

COREQUISITE COMPANION TO PRECALCULUS



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Math 091: Corequisite Companion to Precalculus

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CHAPTER OVERVIEW

1: Simplifying Expressions

Learning Objectives

- Differentiate between the mathematical properties and apply properties correctly.
- Generalize the FOIL method from numerical examples to algebraic examples.
- Categorize special products and apply formulas correctly.
- Assimilate function notation into calculations that use mathematical properties.

[1.1: Exponent Properties and More!](#)

[1.2: FOIL Method and Special Products](#)

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1.1: Exponent Properties and More!



Figure 1.1.1 : “What do you remember most about your math classes?”



Figure 1.1.2 : “All the rules and the x’s and y’s!”

Folks remember, above all else, properties of math! Properties allow us to transform an expression or an equation into an equivalent form. We often need to use properties to move through a problem’s solution. This section introduces several properties. For all the properties in this section, the variables a , b , and c represent real numbers. Here are a few properties, followed by examples for their use:

Property: Commutative Property of Addition

$$a + b = b + a \quad (1.1.1)$$

The order of addition does not matter.

Property: Commutative Property of Multiplication

$$ab = ba \quad (1.1.2)$$

The order of multiplication does not matter.

✓ Example 1.1.1

Evaluate $40 + -6 + 12 + 6 + -40$

Solution

The commutative property of addition allows us to add in any order.

$$\begin{aligned} 40 + -40 + 6 + -6 + 12 & \quad \text{Rearrange the order to group opposites.} \\ = 0 + 0 + 12 & \quad \text{Opposites sum to zero.} \\ = 12 \end{aligned}$$

✓ Example 1.1.2

Why is 20% of 45 the same as 45% of 20?

Solution

Let’s verify the two expressions are equivalent:

$$\begin{aligned} 0.20 \cdot 45 &= 9 \quad \text{20\% of 45 is 9.} \\ 0.45 \cdot 20 &= 9 \quad \text{45\% of 20 is 9.} \end{aligned}$$

The commutative property of multiplication tells us *why* it’s true.

$$20 \cdot 0.01 \cdot 45 = 45 \cdot 0.01 \cdot 20 \quad \text{The order of multiplication is just rearranged.}$$

The commutative property, as demonstrated, shows *why* the equivalency holds.

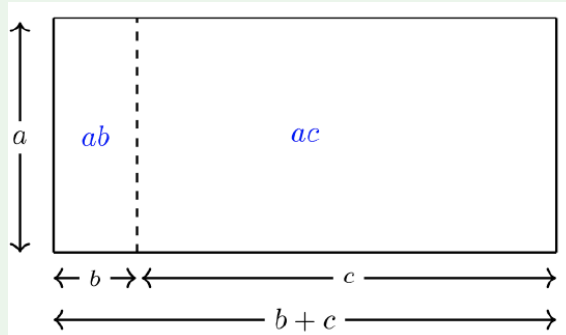
Note

Caution! The order of division and subtraction matters. Therefore, the commutative property only applies to multiplication and addition, NOT to division nor subtraction.

Property: The Distributive Property

$$a(b + c) = ab + ac \quad (1.1.3)$$

The distributive property can be visualized as a rectangular area which is the sum of two smaller rectangular areas:



✓ Example 1.1.3

Simplify $6(2x + 7)$

Solution

$$\begin{aligned} 6(2x + 7) &= 6(2x) + 6(7) \\ &= 12x + 42 \end{aligned}$$

The Product of Powers Property

$$x^a \cdot x^b = x^{a+b} \quad (1.1.4)$$

There are many power properties. Understanding why each property holds will help you decide whether to add or multiply exponents. Here's why the Product Property holds:

$$\underbrace{\underbrace{x \cdot x \cdot x \cdot \dots \cdot x}_{a \text{ number of } x\text{'s}} \cdot \underbrace{x \cdot x \cdot x \cdot \dots \cdot x}_{b \text{ number of } x\text{'s}}}_{a+b \text{ number of } x} = x^a \cdot x^b = x^{a+b}$$

The Power of a Power Property

$$(x^a)^b = x^{ab} \quad (1.1.5)$$

Again, understand why the property holds will help you master the rule:

$$\underbrace{x^a \cdot x^a \cdot x^a \cdot \dots \cdot x^a}_{b \text{ number of } x^a\text{'s}} = \underbrace{x^{a+a+\dots+a}}_{\text{Product of Powers Property}} = \underbrace{x^{ba}}_{\substack{a+a+\dots+a = ba \\ b \text{ number of } a\text{'s}}} = x^{ab}$$

✓ Example 1.1.4

Simplify each of the following expressions.

- a. $5y^8 \cdot y^4 \cdot y$
- b. $9(d^2)^3$
- c. $6u \cdot 3u^4 + (u^3)^2$

Solution

- a. Product of Powers Property

$$\begin{aligned} & 5y^8 \cdot y^4 \cdot y \\ &= 5 \cdot y^{8+4+1} \\ &= 5y^{13} \end{aligned}$$

- b. Power of a Power

$$\begin{aligned} & 9(d^2)^3 \\ &= 9d^{2 \cdot 3} \\ &= 9d^6 \end{aligned}$$

- c. Both properties are used:

$$\begin{aligned} & 6u \cdot 3u^4 + (u^3)^2 \\ &= 6 \cdot 3 \cdot u^{1+4} \cdot u^{3 \cdot 2} \\ &= 18u^5 + u^6 \end{aligned}$$

Note

The sum of powers can be simplified only if the powers are identical in base and exponent. For example, $x^3 + 7x^3 = (1 + 7)x^3 = 8x^3$. Simplify by **combining like terms**. Otherwise, the powers cannot be simplified. See solution to Example 4c.

The Power of a Product Property

$$(xy)^a = x^a \cdot y^a \quad (1.1.6)$$

Each factor inside the parentheses is raised to the power indicated.

Definition: Negative Exponents

$$b^{-n} = \left(\frac{1}{b}\right)^n = \frac{1}{b^n} \quad (1.1.7)$$

Create the reciprocal of base b , then raise b to the positive power, n .

Think of the negative exponent as if it were flagging the power, letting you know that it is in the wrong place.

$$\frac{1}{\boxed{b^{-n}}} = \frac{b^n}{1} = b^n$$



The power b^n is in the wrong place. Move to the top.

✓ Example 1.1.5

Simplify each of the following expressions.

- a. $(3n)^4$
- b. $2(-4x^3y^2z)^2$
- c. $(-2u^{-3}w^5)^{-4}$

Solution

$$\begin{aligned}
 & (3n)^4 \\
 \text{a. } & = 3^n \cdot 4^n \\
 & = 81n^4 \\
 & 2(-4x^3y^2z)^2 \\
 \text{b. } & = 2(-4)^2(x^3)^2(y^2)^2(z)^2 \\
 & = 2 \cdot 16 \cdot x^{3 \cdot 2} \cdot y^{2 \cdot 2} \cdot z^2 \\
 & = 32x^6y^4z^2 \\
 & (-2u^{-3}w^5)^{-4} \\
 & = (-2)^{-4} \cdot (u^{-3})^{-4} \cdot (w^5)^{-4} \\
 \text{c. } & = \frac{1}{-2^4} \cdot u^{(-3)(-4)} \cdot w^{(5)(-4)} \\
 & = \frac{u^{12}}{16w^{20}}
 \end{aligned}$$

The Power of a Quotient Property

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n} \quad (1.1.8)$$

The power is applied to both the numerator and the denominator.

✓ Example 1.1.6

Simplify $\left(\frac{5a^{-1}b^2}{2c^6}\right)^{-2}$

Solution

Let's approach the solution like peeling an onion, beginning with the outer layer:

$$\begin{aligned}
 & = \left(\frac{2c^6}{5a^{-1}b^2}\right)^2 && \text{The reciprocal of the fraction is taken. The power becomes positive.} \\
 & = \left(\frac{2ac^6}{5b^2}\right)^2 && \text{The negative exponent flags the power. } \frac{1}{a^{-1}} = \frac{1}{\frac{1}{a}} = 1 \cdot a = a \\
 & = \left(\frac{(2ac^6)^2}{(5b^2)^2}\right) && \text{The Power of a Quotient Property is applied.} \\
 & = \frac{2^2 \cdot a^2 \cdot (c^6)^2}{5^2 \cdot (b^2)^2} && \text{The Power of a Product Property is applied.} \\
 & = \frac{4a^2c^{12}}{25b^4} && \text{The Power of a Power Property is applied. All is simplified.}
 \end{aligned}$$

The Quotient of Powers Property

$$\frac{x^a}{x^b} = x^{a-b} \quad (1.1.9)$$

The exponents are subtracted is due to the cancellation of common factors. For example, $\frac{x^5}{x^3} = \frac{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot x \cdot x}{\cancel{x} \cdot \cancel{x} \cdot \cancel{x}} = \frac{x \cdot x}{1} = x^2 = x^{5-3}$. Similarly, $\frac{x^3}{x^5} = \frac{\cancel{x} \cdot \cancel{x} \cdot \cancel{x}}{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot x \cdot x} = \frac{1}{x \cdot x} = x^{-2} = x^{3-5}$

✓ Example 1.1.7

Simplify $\left(\frac{4p^3q}{2p^2q^5}\right)^2$

Solution

$$= \left(\frac{4}{2} \cdot p^{3-2} \cdot q^{1-5}\right)^2$$

The Quotient of Powers Property is applied.

$$= (2pq^{-4})^2 \text{ The exponents are simplified.}$$

The Power of a Product Property is applied.

$$= 2^2 \cdot p^2 \cdot (q^{-4})^2$$

The Power of a Power Property is applied.

$$= 4p^2q^{(-4)(2)}$$

The negative exponent flags the power q^8 .

$$= 4p^2q^{-8}$$

$$= \frac{4p^2}{q^8}$$

The power q^8 belongs in the denominator. All is simplified.

Try It! (Exercises)

1. **Name the property** the equation demonstrates.

a. $(5 \cdot 7)^2 = 5^2 \cdot 7^2$

b. $5(-20 + 2) = 5(-20) + 5(2)$

c. $7(x + 1) = (x + 1) \cdot 7$

d. $6^{12} \cdot 6^3 = 6^{15}$

e. $8 + (-9) = -9 + 8$

f. $4^{30} = (4^5)^6$

2. Evaluate the following sum. Rearrange the terms so a calculator isn't necessary.

a. $12 + 84 + (-1) + (-84) + (-12) + 10$

b. $(-1.9) + (-5.8) + 13.2 + 1.9 + 5.8 + (-1.2)$

For #3-11, use the distributive property to simplify.

3. $10(7x + 9)$

4. $2y^2(8 + 2y)$

5. $-7p(4p^4 + 1)$

6. $3b[b^3 + (-2)]$

7. $4u(5u^2 + 9u - 1)$

8. $n^3m(2n^4m - 9nm^2 + 8)$

9. $\frac{1}{2}\left(\frac{3}{4} - z^2\right)$

10. $\frac{2}{5}q^2(5q + 10)$

11. $\frac{7}{8}t(2t^2 - 4t + 3)$

12. The commutative property applies to addition and multiplication. The commutative property does NOT apply to subtraction nor to division. The symbol $\neq$ means "does not equal." Show that the two sides of the inequality produce different results.

a. $6 - 2 \neq 2 - 6$

b. $10 \div 5 \neq 5 \div 10$

For #13-15, use power properties to simplify. Answers should contain positive exponents only.

13. $(3n)^2 + (5n)(6n)$

14. $(5x^32y)^3$

15. $(-3a^5b^{-2}c)^4$

16. **Scientific Notation**, coupled with rules for exponents, eases the computations with very large and very small numbers. Use power properties to evaluate each of the following. Leave your answer in scientific notation. Leave negative powers of 10 as is for the final answer.

- a. $(6.02x10^{24})(1.3x10^{15})$
- b. $(3.14x10^{10})(2.65x10^{18})$
- c. $(7.4x10^{-6})(1.25x10^{-4})$
- d. $(3.8x10^{-16})(2.4x10^{-8})$

For #17-25, use power properties to simplify. Answers should contain positive exponents only. Use fractions, not decimals.

- 17. $\frac{12w^6}{4w}$
- 18. $\frac{5u^2}{10u^3}$
- 19. $\left(\frac{5}{4}\right)^{-1}$
- 20. $\left(\frac{6t}{3}\right)^{-2}$
- 21. $\left(\frac{4n^6}{2n}\right)^{-1}$
- 22. $\left(\frac{7pq^5}{28p^3q}\right)^2$
- 23. $\left(\frac{12x^{10}y^3}{24x^9y^8}\right)^3$
- 24. $\left(\frac{9ab^{12}c^4}{27a^7bc^3}\right)^{-4}$
- 25. $\left(\frac{6r^5s^{-2}}{15r^{-1}s}\right)^{-3}$

For #26-35, simplify using properties.

- 26. $2(3x + 10) + 5(2x - 1)$
- 27. $4a^3(a + 1) - 2a^2(a^2 - a)$
- 28. $(8y^2)2 - 3(7y^3)(2y)$
- 29. $(4d)(9d^3)(2d^2) - (5d^3)^2$
- 30. $\frac{1}{2}b^4(6b - 2) - (3b^2)^2$
- 31. $\frac{3}{4}uv(8u - 4v) + \frac{3}{2}u\left(uv + \frac{2v}{u}\right)$
- 32. $p^2q(4p + 3q) - (2pq)^2$
- 33. $5mn^2(m - n) - 4m^2(n^2 - 1)$
- 34. $10c^8(2c^{-3} + c^{-4}) + c^{-2}(c^6 - 12c^7)$
- 35. $3n^7(3n^{-2})^{-1} + 2n^8(n - 1) + (-n^3)^3$

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1.2: FOIL Method and Special Products

In this section, examples are given for multiplying a **binomial** (2-term polynomial) to another **binomial**. In some cases, the FOIL method yields predictable patterns. We call these “special products.” Recognizing special products will be useful when we turn to solving quadratic equations.

Real numbers inform *how and why* rules work within algebraic expressions. For that reason, let’s examine how to multiply the real numbers: 53×27 .

You might complete the problem using a traditional approach learned in grade school. This approach is shown below.

$$\begin{array}{r} 53 \\ \times 27 \\ \hline 371 \\ 106 \\ \hline 1431 \end{array}$$

Alternatively, we can use total rectangular area to find the product 53×27 . The area total area is the sum of four smaller rectangular areas.

Step 1: Write both 53 and 27 as the sum of tens and ones:

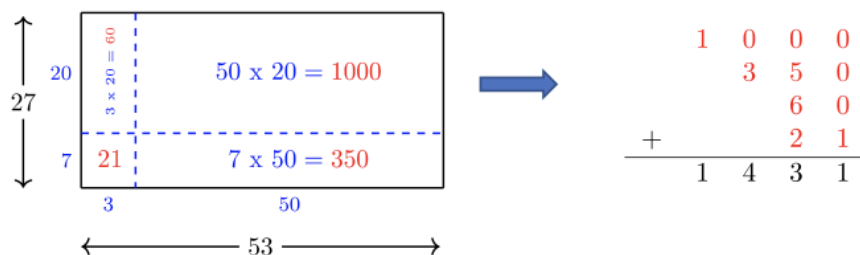
$$53 = 50 + 3$$

$$27 = 20 + 7$$

Step 2: Each side length of the larger rectangle is broken into the sum of tens and ones.

Step 3: Find the area of each of the four smaller rectangles.

Step 4: Sum the four areas to find the total area.



By either method, the correct answer is $53 \times 27 = (50 + 3)(20 + 7) = 1431$.

However, the rectangular area method informs us how binomials are correctly multiplied. We can follow a pattern of multiplication called FOIL: **F**irst **O**uter **I**nner **L**ast.

$$\begin{array}{c} \text{F} \quad \text{O} \\ \text{I} \quad \text{L} \end{array} \quad (50+3)(20+7) = \underbrace{(50)(20)}_{\text{F}} + \underbrace{(50)(7)}_{\text{O}} + \underbrace{(3)(20)}_{\text{I}} + \underbrace{(3)(7)}_{\text{L}} = 1,000 + 350 + 60 + 21 = 1,431$$

Binomials containing algebraic expressions will behave the same way real numbers behave. The FOIL method is required when variables stand in the place of real numbers.

✓ Example 1.2.1

Multiply $(2x + 5)(3x + 2)$.

Solution

Use the FOIL method:

$$(2x+5)(3x+2) = \underbrace{(2x)(3x)}_F + \underbrace{(2x)(2)}_O + \underbrace{(5)(3x)}_I + \underbrace{(5)(2)}_L = 6x^2 + 4x + 15x + 10 = 6x^2 + 19x + 10$$

✓ Example 1.2.2

Multiply $(2x - 5)(x - 4)$.

Solution

Use the FOIL method. Subtraction can be changed to addition, $(2x + (-5))(x + (-4))$, but it's customary to allow subtraction to be perceived as a negative rather than writing it as such. In short, pay attention to your negative values and adjust the operation (add or subtract) accordingly.

$$(2x-5)(x-4) = \underbrace{(2x)(x)}_F - \underbrace{(2x)(4)}_O - \underbrace{(5)(x)}_I + \underbrace{(5)(4)}_L = 2x^2 - 8x - 5x + 20 = 2x^2 - 13x + 20$$

Special Products of Binomials

The FOIL method can be reliably used to multiply all binomials. That is, you are not required to use the following special products if you wish to continue using FOIL. However, getting used to observing mathematical patterns and using patterns is a good math skill to hone.

Case 1: Same terms, but one binomial is a sum, while the other binomial is a difference.

✓ Example 1.2.3

Multiply $(3x + 7)(3x - 7)$

Solution

By the FOIL method: $(3x + 7)(3x - 7) = 9x^2 - 21x + 21x - 49 = 9x^2 - 49$ The two middle terms cancel.

Case 2: Same terms, and same operation: either both are plus, or both are minus.

✓ Example 1.2.4

Multiply $(5x + 2)(5x + 2)$

Solution

By the FOIL method: $(5x + 2)(5x + 2) = 25x^2 + 10x + 10x + 4 = 25x^2 + 20x + 4$ The two middle terms are the same. Double up!

Special Products of Binomials

$$(A + B)(A - B) = A^2 - B^2 \quad (1.2.1)$$

The product is called a **difference of squares**.

$$(A + B)^2 = (A + B)(A + B) = A^2 + 2AB + B^2 \quad (1.2.2)$$

$$(A - B)^2 = (A - B)(A - B) = A^2 - 2AB + B^2$$

The product is called a **perfect square trinomial**.

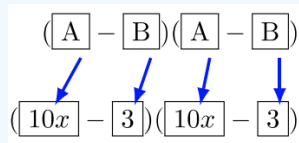
✓ Example 1.2.5

Multiply $(10x - 3)^2$.

Solution

Use the Special Product Formula: $(A - B)^2 = (A - B)(A - B) = A^2 - 2AB + B^2$

Determine the values A and B . The formula will require these substitutions: $A = 10x$ and $B = 3$



$$\begin{array}{c} (\boxed{A} - \boxed{B})(\boxed{A} - \boxed{B}) \\ \swarrow \quad \searrow \quad \swarrow \quad \searrow \\ (\boxed{10x} - \boxed{3})(\boxed{10x} - \boxed{3}) \end{array}$$

$$\begin{aligned} A^2 - 2AB + B^2 & \quad \text{Substitute } A = 10x \text{ and } B = 3 \\ = (10x)^2 - 2(10x)(3) + 3^2 \\ = 100x^2 - 60x + 9 \end{aligned}$$

Answer $(10x - 3)^2 = 100x^2 - 60x + 9$

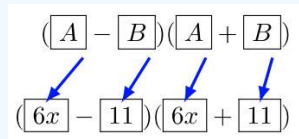
✓ Example 1.2.6

Multiply $(6x - 11)(6x + 11)$

Solution

Use the Special Product Formula: $(A - B)(A + B) = A^2 - B^2$

Determine the values A and B . The formula will require these substitutions: $A = 6x$ and $B = 11$



$$\begin{array}{c} (\boxed{A} - \boxed{B})(\boxed{A} + \boxed{B}) \\ \swarrow \quad \searrow \quad \swarrow \quad \searrow \\ (\boxed{6x} - \boxed{11})(\boxed{6x} + \boxed{11}) \end{array}$$

$$\begin{aligned} A^2 - B^2 & \quad \text{Substitute } A = 6x \text{ and } B = 11 \\ = (6x)^2 - 11^2 \\ = 36x^2 - 121 \end{aligned}$$

Answer: $(6x - 11)(6x + 11) = 36x^2 - 121$

✓ Example 1.2.7

What property of multiplication is demonstrated in the following equation?

$$(6x - 11)(6x + 11) = (6x + 11)(6x - 11)$$

Solution

The quantities $(6x - 11)$ and $(6x + 11)$ stand in the place of real numbers a and b . The order of multiplication does not yield different results. That is, for all real numbers a and b , $ab = ba$. The equation demonstrates the **Commutative Property of Multiplication**.

Multiplying Polynomials of More Than 2 Terms

Finally, let's tackle multiplying polynomials of any number of terms, not just binomials. The FOIL method was developed using area of a rectangle. We'll use the same method to develop a strategy for multiplying polynomials of more than 2 terms to each other.

✓ Example 1.2.8

Multiply $(2x^2 - 4x + 5)(x^2 + 6x - 8)$

Solution

The concept of a rectangle's area will be our guide¹.

$$\underbrace{(2x^2 - 4x + 5)}_{\text{Length}} \underbrace{(x^2 + 6x - 8)}_{\text{Width}} = \text{Total Area}$$

Create a sum of all areas:

$$2x^4 + 12x^3 - 4x^3 - 16x^2 - 12x^2 + 5x^2 + 32x + 30x - 40$$

$$+ 2x^4 + 8x^3 - 35x^2 + 62x - 40$$

Answer: $(2x^2 - 4x + 5)(x^2 + 6x - 8) = 2x^4 + 8x^3 - 35x^2 + 62x - 40$

✓ Example 1.2.9

Multiply $2x(x^3 - 5)(x^2 - 7x + 10)$

Solution

We have 3 quantities, $2x$, $x^3 - 5$, and $x^2 - 7x + 10$. These quantities stand in the place of real numbers. If three real numbers were multiplied, for example: $3 \cdot 4 \cdot 7$, how would you do it? Select any two numbers to multiply! $(3 \cdot 4)^7 = 12 \cdot 7 = 84$. Likewise, algebra follows the same rules.

$[2x(x^3 - 5)](x^2 - 7x + 10)$ Multiply two quantities together.

$(2x^4 - 10x)(x^2 - 7x + 10)$ Use the distributive property to multiply.

Create a table with the terms of each polynomial representing length and width of a rectangle:

	x^2	$-7x$	10
$2x^4$	$2x^6$	$-14x^5$	$20x^4$
$-10x$	$-10x^3$	$70x^2$	$-100x$

Answer $2x(x^3 - 5)(x^2 - 7x + 10) = 2x^6 - 14x^5 + 20x^4 - 10x^3 + 70x^2 - 100x$

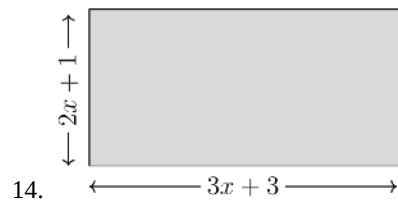
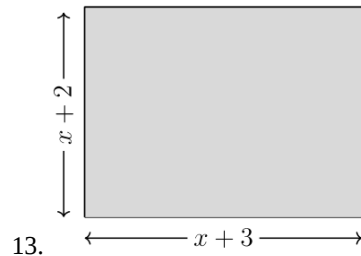
Try It! (Exercises)

For #1-12, multiply using the FOIL method.

- $(x + 7)(x + 6)$
- $(y + 5)(y + 3)$
- $(2t + 9)(t + 1)$
- $(n - 2)(n + 4)$
- $(p + 8)(p - 11)$
- $(3q - 1)(2q + 1)$
- $(10 - m)(12 - m)$
- $(15 - w)(2 + w)$
- $(9 + u)(2 - u)$
- $(5z + 12)(z - 1)$
- $(3r + 7)(2r - 7)$

12. $(6n + 5)(6n - 4)$

For #13-14, find the polynomial that represents the area of each rectangle.



For #15-23, use an appropriate Special Products Formula to multiply.

15. $(y + 5)^2$

16. $(p - 3)(p + 3)$

17. $(t - 7)^2$

18. $(7q - 1)(7q + 1)$

19. $(4n + 9)^2$

20. $(8c + 6)(8c - 6)$

21. $(2u - 2)^2$

22. $(4 - z)^2$

23. $(5 - 3r)^2$

For #24-31, multiply the polynomials.

24. $(x - 8)(x^2 - 3x + 1)$

25. $(2y + 3)(y^2 - 6y - 4)$

26. $(u^2 + 1)(u^2 + 2u - 5)$

27. $(4p^2 - p + 2)(p^2 + 2p - 3)$

28. $2h(3h - 1)(6h + 1)$

29. $5t(t - 4)(t^2 + 3t - 2)$

30. $(2n + 1)^3$

31. $4b(b - 3)^2$

Notes:

1. The process shown is an abstraction of area. Because negative numbers can occur, the product is not true area. Area of a rectangle helps us visualize multiplying each term without duplication.

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1.3: Function Notation and Simplify Expressions

The goal of this section is to practice function notation. Functions have many interconnected mathematical concepts. Therefore, more detail about functions will emerge in your Precalculus course. This section will practice the correspondence of inputs to outputs using function notation, giving further practice with the properties introduced in previous sections.

The following three equations express three different mathematical rules. The rule creates a correspondence between x -values and y -values.

Equation	The Rule In Words
$y = 2x + 6$	Double the x -value then add 6.
$y = \frac{x}{2} - 4$	Take half of the x -value then subtract 4.
$y = x^2 - 10$	Square the x -value then subtract 10.

Definition: Function

A **function** is a rule that assigns each input to exactly one output

A table displays the correspondence between inputs (x -values) and outputs (y -values). The three functions above are shown below using x - y tables. An x -value is chosen, the rule is followed, and a y -value results. Each y -value is computed according to the function's rule.

x	$y = 2x + 6$
-10	$2(-10) + 6 = -14$
-3	$2(-3) + 6 = 0$
0	$2(0) + 6 = 6$
2	$2(2) + 6 = 10$

x	$y = \frac{x}{2} - 4$
-6	$-6 \div 2 - 4 = -7$
0	$0 \div 2 - 4 = -4$
6	$6 \div 2 - 4 = -1$
18	$18 \div 2 - 4 = 5$

x	$y = x^2 - 10$
-1	$(-1)^2 - 10 = -9$
0	$0^2 - 10 = -10$
1	$1^2 - 10 = -9$
4	$4^2 - 10 = 6$

Since the output of the function depends on the input, the y -value is called the **dependent variable**. Function notation emphasizes this dependence by replacing the variable y with $f(x)$.

x - y notation	Function notation	Read out loud
--------------------	-------------------	---------------

$$y = 2x + 6$$

$$f(x) = 2x + 6$$

" f of x equals two x plus six"

✓ Example 1.3.1

Let $f(x) = 2x + 6$. Evaluate each of the following.

- $f(-1)$
- $f\left(\frac{1}{2}\right)$
- $f(3a)$

Solution

In each problem, the input, or x -value, is given inside the parentheses. We substitute the given x -value into the function $f(x) = 2x + 6$.

- $f(-1) = 2(-1) + 6 = -2 + 6 = 4$. Therefore, $f(-1) = 4$.
- $f\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right) + 6 = 1 + 6 = 7$. Therefore, $f\left(\frac{1}{2}\right) = 7$.
- The x -value is replaced by $3a$. This will yield an expression rather than a numerical value: $f(3a) = 2(3a) + 6 = 6a + 6$. Therefore, $f(3a) = 6a + 6$.

Definition: Naming Functions f , g , and h

Functions are typically named with the letters f , g , and h , but functions can take on other letter-names as well. Using different letter-names for functions can help us refer to several different functions in one problem. Example 1.3.2 makes use of this concept.

✓ Example 1.3.2

Let $h(x) = f(x) + g(x)$ where $f(x) = 3x^2$ and $g(x) = 5x - 9$. Evaluate each of the following:

- $h(-4)$
- $h\left(\frac{1}{2}\right)$
- $h(a+1)$

Solution

The function h is defined as the sum of f and g . Plug the given input-value into each function, evaluate to find each output, and take the sum.

$$a. h(-4) = f(-4) + g(-4) = \underbrace{3(-4)^2}_{f(-4)} + \underbrace{5(-4) - 9}_{g(-4)} = 3 \cdot 16 + (-20 - 9) = 48 - 29 = 19$$

Therefore, $h(-4) = 19$.

$$b. h\left(\frac{1}{2}\right) = f\left(\frac{1}{2}\right) + g\left(\frac{1}{2}\right) = \underbrace{3\left(\frac{1}{2}\right)^2}_{f\left(\frac{1}{2}\right)} + \underbrace{5\left(\frac{1}{2}\right) - 9}_{g\left(\frac{1}{2}\right)} = 3\left(\frac{1}{4}\right) + \frac{5}{2} - 9 = \frac{3}{4} + \frac{10}{4} - \frac{36}{4} = -\frac{23}{4}$$

Therefore, $h\left(\frac{1}{2}\right) = -\frac{23}{4}$

- Replace x with $a+1$, then simplify. $h(a+1) = f(a+1) + g(a+1)$

$$\begin{aligned}
 &= \underbrace{3(a+1)^2}_{f(a+1)} + \underbrace{5(a+1) - 9}_{g(a+1)} \\
 &= 3(a+1)(a+1) + 5a + 5 - 9 \\
 &= 3(a^2 + 2a + 1) + 5a - 4 \\
 &= 3a^2 + 6a + 3 + 5a - 4 \\
 &= 3a^2 + 6a + 5a + 3 - 4 \\
 h(a+1) &= 3a^2 + 11a - 1
 \end{aligned}$$

✓ Example 1.3.3

Let $f(x) = 2x^2 - 4x + 1$. Evaluate each of the following:

- $f(-3)$
- $f(a+2)$
- $f(a+2) - f(a)$

Solution

- Replace each x with $x = -3$.

$$f(-3) = 2(-3)^2 - 4(-3) + 1 = 2 \cdot 9 + 12 + 1 = 18 + 12 + 1 = 31$$

Therefore, $f(-3) = 31$.

- Replace each x with $a+2$. Then simplify.

$$\begin{aligned}
 f(a+2) &= 2(a+2)^2 - 4(a+2) + 1 \\
 &= 2(a^2 + 4a + 4) - 4a - 8 + 1 && \text{Use Special Products and the Distributive Property} \\
 &= 2a^2 + 8a + 8 - 4a - 8 + 1 \\
 &= 2a^2 + 8a - 4a + 8 - 8 + 1 && \text{Rearrange terms to combine like terms}
 \end{aligned}$$

$$f(a+2) = 2a^2 + 4a + 1$$

- We will subtract $f(a)$ from the answer to part b.

$$\begin{aligned}
 &\underbrace{2a^2 + 4a + 1}_{f(a+2)} - \underbrace{(2a^2 - 4a + 1)}_{f(a)} && \text{Use answer from part b: } f(a+2) = 2a^2 + 4a + 1 \\
 &= 2a^2 + 4a + 1 - 2a^2 + 4a - 1 && \text{Subtract a quantity, so subtract each term.} \\
 &= 2a^2 - 2a^2 + 4a + 4a + 1 - 1 && \text{Rearrange terms to combine like terms.}
 \end{aligned}$$

Answer $f(a+2) - f(a) = 8a$

✓ Example 1.3.4

Let $f(x) = 3x - 4$. Evaluate each of the following.

- $f(a^2)$
- $[f(a)]^2$

Solution

The order of operations will yield different results.

- $f(a^2)$ indicates $x = a^2$. Therefore, $f(a^2) = 3a^2 - 4$.
- $[f(a)]^2$ indicates $x = a$. After substitution, square the result.

$$\begin{aligned}
 [f(a)]^2 &= (3a - 4)^2 && \text{Substitute } x = a \text{ into the function } f. \\
 [f(a)]^2 &= 9a^2 - 24a + 16 && \text{Use the Special Products Formula.}
 \end{aligned}$$

Try It! (Exercises)

For #1-13, evaluate the functions using f and g as defined below. Do not use a calculator.

$$f(x) = -2x + 1$$

$$g(x) = x^2 + 3x - 2$$

1. $f(3)$
2. $g(1)$
3. $f\left(\frac{1}{2}\right)$
4. $g(-2)$
5. $f(-1) - g(3)$
6. $g(2a)$
7. $f(3a)$
8. $f(a - 4)$
9. $g(a - 1)$
10. $g(a^2)$
11. $[f(a)]^2$
12. $g(a) - f(a + 1)$
13. $g(a) - f(a^2)$

14. Let $h(x) = f(x) + g(x)$. Evaluate each of the following:

- a. $h(-5)$
- b. $h(4a)$
- c. $h(a - 3)$

For #15-34, evaluate the functions using f and g as defined below. Do not use a calculator.

$$f(x) = \frac{x}{x + 1}$$

$$g(x) = \frac{4x}{x + 2}$$

15. $f(3)$
16. $f(-3)$
17. $g(2)$
18. $g(-2)$
19. $f(2) + g(4)$
20. $f(0) - g(6)$
21. $[f(1)]^2$
22. $[g(3)]^2$
23. $f\left(\frac{1}{2}\right)$
24. $g\left(\frac{1}{2}\right)$
25. $f(a - 1)$
26. $g(a - 1)$
27. $[g(1)]^{-2}$
28. $[f(4)]^{-2}$
29. $[f(-2)]^{-1}$
30. $[g(-1)]^{-1}$
31. $\left[g\left(\frac{1}{4}\right)\right]^{-1}$
32. $[f(5)]^{-1} + [g(5)]^{-1}$
33. $[f(5) + g(5)]^{-1}$

34. $\frac{f(a)}{g(a)}$

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CHAPTER OVERVIEW

2: Polynomials

Learning Objectives

- Distinguish between polynomials by terms and degrees, using appropriate vocabulary.
- Use function notation to compute with polynomials.
- Distinguish between factoring techniques.
- Create a factored expression from a given polynomial.

[2.1: The Anatomy of a Polynomial](#)

[2.2: Polynomial Addition, Subtraction, Multiplication and Monomial Division](#)

[2.3: Review of Factoring](#)

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2.1: The Anatomy of a Polynomial

Polynomials are a class of functions that are studied in calculus because they are predictable in their behavior. They are smooth, continuous curves when graphed. They are fairly easy to graph, find roots, and calculate outputs for real-number inputs. Polynomials will continue to be relevant to all of your math and science courses going forward!

Let's start with the prefixes in the following words:

- **Monomial:** Mono means "single" or "one."
- **Binomial:** Bi means "two."
- **Trinomial:** Tri means "three."
- **Polynomial:** Poly means "many" or "multi" or "one or more."

Definition: Monomial

A **monomial** is a single-term expression in which real numbers multiply variables with whole-number exponents.

The following expressions are examples of **monomials**:

$$7x^3 \quad \frac{1}{2}xy \quad 22 \quad pq^5 \quad \pi r^2 \quad -10a^4bc$$

Whole numbers are the counting numbers, starting with zero: 0, 1, 2, 3, 4, ...

The following expressions are *not* monomials because the exponents are not whole numbers.

- × $2x^{-1}$ The exponent on x is -1 .
- × $4u^\pi v$ The exponent on u is π .
- × $5\sqrt{y}$ The exponent on y is $\frac{1}{2}$.
- × $\frac{3}{4t^2}$ The exponent on t is -2 .

Definition: Binomial

A **binomial** is a two-term expression in which two monomials are added or subtracted to form a single expression of 2 terms.

The following expressions are examples of **binomials**:

$$3x + 1 \quad x^4 - y^4 \quad 5y^5 - 5y \quad \pi r^2 + 2\pi rh$$

Definition: Trinomial

A **trinomial** is a three-term expression in which three monomials are added or subtracted.

The following expressions are examples of **trinomials**:

$$x^3 + 4x^2 - 3 \quad p^2q^2 - 5pq + 6 \quad \frac{1}{4}n^2 - mn - \frac{3}{2}n \quad 6t^{10} + 2\pi t^2 + \pi$$

Definition: Polynomial

A **polynomial** is an expression consisting of one or more terms, and each term is a monomial.

Monomials, binomials, and trinomials are special names of polynomials with 1, 2, or 3 terms. You could call a trinomial a polynomial and that is fine! Once the expression extends beyond 3 terms, the polynomial has no special name; it is just a polynomial!

