles. Thus, for $i = 1, 2, \ldots, n$, we select $x_i^* \in [x_{i-1}, x_i]$ such that $x_i^*$ is the midpoint of the interval. That is, $x_i^* = (x_{i-1} + x_i)/2$. Now, for $i = 1, 2, \ldots, n$, construct a rectangle of height $f(x_i^*)$ on $[x_{i-1}, x_i]$. The center of mass of this rectangle is $(x_i^*, (f(x_i^*))/2)$, as shown in the following figure.

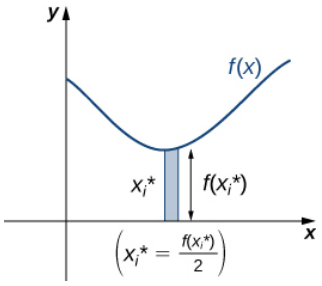



Figure 2.7.5: A representative rectangle of the lamina.

Next, we need to find the total mass of the rectangle. Let $\rho$ represent the density of the lamina (note that $\rho$ is a constant). In this case, $\rho$ is expressed in terms of mass per unit area. Thus, to find the total mass of the rectangle, we multiply the area of the rectangle by $\rho$. Then, the mass of the rectangle is given by $\rho f(x_i^*)\Delta x$.

To get the approximate mass of the lamina, we add the masses of all the rectangles to get

$$m \approx \sum_{i=1}^{n} \rho f(x_i^*)\Delta x. \tag{2.7.10}$$

Equation 2.7.10 is a Riemann sum. Taking the limit as $n \to \infty$ gives the exact mass of the lamina:

$$\begin{aligned} m &= \lim_{n\to\infty} \sum_{i=1}^{n} \rho f(x_i^*)\Delta x \\ &= \rho \int_a^b f(x)dx. \end{aligned}$$

Next, we calculate the moment of the lamina with respect to the x-axis. Returning to the representative rectangle, recall its center of mass is $(x_i^*, (f(x_i^*))/2)$. Recall also that treating the rectangle as if it is a point mass located at the center of mass does not change

the moment. Thus, the moment of the rectangle with respect to the x-axis is given by the mass of the rectangle, $\rho f(x_i^*)\Delta x$, multiplied by the distance from the center of mass to the x-axis: $(f(x_i^*))/2$. Therefore, the moment with respect to the x-axis of the rectangle is $\rho([f(x_i^*)]^2/2)\Delta x$. Adding the moments of the rectangles and taking the limit of the resulting Riemann sum, we see that the moment of the lamina with respect to the x-axis is

$$\begin{aligned} M_x &= \lim_{n\to\infty}\sum_{i=1}^{n}\rho\frac{[f(x_i^*)]^2}{2}\Delta x \\ &= \rho\int_a^b \frac{[f(x)]^2}{2}dx. \end{aligned}$$

We derive the moment with respect to the y-axis similarly, noting that the distance from the center of mass of the rectangle to the *y*-axis is $x_i^*$. Then the moment of the lamina with respect to the *y*-axis is given by

$$\begin{aligned} M_y &= \lim_{n\to\infty}\sum_{i=1}^{n}\rho x_i^* f(x^*)i)\Delta x \\ &= \rho\int_a^b xf(x)dx. \end{aligned}$$

We find the coordinates of the center of mass by dividing the moments by the total mass to give $\bar{x} = M_y/m$ and $\bar{y} = M_x/m$. If we look closely at the expressions for $M_x, M_y$, and $m$, we notice that the constant $\rho$ cancels out when $\bar{x}$ and $\bar{y}$ are calculated.

We summarize these findings in the following theorem.

**Center of Mass of a Thin Plate in the xy-Plane**

Let R denote a region bounded above by the graph of a continuous function $f(x)$, below by the *x*-axis, and on the left and right by the lines $x = a$ and $x = b$, respectively. Let $\rho$ denote the density of the associated lamina. Then we can make the following statements:

i. The mass of the lamina is

$$m = \rho\int_a^b f(x)dx. \tag{2.7.11}$$

ii. The moments $M_x$ and $M_y$ of the lamina with respect to the *x*- and *y*-axes, respectively, are

$$M_x = \rho\int_a^b \frac{[f(x)]^2}{2}dx \tag{2.7.12}$$

and

$$M_y = \rho\int_a^b xf(x)dx. \tag{2.7.13}$$

iii. The coordinates of the center of mass $(\bar{x}, \bar{y})$ are

$$\bar{x} = \frac{M_y}{m} \tag{2.7.14}$$

and

$$\bar{y} = \frac{M_x}{m}. \tag{2.7.15}$$

In the next example, we use this theorem to find the center of mass of a lamina.

**Example 2.7.3: Finding the Center of Mass of a Lamina**

Let R be the region bounded above by the graph of the function $f(x) = \sqrt{x}$ and below by the *x*-axis over the interval $[0, 4]$. Find the centroid of the region.

**Solution**

The region is depicted in the following figure.

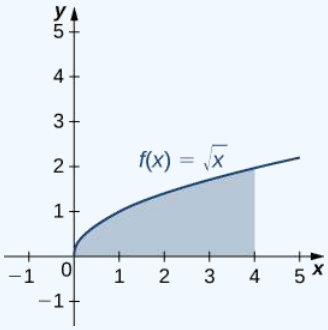



Figure **2.7.6**: Finding the center of mass of a lamina.

Since we are only asked for the centroid of the region, rather than the mass or moments of the associated lamina, we know the density constant $\rho$ cancels out of the calculations eventually. Therefore, for the sake of convenience, let's assume $\rho = 1$.

First, we need to calculate the total mass (Equation 2.7.11):

$$\begin{aligned} m &= \rho \int_a^b f(x)dx \\ &= \int_0^4 \sqrt{x}dx \\ &= \frac{2}{3}x^{3/2}\,|_0^4 \\ &= \frac{2}{3}[8-0] \\ &= \frac{16}{3}. \end{aligned}$$

Next, we compute the moments (Equation 2.7.14):

$$\begin{aligned} M_x &= \rho \int_a^b \frac{[f(x)]^2}{2}dx \\ &= \int_0^4 \frac{x}{2}dx \\ &= \frac{1}{4}x^2\,|_0^4 \\ &= 4 \end{aligned}$$

and (Equation 2.7.13):

$$\begin{aligned} M_y &= \rho \int_a^b xf(x)dx \\ &= \int_0^4 x\sqrt{x}dx \\ &= \int_0^4 x^{3/2}dx \\ &= \frac{2}{5}x^{5/2}\,|_0^4 \\ &= \frac{2}{5}[32-0] \\ &= \frac{64}{5}. \end{aligned}$$

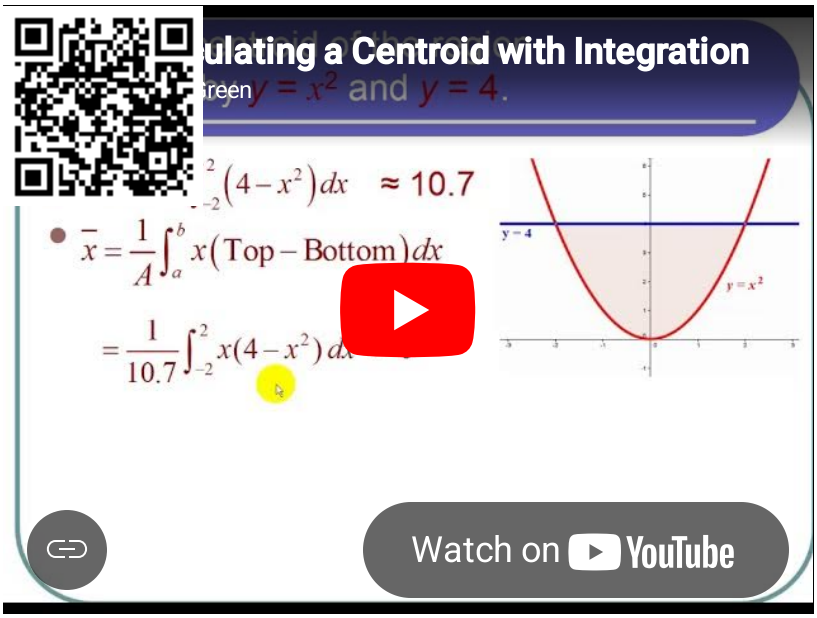
Thus, we have (Equation 2.7.14):

$$\begin{aligned}\bar{x} &= \frac{M_y}{m} \\ &= \frac{64/5}{16/3} \\ &= \frac{64}{5} \cdot \frac{3}{16} \\ &= \frac{12}{5}\end{aligned}$$

and (Equation 2.7.15):

$$\begin{aligned}\bar{y} &= \frac{M_x}{y} \\ &= \frac{4}{16/3} \\ &= 4 \cdot \frac{3}{16} \\ &= \frac{3}{4}.\end{aligned}$$

The centroid of the region is $(12/5, 3/4)$.




**Exercise 2.7.3**

Let $R$ be the region bounded above by the graph of the function $f(x) = x^2$ and below by the $x$-axis over the interval $[0, 2]$. Find the centroid of the region.

**Hint**

Use the process from the previous example.

**Answer**

The centroid of the region is $(3/2, 6/5)$.

We can adapt this approach to find centroids of more complex regions as well. Suppose our region is bounded above by the graph of a continuous function $f(x)$, as before, but now, instead of having the lower bound for the region be the $x$-axis, suppose the region is bounded below by the graph of a second continuous function, $g(x)$, as shown in Figure 2.7.7.

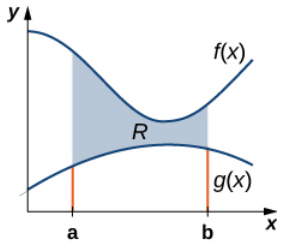



Figure 2.7.7: A region between two functions.

Again, we partition the interval $[a, b]$ and construct rectangles. A representative rectangle is shown in Figure 2.7.8.

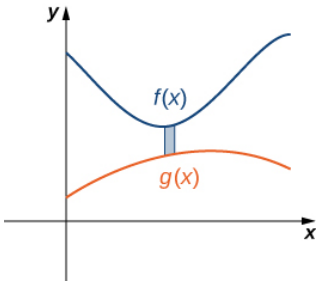



Figure 2.7.8: A representative rectangle of the region between two functions.

Note that the centroid of this rectangle is $(x_i^*, (f(x_i^*) + g(x_i^*))/2)$. We won't go through all the details of the Riemann sum development, but let's look at some of the key steps. In the development of the formulas for the mass of the lamina and the moment with respect to the *y*-axis, the height of each rectangle is given by $f(x_i^*) - g(x_i^*)$, which leads to the expression $f(x) - g(x)$ in the integrands.

In the development of the formula for the moment with respect to the x-axis, the moment of each rectangle is found by multiplying the area of the rectangle, $\rho[f(x_i^*) - g(x_i^*)]\Delta x$, by the distance of the centroid from the $x$-axis, $(f(x_i^*) + g(x_i^*))/2$, which gives $\rho(1/2)[f(x_i^*)]^2 - [g(x_i^*)]^2\Delta x$. Summarizing these findings, we arrive at the following theorem.

### Center of Mass of a Lamina Bounded by Two Functions

Let $R$ denote a region bounded above by the graph of a continuous function $f(x)$, below by the graph of the continuous function $g(x)$, and on the left and right by the lines $x = a$ and $x = b$, respectively. Let $\rho$ denote the density of the associated lamina. Then we can make the following statements:

i. The mass of the lamina is

$$m = \rho \int_a^b [f(x) - g(x)]dx. \tag{2.7.16}$$

ii. The moments $M_x$ and $M_y$ of the lamina with respect to the x- and y-axes, respectively, are

$$M_x = \rho \int_a^b \frac{1}{2}([f(x)]^2 - [g(x)]^2)dx \tag{2.7.17}$$

and

$$M_y = \rho \int_a^b x[f(x) - g(x)]dx. \tag{2.7.18}$$

iii. The coordinates of the center of mass $\bar{x}, \bar{y})$ are

$$\bar{x} = \frac{M_y}{m} \tag{2.7.19}$$

and

$$\bar{y} = \frac{M_x}{m} \tag{2.7.20}$$

We illustrate this theorem in the following example.

**Example 2.7.4: Finding the Centroid of a Region Bounded by Two Functions**

Let $R$ be the region bounded above by the graph of the function $f(x) = 1 - x^2$ and below by the graph of the function $g(x) = x - 1$. Find the centroid of the region.

**Solution**

The region is depicted in the following figure.

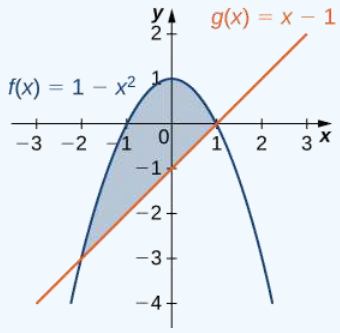



Figure 2.7.9: Finding the centroid of a region between two curves.

The graphs of the functions intersect at $(-2, -3)$ and $(1, 0)$, so we integrate from $-2$ to 1. Once again, for the sake of convenience, assume $\rho = 1$.

First, we need to calculate the total mass:

$$\begin{aligned} m &= \rho \int_a^b [f(x) - g(x)]dx \\ &= \int_{-2}^{1} [1 - x^2 - (x - 1)]dx \\ &= \int_{-2}^{1} (2 - x^2 - x)dx \\ &= \left[2x - \frac{1}{3}x^3 - \frac{1}{2}x^2\right] |_{-2}^{1} \\ &= \left[2 - \frac{1}{3} - \frac{1}{2}\right] - \left[-4 + \frac{8}{3} - 2\right] \\ &= \frac{9}{2}. \end{aligned}$$

Next, we compute the moments:

$$\begin{aligned} M_x &= \rho \int_a^b \frac{1}{2}([f(x)]^2 - [g(x)]^2)dx \\ &= \frac{1}{2} \int_{-2}^{1} ((1 - x^2)^2 - (x - 1)^2)dx \\ &= \frac{1}{2} \int_{-2}^{1} (x^4 - 3x^2 + 2x)dx \\ &= \frac{1}{2} \left[\frac{x^5}{5} - x^3 + x^2\right] |_{-2}^{1} \\ &= -\frac{27}{10} \end{aligned}$$

and

$$
\begin{aligned}
M_y &= \rho \int_a^b x[f(x) - g(x)]dx \\
&= \int_{-2}^{1} x[(1 - x^2) - (x - 1)]dx \\
&= \int_{-2}^{1} x[2 - x^2 - x]dx \\
&= \int_{-2}^{1} (2x - x^4 - x^2)dx \\
&= \left[x^2 - \frac{x^5}{5} - \frac{x^3}{3}\right] |_{-2}^{1} \\
&= -\frac{9}{4}.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
\bar{x} &= \frac{M_y}{m} \\
&= -\frac{9}{4} \cdot \frac{2}{9} \\
&= -\frac{1}{2}
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{y} &= \frac{M_x}{y} \\
&= -\frac{27}{10} \cdot \frac{2}{9} \\
&= -\frac{3}{5}.
\end{aligned}
$$

The centroid of the region is $(-(1/2), -(3/5))$.

**Exercise 2.7.4**

Let $R$ be the region bounded above by the graph of the function $f(x) = 6 - x^2$ and below by the graph of the function $g(x) = 3 - 2x$. Find the centroid of the region.

**Hint**

Use the process from the previous example.

**Answer**

The centroid of the region is $(1, 13/5)$.

## The Symmetry Principle

We stated the symmetry principle earlier, when we were looking at the centroid of a rectangle. The symmetry principle can be a great help when finding centroids of regions that are symmetric. Consider the following example.

**Example 2.7.5: Finding the Centroid of a Symmetric Region**

Let R be the region bounded above by the graph of the function $f(x) = 4 - x^2$ and below by the x-axis. Find the centroid of the region.

**Solution**

The region is depicted in the following figure

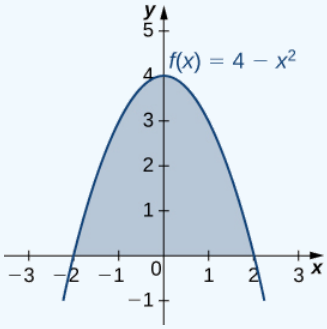



Figure **2.7.10**: We can use the symmetry principle to help find the centroid of a symmetric region.

The region is symmetric with respect to the *y*-axis. Therefore, the x-coordinate of the centroid is zero. We need only calculate $\bar{y}$. Once again, for the sake of convenience, assume $\rho = 1$.

First, we calculate the total mass:

$$\begin{aligned} m &= \rho \int_a^b f(x)dx \\ &= \int_{-2}^{2} (4 - x^2)dx \\ &= \left[4x - \frac{x^3}{3}\right] |_{-2}^{2} \\ &= \frac{32}{3}. \end{aligned}$$

Next, we calculate the moments. We only need $M_x$:

$$\begin{aligned} M_x &= \rho \int_a^b \frac{[f(x)]^2}{2} dx \\ &= \frac{1}{2} \int_{-2}^{2} \left[4 - x^2\right]^2 dx = \frac{1}{2} \int_{-2}^{2} (16 - 8x^2 + x^4)dx \\ &= \frac{1}{2} \left[\frac{x^5}{5} - \frac{8x^3}{3} + 16x\right] |_{-2}^{2} = \frac{256}{15} \end{aligned}$$

Then we have

$$\bar{y} = \frac{M_x}{y} = \frac{256}{15} \cdot \frac{3}{32} = \frac{8}{5}.$$

The centroid of the region is $(0, 8/5)$.

**Exercise 2.7.5**

Let $R$ be the region bounded above by the graph of the function $f(x) = 1 - x^2$ and below by $x$-axis. Find the centroid of the region.

**Hint**

Use the process from the previous example.

**Answer**

The centroid of the region is $(0, 2/5)$.

## The Grand Canyon Skywalk

The Grand Canyon Skywalk opened to the public on March 28, 2007. This engineering marvel is a horseshoe-shaped observation platform suspended 4000 ft above the Colorado River on the West Rim of the Grand Canyon. Its crystal-clear glass floor allows stunning views of the canyon below (see the following figure).



Figure **2.7.11**: The Grand Canyon Skywalk offers magnificent views of the canyon. (credit: 10da_ralta, Wikimedia Commons)

The Skywalk is a cantilever design, meaning that the observation platform extends over the rim of the canyon, with no visible means of support below it. Despite the lack of visible support posts or struts, cantilever structures are engineered to be very stable and the Skywalk is no exception. The observation platform is attached firmly to support posts that extend 46 ft down into bedrock. The structure was built to withstand 100-mph winds and an 8.0-magnitude earthquake within 50 mi, and is capable of supporting more than 70,000,000 lb.

One factor affecting the stability of the Skywalk is the center of gravity of the structure. We are going to calculate the center of gravity of the Skywalk, and examine how the center of gravity changes when tourists walk out onto the observation platform.

The observation platform is U-shaped. The legs of the U are 10 ft wide and begin on land, under the visitors' center, 48 ft from the edge of the canyon. The platform extends 70 ft over the edge of the canyon.

To calculate the center of mass of the structure, we treat it as a lamina and use a two-dimensional region in the xy-plane to represent the platform. We begin by dividing the region into three subregions so we can consider each subregion separately. The first region, denoted $R_1$, consists of the curved part of the U. We model $R_1$ as a semicircular annulus, with inner radius 25 ft and outer radius 35 ft, centered at the origin (Figure **2.7.12**).

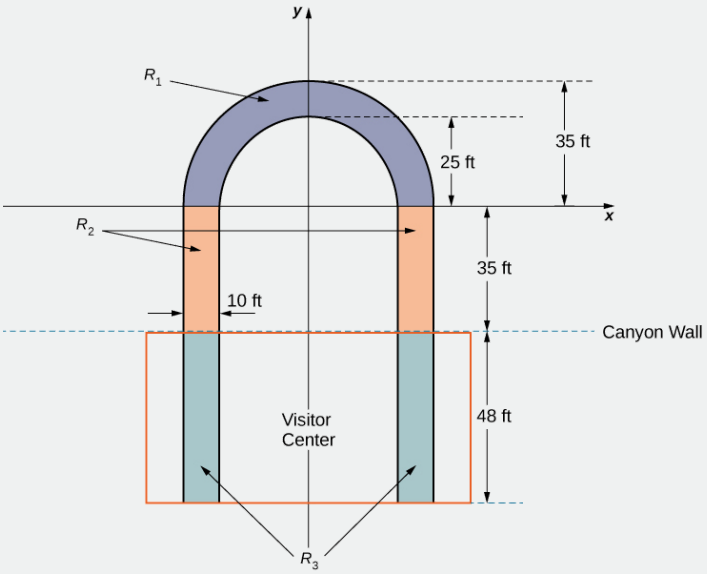



Figure **2.7.12**: We model the Skywalk with three sub-regions.

The legs of the platform, extending 35 ft between $R_1$ and the canyon wall, comprise the second sub-region, $R_2$. Last, the ends of the legs, which extend 48 ft under the visitor center, comprise the third sub-region, $R_3$. Assume the density of the lamina is constant and assume the total weight of the platform is 1,200,000 lb (not including the weight of the visitor center; we will consider that later). Use $g = 32\ ft/sec^2$.

1. Compute the area of each of the three sub-regions. Note that the areas of regions $R_2$ and $R_3$ should include the areas of the legs only, not the open space between them. Round answers to the nearest square foot.
2. Determine the mass associated with each of the three sub-regions.
3. Calculate the center of mass of each of the three sub-regions.
4. Now, treat each of the three sub-regions as a point mass located at the center of mass of the corresponding sub-region. Using this representation, calculate the center of mass of the entire platform.
5. Assume the visitor center weighs 2,200,000 lb, with a center of mass corresponding to the center of mass of $R_3$.Treating the visitor center as a point mass, recalculate the center of mass of the system. How does the center of mass change?
6. Although the Skywalk was built to limit the number of people on the observation platform to 120, the platform is capable of supporting up to 800 people weighing 200 lb each. If all 800 people were allowed on the platform, and all of them went to the farthest end of the platform, how would the center of gravity of the system be affected? (Include the visitor center in the calculations and represent the people by a point mass located at the farthest edge of the platform, 70 ft from the canyon wall.)

## Theorem of Pappus

This section ends with a discussion of the *theorem of Pappus for volume*, which allows us to find the volume of particular kinds of solids by using the centroid. (There is also a theorem of Pappus for surface area, but it is much less useful than the theorem for volume.)

### Theorem of Pappus for Volume

Let $R$ be a region in the plane and let l be a line in the plane that does not intersect $R$. Then the volume of the solid of revolution formed by revolving $R$ around l is equal to the area of $R$ multiplied by the distance $d$ traveled by the centroid of $R$.

### Proof

We can prove the case when the region is bounded above by the graph of a function $f(x)$ and below by the graph of a function $g(x)$ over an interval $[a, b]$, and for which the axis of revolution is the $y$-axis. In this case, the area of the region is $A = \int_a^b [f(x) - g(x)]\,dx$. Since the axis of rotation is the $y$-axis, the distance traveled by the centroid of the region depends only on the $x$-coordinate of the centroid, $\bar{x}$, which is

$$x = \frac{M_y}{m}, \tag{2.7.21}$$

where

$$m = \rho \int_a^b [f(x) - g(x)]dx \tag{2.7.22}$$

and

$$M_y = \rho \int_a^b x[f(x) - g(x)]dx. \tag{2.7.23}$$

Then,

$$d = 2\pi \frac{\rho \int_a^b x[f(x) - g(x)]dx}{\rho \int_a^b [f(x) - g(x)]dx} \tag{2.7.24}$$

and thus

$$d \cdot A = 2\pi \int_a^b x[f(x) - g(x)]dx. \tag{2.7.25}$$

However, using the method of cylindrical shells, we have

$$V = 2\pi \int_a^b x[f(x) - g(x)]dx. \tag{2.7.26}$$

So,

$$V = d \cdot A \tag{2.7.27}$$

and the proof is complete.

□

**Example 2.7.6: Using the Theorem of Pappus for Volume**

Let $R$ be a circle of radius 2 centered at $(4, 0)$. Use the theorem of Pappus for volume to find the volume of the torus generated by revolving $R$ around the $y$-axis.

**Solution**

The region and torus are depicted in the following figure.

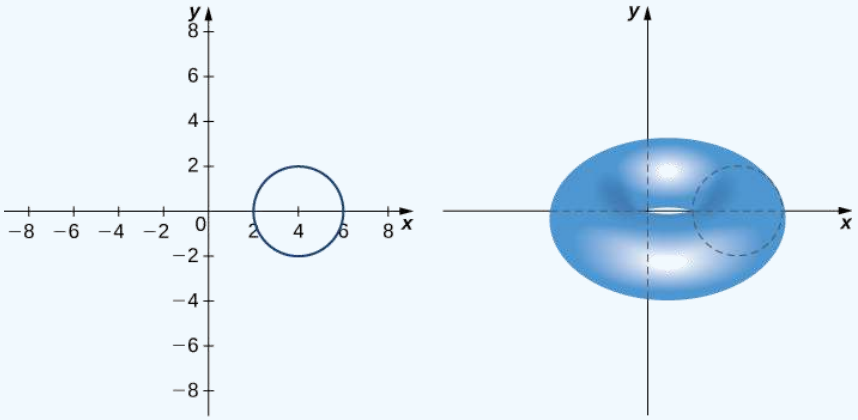



Figure 2.7.13: Determining the volume of a torus by using the theorem of Pappus. (a) A circular region $R$ in the plane; (b) the torus generated by revolving $R$ about the $y$-axis.

The region $R$ is a circle of radius 2, so the area of R is $A = 4\pi$ units$^2$. By the symmetry principle, the centroid of R is the center of the circle. The centroid travels around the $y$-axis in a circular path of radius 4, so the centroid travels $d = 8\pi$ units. Then, the volume of the torus is $A \cdot d = 32\pi^2$ units$^3$.



**Exercise 2.7.6**

Let R be a circle of radius 1 centered at $(3,0)$. Use the theorem of Pappus for volume to find the volume of the torus generated by revolving R around the $y$-axis.

**Hint**

Use the process from the previous example.

**Answer**

$6\pi^2$ units$^3$

## Key Concepts

- Mathematically, the center of mass of a system is the point at which the total mass of the system could be concentrated without changing the moment. Loosely speaking, the center of mass can be thought of as the balancing point of the system.
- For point masses distributed along a number line, the moment of the system with respect to the origin is $M = \sum_{i=1}^{n} m_i x_i$. For point masses distributed in a plane, the moments of the system with respect to the $x$- and $y$-axes, respectively, are $M_x = \sum_{i=1}^{n} m_i y_i$ and $M_y = \sum_{i=}^{n} m_i x_i$, respectively.
- For a lamina bounded above by a function $f(x)$, the moments of the system with respect to the $x$- and $y$-axes, respectively, are $M_x = \rho \int_a^b \frac{[f(x)]^2}{2}\,dx$ and $M_y = \rho \int_a^b x f(x)\,dx$.
- The $x$- and $y$-coordinates of the center of mass can be found by dividing the moments around the $y$-axis and around the $x$-axis, respectively, by the total mass. The symmetry principle says that if a region is symmetric with respect to a line, then the centroid of the region lies on the line.
- The theorem of Pappus for volume says that if a region is revolved around an external axis, the volume of the resulting solid is equal to the area of the region multiplied by the distance traveled by the centroid of the region.

## Key Equations

- **Mass of a lamina**

$$m = \rho \int_a^b f(x)dx$$

- **Moments of a lamina**

$$M_x = \rho \int_a^b \frac{[f(x)]^2}{2}\,dx \text{ and } M_y = \rho \int_a^b xf(x)\,dx$$

- **Center of mass of a lamina**

$$\bar{x} = \frac{M_y}{m} \text{ and } \bar{y} = \frac{M_x}{m}$$

## Glossary

**center of mass**

the point at which the total mass of the system could be concentrated without changing the moment

**centroid**

the centroid of a region is the geometric center of the region; laminas are often represented by regions in the plane; if the lamina has a constant density, the center of mass of the lamina depends only on the shape of the corresponding planar region; in this case, the center of mass of the lamina corresponds to the centroid of the representative region

**lamina**

a thin sheet of material; laminas are thin enough that, for mathematical purposes, they can be treated as if they are two-dimensional

**moment**

if n masses are arranged on a number line, the moment of the system with respect to the origin is given by $M = \sum_{i=1}^{n} m_i x_i$; if, instead, we consider a region in the plane, bounded above by a function $f(x)$ over an interval $[a, b]$, then the moments of the region with respect to the $x$- and $y$-axes are given by $M_x = \rho \int_a^b \frac{[f(x)]^2}{2}\,dx$ and $M_y = \rho \int_a^b xf(x)\,dx$, respectively

**symmetry principle**

the symmetry principle states that if a region $R$ is symmetric about a line $I$, then the centroid of $R$ lies on $I$

**theorem of Pappus for volume**

this theorem states that the volume of a solid of revolution formed by revolving a region around an external axis is equal to the area of the region multiplied by the distance traveled by the centroid of the region

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# 2.7E: Exercises for Section 2.7

**In exercises 1 - 6, calculate the center of mass for the collection of masses given.**

1) $m_1 = 2$ at $x_1 = 1$ and $m_2 = 4$ at $x_2 = 2$

2) $m_1 = 1$ at $x_1 = -1$ and $m_2 = 3$ at $x_2 = 2$

**Answer**

$x = \frac{5}{4}$

3) $m = 3$ at $x = 0, 1, 2, 6$

4) Unit masses at $(x, y) = (1, 0), (0, 1), (1, 1)$

**Answer**

$\left(\frac{2}{3}, \frac{2}{3}\right)$

5) $m_1 = 1$ at $(1, 0)$ and $m_2 = 4$ at $(0, 1)$

6) $m_1 = 1$ at $(1, 0)$ and $m_2 = 3$ at $(2, 2)$

**Answer**

$\left(\frac{7}{4}, \frac{3}{2}\right)$

**In exercises 7 - 16, compute the center of mass $\bar{x}$.**

7) $\rho = 1$ for $x \in (-1, 3)$

8) $\rho = x^2$ for $x \in (0, L)$

**Answer**

$\frac{3L}{4}$

9) $\rho = 1$ for $x \in (0, 1)$ and $\rho = 2$ for $x \in (1, 2)$

10) $\rho = \sin x$ for $x \in (0, \pi)$

**Answer**

$\frac{\pi}{2}$

11) $\rho = \cos x$ for $x \in \left(0, \frac{\pi}{2}\right)$

12) $\rho = e^x$ for $x \in (0, 2)$

**Answer**

$\frac{e^2 + 1}{e^2 - 1}$

13) $\rho = x^3 + xe^{-x}$ for $x \in (0, 1)$

14) $\rho = x \sin x$ for $x \in (0, \pi)$

**Answer**

$\frac{\pi^2 - 4}{\pi}$

15) $\rho = \sqrt{x}$ for $x \in (1, 4)$

16) $\rho = \ln x$ for $x \in (1, e)$

**Answer**

**Answer**

$\frac{\pi}{4}(1+e^2)$

**In exercises 17 - 19, compute the center of mass $(\bar{x},\bar{y})$. Use symmetry to help locate the center of mass whenever possible.**

17) $\rho=7$ in the square $0\le x\le 1,\ 0\le y\le 1$

18) $\rho=3$ in the triangle with vertices $(0,0)$, $(a,0)$, and $(0,b)$

**Answer**

$\left(\frac{a}{3},\ \frac{b}{3}\right)$

19) $\rho=2$ for the region bounded by $y=\cos(x),\ y=-\cos(x),\ x=-\frac{\pi}{2}$, and $x=\frac{\pi}{2}$

**In exercises 20 - 26, use a calculator to draw the region, then compute the center of mass $(\bar{x},\bar{y})$. Use symmetry to help locate the center of mass whenever possible.**

20) [T] The region bounded by $y=\cos(2x),\ x=-\frac{\pi}{4}$, and $x=\frac{\pi}{4}$

**Answer**

$\left(0,\frac{\pi}{8}\right)$

21) [T] The region between $y=2x^2,\ y=0,\ x=0$, and $x=1$

22) [T] The region between $y=\frac{5}{4}x^2$ and $y=5$

**Answer**

$(0,3)$

23) [T] Region between $y=\sqrt{x},\ y=\ln x,\ x=1$, and $x=4$

24) [T] The region bounded by $y=0$ and $\frac{x^2}{4}+\frac{y^2}{9}=1$

**Answer**

$\left(0,\frac{4}{\pi}\right)$

25) [T] The region bounded by $y=0,\ x=0$, and $\frac{x^2}{4}+\frac{y^2}{9}=1$

26) [T] The region bounded by $y=x^2$ and $y=x^4$ in the first quadrant

**Answer**

$\left(\frac{5}{8},\ \frac{1}{3}\right)$

**In exercises 27 - 31, use the theorem of Pappus to determine the volume of the shape.**

27) Rotating $y=mx$ around the $x$-axis between $x=0$ and $x=1$

28) Rotating $y=mx$ around the $y$-axis between $x=0$ and $x=1$

**Answer**

$V=\frac{m\pi}{3}$ units$^3$

29) A general cone created by rotating a triangle with vertices $(0,0)$, $(a,0)$, and $(0,b)$ around the $y$-axis. Does your answer agree with the volume of a cone?

30) A general cylinder created by rotating a rectangle with vertices $(0,0)$, $(a,0)$, $(0,b)$, and $(a,b)$ around the $y$-axis. Does your answer agree with the volume of a cylinder?

**Answer**

$V=\pi a^2 b$ units$^3$

31) A sphere created by rotating a semicircle with radius $a$ around the $y$-axis. Does your answer agree with the volume of a sphere?

**In exercises 32 - 36, use a calculator to draw the region enclosed by the curve. Find the area $M$ and the centroid $(\bar{x}, \bar{y})$ for the given shapes. Use symmetry to help locate the center of mass whenever possible.**

32) [T] Quarter-circle: $y = \sqrt{1 - x^2}$, $y = 0$, and $x = 0$

**Answer**

$\left(\frac{4}{3\pi}, \frac{4}{3\pi}\right)$

33) [T] Triangle: $y = x$, $y = 2 - x$, and $y = 0$

34) [T] Lens: $y = x^2$ and $y = x$

**Answer**

$\left(\frac{1}{2}, \frac{2}{5}\right)$

35) [T] Ring: $y^2 + x^2 = 1$ and $y^2 + x^2 = 4$

36) [T] Half-ring: $y^2 + x^2 = 1$, $y^2 + x^2 = 4$, and $y = 0$

**Answer**

$\left(0, \frac{28}{9\pi}\right)$

37) Find the generalized center of mass in the sliver between $y = x^a$ and $y = x^b$ with $a > b$. Then, use the Pappus theorem to find the volume of the solid generated when revolving around the $y$-axis.

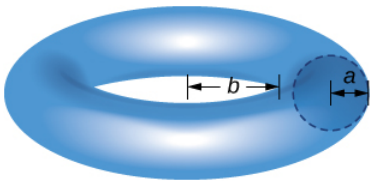
38) Find the generalized center of mass between $y = a^2 - x^2$, $x = 0$, and $y = 0$. Then, use the Pappus theorem to find the volume of the solid generated when revolving around the $y$-axis.

**Answer**

Center of mass: $\left(\frac{a}{6}, \frac{4a^2}{5}\right)$,
Volume: $\frac{2\pi a^4}{9}$ units$^3$

39) Find the generalized center of mass between $y = b\sin(ax)$, $x = 0$, and $x = \frac{\pi}{a}$. Then, use the Pappus theorem to find the volume of the solid generated when revolving around the $y$-axis.

40) Use the theorem of Pappus to find the volume of a torus (pictured here). Assume that a disk of radius $a$ is positioned with the left end of the circle at $x = b$, $b > 0$, and is rotated around the $y$-axis.




**Answer**

Volume: $V = 2\pi^2 a^2 (b + a)$

41) Find the center of mass $(\bar{x}, \bar{y})$ for a thin wire along the semicircle $y = \sqrt{1 - x^2}$ with unit mass. (Hint: Use the theorem of Pappus.)

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# 2.8: Integrals, Exponential Functions, and Logarithms

**Learning Objectives**

- Write the definition of the natural logarithm as an integral.
- Recognize the derivative of the natural logarithm.
- Integrate functions involving the natural logarithmic function.
- Define the number $e$ through an integral.
- Recognize the derivative and integral of the exponential function.
- Prove properties of logarithms and exponential functions using integrals.
- Express general logarithmic and exponential functions in terms of natural logarithms and exponentials.

We already examined exponential functions and logarithms in earlier chapters. However, we glossed over some key details in the previous discussions. For example, we did not study how to treat exponential functions with exponents that are irrational. The definition of the number e is another area where the previous development was somewhat incomplete. We now have the tools to deal with these concepts in a more mathematically rigorous way, and we do so in this section.

For purposes of this section, assume we have not yet defined the natural logarithm, the number $e$, or any of the integration and differentiation formulas associated with these functions. By the end of the section, we will have studied these concepts in a mathematically rigorous way (and we will see they are consistent with the concepts we learned earlier). We begin the section by defining the natural logarithm in terms of an integral. This definition forms the foundation for the section. From this definition, we derive differentiation formulas, define the number $e$, and expand these concepts to logarithms and exponential functions of any base.

## The Natural Logarithm as an Integral

Recall the power rule for integrals:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1.$$

Clearly, this does not work when $n = -1$, as it would force us to divide by zero. So, what do we do with $\int \frac{1}{x}\,dx$? Recall from the Fundamental Theorem of Calculus that $\int_1^x \frac{1}{t}dt$ is an antiderivative of $\frac{1}{x}$. Therefore, we can make the following definition.

**Definition: The Natural Logarithm**

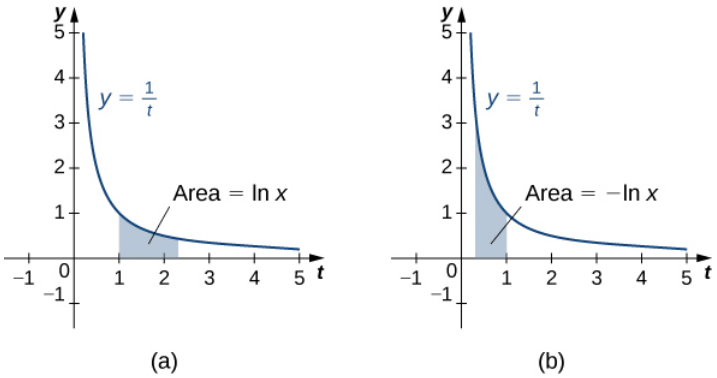
For $x > 0$, define the natural logarithm function by

$$\ln x = \int_1^x \frac{1}{t}\,dt.$$

For $x > 1$, this is just the area under the curve $y = \frac{1}{t}$ from 1 to $x$. For $x < 1$, we have

$$\int_1^x \frac{1}{t}\,dt = -\int_x^1 \frac{1}{t}\,dt,$$

so in this case it is the negative of the area under the curve from $x$ to 1 (see the following figure).




(a) (b)

Figure 2.8.1: (a) When $x > 1$, the natural logarithm is the area under the curve $y = 1/t$ from 1 to $x$. (b) When $x < 1$, the natural logarithm is the negative of the area under the curve from $x$ to 1.

Notice that $\ln 1 = 0$. Furthermore, the function $y = \frac{1}{t} > 0$ for $x > 0$. Therefore, by the properties of integrals, it is clear that $\ln x$ is increasing for $x > 0$.

## Properties of the Natural Logarithm

Because of the way we defined the natural logarithm, the following differentiation formula falls out immediately as a result of to the Fundamental Theorem of Calculus.

**Definition: Derivative of the Natural Logarithm**

For $x > 0$, the derivative of the natural logarithm is given by

$$\frac{d}{dx}\left(\ln x\right) = \frac{1}{x}.$$

**Corollary to the Derivative of the Natural Logarithm**

The function $\ln x$ is differentiable; therefore, it is continuous.

A graph of $\ln x$ is shown in Figure. Notice that it is continuous throughout its domain of $(0, \infty)$.

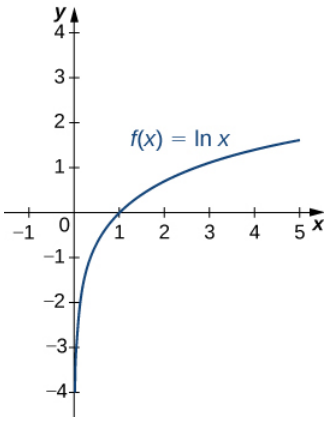



Figure 2.8.2: The graph of $f(x) = \ln x$ shows that it is a continuous function.

**Example 2.8.1: Calculating Derivatives of Natural Logarithms**

Calculate the following derivatives:

a. $\frac{d}{dx}\left(\ln(5x^3 - 2)\right)$
b. $\frac{d}{dx}\left((\ln(3x))^2\right)$

**Solution**

We need to apply the chain rule in both cases.

a. $\frac{d}{dx}\left(\ln(5x^3-2)\right)=\frac{15x^2}{5x^3-2}$

b. $\frac{d}{dx}\left((\ln(3x))^2\right)=\frac{2(\ln(3x))\cdot 3}{3x}=\frac{2(\ln(3x))}{x}$

**? Exercise 2.8.1**

Calculate the following derivatives:

a. $\frac{d}{dx}\left(\ln(2x^2+x)\right)$

b. $\frac{d}{dx}\left((\ln(x^3))^2\right)$

**Hint**

Apply the differentiation formula just provided and use the chain rule as necessary.

**Answer**

a. $\frac{d}{dx}\left(\ln(2x^2+x)\right)=\frac{4x+1}{2x^2+x}$

b. $\frac{d}{dx}\left((\ln(x^3))^2\right)=\frac{6\ln(x^3)}{x}$

Note that if we use the absolute value function and create a new function $\ln|x|$, we can extend the domain of the natural logarithm to include $x<0$. Then $\frac{d}{dx}\left(\ln x\right)=\frac{1}{x}$. This gives rise to the familiar integration formula.

**Integral of $\frac{1}{u}\,du$**

The natural logarithm is the antiderivative of the function $f(u)=\frac{1}{u}$:

$$\int \frac{1}{u}\,du=\ln|u|+C.$$

**✓ Example 2.8.2: Calculating Integrals Involving Natural Logarithms**

Calculate the integral $\int \frac{x}{x^2+4}\,dx$.

**Solution**

Using $u$-substitution, let $u=x^2+4$. Then $du=2x\,dx$ and we have

$$\int \frac{x}{x^2+4}\,dx=\frac{1}{2}\int \frac{1}{u}\,du=\frac{1}{2}\ln|u|+C=\frac{1}{2}\ln|x^2+4|+C=\frac{1}{2}\ln(x^2+4)+C.$$

**? Exercise 2.8.2**

Calculate the integral $\int \frac{x^2}{x^3+6}\,dx$.

**Hint**

Apply the integration formula provided earlier and use u-substitution as necessary.

**Answer**

$$\int \frac{x^2}{x^3+6}\,dx=\frac{1}{3}\ln|\,x^3+6\,|+C$$

Although we have called our function a "logarithm," we have not actually proved that any of the properties of logarithms hold for this function. We do so here.

**Properties of the Natural Logarithm**

If $a, b > 0$ and $r$ is a rational number, then

i. $\ln 1 = 0$
ii. $\ln(ab) = \ln a + \ln b$
iii. $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$
iv. $\ln(a^r) = r \ln a$

**Proof**

i. By definition, $\ln 1 = \int_1^1 \frac{1}{t}\, dt = 0.$

ii. We have

$$\ln(ab) = \int_1^{ab} \frac{1}{t}\, dt = \int_1^a \frac{1}{t}\, dt + \int_a^{ab} \frac{1}{t}\, dt.$$

Use $u-$*substitution* on the last integral in this expression. Let $u = t/a$. Then $du = (1/a)dt$. Furthermore, when $t = a$, $u = 1$, and when $t = ab$, $u = b$. So we get

$$\ln(ab) = \int_1^a \frac{1}{t}\, dt + \int_a^{ab} \frac{1}{t}\, dt = \int_1^a \frac{1}{t}\, dt + \int_1^{ab} \frac{a}{t} \cdot \frac{1}{a}\, dt = \int_1^a \frac{1}{t}\, dt + \int_1^b \frac{1}{u}\, du = \ln a + \ln b.$$

iii. Note that

$$\frac{d}{dx}\left(\ln(x^r)\right) = \frac{rx^{r-1}}{x^r} = \frac{r}{x}.$$

Furthermore,

$$\frac{d}{dx}\left((r \ln x)\right) = \frac{r}{x}.$$

Since the derivatives of these two functions are the same, by the Fundamental Theorem of Calculus, they must differ by a constant. So we have

$$\ln(x^r) = r \ln x + C$$

for some constant $C$. Taking $x = 1$, we get

$$\ln(1^r) = r \ln(1) + C$$

$$0 = r(0) + C$$

$$C = 0.$$

Thus $\ln(x^r) = r \ln x$ and the proof is complete. Note that we can extend this property to irrational values of $r$ later in this section.

Part iii. follows from parts ii. and iv. and the proof is left to you.

□

**Example 2.8.3: Using Properties of Logarithms**

Use properties of logarithms to simplify the following expression into a single logarithm:

$$\ln 9 - 2 \ln 3 + \ln\left(\tfrac{1}{3}\right).$$

**Solution**

We have

$$\ln 9 - 2\ln 3 + \ln\left(\tfrac{1}{3}\right) = \ln(3^2) - 2\ln 3 + \ln(3^{-1}) = 2\ln 3 - 2\ln 3 - \ln 3 = -\ln 3.$$

**Exercise 2.8.3**

Use properties of logarithms to simplify the following expression into a single logarithm:

$$\ln 8 - \ln 2 - \ln\left(\tfrac{1}{4}\right)$$

**Hint**

Apply the properties of logarithms.

**Answer**

$4\ln 2$

## Defining the Number e

Now that we have the natural logarithm defined, we can use that function to define the number $e$.

**Definition: $e$**

The number $e$ is defined to be the real number such that

$$\ln e = 1$$

To put it another way, the area under the curve $y = 1/t$ between $t = 1$ and $t = e$ is 1 (Figure). The proof that such a number exists and is unique is left to you. (Hint: Use the Intermediate Value Theorem to prove existence and the fact that $\ln x$ is increasing to prove uniqueness.)

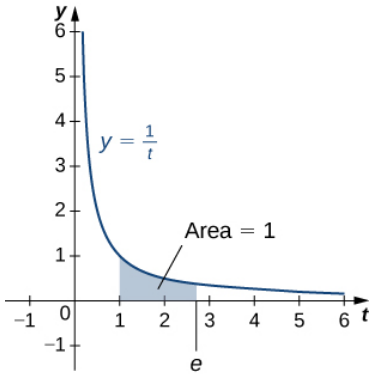



Figure 2.8.3:The area under the curve from 1 to $e$ is equal to one.

The number $e$ can be shown to be *irrational*, although we won't do so here (see the Student Project in Taylor and Maclaurin Series). Its approximate value is given by

$$e \approx 2.71828182846.$$

## The Exponential Function

We now turn our attention to the function $e^x$. Note that the natural logarithm is one-to-one and therefore has an inverse function. For now, we denote this inverse function by $\exp x$. Then,

$$\exp(\ln x) = x$$

for $x > 0$ and

$$\ln(\exp x) = x$$

for all $x$.

The following figure shows the graphs of $\exp x$ and $\ln x$.

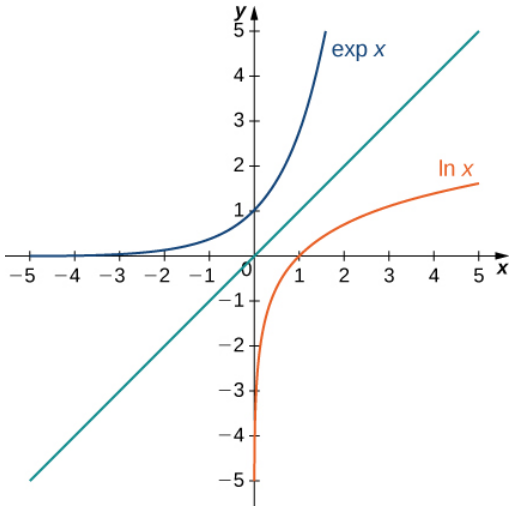



Figure 2.8.4: The graphs of $\ln x$ and $\exp x$.

We hypothesize that $\exp x = e^x$. For rational values of $x$, this is easy to show. If $x$ is rational, then we have $\ln(e^x) = x \ln e = x$. Thus, when $x$ is rational, $e^x = \exp x$. For irrational values of $x$, we simply define $e^x$ as the inverse function of $\ln x$.

**Definition**

For any real number $x$, define $y = e^x$ to be the number for which

$$\ln y = \ln(e^x) = x.$$

Then we have $e^x = \exp x$ for all $x$, and thus

$$e^{\ln x} = x \text{ for } x > 0 \text{ and } \ln(e^x) = x$$

for all $x$.

## Properties of the Exponential Function

Since the exponential function was defined in terms of an inverse function, and not in terms of a power of $e$ we must verify that the usual laws of exponents hold for the function $e^x$.

**Properties of the Exponential Function**

If $p$ and $q$ are any real numbers and $r$ is a rational number, then

i. $e^p e^q = e^{p+q}$
ii. $\dfrac{e^p}{e^q} = e^{p-q}$
iii. $(e^p)^r = e^{pr}$

**Proof**

Note that if $p$ and $q$ are rational, the properties hold. However, if $p$ or $q$ are irrational, we must apply the inverse function definition of $e^x$ and verify the properties. Only the first property is verified here; the other two are left to you. We have

$$\ln(e^p e^q) = \ln(e^p) + \ln(e^q) = p + q = \ln(e^{p+q}).$$

Since $\ln x$ is one-to-one, then

$$e^p e^q = e^{p+q}.$$

□

As with part iv. of the logarithm properties, we can extend property iii. to irrational values of $r$, and we do so by the end of the section.

We also want to verify the differentiation formula for the function $y = e^x$. To do this, we need to use implicit differentiation. Let $y = e^x$. Then

$$\begin{aligned} \ln y &= x \\ \frac{d}{dx}\left(\ln y\right) &= \frac{d}{dx}\left(x\right) \\ \frac{1}{y}\frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= y. \end{aligned}$$

Thus, we see

$$\frac{d}{dx}\left(e^x\right) = e^x$$

as desired, which leads immediately to the integration formula

$$\int e^x\,dx = e^x + C.$$

We apply these formulas in the following examples.

**Example 2.8.4: Using Properties of Exponential Functions**

Evaluate the following derivatives:

a. $\frac{d}{dt}\left(e^{3t}e^{t^2}\right)$
b. $\frac{d}{dx}\left(e^{3x^2}\right)$

**Solution**

We apply the chain rule as necessary.

a. $\frac{d}{dt}\left(e^{3t}e^{t^2}\right) = \frac{d}{dt}\left(e^{3t+t^2}\right) = e^{3t+t^2}(3+2t)$
b. $\frac{d}{dx}\left(e^{3x^2}\right) = e^{3x^2}6x$

**Exercise 2.8.4**

Evaluate the following derivatives:

a. $\frac{d}{dx}\left(\frac{e^{x^2}}{e^{5x}}\right)$
b. $\frac{d}{dt}\left((e^{2t})^3\right)$

**Hint**

Use the properties of exponential functions and the chain rule as necessary.

**Answer**

a. $\frac{d}{dx}\left(\frac{e^{x^2}}{e^{5x}}\right) = e^{x^2-5x}(2x-5)$
b. $\frac{d}{dt}\left((e^{2t})^3\right) = 6e^{6t}$

**Example 2.8.5: Using Properties of Exponential Functions**

Evaluate the following integral: $\int 2xe^{-x^2}\,dx$.

**Solution**

Using $u$-substitution, let $u = -x^2$. Then $du = -2x\,dx$, and we have

$$\int 2xe^{-x^2}\,dx = -\int e^u\,du = -e^u + C = -e^{-x^2} + C.$$

**Exercise 2.8.5**

Evaluate the following integral: $\int \frac{4}{e^{3x}}\,dx$.

**Hint**

Use the properties of exponential functions and $u-$ *substitution* as necessary.

**Answer**

$$\int \frac{4}{e^{3x}}\,dx = -\frac{4}{3}e^{-3x} + C$$

## General Logarithmic and Exponential Functions

We close this section by looking at exponential functions and logarithms with bases other than $e$. Exponential functions are functions of the form $f(x) = a^x$. Note that unless $a = e$, we still do not have a mathematically rigorous definition of these functions for irrational exponents. Let's rectify that here by defining the function $f(x) = a^x$ in terms of the exponential function $e^x$. We then examine logarithms with bases other than e as inverse functions of exponential functions.

**Definition: Exponential Function**

For any $a > 0$, and for any real number $x$, define $y = a^x$ as follows:

$$y = a^x = e^{x\ln a}.$$

Now $a^x$ is defined rigorously for all values of $x$. This definition also allows us to generalize property iv. of logarithms and property iii. of exponential functions to apply to both rational and irrational values of $r$. It is straightforward to show that properties of exponents hold for general exponential functions defined in this way.

Let's now apply this definition to calculate a differentiation formula for $a^x$. We have

$$\frac{d}{dx}\left(a^x\right) = \frac{d}{dx}\left(e^{x\ln a}\right) = e^{x\ln a}\ln a = a^x \ln a.$$

The corresponding integration formula follows immediately.

**Derivatives and Integrals Involving General Exponential Functions**

Let $a > 0$. Then,

$$\frac{d}{dx}\left(a^x\right) = a^x \ln a$$

and

$$\int a^x\,dx = \frac{1}{\ln a}a^x + C.$$

If $a \neq 1$, then the function $a^x$ is one-to-one and has a well-defined inverse. Its inverse is denoted by $\log_a x$. Then,

$$y = \log_a x \text{ if and only if } x = a^y.$$

Note that general logarithm functions can be written in terms of the natural logarithm. Let $y = \log_a x$. Then, $x = a^y$. Taking the natural logarithm of both sides of this second equation, we get

$$\begin{aligned} \ln x &= \ln(a^y) \\ \ln x &= y \ln a \\ y &= \frac{\ln x}{\ln a} \\ \log_a x &= \frac{\ln x}{\ln a}. \end{aligned}$$

Thus, we see that all logarithmic functions are constant multiples of one another. Next, we use this formula to find a differentiation formula for a logarithm with base $a$. Again, let $y = \log_a x$. Then,

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}\left(\log_a x\right) \\ &= \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right) \\ &= \left(\frac{1}{\ln a}\right)\frac{d}{dx}\left(\ln x\right) \\ &= \frac{1}{\ln a} \cdot \frac{1}{x} = \frac{1}{x \ln a} \end{aligned}$$

**Derivatives of General Logarithm Functions**

Let $a > 0$. Then,

$$\frac{d}{dx}\left(\log_a x\right) = \frac{1}{x \ln a}.$$

**Example 2.8.6: Calculating Derivatives of General Exponential and Logarithm Functions**

Evaluate the following derivatives:

a. $\frac{d}{dt}\left(4^t \cdot 2^{t^2}\right)$
b. $\frac{d}{dx}\left(\log_8(7x^2 + 4)\right)$

**Solution**: We need to apply the chain rule as necessary.

a. $\frac{d}{dt}\left(4^t \cdot 2^{t^2}\right) = \frac{d}{dt}\left(2^{2t} \cdot 2^{t^2}\right) = \frac{d}{dt}\left(2^{2t+t^2}\right) = 2^{2t+t^2} \ln(2)(2 + 2t)$
b. $\frac{d}{dx}\left(\log_8(7x^2 + 4)\right) = \frac{1}{(7x^2 + 4)(\ln 8)}(14x)$

**Exercise 2.8.6**

Evaluate the following derivatives:

a. $\frac{d}{dt}\left(4^{t^4}\right)$
b. $\frac{d}{dx}\left(\log_3(\sqrt{x^2 + 1})\right)$

**Hint**

Use the formulas and apply the chain rule as necessary.

**Answer**

a. $\frac{d}{dt}\left(4^{t^4}\right) = 4^{t^4}(\ln 4)(4t^3)$

b. $\frac{d}{dx}\left(\log_3(\sqrt{x^2+1})\right) = \frac{x}{(\ln 3)(x^2+1)}$

### Example 2.8.7: Integrating General Exponential Functions

Evaluate the following integral: $\int \frac{3}{2^{3x}}\,dx$.

**Solution**

Use $u-$ *substitution* and let $u = -3x$. Then $du = -3\,dx$ and we have

$$\int \frac{3}{2^{3x}}\,dx = \int 3\cdot 2^{-3x}\,dx = -\int 2^u\,du = -\frac{1}{\ln 2}2^u + C = -\frac{1}{\ln 2}2^{-3x} + C.$$

### Exercise 2.8.7

Evaluate the following integral: $\int x^2 2^{x^3}\,dx$.

**Hint**

Use the properties of exponential functions and u-substitution

**Answer**

$$\int x^2 2^{x^3}\,dx = \frac{1}{3\ln 2}2^{x^3} + C$$

## Key Concepts

- The earlier treatment of logarithms and exponential functions did not define the functions precisely and formally. This section develops the concepts in a mathematically rigorous way.
- The cornerstone of the development is the definition of the natural logarithm in terms of an integral.
- The function $e^x$ is then defined as the inverse of the natural logarithm. General exponential functions are defined in terms of $e^x$, and the corresponding inverse functions are general logarithms.
- Familiar properties of logarithms and exponents still hold in this more rigorous context.

## Key Equations

- **Natural logarithm function**
- $\ln x = \int_1^x \frac{1}{t}\,dt$
- **Exponential function** $y = e^x$
- $\ln y = \ln(e^x) = x$

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# 2.8E: Exercises for Section 2.8

**In exercises 1 - 3, find the derivative $\frac{dy}{dx}$.**

1) $y = \ln(2x)$

**Answer**

$$\frac{dy}{dx} = \frac{1}{x}$$

2) $y = \ln(2x + 1)$

3) $y = \frac{1}{\ln x}$

**Answer**

$$\frac{dy}{dx} = -\frac{1}{x(\ln x)^2}$$

**In exercises 4 - 5, find the indefinite integral.**

4) $\int \frac{dt}{3t}$

5) $\int \frac{dx}{1+x}$

**Answer**

$$\int \frac{dx}{1+x} = \ln|x+1| + C$$

**In exercises 6 - 15, find the derivative $\frac{dy}{dx}$. (You can use a calculator to plot the function and the derivative to confirm that it is correct.)**

6) [T] $y = \frac{\ln x}{x}$

7) [T] $y = x \ln x$

**Answer**

$$\frac{dy}{dx} = \ln(x) + 1$$

8) [T] $y = \log_{10} x$

9) [T] $y = \ln(\sin x)$

**Answer**

$$\frac{dy}{dx} = \cot x$$

10) [T] $y = \ln(\ln x)$

11) [T] $y = 7\ln(4x)$

**Answer**

$$\frac{dy}{dx} = \frac{7}{x}$$

12) [T] $y = \ln\left((4x)^7\right)$

13) [T] $y = \ln(\tan x)$

**Answer**

$$\frac{dy}{dx} = \csc x \sec x$$

14) [T] $y = \ln(\tan 3x)$

15) [T] $y = \ln(\cos^2 x)$

**Answer**

$$\frac{dy}{dx} = -2\tan x$$

**In exercises 16 - 25, find the definite or indefinite integral.**

16) $\int_0^1 \frac{dx}{3+x}$

17) $\int_0^1 \frac{dt}{3+2t}$

**Answer**

$$\int_0^1 \frac{dt}{3+2t} = \tfrac{1}{2}\ln\left(\tfrac{5}{3}\right)$$

18) $\int_0^2 \frac{x}{x^2+1}\,dx$

19) $\int_0^2 \frac{x^3}{x^2+1}\,dx$

**Answer**

$$\int_0^2 \frac{x^3}{x^2+1}\,dx = 2 - \tfrac{1}{2}\ln(5)$$

20) $\int_2^e \frac{dx}{x\ln x}$

21) $\int_2^e \frac{dx}{(x\ln x)^2}$

**Answer**

$$\int_2^e \frac{dx}{(x\ln x)^2} = \frac{1}{\ln(2)} - 1$$

22) $\int \frac{\cos x}{\sin x}\,dx$

23) $\int_0^{\pi/4} \tan x\,dx$

**Answer**

$$\int_0^{\pi/4} \tan x\,dx = \tfrac{1}{2}\ln(2)$$

24) $\int \cot(3x)\,dx$

25) $\int \frac{(\ln x)^2}{x}\,dx$

**Answer**

$$\int \frac{(\ln x)^2}{x}\,dx = \tfrac{1}{3}(\ln x)^3$$

**In exercises 26 - 35, compute $\frac{dy}{dx}$ by differentiating $\ln y$.**

26) $y = \sqrt{x^2+1}$

27) $y = \sqrt{x^2+1}\sqrt{x^2-1}$

**Answer**

$$\frac{dy}{dx} = \frac{2x^3}{\sqrt{x^2+1}\sqrt{x^2-1}}$$

28) $y = e^{\sin x}$

29) $y = x^{-1/x}$

**Answer**

$$\frac{dy}{dx} = x^{-2-(1/x)}(\ln x - 1)$$

30) $y = e^{ex}$

31) $y = x^e$

**Answer**

$$\frac{dy}{dx} = ex^{e-1}$$

32) $y = x^{(ex)}$

33) $y = \sqrt{x}\sqrt[3]{x}\sqrt[6]{x}$

**Answer**

$$\frac{dy}{dx} = 1$$

34) $y = x^{-1/\ln x}$

35) $y = e^{-\ln x}$

**Answer**

$$\frac{dy}{dx} = -\frac{1}{x^2}$$

**In exercises 36 - 40, evaluate by any method.**

36) $\int_5^{10} \frac{dt}{t} - \int_{5x}^{10x} \frac{dt}{t}$

37) $\int_1^{e^x} \frac{dx}{x} + \int_{-2}^{-1} \frac{dx}{x}$

**Answer**

$$\pi - \ln(2)$$

38) $\frac{d}{dx}\left[\int_x^1 \frac{dt}{t}\right]$

39) $\frac{d}{dx}\left[\int_x^{x^2} \frac{dt}{t}\right]$

**Answer**

$$\frac{1}{x}$$

40) $\frac{d}{dx}\left[\ln(\sec x + \tan x)\right]$

**In exercises 41 - 44, use the function $\ln x$. If you are unable to find intersection points analytically, use a calculator.**

41) Find the area of the region enclosed by $x = 1$ and $y = 5$ above $y = \ln x$.

**Answer**

$(e^5 - 6)$ units$^2$

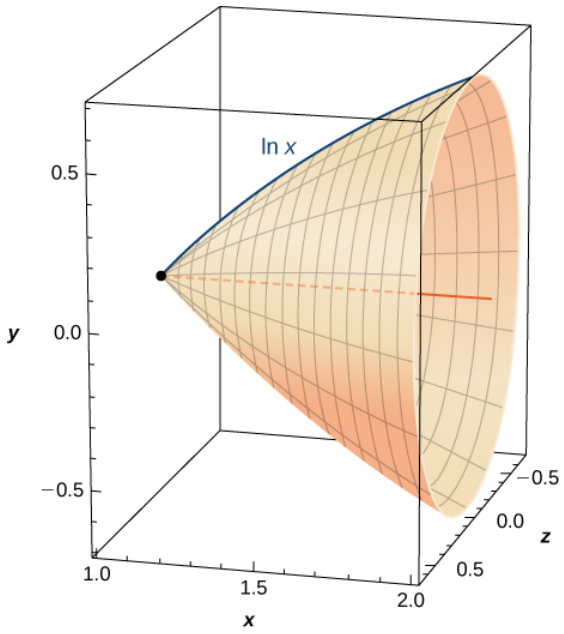
42) [T] Find the arc length of $\ln x$ from $x = 1$ to $x = 2$.

43) Find the area between $\ln x$ and the $x$-axis from $x = 1$ to $x = 2$.

**Answer**

$\ln(4) - 1)$ units$^2$

44) Find the volume of the shape created when rotating this curve from $x = 1$ to $x = 2$ around the $x$-axis, as pictured here.




45) [T] Find the surface area of the shape created when rotating the curve in the previous exercise from $x = 1$ to $x = 2$ around the $x$-axis.

**Answer**

2.8656 units$^2$

**If you are unable to find intersection points analytically in the following exercises, use a calculator.**

46) Find the area of the hyperbolic quarter-circle enclosed by $x = 2$ and $y = 2$ above $y = 1/x$.

47) [T] Find the arc length of $y = 1/x$ from $x = 1$ to $x = 4$.

**Answer**

$s = 3.1502$ units

48) Find the area under $y = 1/x$ and above the $x$-axis from $x = 1$ to $x = 4$.

**In exercises 49 - 53, verify the derivatives and antiderivatives.**

49) $\frac{d}{dx}\left[\ln(x + \sqrt{x^2 + 1})\right] = \frac{1}{\sqrt{1 + x^2}}$

50) $\frac{d}{dx}\left[\ln\left(\frac{x-a}{x+a}\right)\right] = \frac{2a}{(x^2 - a^2)}$

51) $\frac{d}{dx}\left[\ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)\right] = -\frac{1}{x\sqrt{1 - x^2}}$

52) $\frac{d}{dx}\left[\ln(x+\sqrt{x^2-a^2})\right]=\frac{1}{\sqrt{x^2-a^2}}$

53) $\int \frac{dx}{x\ln(x)\ln(\ln x)}=\ln|\ln(\ln x)|+C$

## Contributors

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# 2.9: Exponential Growth and Decay

**Learning Objectives**

- Use the exponential growth model in applications, including population growth and compound interest.
- Explain the concept of doubling time.
- Use the exponential decay model in applications, including radioactive decay and Newton's law of cooling.
- Explain the concept of half-life.

One of the most prevalent applications of exponential functions involves growth and decay models. Exponential growth and decay show up in a host of natural applications. From population growth and continuously compounded interest to radioactive decay and Newton's law of cooling, exponential functions are ubiquitous in nature. In this section, we examine exponential growth and decay in the context of some of these applications.

## Exponential Growth Model

Many systems exhibit exponential growth. These systems follow a model of the form $y = y_0 e^{kt}$, where $y_0$ represents the initial state of the system and $k$ is a positive constant, called the growth constant. Notice that in an exponential growth model, we have

$$y' = ky_0 e^{kt} = ky. \tag{2.9.1}$$

That is, the rate of growth is proportional to the current function value. This is a key feature of exponential growth. Equation 2.9.1 involves derivatives and is called *a differential equation*.

**Exponential Growth**

Systems that exhibit *exponential growth* increase according to the mathematical model

$$y = y_0 e^{kt} \tag{2.9.2}$$

where $y_0$ represents the initial state of the system and $k > 0$ is a constant, called the growth constant.

*Population growth* is a common example of exponential growth. Consider a population of bacteria, for instance. It seems plausible that the rate of population growth would be proportional to the size of the population. After all, the more bacteria there are to reproduce, the faster the population grows. Figure 2.9.1 and Table 2.9.1 represent the growth of a population of bacteria with an initial population of 200 bacteria and a growth constant of 0.02. Notice that after only 2 hours (120 minutes), the population is 10 times its original size!

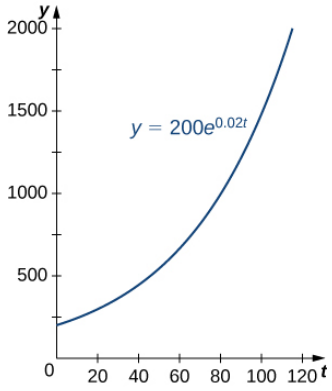



Figure 2.9.1: An example of exponential growth for bacteria.

Table 2.9.1: Exponential Growth of a Bacterial Population

| Time(min) | Population Size (no. of bacteria) |
|---|---|
| 10 | 244 |
| 20 | 298 |
| 30 | 364 |
| 40 | 445 |
| 50 | 544 |
| 60 | 664 |
| 70 | 811 |
| 80 | 991 |
| 90 | 1210 |
| 100 | 1478 |
| 110 | 1805 |
| 120 | 2205 |

Note that we are using a continuous function to model what is inherently discrete behavior. At any given time, the real-world population contains a whole number of bacteria, although the model takes on noninteger values. When using exponential growth models, we must always be careful to interpret the function values in the context of the phenomenon we are modeling.

**Example 2.9.1: Population Growth**

Consider the population of bacteria described earlier. This population grows according to the function $f(t) = 200e^{0.02t}$, where t is measured in minutes. How many bacteria are present in the population after $5$ hours ($300$ minutes)? When does the population reach $100,000$ bacteria?

**Solution**

We have $f(t) = 200e^{0.02t}$. Then

$$f(300) = 200e^{0.02(300)} \approx 80,686.$$

There are $80,686$ bacteria in the population after $5$ hours.

To find when the population reaches $100,000$ bacteria, we solve the equation

$$\begin{aligned} 100,000 &= 200e^{0.02t} \\ 500 &= e^{0.02t} \\ \ln 500 &= 0.02t \\ t &= \frac{\ln 500}{0.02} \approx 310.73. \end{aligned}$$

The population reaches $100,000$ bacteria after $310.73$ minutes.



**Exercise 2.9.1**

Consider a population of bacteria that grows according to the function $f(t) = 500e^{0.05t}$, where $t$ is measured in minutes. How many bacteria are present in the population after 4 hours? When does the population reach 100 million bacteria?

**Answer**

Use the process from the previous example.

**Answer**

There are $81,377,396$ bacteria in the population after $4$ hours. The population reaches $100$ million bacteria after $244.12$ minutes.

Let's now turn our attention to a financial application: **compound interest**. Interest that is not compounded is called simple interest. ***Simple interest*** is paid once, at the end of the specified time period (usually $1$ year). So, if we put $\$1000$ in a savings account earning $2$ simple interest per year, then at the end of the year we have

$$1000(1 + 0.02) = \$1020. \tag{2.9.3}$$

Compound interest is paid multiple times per year, depending on the compounding period. Therefore, if the bank compounds the interest every $6$ months, it credits half of the year's interest to the account after $6$ months. During the second half of the year, the account earns interest not only on the initial $\$1000$, but also on the interest earned during the first half of the year. Mathematically speaking, at the end of the year, we have

$$1000\left(1+\frac{0.02}{2}\right)^2 = \$1020.10. \tag{2.9.4}$$

Similarly, if the interest is compounded every $4$ months, we have

$$1000\left(1+\frac{0.02}{3}\right)^3 = \$1020.13, \tag{2.9.5}$$

and if the interest is compounded daily ($365$ times per year), we have $\$1020.20$. If we extend this concept, so that the interest is compounded continuously, after $t$ years we have

$$1000\lim_{n\to\infty}\left(1+\frac{0.02}{n}\right)^{nt}. \tag{2.9.6}$$

Now let's manipulate this expression so that we have an exponential growth function. Recall that the number $e$ can be expressed as a limit:

$$e = \lim_{m \to \infty} \left(1 + \frac{1}{m}\right)^m. \tag{2.9.7}$$

Based on this, we want the expression inside the parentheses to have the form $(1 + 1/m)$. Let $n = 0.02m$. Note that as $n \to \infty, m \to \infty$ as well. Then we get

$$1000 \lim_{n \to \infty} \left(1 + \frac{0.02}{n}\right)^{nt} = 1000 \lim_{m \to \infty} \left(1 + \frac{0.02}{0.02m}\right)^{0.02mt} = 1000 \left[\lim_{m \to \infty} \left(1 + \frac{1}{m}\right)^m\right]^{0.02t}. \tag{2.9.8}$$

We recognize the limit inside the brackets as the number $e$. So, the balance in our bank account after $t$ years is given by $1000e^{0.02t}$. Generalizing this concept, we see that if a bank account with an initial balance of $\$P$ earns interest at a rate of $r$, compounded continuously, then the balance of the account after $t$ years is

$$\text{Balance } = Pe^{rt}. \tag{2.9.9}$$

**Example 2.9.2: Compound Interest**

A 25-year-old student is offered an opportunity to invest some money in a retirement account that pays $5$ annual interest compounded continuously. How much does the student need to invest today to have $\$1$ million when she retires at age $65$? What if she could earn $6$ annual interest compounded continuously instead?

**Solution**

We have

$$1,000,000 = Pe^{0.05(40)} \tag{2.9.10}$$

$$P = 135,335.28. \tag{2.9.11}$$

She must invest $\$135,335.28$ at $5$ interest.

If, instead, she is able to earn $6$ then the equation becomes

$$1,000,000 = Pe^{0.06(40)} \tag{2.9.12}$$

$$P = 90,717.95. \tag{2.9.13}$$

In this case, she needs to invest only $\$90,717.95$. This is roughly two-thirds the amount she needs to invest at $5$. The fact that the interest is compounded continuously greatly magnifies the effect of the $1$ increase in interest rate.

**Exercise 2.9.2**

Suppose instead of investing at age $25\sqrt{b^2 - 4ac}$, the student waits until age $35$. How much would she have to invest at $5$? At $6$?

**Hint**

Use the process from the previous example.

**Answer**

At $5$ interest, she must invest $\$223,130.16$. At $6$ interest, she must invest $\$165,298.89$.

If a quantity grows exponentially, the time it takes for the quantity to double remains constant. In other words, it takes the same amount of time for a population of bacteria to grow from $100$ to $200$ bacteria as it does to grow from $10,000$ to $20,000$ bacteria. This time is called the doubling time. To calculate the doubling time, we want to know when the quantity reaches twice its original size. So we have

$$2y_0 = y_0 e^{kt}$$
$$2 = e^{kt}$$
$$\ln 2 = kt$$
$$t = \frac{\ln 2}{k}.$$

**Definition: Doubling Time**

If a quantity grows exponentially, the **doubling time** is the amount of time it takes the quantity to double. It is given by

$$\text{Doubling time} = \frac{\ln 2}{k}. \tag{2.9.14}$$

**Example 2.9.3: Using the Doubling Time**

Assume a population of fish grows exponentially. A pond is stocked initially with 500 fish. After 6 months, there are 1000 fish in the pond. The owner will allow his friends and neighbors to fish on his pond after the fish population reaches 10,000. When will the owner's friends be allowed to fish?

**Solution**

We know it takes the population of fish 6 months to double in size. So, if $t$ represents time in months, by the doubling-time formula, we have $6 = (\ln 2)/k$. Then, $k = (\ln 2)/6$. Thus, the population is given by $y = 500e^{((\ln 2)/6)t}$. To figure out when the population reaches 10,000 fish, we must solve the following equation:

$$10,000 = 500e^{(\ln 2/6)t}$$
$$20 = e^{(\ln 2/6)t}$$
$$\ln 20 = \left(\frac{\ln 2}{6}\right) t$$
$$t = \frac{6(\ln 20)}{\ln 2}$$
$$\approx 25.93.$$

The owner's friends have to wait 25.93 months (a little more than 2 years) to fish in the pond.

**Exercise 2.9.3**

Suppose it takes 9 months for the fish population in Example 2.9.3 to reach 1000 fish. Under these circumstances, how long do the owner's friends have to wait?

**Hint**

Use the process from the previous example.

**Answer**

38.90 months

## Exponential Decay Model

Exponential functions can also be used to model populations that shrink (from disease, for example), or chemical compounds that break down over time. We say that such systems exhibit exponential decay, rather than exponential growth. The model is nearly the same, except there is a negative sign in the exponent. Thus, for some positive constant $k$, we have

$$y = y_0 e^{-kt}. \tag{2.9.15}$$

As with exponential growth, there is a differential equation associated with exponential decay. We have

$$y' = -ky_0 e^{-kt} = -ky. \tag{2.9.16}$$

**Exponential Decay**

Systems that exhibit exponential decay behave according to the model

$$y = y_0 e^{-kt}, \tag{2.9.17}$$

where $y_0$ represents the initial state of the system and $k > 0$ is a constant, called the decay constant.

Figure 2.9.2 shows a graph of a representative exponential decay function.

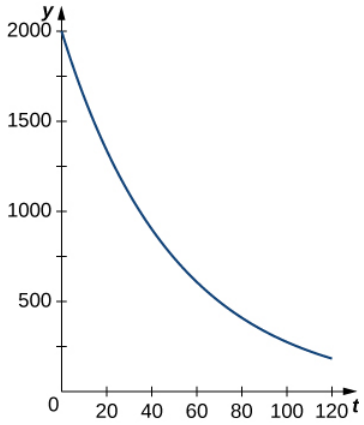



Figure 2.9.2: An example of exponential decay.

Let's look at a physical application of exponential decay. *Newton's law of cooling* says that an object cools at a rate proportional to the difference between the temperature of the object and the temperature of the surroundings. In other words, if $T$ represents the temperature of the object and $T_a$ represents the ambient temperature in a room, then

$$T' = -k(T - T_a). \tag{2.9.18}$$

Note that this is not quite the right model for exponential decay. We want the derivative to be proportional to the function, and this expression has the additional $T_a$ term. Fortunately, we can make a change of variables that resolves this issue. Let $y(t) = T(t) - T_a$. Then $y'(t) = T'(t) - 0 = T'(t)$, and our equation becomes

$$y' = -ky. \tag{2.9.19}$$

From our previous work, we know this relationship between $y$ and its derivative leads to exponential decay. Thus,

$$y = y_0 e^{-kt}, \tag{2.9.20}$$

and we see that

$$T - T_a = (T_0 - T_a)e^{-kt} \tag{2.9.21}$$

$$T = (T_0 - T_a)e^{-kt} + T_a \tag{2.9.22}$$

where $T_0$ represents the initial temperature. Let's apply this formula in the following example.

**Example 2.9.4: Newton's Law of Cooling**

According to experienced baristas, the optimal temperature to serve coffee is between $155\,°F$ and $175\,°F$. Suppose coffee is poured at a temperature of $200\,°F$, and after $2$ minutes in a $70\,°F$ room it has cooled to $180\,°F$. When is the coffee first cool enough to serve? When is the coffee too cold to serve? Round answers to the nearest half minute.

**Solution**

We have

$$
\begin{aligned}
T &= (T_0 - T_a)e^{-kt} + T_a \\
180 &= (200 - 70)e^{-k(2)} + 70 \\
110 &= 130e^{-2k} \\
\frac{11}{13} &= e^{-2k} \\
\ln\frac{11}{13} &= -2k \\
\ln 11 - \ln 13 &= -2k \\
k &= \frac{\ln 13 - \ln 11}{2}
\end{aligned}
$$

Then, the model is

$$T = 130e^{(\ln 11 - \ln 13/2)t} + 70.$$

The coffee reaches $175°F$ when

$$
\begin{aligned}
175 &= 130e^{(\ln 11 - \ln 13/2)t} + 70 \\
105 &= 130e^{(\ln 11 - \ln 13/2)t} \\
\frac{21}{26} &= e^{(\ln 11 - \ln 13/2)t} \\
\ln\frac{21}{26} &= \frac{\ln 11 - \ln 13}{2}t \\
\ln 21 - \ln 26 &= \left(\frac{\ln 11 - \ln 13}{2}\right)t \\
t &= \frac{2(\ln 21 - \ln 26)}{\ln 11 - \ln 13} \\
&\approx 2.56.
\end{aligned}
$$

The coffee can be served about $2.5$ minutes after it is poured. The coffee reaches $155°F$ at

$$
\begin{aligned}
155 &= 130e^{(\ln 11 - \ln 13/2)t} + 70 \\
85 &= 130e^{(\ln 11 - \ln 13)t} \\
\frac{17}{26} &= e^{(\ln 11 - \ln 13)t} \\
\ln 17 - \ln 26 &= \left(\frac{\ln 11 - \ln 13}{2}\right)t \\
t &= \frac{2(\ln 17 - \ln 26)}{\ln 11 - \ln 13} \\
&\approx 5.09.
\end{aligned}
$$

The coffee is too cold to be served about $5$ minutes after it is poured.

## Using Newton's Law of Cooling for Forensic Pathology

You are a forensic pathologist and your task is to determine the time of death of a recently found murdered body, so that the murderer can be found and convicted. You will be given the temperature of the body and the temperature outside. You can safely assume that the victim had a healthy body temperature of 98.6 degrees F at the time of death and that the outside temperature remained constant between the time of death and now.



Click Begin to see the outside temperature and the temperature of the dead body.

Begin

**Exercise 2.9.4**

Suppose the room is warmer $(75°F)$ and, after $2$ minutes, the coffee has cooled only to $185°F$. When is the coffee first cool enough to serve? When is the coffee be too cold to serve? Round answers to the nearest half minute.

**Hint**

Use the process from the previous example.

**Answer**

The coffee is first cool enough to serve about $3.5$ minutes after it is poured. The coffee is too cold to serve about $7$ minutes after it is poured.

Just as systems exhibiting exponential growth have a constant doubling time, systems exhibiting exponential decay have a constant half-life. To calculate the half-life, we want to know when the quantity reaches half its original size. Therefore, we have

$$\begin{aligned} \frac{y_0}{2} &= y_0 e^{-kt} \\ \frac{1}{2} &= e^{-kt} \\ -\ln 2 &= -kt \\ t &= \frac{\ln 2}{k}. \end{aligned}$$

*Note:* This is the same expression we came up with for doubling time.

**Definition: Half-Life**

If a quantity decays exponentially, the half-life is the amount of time it takes the quantity to be reduced by half. It is given by

$$\text{Half-life} = \frac{\ln 2}{k}. \tag{2.9.23}$$

**Example 2.9.5: Radiocarbon Dating**

One of the most common applications of an exponential decay model is carbon dating. Carbon-14 decays (emits a radioactive particle) at a regular and consistent exponential rate. Therefore, if we know how much carbon-14 was originally present in an object and how much carbon-14 remains, we can determine the age of the object. The half-life of carbon-14 is approximately 5730 years—meaning, after that many years, half the material has converted from the original carbon-14 to the new

nonradioactive nitrogen-14. If we have 100 g carbon-14 today, how much is left in 50 years? If an artifact that originally contained 100 g of carbon-14 now contains 10 g of carbon-14, how old is it? Round the answer to the nearest hundred years.

**Solution**

We have

$$5730 = \frac{\ln 2}{k}$$

$$k = \frac{\ln 2}{5730}.$$

So, the model says

$$y = 100e^{-(\ln 2/5730)t}.$$

In $50$ years, we have

$$y = 100e^{-(\ln 2/5730)(50)} \approx 99.40$$

Therefore, in $50$ years, $99.40$ g of carbon-14 remains.

To determine the age of the artifact, we must solve

$$10 = 100e^{-(\ln 2/5730)t}$$

$$\frac{1}{10} = e^{-(\ln 2/5730)t}$$

$$t \approx 19035.$$

The artifact is about $19,000$ years old.

**Exercise 2.9.5**: Carbon-14 Decay

If we have 100 g of carbon-14 , how much is left after 500 years? If an artifact that originally contained 100 g of carbon-14 now contains 20 g of carbon-14, how old is it? Round the answer to the nearest hundred years.

**Hint**

Use the process from the previous example.

**Answer**

A total of 94.13 g of carbon-14 remains after 500 years. The artifact is approximately 13,300 years old.

## Key Concepts

- Exponential growth and exponential decay are two of the most common applications of exponential functions.
- Systems that exhibit exponential growth follow a model of the form $y = y_0e^{kt}$.
- In exponential growth, the rate of growth is proportional to the quantity present. In other words, $y' = ky$.
- Systems that exhibit exponential growth have a constant doubling time, which is given by $(\ln 2)/k$.
- Systems that exhibit exponential decay follow a model of the form $y = y_0e^{-kt}$.
- Systems that exhibit exponential decay have a constant half-life, which is given by $(\ln 2)/k$.

## Glossary

**doubling time**

if a quantity grows exponentially, the doubling time is the amount of time it takes the quantity to double, and is given by $(\ln 2)/k$

**exponential decay**

systems that exhibit exponential decay follow a model of the form $y = y_0e^{-kt}$

**exponential growth**

systems that exhibit exponential growth follow a model of the form $y = y_0 e^{kt}$

**half-life**

if a quantity decays exponentially, the half-life is the amount of time it takes the quantity to be reduced by half. It is given by $(\ln 2)/k$

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# 2.9E: Exercises for Section 2.9

**In exercises 1 - 2, answer *True or False*? If true, prove it. If false, find the true answer.**

1) The doubling time for $y = e^{ct}$ is $\frac{\ln 2}{\ln c}$.

2) If you invest $\$500$, an annual rate of interest of $3\%$ yields more money in the first year than a $2.5\%$ continuous rate of interest.

**Answer**

True

3) If you leave a $100°C$ pot of tea at room temperature ($25°C$) and an identical pot in the refrigerator ($5°C$), with $k = 0.02$, the tea in the refrigerator reaches a drinkable temperature ($70°C$) more than $5$ minutes before the tea at room temperature.

4) If given a half-life of t years, the constant $k$ for $y = e^{kt}$ is calculated by $k = \frac{\ln(\frac{1}{2})}{t}$.

**Answer**

False; $k = \frac{\ln 2}{t}$

**In exercises 5 - 18, use $y = y_0 e^{kt}$.**

5) If a culture of bacteria doubles in $3$ hours, how many hours does it take to multiply by $10$?

6) If bacteria increase by a factor of $10$ in $10$ hours, how many hours does it take to increase by $100$?

**Answer**

$20$ hours

7) How old is a skull that contains one-fifth as much radiocarbon as a modern skull? Note that the half-life of radiocarbon is $5730$ years.

8) If a relic contains $90\%$ as much radiocarbon as new material, can it have come from the time of Christ (approximately $2000$ years ago)? Note that the half-life of radiocarbon is $5730$ years.

**Answer**

No. The relic is approximately $871$ years old.

9) The population of Cairo grew from $5$ million to $10$ million in $20$ years. Use an exponential model to find when the population was $8$ million.

10) The populations of New York and Los Angeles are growing at $1\%$ and $1.4\%$ a year, respectively. Starting from $8$ million (New York) and $6$ million (Los Angeles), when are the populations equal?

**Answer**

$71.92$ years

11) Suppose the value of $\$1$ in Japanese yen decreases at $2\%$ per year. Starting from $\$1 = ¥250$, when will $\$1 = ¥1$?

12) The effect of advertising decays exponentially. If $40\%$ of the population remembers a new product after $3$ days, how long will $20\%$remember it?

**Answer**

$5$ days $6$ hours $27$minutes

13) If $y = 1000$ at $t = 3$ and $y = 3000$ at $t = 4$, what was $y_0$ at $t = 0$?

14) If $y = 100$ at $t = 4$ and $y = 10$ at $t = 8$, when does $y = 1$?

**Answer**

At $t = 12$

15) If a bank offers annual interest of $7.5\%$ or continuous interest of $7.25\%$, which has a better annual yield?

16) What continuous interest rate has the same yield as an annual rate of $9\%$?

**Answer**

$8.618\%$

17) If you deposit $\$5000$ at $8\%$ annual interest, how many years can you withdraw $\$500$ (starting after the first year) without running out of money?

18) You are trying to save $\$50,000$ in $20$ years for college tuition for your child. If interest is a continuous $10\%$, how much do you need to invest initially?

**Answer**

$6766.76

19) You are cooling a turkey that was taken out of the oven with an internal temperature of $165°F$. After $10$ minutes of resting the turkey in a $70°F$ apartment, the temperature has reached $155°F$. What is the temperature of the turkey $20$ minutes after taking it out of the oven?

20) You are trying to thaw some vegetables that are at a temperature of $1°F$. To thaw vegetables safely, you must put them in the refrigerator, which has an ambient temperature of $44°F$. You check on your vegetables $2$ hours after putting them in the refrigerator to find that they are now $12°F$. Plot the resulting temperature curve and use it to determine when the vegetables reach $33°$.

**Answer**

$9$hours $13$minutes

21) You are an archeologist and are given a bone that is claimed to be from a Tyrannosaurus Rex. You know these dinosaurs lived during the Cretaceous Era ($146$ million years to $65$ million years ago), and you find by radiocarbon dating that there is $0.000001\%$ the amount of radiocarbon. Is this bone from the Cretaceous?

22) The spent fuel of a nuclear reactor contains plutonium-239, which has a half-life of $24,000$ years. If $1$ barrel containing $10$ kg of plutonium-239 is sealed, how many years must pass until only $10$ g of plutonium-239 is left?

**Answer**

$239,179$ years

**For exercises 23 - 26, use the following table, which features the world population by decade.**

| Years since 1950 | Population (millions) |
|---|---|
| 0 | 2,556 |
| 10 | 3,039 |
| 20 | 3,706 |
| 30 | 4,453 |
| 40 | 5,279 |
| 50 | 6,083 |
| 60 | 6,849 |

*Source*: http:/www.factmonster.com/ipka/A0762181.html.

23) [T] The best-fit exponential curve to the data of the form $P(t) = ae^{bt}$ is given by $P(t) = 2686e^{0.01604t}$. Use a graphing calculator to graph the data and the exponential curve together.

24) [T] Find and graph the derivative $y'$ of your equation. Where is it increasing and what is the meaning of this increase?

**Answer**

$P'(t) = 43e^{0.01604t}$. The population is always increasing.

25) [T] Find and graph the second derivative of your equation. Where is it increasing and what is the meaning of this increase?

26) [T] Find the predicted date when the population reaches $10$ billion. Using your previous answers about the first and second derivatives, explain why exponential growth is unsuccessful in predicting the future.

**Answer**

The population reaches $10$ billion people in $2027$.

**For exercises 27 - 29, use the following table, which shows the population of San Francisco during the 19th century.**

| Years since 1850 | Population (thousands) |
|---|---|
| 0 | 21.00 |
| 10 | 56.80 |
| 20 | 149.5 |
| 30 | 234.0 |

*Source*: http:/www.sfgenealogy.com/sf/history/hgpop.htm.

27) [T] The best-fit exponential curve to the data of the form $P(t) = ae^{bt}$ is given by $P(t) = 35.26e^{0.06407t}$. Use a graphing calculator to graph the data and the exponential curve together.

28) [T] Find and graph the derivative $y'$ of your equation. Where is it increasing? What is the meaning of this increase? Is there a value where the increase is maximal?

**Answer**

$P'(t) = 2.259e^{0.06407t}$. The population is always increasing.

29) [T] Find and graph the second derivative of your equation. Where is it increasing? What is the meaning of this increase?

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# 2.10: Calculus of the Hyperbolic Functions

**Learning Objectives**

- Apply the formulas for derivatives and integrals of the hyperbolic functions.
- Apply the formulas for the derivatives of the inverse hyperbolic functions and their associated integrals.
- Describe the common applied conditions of a catenary curve.

We were introduced to hyperbolic functions previously, along with some of their basic properties. In this section, we look at differentiation and integration formulas for the hyperbolic functions and their inverses.

## Derivatives and Integrals of the Hyperbolic Functions

Recall that the hyperbolic sine and hyperbolic cosine are defined as

$$\sinh x = \frac{e^x - e^{-x}}{2} \tag{2.10.1}$$

and

$$\cosh x = \frac{e^x + e^{-x}}{2}. \tag{2.10.2}$$

The other hyperbolic functions are then defined in terms of $\sinh x$ and $\cosh x$. The graphs of the hyperbolic functions are shown in Figure 2.10.1.

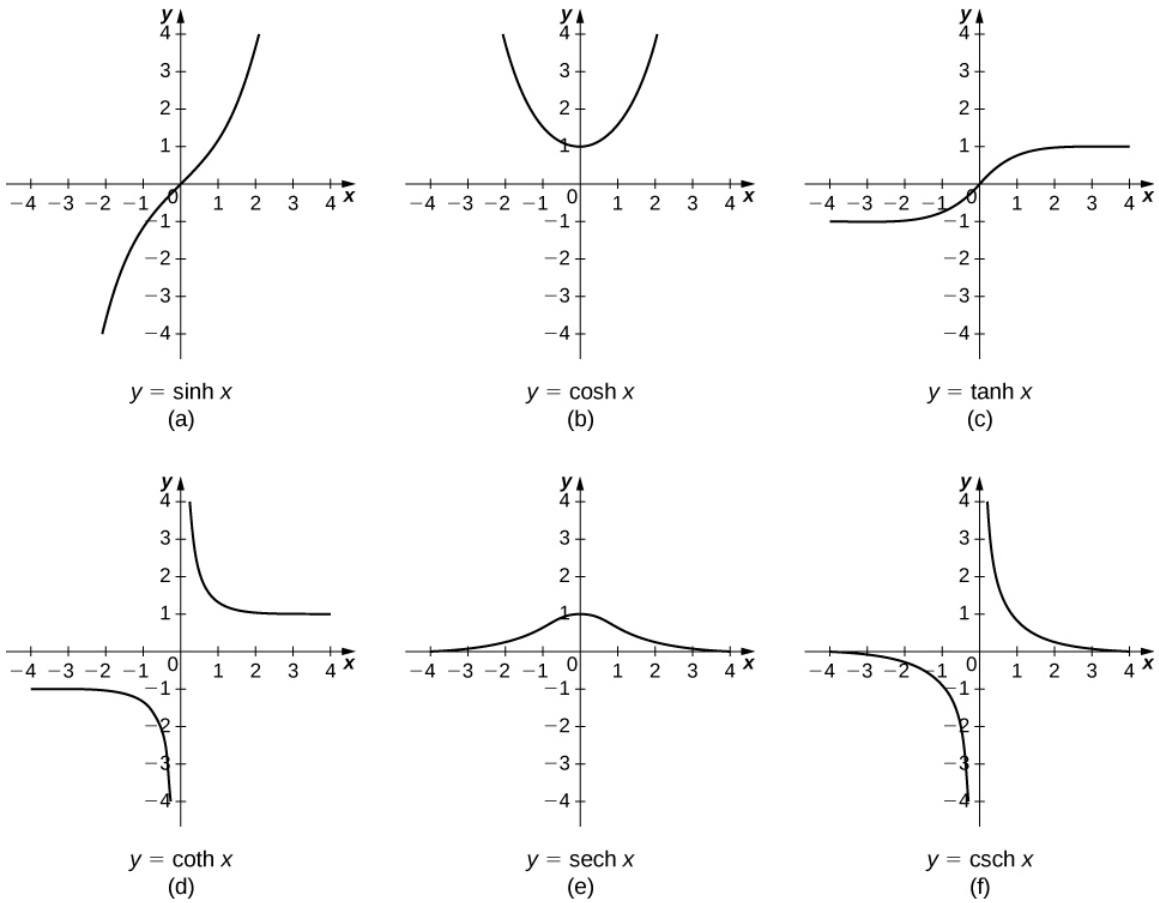



Figure 2.10.1: Graphs of the hyperbolic functions.

It is easy to develop differentiation formulas for the hyperbolic functions. For example, looking at $\sinh x$ we have

$$\begin{aligned}\frac{d}{dx}(\sinh x) &= \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) \\ &= \frac{1}{2}\left[\frac{d}{dx}(e^x) - \frac{d}{dx}(e^{-x})\right] \\ &= \frac{1}{2}[e^x + e^{-x}] \\ &= \cosh x.\end{aligned}$$

Similarly,

$$\frac{d}{dx}\cosh x = \sinh x. \tag{2.10.3}$$

We summarize the differentiation formulas for the hyperbolic functions in Table 2.10.1.

Table 2.10.1: Derivatives of the Hyperbolic Functions

| $f(x)$ | $\frac{d}{dx}f(x)$ |
|---|---|
| $\sinh x$ | $\cosh x$ |
| $\cosh x$ | $\sinh x$ |
| $\tanh x$ | $\operatorname{sech}^2 x$ |
| $\coth x$ | $-\operatorname{csch}^2 x$ |
| $\operatorname{sech} x$ | $-\operatorname{sech} x \tanh x$ |
| $\operatorname{csch} x$ | $-\operatorname{csch} x \coth x$ |

Let's take a moment to compare the derivatives of the hyperbolic functions with the derivatives of the standard trigonometric functions. There are a lot of similarities, but differences as well. For example, the derivatives of the sine functions match:

$$\frac{d}{dx}\sin x = \cos x \tag{2.10.4}$$

and

$$\frac{d}{dx}\sinh x = \cosh x.$$

The derivatives of the cosine functions, however, differ in sign:

$$\frac{d}{dx}\cos x = -\sin x,$$

but

$$\frac{d}{dx}\cosh x = \sinh x.$$

As we continue our examination of the hyperbolic functions, we must be mindful of their similarities and differences to the standard trigonometric functions. These differentiation formulas for the hyperbolic functions lead directly to the following integral formulas.

$$\int \sinh u\, du = \cosh u + C \tag{2.10.5}$$

$$\int \operatorname{csch}^2 u\, du = -\coth u + C \tag{2.10.6}$$

$$\int \cosh u\, du = \sinh u + C \tag{2.10.7}$$

$$\int \operatorname{sech} u \tanh u\, du = -\operatorname{sech}\ u + C - \operatorname{csch} u + C \tag{2.10.8}$$

$$\int \operatorname{sech}^2 u\, du = \tanh u + C \tag{2.10.9}$$

$$\int \operatorname{csch} u \coth u\, du = -\operatorname{csch} u + C \tag{2.10.10}$$

**Example 2.10.1: Differentiating Hyperbolic Functions**

Evaluate the following derivatives:

a. $\frac{d}{dx}(\sinh(x^2))$

b. $\frac{d}{dx}(\cosh x)^2$

**Solution:**

Using the formulas in Table 2.10.1 and the chain rule, we get

a. $\frac{d}{dx}(\sinh(x^2)) = \cosh(x^2) \cdot 2x$

b. $\frac{d}{dx}(\cosh x)^2 = 2\cosh x \sinh x$

**Exercise 2.10.1**

Evaluate the following derivatives:

a. $\frac{d}{dx}(\tanh(x^2 + 3x))$

b. $\frac{d}{dx}\left(\frac{1}{(\sinh x)^2}\right)$

**Hint**

Use the formulas in Table 2.10.1 and apply the chain rule as necessary.

**Answer a**

$\frac{d}{dx}(\tanh(x^2 + 3x)) = (\operatorname{sech}^2(x^2 + 3x))(2x + 3)$

**Answer b**

$\frac{d}{dx}\left(\frac{1}{(\sinh x)^2}\right) = \frac{d}{dx}(\sinh x)^{-2} = -2(\sinh x)^{-3}\cosh x$

**Example 2.10.2: Integrals Involving Hyperbolic Functions**

Evaluate the following integrals:

a. $\int x \cosh(x^2) dx$

b. $\int \tanh x\, dx$

**Solution**

We can use $u$-substitution in both cases.

a. Let $u = x^2$. Then, $du = 2x\, dx$ and

$$\int x\cosh(x^2)dx = \int \frac{1}{2}\cosh u\, du$$
$$= \frac{1}{2}\sinh u + C$$
$$= \frac{1}{2}\sinh(x^2) + C.$$

b. Let $u = \cosh x$. Then, $du = \sinh x\, dx$ and

$$\int \tanh x\, dx = \int \frac{\sinh x}{\cosh x}\, dx$$
$$= \int \frac{1}{u}\, du$$
$$= \ln|u| + C$$
$$= \ln|\cosh x| + C.$$

Note that $\cosh x > 0$ for all $x$, so we can eliminate the absolute value signs and obtain

$$\int \tanh x\, dx = \ln(\cosh x) + C.$$



## Exercise 2.10.2

Evaluate the following integrals:

a. $\int \sinh^3 x \cosh x\, dx$

b. $\int \text{sech}^{\,2}(3x)\, dx$

**Hint**

Use the formulas above and apply $u$-substitution as necessary.

**Answer a**

$$\int \sinh^3 x \cosh x\, dx = \frac{\sinh^4 x}{4} + C$$

**Answer b**

$$\int \operatorname{sech}^2(3x)\,dx = \frac{\tanh(3x)}{3} + C$$

## Calculus of Inverse Hyperbolic Functions

Looking at the graphs of the hyperbolic functions, we see that with appropriate range restrictions, they all have inverses. Most of the necessary range restrictions can be discerned by close examination of the graphs. The domains and ranges of the inverse hyperbolic functions are summarized in Table 2.10.2.

Table 2.10.2: Domains and Ranges of the Inverse Hyperbolic Functions

| Function | Domain | Range |
|---|---|---|
| $\sinh^{-1} x$ | $(-\infty,\infty)$ | $(-\infty,\infty)$ |
| $\cosh^{-1} x$ | $(1,\infty)$ | $[0,\infty)$ |
| $\tanh^{-1} x$ | $(-1,1)$ | $(-\infty,\infty)$ |
| $\coth^{-1} x$ | $(-\infty,1)\cup(1,\infty)$ | $(-\infty,0)\cup(0,\infty)$ |
| $\operatorname{sech}^{-1} x$ | $(0,1)$ | $[0,\infty)$ |
| $\operatorname{csch}^{-1} x$ | $(-\infty,0)\cup(0,\infty)$ | $(-\infty,0)\cup(0,\infty)$ |

The graphs of the inverse hyperbolic functions are shown in the following figure.

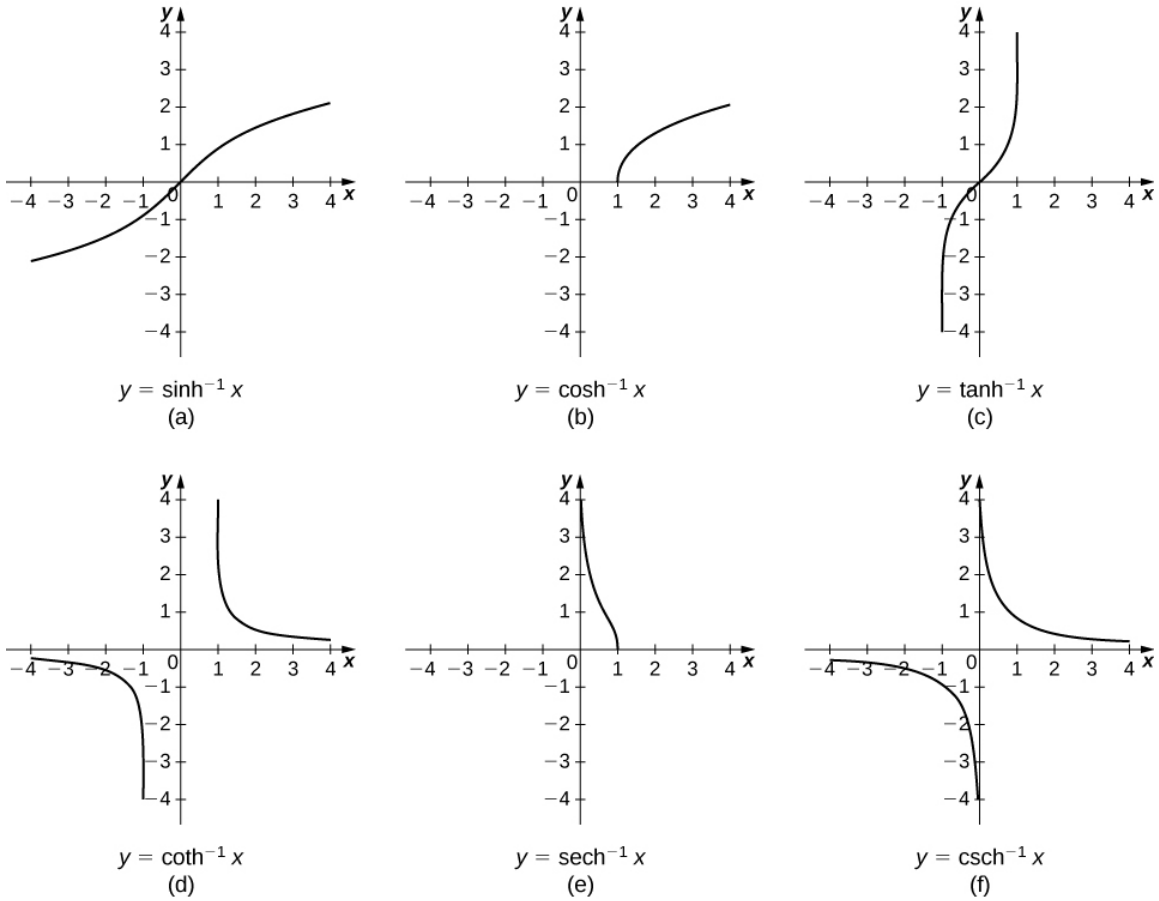



Figure 2.10.3: Graphs of the inverse hyperbolic functions.

To find the derivatives of the inverse functions, we use implicit differentiation. We have

$$y = \sinh^{-1} x \tag{2.10.11}$$

$$\sinh y = x \tag{2.10.12}$$

$$\frac{d}{dx}\sinh y = \frac{d}{dx}x \tag{2.10.13}$$

$$\cosh y \frac{dy}{dx} = 1. \tag{2.10.14}$$

Recall that $\cosh^2 y - \sinh^2 y = 1$, so $\cosh y = \sqrt{1 + \sinh^2 y}$.Then,

$$\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1 + \sinh^2 y}} = \frac{1}{\sqrt{1 + x^2}}. \tag{2.10.15}$$

We can derive differentiation formulas for the other inverse hyperbolic functions in a similar fashion. These differentiation formulas are summarized in Table **2.10.3**.

Table **2.10.3**: Derivatives of the Inverse Hyperbolic Functions

| $f(x)$ | $\frac{d}{dx}f(x)$ |
|---|---|
| $\sinh^{-1} x$ | $\frac{1}{\sqrt{1+x^2}}$ |
| $\cosh^{-1} x$ | $\frac{1}{\sqrt{x^2-1}}$ |
| $\tanh^{-1} x$ | $\frac{1}{1-x^2}$ |
| $\coth^{-1} x$ | $\frac{1}{1-x^2}$ |
| $\text{sech}^{-1} x$ | $\frac{-1}{x\sqrt{1-x^2}}$ |
| $\text{csch}^{-1} x$ | $\frac{-1}{\lvert x\rvert\sqrt{1+x^2}}$ |

Note that the derivatives of $\tanh^{-1} x$ and $\coth^{-1} x$ are the same. Thus, when we integrate $1/(1 - x^2)$, we need to select the proper antiderivative based on the domain of the functions and the values of $x$. Integration formulas involving the inverse hyperbolic functions are summarized as follows.

$$\int \frac{1}{\sqrt{1+u^2}}du = \sinh^{-1} u + C \tag{2.10.16}$$

$$\int \frac{1}{u\sqrt{1-u^2}}du = -\text{sech}^{-1} |u| + C \tag{2.10.17}$$

$$\int \frac{1}{\sqrt{u^2-1}}du = \cosh^{-1} u + C \tag{2.10.18}$$

$$\int \frac{1}{u\sqrt{1+u^2}}du = -\text{csch}^{-1} |u| + C \tag{2.10.19}$$

$$\int \frac{1}{1-u^2}du = \begin{cases} \tanh^{-1} u + C & \text{if } |u| < 1 \\ \coth^{-1} u + C & \text{if } |u| > 1 \end{cases} \tag{2.10.20}$$

**Example 2.10.3: Differentiating Inverse Hyperbolic Functions**

Evaluate the following derivatives:

a. $\frac{d}{dx}\left(\sinh^{-1}\left(\frac{x}{3}\right)\right)$

b. $\frac{d}{dx}\left(\tanh^{-1} x\right)^2$

**Solution**

Using the formulas in Table **2.10.3** and the chain rule, we obtain the following results:

a. $\frac{d}{dx}(\sinh^{-1}(\frac{x}{3})) = \frac{1}{3\sqrt{1+\frac{x^2}{9}}} = \frac{1}{\sqrt{9+x^2}}$

b. $\frac{d}{dx}(\tanh^{-1} x)^2 = \frac{2(\tanh^{-1} x)}{1-x^2}$

## Exercise 2.10.3

Evaluate the following derivatives:

a. $\frac{d}{dx}(\cosh^{-1}(3x))$

b. $\frac{d}{dx}(\coth^{-1} x)^3$

**Hint**

Use the formulas in Table **2.10.3** and apply the chain rule as necessary.

**Answer a**

$$\frac{d}{dx}(\cosh^{-1}(3x)) = \frac{3}{\sqrt{9x^2-1}}$$

**Answer b**

$$\frac{d}{dx}(\coth^{-1} x)^3 = \frac{3(\coth^{-1} x)^2}{1-x^2}$$

## Example 2.10.4: Integrals Involving Inverse Hyperbolic Functions

Evaluate the following integrals:

a. $\int \frac{1}{\sqrt{4x^2-1}}dx$

b. $\int \frac{1}{2x\sqrt{1-9x^2}}dx$

**Solution**

We can use $u$-**substitution** in both cases.

Let $u = 2x$. Then, $du = 2\,dx$ and we have

$$\begin{aligned}\int \frac{1}{\sqrt{4x^2-1}}\,dx &= \int \frac{1}{2\sqrt{u^2-1}}\,du \\ &= \frac{1}{2}\cosh^{-1} u + C \\ &= \frac{1}{2}\cosh^{-1}(2x) + C.\end{aligned}$$

Let $u = 3x$. Then, $du = 3\,dx$ and we obtain

$$\begin{aligned}\int \frac{1}{2x\sqrt{1-9x^2}}dx &= \frac{1}{2}\int \frac{1}{u\sqrt{1-u^2}}du \\ &= -\frac{1}{2}\operatorname{sech}^{-1}|u| + C \\ &= -\frac{1}{2}\operatorname{sech}^{-1}|3x| + C\end{aligned}$$



**Exercise 2.10.4**

Evaluate the following integrals:

a. $\int \frac{1}{\sqrt{x^2-4}}dx, x>2$

b. $\int \frac{1}{\sqrt{1-e^{2x}}}dx$

**Hint**

Use the formulas above and apply $u$-substitution as necessary.

**Answer a**

$$\int \frac{1}{\sqrt{x^2-4}}dx = \cosh^{-1}(\frac{x}{2})+C$$

**Answer b**

$$\int \frac{1}{\sqrt{1-e^{2x}}}dx = -\text{sech}^{-1}(e^x)+C$$

## Applications

One physical application of hyperbolic functions involves hanging cables. If a cable of uniform density is suspended between two supports without any load other than its own weight, the cable forms a curve called a **catenary**. High-voltage power lines, chains hanging between two posts, and strands of a spider's web all form catenaries. The following figure shows chains hanging from a row of posts.



Figure 2.10.3: Chains between these posts take the shape of a catenary. (credit: modification of work by OKFoundryCompany, Flickr)

Hyperbolic functions can be used to model catenaries. Specifically, functions of the form $y = a \cdot \cosh(x/a)$ are catenaries. Figure 2.10.4 shows the graph of $y = 2\cosh(x/2)$.

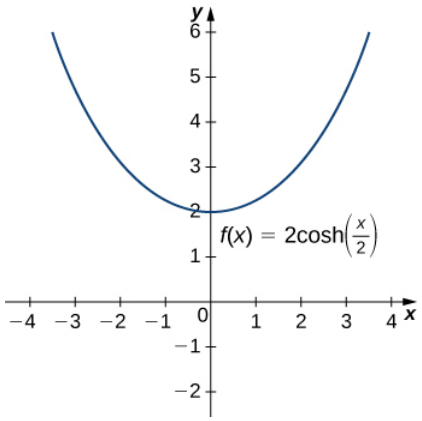



Figure 2.10.4: A hyperbolic cosine function forms the shape of a catenary.

**Example 2.10.5: Using a Catenary to Find the Length of a Cable**

Assume a hanging cable has the shape $10\cosh(x/10)$ for $-15 \le x \le 15$, where $x$ is measured in feet. Determine the length of the cable (in feet).

**Solution**

Recall from Section 6.4 that the formula for arc length is

$$\underbrace{\int_a^b \sqrt{1+[f'(x)]^2}dx}_{\text{Arc Length}}.$$

We have $f(x) = 10\cosh(x/10)$, so $f'(x) = \sinh(x/10)$. Then the arc length is

$$\int_a^b \sqrt{1+[f'(x)]^2}dx = \int_{-15}^{15} \sqrt{1+\sinh^2\left(\frac{x}{10}\right)}dx.$$

Now recall that

$$1 + \sinh^2 x = \cosh^2 x,$$

so we have

$$\begin{aligned}\text{Arc Length} &= \int_{-15}^{15} \sqrt{1+\sinh^2\left(\frac{x}{10}\right)}\,dx \\ &= \int_{-15}^{15} \cosh\left(\frac{x}{10}\right)\,dx \\ &= 10\sinh\left(\frac{x}{10}\right)\Big|_{-15}^{15} \\ &= 10\left[\sinh\left(\frac{3}{2}\right) - \sinh\left(-\frac{3}{2}\right)\right] \\ &= 20\sinh\left(\frac{3}{2}\right) \\ &\approx 42.586 \text{ ft.}\end{aligned}$$

## Finding the Length of a Hyperbolic Cosine Shaped Bridge

You are a structural engineer in charge of the design of the bridge shown below. As pictured, the shape of the bridge will be the same as the shape of the hanging rope that holds it together, or the catenary $y = a \cdot \cosh(\frac{x}{a})$ where $a$ represents the number of feet the lowest point of the bridge will be above the river. You will be given $a$ and how far the bridge will extend from the center of the river, and it will be your job to determine the total length of the bridge, so that the appropriate amount of material is ordered. Round your answer to two decimal places.



Click Begin to see the height of the bridge and the horizontal distance from the beginning and end to the center.

Begin

**Exercise 2.10.5:**

Assume a hanging cable has the shape $15\cosh(x/15)$ for $-20 \le x \le 20$. Determine the length of the cable (in feet).

**Answer**

52.95 ft

## Key Concepts

- Hyperbolic functions are defined in terms of exponential functions.
- Term-by-term differentiation yields differentiation formulas for the hyperbolic functions. These differentiation formulas give rise, in turn, to integration formulas.
- With appropriate range restrictions, the hyperbolic functions all have inverses.
- Implicit differentiation yields differentiation formulas for the inverse hyperbolic functions, which in turn give rise to integration formulas.

- The most common physical applications of hyperbolic functions are calculations involving catenaries.

## Glossary

**catenary**

a curve in the shape of the function $y = a \cdot \cosh(x/a)$ is a catenary; a cable of uniform density suspended between two supports assumes the shape of a catenary

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# 2.10E: Exercises for Section 2.10

1) [T] Find expressions for $\cosh x + \sinh x$ and $\cosh x - \sinh x$. Use a calculator to graph these functions and ensure your expression is correct.

**Answer**

$e^x$ and $e^{-x}$

2) From the definitions of $\cosh(x)$ and $\sinh(x)$, find their antiderivatives.

3) Show that $\cosh(x)$ and $\sinh(x)$ satisfy $y'' = y$.

**Answer**

Answers may vary

4) Use the quotient rule to verify that $\frac{d}{dx}\left(\tanh(x)\right) = \operatorname{sech}^2(x)$.

5) Derive $\cosh^2(x) + \sinh^2(x) = \cosh(2x)$ from the definition.

**Answer**

Answers may vary

6) Take the derivative of the previous expression to find an expression for $\sinh(2x)$.

7) Prove $\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$ by changing the expression to exponentials.

**Answer**

Answers may vary

8) Take the derivative of the previous expression to find an expression for $\cosh(x+y)$.

**In exercises 9 - 18, find the derivatives of the given functions and graph along with the function to ensure your answer is correct.**

9) [T] $\cosh(3x+1)$

**Answer**

$3\sinh(3x+1)$

10) [T] $\sinh(x^2)$

11) [T] $\frac{1}{\cosh(x)}$

**Answer**

$-\tanh(x)\operatorname{sech}(x)$

12) [T] $\sinh(\ln(x))$

13) [T] $\cosh^2(x) + \sinh^2(x)$

**Answer**

$4\cosh(x)\sinh(x)$

14) [T] $\cosh^2(x) - \sinh^2(x)$

15) [T] $\tanh(\sqrt{x^2+1})$

**Answer**

$$\frac{x\text{sech}^2(\sqrt{x^2+1})}{\sqrt{x^2+1}}$$

16) [T] $\frac{1+\tanh(x)}{1-\tanh(x)}$

17) [T] $\sinh^6(x)$

**Answer**

$6\sinh^5(x)\cosh(x)$

18) [T] $\ln(\text{sech}(x)+\tanh(x))$

**In exercises 19 - 28, find the antiderivatives for the given functions.**

19) $\cosh(2x+1)$

**Answer**

$\frac{1}{2}\sinh(2x+1)+C$

20) $\tanh(3x+2)$

21) $x\cosh(x^2)$

**Answer**

$\frac{1}{2}\sinh^2(x^2)+C$

22) $3x^3\tanh(x^4)$

23) $\cosh^2(x)\sinh(x)$

**Answer**

$\frac{1}{3}\cosh^3(x)+C$

24) $\tanh^2(x)\text{sech}^2(x)$

25) $\frac{\sinh(x)}{1+\cosh(x)}$

**Answer**

$\ln(1+\cosh(x))+C$

26) $\coth(x)$

27) $\cosh(x)+\sinh(x)$

**Answer**

$\cosh(x)+\sinh(x)+C$

28) $(\cosh(x)+\sinh(x))^n$

**In exercises 29 - 35, find the derivatives for the functions.**

29) $\tanh^{-1}(4x)$

**Answer**

$\frac{4}{1-16x^2}$

30) $\sinh^{-1}(x^2)$

31) $\sinh^{-1}(\cosh(x))$

**Answer**

$$\frac{\sinh(x)}{\sqrt{\cosh^2(x)+1}}$$

32) $\cosh^{-1}(x^3)$

33) $\tanh^{-1}(\cos(x))$

**Answer**

$-\csc(x)$

34) $e^{\sinh^{-1}(x)}$

35) $\ln(\tanh^{-1}(x))$

**Answer**

$$-\frac{1}{(x^2-1)\tanh^{-1}(x)}$$

**In exercises 36 - 42, find the antiderivatives for the functions.**

36) $\int \frac{dx}{4-x^2}$

37) $\int \frac{dx}{a^2-x^2}$

**Answer**

$\frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right)+C$

38) $\int \frac{dx}{\sqrt{x^2+1}}$

39) $\int \frac{xdx}{\sqrt{x^2+1}}$

**Answer**

$\sqrt{x^2+1}+C$

40) $\int -\frac{dx}{x\sqrt{1-x^2}}$

41) $\int \frac{e^x}{\sqrt{e^{2x}-1}}$

**Answer**

$\cosh^{-1}(e^x)+C$

42) $\int -\frac{2x}{x^4-1}$

**In exercises 43 - 45, use the fact that a falling body with friction equal to velocity squared obeys the equation $\frac{dv}{dt}=g-v^2$.**

43) Show that $v(t)=\sqrt{g}\tanh(\sqrt{g}t)$ satisfies this equation.

**Answer**

Answers may vary

44) Derive the previous expression for $v(t)$ by integrating $\frac{dv}{g-v^2}=dt$.

45) [T] Estimate how far a body has fallen in $12$seconds by finding the area underneath the curve of $v(t)$.

**Answer**

37.30

**In exercises 46 - 48, use this scenario: A cable hanging under its own weight has a slope $S = \frac{dy}{dx}$ that satisfies $\frac{dS}{dx} = c\sqrt{1+S^2}$. The constant $c$ is the ratio of cable density to tension.**

46) Show that $S = \sinh(cx)$ satisfies this equation.

47) Integrate $\frac{dy}{dx} = \sinh(cx)$ to find the cable height $y(x)$ if $y(0) = 1/c$.

**Answer**

$y = \frac{1}{c}\cosh(cx)$

48) Sketch the cable and determine how far down it sags at $x = 0$.

**In exercises 49 - 52, solve each problem.**

49) [T] A chain hangs from two posts 2m apart to form a catenary described by the equation $y = 2\cosh(x/2) - 1$. Find the slope of the catenary at the left fence post.

**Answer**

$-0.521095$

50) [T] A chain hangs from two posts four meters apart to form a catenary described by the equation $y = 4\cosh(x/4) - 3$. Find the total length of the catenary (arc length).

51) [T] A high-voltage power line is a catenary described by $y = 10\cosh(x/10)$. Find the ratio of the area under the catenary to its arc length. What do you notice?

**Answer**

10

52) A telephone line is a catenary described by $y = a\cosh(x/a)$. Find the ratio of the area under the catenary to its arc length. Does this confirm your answer for the previous question?

53) Prove the formula for the derivative of $y = \sinh^{-1}(x)$ by differentiating $x = \sinh(y)$.

(Hint: Use hyperbolic trigonometric identities.)

54) Prove the formula for the derivative of $y = \cosh^{-1}(x)$ by differentiating $x = \cosh(y)$.

(Hint: Use hyperbolic trigonometric identities.)

55) Prove the formula for the derivative of $y = \operatorname{sech}^{-1}(x)$ by differentiating $x = \operatorname{sech}(y)$.

(Hint: Use hyperbolic trigonometric identities.)

56) Prove that $\cosh(x) + \sinh(x))^n = \cosh(nx) + \sinh(nx)$.

57) Prove the expression for $\sinh^{-1}(x)$. Multiply $x = \sinh(y) = \frac{e^y - e^{-y}}{2}$ by $2e^y$ and solve for $y$. Does your expression match the textbook?

58) Prove the expression for $\cosh^{-1}(x)$. Multiply $x = \cosh(y) = \frac{e^y + e^{-y}}{2}$ by $2e^y$ and solve for $y$. Does your expression match the textbook?

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# 2.11: Chapter 2 Review Exercises

***True or False?* Justify your answer with a proof or a counterexample.**

1) The amount of work to pump the water out of a half-full cylinder is half the amount of work to pump the water out of the full cylinder.

**Answer**

False

2) If the force is constant, the amount of work to move an object from $x = a$ to $x = b$ is $F(b - a)$.

3) The disk method can be used in any situation in which the washer method is successful at finding the volume of a solid of revolution.

**Answer**

False

4) If the half-life of $seaborgium - 266$ is $360$ ms, then $k = \dfrac{\ln 2}{360}$.

**For exercises 5 - 8, use the requested method to determine the volume of the solid.**

5) The volume that has a base of the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$ and cross-sections of an equilateral triangle perpendicular to the $y$-axis. Use the method of slicing.

**Answer**

$V = 32\sqrt{3}\text{ units}^3$

6) $y = x^2 - x$, from $x = 1$ to $x = 4$, rotated around the $y$-axis using the washer method

7) $x = y^2$ and $x = 3y$ rotated around the $y$-axis using the washer method

**Answer**

$V = \frac{162\pi}{5}\text{ units}^3$

8) $x = 2y^2 - y^3,\ x = 0$,and $y = 0$ rotated around the $x$-axis using cylindrical shells

**For exercises 9 - 14, find**

**a. the area of the region,**

**b.the volume of the solid when rotated around the $x$-axis, and**

**c. the volume of the solid when rotated around the $y$-axis. Use whichever method seems most appropriate to you.**

9) $y = x^3, x = 0, y = 0$, and $x = 2$

**Answer**

a. $A = 4$ units$^2$
b. $V = \frac{128\pi}{7}$ units$^3$
c. $V = \frac{64\pi}{5}$ units$^3$

10) $y = x^2 - x$ and $x = 0$

11) [T] $y = \ln(x) + 2$ and $y = x$

**Answer**

a. $A \approx 1.949$ units$^2$
b. $V \approx 21.952$ units$^3$

c. $V = \approx 17.099$ units$^3$

12) $y = x^2$ and $y = \sqrt{x}$

13) $y = 5 + x, y = x^2, x = 0$, and $x = 1$

**Answer**

a. $A = \frac{31}{6}$ units$^2$
b. $V = \frac{452\pi}{15}$ units$^3$
c. $V = \frac{31\pi}{6}$ units$^3$

14) Below $x^2 + y^2 = 1$ and above $y = 1 - x$

15) Find the mass of $\rho = e^{-x}$ on a disk centered at the origin with radius $4$.

**Answer**

$m \approx 245.282$

16) Find the center of mass for $\rho = \tan^2 x$ on $x \in (-\frac{\pi}{4}, \frac{\pi}{4})$.

17) Find the mass and the center of mass of $\rho = 1$ on the region bounded by $y = x^5$ and $y = \sqrt{x}$.

**Answer**

Mass: $\frac{1}{2}$,
Center of mass: $(\frac{18}{35}, \frac{9}{11})$

**For exercises 18 - 19, find the requested arc lengths.**

18) The length of $x$ for $y = \cosh(x)$ from $x = 0$ to $x = 2$.

19) The length of $y$ for $x = 3 - \sqrt{y}$ from $y = 0$ to $y = 4$

**Answer**

$s = \left[\sqrt{17} + \frac{1}{8}\ln(33 + 8\sqrt{17})\right]$ units

**For exercises 20 - 21, find the surface area and volume when the given curves are revolved around the specified axis.**

20) The shape created by revolving the region between $y = 4 + x,\ y = 3 - x,\ x = 0$, and $x = 2$ rotated around the $y$-axis.

21) The loudspeaker created by revolving $y = \dfrac{1}{x}$ from $x = 1$ to $x = 4$ around the $x$-axis.

**Answer**

Volume: $V = \frac{3\pi}{4}$ units$^3$
Surface area: $A = \pi\left(\sqrt{2} - \sinh^{-1}(1) + \sinh^{-1}(16) - \frac{\sqrt{257}}{16}\right)$ units$^2$

**For exercise 22, consider the Karun-3 dam in Iran. Its shape can be approximated as an isosceles triangle with height 205 m and width 388 m. Assume the current depth of the water is 180 m. The density of water is 1000 kg/m$^3$.**

22) Find the total force on the wall of the dam.

23) You are a crime scene investigator attempting to determine the time of death of a victim. It is noon and $45$ °F outside and the temperature of the body is $78$ °F. You know the cooling constant is $k = 0.00824$ °F/min. When did the victim die, assuming that a human's temperature is $98$ °F?

**Answer**

11:02 a.m.

**For the following exercise, consider the stock market crash in 1929 in the United States. The table lists the Dow Jones industrial average per year leading up to the crash.**

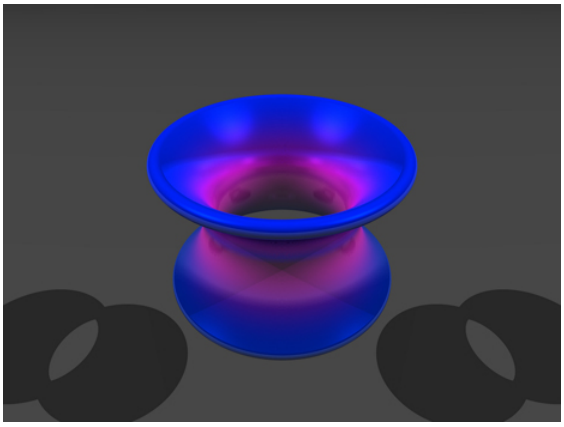
| Year after 1920 | Value ($) |
|---|---|
| 1 | 63.90 |
| 3 | 100 |
| 5 | 110 |
| 7 | 160 |
| 9 | 381.17 |

*Source*: http:/stockcharts.com/freecharts/hi...a19201940.html

24) [T] The best-fit exponential curve to these data is given by $y = 40.71 + 1.224^x$. Why do you think the gains of the market were unsustainable? Use first and second derivatives to help justify your answer. What would this model predict the Dow Jones industrial average to be in **2014** ?

**For exercises 25 - 26, consider the catenoid, the only solid of revolution that has a minimal surface, or zero mean curvature. A catenoid in nature can be found when stretching soap between two rings.**

25) Find the volume of the catenoid $y = \cosh(x)$ from $x = -1$ to $x = 1$ that is created by rotating this curve around the $x$-**axis,** as shown here.



**Answer**

$V = \pi\left(1 + \sinh(1)\cosh(1)\right)$ units$^3$

26) Find surface area of the catenoid $y = \cosh(x)$ from $x = -1$ to $x = 1$ that is created by rotating this curve around the $x$-**axis.**

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# CHAPTER OVERVIEW

## 3: Techniques of Integration

We saw in the previous chapter how important integration can be for all kinds of different topics—from calculations of volumes to flow rates, and from using a velocity function to determine a position to locating centers of mass. It is no surprise, then, that techniques for finding antiderivatives (or indefinite integrals) are important to know for everyone who uses them. We have already discussed some basic integration formulas and the method of integration by substitution. In this chapter, we study some additional techniques, including some ways of approximating definite integrals when normal techniques do not work.


3.1: Prelude to Techniques of Integration

3.2: Integration by Parts

3.2E: Exercises for Section 3.2

3.3: Trigonometric Integrals

3.3E: Exercises for Section 3.3

3.4: Trigonometric Substitution

3.4E: Exercises for Section 3.4

3.5: Partial Fractions

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3.6: Other Strategies for Integration

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3.7: Numerical Integration

3.7E: Exercises for Section 3.7

3.8: Improper Integrals

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3.9: Chapter 3 Review Exercises


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# 3.1: Prelude to Techniques of Integration

In a large city, accidents occurred at an average rate of one every three months at a particularly busy intersection. After residents complained, changes were made to the traffic lights at the intersection. It has now been eight months since the changes were made and there have been no accidents. Were the changes effective or is the eight-month interval without an accident a result of chance? We explore this question later in this chapter and see that integration is an essential part of determining the answer.



Figure **3.1.1**: Careful planning of traffic signals can prevent or reduce the number of accidents at busy intersections. (credit: modification of work by David McKelvey, Flickr)

We saw in the previous chapter how important integration can be for all kinds of different topics—from calculations of volumes to flow rates, and from using a velocity function to determine a position to locating centers of mass. It is no surprise, then, that techniques for finding antiderivatives (or indefinite integrals) are important to know for everyone who uses them. We have already discussed some basic integration formulas and the method of integration by substitution. In this chapter, we study some additional techniques, including some ways of approximating definite integrals when normal techniques do not work.

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# 3.2: Integration by Parts

**Learning Objectives**

- Recognize when to use integration by parts.
- Use the integration-by-parts formula to solve integration problems.
- Use the integration-by-parts formula for definite integrals.

By now we have a fairly thorough procedure for how to evaluate many basic integrals. However, although we can integrate $\int x \sin(x^2)\, dx$ by using the substitution, $u = x^2$, something as simple looking as $\int x \sin x\, dx$ defies us. Many students want to know whether there is a product rule for integration. There is not, but there is a technique based on the product rule for differentiation that allows us to exchange one integral for another. We call this technique **integration by parts.**

## The Integration-by-Parts Formula

If, $h(x) = f(x)g(x)$, then by using the product rule, we obtain

$$h'(x) = f'(x)g(x) + g'(x)f(x). \tag{3.2.1}$$

Although at first it may seem counterproductive, let's now integrate both sides of Equation 3.2.1:

$$\int h'(x)\, dx = \int (g(x)f'(x) + f(x)g'(x))\, dx.$$

This gives us

$$h(x) = f(x)g(x) = \int g(x)f'(x)\, dx + \int f(x)g'(x)\, dx.$$

Now we solve for $\int f(x)g'(x)\, dx$ :

$$\int f(x)g'(x)\, dx = f(x)g(x) - \int g(x)f'(x)\, dx.$$

By making the substitutions $u = f(x)$ and $v = g(x)$, which in turn make $du = f'(x)\, dx$ and $dv = g'(x)\, dx$, we have the more compact form

$$\int u\, dv = uv - \int v\, du.$$

**Integration by Parts**

Let $u = f(x)$ and $v = g(x)$ be functions with continuous derivatives. Then, the integration-by-parts formula for the integral involving these two functions is:

$$\int u\, dv = uv - \int v\, du. \tag{3.2.2}$$

The advantage of using the integration-by-parts formula is that we can use it to exchange one integral for another, possibly easier, integral. The following example illustrates its use.

**Example 3.2.1: Using Integration by Parts**

Use integration by parts with $u = x$ and $dv = \sin x\, dx$ to evaluate

$$\int x \sin x\, dx.$$

**Solution**

By choosing $u = x$, we have $du = 1\, dx$. Since $dv = \sin x\, dx$, we get

$$v = \int \sin x \; dx = -\cos x.$$

It is handy to keep track of these values as follows:

- $u = x$
- $dv = \sin x \; dx$
- $du = 1 \, dx$
- $v = \int \sin x \; dx = -\cos x.$

Applying the integration-by-parts formula (Equation 3.2.2) results in

$$\begin{aligned} \int x \sin x \; dx &= (x)(-\cos x) - \int (-\cos x)(1 \; dx) && \text{(Substitute)} \\ &= -x \cos x + \int \cos x \; dx && \text{(Simplify)} \end{aligned}$$

Then use

$$\int \cos x \; dx = \sin x + C.$$

to obtain

$$\int x \sin x \; dx = -x \cos x + \sin x + C.$$

**Analysis**

At this point, there are probably a few items that need clarification. First of all, you may be curious about what would have happened if we had chosen $u = \sin x$ and $dv = x$. If we had done so, then we would have $du = \cos x$ and $v = \frac{1}{2}x^2$. Thus, after applying integration by parts (Equation 3.2.2), we have

$$\int x \sin x \; dx = \frac{1}{2}x^2 \sin x - \int \frac{1}{2}x^2 \cos x \; dx.$$

Unfortunately, with the new integral, we are in no better position than before. It is important to keep in mind that when we apply integration by parts, we may need to try several choices for $u$ and $dv$ before finding a choice that works.

Second, you may wonder why, when we find $v = \int \sin x \; dx = -\cos x$, we do not use $v = -\cos x + K$. To see that it makes no difference, we can rework the problem using $v = -\cos x + K$:

$$\begin{aligned} \int x \sin x \; dx &= (x)(-\cos x + K) - \int (-\cos x + K)(1 \; dx) \\ &= -x \cos x + Kx + \int \cos x \; dx - \int K \; dx \\ &= -x \cos x + Kx + \sin x - Kx + C \\ &= -x \cos x + \sin x + C. \end{aligned}$$

As you can see, it makes no difference in the final solution.

Last, we can check to make sure that our antiderivative is correct by differentiating $-x \cos x + \sin x + C$ :

$$\begin{aligned} \frac{d}{dx}(-x \cos x + \sin x + C) &= \cancel{(-1)\cos x} + (-x)(-\sin x) + \cancel{\cos x} \\ &= x \sin x \end{aligned}$$

Therefore, the antiderivative checks out.



**Exercise 3.2.1**

Evaluate $\int xe^{2x}\,dx$ using the integration-by-parts formula (Equation 3.2.2) with $u = x$ and $dv = e^{2x}\,dx$.

**Hint**

Find $du$ and $v$, and use the previous example as a guide.

**Answer**

$$\int xe^{2x}\,dx = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$$

The natural question to ask at this point is: How do we know how to choose $u$ and $dv$? Sometimes it is a matter of trial and error; however, the acronym **LIATE** can often help to take some of the guesswork out of our choices. This acronym stands for **L**ogarithmic Functions, **I**nverse Trigonometric Functions, **A**lgebraic Functions, **T**rigonometric Functions, and **E**xponential Functions. This mnemonic serves as an aid in determining an appropriate choice for $u$. The type of function in the integral that appears first in the list should be our first choice of $u$.

For example, if an integral contains a **logarithmic function** and an **algebraic function,** we should choose $u$ to be the logarithmic function, because L comes before A in LIATE. The integral in Example 3.2.1 has a trigonometric function ($\sin x$) and an algebraic function ($x$). Because A comes before T in LIATE, we chose $u$ to be the algebraic function. When we have chosen $u$, $dv$ is selected to be the remaining part of the function to be integrated, together with $dx$.

Why does this mnemonic work? Remember that whatever we pick to be $dv$ must be something we can integrate. Since we do not have integration formulas that allow us to integrate simple logarithmic functions and inverse trigonometric functions, it makes sense that they should not be chosen as values for $dv$. Consequently, they should be at the head of the list as choices for $u$. Thus, we put LI at the beginning of the mnemonic. (We could just as easily have started with IL, since these two types of functions won't appear together in an integration-by-parts problem.) The exponential and trigonometric functions are at the end of our list because they are fairly easy to integrate and make good choices for $dv$. Thus, we have TE at the end of our mnemonic. (We could just as easily have used ET at the end, since when these types of functions appear together it usually doesn't really matter which one is $u$ and which one is $dv$.) Algebraic functions are generally easy both to integrate and to differentiate, and they come in the middle of the mnemonic.

**Example 3.2.2: Using Integration by Parts**

Evaluate

$$\int \frac{\ln x}{x^3}\, dx.$$

**Solution**

Begin by rewriting the integral:

$$\int \frac{\ln x}{x^3}\, dx = \int x^{-3} \ln x\, dx.$$

Since this integral contains the algebraic function $x^{-3}$ and the logarithmic function $\ln x$, choose $u = \ln x$, since $L$ comes before A in LIATE. After we have chosen $u = \ln x$, we must choose $dv = x^{-3}\, dx$.

Next, since $u = \ln x$, we have $du = \frac{1}{x}\, dx$. Also, $v = \int x^{-3}\, dx = -\frac{1}{2}x^{-2}$. Summarizing,

- $u = \ln x$
- $du = \frac{1}{x}\, dx$
- $dv = x^{-3}\, dx$
- $v = \int x^{-3}\, dx = -\frac{1}{2}x^{-2}$.

Substituting into the integration-by-parts formula (Equation 3.2.2) gives

$$\begin{aligned}\int \frac{\ln x}{x^3}\, dx = \int x^{-3} \ln x\, dx &= (\ln x)(-\tfrac{1}{2}x^{-2}) - \int (-\tfrac{1}{2}x^{-2})(\tfrac{1}{x}\, dx) \\ &= -\frac{1}{2}x^{-2}\ln x + \int \frac{1}{2}x^{-3}\, dx \\ &= -\frac{1}{2}x^{-2}\ln x - \frac{1}{4}x^{-2} + C \\ &= -\frac{1}{2x^2}\ln x - \frac{1}{4x^2} + C\end{aligned}$$

**Exercise 3.2.2**

Evaluate

$$\int x \ln x\, dx.$$

**Hint**

Use $u = \ln x$ and $dv = x\ dx$.

**Answer**

$$\int x \ln x\, dx = \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C$$



In some cases, as in the next two examples, it may be necessary to apply integration by parts more than once.

**Example 3.2.3A: Applying Integration by Parts More Than Once**

Evaluate

$$\int x^2 e^{3x}\,dx.$$

**Solution**

Using LIATE, choose $u = x^2$ and $dv = e^{3x}\,dx$. Thus, $du = 2x\,dx$ and $v = \int e^{3x}\,dx = \left(\frac{1}{3}\right)e^{3x}$. Therefore,

- $u = x^2$
- $du = 2x\,dx$
- $dv = e^{3x}\,dx$
- $v = \int e^{3x}\,dx = \frac{1}{3}e^{3x}$.

Substituting into Equation 3.2.2 produces

$$\int x^2 e^{3x}\,dx = \frac{1}{3}x^2 e^{3x} - \int \frac{2}{3}xe^{3x}\,dx. \tag{3.2.3}$$

We still cannot integrate $\int \frac{2}{3}xe^{3x}\,dx$ directly, but the integral now has a lower power on $x$. We can evaluate this new integral by using integration by parts again. To do this, choose

$$u = x$$

and

$$dv = \frac{2}{3}e^{3x}\,dx.$$

Thus,

$$du = dx$$

and

$$v = \int \left(\frac{2}{3}\right)e^{3x}\,dx = \left(\frac{2}{9}\right)e^{3x}.$$

Now we have

- $u = x$
- $du = dx$
- $dv = \frac{2}{3}e^{3x}\,dx$
- $v = \int \frac{2}{3}e^{3x}\,dx = \frac{2}{9}e^{3x}$.

Substituting back into Equation 3.2.3 yields

$$\int x^2 e^{3x}\,dx = \frac{1}{3}x^2e^{3x} - \left(\frac{2}{9}xe^{3x} - \int \frac{2}{9}e^{3x}\,dx\right).$$

After evaluating the last integral and simplifying, we obtain

$$\int x^2 e^{3x}\,dx = \frac{1}{3}x^2e^{3x} - \frac{2}{9}xe^{3x} + \frac{2}{27}e^{3x} + C.$$

## First Step For Integration By Parts Activity

An integral will appear at the top and slowly fall down. Your task is to move the integral to the correct position so that it lands on the correct first step, or on "Impossible" if it cannot be done using substitution or integration by parts. Click on Get Started to begin.