Definition: Degree of a Polynomial

To find the **degree of a polynomial**, inspect each term's exponents. Each term of the polynomial has its own degree. A term's degree is found by summing the exponents of the term. The term with the highest exponent-sum becomes the degree of the polynomial.

✓ Example 2.1.1

Find the degree of each polynomial.

a. $p^2q^2 - 5pq + 6$

b. $2y^5 - 4x^4y^3 + 10y^6 - y$

Solution

a) The degree of the trinomial is determined by the term of highest degree

$$\underbrace{p^2q^2}_{\text{Term1}} - \underbrace{5pq}_{\text{Term2}} + \underbrace{6}_{\text{Term3}}$$

Term 1	p^2q^2	Degree = $2 + 2 = 4$ ✓
Term 2	$-5p^1q^1$	Degree = $1 + 1 = 2$
Term 3	$6p^0q^0$	Degree = $0 + 0 = 0$

Answer The degree of the trinomial is 4.

b) Determine the degree of each of the 4 terms. The term of highest degree is the degree of the polynomial.

$$\underbrace{2y^5}_{\text{Term1}} - \underbrace{4x^4y^3}_{\text{Term2}} + \underbrace{10y^6}_{\text{Term3}} - \underbrace{y}_{\text{Term4}}$$

Term 1	$2y^5$	Degree = 5
Term 2	$4x^4y^3$	Degree = $4 + 3 = 7$ ✓
Term 3	$10y^6$	Degree = 6
Term 4	y	Degree = 1

Answer The degree of the trinomial is 7.

Associated Vocabulary

The terms p^2q^2 and $-5pq$ are **variable terms**, and the term "6" is called a **constant term**. That is, a term without variables is a constant term. Each variable term has a numerical factor which is called a **coefficient** of the term. A polynomial often has terms stated in the descending order of degree. Therefore, the term of highest degree is often stated first, and this is why the term of highest degree is called the **leading term** of the polynomial. The coefficient of the leading term is called the **leading coefficient**.

The table below summarizes polynomial vocabulary and key concepts:

	Polynomial	Descending Order	Leading Coefficient	Degree of Polynomial
Monomials	$-8x^2y$	$-8x^2y$	-8	3
	15	15	15	0
Binomials	$-6n^2m^3 + 3n^7$	$3n^7 - 6n^2m^3$	3	7
	$\frac{2}{3} - \frac{1}{3}b$	$-\frac{1}{3}b + \frac{2}{3}$	$-\frac{1}{3}$	1

	Polynomial	Descending Order	Leading Coefficient	Degree of Polynomial
Trinomials	$3 - 5c - c^2$	$-c^2 - 5c + 3$	-1	2
	$2\pi r^3 h - r^2 + \pi r^3 h^2$	$\pi r^3 h^2 + 2\pi r^3 h - r^2$	π	5

A Single Variable Polynomial and the Function $p(x)$

Most of your work will be with polynomials of a single variable. The following is a formal definition of a single variable polynomial function, $p(x)$.

Definition: Polynomial Function

A polynomial function $p(x)$ is a sum of the terms $a_n x^n$ where $a_0, a_1, a_2, \dots, a_n$ are real numbers and n is a nonnegative integer.

$$p(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n \quad (2.1.1)$$

The above function notation may seem unnecessarily complicated at first glance. However, notice that if we used the alphabet letters A through Z for coefficients, we are limited to 26 terms. Therefore, by using the letter a with a numerical subscript rather than using letters A-Z, we don't have any limitation on the number of terms.

Try It! (Exercises)

For #1-9, say whether each expression is a polynomial. If it is, identify it as a monomial, binomial, or trinomial.

- $12t^4 s^2$
- $1 - \frac{1}{6}p$
- $\frac{4}{c} - 1$
- $3y^4 - 2x^2$
- $4v - u^2 + uv$
- $8r^\pi$
- $-3.8 - 6.2x + 0.4x^2$
- $6\pi r^2 d$
- $5a^2 - 4a + a - 1$
- Create your own example of a monomial of degree eight.
- Create your own example of a trinomial of degree five.
- Create your own example of a binomial of degree four.
- The degree of a nonzero constant term such as 3 is zero. Explain why.

For #14-19, fill in the table appropriately for each polynomial.

	Polynomial	Leading Term	Leading Coefficient	Degree
14.	$1 - x^3$			
15.	$-6a^5 b^3$			
16.	$3p^5 q^4 - 5p^5 + 6p^6 q$			
17.	$2 + \frac{3}{4}x$			
18.	$5\pi - y - 7y^3 + 4\pi y^4 - y^2$			
19.	$1 + 0.25xy + 0.65y$			

- Fill in the blank with the correct vocabulary.

- a. A _____ is a polynomial of a single term.
 - b. For the term bx^n , b is called _____ the of the term and n is the _____ of the term.
 - c. The term with the highest degree is called the _____ term.
 - d. All linear functions of the form $f(x) = mx + b$ are polynomials that have degree = _____.
-

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2.2: Polynomial Addition, Subtraction, Multiplication and Monomial Division

Much of Section 2.2 is mathematics you already know, like combining like terms when adding polynomials, but we'll keep you on your toes with the introduction of function notation! You'll also get the added practice you crave for the various exponent properties. This section aims to strengthen your ability to work with polynomial functions using function notation.

 Adding Functions f and g may be notated $(f+g)(x)$

$$(f+g)(x) = f(x) + g(x) \quad (2.2.1)$$

✓ Example 2.2.1

$f(x) = 10x^3 - 6x^2 + 5$ and $g(x) = 1 - 4x^3$. Find $(f+g)(x)$.

Solution

$$\begin{aligned} (f+g)(x) &= \left(\underbrace{10x^3 - 6x^2 + 5}_{f(x)} \right) + \left(\underbrace{1 - 4x^3}_{g(x)} \right) \\ &= 10x^3 + (-6x^2) + 5 + 1 + (-4x^3) \\ &= 10x^3 + (-4x^3) + (-6x^2) + 5 + 1 \\ &= 6x^3 - 6x^2 + 6 \end{aligned}$$

Turn subtraction into addition: $A - B = A + (-B)$.

Rearrange terms so like terms are side-by-side.

Simplify by combining like terms.

 Subtracting Functions f and g may be notated $(f-g)(x)$

$$(f-g)(x) = f(x) - g(x) \quad (2.2.2)$$

Caution

Subtracting functions must be handled with greater care than adding functions. Remember that the quantity $g(x)$ must be subtracted from $f(x)$. Place parentheses around $g(x)$ and subtract the quantity.

✓ Example 2.2.2

$f(x) = 2x^2 - 8x - 4$ and $g(x) = 3x - 9$. Find $(f-g)(x)$.

Solution

$$\begin{aligned} (f-g)(x) &= \left(\underbrace{2x^2 - 8x - 4}_{f(x)} \right) - \left(\underbrace{3x - 9}_{g(x)} \right) \\ &= \underbrace{2x^2 - 8x - 4}_{f(x)} - \underbrace{(3x + (-9))}_{g(x)} \end{aligned}$$

Consider the sign value of each term.

$$= 2x^2 + (-8x) + (-4) - 3x - (-9)$$

Subtract each term of g . Then simplify.

$$= 2x^2 + (-8x) + (-4) - 3x + 9$$

$$A - -(B) = A + B.$$

$$= 2x^2 + (-8x) + (-3x) + (-4) + 9$$

Rearrange terms so like terms are side-by-side.

$$= 2x^2 - 11x + 5$$

Simplify.

 Multiplying Functions f and g may be notated $(f \cdot g)(x)$

$$(f \cdot g)(x) = f(x) \cdot g(x) \quad (2.2.3)$$

✓ Example 2.2.3

$f(x) = 6 - 2x$ and $g(x) = 3x^2 - 8x - 4$. Find $(f \cdot g)(x)$.

Solution

$$(f \cdot g)(x) = \underbrace{\left(\begin{matrix} 6 \\ -2x \end{matrix} \right)}_{f(x)} \cdot \underbrace{\left(\begin{matrix} 3x^2 \\ -8x \\ -4 \end{matrix} \right)}_{g(x)}$$

The table organizes terms. It's clear what terms must be multiplied. Organization means fewer mistakes!

	$3x^2$	$-8x$	-4
6	$18x^2$	$-48x$	-24
$-2x$	$-6x^3$	$16x^2$	$8x$

$$\begin{aligned} &= -6x^3 + 18x^2 + 16x^2 - 48x + 8x - 24 \quad \text{Gather like terms. Then simplify.} \\ &= -6x^3 + 34x^2 - 40x + 8x \end{aligned}$$

Multiplying Functions f and g may be notated $(f/g)(x)$

$$\left(\frac{f}{g} \right)(x) = \frac{f(x)}{g(x)} \quad (2.2.4)$$

✓ Example 2.2.4

$f(x) = 8x^4$ and $g(x) = 2x^9$. Find $\left(\frac{f}{g} \right)(x)$.

Solution

$$\left(\frac{f}{g} \right)(x) = \frac{8x^4}{2x^9} \quad \text{The numerator and denominator are each monomials.}$$

$$\frac{8x^4}{2x^9} = \frac{8}{2} \cdot \frac{x^4}{x^9} = 4x^{4-9} = 4x^{-5} \quad \text{Use the Quotient of Powers Property to simplify.}$$

$$\frac{4}{x^5} \quad \text{Answer with positive exponents.}$$

Definition: Rational Function

A **rational function** is a function $\frac{f(x)}{g(x)}$, where both $f(x)$ and $g(x)$ are polynomial functions and $g(x)$ is not zero itself.

We will study rational function in more depth in Chapter 5.

Try It! (Exercises)

1. Let $f(x) = 8x + 2$ and $g(x) = 3 - 4x$. Find each of the following:

- $(f + g)(x)$
- $(f - g)(x)$
- $(f \cdot g)(x)$

2. Find $\left(\frac{f}{g} \right)(x)$ given the functions f and g below.

- $f(x) = 10x^{20}$ and $g(x) = 20x^8$

b. $f(x) = \frac{3}{4}x^5$ and $g(x) = \frac{1}{8}x^7$

3. Let $f(x) = x^3 - 6x^2 + 2$ and $g(x) = 5x^2 - 4$. Find each of the following:

a. $(f + g)(x)$

b. $(f - g)(x)$

c. $(f \cdot g)(x)$

4. In general, Is $(f - g)(x)$ the same as $(g - f)(x)$? Explain.

5a. The width of a rectangle is described by the function $f(x) = 2x - 5$. The length of the rectangle is described by the function $g(x) = 3x + 1$. Which of the following describes the area of the rectangle?

i. $(f + g)(x)$

ii. $(f - g)(x)$

iii. $(f \cdot g)(x)$

5b. Find the area of the rectangle as a polynomial function.

6) Since $(f + g)(x) = f(x) + g(x)$, the notation also applies to specific values of x . That is, $(f + g)(2) = f(2) + g(2)$. Evaluate each of the following given $f(x) = -4x^2 + 2x$ and $g(x) = 1 - x^2$.

a. $(f + g)(2)$

b. $(f + g)(0)$

c. $(f + g)(-1)$

d. $(f + g)(-2)$

7) Since $(f \cdot g)(x) = f(x) \cdot g(x)$, the notation also applies to specific values of x . That is, $(f \cdot g)(2) = f(2) \cdot g(2)$. Evaluate each of the following given $f(x) = -4x^2 + 2x$ and $g(x) = 1 - x^2$.

a. $(f \cdot g)(2)$

b. $(f \cdot g)(0)$

c. $(f \cdot g)(-1)$

d. $(f \cdot g)(-2)$

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2.3: Review of Factoring

Factoring is a necessary skill to have in precalculus and all math classes beyond precalculus. Factoring is a tool to simplify expressions and eases calculations. It helps us identify the roots of polynomials, making quick work of graphing without any need for computers or graphing calculators. In the same way arithmetic rules inform algebraic properties, prime factorization of real numbers informs polynomials factorization. The following real numbers have been factored into a product of prime numbers:

$$36 = 2 \cdot 2 \cdot 3 \cdot 3$$

$$40 = 2 \cdot 2 \cdot 2 \cdot 5$$

$$45 = 3 \cdot 3 \cdot 5$$

$$455 = 5 \cdot 7 \cdot 13$$

Polynomial factorization models the same concept: break a polynomial down into the product of its prime polynomials.

This section will describe the following factoring techniques:

- Greatest Common Factor (GCF)
- Trinomial Factorization: $x^2 - bx + c$
- Trinomial Factorization: $ax^2 - bx + c$ where $a \neq 1$.
- Difference of Squares
- Perfect Square Trinomials
- Difference of Cubes
- Sum of Cubes

Greatest Common Factor (GCF)

Think of GCF factoring as reversing (or undoing) the distributive property.

$$\begin{array}{c} \xrightarrow{\text{multiply}} \\ a(b + c + d + \dots) = a(b) + a(c) + a(d) + \dots \\ \xleftarrow{\text{factor}} \end{array}$$

To find the GCF, list the factors of each term. Identify the factors in common. Then factor using the GCF method, as shown in the example below.

✓ Example 2.3.1

Factor the trinomial $16x^5 + 8x^3 - 4x^2$.

Solution

Break down each of the 3 terms of the trinomial into its factors as follows

$$16x^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot x \cdot x \cdot x \cdot x \cdot x$$

$$8x^3 = 2 \cdot 2 \cdot 2 \cdot x \cdot x \cdot x$$

$$4x^2 = 2 \cdot 2 \cdot x \cdot x$$

$$2 \cdot 2 \cdot x \cdot x = 4x^2 = \text{GCF}$$

Next step: factor out the GCF from each of the terms.

Inside the parentheses, preserve placement of the 3 terms and the operations addition and subtraction:

$$\text{GCF}(\text{Term 1} + \text{Term 2} - \text{Term 3}).$$

Answer The factored form of the polynomial is $4x^2(4x^3 + 2x - 1)$

Note: You can check your answer using the distributive property. This step is optional, but by multiplying out the factored form, you will internalize the concept that the product is the original polynomial.



$$4x^2(4x^3 + 2x - 1) = 4x^2(4x^3) + 4x^2(2x) - 4x^2(1) = 16x^5 + 8x^3 - 4x^2$$

Tip: GCF factorization is always the first technique to look for in any polynomial. Sometimes a polynomial will require several different factoring techniques to completely factor the polynomial.

Trinomial Factorization $x^2 + bx + c$

If the leading coefficient of a 2nd degree trinomial is 1, we can factor the trinomial into the product of two binomials: $(x + m)(x + n)$. Two conditions on m and n must hold:

$$\left. \begin{array}{l} mn = c \\ m + n = b \end{array} \right\} \text{Factoring Technique} \implies x^2 + bx + c = (x + m)(x + n)$$

✓ Example 2.3.2

Factor the trinomial $x^2 - x - 12$.

Solution

Think of a pair of integers, m and n , such that the two conditions hold:

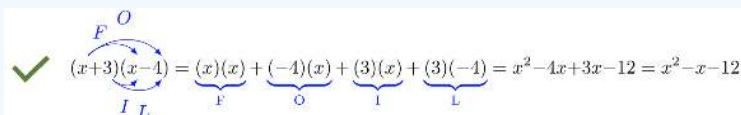
$$\left. \begin{array}{l} mn = -12 \\ m + n = -1 \end{array} \right\}$$

Possible values $m, (n)$:

Product is -12	Success if the sum is -1 ,
$(-1)(12) = -12$	$\times -1 + 12 = 11$
$(1)(-12) = -12$	$\times 12 + (-1) = 11$
$(-2)(6) = -12$	$\times -2 + 6 = 4$
$(2)(-6) = -12$	$\times 2 + (-6) = -4$
$(-3)(4) = -12$	$\times -3 + 4 = 1$
$(3)(-4) = -12$	$\checkmark 3 + (-4) = -1$

Answer The factored form of $x^2 - x - 12$ is $(x + 3)(x - 4)$.

Note: You can check your answer using the FOIL method. This step is optional, but by multiplying out the factored form, you will internalize the concept that the product is the original polynomial.



$$(x+3)(x-4) = (x)(x) + (-4)(x) + (3)(x) + (3)(-4) = x^2 - 4x + 3x - 12 = x^2 - x - 12$$

👋 Note

Some polynomials are **prime**, which means the polynomial cannot be factored. If every possible factoring technique fails, the polynomial is a **prime polynomial**.

Trinomial Factorization $ax^2 + bx + c$ where $a \neq 1$

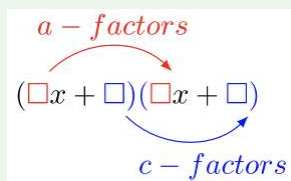
If the leading coefficient of a 2nd degree trinomial is not 1, it will require more trial-and-error to factor the polynomial. The goal, as with the case $a = 1$, is to create two binomials that, when multiplied by the FOIL method, produce the given polynomial. However, the a -value will add more attempted trials. Don't forget that some polynomials are prime and cannot be factored. The steps below guide your approach to factoring.

✎ Trial-and-Error Method for Factoring $ax^2 + bx + c$ and $a \neq 1$

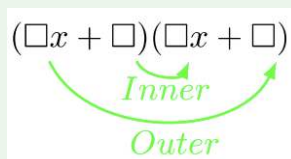
Step 1: Is there a GCF in the trinomial? Factor out a common value if you find one.

Step 2: List the pairs of **factors of a** and the pairs of **factors of c** .

Step 3: Construct the binomials. Fill in the 4 boxes appropriately.



Step 4: FOIL: Outer = big smiley-face. Inner = small smiley-face. Multiply to find Outer and Inner of FOIL. Find a successful combination such that **Outer + Inner** = bx = Target Sum.



Step 5: Keep a -factors fixed, but consider reversing c -factors if the trial fails to produce bx .

Step 6: If all combinations of factors fail to produce the Target Sum bx , the trinomial is prime.

✓ Example 2.3.3

Factor $2x^2 + 3x + 1$

Solution

The first term of the trinomial is $2x^2$. Since $a \neq 1$, let's walk through each step:

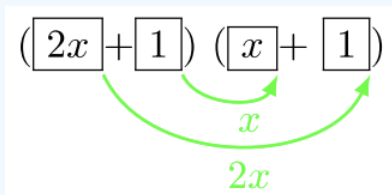
Step 1: GCF = 1, so no common factors.

Step 2: List pairs of factors of a and of c :

$a = 2$	$c = 1$
$2 \cdot 1$	$1 \cdot 1$

Step 3: $(2x + 1)(1x + 1)$

Step 4: Let's check Outer and Inner to make sure their sum is $bx = 3x$.



$$\text{Outer} + \text{Inner} = 2x + x = 3x$$

Success! We found the correct factorization!

Answer The factored form of the trinomial $2x^2 + 3x + 1 = (2x + 1)(x + 1)$

✓ Example 2.3.4

Factor $6t^2 - 28t + 16$

Solution

Step 1: GCF = 2. Factor out the common factor: $2(3t^2 - 14t + 8)$

Step 2: List pairs of factors of a and of c :

Notice the Target Sum is negative.

$$bx = -14t$$

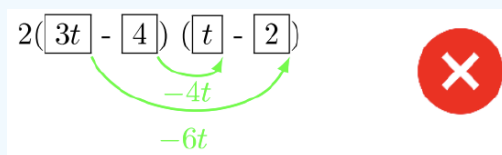
However, c is positive. The c -factors must both be negative (see the table below)

$a = 3$	$c = 8$
$3 \cdot 1$	$-8 \cdot -1$
	$-4 \cdot -2$

Step 3: Place the a -factors, then choose a pair of c -factors. Since the target sum is $-14t$, it's more likely that the smaller numbers will be successful.

Trial 1: $2(3t - 4)(1t - 2)$

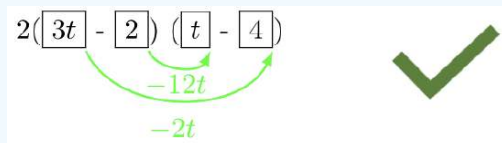
Step 4: Let's check the Outer and Inner.



Trial 1 fails. $-4t - 6t \neq -14t$

Trial 2: $2(3t - 2)(1t - 4)$

Step 5: Let's check the Outer and Inner



Trial 2 is a success! $-12t - 2t = -14t$

Answer $6t^2 - 28t + 16 = 2(3t - 2)(t - 4)$. The trinomial is completely factored.

Difference of Squares

Below are three examples that demonstrate a pattern after using the FOIL method.

Multiply the Binomials		FOIL Method		Simplified Result
$(x + 3)(x - 3)$	=	$x^2 + \underbrace{-3x + 3x}_{\text{O + I cancels}} - 3 \cdot 3$	=	$x^2 - 9$
$(y - 10)(y + 10)$	=	$y^2 + \underbrace{+10y - 10y}_{\text{O + I cancels}} - 10 \cdot 10$	=	$y^2 - 100$
$(n^2 + 4)(n^2 - 4)$	=	$n^4 + \underbrace{-4n^2 + 4n^2}_{\text{O + I cancels}} - 4 \cdot 4$	=	$n^4 - 16$

Two binomials are multiplied. One is the conjugate of the other. That is, in one binomial x and a are added, and in the other x and a are subtracted. Notice that the outer and inner terms always cancel each other under this pattern.

Special Product = Difference of Squares: $(A + B)(A - B) = A^2 - B^2$

Factoring is the reverse of multiplying. In order to factor, we have to unsquare the perfect squares, creating a product:

$$\begin{array}{c}
 \xrightarrow{\text{multiply}} \\
 (x+a)(x-a) = \underbrace{x^2 - a^2}_{\text{difference of squares}} \\
 \xleftarrow{\text{factor}}
 \end{array}$$

✓ Example 2.3.5

Factor: $n^2 - 64$

Solution

Both terms are perfect squares. We'll use the Special Product formula in reverse: $A^2 - B^2 = (A+B)(A-B)$. In order to factor, we need the unsquare of each term!

$$\begin{array}{cc}
 n^2 & - & 64 \\
 \downarrow & & \downarrow \\
 n & & 8
 \end{array}
 \quad
 \boxed{\text{Unsquare!}}$$

Here, $A = n$ and $B = 8$. The formula guides our factoring:

$$\begin{array}{l}
 (A+B)(A-B) \\
 (n+8)(n-8)
 \end{array}$$

Answer $n^2 - 64 = (n+8)(n-8)$

✓ Example 2.3.6

Factor completely. $4x^2y - 81y$

Solution

When factoring, the first thing to look for is a Greatest Common Factor (GCF). Both terms of the polynomial share the factor y . After factoring out the GCF, we factor the parentheses as a difference of squares.

$$4x^2y - 81y$$

GCF = y Factor out GCF

$$= y(4x^2 - 81)$$

Difference of Squares $A = 2x$ and $B = 9$

$$= y(2x+9)(2x-9)$$

👋 Note

The word “difference” translates to “subtraction.” The Difference of Squares requires **subtraction** between the two squared terms: $A^2 - B^2$. Aside from common factors, the **sum** of squares $A^2 + B^2$ is not factorable.

✓ Example 2.3.7

Factor $n^2 + 25$.

Solution

Although both terms are perfect squares, this is not a difference of squares. The operation between the squares is not subtraction. The GCF = 1. Since we cannot factor out a common factor, we conclude that $n^2 + 25$ is a **prime polynomial**.

Perfect Square Trinomials

Another pattern emerges when two identical binomials are multiplied. Below are three examples that demonstrate a pattern after using the FOIL method.

Multiply the Binomials		FOIL Method		Simplified Result
$(t + 5)(t + 5)$	=	$t^2 + \underbrace{-5t + 5t}_{O + I = 2(5t)} + 5^2$	=	$t^2 + 10t + 25$
$(2y - 9)(2y - 9)$	=	$(2y)^2 - \underbrace{18y - 18y}_{O + I = 2(2y)(-10)} + 9^2$	=	$4y^2 - 36y + 81$
$(pq + 7)(pq + 7)$	=	$(pq)^2 + \underbrace{7pq + 7pq}_{O + I = 2(7pq)} + 7^2$	=	$p^2q^2 + 14pq + 49$

Special Product = Perfect Square Trinomial

$$(A + B)^2 = (A + B)(A + B) = A^2 + 2AB + B^2$$

$$(A - B)^2 = (A - B)(A - B) = A^2 - 2AB + B^2$$

Factoring is the reverse of multiplying. In order to factor, identify the **unsquare** of the first and last term. That is, find A and B , then follow the formulas above.

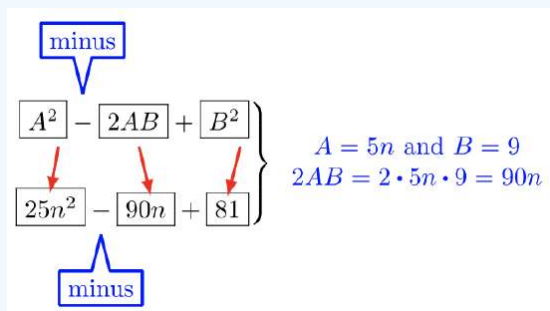
✓ Example 2.3.8

Factor $25n^2 - 90n + 81$

Solution

The trinomial has GCF = 1. However, the first and last terms are perfect squares, and the polynomial is a trinomial. Let's see if we were given a Perfect Square Trinomial.

- Match the formula with the given trinomial.
- Determine the value of A and B .
- Check that $2AB$ also matches.
- The minus symbol is a major clue telling us which special product formula we'll use.
- $A^2 - 2AB + B^2 = (A - B)^2$



$$\left. \begin{array}{c} A^2 - 2AB + B^2 \\ \downarrow \quad \downarrow \quad \downarrow \\ 25n^2 - 90n + 81 \end{array} \right\} \begin{array}{l} A = 5n \text{ and } B = 9 \\ 2AB = 2 \cdot 5n \cdot 9 = 90n \end{array}$$

Be sure to verify that $2AB$ also matches the given middle term. In Example 2.3.8, we were given a Perfect Square Trinomial because $2AB$ matched the middle term, $90n$. Therefore, we can factor this trinomial using the formula for factoring Perfect Square Trinomials.

Answer: $25n^2 - 90n + 81 = (5n - 9)^2$

👋 Note

Caution: If a polynomial cannot be factored using one factoring technique, it doesn't necessarily mean the polynomial is prime. Don't give up! Exhaust all possible factoring techniques before concluding that the polynomial is prime.

✓ Example 2.3.9

Factor $9y^2 + 15y + 4$

Solution

The $GCF = 1$. However, the first and last terms are perfect squares, and the polynomial is a trinomial. Let's see if we were given a Perfect Square Trinomial.

$$\left. \begin{array}{ccc} A^2 & + & 2AB & + & B^2 \\ \downarrow & & \downarrow & & \downarrow \\ 9y^2 & + & 15y & + & 4 \end{array} \right\} \begin{array}{l} A = 3y \text{ and } B = 2 \\ 2AB = 2 \cdot 3y \cdot 2 = 12y \neq 15y \end{array}$$

By matching the terms, we determine $A = 3y$ and $B = 2$. However, $2AB = 12y$ does not match the given middle term $15y$. The trinomial is not a Perfect Square Trinomial.

The trinomial is not prime, however. It can be factored using the trial-and-error method.

Step 1: $GCF = 1$. There is no common factor.

Step 2: List pairs of factors of a and of c : (see the chart below).

$a = 9$	$c = 4$
$9 \cdot 1$	$4 \cdot 1$
$3 \cdot 3$	$2 \cdot 2$

$$bx = 15y = \text{Target Sum}$$

Step 3: Place the a -factors, then choose a pair of c -factors. We already know that $(3y + 2)(3y + 2)$ does not work. Let's try: $(3y + 4)(3y + 1)$

Step 4: Let's check the Outer and Inner

$$\begin{array}{c} (3y + 4)(3y + 1) \\ \text{Outer: } 3y \cdot 1 = 3y \\ \text{Inner: } 4 \cdot 3y = 12y \\ \text{Sum: } 3y + 12y = 15y \end{array} \quad \checkmark$$

This trial is a success! $12y + 3y = 15y$

Other trials that fail:

$(3y + 2)(3y + 2)$	$6y + 6y = 12y$	$\neq 15y$
$(9y + 4)(y + 1)$	$4y + 9y = 13y$	$\neq 15y$
$(9y + 1)(y + 4)$	$y + 36y = 37y$	$\neq 15y$

Answer $9y^2 + 15y + 4 = (3y + 4)(3y + 1)$.

Factor Sum of Cubes and Difference of Cubes

In order to identify the correct strategy for factoring, it's a good idea to have at least the first five cubes memorized. The formulas shown below guide our factoring.

$$\begin{aligned}1^3 &= 1 \\2^3 &= 8 \\3^3 &= 27 \\4^3 &= 64 \\5^3 &= 125\end{aligned}$$

Factoring Sums and Differences of Cubes

$$\begin{aligned}A^3 + B^3 &= (A + B)(A^2 - AB + B^2) \\A^3 - B^3 &= (A - B)(A^2 + AB + B^2)\end{aligned}$$

✓ Example 2.3.10

Factor $27 - 8m^3$

Solution

The binomial is a difference of cubes.

Use the formula with $A = 3$ and $B = 2m$.

$$\left. \begin{array}{c} \boxed{A^3} - \boxed{B^3} \\ \downarrow \quad \downarrow \\ \boxed{27} - \boxed{8m^3} \end{array} \right\} A = 3 \text{ and } B = 2m$$

$$27 - 8m^3 = (3 - 2m)(9 + 6m + 4m^2)$$

Try It! (Exercises)

Factor #1-3 following using GCF factorization:

1. $12u - 6u^2$
2. $10x^3y + 15xy^2$
3. $12z^3 - 9z^2 - 3z$

Factor #4-6 using trinomial factorization:

4. $t^2 + 2t - 15$
5. $h^2 - 12h + 20$
6. $r^2 + 14r + 24$

Factor #7-12 using both GCF and trinomial factorization:

7. $3b^2 + 12b + 9$
8. $4q^2 - 4q - 80$
9. $x^3 - 11x^2 + 10x$
10. $2a^2b + 24ab - 56b$
11. $7x^4 + 49x^3 + 70x^2$
12. $3n^2m - 12mn - 63m$

Factor #13-18 using trinomial factorization where the leading coefficient $\neq 1$.

13. $2d^2 - 9d - 18$
14. $3h^2 - 29h + 18$
15. $7t^2 + 17t - 12$
16. $4y^2 - 3y - 7$
17. $10n^2 - 11n - 8$
18. $8x^2 + 6x - 9$

Factor #19-24 using difference of squares factorization:

19. $r^2 - 100$
20. $81 - h^2$
21. $4t^2 - 1$
22. $25 - n^4$
23. $v^4 - 36$
24. $16p^2 - 25q^2$

Factor #25-27 using sum of cubes formula:

25. $c^3 + 125$
26. $27 + 64n^3$
27. $1 + a^6$

Factor #28-30 using difference of cubes formula:

28. $q^3 - 8$
29. $125d^3 - 27$
30. $64u^3 - v^3$

Factor #31-36 using any appropriate method. Factor completely.

31. $30m^2 - 5m - 5$
32. $2c^2 - 8$
33. $h^4 - 81$
34. $6p^3q + 6q$
35. $15k^3 - 2k^2 - k$
36. $a^2b^2 - a^2$
37. What is a prime polynomial? Give 3 examples of prime trinomials.
38. Give an example of a sum of squares. Is it factorable? Explain.
39. The number 64 is both a perfect square and a cube. What other numbers are both a square and a cube? Do you think there are there only a few or infinitely many?

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CHAPTER OVERVIEW

3: Graphs

Learning Objectives

- Interpret, from function notation, an action or several actions to be taken on a known function, f , in order to sketch the transformed function.
- Sketch transformations from 9 familiar functions.
- Write the equation of a function associated with descriptive transformations such as shifting left, right, up, or down, reflecting, stretching, and compressing of its parent function.

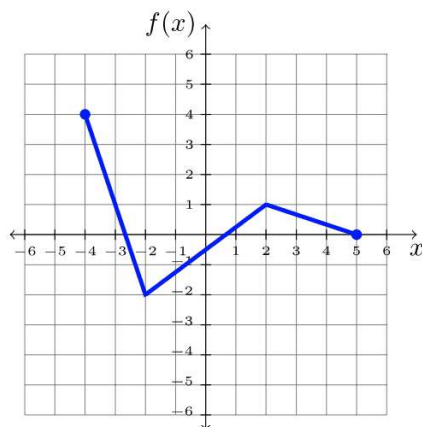
[3.1: Transformations of \$f\(x\)\$](#)

[3.2: Transformations of Common Graphs](#)

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3.1: Transformations of $f(x)$

In this section, you will practice manipulating a given graph, according to the corresponding function notation. We'll use the function f , graphed below, for demonstration throughout this section. But any graph will do!



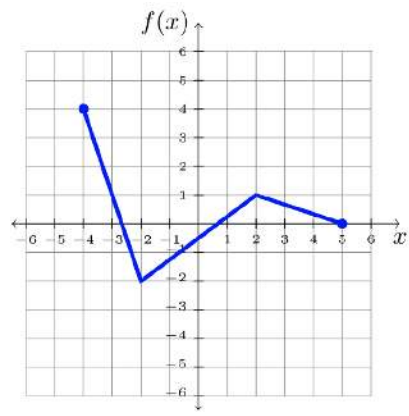
By the end of this section, you'll learn to read function notation like a blueprint, reposition the graph according to the instructions it gives. You will learn how to apply these transformations.

Transformation	Function Notation ($c > 0$)	Notes
Shift up c units	$y = f(x) + c$	If addition or subtraction is <i>outside</i> of the function, shift up or down.
Shift down c units	$y = f(x) - c$	
Shift right c units	$y = f(x - c)$	If addition or subtraction is <i>inside</i> the function, shift left or right.
Shift left c units	$y = f(x + c)$	
Reflection across the x -axis	$y = -f(x)$	If the negative is <i>outside</i> of the function, there is an x -axis reflection.
Reflection across the y -axis	$y = f(-x)$	
Vertical scale change	$y = cf(x)$	If the multiplier, c , is <i>outside</i> of the function, there is a vertical scale change.
Horizontal scale change	$y = f(cx)$	

The examples demonstrate the appropriate manipulation to the graph of f . The original graph f is referred to as **the parent graph**. The concept can be applied to any function, but the same parent function is used in each example below.

✓ Example 3.1.1

Using the graph of f , transform the graph appropriately.