Begin

Left Right Slower Faster

0 Correct and 0 Incorrect

### Example 3.2.3B: Applying Integration by Parts When LIATE Does not Quite Work

Evaluate

$$\int t^3 e^{t^2}\,dt.$$

**Solution**

If we use a strict interpretation of the mnemonic LIATE to make our choice of $u$, we end up with $u = t^3$ and $dv = e^{t^2}dt$. Unfortunately, this choice won't work because we are unable to evaluate $\int e^{t^2}dt$. However, since we can evaluate $\int te^{t^2}\,dx$, we can try choosing $u = t^2$ and $dv = te^{t^2}dt$. With these choices we have

- $u = t^2$
- $du = 2tdt$
- $dv = te^{t^2}dt$
- $v = \int te^{t^2}dt = \frac{1}{2}e^{t^2}$.

Thus, we obtain

$$\begin{aligned}\int t^3 e^{t^2}dt &= \frac{1}{2}t^2e^{t^2} - \int \frac{1}{2}e^{t^2}2t\,dt \\ &= \frac{1}{2}t^2e^{t^2} - \frac{1}{2}e^{t^2} + C.\end{aligned}$$

### Example 3.2.3C: Applying Integration by Parts More Than Once

Evaluate

$$\int \sin(\ln x)\,dx.$$

**Solution**

This integral appears to have only one function—namely, $\sin(\ln x)$—however, we can always use the constant function 1 as the other function. In this example, let's choose $u = \sin(\ln x)$ and $dv = 1\,dx$. (The decision to use $u = \sin(\ln x)$ is easy. We can't choose $dv = \sin(\ln x)\,dx$ because if we could integrate it, we wouldn't be using integration by parts in the first place!) Consequently, $du = (1/x)\cos(\ln x)\,dx$ and $v = \int 1\,dx = x$. After applying integration by parts to the integral and simplifying, we have

$$\int \sin(\ln x)\,dx = x\sin(\ln x) - \int \cos(\ln x)\,dx.$$

Unfortunately, this process leaves us with a new integral that is very similar to the original. However, let's see what happens when we apply integration by parts again. This time let's choose $u = \cos(\ln x)$ and $dv = 1\,dx$, making $du = -(1/x)\sin(\ln x)\,dx$ and $v = \int 1\,dx = x$.

Substituting, we have

$$\int \sin(\ln x)\,dx = x\sin(\ln x) - (x\cos(\ln x) - \int -\sin(\ln x)\,dx).$$

After simplifying, we obtain

$$\int \sin(\ln x)\,dx = x\sin(\ln x) - x\cos(\ln x) - \int \sin(\ln x)\,dx.$$

The last integral is now the same as the original. It may seem that we have simply gone in a circle, but now we can actually evaluate the integral. To see how to do this more clearly, substitute $I = \int \sin(\ln x)\,dx$. Thus, the equation becomes

$$I = x\sin(\ln x) - x\cos(\ln x) - I.$$

First, add $I$ to both sides of the equation to obtain

$$2I = x\sin(\ln x) - x\cos(\ln x).$$

Next, divide by 2:

$$I = \frac{1}{2}x\sin(\ln x) - \frac{1}{2}x\cos(\ln x).$$

Substituting $I = \int \sin(\ln x)\,dx$ again, we have

$$\int \sin(\ln x)\,dx = \frac{1}{2}x\sin(\ln x) - \frac{1}{2}x\cos(\ln x).$$

From this we see that $(1/2)x\sin(\ln x) - (1/2)x\cos(\ln x)$ is an antiderivative of $\sin(\ln x)\,dx$. For the most general antiderivative, add $+C$:

$$\int \sin(\ln x)\,dx = \frac{1}{2}x\sin(\ln x) - \frac{1}{2}x\cos(\ln x) + C.$$

**Analysis**

If this method feels a little strange at first, we can check the answer by differentiation:

$$\begin{aligned}\frac{d}{dx}\left(\frac{1}{2}x\sin(\ln x) - \frac{1}{2}x\cos(\ln x)\right)& \\ &= \frac{1}{2}(\sin(\ln x)) + \cos(\ln x)\cdot\frac{1}{x}\cdot\frac{1}{2}x \\ &\quad - \left(\frac{1}{2}\cos(\ln x) - \sin(\ln x)\cdot\frac{1}{x}\cdot\frac{1}{2}x\right) \\ &= \sin(\ln x).\end{aligned}$$

**Exercise 3.2.3**

Evaluate

$$\int x^2\sin x\,dx.$$

**Hint**

This is similar to Examples 3.2.3*A* - 3.2.3*C*.

**Answer**

$$\int x^2\sin x\,dx = -x^2\cos x + 2x\sin x + 2\cos x + C$$



## Integration by Parts for Definite Integrals

Now that we have used integration by parts successfully to evaluate *indefinite integrals*, we turn our attention to definite integrals. The integration technique is really the same, only we add a step to evaluate the integral at the upper and lower limits of integration.

### Integration by Parts for Definite Integrals

Let $u = f(x)$ and $v = g(x)$ be functions with continuous derivatives on $[a, b]$. Then

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du \tag{3.2.4}$$

### Example 3.2.4A: Finding the Area of a Region

Find the area of the region bounded above by the graph of $y = \tan^{-1} x$ and below by the $x$-axis over the interval $[0, 1]$.

**Solution**

This region is shown in Figure 3.2.1. To find the area, we must evaluate

$$\int_0^1 \tan^{-1} x\,dx.$$

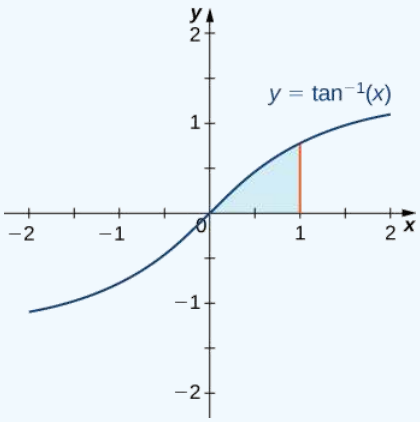



Figure 3.2.1: To find the area of the shaded region, we have to use integration by parts.

For this integral, let's choose $u = tan^{-1}x$ and $dv = dx$, thereby making $du = \frac{1}{x^2+1}\,dx$ and $v = x$. After applying the integration-by-parts formula (Equation 3.2.2) we obtain

$$\text{Area} = x\tan^{-1}x\Big|_0^1 - \int_0^1 \frac{x}{x^2+1}\,dx.$$

Use $u$-substitution to obtain

$$\int_0^1 \frac{x}{x^2+1}\,dx = \frac{1}{2}\ln(x^2+1)\Big|_0^1.$$

Thus,

$$\text{Area} = x\tan^{-1}x\Big|_0^1 - \frac{1}{2}\ln(x^2+1)\Big|_0^1 = \left(\frac{\pi}{4} - \frac{1}{2}\ln 2\right)\ \text{units}^2.$$

At this point it might not be a bad idea to do a "reality check" on the reasonableness of our solution. Since $\frac{\pi}{4} - \frac{1}{2}\ln 2 \approx 0.4388$ $\text{units}^2$, and from Figure 3.2.1 we expect our area to be slightly less than $0.5$ $\text{units}^2$, this solution appears to be reasonable.

## Example 3.2.4B: Finding a Volume of Revolution

Find the volume of the solid obtained by revolving the region bounded by the graph of $f(x) = e^{-x}$, the $x$-axis, the $y$-axis, and the line $x = 1$ about the $y$-axis.

**Solution**

The best option to solving this problem is to use the shell method. Begin by sketching the region to be revolved, along with a typical rectangle (Figure 3.2.2).

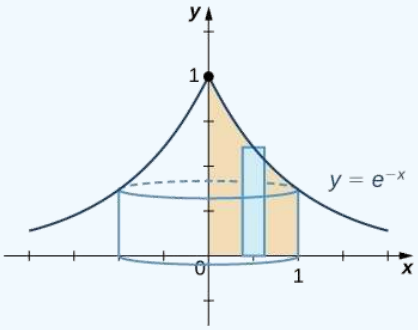



Figure 3.2.2: We can use the shell method to find a volume of revolution.

To find the volume using shells, we must evaluate

$$2\pi\int_0^1 xe^{-x}\,dx.$$

To do this, let $u = x$ and $dv = e^{-x}$. These choices lead to $du = dx$ and $v = \int e^{-x}\,dx = -e^{-x}$. Using the Shell Method formula, we obtain

$$\text{Volume} = 2\pi \int_0^1 xe^{-x}\,dx$$

$$= 2\pi \left(-xe^{-x}\Big|_0^1 + \int_0^1 e^{-x}\,dx\right) \quad \text{(Use integration by parts)}$$

$$= 2\pi\left(-e^{-1} + 0 - e^{-x}\Big|_0^1\right)$$

$$= 2\pi\left(-e^{-1} - e^{-1} + 1\right)$$

$$= 2\pi\left(1 - \frac{2}{e}\right) \text{ units}^3. \quad \text{(Evaluate and simplify)}$$

**Analysis**

Again, it is a good idea to check the reasonableness of our solution. We observe that the solid has a volume slightly less than that of a cylinder of radius $1$ and height of $1/e$ added to the volume of a cone of base radius $1$ and height of $1 - \frac{1}{e}$. Consequently, the solid should have a volume a bit less than

$$\pi(1)^2\frac{1}{e} + \left(\frac{\pi}{3}\right)(1)^2\left(1 - \frac{1}{e}\right) = \frac{2\pi}{3e} + \frac{\pi}{3} \approx 1.8177 \text{ units}^3.$$

Since $2\pi - \frac{4\pi}{e} \approx 1.6603$, we see that our calculated volume is reasonable.

**Exercise 3.2.4**

Evaluate

$$\int_0^{\pi/2} x\cos x\,dx.$$

**Hint**

Use Equation 3.2.2 with $u = x$ and $dv = \cos x\,dx$.

**Answer**

$$\int_0^{\pi/2} x\cos x\,dx = \frac{\pi}{2} - 1$$

## Key Concepts

- The integration-by-parts formula (Equation 3.2.2) allows the exchange of one integral for another, possibly easier, integral.
- Integration by parts applies to both definite and indefinite integrals.

## Key Equations

- **Integration by parts formula**

$$\int u\,dv = uv - \int v\,du$$

- **Integration by parts for definite integrals**

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

## Glossary

**integration by parts**

a technique of integration that allows the exchange of one integral for another using the formula $\int u\,dv = uv - \int v\,du$

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# 3.2E: Exercises for Section 3.2

**In using the technique of integration by parts, you must carefully choose which expression is $u$. For each of the following problems, use the guidelines in this section to choose $u$. Do not evaluate the integrals.**

1) $\int x^3 e^{2x}\,dx$

**Answer**

$u = x^3$

2) $\int x^3 \ln(x)\,dx$

3) $\int y^3 \cos y\,dy$

**Answer**

$u = y^3$

4) $\int x^2 \arctan x\,dx$

5) $\int e^{3x} \sin(2x)\,dx$

**Answer**

$u = \sin(2x)$

**In exercises 6 - 37, find the integral by using the simplest method. Not all problems require integration by parts.**

6) $\int v \sin v\,dv$

7) $\int \ln x\,dx$ (Hint: $\int \ln x\,dx$ is equivalent to $\int 1 \cdot \ln(x)\,dx$.)

**Answer**

$$\int \ln x\,dx \quad = \quad -x + x\ln x + C$$

8) $\int x \cos x\,dx$

9) $\int \tan^{-1} x\,dx$

**Answer**

$$\int \tan^{-1} x\,dx \quad = \quad x\tan^{-1} x - \tfrac{1}{2}\ln(1+x^2) + C$$

10) $\int x^2 e^x\,dx$

11) $\int x \sin(2x)\,dx$

**Answer**

$$\int x \sin(2x)\,dx \quad = \quad -\tfrac{1}{2}x\cos(2x) + \tfrac{1}{4}\sin(2x) + C$$

12) $\int x e^{4x}\,dx$

13) $\int x e^{-x}\,dx$

**Answer**

$$\int xe^{-x}\,dx \quad = \quad e^{-x}(-1-x)+C$$

14) $\int x\cos 3x\,dx$

15) $\int x^2\cos x\,dx$

**Answer**

$$\int x^2\cos x\,dx \quad = \quad 2x\cos x+(-2+x^2)\sin x+C$$

16) $\int x\ln x\,dx$

17) $\int \ln(2x+1)\,dx$

**Answer**

$$\int \ln(2x+1)\,dx \quad = \quad \tfrac{1}{2}(1+2x)(-1+\ln(1+2x))+C$$

18) $\int x^2e^{4x}\,dx$

19) $\int e^x\sin x\,dx$

**Answer**

$$\int e^x\sin x\,dx \quad = \quad \tfrac{1}{2}e^x(-\cos x+\sin x)+C$$

20) $\int e^x\cos x\,dx$

21) $\int xe^{-x^2}\,dx$

**Answer**

$$\int xe^{-x^2}\,dx \quad = \quad -\frac{e^{-x^2}}{2}+C$$

22) $\int x^2e^{-x}\,dx$

23) $\int \sin(\ln(2x))\,dx$

**Answer**

$$\int \sin(\ln(2x))\,dx \quad = \quad -\tfrac{1}{2}x\cos[\ln(2x)]+\tfrac{1}{2}x\sin[\ln(2x)]+C$$

24) $\int \cos(\ln x)\,dx$

25) $\int (\ln x)^2\,dx$

**Answer**

$$\int (\ln x)^2\,dx \quad = \quad 2x-2x\ln x+x(\ln x)^2+C$$

26) $\int \ln(x^2)\,dx$

27) $\int x^2\ln x\,dx$

**Answer**

$$\int x^2 \ln x \, dx = -\frac{x^3}{9} + \frac{1}{3}x^3 \ln x + C$$

28) $\int \sin^{-1} x \, dx$

29) $\int \cos^{-1}(2x) \, dx$

**Answer**

$$\int \cos^{-1}(2x) \, dx = -\frac{1}{2}\sqrt{1-4x^2} + x\cos^{-1}(2x) + C$$

30) $\int x \arctan x \, dx$

31) $\int x^2 \sin x \, dx$

**Answer**

$$\int x^2 \sin x \, dx = -(-2+x^2)\cos x + 2x \sin x + C$$

32) $\int x^3 \cos x \, dx$

33) $\int x^3 \sin x \, dx$

**Answer**

$$\int x^3 \sin x \, dx = -x(-6+x^2)\cos x + 3(-2+x^2)\sin x + C$$

34) $\int x^3 e^x \, dx$

35) $\int x \sec^{-1} x \, dx$

**Answer**

$$\int x \sec^{-1} x \, dx = \frac{1}{2}x\left(-\sqrt{1-\frac{1}{x^2}} + x \cdot \sec^{-1} x\right) + C$$

36) $\int x \sec^2 x \, dx$

37) $\int x \cosh x \, dx$

**Answer**

$$\int x \cosh x \, dx = -\cosh x + x \sinh x + C$$

**In exercises 38 - 46, compute the definite integrals. Use a graphing utility to confirm your answers.**

38) $\int_{1/e}^{1} \ln x \, dx$

39) $\int_0^1 xe^{-2x} \, dx$ (Express the answer in exact form.)

**Answer**

$$\int_0^1 xe^{-2x} \, dx = \frac{1}{4} - \frac{3}{4e^2}$$

40) $\int_0^1 e^{\sqrt{x}} \, dx \quad (\text{let } u = \sqrt{x})$

41) $\int_1^e \ln(x^2) \, dx$

**Answer**

$$\int_1^e \ln(x^2)\,dx \quad = \quad 2$$

42) $\int_0^{\pi} x\cos x\,dx$

43) $\int_{-\pi}^{\pi} x\sin x\,dx$ (Express the answer in exact form.)

**Answer**

$$\int_{-\pi}^{\pi} x\sin x\,dx \quad = \quad 2\pi$$

44) $\int_0^3 \ln(x^2+1)\,dx$ (Express the answer in exact form.)

45) $\int_0^{\pi/2} x^2\sin x\,dx$ (Express the answer in exact form.)

**Answer**

$$\int_0^{\pi/2} x^2\sin x\,dx \quad = \quad -2+\pi$$

46) $\int_0^1 x5^x\,dx$ (Express the answer using five significant digits.)

47) Evaluate $\int \cos x\ln(\sin x)\,dx$

**Answer**

$$\int \cos x\ln(\sin x)\,dx \quad = \quad -\sin(x)+\ln[\sin(x)]\sin x + C$$

**In exercises 48 - 50, derive the following formulas using the technique of integration by parts. Assume that $n$ is a positive integer. These formulas are called *reduction formulas* because the exponent in the $x$ term has been reduced by one in each case. The second integral is simpler than the original integral.**

48) $\int x^n e^x\,dx = x^n e^x - n\int x^{n-1}e^x\,dx$

49) $\int x^n\cos x\,dx = x^n\sin x - n\int x^{n-1}\sin x\,dx$

**Answer**

Answers vary

50) $\int x^n\sin x\,dx =$_____

51) Integrate $\int 2x\sqrt{2x-3}\,dx$ using two methods:

a. Using parts, letting $dv = \sqrt{2x-3}\,dx$

b. Substitution, letting $u = 2x-3$

**Answer**

a. $\int 2x\sqrt{2x-3}\,dx \quad = \quad \frac{2}{5}(1+x)(-3+2x)^{3/2}+C$

b. $\int 2x\sqrt{2x-3}\,dx \quad = \quad \frac{2}{5}(1+x)(-3+2x)^{3/2}+C$

**In exercises 52 - 57, state whether you would use integration by parts to evaluate the integral. If so, identify $u$ and $dv$. If not, describe the technique used to perform the integration without actually doing the problem.**

52) $\int x\ln x\,dx$

53) $\int \frac{\ln^2 x}{x}\,dx$

**Answer**

Do not use integration by parts. Choose $u$ to be $\ln x$, and the integral is of the form $\int u^2\,du$.

54) $\int xe^x\,dx$

55) $\int xe^{x^2-3}\,dx$

**Answer**

Do not use integration by parts. Let $u = x^2 - 3$, and the integral can be put into the form $\int e^u\,du$.

56) $\int x^2 \sin x\,dx$

57) $\int x^2 \sin(3x^3 + 2)\,dx$

**Answer**

Do not use integration by parts. Choose $u$ to be $u = 3x^3 + 2$ and the integral can be put into the form $\int \sin(u)\,du$.

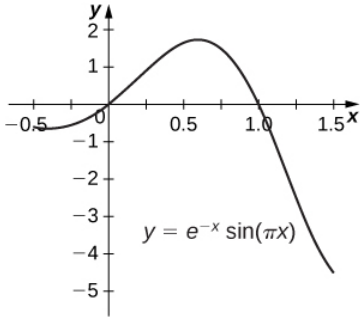
**In exercises 58-59, sketch the region bounded above by the curve, the $x$-axis, and $x = 1$, and find the area of the region. Provide the exact form or round answers to the number of places indicated.**

58) $y = 2xe^{-x}$ (Approximate answer to four decimal places.)

59) $y = e^{-x}\sin(\pi x)$ (Approximate answer to five decimal places.)

**Answer**

The area under graph is $0.39535\ \text{units}^2$.




**In exercises 60 - 61, find the volume generated by rotating the region bounded by the given curves about the specified line. Express the answers in exact form or approximate to the number of decimal places indicated.**

60) $y = \sin x$, $y = 0$, $x = 2\pi$, $x = 3\pi$; about the $y$-axis (Express the answer in exact form.)

61) $y = e^{-x}$, $y = 0$, $x = -1$, $x = 0$; about $x = 1$ (Express the answer in exact form.)

**Answer**

$V = 2\pi e\ \text{units}^3$

62) A particle moving along a straight line has a velocity of $v(t) = t^2e^{-t}$ after $t$ sec. How far does it travel in the first 2 sec? (Assume the units are in feet and express the answer in exact form.)

63) Find the area under the graph of $y = \sec^3 x$ from $x = 0$ to $x = 1$. (Round the answer to two significant digits.)

**Answer**

$A = 2.05 \text{ units}^2$

64) Find the area between $y = (x - 2)e^x$ and the $x$-axis from $x = 2$ to $x = 5$. (Express the answer in exact form.)

65) Find the area of the region enclosed by the curve $y = x \cos x$ and the $x$-axis for $\frac{11\pi}{2} \leq x \leq \frac{13\pi}{2}$. (Express the answer in exact form.)

**Answer**

$A = 12\pi \text{ units}^2$

66) Find the volume of the solid generated by revolving the region bounded by the curve $y = \ln x$, the $x$-axis, and the vertical line $x = e^2$ about the $x$-axis. (Express the answer in exact form.)

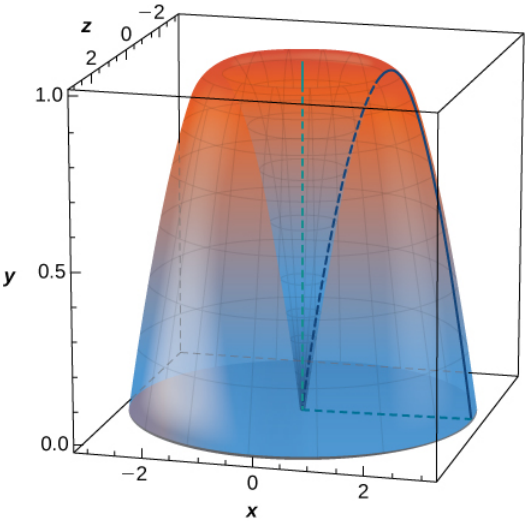
67) Find the volume of the solid generated by revolving the region bounded by the curve $y = 4 \cos x$ and the $x$-axis, $\frac{\pi}{2} \leq x \leq \frac{3\pi}{2}$, about the $x$-axis. (Express the answer in exact form.)

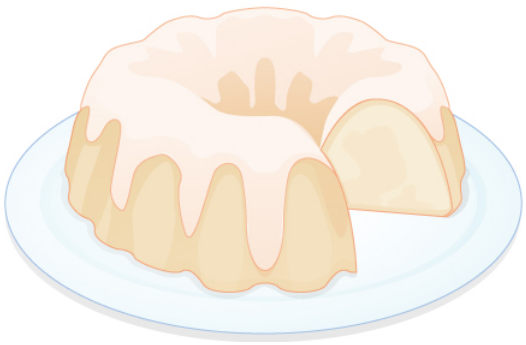
**Answer**

$V = 8\pi^2 \text{ units}^3$

68) Find the volume of the solid generated by revolving the region in the first quadrant bounded by $y = e^x$ and the $x$-axis, from $x = 0$ to $x = \ln(7)$, about the $y$-axis. (Express the answer in exact form.)

69) What is the volume of the Bundt cake that comes from rotating $y = \sin x$ around the $y$-axis from $x = 0$ to $x = \pi$?



**Answer**

$V = 2\pi^2 \text{ units}^3$


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# 3.3: Trigonometric Integrals

**Learning Objectives**

- Solve integration problems involving products and powers of $\sin x$ and $\cos x$.
- Solve integration problems involving products and powers of $\tan x$ and $\sec x$.
- Use reduction formulas to solve trigonometric integrals.

In this section we look at how to integrate a variety of products of trigonometric functions. These integrals are called **trigonometric integrals**. They are an important part of the integration technique called *trigonometric substitution*, which is featured in Trigonometric Substitution. This technique allows us to convert algebraic expressions that we may not be able to integrate into expressions involving trigonometric functions, which we may be able to integrate using the techniques described in this section. In addition, these types of integrals appear frequently when we study polar, cylindrical, and spherical coordinate systems later. Let's begin our study with products of $\sin x$ and $\cos x$.

## Integrating Products and Powers of sin x and cos x

A key idea behind the strategy used to integrate combinations of products and powers of $\sin x$ and $\cos x$ involves rewriting these expressions as sums and differences of integrals of the form $\int \sin^j x \cos x \, dx$ or $\int \cos^j x \sin x \, dx$. After rewriting these integrals, we evaluate them using $u$-substitution. Before describing the general process in detail, let's take a look at the following examples.

**Example 3.3.1: Integrating $\int \cos^j x \sin x \, dx$**

Evaluate $\int \cos^3 x \sin x \, dx$.

**Solution**

Use $u$-substitution and let $u = \cos x$. In this case, $du = -\sin x \, dx$.

Thus,

$$\int \cos^3 x \sin x \, dx = -\int u^3 \, du = -\frac{1}{4}u^4 + C = -\frac{1}{4}\cos^4 x + C.$$



**Exercise 3.3.1**

Evaluate $\int \sin^4 x \cos x \, dx$.

**Hint**

Let $u = \sin x$.

**Answer**

$$\int \sin^4 x \cos x \, dx = \frac{1}{5}\sin^5 x + C$$

**Example 3.3.2: A Preliminary Example: Integrating $\int \cos^j x \sin^k x \, dx$ where $k$ is Odd**

Evaluate $\int \cos^2 x \sin^3 x \, dx$.

**Solution**

To convert this integral to integrals of the form $\int \cos^j x \sin x \, dx$, rewrite $\sin^3 x = \sin^2 x \sin x$ and make the substitution $\sin^2 x = 1 - \cos^2 x$.

Thus,

Erroneous nesting of equation structures

**Exercise 3.3.2**

Evaluate $\int \cos^3 x \sin^2 x \, dx$.

**Hint**

Write $\cos^3 x = \cos^2 x \cos x = (1 - \sin^2 x)\cos x$ and let $u = \sin x$.

**Answer**

$$\int \cos^3 x \sin^2 x \, dx = \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C$$

In the next example, we see the strategy that must be applied when there are only even powers of $\sin x$ and $\cos x$. For integrals of this type, the identities

$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x) = \frac{1 - \cos(2x)}{2} \tag{3.3.1}$$

and

$$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos(2x) = \frac{1 + \cos(2x)}{2} \tag{3.3.2}$$

are invaluable. These identities are sometimes known as **power-reducing identities** and they may be derived from the double-angle identity $\cos(2x) = \cos^2 x - \sin^2 x$ and the Pythagorean identity $\cos^2 x + \sin^2 x = 1$.

**Example 3.3.3: Integrating an Even Power of $\sin x$**

Evaluate $\int \sin^2 x \, dx$.

**Solution**

To evaluate this integral, let's use the trigonometric identity $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x)$. Thus,

$$\int \sin^2 x\,dx = \int \left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right) dx = \frac{1}{2}x - \frac{1}{4}\sin(2x) + C.$$



**Exercise 3.3.3**

Evaluate $\int \cos^2 x\,dx$.

**Hint**

$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos(2x)$

**Answer**

$$\int \cos^2 x\,dx = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C$$

The general process for integrating products of powers of $\sin x$ and $\cos x$ is summarized in the following set of guidelines.

**Problem-Solving Strategy: Integrating Products and Powers of $\sin x$ and $\cos x$**

To integrate $\int \cos^j x \sin^k x\,dx$ use the following strategies:

1. If $k$ is odd, rewrite $\sin^k x = \sin^{k-1} x \sin x$ and use the identity $\sin^2 x = 1 - \cos^2 x$ to rewrite $\sin^{k-1} x$ in terms of $\cos x$. Integrate using the substitution $u = \cos x$. This substitution makes $du = -\sin x\,dx$.
2. If $j$ is odd, rewrite $\cos^j x = \cos^{j-1} x \cos x$ and use the identity $\cos^2 x = 1 - \sin^2 x$ to rewrite $\cos^{j-1} x$ in terms of $\sin x$. Integrate using the substitution $u = \sin x$. This substitution makes $du = \cos x\,dx$. (Note: If both $j$ and $k$ are odd, either strategy 1 or strategy 2 may be used.)
3. If both $j$ and $k$ are even, use $\sin^2 x = \frac{1 - \cos(2x)}{2}$ and $\cos^2 x = \frac{1 + \cos(2x)}{2}$. After applying these formulas, simplify and reapply strategies 1 through 3 as appropriate.

**Example 3.3.4: Integrating $\int \cos^j x \sin^k x\,dx$ where $k$ is Odd**

Evaluate $\int \cos^8 x \sin^5 x\,dx$.

**Solution**

Since the power on $\sin x$ is odd, use strategy 1. Thus,

Erroneous nesting of equation structures

### Example 3.3.5: Integrating $\int \cos^j x \sin^k x\,dx$ where $k$ and $j$ are Even

Evaluate $\int \sin^4 x\,dx$.

**Solution**: Since the power on $\sin x$ is even ($k = 4$) and the power on $\cos x$ is even ($j = 0$), we must use strategy 3. Thus,

$$\begin{aligned}
\int \sin^4 x\,dx &= \int (\sin^2 x)^2\,dx && \text{Rewrite } \sin^4 x = (\sin^2 x)^2.\\
&= \int \left(\frac{1}{2}-\frac{1}{2}\cos(2x)\right)^2 dx && \text{Substitute } \sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x).\\
&= \int \left(\frac{1}{4} - \frac{1}{2}\cos(2x) + \frac{1}{4}\cos^2(2x)\right) dx && \text{Expand } \left(\frac{1}{2}-\frac{1}{2}\cos(2x)\right)^2.\\
&= \int \left(\frac{1}{4} - \frac{1}{2}\cos(2x) + \frac{1}{4}\left(\frac{1}{2} + \frac{1}{2}\cos(4x)\right)\right) dx && \cos^2(2x) \text{ has an even power, substitute } \cos^2(2x) = \frac{1}{2} + \frac{1}{2}\cos(4x).\\
&= \int \left(\frac{3}{8} - \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x)\right) dx && \text{Simplify.}\\
&= \frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C && \text{Evaluate the integral.}
\end{aligned}$$

### Exercise 3.3.4

Evaluate $\int \cos^3 x\,dx$.

**Hint**

Use strategy 2. Write $\cos^3 x = \cos^2 x \cos x$ and substitute $\cos^2 x = 1 - \sin^2 x$.

**Answer**

$$\int \cos^3 x\,dx = \sin x - \frac{1}{3}\sin^3 x + C$$

### Exercise 3.3.5

Evaluate $\int \cos^2(3x)\,dx$.

**Hint**

Use strategy 3. Substitute $\cos^2(3x) = \frac{1}{2} + \frac{1}{2}\cos(6x)$

**Answer**

$$\int \cos^2(3x)\,dx = \frac{1}{2}x + \frac{1}{12}\sin(6x) + C$$

In some areas of physics, such as quantum mechanics, signal processing, and the computation of Fourier series, it is often necessary to integrate products that include $sin(ax), sin(bx), cos(ax)$, and $cos(bx)$. These integrals are evaluated by applying trigonometric identities, as outlined in the following rule.

### Rule: Integrating Products of Sines and Cosines of Different Angles

To integrate products involving $\sin(ax)$, $\sin(bx)$, $\cos(ax)$, and $\cos(bx)$, use the substitutions

$$\sin(ax)\sin(bx) = \frac{1}{2}\cos((a-b)x) - \frac{1}{2}\cos((a+b)x) \tag{3.3.3}$$

$$\sin(ax)\cos(bx) = \frac{1}{2}\sin((a-b)x) + \frac{1}{2}\sin((a+b)x) \tag{3.3.4}$$

$$\cos(ax)\cos(bx) = \frac{1}{2}\cos((a-b)x) + \frac{1}{2}\cos((a+b)x) \tag{3.3.5}$$

These formulas may be derived from the sum-of-angle formulas for sine and cosine.

**Example 3.3.6: Evaluating $\int \sin(ax)\cos(bx)\,dx$**

Evaluate $\int \sin(5x)\cos(3x)\,dx$.

Solution: Apply the identity $\sin(5x)\cos(3x) = \frac{1}{2}\sin(2x) + \frac{1}{2}\sin(8x)$. Thus,

$$\int \sin(5x)\cos(3x)\,dx = \int \frac{1}{2}\sin(2x) + \frac{1}{2}\sin(8x)\,dx = -\frac{1}{4}\cos(2x) - \frac{1}{16}\cos(8x) + C.$$

**Exercise 3.3.6**

Evaluate $\int \cos(6x)\cos(5x)\,dx$.

**Hint**

Substitute $\cos(6x)\cos(5x) = \frac{1}{2}\cos x + \frac{1}{2}\cos(11x)$.

**Answer**

$$\int \cos(6x)\cos(5x)\,dx = \frac{1}{2}\sin x + \frac{1}{22}\sin(11x) + C$$

## Integrating Products and Powers of $\tan x$ and $\sec x$

Before discussing the integration of products and powers of $\tan x$ and $\sec x$, it is useful to recall the integrals involving $\tan x$ and $\sec x$ we have already learned:

1. $\int \sec^2 x\,dx = \tan x + C$
2. $\int \sec x \tan x\,dx = \sec x + C$
3. $\int \tan x\,dx = \ln|\sec x| + C$
4. $\int \sec x\,dx = \ln|\sec x + \tan x| + C.$

For most integrals of products and powers of $\tan x$ and $\sec x$, we rewrite the expression we wish to integrate as the sum or difference of integrals of the form $\int \tan^j x \sec^2 x\,dx$ or $\int \sec^j x \tan x\,dx$. As we see in the following example, we can evaluate these new integrals by using u-substitution.

**Example 3.3.7: Evaluating $\int \sec^j x \tan x\,dx$**

Evaluate $\int \sec^5 x \tan x\,dx$.

Solution: Start by rewriting $\sec^5 x \tan x$ as $\sec^4 x \sec x \tan x$.

Erroneous nesting of equation structures

You can read some interesting information at this website to learn about a common integral involving the secant.



**Exercise 3.3.7**

Evaluate $\int \tan^5 x \sec^2 x \, dx$.

**Hint**

Let $u = \tan x$ and $du = \sec^2 x$.

**Answer**

$\int \tan^5 x \sec^2 x \, dx = \frac{1}{6}\tan^6 x + C$

We now take a look at the various strategies for integrating products and powers of $\sec x$ and $\tan x$.

**Problem-Solving Strategy: Integrating $\int \tan^k x \sec^j x \, dx$**

To integrate $\int \tan^k x \sec^j x \, dx$, use the following strategies:

1. If $j$ is even and $j \geq 2$, rewrite $\sec^j x = \sec^{j-2} x \sec^2 x$ and use $\sec^2 x = \tan^2 x + 1$ to rewrite $\sec^{j-2} x$ in terms of $\tan x$. Let $u = \tan x$ and $du = \sec^2 x$.

2. If $k$ is odd and $j \geq 1$, rewrite $\tan^k x \sec^j x = \tan^{k-1} x \sec^{j-1} x \sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$ to rewrite $\tan^{k-1} x$ in terms of $\sec x$. Let $u = \sec x$ and $du = \sec x \tan x \, dx$. (Note: If $j$ is even and $k$ is odd, then either strategy 1 or strategy 2 may be used.)

3. If $k$ is odd where $k \geq 3$ and $j = 0$, rewrite $\tan^k x = \tan^{k-2} x \tan^2 x = \tan^{k-2} x(\sec^2 x - 1) = \tan^{k-2} x \sec^2 x - \tan^{k-2} x$. It may be necessary to repeat this process on the $\tan^{k-2} x$ term.

4. If $k$ is even and $j$ is odd, then use $\tan^2 x = \sec^2 x - 1$ to express $\tan^k x$ in terms of $\sec x$. Use integration by parts to integrate odd powers of $\sec x$.

**Example 3.3.8: Integrating $\int \tan^k x \sec^j x \, dx$ when $j$ is Even**

Evaluate $\int \tan^6 x \sec^4 x \, dx$.

**Solution**

Since the power on $\sec x$ is even, rewrite $\sec^4 x = \sec^2 x \sec^2 x$ and use $\sec^2 x = \tan^2 x + 1$ to rewrite the first $\sec^2 x$ in terms of $\tan x$. Thus,

$$\begin{aligned}
\int \tan^6 x \sec^4 x\, dx &= \int \tan^6 x(\tan^2 x + 1)\sec^2 x\, dx \\
&= \int u^6(u^2+1)\, du && \text{Let } u = \tan x \text{ and } du = \sec^2 x. \\
&= \int (u^8 + u^6)\, du && \text{Expand.} \\
&= \frac{1}{9}u^9 + \frac{1}{7}u^7 + C && \text{Evaluate the integral.} \\
&= \frac{1}{9}\tan^9 x + \frac{1}{7}\tan^7 x + C. && \text{Substitute } \tan x = u.
\end{aligned}$$

## Example 3.3.9: Integrating $\int \tan^k x \sec^j x\, dx$ when $k$ is Odd

Evaluate $\int \tan^5 x \sec^3 x\, dx$.

**Solution**

Since the power on $\tan x$ is odd, begin by rewriting $\tan^5 x \sec^3 x = \tan^4 x \sec^2 x \sec x \tan x$. Thus,

$$\begin{aligned}
\int \tan^5 x \sec^3 x\, dx &= \tan^4 x \sec^2 x \sec x \tan x. \\
&= \int (\tan^2 x)^2 \sec^2 x \sec x \tan x\, dx && \text{Write } \tan^4 x = (\tan^2 x)^2. \\
&= \int (\sec^2 x - 1)^2 \sec^2 x \sec x \tan x\, dx && \text{Use } \tan^2 x = \sec^2 x - 1. \\
&= \int (u^2-1)^2 u^2 du && \text{Let } u = \sec x \text{ and } du = \sec x \tan x\, dx \\
&= \int (u^6 - 2u^4 + u^2) du && \text{Expand.} \\
&= \frac{1}{7}u^7 - \frac{2}{5}u^5 + \frac{1}{3}u^3 + C && \text{Integrate.} \\
&= \frac{1}{7}\sec^7 x - \frac{2}{5}\sec^5 x + \frac{1}{3}\sec^3 x + C && \text{Substitute } \sec x = u.
\end{aligned}$$

## Example 3.3.10: Integrating $\int \tan^k x\, dx$ where $k$ is Odd and $k \geq 3$

Evaluate $\int \tan^3 x\, dx$.

**Solution**

Begin by rewriting $\tan^3 x = \tan x \tan^2 x = \tan x(\sec^2 x - 1) = \tan x \sec^2 x - \tan x$. Thus,

$$\begin{aligned}
\int \tan^3 x\, dx &= \int (\tan x \sec^2 x - \tan x)\, dx \\
&= \int \tan x \sec^2 x\, dx - \int \tan x\, dx \\
&= \frac{1}{2}\tan^2 x - \ln|\sec x| + C.
\end{aligned}$$

For the first integral, use the substitution $u = \tan x$. For the second integral, use the formula.

### Example 3.3.11: Integrating $\int \sec^3 x\,dx$

Integrate $\int \sec^3 x\,dx$.

**Solution**

This integral requires integration by parts. To begin, let $u = \sec x$ and $dv = \sec^2 x$. These choices make $du = \sec x \tan x$ and $v = \tan x$. Thus,

$$\begin{aligned}
\int \sec^3 x\,dx &= \sec x \tan x - \int \tan x \sec x \tan x\,dx \\
&= \sec x \tan x - \int \tan^2 x \sec x\,dx && \text{Simplify.} \\
&= \sec x \tan x - \int (\sec^2 x - 1)\sec x\,dx && \text{Substitute } \tan^2 x = \sec^2 x - 1. \\
&= \sec x \tan x + \int \sec x\,dx - \int \sec^3 x\,dx && \text{Rewrite.} \\
&= \sec x \tan x + \ln|\sec x + \tan x| - \int \sec^3 x\,dx. && \text{Evaluate } \int \sec x\,dx.
\end{aligned}$$

We now have

$$\int \sec^3 x\,dx = \sec x \tan x + \ln|\sec x + \tan x| - \int \sec^3 x\,dx.$$

Since the integral $\int \sec^3 x\,dx$ has reappeared on the right-hand side, we can solve for $\int \sec^3 x\,dx$ by adding it to both sides. In doing so, we obtain

$$2\int \sec^3 x\,dx = \sec x \tan x + \ln|\sec x + \tan x|.$$

Dividing by 2, we arrive at

$$\int \sec^3 x\,dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C$$

### Exercise 3.3.8

Evaluate $\int \tan^3 x \sec^7 x\,dx$.

**Hint**

Use Example **3.3.9** as a guide.

**Answer**

$$\int \tan^3 x \sec^7 x\,dx = \frac{1}{9}\sec^9 x - \frac{1}{7}\sec^7 x + C$$

## Reduction Formulas

Evaluating $\int \sec^n x\,dx$ for values of $n$ where $n$ is odd requires integration by parts. In addition, we must also know the value of $\int \sec^{n-2} x\,dx$ to evaluate $\int \sec^n x\,dx$. The evaluation of $\int \tan^n x\,dx$ also requires being able to integrate $\int \tan^{n-2} x\,dx$. To make the process easier, we can derive and apply the following power reduction formulas. These rules allow us to replace the integral of a power of $\sec x$ or $\tan x$ with the integral of a lower power of $\sec x$ or $\tan x$.

**Rule: Reduction Formulas for $\int \sec^n x\,dx$ and $\int \tan^n x\,dx$**

$$\int \sec^n x\,dx = \frac{1}{n-1}\sec^{n-2} x\tan x + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx \tag{3.3.6}$$

$$\int \tan^n x\,dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x\,dx \tag{3.3.7}$$

The first power reduction rule may be verified by applying integration by parts. The second may be verified by following the strategy outlined for integrating odd powers of $\tan x$.

**Example 3.3.12: Revisiting $\int \sec^3 x\,dx$**

Apply a reduction formula to evaluate $\int \sec^3 x\,dx$.

Solution: By applying the first reduction formula, we obtain

$$\begin{aligned}\int \sec^3 x\,dx &= \frac{1}{2}\sec x\tan x + \frac{1}{2}\int \sec x\,dx \\ &= \frac{1}{2}\sec x\tan x + \frac{1}{2}\ln|\sec x + \tan x| + C.\end{aligned}$$

**Example 3.3.13: Using a Reduction Formula**

Evaluate $\int \tan^4 x\,dx$.

Solution: Applying the reduction formula for $\int \tan^4 x\,dx$ we have

$$\begin{aligned}\int \tan^4 x\,dx &= \frac{1}{3}\tan^3 x - \int \tan^2 x\,dx \\ &= \frac{1}{3}\tan^3 x - (\tan x - \int \tan^0 x\,dx) && \text{Apply the reduction formula to } \int \tan^2 x\,dx. \\ &= \frac{1}{3}\tan^3 x - \tan x + \int 1\,dx && \text{Simplify.} \\ &= \frac{1}{3}\tan^3 x - \tan x + x + C && \text{Evaluate } \int 1\,dx\end{aligned}$$

**Exercise 3.3.9**

Apply the reduction formula to $\int \sec^5 x\,dx$.

**Hint**

Use reduction formula 1 and let $n = 5$.

**Answer**

$$\int \sec^5 x\,dx = \frac{1}{4}\sec^3 x\tan x + \frac{3}{4}\int \sec^3 x$$

## Key Concepts

Integrals of trigonometric functions can be evaluated by the use of various strategies. These strategies include

1. Applying trigonometric identities to rewrite the integral so that it may be evaluated by $u$-substitution
2. Using integration by parts
3. Applying trigonometric identities to rewrite products of sines and cosines with different arguments as the sum of individual sine and cosine functions
4. Applying reduction formulas

## Key Equations

To integrate products involving $\sin(ax)$, $\sin(bx)$, $\cos(ax)$, and $\cos(bx)$, use the substitutions.

- **Sine Products**

  $$\sin(ax)\sin(bx) = \tfrac{1}{2}\cos((a-b)x) - \tfrac{1}{2}\cos((a+b)x)$$

- **Sine and Cosine Products**

  $$\sin(ax)\cos(bx) = \tfrac{1}{2}\sin((a-b)x) + \tfrac{1}{2}\sin((a+b)x)$$

- **Cosine Products**

  $$\cos(ax)\cos(bx) = \tfrac{1}{2}\cos((a-b)x) + \tfrac{1}{2}\cos((a+b)x)$$

- **Power Reduction Formula**

  $$\int \sec^n x\,dx = \frac{1}{n-1}\sec^{n-2} x\tan x + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx$$

- **Power Reduction Formula**

  $$\int \tan^n x\,dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x\,dx$$

## Glossary

**power reduction formula**
a rule that allows an integral of a power of a trigonometric function to be exchanged for an integral involving a lower power

**trigonometric integral**
an integral involving powers and products of trigonometric functions

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# 3.3E: Exercises for Section 3.3

**Fill in the blank to make a true statement.**

1) $\sin^2 x + \_\_\_\_\_ = 1$

**Answer**

$\cos^2 x$

2) $\sec^2 x - 1 = \_\_\_\_\_$

**Answer**

$\tan^2 x$

**Use an identity to reduce the power of the trigonometric function to a trigonometric function raised to the first power.**

3) $\sin^2 x = \_\_\_\_\_$

**Answer**

$\frac{1-\cos(2x)}{2}$

4) $\cos^2 x = \_\_\_\_\_$

**Answer**

$\frac{1+\cos(2x)}{2}$

**Evaluate each of the following integrals by $u$-substitution.**

5) $\int \sin^3 x \cos x \, dx$

**Answer**

$$\int \sin^3 x \cos x \, dx = \frac{\sin^4 x}{4} + C$$

6) $\int \sqrt{\cos x} \sin x \, dx$

7) $\int \tan^5(2x) \sec^2(2x) \, dx$

**Answer**

$$\int \tan^5(2x) \sec^2(2x) \, dx = \tfrac{1}{12}\tan^6(2x) + C$$

8) $\int \sin^7(2x) \cos(2x) \, dx$

9) $\int \tan(\frac{x}{2}) \sec^2(\frac{x}{2}) \, dx$

**Answer**

$$\int \tan(\frac{x}{2}) \sec^2(\frac{x}{2}) \, dx = \tan^2(\frac{x}{2}) + C$$

10) $\int \tan^2 x \sec^2 x \, dx$

**Compute the following integrals using the guidelines for integrating powers of trigonometric functions. Use a CAS to check the solutions. (*Note*: Some of the problems may be done using techniques of integration learned previously.)**

11) $\int \sin^3 x \, dx$

**Answer**

$$\int \sin^3 x\,dx \quad = \quad -\frac{3\cos x}{4} + \tfrac{1}{12}\cos(3x) + C = -\cos x + \frac{\cos^3 x}{3} + C$$

12) $\int \cos^3 x\,dx$

13) $\int \sin x \cos x\,dx$

**Answer**

$$\int \sin x \cos x\,dx \quad = \quad -\tfrac{1}{2}\cos^2 x + C$$

14) $\int \cos^5 x\,dx$

15) $\int \sin^5 x \cos^2 x\,dx$

**Answer**

$$\int \sin^5 x \cos^2 x\,dx \quad = \quad -\frac{5\cos x}{64} - \tfrac{1}{192}\cos(3x) + \tfrac{3}{320}\cos(5x) - \tfrac{1}{448}\cos(7x) + C$$

16) $\int \sin^3 x \cos^3 x\,dx$

17) $\int \sqrt{\sin x}\cos x\,dx$

**Answer**

$$\int \sqrt{\sin x}\cos x\,dx \quad = \quad \tfrac{2}{3}(\sin x)^{3/2} + C$$

18) $\int \sqrt{\sin x}\cos^3 x\,dx$

19) $\int \sec x \tan x\,dx$

**Answer**

$$\int \sec x \tan x\,dx \quad = \quad \sec x + C$$

20) $\int \tan(5x)\,dx$

21) $\int \tan^2 x \sec x\,dx$

**Answer**

$$\int \tan^2 x \sec x\,dx \quad = \quad \tfrac{1}{2}\sec x \tan x - \tfrac{1}{2}\ln(\sec x + \tan x) + C$$

22) $\int \tan x \sec^3 x\,dx$

23) $\int \sec^4 x\,dx$

**Answer**

$$\int \sec^4 x\,dx \quad = \quad \frac{2\tan x}{3} + \tfrac{1}{3}\sec^2 x \tan x = \tan x + \frac{\tan^3 x}{3} + C$$

24) $\int \cot x\,dx$

25) $\int \csc x\,dx$

**Answer**

$$\int \csc x\,dx \quad = \quad -\ln|\cot x + \csc x| + C$$

26) $\int \frac{\tan^3 x}{\sqrt{\sec x}}\,dx$

**For exercises 27 - 28, find a general formula for the integrals.**

27) $\int \sin^2 ax \cos ax\,dx$

**Answer**

$$\int \sin^2 ax \cos ax\,dx \quad = \quad \frac{\sin^3(ax)}{3a} + C$$

28) $\int \sin ax \cos ax\,dx.$

**Use the double-angle formulas to evaluate the integrals in exercises 29 - 34.**

29) $\int_0^{\pi} \sin^2 x\,dx$

**Answer**

$$\int_0^{\pi} \sin^2 x\,dx \quad = \quad \frac{\pi}{2}$$

30) $\int_0^{\pi} \sin^4 x\,dx$

31) $\int \cos^2 3x\,dx$

**Answer**

$$\int \cos^2 3x\,dx \quad = \quad \frac{x}{2} + \tfrac{1}{12}\sin(6x) + C$$

32) $\int \sin^2 x \cos^2 x\,dx$

33) $\int \sin^2 x\,dx + \int \cos^2 x\,dx$

**Answer**

$$\int \sin^2 x\,dx + \int \cos^2 x\,dx \quad = \quad x + C$$

34) $\int \sin^2 x \cos^2(2x)\,dx$

**For exercises 35 - 43, evaluate the definite integrals. Express answers in exact form whenever possible.**

35) $\int_0^{2\pi} \cos x \sin 2x\,dx$

**Answer**

$$\int_0^{2\pi} \cos x \sin 2x\,dx \quad = \quad 0$$

36) $\int_0^{\pi} \sin 3x \sin 5x\,dx$

37) $\int_0^{\pi} \cos(99x)\sin(101x)\,dx$

**Answer**

$$\int_0^{\pi} \cos(99x)\sin(101x)\,dx \quad = \quad 0$$

38) $\int_{-\pi}^{\pi} \cos^2(3x)\,dx$

39) $\int_0^{2\pi} \sin x \sin(2x) \sin(3x)\,dx$

**Answer**

$$\int_0^{2\pi} \sin x \sin(2x) \sin(3x)\,dx \quad = \quad 0$$

40) $\int_0^{4\pi} \cos(x/2) \sin(x/2)\,dx$

41) $\int_{\pi/6}^{\pi/3} \frac{\cos^3 x}{\sqrt{\sin x}}\,dx$ (Round this answer to three decimal places.)

**Answer**

$$\int_{\pi/6}^{\pi/3} \frac{\cos^3 x}{\sqrt{\sin x}}\,dx \quad \approx \quad 0.239$$

42) $\int_{-\pi/3}^{\pi/3} \sqrt{\sec^2 x - 1}\,dx$

43) $\int_0^{\pi/2} \sqrt{1 - \cos(2x)}\,dx$

**Answer**

$$\int_0^{\pi/2} \sqrt{1 - \cos(2x)}\,dx \quad = \quad \sqrt{2}$$

44) Find the area of the region bounded by the graphs of the equations $y = \sin x$, $y = \sin^3 x$, $x = 0$, and $x = \frac{\pi}{2}$.

45) Find the area of the region bounded by the graphs of the equations $y = \cos^2 x$, $y = \sin^2 x$, $x = -\frac{\pi}{4}$, and $x = \frac{\pi}{4}$.

**Answer**

$A = 1 \text{ unit}^2$

46) A particle moves in a straight line with the velocity function $v(t) = \sin(\omega t)\cos^2(\omega t)$. Find its position function $x = f(t)$ if $f(0) = 0$.

47) Find the average value of the function $f(x) = \sin^2 x \cos^3 x$ over the interval $[-\pi, \pi]$.

**Answer**

0

**For exercises 48 - 49, solve the differential equations.**

48) $\frac{dy}{dx} = \sin^2 x$. The curve passes through point $(0, 0)$.

49) $\frac{dy}{d\theta} = \sin^4(\pi\theta)$

**Answer**

$$f(x) = \frac{3\theta}{8} - \frac{1}{4\pi}\sin(2\pi\theta) + \frac{1}{32\pi}\sin(4\pi\theta) + C$$

50) Find the length of the curve $y = \ln(\csc x)$, for $\frac{\pi}{4} \le x \le \frac{\pi}{2}$.

51) Find the length of the curve $y = \ln(\sin x)$, for $\frac{\pi}{3} \le x \le \frac{\pi}{2}$.

**Answer**

$s = \ln(\sqrt{3})$

52) Find the volume generated by revolving the curve $y = \cos(3x)$ about the $x$-axis, for $0 \le x \le \frac{\pi}{36}$.

**For exercises 53 - 54, use this information: The inner product of two functions $f$ and $g$ over $[a, b]$ is defined by $f(x) \cdot g(x) = \langle f, g \rangle = \int_a^b f \cdot g \, dx$. Two distinct functions $f$ and $g$ are said to be orthogonal if $\langle f, g \rangle = 0$.**

53) Show that $\sin(2x)$, $\cos(3x)$ are orthogonal over the interval $[-\pi, \pi]$.

**Answer**

$$\int_{-\pi}^{\pi} \sin(2x)\cos(3x)\,dx = 0$$

54) Evaluate $\int_{-\pi}^{\pi} \sin(mx)\cos(nx)\,dx$.

55) Integrate $y' = \sqrt{\tan x}\sec^4 x$.

**Answer**

$$y = \int \sqrt{\tan x}\sec^4 x\,dx \quad = \quad \tfrac{2}{3}(\tan x)^{3/2} + \tfrac{2}{7}(\tan x)^{7/2} + C = \tfrac{2}{21}(\tan x)^{3/2}\left[7 + 3\tan^2 x\right] + C$$

**For each pair of integrals in exercises 56 - 57, determine which one is more difficult to evaluate. Explain your reasoning.**

56) $\int \sin^{456} x \cos x\,dx$ or $\int \sin^2 x \cos^2 x\,dx$

57) $\int \tan^{350} x \sec^2 x\,dx$ or $\int \tan^{350} x \sec x\,dx$

**Answer**

The second integral is more difficult because the first integral is simply a $u$-substitution type.

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# 3.4: Trigonometric Substitution

**Learning Objectives**

- Solve integration problems involving the square root of a sum or difference of two squares.

In this section, we explore integrals containing expressions of the form $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, and $\sqrt{x^2-a^2}$, where the values of $a$ are positive. We have already encountered and evaluated integrals containing some expressions of this type, but many still remain inaccessible. The technique of *trigonometric substitution* comes in very handy when evaluating these integrals. This technique uses substitution to rewrite these integrals as trigonometric integrals.

## Integrals Involving $\sqrt{a^2-x^2}$

Before developing a general strategy for integrals containing $\sqrt{a^2-x^2}$, consider the integral $\int \sqrt{9-x^2}dx$. This integral cannot be evaluated using any of the techniques we have discussed so far. However, if we make the substitution $x=3\sin\theta$, we have $dx=3\cos\theta\,d\theta$. After substituting into the integral, we have

$$\int \sqrt{9-x^2}\,dx = \int \sqrt{9-(3\sin\theta)^2}\cdot 3\cos\theta\,d\theta. \tag{3.4.1}$$

After simplifying, we have

$$\int \sqrt{9-x^2}\,dx = \int 9\sqrt{1-\sin^2\theta}\cdot\cos\theta\,d\theta. \tag{3.4.2}$$

Letting $1-\sin^2\theta=\cos^2\theta$, we now have

$$\int \sqrt{9-x^2}\,dx = \int 9\sqrt{\cos^2\theta}\cos\theta\,d\theta. \tag{3.4.3}$$

Assuming that $\cos\theta\geq 0$, we have

$$\int \sqrt{9-x^2}\,dx = \int 9\cos^2\theta\,d\theta. \tag{3.4.4}$$

At this point, we can evaluate the integral using the techniques developed for integrating powers and products of trigonometric functions. Before completing this example, let's take a look at the general theory behind this idea.

To evaluate integrals involving $\sqrt{a^2-x^2}$, we make the substitution $x=a\sin\theta$ and $dx=a\cos\theta$. To see that this actually makes sense, consider the following argument: The domain of $\sqrt{a^2-x^2}$ is $[-a,a]$. Thus,

$$-a\leq x\leq a. \tag{3.4.5}$$

Consequently,

$$-1\leq \frac{x}{a}\leq 1. \tag{3.4.6}$$

Since the range of $\sin x$ over $[-(\pi/2),\pi/2]$ is $[-1,1]$, there is a unique angle $\theta$ satisfying $-(\pi/2)\leq\theta\leq\pi/2$ so that $\sin\theta=x/a$, or equivalently, so that $x=a\sin\theta$. If we substitute $x=a\sin\theta$ into $\sqrt{a^2-x^2}$, we get

$$\begin{aligned}
\sqrt{a^2-x^2} &= \sqrt{a^2-(a\sin\theta)^2} && \text{Let } x=a\sin\theta \text{ where } -\frac{\pi}{2}\leq\theta\leq\frac{\pi}{2}.\\
&&& \text{Simplify.}\\
&= \sqrt{a^2-a^2\sin^2\theta} && \text{Factor out } a^2.\\
&= \sqrt{a^2(1-\sin^2\theta)} && \text{Substitute } 1-\sin^2 x=\cos^2 x.\\
&= \sqrt{a^2\cos^2\theta} && \text{Take the square root.}\\
&= |a\cos\theta|\\
&= a\cos\theta
\end{aligned}$$

Since $\cos x \geq 0$ on $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ and $a > 0, |a\cos\theta| = a\cos\theta$. We can see, from this discussion, that by making the substitution $x = a\sin\theta$, we are able to convert an integral involving a radical into an integral involving trigonometric functions. After we evaluate the integral, we can convert the solution back to an expression involving $x$. To see how to do this, let's begin by assuming that $0 < x < a$. In this case, $0 < \theta < \frac{\pi}{2}$. Since $\sin\theta = \frac{x}{a}$, we can draw the reference triangle in Figure 3.4.1 to assist in expressing the values of $\cos\theta$, $\tan\theta$, and the remaining trigonometric functions in terms of x. It can be shown that this triangle actually produces the correct values of the trigonometric functions evaluated at $\theta$ for all $\theta$ satisfying $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. It is useful to observe that the expression $\sqrt{a^2 - x^2}$ actually appears as the length of one side of the triangle. Last, should $\theta$ appear by itself, we use $\theta = \sin^{-1}\left(\frac{x}{a}\right)$.

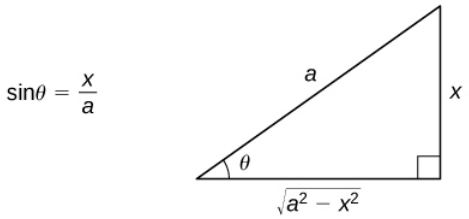



Figure 3.4.1: A reference triangle can help express the trigonometric functions evaluated at $\theta$ in terms of $x$.

The essential part of this discussion is summarized in the following problem-solving strategy.

### Problem-Solving Strategy: Integrating Expressions Involving $\sqrt{a^2 - x^2}$

1. It is a good idea to make sure the integral cannot be evaluated easily in another way. For example, although this method can be applied to integrals of the form $\int \frac{1}{\sqrt{a^2 - x^2}}dx$, $\int \frac{x}{\sqrt{a^2 - x^2}}dx$, and $\int x\sqrt{a^2 - x^2}\,dx$, they can each be integrated directly either by formula or by a simple $u$-substitution.
2. Make the substitution $x = a\sin\theta$ and $dx = a\cos\theta\,d\theta$. Note: This substitution yields $\sqrt{a^2 - x^2} = a\cos\theta$.
3. Simplify the expression.
4. Evaluate the integral using techniques from the section on trigonometric integrals.
5. Use the reference triangle from Figure 1 to rewrite the result in terms of $x$. You may also need to use some trigonometric identities and the relationship $\theta = \sin^{-1}\left(\frac{x}{a}\right)$.

The following example demonstrates the application of this problem-solving strategy.

### Example 3.4.1: Integrating an Expression Involving $\sqrt{a^2 - x^2}$

Evaluate

$$\int \sqrt{9 - x^2}dx. \tag{3.4.7}$$

**Solution**

Begin by making the substitutions $x = 3\sin\theta$ and $dx = 3\cos\theta\,d\theta$. Since $\sin\theta = \frac{x}{3}$, we can construct the reference triangle shown in Figure 2.

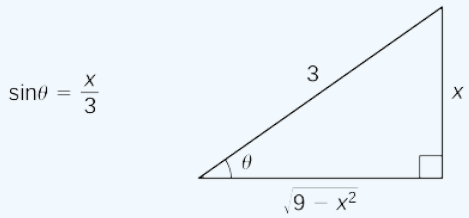



Figure 3.4.2: A reference triangle can be constructed for Example 3.4.1.

Thus,

$$\int \sqrt{9 - x^2}\,dx = \int \sqrt{9 - (3\sin\theta)^2}3\cos\theta\,d\theta \tag{3.4.8}$$

Substitute $x = 3\sin\theta$ and $dx = 3\cos\theta\, d\theta$.

$$= \int \sqrt{9(1-\sin^2\theta)} \cdot 3\cos\theta\, d\theta \text{ Simplify.}$$

$$= \int \sqrt{9\cos^2\theta} \cdot 3\cos\theta\, d\theta \text{ Substitute } \cos^2\theta = 1-\sin^2\theta.$$

$$= \int 3|\cos\theta| 3\cos\theta\, d\theta \text{ Take the square root.}$$

$$= \int 9\cos^2\theta\, d\theta \text{ Simplify. Since } -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, \cos\theta \ge 0 \text{ and } |\cos\theta| = \cos\theta.$$

$$= \int 9\left(\frac{1}{2} + \frac{1}{2}\cos(2\theta)\right) d\theta \text{ Use the strategy for integrating an even power of } \cos\theta.$$

$$= \frac{9}{2}\theta + \frac{9}{4}\sin(2\theta) + C \text{ Evaluate the integral.}$$

$$= \frac{9}{2}\theta + \frac{9}{4}(2\sin\theta\cos\theta) + C$$

Substitute $\sin(2\theta) = 2\sin\theta\cos\theta$.

$$= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) + \frac{9}{2} \cdot \frac{x}{3} \cdot \frac{\sqrt{9-x^2}}{3} + C \text{ Substitute } \sin^{-1}\left(\frac{x}{3}\right) = \theta \text{ and } \sin\theta = \frac{x}{3}.$$ Use the reference triangle to see that $\cos\theta = \frac{\sqrt{9-x^2}}{3}$ and make this substitution. Simplify.

$$= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) + \frac{x\sqrt{9-x^2}}{2} + C. \text{ Simplify.}$$



## Example 3.4.2: Integrating an Expression Involving $\sqrt{a^2-x^2}$

Evaluate

$$\int \frac{\sqrt{4-x^2}}{x} dx. \tag{3.4.9}$$

**Solution**

First make the substitutions $x = 2\sin\theta$ and $dx = 2\cos\theta\, d\theta$. Since $\sin\theta = \frac{x}{2}$, we can construct the reference triangle shown in Figure 3.4.3.

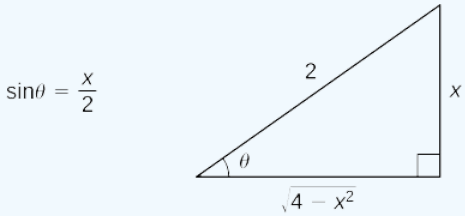



Figure 3.4.3: A reference triangle can be constructed for Example 3.4.2.

Thus,

$$\int \frac{\sqrt{4-x^2}}{x}dx = \int \frac{\sqrt{4-(2\sin\theta)^2}}{2\sin\theta}2\cos\theta\,d\theta \text{ Substitute } x = 2\sin\theta \text{ and } dx = 2\cos\theta\,d\theta.$$

$$= \int \frac{2\cos^2\theta}{\sin\theta}\,d\theta \text{ Substitute } \cos^2\theta = 1-\sin^2\theta \text{ and simplify.}$$

$$= \int \frac{2(1-\sin^2\theta)}{\sin\theta}\,d\theta \text{ Substitute } \cos^2\theta = 1-\sin^2\theta.$$

$$= \int (2\csc\theta - 2\sin\theta)\,d\theta \text{ Separate the numerator, simplify, and use } \csc\theta = \frac{1}{\sin\theta}.$$

$$= 2\ln|\csc\theta - \cot\theta| + 2\cos\theta + C \text{ Evaluate the integral.}$$

$$= 2\ln\left|\frac{2}{x} - \frac{\sqrt{4-x^2}}{x}\right| + \sqrt{4-x^2} + C. \text{ Use the reference triangle to rewrite the expression in terms of } x \text{ and simplify.}$$

In the next example, we see that we sometimes have a choice of methods.

## Example 3.4.3: Integrating an Expression Involving $\sqrt{a^2-x^2}$ Two Ways

Evaluate $\int x^3\sqrt{1-x^2}dx$ two ways: first by using the substitution $u = 1-x^2$ and then by using a trigonometric substitution.

**Method 1**

Let $u = 1-x^2$ and hence $x^2 = 1-u$. Thus, $du = -2x\,dx$. In this case, the integral becomes

$$\int x^3\sqrt{1-x^2}\,dx = -\frac{1}{2}\int x^2\sqrt{1-x^2}(-2x\,dx) \text{ Make the substitution.}$$

$$= -\frac{1}{2}\int (1-u)\sqrt{u}\,du \text{ Expand the expression.}$$

$$= -\frac{1}{2}\int (u^{1/2} - u^{3/2})\,du \text{ Evaluate the integral.}$$

$$= -\frac{1}{2}\left(\frac{2}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right) + C \text{ Rewrite in terms of x.}$$

$$= -\frac{1}{3}(1-x^2)^{3/2} + \frac{1}{5}(1-x^2)^{5/2} + C.$$

**Method 2**

Let $x = \sin\theta$. In this case, $dx = \cos\theta\,d\theta$. Using this substitution, we have

$$\int x^3\sqrt{1-x^2}dx = \int \sin^3\theta\cos^2\theta\,d\theta$$

$$= \int (1-\cos^2\theta)\cos^2\theta\sin\theta\,d\theta \text{ Let } u = \cos\theta. \text{Thus,} du = -\sin\theta\,d\theta.$$

$$= \int (u^4 - u^2)\,du$$

$$= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C \text{ Substitute } \cos\theta = u.$$

$$= \frac{1}{5}\cos^5\theta - \frac{1}{3}\cos^3\theta + C \text{ Use a reference triangle to see that } \cos\theta = \sqrt{1-x^2}.$$

$$= \frac{1}{5}(1-x^2)^{5/2} - \frac{1}{3}(1-x^2)^{3/2} + C.$$

## Exercise 3.4.1

Rewrite the integral $\int \frac{x^3}{\sqrt{25-x^2}}\,dx$ using the appropriate trigonometric substitution (do not evaluate the integral).

**Hint**

Substitute $x = 5\sin\theta$ and $dx = 5\cos\theta\,d\theta$.

**Answer**

$$\int 125\sin^3\theta\,d\theta$$

## Integrating Expressions Involving $\sqrt{a^2+x^2}$

For integrals containing $\sqrt{a^2+x^2}$,let's first consider the domain of this expression. Since $\sqrt{a^2+x^2}$ is defined for all real values of $x$, we restrict our choice to those trigonometric functions that have a range of all real numbers. Thus, our choice is restricted to selecting either $x = a\tan\theta$ or $x = a\cot\theta$. Either of these substitutions would actually work, but the standard substitution is $x = a\tan\theta$ or, equivalently, $\tan\theta = x/a$. With this substitution, we make the assumption that $-(\pi/2) < \theta < \pi/2$, so that we also have $\theta = \tan^{-1}(x/a)$. The procedure for using this substitution is outlined in the following problem-solving strategy.

### Problem-Solving Strategy: Integrating Expressions Involving $\sqrt{a^2+x^2}$

1. Check to see whether the integral can be evaluated easily by using another method. In some cases, it is more convenient to use an alternative method.
2. Substitute $x = a\tan\theta$ and $dx = a\sec^2\theta\, d\theta$. This substitution yields $\sqrt{a^2+x^2} = \sqrt{a^2+(a\tan\theta)^2} = \sqrt{a^2(1+\tan^2\theta)} = \sqrt{a^2sec^2\theta} = |a\sec\theta| = a\sec\theta$. (Since $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ and $\sec\theta > 0$ over this interval, $|a\sec\theta| = a\sec\theta$.)
3. Simplify the expression.
4. Evaluate the integral using techniques from the section on trigonometric integrals.
5. Use the reference triangle from Figure 3.4.4 to rewrite the result in terms of $x$. You may also need to use some trigonometric identities and the relationship $\theta = \tan^{-1}\left(\frac{x}{a}\right)$. (Note: The reference triangle is based on the assumption that $x > 0$; however, the trigonometric ratios produced from the reference triangle are the same as the ratios for which $x \le 0$.)

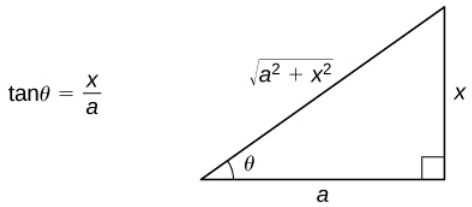



Figure 3.4.4: A reference triangle can be constructed to express the trigonometric functions evaluated at $\theta$ in terms of $x$.

### Example 3.4.4: Integrating an Expression Involving $\sqrt{a^2+x^2}$

Evaluate $\int \frac{dx}{\sqrt{1+x^2}}$ and check the solution by differentiating.

**Solution**

Begin with the substitution $x = \tan\theta$ and $dx = sec^2\theta\, d\theta$. Since $\tan\theta = x$, draw the reference triangle in Figure 3.4.5.

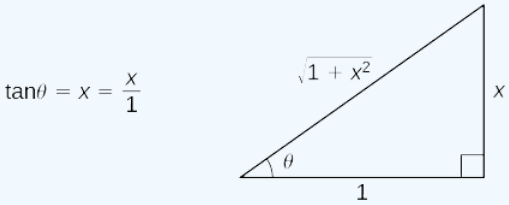



Figure 3.4.5: The reference triangle for Example 3.4.4.

Thus,

Erroneous nesting of equation structures

To check the solution, differentiate:

$$\frac{d}{dx}\left(\ln\left|\sqrt{1+x^2}+x\right|\right) = \frac{1}{\sqrt{1+x^2}+x}\cdot\left(\frac{x}{\sqrt{1+x^2}}+1\right) = \frac{1}{\sqrt{1+x^2}+x}\cdot\frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}} = \frac{1}{\sqrt{1+x^2}}.$$

Since $\sqrt{1+x^2}+x > 0$ for all values of $x$, we could rewrite $\ln\left|\sqrt{1+x^2}+x\right| + C = \ln(\sqrt{1+x^2}+x) + C$, if desired.


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### Example 3.4.5: Evaluating $\int \frac{dx}{\sqrt{1+x^2}}$ Using a Different Substitution

Use the substitution $x = \sinh\theta$ to evaluate $\int \frac{dx}{\sqrt{1+x^2}}$.

**Solution**

Because $\sinh\theta$ has a range of all real numbers, and $1 + \sinh^2\theta = \cosh^2\theta$, we may also use the substitution $x = \sinh\theta$ to evaluate this integral. In this case, $dx = \cosh\theta\, d\theta$. Consequently,

Erroneous nesting of equation structures

**Analysis**

This answer looks quite different from the answer obtained using the substitution $x = \tan\theta$. To see that the solutions are the same, set $y = \sinh^{-1}x$. Thus, $\sinh y = x$. From this equation we obtain:

$$\frac{e^y - e^{-y}}{2} = x. \tag{3.4.10}$$

After multiplying both sides by $2e^y$ and rewriting, this equation becomes:

$$e^{2y} - 2xe^y - 1 = 0. \tag{3.4.11}$$

Use the quadratic equation to solve for $e^y$:

$$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2}. \tag{3.4.12}$$

Simplifying, we have:

$$e^y = x \pm \sqrt{x^2+1}. \tag{3.4.13}$$

Since $x - \sqrt{x^2+1} < 0$, it must be the case that $e^y = x + \sqrt{x^2+1}$. Thus,

$$y = \ln(x + \sqrt{x^2+1}). \tag{3.4.14}$$

Last, we obtain

$$\sinh^{-1}x = \ln(x + \sqrt{x^2+1}). \tag{3.4.15}$$

After we make the final observation that, since $x + \sqrt{x^2+1} > 0$,

$$\ln(x + \sqrt{x^2+1}) = \ln|\sqrt{1+x^2} + x|, \tag{3.4.16}$$

we see that the two different methods produced equivalent solutions.

**Example 3.4.6: Finding an Arc Length**

Find the length of the curve $y = x^2$ over the interval $[0, \frac{1}{2}]$.

**Solution**

Because $\frac{dy}{dx} = 2x$, the arc length is given by

$$\int_0^{1/2} \sqrt{1 + (2x)^2}dx = \int_0^{1/2} \sqrt{1 + 4x^2}dx. \tag{3.4.17}$$

To evaluate this integral, use the substitution $x = \frac{1}{2}\tan\theta$ and $dx = \frac{1}{2}\sec^2\theta\, d\theta$. We also need to change the limits of integration. If $x = 0$, then $\theta = 0$ and if $x = \frac{1}{2}$, then $\theta = \frac{\pi}{4}$. Thus,

$\int_0^{1/2} \sqrt{1 + 4x^2}dx = \int_0^{\pi/4} \sqrt{1 + \tan^2\theta} \cdot \frac{1}{2}\sec^2\theta\, d\theta$ After substitution, $\sqrt{1 + 4x^2} = \sec\theta$. (Substitute $1 + \tan^2\theta = \sec^2\theta$ and simplify.)

$= \frac{1}{2}\int_0^{\pi/4} \sec^3\theta\, d\theta$ We derived this integral in the previous section.

$= \frac{1}{2}(\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta|)\,|_0^{\pi/4}$ Evaluate and simplify.

$= \frac{1}{4}(\sqrt{2} + \ln(\sqrt{2} + 1))$.



**Exercise 3.4.2**

Rewrite $\int x^3\sqrt{x^2 + 4}dx$ by using a substitution involving $\tan\theta$.

**Hint**

Use $x = 2\tan\theta$ and $dx = 2\sec^2\theta\, d\theta$.

**Answer**

$$\int 32\tan^3\theta\sec^3\theta\, d\theta \tag{3.4.18}$$

## Integrating Expressions Involving $\sqrt{x^2 - a^2}$

The domain of the expression $\sqrt{x^2 - a^2}$ is $(-\infty, -a] \cup [a, +\infty)$. Thus, either $x \le -a$ or $x \ge a$. Hence, $\frac{x}{a} \le -1$ or $\frac{x}{a} \ge 1$. Since these intervals correspond to the range of $\sec\theta$ on the set $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$, it makes sense to use the substitution $\sec\theta = \frac{x}{a}$ or, equivalently, $x = a\sec\theta$, where $0 \le \theta < \frac{\pi}{2}$ or $\frac{\pi}{2} < \theta \le \pi$. The corresponding substitution for $dx$ is $dx = a\sec\theta\tan\theta\, d\theta$. The procedure for using this substitution is outlined in the following problem-solving strategy.

## Problem-Solving Strategy: Integrals Involving $\sqrt{x^2-a^2}$

1. Check to see whether the integral cannot be evaluated using another method. If so, we may wish to consider applying an alternative technique.
2. Substitute $x = a\sec\theta$ and $dx = a\sec\theta\tan\theta\,d\theta$. This substitution yields

$$\sqrt{x^2-a^2} = \sqrt{(a\sec\theta)^2 - a^2} = \sqrt{a^2(\sec^2\theta - 1)} = \sqrt{a^2\tan^2\theta} = |a\tan\theta|. \tag{3.4.19}$$

   For $x \geq a, |a\tan\theta| = a\tan\theta$ and for $x \leq -a, |a\tan\theta| = -a\tan\theta$.
3. Simplify the expression.
4. Evaluate the integral using techniques from the section on trigonometric integrals.
5. Use the reference triangles from Figure 3.4.6 to rewrite the result in terms of $x$.
6. You may also need to use some trigonometric identities and the relationship $\theta = \sec^{-1}\left(\frac{x}{a}\right)$. (Note: We need both reference triangles, since the values of some of the trigonometric ratios are different depending on whether $x > a$ or $x < -a$.)

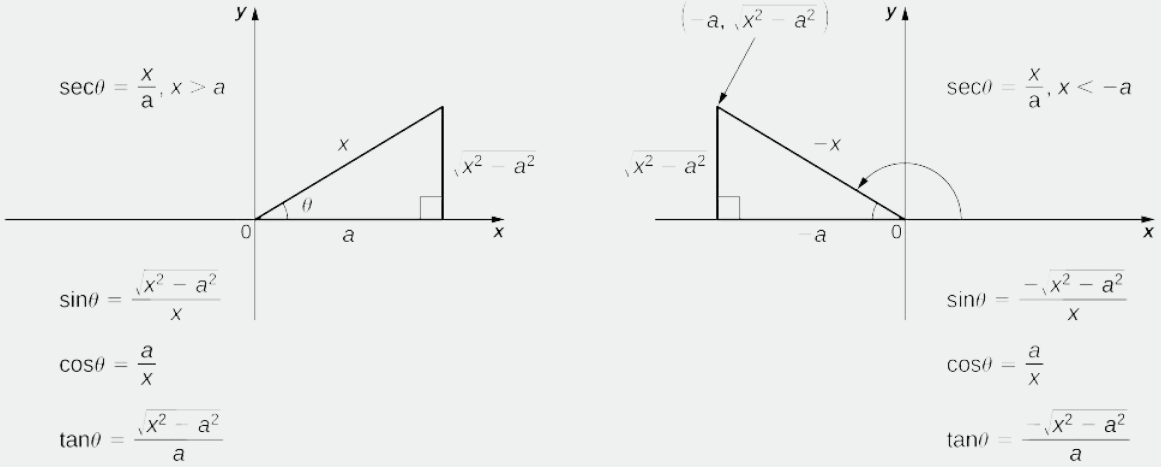



Figure 3.4.6: Use the appropriate reference triangle to express the trigonometric functions evaluated at $\theta$ in terms of $x$.

## Example 3.4.7: Finding the Area of a Region

Find the area of the region between the graph of $f(x) = \sqrt{x^2-9}$ and the x-axis over the interval $[3, 5]$.

**Solution**

First, sketch a rough graph of the region described in the problem, as shown in the following figure.

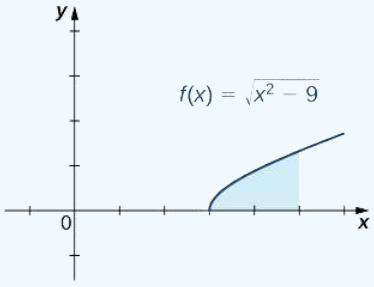



Figure 3.4.7: Calculating the area of the shaded region requires evaluating an integral with a trigonometric substitution.

We can see that the area is $A = \int_3^5 \sqrt{x^2-9}dx$. To evaluate this definite integral, substitute $x = 3\sec\theta$ and $dx = 3\sec\theta\tan\theta\,d\theta$. We must also change the limits of integration. If $x = 3$, then $3 = 3\sec\theta$ and hence $\theta = 0$. If $x = 5$, then $\theta = \sec^{-1}(\frac{5}{3})$. After making these substitutions and simplifying, we have

$$\begin{aligned}\textbf{Area} &= \int_3^5 \sqrt{x^2-9}dx \\ &= \int_0^{\sec^{-1}(5/3)} 9\tan^2\theta\sec\theta\,d\theta \text{ Use } \tan^2\theta = \sec^2\theta - 1.\end{aligned}$$

$= \int_0^{\sec^{-1}(5/3)} 9(\sec^2\theta - 1)\sec\theta\, d\theta$ Expand.

$= \int_0^{\sec^{-1}(5/3)} 9(\sec^3\theta - \sec\theta)\, d\theta$ Evaluate the integral.

$= (\frac{9}{2}\ln|\sec\theta + \tan\theta| + \frac{9}{2}\sec\theta\tan\theta) - 9\ln|\sec\theta + \tan\theta|\,|_0^{\sec^{-1}(5/3)}$ Simplify.

$= \frac{9}{2}\sec\theta\tan\theta - \frac{9}{2}\ln|\sec\theta + \tan\theta|\,|_0^{\sec^{-1}(5/3)}$ Evaluate. Use $\sec(\sec^{-1}\frac{5}{3}) = \frac{5}{3}$ and $\tan(\sec^{-1}\frac{5}{3}) = \frac{4}{3}$.

$= \frac{9}{2}\cdot\frac{5}{3}\cdot\frac{4}{3} - \frac{9}{2}\ln|\frac{5}{3} + \frac{4}{3}| - (\frac{9}{2}\cdot 1\cdot 0 - \frac{9}{2}\ln|1+0|)$

$= 10 - \frac{9}{2}\ln 3$



**Exercise 3.4.3**

Evaluate

$$\int \frac{dx}{\sqrt{x^2-4}}. \tag{3.4.20}$$

Assume that $x > 2$.

**Hint**

Substitute $x = 2\sec\theta$ and $dx = 2\sec\theta\tan\theta\, d\theta$.

**Answer**

$$\ln\left|\frac{x}{2} + \frac{\sqrt{x^2-4}}{2}\right| + C \tag{3.4.21}$$

## Key Concepts

- For integrals involving $\sqrt{a^2-x^2}$, use the substitution $x = a\sin\theta$ and $dx = a\cos\theta\, d\theta$.
- For integrals involving $\sqrt{a^2+x^2}$, use the substitution $x = a\tan\theta$ and $dx = a\sec^2\theta\, d\theta$.
- For integrals involving $\sqrt{x^2-a^2}$, substitute $x = a\sec\theta$ and $dx = a\sec\theta\tan\theta\, d\theta$.

## Glossary

**trigonometric substitution**

an integration technique that converts an algebraic integral containing expressions of the form $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, or $\sqrt{x^2-a^2}$ into a trigonometric integral

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# 3.4E: Exercises for Section 3.4

**Simplify the expressions in exercises 1 - 5 by writing each one using a single trigonometric function.**

1) $4-4\sin^2\theta$

2) $9\sec^2\theta-9$

**Answer**

$$9\sec^2\theta-9 \quad = \quad 9\tan^2\theta$$

3) $a^2+a^2\tan^2\theta$

4) $a^2+a^2\sinh^2\theta$

**Answer**

$$a^2+a^2\sinh^2\theta \quad = \quad a^2\cosh^2\theta$$

5) $16\cosh^2\theta-16$

**Use the technique of completing the square to express each trinomial in exercises 6 - 8 as the square of a binomial.**

6) $4x^2-4x+1$

**Answer**

$4(x-\frac{1}{2})^2$

7) $2x^2-8x+3$

8) $-x^2-2x+4$

**Answer**

$-(x+1)^2+5$

**In exercises 9 - 28, integrate using the method of trigonometric substitution. Express the final answer in terms of the original variable.**

9) $\int \frac{dx}{\sqrt{4-x^2}}$

10) $\int \frac{dx}{\sqrt{x^2-a^2}}$

**Answer**

$$\int \frac{dx}{\sqrt{x^2-a^2}} \quad = \quad \ln|x+\sqrt{-a^2+x^2}|+C$$

11) $\int \sqrt{4-x^2}\,dx$

12) $\int \frac{dx}{\sqrt{1+9x^2}}$

**Answer**

$$\int \frac{dx}{\sqrt{1+9x^2}} \quad = \quad \tfrac{1}{3}\ln|\sqrt{9x^2+1}+3x|+C$$

13) $\int \frac{x^2\,dx}{\sqrt{1-x^2}}$

14) $\int \frac{dx}{x^2\sqrt{1-x^2}}$

**Answer**

$$\int \frac{dx}{x^2\sqrt{1-x^2}} = -\frac{\sqrt{1-x^2}}{x} + C$$

15) $\int \frac{dx}{(1+x^2)^2}$

16) $\int \sqrt{x^2+9}\,dx$

**Answer**

$$\int \sqrt{x^2+9}\,dx = 9\left[\frac{x\sqrt{x^2+9}}{18} + \tfrac{1}{2}\ln\left|\frac{\sqrt{x^2+9}}{3} + \frac{x}{3}\right|\right] + C$$

17) $\int \frac{\sqrt{x^2-25}}{x}\,dx$

18) $\int \frac{\theta^3}{\sqrt{9-\theta^2}}\,d\theta$

**Answer**

$$\int \frac{\theta^3 d\theta}{\sqrt{9-\theta^2}}\,d\theta = -\tfrac{1}{3}\sqrt{9-\theta^2}(18+\theta^2) + C$$

19) $\int \frac{dx}{\sqrt{x^6-x^2}}$

20) $\int \sqrt{x^6-x^8}\,dx$

**Answer**

$$\int \sqrt{x^6-x^8}\,dx = \frac{(-1+x^2)(2+3x^2)\sqrt{x^6-x^8}}{15x^3} + C$$

21) $\int \frac{dx}{(1+x^2)^{3/2}}$

22) $\int \frac{dx}{(x^2-9)^{3/2}}$

**Answer**

$$\int \frac{dx}{(x^2-9)^{3/2}} = -\frac{x}{9\sqrt{x^2-9}} + C$$

23) $\int \frac{\sqrt{1+x^2}}{x}\,dx$

24) $\int \frac{x^2}{\sqrt{x^2-1}}\,dx$

**Answer**

$$\int \frac{x^2}{\sqrt{x^2-1}}\,dx = \tfrac{1}{2}(\ln|x+\sqrt{x^2-1}| + x\sqrt{x^2-1}) + C$$

25) $\int \frac{x^2}{x^2+4}\,dx$

26) $\int \frac{dx}{x^2\sqrt{x^2+1}}$

**Answer**

$$\int \frac{dx}{x^2\sqrt{x^2+1}} = -\frac{\sqrt{1+x^2}}{x} + C$$

27) $\int \frac{x^2}{\sqrt{1+x^2}}\,dx$

28) $\int_{-1}^{1}(1-x^2)^{3/2}\,dx$

**Answer**

$$\int_{-1}^{1}(1-x^2)^{3/2}\,dx \quad = \quad \tfrac{1}{8}\left(x(5-2x^2)\sqrt{1-x^2}+3\arcsin x\right)+C$$

**In exercises 29 - 34, use the substitutions $x=\sinh\theta$, $\cosh\theta$, or $\tanh\theta$. Express the final answers in terms of the variable $x$.**

29) $\int \frac{dx}{\sqrt{x^2-1}}$

30) $\int \frac{dx}{x\sqrt{1-x^2}}$

**Answer**

$$\int \frac{dx}{x\sqrt{1-x^2}} \quad = \quad \ln x - \ln\left|1+\sqrt{1-x^2}\right|+C$$

31) $\int \sqrt{x^2-1}\,dx$

32) $\int \frac{\sqrt{x^2-1}}{x^2}\,dx$

**Answer**

$$\int \frac{\sqrt{x^2-1}}{x^2}\,dx \quad = \quad -\frac{\sqrt{-1+x^2}}{x}+\ln\left|x+\sqrt{-1+x^2}\right|+C$$

33) $\int \frac{dx}{1-x^2}$

34) $\int \frac{\sqrt{1+x^2}}{x^2}\,dx$

**Answer**

$$\int \frac{\sqrt{1+x^2}}{x^2}\,dx \quad = \quad -\frac{\sqrt{1+x^2}}{x}+\operatorname{arcsinh} x+C$$

**Use the technique of completing the square to evaluate the integrals in exercises 35 - 39.**

35) $\int \frac{1}{x^2-6x}\,dx$

36) $\int \frac{1}{x^2+2x+1}\,dx$

**Answer**

$$\int \frac{1}{x^2+2x+1}\,dx \quad = \quad -\frac{1}{1+x}+C$$

37) $\int \frac{1}{\sqrt{-x^2+2x+8}}\,dx$

38) $\int \frac{1}{\sqrt{-x^2+10x}}\,dx$

**Answer**

$$\int \frac{1}{\sqrt{-x^2+10x}}\,dx \quad = \quad \arcsin\left(\frac{x-5}{5}\right)+C$$

39) $\int \frac{1}{\sqrt{x^2+4x-12}}\,dx$

40) Evaluate the integral without using calculus: $\int_{-3}^{3}\sqrt{9-x^2}\,dx$.

**Answer**

$\int_{-3}^{3} \sqrt{9-x^2}\,dx = \frac{9\pi}{2}$; area of a semicircle with radius 3

41) Find the area enclosed by the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$.

42) Evaluate the integral $\int \frac{dx}{\sqrt{1-x^2}}$ using two different substitutions. First, let $x=\cos\theta$ and evaluate using trigonometric substitution. Second, let $x=\sin\theta$ and use trigonometric substitution. Are the answers the same?

**Answer**

$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin(x)+C$ is the common answer.

43) Evaluate the integral $\int \frac{dx}{x\sqrt{x^2-1}}$ using the substitution $x=\sec\theta$. Next, evaluate the same integral using the substitution $x=\csc\theta$. Show that the results are equivalent.

44) Evaluate the integral $\int \frac{x}{x^2+1}\,dx$ using the form $\int \frac{1}{u}\,du$. Next, evaluate the same integral using $x=\tan\theta$. Are the results the same?

**Answer**

$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\ln(1+x^2)+C$ is the result using either method.

45) State the method of integration you would use to evaluate the integral $\int x\sqrt{x^2+1}\,dx$. Why did you choose this method?

46) State the method of integration you would use to evaluate the integral $\int x^2\sqrt{x^2-1}\,dx$. Why did you choose this method?

**Answer**

Use trigonometric substitution. Let $x=\sec(\theta)$.

47) Evaluate $\int_{-1}^{1} \frac{x}{x^2+1}\,dx$

48) Find the length of the arc of the curve over the specified interval: $y=\ln x, \quad [1,5]$. Round the answer to three decimal places.

**Answer**

$s=4.367$ units

49) Find the surface area of the solid generated by revolving the region bounded by the graphs of $y=x^2$, $y=0$, $x=0$, and $x=\sqrt{2}$ about the $x$-axis. (Round the answer to three decimal places).

50) The region bounded by the graph of $f(x)=\frac{1}{1+x^2}$ and the $x$-axis between $x=0$ and $x=1$ is revolved about the $x$-axis. Find the volume of the solid that is generated.

**Answer**

$V=\left(\frac{\pi^2}{8}+\frac{\pi}{4}\right)$ units$^3$

**In exercises 51 - 52, solve the initial-value problem for $y$ as a function of $x$.**

51) $(x^2+36)\frac{dy}{dx}=1, \quad y(6)=0$

52) $(64-x^2)\frac{dy}{dx}=1, \quad y(0)=3$

**Answer**

$y=\frac{1}{16}\ln\left|\frac{x+8}{x-8}\right|+3$

53) Find the area bounded by $y = \dfrac{2}{\sqrt{64 - 4x^2}}$, $x = 0$, $y = 0$, and $x = 2$.

54) An oil storage tank can be described as the volume generated by revolving the area bounded by $y = \dfrac{16}{\sqrt{64 + x^2}}$, $x = 0$, $y = 0$, $x = 2$ about the $x$-axis. Find the volume of the tank (in cubic meters).

**Answer**

$V = 24.6$ m$^3$

55) During each cycle, the velocity $v$ (in feet per second) of a robotic welding device is given by $v = 2t - \dfrac{14}{4 + t^2}$, where $t$ is time in seconds. Find the expression for the displacement $s$ (in feet) as a function of $t$ if $s = 0$ when $t = 0$.

56) Find the length of the curve $y = \sqrt{16 - x^2}$ between $x = 0$ and $x = 2$.

**Answer**

$s = \frac{2\pi}{3}$ units

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# 3.5: Partial Fractions

**Learning Objectives**

- Integrate a rational function using the method of partial fractions.
- Recognize simple linear factors in a rational function.
- Recognize repeated linear factors in a rational function.
- Recognize quadratic factors in a rational function.

We have seen some techniques that allow us to integrate specific rational functions. For example, we know that

$$\int \frac{du}{u} = \ln|u| + C$$

and

$$\int \frac{du}{u^2 + a^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C.$$

However, we do not yet have a technique that allows us to tackle arbitrary quotients of this type. Thus, it is not immediately obvious how to go about evaluating

$$\int \frac{3x}{x^2 - x - 2}\, dx.$$

However, we know from material previously developed that

$$\int \left(\frac{1}{x+1} + \frac{2}{x-2}\right) dx = \ln|x+1| + 2\ln|x-2| + C.$$

In fact, by getting a common denominator, we see that

$$\frac{1}{x+1} + \frac{2}{x-2} = \frac{3x}{x^2 - x - 2}.$$

Consequently,

$$\int \frac{3x}{x^2 - x - 2}\, dx = \int \left(\frac{1}{x+1} + \frac{2}{x-2}\right) dx.$$

In this section, we examine the method of **partial fraction decomposition**, which allows us to decompose **rational functions** into sums of simpler, more easily integrated rational functions. Using this method, we can rewrite an expression such as:

$$\frac{3x}{x^2 - x - 2}$$

as an expression such as

$$\frac{1}{x+1} + \frac{2}{x-2}.$$

The key to the method of partial fraction decomposition is being able to anticipate the form that the decomposition of a rational function will take. As we shall see, this form is both predictable and highly dependent on the factorization of the denominator of the rational function. It is also extremely important to keep in mind that partial fraction decomposition can be applied to a rational function $\frac{P(x)}{Q(x)}$ only if $deg(P(x)) < deg(Q(x))$. In the case when $deg(P(x)) \geq deg(Q(x))$, we must first perform long division to rewrite the quotient $\frac{P(x)}{Q(x)}$ in the form $A(x) + \frac{R(x)}{Q(x)}$, where $deg(R(x)) < deg(Q(x))$. We then do a partial fraction decomposition on $\frac{R(x)}{Q(x)}$. The following example, although not requiring partial fraction decomposition, illustrates our approach to integrals of rational functions of the form $\int \frac{P(x)}{Q(x)}\, dx$, where $deg(P(x)) \geq deg(Q(x))$.

### Example 3.5.1: Integrating $\int \frac{P(x)}{Q(x)}\,dx$, where $deg(P(x)) \geq deg(Q(x))$

Evaluate

$$\int \frac{x^2+3x+5}{x+1}\,dx.$$

**Solution**

Since $deg(x^2+3x+5) \geq deg(x+1)$, we perform long division to obtain

$$\frac{x^2+3x+5}{x+1} = x+2+\frac{3}{x+1}.$$

Thus,

$$\int \frac{x^2+3x+5}{x+1}\,dx = \int \left(x+2+\frac{3}{x+1}\right)\,dx = \frac{1}{2}x^2+2x+3\ln|x+1|+C.$$

Visit this website for a review of long division of polynomials.

### Exercise 3.5.1

Evaluate

$$\int \frac{x-3}{x+2}\,dx.$$

**Hint**

Use long division to obtain $\frac{x-3}{x+2} = 1-\frac{5}{x+2}$.

**Answer**

$$x-5\ln|x+2|+C$$

To integrate $\int \frac{P(x)}{Q(x)}\,dx$, where $deg(P(x)) < deg(Q(x))$, we must begin by factoring $Q(x)$.

## Nonrepeated Linear Factors

If $Q(x)$ can be factored as $(a_1x+b_1)(a_2x+b_2)\ldots(a_nx+b_n)$, where each linear factor is distinct, then it is possible to find constants $A_1, A_2, \ldots A_n$ satisfying

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x+b_1}+\frac{A_2}{a_2x+b_2}+\cdots+\frac{A_n}{a_nx+b_n}. \tag{3.5.1}$$

The proof that such constants exist is beyond the scope of this course.

In this next example, we see how to use partial fractions to integrate a rational function of this type.

### Example 3.5.2: Partial Fractions with Nonrepeated Linear Factors

Evaluate $\int \frac{3x+2}{x^3-x^2-2x}\,dx$.

**Solution**

Since $deg(3x+2) < deg(x^3-x^2-2x)$, we begin by factoring the denominator of $\frac{3x+2}{x^3-x^2-2x}$. We can see that $x^3-x^2-2x = x(x-2)(x+1)$. Thus, there are constants $A$, $B$, and $C$ satisfying Equation 3.5.1 such that

$$\frac{3x+2}{x(x-2)(x+1)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+1}.$$

We must now find these constants. To do so, we begin by getting a common denominator on the right. Thus,

$$\frac{3x+2}{x(x-2)(x+1)} = \frac{A(x-2)(x+1) + Bx(x+1) + Cx(x-2)}{x(x-2)(x+1)}.$$

Now, we set the numerators equal to each other, obtaining

$$3x+2 = A(x-2)(x+1) + Bx(x+1) + Cx(x-2). \tag{3.5.2}$$

There are two different strategies for finding the coefficients $A$, $B$, and $C$. We refer to these as the *method of equating coefficients* and the *method of strategic substitution*.

**Strategy one: Method of Equating Coefficients**

Rewrite Equation 3.5.2 in the form

$$3x+2 = (A+B+C)x^2 + (-A+B-2C)x + (-2A).$$

Equating coefficients produces the system of equations

$$\begin{aligned} A+B+C &= 0 \\ -A+B-2C &= 3 \\ -2A &= 2. \end{aligned}$$

To solve this system, we first observe that $-2A = 2 \Rightarrow A = -1$. Substituting this value into the first two equations gives us the system

$$\begin{aligned} B+C &= 1 \\ B-2C &= 2. \end{aligned}$$

Multiplying the second equation by $-1$ and adding the resulting equation to the first produces

$$-3C = 1,$$

which in turn implies that $C = -\frac{1}{3}$. Substituting this value into the equation $B + C = 1$ yields $B = \frac{4}{3}$. Thus, solving these equations yields $A = -1, B = \frac{4}{3}$, and $C = -\frac{1}{3}$.

It is important to note that the system produced by this method is consistent if and only if we have set up the decomposition correctly. If the system is inconsistent, there is an error in our decomposition.

**Strategy two: Method of Strategic Substitution**

The method of strategic substitution is based on the assumption that we have set up the decomposition correctly. If the decomposition is set up correctly, then there must be values of $A, B$, and $C$ that satisfy Equation 3.5.2 for all values of $x$. That is, this equation must be true for any value of $x$ we care to substitute into it. Therefore, by choosing values of $x$ carefully and substituting them into the equation, we may find $A, B$, and $C$ easily. For example, if we substitute $x = 0$, the equation reduces to $2 = A(-2)(1)$. Solving for $A$ yields $A = -1$. Next, by substituting $x = 2$, the equation reduces to $8 = B(2)(3)$, or equivalently $B = 4/3$. Last, we substitute $x = -1$ into the equation and obtain $-1 = C(-1)(-3)$. Solving, we have $C = -\frac{1}{3}$.

It is important to keep in mind that if we attempt to use this method with a decomposition that has not been set up correctly, we are still able to find values for the constants, but these constants are meaningless. If we do opt to use the method of strategic substitution, then it is a good idea to check the result by recombining the terms algebraically.

Now that we have the values of $A, B$, and $C$, we rewrite the original integral:

$$\int \frac{3x+2}{x^3-x^2-2x}\,dx = \int \left(-\frac{1}{x} + \frac{4}{3}\cdot\frac{1}{x-2} - \frac{1}{3}\cdot\frac{1}{x+1}\right)dx.$$

Evaluating the integral gives us

$$\int \frac{3x+2}{x^3-x^2-2x}\,dx = -\ln|x| + \frac{4}{3}\ln|x-2| - \frac{1}{3}\ln|x+1| + C.$$



In the next example, we integrate a rational function in which the degree of the numerator is not less than the degree of the denominator.

**Example 3.5.3: Dividing before Applying Partial Fractions**

Evaluate $\int \frac{x^2+3x+1}{x^2-4}\,dx$.

**Solution**

Since $deg(x^2+3x+1) \geq deg(x^2-4)$, we must perform long division of polynomials. This results in

$$\frac{x^2+3x+1}{x^2-4} = 1 + \frac{3x+5}{x^2-4}$$

Next, we perform partial fraction decomposition on $\frac{3x+5}{x^2-4} = \frac{3x+5}{(x+2)(x-2)}$. We have

$$\frac{3x+5}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}.$$

Thus,

$$3x+5 = A(x+2) + B(x-2).$$

Solving for $A$ and $B$ using either method, we obtain $A = 11/4$ and $B = 1/4$.

Rewriting the original integral, we have

$$\int \frac{x^2+3x+1}{x^2-4}\,dx = \int \left(1 + \frac{11}{4}\cdot\frac{1}{x-2} + \frac{1}{4}\cdot\frac{1}{x+2}\right)\,dx.$$

Evaluating the integral produces

$$\int \frac{x^2+3x+1}{x^2-4}\,dx = x + \frac{11}{4}\ln|x-2| + \frac{1}{4}\ln|x+2| + C.$$

As we see in the next example, it may be possible to apply the technique of partial fraction decomposition to a nonrational function. The trick is to convert the nonrational function to a rational function through a substitution.

**Example 3.5.4: Applying Partial Fractions after a Substitution**

Evaluate $\int \frac{\cos x}{\sin^2 x - \sin x}\, dx$.

**Solution**

Let's begin by letting $u = \sin x$. Consequently, $du = \cos x\, dx$. After making these substitutions, we have

$$\int \frac{\cos x}{\sin^2 x - \sin x}\, dx = \int \frac{du}{u^2 - u} = \int \frac{du}{u(u-1)}.$$

Applying partial fraction decomposition to $\frac{1}{u(u-1)}$ gives $\frac{1}{u(u-1)} = -\frac{1}{u} + \frac{1}{u-1}$.

Thus,

$$\int \frac{\cos x}{\sin^2 x - \sin x}\, dx = -\ln|u| + \ln|u-1| + C = -\ln|\sin x| + \ln|\sin x - 1| + C.$$

**Exercise 3.5.2**

Evaluate $\int \frac{x+1}{(x+3)(x-2)}\, dx$.

**Hint**

$$\frac{x+1}{(x+3)(x-2)} = \frac{A}{x+3} + \frac{B}{x-2}$$

**Answer**

$$\frac{2}{5}\ln|x+3| + \frac{3}{5}\ln|x-2| + C$$

## Repeated Linear Factors

For some applications, we need to integrate rational expressions that have denominators with repeated linear factors—that is, rational functions with at least one factor of the form $(ax+b)^n$, where $n$ is a positive integer greater than or equal to 2. If the denominator contains the repeated linear factor $(ax+b)^n$, then the decomposition must contain

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}. \tag{3.5.3}$$

As we see in our next example, the basic technique used for solving for the coefficients is the same, but it requires more algebra to determine the numerators of the partial fractions.

**Example 3.5.5: Partial Fractions with Repeated Linear Factors**

Evaluate $\int \frac{x-2}{(2x-1)^2(x-1)}\, dx$.

**Solution**

We have $deg(x-2) < deg((2x-1)^2(x-1))$, so we can proceed with the decomposition. Since $(2x-1)^2$ is a repeated linear factor, include

$$\frac{A}{2x-1} + \frac{B}{(2x-1)^2}$$

in the decomposition in Equation 3.5.3. Thus,

$$\frac{x-2}{(2x-1)^2(x-1)} = \frac{A}{2x-1} + \frac{B}{(2x-1)^2} + \frac{C}{x-1}.$$

After getting a common denominator and equating the numerators, we have

$$x-2=A(2x-1)(x-1)+B(x-1)+C(2x-1)^2. \tag{3.5.4}$$

We then use the method of equating coefficients to find the values of $A, B$, and $C$.

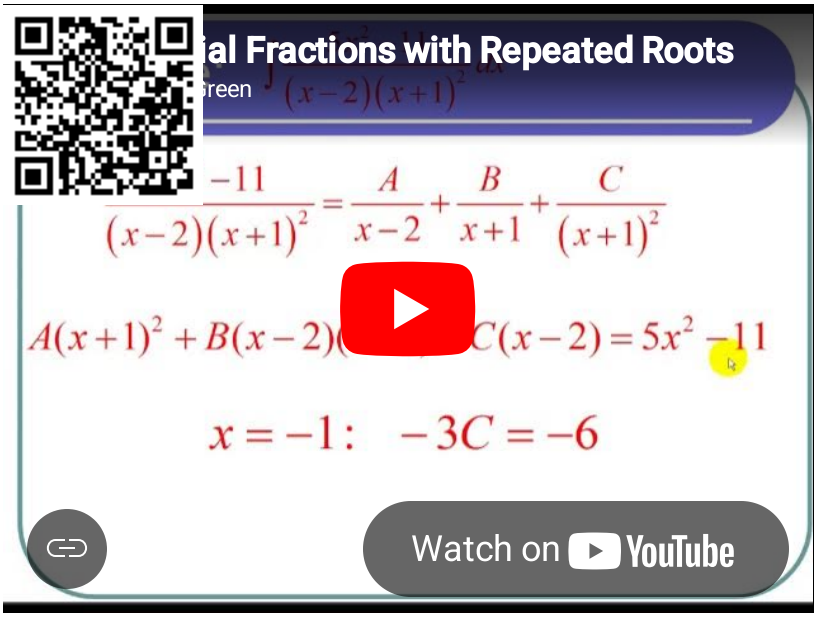
$$x-2=(2A+4C)x^2+(-3A+B-4C)x+(A-B+C).$$

Equating coefficients yields $2A+4C=0$, $-3A+B-4C=1$, and $A-B+C=-2$. Solving this system yields $A=2, B=3$, and $C=-1$.

Alternatively, we can use the method of strategic substitution. In this case, substituting $x=1$ and $x=1/2$ into Equation 3.5.4 easily produces the values $B=3$ and $C=-1$. At this point, it may seem that we have run out of good choices for $x$, however, since we already have values for $B$ and $C$, we can substitute in these values and choose any value for $x$ not previously used. The value $x=0$ is a good option. In this case, we obtain the equation $-2=A(-1)(-1)+3(-1)+(-1)(-1)^2$ or, equivalently, $A=2$.

Now that we have the values for $A, B$, and $C$, we rewrite the original integral and evaluate it:

$$\begin{aligned}\int \frac{x-2}{(2x-1)^2(x-1)}\,dx &= \int \left(\frac{2}{2x-1}+\frac{3}{(2x-1)^2}-\frac{1}{x-1}\right)dx \\ &= \ln|2x-1|-\frac{3}{2(2x-1)}-\ln|x-1|+C.\end{aligned}$$




**Exercise 3.5.3**

Set up the partial fraction decomposition for

$$\int \frac{x+2}{(x+3)^3(x-4)^2}\,dx.$$

(Do not solve for the coefficients or complete the integration.)

**Hint**

Use the problem-solving method of Example 3.5.5 for guidance.

**Answer**

$$\frac{x+2}{(x+3)^3(x-4)^2}=\frac{A}{x+3}+\frac{B}{(x+3)^2}+\frac{C}{(x+3)^3}+\frac{D}{(x-4)}+\frac{E}{(x-4)^2}$$

## The General Method

Now that we are beginning to get the idea of how the technique of partial fraction decomposition works, let's outline the basic method in the following problem-solving strategy.

### Problem-Solving Strategy: Partial Fraction Decomposition

To decompose the rational function $P(x)/Q(x)$, use the following steps:

1. Make sure that $deg(P(x)) < deg(Q(x))$. If not, perform long division of polynomials.
2. Factor $Q(x)$ into the product of linear and irreducible quadratic factors. An irreducible quadratic is a quadratic that has no real zeros.
3. Assuming that $deg(P(x)) < deg(Q(x))$, the factors of $Q(x)$ determine the form of the decomposition of $P(x)/Q(x)$.
   a. If $Q(x)$ can be factored as $(a_1x + b_1)(a_2x + b_2) \ldots (a_nx + b_n)$, where each linear factor is distinct, then it is possible to find constants $A_1, A_2, \ldots A_n$ satisfying

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_n}{a_nx + b_n}. \tag{3.5.5}$$

   b. If $Q(x)$ contains the repeated linear factor $(ax + b)^n$, then the decomposition must contain

$$\frac{A_1}{ax + b} + \frac{A_2}{(ax + b)^2} + \cdots + \frac{A_n}{(ax + b)^n}. \tag{3.5.6}$$

   c. For each irreducible quadratic factor $ax^2 + bx + c$ that $Q(x)$ contains, the decomposition must include

$$\frac{Ax + B}{ax^2 + bx + c}. \tag{3.5.7}$$

   d. For each repeated irreducible quadratic factor $(ax^2 + bx + c)^n$, the decomposition must include

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_nx + B_n}{(ax^2 + bx + c)^n}. \tag{3.5.8}$$

   e. After the appropriate decomposition is determined, solve for the constants.
   f. Last, rewrite the integral in its decomposed form and evaluate it using previously developed techniques or integration formulas.

## Simple Quadratic Factors

Now let's look at integrating a rational expression in which the denominator contains an irreducible quadratic factor. Recall that the quadratic $ax^2 + bx + c$ is irreducible if $ax^2 + bx + c = 0$ has no real zeros—that is, if $b^2 - 4ac < 0$.

### Example 3.5.6: Rational Expressions with an Irreducible Quadratic Factor

Evaluate

$$\int \frac{2x - 3}{x^3 + x}\, dx.$$

**Solution**

Since $deg(2x - 3) < deg(x^3 + x)$, factor the denominator and proceed with partial fraction decomposition. Since $x^3 + x = x(x^2 + 1)$ contains the irreducible quadratic factor $x^2 + 1$, include $\frac{Ax + B}{x^2 + 1}$ as part of the decomposition, along with $\frac{C}{x}$ for the linear term $x$. Thus, the decomposition has the form

$$\frac{2x - 3}{x(x^2 + 1)} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x}.$$

After getting a common denominator and equating the numerators, we obtain the equation

$$2x - 3 = (Ax + B)x + C(x^2 + 1).$$

Solving for $A, B$, and $C$, we get $A = 3, B = 2$, and $C = -3$.

Thus,

$$\frac{2x-3}{x^3+x} = \frac{3x+2}{x^2+1} - \frac{3}{x}.$$

Substituting back into the integral, we obtain

$$\begin{aligned}\int \frac{2x-3}{x^3+x}\,dx &= \int \left(\frac{3x+2}{x^2+1} - \frac{3}{x}\right) dx \\ &= 3\int \frac{x}{x^2+1}\,dx + 2\int \frac{1}{x^2+1}\,dx - 3\int \frac{1}{x}\,dx \qquad \text{Split up the integral} \\ &= \frac{3}{2}\ln\left|x^2+1\right| + 2\tan^{-1}x - 3\ln|x| + C. \qquad \text{Evaluate each integral}\end{aligned}$$

Note: We may rewrite $\ln\left|x^2+1\right| = \ln(x^2+1)$, if we wish to do so, since $x^2+1 > 0$.

**Example 3.5.7: Partial Fractions with an Irreducible Quadratic Factor**

Evaluate $\int \frac{dx}{x^3-8}$.

Solution: We can start by factoring $x^3 - 8 = (x-2)(x^2+2x+4)$. We see that the quadratic factor $x^2+2x+4$ is irreducible since $2^2 - 4(1)(4) = -12 < 0$. Using the decomposition described in the problem-solving strategy, we get

$$\frac{1}{(x-2)(x^2+2x+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+2x+4}.$$

After obtaining a common denominator and equating the numerators, this becomes

$$1 = A(x^2+2x+4) + (Bx+C)(x-2).$$

Applying either method, we get $A = \frac{1}{12}, B = -\frac{1}{12}$, and $C = -\frac{1}{3}$.

Rewriting $\int \frac{dx}{x^3-8}$, we have

$$\int \frac{dx}{x^3-8} = \frac{1}{12}\int \frac{1}{x-2}\,dx - \frac{1}{12}\int \frac{x+4}{x^2+2x+4}\,dx.$$

We can see that

$$\int \frac{1}{x-2}\,dx = \ln|x-2| + C,$$

but

$$\int \frac{x+4}{x^2+2x+4}\,dx$$

requires a bit more effort. Let's begin by **completing the square** on $x^2+2x+4$ to obtain

$$x^2+2x+4 = (x+1)^2 + 3.$$

By letting $u = x+1$ and consequently $du = dx$, we see that

$$\begin{aligned}
\int \frac{x+4}{x^2+2x+4}\,dx &= \int \frac{x+4}{(x+1)^2+3}\,dx && \text{Complete the square on the denominator}\\
&= \int \frac{u+3}{u^2+3}\,du && \text{Substitute } u=x+1,\ x=u-1, \text{ and } du=dx\\
&= \int \frac{u}{u^2+3}du + \int \frac{3}{u^2+3}du && \text{Split the numerator apart}\\
&= \frac{1}{2}\ln\left|u^2+3\right| && \text{Evaluate each integral}\\
&\quad + \frac{3}{\sqrt{3}}\tan^{-1}\frac{u}{\sqrt{3}} + C\\
&= \frac{1}{2}\ln\left|x^2+2x+4\right| && \text{Rewrite in terms of } x \text{ and simplify}\\
&\quad + \sqrt{3}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right) + C
\end{aligned}$$

Substituting back into the original integral and simplifying gives

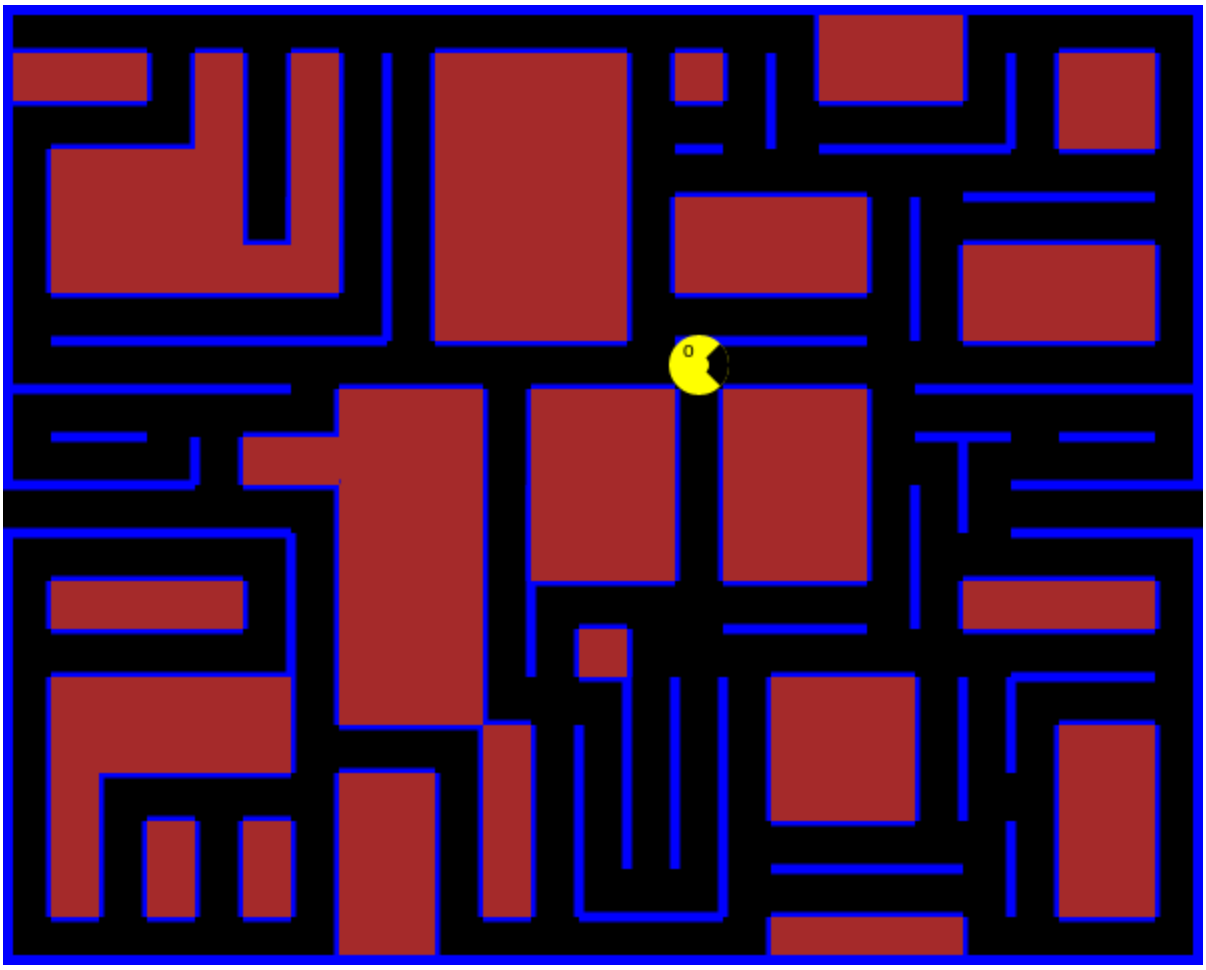
$$\int \frac{dx}{x^3-8} = \frac{1}{12}\ln|x-2| - \frac{1}{24}\ln\left|x^2+2x+4\right| - \frac{\sqrt{3}}{12}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right) + C.$$

Here again, we can drop the absolute value if we wish to do so, since $x^2+2x+4>0$ for all $x$.



## Integration Math-Man

This is a special Math-Man game where your task is to intercept with the correct integration technique ghost without colliding with the incorrect integration technique ghost. After hitting the Start button, look at the integral and decide on the technique to use. Then hit the play button. You will need to navigate by either hitting the arrow buttons or by using the arrow keys on your keypad.



Start ↑ ↓ ← → Slow Fast

Click on Start to begin.

Partial Fractions By Parts U-Substitution Trigonometric Substitution

**Example 3.5.8: Finding a Volume**

Find the volume of the solid of revolution obtained by revolving the region enclosed by the graph of $f(x) = \dfrac{x^2}{(x^2+1)^2}$ and the $x$-axis over the interval $[0, 1]$ about the $y$-axis.

**Solution**

Let's begin by sketching the region to be revolved (see Figure 3.5.1). From the sketch, we see that the shell method is a good choice for solving this problem.

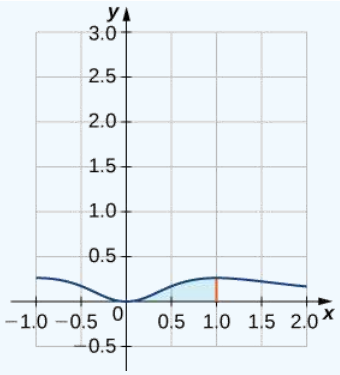



Figure 3.5.1: We can use the shell method to find the volume of revolution obtained by revolving the region shown about the $y$-axis.

The volume is given by

$$V = 2\pi \int_0^1 x \cdot \frac{x^2}{(x^2+1)^2}\, dx = 2\pi \int_0^1 \frac{x^3}{(x^2+1)^2}\, dx.$$

Since $deg((x^2+1)^2) = 4 > 3 = deg(x^3)$, we can proceed with partial fraction decomposition. Note that $(x^2+1)^2$ is a repeated irreducible quadratic. Using the decomposition described in the problem-solving strategy, we get

$$\frac{x^3}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}.$$

Finding a common denominator and equating the numerators gives

$$x^3 = (Ax+B)(x^2+1) + Cx + D.$$

Solving, we obtain $A = 1, B = 0, C = -1$, and $D = 0$. Substituting back into the integral, we have

$$\begin{aligned} V &= 2\pi \int_0^1 \frac{x^3}{(x^2+1)^2}\, dx = 2\pi \int_0^1 \left( \frac{x}{x^2+1} - \frac{x}{(x^2+1)^2} \right) dx \\ &= 2\pi \left( \frac{1}{2}\ln(x^2+1) + \frac{1}{2} \cdot \frac{1}{x^2+1} \right) \Big|_0^1 = \pi \left( \ln 2 - \tfrac{1}{2} \right). \end{aligned}$$

## Exercise 3.5.4

Set up the partial fraction decomposition for

$$\int \frac{x^2+3x+1}{(x+2)(x-3)^2(x^2+4)^2}\, dx.$$

**Hint**

Use the problem-solving strategy.

**Answer**

$$\frac{x^2+3x+1}{(x+2)(x-3)^2(x^2+4)^2} = \frac{A}{x+2} + \frac{B}{x-3} + \frac{C}{(x-3)^2} + \frac{Dx+E}{x^2+4} + \frac{Fx+G}{(x^2+4)^2}$$

## Key Concepts

- Partial fraction decomposition is a technique used to break down a rational function into a sum of simple rational functions that can be integrated using previously learned techniques.
- When applying partial fraction decomposition, we must make sure that the degree of the numerator is less than the degree of the denominator. If not, we need to perform long division before attempting partial fraction decomposition.

- The form the decomposition takes depends on the type of factors in the denominator. The types of factors include nonrepeated linear factors, repeated linear factors, nonrepeated irreducible quadratic factors, and repeated irreducible quadratic factors.

## Glossary

**partial fraction decomposition**
a technique used to break down a rational function into the sum of simple rational functions

## Contributors and Attributions

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# 3.5E: Exercises for Section 3.5

**Use partial fraction decomposition (or a simpler technique) to express the rational function as a sum or difference of two or more simpler rational expressions.**

1) $\frac{1}{(x-3)(x-2)}$

2) $\frac{x^2+1}{x(x+1)(x+2)}$

**Answer**

$$\frac{x^2+1}{x(x+1)(x+2)} = -\frac{2}{x+1}+\frac{5}{2(x+2)}+\frac{1}{2x}$$

3) $\frac{1}{x^3-x}$

4) $\frac{3x+1}{x^2}$

**Answer**

$$\frac{3x+1}{x^2} = \frac{1}{x^2}+\frac{3}{x}$$

5) $\frac{3x^2}{x^2+1}$ (Hint: Use long division first.)

6) $\frac{2x^4}{x^2-2x}$

**Answer**

$$\frac{2x^4}{x^2-2x} = 2x^2+4x+8+\frac{16}{x-2}$$

7) $\frac{1}{(x-1)(x^2+1)}$

8) $\frac{1}{x^2(x-1)}$

**Answer**

$$\frac{1}{x^2(x-1)} = -\frac{1}{x^2}-\frac{1}{x}+\frac{1}{x-1}$$

9) $\frac{x}{x^2-4}$

10) $\frac{1}{x(x-1)(x-2)(x-3)}$

**Answer**

$$\frac{1}{x(x-1)(x-2)(x-3)} = -\frac{1}{2(x-2)}+\frac{1}{2(x-1)}-\frac{1}{6x}+\frac{1}{6(x-3)}$$

11) $\frac{1}{x^4-1} = \frac{1}{(x+1)(x-1)(x^2+1)}$

12) $\frac{3x^2}{x^3-1} = \frac{3x^2}{(x-1)(x^2+x+1)}$

**Answer**

$$\frac{3x^2}{x^3-1} = \frac{1}{x-1}+\frac{2x+1}{x^2+x+1}$$

13) $\frac{2x}{(x+2)^2}$

14) $\dfrac{3x^4+x^3+20x^2+3x+31}{(x+1)(x^2+4)^2}$

**Answer**

$$\frac{3x^4+x^3+20x^2+3x+31}{(x+1)(x^2+4)^2} = \frac{2}{x+1}+\frac{x}{x^2+4}-\frac{1}{(x^2+4)^2}$$

**In exercises 15 - 25, use the method of partial fractions to evaluate each of the following integrals.**

15) $\displaystyle\int \frac{dx}{(x-3)(x-2)}$

16) $\displaystyle\int \frac{3x}{x^2+2x-8}\,dx$

**Answer**

$$\int \frac{3x}{x^2+2x-8}\,dx = 2\ln|x+4|+\ln|x-2|+C = \ln\left|(x+4)^2(x-2)\right|+C$$

17) $\displaystyle\int \frac{dx}{x^3-x}$

18) $\displaystyle\int \frac{x}{x^2-4}\,dx$

**Answer**

Note that you don't need Partial Fractions here. We use a simple $u$-substitution.

$$\int \frac{x}{x^2-4}\,dx = \tfrac{1}{2}\ln\left|4-x^2\right|+C$$

19) $\displaystyle\int \frac{dx}{x(x-1)(x-2)(x-3)}$

20) $\displaystyle\int \frac{2x^2+4x+22}{x^2+2x+10}\,dx$

**Answer**

Note that since the degree of the numerator is equal to the degree of the denominator, we need to start with long division. Then note that we will need to use completing the square to continue since we cannot factor the trinomial in the denominator.

$$\int \frac{2x^2+4x+22}{x^2+2x+10}\,dx = 2\left(x+\tfrac{1}{3}\arctan\left(\frac{1+x}{3}\right)\right)+C$$

21) $\displaystyle\int \frac{dx}{x^2-5x+6}$

22) $\displaystyle\int \frac{2-x}{x^2+x}\,dx$

**Answer**

$$\int \frac{2-x}{x^2+x}\,dx = 2\ln|x|-3\ln|1+x|+C = \ln\left|\frac{x^2}{(1+x)^3}\right|+C$$

23) $\displaystyle\int \frac{2}{x^2-x-6}\,dx$

24) $\displaystyle\int \frac{dx}{x^3-2x^2-4x+8}$

**Answer**

$$\int \frac{dx}{x^3-2x^2-4x+8} = \tfrac{1}{16}\left(-\frac{4}{-2+x}-\ln|-2+x|+\ln|2+x|\right)+C = \tfrac{1}{16}\left(-\frac{4}{-2+x}+\ln\left|\frac{x+2}{x-2}\right|\right)+C$$

25) $\displaystyle\int \frac{dx}{x^4-10x^2+9}$

**In exercises 26 - 29, evaluate the integrals with irreducible quadratic factors in the denominators.**

26) $\int \frac{2}{(x-4)(x^2+2x+6)}\,dx$

**Answer**

$$\int \frac{2}{(x-4)(x^2+2x+6)}\,dx \quad = \quad \tfrac{1}{30}\left(-2\sqrt{5}\arctan\left[\frac{1+x}{\sqrt{5}}\right]+2\ln|-4+x|-\ln\left|6+2x+x^2\right|\right)+C$$

27) $\int \frac{x^2}{x^3-x^2+4x-4}\,dx$

28) $\int \frac{x^3+6x^2+3x+6}{x^3+2x^2}\,dx$

**Answer**

Note that we need to use long division first, since the degree of the numerator is greater than the degree of the denominator.

$$\int \frac{x^3+6x^2+3x+6}{x^3+2x^2}\,dx \quad = \quad -\frac{3}{x}+4\ln|x+2|+x+C$$

29) $\int \frac{x}{(x-1)(x^2+2x+2)^2}\,dx$

**In exercises 30 - 32, use the method of partial fractions to evaluate the integrals.**

30) $\int \frac{3x+4}{(x^2+4)(3-x)}\,dx$

**Answer**

$$\int \frac{3x+4}{(x^2+4)(3-x)}\,dx \quad = \quad -\ln|3-x|+\tfrac{1}{2}\ln\left|x^2+4\right|+C$$

31) $\int \frac{2}{(x+2)^2(2-x)}\,dx$

32) $\int \frac{3x+4}{x^3-2x-4}\,dx$ (Hint: Use the rational root theorem.)

**Answer**

$$\int \frac{3x+4}{x^3-2x-4}\,dx \quad = \quad \ln|x-2|-\tfrac{1}{2}\ln\left|x^2+2x+2\right|+C$$

**In exercises 33 - 46, use substitution to convert the integrals to integrals of rational functions. Then use partial fractions to evaluate the integrals.**

33) $\int_0^1 \frac{e^x}{36-e^{2x}}\,dx$ (Give the exact answer and the decimal equivalent. Round to five decimal places.)

34) $\int \frac{e^x}{e^{2x}-e^x}\,dx$

**Answer**

$$\int \frac{e^x}{e^{2x}-e^x}\,dx \quad = \quad -x+\ln|1-e^x|+C$$

35) $\int \frac{\sin x\,dx}{1-\cos^2 x}$

36) $\int \frac{\sin x}{\cos^2 x+\cos x-6}\,dx$

**Answer**

$$\int \frac{\sin x}{\cos^2 x+\cos x-6}\,dx \quad = \quad \tfrac{1}{5}\ln\left|\frac{\cos x+3}{\cos x-2}\right|+C$$

37) $\int \frac{1-\sqrt{x}}{1+\sqrt{x}}\,dx$

38) $\int \frac{dt}{(e^t - e^{-t})^2}$

**Answer**

$$\int \frac{dt}{(e^t - e^{-t})^2} = \frac{1}{2 - 2e^{2t}} + C$$

39) $\int \frac{1 + e^x}{1 - e^x}\, dx$

40) $\int \frac{dx}{1 + \sqrt{x+1}}$

**Answer**

$$\int \frac{dx}{1 + \sqrt{x+1}} = 2\sqrt{1+x} - 2\ln\left|1 + \sqrt{1+x}\right| + C$$

41) $\int \frac{dx}{\sqrt{x} + \sqrt[4]{x}}$

42) $\int \frac{\cos x}{\sin x(1 - \sin x)}\, dx$

**Answer**

$$\int \frac{\cos x}{\sin x(1 - \sin x)}\, dx = \ln\left|\frac{\sin x}{1 - \sin x}\right| + C$$

43) $\int \frac{e^x}{(e^{2x} - 4)^2}\, dx$

44) $\int_1^2 \frac{1}{x^2\sqrt{4 - x^2}}\, dx$

**Answer**

$$\int_1^2 \frac{1}{x^2\sqrt{4 - x^2}}\, dx = \frac{\sqrt{3}}{4}$$

45) $\int \frac{1}{2 + e^{-x}}\, dx$

46) $\int \frac{1}{1 + e^x}\, dx$

**Answer**

$$\int \frac{1}{1 + e^x}\, dx = x - \ln(1 + e^x) + C$$

**In exercises 47 - 48, use the given substitution to convert the integral to an integral of a rational function, then evaluate.**

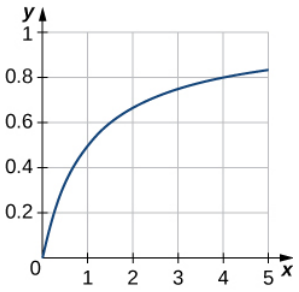
47) $\int \frac{1}{t - \sqrt[3]{t}}\, dt; \quad t = x^3$

48) $\int \frac{1}{\sqrt{x} + \sqrt[3]{x}}\, dx; \quad x = u^6$

**Answer**

$$\int \frac{1}{\sqrt{x} + \sqrt[3]{x}}\, dx = 6x^{1/6} - 3x^{1/3} + 2\sqrt{x} - 6\ln(1 + x^{1/6}) + C$$

49) Graph the curve $y = \frac{x}{1 + x}$ over the interval $[0, 5]$. Then, find the area of the region bounded by the curve, the $x$-axis, and the line $x = 4$.




50) Find the volume of the solid generated when the region bounded by $y = \dfrac{1}{\sqrt{x(3-x)}}$, $y = 0$, $x = 1$, and $x = 2$ is revolved about the $x$-axis.

**Answer**

$V = \frac{4}{3}\pi \operatorname{arctanh}\left[\frac{1}{3}\right] = \frac{1}{3}\pi \ln 4 \text{ units}^3$

51) The velocity of a particle moving along a line is a function of time given by $v(t) = \dfrac{88t^2}{t^2+1}$. Find the distance that the particle has traveled after $t = 5$ sec.

**In exercises 52 - 54, solve the initial-value problem for $x$ as a function of $t$.**

52) $(t^2 - 7t + 12)\dfrac{dx}{dt} = 1, \quad t > 4, \; x(5) = 0$

**Answer**

$x = -\ln|t-3| + \ln|t-4| + \ln 2 = \ln\left|\dfrac{2(t-4)}{t-3}\right|$

53) $(t+5)\dfrac{dx}{dt} = x^2 + 1, \quad t > -5, \; x(1) = \tan 1$

54) $(2t^3 - 2t^2 + t - 1)\dfrac{dx}{dt} = 3, \quad x(2) = 0$

**Answer**

$x = \ln|t-1| - \sqrt{2}\arctan(\sqrt{2}t) - \frac{1}{2}\ln(t^2 + \frac{1}{2}) + \sqrt{2}\arctan(2\sqrt{2}) + \frac{1}{2}\ln 4.5$

55) Find the $x$-coordinate of the centroid of the area bounded by $y(x^2 - 9) = 1$, $y = 0$, $x = 4$, and $x = 5$. (Round the answer to two decimal places.)

56) Find the volume generated by revolving the area bounded by $y = \dfrac{1}{x^3 + 7x^2 + 6x}$, $x = 1$, $x = 7$, and $y = 0$ about the $y$-axis.

**Answer**

$V = \frac{2}{5}\pi \ln \frac{28}{13} \text{ units}^3$

57) Find the area bounded by $y = \dfrac{x - 12}{x^2 - 8x - 20}$, $y = 0$, $x = 2$, and $x = 4$. (Round the answer to the nearest hundredth.)

58) Evaluate the integral $\displaystyle\int \frac{dx}{x^3 + 1}$.

**Answer**

$$\int \frac{dx}{x^3+1} = \frac{\arctan\left[\frac{-1+2x}{\sqrt{3}}\right]}{\sqrt{3}} + \frac{1}{3}\ln|1+x| - \frac{1}{6}\ln|1 - x + x^2| + C$$

**For problems 59 - 62, use the substitutions $\tan(\frac{x}{2}) = t$, $dx = \dfrac{2}{1+t^2}\,dt$, $\sin x = \dfrac{2t}{1+t^2}$, and $\cos x = \dfrac{1-t^2}{1+t^2}$.**

59) $\displaystyle\int \frac{dx}{3 - 5\sin x}$

60) Find the area under the curve $y = \dfrac{1}{1+\sin x}$ between $x = 0$ and $x = \pi$. (Assume the dimensions are in inches.)

**Answer**

2.0 in.$^2$

61) Given $\tan\left(\frac{x}{2}\right) = t$, derive the formulas $dx = \dfrac{2}{1+t^2}dt$, $\sin x = \dfrac{2t}{1+t^2}$, and $\cos x = \dfrac{1-t^2}{1+t^2}$.

62) Evaluate $\displaystyle\int \frac{\sqrt[3]{x-8}}{x}\,dx$.

**Answer**

$$\int \frac{\sqrt[3]{x-8}}{x}\,dx = 3(-8+x)^{1/3} - 2\sqrt{3}\arctan\left[\frac{-1+(-8+x)^{1/3}}{\sqrt{3}}\right] - 2\ln\left[2+(-8+x)^{1/3}\right] + \ln\left[4-2(-8+x)^{1/3}+(-8+x)^{2/3}\right] + C$$

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# 3.6: Other Strategies for Integration

**Learning Objectives**

- Use a table of integrals to solve integration problems.
- Use a computer algebra system (CAS) to solve integration problems.

In addition to the techniques of integration we have already seen, several other tools are widely available to assist with the process of integration. Among these tools are **integration tables**, which are readily available in many books, including the appendices to this one. Also widely available are **computer algebra systems (CAS)**, which are found on calculators and in many campus computer labs, and are free online.

## Tables of Integrals

Integration tables, if used in the right manner, can be a handy way either to evaluate or check an integral quickly. Keep in mind that when using a table to check an answer, it is possible for two completely correct solutions to look very different. For example, in Trigonometric Substitution, we found that, by using the substitution $x = \tan\theta$, we can arrive at

$$\int \frac{dx}{\sqrt{1+x^2}} = \ln\left|x+\sqrt{x^2+1}\right| + C.$$

However, using $x = \sinh\theta$, we obtained a different solution—namely,

$$\int \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1} x + C.$$

We later showed algebraically that the two solutions are equivalent. That is, we showed that $\sinh^{-1} x = \ln\left|x+\sqrt{x^2+1}\right|$. In this case, the two antiderivatives that we found were actually equal. This need not be the case. However, as long as the difference in the two antiderivatives is a constant, they are equivalent.

**Example 3.6.1: Using a Formula from a Table to Evaluate an Integral**

Use the table formula

$$\int \frac{\sqrt{a^2-u^2}}{u^2}du = -\frac{\sqrt{a^2-u^2}}{u} - \sin^{-1}\frac{u}{a} + C$$

to evaluate $\displaystyle\int \frac{\sqrt{16-e^{2x}}}{e^x}dx$.

**Solution**

If we look at integration tables, we see that several formulas contain expressions of the form $\sqrt{a^2-u^2}$. This expression is actually similar to $\sqrt{16-e^{2x}}$, where $a = 4$ and $u = e^x$. Keep in mind that we must also have $du = e^x$. Multiplying the numerator and the denominator of the given integral by $e^x$ should help to put this integral in a useful form. Thus, we now have

$$\int \frac{\sqrt{16-e^{2x}}}{e^x}dx = \int \frac{\sqrt{16-e^{2x}}}{e^{2x}}e^x dx.$$

Substituting $u = e^x$ and $du = e^x\,dx$ produces $\displaystyle\int \frac{\sqrt{a^2-u^2}}{u^2}du$. From the integration table (#88 in Appendix A),

$$\int \frac{\sqrt{a^2-u^2}}{u^2}du = -\frac{\sqrt{a^2-u^2}}{u} - \sin^{-1}\frac{u}{a} + C.$$

Thus,

$$\begin{aligned}\int \frac{\sqrt{16-e^{2x}}}{e^x}dx &= \int \frac{\sqrt{16-e^{2x}}}{e^{2x}}e^x dx \text{ Substitute } u = e^x \text{ and } du = e^x dx.\\ &= \int \frac{\sqrt{4^2-u^2}}{u^2}du \text{ Apply the formula using } a = 4.\\ &= -\frac{\sqrt{4^2-u^2}}{u} - \sin^{-1}\frac{u}{4} + C \text{ Substitute } u = e^x.\end{aligned}$$

$$= -\frac{\sqrt{16 - e^{2x}}}{e^x} - \sin^{-1}\left(\frac{e^x}{4}\right) + C$$

## Computer Algebra Systems

If available, a CAS is a faster alternative to a table for solving an integration problem. Many such systems are widely available and are, in general, quite easy to use.

### Example 3.6.2: Using a Computer Algebra System to Evaluate an Integral

Use a computer algebra system to evaluate $\int \frac{dx}{\sqrt{x^2-4}}$. Compare this result with $\ln\left|\frac{\sqrt{x^2-4}}{2} + \frac{x}{2}\right| + C$, a result we might have obtained if we had used trigonometric substitution.

**Solution**

Using Wolfram Alpha, we obtain

$$\int \frac{dx}{\sqrt{x^2-4}} = \ln\left|\sqrt{x^2-4} + x\right| + C.$$

Notice that

$$\ln\left|\frac{\sqrt{x^2-4}}{2} + \frac{x}{2}\right| + C = \ln\left|\frac{\sqrt{x^2-4}+x}{2}\right| + C = \ln\left|\sqrt{x^2-4}+x\right| - \ln 2 + C.$$

Since these two antiderivatives differ by only a constant, the solutions are equivalent. We could have also demonstrated that each of these antiderivatives is correct by differentiating them.

You can access an integral calculator for more examples.

### Example 3.6.3: Using a CAS to Evaluate an Integral

Evaluate $\int \sin^3 x\, dx$ using a CAS. Compare the result to $\frac{1}{3}\cos^3 x - \cos x + C$, the result we might have obtained using the technique for integrating odd powers of $\sin x$ discussed earlier in this chapter.

**Solution**

Using Wolfram Alpha, we obtain

$$\int \sin^3 x\, dx = \frac{1}{12}(\cos(3x) - 9\cos x) + C.$$

This looks quite different from $\frac{1}{3}\cos^3 x - \cos x + C$. To see that these antiderivatives are equivalent, we can make use of a few trigonometric identities:

$$\begin{aligned}
\frac{1}{12}(\cos(3x) - 9\cos x) &= \frac{1}{12}(\cos(x+2x) - 9\cos x) \\
&= \frac{1}{12}(\cos(x)\cos(2x) - \sin(x)\sin(2x) - 9\cos x) \\
&= \frac{1}{12}(\cos x(2\cos^2 x - 1) - \sin x(2\sin x\cos x) - 9\cos x) \\
&= \frac{1}{12}(2\cos^3 x - \cos x - 2\cos x(1 - \cos^2 x) - 9\cos x) \\
&= \frac{1}{12}(4\cos^3 x - 12\cos x) \\
&= \frac{1}{3}\cos^3 x - \cos x.
\end{aligned}$$

Thus, the two antiderivatives are identical.