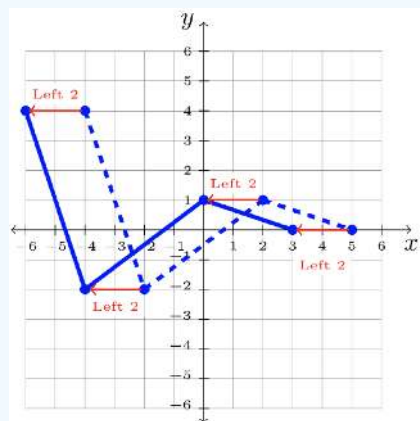
- a. $y = f(x + 2)$
- b. $y = f(x) + 2$
- c. $y = f(x - 2)$

Solution

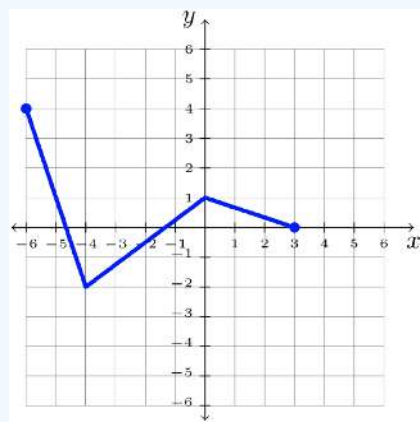
- a. **Step 1:** Identify the transformation on the parent graph, f .

$$y = f(x + 2) \quad \text{Plus 2 Inside Function; Shift Left 2}$$

Step 2: Shift each point 2 units left:



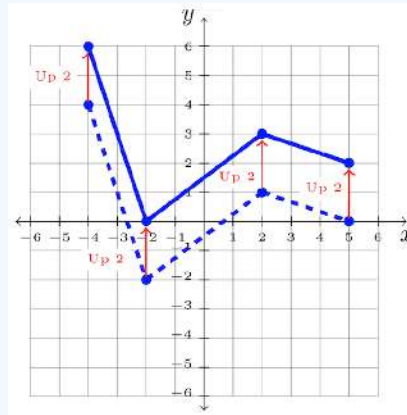
Step 3: Answer: $y = f(x + 2)$



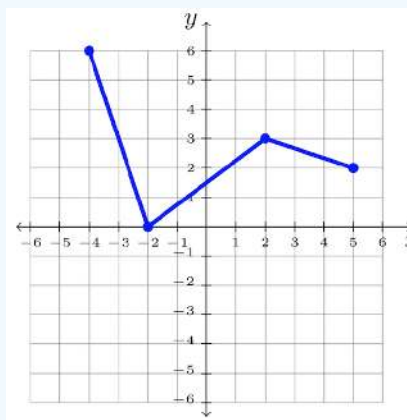
- b. **Step 1:** Identify the transformation on the parent graph, f .

$$y = f(x) + 2 \quad \text{Plus 2 Outside Function; Shift Up 2}$$

Step 2: Shift each point 2 units up:



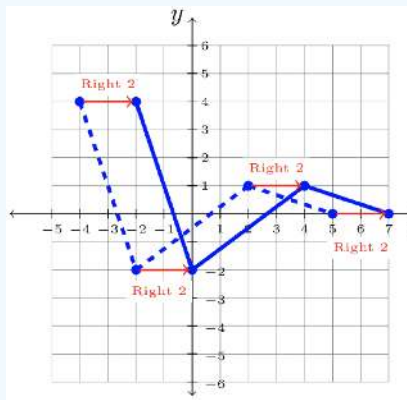
Step 3: Answer: $y = f(x) + 2$



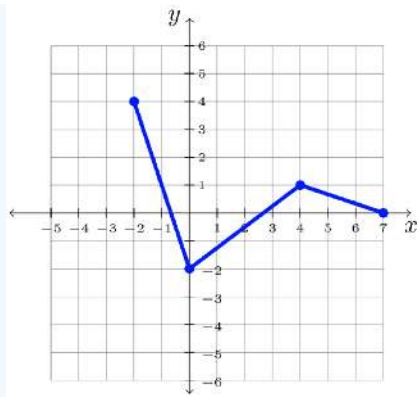
c. **Step 1:** Identify the transformation on the parent graph, f .

$$y = f(x - 2) \quad \text{Minus 2 Inside Function; Shift Right 2}$$

Step 2: Shift each point 2 units right:

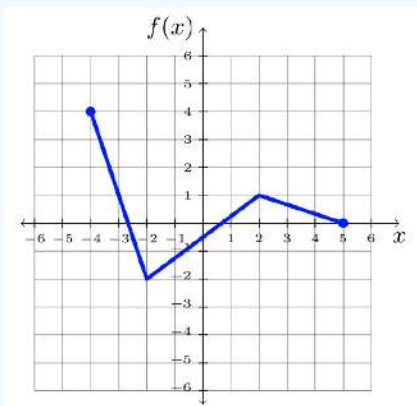


Step 3: Answer: $y = f(x - 2)$



✓ Example 3.1.2

Using the graph of f , transform the graph appropriately.



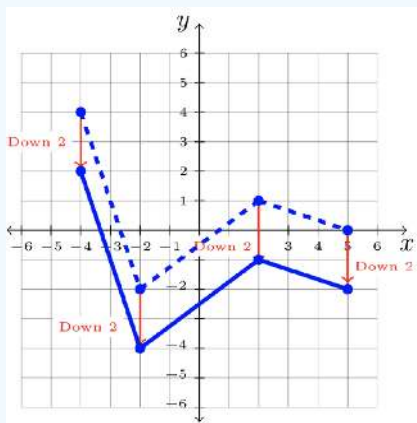
- a. $y = f(x) - 2$
- b. $y = -f(x)$
- c. $y = f(-x)$

Solution

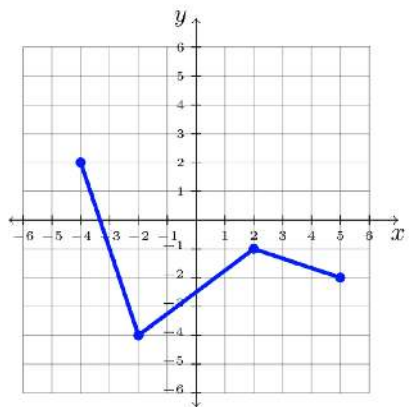
- a. **Step 1:** Identify the transformation on the parent graph, f .

$$y = f(x) - 2 \quad \text{Minus 2 Outside Function; Shift Down 2}$$

Step 2: Shift each point 2 units down:



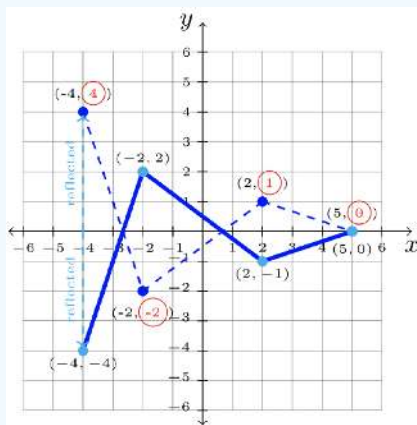
Step 3: Answer: $y = f(x) - 2$



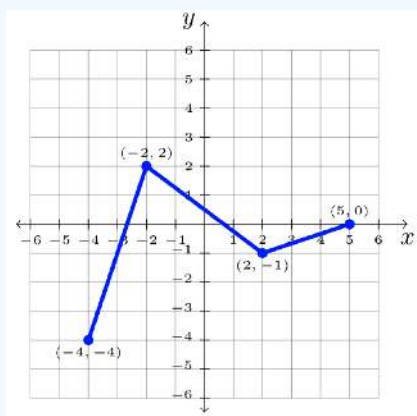
b. **Step 1:** Identify the transformation on the parent graph, f .

$$y = -f(x) \quad \text{Negative Outside Function; } x\text{-axis Reflection}$$

Step 2: Change each y -value to its opposite.



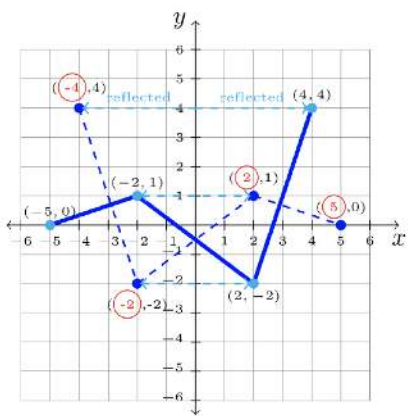
Step 3: Answer: $y = -f(x)$



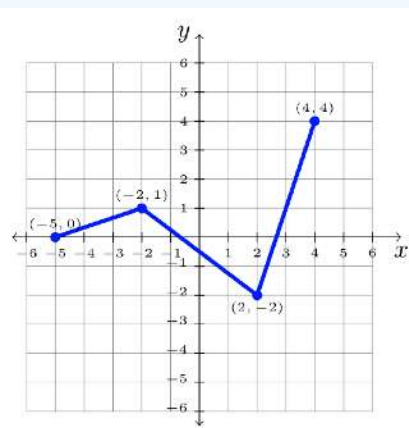
c. **Step 1:** Identify the transformation on the parent graph, f .

$$y = f(-x) \quad \text{Negative Inside Function; } y\text{-axis Reflection}$$

Step 2: Change each x -value to its opposite.

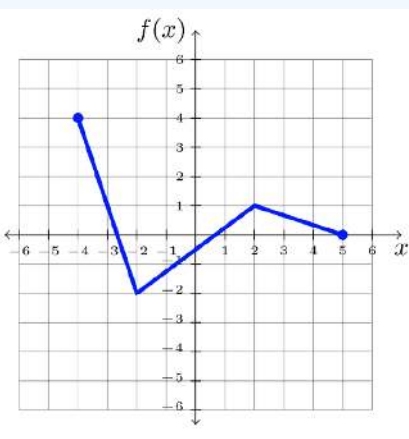


Step 3: Answer: $y = f(x) - 2$:



✓ Example 3.1.3

Using the graph of f , transform the graph appropriately.



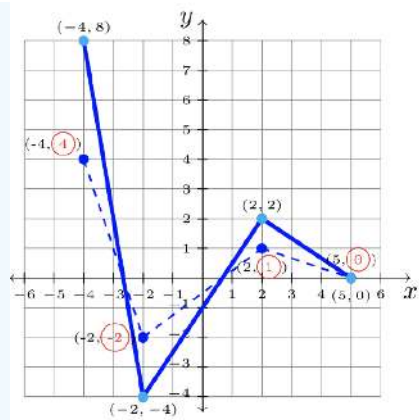
- a. $y = 2f(x)$
- b. $y = f(2x)$

Solution

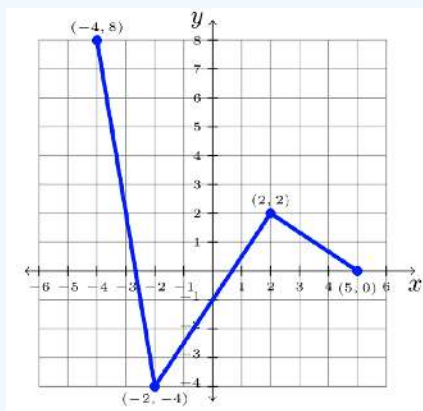
- a. **Step 1:** Identify the transformation on the parent graph, f .

$$y = 2f(x) \quad \text{Times 2 Outside Function; Vertical Scale Change 2}$$

Step 2: Multiply each y -value by 2.



Step 3: Answer: $y = 2f(x)$:

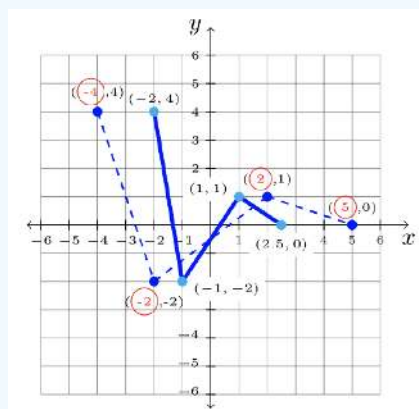


Caution: Horizontal scale change is counter-intuitive! For $y = f(cx)$, the graph of f is scaled horizontally by a factor of $\frac{1}{c}$. In other words, if $c > 1$, then the graph is compressed. If $0 < c < 1$, (a proper fraction) then the graph is *stretched* horizontally

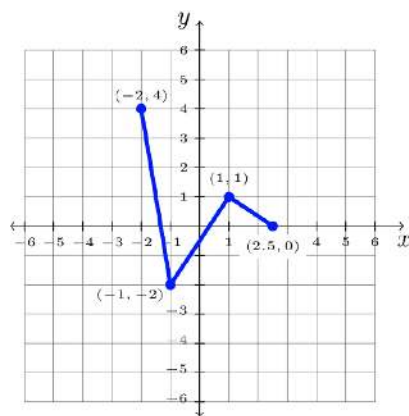
b. **Step 1:** Identify the transformation on the parent graph, f .

$$y = -f(x) \quad \text{Minus 2 Outside Function; Shift Down 2}$$

Step 2: Multiply each x -value by $\frac{1}{2}$.



Step 3: Answer: $y = f(2x)$:

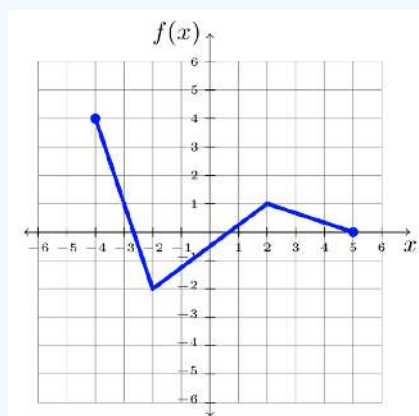


Combining Transformations

In many cases, graphing a function will require more than one transformation. Perform transformations in the same order as PEMDAS, the order of operations. The next example demonstrates the manipulation of a graph using multiple transformations.

✓ Example 3.1.4

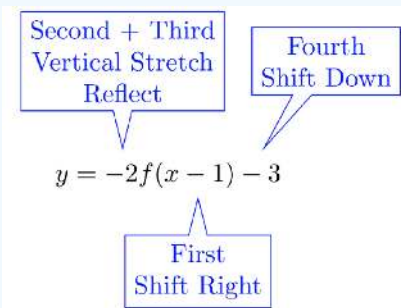
Using the graph of f ,



Sketch the graph: $y = -2f(x - 1) - 3$

Solution

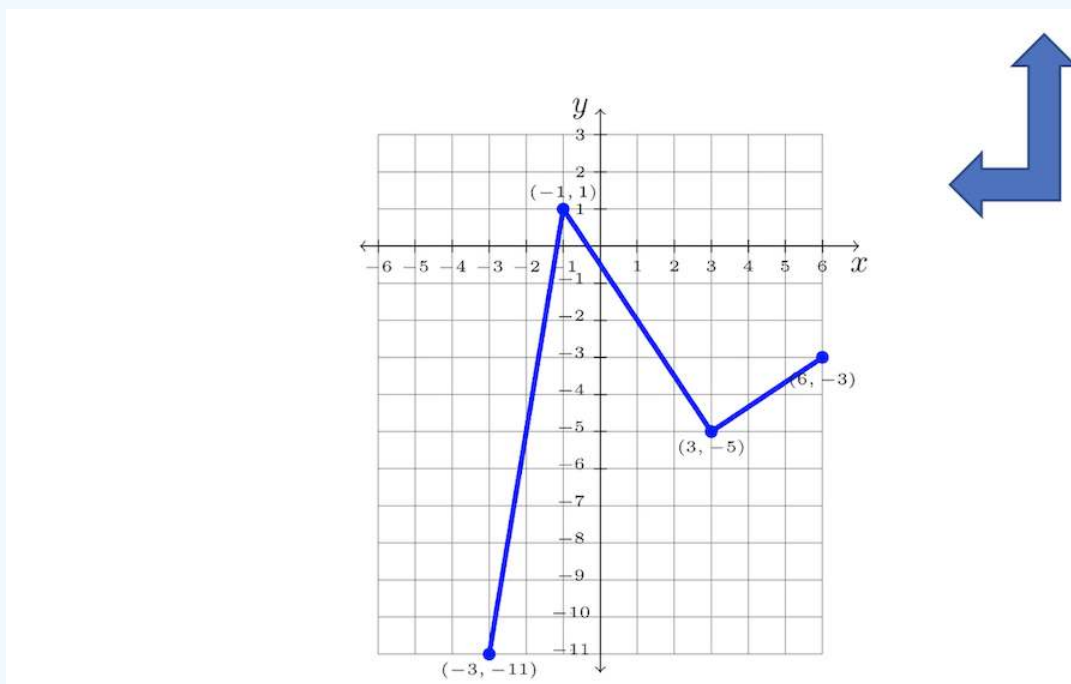
Since transformations are to be performed in the order of PEMDAS, each transformation is noted then ordered.



The transformations of 4 points of f are charted below. After completing all transformations, plot the transformed points stated in the final column. Connect the points to create the graph.

f 's Point	Shift Right: Add one to each x -value.	Vertical Stretch & Reflect: Multiply y -values by -2 .	Shift Down: Subtract 3 from each y -value
$(-4, 4)$	$(-3, 4)$	$(-3, -8)$	$(-3, -11)$
$(-2, -2)$	$(-1, -2)$	$(-1, 4)$	$(-1, 1)$
$(2, 1)$	$(3, 1)$	$(3, -2)$	$(3, -5)$
$(5, 0)$	$(6, 0)$	$(6, 0)$	$(6, -3)$

The graph $y = -2f(x - 1) - 3$ is pictured below:



Try It! (Exercises)

For exercises 1-17, use the graph of f at right, along with the appropriate transformation(s), to sketch the transformed function. Use graph paper.

1. $y = -f(x)$
2. $y = f(x) + 1$
3. $y = f(x + 2)$
4. $y = f(x - 3)$
5. $y = f(x) - 4$
6. $y = f(-x)$
7. $y = 3f(x)$
8. $y = f(2x)$
9. $y = \frac{1}{2}f(x)$
10. $y = f\left(\frac{1}{2}x\right)$
11. $y = -f(x)$
12. $y = 2f(x + 1)$
13. $y = 4 + f(x - 2)$
14. $y = 3 - f(x)$
15. $y = f(x - 3) - 3$

16. $y = -3f\left(\frac{1}{3}x\right)$

17. $y = 2f(-x) - 1$

For #18-24, write the function that would correspond to the described transformations on f .

18. Reflect f through the y -axis, then move the graph up 5 units.

19. Stretch f vertically by a factor of 6, then shift the graph left 8 units.

20. Shift the graph of f to the right 10 units, then shift the graph down 7 units.

21. Reflect f through the x -axis, move the graph 3 units right, then shift the graph up 12 units.

22. Reflect f through the x -axis, stretch it vertically by a factor of 4, then shift it down 5 units.

23. Reflect f through the y -axis, shift it left (9) units, then shift it up 8 units.

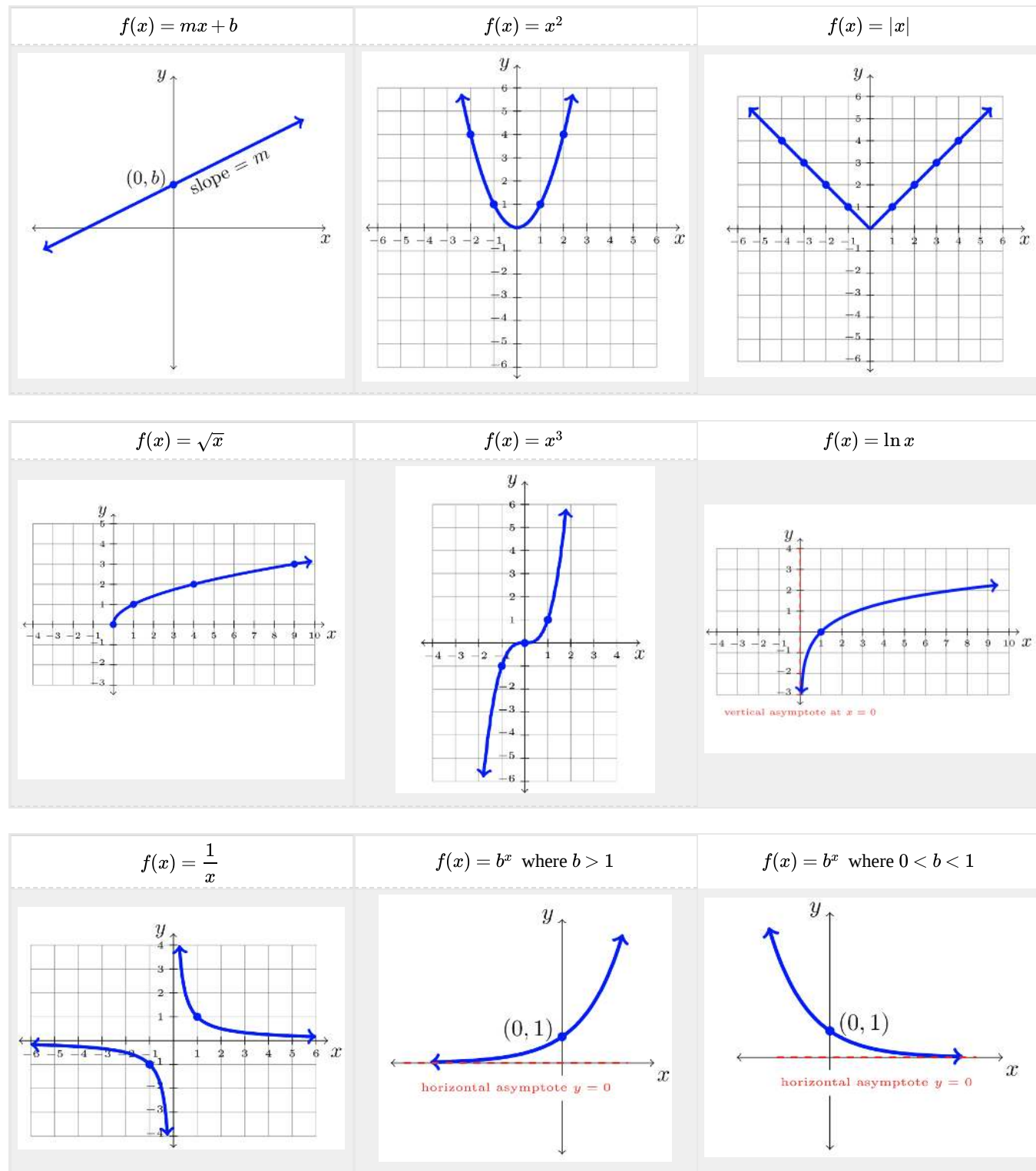
24. Compress f horizontally by a factor of $\frac{1}{4}$, reflect it through the x -axis, then shift down 1 unit.

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3.2: Transformations of Common Graphs

The transformation of graphs, using common functions, will be a skill that will bring insight to graphing functions quickly and painlessly. Anticipating how a graph of a function will look, and transforming old graphs to new graphs, is a skill we will explore in this section. Mastering this skill will give you a leg up on understanding analytic geometry, a key component to calculus.

Here are nine common functions and their graphs. Each of these functions will be referred to as “the parent function” for its family of transformed functions.



Using the transformations described in Section 3.1, the nine graphs shown on the previous page can be shifted, reflected, stretched, and compressed. The take-away for this section is to recognize the parent function from function notation and apply the appropriate transformations to quickly sketch a graph or visualize the graph in your head. No technology is necessary! It's enough to have an understanding for functions and their shifting graphs.

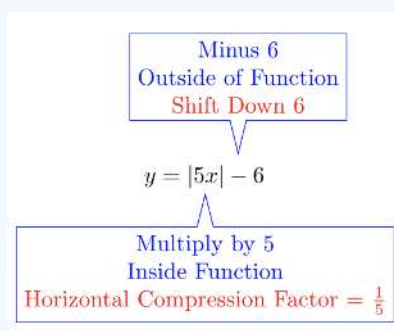
✓ Example 3.2.1

Name the parent function of the given function. Then describe in words the transformation(s) on the parent function.

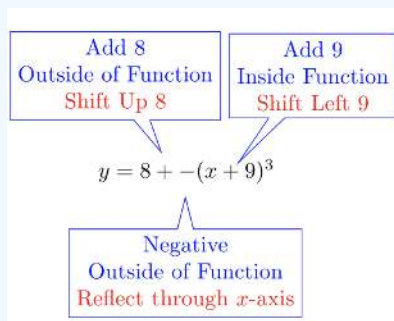
- $y = |5x| - 6$
- $y = 8 - (x + 9)^3$
- $y = \frac{2}{x - 10}$

Solution

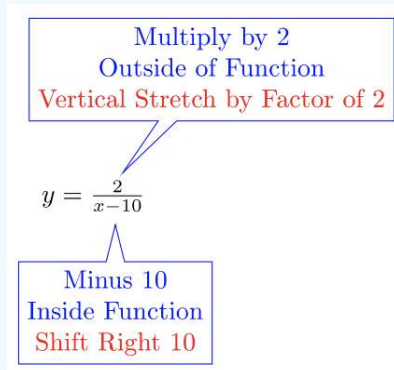
- The parent function is $y = |x|$:



- The parent function is $y = x^3$:



- The parent function is $y = \frac{1}{x}$:



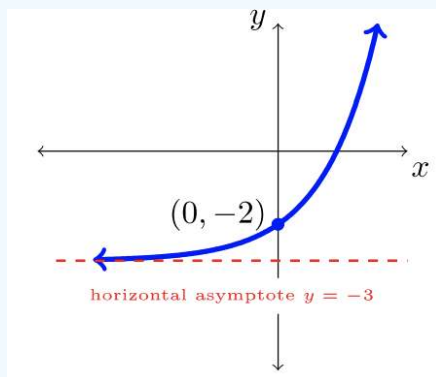
✓ Example 3.2.2

Use the parent function and its transformation(s) to sketch the graph of the given function. It is much more important to create a rough but accurate sketch without technology than a detailed sketch with technology.

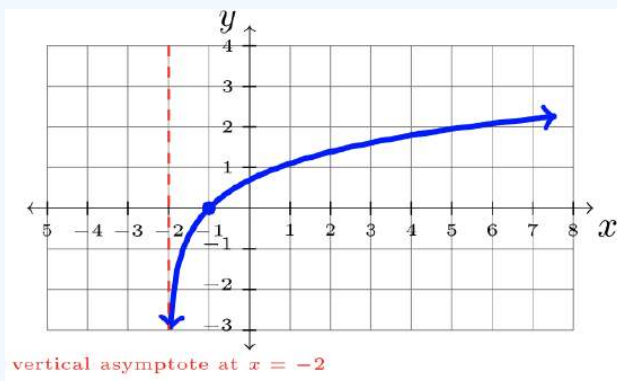
- a. $y = 2^x - 3$
- b. $y = \ln(x + 2)$

Solution

- a. The parent function is $y = b^x$ where $b = 2 > 1$. Shift this graph, including its horizontal asymptote, down 3 units.



- b. The parent function is $y = \ln x$. Shift this graph, including its vertical asymptote, left 2 units.



Try It! (Exercises)

For exercises #1-18, use the parent functions (see graphs within this section), along with appropriate transformations, to create a sketch of the graph of the given function. Do not use graphing technology.

1. $y = \frac{1}{x+5}$
2. $y = e^{x-4}$
3. $y = |x| - 6$
4. $y = -(x+2)^3$
5. $y = 4 + x^2$
6. $y = \sqrt{x-4}$
7. $y = \left(\frac{1}{2}\right)^x + 3$
8. $y = 1 - \sqrt{x}$
9. $y = |x-5| + 2$
10. $y = 1 + (x-6)^2$
11. $y = 4 - \frac{1}{x}$

12. $y = 3 + \ln(-x)$
13. $y = 2\sqrt{x} - 4$
14. $y = -2(x - 5)^2$
15. $y = 1 + x^3$
16. $y = -3|x - 7|$
17. $y = 5 + \sqrt{-x}$
18. $y = 3^{x+2} + 4$
19. Use the graph of $y = |x|$ and transformations to discuss why the graph of the transformed function $y = |-x|$ is the same graph.
20. Use the graph of $y = x^2$ and transformations to discuss why the graph of the transformed function $y = (-x)^2$ is the same graph.
21. Using $f(x) = x^2$ as the parent function, write the function that would correspond to the transformations on f . No need to graph.
 - a. Reflected through the x -axis and shifted down 12 units.
 - b. Shifted 10 units left and 25 units up.
22. Discuss two different approaches to transform the parent function $y = \frac{1}{x}$ to graph $y = \frac{1}{2x}$. What are the transformations?
Discuss how and why each approach will produce the same graph for the new function $y = \frac{1}{2x}$.

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CHAPTER OVERVIEW

4: Inequalities

Learning Objectives

- Verbalize, visualize, and sketch number line graphs for inequalities.
- Translate compound inequalities to sketches and vice-versa.
- Find solutions to linear inequalities and visualize as number line graphs.
- Find solutions to absolute value inequalities and visualize as number line graphs.

[4.1: Number Line Graphs](#)

[4.2: Interval Notation](#)

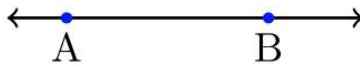
[4.3: Solving Linear Inequalities](#)

[4.4: Absolute Value Equations and Inequalities as Applied to Distance](#)

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4.1: Number Line Graphs

A real number line is a visual approach to ordering all real numbers:



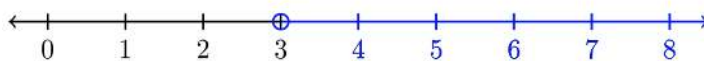
Reading Real Number Lines

Any real number A plotted left of another real number B has the relation: $A < B$, or equivalently, $B > A$. We read aloud, “ A is less than B ” or equivalently, “ B is greater than A .”

Graphing Inequalities

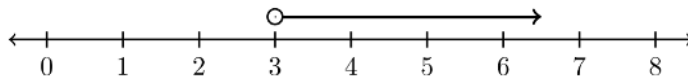
“Graph the solution set, $x > 3$.”

The solution set to an inequality is the set of real numbers that make the inequality a true statement. All values that lie to the right of 3 on the number line are greater than 3. The number 3 itself is not greater than 3. A graph quickly conveys the solution set. To visually denote that 3 is not greater than 3, we will use an open circle; a circle that is not filled. The blue portion of this number line indicates values greater than 3.



Finally, we shift the entire blue line above the number line. Now we don't need colors:

All values $x > 3$ are graphed below:

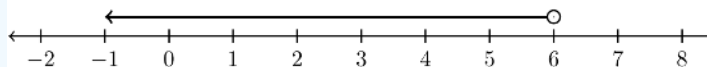


✓ Example 4.1.1

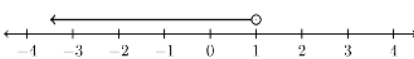
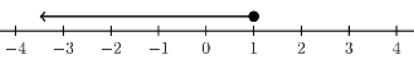
Graph the solution set $y < 6$.

Solution

The solution set is the set of all real numbers strictly less than 6. The number line graph conveys the solution:



How to Decide: Closed or Open Circle?

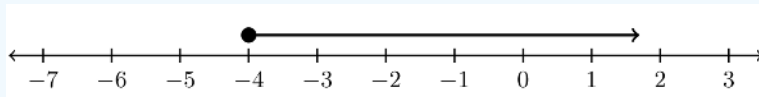
Inequality	Associated Circle	Example Graphs
Either $<$ or $>$		$x < 1$ 
Either $\leq$ or $\geq$		$x \leq 1$ 

✓ Example 4.1.2

Graph the solution set of $-4 \leq t$.

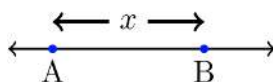
Solution

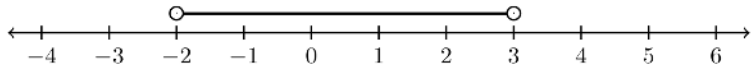
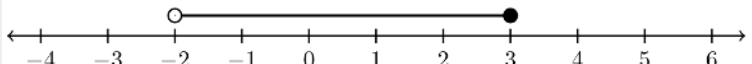
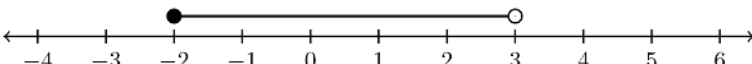
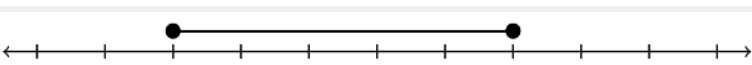
The given inequality is equivalent to $t \geq -4$. The graph will express all values greater than or equal to -4 . In this case, -4 is included as a solution. Use a filled circle to denote its inclusion in the solution set.



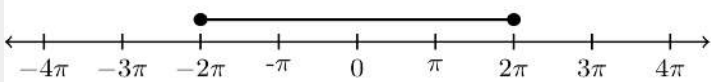
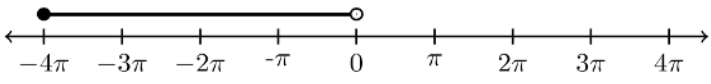
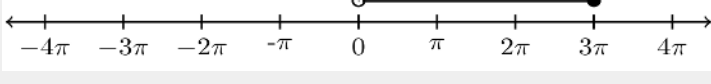
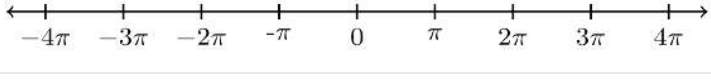
All Values Between A and B: Compound Inequalities

If we want to describe *all numbers between A and B*, there is an elegant way to do this. Let x = all real numbers between A and B . The examples below use $A = -2$ and $B = 3$. The open and closed circles correspond with the appropriate inequality.



Inequality	Graph
$-2 < x < 3$	
$-2 < x \leq 3$	
$-2 \leq x < 3$	
$-2 \leq x \leq 3$	

These inequalities are handled no differently, yet the unit is π :

Inequality	Graph
$-2\pi \leq x \leq 2\pi$	
$-4\pi \leq x < 0$	
$0 < x \leq 3\pi$	
$-\pi < x < \pi$	

Compound Inequalities

The above inequalities are examples of compound inequalities: inequalities that express two or more inequalities at once. We can split each inequality above into two inequalities, using an “and” statement between each inequality:

$$A < x < B \text{ is equivalent to } A < x \text{ and } x < B$$

Notice $A < B$. When the inequalities are uncoupled, the middle variable is expressed in both inequalities, and the inequality symbol is maintained.

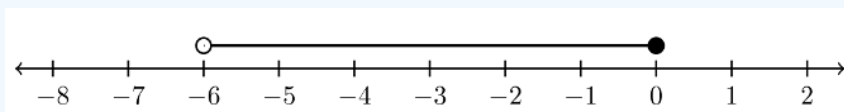
We’ll revisit compound inequalities in the context of absolute value inequalities in Section 4.4.

✓ Example 4.1.3

Graph the solution set of the compound inequality: $x > -6$ and $x \leq 0$.

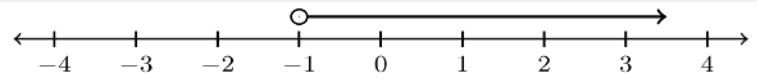
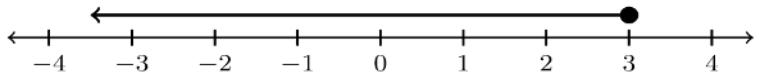
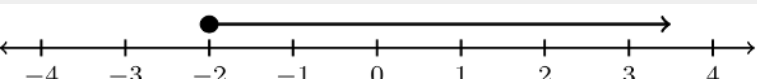
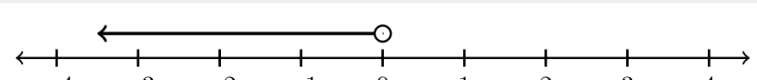
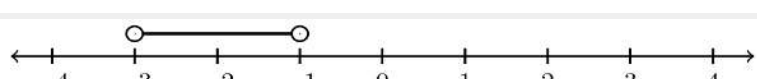
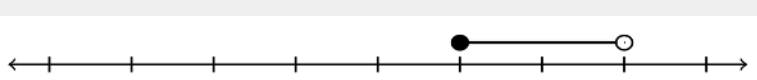
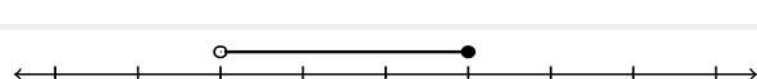
Solution

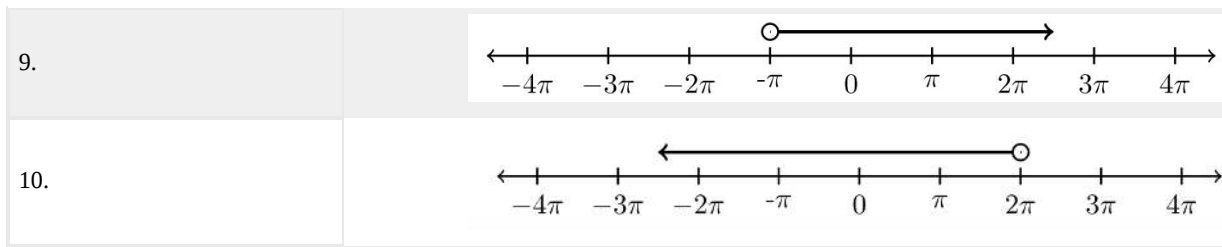
The two inequalities have an “and” statement between them. The first inequality $x > -6$ can be restated in its equivalent form: $-6 < x$. The real value $A = -6$ is the left endpoint, whereas $B = 0$ is the right endpoint. The graph is shown below for $-6 < x \leq 0$.



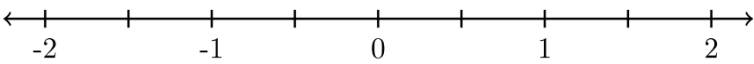
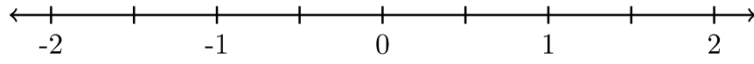
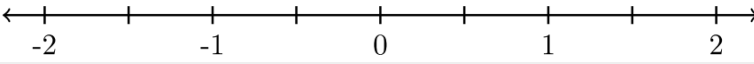
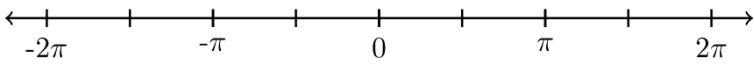
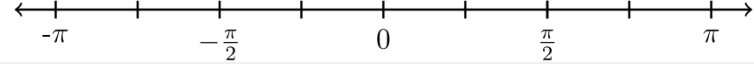
Try It! (Exercises)

For exercises #1-10, state the inequality that is represented by the graph.