We may also use a CAS to compare the graphs of the two functions, as shown in the following figure.

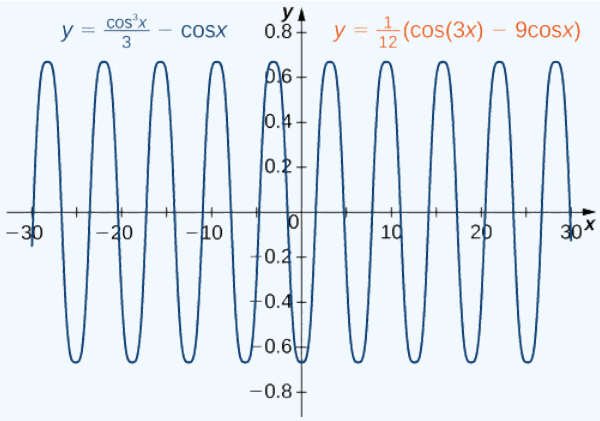



Figure 3.6.1: The graphs of $y = \frac{1}{3}\cos^3 x - \cos x$ and $y = \frac{1}{12}(\cos(3x) - 9\cos x)$ are identical.

**? Exercise 3.6.1**

Use a CAS to evaluate $\int \frac{dx}{\sqrt{x^2+4}}$.

**Hint**

Answers may vary.

**Answer**

Possible solutions include $\sinh^{-1}\left(\frac{x}{2}\right) + C$ and $\ln\left|\sqrt{x^2+4} + x\right| + C$.

## Key Concepts

- An integration table may be used to evaluate indefinite integrals.
- A CAS (or computer algebra system) may be used to evaluate indefinite integrals.
- It may require some effort to reconcile equivalent solutions obtained using different methods.

## Glossary

**computer algebra system (CAS)**

technology used to perform many mathematical tasks, including integration

**integration table**

a table that lists integration formulas

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# 3.6E: Exercises for Section 3.6

**Use a table of integrals to evaluate the following integrals.**

1) $\int_0^4 \frac{x}{\sqrt{1+2x}}\,dx$

2) $\int \frac{x+3}{x^2+2x+2}\,dx$

**Answer**

$$\int \frac{x+3}{x^2+2x+2}\,dx = \tfrac{1}{2}\ln\left|x^2+2x+2\right| + 2\arctan(x+1) + C$$

3) $\int x^3\sqrt{1+2x^2}\,dx$

4) $\int \frac{1}{\sqrt{x^2+6x}}\,dx$

**Answer**

$$\int \frac{1}{\sqrt{x^2+6x}}\,dx = \cosh^{-1}\left(\frac{x+3}{3}\right) + C$$

5) $\int \frac{x}{x+1}\,dx$

6) $\int x\cdot 2^{x^2}\,dx$

**Answer**

$$\int x\cdot 2^{x^2}\,dx = \frac{2^{x^2-1}}{\ln 2} + C$$

7) $\int \frac{1}{4x^2+25}\,dx$

8) $\int \frac{dy}{\sqrt{4-y^2}}$

**Answer**

$$\int \frac{dy}{\sqrt{4-y^2}} = \arcsin\left(\frac{y}{2}\right) + C$$

9) $\int \sin^3(2x)\cos(2x)\,dx$

10) $\int \csc(2w)\cot(2w)\,dw$

**Answer**

$$\int \csc(2w)\cot(2w)\,dw = -\tfrac{1}{2}\csc(2w) + C$$

11) $\int 2^y\,dy$

12) $\int_0^1 \frac{3x}{\sqrt{x^2+8}}\,dx$

**Answer**

$$\int_0^1 \frac{3x}{\sqrt{x^2+8}}\,dx = 9 - 6\sqrt{2}$$

13) $\int_{-1/4}^{1/4} \sec^2(\pi x)\tan(\pi x)\,dx$

14) $\int_0^{\pi/2} \tan^2\left(\frac{x}{2}\right)\,dx$

**Answer**

$$\int_0^{\pi/2} \tan^2\left(\frac{x}{2}\right)\,dx = 2 - \frac{\pi}{2}$$

15) $\int \cos^3 x\,dx$

16) $\int \tan^5(3x)\,dx$

**Answer**

$$\int \tan^5(3x)\,dx = \tfrac{1}{12}\tan^4(3x) - \tfrac{1}{6}\tan^2(3x) + \tfrac{1}{3}\ln|\sec 3x| + C$$

17) $\int \sin^2 y \cos^3 y\,dy$

**Use a CAS to evaluate the following integrals. Tables can also be used to verify the answers.**

18) [T] $\int \frac{dw}{1+\sec\left(\frac{w}{2}\right)}$

**Answer**

$$\int \frac{dw}{1+\sec\left(\frac{w}{2}\right)} = 2\cot\left(\tfrac{w}{2}\right) - 2\csc\left(\tfrac{w}{2}\right) + w + C$$

19) [T] $\int \frac{dw}{1-\cos(7w)}$

20) [T] $\int_0^t \frac{dt}{4\cos t + 3\sin t}$

**Answer**

$$\int_0^t \frac{dt}{4\cos t + 3\sin t} = \tfrac{1}{5}\ln\left|\frac{2(5+4\sin t - 3\cos t)}{4\cos t + 3\sin t}\right|$$

21) [T] $\int \frac{\sqrt{x^2-9}}{3x}\,dx$

22) [T] $\int \frac{dx}{x^{1/2}+x^{1/3}}$

**Answer**

$$\int \frac{dx}{x^{1/2}+x^{1/3}} = 6x^{1/6} - 3x^{1/3} + 2\sqrt{x} - 6\ln[1+x^{1/6}] + C$$

23) [T] $\int \frac{dx}{x\sqrt{x-1}}$

24) [T] $\int x^3 \sin x\,dx$

**Answer**

$$\int x^3 \sin x\,dx = -x^3\cos x + 3x^2\sin x + 6x\cos x - 6\sin x + C$$

25) [T] $\int x\sqrt{x^4-9}\,dx$

26) [T] $\int \frac{x}{1+e^{-x^2}}\,dx$

**Answer**

$$\int \frac{x}{1+e^{-x^2}}\,dx = \tfrac{1}{2}\left(x^2 + \ln\left|1+e^{-x^2}\right|\right) + C$$

27) [T] $\int \frac{\sqrt{3-5x}}{2x}\,dx$

28) [T] $\int \frac{dx}{x\sqrt{x-1}}$

**Answer**

$$\int \frac{dx}{x\sqrt{x-1}} = 2\arctan\left(\sqrt{x-1}\right) + C$$

29) [T] $\int e^x \cos^{-1}(e^x)\,dx$

**Use a calculator or CAS to evaluate the following integrals.**

30) [T] $\int_0^{\pi/4} \cos 2x\,dx$

**Answer**

$$\int_0^{\pi/4} \cos 2x\,dx = 0.5 = \frac{1}{2}$$

31) [T] $\int_0^1 x \cdot e^{-x^2}\,dx$

32) [T] $\int_0^8 \frac{2x}{\sqrt{x^2+36}}\,dx$

**Answer**

$$\int_0^8 \frac{2x}{\sqrt{x^2+36}}\,dx = 8.0$$

33) [T] $\int_0^{2/\sqrt{3}} \frac{1}{4+9x^2}\,dx$

34) [T] $\int \frac{dx}{x^2+4x+13}$

**Answer**

$$\int \frac{dx}{x^2+4x+13} = \tfrac{1}{3}\arctan\left(\tfrac{1}{3}(x+2)\right) + C$$

35) [T] $\int \frac{dx}{1+\sin x}$

**Use tables to evaluate the integrals. You may need to complete the square or change variables to put the integral into a form given in the table.**

36) $\int \frac{dx}{x^2+2x+10}$

**Answer**

$$\int \frac{dx}{x^2+2x+10} = \tfrac{1}{3}\arctan\left(\frac{x+1}{3}\right) + C$$

37) $\int \frac{dx}{\sqrt{x^2-6x}}$

38) $\int \frac{e^x}{\sqrt{e^{2x}-4}}\,dx$

**Answer**

$$\int \frac{e^x}{\sqrt{e^{2x}-4}}\,dx = \ln\left(e^x + \sqrt{4+e^{2x}}\right) + C$$

39) $\int \frac{\cos x}{\sin^2 x + 2\sin x}\,dx$

40) $\int \frac{\arctan(x^3)}{x^4}\,dx$

**Answer**

$$\int \frac{\arctan(x^3)}{x^4}\,dx = \ln x - \tfrac{1}{6}\ln(x^6+1) - \frac{\arctan(x^3)}{3x^3} + C$$

41) $\int \frac{\ln|x|\arcsin(\ln|x|)}{x}\,dx$

**Use tables to perform the integration.**

42) $\int \frac{dx}{\sqrt{x^2+16}}$

**Answer**

$$\int \frac{dx}{\sqrt{x^2+16}} = \ln|x| + \sqrt{16+x^2}\,| + C$$

43) $\int \frac{3x}{2x+7}\,dx$

44) $\int \frac{dx}{1-\cos 4x}$

**Answer**

$$\int \frac{dx}{1-\cos 4x} = -\frac{1}{4}\cot 2x + C$$

45) $\int \frac{dx}{\sqrt{4x+1}}$

46) Find the area bounded by $y(4+25x^2)=5,\ x=0,\ y=0,$ and $x=4.$ Use a table of integrals or a CAS.

**Answer**

$\frac{1}{2}\arctan 10$ units$^2$

47) The region bounded between the curve $y=\frac{1}{\sqrt{1+\cos x}},\ 0.3\le x\le 1.1,$ and the $x$-axis is revolved about the $x$-axis to generate a solid. Use a table of integrals to find the volume of the solid generated. (Round the answer to two decimal places.)

48) Use substitution and a table of integrals to find the area of the surface generated by revolving the curve $y=e^x,\ 0\le x\le 3,$ about the $x$-axis. (Round the answer to two decimal places.)

**Answer**

1276.14 units$^2$

49) [T] Use an integral table and a calculator to find the area of the surface generated by revolving the curve $y=\frac{x^2}{2},\ 0\le x\le 1,$ about the $x$-axis. (Round the answer to two decimal places.)

50) [T] Use a CAS or tables to find the area of the surface generated by revolving the curve $y=\cos x,\ 0\le x\le \frac{\pi}{2},$ about the $x$-axis. (Round the answer to two decimal places.)

**Answer**

7.21 units$^2$

51) Find the length of the curve $y=\frac{x^2}{4}$ over $[0,8]$.

52) Find the length of the curve $y=e^x$ over $[0,\ \ln(2)]$.

**Answer**

$\left(\sqrt{5}-\sqrt{2}+\ln\left|\frac{2+2\sqrt{2}}{1+\sqrt{5}}\right|\right)$ units

53) Find the area of the surface formed by revolving the graph of $y = 2\sqrt{x}$ over the interval $[0, 9]$ about the $x$-axis.

54) Find the average value of the function $f(x) = \dfrac{1}{x^2 + 1}$ over the interval $[-3, 3]$.

**Answer**

$\frac{1}{3}\arctan(3) \approx 0.416$

55) Approximate the arc length of the curve $y = \tan \pi x$ over the interval $\left[0, \frac{1}{4}\right]$. (Round the answer to three decimal places.)

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# 3.7: Numerical Integration

**Learning Objectives**

- Approximate the value of a definite integral by using the midpoint and trapezoidal rules.
- Determine the absolute and relative error in using a numerical integration technique.
- Estimate the absolute and relative error using an error-bound formula.
- Recognize when the midpoint and trapezoidal rules over- or underestimate the true value of an integral.
- Use Simpson's rule to approximate the value of a definite integral to a given accuracy.

The antiderivatives of many functions either cannot be expressed or cannot be expressed easily in closed form (that is, in terms of known functions). Consequently, rather than evaluate definite integrals of these functions directly, we resort to various techniques of **numerical integration** to approximate their values. In this section, we explore several of these techniques. In addition, we examine the process of estimating the error in using these techniques.

## The Midpoint Rule

Earlier in this text we defined the definite integral of a function over an interval as the limit of **Riemann sums**. In general, any Riemann sum of a function $f(x)$ over an interval $[a,b]$ may be viewed as an estimate of $\int_a^b f(x)\,dx$. Recall that a Riemann sum of a function $f(x)$ over an interval $[a,b]$ is obtained by selecting a partition

$$P = \{x_0, x_1, x_2, \ldots, x_n\}$$

where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$

and a set

$$S = \{x_1^*, x_2^*, \ldots, x_n^*\} \tag{3.7.1}$$

where $x_{i-1} \le x_i^* \le x_i \quad \text{for all } i.$

The Riemann sum corresponding to the partition $P$ and the set $S$ is given by $\sum_{i=1}^{n} f(x_i^*)\Delta x_i$, where $\Delta x_i = x_i - x_{i-1}$, the length of the $i^{\text{th}}$ subinterval.

The midpoint rule for estimating a definite integral uses a Riemann sum with subintervals of equal width and the midpoints, $m_i$, of each subinterval in place of $x_i^*$. Formally, we state a theorem regarding the convergence of the midpoint rule as follows.

**The Midpoint Rule**

Assume that $f(x)$ is continuous on $[a,b]$. Let $n$ be a positive integer and $\Delta x = \dfrac{b-a}{n}$. If $[a,b]$ is divided into $n$ subintervals, each of length $\Delta x$, and $m_i$ is the midpoint of the $i^{\text{th}}$ subinterval, set

$$M_n = \sum_{i=1}^{n} f(m_i)\Delta x. \tag{3.7.2}$$

Then $\lim_{n\to\infty} M_n = \int_a^b f(x)\,dx$.

As we can see in Figure 3.7.1, if $f(x) \ge 0$ over $[a,b]$, then $\sum_{i=1}^{n} f(m_i)\Delta x$ corresponds to the sum of the areas of rectangles approximating the area between the graph of $f(x)$ and the $x$-axis over $[a,b]$. The graph shows the rectangles corresponding to $M_4$ for a nonnegative function over a closed interval $[a,b]$.

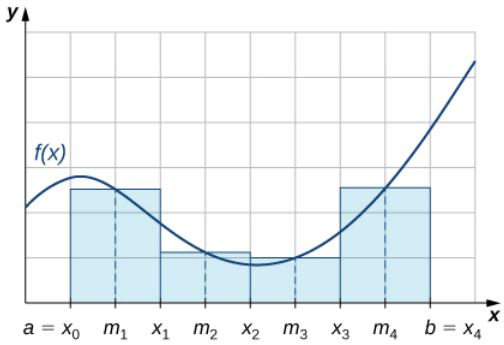
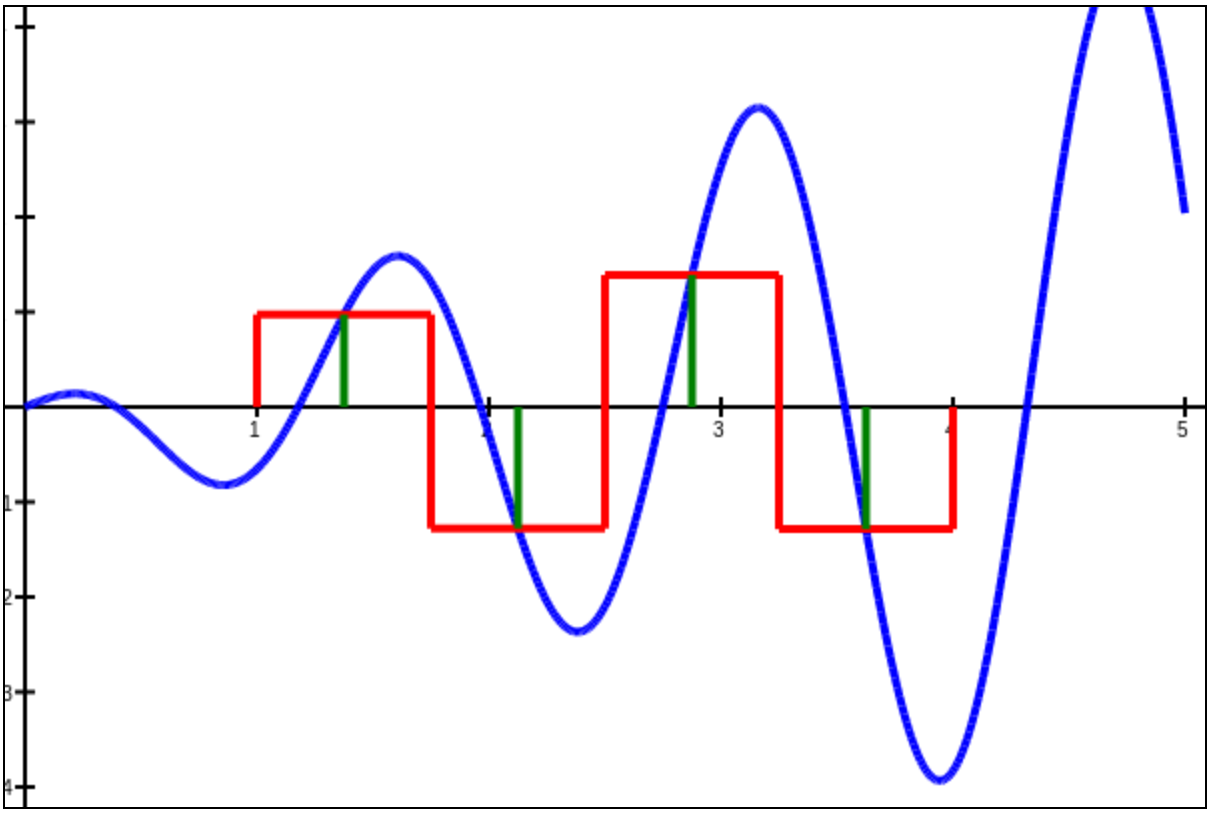



Figure **3.7.1**: The midpoint rule approximates the area between the graph of $f(x)$ and the $x$-axis by summing the areas of rectangles with midpoints that are points on $f(x)$.

## Visualize the Midpoint Rule

Move the slider to see the midpoint rule being used to approximate $\int_1^4 x\cos(4x)dx = -0.1177...$ using the selected number of rectangles.




n = 4

Area ≈ 0.7308 + -0.9595 + 1.0421 + -0.9650 = -0.1516

**Example 3.7.1: Using the Midpoint Rule with $M_4$**

Use the midpoint rule to estimate $\int_0^1 x^2\,dx$ using four subintervals. Compare the result with the actual value of this integral.

Solution: Each subinterval has length $\Delta x = \frac{1-0}{4} = \frac{1}{4}$. Therefore, the subintervals consist of

$$\left[0, \tfrac{1}{4}\right], \left[\tfrac{1}{4}, \tfrac{1}{2}\right], \left[\tfrac{1}{2}, \tfrac{3}{4}\right], \text{ and } \left[\tfrac{3}{4}, 1\right].$$

The midpoints of these subintervals are $\left\{\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}\right\}$. Thus,

$$\begin{aligned} M_4 &= \frac{1}{4} \cdot f\left(\frac{1}{8}\right) + \frac{1}{4} \cdot f\left(\frac{3}{8}\right) + \frac{1}{4} \cdot f\left(\frac{5}{8}\right) + \frac{1}{4} \cdot f\left(\frac{7}{8}\right) \\ &= \frac{1}{4} \cdot \frac{1}{64} + \frac{1}{4} \cdot \frac{9}{64} + \frac{1}{4} \cdot \frac{25}{64} + \frac{1}{4} \cdot \frac{49}{64} \\ &= \frac{21}{64} = 0.328125. \end{aligned}$$

Since

$$\int_0^1 x^2\, dx = \frac{1}{3},$$

the absolute error in this approximation is:

$$\left|\frac{1}{3} - \frac{21}{64}\right| = \frac{1}{192} \approx 0.0052,$$

and we see that the midpoint rule produces an estimate that is somewhat close to the actual value of the definite integral.

### Example 3.7.2: Using the Midpoint Rule with $M_6$

Use $M_6$ to estimate the length of the curve $y = \frac{1}{2}x^2$ on $[1, 4]$.

Solution: The length of $y = \frac{1}{2}x^2$ on $[1, 4]$ is

$$s = \int_1^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Since $\frac{dy}{dx} = x$, this integral becomes $\int_1^4 \sqrt{1 + x^2}\, dx$.

If $[1, 4]$ is divided into six subintervals, then each subinterval has length $\Delta x = \frac{4-1}{6} = \frac{1}{2}$ and the midpoints of the subintervals are $\left\{\frac{5}{4}, \frac{7}{4}, \frac{9}{4}, \frac{11}{4}, \frac{13}{4}, \frac{15}{4}\right\}$. If we set $f(x) = \sqrt{1 + x^2}$,

$$\begin{aligned} M_6 &= \tfrac{1}{2} \cdot f\left(\frac{5}{4}\right) + \tfrac{1}{2} \cdot f\left(\frac{7}{4}\right) + \frac{1}{2} \cdot f\left(\frac{9}{4}\right) + \frac{1}{2} \cdot f\left(\frac{11}{4}\right) + \frac{1}{2} \cdot f\left(\frac{13}{4}\right) + \frac{1}{2} \cdot f\left(\frac{15}{4}\right) \\ &\approx \frac{1}{2}(1.6008 + 2.0156 + 2.4622 + 2.9262 + 3.4004 + 3.8810) = 8.1431 \text{ units}. \end{aligned}$$

### Exercise 3.7.1

Use the midpoint rule with $n = 2$ to estimate $\int_1^2 \frac{1}{x}\, dx$.

**Hint**

$\Delta x = \frac{1}{2}$, $m_1 = \frac{5}{4}$, and $m_2 = \frac{7}{4}$.

**Answer**

$\frac{24}{35} \approx 0.685714$

## The Trapezoidal Rule

We can also approximate the value of a definite integral by using trapezoids rather than rectangles. In Figure **3.7.2**, the area beneath the curve is approximated by trapezoids rather than by rectangles.

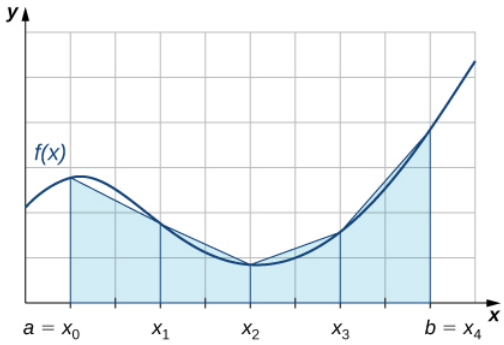
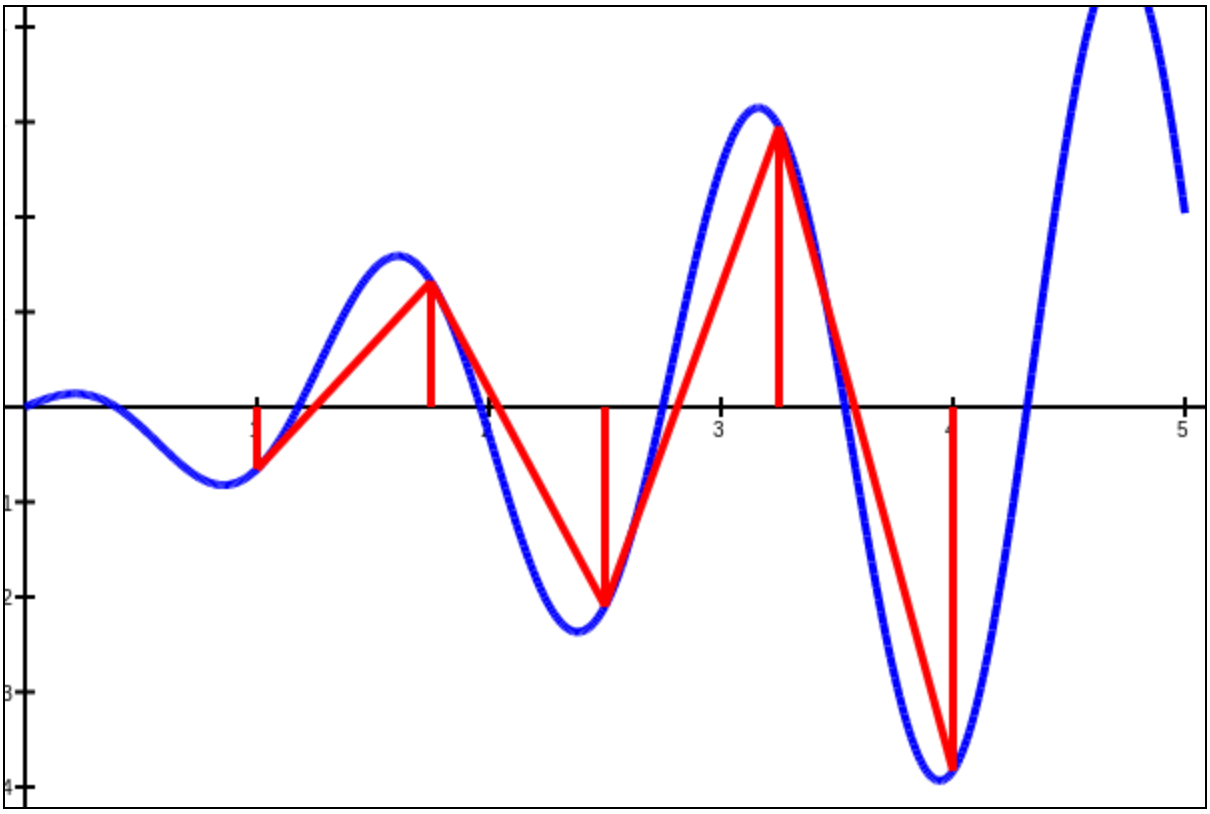



Figure 3.7.2: Trapezoids may be used to approximate the area under a curve, hence approximating the definite integral.

## Visualize the Trapezoidal Rule

Move the slider to see the trapezoidal rule being used to approximate $\int_1^4 x\cos(4x)dx = -0.1177...$ using the selected number of trapezoids.




n = 4

Area ≈ 0.2496 + -0.2919 + 0.3193 + -0.3305 = -0.0535

The **trapezoidal rule** for estimating definite integrals uses trapezoids rather than rectangles to approximate the area under a curve. To gain insight into the final form of the rule, consider the trapezoids shown in Figure 3.7.2. We assume that the length of each subinterval is given by $\Delta x$. First, recall that the area of a trapezoid with a height of $h$ and bases of length $b_1$ and $b_2$ is given by $\text{Area} = \frac{1}{2}h(b_1 + b_2)$. We see that the first trapezoid has a height $\Delta x$ and parallel bases of length $f(x_0)$ and $f(x_1)$. Thus, the area of the first trapezoid in Figure 3.7.2 is

$$\frac{1}{2}\Delta x\Big(f(x_0)+f(x_1)\Big).$$

The areas of the remaining three trapezoids are

$$\frac{1}{2}\Delta x\Big(f(x_1)+f(x_2)\Big),\ \frac{1}{2}\Delta x\Big(f(x_2)+f(x_3)\Big),\text{ and }\frac{1}{2}\Delta x\Big(f(x_3)+f(x_4)\Big).$$

Consequently,

$$\int_a^b f(x)\,dx$$
$$\approx \frac{1}{2}\Delta x\Big(f(x_0)+f(x_1)\Big)+\frac{1}{2}\Delta x\Big(f(x_1)+f(x_2)\Big)+\frac{1}{2}\Delta x\Big(f(x_2)+f(x_3)\Big)+\frac{1}{2}\Delta x\Big(f(x_3)+f(x_4)\Big).$$

After taking out a common factor of $\frac{1}{2}\Delta x$ and combining like terms, we have

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}\Big[f(x_0)+2f(x_1)+2f(x_2)+2f(x_3)+f(x_4)\Big].$$

Generalizing, we formally state the following rule.

**The Trapezoidal Rule**

Assume that $f(x)$ is continuous over $[a,b]$. Let $n$ be a positive integer and $\Delta x=\frac{b-a}{n}$. Let $[a,b]$ be divided into $n$ subintervals, each of length $\Delta x$, with endpoints at $P=\{x_0,x_1,x_2\ldots,x_n\}$.

Set

$$T_n=\frac{\Delta x}{2}\Big[f(x_0)+2f(x_1)+2f(x_2)+\cdots+2f(x_{n-1})+f(x_n)\Big]. \tag{3.7.3}$$

Then, $\lim\limits_{n\to+\infty} T_n=\int_a^b f(x)\,dx$.

Before continuing, let's make a few observations about the trapezoidal rule. First of all, it is useful to note that

$$T_n=\frac{1}{2}(L_n+R_n)\text{ where }L_n=\sum_{i=1}^{n}f(x_{i-1})\Delta x\text{ and }R_n=\sum_{i=1}^{n}f(x_i)\Delta x.$$

That is, $L_n$ and $R_n$ approximate the integral using the left-hand and right-hand endpoints of each subinterval, respectively. In addition, a careful examination of Figure **3.7.3** leads us to make the following observations about using the trapezoidal rules and midpoint rules to estimate the definite integral of a nonnegative function. The trapezoidal rule tends to overestimate the value of a definite integral systematically over intervals where the function is concave up and to underestimate the value of a definite integral systematically over intervals where the function is concave down. On the other hand, the midpoint rule tends to average out these errors somewhat by partially overestimating and partially underestimating the value of the definite integral over these same types of intervals. This leads us to hypothesize that, in general, the midpoint rule tends to be more accurate than the trapezoidal rule.

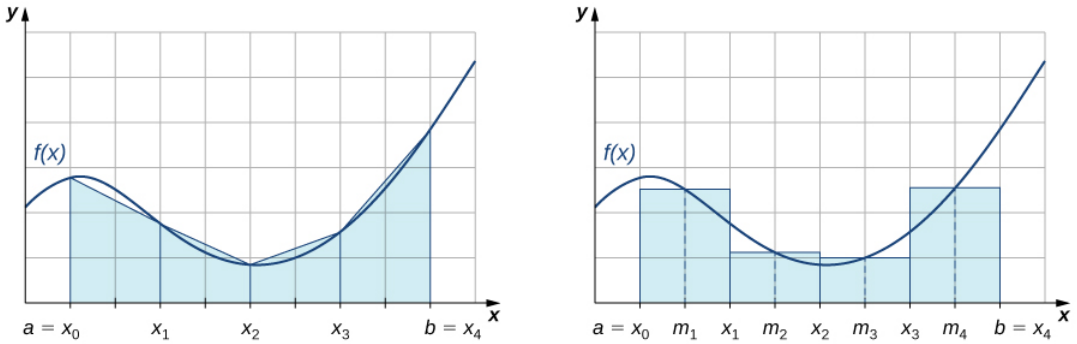



Figure **3.7.3**:The trapezoidal rule tends to be less accurate than the midpoint rule.


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**Example 3.7.3: Using the Trapezoidal Rule**

Use the trapezoidal rule to estimate $\int_0^1 x^2\,dx$ using four subintervals.

**Solution**

The endpoints of the subintervals consist of elements of the set $P = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$ and $\Delta x = \frac{1-0}{4} = \frac{1}{4}$. Thus,

$$\begin{aligned}\int_0^1 x^2 dx &\approx \frac{1}{2}\cdot\frac{1}{4}\left[f(0)+2f\left(\tfrac{1}{4}\right)+2f\left(\tfrac{1}{2}\right)+2f\left(\tfrac{3}{4}\right)+f(1)\right]\\ &= \tfrac{1}{8}\left(0+2\cdot\tfrac{1}{16}+2\cdot\tfrac{1}{4}+2\cdot\tfrac{9}{16}+1\right)\\ &= \frac{11}{32} = 0.34375\end{aligned}$$



**Exercise 3.7.2**

Use the trapezoidal rule with $n = 2$ to estimate $\int_1^2 \frac{1}{x}\,dx$.

**Hint**

Set $\Delta x = \frac{1}{2}$. The endpoints of the subintervals are the elements of the set $P = \left\{1, \frac{3}{2}, 2\right\}$.

**Answer**

$\frac{17}{24} \approx 0.708333$

## Absolute and Relative Error

An important aspect of using these numerical approximation rules consists of calculating the error in using them for estimating the value of a definite integral. We first need to define absolute error and relative error.

**Definition: absolute and relative error**

If $B$ is our estimate of some quantity having an actual value of $A$, then the **absolute error** is given by $|A - B|$.

The **relative error** is the error as a percentage of the actual value and is given by

$$\left|\frac{A-B}{A}\right| \cdot 100\%. \tag{3.7.4}$$

**Example 3.7.4: Calculating Error in the Midpoint Rule**

Calculate the absolute and relative error in the estimate of $\int_0^1 x^2\,dx$ using the midpoint rule, found in Example 3.7.1.

Solution: The calculated value is $\int_0^1 x^2\,dx = \frac{1}{3}$ and our estimate from the example is $M_4 = \frac{21}{64}$. Thus, the absolute error is given by $\left|\frac{1}{3} - \frac{21}{64}\right| = \frac{1}{192} \approx 0.0052$.

The relative error is

$$\frac{1/192}{1/3} = \frac{1}{64} \approx 0.015625 \approx 1.6\%.$$

**Example 3.7.5: Calculating Error in the Trapezoidal Rule**

Calculate the absolute and relative error in the estimate of $\int_0^1 x^2\,dx$ using the trapezoidal rule, found in Example 3.7.3.

Solution: The calculated value is $\int_0^1 x^2\,dx = \frac{1}{3}$ and our estimate from the example is $T_4 = \frac{11}{32}$. Thus, the absolute error is given by $\left|\frac{1}{3} - \frac{11}{32}\right| = \frac{1}{96} \approx 0.0104$.

The relative error is given by

$$\frac{1/96}{1/3} = 0.03125 \approx 3.1\%.$$

**Exercise 3.7.3**

In an earlier checkpoint, we estimated $\int_1^2 \frac{1}{x}\,dx$ to be $\frac{24}{35}$ using $M_2$. The actual value of this integral is $\ln 2$. Using $\frac{24}{35} \approx 0.6857$ and $\ln 2 \approx 0.6931$, calculate the absolute error and the relative error.

**Hint**

Use the previous examples as a guide.

**Answer**

absolute error $\approx 0.0074$, and relative error $\approx 1.1\%$

## Error Bounds on the Midpoint and Trapezoidal Rules

In the two previous examples, we were able to compare our estimate of an integral with the actual value of the integral; however, we do not typically have this luxury. In general, if we are approximating an integral, we are doing so because we cannot compute the exact value of the integral itself easily. Therefore, it is often helpful to be able to determine an upper bound for the error in an approximation of an integral. The following theorem provides error bounds for the midpoint and trapezoidal rules. The theorem is stated without proof.

**Error Bounds for the Midpoint and Trapezoidal Rules**

Let $f(x)$ be a continuous function over $[a, b]$, having a second derivative $f''(x)$ over this interval. If $M$ is the maximum value of $|f''(x)|$ over $[a, b]$, then the upper bounds for the error in using $M_n$ and $T_n$ to estimate $\int_a^b f(x)\,dx$ are

$$\text{Error in } M_n \leq \frac{M(b-a)^3}{24n^2} \tag{3.7.5}$$

and

$$\text{Error in } T_n \leq \frac{M(b-a)^3}{12n^2} \tag{3.7.6}$$

.

We can use these bounds to determine the value of $n$ necessary to guarantee that the error in an estimate is less than a specified value.

### Example 3.7.6: Determining the Number of Intervals to Use

What value of $n$ should be used to guarantee that an estimate of $\int_0^1 e^{x^2}\,dx$ is accurate to within $0.01$ if we use the midpoint rule?

**Solution**

We begin by determining the value of $M$, the maximum value of $|f''(x)|$ over $[0,1]$ for $f(x) = e^{x^2}$. Since $f'(x) = 2xe^{x^2}$, we have

$$f''(x) = 2e^{x^2} + 4x^2e^{x^2}.$$

Thus,

$$|f''(x)| = 2e^{x^2}(1+2x^2) \leq 2 \cdot e \cdot 3 = 6e.$$

From the error-bound Equation 3.7.5, we have

$$\text{Error in } M_n \leq \frac{M(b-a)^3}{24n^2} \leq \frac{6e(1-0)^3}{24n^2} = \frac{6e}{24n^2}.$$

Now we solve the following inequality for $n$:

$$\frac{6e}{24n^2} \leq 0.01.$$

Thus, $n \geq \sqrt{\frac{600e}{24}} \approx 8.24$. Since $n$ must be an integer satisfying this inequality, a choice of $n = 9$ would guarantee that

$$\left|\int_0^1 e^{x^2}\,dx - M_n\right| < 0.01.$$

**Analysis**

We might have been tempted to round $8.24$ down and choose $n = 8$, but this would be incorrect because we must have an integer greater than or equal to $8.24$. We need to keep in mind that the error estimates provide an upper bound only for the error. The actual estimate may, in fact, be a much better approximation than is indicated by the error bound.



**Exercise 3.7.4**

Use Equation 3.7.5 to find an upper bound for the error in using $M_4$ to estimate $\int_0^1 x^2\,dx$.

**Hint**

$f''(x) = 2$, so $M = 2$.

**Answer**

$\frac{1}{192}$

## Simpson's Rule

With the midpoint rule, we estimated areas of regions under curves by using rectangles. In a sense, we approximated the curve with piecewise constant functions. With the trapezoidal rule, we approximated the curve by using piecewise linear functions. What if we were, instead, to approximate a curve using piecewise quadratic functions? With **Simpson's rule,** we do just this. We partition the interval into an even number of subintervals, each of equal width. Over the first pair of subintervals we approximate $\int_{x_0}^{x_2} f(x)\,dx$ with $\int_{x_0}^{x_2} p(x)\,dx$, where $p(x) = Ax^2 + Bx + C$ is the quadratic function passing through $(x_0, f(x_0))$, $(x_1, f(x_1))$, and $(x_2, f(x_2))$ (Figure 3.7.4). Over the next pair of subintervals we approximate $\int_{x_2}^{x_4} f(x)\,dx$ with the integral of another quadratic function passing through $(x_2, f(x_2))$, $(x_3, f(x_3))$, and $(x_4, f(x_4))$. This process is continued with each successive pair of subintervals.

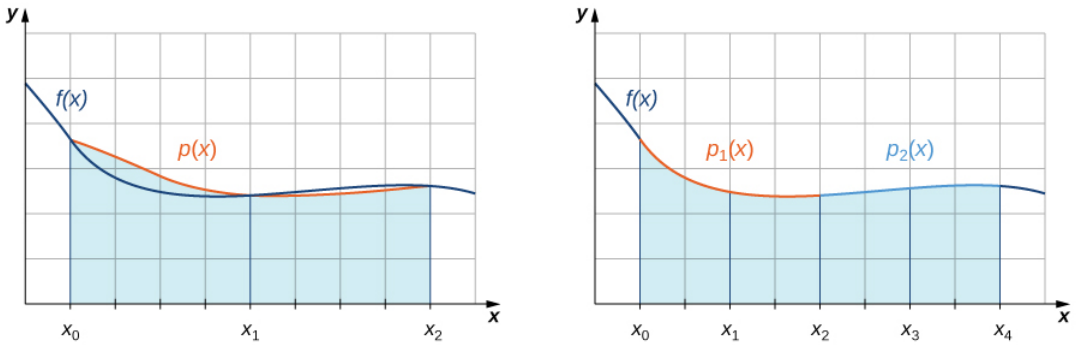



Figure 3.7.4: With Simpson's rule, we approximate a definite integral by integrating a piecewise quadratic function.

To understand the formula that we obtain for Simpson's rule, we begin by deriving a formula for this approximation over the first two subintervals. As we go through the derivation, we need to keep in mind the following relationships:

$$f(x_0) = p(x_0) = Ax_0^2 + Bx_0 + C \tag{3.7.7}$$

$$f(x_1) = p(x_1) = Ax_1^2 + Bx_1 + C \tag{3.7.8}$$

$$f(x_2) = p(x_2) = Ax_2^2 + Bx_2 + C \tag{3.7.9}$$

$x_2 - x_0 = 2\Delta x$, where $\Delta x$ is the length of a subinterval.

$x_2 + x_0 = 2x_1$, since $x_1 = \dfrac{(x_2 + x_0)}{2}$.

Thus,

$$\begin{aligned}
\int_{x_0}^{x_2} f(x)\,dx &\approx \int_{x_0}^{x_2} p(x)\,dx \\
&= \int_{x_0}^{x_2} (Ax^2 + Bx + C)\,dx \\
&= \left(\frac{A}{3}x^3 + \frac{B}{2}x^2 + Cx\right)\Big|_{x_0}^{x_2} && \text{Find the antiderivative.} \\
&= \frac{A}{3}(x_2^3 - x_0^3) + \frac{B}{2}(x_2^2 - x_0^2) + C(x_2 - x_0) && \text{Evaluate the antiderivative.} \\
&= \frac{A}{3}(x_2 - x_0)(x_2^2 + x_2x_0 + x_0^2) \\
&\quad + \frac{B}{2}(x_2 - x_0)(x_2 + x_0) + C(x_2 - x_0) \\
&= \frac{x_2 - x_0}{6} && \text{Factor out } \frac{x_2 - x_0}{6}. \\
&\quad \Big(2A(x_2^2 + x_2x_0 + x_0^2) + 3B(x_2 + x_0) + 6C\Big) \\
&= \frac{\Delta x}{3}\Big((Ax_2^2 + Bx_2 + C) + (Ax_0^2 + Bx_0 + C) && \text{Rearrange the terms. Note:} \\
&\quad + A(x_2^2 + 2x_2x_0 + x_0^2) + 2B(x_2 + x_0) + 4C\Big) && \Delta x = \frac{x_2 - x_0}{2} \\
&= \frac{\Delta x}{3}\Big(f(x_2) + f(x_0) + A(x_2 + x_0)^2 && \text{Factor and substitute:} \\
&\quad + 2B(x_2 + x_0) + 4C\Big) \\
& && f(x_2) = Ax_2^2 + Bx_2 + C \text{ and} \\
& && f(x_0) = Ax_0^2 + Bx_0 + C. \\
&= \frac{\Delta x}{3}\Big(f(x_2) + f(x_0) + A(2x_1)^2 + 2B(2x_1) + 4C\Big) && \text{Substitute } x_2 + x_0 = 2x_1. \\
& && \text{Note: } x_1 = \frac{x_2 + x_0}{2} \text{ is} \\
& && \text{the midpoint.} \\
&= \frac{\Delta x}{3}\Big(f(x_2) + 4f(x_1) + f(x_0)\Big). && \text{Expand and substitute} \\
& && f(x_1) = Ax_1^2 + Bx_1 + C.
\end{aligned}$$

If we approximate $\int_{x_2}^{x_4} f(x)\,dx$ using the same method, we see that we have

$$\int_{x_2}^{x_4} f(x)\,dx \approx \frac{\Delta x}{3}(f(x_4) + 4f(x_3) + f(x_2)).$$

Combining these two approximations, we get

$$\int_{x_0}^{x_4} f(x)\,dx \approx \frac{\Delta x}{3}(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)).$$

The pattern continues as we add pairs of subintervals to our approximation. The general rule may be stated as follows.

**Simpson's Rule**

Assume that $f(x)$ is continuous over $[a,b]$. Let $n$ be a positive even integer and $\Delta x = \frac{b-a}{n}$. Let $[a,b]$ be divided into $n$ subintervals, each of length $\Delta x$, with endpoints at $P = \{x_0, x_1, x_2, \ldots, x_n\}$. Set

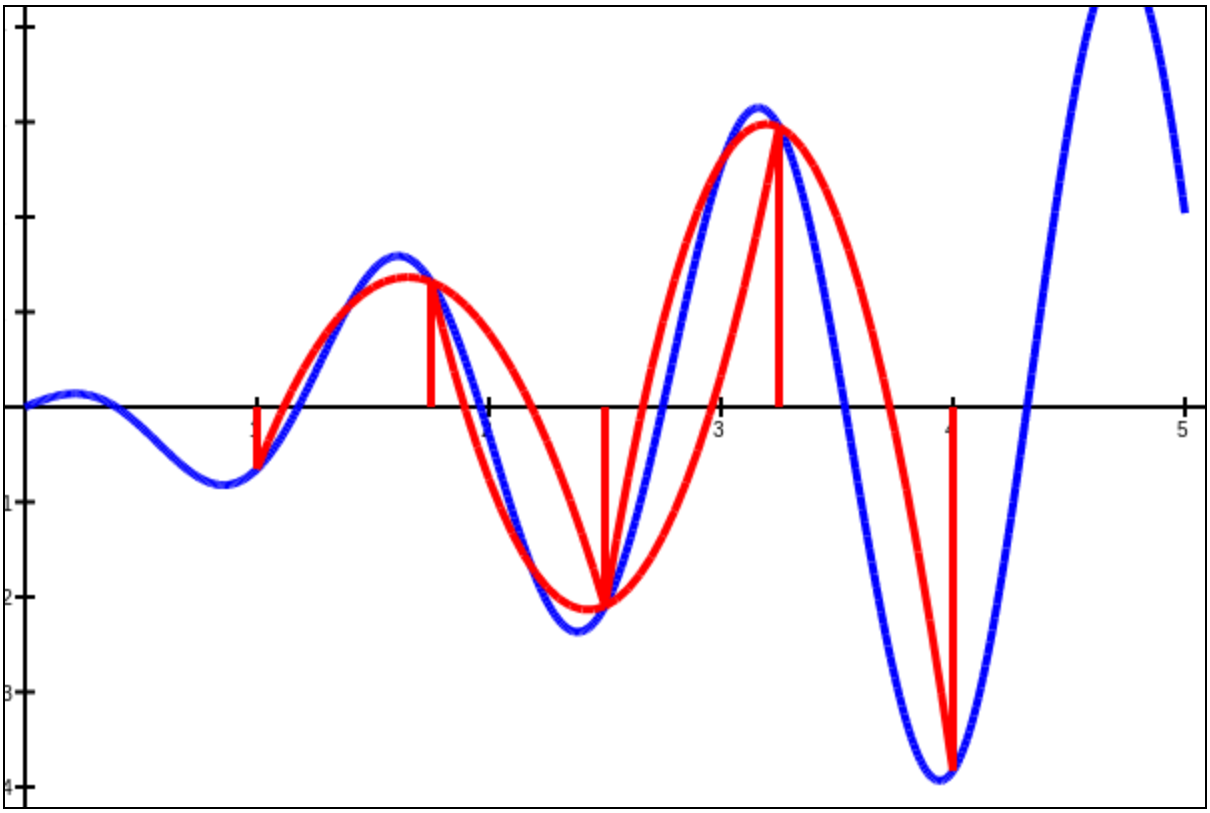
$$S_n = \frac{\Delta x}{3}\Big[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\Big]. \tag{3.7.10}$$

Then,

$$\lim_{n\to+\infty} S_n = \int_a^b f(x)\,dx.$$

## Visualize Simpson's Rule

Move the slider to see the Simpson's rule being used to approximate $\int_1^4 x\cos(4x)dx = -0.1177...$ using the selected number of partitions.




n = 4

Area ≈ (4 - 1)/(3*4)*(-0.6536 + 5.2773 + -4.1954 + 11.7968 + -3.8306) = 2.0986

Just as the trapezoidal rule is the average of the left-hand and right-hand rules for estimating definite integrals, Simpson's rule may be obtained from the midpoint and trapezoidal rules by using a weighted average. It can be shown that $S_{2n} = \left(\frac{2}{3}\right)M_n + \left(\frac{1}{3}\right)T_n$.

It is also possible to put a bound on the error when using Simpson's rule to approximate a definite integral. The bound in the error is given by the following rule:

**Rule: Error Bound for Simpson's Rule**

Let $f(x)$ be a continuous function over $[a,b]$ having a fourth derivative, $f^{(4)}(x)$, over this interval. If $M$ is the maximum value of $|f^{(4)}(x)|$ over $[a,b]$, then the upper bound for the error in using $S_n$ to estimate $\int_a^b f(x)\,dx$ is given by

$$\text{Error in } S_n \le \frac{M(b-a)^5}{180n^4}. \tag{3.7.11}$$

**Example 3.7.7: Applying Simpson's Rule 1**

Use $S_2$ to approximate $\int_0^1 x^3\,dx$. Estimate a bound for the error in $S_2$.

**Solution**

Since $[0,1]$ is divided into two intervals, each subinterval has length $\Delta x = \frac{1-0}{2} = \frac{1}{2}$. The endpoints of these subintervals are $\{0, \frac{1}{2}, 1\}$. If we set $f(x) = x^3$, then

$$S_2 = \frac{1}{3}\cdot\frac{1}{2}(f(0) + 4f(\tfrac{1}{2}) + f(1)) = \frac{1}{6}(0 + 4\cdot\frac{1}{8} + 1) = \frac{1}{4}.$$

Since $f^{(4)}(x) = 0$ and consequently $M = 0$, we see that

$$\text{Error in } S_2 \le \tfrac{0(1)^5}{180\cdot 2^4} = 0.$$

This bound indicates that the value obtained through Simpson's rule is exact. A quick check will verify that, in fact, $\int_0^1 x^3\,dx = \frac{1}{4}$.



**Example 3.7.8: Applying Simpson's Rule 2**

Use $S_6$ to estimate the length of the curve $y = \frac{1}{2}x^2$ over $[1,4]$.

**Solution**

The length of $y = \frac{1}{2}x^2$ over $[1,4]$ is $\int_1^4 \sqrt{1+x^2}\,dx$. If we divide $[1,4]$ into six subintervals, then each subinterval has length $\Delta x = \frac{4-1}{6} = \frac{1}{2}$, and the endpoints of the subintervals are $\{1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, 4\}$. Setting $f(x) = \sqrt{1+x^2}$,

$$S_6 = \frac{1}{3}\cdot\frac{1}{2}(f(1) + 4f(\tfrac{3}{2}) + 2f(2) + 4f(\tfrac{5}{2}) + 2f(3) + 4f(\tfrac{7}{2}) + f(4)).$$

After substituting, we have

$$S_6 = \frac{1}{6}(1.4142 + 4 \cdot 1.80278 + 2 \cdot 2.23607 + 4 \cdot 2.69258 + 2 \cdot 3.16228 + 4 \cdot 3.64005 + 4.12311) \approx 8.14594 \text{ units.} \tag{3.7.12}$$



**Exercise 3.7.5**

Use $S_2$ to estimate $\int_1^2 \frac{1}{x}\,dx$.

**Hint**

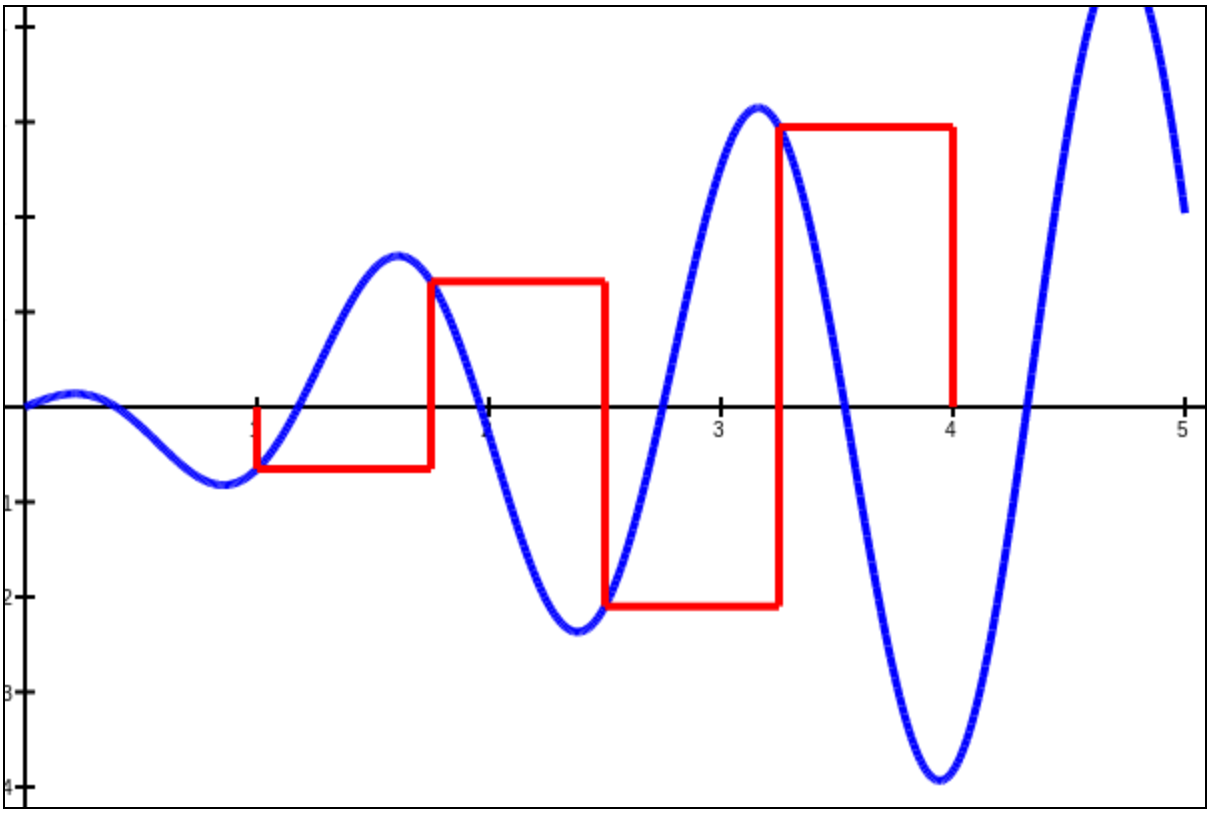
$$S_2 = \frac{1}{3}\Delta x\,(f(x_0) + 4f(x_1) + f(x_2)) \tag{3.7.13}$$

**Answer**

$\frac{25}{36} \approx 0.694444$

## Visualize the Different Integration Approximation Techniques

Move the slider to change the number of partitions and pick an approximation technique to approximate ∫xcos(4x)dx=−0.1177... .




n = 4

◉ Left Endpoint ○ Right Endpoint ○ Midpoint ○ Trapezoid ○ Simpson's

Area ≈ -0.4902 + 0.9895 + -1.5733 + 2.2119 = 1.1379

## Key Concepts

- We can use numerical integration to estimate the values of definite integrals when a closed form of the integral is difficult to find or when an approximate value only of the definite integral is needed.
- The most commonly used techniques for numerical integration are the midpoint rule, trapezoidal rule, and Simpson's rule.
- The midpoint rule approximates the definite integral using rectangular regions whereas the trapezoidal rule approximates the definite integral using trapezoidal approximations.
- Simpson's rule approximates the definite integral by first approximating the original function using piecewise quadratic functions.

## Key Equations

- **Midpoint rule**

$$M_n = \sum_{i=1}^{n} f(m_i)\Delta x$$

- **Trapezoidal rule**

$$T_n = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$$

- **Simpson's rule**

$$S_n = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$$

- **Error bound for midpoint rule**

  Error in $M_n \leq \frac{M(b-a)^3}{24n^2}$, where $M$ is the maximum value of $|f''(x)|$ over $[a,b]$.

- **Error bound for trapezoidal rule**

  Error in $T_n \leq \frac{M(b-a)^3}{12n^2}$, where $M$ is the maximum value of $|f''(x)|$ over $[a,b]$.

- **Error bound for Simpson's rule**

  Error in $S_n \leq \frac{M(b-a)^5}{180n^4}$, where $M$ is the maximum value of $|f^{(4)}(x)|$ over $[a,b]$.

## Glossary

**absolute error**

if $B$ is an estimate of some quantity having an actual value of $A$, then the absolute error is given by $|A-B|$

**midpoint rule**

a rule that uses a Riemann sum of the form $M_n = \sum_{i=1}^{n} f(m_i)\Delta x$, where $m_i$ is the midpoint of the $i^{\text{th}}$ subinterval to approximate $\int_a^b f(x)\,dx$

**numerical integration**

the variety of numerical methods used to estimate the value of a definite integral, including the midpoint rule, trapezoidal rule, and Simpson's rule

**relative error**

error as a percentage of the actual value, given by

$$\text{relative error} = \left|\frac{A-B}{A}\right| \cdot 100\%$$

**Simpson's rule**

a rule that approximates $\int_a^b f(x)\,dx$ using the area under a piecewise quadratic function.

The approximation $S_n$ to $\int_a^b f(x)\,dx$ is given by

$$S_n = \frac{\Delta x}{3}\big(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\big).$$

**trapezoidal rule**

a rule that approximates $\int_a^b f(x)\,dx$ using the area of trapezoids.

The approximation $T_n$ to $\int_a^b f(x)\,dx$ is given by

$$T_n = \frac{\Delta x}{2}\big(f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\big).$$

## Contributors and Attributions


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- Edited by Paul Seeburger (Monroe Community College). Notes added to development of area under a parabola and typos fixed in original text.

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# 3.7E: Exercises for Section 3.7

**In exercises 1 - 5, approximate the following integrals using either the midpoint rule, trapezoidal rule, or Simpson's rule as indicated. (Round answers to three decimal places.)**

1) $\int_1^2 \frac{dx}{x}$; trapezoidal rule; $n = 5$

**Answer**

0.696

2) $\int_0^3 \sqrt{4+x^3}\, dx$; trapezoidal rule; $n = 6$

3) $\int_0^3 \sqrt{4+x^3}\, dx$; Simpson's rule; $n = 6$

**Answer**

9.279

4) $\int_0^{12} x^2\, dx$; midpoint rule; $n = 6$

5) $\int_0^1 \sin^2(\pi x)\, dx$; midpoint rule; $n = 3$

**Answer**

0.500

6) Use the midpoint rule with eight subdivisions to estimate $\int_2^4 x^2\, dx$.

7) Use the trapezoidal rule with four subdivisions to estimate $\int_2^4 x^2\, dx$.

**Answer**

$T_4 = 18.75$

8) Find the exact value of $\int_2^4 x^2\, dx$. Find the error of approximation between the exact value and the value calculated using the trapezoidal rule with four subdivisions. Draw a graph to illustrate.

**Approximate the integral to four decimal places using the indicated rule.**

9) $\int_0^1 \sin^2(\pi x)\, dx$; trapezoidal rule; $n = 6$

**Answer**

0.5000

10) $\int_0^3 \frac{1}{1+x^3}\, dx$; trapezoidal rule; $n = 6$

11) $\int_0^3 \frac{1}{1+x^3}\, dx$; Simpson's rule; $n = 6$

**Answer**

1.1614

12) $\int_0^{0.8} e^{-x^2}\, dx$; trapezoidal rule; $n = 4$

13) $\int_0^{0.8} e^{-x^2}\, dx$; Simpson's rule; $n = 4$

**Answer**

0.6577

14) $\int_0^{0.4} \sin(x^2)\,dx$; trapezoidal rule; $n = 4$

15) $\int_0^{0.4} \sin(x^2)\,dx$; Simpson's rule; $n = 4$

**Answer**

0.0213

16) $\int_{0.1}^{0.5} \frac{\cos x}{x}\,dx$; trapezoidal rule; $n = 4$

17) $\int_{0.1}^{0.5} \frac{\cos x}{x}\,dx$; Simpson's rule; $n = 4$

**Answer**

1.5629

18) Evaluate $\int_0^1 \frac{dx}{1+x^2}$ exactly and show that the result is $\pi/4$. Then, find the approximate value of the integral using the trapezoidal rule with $n = 4$ subdivisions. Use the result to approximate the value of $\pi$.

19) Approximate $\int_2^4 \frac{1}{\ln x}\,dx$ using the midpoint rule with four subdivisions to four decimal places.

**Answer**

1.9133

20) Approximate $\int_2^4 \frac{1}{\ln x}\,dx$ using the trapezoidal rule with eight subdivisions to four decimal places.

21) Use the trapezoidal rule with four subdivisions to estimate $\int_0^{0.8} x^3\,dx$ to four decimal places.

**Answer**

$T(4) = 0.1088$

22) Use the trapezoidal rule with four subdivisions to estimate $\int_0^{0.8} x^3\,dx$. Compare this value with the exact value and find the error estimate.

23) Using Simpson's rule with four subdivisions, find $\int_0^{\pi/2} \cos(x)\,dx$.

**Answer**

$\int_0^{\pi/2} \cos(x)\,dx \approx \quad 1.0$

24) Show that the exact value of $\int_0^1 xe^{-x}\,dx = 1 - \frac{2}{e}$. Find the absolute error if you approximate the integral using the midpoint rule with 16 subdivisions.

25) Given $\int_0^1 xe^{-x}\,dx = 1 - \frac{2}{e}$, use the trapezoidal rule with 16 subdivisions to approximate the integral and find the absolute error.

**Answer**

Approximate error is 0.000325.

26) Find an upper bound for the error in estimating $\int_0^3 (5x+4)\,dx$ using the trapezoidal rule with six steps.

27) Find an upper bound for the error in estimating $\int_4^5 \frac{1}{(x-1)^2}\,dx$ using the trapezoidal rule with seven subdivisions.

**Answer**

$\frac{1}{7938}$

28) Find an upper bound for the error in estimating $\int_0^3 (6x^2-1)\,dx$ using Simpson's rule with $n=10$ steps.

29) Find an upper bound for the error in estimating $\int_2^5 \frac{1}{x-1}\,dx$ using Simpson's rule with $n=10$ steps.

**Answer**

$\frac{81}{25{,}000}$

30) Find an upper bound for the error in estimating $\int_0^{\pi} 2x\cos(x)\,dx$ using Simpson's rule with four steps.

31) Estimate the minimum number of subintervals needed to approximate the integral $\int_1^4 (5x^2+8)\,dx$ with an error magnitude of less than 0.0001 using the trapezoidal rule.

**Answer**

475

32) Determine a value of n such that the trapezoidal rule will approximate $\int_0^1 \sqrt{1+x^2}\,dx$ with an error of no more than 0.01.

33) Estimate the minimum number of subintervals needed to approximate the integral $\int_2^3 (2x^3+4x)\,dx$ with an error of magnitude less than 0.0001 using the trapezoidal rule.

**Answer**

174

34) Estimate the minimum number of subintervals needed to approximate the integral $\int_3^4 \frac{1}{(x-1)^2}\,dx$ with an error magnitude of less than 0.0001 using the trapezoidal rule.

35) Use Simpson's rule with four subdivisions to approximate the area under the probability density function $y=\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ from $x=0$ to $x=0.4$.

**Answer**

0.1554

36) Use Simpson's rule with $n=14$ to approximate (to three decimal places) the area of the region bounded by the graphs of $y=0, x=0$, and $x=\pi/2$.

37) The length of one arch of the curve $y=3\sin(2x)$ is given by $L=\int_0^{\pi/2}\sqrt{1+36\cos^2(2x)}\,dx$. Estimate L using the trapezoidal rule with $n=6$.

**Answer**

6.2807

38) The length of the ellipse $x=a\cos(t), y=b\sin(t), 0\le t\le 2\pi$ is given by $L=4a\int_0^{\pi/2}\sqrt{1-e^2\cos^2(t)}dt$, where e is the eccentricity of the ellipse. Use Simpson's rule with $n=6$ subdivisions to estimate the length of the ellipse when $a=2$ and $e=1/3$.

39) Estimate the area of the surface generated by revolving the curve $y=\cos(2x), 0\le x\le \frac{\pi}{4}$ about the x-axis. Use the trapezoidal rule with six subdivisions.

**Answer**

4.606

40) Estimate the area of the surface generated by revolving the curve $y = 2x^2, 0 \le x \le 3$ about the x-axis. Use Simpson's rule with $n = 6$.

41) The growth rate of a certain tree (in feet) is given by $y = \frac{2}{t+1} + e^{-t^2/2}$, where t is time in years. Estimate the growth of the tree through the end of the second year by using Simpson's rule, using two subintervals. (Round the answer to the nearest hundredth.)

**Answer**

3.41 ft

42) [T] Use a calculator to approximate $\int_0^1 \sin(\pi x)\, dx$ using the midpoint rule with 25 subdivisions. Compute the relative error of approximation.

43) [T] Given $\int_1^5 (3x^2 - 2x)\, dx = 100$, approximate the value of this integral using the midpoint rule with 16 subdivisions and determine the absolute error.

**Answer**

$T_{16} = 100.125$; absolute error = 0.125

44) Given that we know the Fundamental Theorem of Calculus, why would we want to develop numerical methods for definite integrals?

45) The table represents the coordinates $(x, y)$ that give the boundary of a lot. The units of measurement are meters. Use the trapezoidal rule to estimate the number of square meters of land that is in this lot.

| $x$ | $y$ | $x$ | $y$ |
|---|---|---|---|
| 0 | 125 | 600 | 95 |
| 100 | 125 | 700 | 88 |
| 200 | 120 | 800 | 75 |
| 300 | 112 | 900 | 35 |
| 400 | 90 | 1000 | 0 |
| 500 | 90 | | |

**Answer**

about 89,250 $m^2$

46) Choose the correct answer. When Simpson's rule is used to approximate the definite integral, it is necessary that the number of partitions be____

a. an even number

b. odd number

c. either an even or an odd number

d. a multiple of 4

47) The "Simpson" sum is based on the area under a ____.

**Answer**

parabola

48) The error formula for Simpson's rule depends on___.

a. $f(x)$

b. $f'(x)$

c. $f^{(4)}(x)$

d. the number of steps

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# 3.8: Improper Integrals

**Learning Objectives**

- Evaluate an integral over an infinite interval.
- Evaluate an integral over a closed interval with an infinite discontinuity within the interval.
- Use the comparison theorem to determine whether a definite integral is convergent.

Is the area between the graph of $f(x) = \frac{1}{x}$ and the $x$-axis over the interval $[1, +\infty)$ finite or infinite? If this same region is revolved about the $x$-axis, is the volume finite or infinite? Surprisingly, the area of the region described is infinite, but the volume of the solid obtained by revolving this region about the $x$-axis is finite.

In this section, we define integrals over an infinite interval as well as integrals of functions containing a discontinuity on the interval. Integrals of these types are called improper integrals. We examine several techniques for evaluating improper integrals, all of which involve taking limits.

## Integrating over an Infinite Interval

How should we go about defining an integral of the type $\int_a^{+\infty} f(x)\,dx$? We can integrate $\int_a^t f(x)\,dx$ for any value of $t$, so it is reasonable to look at the behavior of this integral as we substitute larger values of $t$. Figure 3.8.1 shows that $\int_a^t f(x)\,dx$ may be interpreted as area for various values of $t$. In other words, we may define an improper integral as a limit, taken as one of the limits of integration increases or decreases without bound.

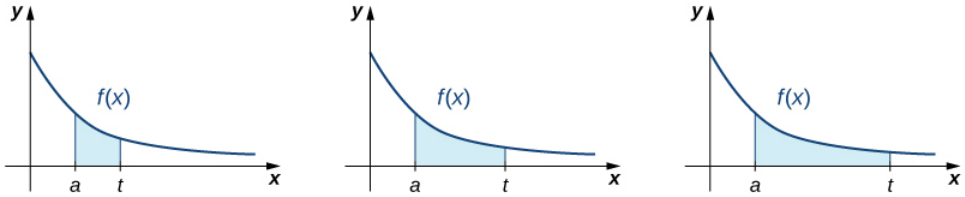



Figure 3.8.1: To integrate a function over an infinite interval, we consider the limit of the integral as the upper limit increases without bound.

**Definition: improper integral**

1. Let $f(x)$ be continuous over an interval of the form $[a, +\infty)$. Then

$$\int_a^{+\infty} f(x)\,dx = \lim_{t \to +\infty} \int_a^t f(x)\,dx, \tag{3.8.1}$$

provided this limit exists.

2. Let $f(x)$ be continuous over an interval of the form $(-\infty, b]$. Then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t \to -\infty} \int_t^b f(x)\,dx, \tag{3.8.2}$$

provided this limit exists.

In each case, if the limit exists, then the *improper integral* is said to converge. If the limit does not exist, then the improper integral is said to diverge.

3. Let $f(x)$ be continuous over $(-\infty, +\infty)$. Then

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^0 f(x)\,dx + \int_0^{+\infty} f(x)\,dx, \tag{3.8.3}$$

provided that $\int_{-\infty}^{0} f(x)\,dx$ and $\int_{0}^{+\infty} f(x)\,dx$ both converge. If either of these two integrals diverge, then $\int_{-\infty}^{+\infty} f(x)\,dx$ diverges. (It can be shown that, in fact, $\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{+\infty} f(x)\,dx$ for any value of a.).

In our first example, we return to the question we posed at the start of this section: Is the area between the graph of $f(x) = \frac{1}{x}$ and the $x$-axis over the interval $[1, +\infty)$ finite or infinite?

### Example 3.8.1: Finding an Area

Determine whether the area between the graph of $f(x) = \dfrac{1}{x}$ and the $x$-axis over the interval $[1, +\infty)$ is finite or infinite.

**Solution**

We first do a quick sketch of the region in question, as shown in Figure 3.8.2.

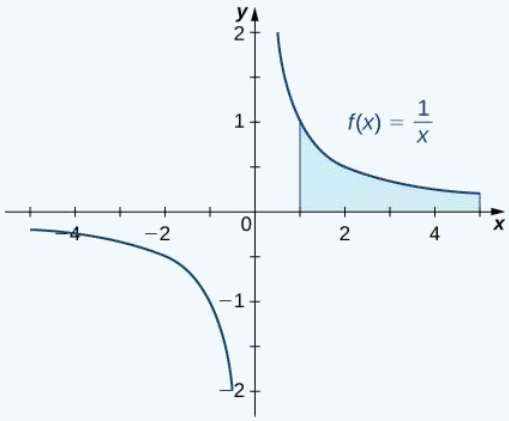



Figure 3.8.2: We can find the area between the curve $f(x) = 1/x$ and the $x$-axis on an infinite interval.