Inequality	Graph
1.	
2.	
3.	
4.	
5.	
6.	
7.	
8.	



For exercises #11-15, the number line is equally divided. Label the tick marks. Use fractions. Then sketch the number line graph that represents each inequality.

Inequality	Graph
11. $-\frac{1}{2} < x \leq \frac{3}{2}$	
12. $x \geq \frac{1}{2}$	
13. $x < -\frac{3}{2}$	
14. $-\frac{3\pi}{2} \leq x < \frac{3\pi}{2}$	
15. $-\frac{\pi}{4} < x < \frac{3\pi}{4}$	

For exercises #16-22, Sketch a number line graph that corresponds with the given compound inequality.

16. $x > -2$ and $x \leq 5$
17. $x \geq 1$ and $x \leq 10$
18. $x \leq -1$ and $x > -3$
19. $x < 8$ and $x \geq -8$
20. $x > -\frac{3}{2}$ and $x < \frac{1}{2}$
21. $x \geq 2\pi$ and $x \leq 4\pi$
22. $x > -\frac{\pi}{2}$ and $x < \frac{\pi}{2}$

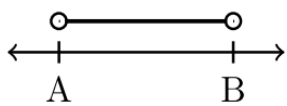
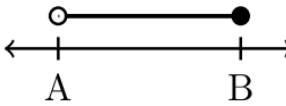
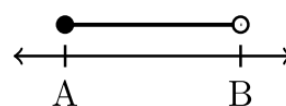
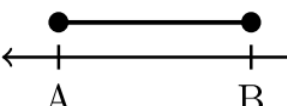
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4.2: Interval Notation

Inequalities slice and dice the real number line into segments of interest or intervals. An **interval** is a continuous, uninterrupted subset of real numbers. How can we notate intervals with simplicity? The table below introduces **Interval Notation**.

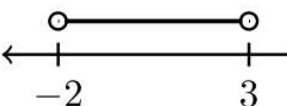
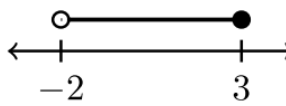
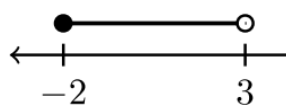
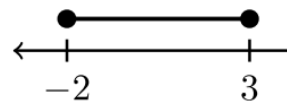
Inequality	Associated Circle	Associated Endpoint Closures
Either $<$ or $>$		Left Parenthesis: (or Right Parenthesis:)
Either $\leq$ or $\geq$		Left Square bracket: [or Right Square Bracket:]

Inequalities have 4 possible interval closures:

(A, B)	$(A, B]$	$[A, B)$	$[A, B]$
			

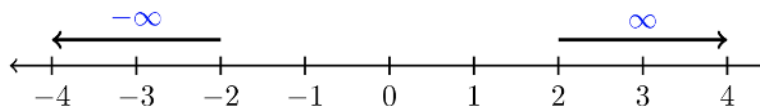
The least number in the interval, A , is always stated first. A comma is placed. The largest number in the interval, B , is stated after the comma. The appropriate closure is considered for each value A and B .

Four Examples of Interval Notation

$-2 < x < 3$	$-2 < x \leq 3$	$-2 \leq x < 3$	$-2 \leq x \leq 3$
			
$(-2, 3)$	$(-2, 3]$	$[-2, 3)$	$[-2, 3]$

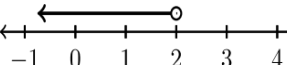
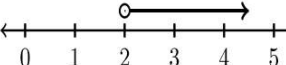
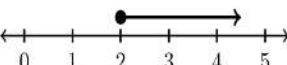
The Infinities

There are two infinities: positive and negative. Each define a direction on the number line:



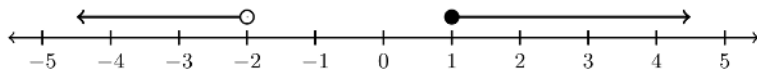
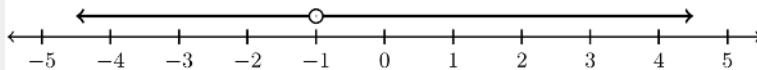
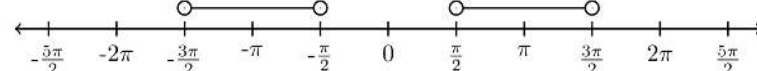
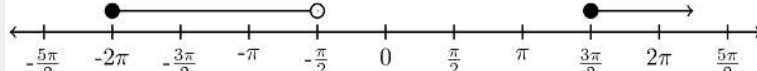
Infinity is not a real number. It indicates a direction. Therefore, when using interval notation, always enclose ∞ and $-\infty$ with parenthesis. We never enclose infinities with square bracket.

The table below shows four examples of interval notation that require the use of infinity.

$x < 2$	$x \leq 2$	$x > 2$	$x \geq 2$
			
$(-\infty, 2)$	$(-\infty, 2]$	$(2, \infty)$	$[2, \infty)$

Combinations of Intervals

If two or more intervals are interrupted with a gap in the number line, set notation is used to stitch the intervals together, symbolically. The symbol we use to combine intervals is the union symbol: $\cup$. The table below shows four examples:

Interval Notation	Graph
$(-\infty, -2) \cup [1, \infty)$	
$(-\infty, -1) \cup (-1, \infty)$	
$\left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$	
$\left[-2\pi, -\frac{\pi}{2}\right) \cup \left[\frac{3\pi}{2}, \infty\right)$	

Compound Inequalities

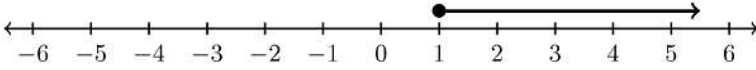
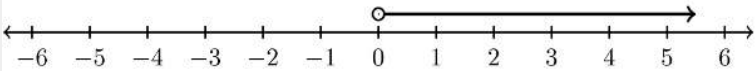
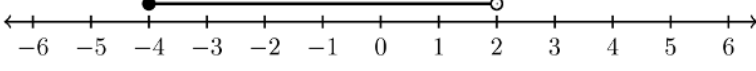
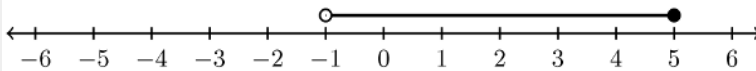
Intervals that have gaps, like the ones shown above, translate to compound inequalities. Real solutions belong in one interval or another. The word “or” plays a key role when translating. For example: the interval $(-\infty, -2) \cup [1, \infty)$ translates to its associated compound inequality:

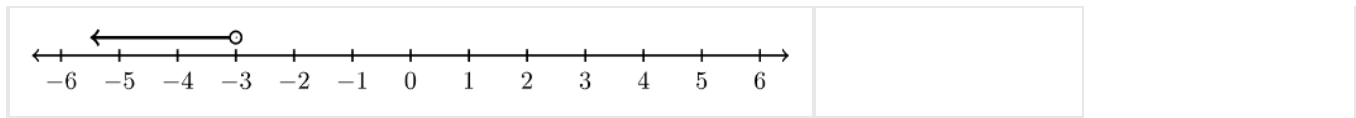
$$x < -2 \text{ or } x \geq 1$$

The word “and” cannot be used between the inequalities because a number cannot belong to both intervals at once. For example, $x = 5$ is a solution because 5 belongs in the interval $x \geq 1$, but 5 does not belong in the interval $x < -2$. Nevertheless, because of the word “or,” $x = 5$ is a solution to the interval $(-\infty, -2) \cup [1, \infty)$.

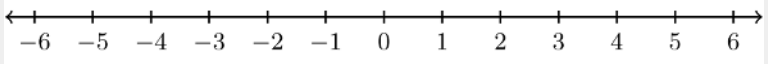
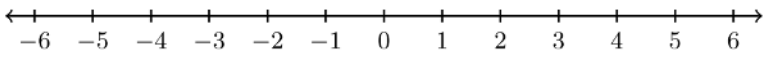
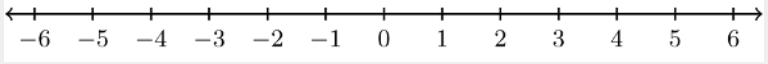
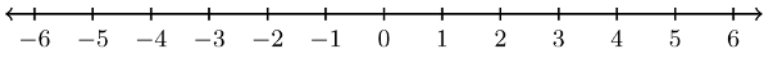
Try It! (Exercises)

For exercises #1-6, state the inequality and the interval notation associated with the graph.

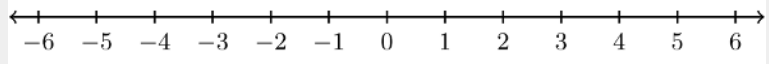
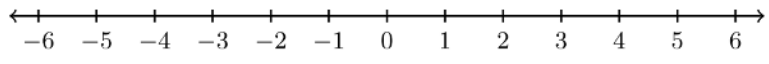
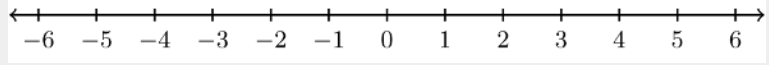
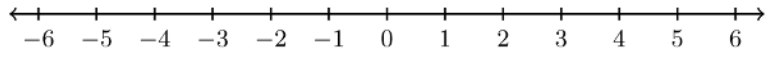
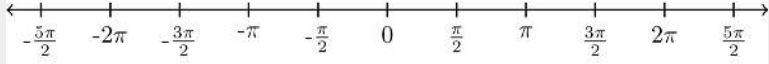
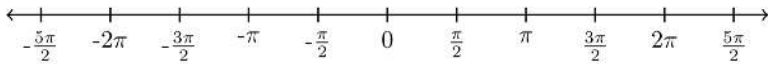
Graph	Inequality	Interval Notation
		
		
		
		
		

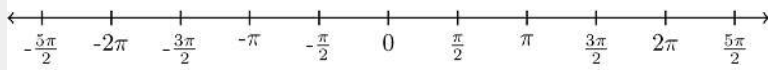


For exercises #7-10, state the interval notation and sketch the graph associated with the inequality.

Graph	Inequality	Interval Notation
	$-3 \leq x \leq 1$	
	$x < 4$	
	$x \geq -2$	
	$0 \leq x < 3$	

For exercises #11-17, sketch the graph associated with the given interval notation.

Graph	Interval Notation
	$(-\infty, 4)$
	$(-\infty, -3) \cup [0, \infty)$
	$[-1, 1) \cup [2, \infty)$
	$(-\infty, -5] \cup (-1, 5)$
	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
	$(-\infty, -\pi] \cup [\pi, \infty)$



$$\left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right) \cup \left(-\frac{\pi}{2}, 0\right)$$

For #18-21

- Sketch a graph of the compound inequality.
- State the interval using interval notation.

- $x \geq 4$ or $x \leq 0$
- $x \leq -2\pi$ or $x > \pi$
- $-1 > x$ or $2 \leq x$
- $x > 3\pi$ or $x < -\pi$

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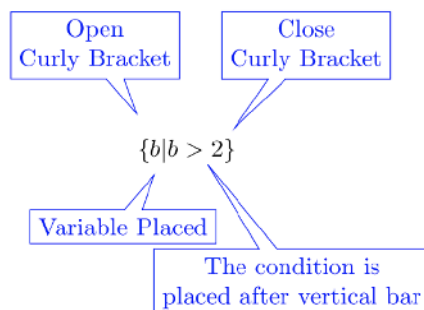
4.3: Solving Linear Inequalities

Linear equations have just one solution. Linear inequalities have infinitely many solutions, requiring intervals to express solutions. Let's compare:

Linear Equation	Linear Inequality
$4b - 3 = 5$	$4b - 3 > 5$
$4b - 3 + 3 = 5 + 3$	$4b - 3 + 3 > 5 + 3$
$4b = 8$	$4b > 8$
$b = 2$	$b > 2$
Solutions set: $b = \{2\}$	Solution set: $\{b b > 2\}$ or Interval $(2, \infty)$

Set-builder Notation: To express solutions using set notation, the curly-brackets are used.

The **condition** of the set is descriptive. The inequality describes the condition. A number must meet the condition to qualify as a solution.



In this case, all numbers larger than two qualify as solutions, and those numbers belong to the solution set.

Note

Although addition and subtraction properties of inequality are the same for inequalities as they are for equalities, multiplication and division properties differ when multiplying or dividing by a negative. **Dividing or multiplying both sides of an inequality by a negative reverses the inequality sign.**

✓ Example 4.3.1

Solve the inequality and graph the solution set. State the answer in both set builder notation and in interval notation.

$$-6x - 5 \geq 13$$

Solution

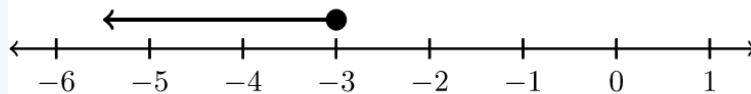
$$\begin{aligned}
 -6x - 5 &\geq 13 && \text{Add 5 to each side} \\
 -6x &\geq 18 \\
 \frac{-6x}{-6} &\leq \frac{18}{-6} && \text{Dividing by a negative reverses the inequality} \\
 x &\leq -3
 \end{aligned}$$

Answers:

Set Builder Notation: $\{x|x \leq -3\}$

Interval Notation: $(-\infty, -3]$

Graph:



✓ Example 4.3.2

Solve the inequality and graph the solution set. State the answer in both set builder notation and in interval notation.

$$10 - (2y + 1) \leq -4(3y + 2) - 3$$

Solution

Stay vigilant for any step in the solution where you multiply both sides by a negative or divide both sides by a negative. Otherwise, solve the inequality in the same manner you solve equalities.

$$10 - 2y - 1 \leq -12y - 8 - 3$$

Simplify each side by clearing parentheses.

$$-2y + 9 \leq -12y - 11$$

Simplify each side by combining like terms.

$$-2y + 12y + 9 \leq -12y + 12y + (-11)$$

Add $12y$ to both sides.

$$10y + 9 \leq -11$$

Simplify by combining like terms.

$$10y + 9 - 9 \leq -11 - 9$$

Subtract 9 to both sides.

$$10y \leq -20$$

Simplify

$$\frac{10y}{10} \leq \frac{-20}{10}$$

$$\left. \begin{array}{l} \frac{10y}{10} \leq \frac{-20}{10} \\ y \leq -2 \end{array} \right\} \text{Dividing by a positive}$$

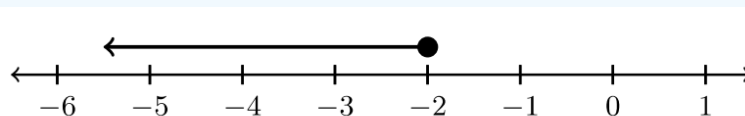
The inequality is not reversed.

Answers:

Set Builder Notation: $\{y | y \leq -2\}$

Interval Notation: $(-\infty, -2]$

Graph:



Taking Different Steps Should Lead to the Same Answer

In many cases, arriving at an answer can come from different paths! No matter the path, if the math performed along the way is sound, the answer should be the same.

✓ Example 4.3.3

Solve the inequality for w : $6w - 7 < 9w + 5$

Solution

Method #1

$$6w - 7 < 9w + 5$$

$$6w - 6q - 7 < 9q - 6q + 5 \quad \leftarrow \text{Compare these steps!} \rightarrow$$

$$-7 < 3w + 5$$

$$-7 - 5 < 3w + 5 - 5$$

$$-12 < 3w$$

$$\frac{-12}{3} < \frac{3w}{3} \quad \leftarrow \text{Compare these steps!} \rightarrow$$

$$-4 < w$$

Method #2

$$6w - 7 < 9w + 5$$

$$6w - 9w - 7 < 9w - 9w + 5$$

$$-3w - 7 < 5$$

$$-3w - 7 + 7 < 5 + 7$$

$$-3w < 12$$

$$\frac{-3w}{-3} > \frac{12}{-3}$$

$$w > -4$$

The answers are the same!

Try It! (Exercises)

For each of the exercises #1-20,

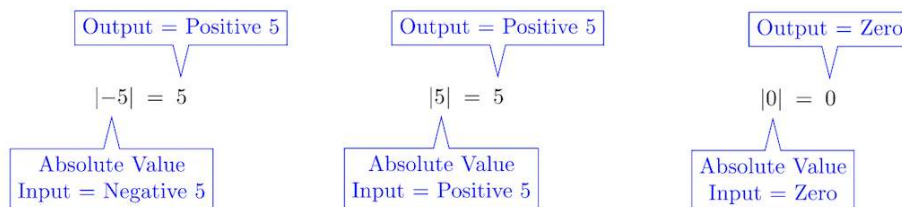
- Solve the inequality for the variable indicated.
- Express the solution set in both set-builder notation and in interval notation.
- Graph the solution set.

1. $-2p < 10$
2. $-9 < 3y$
3. $16 \leq 5t + 1$
4. $-14 \geq 4x + 2$
5. $6y - 10 > 4y$
6. $8b + 3 < 7b - 1$
7. $-2a - 3 \leq 4a + 3$
8. $9 - u \leq 11 + u$
9. $7 - 5p \geq 8 - 10p$
10. $5(v + 1) > 3(y - 1)$
11. $2(11 - 3t) < 4(t - 2)$
12. $6(4y - 3) \leq -3(6 - 7y)$
13. $8a - (6 + 2a) \geq 3(4a - 6)$
14. $10 - 2(7 + x) > 7 - 3(3x - 1)$
15. $3 + 5(3 - 2d) \leq 18 - 6(4d - 7)$
16. $7p - (10 - 25p) > 4p - 9(4p - 6)$
17. $\frac{1}{2}x - \frac{3}{4} < x + \frac{1}{4}$
18. $\frac{3}{5}(y + 5) > \frac{2}{5}(y - 10)$
19. $\frac{1}{6}(q - 2) \leq \frac{1}{3}(q + 2)$
20. $1 - \frac{1}{4}(2t - 1) \geq \frac{3}{4} + \frac{1}{2}(t + 1)$

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4.4: Absolute Value Equations and Inequalities as Applied to Distance

The absolute value function, denoted $y = |x|$, takes any negative real number input and outputs the positive version of that number. Nonnegative numbers are left unchanged. Here's a few examples:

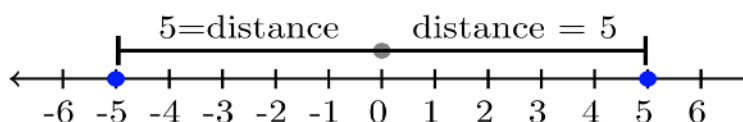


Measuring distance is a good application to demonstrate the usefulness of this function. Distance is never negative. In fact, let's view the absolute value function through the lens of measuring distance to better understand absolute value equations and inequalities:

Geometric Definition of the Absolute Value of a Real Number

The absolute value of a real number b , denoted $|b|$, is the distance between zero and b on the number line.

The above examples can be geometrically interpreted by this graph:



✓ Example 4.4.1

Solve for x : $|x| = 3$.

Solution

What numbers are a distance of 3 units from zero? A number line graph tells us! Both 3 and -3 are a distance of 3 units from zero.

Answer $x = \{-3, 3\}$

The Distance Between a and b

Absolute value is often used to measure the distance between two numbers, a and b . If we use absolute value bars around the subtracted quantity, then the order of subtraction does not matter.

Distance From a to b

The distance from a to b is the absolute value of the difference.

$$|a - b| = |b - a| = \text{distance between } a \text{ and } b$$

Draw a number line and practice finding the distance between any two points. Convince yourself that the above formula works!

✓ Example 4.4.2

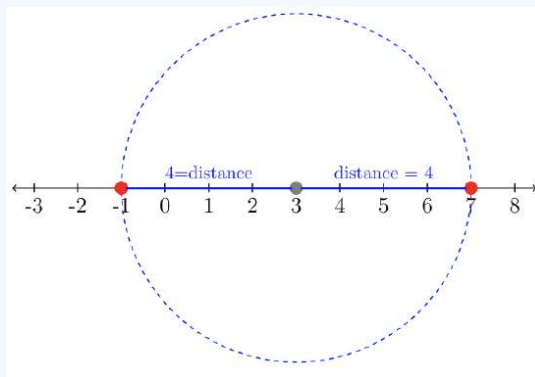
Solve for x : $|x - 3| = 4$

Use the geometric concept of distance to show solution.

Solution

Let's break down the equation into meaningful words:

distance from x to 3 is $|x - 3|$ $\stackrel{\text{is}}{=} 4$: “The distance from the number 3 to another number x is 4 units.”



It's quite effective to think of the value 3 as the center of a circle of radius 4. The circle crosses the number line, marking the distance of 4 units in each direction. The visual tells us the two answers are -1 and 7 .

Answer $x = \{-1, 7\}$

✓ Confirm the visual solution matches the math:

$$\begin{array}{rcl} \text{Let } x = -1 & & \\ |x - 3| & = & 4 \\ |-1 - 3| & \stackrel{?}{=} & 4 \\ |-4| & \stackrel{?}{=} & 4 \\ 4 & = & 4 \quad \checkmark \end{array}$$

$$\begin{array}{rcl} \text{Let } x = 7 & & \\ |x - 3| & = & 4 \\ |7 - 3| & \stackrel{?}{=} & 4 \\ |4| & \stackrel{?}{=} & 4 \\ 4 & = & 4 \quad \checkmark \end{array}$$

Absolute Value Inequalities

Now that you have the hang of solving absolute value equations describing a distance, let's continue using the number line to solve absolute value inequalities. Place a circle whose radius is the given distance and center is a given value in the equation or inequality. The answer will leap out at you visually!

✓ Example 4.4.3

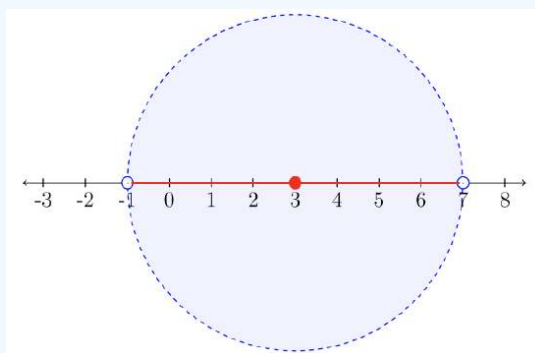
Solve for x : $|x - 3| < 4$

Solution

Compare this inequality to the equation in Example 4.4.2.

Let's begin the solution similarly: make a meaningful sentence of the inequality.

“The distance from the number 3 to a number x is less than 4 units.”



The solution set will be all numbers within a distance of 4 units from the number 3. The circle marks the boundary of our inequality. The solution set is the numbers within the boundary. This time, the solution is an interval, describing all real numbers between -1 and 7 , not including -1 nor 7 .

Solution set: $\{x \mid -1 < x < 7\}$ or the interval $(-1, 7)$.

✓ Example 4.4.4

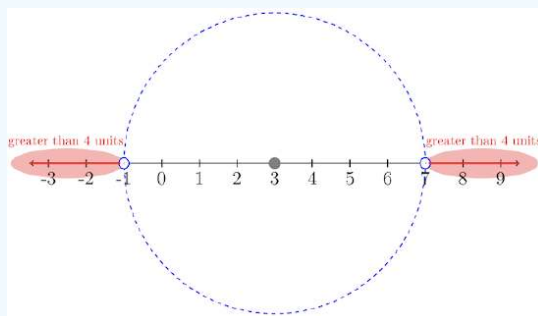
Solve for t : $|t - 3| > 4$

Solution

Compare this inequality to the inequality in Example 4.4.3.

Again, the meaningful application of distance is used to find the solution visually.

“The distance from the number 3 to another number t is greater than 4 units.”



The solution set will be all numbers beyond a distance of 4 units from the number 3. The circle marks the boundary of our inequality. The solution set is the numbers outside the circle's boundary. The solution set is a compound inequality since the numbers excluded from the solution set are within the circle's boundary.

Solution Set: $\{t \mid t < -1 \text{ or } t > 7\}$

Interval Notation: $(-\infty, -1) \cup (7, \infty)$

✓ Example 4.4.5

Solve for y : $|y + 2| \leq 5$

Solution

If we are to use distance to find a solution, our inequality must be converted to the absolute value of a difference.

We will use the following property of arithmetic to convert to a difference:

$$A + B = A - (-B)$$

Therefore:

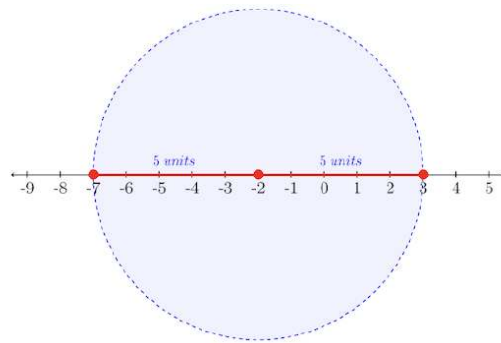
$$y + 2 = y - (-2)$$

We revise the stated inequality:

$$|y - (-2)| \leq 5$$

In words,

“The distance from -2 to another number y is less than or equal to 5 units.”



The solution set will be all numbers within a distance of 5 units from -2 , including the values exactly 5 units from -2 . The visual solution leads us to our answer:

Solution Set: $\{y \mid -7 \leq y \leq 3\}$

Interval Notation: $[-7, 3]$

Try It! (Exercises)

For each of the exercises #1-23,

- Interpret the equation or inequality as distance, using a sentence.
- Solve the inequality for the variable indicated, using distance.
- Express the solution set in both set-builder notation and in interval notation.

1. $|x| = 6$
2. $|y| = 1$
3. $|b| = 3.26$
4. $|x| < 5$
5. $|y| > 4$
6. $|c| \leq \frac{3}{4}$
7. $|q| \geq 10.45$
8. $|p - 1| \leq 2$
9. $|u - 2| > 3$
10. $|t - 4| \leq 1$
11. $|x - 6.3| \geq 4$
12. $|y - 3| < \frac{3}{2}$
13. $|h - 5| > \frac{3}{2}$
14. $|q - \frac{7}{2}| \geq \frac{1}{2}$
15. $|w - 7| < 2.3$
16. $|r - 10| \geq 0.15$
17. $|d - \pi| \leq 2\pi$
18. $|p - 2\pi| > \pi$
19. $|u - \frac{\pi}{2}| < \frac{\pi}{2}$
20. $|t + 4| \leq 2$
21. $|x + 5| > 3$
22. $|r + 3| \geq \frac{1}{4}$
23. $|a + \frac{1}{2}| < \frac{5}{2}$

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CHAPTER OVERVIEW

5: Rational Expression

Learning Objectives

- Identify a rational expression.
- Identify opposite quantities within a rational expression.
- Detect cancelable factors within a rational expression and perform cancellations.
- Perform operations (add, subtract, multiply, divide) on rational functions.

[5.1: Simplify Rational Expressions](#)

[5.2: Multiply Rational Expressions](#)

[5.3: Divide Rational Expressions](#)

[5.4: Add and Subtract Rational Expressions](#)

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5.1: Simplify Rational Expressions

Rational numbers are sometimes informally called fractions. Numbers such as $\frac{3}{4}$ and $-\frac{1}{5}$ are rational numbers.

Definition: Rational Numbers

The set of **rational numbers** is the set of all numbers of the form $\frac{p}{q}$ where both p and q are integers and $q \neq 0$.

Numbers like $\frac{\sqrt{3}}{2}$ and $\frac{\pi}{4}$ are therefore not rational numbers! The numerators, in both cases, are not integers. Although $\frac{\sqrt{3}}{2}$ and $\frac{\pi}{4}$ are irrational numbers due to their irrational numerators, we commonly refer to them as fractions, even though that's a bit of an allowance. But let's agree $\frac{\sqrt{3}}{2}$ and $\frac{\pi}{4}$ are not rational numbers!

Whereas a rational *number* describes a type of *number*, a rational *expression* describes a type of algebraic *expression*.

Definition: Rational Expressions

Rational expressions are ratios of polynomials, $p(x)$ and $q(x)$:

$$\frac{p(x)}{q(x)} \quad (5.1.1)$$

where $q(x)$ is not zero itself.

✓ Examples of rational expressions: $\frac{2x+1}{3x+4}$, $-\frac{1}{x^2+1}$, $\frac{2-x^2}{5x}$, or $\frac{5x^9-3x^4+x}{2x^2-7x}$.

✗ Examples of expressions that are not rational expressions: $\frac{6x^2}{0}$, $\frac{x^\pi}{3x}$, $\frac{\sqrt{2x+9}}{5x}$, or $\frac{x^2+2}{2^x}$

How do these last 4 expressions fail to be rational expressions? You might have to revisit the definition of a polynomial from Chapter 2 to articulate why they are not rational expressions!

Canceling Quantities

Fractions denote division. $\frac{5}{5} = 5 \div 5 = 1$. Algebraically, $\frac{a}{a} = a \div a = 1$ for all $a \neq 0$. Canceling within fractions uses this property. Although the word “cancel” suggests the number or quantity “disappears,” it’s not a magic act! We should probably say “cancels to one” instead, so that we don’t lose sight of the mathematical sense.

✓ Example 5.1.1

Simplify $\frac{3a^3b^2}{12a^2b^5}$

Solution

When simplifying rational expressions, look for groups of variables or numbers that can be canceled to one. The numbers 3 and 12 have a common factor. The variable a is repetitively multiplied to create a power. The variable b is also repetitively multiplied to create a power. Cancel as many times as permits within a rational expression.

$$\frac{3}{12} \cdot \frac{a^3}{a^2} \cdot \frac{b^2}{b^5}$$

$$= \frac{3}{3 \cdot 4} \cdot \frac{\cancel{a} \cdot \cancel{a} \cdot a}{\cancel{a} \cdot \cancel{a}} \cdot \frac{\cancel{b} \cdot \cancel{b}}{\cancel{b} \cdot \cancel{b} \cdot b \cdot b \cdot b}$$

$$= \frac{1}{4} \cdot \frac{a}{1} \cdot \frac{1}{b \cdot b \cdot b}$$

$$= \frac{a}{4b^3}$$

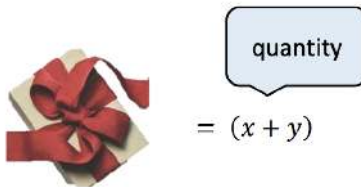
Separating into groups sets up for cancelling.

You could also use rules of exponents. See Chapter 1.

The cancelations are complete.

Simplify powers.

The same property of division applies to quantities. Parentheses group algebraic expressions, bundling them together like a packaged box. The contents within the parentheses act as a single value, which is referred to as a quantity.



✓ Example 5.1.2

Simplify $\frac{10(2y-1)^2(y+6)}{6(2y-1)(y+6)}$

Solution

Since quantities behave like single values, the problem is logically equivalent to: $\frac{10A^2B}{6AB}$. By refocusing the lens, this problem is very much like Example 5.1.1.

$$\frac{5 \cdot 2}{3 \cdot 2} \cdot \frac{\cancel{(2y-1)} \cdot \cancel{(2y-1)}}{\cancel{(2y-1)}} \cdot \frac{\cancel{y+6}}{\cancel{y+6}}$$

Separating into groups sets up for cancelling.

$$= \frac{5}{3} \cdot \frac{2y-1}{1} \cdot 1$$

The cancelations are complete.

$$= \frac{5(2y-1)}{3}$$

Write as a single fraction. This answer is okay!

$$= \frac{10y-5}{3}$$

When multiplication is simple to do, it is often performed.

The final simplified answer can be multiplied out, or not! If it isn't a quick multiplication, it's often preferred that you leave the answer as powers or factored form.

✓ Example 5.1.3

Simplify $\frac{x^2+2x-15}{x^2-5x+6}$

Solution

Cancelations occur only over factored quantities. The given numerator and denominator are not in factored form. For factoring review, see chapter 2.

$$\begin{aligned}\frac{x^2 + 2x - 15}{x^2 - 5x + 6} &= \frac{(x+5) \cancel{(x-3)}}{(x-2) \cancel{(x-3)}} && \text{Factor the numerator and the denominator.} \\ &= \frac{(x+5)}{(x-2)} \cdot 1 && \text{The quantities } (x-3) \text{ cancel to one.} \\ &= \frac{x+5}{x-2} && \text{The rational expression is fully simplified.}\end{aligned}$$

The horizontal fraction bar groups the numerator and groups the denominator. The parentheses become redundant. Therefore, removing the parentheses is preferred.

Note

Although $\frac{ab}{cd} = \frac{a}{c} \cdot \frac{b}{d}$, addition and subtraction are more persnickety. You can make mistakes by incorrectly applying cancellation properties to addition or subtraction. As Example 5.1.3 demonstrates, cancellation can be applied *after* factoring.

✓ Example 5.1.4

Simplify $\frac{2p+14}{p+7}$

Solution

Both the numerator and the denominator are expressions involving addition. If the quantities are factorable, then we factor to compare quantities. Identical quantities can be cancelled after factoring.

$$\begin{aligned}\frac{2p+14}{p+7} &= \frac{2(p+7)}{p+7} && \text{Factor the numerator using GCF} = 2. \\ &= 2 \cdot \frac{p+7}{p+7} && \text{The quantities } (p+7) \text{ cancel.} \\ &= 2 \cdot 1 && \text{The value after cancelling is 1.} \\ &= 2\end{aligned}$$

✓ Example 5.1.5

Simplify $\frac{12}{6t-4}$

Solution

The denominator is a factorable expression. The numerator is a factorable number.

$$\begin{aligned}\frac{12}{6t-4} &= \frac{\cancel{2} \cdot 2 \cdot 3}{\cancel{2}(3t-2)} && \text{Factor numerator and denominator. Cancel the common factor.} \\ &= \frac{6}{3t-2} && \text{Simplify.}\end{aligned}$$

It's tempting to inspect the individual terms of the quantity $(3t-2)$ for further cancellations, but because the terms are subtracted, not multiplied, canceling is not possible.

Try It! (Exercises)

For #1-6, use the definition of a rational number to determine if the number is rational or irrational.

1. $\frac{5}{9}$
2. 1
3. -4

4. $\frac{\sqrt{2}}{2}$

5. π

6. $\frac{3\pi}{4}$

For #7-12, use the definition of a rational expression to determine if the expression is a rational expression or something else.

7. $\frac{1}{x}$

8. $\frac{x}{5^x}$

9. $\frac{3x^5}{2x+1}$

10. $\frac{1}{\sqrt{x}}$

11. $\frac{4(2x-5)^2}{(x-7)(x+5)}$

12. $7x^{-3}$

For #13-34, simplify the rational expression.

13. $\frac{15a^4b^2}{6a^5b}$

14. $\frac{7p^6q}{35p^7q^2}$

15. $\frac{16x^4y^5z^2}{6x^3y^6z^2}$

16. $-\frac{30u^7v^3w^9}{u^8v^8w^8}$

17. $\frac{4(t-2)(3t+7)^3}{10(t-2)^2(3t+7)}$

18. $\frac{40(6u+1)^2(4u-1)^5}{8(4u-1)^3(6u-1)}$

19. $-\frac{2(8a-b)^3(2a+1)^2}{12(8a-b)^3(2a+1)^3}$

20. $\frac{x^2+x-20}{x^2-5x+4}$

21. $\frac{3q-6}{q^2-4}$

22. $\frac{y^2-11y+30}{y^2-25}$

23. $\frac{3c^2-14c-5}{3c^2+c}$

24. $\frac{a^2+12a+36}{a^2+3a-18}$

25. $\frac{5r^2-12r+4}{r^2-4r+4}$

26. $\frac{w^2-81}{2w^2+18w}$

27. $\frac{20y^2-5}{10y^2+35y-20}$

28. $\frac{m^3-27}{2m^2-7m+3}$

29. $\frac{10b^3+10}{10b^2-10}$

30. $\frac{3p^4 - 48}{6p^2 - 24}$
31. $\frac{2x^2 + 7x + 6}{4x^2 + 3x - 6}$
32. $\frac{5d(d+1)^3}{5d^3 - 5d}$
33. $\frac{7v^3 - 28v}{7v(v-2)^3}$
34. $\frac{3x^5 - 6x^4 - 105x^3}{6x^2 + 30x}$

For #35-36, simplify the numerator and denominator, then simplify the rational expression.

35. $\frac{(3x+4) - (2x-4)}{(x+10) + (x+6)}$
36. $\frac{(4x^2 + 5x + 1) - (3x^2 + 3x + 4)}{(2x^2 + x + 8) - (x^2 - 6x - 4)}$

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5.2: Multiply Rational Expressions

Multiplying rational expressions is very much the same as fraction multiplication in arithmetic. The numerators are multiplied to numerators. The denominators are multiplied to denominators. The common factors in the numerator and denominator are canceled before multiplying. That's it! Two examples are shown below. Compare the similarities!