We can see that the area of this region is given by

$$A = \int_{1}^{\infty} \frac{1}{x}\,dx.$$

which can be evaluated using Equation 3.8.1:

$$\begin{aligned} A &= \int_{1}^{\infty} \frac{1}{x}\,dx \\ &= \lim_{t\to+\infty} \int_{1}^{t} \frac{1}{x}\,dx && \text{(Rewrite the improper integral as a limit)} \\ &= \lim_{t\to+\infty} \ln|x|\Big|_{1}^{t} && \text{(Find the antiderivative)} \\ &= \lim_{t\to+\infty} (\ln|t| - \ln 1) && \text{(Evaluate the antiderivative)} \\ &= +\infty. && \text{(Evaluate the limit.)} \end{aligned}$$

Since the improper integral diverges to $+\infty$, the area of the region is infinite.



**Example 3.8.2: Finding a Volume**

Find the volume of the solid obtained by revolving the region bounded by the graph of $f(x) = \frac{1}{x}$ and the $x$-axis over the interval $[1, +\infty)$ about the $x$-axis.

**Solution**

The solid is shown in Figure 3.8.3. Using the disk method, we see that the volume $V$ is

$$V = \pi \int_1^{+\infty} \frac{1}{x^2}\, dx.$$

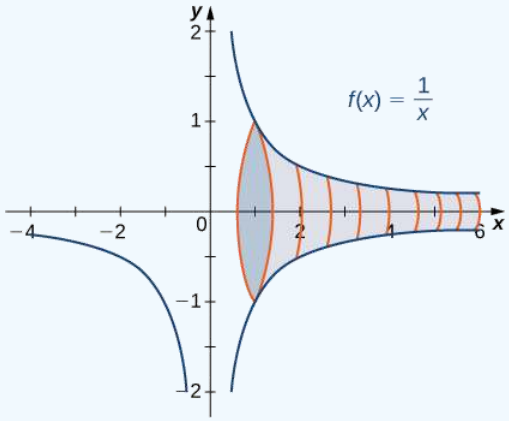



Figure 3.8.3: The solid of revolution can be generated by rotating an infinite area about the $x$-axis.

Then we have

Erroneous nesting of equation structures

The improper integral converges to $\pi$. Therefore, the volume of the solid of revolution is $\pi$.

In conclusion, although the area of the region between the $x$-axis and the graph of $f(x) = 1/x$ over the interval $[1, +\infty)$ is infinite, the volume of the solid generated by revolving this region about the $x$-axis is finite. The solid generated is known as Gabriel's Horn.

Note: Gabriel's horn (also called Torricelli's trumpet) is a geometric figure which has *infinite* surface area, but *finite* volume. The name refers to the tradition identifying the Archangel Gabriel as the angel who blows the horn to announce Judgment Day,

associating the divine, or infinite, with the finite. The properties of this figure were first studied by Italian physicist and mathematician Evangelista Torricelli in the 17th century.

### Example 3.8.3: Traffic Accidents in a City

Suppose that at a busy intersection, traffic accidents occur at an average rate of one every three months. After residents complained, changes were made to the traffic lights at the intersection. It has now been eight months since the changes were made and there have been no accidents. Were the changes effective or is the 8-month interval without an accident a result of chance?



Figure 3.8.4: Modification of work by David McKelvey, Flickr.

Probability theory tells us that if the average time between events is $k$, the *probability* that $X$, the time between events, is between $a$ and $b$ is given by

$$(P(a \le x \le b) = \int_a^b f(x)\,dx$$

where

$$f(x) = \begin{cases} 0, \text{if } x < 0 \\ ke^{-kx}, \text{if } x \ge 0 \end{cases}.$$

Thus, if accidents are occurring at a rate of one every 3 months, then the probability that $X$, the time between accidents, is between $a$ and $b$ is given by

$$P(a \le x \le b) = \int_a^b f(x)\,dx$$

where

$$f(x) = \begin{cases} 0, \text{if } x < 0 \\ 3e^{-3x}, \text{if } x \ge 0 \end{cases}.$$

To answer the question, we must compute $P(X \ge 8) = \int_8^{+\infty} 3e^{-3x}\,dx$ and decide whether it is likely that 8 months could have passed without an accident if there had been no improvement in the traffic situation.

**Solution**

We need to calculate the probability as an improper integral:

Erroneous nesting of equation structures

The value $3.8 \times 10^{-11}$ represents the probability of no accidents in 8 months under the initial conditions. Since this value is very, very small, it is reasonable to conclude the changes were effective.

### Example 3.8.4: Evaluating an Improper Integral over an Infinite Interval

Evaluate $\int_{-\infty}^0 \frac{1}{x^2+4}\,dx$. State whether the improper integral converges or diverges.

**Solution**

Begin by rewriting $\int_{-\infty}^0 \frac{1}{x^2+4}\,dx$ as a limit using Equation 3.8.2 from the definition. Thus,

$$\int_{-\infty}^{0} \frac{1}{x^2+4}\,dx = \lim_{t\to-\infty}\int_{t}^{0} \frac{1}{x^2+4}\,dx \quad \text{Rewrite as a limit.}$$

$$= \lim_{t\to-\infty} \frac{1}{2}\tan^{-1}\frac{x}{2}\,|_{t}^{0} \quad \text{Find the antiderivative.}$$

$$= \lim_{t\to-\infty} \left(\frac{1}{2}\tan^{-1}0 - \frac{1}{2}\tan^{-1}\frac{t}{2}\right) \quad \text{Evaluate the antiderivative.}$$

$$= \frac{\pi}{4}. \quad \text{Evaluate the limit and simplify.}$$

The improper integral converges to $\frac{\pi}{4}$.

## Example 3.8.5: Evaluating an Improper Integral on $(-\infty, +\infty)$

Evaluate $\int_{-\infty}^{+\infty} xe^x\,dx$. State whether the improper integral converges or diverges.

**Solution**

Start by splitting up the integral:

$$\int_{-\infty}^{+\infty} xe^x\,dx = \int_{-\infty}^{0} xe^x\,dx + \int_{0}^{+\infty} xe^x\,dx. \tag{3.8.4}$$

If either $\int_{-\infty}^{0} xe^x\,dx$ or $\int_{0}^{+\infty} xe^x\,dx$ diverges, then $\int_{-\infty}^{+\infty} xe^x\,dx$ diverges. Compute each integral separately. For the first integral,

$\int_{-\infty}^{0} xe^x\,dx = \lim_{t\to-\infty}\int_{t}^{0} xe^x\,dx$ Rewrite as a limit.

$= \lim_{t\to-\infty}(xe^x - e^x)\,|_{t}^{0}$ Use integration by parts to find the antiderivative. (Here $u = x$ and $dv = e^x$.)

$= \lim_{t\to-\infty}(-1 - te^t + e^t)$ Evaluate the antiderivative.

$= -1.$

Evaluate the limit. Note: $\lim_{t\to-\infty} te^t$ is indeterminate of the form $0\cdot\infty$.Thus, $\lim_{t\to-\infty} te^t = \lim_{t\to-\infty} \frac{t}{e^{-t}} = \lim_{t\to-\infty} \frac{-1}{e^{-t}} = \lim_{t\to-\infty} -e^t = 0$ by L'Hôpital's Rule.

The first improper integral converges. For the second integral,

$\int_{0}^{+\infty} xe^x\,dx = \lim_{t\to+\infty}\int_{0}^{t} xe^x\,dx$ Rewrite as a limit.

$= \lim_{t\to+\infty}(xe^x - e^x)\,|_{0}^{t}$ Find the antiderivative.

$= \lim_{t\to+\infty}(te^t - e^t + 1)$ Evaluate the antiderivative.

$= \lim_{t\to+\infty}((t-1)e^t + 1)$ Rewrite. ($te^t - e^t$ is indeterminate.)

$= +\infty$. Evaluate the limit.

Thus, $\int_{0}^{+\infty} xe^x\,dx$ diverges. Since this integral diverges, $\int_{-\infty}^{+\infty} xe^x\,dx$ diverges as well.

## Exercise 3.8.1

Evaluate $\int_{-3}^{+\infty} e^{-x}\,dx$. State whether the improper integral converges or diverges.

**Hint**

$$\int_{-3}^{+\infty} e^{-x}\,dx = \lim_{t\to+\infty}\int_{-3}^{t} e^{-x}\,dx \tag{3.8.5}$$

**Answer**

It converges to $e^3$.

## Integrating a Discontinuous Integrand

Now let's examine integrals of functions containing an infinite discontinuity in the interval over which the integration occurs. Consider an integral of the form $\int_a^b f(x)\,dx$, where $f(x)$ is continuous over $[a,b)$ and discontinuous at $b$. Since the function $f(x)$ is continuous over $[a,t]$ for all values of $t$ satisfying $a \le t < b$, the integral $\int_a^t f(x)\,dx$ is defined for all such values of $t$. Thus, it makes sense to consider the values of $\int_a^t f(x)\,dx$ as $t$ approaches $b$ for $a \le t < b$. That is, we define $\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx$, provided this limit exists. Figure 3.8.5 illustrates $\int_a^t f(x)\,dx$ as areas of regions for values of $t$ approaching $b$.

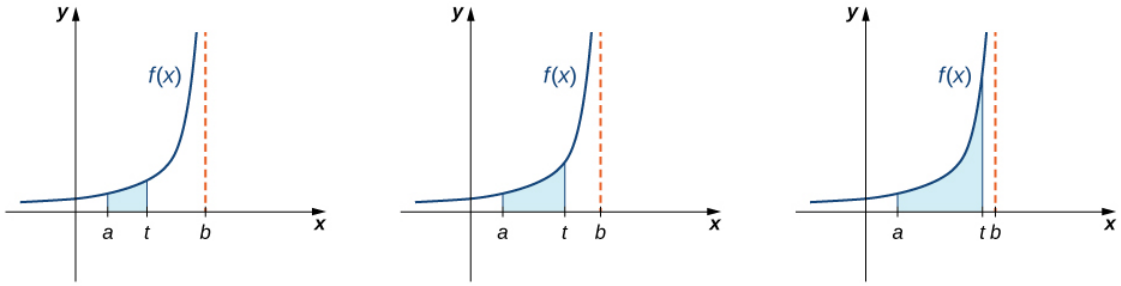



Figure 3.8.5: As t approaches b from the left, the value of the area from a to t approaches the area from a to b.

We use a similar approach to define $\int_a^b f(x)\,dx$, where $f(x)$ is continuous over $(a,b]$ and discontinuous at $a$. We now proceed with a formal definition.

**Definition: Converging and Diverging Improper Integral**

1. Let $f(x)$ be continuous over $[a,b)$. Then,

$$\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx. \tag{3.8.6}$$

2. Let $f(x)$ be continuous over $(a,b]$. Then,

$$\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx. \tag{3.8.7}$$

In each case, if the limit exists, then the improper integral is said to converge. If the limit does not exist, then the improper integral is said to diverge.

3. If $f(x)$ is continuous over $[a,b]$ except at a point $c$ in $(a,b)$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx, \tag{3.8.8}$$

provided both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ converge. If either of these integrals diverges, then $\int_a^b f(x)\,dx$ diverges.

The following examples demonstrate the application of this definition.

**Example 3.8.6: Integrating a Discontinuous Integrand**

Evaluate $\int_0^4 \frac{1}{\sqrt{4-x}}\,dx$, if possible. State whether the integral converges or diverges.

**Solution**

The function $f(x) = \dfrac{1}{\sqrt{4-x}}$ is continuous over $[0, 4)$ and discontinuous at 4. Using Equation 3.8.6 from the definition, rewrite $\displaystyle\int_0^4 \frac{1}{\sqrt{4-x}}\,dx$ as a limit:

Erroneous nesting of equation structures

The improper integral converges.

## Example 3.8.7: Integrating a Discontinuous Integrand

Evaluate $\displaystyle\int_0^2 x \ln x\,dx$. State whether the integral converges or diverges.

**Solution**

Since $f(x) = x \ln x$ is continuous over $(0, 2]$ and is discontinuous at zero, we can rewrite the integral in limit form using Equation 3.8.7:

Erroneous nesting of equation structures

Therefore

$\lim_{t\to 0^+} t^2 \ln t$ **is indeterminate.**

To evaluate it, rewrite as a quotient and apply L'Hôpital's rule.

The improper integral converges.

## Example 3.8.8: Integrating a Discontinuous Integrand

Evaluate $\displaystyle\int_{-1}^1 \frac{1}{x^3}\,dx$. State whether the improper integral converges or diverges.

**Solution**

Since $f(x) = 1/x^3$ is discontinuous at zero, using Equation 3.8.8, we can write

$$\int_{-1}^1 \frac{1}{x^3}\,dx = \int_{-1}^0 \frac{1}{x^3}\,dx + \int_0^1 \frac{1}{x^3}\,dx.$$

If either of the two integrals diverges, then the original integral diverges. Begin with $\displaystyle\int_{-1}^0 \frac{1}{x^3}\,dx$:

$$\begin{aligned}
\int_{-1}^0 \frac{1}{x^3}\,dx &= \lim_{t\to 0^-} \int_{-1}^t \frac{1}{x^3}\,dx \text{ Rewrite as a limit.}\\
&= \lim_{t\to 0^-} \left(-\tfrac{1}{2x^2}\right)\Big|_{-1}^t \text{ Find the antiderivative.}\\
&= \lim_{t\to 0^-} \left(-\tfrac{1}{2t^2} + \tfrac{1}{2}\right) \text{ Evaluate the antiderivative.}\\
&= +\infty. \text{ Evaluate the limit.}
\end{aligned}$$

Therefore, $\displaystyle\int_{-1}^0 \frac{1}{x^3}\,dx$ diverges. Since $\displaystyle\int_{-1}^0 \frac{1}{x^3}\,dx$ diverges, $\displaystyle\int_{-1}^1 \frac{1}{x^3}\,dx$ diverges.



**Exercise 3.8.2**

Evaluate $\int_0^2 \frac{1}{x}\,dx$. State whether the integral converges or diverges.

**Hint**

Write $\int_0^2 \frac{1}{x}\,dx$ in limit form using Equation 3.8.7.

**Answer**

$+\infty$, It diverges.

## A Comparison Theorem

It is not always easy or even possible to evaluate an improper integral directly; however, by comparing it with another carefully chosen integral, it may be possible to determine its convergence or divergence. To see this, consider two continuous functions $f(x)$ and $g(x)$ satisfying $0 \le f(x) \le g(x)$ for $x \ge a$ (Figure 3.8.6). In this case, we may view integrals of these functions over intervals of the form $[a, t]$ as areas, so we have the relationship

$$0 \le \int_a^t f(x)\,dx \le \int_a^t g(x)\,dx \tag{3.8.9}$$

for $t \ge a$.

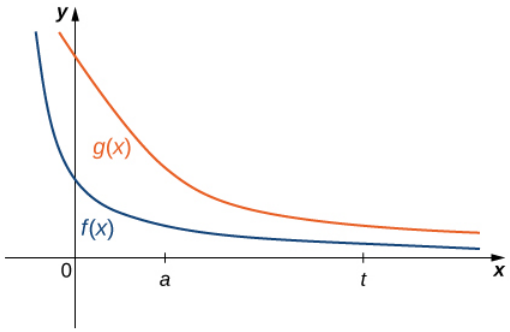



Figure 3.8.6: If $0 \le f(x) \le g(x)$ for $x \ge a$, then for $t \ge a$, $\int_a^t f(x)\,dx \le \int_a^t g(x)\,dx$.

Thus, if

$$\int_a^{+\infty} f(x)\,dx = \lim_{t\to+\infty}\int_a^t f(x)\,dx = +\infty, \tag{3.8.10}$$

then

$$\int_a^{+\infty} g(x)\,dx = \lim_{t\to+\infty}\int_a^t g(x)\,dx = +\infty \tag{3.8.11}$$

as well. That is, if the area of the region between the graph of $f(x)$ and the $x$-axis over $[a,+\infty)$ is infinite, then the area of the region between the graph of $g(x)$ and the $x$-axis over $[a,+\infty)$ is infinite too.

On the other hand, if

$$\int_a^{+\infty} g(x)\,dx = \lim_{t\to+\infty}\int_a^t g(x)\,dx = L \tag{3.8.12}$$

for some real number $L$, then

$$\int_a^{+\infty} f(x)\,dx = \lim_{t\to+\infty}\int_a^t f(x)\,dx \tag{3.8.13}$$

must converge to some value less than or equal to $L$, since $\int_a^t f(x)\,dx$ increases as $t$ increases and $\int_a^t f(x)\,dx \le L$ for all $t \ge a$.

If the area of the region between the graph of $g(x)$ and the $x$-axis over $[a,+\infty)$ is finite, then the area of the region between the graph of $f(x)$ and the $x$-axis over $[a,+\infty)$ is also finite.

These conclusions are summarized in the following theorem.

### A Comparison Theorem

Let $f(x)$ and $g(x)$ be continuous over $[a,+\infty)$. Assume that $0 \le f(x) \le g(x)$ for $x \ge a$.

i. If

$$\int_a^{+\infty} f(x)\,dx = \lim_{t\to+\infty}\int_a^t f(x)\,dx = +\infty, \tag{3.8.14}$$

then

$$\int_a^{+\infty} g(x)\,dx = \lim_{t\to+\infty}\int_a^t g(x)\,dx = +\infty. \tag{3.8.15}$$

ii. If

$$\int_a^{+\infty} g(x)\,dx = \lim_{t\to+\infty}\int_a^t g(x)\,dx = L, \tag{3.8.16}$$

where $L$ is a real number, then

$$\int_a^{+\infty} f(x)\,dx = \lim_{t\to+\infty}\int_a^t f(x)\,dx = M \tag{3.8.17}$$

for some real number $M \le L$.

### Example 3.8.9: Applying the Comparison Theorem

Use a comparison to show that

$$\int_1^{+\infty} \frac{1}{xe^x}\,dx$$

converges.

**Solution**

We can see that

$$0 \le \frac{1}{xe^x} \le \frac{1}{e^x} = e^{-x}, \tag{3.8.18}$$

so if $\int_1^{+\infty} e^{-x}\,dx$ converges, then so does $\int_1^{+\infty} \frac{1}{xe^x}\,dx$. To evaluate $\int_1^{+\infty} e^{-x}\,dx$, first rewrite it as a limit:

$$\begin{aligned}\int_1^{+\infty} e^{-x}\,dx &= \lim_{t\to+\infty}\int_1^t e^{-x}\,dx \\ &= \lim_{t\to+\infty}(-e^{-x})\,|_1^t \\ &= \lim_{t\to+\infty}(-e^{-t}+e^{-1}) \\ &= e^{-1}.\end{aligned}$$

Since $\int_1^{+\infty} e^{-x}\,dx$ converges, so does $\int_1^{+\infty} \frac{1}{xe^x}\,dx$.

**Example 3.8.10: Applying the Comparison Theorem**

Use the comparison theorem to show that $\int_1^{+\infty} \frac{1}{x^p}\,dx$ diverges for all $p < 1$.

**Solution**

For $p < 1, 1/x \le 1/(x^p)$ over $[1, +\infty)$. In Example 3.8.1, we showed that $\int_1^{+\infty} \frac{1}{x}\,dx = +\infty$. Therefore, $\int_1^{+\infty} \frac{1}{x^p}\,dx$ diverges for all $p < 1$.

**Exercise 3.8.3**

Use a comparison to show that $\int_e^{+\infty} \frac{\ln x}{x}\,dx$ diverges.

**Hint**

$\frac{1}{x} \le \frac{\ln x}{x}$ on $[e, +\infty)$

**Answer**

Since $\int_e^{+\infty} \frac{1}{x}\,dx = +\infty$, $\int_e^{+\infty} \frac{\ln x}{x}\,dx$ diverges.

## Laplace Transforms

In the last few chapters, we have looked at several ways to use integration for solving real-world problems. For this next project, we are going to explore a more advanced application of integration: integral transforms. Specifically, we describe the Laplace transform and some of its properties. The **Laplace transform** is used in engineering and physics to simplify the computations needed to solve some problems. It takes functions expressed in terms of time and *transforms* them to functions expressed in terms of frequency. It turns out that, in many cases, the computations needed to solve problems in the frequency domain are much simpler than those required in the time domain.

The Laplace transform is defined in terms of an integral as

$$Lf(t) = F(s) = \int_0^{\infty} e^{-st} f(t)dt. \tag{3.8.19}$$

Note that the input to a Laplace transform is a function of time, $f(t)$, and the output is a function of frequency, $F(s)$. Although many real-world examples require the use of complex numbers (involving the imaginary number $i = \sqrt{-1}$), in this project we limit ourselves to functions of real numbers.

Let's start with a simple example. Here we calculate the Laplace transform of $f(t) = t$. We have

$$Lt = \int_0^\infty te^{-st}dt. \tag{3.8.20}$$

This is an improper integral, so we express it in terms of a limit, which gives

$$Lt = \int_0^\infty te^{-st}dt = \lim_{z\to\infty}\int_0^z te^{-st}dt. \tag{3.8.21}$$

Now we use integration by parts to evaluate the integral. Note that we are integrating with respect to t, so we treat the variable s as a constant. We have

$$u = t\ du = dt\ dv = e^{-st}dt\ v = -\tfrac{1}{s}e^{-st}.$$

Then we obtain

$$\begin{aligned}
\lim_{z\to\infty}\int_0^z te^{-st}dt &= \lim_{z\to\infty}[[-\frac{t}{s}e^{-st}]\,|_0^z + \frac{1}{s}\int_0^z e^{-st}dt] \\
&= \lim_{z\to\infty}[[-\frac{z}{s}e^{-sz} + \frac{0}{s}e^{-0s}] + \frac{1}{s}\int_0^z e^{-st}dt] \\
&= \lim_{z\to\infty}[[-\frac{z}{s}e^{-sz} + 0] - \frac{1}{s}[\frac{e^{-st}}{s}]\,|_0^z] \\
&= \lim_{z\to\infty}[[-\frac{z}{s}e^{-sz}] - \frac{1}{s^2}[e^{-sz} - 1]] \\
&= \lim_{z\to\infty}[-\frac{z}{se^{sz}}] - \lim_{z\to\infty}[\frac{1}{s^2e^{sz}}] + \lim_{z\to\infty}\frac{1}{s^2} \\
&= 0 - 0 + \frac{1}{s^2} \\
&= \frac{1}{s^2}.
\end{aligned}$$

1. Calculate the Laplace transform of $f(t) = 1$.
2. Calculate the Laplace transform of $f(t) = e^{-3t}$.
3. Calculate the Laplace transform of $f(t) = t^2$. (Note, you will have to integrate by parts twice.)

Laplace transforms are often used to solve differential equations. Differential equations are not covered in detail until later in this book; but, for now, let's look at the relationship between the Laplace transform of a function and the Laplace transform of its derivative.

Let's start with the definition of the Laplace transform. We have

$$Lf(t) = \int_0^\infty e^{-st}f(t)dt = \lim_{z\to\infty}\int_0^z e^{-st}f(t)dt. \tag{3.8.22}$$

Use integration by parts to evaluate $\lim_{z\to\infty}\int_0^z e^{-st}f(t)dt$. (Let $u = f(t)$ and $dv = e^{-st}dt$.)

After integrating by parts and evaluating the limit, you should see that

$$Lf(t) = \frac{f(0)}{s} + \frac{1}{s}[Lf'(t)]. \tag{3.8.23}$$

Then,

$$Lf'(t) = sLf(t) - f(0). \tag{3.8.24}$$

Thus, differentiation in the time domain simplifies to multiplication by s in the frequency domain.

The final thing we look at in this project is how the Laplace transforms of $f(t)$ and its antiderivative are related. Let $g(t) = \int_0^t f(u)du$. Then,

$$Lg(t) = \int_0^\infty e^{-st}g(t)dt = \lim_{z\to\infty}\int_0^z e^{-st}g(t)dt. \tag{3.8.25}$$

Use integration by parts to evaluate $\lim_{z\to\infty}\int_0^z e^{-st}g(t)dt$. (Let $u=g(t)$ and $dv=e^{-st}dt$. Note, by the way that we have defined $g(t), du=f(t)dt$.)

As you might expect, you should see that

$$Lg(t)=\frac{1}{s}\cdot Lf(t). \tag{3.8.26}$$

Integration in the time domain simplifies to division by $s$ in the frequency domain.

## Key Concepts

- Integrals of functions over infinite intervals are defined in terms of limits.
- Integrals of functions over an interval for which the function has a discontinuity at an endpoint may be defined in terms of limits.
- The convergence or divergence of an improper integral may be determined by comparing it with the value of an improper integral for which the convergence or divergence is known.

## Key Equations

- **Improper integrals**

$$\int_a^{+\infty} f(x)\,dx=\lim_{t\to+\infty}\int_a^t f(x)\,dx$$
$$\int_{-\infty}^b f(x)\,dx=\lim_{t\to-\infty}\int_t^b f(x)\,dx$$
$$\int_{-\infty}^{+\infty} f(x)\,dx=\int_{-\infty}^0 f(x)\,dx+\int_0^{+\infty} f(x)\,dx$$

## Glossary

**improper integral**

an integral over an infinite interval or an integral of a function containing an infinite discontinuity on the interval; an improper integral is defined in terms of a limit. The improper integral converges if this limit is a finite real number; otherwise, the improper integral diverges

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# 3.8E: Exercises for Section 3.8

**In exercises 1 - 8, evaluate the following integrals. If the integral is not convergent, answer "It diverges."**

1) $\int_2^4 \frac{dx}{(x-3)^2}$

**Answer**

It diverges.

2) $\int_0^\infty \frac{1}{4+x^2}\,dx$

3) $\int_0^2 \frac{1}{\sqrt{4-x^2}}\,dx$

**Answer**

Converges to $\frac{\pi}{2}$

4) $\int_1^\infty \frac{1}{x\ln x}\,dx$

5) $\int_1^\infty xe^{-x}\,dx$

**Answer**

Converges to $\frac{2}{e}$

6) $\int_{-\infty}^\infty \frac{x}{x^2+1}\,dx$

7) Without integrating, determine whether the integral $\int_1^\infty \frac{1}{\sqrt{x^3+1}}\,dx$ converges or diverges by comparing the function $f(x) = \frac{1}{\sqrt{x^3+1}}$ with $g(x) = \frac{1}{\sqrt{x^3}}$.

**Answer**

It converges.

8) Without integrating, determine whether the integral $\int_1^\infty \frac{1}{\sqrt{x+1}}\,dx$ converges or diverges.

**In exercises 9 - 25, determine whether the improper integrals converge or diverge. If possible, determine the value of the integrals that converge.**

9) $\int_0^\infty e^{-x}\cos x\,dx$

**Answer**

Converges to $\frac{1}{2}$.

10) $\int_1^\infty \frac{\ln x}{x}\,dx$

11) $\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx$

**Answer**

Converges to $-4$.

12) $\int_0^1 \ln x\,dx$

13) $\int_{-\infty}^\infty \frac{1}{x^2+1}\,dx$

**Answer**

Converges to $\pi$.

14) $\int_1^5 \frac{dx}{\sqrt{x-1}}$

15) $\int_{-2}^2 \frac{dx}{(1+x)^2}$

**Answer**

It diverges.

16) $\int_0^\infty e^{-x}\,dx$

17) $\int_0^\infty \sin x\,dx$

**Answer**

It diverges.

18) $\int_{-\infty}^\infty \frac{e^x}{1+e^{2x}}\,dx$

19) $\int_0^1 \frac{dx}{\sqrt[3]{x}}$

**Answer**

Converges to $1.5$.

20) $\int_0^2 \frac{dx}{x^3}$

21) $\int_{-1}^2 \frac{dx}{x^3}$

**Answer**

It diverges.

22) $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

23) $\int_0^3 \frac{1}{x-1}\,dx$

**Answer**

It diverges.

24) $\int_1^\infty \frac{5}{x^3}\,dx$

25) $\int_3^5 \frac{5}{(x-4)^2}\,dx$

**Answer**

It diverges.

**In exercises 26 and 27, determine the convergence of each of the following integrals by comparison with the given integral. If the integral converges, find the number to which it converges.**

26) $\int_1^\infty \frac{dx}{x^2+4x}$; compare with $\int_1^\infty \frac{dx}{x^2}$.

27) $\int_1^\infty \frac{dx}{\sqrt{x}+1}$; compare with $\int_1^\infty \frac{dx}{2\sqrt{x}}$.

**Answer**

Both integrals diverge.

**In exercises 28 - 38, evaluate the integrals. If the integral diverges, answer "It diverges."**

28) $\int_1^{\infty} \frac{dx}{x^e}$

29) $\int_0^1 \frac{dx}{x^{\pi}}$

**Answer**

It diverges.

30) $\int_0^1 \frac{dx}{\sqrt{1-x}}$

31) $\int_0^1 \frac{dx}{1-x}$

**Answer**

It diverges.

32) $\int_{-\infty}^0 \frac{dx}{x^2+1}$

33) $\int_{-1}^1 \frac{dx}{\sqrt{1-x^2}}$

**Answer**

Converges to $\pi$.

34) $\int_0^1 \frac{\ln x}{x}\, dx$

35) $\int_0^e \ln(x)\, dx$

**Answer**

Converges to **0**.

36) $\int_0^{\infty} xe^{-x}\, dx$

37) $\int_{-\infty}^{\infty} \frac{x}{(x^2+1)^2}\, dx$

**Answer**

Converges to **0**.

38) $\int_0^{\infty} e^{-x}\, dx$

**In exercises 39 - 44, evaluate the improper integrals. Each of these integrals has an infinite discontinuity either at an endpoint or at an interior point of the interval.**

39) $\int_0^9 \frac{dx}{\sqrt{9-x}}$

**Answer**

Converges to **6**.

40) $\int_{-27}^1 \frac{dx}{x^{2/3}}$

41) $\int_0^3 \frac{dx}{\sqrt{9-x^2}}$

**Answer**

Converges to $\frac{\pi}{2}$.

42) $\int_6^{24} \frac{dt}{t\sqrt{t^2-36}}$

43) $\int_0^4 x\ln(4x)\,dx$

**Answer**

Converges to $8\ln(16)-4$.

44) $\int_0^3 \frac{x}{\sqrt{9-x^2}}\,dx$

45) Evaluate $\int_{.5}^t \frac{dx}{\sqrt{1-x^2}}$. (Be careful!) (Express your answer using three decimal places.)

**Answer**

Converges to about $1.047$.

46) Evaluate $\int_1^4 \frac{dx}{\sqrt{x^2-1}}$. (Express the answer in exact form.)

47) Evaluate $\int_2^\infty \frac{dx}{(x^2-1)^{3/2}}$.

**Answer**

Converges to $-1+\frac{2}{\sqrt{3}}$.

48) Find the area of the region in the first quadrant between the curve $y=e^{-6x}$ and the $x$-axis.

49) Find the area of the region bounded by the curve $y=\frac{7}{x^2}$, the $x$-axis, and on the left by $x=1$.

**Answer**

$A=7.0$ units.$^2$

50) Find the area under the curve $y=\frac{1}{(x+1)^{3/2}}$, bounded on the left by $x=3$.

51) Find the area under $y=\frac{5}{1+x^2}$ in the first quadrant.

**Answer**

$A=\frac{5\pi}{2}$ units.$^2$

52) Find the volume of the solid generated by revolving about the $x$-axis the region under the curve $y=\frac{3}{x}$ from $x=1$ to $x=\infty$.

53) Find the volume of the solid generated by revolving about the $y$-axis the region under the curve $y=6e^{-2x}$ in the first quadrant.

**Answer**

$V=3\pi$ units$^3$

54) Find the volume of the solid generated by revolving about the $x$-axis the area under the curve $y=3e^{-x}$ in the first quadrant.

The **Laplace transform** of a continuous function over the interval $[0,\infty)$ is defined by $F(s)=\int_0^\infty e^{-sx}f(x)\,dx$ (see the Student Project). This definition is used to solve some important initial-value problems in differential equations, as discussed later. The

domain of $F$ is the set of all real numbers s such that the improper integral converges. Find the Laplace transform $F$ of each of the following functions and give the domain of $F$.

55) $f(x) = 1$

**Answer**

$\frac{1}{s}, \quad s > 0$

56) $f(x) = x$

57) $f(x) = \cos(2x)$

**Answer**

$\frac{s}{s^2 + 4}, \quad s > 0$

58) $f(x) = e^{ax}$

59) Use the formula for arc length to show that the circumference of the circle $x^2 + y^2 = 1$ is $2\pi$.

**Answer**

Answers will vary.

A function is a **probability density function** if it satisfies the following definition: $\int_{-\infty}^{\infty} f(t)\,dt = 1$. The probability that a random variable $x$ lies between a and b is given by $P(a \le x \le b) = \int_a^b f(t)\,dt$.

60) Show that $f(x) = \begin{cases} 0, & \text{if } x < 0 \\ 7e^{-7x}, & \text{if } x \ge 0 \end{cases}$ is a probability density function.

61) Find the probability that $x$ is between $0$ and $0.3$. (Use the function defined in the preceding problem.) Use four-place decimal accuracy.

**Answer**

0.8775

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# 3.9: Chapter 3 Review Exercises

**In exercises 1 - 4, determine whether the statement is true or false. Justify your answer with a proof or a counterexample.**

1) $\int e^x \sin(x)\,dx$ cannot be integrated by parts.

2) $\int \frac{1}{x^4+1}\,dx$ cannot be integrated using partial fractions.

**Answer**

False

3) In numerical integration, increasing the number of points decreases the error.

4) Integration by parts can always yield the integral.

**Answer**

False

**In exercises 5 - 10, evaluate the integral using the specified method.**

5) $\int x^2 \sin(4x)\,dx$, using integration by parts

6) $\int \frac{1}{x^2\sqrt{x^2+16}}\,dx$, using trigonometric substitution

**Answer**

$$\int \frac{1}{x^2\sqrt{x^2+16}}\,dx = -\frac{\sqrt{x^2+16}}{16x} + C$$

7) $\int \sqrt{x}\ln x\,dx$, using integration by parts

8) $\int \frac{3x}{x^3+2x^2-5x-6}\,dx$, using partial fractions

**Answer**

$$\int \frac{3x}{x^3+2x^2-5x-6}\,dx = \frac{1}{10}\left(4\ln|2-x| + 5\ln|x+1| - 9\ln|x+3|\right) + C$$

9) $\int \frac{x^5}{(4x^2+4)^{5/2}}\,dx$, using trigonometric substitution

10) $\int \frac{\sqrt{4-\sin^2(x)}}{\sin^2(x)}\cos(x)\,dx$, using a table of integrals or a CAS

**Answer**

$$\int \frac{\sqrt{4-\sin^2(x)}}{\sin^2(x)}\cos(x)\,dx = -\frac{\sqrt{4-\sin^2(x)}}{\sin(x)} - \frac{x}{2} + C$$

**In exercises 11 - 15, integrate using whatever method you choose.**

11) $\int \sin^2 x \cos^2 x\,dx$

12) $\int x^3\sqrt{x^2+2}\,dx$

**Answer**

$$\int x^3\sqrt{x^2+2}\,dx = \frac{1}{15}(x^2+2)^{3/2}(3x^2-4) + C$$

13) $\int \frac{3x^2+1}{x^4-2x^3-x^2+2x}\,dx$

14) $\int \frac{1}{x^4+4}\,dx$

**Answer**

$$\int \frac{1}{x^4+4}\,dx = \frac{1}{16}\ln\left(\frac{x^2+2x+2}{x^2-2x+2}\right) - \frac{1}{8}\tan^{-1}(1-x) + \frac{1}{8}\tan^{-1}(x+1) + C$$

15) $\int \frac{\sqrt{3+16x^4}}{x^4}\,dx$

**In exercises 16 - 18, approximate the integrals using the midpoint rule, trapezoidal rule, and Simpson's rule using four subintervals, rounding to three decimals.**

16) [T] $\int_1^2 \sqrt{x^5+2}\,dx$

**Answer**

$M_4 = 3.312,$
$T_4 = 3.354,$
$S_4 = 3.326$

17) [T] $\int_0^{\sqrt{\pi}} e^{-\sin(x^2)}\,dx$

18) [T] $\int_1^4 \frac{\ln(1/x)}{x}\,dx$

**Answer**

$M_4 = -0.982,$
$T_4 = -0.917,$
$S_4 = -0.952$

**In exercises 19 - 20, evaluate the integrals, if possible.**

19) $\int_1^\infty \frac{1}{x^n}\,dx$, for what values of $n$ does this integral converge or diverge?

20) $\int_1^\infty \frac{e^{-x}}{x}\,dx$

**Answer**

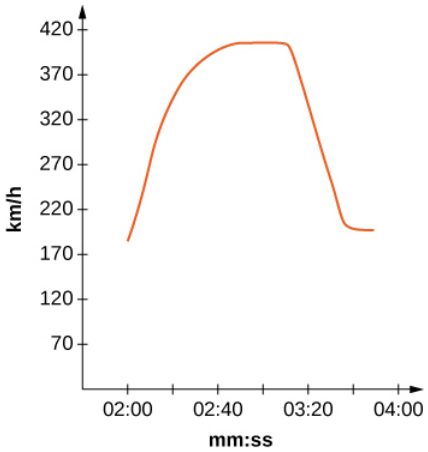
approximately 0.2194

**In exercises 21 - 22, consider the gamma function given by $\Gamma(a) = \int_0^\infty e^{-y}y^{a-1}\,dy$.**

21) Show that $\Gamma(a) = (a-1)\Gamma(a-1)$.

22) Extend to show that $\Gamma(a) = (a-1)!$, assuming $a$ is a positive integer.

The fastest car in the world, the Bugati Veyron, can reach a top speed of 408 km/h. The graph represents its velocity.




23) [T] Use the graph to estimate the velocity every 20 sec and fit to a graph of the form $v(t) = ae^{bx}\sin(cx) + d$. (Hint: Consider the time units.)

24) [T] Using your function from the previous problem, find exactly how far the Bugati Veyron traveled in the 1 min 40 sec included in the graph.

**Answer**

Answers may vary. Ex: **9.405** km

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# CHAPTER OVERVIEW

## 4: Introduction to Differential Equations

Many real-world phenomena can be modeled mathematically by using differential equations. Population growth, radioactive decay, predator-prey models, and spring-mass systems are four examples of such phenomena. In this chapter we study some of these applications. A goal of this chapter is to develop solution techniques for different types of differential equations. As the equations become more complicated, the solution techniques also become more complicated, and in fact an entire course could be dedicated to the study of these equations. In this chapter we study several types of differential equations and their corresponding methods of solution.


4.1: Prelude to Differential Equations
4.2: Basics of Differential Equations
4.2E: Exercises for Section 4.2
4.3: Direction Fields and Numerical Methods
4.3E: Exercises for Section 4.3
4.4: Separable Equations
4.4E: Exercises for Section 4.4
4.5: The Logistic Equation
4.5E: Exercises for Section 4.5
4.6: Chapter 4 Review Exercises


*Thumbnail: An exponential growth model of population.*

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# 4.1: Prelude to Differential Equations

Many real-world phenomena can be modeled mathematically by using differential equations. Population growth, radioactive decay, predator-prey models, and spring-mass systems are four examples of such phenomena. In this chapter we study some of these applications. Suppose we wish to study a population of deer over time and determine the total number of animals in a given area. We can first observe the population over a period of time, estimate the total number of deer, and then use various assumptions to derive a mathematical model for different scenarios. Some factors that are often considered are environmental impact, threshold population values, and predators. In this chapter we see how differential equations can be used to predict populations over time.



Figure **4.1.1**: The white-tailed deer (Odocoileus virginianus) of the eastern United States. Differential equations can be used to study animal populations. (credit: modification of work by Rachel Kramer, Flickr)

Another goal of this chapter is to develop solution techniques for different types of differential equations. As the equations become more complicated, the solution techniques also become more complicated, and in fact an entire course could be dedicated to the study of these equations. In this chapter we study several types of differential equations and their corresponding methods of solution.

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# 4.2: Basics of Differential Equations

**Learning Objectives**

- Identify the order of a differential equation.
- Explain what is meant by a solution to a differential equation.
- Distinguish between the general solution and a particular solution of a differential equation.
- Identify an initial-value problem.
- Identify whether a given function is a solution to a differential equation or an initial-value problem.

Calculus is the mathematics of change, and rates of change are expressed by derivatives. Thus, one of the most common ways to use calculus is to set up an equation containing an unknown function $y = f(x)$ and its derivative, known as a differential equation. Solving such equations often provides information about how quantities change and frequently provides insight into how and why the changes occur.

Techniques for solving differential equations can take many different forms, including direct solution, use of graphs, or computer calculations. We introduce the main ideas in this chapter and describe them in a little more detail later in the course. In this section we study what differential equations are, how to verify their solutions, some methods that are used for solving them, and some examples of common and useful equations.

## General Differential Equations

Consider the equation $y' = 3x^2$, which is an example of a differential equation because it includes a derivative. There is a relationship between the variables $x$ and $y$ : $y$ is an unknown function of $x$. Furthermore, the left-hand side of the equation is the derivative of $y$. Therefore we can interpret this equation as follows: Start with some function $y = f(x)$ and take its derivative. The answer must be equal to $3x^2$. What function has a derivative that is equal to $3x^2$? One such function is $y = x^3$, so this function is considered a *solution* to a differential equation.

**Definition: differential equation**

A *differential equation* is an equation involving an unknown function $y = f(x)$ and one or more of its derivatives. A solution to a differential equation is a function $y = f(x)$ that satisfies the differential equation when $f$ and its derivatives are substituted into the equation.

Go to this website to explore more on this topic.

Some examples of differential equations and their solutions appear in Table 4.2.1.

Table 4.2.1: Examples of Differential Equations and Their Solutions

| Equation | Solution |
|---|---|
| $y' = 2x$ | $y = x^2$ |
| $y' + 3y = 6x + 11$ | $y = e^{-3x} + 2x + 3$ |
| $y'' - 3y' + 2y = 24e^{-2x}$ | $y = 3e^x - 4e^{2x} + 2e^{-2x}$ |

Note that a solution to a differential equation is not necessarily unique, primarily because the derivative of a constant is zero. For example, $y = x^2 + 4$ is also a solution to the first differential equation in Table 4.2.1. We will return to this idea a little bit later in this section. For now, let's focus on what it means for a function to be a solution to a differential equation.

**Example 4.2.1: Verifying Solutions of Differential Equations**

Verify that the function $y = e^{-3x} + 2x + 3$ is a solution to the differential equation $y' + 3y = 6x + 11$.

**Solution**

To verify the solution, we first calculate $y'$ using the chain rule for derivatives. This gives $y' = -3e^{-3x} + 2$. Next we substitute $y$ and $y'$ into the left-hand side of the differential equation:

$$(-3e^{-2x} + 2) + 3(e^{-2x} + 2x + 3).$$

The resulting expression can be simplified by first distributing to eliminate the parentheses, giving

$$-3e^{-2x} + 2 + 3e^{-2x} + 6x + 9.$$

Combining like terms leads to the expression $6x + 11$, which is equal to the right-hand side of the differential equation. This result verifies that $y = e^{-3x} + 2x + 3$ is a solution of the differential equation.



**Exercise 4.2.1**

Verify that $y = 2e^{3x} - 2x - 2$ is a solution to the differential equation $y' - 3y = 6x + 4$.

**Hint**

First calculate $y'$ then substitute both $y'$ and $y$ into the left-hand side.

It is convenient to define characteristics of differential equations that make it easier to talk about them and categorize them. The most basic characteristic of a differential equation is its order.

**Definition: order of a differential equation**

The *order of a differential equation* is the highest order of any derivative of the unknown function that appears in the equation.

**Example 4.2.2: Identifying the Order of a Differential Equation**

The highest derivative in the equation is $y'$,

What is the order of each of the following differential equations?

a. $y' - 4y = x^2 - 3x + 4$
b. $x^2y''' - 3xy'' + xy' - 3y = \sin x$
c. $\frac{4}{x}y^{(4)} - \frac{6}{x^2}y'' + \frac{12}{x^4}y = x^3 - 3x^2 + 4x - 12$

**Solution**

a. The highest derivative in the equation is $y'$,so the order is $1$.

b. The highest derivative in the equation is $y'''$, so the order is $3$.
c. The highest derivative in the equation is $y^{(4)}$, so the order is $4$.

**Exercise 4.2.2**

What is the order of the following differential equation?

$$(x^4 - 3x)y^{(5)} - (3x^2 + 1)y' + 3y = \sin x \cos x$$

**Hint**

What is the highest derivative in the equation?

**Answer**

$5$

## General and Particular Solutions

We already noted that the differential equation $y' = 2x$ has at least two solutions: $y = x^2$ and $y = x^2 + 4$. The only difference between these two solutions is the last term, which is a constant. What if the last term is a different constant? Will this expression still be a solution to the differential equation? In fact, any function of the form $y = x^2 + C$, where $C$ represents any constant, is a solution as well. The reason is that the derivative of $x^2 + C$ is $2x$, regardless of the value of $C$. It can be shown that any solution of this differential equation must be of the form $y = x^2 + C$. This is an example of a **general solution** to a differential equation. A graph of some of these solutions is given in Figure 4.2.1. (Note: in this graph we used even integer values for C ranging between $-4$ and $4$. In fact, there is no restriction on the value of $C$; it can be an integer or not.)

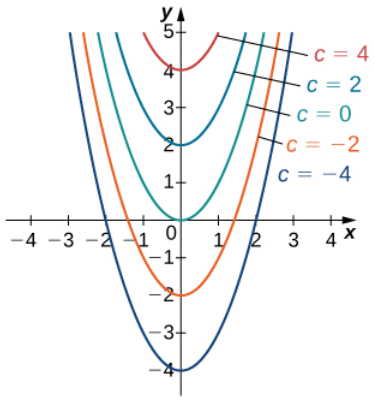



Figure 4.2.1: Family of solutions to the differential equation $y' = 2x$.

In this example, we are free to choose any solution we wish; for example, $y = x^2 - 3$ is a member of the family of solutions to this differential equation. This is called a particular solution to the differential equation. A particular solution can often be uniquely identified if we are given additional information about the problem.

**Example 4.2.3: Finding a Particular Solution**

Find the particular solution to the differential equation $y' = 2x$ passing through the point $(2, 7)$.

**Solution**

Any function of the form $y = x^2 + C$ is a solution to this differential equation. To determine the value of $C$, we substitute the values $x = 2$ and $y = 7$ into this equation and solve for $C$:

$$\begin{aligned} y &= x^2 + C \\ 7 &= 2^2 + C \\ &= 4 + C \\ C &= 3. \end{aligned}$$

Therefore the particular solution passing through the point $(2, 7)$ is $y = x^2 + 3$.

**Exercise 4.2.3**

Find the particular solution to the differential equation

$$y' = 4x + 3$$

passing through the point $(1, 7)$, given that $y = 2x^2 + 3x + C$ is a general solution to the differential equation.

**Hint**

First substitute $x = 1$ and $y = 7$ into the equation, then solve for $C$.

**Answer**

$$y = 2x^2 + 3x + 2$$

## Initial-Value Problems

Usually a given differential equation has an infinite number of solutions, so it is natural to ask which one we want to use. To choose one solution, more information is needed. Some specific information that can be useful is an *initial value*, which is an ordered pair that is used to find a particular solution.

A differential equation together with one or more initial values is called an *initial-value problem*. The general rule is that the number of initial values needed for an initial-value problem is equal to the order of the differential equation. For example, if we have the differential equation $y' = 2x$, then $y(3) = 7$ is an initial value, and when taken together, these equations form an initial-value problem. The differential equation $y'' - 3y' + 2y = 4e^x$ is second order, so we need two initial values. With initial-value problems of order greater than one, the same value should be used for the independent variable. An example of initial values for this second-order equation would be $y(0) = 2$ and $y'(0) = -1$. These two initial values together with the differential equation form an initial-value problem. These problems are so named because often the independent variable in the unknown function is $t$, which represents time. Thus, a value of $t = 0$ represents the beginning of the problem.

**Example 4.2.4: Verifying a Solution to an Initial-Value Problem**

Verify that the function $y = 2e^{-2t} + e^t$ is a solution to the initial-value problem

$$y' + 2y = 3e^t, \quad y(0) = 3.$$

**Solution**

For a function to satisfy an initial-value problem, it must satisfy both the differential equation and the initial condition. To show that $y$ satisfies the differential equation, we start by calculating $y'$. This gives $y' = -4e^{-2t} + e^t$. Next we substitute both $y$ and $y'$ into the left-hand side of the differential equation and simplify:

$$\begin{aligned} y' + 2y &= (-4e^{-2t} + e^t) + 2(2e^{-2t} + e^t) \\ &= -4e^{-2t} + e^t + 4e^{-2t} + 2e^t = 3e^t. \end{aligned}$$

This is equal to the right-hand side of the differential equation, so $y = 2e^{-2t} + e^t$ solves the differential equation. Next we calculate $y(0)$:

$$y(0) = 2e^{-2(0)} + e^0 = 2 + 1 = 3.$$

This result verifies the initial value. Therefore the given function satisfies the initial-value problem.

**Exercise 4.2.4**

Verify that $y = 3e^{2t} + 4\sin t$ is a solution to the initial-value problem

$$y' - 2y = 4\cos t - 8\sin t, y(0) = 3.$$

**Hint**

First verify that $y$ solves the differential equation. Then check the initial value.

In Example 4.2.4, the initial-value problem consisted of two parts. The first part was the differential equation $y' + 2y = 3e^x$, and the second part was the initial value $y(0) = 3$. These two equations together formed the initial-value problem.

The same is true in general. An initial-value problem will consists of two parts: the differential equation and the initial condition. The differential equation has a family of solutions, and the initial condition determines the value of $C$. The family of solutions to the differential equation in Example 4.2.4 is given by $y = 2e^{-2t} + Ce^t$. This family of solutions is shown in Figure 4.2.2, with the particular solution $y = 2e^{-2t} + e^t$ labeled.

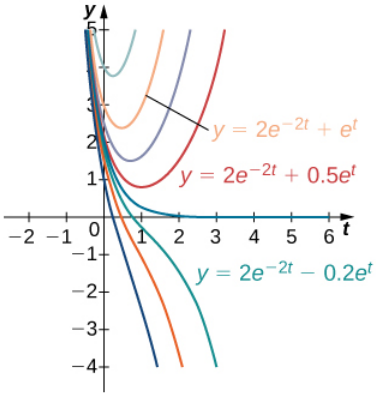



Figure 4.2.2: A family of solutions to the differential equation $y' + 2y = 3e^t$. The particular solution $y = 2e^{-2t} + e^t$ is labeled.

**Example 4.2.5: Solving an Initial-value Problem**

Solve the following initial-value problem:

$$y' = 3e^x + x^2 - 4, y(0) = 5.$$

**Solution**

The first step in solving this initial-value problem is to find a general family of solutions. To do this, we find an antiderivative of both sides of the differential equation

$$\int y'\,dx = \int (3e^x + x^2 - 4)\,dx,$$

namely,

$$y + C_1 = 3e^x + \tfrac{1}{3}x^3 - 4x + C_2.$$

We are able to integrate both sides because the y term appears by itself. Notice that there are two integration constants: $C_1$ and $C_2$. Solving this equation for $y$ gives

$$y = 3e^x + \tfrac{1}{3}x^3 - 4x + C_2 - C_1.$$

Because $C_1$ and $C_2$ are both constants, $C_2 - C_1$ is also a constant. We can therefore define $C = C_2 - C_1$, which leads to the equation

$$y = 3e^x + \tfrac{1}{3}x^3 - 4x + C.$$

Next we determine the value of $C$. To do this, we substitute $x = 0$ and $y = 5$ into this equation and solve for $C$:

$$5 = 3e^0 + \frac{1}{3}0^3 - 4(0) + C$$
$$5 = 3 + C$$
$$C = 2$$

Now we substitute the value $C = 2$ into the general equation. The solution to the initial-value problem is $y = 3e^x + \frac{1}{3}x^3 - 4x + 2$.

**Analysis**

The difference between a general solution and a particular solution is that a general solution involves a family of functions, either explicitly or implicitly defined, of the independent variable. The initial value or values determine which particular solution in the family of solutions satisfies the desired conditions.



**Exercise 4.2.5**

Solve the initial-value problem

$$y' = x^2 - 4x + 3 - 6e^x, y(0) = 8.$$

**Hint**

First take the antiderivative of both sides of the differential equation. Then substitute $x = 0$ and $y = 8$ into the resulting equation and solve for $C$.

**Answer**

$y = \frac{1}{3}x^3 - 2x^2 + 3x - 6e^x + 14$

In physics and engineering applications, we often consider the forces acting upon an object, and use this information to understand the resulting motion that may occur. For example, if we start with an object at Earth's surface, the primary force acting upon that object is gravity. Physicists and engineers can use this information, along with Newton's **second law of motion** (in equation form $F = ma$, where $F$ represents force, $m$ represents mass, and $a$ represents acceleration), to derive an equation that can be solved.

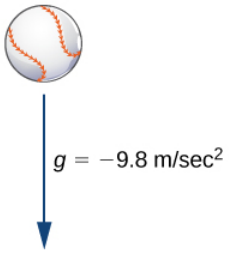



Figure **4.2.3**: For a baseball falling in air, the only force acting on it is gravity (neglecting air resistance).

In Figure 4.2.3 we assume that the only force acting on a baseball is the force of gravity. This assumption ignores air resistance. (The force due to air resistance is considered in a later discussion.) The acceleration due to gravity at Earth's surface, g, is approximately $9.8\ \text{m/s}^2$. We introduce a frame of reference, where Earth's surface is at a height of 0 meters. Let $v(t)$ represent the velocity of the object in meters per second. If $v(t) > 0$, the ball is rising, and if $v(t) < 0$, the ball is falling (Figure).




Figure 4.2.4: Possible velocities for the rising/falling baseball.

Our goal is to solve for the velocity $v(t)$ at any time $t$. To do this, we set up an initial-value problem. Suppose the mass of the ball is $m$, where $m$ is measured in kilograms. We use Newton's second law, which states that the force acting on an object is equal to its mass times its acceleration ($F = ma$). Acceleration is the derivative of velocity, so $a(t) = v'(t)$. Therefore the force acting on the baseball is given by $F = mv'(t)$. However, this force must be equal to the force of gravity acting on the object, which (again using Newton's second law) is given by $F_g = -mg$, since this force acts in a downward direction. Therefore we obtain the equation $F = F_g$, which becomes $mv'(t) = -mg$. Dividing both sides of the equation by $m$ gives the equation

$$v'(t) = -g. \tag{4.2.1}$$

Notice that this differential equation remains the same regardless of the mass of the object.

We now need an initial value. Because we are solving for velocity, it makes sense in the context of the problem to assume that we know the **initial velocity**, or the velocity at time $t = 0$. This is denoted by $v(0) = v_0$.

**Example 4.2.6: Velocity of a Moving Baseball**

A baseball is thrown upward from a height of 3 meters above Earth's surface with an initial velocity of 10 m/s, and the only force acting on it is gravity. The ball has a mass of 0.15 kg at Earth's surface.

a. Find the velocity $v(t)$ of the basevall at time $t$.
b. What is its velocity after 2 seconds?

**Solution**

a. From the preceding discussion, the differential equation that applies in this situation is

$$v'(t) = -g,$$

where $g = 9.8\ \text{m/s}^2$. The initial condition is $v(0) = v_0$, where $v_0 = 10$ m/s. Therefore the initial-value problem is $v'(t) = -9.8\ \text{m/s}^2$, $v(0) = 10$ m/s.

The first step in solving this initial-value problem is to take the antiderivative of both sides of the differential equation. This gives

$$\int v'(t)\,dt = \int -9.8\,dt$$

$$v(t) = -9.8t + C.$$

The next step is to solve for $C$. To do this, substitute $t = 0$ and $v(0) = 10$:

$$v(t) = -9.8t + C$$
$$v(0) = -9.8(0) + C$$
$$10 = C.$$

Therefore $C = 10$ and the velocity function is given by $v(t) = -9.8t + 10$.

b. To find the velocity after $2$ seconds, substitute $t = 2$ into $v(t)$.

$$v(t) = -9.8t + 10$$
$$v(2) = -9.8(2) + 10$$
$$v(2) = -9.6$$

The units of velocity are meters per second. Since the answer is negative, the object is falling at a speed of $9.6$ m/s.

**Exercise 4.2.6**

Suppose a rock falls from rest from a height of $100$ meters and the only force acting on it is gravity. Find an equation for the velocity $v(t)$ as a function of time, measured in meters per second.

**Hint**

What is the initial velocity of the rock? Use this with the differential equation in Example $4.2.6$ to form an initial-value problem, then solve for $v(t)$.

**Answer**

$v(t) = -9.8t$

A natural question to ask after solving this type of problem is how high the object will be above Earth's surface at a given point in time. Let $s(t)$ denote the height above Earth's surface of the object, measured in meters. Because velocity is the derivative of position (in this case height), this assumption gives the equation $s'(t) = v(t)$. An initial value is necessary; in this case the initial height of the object works well. Let the initial height be given by the equation $s(0) = s_0$. Together these assumptions give the initial-value problem

$$s'(t) = v(t), s(0) = s_0. \tag{4.2.2}$$

If the velocity function is known, then it is possible to solve for the position function as well.

**Example 4.2.7: Height of a Moving Baseball**

A baseball is thrown upward from a height of $3$ meters above Earth's surface with an initial velocity of $10m/s$, and the only force acting on it is gravity. The ball has a mass of $0.15$ kilogram at Earth's surface.

a. Find the position $s(t)$ of the baseball at time $t$.
b. What is its height after $2$ seconds?

**Solution**

We already know the velocity function for this problem is $v(t) = -9.8t + 10$. The initial height of the baseball is $3$ meters, so $s_0 = 3$. Therefore the initial-value problem for this example is

To solve the initial-value problem, we first find the antiderivatives:

$$\int s'(t)\,dt = \int (-9.8t + 10)\,dt$$
$$s(t) = -4.9t^2 + 10t + C.$$

Next we substitute $t = 0$ and solve for $C$:

$$s(t) = -4.9t^2 + 10t + C$$

$$s(0) = -4.9(0)^2 + 10(0) + C$$

$$3 = C.$$

Therefore the position function is $s(t) = -4.9t^2 + 10t + 3$.

b. The height of the baseball after $2$ sec is given by $s(2)$ :

$$s(2) = -4.9(2)^2 + 10(2) + 3 = -4.9(4) + 23 = 3.4.$$

Therefore the baseball is $3.4$ meters above Earth's surface after $2$ seconds. It is worth noting that the mass of the ball cancelled out completely in the process of solving the problem.

## Key Concepts

- A differential equation is an equation involving a function $y = f(x)$ and one or more of its derivatives. A solution is a function $y = f(x)$ that satisfies the differential equation when $f$ and its derivatives are substituted into the equation.
- The order of a differential equation is the highest order of any derivative of the unknown function that appears in the equation.
- A differential equation coupled with an initial value is called an initial-value problem. To solve an initial-value problem, first find the general solution to the differential equation, then determine the value of the constant. Initial-value problems have many applications in science and engineering.

## Glossary

**differential equation**
an equation involving a function $y = y(x)$ and one or more of its derivatives

**general solution (or family of solutions)**
the entire set of solutions to a given differential equation

**initial value(s)**
a value or set of values that a solution of a differential equation satisfies for a fixed value of the independent variable

**initial velocity**
the velocity at time $t = 0$

**initial-value problem**
a differential equation together with an initial value or values

**order of a differential equation**
the highest order of any derivative of the unknown function that appears in the equation

**particular solution**
member of a family of solutions to a differential equation that satisfies a particular initial condition

**solution to a differential equation**
a function $y = f(x)$ that satisfies a given differential equation

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# 4.2E: Exercises for Section 4.2

**In exercises 1 - 7, determine the order of each differential equation.**

1) $y'+y=3y^2$

**Answer**

1st-order

2) $(y')^2=y'+2y$

3) $y'''+y''y'=3x^2$

**Answer**

3rd-order

4) $y'=y''+3t^2$

5) $\frac{dy}{dt}=t$

**Answer**

1st-order

6) $\frac{dy}{dx}+\frac{d^2y}{dx^2}=3x^4$

7) $\left(\frac{dy}{dt}\right)^2+8\frac{dy}{dt}+3y=4t$

**Answer**

1st-order

**In exercises 8 - 17, verify that the given function is a solution to the given differential equation.**

8) $y=\frac{x^3}{3}$ solves $y'=x^2$

9) $y=2e^{-x}+x-1$ solves $y'=x-y$

10) $y=e^{3x}-\frac{e^x}{2}$ solves $y'=3y+e^x$

11) $y=\frac{1}{1-x}$ solves $y'=y^2$

12) $y=\frac{e^{x^2}}{2}$ solves $y'=2xy$

13) $y=4+\ln x$ solves $xy'=1$

14) $y=3-x+x\ln x$ solves $y'=\ln x$

15) $y=2e^x-x-1$ solves $y'=y+x$

16) $y=e^x+\frac{\sin x}{2}-\frac{\cos x}{2}$ solves $y'=\cos x+y$

17) $y=\pi e^{-\cos x}$ solves $y'=y\sin x$

**In exercises 18 - 27, verify the given general solution and find the particular solution.**

18) Find the particular solution to the differential equation $y'=4x^2$ that passes through $(-3,-30)$, given that $y=C+\frac{4x^3}{3}$ is a general solution.

19) Find the particular solution to the differential equation $y'=3x^3$ that passes through $(1,4.75)$, given that $y=C+\frac{3x^4}{4}$ is a general solution.

**Answer**

$y = 4 + \frac{3x^4}{4}$

20) Find the particular solution to the differential equation $y' = 3x^2y$ that passes through $(0, 12)$, given that $y = Ce^{x^3}$ is a general solution.

21) Find the particular solution to the differential equation $y' = 2xy$ that passes through $\left(0, \frac{1}{2}\right)$, given that $y = Ce^{x^2}$ is a general solution.

**Answer**

$y = \frac{1}{2}e^{x^2}$

22) Find the particular solution to the differential equation $y' = \left(2xy\right)^2$ that passes through $\left(1, -\frac{1}{2}\right)$, given that $y = -\frac{3}{C + 4x^3}$ is a general solution.

23) Find the particular solution to the differential equation $y'x^2 = y$ that passes through $\left(1, \frac{2}{e}\right)$, given that $y = Ce^{-1/x}$ is a general solution.

**Answer**

$y = 2e^{-1/x}$

24) Find the particular solution to the differential equation $8\frac{dx}{dt} = -2\cos(2t) - \cos(4t)$ that passes through $(\pi, \pi)$, given that $x = C - \frac{1}{8}\sin(2t) - \frac{1}{32}\sin(4t)$ is a general solution.

25) Find the particular solution to the differential equation $\frac{du}{dt} = \tan u$ that passes through $\left(1, \frac{\pi}{2}\right)$, given that $u = \sin^{-1}\left(e^{C+t}\right)$ is a general solution.

**Answer**

$u = \sin^{-1}\left(e^{-1+t}\right)$

26) Find the particular solution to the differential equation $\frac{dy}{dt} = e^{t+y}$ that passes through $(1, 0)$, given that $y = -\ln(C - e^t)$ is a general solution.

27) Find the particular solution to the differential equation $y'(1 - x^2) = 1 + y$ that passes through $(0, -2)$, given that $y = C\frac{\sqrt{x+1}}{\sqrt{1-x}} - 1$ is a general solution.

**Answer**

$y = -\frac{\sqrt{x+1}}{\sqrt{1-x}} - 1$

**In exercises 28 - 37, find the general solution to the differential equation.**

28) $y' = 3x + e^x$

29) $y' = \ln x + \tan x$

**Answer**

$y = C - x + x\ln x - \ln(\cos x)$

30) $y' = \sin x e^{\cos x}$

31) $y' = 4^x$

**Answer**

$y = C + \frac{4^x}{\ln 4}$

32) $y' = \sin^{-1}(2x)$

33) $y' = 2t\sqrt{t^2+16}$

**Answer**

$y = \frac{2}{3}\sqrt{t^2+16}\left(t^2+16\right) + C$

34) $x' = \coth t + \ln t + 3t^2$

35) $x' = t\sqrt{4+t}$

**Answer**

$x = \frac{2}{15}\sqrt{4+t}\left(3t^2+4t-32\right) + C$

36) $y' = y$

37) $y' = \frac{y}{x}$

**Answer**

$y = Cx$

**In exercises 38 - 42, solve the initial-value problems starting from $y(t=0) = 1$ and $y(t=0) = -1$. Draw both solutions on the same graph.**

38) $\frac{dy}{dt} = 2t$

39) $\frac{dy}{dt} = -t$

**Answer**

$y = 1 - \frac{t^2}{2}$, and $y = -\frac{t^2}{2} - 1$

40) $\frac{dy}{dt} = 2y$

41) $\frac{dy}{dt} = -y$

**Answer**

$y = e^{-t}$ and $y = -e^{-t}$

42) $\frac{dy}{dt} = 2$

**In exercises 43 - 47, solve the initial-value problems starting from $y_0 = 10$. At what time does $y$ increase to 100 or drop to 1?**

43) $\frac{dy}{dt} = 4t$

**Answer**

$y = 2(t^2+5)$, When $t = 3\sqrt{5}$, $y$ will increase to $100$.

44) $\frac{dy}{dt} = 4y$

45) $\frac{dy}{dt} = -2y$

**Answer**

$y = 10e^{-2t}$, When $t = -\frac{1}{2}\ln\left(\frac{1}{10}\right)$, $y$ will decrease to $1$.

46) $\frac{dy}{dt} = e^{4t}$

47) $\frac{dy}{dt} = e^{-4t}$

**Answer**

$y = \frac{1}{4}(41 - e^{-4t})$, Neither condition will ever happen.

**Recall that a family of solutions includes solutions to a differential equation that differ by a constant. For exercises 48 - 52, use your calculator to graph a family of solutions to the given differential equation. Use initial conditions from $y(t = 0) = -10$ to $y(t = 0) = 10$ increasing by 2. Is there some critical point where the behavior of the solution begins to change?**

48) [T] $y' = y(x)$

49) [T] $xy' = y$

**Answer**

Solution changes from increasing to decreasing at $y(0) = 0$.

50) [T] $y' = t^3$

51) [T] $y' = x + y$ (Hint: $y = Ce^x - x - 1$ is the general solution)

**Answer**

Solution changes from increasing to decreasing at $y(0) = 0$.

52) [T] $y' = x \ln x + \sin x$

53) Find the general solution to describe the velocity of a ball of mass $1$ lb that is thrown upward at a rate of $a$ ft/sec.

**Answer**

$v(t) = -32t + a$

54) In the preceding problem, if the initial velocity of the ball thrown into the air is $a = 25$ ft/s, write the particular solution to the velocity of the ball. Solve to find the time when the ball hits the ground.

55) You throw two objects with differing masses $m_1$ and $m_2$ upward into the air with the same initial velocity of $a$ ft/s. What is the difference in their velocity after $1$ second?

**Answer**

$0$ ft/s

56) [T] You throw a ball of mass $1$ kilogram upward with a velocity of $a = 25$ m/s on Mars, where the force of gravity is $g = -3.711$ m/s$^2$. Use your calculator to approximate how much longer the ball is in the air on Mars.

57) [T] For the previous problem, use your calculator to approximate how much higher the ball went on Mars.

**Answer**

$4.86$ meters

58) [T] A car on the freeway accelerates according to $a = 15\cos(\pi t)$, where $t$ is measured in hours. Set up and solve the differential equation to determine the velocity of the car if it has an initial speed of $51$ mph. After $40$ minutes of driving, what is the driver's velocity?

59) [T] For the car in the preceding problem, find the expression for the distance the car has traveled in time $t$, assuming an initial distance of $0$. How long does it take the car to travel $100$ miles? Round your answer to hours and minutes.

**Answer**

$x = 50t - \frac{15}{\pi^2}\cos(\pi t) + \frac{3}{\pi^2}$, $2$ hours $1$ minute

60) [T] For the previous problem, find the total distance traveled in the first hour.

61) Substitute $y = Be^{3t}$ into $y' - y = 8e^{3t}$ to find a particular solution.

**Answer**

$y = 4e^{3t}$

62) Substitute $y = a\cos(2t) + b\sin(2t)$ into $y' + y = 4\sin(2t)$ to find a particular solution.

63) Substitute $y = a + bt + ct^2$ into $y' + y = 1 + t^2$ to find a particular solution.

**Answer**

$y = 1 - 2t + t^2$

64) Substitute $y = ae^t \cos t + be^t \sin t$ into $y' = 2e^t \cos t$ to find a particular solution.

65) Solve $y' = e^{kt}$ with the initial condition $y(0) = 0$ and solve $y' = 1$ with the same initial condition. As $k$ approaches $0$, what do you notice?

**Answer**

$y = \frac{1}{k}(e^{kt} - 1)$ and $y = t$

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# 4.3: Direction Fields and Numerical Methods

**Learning Objectives**

- Draw the direction field for a given first-order differential equation.
- Use a direction field to draw a solution curve of a first-order differential equation.
- Use Euler's Method to approximate the solution to a first-order differential equation.

For the rest of this chapter we will focus on various methods for solving differential equations and analyzing the behavior of the solutions. In some cases it is possible to predict properties of a solution to a differential equation without knowing the actual solution. We will also study numerical methods for solving differential equations, which can be programmed by using various computer languages or even by using a spreadsheet program, such as Microsoft Excel.

## Creating Direction Fields

Direction fields (also called slope fields) are useful for investigating first-order differential equations. In particular, we consider a first-order differential equation of the form

$$y' = f(x, y). \tag{4.3.1}$$

An applied example of this type of differential equation appears in Newton's law of cooling, which we will solve explicitly later in this chapter. First, though, let us create a direction field for the differential equation

$$T'(t) = -0.4(T - 72). \tag{4.3.2}$$

Here $T(t)$ represents the temperature (in degrees Fahrenheit) of an object at time $t$, and the ambient temperature is $72°F$. Figure 4.3.1 shows the direction field for this equation.