Multiply rational numbers

$$\begin{aligned} \frac{3}{20} \cdot \frac{4}{9} \\ \frac{\cancel{3}}{\cancel{4} \cdot 5} \cdot \frac{\cancel{4}}{\cancel{3} \cdot 3} \\ = \frac{1}{5} \cdot \frac{1}{3} \\ \frac{1 \cdot 1}{5 \cdot 3} \\ \frac{1}{15} \end{aligned}$$

Factor & Cancel

What's left?

Multiply

Answer!

Multiply rational expressions

$$\begin{aligned} \frac{2x+6}{x+4} \cdot \frac{x^2+3x-4}{x^2-9} \\ \frac{2(\cancel{x+3})}{\cancel{x+4}} \cdot \frac{(\cancel{x+4})(x-1)}{(\cancel{x+3})(x-3)} \\ \frac{2}{1} \cdot \frac{x-1}{x-3} \\ \frac{2(x-1)}{1(x-3)} \\ \frac{2x-2}{x-3} \end{aligned}$$

Factoring is key to multiplying rational expressions! Parentheses group expressions. The whole group is seen as a single number, like a package deal. If numerators and denominators are factored, you can cancel common factors, just like with real numbers.

✓ Example 5.2.1

Multiply $\frac{12a^2b}{5c^4} \cdot \frac{10c}{9a^2b^2}$

Solution

The given rational expressions are factored. Find the common factors that cancel. Stay organized to avoid losing track of what's left after cancelling.

$$\begin{aligned} &= \frac{3 \cdot 4 \cdot a^2 \cdot b}{5 \cdot c \cdot c^3} \cdot \frac{5 \cdot 2 \cdot c}{3 \cdot 3a^2 \cdot b \cdot b} \\ &= \frac{4}{c^3} \cdot \frac{2}{3b} \\ &= \frac{4 \cdot 2}{c^3 \cdot 3b} \\ &= \frac{8}{3c^3b} \end{aligned}$$

Multiplication shows cancelable factors.

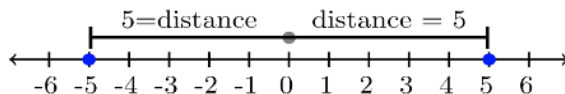
These are the factors left after canceling.

Multiply numerators. Multiply denominators.

Final inspection. Did we cancel everything? Yes!

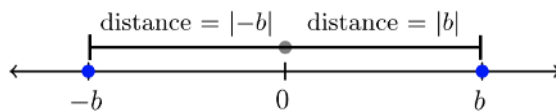
Opposites

Two numbers are opposites if both numbers have the same distance from zero. For example, 5 and -5 are opposites.



Algebraic Opposites

Algebra generalizes mathematical patterns using variables. If b is any real number except zero, b and $-b$ are opposites.



Opposite of a Quantity

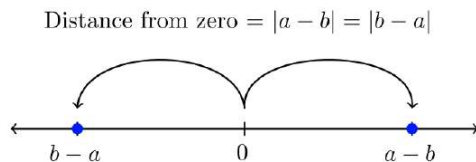
Quantities act like one real number. That means the quantity $(a - b)$ should also have an opposite, just like 5 has its opposite, -5. Find the opposite of $(a - b)$ in 3 steps:

1. Put a negative on the entire quantity: $-(a - b)$
2. Distribute through the negative: $-a + b$
3. Rearrange the terms: $b - a$

Definition: The Opposite of a Difference

$a - b$ and $b - a$ are opposites.

$$|a - b| = |b - a| \quad (5.2.1)$$



Note

It's arbitrary which you assign a positive value: $(a - b) > 0$ is pictured above, but you could correctly assign $(b - a) > 0$ making $(a - b) < 0$. In that case, you would swap their marked positions on the number line. Nevertheless, $|a - b| = |b - a|$.

The Ratio of Opposites

What happens when we take the ratio of opposites? In arithmetic, $-\frac{5}{5} = -\frac{1 \cdot 5}{5} = -1$. Now let's use $b =$ any real number except zero:

$$\checkmark \frac{-b}{b} = \frac{-1 \cdot b}{b} = -1 \cdot \frac{b}{b} = -1 \cdot 1 = -1$$

Since zero in the denominator doesn't compute, $b \neq 0$.

Definition: The Ratio of Opposites Equals

$$\begin{aligned} \frac{-b}{b} &= \frac{b}{-b} = -1 \\ \frac{a-b}{b-a} &= \frac{b-a}{a-b} = -1 \end{aligned} \quad (5.2.2)$$

Spotting opposites gives us another cancelation opportunity. Opposites in the numerator and denominator cancel to negative 1.

✓ Example 5.2.2

Multiply $\frac{y^2 - 4}{y^2 - 4y - 12} \cdot \frac{y^2 - 7y + 6}{2 - y}$

Solution

The given rational expressions are not factored. We can't cancel before factoring.

$$\frac{y^2 - 4}{y^2 - 4y - 12} \cdot \frac{y^2 - 7y + 6}{2 - y} = \frac{(y+2)(y-2)}{(y+2)(y-6)} \cdot \frac{(y-6)(y-1)}{2-y} \quad \text{Factor wherever possible. Cancel.}$$

$$= \frac{y-2}{1} \cdot \frac{y-1}{2-y}$$

$$= -1 \cdot (y-1)$$

$$= -y + 1$$

$$= 1 - y$$

Notice the opposites! $\frac{y-2}{2-y} = -1$.

Treat the numerator as a quantity. Use parentheses.

Use the distributive property.

A positive leading term is preferred.

✓ Example 5.2.3

Multiply $\frac{9 - 4t^2}{16t^2 - 20t - 6} \cdot (1 + 4t)$

Solution

When multiplying fractions to an integer, we create a fraction of the integer by placing a one in the denominator. Let's do the same for rational expressions!

$$\begin{aligned}
 \frac{9-4t^2}{16t^2-20t-6} \cdot (1+4t) &= \frac{9-4t^2}{16t^2-20t-6} \cdot \frac{1+4t}{1} \\
 &\quad \text{Difference of Squares} \\
 &= \frac{\overbrace{(3-2t)(3+2t)}^{\text{GCF}=2}}{2(8t^2-10t-3)} \cdot \frac{1+4t}{1} \\
 &= \frac{(3-2t)(3+2t)}{2(\cancel{4t+1})(2t-3)} \cdot \frac{1+\cancel{4t}}{1} \\
 &= \frac{(3-2t)(3+2t)}{2(2t-3)} \\
 &= -1 \cdot \frac{3+2t}{2} \\
 &= \frac{-1(3+2t)}{2} = \frac{-3-2t}{2} \quad \text{OR} \quad -1 \cdot \frac{3+2t}{2} = -\frac{3+2t}{2} \\
 &\quad \text{Answer \#1} \qquad \qquad \qquad \text{Answer \#2}
 \end{aligned}$$

Create a fraction. Put a "1" under it!

Factor. Notice the denominator can be further factored.

Cancel common factors. $4t+1 = 1+4t$.

Notice the opposites! $\frac{3-2t}{2t-3} = -1$.

The cancelation of opposites creates a multiplier of -1 .

Which answer do you prefer?

Both Answer #1 and Answer #2 are perfectly good answers. Your final answer is your choice!

Try It! (Exercises)

For #1-5, answer the questions about opposites:

1. The opposite of 3 is
2. The opposite of -4 is
3. The opposite of $(6-4c)$ is
4. The opposite of $(2t+5)$ is
5. Opposites in a rational expression cancel to what number?

For #6-8, multiply the rational numbers without using a calculator. Give answer in reduced fraction form.

6. $\frac{2^3 \cdot 5^4}{11^6} \cdot \frac{11^7 \cdot 3^2}{2^5 \cdot 5^3}$
7. $6^5 \cdot \frac{5^2}{3 \cdot 6^6}$
8. $\frac{4^3 \cdot 2^{10}}{3} \cdot \frac{6}{4^2 \cdot 2^{12}} \cdot 2^3$

For #9-30, multiply and simplify.

9. $\frac{4x^5}{9y^4} \cdot \frac{27y^2z^3}{8x^5}$
10. $49u^6 \cdot \frac{w^3}{7u^3}$
11. $\frac{10p^8}{9q^2} \cdot \frac{18q^5}{25p^9} \cdot 5p$
12. $\frac{(2r-3)^3}{5(4r+1)^2} \cdot \frac{15(4r+1)^3}{2r-3}$
13. $12(3c-5)^4 \cdot \frac{c+1}{6(3c-5)^3}$
14. $\frac{4x+24}{x^2+x-2} \cdot \frac{x^2-4}{6x+36}$
15. $\frac{z^2+2z+1}{3z-15} \cdot \frac{z^2+z}{z^2-6z+5}$
16. $\frac{y^2-6y-16}{6y-3} \cdot \frac{2y^2+y-1}{64-y^2}$
17. $\frac{b^2+5b-14}{16h^2+24h+9} \cdot \frac{b^2-3b-10}{6h^2-24h}$
18. $\frac{h^2-h-12}{6a^2+7a-3} \cdot \frac{8h^2+6h}{2a+6}$
19. $\frac{9a^2-1}{5n} \cdot \frac{2a+6}{3p^2+13p+14}$
20. $(2n+1) \cdot \frac{2p}{10n^2+n-2}$
21. $(3p+7) \cdot \frac{2p}{3p^2+13p+14}$

22. $\frac{4d^2 - 4}{2d^2 - 20d + 18} \cdot (d - 9)$
23. $\frac{4t}{10t - t^2} \cdot (t^2 - 12t + 20)$
24. $\frac{7w^2 - 7}{w^2 + 6w + 5} \cdot \frac{w^2 + 5w}{14w}$
25. $\frac{1 - x}{3x} \cdot \frac{x^2 - x - 42}{3x + 18} \cdot \frac{27x^2}{x^2 - 8x + 7}$
26. $\frac{4a^2 + 44a}{9a^2} \cdot \frac{63a^3}{a^2 + a - 20} \cdot \frac{a + 5}{a^3 + 11a^2}$
27. $\frac{v^3 + 1}{v^2 - 1} \cdot \frac{v^2 - 8v - 9}{v^2 - 2v + 1}$
28. $\frac{q^3 - 8}{q^3 - 2q^2} \cdot \frac{8q^3}{q^2 + 16q + 4}$
29. $\frac{27 - 8m^3}{m^2 - 64} \cdot \frac{8 + m}{2m - 3}$
30. $\frac{a^3}{a^2 + ab} \cdot \frac{a^2 - b^2}{ab - a^2}$

For #31-36, one numerator is left blank. What expression goes in the blank to make the equation true?

31. $\frac{x + 1}{8x - 1} \cdot \frac{?}{x + 1} = 1$
32. $\frac{x + 1}{8x - 1} \cdot \frac{?}{x + 1} = -1$
33. $\frac{y^2 - 4}{y^2 - y - 6} \cdot \frac{?}{y - 2} = 1$
34. $\frac{y^2 - 4}{y^2 - y - 6} \cdot \frac{?}{y - 2} = -1$
35. $\frac{5u^2}{u - 4} \cdot \frac{?}{10u^3} = 1$
36. $\frac{5u^2}{u - 4} \cdot \frac{?}{10u^3} = -1$

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5.3: Divide Rational Expressions

Dividing rational expressions is very much the same as fraction division in arithmetic. The first step is to change the division to multiplication and take the reciprocal of the second fraction. Two examples are shown below. Compare the similarities!

Divide rational numbers

$$\begin{aligned}\frac{3}{4} \div \frac{9}{20} \\ \frac{3}{4} \cdot \frac{20}{9} \\ = \frac{3}{4} \cdot \frac{4 \cdot 5}{3 \cdot 3} \\ \frac{5}{3}\end{aligned}$$

This answer is in simplest form.

Multiply by the reciprocal

Factor and cancel

What's left?

Simplify answer

Divide rational expressions

$$\begin{aligned}\frac{x-6}{x^2-16} \div \frac{4x-24}{x^2+x-12} \\ \frac{x-6}{x^2-16} \cdot \frac{x^2+x-12}{4x-24} \\ \frac{x-6}{(x-4)(x+4)} \cdot \frac{(x+4)(x-3)}{4(x-6)} \\ \frac{x-3}{4(x-4)} \\ \frac{x-3}{4x-16}\end{aligned}$$

Complex Fractions

There are two dueling notations for dividing rational expressions. The above notation uses the symbol $\div$ for division. The competing notation is called a complex fraction, which maintains fraction notation. A horizontal fraction bar indicates division.

A **complex fraction** is a fraction in which either the numerator is a fraction, or the denominator is a fraction, or both. To simplify complex fractions, translate the main fraction bar to division.

$$\begin{aligned}\frac{\frac{a}{b}}{\frac{c}{d}} &= \frac{a}{b} \div \frac{c}{d} && \text{Step 1: Translate to } \div \text{ notation.} \\ &= \frac{a}{b} \cdot \frac{d}{c} && \text{Step 2: Multiply by the reciprocal.} \\ &= \frac{ad}{bc} && \text{Step 3: Multiply and simplify.}\end{aligned}$$

✓ Example 5.3.1

Simplify $\frac{\frac{8u^3}{35w^4}}{\frac{24u}{5w^3}}$

Solution

Do you see the main fraction bar? $\frac{\frac{8u^3}{35w^4}}{\frac{24u}{5w^3}} \leftarrow$ Change fraction bar to $\div$

$$\frac{8u^3}{35w^4} \div \frac{24u}{5w^3} = \frac{8\cancel{u^3}^{u^2}}{35\cancel{w^4}^{7w}} \cdot \frac{5\cancel{w^3}^1}{24\cancel{u}^3} = \frac{u^2}{21w}$$

✓ Example 5.3.2

Simplify $\frac{\frac{2}{h}}{10h}$

Solution

Let's convert $10h$ to fraction form: $\frac{\frac{2}{h}}{\frac{10h}{1}} \leftarrow$ Change fraction bar to $\div$

$$\frac{2}{h} \div \frac{10h}{1} = \frac{\cancel{2}^1}{h} \cdot \frac{1}{\cancel{10h}^{5h}} = \frac{1}{5h^2}$$

✓ Example 5.3.3

Simplify $\frac{\frac{t^2-1}{t+2}}{\frac{5-5t}{t^2+3t+2}}$

Solution

The given expression is a complex fraction.

$$\frac{\frac{t^2-1}{t+2}}{\frac{5-5t}{t^2+3t+2}} = \frac{t^2-1}{t+2} \div \frac{5-5t}{t^2+3t+2}$$

Change the main fraction bar to $\div$

$$= \frac{t^2-1}{t+2} \cdot \frac{t^2+3t+2}{5-5t}$$

Multiply by the reciprocal.

$$= \frac{(t+1)(t-1)}{(t+2)} \cdot \frac{(t+2)(t+1)}{5(1-t)} = \frac{(t+1)(\cancel{t-1})}{(\cancel{t+2})} \cdot \frac{(\cancel{t+2})(t+1)}{5(\cancel{1-t})^{-1}}$$

Factor and cancel. Opposites, too!

$$= -\frac{(t+1)^2}{5}$$

In our final example (below), we divide three rational expressions. As a student, allow yourself to apply your knowledge of fractions. The process is the same for rational expressions. Before doing example 4, you might try to simplify the following:

$$\frac{1}{2} \div \frac{3}{4} \div \frac{3}{5}$$

Order of operations requires us to move from left to right, taking two fractions at a time.

$$\left(\frac{1}{2} \div \frac{3}{4}\right) \div \frac{3}{5} = \left(\frac{1}{2} \cdot \frac{4}{3}\right) \div \frac{3}{5} = \left(\frac{2}{3}\right) \div \frac{3}{5} = \frac{2}{3} \cdot \frac{5}{3} = \frac{10}{9}$$

✓ Example 5.3.4

Simplify $\frac{2y}{24-6y} \div \frac{y-2}{y^2-3y-4} \div \frac{y^2+y}{3}$

Solution

$$\begin{aligned} \left(\frac{2y}{24-6y} \div \frac{y-2}{y^2-3y-4}\right) \div \frac{y^2+y}{3} &= \left(\frac{2}{6} \cdot \frac{y}{\cancel{4-y}^{-1}} \cdot \frac{(y-4)(y+1)}{y-2}\right) \div \frac{y^2+y}{3} \\ &= \left(\frac{1}{\cancel{3}} \cdot \frac{\cancel{y}}{-1} \cdot \frac{y+1}{y-2}\right) \cdot \frac{\cancel{3}}{\cancel{y}(y+1)} \\ &= -\frac{1}{y-2} \\ &= \frac{1}{2-y} \end{aligned}$$

Try It! (Exercises)

For #1-3, divide the rational numbers without using a calculator. Give answer in reduced fraction form.

1. $\frac{7^{10}}{4^5} \div \frac{7^{12}}{4^6}$

$$2. \frac{13^8 \cdot 5^{10}}{6^7} \div \frac{13^8 \cdot 5^8}{3 \cdot 6^6 \cdot 5}$$

$$3. 2^9 \div \frac{4 \cdot 2^6}{8^2} \div \frac{2^3}{6}$$

For #4-20, divide and simplify.

$$4. \frac{24u^2}{9w^8} \div \frac{12u}{45w^{10}}$$

$$5. \frac{\frac{36a^2b^2}{25c^3}}{\frac{72ab}{5c^5}}$$

$$6. \frac{60x^6y^{10}}{11z^5} \div \frac{9x^8}{44z^2} \div \frac{8y^7}{x}$$

$$7. \frac{10p(p-q)^4}{3q(q-7)^2} \div \frac{2p^2(p-q)^5}{15q(q-7)^4}$$

$$8. \frac{40v(v-2)^{12}}{33(5v-6)^9} \div \frac{24v^3(v-12)^{10}}{11(5v-6)^{10}}$$

$$9. \frac{\frac{d-5}{20d}}{\frac{(d-5)^3}{25d^2} \div \frac{4r^2-9}{r}}$$

$$10. \frac{2r+3}{x^2+3x-18} \div \frac{x^2+12x+36}{x^2-2x-3}$$

$$11. \frac{x^2-2x-3}{4m^3+3m^2} \div \frac{x^2-6x-7}{4m^3+7m^2+3m}$$

$$12. \frac{8m^2}{\frac{12t^3}{t^2-4}} \div \frac{4m}{\frac{8t^2}{4t-16}}$$

$$13. \frac{5n+6}{\frac{5n^2-4n-12}{4-n^2}}$$

$$14. \frac{6p^2+6p-72}{3-p} \div (3p+12)$$

$$15. \frac{y^2+2y}{4y-32} \div \frac{y+2}{12y-24} \div \frac{6y-12}{5y}$$

$$16. \frac{w^2-25}{w^2-2w-35} \div \frac{4-4w}{w^4+2w^3} \div \frac{w^3-5w^2}{w^2-8w+7}$$

$$17. \frac{b^4-1}{b^2-5b-6} \div \frac{b^4+b^2}{b^2-12b+36}$$

$$18. \frac{\frac{8a^3-1}{2a^2+7a-4}}{\frac{16a}{4a+16} \div \frac{64c^3+27}{16c^3}}$$

$$19. \frac{16c^3}{16c^2-9} \div \frac{32c^6}{32c^6}$$

$$20. \frac{16c^2-9}{32c^6}$$

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5.4: Add and Subtract Rational Expressions

A fraction is a proportion. The fraction communicates the number of parts out of the whole.

$$\frac{a}{b} \quad \begin{array}{l} \leftarrow \text{The part is the numerator.} \\ \leftarrow \text{The whole is the denominator.} \end{array}$$

For example, the egg carton pictured below contains 10 eggs. Part of the eggs are brown (7 eggs) while the rest are white (3 eggs). A fraction $\frac{a}{b}$ quickly communicates the proportion of eggs that are brown or are white. Proportions can also be given as a decimal or percentage.



Proportion of brown eggs: $\frac{7}{10} = 7 \div 10 = 0.7 = 70\%$

Proportion of white eggs: $\frac{3}{10} = 3 \div 10 = 0.3 = 30\%$

The proportion of brown and white eggs describes all of the eggs. Intuitively, we know this proportion is 100%. The math should match our intuition, so let's investigate.

$$\begin{array}{rcl} 70\% + 30\% & = & 100\% \\ \frac{7}{10} + \frac{3}{10} & = & \frac{10}{10} \end{array}$$

Our investigation leads us to deduce that we add the numerators, but not the denominators.

To Add or Subtract Fractions (Denominator is the Same)

1. Add or subtract the numerators.
2. Keep the same denominator.
3. Reduce if necessary.

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c} \quad (5.4.1)$$

What works for real numbers also works for algebraic quantities. Algebra follows the same rules, which is an excellent anchor from which to perform your own algebraic investigations.

✓ Example 5.4.1

Add $\frac{2x}{x+3} + \frac{x}{x+3}$

Solution

The denominators match. Add the two numerators, but keep the denominator.

$$\frac{2x}{x+3} + \frac{x}{x+3} = \frac{2x+x}{x+3} = \frac{3x}{x+3}$$

Since the result cannot be further simplified, the answer is $\frac{3x}{x+3}$

✓ Example 5.4.2

Subtract $\frac{y^2+2y+8}{y^2-2y-15} - \frac{y^2+2}{y^2-2y-15}$

Solution

The denominators are the same. Subtract the numerators. Remember that each numerator is a quantity.

$$\begin{aligned} \frac{y^2+2y+8}{y^2-2y-15} - \frac{y^2+2}{y^2-2y-15} &= \frac{y^2+2y+8-(y^2+2)}{y^2-2y-15} && \text{Numerator: subtract the quantity } (y^2+2). \\ &= \frac{y^2+2y+8-y^2-2}{y^2-2y-15} && \text{Each term of the quantity is subtracted.} \\ &= \frac{2y+6}{y^2-2y-15} && \text{The subtraction is complete.} \\ &= \frac{2(y+3)}{(y-5)(y+3)} && \text{Factor numerator and denominator. Cancel.} \\ &= \frac{2}{y-5} && \text{The answer is most simplified.} \end{aligned}$$

Addition and Subtraction with Different Denominators

In order to add rational numbers or rational expressions, we need to find a common denominator. Create common denominators using equivalent fractions. That is, $\frac{a \cdot d}{b \cdot d} = \frac{ad}{bd}$. Then use the process for adding fractions with common denominators.

✓ Example 5.4.3

Add $\frac{7}{75a^2b} + \frac{13}{180ab}$

Solution

To find the least common denominator, we play a game of “who has more?” for each denominator. Break down denominators into prime factors. Imagine that each denominator is a set of cards, and each card is a factor of the denominator. Because we have two denominators, we have two players who will play the game, “who has more?”. Whichever player has the most of a factor “wins,” and they place those cards into the winning pile. Ready to play?

Player #1: $75a^2 \rightarrow \boxed{3} \boxed{5} \boxed{5} \boxed{a} \boxed{a} \boxed{b}$

Player #2: $180ab \rightarrow \boxed{2} \boxed{2} \boxed{3} \boxed{3} \boxed{5} \boxed{a} \boxed{b}$

Who has more?	Player #1	Player #2	Winning Pile
2's		✓	$2 \cdot 2$
3's		✓	$3 \cdot 3$
5's	✓		$5 \cdot 5$
a's	✓		$a \cdot a$
b's	TIE	TIE	b

The Winning Pile = $2^2 \cdot 3^2 \cdot 5^2 \cdot a^2 \cdot b$

The Least Common Denominator = $900a^2b$

Use the least common denominator to create a common denominator. Determine what factors are missing in each denominator. Multiply the missing factors to the denominator and also to the numerator. Then add.

$$\frac{7}{75a^2b} + \frac{13}{180ab} = \frac{7 \cdot 12}{75a^2b \cdot 12} + \frac{13 \cdot 5a}{180ab \cdot 5a} = \frac{84}{900a^2b} + \frac{65a}{900a^2b} = \frac{84 + 65a}{900a^2b}$$

The answer is: $\frac{84 + 65a}{900a^2b}$

✓ Example 5.4.4

Add $\frac{2y+1}{(4y+1)^2(y+6)} + \frac{5}{(4y+1)(y+6)}$

Solution

In the game of “who has more?” the quantities $(4y+1)$ and $(y+6)$ act as single numbers.

Player #1: $(4y+1)^2(y+6) \rightarrow \boxed{4y+1} \boxed{4y+1} \boxed{y+6}$

Player #2: $(4y+1)(y+6) \rightarrow \boxed{4y+1} \boxed{y+6}$

Winning Pile = $(4y+1)^2(y+6)$ = Least Common Denominator

$$\begin{aligned} \frac{2y+1}{(4y+1)^2(y+6)} + \frac{5}{(4y+1)(y+6)} &= \frac{2y+1}{(4y+1)^2(y+6)} + \frac{5 \cdot (4y+1)}{(4y+1)(y+6) \cdot (4y+1)} \\ &= \frac{2y+1 + (20y+5)}{(4y+1)^2(y+6)} \\ &= \frac{22y+6}{(4y+1)^2(y+6)} \end{aligned}$$

Simplify the numerator, but leave the denominator in factored form.

The numerator is simplified. This is the answer!

Tip: The numerator can be factored using GCF = 2: $22y+6 = 2(11y+3)$. However, this factorization does not lead to a cancellation. Therefore, the stated numerator is preferred.

✓ Example 5.4.5

Subtract $\frac{u^2+2u}{u^2+3u-10} - \frac{2u}{3u-6}$

Solution

To compare the factors of each denominator, we need to factor.

$$\frac{u^2+2u}{u^2+3u-10} - \frac{2u}{3u-6} = \frac{u^2+2u}{(u+5)(u-2)} - \frac{2u}{3(u-2)}$$

The denominators are factored.

Play the game of “who has more?” to find the Least Common Denominator. Quantities act as single numbers.

Player #1: $(u+5)$ $(u-2)$

Player #2: (3) $(u-2)$

Winning Pile = $3(u+5)(u-2)$ = **Least Common Denominator**

$$\begin{aligned}\frac{u^2+2u}{(u+5)(u-2)} - \frac{2u}{3(u-2)} &= \frac{3(u^2+2u)}{3(u+5)(u-2)} - \frac{2u(u+5)}{3(u-2)(u+5)} \\ &= \frac{3u^2+6u}{3(u+5)(u-2)} - \frac{2u^2+10u}{3(u+5)(u-2)} \\ &= \frac{3u^2+6u-(2u^2+10u)}{3(u+5)(u-2)} \\ &= \frac{3u^2+6u-2u^2-10u}{3(u+5)(u-2)} \\ &= \frac{u^2-4u}{3(u+5)(u-2)}\end{aligned}$$

Determine the missing factors & multiply.

Simplify numerators.

Numerator: subtract the quantity.

Subtract each term of the quantity.

The numerator is simplified. This is the answer!

Tip: The numerator can be factored using GCF = u : $u^2 - 4u = u(u - 4)$. However, this factorization does not lead to a cancellation. Therefore, the stated numerator is preferred.

✓ Example 5.4.6

Subtract $\frac{3c}{49-c^2} - \frac{1}{c^2-7c}$

Solution

To compare the factors of each denominator, we need to factor.

$$\frac{3c}{(7-c)(7+c)} - \frac{1}{c(c-7)} \quad \text{The denominators are factored.}$$

Play the game of “who has more?” to find the Least Common Denominator. Quantities act as single numbers.

Player #1: $(7+c)$ $(7-c)$

Player #2: (c) $(c-7)$ ↑ opposites

Note: $7-c$ and $c-7$ are opposites. Since they are opposites, one quantity can be made to “look like” the other.

$$7-c = -(c-7)$$

Next, bounce the negative to the numerator. Voilà! This is how we make a *Least Common Denominator*.

Winning Pile = $c(c+7)(c-7)$ = **Least Common Denominator**

$$\begin{aligned}\frac{3c}{(7-c)(7+c)} - \frac{1}{c(c-7)} &= \frac{3c}{-(c-7)(7+c)} - \frac{1}{c(c-7)} \\ &= \frac{-3c}{(c-7)(7+c)} - \frac{1}{c(c-7)} \\ &= \frac{-3c \cdot c}{c(c-7)(c+7)} - \frac{1 \cdot (c+7)}{c(c-7)(c+7)} \\ &= \frac{-3c^2 - (c+7)}{c(c-7)(c+7)} \\ &= \frac{-3c^2 - c - 7}{c(c-7)(c+7)} \\ &= \frac{-3c^2 - c + 7}{c(c-7)(c+7)}\end{aligned}$$

Replace $7-c$ with $-(c-7)$.

Since $\frac{a}{-b} = \frac{-a}{b}$, put the negative on top.

Construct the LCD. Note: $7+c = c+7$.

Numerator: subtract the quantity.

The numerator has been simplified.

Factor out -1 .

Either of the last two lines could be your final answer. Which do you prefer? It's your choice!

📌 Note

After simplifying the numerator, pause and ask yourself, “what more can I do to simplify?” If the numerator is factorable, check to see if the factors are the same as those in the denominator. If so, cancel the common factors. Keep the denominator factored to remind yourself (and your reader) what factors do not cancel.

Try It! (Exercises)

For #1-20, add or subtract the rational expressions as indicated.

1. $\frac{5x}{3y^2} + \frac{x}{3y^2}$
2. $\frac{14a}{4r} - \frac{5b}{4r}$
3. $\frac{5b}{9qp} + \frac{1}{3q^2p}$

4. $\frac{1-n}{n+2} + \frac{2n+1}{3n}$
5. $\frac{12m^3}{t+6} - \frac{3m^3n}{t+4}$
6. $\frac{t+5}{6c+1} + \frac{t+4}{t+5}$
7. $\frac{c-1}{(5c+2)^3} - \frac{c-1}{(5c+2)^3}$
8. $\frac{2}{x-3} + \frac{3}{10}$
9. $\frac{y+2}{3} + \frac{3y+6}{3}$
10. $\frac{x+3}{h-2} - \frac{x-3}{2}$
11. $\frac{(h+1)^2}{5} + \frac{3h+3}{3}$
12. $\frac{y-1}{-2} - \frac{y^2-1}{3}$
13. $\frac{a^2-5a+6}{4} + \frac{a-3}{5d-12}$
14. $\frac{d-3}{5} - \frac{d^2-6d+9}{v}$
15. $\frac{v^2-5v}{1-n} - \frac{5v-25}{n-2}$
16. $\frac{n^2-4}{x} - \frac{n^2-2n-8}{n^2-2n-8}$
17. $3 + \frac{x+2}{3p-1}$
18. $\frac{p^2+5p+6}{3} + \frac{2p-3}{p^2+3p+2}$
19. $\frac{q+3}{w+2} + \frac{q^2+8q+15}{2q}$
20. $\frac{4w+16}{w^2+4w} - \frac{w^2+4w}{w^2+4w}$

For #21-30, opposite quantities occur. Add or subtract the rational expressions as indicated.

21. $\frac{3}{8} + \frac{5}{-8}$
22. $\frac{y^2}{y-5} + \frac{4y+5}{5-y}$
23. $\frac{6}{w-2} + \frac{w+3}{2-w}$
24. $\frac{2a-7}{5a-8} + \frac{6+10a}{8-5a}$
25. $\frac{u^2-2}{2u^2-u-3} + \frac{u-2}{3-2u}$
26. $\frac{b^2-b-20}{b-23} - \frac{5-b}{2}$
27. $\frac{t+1}{t-1} - \frac{3t^2-1}{1-t^2}$
28. $\frac{2}{6-3x} + \frac{x}{x^2-4}$
29. $\frac{1-2r}{2y} - \frac{2r+1}{2r+1} + \frac{3}{4r^2-1}$
30. $\frac{2y}{y^2-9} - 1 + \frac{4}{3-y}$

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CHAPTER OVERVIEW

6: Radical Expressions

Learning Objectives

- Evaluate n^{th} -root radicals.
- Simplify radicals by applying appropriate properties.
- Distinguish between like radicals and unlike radicals.
- Add and subtract like radicals.
- Multiply and divide radicals, simplifying results.

[6.1: Simplify Radical Expressions](#)

[6.2: Add and Subtract Radical Expressions](#)

[6.3: Rationalize Denominators](#)

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6.1: Simplify Radical Expressions

Who doesn't love the "undo" command on a computer? Radicals are like an undo command to all powers. If taking powers is a forward-action, then radicals require us to think backwards. To prepare for this section, memorize the first twelve perfect squares and the first five cubes. It will be very helpful to recognize powers of 2 and 3.

Definition: Perfect Squares and Perfect Cubes to Memorize

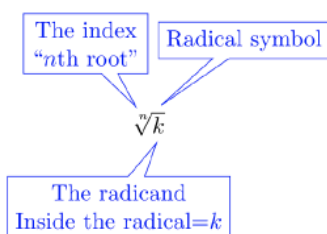
Perfect Squares to memorize (first twelve):

1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, ...

Cubes to memorize (first five):

1, 8, 27, 64, 125, ...

Let's look at the anatomy of a radical:



Index n	Example	Read Out Loud
Default Index $n = 2$	$\sqrt{4}$	The square root of 4.
$n = 3$	$\sqrt[3]{8}$	The cube root of 8.
$n = 4$	$\sqrt[4]{16}$	The 4 th root of 16.
$n = 5$	$\sqrt[5]{32}$	The 5 th root of 32.

The index, n , indicates the exponent of the related power.

$$\begin{aligned}\sqrt{4} &= 2 && \text{Why? Because } 2^2 = 4. \\ \sqrt[3]{8} &= 2 && \text{Why? Because } 2^3 = 8. \\ \sqrt[4]{16} &= 2 && \text{Why? Because } 2^4 = 16. \\ \sqrt[5]{32} &= 2 && \text{Why? Because } 2^5 = 32.\end{aligned}$$

All four examples above have a value of 2.

✓ Example 6.1.1

Simplify.

- $\sqrt{\frac{9}{64}}$
- $\sqrt[3]{-64}$
- $\sqrt{-81}$
- $\sqrt[5]{100000}$

Solution

- $\sqrt{\frac{9}{64}} = \frac{3}{8}$
- $\sqrt[3]{-64} = -4$

$$\begin{aligned}\sqrt{\frac{9}{64}} &= \frac{3}{8} \text{ because } \frac{3}{8} \geq 0 \text{ and } \left(\frac{3}{8}\right)^2 = \frac{9}{64}. \\ \sqrt[3]{-64} &= -4 \text{ because } n = 3 \text{ is odd and } (-4)^3 = -64.\end{aligned}$$

- c. $\sqrt{-81}$ is not a real number. No real number, when squared, equals -81 .
 d. $\sqrt[5]{100000} = 10$ $\sqrt[5]{100000} = 10$ because $10^5 = 100000$.

The properties and examples (below) will involve a radicand with variables. From here on out, we will assume that all variables of even-indexed radicals are nonnegative values. That is, for $\sqrt[n]{x^p}$ and n is even, we will assume $x \geq 0$.

Property	Examples
1. $\sqrt[n]{a^n} = a$	$\sqrt[4]{x^4} = x$ or $\sqrt[3]{(-5)^3} = -5$
2. Product Property: $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$	$\sqrt[5]{6} \cdot \sqrt[5]{x^2} = \sqrt[5]{6x^2}$ or $\sqrt{100x^4} = \sqrt{100} \cdot \sqrt{x^4} = 10x^2$
3. Quotient Property: $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$	$\sqrt{\frac{x^2}{100}} = \frac{\sqrt{x^2}}{\sqrt{100}} = \frac{x}{10}$ or $\frac{\sqrt[3]{54}}{\sqrt[3]{16}} = \sqrt[3]{\frac{54}{16}} = \sqrt[3]{\frac{27}{8}} = \frac{3}{2}$

Definition

Let m and n be integers such that m/n is a rational number in lowest terms and $n > 1$. Then,

$$a^{1/n} = \sqrt[n]{a} \text{ and } a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$$

If n is even, then we require $a \geq 0$.

✓ Example 6.1.2

Write each expression in radical notation, then simplify.

- a. $(81x^2)^{-1/2}$
 b. $(243)^{3/5}$

Solution

a. $(81x^2)^{-1/2} = \frac{1}{\sqrt{81x^2}} = \frac{1}{9x}$
 b. $(243)^{3/5} = \sqrt[5]{243^3} = \sqrt[5]{(3^5)^3} = \sqrt[5]{(3^3)^5} = 3^3 = 27$

Simplifying Radicals

Use as often as possible the property $\sqrt[n]{a^n} = a$ to simplify radicals. Factor into chunks where powers equal the index n , then set those numbers or variable free from the radical! Again, you may assume in all problems that variables represent positive real numbers.

✓ Example 6.1.3

- a. $\sqrt{12x^5y^3}$
 b. $5\sqrt[3]{216u^6v^5}$
 c. $-\sqrt[4]{16a^{23}}$

Solution

a.
$$\begin{aligned} \sqrt{12x^5y^3} &= \sqrt{12} \cdot \sqrt{x^5} \cdot \sqrt{y^3} \\ &= \sqrt{2^2 \cdot 3} \cdot \sqrt{x^2 \cdot x^2 \cdot x} \cdot \sqrt{y^2 \cdot y} \\ &= \sqrt{2^2 \cdot x^2 \cdot x^2 \cdot y^2} \cdot \sqrt{3xy} \\ &= 2 \cdot x \cdot x \cdot y \cdot \sqrt{3xy} \\ &= 2x^2y\sqrt{3xy} \end{aligned}$$

The product property: simplify 3 separate radicals.

The index $n = 2$. Find powers of 2 to simplify.

Group the squares. Group the non-squares.

Simplify using $\sqrt[n]{a^n} = a$.

Combine powers to simplify.

$$\begin{aligned}
 5\sqrt[3]{216u^6v^5} &= 5\sqrt[3]{216} \cdot \sqrt[3]{u^6} \cdot \sqrt[3]{v^5} \\
 &= 5 \cdot \sqrt[3]{6^3} \cdot \sqrt[3]{(u^2)^3} \cdot \sqrt[3]{v^3 \cdot v^2} \\
 \text{b.} \quad &= 5 \cdot \sqrt[3]{6^3 \cdot (u^2)^3 \cdot v^3} \cdot \sqrt[3]{v^2} \\
 &= 5 \cdot 6 \cdot u^2 \cdot v \cdot \sqrt[3]{v^2} \\
 &= 30u^2v\sqrt[3]{v^2} \\
 -\sqrt[4]{16a^{23}} &= -1 \cdot \sqrt[4]{16} \cdot \sqrt[4]{a^{23}} \\
 &= -1 \cdot \sqrt[4]{2^4} \cdot \sqrt[4]{(a^5)^4 \cdot a^3} \\
 \text{c.} \quad &= -1 \cdot \sqrt[4]{4^2 \cdot (a^5)^4} \cdot \sqrt[4]{a^3} \\
 &= -1 \cdot 2 \cdot a^5 \cdot \sqrt[4]{a^3} \\
 &= -2a^5\sqrt[4]{a^3}
 \end{aligned}$$

The product property: simplify 3 separate radicals.

The index $n = 3$. Find powers of 3 to simplify.

Group the cubes. Group the non-cubes.

Simplify using $\sqrt[n]{a^n} = a$.

Combine powers to simplify.

The product property: simplify 2 separate radicals.

How many times does 4 go into 23 evenly? 5. $R = 3$.

Group powers of 4. Group non-powers of 4.

Simplify using $\sqrt[n]{a^n} = a$.

Combine powers to simplify.

✓ Example 6.1.4

Multiply and simplify $\sqrt[3]{4p^2q^3} \cdot \sqrt[3]{6pq}$

Solution

$$\begin{aligned}
 \sqrt[3]{4p^2q^3} \cdot \sqrt[3]{6pq} &= \sqrt[3]{4 \cdot 6 \cdot p^2 \cdot p \cdot q \cdot q} \\
 &= \sqrt[3]{24 \cdot p^3 \cdot q^2} \\
 &= \sqrt[3]{2^3 \cdot 3 \cdot p^3} \cdot \sqrt[3]{q^2} \\
 &= -\sqrt[3]{2^3 \cdot p^3} \cdot \sqrt[3]{3q^2} \\
 &= 2p \cdot \sqrt[3]{3q^2}
 \end{aligned}$$

The product property consolidates the radical.

Consolidate the powers using power properties.

The product property: simplify 3 separate radicals.

Group the cubes. Group the non-cubes.

Simplify using $\sqrt[n]{a^n} = a$.

Try It! (Exercises)

1. In Example 6.1.2b of this section, it is stated: $\sqrt[5]{(3^5)^3} = \sqrt[5]{(3^3)^5}$.

Name the properties which allows us to equate $(3^5)^3 = (3^3)^5$. Explain.

2. Use the equivalency $x^{ab} = x^{ba}$ to simplify each of the following without using a calculator.

a. $\sqrt[3]{(7^3)^{10}}$

b. $\sqrt[4]{(5^4)^3}$

c. $8\sqrt[5]{(-10^5)^4}$

3. For each of the following, replace the stated number as a power of the index. Use the equivalency $x^{ab} = x^{ba}$ to simplify each of the following.

For example: $\sqrt[3]{8^5} = \sqrt[3]{(2^3)^5} = \sqrt[3]{(2^5)^3} = 2^5 = 32$.

a. $\sqrt{16^5}$

b. $-5\sqrt[4]{81^7}$

c. $-9\sqrt[5]{32^6}$

For 4-11, write each expression in radical notation, then simplify without a calculator.

4. $121^{-1/2}$

5. $32^{2/5}$

6. $125^{-2/3}$

7. $(-125)^{2/3}$

8. $\left(\frac{81}{100}\right)^{1/2}$

9. $\left(\frac{64}{125}\right)^{2/3}$

10. $\left(-\frac{64}{125}\right)^{2/3}$

11. $\left(\frac{64}{125}\right)^{-2/3}$

For 12-23, simplify the expression. Assume that variables represent positive real numbers.

12. $\sqrt{18y^3}$

13. $\sqrt[3]{250b^5}$

14. $\sqrt[4]{48x^9}$

15. $\sqrt[5]{-243c^{10}}$

16. $-2a^{10}\sqrt[3]{80a}$

17. $\frac{\sqrt[3]{500u^5}}{45u}$

18. $\sqrt[4]{\frac{1250p^9}{p^{13}}}$

19. $\frac{3}{4}\sqrt{\frac{96z^{20}}{6}}$

20. $\sqrt{128n^{10}m^3}$

21. $\sqrt[4]{\frac{405x^{14}y^6}{5x^4y}}$

22. $(\sqrt[7]{a^5b^6})^{35}$

23. $\left(\frac{\sqrt[3]{8u^6v^{12}}}{6u^2v}\right)^2$

For 24-31, multiply and simplify. Assume that variables represent positive real numbers.

24. $-3x\sqrt{5x} \cdot \sqrt{20x^3}$

25. $2q\sqrt[3]{54q^2} \cdot \sqrt[3]{4q^4}$

26. $6t^3(\sqrt[4]{75t^6} \cdot \sqrt[4]{100t^3})$

27. $\sqrt[5]{64u^5v^8} \cdot \sqrt[5]{112v^4} \cdot 7v$

28. $\frac{\sqrt{5x}}{10} \cdot \frac{\sqrt{15x}}{3x}$

29. $\frac{1}{2}\left(\frac{\sqrt[3]{18a^2}}{2}\right)\left(\frac{\sqrt[3]{6a^4}}{3a}\right)$

30. $\sqrt[3]{\frac{56n^4}{27m^5}} \cdot \sqrt[3]{\frac{49m^2}{8n}}$

31. $\left(\frac{24w^{11}}{13}\right)^3 \left(\frac{13w}{6}\right)^3$

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6.2: Add and Subtract Radical Expressions

Rules for radical multiplication and division have a simplicity and ease which lulls students into thinking addition and subtraction will follow suit. However, rules for addition and subtraction have more complication and less flexibility. Let's start with the most common pitfall:

$$\sqrt[n]{a} + \sqrt[n]{b} \neq \sqrt[n]{a+b}$$

Note: The symbol $\neq$ says "does not equal."

These two examples have even more temptation to go wrong due to the squaring and the variables:

$$\sqrt{x^2 + y^2} \neq x + y$$

Also,

$$\sqrt{x^2 - y^2} \neq x - y$$

This doesn't change any property that was previously introduced! **The following still holds true!** ($x \geq 0$ and $y \geq 0$ if n is even)

$$\sqrt{x^2} + \sqrt{y^2} = x + y$$

And

$$\sqrt{x^2} - \sqrt{y^2} = x - y$$

And in general,

$$\sqrt[n]{x^n} + \sqrt[n]{y^n} = x \pm y$$

Use a calculator to convince yourself:

$$\sqrt{6} + \sqrt{2} \neq \sqrt{8}$$

On the left side, find the value $\sqrt{6} + \sqrt{2}$ to the nearest hundredth:

$$\sqrt{6} + \sqrt{2} \approx 3.86$$

On the right side,

$$\sqrt{8} \approx 2.83$$

Now we see radicands cannot be added:

$$3.86 \neq 2.83$$

Use the order of operations as the problem is presented. Any other order of computation could be a pitfall.

$$\sqrt[3]{27} + \sqrt[3]{8} = 3 + 2 = 5$$

The order of operations tells us to take the cube root first, then add. Use the order shown and it will be correct!

Using "Like Terms" to Add or Subtract "Like Radicands"

Adding and subtracting radicals starts with simplifying radicals. Use as often as possible the property $\sqrt[n]{a^n} = a$ to simplify radicals. We will continue to assume the variables take on nonnegative values only. Let's explore how gathering like terms can be used with radicals.

$$2x + x = 3x \quad x = \text{any real number}$$

Gathering like terms is a technique you've used countless times before. Why not use with radicals? Radicals are real numbers, too! This approach opens our toolbox for adding or subtracting radical expressions. We can add radicals if the radicand is identical and index n also matches.

$$\begin{array}{c} \boxed{\text{Substitute}} \quad \boxed{\text{Substitute}} \\ x = \sqrt{5} \quad x = \sqrt{5} \\ \downarrow \quad \downarrow \\ 2x + x = 3x \\ \uparrow \\ \boxed{\text{Substitute}} \\ x = \sqrt{5} \end{array}$$

Suppose $x = \sqrt{5}$. Gather like radicals:

$$2\sqrt{5} + \sqrt{5} = 3\sqrt{5}$$

Definition: Like Radicals

Two radical terms are said to be **like radicals** if they have the same index and the same radicand.

Example 6.2.1

Which of the three expressions can be simplified? Explain.

a. $4\sqrt[3]{10} + 7\sqrt[3]{10}$

b. $3\sqrt{2} + 3\sqrt{11}$
 c. $\sqrt[3]{6} + \sqrt{6}$

Solution

a. This is the only given expression of the three that can be simplified.

$$4\sqrt[3]{10} + 7\sqrt[3]{10} = 11\sqrt[3]{10}$$

$4\sqrt[3]{10}$ and $7\sqrt[3]{10}$ both contain $\sqrt[3]{10}$ as the like radicals.

b. Cannot be simplified.

$$3\sqrt{2} \neq 3\sqrt{11}$$

The radicands are not equal.

c. Cannot be simplified.

$$\sqrt[3]{6} \neq \sqrt{6}$$

The indices are not equal.

Simplify First

Don't see a match? Make sure your radicals are completely simplified before you attempt to add or subtract radicals. The following examples demonstrate how simplification can help you find like radicals.

✓ Example 6.2.2

Simplify $\sqrt{75} + 2\sqrt{12} - \sqrt{3}$

Solution

$$\begin{aligned}\sqrt{75} + 2\sqrt{12} - \sqrt{3} &= \sqrt{25 \cdot 3} + 2\sqrt{4 \cdot 3} - \sqrt{3} \\ &= 5\sqrt{3} + 2 \cdot 2\sqrt{3} - \sqrt{3} \\ &= 5\sqrt{3} + 4\sqrt{3} - \sqrt{3} \\ (5 + 4 - 1)\sqrt{3} &= 8\sqrt{3}\end{aligned}$$

At first glance, none of the radicals are like radicals. However, both $\sqrt{75}$ and $\sqrt{12}$ can be simplified.

Factor perfect squares within each radicand.

Simplify.

Combine like radicals.

✓ Example 6.2.3

Add $\sqrt[3]{54x^2} + \sqrt[3]{3x^2} + \sqrt[3]{16x^2}$

Solution

$$\begin{aligned}\sqrt[3]{54x^2} + \sqrt[3]{3x^2} + \sqrt[3]{16x^2} &= \sqrt[3]{27 \cdot 2x^2} + \sqrt[3]{3x^2} + \sqrt[3]{8 \cdot 2x^2} \\ &= 3\sqrt[3]{2x^2} + \sqrt[3]{3x^2} + 2\sqrt[3]{2x^2} \\ &= 5\sqrt[3]{2x^2} + \sqrt[3]{3x^2}\end{aligned}$$

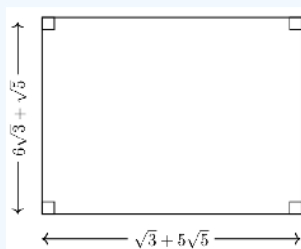
Simplify each radical.

Factor the radicands.

Two terms contain like radicals. Simplify.

✓ Example 6.2.4

Find the area and the perimeter of the rectangle shown.



Length = $(6\sqrt{3} + \sqrt{5})$ feet

Width = $(\sqrt{3} + 5\sqrt{5})$ feet

Solution

Use FOIL:

$$\begin{aligned}\text{Area} &= (6\sqrt{3} + \sqrt{5})(\sqrt{3} + 5\sqrt{5}) \\ &= \underbrace{(6\sqrt{3})(\sqrt{3})}_F + \underbrace{(6\sqrt{3})(5\sqrt{5})}_O + \underbrace{(\sqrt{5})(\sqrt{3})}_I + \underbrace{(\sqrt{5})(5\sqrt{5})}_L \\ &= 6 \cdot 3 + 6 \cdot 5 \cdot \sqrt{3 \cdot 5} + \sqrt{5 \cdot 3} + 5 \cdot 5 \\ &= 18 + 30\sqrt{15} + \sqrt{15} + 25 \\ &= 43 + 31\sqrt{15} \text{ ft}^2\end{aligned}$$

$$\begin{aligned}
 \text{Perimeter} &= \overbrace{2(6\sqrt{3} + \sqrt{5}) + 2(\sqrt{3} + 5\sqrt{5})}^{P=2L+2W} = \underbrace{(2 \cdot 6\sqrt{3} + 2 \cdot \sqrt{5})}_{\text{Distributive Property}} + \underbrace{(2 \cdot \sqrt{3} + 2 \cdot 5\sqrt{5})}_{\text{Distributive Property}} \\
 &= (12\sqrt{3} + 2\sqrt{5}) + (2\sqrt{3} + 10\sqrt{5}) \\
 &= \underbrace{12\sqrt{3} + 2\sqrt{3}}_{\text{Like Radicals}} + \underbrace{2\sqrt{5} + 10\sqrt{5}}_{\text{Like Radicals}} \\
 &= 14\sqrt{3} + 12\sqrt{5} \text{ ft}^2
 \end{aligned}$$

Try It! (Exercises)

For #1-4, add or subtract the expressions, if possible. Assume the variables represent nonnegative numbers.

1. $7\sqrt{5} - 2\sqrt{5} + \sqrt{5}$
2. $3\sqrt{3x} - 4\sqrt{3x} + 6\sqrt{6x} - 8\sqrt{6x}$
3. $\sqrt[3]{9y} + \sqrt[4]{9y} - \sqrt[3]{2y} + 2\sqrt[4]{9y}$
4. $7b\sqrt[5]{16b^3} - 4b\sqrt[3]{16b^2} - 3b\sqrt[5]{3b^2}$

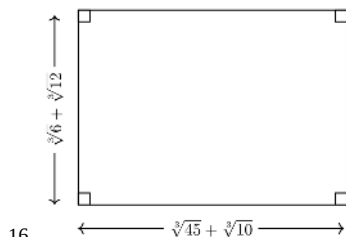
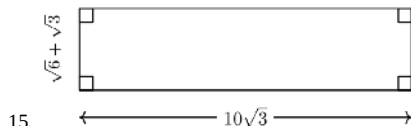
For #5-8, simplify each radical expression, then add or subtract the expressions, if possible. Assume the variables represent nonnegative numbers.

5. $2\sqrt{3a^3} + 5a\sqrt{3a} - \sqrt{27a^3}$
6. $\sqrt[3]{56c^6} - 2c^2 \cdot \sqrt[3]{189} + \sqrt[3]{7c^2}$
7. $3\sqrt[4]{4p^7} - 3p\sqrt[4]{324p^3} + 2\sqrt[4]{64p^7}$
8. $2b\sqrt[5]{192b^3} - \sqrt[5]{64b} + 3\sqrt[5]{6b^8} + \sqrt[5]{486b}$

For #9-14, evaluate each of the following. Simplify wherever possible. Assume the variables represent nonnegative numbers.

9. $\sqrt{12}(\sqrt{2} - \sqrt{18})$
10. $(7\sqrt[3]{10} + 3\sqrt[3]{4})(2\sqrt[3]{20} - \sqrt[3]{50})$
11. $(9\sqrt{2x} + 6\sqrt{3x})^2$
12. $(\sqrt{26} - 4\sqrt{10})(\sqrt{26} + 4\sqrt{10})$
13. $(\sqrt[3]{24} - \sqrt[3]{54})^2$
14. $5(3\sqrt{7} - 1)(4\sqrt{14} + 1)$

For #15-16, find the area and the perimeter of each rectangle. The length and width are each marked in the figures and the units are feet.



17. A square has side lengths = $(\sqrt[4]{4} + 2)$ feet. Find each of the area and the perimeter of the square.
18. If a square has area = 540 ft^2 . What is the perimeter of the square?

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6.3: Rationalize Denominators

Suppose a fraction $\frac{a}{b}$ contains a radical in the denominator. Rationalizing the denominator is a method of simplification that eliminates radicals from the denominator. The numerator may contain radicals, but we generally don't worry about that. Only the denominator is rationalized. Sometimes the denominator becomes rational in the simplification steps. Let's look at the easiest type of problems in the first example.

✓ Example 6.3.1

Simplify the expression.

a. $\frac{5}{\sqrt[3]{27}}$

b. $\sqrt{\frac{10}{49}}$

c. $\sqrt{\frac{15}{12}}$

d. $\frac{\sqrt[3]{21}}{\sqrt[3]{24}}$

Solution

a. $\frac{5}{\sqrt[3]{27}} = \frac{5}{3}$ The denominator of the simplified fraction is rational.

b. $\sqrt{\frac{10}{49}} = \frac{\sqrt{10}}{\sqrt{49}}$ The quotient property isolates the perfect square: 49.

$= \frac{\sqrt{10}}{7}$ The denominator of the simplified fraction is rational.

c. $\sqrt{\frac{15}{12}} = \sqrt{\frac{3 \cdot 5}{3 \cdot 4}}$ The order of operations allow us to reduce $\frac{15}{12}$ in the first step.

$= \sqrt{\frac{5}{4}}$ Now that the fraction is reduced, we spot the perfect square, 4.

$= \frac{\sqrt{5}}{\sqrt{4}}$ The quotient property isolates the perfect square: 4.

$= \frac{\sqrt{5}}{2}$ The denominator of the simplified fraction is rational.

d. $\frac{\sqrt[3]{21}}{\sqrt[3]{24}} = \sqrt[3]{\frac{21}{24}}$ The quotient property changes the order of operations.

$= \sqrt[3]{\frac{3 \cdot 7}{3 \cdot 8}}$ The fraction is reducible. Cancel the common factor: 3.

$= \sqrt[3]{\frac{7}{8}}$ Conveniently, the cube is in the denominator.

$= \frac{\sqrt[3]{7}}{\sqrt[3]{8}}$ The quotient property isolates the cube: 8.

$= \frac{\sqrt[3]{7}}{2}$ The denominator of the simplified fraction is rational.

More commonly, simplification will not eliminate the radical from the denominator. That is, we wish to use the property $\sqrt[n]{a^n} = a$ ($a \geq 0$), but the radicand is short a necessary factor or two for that to occur. For problems of this type, we will build the fraction to rationalize the denominator.

Play the Game of “n-of-a-kind.”

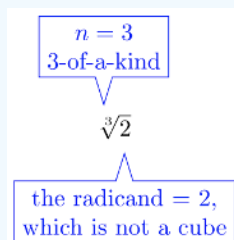
The goal is to use the property $\sqrt[n]{a^n} = a$ ($a \geq 0$) to rationalize the denominator. Whatever factors are missing, multiply to top and bottom. Simply build up the denominator (and consequently the numerator) to make it happen.

✓ Example 6.3.2

Simplify $\frac{1}{\sqrt[3]{2}}$ by rationalizing the denominator.

Solution

The denominator is irrational, and no simplification of the radical is immediately evident.



The index of the radical, $n = 3$, tells us how many of-a-kind we need. The radicand = 2. We have one 2 out of the necessary three 2's to simplify. We need two more 2's to make 3-of-a-kind.

$$\frac{1}{\sqrt[3]{2}} \cdot \frac{\sqrt[3]{2 \cdot 2}}{\sqrt[3]{2 \cdot 2}} = \frac{\sqrt[3]{4}}{\sqrt[3]{8}} = \frac{\sqrt[3]{4}}{2}$$

$\boxed{2}$ $\boxed{2 \ 2}$
 two more 2's

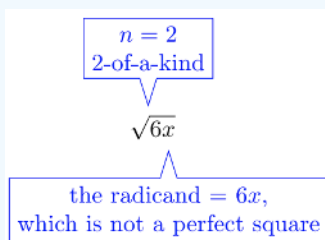
Therefore, the given fraction after rationalizing the denominator is equal to $\frac{\sqrt[3]{4}}{2}$.

✓ Example 6.3.3

Simplify $\frac{3}{2\sqrt{6x}}$ ($x > 0$), by rationalizing the denominator.

Solution

The denominator contains $2\sqrt{6x}$, but only $\sqrt{6x}$ is irrational. Therefore, we'll only rationalize the square root.



The index of the radical, $n = 2$, tells us how many of-a-kind we need. The radicand = $6x$. We have one $6x$ out of the necessary two $6x$'s to simplify.

$$\frac{3}{2\sqrt{6x}} \cdot \frac{\sqrt{6x}}{\sqrt{6x}} = \frac{3\sqrt{6x}}{2 \cdot 6x} = \frac{\cancel{3}\sqrt{6x}}{2 \cdot 2 \cdot \cancel{3} \cdot x} = \frac{\sqrt{6x}}{4x}$$

$\boxed{6x}$ $\boxed{6x}$
 one more $6x$

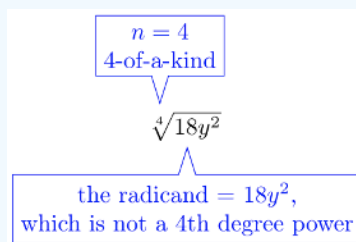
Therefore, $\frac{3}{2\sqrt{6x}} = \frac{\sqrt{6x}}{4x}$. The denominator is rationalized.

✓ Example 6.3.4

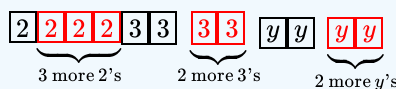
Simplify $\frac{2\sqrt[4]{5}}{3\sqrt[4]{18y^2}}$ ($y > 0$) by rationalizing the denominator.

Solution

The denominator contains $3\sqrt[4]{18y^2}$, but only $\sqrt[4]{18y^2}$ is irrational. Therefore, we'll only rationalize the 4th root.



Take inventory of the factors of the radicand $18y^2$. We need powers of 4 to simplify.



$$\begin{aligned} \frac{2\sqrt[4]{5}}{3\sqrt[4]{18y^2}} &= \frac{2\sqrt[4]{5} \cdot \sqrt[4]{2^4 \cdot 3^2 \cdot y^2}}{3\sqrt[4]{18y^2} \cdot \sqrt[4]{2^4 \cdot 3^2 \cdot y^2}} \\ &= \frac{2\sqrt[4]{5 \cdot 8 \cdot 9 \cdot y^2}}{3\sqrt[4]{18 \cdot 8 \cdot 9 \cdot y^2 \cdot y^2}} \\ &= \frac{2\sqrt[4]{360y^2}}{3\sqrt[4]{2^4 \cdot 3^4 y^4}} \\ &= \frac{\cancel{2}\sqrt[4]{360y^2}}{3 \cdot \cancel{2} \cdot 3 \cdot y} \\ &= \frac{\sqrt[4]{360y^2}}{9y} \end{aligned}$$

Multiply necessary factors to numerator and denominator.

Use the product property of radicals.

The denominator's radicand indicates powers of 4.

Simplify the denominator's radical. Reduce the fraction.

Simplify completely. The denominator is rationalized.

Rationalize Using Conjugates

We will borrow from the Special Products Formula (below) to rationalize denominators that involve two terms, where one or both terms contain square roots. Recall from Section 2.3 the multiplication of conjugates results in a difference of squares:

Special Product = Difference of Squares: $(A + B)(A - B) = A^2 - B^2$

✓ Example 6.3.5

Rationalize the denominator: $\frac{4}{3 + \sqrt{x}}$ where $x \geq 0$.

Solution

$$\begin{aligned}\frac{4}{3+\sqrt{x}} &= \frac{4(3-\sqrt{x})}{(3+\sqrt{x})(3-\sqrt{x})} \\ &= \frac{12-4\sqrt{x}}{3^2-(\sqrt{x})^2} \\ &= \frac{12-4\sqrt{x}}{9-x}\end{aligned}$$

Multiply numerator and denominator by the conjugate.

Use the Special Products Formula.

The denominator is rationalized.

Try It! (Exercises)

For 1-30, rationalize the denominator. Assume all variables are nonnegative values. Simplify completely.

1. $\frac{1}{\sqrt{36}}$

2. $\frac{6}{\sqrt[3]{64}}$

3. $\frac{10}{\sqrt[4]{16}}$

4. $\frac{\sqrt[3]{4}}{\sqrt[3]{32}}$

5. $\frac{\sqrt{45x}}{\sqrt{5x}}$

6. $\sqrt{\frac{6}{200}}$

7. $\sqrt[5]{\frac{972}{128}}$

8. $\frac{\sqrt[3]{2y^2}}{\sqrt[3]{5}}$

9. $\frac{8t}{\sqrt[3]{4t}}$

10. $\frac{2h}{\sqrt[4]{6h^3}}$

11. $\frac{10}{3\sqrt{5p}}$

12. $\frac{15\sqrt{2w}}{7w\sqrt{5w}}$

13. $\frac{24c \cdot \sqrt[3]{4c}}{3c^2 \cdot \sqrt[3]{6c}}$

14. $\sqrt[3]{\frac{55}{21q^2}}$

15. $\sqrt[4]{\frac{25}{12u^3}}$

16. $\sqrt[5]{\frac{60a^3}{45a^7}}$

17. $\frac{26n\sqrt[4]{5n}}{5\sqrt[4]{8n^5}}$

18. $\frac{2r}{3\sqrt{2\pi r}}$

19. $\sqrt[3]{\frac{2b}{a^2}}$

20. $\frac{54y\sqrt[4]{2x^3}}{\sqrt[4]{27y^2}}$

21. $-\sqrt{\frac{20n^5m^3}{65n^9m^4}}$

22. $\sqrt[3]{\frac{-21}{16x^2y}}$

23. $\frac{-75pq}{\sqrt[5]{-3p^3}}$

24. $\frac{-12uv^2}{5\sqrt{36uv}}$

25. $\frac{9}{x-\sqrt{2}}$

26. $\frac{\sqrt{y}}{1\sqrt{2y}}$

27. $\frac{z+\sqrt{3}}{z-\sqrt{3}}$

28. $\frac{\sqrt{10}-\sqrt{5a}}{\sqrt{2a}}$

29. $\frac{\sqrt{10}-\sqrt{5a}}{\sqrt{u}-\sqrt{v}}$

30. $\frac{v-\sqrt{u}}{v-\sqrt{u}}$

For #31-36, use the order of operations. Rationalize denominators where appropriate.

31. $\frac{1}{\sqrt{3}} + \frac{\sqrt{3}}{3}$

32. $16 \cdot \frac{1}{\sqrt[3]{2}} + 6\sqrt[3]{4}$

33. $\frac{2-\sqrt{3}}{\sqrt{3}(\sqrt{3}+4)}$

34. $\frac{3}{4} \left(\frac{\sqrt{2}}{3-\sqrt{2}} \right)$

35. $\left(\frac{2\sqrt[3]{25}}{\sqrt[3]{16}} \right)^3 - \left(\frac{5}{\sqrt{2}} \right)^2$

36. $\left(\frac{2}{\sqrt{10}} \right)^4 - \left(\frac{1}{\sqrt[3]{10}} \right)^6$

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CHAPTER OVERVIEW

7: Trigonometry

Learning Objectives

- Sketch angles in standard position measured in either degrees or radians.
- Identify coterminal angles and produce a sketch.
- Identify reference angles and produce a sketch.
- Use the equation of a circle to compute points on the circle as ordered pairs.
- Understand how special right triangles, together with symmetry, yields the ordered pairs on the unit circle.

[7.1: The Unit Circle](#)

[7.2: Reference Angles](#)

[7.3: Radians](#)

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7.1: The Unit Circle

The core concepts of trigonometry are developed from a circle with radius equal to 1 unit, drawn in the xy -coordinate plane, centered at the origin. This circle is given a name: the **unit circle** (Figure 7.1.1 below). Just like a 12-hour clock with values of time from 1 to 12, trigonometric functions are periodic, meaning the same values are reproduced with each 360° revolution.

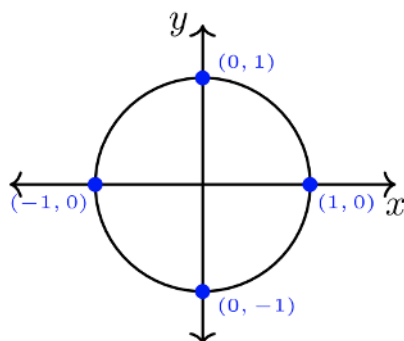


Figure 7.1.1 : The Unit Circle

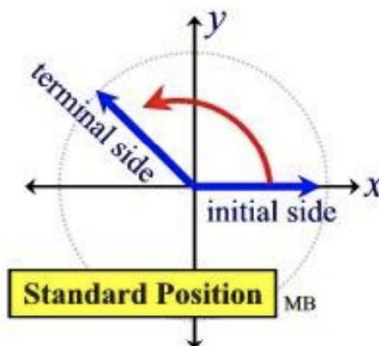


Figure 7.1.2 : Angles in Standard Position

An angle is in **standard position** (see Figure 7.1.2 above) if its initial side is along the positive x -axis and its vertex is at the **origin**: point $(0, 0)$. The following angles are in standard position. An angle that rotates in the counter-clockwise direction is a **positive angle**. An angle that rotates in the clockwise direction is a **negative angle**.

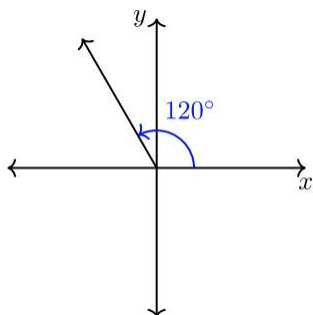


Figure 7.1.3 : Positive angle in standard position.

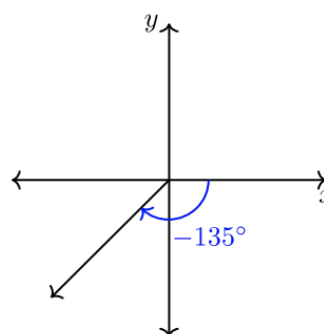


Figure 7.1.4 : Negative angle in standard position.

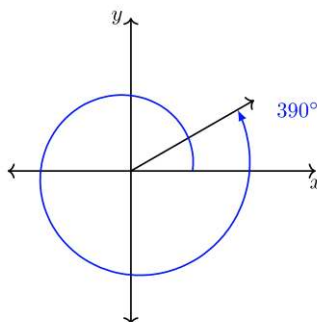
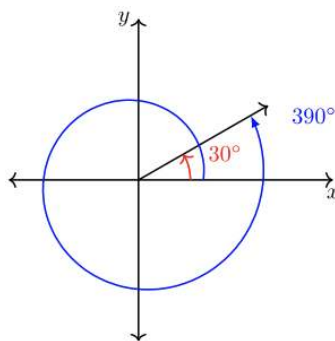


Figure 7.1.5 : Positive angle in standard position.

Coterminal Angles

Two or more standard angles that share common terminal sides are said to be **coterminal angles**. For example, 30° and 390° are coterminal angles.



More formally: Every angle B is coterminal with angle A where $B = A + 360^\circ k$, $k = \text{any integer}$.

✓ Example 7.1.1

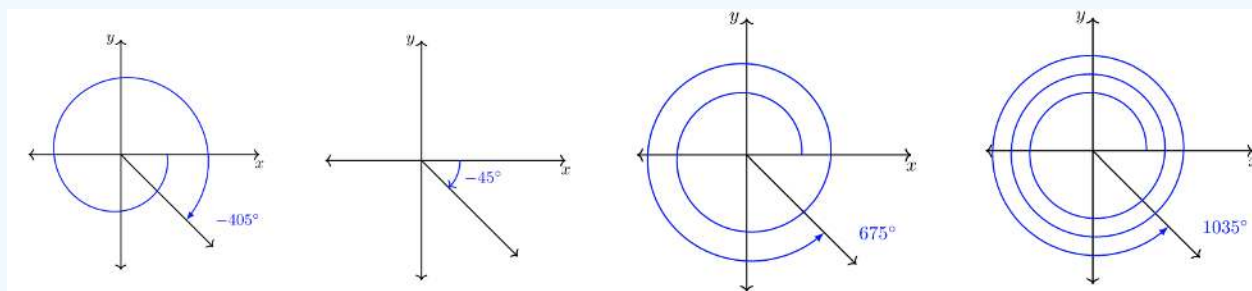
The expression $315^\circ + 360^\circ k$ gives the angles coterminal with 315° . State the coterminal angles the equation generates using the following k -values: $k = -2, -1, 1, 2$. Then sketch the angles.

Solution

We substitute the given k -values into the expression $315^\circ + 360^\circ k$.

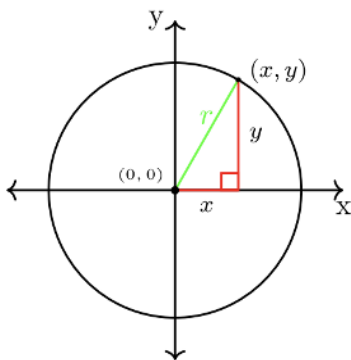
$k = -2$	$k = -1$	$k = 1$	$k = 2$
$315^\circ + 360^\circ(-2) = -405^\circ$	$315^\circ + 360^\circ(-1) = -45^\circ$	$315^\circ + 360^\circ(1) = 675^\circ$	$315^\circ + 360^\circ(2) = 1035^\circ$

Below, the angles are sketched. Do you see that each angle shares the same terminal side? All four angles are coterminal with 315° , and coterminal with each other.



Circle Centered at the Origin

Every ordered pair (x, y) on a circle is associated with a right triangle. The right triangle has horizontal distance x , vertical distance y , and hypotenuse = radius = r .



Equation of a Circle

The **equation of a circle** of radius r centered at the origin:

$$x^2 + y^2 = r^2 \quad (7.1.1)$$

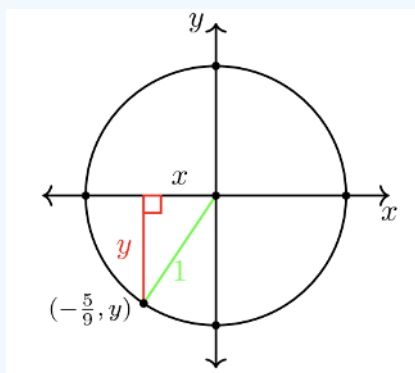
Note: the x -coordinate and the y -coordinate can take on negative values, depending on the quadrant of the terminal side of the angle.

✓ Example 7.1.2

Find the y -coordinate of point A $\left(-\frac{5}{9}, y\right)$ if point A lies in QIII on the unit circle.

Solution

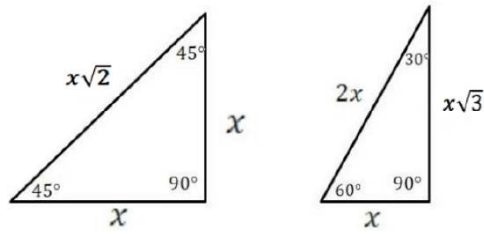
The unit circle has radius $r = 1$. Trigonometry weds algebra and geometry with visual sketches. Create a sketch before jumping into a solution. This helps you see the answer.