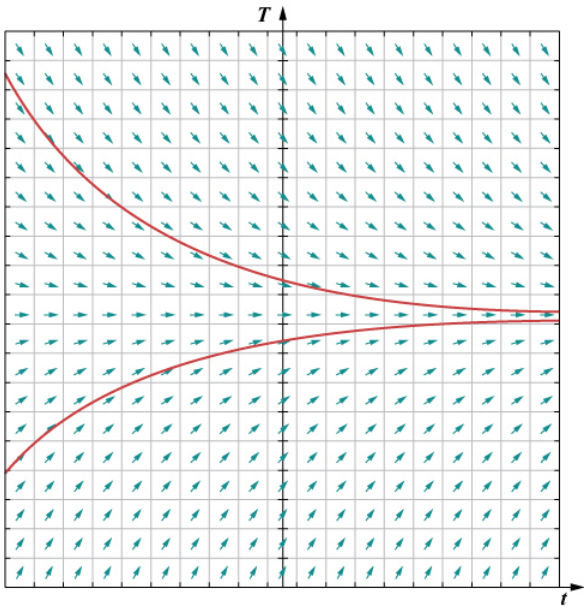



Figure 4.3.1: Direction field for the differential equation $T'(t) = -0.4(T - 72)$. Two solutions are plotted: one with initial temperature less than $72°F$ and the other with initial temperature greater than $72°F$.

The idea behind a direction field is the fact that the derivative of a function evaluated at a given point is the slope of the tangent line to the graph of that function at the same point. Other examples of differential equations for which we can create a direction field include

$$y' = 3x + 2y - 4 \tag{4.3.3}$$

$$y' = x^2 - y^2 \tag{4.3.4}$$

$$y' = \frac{2x+4}{y-2}. \tag{4.3.5}$$

To create a direction field, we start with the first equation: $y' = 3x + 2y - 4$. We let $(x_0, y_0)$ be any ordered pair, and we substitute these numbers into the right-hand side of the differential equation. For example, if we choose $x = 1$ and $y = 2$, substituting into the right-hand side of the differential equation yields

$$y' = 3x + 2y - 4 = 3(1) + 2(2) - 4 = 3.$$

This tells us that if a solution to the differential equation $y' = 3x + 2y - 4$ passes through the point $(1, 2)$, then the slope of the solution at that point must equal 3. To start creating the direction field, we put a short line segment at the point $(1, 2)$ having slope 3. We can do this for any point in the domain of the function $f(x, y) = 3x + 2y - 4$, which consists of all ordered pairs $(x, y)$ in $R^2$. Therefore any point in the Cartesian plane has a slope associated with it, assuming that a solution to the differential equation passes through that point. The direction field for the differential equation $y' = 3x + 2y - 4$ is shown in Figure 4.3.2.

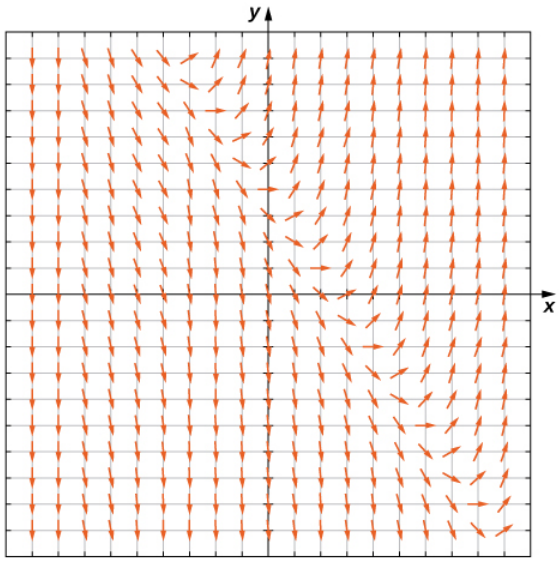
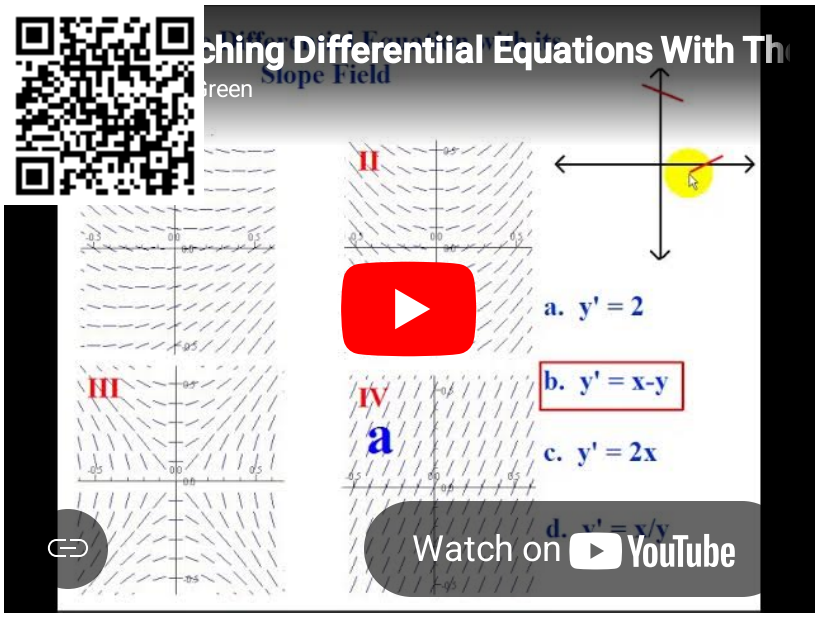



Figure 4.3.2: Direction field for the differential equation $y' = 3x + 2y - 4$.

We can generate a direction field of this type for any differential equation of the form $y' = f(x, y)$.



**Definition: Direction Field (Slope Field)**

A direction field (slope field) is a mathematical object used to graphically represent solutions to a first-order differential equation. At each point in a direction field, a line segment appears whose slope is equal to the slope of a solution to the differential equation passing through that point.

## Using Direction Fields

We can use a direction field to predict the behavior of solutions to a differential equation without knowing the actual solution. For example, the direction field in Figure 4.3.3 serves as a guide to the behavior of solutions to the differential equation $y' = 3x + 2y - 4$.

To use a direction field, we start by choosing any point in the field. The line segment at that point serves as a signpost telling us what direction to go from there. For example, if a solution to the differential equation passes through the point $(0, 1)$, then the slope of the solution passing through that point is given by $y' = 3(0) + 2(1) - 4 = -2$. Now let $x$ increase slightly, say to $x = 0.1$. Using the method of linear approximations gives a formula for the approximate value of $y$ for $x = 0.1$. In particular,

$$L(x) = y_0 + f'(x_0)(x - x_0) = 1 - 2(x_0 - 0) = 1 - 2x_0. \tag{4.3.6}$$

Substituting $x_0 = 0.1$ into $L(x)$ gives an approximate $y$ value of $0.8$.

At this point the slope of the solution changes (again according to the differential equation). We can keep progressing, recalculating the slope of the solution as we take small steps to the right, and watching the behavior of the solution. Figure 4.3.3 shows a graph of the solution passing through the point $(0, 1)$.

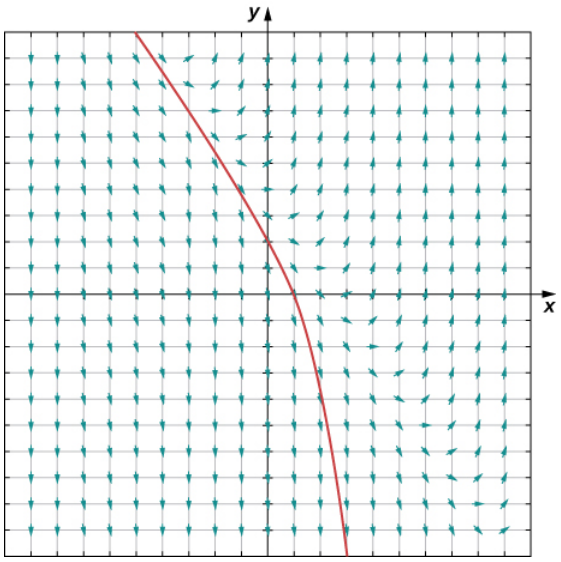



Figure 4.3.3: Direction field for the differential equation $y' = 3x + 2y - 4$ with the solution passing through the point $(0, 1)$.

The curve is the graph of the solution to the initial-value problem

$$y' = 3x + 2y - 4, y(0) = 1. \tag{4.3.7}$$

This curve is called a solution curve passing through the point $(0, 1)$. The exact solution to this initial-value problem is

$$y = -\frac{3}{2}x + \frac{5}{4} - \frac{1}{4}e^{2x}, \tag{4.3.8}$$

and the graph of this solution is identical to the curve in Figure 4.3.3.

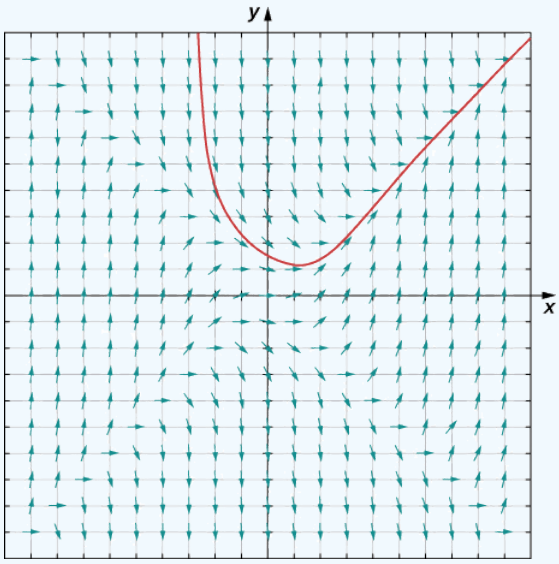
**Exercise 4.3.1**

Create a direction field for the differential equation $y' = x^2 - y^2$ and sketch a solution curve passing through the point $(-1, 2)$.

**Hint**

Use $x$ and $y$ values ranging from $-5$ to $5$. For each coordinate pair, calculate $y'$ using the right-hand side of the differential equation.

**Answer**




Go to this Java applet and this website to see more about slope fields.

Now consider the direction field for the differential equation $y' = (x-3)(y^2-4)$, shown in Figure 4.3.4. This direction field has several interesting properties. First of all, at $y = -2$ and $y = 2$, horizontal dashes appear all the way across the graph. This means that if $y = -2$, then $y' = 0$. Substituting this expression into the right-hand side of the differential equation gives

$$(x-3)(y^2-4) = (x-3)((-2)^2-4) = (x-3)(0) = 0 = y'. \tag{4.3.9}$$

Therefore $y = -2$ is a solution to the differential equation. Similarly, $y = 2$ is a solution to the differential equation. These are the only constant-valued solutions to the differential equation, as we can see from the following argument. Suppose $y = k$ is a constant solution to the differential equation. Then $y' = 0$. Substituting this expression into the differential equation yields $0 = (x-3)(k^2-4)$. This equation must be true for all values of $x$, so the second factor must equal zero. This result yields the equation $k^2 - 4 = 0$. The solutions to this equation are $k = -2$ and $k = 2$, which are the constant solutions already mentioned. These are called the equilibrium solutions to the differential equation.

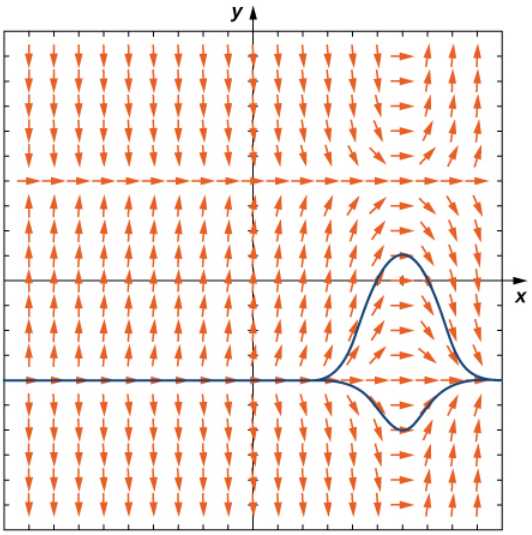



Figure 4.3.4: Direction field for the differential equation $y' = (x-3)(y^2-4)$ showing two solutions. These solutions are very close together, but one is barely above the equilibrium solution $x = -2$ and the other is barely below the same equilibrium solution.

**Definition: Equilibrium Solutions**

Consider the differential equation $y' = f(x,y)$. An *equilibrium solution* is any solution to the differential equation of the form $y = c$, where $c$ is a constant.

To determine the equilibrium solutions to the differential equation $y' = f(x,y)$, set the right-hand side equal to zero. An equilibrium solution of the differential equation is any function of the form $y = k$ such that $f(x,k) = 0$ for all values of $x$ in the domain of $f$.

An important characteristic of equilibrium solutions concerns whether or not they approach the line $y = k$ as an asymptote for large values of $x$.

**Definition: asymptotically Stable, Unstable and Semi-Stable Solutions**

Consider the differential equation $y' = f(x,y)$, and assume that all solutions to this differential equation are defined for $x \geq x_0$. Let $y = k$ be an equilibrium solution to the differential equation.

1. $y = k$ is an *asymptotically stable solution* to the differential equation if there exists $\varepsilon > 0$ such that for any value $c \in (k-\varepsilon, k+\varepsilon)$ the solution to the initial-value problem
$$y' = f(x,y), y(x_0) = c \tag{4.3.10}$$
approaches $k$ as $x$ approaches infinity.
2. $y = k$ is an *asymptotically unstable solution* to the differential equation if there exists $\varepsilon > 0$ such that for any value $c \in (k-\varepsilon, k+\varepsilon)$ the solution to the initial-value problem
$$y' = f(x,y), y(x_0) = c \tag{4.3.11}$$
never approaches $k$ as $x$ approaches infinity.
3. $y = k$ is an *asymptotically semi-stable solution* to the differential equation if it is neither asymptotically stable nor asymptotically unstable.

Now we return to the differential equation $y' = (x-3)(y^2-4)$, with the initial condition $y(0) = 0.5$. The direction field for this initial-value problem, along with the corresponding solution, is shown in Figure 4.3.5.

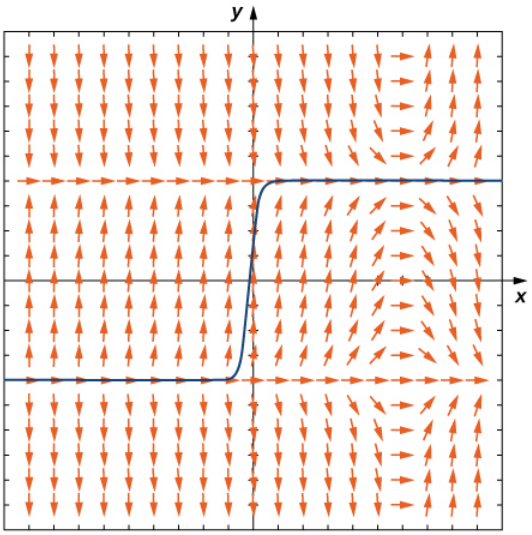



Figure 4.3.5: Direction field for the initial-value problem $y' = (x-3)(y^2-4), y(0) = 0.5$.

The values of the solution to this initial-value problem stay between $y = -2$ and $y = 2$, which are the equilibrium solutions to the differential equation. Furthermore, as $x$ approaches infinity, $y$ approaches 2. The behavior of solutions is similar if the initial value is higher than 2, for example, $y(0) = 2.3$. In this case, the solutions decrease and approach $y = 2$ as $x$ approaches infinity. Therefore $y = 2$ is an asymptotically stable solution to the differential equation.

What happens when the initial value is below $y = -2$? This scenario is illustrated in Figure 4.3.6, with the initial value $y(0) = -3$.

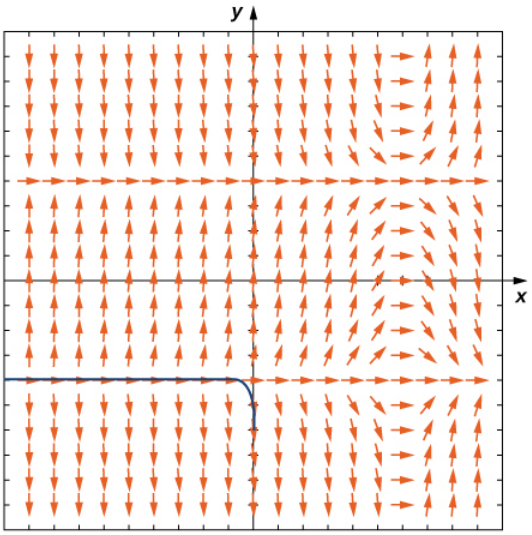



Figure 4.3.6: Direction field for the initial-value problem $y' = (x-3)(y^2-4), y(0) = -3$.

The solution decreases rapidly toward negative infinity as $x$ approaches infinity. Furthermore, if the initial value is slightly higher than $-2$, then the solution approaches 2, which is the other equilibrium solution. Therefore in neither case does the solution approach $y = -2$, so $y = -2$ is called an asymptotically unstable, or unstable, equilibrium solution.

**Example 4.3.1: Stability of an Equilibrium Solution**

Create a direction field for the differential equation $y' = (y-3)^2(y^2+y-2)$ and identify any equilibrium solutions. Classify each of the equilibrium solutions as stable, unstable, or semi-stable.

**Solution**

The direction field is shown in Figure 4.3.7.

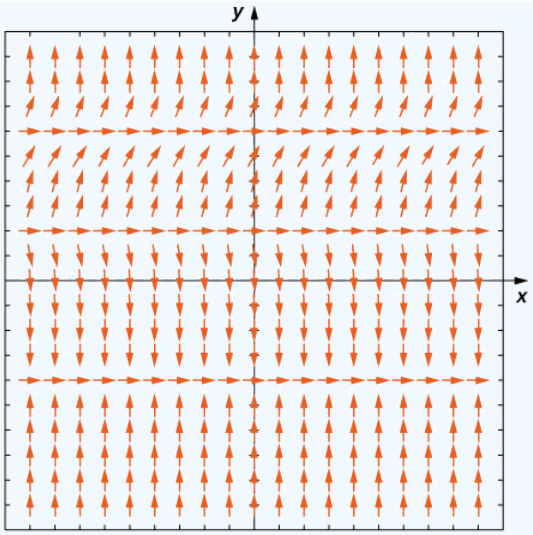



Figure 4.3.7: Direction field for the differential equation $y' = (y-3)^2(y^2+y-2)$.

The equilibrium solutions are $y = -2, y = 1$, and $y = 3$. To classify each of the solutions, look at an arrow directly above or below each of these values. For example, at $y = -2$ the arrows directly below this solution point up, and the arrows directly above the solution point down. Therefore all initial conditions close to $y = -2$ approach $y = -2$, and the solution is stable. For the solution $y = 1$, all initial conditions above and below $y = 1$ are repelled (pushed away) from $y = 1$, so this solution is unstable. The solution $y = 3$ is semi-stable, because for initial conditions slightly greater than 3, the solution approaches infinity, and for initial conditions slightly less than 3, the solution approaches $y = 1$.

**Analysis**

It is possible to find the equilibrium solutions to the differential equation by setting the right-hand side equal to zero and solving for $y$. This approach gives the same equilibrium solutions as those we saw in the direction field.

## Exercise 4.3.2

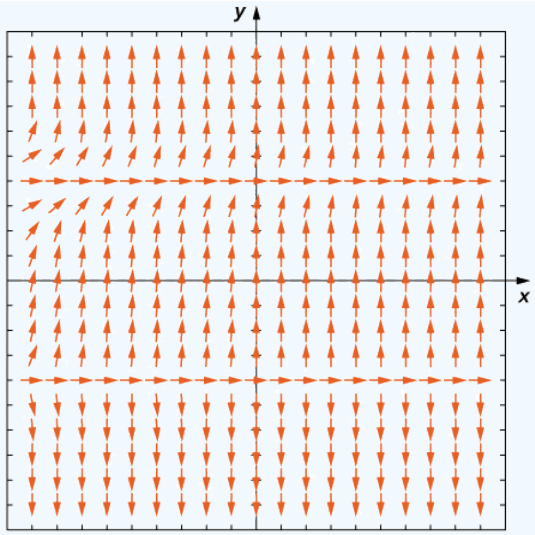
Create a direction field for the differential equation $y' = (x+5)(y+2)(y^2-4y+4)$ and identify any equilibrium solutions. Classify each of the equilibrium solutions as stable, unstable, or semi-stable.

**Hint**

First create the direction field and look for horizontal dashes that go all the way across. Then examine the slope lines directly above and below the equilibrium solutions.

**Answer**

The equilibrium solutions are $y = -2$ and $y = 2$. For this equation, $y = -2$ is an unstable equilibrium solution, and $y = 2$ is a semi-stable equilibrium solution.




## Euler's Method

Consider the initial-value problem

$$y' = 2x - 3, y(0) = 3. \tag{4.3.12}$$

Integrating both sides of the differential equation gives $y = x^2 - 3x + C$, and solving for $C$ yields the particular solution $y = x^2 - 3x + 3$. The solution for this initial-value problem appears as the parabola in Figure 4.3.8.

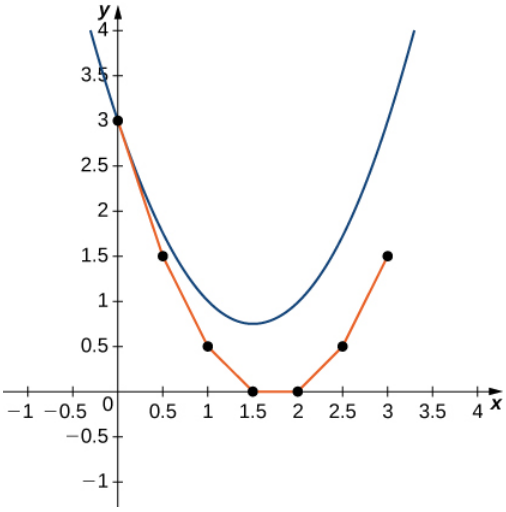



Figure 4.3.8: Euler's Method for the initial-value problem $y' = 2x - 3, y(0) = 3$.

The red graph consists of line segments that approximate the solution to the initial-value problem. The graph starts at the same initial value of $(0, 3)$. Then the slope of the solution at any point is determined by the right-hand side of the differential equation, and the length of the line segment is determined by increasing the $x$ value by $0.5$ each time (the step size). This approach is the basis of Euler's Method.

Before we state Euler's Method as a theorem, let's consider another initial-value problem:

$$y' = x^2 - y^2, y(-1) = 2. \tag{4.3.13}$$

The idea behind direction fields can also be applied to this problem to study the behavior of its solution. For example, at the point $(-1, 2)$, the slope of the solution is given by $y' = (-1)^2 - 2^2 = -3$, so the slope of the tangent line to the solution at that point is also equal to $-3$. Now we define $x_0 = -1$ and $y_0 = 2$. Since the slope of the solution at this point is equal to $-3$, we can use the method of linear approximation to approximate y near $(-1, 2)$.

$$L(x) = y_0 + f'(x_0)(x - x_0). \tag{4.3.14}$$

Here $x_0 = -1, y_0 = 2$, and $f'(x_0) = -3$, so the linear approximation becomes

$$L(x) = 2 - 3(x - (-1)) = 2 - 3x - 3 = -3x - 1. \tag{4.3.15}$$

Now we choose a **step size**. The step size is a small value, typically $0.1$ or less, that serves as an increment for $x$; it is represented by the variable $h$. In our example, let $h = 0.1$. Incrementing $x_0$ by $h$ gives our next $x$ value:

$$x_1 = x_0 + h = -1 + 0.1 = -0.9. \tag{4.3.16}$$

We can substitute $x_1 = -0.9$ into the linear approximation to calculate $y_1$.

$$y_1 = L(x_1) = -3(-0.9) - 1 = 1.7. \tag{4.3.17}$$

Therefore the approximate $y$ value for the solution when $x = -0.9$ is $y = 1.7$. We can then repeat the process, using $x_1 = -0.9$ and $y_1 = 1.7$ to calculate $x_2$ and $y_2$. The new slope is given by $y' = (-0.9)^2 - (1.7)^2 = -2.08$. First, $x_2 = x_1 + h = -0.9 + 0.1 = -0.8$. Using linear approximation gives

$$\begin{aligned} L(x) &= y_1 + f'(x_1)(x - x_1) \\ &= 1.7 - 2.08(x - (-0.9)) \\ &= 1.7 - 2.08x - 1.872 \\ &= -2.08x - 0.172. \end{aligned}$$

Finally, we substitute $x_2 = -0.8$ into the linear approximation to calculate $y_2$.

$$\begin{aligned} y_2 &= L(x_2) \\ &= -2.08x_2 - 0.172 \\ &= -2.08(-0.8) - 0.172 \\ &= 1.492. \end{aligned}$$

Therefore the approximate value of the solution to the differential equation is $y = 1.492$ when $x = -0.8$.

What we have just shown is the idea behind **Euler's Method**. Repeating these steps gives a list of values for the solution. These values are shown in the table, rounded off to four decimal places.

Using Euler's Method to Approximate Solutions to a Differential Equation

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $x_n$ | −1 | −0.9 | −0.8 | −0.7 | −0.6 | −0.5 |
| $y_n$ | 2 | 1.7 | 1.492 | 1.3334 | 1.2046 | 1.0955 |
| $n$ | 6 | 7 | 8 | 9 | 10 | |
| $x_n$ | −0.4 | −0.3 | −0.2 | −0.1 | 0 | |
| $y_n$ | 1.0004 | 1.9164 | 1.8414 | 1.7746 | 1.7156 | |

**Euler's method**

Consider the initial-value problem

$$y' = f(x, y), y(x_0) = y_0. \tag{4.3.18}$$

To approximate a solution to this problem using Euler's method, define

$$\begin{aligned} x_n &= x_0 + nh \\ y_n &= y_{n-1} + hf(x_{n-1}, y_{n-1}). \end{aligned}$$

Here $h > 0$ represents the step size and $n$ is an integer, starting with $1$. The number of steps taken is counted by the variable $n$.

Typically $h$ is a small value, say $0.1$ or $0.05$. The smaller the value of $h$, the more calculations are needed. The higher the value of $h$, the fewer calculations are needed. However, the tradeoff results in a lower degree of accuracy for larger step size, as illustrated in Figure 4.3.9.

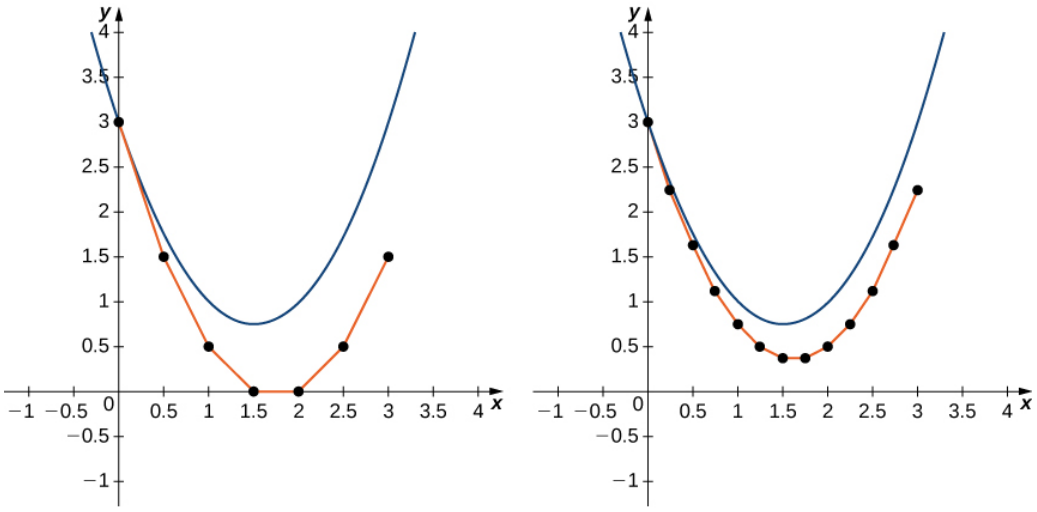
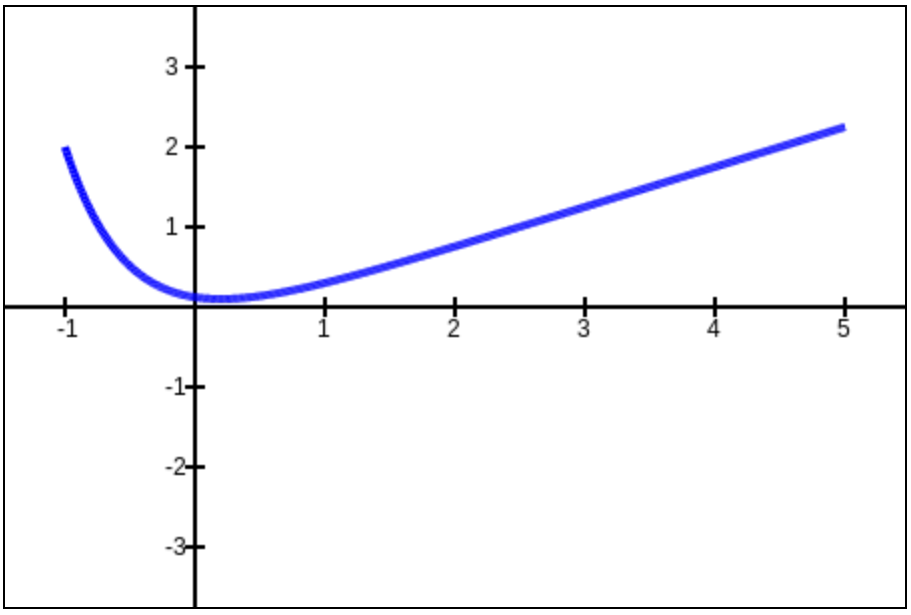



Figure 4.3.9: Euler's method for the initial-value problem $y' = 2x - 3, y(0) = 3$ with (a) a step size of $h = 0.5$; and (b) a step size of $h = 0.25$.

## Euler's Method Visualization

This activity allows you to interactively use Euler's Method to approximate a solution to the initial value problem with $y' = x - 2y$. You can change the initial value and the step size and then click on the Next Step button to see Euler's Method in action, up to at most 16 steps.

$$x_n \quad y_n \quad f'_n = x_n - 2y_n \quad x_{n+1} = x_n + h \quad y_{n+1} = y_n + hf'_n$$



$x_0$: -1 $y_0$: 2 h: 0.5

**Example 4.3.2: Using Euler's Method**

Consider the initial-value problem

$$y' = 3x^2 - y^2 + 1, y(0) = 2. \tag{4.3.19}$$

Use Euler's method with a step size of $0.1$ to generate a table of values for the solution for values of $x$ between $0$ and $1$.

**Solution**

We are given $h = 0.1$ and $f(x, y) = 3x^2 - y^2 + 1$. Furthermore, the initial condition $y(0) = 2$ gives $x_0 = 0$ and $y_0 = 2$. Using Equation with $n = 0$, we can generate this table.

Using Euler's Method to Approximate Solutions to a Differential Equation

| $n$ | $x_n$ | $y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$ |
|---|---|---|
| 0 | 0 | 2 |
| 1 | 0.1 | $y_1 = y_0 + hf(x_0, y_0) = 1.7$ |
| 2 | 0.2 | $y_2 = y_1 + hf(x_1, y_1) = 1.514$ |
| 3 | 0.3 | $y_3 = y_2 + hf(x_2, y_2) = 1.3968$ |
| 4 | 0.4 | $y_4 = y_3 + hf(x_3, y_3) = 1.3287$ |
| 5 | 0.5 | $y_5 = y_4 + hf(x_4, y_4) = 1.3001$ |
| 6 | 0.6 | $y_6 = y_5 + hf(x_5, y_5) = 1.3061$ |
| 7 | 0.7 | $y_7 = y_6 + hf(x_6, y_6) = 1.3435$ |
| 8 | 0.8 | $y_8 = y_7 + hf(x_7, y_7) = 1.4100$ |
| 9 | 0.9 | $y_9 = y_8 + hf(x_8, y_8) = 1.5032$ |
| 10 | 1.0 | $y_{10} = y_9 + hf(x_9, y_9) = 1.6202$ |

With ten calculations, we are able to approximate the values of the solution to the initial-value problem for values of $x$ between $0$ and $1$.

Go to this website for more information on Euler's method.



**Exercise 4.3.3**

Consider the initial-value problem

$$y' = x^3 + y^2, y(1) = -2. \tag{4.3.20}$$

Using a step size of $0.1$, generate a table with approximate values for the solution to the initial-value problem for values of $x$ between $1$ and $2$.

**Hint**

Start by identifying the value of $h$, then figure out what $f(x, y)$ is. Then use the formula for Euler's Method to calculate $y_1, y_2$, and so on.

**Answer**

| $n$ | \( x_n) | $y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$ |
|---|---|---|
| 0 | 1 | $-2$ |
| 1 | 1.1 | $y_1 = y_0 + hf(x_0, y_0) = -1.5$ |
| 2 | 1.2 | $y_2 = y_1 + hf(x_1, y_1) = -1.1419$ |
| 3 | 1.3 | $y_3 = y_2 + hf(x_2, y_2) = -0.8387$ |
| 4 | 1.4 | $y_4 = y_3 + hf(x_3, y_3) = -0.5487$ |
| 5 | 1.5 | $y_5 = y_4 + hf(x_4, y_4) = -0.2442$ |
| 6 | 1.6 | $y_6 = y_5 + hf(x_5, y_5) = 0.0993$ |
| 7 | 1.7 | $y_7 = y_6 + hf(x_6, y_6) = 0.5099$ |
| 8 | 1.8 | $y_8 = y_7 + hf(x_7, y_7) = 1.0272$ |
| 9 | 1.9 | $y_9 = y_8 + hf(x_8, y_8) = 1.7159$ |
| 10 | 2 | $y_{10} = y_9 + hf(x_9, y_9) = 2.6962$ |

Visit this website for a practical application of the material in this section.

## Key Concepts

- A direction field is a mathematical object used to graphically represent solutions to a first-order differential equation.
- Euler's Method is a numerical technique that can be used to approximate solutions to a differential equation.

## Key Equations

- **Euler's Method**

  $x_n = x_0 + nh$

  $y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$,where $h$ is the step size

## Glossary

**asymptotically semi-stable solution**

$y = k$ if it is neither asymptotically stable nor asymptotically unstable

**asymptotically stable solution**

$y = k$ if there exists $\varepsilon > 0$ such that for any value $c \in (k - \varepsilon, k + \varepsilon)$ the solution to the initial-value problem $y' = f(x, y), y(x_0) = c$ approaches $k$ as $x$ approaches infinity

**asymptotically unstable solution**

$y = k$ if there exists $\varepsilon > 0$ such that for any value $c \in (k - \varepsilon, k + \varepsilon)$ the solution to the initial-value problem $y' = f(x, y), y(x_0) = c$ never approaches $k$ as $x$approaches infinity

**direction field (slope field)**

a mathematical object used to graphically represent solutions to a first-order differential equation; at each point in a direction field, a line segment appears whose slope is equal to the slope of a solution to the differential equation passing through that point

**equilibrium solution**

any solution to the differential equation of the form $y = c$, where $c$ is a constant

**Euler's Method**

a numerical technique used to approximate solutions to an initial-value problem

**solution curve**

a curve graphed in a direction field that corresponds to the solution to the initial-value problem passing through a given point in the direction field

**step size**

the increment hh that is added to the xx value at each step in Euler's Method

## Contributors and Attributions

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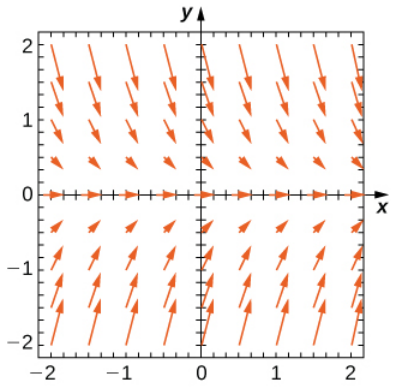
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# 4.3E: Exercises for Section 4.3

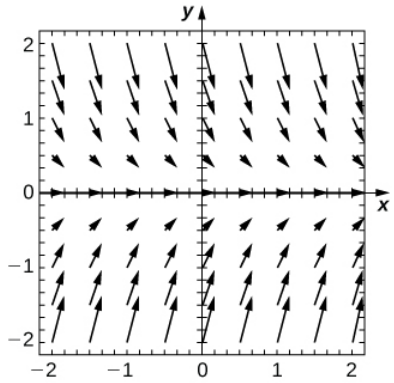
**For exercises 1 - 3, use the direction field below from the differential equation $y' = -2y$. Sketch the graph of the solution for the given initial conditions.**




1) $y(0) = 1$

2) $y(0) = 0$

**Answer**



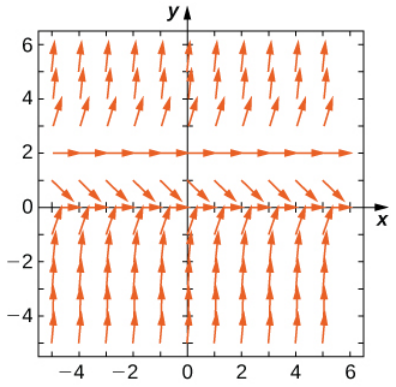

3) $y(0) = -1$

4) Are there any equilibria among the solutions of the differential equation from exercises 1 - 3? List any equilibria along with their stabilities.

**Answer**

$y = 0$ is a stable equilibrium

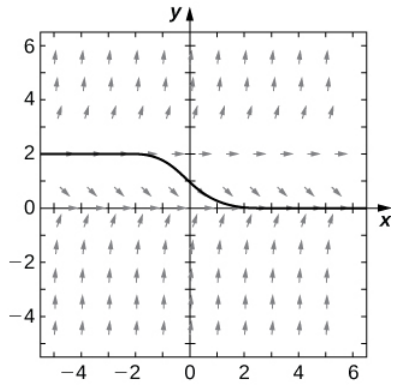
**For exercises 5 - 7, use the direction field below from the differential equation $y' = y^2 - 2y$. Sketch the graph of the solution for the given initial conditions.**




5) $y(0) = 3$

6) $y(0) = 1$

**Answer**




7) $y(0) = -1$

8) Are there any equilibria among the solutions of the differential equation from exercises 5 - 7? List any equilibria along with their stabilities.

**Answer**

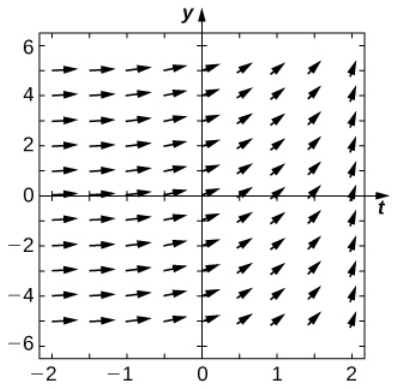
$y = 0$ is a stable equilibrium and $y = 2$ is unstable

**In exercises 9 - 13, draw the direction field for the following differential equations, then solve the differential equation. Draw your solution on top of the direction field. Does your solution follow along the arrows on your direction field?**

9) $y' = t^3$

10) $y' = e^t$

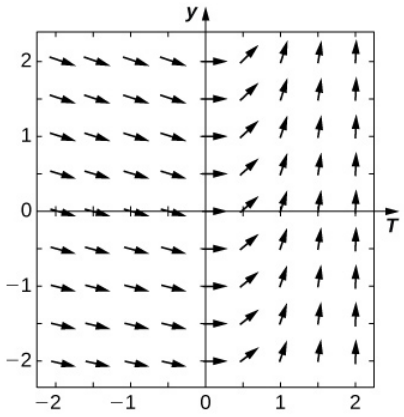
**Answer**




11) $\frac{dy}{dx} = x^2 \cos x$

12) $\frac{dy}{dt} = te^t$

**Answer**



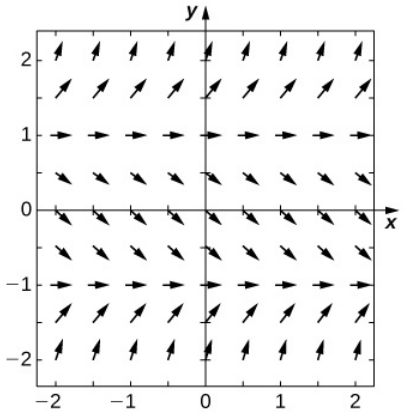

13) $\frac{dx}{dt} = \cosh(t)$

**In exercises 14 - 18, draw the directional field for the following differential equations. What can you say about the behavior of the solution? Are there equilibria? What stability do these equilibria have?**

14) $y' = y^2 - 1$

**Answer**

There appear to be equlibria at $y = -1$ (stable) and $y = 1$ (unstable).

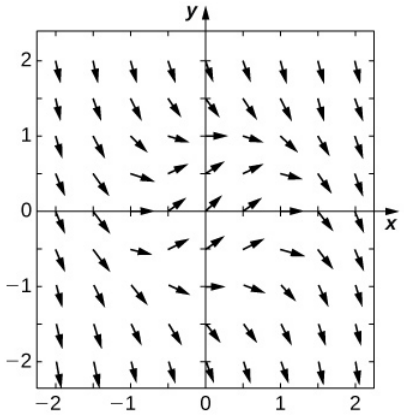


15) $y' = y - x$

16) $y' = 1 - y^2 - x^2$

**Answer**

There do not appear to be any equilibria.

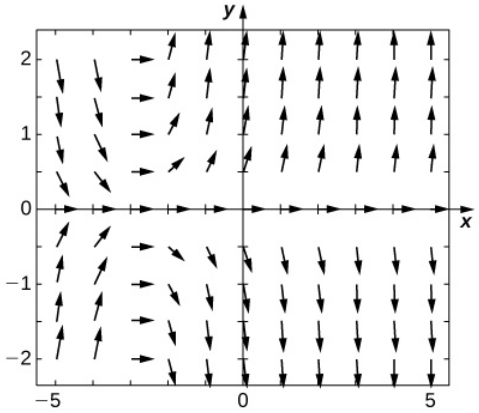



17) $y' = t^2 \sin y$

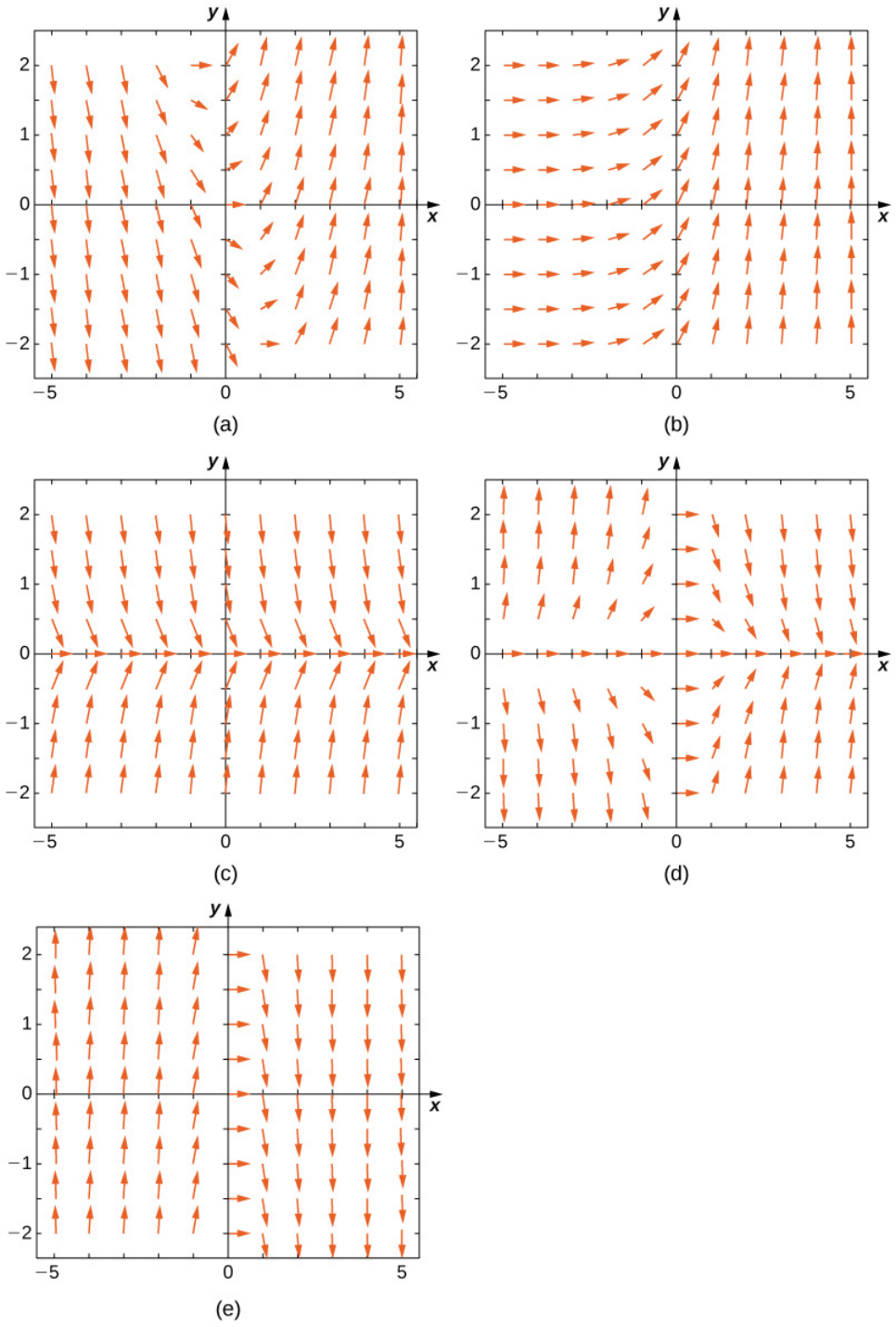
18) $y' = 3y + xy$

**Answer**

There appears to be an unstable equilibrium at $y = 0$.




**Match the direction field with the given differential equations. Explain your selections.**




19) $y' = -3y$

20) $y' = -3t$

**Answer**

$E$

21) $y' = e^t$

22) $y' = \frac{1}{2}y + t$

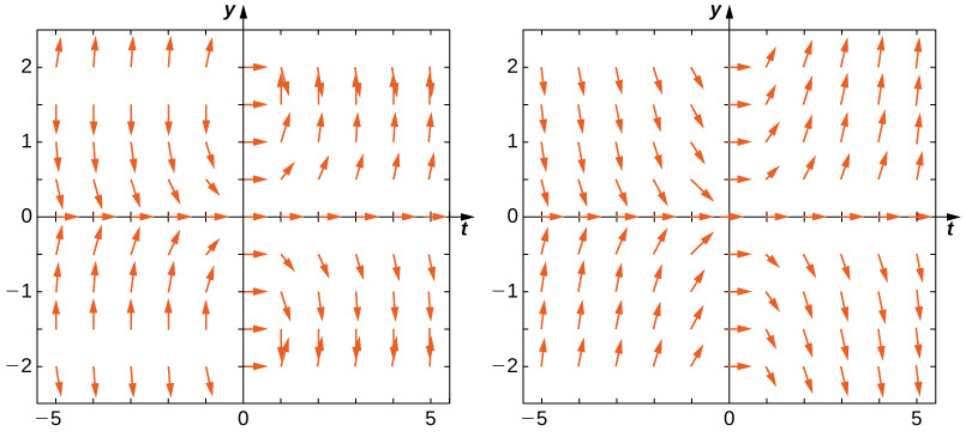

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**Answer**

$A$

23) $y' = -ty$

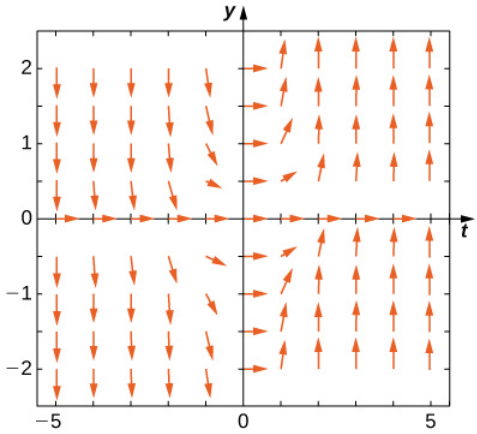
**Match the direction field with the given differential equations. Explain your selections.**



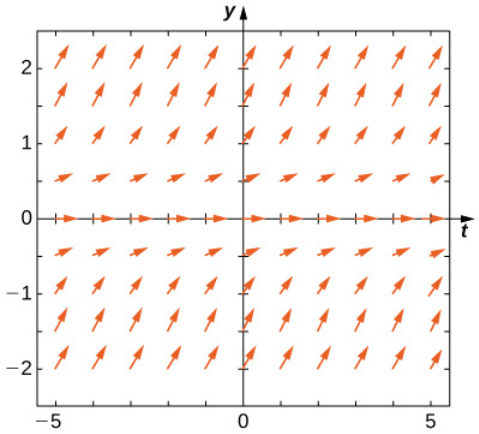

(a)



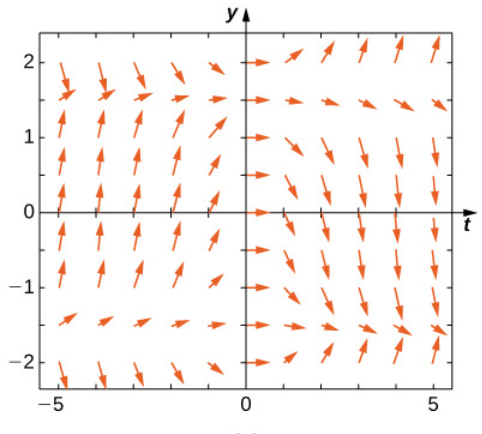

(b)




(c)




(d)




(e)

24) $y' = t \sin y$

**Answer**

$B$


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25) $y' = -t\cos y$

26) $y' = t\tan y$

**Answer**

$A$

27) $y' = \sin^2 y$

28) $y' = y^2 t^3$

**Answer**

$C$

**Estimate the following solutions using Euler's method with $n = 5$ steps over the interval $t = [0, 1]$. If you are able to solve the initial-value problem exactly, compare your solution with the exact solution. If you are unable to solve the initial-value problem, the exact solution will be provided for you to compare with Euler's method. How accurate is Euler's method?**

29) $y' = -3y, \quad y(0) = 1$

30) $y' = t^2, \quad y(0) = 1$

**Answer**

$1.24$, exact: $\frac{4}{3} \approx 1.33$

Solution:

31) $y' = 3t - y, \quad y(0) = 1$. Exact solution is $y = 3t + 4e^{-t} - 3$

32) $y' = y + t^2, \quad y(0) = 3$. Exact solution is $y = 5e^t - 2 - t^2 - 2t$

**Answer**

$7.739364$, exact: $5(e - 1)$

33) $y' = 2t, \quad y(0) = 0$

34) [T] $y' = e^{x+y}, y(0) = -1$. Exact solution is $y = -\ln(e + 1 - e^x)$

**Answer**

$-0.2535$, exact: $0$

35) $y' = y^2 \ln(x + 1), \quad y(0) = 1$. Exact solution is $y = -\dfrac{1}{(x + 1)(\ln(x + 1) - 1)}$

36) $y' = 2^x, \quad y(0) = 0$. Exact solution is $y = \dfrac{2^x - 1}{\ln 2}$

**Answer**

$1.345$, exact: $\frac{1}{\ln(2)}$

37) $y' = y, \quad y(0) = -1$. Exact solution is $y = -e^x$.

38) $y' = -5t, \quad y(0) = -2$. Exact solution is $y = -\frac{5}{2}t^2 - 2$

**Answer**

$-4$, exact: $-9/2 = -4.5$

Differential equations can be used to model **disease epidemics.** In the next set of problems, we examine the change of size of two sub-populations of people living in a city: individuals who are infected and individuals who are susceptible to infection. $S$ represents the size of the susceptible population, and $I$ represents the size of the infected population. We assume that if a

susceptible person interacts with an infected person, there is a probability $c$ that the susceptible person will become infected. Each infected person recovers from the infection at a rate $r$ and becomes susceptible again. We consider the case of influenza, where we assume that no one dies from the disease, so we assume that the total population size of the two sub-populations is a constant number, $N$. The differential equations that model these population sizes are

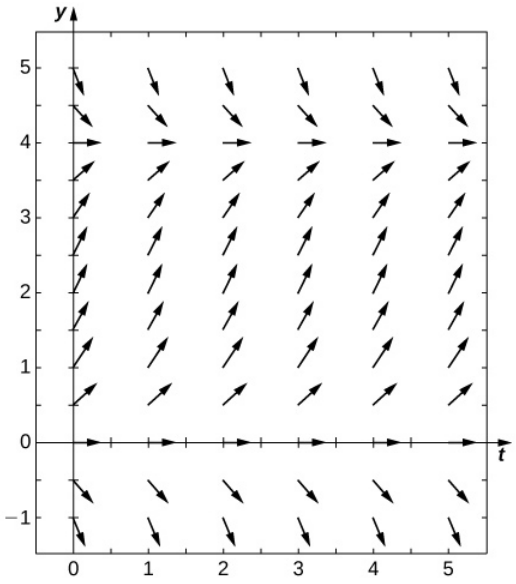
$$S' = rI - cSI \text{ and } I' = cSI - rI.$$

Here $c$ represents the contact rate and $r$ is the recovery rate.

39) Show that, by our assumption that the total population size is constant $(S + I = N)$, you can reduce the system to a single differential equation in $I$ : $I' = c(N - I)I - rI$.

40) Assuming the parameters are $c = 0.5, N = 5,$ and $r = 0.5$, draw the resulting directional field.

**Answer**




41) [T] Use computational software or a calculator to compute the solution to the initial-value problem $y' = ty, y(0) = 2$ using Euler's Method with the given step size $h$. Find the solution at $t = 1$. For a hint, here is "pseudo-code" for how to write a computer program to perform Euler's Method for $y' = f(t, y), y(0) = 2$ :

Create function $f(t, y)$

Define parameters $y(1) = y_0, t(0) = 0,$ step size $h$, and total number of steps, $N$

Write a for-loop:

for $k = 1$ to $N$

$fn = f(t(k), y(k))$

$y(k + 1) = y(k) + h * fn$

$t(k + 1) = t(k) + h$

42) Solve the initial-value problem for the exact solution.

**Answer**

$y' = 2e^{t^2/2}$

43) Draw the directional field

44) $h = 1$

**Answer**

$2$

45) [T] $h = 10$

46) [T] $h = 100$

**Answer**

$3.2756$

47) [T] $h = 1000$

48) [T] Evaluate the exact solution at $t = 1$. Make a table of errors for the relative error between the Euler's method solution and the exact solution. How much does the error change? Can you explain?

**Answer**

Exact solution: y =$2\sqrt{e}$.

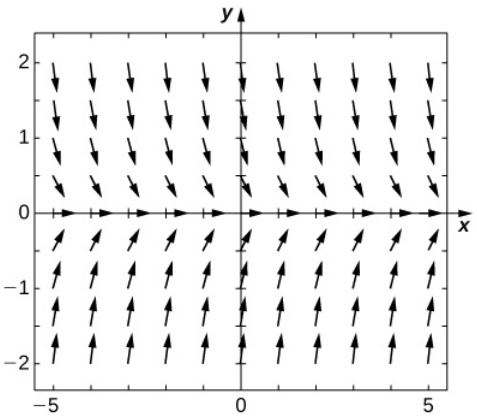
| Step Size | Error |
|---|---|
| $h = 1$ | 0.3935 |
| $h = 10$ | 0.06163 |
| $h = 100$ | 0.006612 |
| $h = 10000$ | 0.0006661 |

**For exercises 49 - 53, consider the initial-value problem $y' = -2y$, with $y(0) = 2$.**

49) Show that $y = 2e^{-2x}$ solves this initial-value problem.

50) Draw the directional field of this differential equation.

**Answer**




51) [T] By hand or by calculator or computer, approximate the solution using Euler's Method at $t = 10$ using $h = 5$.

52) [T] By calculator or computer, approximate the solution using Euler's Method at $t = 10$ using $h = 100$.

**Answer**

$4.0741e^{-10}$

53) [T] Plot exact answer and each Euler approximation (for $h = 5$ and $h = 100$) at each h on the directional field. What do you notice?

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# 4.4: Separable Equations

**Learning Objectives**

- Use separation of variables to solve a differential equation.
- Solve applications using separation of variables.

We now examine a solution technique for finding exact solutions to a class of differential equations known as separable differential equations. These equations are common in a wide variety of disciplines, including physics, chemistry, and engineering. We illustrate a few applications at the end of the section.

## Separation of Variables

We start with a definition and some examples.

**Definition: Separable Differential Equations**

A *separable differential equation* is any equation that can be written in the form

$$y' = f(x)g(y). \tag{4.4.1}$$

The term 'separable' refers to the fact that the right-hand side of Equation 4.4.1 can be separated into a function of $x$ times a function of $y$. Examples of separable differential equations include

$$y' = (x^2 - 4)(3y + 2) \tag{4.4.2}$$
$$y' = 6x^2 + 4x \tag{4.4.3}$$
$$y' = \sec y + \tan y \tag{4.4.4}$$
$$y' = xy + 3x - 2y - 6. \tag{4.4.5}$$

Equation 4.4.3 is separable with $f(x) = 6x^2 + 4x$ and $g(y) = 1$, Equation 4.4.4 is separable with $f(x) = 1$ and $g(y) = \sec y + \tan y$, and the right-hand side of Equation 4.4.5 can be factored as $(x + 3)(y - 2)$, so it is separable as well. Equation 4.4.4 is also called an *autonomous differential equation* because the right-hand side of the equation is a function of $y$ alone. If a differential equation is separable, then it is possible to solve the equation using the method of *separation of variables*.

**Problem-Solving Strategy: Separation of Variables**

1. Check for any values of $y$ that make $g(y) = 0$. These correspond to constant solutions.
2. Rewrite the differential equation in the form

$$\frac{dy}{g(y)} = f(x)dx. \tag{4.4.6}$$

3. Integrate both sides of the equation.
4. Solve the resulting equation for $y$ if possible.
5. If an initial condition exists, substitute the appropriate values for $x$ and $y$ into the equation and solve for the constant.

Note that Step 4 states "Solve the resulting equation for $y$ if possible." It is not always possible to obtain $y$ as an explicit function of $x$. Quite often we have to be satisfied with finding y as an implicit function of $x$.

**Example 4.4.1: Using Separation of Variables**

Find a general solution to the differential equation $y' = (x^2 - 4)(3y + 2)$ using the method of separation of variables.

**Solution**

Follow the five-step method of separation of variables.

1. In this example, $f(x) = x^2 - 4$ and $g(y) = 3y + 2$. Setting $g(y) = 0$ gives $y = -\frac{2}{3}$ as a constant solution.

2. Rewrite the differential equation in the form

$$\frac{dy}{3y+2} = (x^2-4)\,dx.$$

3. Integrate both sides of the equation:

$$\int \frac{dy}{3y+2} = \int (x^2-4)\,dx.$$

Let $u = 3y+2$. Then $du = 3\dfrac{dy}{dx}\,dx$, so the equation becomes

$$\frac{1}{3}\int \frac{1}{u}\,du = \frac{1}{3}x^3 - 4x + C$$

$$\frac{1}{3}\ln|u| = \frac{1}{3}x^3 - 4x + C$$

$$\frac{1}{3}\ln|3y+2| = \frac{1}{3}x^3 - 4x + C.$$

4. To solve this equation for $y$, first multiply both sides of the equation by $3$.

$$\ln|3y+2| = x^3 - 12x + 3C$$

Now we use some logic in dealing with the constant $C$. Since $C$ represents an arbitrary constant, $3C$ also represents an arbitrary constant. If we call the second arbitrary constant $C_1$, where $C_1 = 3C$, the equation becomes

$$\ln|3y+2| = x^3 - 12x + C_1.$$

Now exponentiate both sides of the equation (i.e., make each side of the equation the exponent for the base $e$).

$$e^{\ln|3y+2|} = e^{x^3-12x+C_1}$$
$$|3y+2| = e^{C_1}e^{x^3-12x}$$

Again define a new constant $C_2 = e^{C_1}$ (note that $C_2 > 0$):

$$|3y+2| = C_2e^{x^3-12x}.$$

Because of the absolute value on the left side of the equation, this corresponds to two separate equations:

$$3y+2 = C_2e^{x^3-12x}$$

and

$$3y+2 = -C_2e^{x^3-12x}.$$

The solution to either equation can be written in the form

$$y = \frac{-2 \pm C_2e^{x^3-12x}}{3}.$$

Since $C_2 > 0$, it does not matter whether we use plus or minus, so the constant can actually have either sign. Furthermore, the subscript on the constant $C$ is entirely arbitrary, and can be dropped. Therefore the solution can be written as

$$y = \frac{-2 + Ce^{x^3-12x}}{3}, \text{ where } C = \pm C_2 \text{ or } C = 0.$$

Note that in writing a single general solution in this way, we are also allowing $C$ to equal $0$. This gives us the singular solution, $y = -\dfrac{2}{3}$, for the given differential equation. Check that this is indeed a solution of this differential equation!

5. No initial condition is imposed, so we are finished.

**Exercise 4.4.1**

Use the method of separation of variables to find a general solution to the differential equation

$$y' = 2xy + 3y - 4x - 6.$$

**Hint**

First factor the right-hand side of the equation by grouping, then use the five-step strategy of separation of variables.

**Answer**

$$y = 2 + Ce^{x^2+3x}$$

**Example 4.4.2: Solving an Initial-Value Problem**

Using the method of separation of variables, solve the initial-value problem

$$y' = (2x+3)(y^2-4), \quad y(0) = -1.$$

**Solution**

Follow the five-step method of separation of variables.

1. In this example, $f(x) = 2x + 3$ and $g(y) = y^2 - 4$. Setting $g(y) = 0$ gives $y = \pm 2$ as constant solutions.

2. Divide both sides of the equation by $y^2 - 4$ and multiply by $dx$. This gives the equation

$$\frac{dy}{y^2-4} = (2x+3)\,dx.$$

3. Next integrate both sides:

$$\int \frac{1}{y^2-4}dy = \int (2x+3)\,dx. \tag{4.4.7}$$

To evaluate the left-hand side, use the method of partial fraction decomposition. This leads to the identity

$$\frac{1}{y^2-4} = \frac{1}{4}\left(\frac{1}{y-2} - \frac{1}{y+2}\right).$$

Then Equation 4.4.7 becomes

$$\frac{1}{4}\int \left(\frac{1}{y-2} - \frac{1}{y+2}\right)dy = \int (2x+3)\,dx$$

$$\frac{1}{4}\left(\ln|y-2| - \ln|y+2|\right) = x^2 + 3x + C.$$

Multiplying both sides of this equation by $4$ and replacing $4C$ with $C_1$ gives

$$\ln|y-2| - \ln|y+2| = 4x^2 + 12x + C_1$$

$$\ln\left|\frac{y-2}{y+2}\right| = 4x^2 + 12x + C_1.$$

4. It is possible to solve this equation for $y$. First exponentiate both sides of the equation and define $C_2 = e^{C_1}$:

$$\left|\frac{y-2}{y+2}\right| = C_2 e^{4x^2+12x}.$$

Next we can remove the absolute value and let a new constant $C_3$ be positive, negative, or zero, i.e., $C_3 = \pm C_2$ or $C_3 = 0$.

Then multiply both sides by $y+2$.

$$y - 2 = C_3(y+2)e^{4x^2+12x}$$

$$y-2=C_3ye^{4x^2+12x}+2C_3e^{4x^2+12x}.$$

Now collect all terms involving $y$ on one side of the equation, and solve for $y$:

$$y-C_3ye^{4x^2+12x}=2+2C_3e^{4x^2+12x}$$

$$y\left(1-C_3e^{4x^2+12x}\right)=2+2C_3e^{4x^2+12x}$$

$$y=\frac{2+2C_3e^{4x^2+12x}}{1-C_3e^{4x^2+12x}}.$$

5. To determine the value of $C_3$, substitute $x=0$ and $y=-1$ into the general solution. Alternatively, we can put the same values into an earlier equation, namely the equation $\frac{y-2}{y+2}=C_3e^{4x^2+12}$. This is much easier to solve for $C_3$:

$$\frac{y-2}{y+2}=C_3e^{4x^2+12x}$$

$$\frac{-1-2}{-1+2}=C_3e^{4(0)^2+12(0)}$$

$$C_3=-3.$$

Therefore the solution to the initial-value problem is

$$y=\frac{2-6e^{4x^2+12x}}{1+3e^{4x^2+12x}}.$$

A graph of this solution appears in Figure 4.4.1.

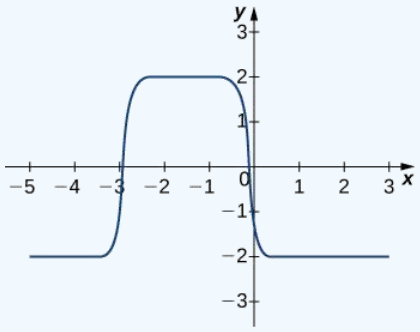



Figure 4.4.1: Graph of the solution to the initial-value problem $y'=(2x+3)(y^2-4), y(0)=-1$.


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**Exercise 4.4.2**

Find the solution to the initial-value problem

$$6y' = (2x+1)(y^2 - 2y - 8)$$

with $y(0) = -3$ using the method of separation of variables.

**Hint**

Follow the steps for separation of variables to solve the initial-value problem.

**Answer**

$$y = \frac{4 + 14e^{x^2+x}}{1 - 7e^{x^2+x}}$$

## Applications of Separation of Variables

Many interesting problems can be described by separable equations. We illustrate two types of problems: solution concentrations and Newton's law of cooling.

### Solution concentrations

Consider a tank being filled with a salt solution. We would like to determine the amount of salt present in the tank as a function of time. We can apply the process of separation of variables to solve this problem and similar problems involving *solution concentrations*.

**Example 4.4.3: Determining Salt Concentration over Time**

A tank containing $100$ L of a brine solution initially has $4$ kg of salt dissolved in the solution. At time $t = 0$, another brine solution flows into the tank at a rate of $2$ L/min. This brine solution contains a concentration of $0.5$ kg/L of salt. At the same time, a stopcock is opened at the bottom of the tank, allowing the combined solution to flow out at a rate of $2$ L/min, so that the level of liquid in the tank remains constant (Figure 4.4.2). Find the amount of salt in the tank as a function of time (measured in minutes), and find the limiting amount of salt in the tank, assuming that the solution in the tank is well mixed at all times.

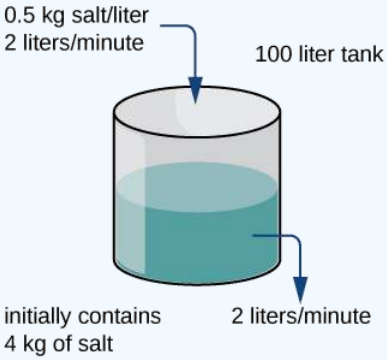



Figure 4.4.2: A brine tank with an initial amount of salt solution accepts an input flow and delivers an output flow. How does the amount of salt change with time?

**Solution**

First we define a function $u(t)$ that represents the amount of salt in kilograms in the tank as a function of time. Then $\frac{du}{dt}$ represents the rate at which the amount of salt in the tank changes as a function of time. Also, $u(0)$ represents the amount of salt in the tank at time $t = 0$, which is $4$ kilograms.

The general setup for the differential equation we will solve is of the form

$$\frac{du}{dt} = \text{INFLOW RATE} \; - \; \text{OUTFLOW RATE}. \tag{4.4.8}$$

INFLOW RATE represents the rate at which salt enters the tank, and OUTFLOW RATE represents the rate at which salt leaves the tank. Because solution enters the tank at a rate of $2$ L/min, and each liter of solution contains $0.5$ kilogram of salt, every minute $2(0.5) = 1$ kilogram of salt enters the tank. Therefore INFLOW RATE = $1$.

To calculate the rate at which salt leaves the tank, we need the concentration of salt in the tank at any point in time. Since the actual amount of salt varies over time, so does the concentration of salt. However, the volume of the solution remains fixed at 100 liters. The number of kilograms of salt in the tank at time $t$ is equal to $u(t)$. Thus, the concentration of salt is $\frac{u(t)}{100}$ kg/L, and the solution leaves the tank at a rate of $2$ L/min. Therefore salt leaves the tank at a rate of $\frac{u(t)}{100} \cdot 2 = \frac{u(t)}{50}$ kg/min, and OUTFLOW RATE is equal to $\frac{u(t)}{50}$. Therefore the differential equation becomes $\frac{du}{dt} = 1 - \frac{u}{50}$, and the initial condition is $u(0) = 4$. The initial-value problem to be solved is

$$\frac{du}{dt} = 1 - \frac{u}{50}, u(0) = 4.$$

The differential equation is a separable equation, so we can apply the five-step strategy for solution.

Step 1. Setting $1 - \frac{u}{50} = 0$ gives $u = 50$ as a constant solution. Since the initial amount of salt in the tank is $4$ kilograms, this solution does not apply.

Step 2. Rewrite the equation as

$$\frac{du}{dt} = \frac{50 - u}{50}.$$

Then multiply both sides by $dt$ and divide both sides by $50 - u$:

$$\frac{du}{50 - u} = \frac{dt}{50}.$$

Step 3. Integrate both sides:

$$\int \frac{du}{50 - u} = \int \frac{dt}{50}$$
$$-\ln|50 - u| = \frac{t}{50} + C.$$

Step 4. Solve for $u(t)$:

$$\ln|50-u| = -\frac{t}{50} - C$$

$$e^{\ln|50-u|} = e^{-(t/50)-C}$$

$$|50-u| = C_1 e^{-t/50}, \text{ where } C_1 = e^{-C}.$$

Eliminate the absolute value by allowing the constant to be positive, negative, or zero, i.e., $C_1 = \pm e^{-C}$ or $C_1 = 0$:

$$50-u = C_1 e^{-t/50}.$$

Finally, solve for $u(t)$:

$$u(t) = 50 - C_1 e^{-t/50}.$$

Step 5. Solve for $C_1$:

$$\begin{aligned} u(0) &= 50 - C_1 e^{-0/50} \\ 4 &= 50 - C_1 \\ C_1 &= 46. \end{aligned}$$

The solution to the initial value problem is $u(t) = 50 - 46e^{-t/50}$. To find the limiting amount of salt in the tank, take the limit as $t$ approaches infinity:

$$\begin{aligned} \lim_{t\to\infty} u(t) &= 50 - 46e^{-t/50} \\ &= 50 - 46(0) = 50. \end{aligned}$$

Note that this was the constant solution to the differential equation. If the initial amount of salt in the tank is $50$ kilograms, then it remains constant. If it starts at less than $50$ kilograms, then it approaches $50$ kilograms over time.