$\left(-\frac{5}{9}\right)^2 + y^2 = 1^2$	Substitute $x = -\frac{5}{9}$ and $r = 1$ into equation of a circle.
$\frac{25}{81} + y^2 = 1$	Simplify.
$y^2 = 1 - \frac{25}{81}$	Subtract $\frac{25}{81}$ to each side.
$y^2 = \frac{81}{81} - \frac{25}{81}$	Find the LCD.
$y^2 = \frac{56}{81}$	Simplify.
$\sqrt{y^2} = \pm \sqrt{\frac{56}{81}}$	Square root both sides.
$y = -\frac{\sqrt{56}}{9}$	Choose the correct sign value. $y < 0$ in QIII.
$y = -\frac{2\sqrt{14}}{9}$	Simplify the radical.

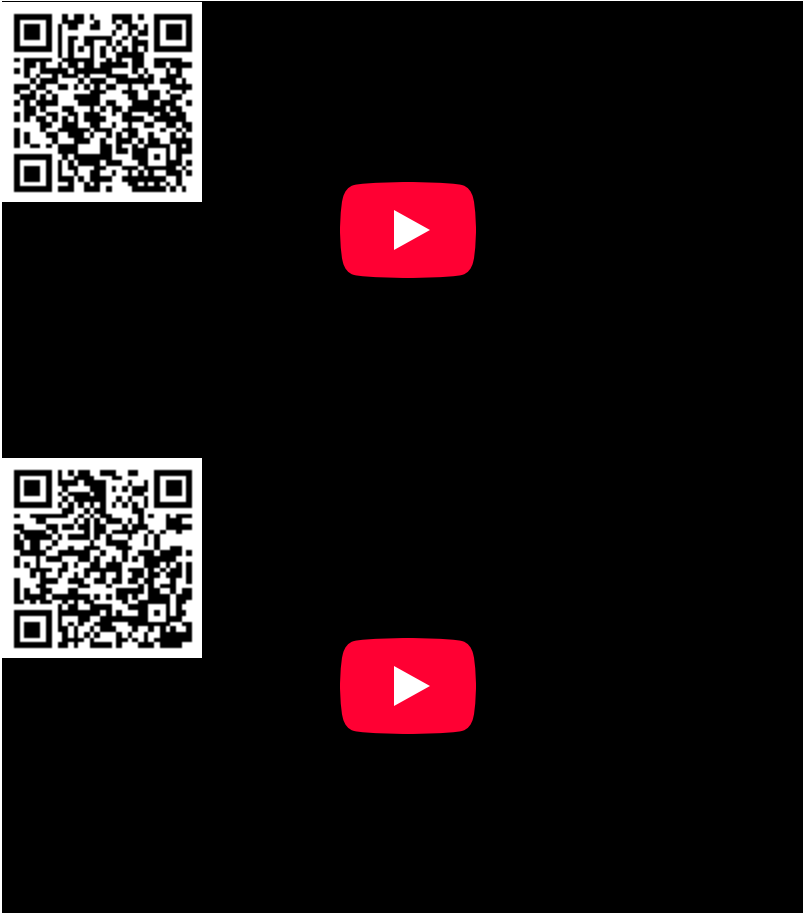
Special Right Triangles

Using the Pythagorean Theorem, one can drive the following two templates for special right triangles: 45° - 45° - 90° triangles and 30° - 60° - 90° triangles.



Since right triangles and circles are inextricably tied to each other, the acute angles 30° , 45° , 60° are frequently useful for finding exact values in trigonometry; these solutions do not require the use of a calculator.

Examples of special right triangles and their solutions, can be viewed in these videos:



Try It! (Exercises)

For #1-5, draw the angle θ in standard position.

1. $\theta = 210^\circ$
2. $\theta = -300^\circ$
3. $\theta = 150^\circ$
4. $\theta = 270^\circ$
5. $\theta = -135^\circ$

For #6-10, state any two angles coterminal with the given angle θ . Give one positive coterminal angle and one negative coterminal angle.

6. $\theta = 30^\circ$
7. $\theta = 60^\circ$
8. $\theta = 90^\circ$

9. $\theta = 180^\circ$

10. $\theta = 240^\circ$

For #11-15, state the equation of the circle with the given radius.

11. $r = 4$

12. $r = \frac{1}{2}$

13. $r = 3\sqrt{2}$

14. $r = \frac{\sqrt{6}}{2}$

15. $r = \sqrt{\frac{101}{62}}$

16. **Fill in the blanks:** A *unit circle* is a circle with _____ equal to one unit. The circle is centered at _____. The circle has equation: _____.

For #17-24, The given point lies on a circle with given radius and terminal side in the given quadrant. Find the missing coordinate of the given ordered pair.

Point on Circle	Radius	Quadrant in which θ terminates.
17. $(2, y)$	$r = 5$	$\theta \in \text{QI}$
18. $(x, \sqrt{6})$	$r = 3$	$\theta \in \text{QII}$
19. $(-3\sqrt{7}, y)$	$r = \sqrt{77}$	$\theta \in \text{QIII}$
20. $(x, -4\sqrt{5})$	$r = 3\sqrt{15}$	$\theta \in \text{QIV}$
21. $\left(\frac{3}{4}, y\right)$	$r = 1$	$\theta \in \text{QI}$
22. $\left(x, \frac{3}{16}\right)$	$r = 1$	$\theta \in \text{QII}$
23. $\left(-\frac{\sqrt{5}}{4}, y\right)$	$r = 1$	$\theta \in \text{QIII}$
24. $\left(x, -\frac{3\sqrt{2}}{8}\right)$	$r = 1$	$\theta \in \text{QIV}$

For #25-29, find the fraction of a full revolution. Then sketch the angle.

25. $\frac{1}{4} \cdot 360^\circ$

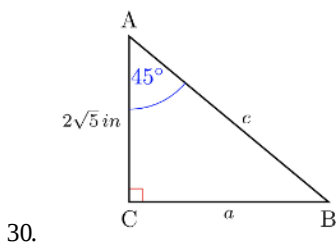
26. $\frac{1}{2} \cdot 360^\circ$

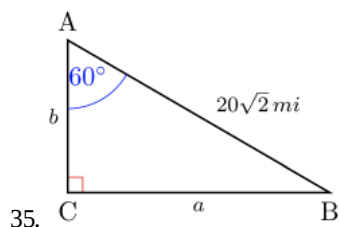
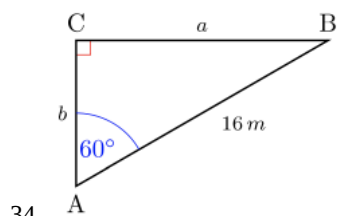
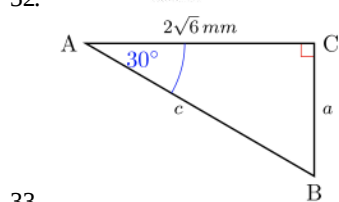
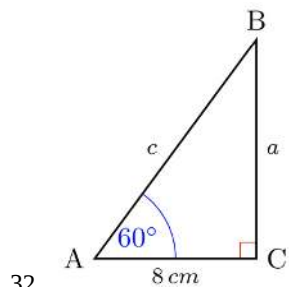
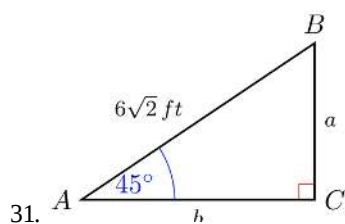
27. $\frac{3}{4} \cdot 360^\circ$

28. $\frac{1}{8} \cdot 360^\circ$

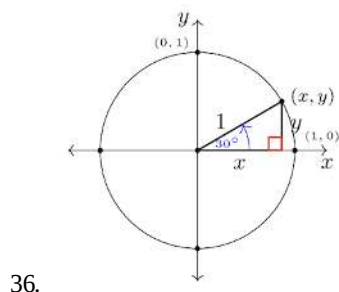
29. $\frac{1}{6} \cdot 360^\circ$

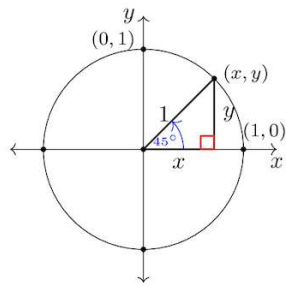
For #30-35, solve for the 2 missing side lengths. Give exact answers. No calculators.



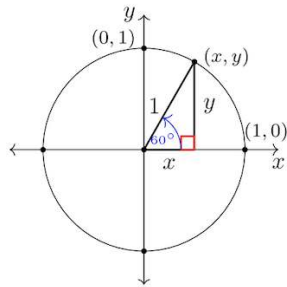


For #36-38, Find the coordinates of the ordered pair (x, y) on the unit circle with the given standard angle. Use special right triangle relationships. Give exact values of x and y .





37.



38.

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7.2: Reference Angles

The Special Right Triangles, 30° - 60° - 90° and 45° - 45° - 90° , allow us to obtain exact values of the ordered pairs (x, y) on a unit circle with standard angles 30° , 45° , or 60° .

If we use symmetry across the y -axis and the x -axis, we can populate the known ordered pairs from QI into Quadrants II, III, and IV. Use the figures below to follow how this is done. To keep the pictures simple, 30° angles are marked, while 45° and 60° can be surmised by their larger magnitudes.

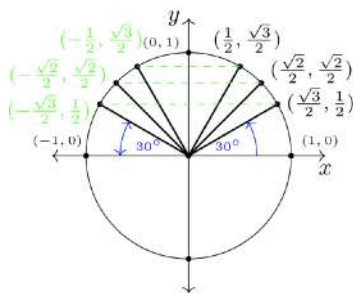


Figure 7.2.1 : QI $\rightarrow$ QII
 $(x, y) \rightarrow (-x, y)$

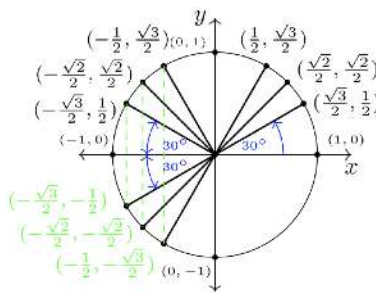


Figure 7.2.2 : QII $\rightarrow$ QIII
 $(-x, y) \rightarrow (-x, -y)$

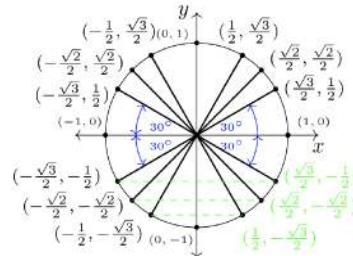


Figure 7.2.3 : QIII $\rightarrow$ QIV
 $(-x, -y) \rightarrow (x, -y)$

Notice the four 30° angles create a bow-tie look in Figure 7.2.3. These angles are called **reference angles**.

Definition: Reference Angle

Let θ be a standard angle. A **reference angle**, denoted $\hat{\theta}$, is the positive acute angle between the terminal side of θ and the x -axis.

The word *reference* is used because all angles can *refer* to QI. That is, memorization of ordered pairs is confined to QI of the unit circle. If a standard angle θ has a reference angle of 30° , 45° , or 60° , the unit circle's ordered pair is duplicated, but the sign value of x or y may need adjustment, depending on the quadrant of the terminal side of θ .

Example 7.2.1

The terminal side of a standard angle $\theta = 225^\circ$ intersects the unit circle. State the ordered pair of the intersection.

Solution

The unit circle has radius $r = 1$. Trigonometry weds algebra and geometry with visual sketches. We start with a sketch of the angle $\theta = 225^\circ$. All standard angles begin on the positive-side of the x -axis. In which quadrant is the terminal side of θ ? That is, where does this angle come to a stop?

Where is the terminal side of $\theta = 225^\circ$? Since $180^\circ < 225^\circ < 270^\circ$, the angle's terminal side is in QIII.

The reference angle is computed by finding The difference between 225° and 180° .

Note: The reference angle is never negative.

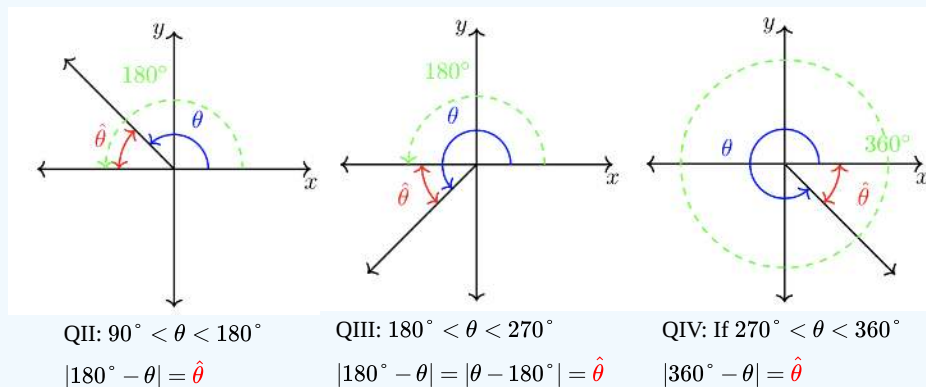
$$|225^\circ - 180^\circ| = |180^\circ - 225^\circ| = \hat{\theta}$$

$$45^\circ = \hat{\theta}$$

In QIII, all ordered pairs (x, y) are such that $x < 0$ and $y < 0$. Referring back to QI, using standard angle of 45° on the unit circle, the ordered pair $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ must be adjusted for negative x and y coordinates.

Answer The terminal side of $\theta = 225^\circ$ intersects the unit circle at $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$.

Tip: When a standard angle is greater than 90° , use markers 180° or 360° to calculate the reference angle. Drawing a picture before computing is always recommended!

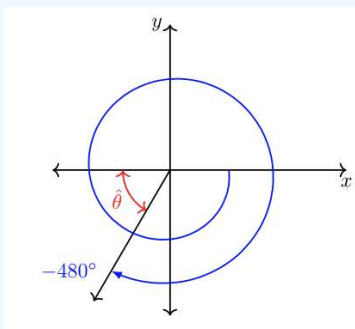


✓ Example 7.2.2

The terminal side of a standard angle $\theta = -480^\circ$ intersects the unit circle. State the ordered pair of the intersection.

Solution

Negative angles rotate clockwise. Sketch θ . Find a positive coterminal angle: $-480^\circ + 360^\circ(2) = 240^\circ$ Then apply the tips above or analyze visually: $|180^\circ - 240^\circ| = |-60^\circ| = 60^\circ = \hat{\theta}$



The ordered pair on the unit circle is $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.

Try It! (Exercises)

For #1-10, state the reference angle, $\hat{\theta}$, of the given standard angle.

1. $\theta = 210^\circ$
2. $\theta = 350^\circ$
3. $\theta = 110^\circ$
4. $\theta = 240.5^\circ$
5. $\theta = 142.75^\circ$
6. $\theta = -315^\circ$
7. $\theta = -230^\circ$
8. $\theta = 500^\circ$
9. $\theta = 615^\circ$
10. $\theta = -835^\circ$

For #11-20, the terminal side of the given standard angle, θ , intersects the unit circle at a point. State the ordered pair of the intersection.

11. $\theta = 135^\circ$

12. $\theta = 300^\circ$
13. $\theta = -240^\circ$
14. $\theta = -150^\circ$
15. $\theta = 420^\circ$
16. $\theta = 570^\circ$
17. $\theta = 840^\circ$
18. $\theta = -765^\circ$
19. $\theta = -930^\circ$
20. $\theta = 1560^\circ$

For #21-28, a standard angle's rotation is described in words. You are given several hints about its rotation. Note: a full revolution is 360° (counter-clockwise) or -360° (clockwise). **Find the measure of the angle described.**

21. An angle has a counter-clockwise rotation. The angle does not make a full revolution. The angle's terminal side is in QIV. The reference angle $\hat{\theta} = 30^\circ$.
22. An angle has a counter-clockwise rotation. The angle does not make a full revolution. The angle's terminal side is in QII. The reference angle $\hat{\theta} = 20^\circ$.
23. An angle has a clockwise rotation. The angle does not make a full revolution. The angle's terminal side is in QIII. The reference angle $\hat{\theta} = 10^\circ$.
24. An angle has a clockwise rotation. The angle does not make a full revolution. The angle's terminal side is in QII. The reference angle $\hat{\theta} = 72^\circ$.
25. An angle has a counter-clockwise rotation. The angle goes just beyond one full revolution. The angle's terminal side is in QI. The reference angle $\hat{\theta} = 55^\circ$.
26. An angle has a counter-clockwise rotation. The angle goes beyond two full revolutions. The angle's terminal side is in QII. The reference angle $\hat{\theta} = 24^\circ$.
27. An angle has a clockwise rotation. The angle goes beyond one full revolution. The angle's terminal side is in QIII. The reference angle $\hat{\theta} = 18^\circ$.
28. An angle has a clockwise rotation. The angle goes beyond two full revolutions. The angle's terminal side is in QIV. The reference angle $\hat{\theta} = 39^\circ$.

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7.3: Radians

Angles can be measured in units of degrees or radians.

Definition: Radians

Radians are a unit of angular measure defined such that

$$2\pi \text{ radians} = 360^\circ \quad (7.3.1)$$

For the crafty at heart, an angle of 1 radian can be created with a circle (use any lid from a jar) and a piece of string.



Figure 7.3.1 : From the center of the circle, lay your string down to create a radius. Then cut the string to the size of its radius. String measure = radius measure



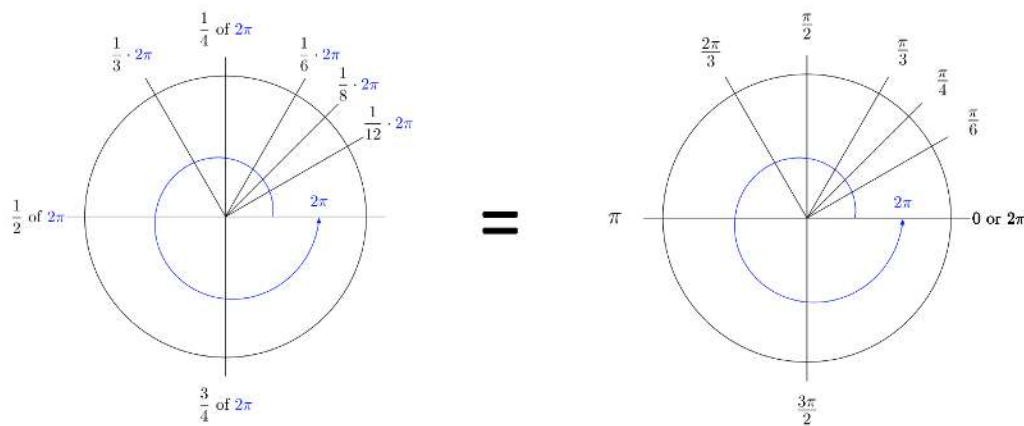
Figure 7.3.2 : Next, take the cut string and paste it to the edge of the circle. The center of the circle is the angle's vertex. Connect the endpoints of the strings to the center of the circle. The angle is one radian!

Recall that the circumference of a circle (which is the perimeter of a circle) has a formula:

$$\text{Circumference} = C = 2\pi r \text{ where } r = \text{radius of the circle}$$

How many strings of length r are needed to wrap around the entire circle? We divide the circumference r ways: $\frac{2\pi r}{r} = 2\pi \approx 6$.
About 6 strings of length r , or exactly 2π strings.

An angle's counter-clockwise rotation can be measured as a fraction of a full rotation.

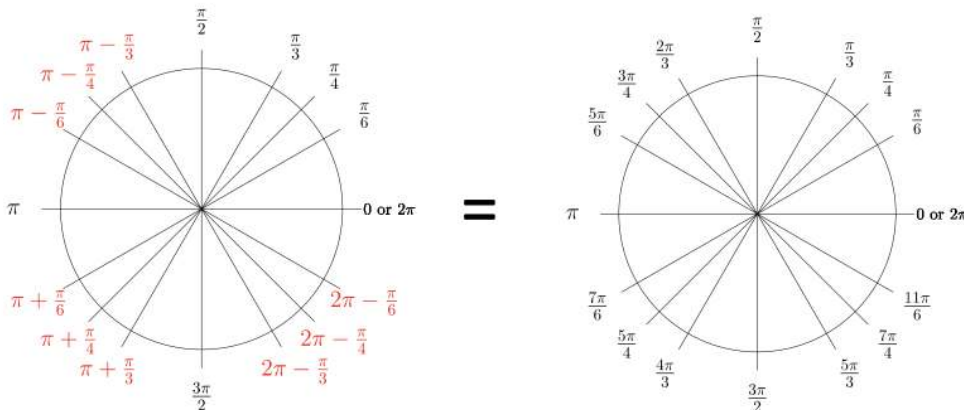


Once you get the hang of radians, your trigonometric calculations are quick and easy. Using radians for angles, rather than degrees, is the key to gain fluidity in trigonometry. They are simpler to use than degrees. Notice the shared denominators from one quadrant to the next. Also notice the denominator of an angle matches its own reference angle's denominator.

$$30^\circ = \frac{\pi}{6} \text{ radians}$$

$$45^\circ = \frac{\pi}{4} \text{ radians}$$

$$60^\circ = \frac{\pi}{3} \text{ radians}$$



Reference angles refer to QI, where your memorization comes into play. Memorize only QI and the unit circle's ordered pairs in that quadrant (see Section 7.2). Avoid memorizing too much! Use intuitive computations to rehearse math concepts. If a radian value is not one of the above values, nor coterminal to the ones shown, then you will need a calculator for trigonometric computations.

✓ Example 7.3.1

Give the reference angle $\hat{\theta}$ (in radians) for the given standard angle θ . Then sketch both θ and $\hat{\theta}$.

- $\theta = \frac{7\pi}{6}$
- $\theta = -\frac{2\pi}{3}$
- $\theta = \frac{11\pi}{4}$
- $\theta = 6 \text{ radians}$

Solution

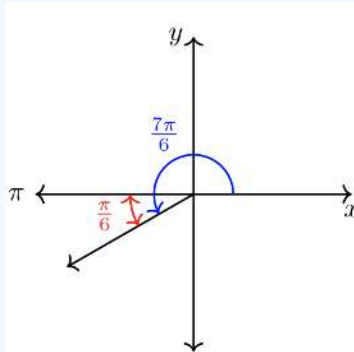
- $\theta = \frac{7\pi}{6}$. Decide whether $\frac{7\pi}{6}$ is closer to π or 2π .

Since $\pi = \frac{6\pi}{6}$ and $2\pi = \frac{12\pi}{6}$, it is clear that $\frac{7\pi}{6} > \frac{6\pi}{6}$ and closer to π .

The reference angle is $\hat{\theta} = \frac{\pi}{6}$.

$$\frac{7\pi}{6} = \pi + \frac{\pi}{6}.$$

The sketch is shown below.



b. $\theta = -\frac{2\pi}{3}$. The angle rotates clockwise.

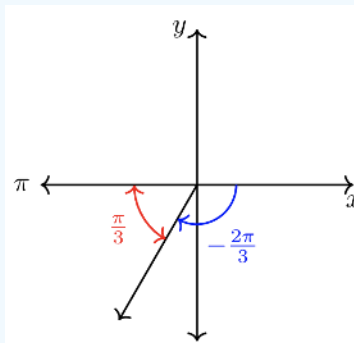
$$\left| -\frac{2\pi}{3} \right| = \frac{2\pi}{3} \text{ which terminates in QII.}$$

The clockwise rotation of $-\frac{2\pi}{3}$ terminates in QII.

Both $\frac{2\pi}{3}$ and $-\frac{2\pi}{3}$ have the same reference angle.

$$\hat{\theta} = \frac{\pi}{3}.$$

The sketch is shown below.

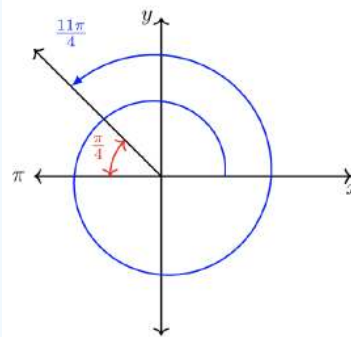


c. $\theta = \frac{11\pi}{4}$. The denominator is 4.

Since $\frac{8\pi}{4} = 2\pi$ and $\frac{11\pi}{4} > \frac{8\pi}{4}$, this angle rotates beyond one full revolution. π is coterminal with $3\pi, 5\pi, 7\pi \dots$ while 2π is coterminal with $4\pi, 6\pi, 8\pi, \dots$

Compare $\frac{11\pi}{4}$ to $3\pi = \frac{12\pi}{4}$. They are very close in value! $\frac{11\pi}{4} < \frac{12\pi}{4}$ and $\frac{11\pi}{4}$ therefore terminates in QII. The reference angle $\hat{\theta} = \frac{\pi}{4}$.

The sketch of the angle is shown below.

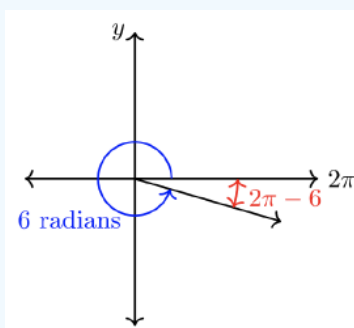


d. $\theta = 6$ radians. Decide whether 6 radians is closer to π or 2π .

Since $\pi \approx 3.14$ and $2\pi \approx 6.28$, 6 radians is closer to 2π . $6 < 2\pi$, therefore the angle terminates in QIV.

The reference angle $\hat{\theta} = 2\pi - 6 \approx 0.28$ radians.

The sketch of the angle is shown below.



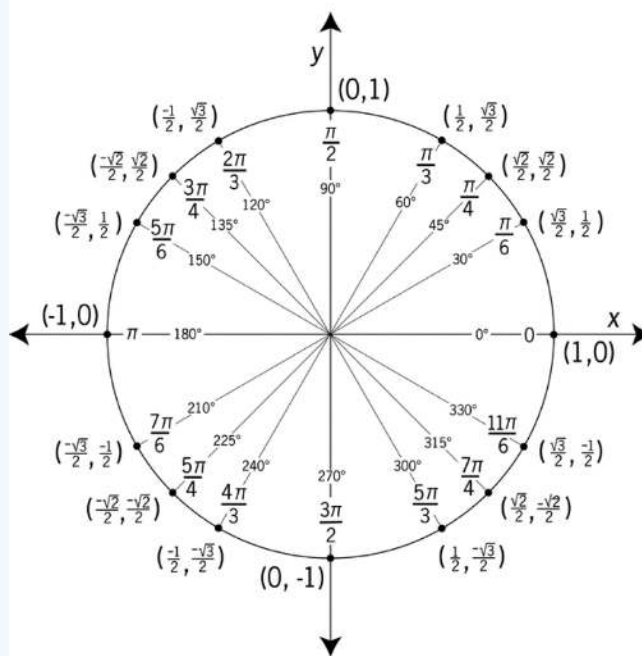
✓ Example 7.3.2

The terminal side of a standard angle $\theta = \frac{3\pi}{4}$ radians intersects the unit circle. State the ordered pair of the intersection.

Solution

Since $\frac{3\pi}{4} = \frac{4\pi}{4} - \frac{\pi}{4}$, the angle $\theta = \frac{3\pi}{4}$ terminates in QII with reference angle $\hat{\theta} = \frac{\pi}{4}$.

The unit circle was introduced in Section 7.2 for degree-angles. The figure shown below includes radian-angles.



The ordered pair on the unit circle, intersecting the terminal side of $\theta = \frac{3\pi}{4}$ is $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

✓ Example 7.3.3

The terminal side of a standard angle $\theta = \frac{5\pi}{2}$ radians intersects the unit circle. State the ordered pair of the intersection.

Solution

Since $\frac{5\pi}{2} = \frac{\pi}{2} + 2\pi$, the angles $\frac{5\pi}{2}$ and $\frac{\pi}{2}$ are coterminal angles. Coterminal angles land in the same spot, so the ordered pair will be the same. The ordered pair on the unit circle, intersecting the terminal side of $\theta = \frac{5\pi}{2}$ is $(0, 1)$.

✓ Example 7.3.4

The terminal side of a standard angle $\theta = 10\pi$ radians intersects the unit circle. State the ordered pair of the intersection.

Solution

Since $10\pi = 2\pi(5)$, the angles 10π and 2π are coterminal angles. Therefore, the ordered pair on the unit circle, intersecting the terminal side of $\theta = 10\pi$ is $(1, 0)$.

Try It! (Exercises)

For #1-10, A standard angle θ is given in radians. State the reference angle $\hat{\theta}$ in radians.

1. $\theta = \frac{11\pi}{6}$
2. $\theta = \frac{7\pi}{9}$
3. $\theta = \frac{11\pi}{7}$
4. $\theta = 2$ radians
5. $\theta = -\frac{5\pi}{6}$

6. $\theta = -\frac{5\pi}{4}$
7. $\theta = \frac{9\pi}{4}$
8. $\theta = \frac{17\pi}{6}$
9. $\theta = 10$ radians
10. $\theta = -\frac{6\pi}{5}$

For #11-20, the terminal side of the given standard angle, θ , intersects the unit circle at a point. State the ordered pair of the intersection.

11. $\theta = \frac{7\pi}{6}$
12. $\theta = \frac{4\pi}{3}$
13. $\theta = \frac{11\pi}{3}$
14. $\theta = \frac{7\pi}{2}$
15. $\theta = 5\pi$
16. $\theta = \frac{13\pi}{4}$
17. $\theta = -\frac{7\pi}{6}$
18. $\theta = -4\pi$
19. $\theta = -\frac{11\pi}{4}$
20. $\theta = -\frac{5\pi}{2}$

For #21-27, a standard angle's rotation is described in words. You are given several hints about its rotation. Note: a full revolution is 2π radians (if counter-clockwise) or -2π radians (if clockwise). Find the measure of the angle described using radians.

21. An angle has a counter-clockwise rotation. The angle does not make a full revolution. The angle's terminal side is in QIII. The reference angle $\hat{\theta} = \frac{\pi}{3}$.
22. An angle has a counter-clockwise rotation. The angle does not make a full revolution. The angle's terminal side is in QIV. The reference angle $\hat{\theta} = \frac{\pi}{4}$.
23. An angle has a counter-clockwise rotation. The angle does not make a full revolution. The angle's terminal side is in QII. The reference angle $\hat{\theta} = \frac{\pi}{6}$.
24. An angle has a clockwise rotation. The angle does not make a full revolution. The angle's terminal side is in QIV. The reference angle $\hat{\theta} = \frac{\pi}{3}$.
25. An angle has a clockwise rotation. The angle does not make a full revolution. The terminal side of the angle intersects the unit circle at point $(-1, 0)$.
26. An angle has a counter-clockwise rotation. The angle rotates more than one full revolution. The terminal side of the angle intersects the unit circle at point $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$.
27. An angle has a counter-clockwise rotation. The angle rotates more than two full revolutions. The terminal side of the angle intersects the unit circle at point $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

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CHAPTER OVERVIEW

8: Problem Solving

Learning Objectives

- Recognize and distinguish between arithmetic, geometric, and quadratic sequences.
- Create a general term for arithmetic, geometric, and quadratic sequences.
- Recognize a pattern and infer the pattern on subsequent terms of a sequence.
- Identify a sequence pattern within real-life observations.

[8.1: Arithmetic Sequences](#)

[8.2: Problem Solving with Arithmetic Sequences](#)

[8.3: Geometric Sequences](#)

[8.4: Quadratic Sequences](#)

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8.1: Arithmetic Sequences

The following sequence of numbers has a pattern you are bound to recognize:

$$2, 4, 6, 8, 10, 12, 14, 16, 18, \dots$$

Likely, you would describe the sequence in words: *the sequence of even numbers*. Alternatively, can we describe the sequence mathematically? That is, can we describe the pattern of the sequence of even numbers using a formula? Absolutely! This section will explore arithmetic sequences, how to identify them, mathematically describe their terms, and the relationship between arithmetic sequences and linear functions. Let's get started!

Definition: Sequence

A sequence is a list of numbers: $a_1, a_2, a_3, a_4, \dots, a_n, \dots$. A sequence can be a finite or infinite list. We call a_1 the first term, a_2 the second term, and a_n the "general term" or the n^{th} term. Sequences have a pattern. We describe the pattern in the general term a_n .

For the sequence of even numbers: $2, 4, 6, 8, 10, \dots$ the general term $a_n = 2n$.

$$\begin{array}{cccccccccc} n = & \boxed{1} & \boxed{2} & \boxed{3} & \boxed{4} & \boxed{5} & \boxed{6} & \boxed{7} & \boxed{8} & \boxed{9} \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 2n = & \boxed{2} & \boxed{4} & \boxed{6} & \boxed{8} & \boxed{10} & \boxed{12} & \boxed{14} & \boxed{16} & \boxed{18} \end{array} \quad a_n = f(n) = 2n$$

The general term a_n of a sequence is simply a function of n , indicated above as $f(n)$, where n is a natural number (or whole number if n starts with zero).

✓ Example 8.1.1

In the sequence of even numbers, what is the 20^{th} term in the sequence?

Solution

The general term of the sequence of even numbers is $a_n = 2n$. Since n = the term number, we are asked to find a_{20} .

$$a_{20} = 2(20) = 40 \quad \text{Plug in the term-number } n = 20 \text{ into the formula } a_n = 2n$$

Answer The 20^{th} term of the sequence of even numbers is the number 40.

Definition: Arithmetic Sequence

If the sequence: $a_1, a_2, a_3, a_4, \dots, a_{n-1}, a_n, \dots$ exhibits a pattern such that

$$a_n - a_{n-1} = d \quad (8.1.1)$$

For all n , then the real number d is called the **common difference**, and the sequence is an **arithmetic sequence**.

✓ Example 8.1.2

A sequence is given. If the sequence is an arithmetic sequence, give the common difference. If the sequence is not an arithmetic sequence, explain how it fails to be arithmetic.

- 25, 32, 39, 46, 53, 60, ...
- 2, 4, 8, 16, 32, ...
- $3^2, 3^4, 3^6, 3^8, 3^{10}, \dots$
- 0, 1, 0, 1, 0, 1, ...

Solution

- Is the sequence

25, 32, 39, 46, 53, 60, ...

an arithmetic sequence?

$$a_2 - a_1 = 32 - 25 = 7$$

$$a_3 - a_2 = 39 - 32 = 7$$

$$a_4 - a_3 = 46 - 39 = 7$$

$$a_5 - a_4 = 53 - 46 = 7$$

$$a_6 - a_5 = 60 - 53 = 7$$

The sequence is arithmetic and the common difference is 7.

b. Is the sequence

2, 4, 8, 16, 32, ...

an arithmetic sequence?

$$a_2 - a_1 = 4 - 2 = 2$$

$$a_3 - a_2 = 8 - 4 = 4$$

$$a_4 - a_3 = 16 - 8 = 8$$

$$a_5 - a_4 = 32 - 16 = 16$$

$$2 \neq 4 \neq 8 \neq 16$$

The sequence is not arithmetic. $a_n - a_{n-1}$ does not yield a common difference.

c. Is the sequence

$3^2, 3^4, 3^6, 3^8, 3^{10}, \dots$

an arithmetic sequence?

$$3^4 - 3^2 = 3^2(3^2 - 1) = 9 \cdot 8 = 72$$

$$3^6 - 3^4 = 3^4(3^2 - 1) = 81 \cdot 8 = 648$$

Since $a_3 - a_2 \neq a_2 - a_1$, we conclude the sequence is not arithmetic.

d. Is the sequence

0, 1, 0, 1, 0, 1, ...

an arithmetic sequence?

$$-0 = 1$$

$$0 - 1 = -1$$

Since $a_3 - a_2 \neq a_2 - a_1$, the sequence is not arithmetic.

Find the General Term of an Arithmetic Sequence

If a sequence is arithmetic, the general term a_n is determined using the common difference, d , of the sequence. Functions of the form $y = mx + b$, known as linear functions, have a strong relationship to arithmetic sequences. The slope m of a linear function is equivalent to the common difference d of an arithmetic sequence. Let's compare arithmetic sequences to linear functions to build a_n , the general term of an arithmetic sequence.

✓ Example 8.1.3

Find the general term a_n of each arithmetic sequence:

a. 4, 7, 10, 13, 16, ...

b. 100, 80, 60, 40, 20, ...