**Exercise 4.4.3**

A tank contains $3$ kilograms of salt dissolved in $75$ liters of water. A salt solution of $0.4$ kg salt/L is pumped into the tank at a rate of $6$ L/min and is drained at the same rate. Solve for the salt concentration at time $t$. Assume the tank is well mixed at all times.

**Hint**

Follow the steps in Example $4.4.3$ and determine an expression for INFLOW and OUTFLOW. Formulate an initial-value problem, and then solve it.

*Initial value problem:*

$$\frac{du}{dt} = 2.4 - \frac{2u}{25},\ u(0) = 3$$

**Answer**

$$u(t) = 30 - 27e^{-t/50}$$

## Newton's Law of Cooling

*Newton's law of cooling* states that the rate of change of an object's temperature is proportional to the difference between its own temperature and the ambient temperature (i.e., the temperature of its surroundings). If we let $T(t)$ represent the temperature of an object as a function of time, then $\frac{dT}{dt}$ represents the rate at which that temperature changes. The temperature of the object's surroundings can be represented by $T_s$. Then Newton's law of cooling can be written in the form

$$\frac{dT}{dt} = k(T(t) - T_s) \tag{4.4.9}$$

or simply

$$\frac{dT}{dt} = k(T - T_s). \tag{4.4.10}$$

The temperature of the object at the beginning of any experiment is the initial value for the initial-value problem. We call this temperature $T_0$. Therefore the initial-value problem that needs to be solved takes the form

$$\frac{dT}{dt} = k(T - T_s) \tag{4.4.11}$$

with $T(0) = T_0$, where $k$ is a constant that needs to be either given or determined in the context of the problem. We use these equations in Example 4.4.4.

**Example 4.4.4: Waiting for a Pizza to Cool**

A pizza is removed from the oven after baking thoroughly, and the temperature of the oven is $350°F$. The temperature of the kitchen is $75°F$, and after $5$ minutes the temperature of the pizza is $340°F$. We would like to wait until the temperature of the pizza reaches $300°F$ before cutting and serving it (Figure 4.4.3). How much longer will we have to wait?

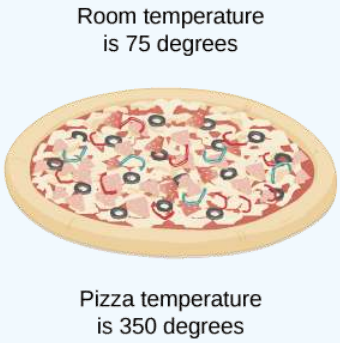



Figure 4.4.3: From Newton's law of cooling, if the pizza cools $10°F$ in $5$ minutes, how long before it cools to $300°F$?

**Solution**

The ambient temperature (surrounding temperature) is $75°F$, so $T_s = 75$. The temperature of the pizza when it comes out of the oven is $350°F$, which is the initial temperature (i.e., initial value), so $T_0 = 350$. Therefore Equation 4.4.11 becomes

$$\frac{dT}{dt} = k(T - 75)$$

with $T(0) = 350$.

To solve the differential equation, we use the five-step technique for solving separable equations.

1. Setting the right-hand side equal to zero gives $T = 75$ as a constant solution. Since the pizza starts at $350°F$, this is not the solution we are seeking.

2. Rewrite the differential equation by multiplying both sides by $dt$ and dividing both sides by $T - 75$:

$$\frac{dT}{T - 75} = k\,dt.$$

3. Integrate both sides:

$$\int \frac{dT}{T - 75} = \int k\,dt$$
$$\ln|T - 75| = kt + C.$$

4. Solve for $T$ by first exponentiating both sides:

$$
\begin{aligned}
e^{\ln|T-75|} &= e^{kt+C} \\
|T-75| &= C_1 e^{kt}, && \text{where } C_1 = e^C. \\
T-75 &= \pm C_1 e^{kt} \\
T-75 &= Ce^{kt}, && \text{where } C = \pm C_1 \text{ or } C = 0. \\
T(t) &= 75 + Ce^{kt}.
\end{aligned}
$$

5. Solve for $C$ by using the initial condition $T(0) = 350$ :

$$
\begin{aligned}
T(t) &= 75 + Ce^{kt} \\
T(0) &= 75 + Ce^{k(0)} \\
350 &= 75 + C \\
C &= 275.
\end{aligned}
$$

Therefore the solution to the initial-value problem is

$$T(t) = 75 + 275e^{kt}.$$

To determine the value of $k$, we need to use the fact that after $5$ minutes the temperature of the pizza is $340°F$. Therefore $T(5) = 340$. Substituting this information into the solution to the initial-value problem, we have

$$T(t) = 75 + 275e^{kt}$$

$$T(5) = 340 = 75 + 275e^{5k}$$

$$265 = 275e^{5k}$$

$$e^{5k} = \frac{53}{55}$$

$$\ln e^{5k} = \ln(\frac{53}{55})$$

$$5k = \ln(\frac{53}{55})$$

$$k = \frac{1}{5}\ln(\frac{53}{55}) \approx -0.007408.$$

So now we have $T(t) = 75 + 275e^{-0.007048t}$. When is the temperature $300°F$? Solving for $t$, we find

$$T(t) = 75 + 275e^{-0.007048t}$$

$$300 = 75 + 275e^{-0.007048t}$$

$$225 = 275e^{-0.007048t}$$

$$e^{-0.007048t} = \frac{9}{11}$$

$$\ln e^{-0.007048t} = \ln \frac{9}{11}$$

$$-0.007048t = \ln \frac{9}{11}$$

$$t = -\frac{1}{0.007048}\ln \frac{9}{11} \approx 28.5.$$

Therefore we need to wait an additional $23.5$ minutes (after the temperature of the pizza reached $340°F$). That should be just enough time to finish this calculation.



**Exercise 4.4.4**

A cake is removed from the oven after baking thoroughly, and the temperature of the oven is $450\,°F$. The temperature of the kitchen is $70°F$, and after $10$ minutes the temperature of the cake is $430°F$.

a. Write the appropriate initial-value problem to describe this situation.
b. Solve the initial-value problem for $T(t)$.
c. How long will it take until the temperature of the cake is within $5°F$ of room temperature?

**Hint**

Determine the values of $T_s$ and $T_0$ then use Equation 4.4.11.

**Answer a**

Initial-value problem

$$\frac{dT}{dt} = k(T - 70), \quad T(0) = 450$$

**Answer b**

$$T(t) = 70 + 380e^{kt}$$

**Answer c**

Approximately $114$ minutes.

## Key Concepts

- A separable differential equation is any equation that can be written in the form $y' = f(x)g(y)$.
- The method of separation of variables is used to find the general solution to a separable differential equation.

## Key Equations

- **Separable differential equation**

  $y' = f(x)g(y)$

- **Solution concentration**

  $\frac{du}{dt} = \text{INFLOW RATE} - \text{OUTFLOW RATE}$

- **Newton's law of cooling**

$$\frac{dT}{dt} = k(T - T_s)$$

## Glossary

**autonomous differential equation**
an equation in which the right-hand side is a function of $y$ alone

**separable differential equation**
any equation that can be written in the form $y' = f(x)g(y)$

**separation of variables**
a method used to solve a separable differential equation

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# 4.4E: Exercises for Section 4.4

**In exercises 1 - 4, solve the following initial-value problems with the initial condition $y_0 = 0$ and graph the solution.**

1) $\frac{dy}{dt} = y + 1$

**Answer**

$y = e^t - 1$

2) $\frac{dy}{dt} = y - 1$

3) $\frac{dy}{dt} = -y + 1$

**Answer**

$y = 1 - e^{-t}$

4) $\frac{dy}{dt} = -y - 1$

**In exercises 5 - 14, find the general solution to the differential equation.**

5) $x^2 y' = (x + 1)y$

**Answer**

$y = Cxe^{-1/x}$

6) $y' = \tan(y)x$

7) $y' = 2xy^2$

**Answer**

$y = \frac{1}{C - x^2}$

8) $\frac{dy}{dt} = y\cos(3t + 2)$

9) $2x\frac{dy}{dx} = y^2$

**Answer**

$y = -\frac{2}{C + \ln|x|}$

10) $y' = e^y x^2$

11) $(1 + x)y' = (x + 2)(y - 1)$

**Answer**

$y = Ce^x(x + 1) + 1$

12) $\frac{dx}{dt} = 3t^2(x^2 + 4)$

13) $t\frac{dy}{dt} = \sqrt{1 - y^2}$

**Answer**

$y = \sin(\ln|t| + C)$

14) $y' = e^x e^y$

**In exercises 15 - 24, find the solution to the initial-value problem.**

15) $y' = e^{y-x}, \quad y(0) = 0$

**Answer**

$y = -\ln(e^{-x})$ which simplifies to $y = x$

16) $y' = y^2(x+1), \quad y(0) = 2$

17) $\frac{dy}{dx} = y^3 x e^{x^2}, \quad y(0) = 1$

**Answer**

$$y = \frac{1}{\sqrt{2 - e^{x^2}}}$$

18) $\frac{dy}{dt} = y^2 e^x \sin(3x), \quad y(0) = 1$

19) $y' = \frac{x}{\operatorname{sech}^2 y}, \quad y(0) = 0$

**Answer**

$$y = \tanh^{-1}\left(\frac{x^2}{2}\right)$$

20) $y' = 2xy(1+2y), \quad y(0) = -1$

21) $\frac{dx}{dt} = \ln(t)\sqrt{1-x^2}, \quad x(1) = 0$

**Answer**

$x = \sin(1 - t + t\ln t)$

22) $y' = 3x^2(y^2+4), \quad y(0) = 0$

23) $y' = e^y 5^x, \quad y(0) = \ln(\ln(5))$

**Answer**

$y = \ln(\ln(5)) - \ln(2 - 5^x)$

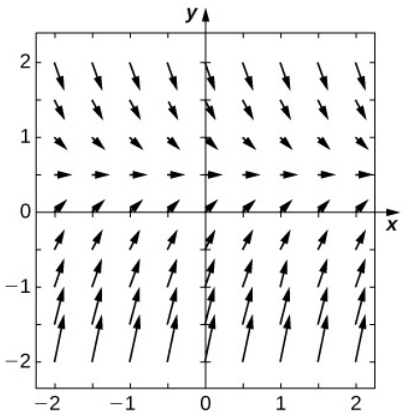
24) $y' = -2x\tan(y), \quad y(0) = \frac{\pi}{2}$

**For problems 25 - 29, use a software program or your calculator to generate the directional fields. Solve explicitly and draw solution curves for several initial conditions. Are there some critical initial conditions that change the behavior of the solution?**

25) [T] $y' = 1 - 2y$

**Answer**

$$y = Ce^{-2x} + \frac{1}{2}$$

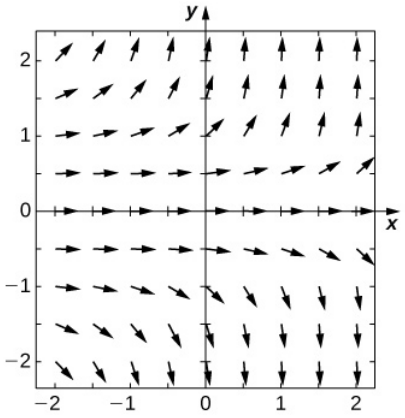



26) [T] $y' = y^2x^3$

27) [T] $y' = y^3e^x$

**Answer**

$$y = \frac{1}{\sqrt{2}\sqrt{C - e^x}}$$

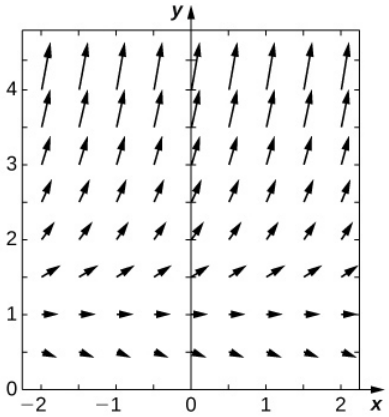



28) [T] $y' = e^y$

29) [T] $y' = y\ln(x)$

**Answer**

$$y = Ce^{-x}x^x$$





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30) Most **drugs in the bloodstream** decay according to the equation $y' = cy$, where $y$ is the concentration of the drug in the bloodstream. If the half-life of a drug is $2$ hours, what fraction of the initial dose remains after $6$ hours?

31) A drug is administered intravenously to a patient at a rate $r$ mg/h and is cleared from the body at a rate proportional to the amount of drug still present in the body, $d$. Set up and solve the differential equation, assuming there is no drug initially present in the body.

**Answer**

$y = \frac{r}{d}(1 - e^{-dt})$

32) [T] How often should a drug be taken if its dose is $3$ mg, it is cleared at a rate $c = 0.1$ mg/h, and $1$ mg is required to be in the bloodstream at all times?

33) A tank contains $1$ kilogram of salt dissolved in $100$ liters of water. A salt solution of $0.1$ kg salt/L is pumped into the tank at a rate of $2$ L/min and is drained at the same rate. Solve for the salt concentration at time $t$. Assume the tank is well mixed.

**Answer**

Since the amount of salt at time $t$ is given by the equation $A(t) = 10 - 9e^{-t/50}$, and the tank volume stays constant at $100$ liters, the concentration at time $t$ would be $c(t) = \frac{1}{100}\left(10 - 9e^{-t/50}\right)$.

34) A tank containing $10$ kilograms of salt dissolved in $1000$ liters of water has two salt solutions pumped in. The first solution of $0.2$ kg salt/L is pumped in at a rate of $20$ L/min and the second solution of $0.05$ kg salt/L is pumped in at a rate of $5$ L/min. The tank drains at $25$ L/min. Assume the tank is well mixed. Solve for the salt concentration at time $t$.

35) [T] For the preceding problem, find how much salt is in the tank $1$ hour after the process begins.

**Answer**

$134.3$ kilograms

36) Torricelli's law states that for a water tank with a hole in the bottom that has a cross-section of $A$ and with a height of water $h$ above the bottom of the tank, the rate of change of volume of water flowing from the tank is proportional to the square root of the height of water, according to $\frac{dV}{dt} = -A\sqrt{2gh}$, where $g$ is the acceleration due to gravity. Note that $\frac{dV}{dt} = A\frac{dh}{dt}$. Solve the resulting initial-value problem for the height of the water, assuming a tank with a hole of radius $2$ ft. The initial height of water is $100$ ft.

37) For the preceding problem, determine how long it takes the tank to drain.

**Answer**

$720$ seconds

**For problems 38 - 44, use Newton's law of cooling.**

38) The liquid base of an ice cream has an initial temperature of $200°F$ before it is placed in a freezer with a constant temperature of $0°F$. After $1$ hour, the temperature of the ice-cream base has decreased to $140°F$. Formulate and solve the initial-value problem to determine the temperature of the ice cream.

39) [T] The liquid base of an ice cream has an initial temperature of $210°F$ before it is placed in a freezer with a constant temperature of $20°F$. After $2$ hours, the temperature of the ice-cream base has decreased to $170°F$. At what time will the ice cream be ready to eat? (Assume $30°F$ is the optimal eating temperature.)

**Answer**

$24$ hours $55$ minutes

40) [T] You are organizing an ice cream social. The outside temperature is $80°F$ and the ice cream is at $10°F$. After $10$ minutes, the ice cream temperature has risen by $10°F$. How much longer can you wait before the ice cream melts at $40°F$?

41) You have a cup of coffee at temperature $70°C$ and the ambient temperature in the room is $20°C$. Assuming a cooling rate $k$ of $0.125$, write and solve the differential equation to describe the temperature of the coffee with respect to time.

**Answer**

$T(t) = 20 + 50e^{-0.125t}$

42) [T] You have a cup of coffee at temperature $70°C$ that you put outside, where the ambient temperature is $0°C$. After $5$ minutes, how much colder is the coffee?

43) You have a cup of coffee at temperature $70°C$ and you immediately pour in $1$ part milk to $5$ parts coffee. The milk is initially at temperature $1°C$. Write and solve the differential equation that governs the temperature of this coffee.

**Answer**

$T(t) = 20 + 38.5e^{-0.125t}$

44) You have a cup of coffee at temperature $70°C$, which you let cool $10$ minutes before you pour in the same amount of milk at $1°C$ as in the preceding problem. How does the temperature compare to the previous cup after $10$ minutes?

45) Solve the generic problem $y' = ay + b$ with initial condition $y(0) = c$.

**Answer**

$y = (c + ba)e^{ax} - \frac{b}{a}$

46) Prove the basic continual compounded interest equation. Assuming an initial deposit of $P_0$ and an interest rate of $r$, set up and solve an equation for continually compounded interest.

47) Assume an initial nutrient amount of $I$ kilograms in a tank with $L$ liters. Assume a concentration of $c$ kg/L being pumped in at a rate of $r$ L/min. The tank is well mixed and is drained at a rate of $r$ L/min. Find the equation describing the amount of nutrient in the tank.

**Answer**

$y(t) = cL + (I - cL)e^{-rt/L}$

48) Leaves accumulate on the forest floor at a rate of $2$ g/cm$^2$/yr and also decompose at a rate of $90\%$ per year. Write a differential equation governing the number of grams of leaf litter per square centimeter of forest floor, assuming at time $0$ there is no leaf litter on the ground. Does this amount approach a steady value? What is that value?

49) Leaves accumulate on the forest floor at a rate of $4$ g/cm$^2$/yr. These leaves decompose at a rate of $10\%$ per year. Write a differential equation governing the number of grams of leaf litter per square centimeter of forest floor. Does this amount approach a steady value? What is that value?

**Answer**

Differential Equation: $\frac{dy}{dt} = 4 - 0.1y$
Solution, the model for this situation: $y = 40(1 - e^{-0.1t})$,
Amount approaches a steady value of 40 g/cm$^2$

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# 4.5: The Logistic Equation

**Learning Objectives**

- Describe the concept of environmental carrying capacity in the logistic model of population growth.
- Draw a direction field for a logistic equation and interpret the solution curves.
- Solve a logistic equation and interpret the results.

Differential equations can be used to represent the size of a population as it varies over time. We saw this in an earlier chapter in the section on exponential growth and decay, which is the simplest model. A more realistic model includes other factors that affect the growth of the population. In this section, we study the logistic differential equation and see how it applies to the study of population dynamics in the context of biology.

## Population Growth and Carrying Capacity

To model population growth using a differential equation, we first need to introduce some variables and relevant terms. The variable $t$. will represent time. The units of time can be hours, days, weeks, months, or even years. Any given problem must specify the units used in that particular problem. The variable $P$ will represent population. Since the population varies over time, it is understood to be a function of time. Therefore we use the notation $P(t)$ for the population as a function of time. If $P(t)$ is a differentiable function, then the first derivative $\frac{dP}{dt}$ represents the instantaneous rate of change of the population as a function of time.

In Exponential Growth and Decay, we studied the exponential growth and decay of populations and radioactive substances. An example of an exponential growth function is $P(t) = P_0e^{rt}$. In this function, $P(t)$ represents the population at time $t, P_0$ represents the **initial population** (population at time $t = 0$), and the constant $r > 0$ is called the **growth rate**. Figure 4.5.1 shows a graph of $P(t) = 100e^{0.03t}$. Here $P_0 = 100$ and $r = 0.03$.

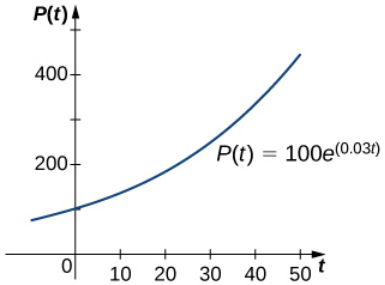



Figure 4.5.1: An exponential growth model of population.

We can verify that the function $P(t) = P_0e^{rt}$ satisfies the initial-value problem

$$\frac{dP}{dt} = rP \tag{4.5.1}$$

with $P(0) = P_0$.

This differential equation has an interesting interpretation. The left-hand side represents the rate at which the population increases (or decreases). The right-hand side is equal to a positive constant multiplied by the current population. Therefore the differential equation states that the rate at which the population increases is proportional to the population at that point in time. Furthermore, it states that the constant of proportionality never changes.

One problem with this function is its prediction that as time goes on, the population grows without bound. This is unrealistic in a real-world setting. Various factors limit the rate of growth of a particular population, including birth rate, death rate, food supply, predators, and so on. The growth constant $r$ usually takes into consideration the birth and death rates but none of the other factors, and it can be interpreted as a net (birth minus death) percent growth rate per unit time. A natural question to ask is whether the population growth rate stays constant, or whether it changes over time. Biologists have found that in many biological systems, the population grows until a certain steady-state population is reached. This possibility is not taken into account with exponential

growth. However, the concept of carrying capacity allows for the possibility that in a given area, only a certain number of a given organism or animal can thrive without running into resource issues.

**Definition: Carrying Capacity**

The *carrying capacity* of an organism in a given environment is defined to be the maximum population of that organism that the environment can sustain indefinitely.

We use the variable $K$ to denote the carrying capacity. The growth rate is represented by the variable $r$. Using these variables, we can define the logistic differential equation.

**Definition: Logistic Differential Equation**

Let $K$ represent the carrying capacity for a particular organism in a given environment, and let $r$ be a real number that represents the growth rate. The function $P(t)$ represents the population of this organism as a function of time $t$, and the constant $P_0$ represents the initial population (population of the organism at time $t = 0$). Then the *logistic differential equation* is

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right). \tag{4.5.2}$$

The logistic equation was first published by Pierre Verhulst in 1845. This differential equation can be coupled with the initial condition $P(0) = P_0$ to form an initial-value problem for $P(t)$.

Suppose that the initial population is small relative to the carrying capacity. Then $\frac{P}{K}$ is small, possibly close to zero. Thus, the quantity in parentheses on the right-hand side of Equation 4.5.2 is close to 1, and the right-hand side of this equation is close to $rP$. If $r > 0$, then the population grows rapidly, resembling exponential growth.

However, as the population grows, the ratio $\frac{P}{K}$ also grows, because $K$ is constant. If the population remains below the carrying capacity, then $\frac{P}{K}$ is less than 1, so $1 - \frac{P}{K} > 0$. Therefore the right-hand side of Equation 4.5.2 is still positive, but the quantity in parentheses gets smaller, and the growth rate decreases as a result. If $P = K$ then the right-hand side is equal to zero, and the population does not change.

Now suppose that the population starts at a value higher than the carrying capacity. Then $\frac{P}{K} > 1$, and $1 - \frac{P}{K} < 0$. Then the right-hand side of Equation 4.5.2 is negative, and the population decreases. As long as $P > K$, the population decreases. It never actually reaches K because $\frac{dP}{dt}$ will get smaller and smaller, but the population approaches the carrying capacity as $t$ approaches infinity. This analysis can be represented visually by way of a phase line. A **phase line** describes the general behavior of a solution to an autonomous differential equation, depending on the initial condition. For the case of a carrying capacity in the logistic equation, the phase line is as shown in Figure 4.5.2.




Figure 4.5.2: A phase line for the differential equation $\frac{dP}{dt} = rP(1 - \frac{P}{K})$.

This phase line shows that when $P$ is less than zero or greater than $K$, the population decreases over time. When $P$ is between 0 and $K$, the population increases over time.

## Example 4.5.1: Examining the Carrying Capacity of a Deer Population

Let's consider the population of white-tailed deer (*Odocoileus virginianus*) in the state of Kentucky. The Kentucky Department of Fish and Wildlife Resources (KDFWR) sets guidelines for hunting and fishing in the state. Before the hunting season of 2004, it estimated a population of 900,000 deer. Johnson notes: "A deer population that has plenty to eat and is not hunted by humans or other predators will double every three years." (George Johnson, "The Problem of Exploding Deer Populations Has No Attractive Solutions," January 12,2001, accessed April 9, 2015)



Figure 4.5.3: (credit: modification of work by Rachel Kramer, Flickr)

This observation corresponds to a rate of increase $r = \frac{\ln(2)}{3} = 0.2311$, so the approximate growth rate is 23.11% per year. (This assumes that the population grows exponentially, which is reasonable—at least in the short term—with plentiful food supply and no predators.) The KDFWR also reports deer population densities for 32 counties in Kentucky, the average of which is approximately 27 deer per square mile. Suppose this is the deer density for the whole state (39,732 square miles). The carrying capacity $K$ is 39,732 square miles times 27 deer per square mile, or 1,072,764 deer.

a. For this application, we have $P_0 = 900,000, K = 1,072,764$, and $r = 0.2311$. Substitute these values into Equation 4.5.2 and form the initial-value problem.
b. Solve the initial-value problem from part a.
c. According to this model, what will be the population in 3 years? Recall that the doubling time predicted by Johnson for the deer population was 3 years. How do these values compare?

Suppose the population managed to reach **1,200,000** What does the logistic equation predict will happen to the population in this scenario?

**Solution**

a. The initial value problem is

$$\frac{dP}{dt} = 0.2311P\left(1 - \frac{P}{1,072,764}\right), \; P(0) = 900,000.$$

b. The logistic equation is an autonomous differential equation, so we can use the method of separation of variables.

Step 1: Setting the right-hand side equal to zero gives $P = 0$ and $P = 1,072,764$. This means that if the population starts at zero it will never change, and if it starts at the carrying capacity, it will never change.

Step 2: Rewrite the differential equation and multiply both sides by:

$$\frac{dP}{dt} = 0.2311P\left(\frac{1,072,764 - P}{1,072,764}\right)$$

$$dP = 0.2311P\left(\frac{1,072,764 - P}{1,072,764}\right)dt$$

$$\frac{dP}{P(1,072,764 - P)} = \frac{0.2311}{1,072,764}dt.$$

Step 3: Integrate both sides of the equation using partial fraction decomposition:

$$\int \frac{dP}{P(1,072,764-P)} = \int \frac{0.2311}{1,072,764}dt$$
$$\frac{1}{1,072,764}\int \left(\frac{1}{P}+\frac{1}{1,072,764-P}\right)dP = \frac{0.2311t}{1,072,764}+C$$
$$\frac{1}{1,072,764}\left(\ln|P|-\ln|1,072,764-P|\right) = \frac{0.2311t}{1,072,764}+C.$$

Step 4: Multiply both sides by 1,072,764 and use the quotient rule for logarithms:

$$\ln\left|\frac{P}{1,072,764-P}\right| = 0.2311t + C_1.$$

Here $C_1 = 1,072,764C$. Next exponentiate both sides and eliminate the absolute value:

$$e^{\ln\left|\frac{P}{1,072,764-P}\right|} = e^{0.2311t+C_1}$$
$$\left|\frac{P}{1,072,764-P}\right| = C_2e^{0.2311t}$$
$$\frac{P}{1,072,764-P} = C_2e^{0.2311t}.$$

Here $C_2 = e^{C_1}$ but after eliminating the absolute value, it can be negative as well. Now solve for:

$$P = C_2e^{0.2311t}(1,072,764-P)$$
$$P = 1,072,764C_2e^{0.2311t} - C_2Pe^{0.2311t}$$
$$P + C_2Pe^{0.2311t} = 1,072,764C_2e^{0.2311t}$$
$$P(1+C_2e^{0.2311t} = 1,072,764C_2e^{0.2311t}$$
$$P(t) = \frac{1,072,764C_2e^{0.2311t}}{1+C_2e^{0.2311t}}.$$

Step 5: To determine the value of $C_2$, it is actually easier to go back a couple of steps to where $C_2$ was defined. In particular, use the equation

$$\frac{P}{1,072,764-P} = C_2e^{0.2311t}.$$

The initial condition is $P(0) = 900,000$. Replace $P$ with $900,000$ and $t$ with zero:

$$\frac{P}{1,072,764-P} = C_2e^{0.2311t}$$
$$\frac{900,000}{1,072,764-900,000} = C_2e^{0.2311(0)}$$
$$\frac{900,000}{172,764} = C_2$$
$$C_2 = \frac{25,000}{4,799}$$
$$\approx 5.209.$$

Therefore

$$P(t) = \frac{1,072,764\left(\frac{25000}{4799}\right)e^{0.2311t}}{1+(250004799)e^{0.2311t}}$$
$$= \frac{1,072,764(25000)e^{0.2311t}}{4799+25000e^{0.2311t}.}$$

Dividing the numerator and denominator by **25,000** gives

$$P(t) = \frac{1,072,764e^{0.2311t}}{0.19196 + e^{0.2311t}}.$$

Figure is a graph of this equation.

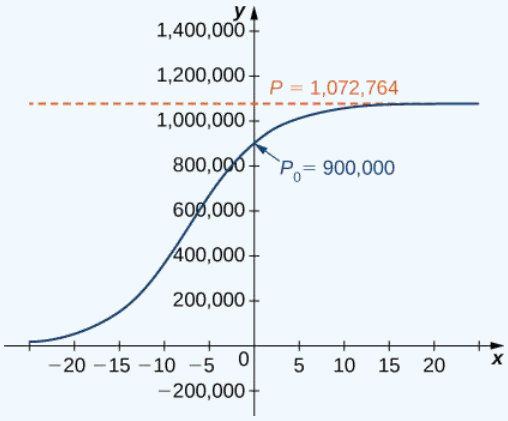



Figure **4.5.4**: Logistic curve for the deer population with an initial population of **900,000** deer.

c. Using this model we can predict the population in **3** years.

$$P(3) = \frac{1,072,764e^{0.2311(3)}}{0.19196 + e^{0.2311(3)}} \approx 978,830\, deer$$

This is far short of twice the initial population of $900,000$. Remember that the doubling time is based on the assumption that the growth rate never changes, but the logistic model takes this possibility into account.

d. If the population reached **1,200,000** deer, then the new initial-value problem would be

$$\frac{dP}{dt} = 0.2311P\left(1 - \frac{P}{1,072,764}\right),\ P(0) = 1,200,000.$$

The general solution to the differential equation would remain the same.

$$P(t) = \frac{1,072,764C_2e^{0.2311t}}{1 + C_2e^{0.2311t}}$$

To determine the value of the constant, return to the equation

$$\frac{P}{1,072,764 - P} = C_2e^{0.2311t}.$$

Substituting the values $t = 0$ and $P = 1,200,000$, you get

$$C_2e^{0.2311(0)} = \frac{1,200,000}{1,072,764 - 1,200,000}$$

$$C_2 = -\frac{100,000}{10,603} \approx -9.431.$$

Therefore

$$P(t) = \frac{1,072,764C_2e^{0.2311t}}{1 + C_2e^{0.2311t}}$$

$$= \frac{1,072,764\left(-\frac{100,000}{10,603}\right)e^{0.2311t}}{1 + \left(-\frac{100,000}{10,603}\right)e^{0.2311t}}$$

$$= -\frac{107,276,400,000e^{0.2311t}}{100,000e^{0.2311t} - 10,603}$$

$$\approx \frac{10,117,551e^{0.2311t}}{9.43129e^{0.2311t} - 1}$$

This equation is graphed in Figure 4.5.5.

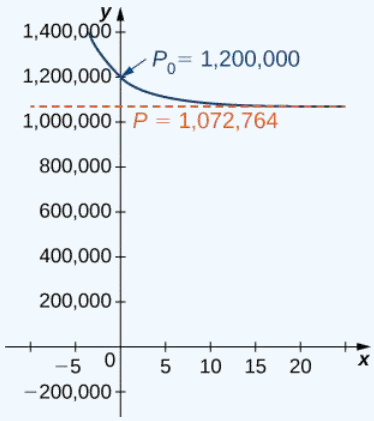



Figure 4.5.5: Logistic curve for the deer population with an initial population of **1,200,000** deer.



## Solving the Logistic Differential Equation

The logistic differential equation is an autonomous differential equation, so we can use separation of variables to find the general solution, as we just did in Example 4.5.1.

Step 1: Setting the right-hand side equal to zero leads to $P = 0$ and $P = K$ as constant solutions. The first solution indicates that when there are no organisms present, the population will never grow. The second solution indicates that when the population starts at the carrying capacity, it will never change.

Step 2: Rewrite the differential equation in the form

$$\frac{dP}{dt} = \frac{rP(K-P)}{K}. \tag{4.5.3}$$

Then multiply both sides by $dt$ and divide both sides by $P(K-P)$. This leads to

$$\frac{dP}{P(K-P)} = \frac{r}{K}dt. \tag{4.5.4}$$

Multiply both sides of the equation by $K$ and integrate:

$$\int \frac{K}{P(K-P)}dP = \int r dt. \tag{4.5.5}$$

The left-hand side of this equation can be integrated using partial fraction decomposition. We leave it to you to verify that

$$\frac{K}{P(K-P)} = \frac{1}{P} + \frac{1}{K-P}. \tag{4.5.6}$$

Then the Equation 4.5.5 becomes

$$\int \frac{1}{P} + \frac{1}{K-P} dP = \int r dt \tag{4.5.7}$$

$$\ln|P| - \ln|K-P| = rt + C \tag{4.5.8}$$

$$\ln\left|\frac{P}{K-P}\right| = rt + C. \tag{4.5.9}$$

Now exponentiate both sides of the equation to eliminate the natural logarithm:

$$e^{\ln\left|\frac{P}{K-P}\right|} = e^{rt+C} \tag{4.5.10}$$

$$\left|\frac{P}{K-P}\right| = e^{C}e^{rt}. \tag{4.5.11}$$

We define $C_1 = e^c$ so that the equation becomes

$$\frac{P}{K-P} = C_1 e^{rt}. \tag{4.5.12}$$

To solve this equation for $P(t)$, first multiply both sides by $K-P$ and collect the terms containing $P$ on the left-hand side of the equation:

$$\begin{aligned} P &= C_1 e^{rt}(K-P) \\ &= C_1 K e^{rt} - C_1 P e^{rt} \\ P + C_1 P e^{rt} &= C_1 K e^{rt}. \end{aligned}$$

Next, factor $P$ from the left-hand side and divide both sides by the other factor:

$$\begin{aligned} P(1 + C_1 e^{rt}) &= C_1 K e^{rt} \\ P(t) &= \frac{C_1 K e^{rt}}{1 + C_1 e^{rt}}. \end{aligned}$$

The last step is to determine the value of $C_1$. The easiest way to do this is to substitute $t = 0$ and $P_0$ in place of $P$ in Equation and solve for $C_1$:

$$\begin{aligned} \frac{P}{K-P} &= C_1 e^{rt} \\ \frac{P_0}{K-P_0} &= C_1 e^{r(0)} \\ C_1 &= \frac{P_0}{K-P_0}. \end{aligned}$$

Finally, substitute the expression for $C_1$ into Equation 4.5.12:

$$P(t) = \frac{C_1 K e^{rt}}{1 + C_1 e^{rt}} = \frac{\frac{P_0}{K-P_0} K e^{rt}}{1 + \frac{P_0}{K-P_0} e^{rt}} \tag{4.5.13}$$

Now multiply the numerator and denominator of the right-hand side by $(K-P_0)$ and simplify:

$$P(t) = \frac{\frac{P_0}{K-P_0}Ke^{rt}}{1+\frac{P_0}{K-P_0}e^{rt}}$$

$$= \frac{\frac{P_0}{K-P_0}Ke^{rt}}{1+\frac{P_0}{K-P_0}e^{rt}} \cdot \frac{K-P_0}{K-P_0} = \frac{P_0Ke^{rt}}{(K-P_0)+P_0e^{rt}}.$$

We state this result as a theorem.

**Solution of the Logistic Differential Equation**

Consider the logistic differential equation subject to an initial population of $P_0$ with carrying capacity $K$ and growth rate $r$. The solution to the corresponding initial-value problem is given by

$$P(t) = \frac{P_0Ke^{rt}}{(K-P_0)+P_0e^{rt}} \tag{4.5.14}$$

.

Now that we have the solution to the initial-value problem, we can choose values for $P_0, r$, and $K$ and study the solution curve. For example, in Example we used the values $r = 0.2311, K = 1,072,764,$ and an initial population of $900,000$ deer. This leads to the solution

$$P(t) = \frac{P_0Ke^{rt}}{(K-P_0)+P_0e^{rt}}$$

$$= \frac{900,000(1,072,764)e^{0.2311t}}{(1,072,764-900,000)+900,000e^{0.2311t}}$$

$$= \frac{900,000(1,072,764)e^{0.2311t}}{172,764+900,000e^{0.2311t}}.$$

Dividing top and bottom by $900,000$ gives

$$P(t) = \frac{1,072,764e^{0.2311t}}{0.19196+e^{0.2311t}}. \tag{4.5.15}$$

This is the same as the original solution. The graph of this solution is shown again in blue in Figure 4.5.6, superimposed over the graph of the exponential growth model with initial population $900,000$ and growth rate $0.2311$ (appearing in green). The red dashed line represents the carrying capacity, and is a horizontal asymptote for the solution to the logistic equation.

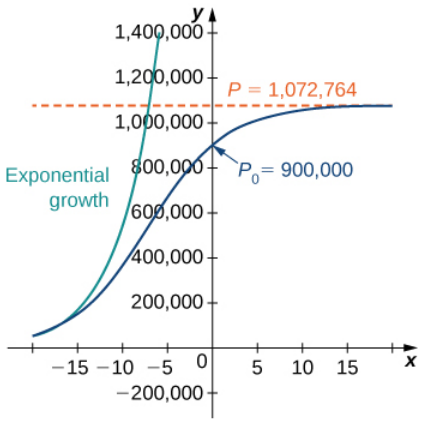



Figure 4.5.6: A comparison of exponential versus logistic growth for the same initial population of $900,000$ organisms and growth rate of $23.11$

Working under the assumption that the population grows according to the logistic differential equation, this graph predicts that approximately $20$ years earlier $(1984)$, the growth of the population was very close to exponential. The net growth rate at that time

would have been around 23.1 per year. As time goes on, the two graphs separate. This happens because the population increases, and the logistic differential equation states that the growth rate decreases as the population increases. At the time the population was measured (2004), it was close to carrying capacity, and the population was starting to level off.

The solution to the logistic differential equation has a point of inflection. To find this point, set the second derivative equal to zero:

$$P(t) = \frac{P_0 K e^{rt}}{(K - P_0) + P_0 e^{rt}}$$

$$P'(t) = \frac{rP_0 K(K - P0)e^{rt}}{((K - P_0) + P_0 e^{rt})^2}$$

$$P''(t) = \frac{r^2 P_0 K(K - P_0)^2 e^{rt} - r^2 P_0^2 K(K - P_0)e^{2rt}}{((K - P_0) + P_0 e^{rt})^3}$$

$$= \frac{r^2 P_0 K(K - P_0)e^{rt}((K - P_0) - P_0 e^{rt})}{((K - P_0) + P_0 e^{rt})^3}.$$

Setting the numerator equal to zero,

$$r^2 P_0 K(K - P_0)e^{rt}((K - P_0) - P_0 e^{rt}) = 0.$$

As long as $P_0 \neq K$, the entire quantity before and including $e^{rt}$is nonzero, so we can divide it out:

$$(K - P_0) - P_0 e^{rt} = 0.$$

Solving for $t$,

$$P_0 e^{rt} = K - P_0$$

$$e^{rt} = \frac{K - P_0}{P_0}$$

$$\ln e^{rt} = \ln \frac{K - P_0}{P_0}$$

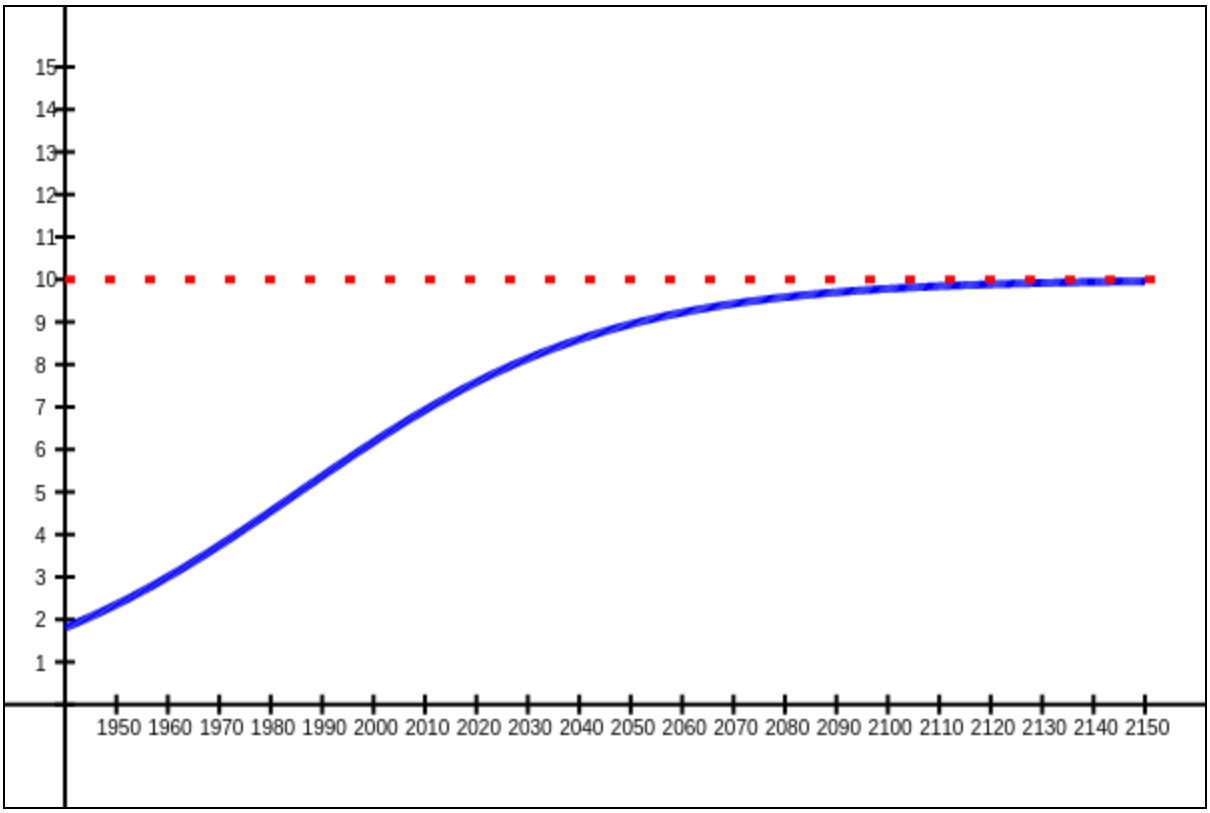
$$rt = \ln \frac{K - P_0}{P_0}$$

$$t = \frac{1}{r} \ln \frac{K - P_0}{P_0}.$$

Notice that if $P_0 > K$, then this quantity is undefined, and the graph does not have a point of inflection. In the logistic graph, the point of inflection can be seen as the point where the graph changes from concave up to concave down. This is where the "leveling off" starts to occur, because the net growth rate becomes slower as the population starts to approach the carrying capacity.

## Logistics Growth and the World Population Depending on the Carrying Capacity

There were 3 billion people on earth in 1960. In the year 2000, there were 6.1 billion people on earth. The graph below shows what the population will be given logistics growth and the carrying capacity that you can change by moving the slider below the graph. This will allow you to forecast the world population over time in the the future.




K = 10 Billion People

Move the slider to change the carrying Capacity, K

**Exercise 4.5.1**

A population of rabbits in a meadow is observed to be 200 rabbits at time $t = 0$. After a month, the rabbit population is observed to have increased by 4. Using an initial population of 200 and a growth rate of 0.04, with a carrying capacity of 750 rabbits,

a. Write the logistic differential equation and initial condition for this model.
b. Draw a slope field for this logistic differential equation, and sketch the solution corresponding to an initial population of 200 rabbits.
c. Solve the initial-value problem for $P(t)$.
d. Use the solution to predict the population after 1 year.

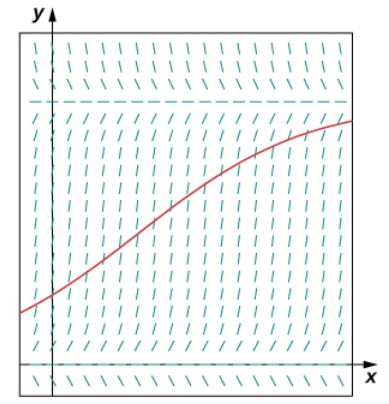
**Hint**

First determine the values of $r, K$, and $P_0$. Then create the initial-value problem, draw the direction field, and solve the problem.

**Answer**

a. $\frac{dP}{dt} = 0.04(1 - \frac{P}{750}), P(0) = 200$

b.




c. $P(t) = \dfrac{3000e^{.04t}}{11 + 4e^{.04t}}$

d. After $12$ months, the population will be $P(12) \approx 278$ rabbits.

### Student Project: Logistic Equation with a Threshold Population

An improvement to the logistic model includes a **threshold population.** The threshold population is defined to be the minimum population that is necessary for the species to survive. We use the variable $T$ to represent the threshold population. A differential equation that incorporates both the threshold population $T$ and carrying capacity $K$ is

$$\frac{dP}{dt} = -rP\left(1 - \frac{P}{K}\right)\left(1 - \frac{P}{T}\right) \tag{4.5.16}$$

where $r$ represents the growth rate, as before.

1. The threshold population is useful to biologists and can be utilized to determine whether a given species should be placed on the endangered list. A group of Australian researchers say they have determined the threshold population for any species to survive: $5000$ adults. (Catherine Clabby, “A Magic Number,” American Scientist 98(1): 24, doi:10.1511/2010.82.24. accessed April 9, 2015, www.americanscientist.org/iss...a-magic-number). Therefore we use $T = 5000$ as the threshold population in this project. Suppose that the environmental carrying capacity in Montana for elk is $25,000$. Set up Equation using the carrying capacity of $25,000$ and threshold population of $5000$. Assume an annual net growth rate of 18%.
2. Draw the direction field for the differential equation from step $1$, along with several solutions for different initial populations. What are the constant solutions of the differential equation? What do these solutions correspond to in the original population model (i.e., in a biological context)?
3. What is the limiting population for each initial population you chose in step $2$? (Hint: use the slope field to see what happens for various initial populations, i.e., look for the horizontal asymptotes of your solutions.)
4. This equation can be solved using the method of separation of variables. However, it is very difficult to get the solution as an explicit function of $t$. Using an initial population of $18,000$ elk, solve the initial-value problem and express the solution as an implicit function of t, or solve the general initial-value problem, finding a solution in terms of $r, K, T$, and $P_0$.

## Key Concepts

- When studying population functions, different assumptions—such as exponential growth, logistic growth, or threshold population—lead to different rates of growth.
- The logistic differential equation incorporates the concept of a carrying capacity. This value is a limiting value on the population for any given environment.
- The logistic differential equation can be solved for any positive growth rate, initial population, and carrying capacity.

## Key Equations

- **Logistic differential equation and initial-value problem**

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right), \quad P(0) = P_0$$

- **Solution to the logistic differential equation/initial-value problem**

$$P(t) = \frac{P_0 K e^{rt}}{(K - P_0) + P_0 e^{rt}}$$

- **Threshold population model**

$$\frac{dP}{dt} = -rP\left(1 - \frac{P}{K}\right)\left(1 - \frac{P}{T}\right)$$

## Glossary

**carrying capacity**
the maximum population of an organism that the environment can sustain indefinitely

**growth rate**
the constant $r > 0$ in the exponential growth function $P(t) = P_0 e^{rt}$

**initial population**
the population at time $t = 0$

**logistic differential equation**
a differential equation that incorporates the carrying capacity $K$ and growth rate rr into a population model

**phase line**
a visual representation of the behavior of solutions to an autonomous differential equation subject to various initial conditions

**threshold population**
the minimum population that is necessary for a species to survive

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# 4.5E: Exercises for Section 4.5

## Basic Logistic Model

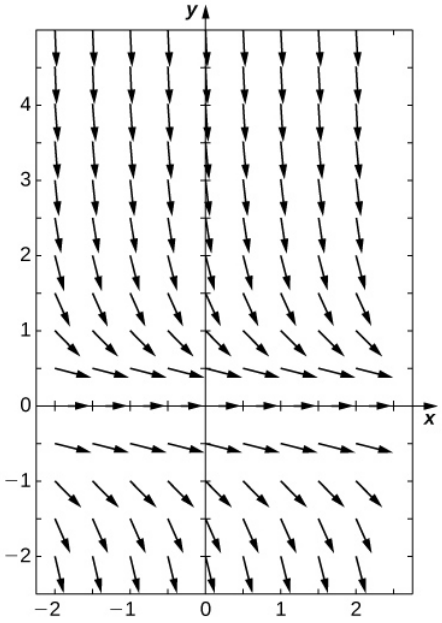
**For problems 1 - 11, consider the logistic equation in the form $P' = CP - P^2$. Draw the directional field and find the stability of the equilibria.**

1) $C = 3$

2) $C = 0$

**Answer**

$P = 0$ semi-stable




3) $C = -3$

4) Solve the logistic equation for $C = 10$ and an initial condition of $P(0) = 2$.

**Answer**

$$P = \frac{10e^{10x}}{e^{10x} + 4}$$

5) Solve the logistic equation for $C = -10$ and an initial condition of $P(0) = 2$.

6) A population of deer inside a park has a carrying capacity of $200$ and a growth rate of $2$. If the initial population is $50$ deer, what is the population of deer at any given time?

**Answer**

$$P(t) = \frac{10000e^{0.02t}}{150 + 50e^{0.02t}}$$

7) A population of frogs in a pond has a growth rate of $5$ If the initial population is $1000$ frogs and the carrying capacity is $6000$, what is the population of frogs at any given time?

8) [T] Bacteria grow at a rate of $20$ per hour in a petri dish. If there is initially one bacterium and a carrying capacity of $1$ million cells, how long does it take to reach $500,000$ cells?

**Answer**

69 hours 5 minutes

9) [T] Rabbits in a park have an initial population of 10 and grow at a rate of 4 per year. If the carrying capacity is 500, at what time does the population reach 100 rabbits?

10) [T] Two monkeys are placed on an island. After 5 years, there are 8 monkeys, and the estimated carrying capacity is 25 monkeys. When does the population of monkeys reach 16 monkeys?

**Answer**

8 years 11 months

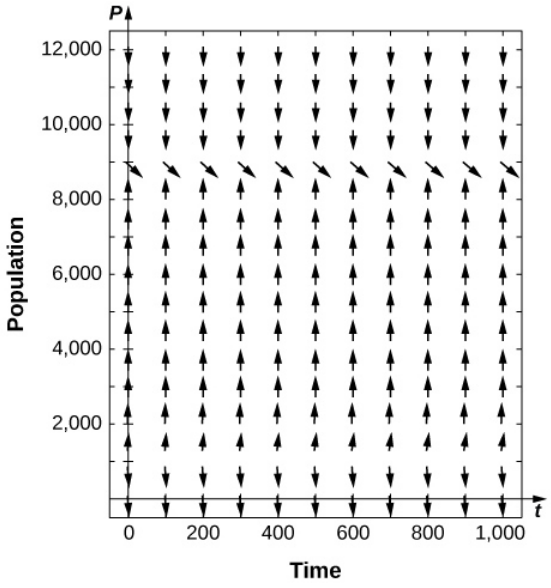
11) [T] A butterfly sanctuary is built that can hold 2000 butterflies, and 400 butterflies are initially moved in. If after 2 months there are now 800 butterflies, when does the population get to 1500 butterflies?

## Logistic Population Model with Depletion

**The following problems consider the logistic equation with an added term for depletion, either through death or emigration.**

12) [T] The population of trout in a pond is given by $P' = 0.4P\left(1 - \frac{P}{10000}\right) - 400$, where 400 trout are caught per year. Use your calculator or computer software to draw a directional field and draw a few sample solutions. What do you expect for the behavior?

**Answer**

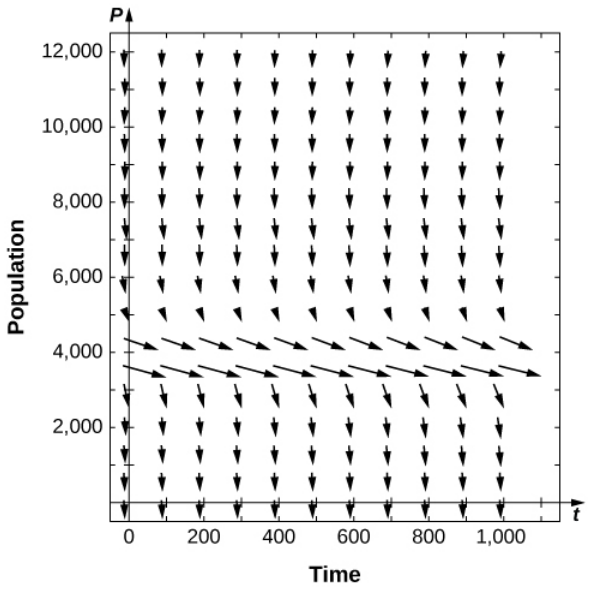



13) In the preceding problem, what are the stabilities of the equilibria $0 < P_1 < P_2$?

14) [T] For the preceding problem, use software to generate a directional field for the value $f = 400$. What are the stabilities of the equilibria?

**Answer**

$P_1$ semi-stable

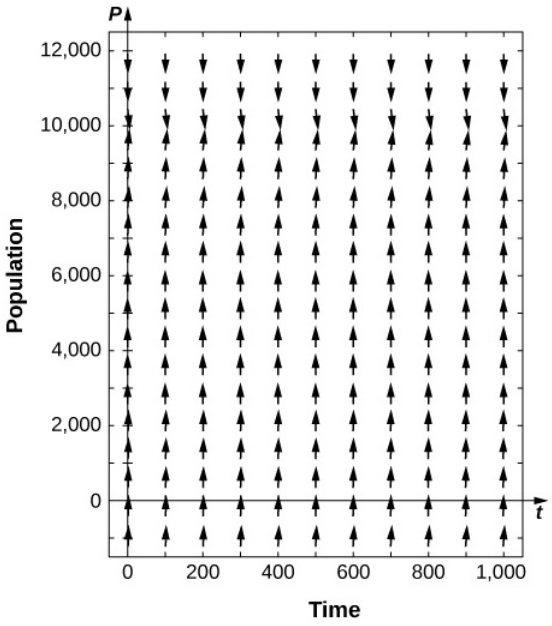



15) [T] For the preceding problems, use software to generate a directional field for the value $f = 600$. What are the stabilities of the equilibria?

16) [T] For the preceding problems, consider the case where a certain number of fish are added to the pond, or $f = -200$. What are the nonnegative equilibria and their stabilities?

**Answer**

$P_2 > 0$ stable



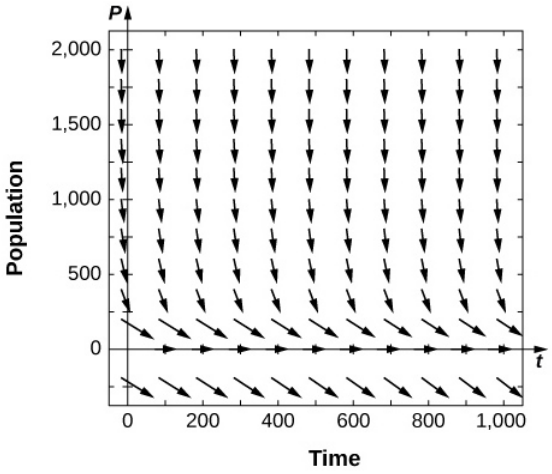

It is more likely that the amount of fishing is governed by the current number of fish present, so instead of a constant number of fish being caught, the rate is proportional to the current number of fish present, with proportionality constant $k$, as $P' = 0.4P\left(1 - \frac{P}{10000}\right) - kP$.

17) [T] For the previous fishing problem, draw a directional field assuming $k = 0.1$. Draw some solutions that exhibit this behavior. What are the equilibria and what are their stabilities?

18) [T] Use software or a calculator to draw directional fields for $k = 0.4$. What are the nonnegative equilibria and their stabilities?

**Answer**

$P_1 = 0$ is semi-stable




19) [T] Use software or a calculator to draw directional fields for $k = 0.6$. What are the equilibria and their stabilities?

20) Solve this equation, assuming a value of $k = 0.05$ and an initial condition of $2000$ fish.

**Answer**

$$y = \frac{-20}{4 \times 10^{-6} - 0.002e^{0.01t}}$$

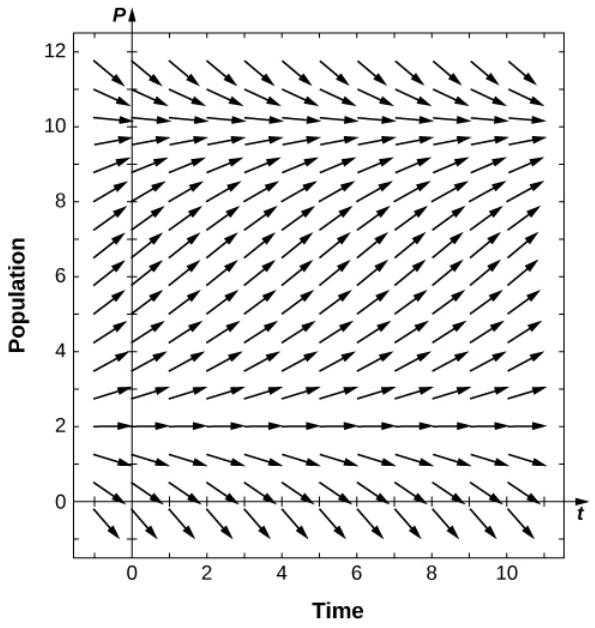
21) Solve this equation, assuming a value of $k = 0.05$ and an initial condition of $5000$ fish.

## Minimal Sustainable Population Thresholds

**The following problems add in a minimal threshold value for the species to survive, $T$, which changes the differential equation to $P'(t) = rP\left(1 - \frac{P}{K}\right)\left(1 - \frac{T}{P}\right)$.**

22) Draw the directional field of the threshold logistic equation, assuming $K = 10, r = 0.1, T = 2$. When does the population survive? When does it go extinct?

**Answer**




23) For the preceding problem, solve the logistic threshold equation, assuming the initial condition $P(0) = P_0$.

24) Bengal tigers in a conservation park have a carrying capacity of $100$ and need a minimum of $10$ to survive. If they grow in population at a rate of $1$ per year, with an initial population of $15$ tigers, solve for the number of tigers present.

**Answer**

$$P(t) = \frac{850 + 500e^{0.009t}}{85 + 5e^{0.009t}}$$

25) A forest containing ring-tailed lemurs in Madagascar has the potential to support $5000$ individuals, and the lemur population grows at a rate of $5$ per year. A minimum of 500 individuals is needed for the lemurs to survive. Given an initial population of $600$ lemurs, solve for the population of lemurs.

26) The population of mountain lions in Northern Arizona has an estimated carrying capacity of $250$ and grows at a rate of $0.25$ per year and there must be $25$ for the population to survive. With an initial population of $30$ mountain lions, how many years will it take to get the mountain lions off the endangered species list (at least $100$)?

**Answer**

$13$ years months

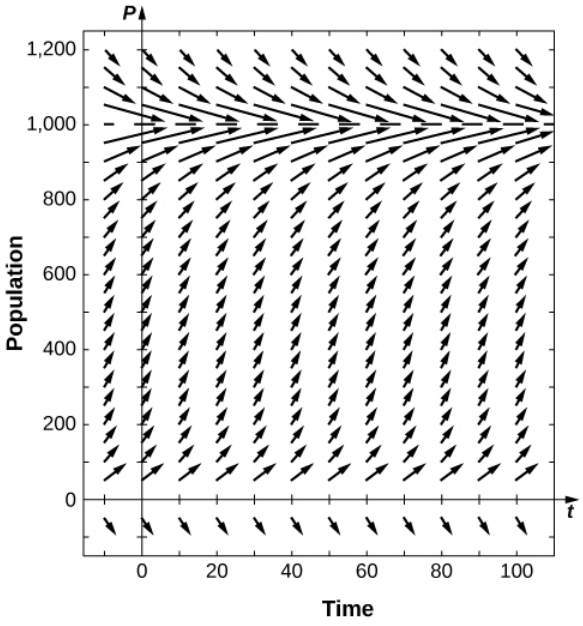
## The Gompertz Equation

**The following questions consider the Gompertz equation, a modification for logistic growth, which is often used for modeling cancer growth, specifically the number of tumor cells.**

27) The Gompertz equation is given by $P(t)' = \alpha \ln\left(\frac{K}{P(t)}\right) P(t)$. Draw the directional fields for this equation assuming all parameters are positive, and given that $K = 1$.

28) Assume that for a population, $K = 1000$ and $\alpha = 0.05$. Draw the directional field associated with this differential equation and draw a few solutions. What is the behavior of the population?

**Answer**




29) Solve the Gompertz equation for generic $\alpha$ and $K$ and $P(0) = P_0$.

30) [T] The Gompertz equation has been used to model tumor growth in the human body. Starting from one tumor cell on day $1$ and assuming $\alpha = 0.1$ and a carrying capacity of $10$ million cells, how long does it take to reach "detection" stage at $5$ million cells?

**Answer**

$31.465$ days

31) [T] It is estimated that the world human population reached $3$ billion people in $1959$ and $6$ billion in $1999$. Assuming a carrying capacity of $16$ billion humans, write and solve the differential equation for logistic growth, and determine what year the population reached $7$ billion.

32) [T] It is estimated that the world human population reached $3$ billion people in $1959$ and $6$ billion in $1999$. Assuming a carrying capacity of $16$ billion humans, write and solve the differential equation for Gompertz growth, and determine what year the population reached $7$ billion. Was logistic growth or Gompertz growth more accurate, considering world population reached $7$ billion on October $31, 2011$?

**Answer**

September $2008$

33) Show that the population grows fastest when it reaches half the carrying capacity for the logistic equation $P' = rP\left(1 - \frac{P}{K}\right)$.

34) When does population increase the fastest in the threshold logistic equation $P'(t) = rP\left(1 - \frac{P}{K}\right)\left(1 - \frac{T}{P}\right)$?

**Answer**

$\frac{K+T}{2}$

35) When does population increase the fastest for the Gompertz equation $P(t)' = \alpha \ln\left(\frac{K}{P(t)}\right) P(t)$?


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Below is a table of the populations of whooping cranes in the wild from $1940$ to $2000$. The population rebounded from near extinction after conservation efforts began. The following problems consider applying population models to fit the data. Assume a carrying capacity of $10,000$ cranes. Fit the data assuming years since $1940$ (so your initial population at time $0$ would be $22$ cranes).

| Year (years since conservation began) | Whooping Crane Population |
|---|---|
| 1940(0) | 22 |
| 1950(10) | 31 |
| 1960(20) | 36 |
| 1970(30) | 57 |
| 1980(40) | 91 |
| 1990(50) | 159 |
| 2000(60) | 256 |

Source: www.savingcranes.org/images/...wc_numbers.pdf

36) Find the equation and parameter $r$ that best fit the data for the logistic equation.

**Answer**

$r = 0.0405$

37) Find the equation and parameters $r$ and $T$ that best fit the data for the threshold logistic equation.

38) Find the equation and parameter $\alpha$ that best fit the data for the Gompertz equation.

**Answer**

$\alpha = 0.0081$

39) Graph all three solutions and the data on the same graph. Which model appears to be most accurate?

40) Using the three equations found in the previous problems, estimate the population in $2010$ (year $70$ after conservation). The real population measured at that time was $437$. Which model is most accurate?

**Answer**

Logistic: $361$, Threshold: $436$, Gompertz: $309$.

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# 4.6: Chapter 4 Review Exercises

**True or False? Justify your answer with a proof or a counterexample.**

1) The differential equation $y' = 3x^2y - \cos(x)y''$ is linear.

2) The differential equation $y' = x - y$ is separable.

**Answer**

$F$

3) You can explicitly solve all first-order differential equations by separation or by the method of integrating factors.

4) You can determine the behavior of all first-order differential equations using directional fields or Euler's method.

**Answer**

$T$

**For the following problems, find the general solution to the differential equations.**

5) $y' = x^2 + 3e^x - 2x$

6) $y' = 2^x + \cos^{-1} x$

**Answer**

$y(x) = \frac{2^x}{ln(2)} + xcos^{-1}x - \sqrt{1-x^2} + C$

7) $y' = y(x^2 + 1)$

8) $y' = e^{-y} \sin x$

**Answer**

$y(x) = \ln(C - \cos x)$

9) $y' = 3x - 2y$

10) $y' = y \ln y$

**Answer**

$y(x) = e^{e^{C+x}}$

**For the following problems, find the solution to the initial value problem.**

11) $y' = 8x - \ln x - 3x^4, \quad y(1) = 5$

12) $y' = 3x - \cos x + 2, \quad y(0) = 4$

**Answer**

$y(x) = 4 + \frac{3}{2}x^2 + 2x - \sin x$

13) $xy' = y(x - 2), \quad y(1) = 3$

14) $y' = 3y^2(x + \cos x), \quad y(0) = -2$

**Answer**

$$y(x) = -\frac{2}{1 + 3(x^2 + 2\sin x)}$$

15) $(x - 1)y' = y - 2, \quad y(0) = 0$

16) $y' = 3y - x + 6x^2, \quad y(0) = -1$

**Answer**

$y(x) = -2x^2 - 2x - \frac{1}{3} - \frac{2}{3}e^{3x}$

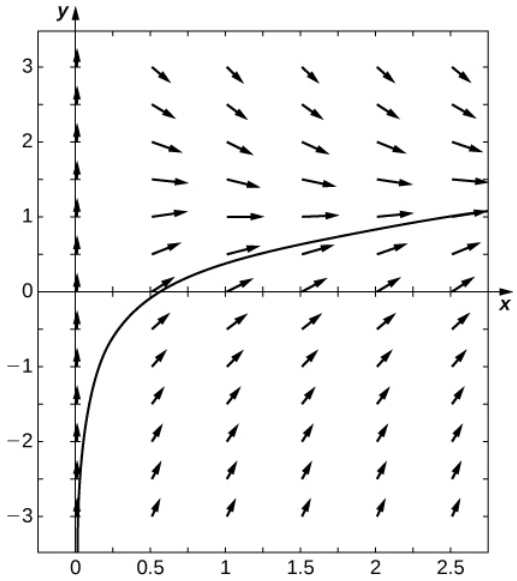
**For the following problems, draw the directional field associated with the differential equation, then solve the differential equation. Draw a sample solution on the directional field.**

17) $y' = 2y - y^2$

18) $y' = \dfrac{1}{x} + \ln x - y$, for $x > 0$

**Answer**

$y(x) = Ce^{-x} + \ln x$




**For the following problems, use Euler's Method with $n = 5$ steps over the interval $t = [0, 1]$. Then solve the initial-value problem exactly. How close is your Euler's Method estimate?**

19) $y' = -4yx, \quad y(0) = 1$

20) $y' = 3^x - 2y, \quad y(0) = 0$

**Answer**

Euler: $0.6939$,

Exact solution: $y(x) = \dfrac{3^x - e^{-2x}}{2 + \ln(3)}$

**For the following problems, set up and solve the differential equations.**

21) A car drives along a freeway, accelerating according to $a = 5\sin(\pi t)$, where $t$ represents time in minutes. Find the velocity at any time $t$, assuming the car starts with an initial speed of $60$ mph.

22) You throw a ball of mass $2$ kilograms into the air with an upward velocity of $8$ m/s. Find exactly the time the ball will remain in the air, assuming that gravity is given by $g = 9.8\,\text{m/s}^2$.

**Answer**

$\frac{40}{49}$ second

23) You drop a ball with a mass of $5$ kilograms out an airplane window at a height of $5000$ m. How long does it take for the ball to reach the ground?

24) You drop the same ball of mass $5$ kilograms out of the same airplane window at the same height, except this time you assume a drag force proportional to the ball's velocity, using a proportionality constant of $3$ and the ball reaches terminal velocity. Solve for the distance fallen as a function of time. How long does it take the ball to reach the ground?

**Answer**

$x(t) = 5000 + \frac{245}{9} - \frac{49}{3}t - \frac{245}{9}e^{-5/3t}, \quad t = 307.8$ seconds

25) A drug is administered to a patient every $24$ hours and is cleared at a rate proportional to the amount of drug left in the body, with proportionality constant $0.2$. If the patient needs a baseline level of $5$ mg to be in the bloodstream at all times, how large should the dose be?

26) A $1000$-liter tank contains pure water and a solution of $0.2$ kg salt/L is pumped into the tank at a rate of $1$ L/min and is drained at the same rate. Solve for total amount of salt in the tank at time $t$.

**Answer**

$T(t) = 200\left(1 - e^{-t/1000}\right)$

27) You boil water to make tea. When you pour the water into your teapot, the temperature is $100°C$. After $5$ minutes in your $15°C$ room, the temperature of the tea is $85°C$. Solve the equation to determine the temperatures of the tea at time $t$. How long must you wait until the tea is at a drinkable temperature ($72°C$)?

28) The human population (in thousands) of Nevada in $1950$ was roughly $160$. If the carrying capacity is estimated at $10$ million individuals, and assuming a growth rate of $2\%$ per year, develop a logistic growth model and solve for the population in Nevada at any time (use $1950$ as time = 0). What population does your model predict for $2000$? How close is your prediction to the true value of $1,998,257$?

**Answer**

$P(t) = \frac{1600000e^{0.02t}}{9840 + 160e^{0.02t}}$

29) Repeat the previous problem but use Gompertz growth model. Which is more accurate?

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# Glossary

**Sample Word 1** | Sample Definition 1

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