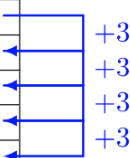
Solution

We will create a table of values for each sequence. The first column will be the term number, n , starting with $n = 1$. The second column will list the terms of the sequence. The common difference is shown on the side of the second column.

a. The sequence 4, 7, 10, 13, 16, ... has the common difference $d = 3$. But it's also the slope m of the linear function $f(x) = mx + b$.

$$m = \frac{\delta y}{\delta x} = \frac{a_n - a_{n-1}}{n - (n-1)} = \frac{d}{1} = d \quad (8.1.2)$$

n	a_n
1	4
2	7
3	10
4	13
5	16



The above table essentially mimics any linear function, $f(x) = mx + b$.

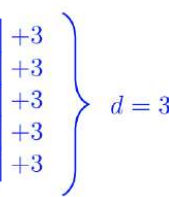
- Instead of x , sequences use n -values.
- Instead of $m = \text{slope}$ in linear functions, sequences use $d = \text{common difference}$.
- Instead of b , a sequence notates the same value with a_0 .

If a_1 denotes the first term of a sequence, then the general term of a sequence is:

$$a_n = f(n) = d \cdot n + a_0 \quad (8.1.3)$$

To find the general term, a_n , we will need to find the value a_0 . There are several ways to do this, but perhaps the simplest is to create an extra row where $n = 0$, then use the common difference to find a_0 . The common difference pattern is maintained and $a_0 + d = a_1$.

n	a_n
0	$a_0 = ?$
1	4
2	7
3	10
4	13
5	16



Find the value a_0 :

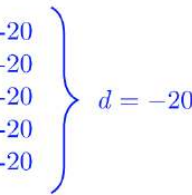
$$\begin{aligned} a_0 + 3 &= 4 \\ a_0 + 3 - 3 &= 4 - 3 \\ a_0 &= 1 \end{aligned}$$

The general term of the sequence is:

$$a_n = 3n + 1$$

b. Use the same strategy for Example 8.1.3a to solve Example 8.1.3b. Create a table, find the common difference, d , and find the a_0 term of the sequence 100, 80, 60, 40, 20, ...

n	a_n
0	$a_0 = ?$
1	100
2	80
3	60
4	40
5	20



The common difference $d = -20$. Find the value a_0 .

$$\begin{aligned}a_0 - 20 &= 100 \\a_0 - 20 + 20 &= 100 + 20 \\a_0 &= 120\end{aligned}$$

The general term of the sequence is:

$$a_n = -20n + 120$$

Try It! (Exercises)

For #1-5, the general term of a sequence is given.

- List the first 5 terms of the sequence: a_1, a_2, a_3, a_4, a_5 .
- Is the sequence arithmetic?

- $a_n = n^2$
- $a_n = 4 - 5n$
- $a_n = 2^n$
- $a_n = \frac{1}{2}n$
- $a_n = 0.3n + 1$

For #6-9, a table of values is given. State the general term, a_n , of the arithmetic sequence.

6.

n	a_n
1	9
2	15
3	21
4	25

7.

n	a_n
1	42
2	38
3	34
4	30

8.

n	a_n
1	7
2	7.25
3	7.5
4	7.75

9.

n	a_n
1	65.4
2	52.2
3	39
4	25.8

For #10-15, find the general term of the arithmetic sequence. Assume the first term is a_1 .

10. 8, 15, 22, 29, ...

11. 110, 85, 60, 35, ...

12. 9, 7.4, 5.8, 4.2, ...

13. $\frac{17}{2}$, 8, $\frac{15}{2}$, 7, ...

14. -20, -8, 4, 16, 28, ...

15. $4\frac{1}{2}$, $5\frac{1}{4}$, 6, $6\frac{3}{4}$, ...

For #16-20, an arithmetic sequence is described. Find the general term a_n .

16. The arithmetic sequence has common difference $d = 8$. The first term $a_1 = 28$.

17. The arithmetic sequence has first term $a_1 = 40$ and second term $a_2 = 36$.

18. The arithmetic sequence has first term $a_1 = 6$ and third term $a_3 = 24$.

19. The arithmetic sequence has common difference $d = -2$ and third term $a_3 = 15$.

20. The arithmetic sequence has common difference $d = 3.6$ and fifth term $a_5 = 10.2$.

21. Explain how the formula for the general term given in this section: $a_n = d \cdot n + a_0$ is equivalent to the following formula:

$$a_n = a_1 + d(n - 1)$$

22. Some sequences have a finite number of terms. Find the number of terms in the finite arithmetic sequence: 3, 17, 31, ..., 143

23. Some sequences have a finite number of terms. Find the number of terms in the finite arithmetic sequence: 80, 69, 58, ..., -52

24. Create a formula for finding the number of terms a finite arithmetic sequence when given the first and the last term of the sequence. Assume the first term is a_1 and the last term is a_k .

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8.2: Problem Solving with Arithmetic Sequences

Arithmetic sequences, introduced in Section 8.1, have many applications in mathematics and everyday life. This section explores those applications.

✓ Example 8.2.1

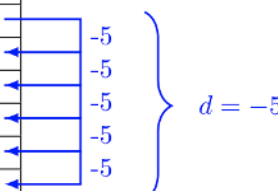
A water tank develops a leak. Each week, the tank loses 5 gallons of water due to the leak. Initially, the tank is full and contains 1500 gallons.

- How many gallons are in the tank 20 weeks later?
- How many weeks until the tank is half-full?
- How many weeks until the tank is empty?

Solution

This problem can be viewed as either a linear function or as an arithmetic sequence. The table of values give us a few clues towards a formula.

n = week	a_n = gallons
0	1500
1	1495
2	1490
3	1485
4	1480
5	1475



The problem allows us to begin the sequence at whatever n -value we wish. It's most convenient to begin at $n = 0$ and set $a_0 = 1500$.

Therefore, $a_n = -5n + 1500$

- How many gallons are in the tank 20 weeks later?

Since the leak is first noticed in week one, 20 weeks after the initial week corresponds with $n = 20$. Use the formula where $n = 20$:

$$a_{20} = -5(20) + 1500 = -100 + 1500 = 1400$$

Therefore, 20 weeks later, the tank contains 1400 gallons of water.

- How many weeks until the tank is half-full? A half-full tank would be 750 gallons. We need to find n when $a_n = 750$.

750	$= -5n + 1500$	Substitute $a_n = 750$ into the general term.
$750 - 1500$	$= -5n + 1500 - 1500$	Subtract 1500 from each side of the equation.
-750	$= -5n$	Simplify each side of the equation.
$\frac{-750}{-5}$	$= \frac{-5n}{-5}$	
150	$= n$	Divide both sides by -5 .

Since n is the week-number, this answer tells us that on week 150, the tank is half full. However, most people would better understand the answer if stated in the following way, **"The tank is half full after 150 weeks."** This answer sounds more natural and is preferred.

- How many weeks until the tank is empty? The tank is empty when $a_n = 0$ gallons. Find n such that $a_n = 0$.

$$\begin{array}{rcl}
 0 & = & -5n + 1500 \\
 0 - 1500 & = & -5n + 1500 - 1500 \\
 -1500 & = & -5n \\
 \frac{-1500}{-5} & = & \frac{-5n}{-5} \\
 300 & = & n
 \end{array}$$

Substitute $a_n = 0$ into the general term.
 Subtract 1500 from each side of the equation.
 Simplify.
 Divide both sides by -5 .

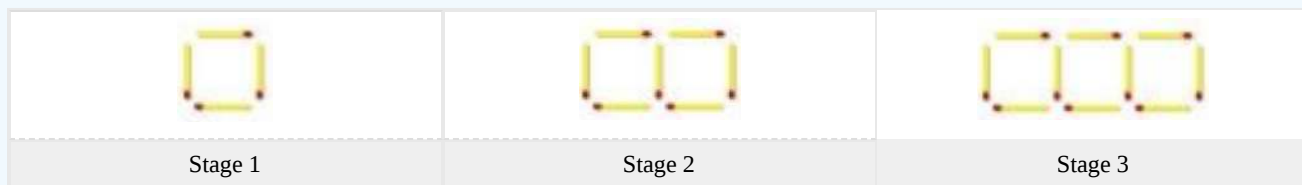
Since n is the week-number, this answer tells us that on week 300, the tank is empty. However, most people would better understand the answer if stated in the following way, “**The tank is empty after 300 weeks.**” This answer sounds more natural and is preferred.

✓ Example 8.2.2

Three stages of a pattern are shown below, using matchsticks. Each stage requires a certain number of matchsticks. If we keep up the pattern...

- How many matchsticks are required to make the figure in stage 34?
- What stage would require 220 matchsticks?

Solution



Let's create a table of values. Let n = stage number, and let a_n = the number of matchsticks used in that stage. Then note the common difference.

n	a_n
0	$a_0 = ?$
1	4
2	7
3	10

$\left. \begin{array}{l} \leftarrow +3 \\ \leftarrow +3 \\ \leftarrow +3 \end{array} \right\} d = 3$

Find the value a_0 :

$$\begin{array}{rcl}
 a_0 + 3 & = & 4 \\
 a_0 + 3 - 3 & = & 4 - 3 \\
 a_0 & = & 1
 \end{array}$$

The general term of the sequence is:

$$a_n = 3n + 1$$

- Compute a_{34} to find the number of matchsticks in stage 34:

$$a_{34} = 3(\mathbf{34}) + 1 = 103.$$

There are 103 matchsticks in stage 34.

- What stage would require 220 matchsticks? We are looking for the stage-number, given the number of matchsticks. Find n if $a_n = 220$.

$$\begin{array}{rcl}
 220 & = & 3n + 1 \\
 219 & = & 3n \\
 73 & = & n
 \end{array}$$

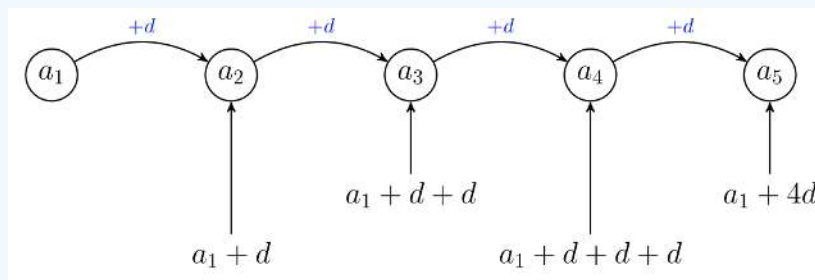
Answer Stage 73 would require 220 matchsticks.

✓ Example 8.2.3

Cory buys 5 items at the grocery store with prices a_1, a_2, a_3, a_4, a_5 which is an arithmetic sequence. The least expensive item is \$1.89, while the total cost of the 5 items is \$12.95. What is the cost of each item?

Solution

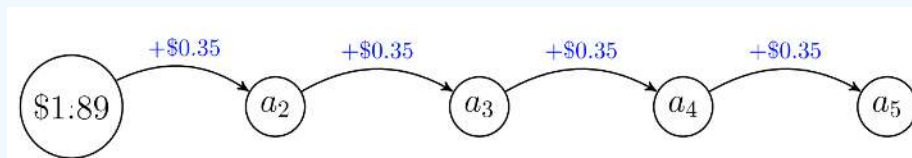
Put the 5 items in order of expense: least to most and left to right. Because it is an arithmetic sequence, each item is d more dollars than the previous item. Each item's price can be written in terms of the price of the least expensive item, a_1 , and $a_1 = \$1.89$.



The diagram above gives 5 expressions for the costs of the 5 items in terms of a_1 and the common difference is d .

$a_1 + a_2 + a_3 + a_4 + a_5$	$= 12.95$	Total cost of 5 items is \$12.95.
$a_1 + (a_1 + d) + (a_1 + 2d) + (a_1 + 3d) + (a_1 + 4d)$	$= 12.95$	See diagram for substitutions.
$5a_1 + 10d$	$= 12.95$	Gather like terms.
$5(1.89) + 10d$	$= 12.95$	$a_1 = 1.89$.
$9.45 + 10d$	$= 12.95$	Simplify.
$9.45 + 10d - 9.45$	$= 12.95 - 9.45$	Subtract 9.45 from each side of equation.
$10d$	$= 3.50$	Simplify. Then divide both sides by 10.
d	$= 0.35$	The common difference is \$0.35.

Now that we know the common difference, $d = \$0.35$, we can answer the question.



The price of each item is as follows: \$1.89, \$2.24, \$2.59, \$2.94, \$3.29

Try It! (Exercises)

1. ZKconnect cable company requires customers sign a 2-year contract to use their services. The following describes the penalty for breaking contract: Your services are subject to a minimum term agreement of 24 months. If the contract is terminated before the end of the 24-month contract, an early termination fee is assessed in the following manner: \$230 termination fee is assessed if contract is terminated in the first 30 days of service. Thereafter, the termination fee decreases by \$10 per month of contract.

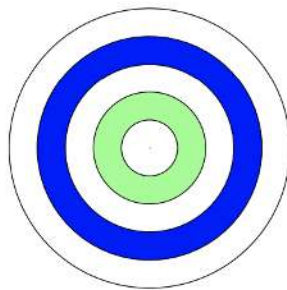
- If Jack enters contract with ZKconnect on April 1st of 2021, but terminates the service on January 10th of 2022, what are Jack's early termination fees?
- The general term a_n describes the termination fees for the stated contract. Describe the meaning of the variable n in the context of this problem. Find the general term a_n .
- Is the early termination fee a finite sequence or an infinite sequence? Explain.
- Find the value of a_{13} and interpret its meaning in words.

2. A drug company has manufactured 4 million doses of a vaccine to date. They promise additional production at a rate of 1.2 million doses/month over the next year.

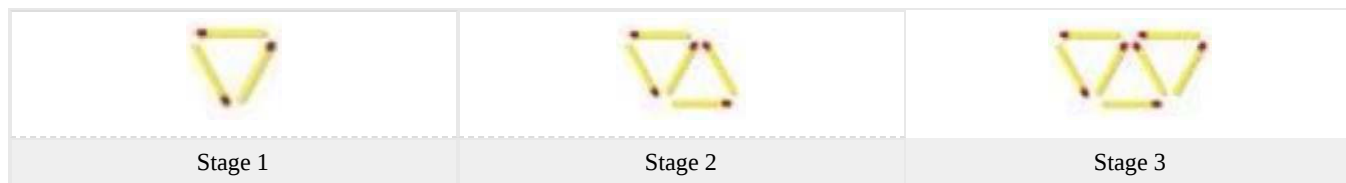
- How many doses of the vaccine, in total, will have been produced after a year?
 - The general term a_n describes the total number of doses of the vaccine produced. Describe the meaning of the variable n in the context of this problem. Find the general term a_n .
 - Find the value of a_8 and interpret its meaning in words.
3. The theater shown at right has 22 seats in the first row of the “A Center” section. Each row behind the first row gains two additional seats.
- Let $a_n = 22 + 2n$, starting with $n = 0$. Give the first 10 values of this sequence.
 - Using $a_n = 22 + 2n$, Find the value of a_{10} and interpret its meaning in words in the context of this problem. Careful! Does n = row number?
 - How many seats, in total, are in “A Center” section if there are 12 rows in the section?



- 4) Logs are stacked in a pile with 48 logs on the bottom row and 24 on the top row. Each row decreases by three logs.
- The stack, as described, has how many rows of logs?
 - Write the general term a_n to describe the number of logs in a row in two different ways. Each general term should produce the same sequence, regardless of its starting n -value.
 - Start with $n = 0$.
 - Start with $n = 1$.
- 5) The radii of the target circle are an arithmetic sequence. If the area of the innermost circle is πun^2 and the area of the entire target is $49\pi \text{un}^2$, what is the area of the blue ring? [The formula for area of a circle is $A = \pi r^2$].

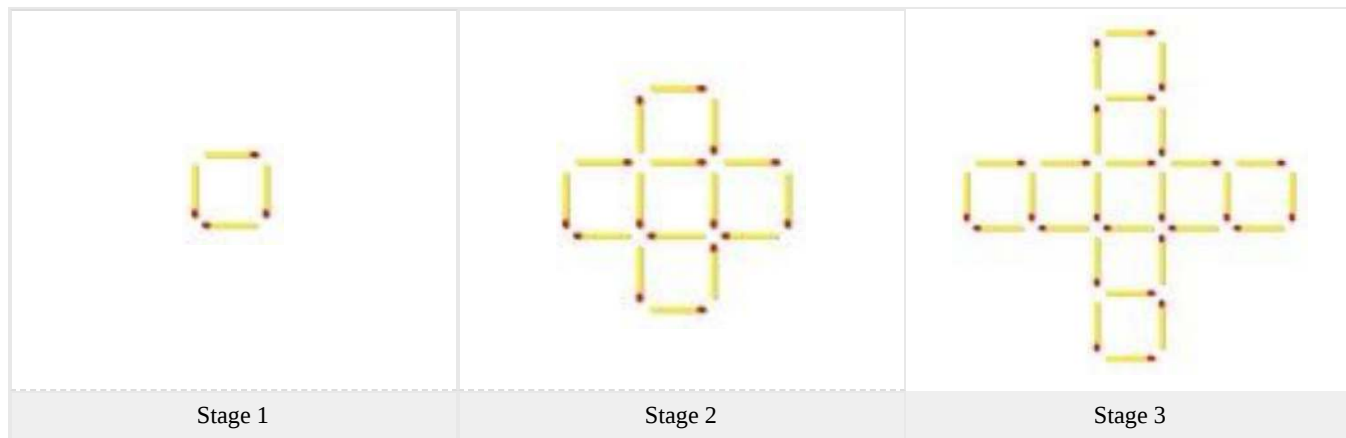


- 6) Three stages of a pattern are shown below, using matchsticks. Each stage adds another triangle and requires a certain number of matchsticks. If we keep up the pattern...
- How many matchsticks are required to make the figure in stage 34?
 - What stage would require 325 matchsticks?



- 7) Three stages of a pattern are shown below, using matchsticks. Each stage requires a certain number of matchsticks. If we keep up the pattern...

- How many matchsticks are required to make the figure in stage 22?
- What stage would require 424 matchsticks?



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8.3: Geometric Sequences

Geometric sequences have a common ratio. Each term after the first term is obtained by multiplying the previous term by r , the common ratio. As an example, the following sequence does not have a common difference, so it is not an arithmetic sequence. Instead, this sequence has a **common ratio**, r :

$$\frac{a_n}{a_{n-1}} = r = 2$$

and it is a geometric sequence:

$$2, 4, 8, 16, 32, 64, 128$$

Notice that each term is double the previous term. By multiplying any term by 2, we obtain the subsequent term. A common ratio is the hallmark signature of a geometric sequence.

Definition: Geometric Sequence

If the sequence: $a_1, a_2, a_3, a_4, \dots, a_{n-1}, a_n, \dots$ exhibits a pattern ($a \neq 0$ and $r \neq 0$) such that

$$a_1, a_1 r, a_1 r^2, a_1 r^3, \dots, a_1 r^{n-1}, a_1 r^n, \dots \quad (8.3.1)$$

Then the sequence is geometric and r is called the common ratio, where $\frac{a_n}{a_{n-1}} = r$

A geometric sequence is analogous to an exponential function, $f(x) = ab^x$, where a and b are constants, $a =$ any real number and $b > 0$. The general term a_n for a geometric sequence will mimic the exponential function formula, but modified in the following way:

- Instead of $x =$ any real number, the domain of the geometric sequence function is the set of natural numbers n .
- The constant a will become the first term, or a_1 , of the geometric sequence.
- The constant b is replaced by the common ratio r , but r can be positive or negative.

The General Term of a Geometric Sequence

The geometric sequence with common ratio r :

$$a_1, a_1 r, a_1 r^2, a_1 r^3, \dots, a_1 r^{n-1}, \dots \quad (8.3.2)$$

has general term

$$f(n) = a_n = a_1 r^{n-1} \quad (8.3.3)$$

The first term of the geometric sequence is a_1 , or $a_1 r^0$. Recall that $r^0 = 1$. It's worth mentioning that, in some cases, the first term is better notated as a_0 rather than a_1 . If we use $a_0 =$ first term (starting the sequence at $n = 0$), then the geometric sequence would be notated: $a_0, a_0 r, a_0 r^2, a_0 r^3, \dots$ and the general term is $a_n = a_0 r^n$. Although the term-number no longer matches the subscript (i.e. $a_1 =$ second term, $a_2 =$ third term, etc.), the exponent on r tells us how many times r was applied. In real-life problems that have an initial value to which r is repetitively multiplied, allow yourself the flexibility to call the initial amount a_0 .

✓ Example 8.3.1

Determine the common ratio of the geometric sequence: 15, 45, 135, 405, ... and give the general term, a_n . Then find the 10th term of the sequence, or a_{10} .

Solution

Finding the common ratio is a matter of dividing any term by its previous term:

$$\frac{45}{15} = 3 = r.$$

Therefore, the general term of the sequence is:

$$a_n = 15 \cdot 3^{n-1}$$

The general term gives us a formula to find a_{10} . Plug $n = 10$ into the general term a_n .

$$a_{10} = 15 \cdot 3^{10-1} = 15 \cdot 3^9 = 295245$$

✓ Example 8.3.2

Determine the common ratio of the geometric sequence: 8, -12, 18, -27, ... and give the general term a_n . Then find the 7th term of the sequence.

Solution

Notice the sequence alternates in sign value: positive, negative, positive, negative, ... An alternating sequence occurs when $r < 0$. We expect the r -value to be negative.

The common ratio is found by dividing two consecutive terms. Let's divide $\frac{a_2}{a_1}$.

$$\frac{-12}{8} = -\frac{3}{2} = -1.5 = r$$

Therefore, the general term of the sequence is:

$$a_n = -1.5 \cdot 8^{n-1}$$

The general term gives us a formula to find a_7 . Plug into $n = 7$ in a_n to find the 7th term:

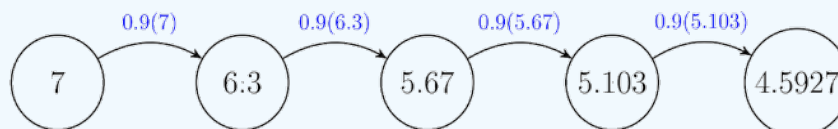
$$a_7 = 8(-1.5)^{7-1} = 8(-1.5)^6 = 91.125$$

✓ Example 8.3.3

A filtering process can reduce Chemical B by 10%. The process can be repeated and have the same reduction rate of 10% each time. Initially, there is 7 mg of Chemical B before filtering. How much of Chemical B remains after 4 filtering processes? Round the answer to 2 decimal places.

Solution

If chemical B is reduced by 10%, then 90% remains after filtering. The sequence would end after 4 filters. The common ratio $r = 0.9$ and we apply this common ratio 4 times to the initial value, 7 mg:



Rather than performing each multiplication separately, it's easier to compute the remaining quantity of Chemical B using the formula for the general term:

$$7 \cdot (0.9)^4 = 4.5927$$

Answer After 4 filtering processes, 4.59 mg of Chemical B remains.

✓ Example 8.3.4

The average annual inflation was 1.75% between 2011 and 2020. The cost of a Big Mac rose with inflation. If inflation was the only factor for an increase in the price of a Big Mac, how much would a Big Mac cost in 2020 if nine years earlier, in 2011, it cost \$4.20?

Solution

The common ratio is a value greater than one. Each year, inflation drove up the price of a Big Mac so that consumers were paying $100\% + 1.75\% = 101.75\%$ for the burger over the previous year's cost. The common ratio is the percentage as a decimal. $r = 1.0175$.

$$4.20(1.0175)^9 \approx 4.91 \quad a_0 = 4.20. \text{ Apply the common ratio to each year 2011-2020.}$$

Answer The cost of a Big Mac would have cost \$4.91 in 2020.

Percentage Increase or Decrease

As demonstrated in Example 8.3.3 (percentage decrease) and Example 8.3.4 (percentage increase), if the change to the initial quantity is given as a constant percentage (the percentage itself doesn't change), the sequence will be geometric. The sequence is also sometimes called a geometric progression.

The geometric progression in Example 8.3.3 is a decreasing sequence. The r -value is calculated by considering the offset from 100%. If the percentage, $p\%$, is a decrease, subtract the percentage from 100%: $(100\% - p\%)$ then change the value to a decimal by moving the decimal two places left (or divide by 100). If the sequence decreases, $0 < r < 1$. The geometric progression in Example 8.3.4 is an increasing sequence. The r -value is calculated by considering the offset from 100%. If the percentage, $p\%$, is an increase, add the percentage to 100%: $(100\% + p\%)$ then change the value to a decimal by moving the decimal two places left (or divide by 100). If the sequence increases, $r > 1$.

✓ Example 8.3.5

Alice puts 1 grain of rice on the first square of an 8×8 chess board. The White Rabbit tells her to put double the amount of rice in each successive square thereafter. The Cheshire Cat chuckles and tells Alice, "you should join the Mad Hatters if you think you can accomplish this task." How many grains of rice would be on the 64th square?



Solution

If one grain of rice occupies the first square, 2 the second square, 4 the third square, then the common ratio is $r = 2$, doubling the grains of rice 63 times. Remember, we don't start doubling until the 2nd square, so although there are 64 squares, we double the quantity of rice 63 times.

$$1 \cdot 2^{63} \approx 9 \times 10^{18} \text{ grains of rice on the 64}^{\text{th}} \text{ square!}$$

It's hard to fathom such a large quantity!

Try It! (Exercises)

For #1-6, the sequence is either arithmetic, geometric, or neither. If arithmetic, give the common difference. If geometric, give the common ratio. If neither, show how it fails to have a common difference or a common ratio.

1. 100, 200, 300, 400, ...
2. 10, 100, 103, 104, ...
3. 12, 14, 18, 116, ...
4. 12, 23, 34, 45, ...
5. 8, -6, 4.5, -3.375, ...
6. 1, -3, -7, -11, ...

For #7-12, the general term of the sequence is given.

- a. Is the sequence geometric or arithmetic?
 - b. State the first five terms, starting with $n = 1$.
 - c. Find the value a_9 .
7. $a_n = 6 \cdot 2^{n-1}$

8. $a_n = 52 - 13n$

9. $a_n = 5n$

10. $a_n = 5\left(\frac{2}{3}\right)^n$

11. $a_n = 2 + \frac{n}{4}$

12. $a_n = 3(-1)^n$

For #13-15, a term of a geometric sequence is given along with the common ratio r . Find the first 5 terms of the geometric sequence, starting with $n = 1$, and state the general term, a_n .

13. $a_1 = 2.5, r = 4$

14. $a_3 = 162, r = 3$

15. $a_5 = \frac{1}{8}, r = \frac{1}{2}$

For #16-18, two terms of a geometric sequence are given. Find the first 5 terms of the geometric sequence, starting with $n = 1$, and state the general term, a_n .

16. $a_2 = 3, a_3 = \frac{9}{10}$

17. $a_1 = \frac{2}{5}, a_4 = -\frac{16}{5}$

18. $a_2 = 40, a_4 = \frac{32}{5}$

19. A solar company guarantees that the solar panels installed on a home can produce 3500 kWh for the first year. Every year thereafter, the guaranteed annual production (in kWh) declines by 2%. How many annual kWh of energy would be guaranteed 8 years after installation? Round answer to the nearest whole kWh.
20. Global carbon emissions have risen by approximately 2.6% annually from 1960 to 2010. If 2500 million metric tons of carbon were emitted in 1960, how many metric tons of carbon were emitted in 2010? Round the answer to the nearest whole million metric ton.
21. When the IRS makes a mistake in your favor, the IRS must pay you back, plus interest, on the overpayment. How much, in total, would the IRS owe you if you overpaid \$1,500 to the IRS and that amount earned interest over 3 years at a constant rate of 3% annual interest? Round your answer to the nearest cent.
22. If there are 30 years in a generation, how many direct ancestors did each of us have 150 years ago? By direct ancestor we mean birth parents, grandparents, great-grandparents, and so on.
23. Organisms in nature draw Carbon-14 from the environment, but once the organism dies, it starts to lose carbon-14 at an exponential rate. Every 5730 years, the amount Carbon-14 is cut in half. Carbon dating can be used on samples of bone, cloth, wood, and plant fibers.
 - a. How many years of decay before $\frac{1}{4}$ remains?
 - b. How many years of decay before $\frac{1}{8}$ remains?
 - c. How many years of decay before $\frac{1}{2^6}$ remains?
24. If the midpoints of the sides of an equilateral triangle are joined by straight lines, the new figure will be an equilateral triangle with a perimeter equal to half the original. If we start with an equilateral triangle with perimeter 3 cm, what is the perimeter of the 5th "nested" triangle, as described?
25. Reservoirs can be the source of water supply for millions of people. Changes can occur to any water supply due to inflow and outflow, but evaporation is one of the factors of water depletion. Suppose a reservoir contains an average of 1.4 billion gallons of water and loses water due to evaporation at a rate of 2% per month. Without considering any other changes to the reservoir's volume, how much water will have evaporated over a one-year period?
26. Zeno's Paradox is an observation which seems absurd, yet it starts sounding logically acceptable in relation to geometric sequences! Zeno's Paradox reads:

Suppose Atalanta wishes to walk to the end of a path. Before she can get there, she must get halfway there. Before she can get halfway there, she must get a quarter of the way there. Before traveling a quarter, she must travel one-eighth; before an eighth, one-sixteenth; and so on.

Zeno's paradox questions the conclusion of a geometric sequence, which paradoxically questions Atalanta's ability to complete her walk to the end of the path! Our brain battles the fact that the sequence is infinite against our observable experience – of course Atalanta can walk to the end of the path! A related paradox to ponder: when would you say that the perimeter of a nested triangle in Problem #24 is equal to zero? This question might seem absurd, just like Zeno's Paradox! Use your own thoughts to contemplate the question and debate your conclusion with a logical argument.

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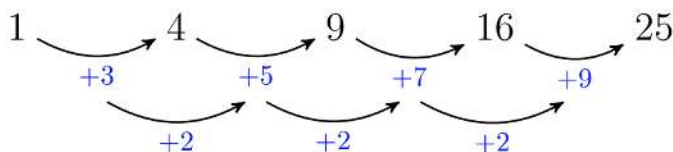
8.4: Quadratic Sequences

Quadratic functions are polynomial functions of degree two. For example, $f(x) = x^2$ is a quadratic function. This section will explore patterns in quadratic functions and sequences. Identifying patterns within a function table gives us valuable clues to build the right function to match the mathematical pattern.

How to Detect a Quadratic Sequence:

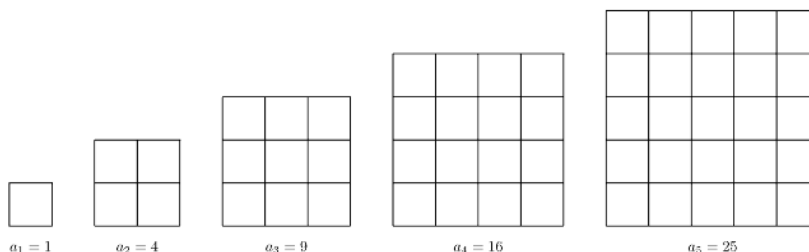
Unlike an arithmetic sequence which has a common difference $d = a_n - a_{n-1}$, the quadratic sequence will not have a common difference until the second difference is taken, or the difference of the difference!

Consider the sequence: 1, 4, 9, 16, 25, ... which has general term $a_n = n^2$.



The first difference was taken, but we did not find a common difference. The first difference gives the uncommon values: 3, 5, 7, 9. However, we persisted and took a difference of the differences: $5 - 3 = 2$, $7 - 5 = 2$, and $9 - 7 = 2$. The second layer of differences revealed a common value: 2. *The sequence is a quadratic sequence if the first difference does not have a common difference, but the second difference does!*

The sequence of squares is another beautiful visual. What's more, the visual truly justifies why we call these numbers "squares."

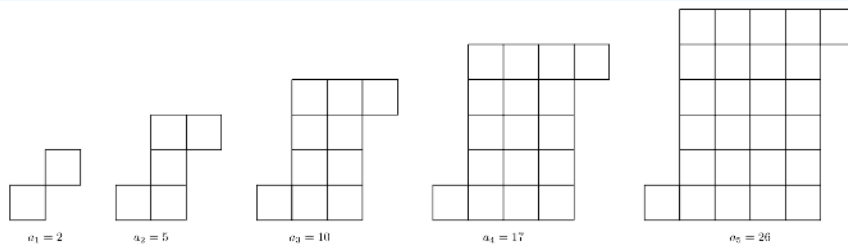


It is also remarkable that the sequence of squares can be created by the sequence of partial sums of odd numbers. The wonder of mathematics continues to astound us!

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 1 + 3 = 4 \\ a_3 &= 1 + 3 + 5 = 9 \\ a_4 &= 1 + 3 + 5 + 7 = 16 \\ a_5 &= 1 + 3 + 5 + 7 + 9 = 25 \\ &\text{etc.} \end{aligned}$$

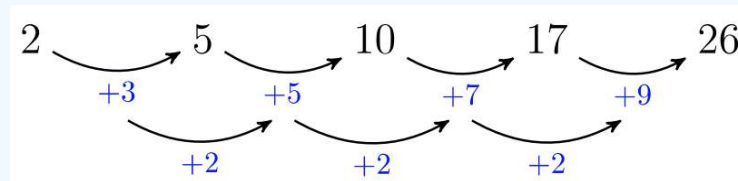
✓ Example 8.4.1

Consider the pattern below. How many small square tiles will be in the 10th stage (a_{10}) of this pattern?

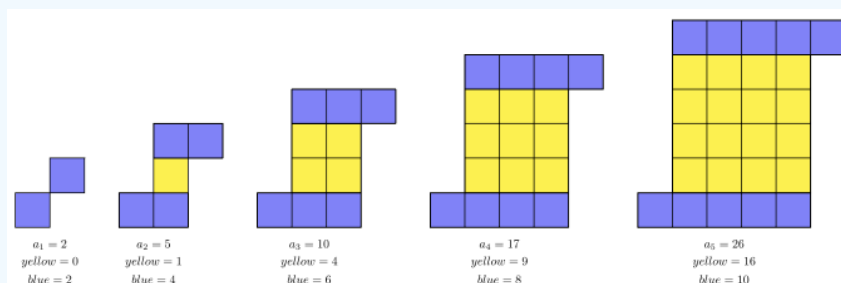


Solution

The pattern is a quadratic sequence:



We can view 2 separate patterns. The 2nd, 3rd, 4th, and 5th figures contain a pattern of squares: 1, 4, 9, 16, ... Below, the squares are shaded yellow that create the Pattern of squares. The top and bottom rows create a linear pattern (blue), which is an arithmetic sequence.



The blue sequence is 2, 4, 6, 8, 10, ... which has general term $b_n = 2n$

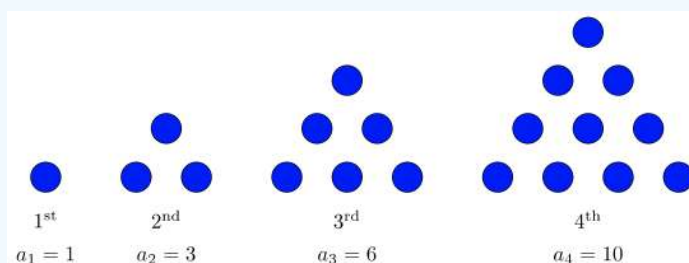
The yellow sequence is 0, 1, 4, 9, 16, ... which has general term $y_n = (n-1)^2$

The blue and the yellow sequence together make the overall figure's sequence, a_n .

$$\begin{aligned}
 a_n &= b_n + y_n && \text{blue sequence} + \text{yellow sequence} = a_n \\
 a_n &= 2n + (n-1)^2 && \text{Substitute the sequence general term.} \\
 a_n &= \underbrace{2n}_{\text{blue}} + \underbrace{n^2 - 2n + 1}_{\text{yellow}} && \text{Expand, using FOIL.} \\
 a_n &= n^2 + 1 && \text{Simplify.}
 \end{aligned}$$

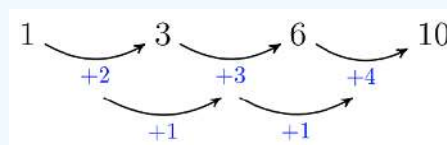
✓ Example 8.4.2

The first 4 stages of a pattern are shown below. How many circles will be required to construct the 25th stage (a_{25})?



Solution

The number of circles in each figure creates a quadratic sequence, which we can see by finding the common difference in the second difference.



The challenge here is to find the value (a_{25}). Although there are many approaches one could take, this is one method to find (a_{25}).

Notice that the sequence 1, 3, 6, 10, ... is the sequence of partial sums of consecutive numbers:

Therefore, $a_{25} = 1 + 2 + 3 + \dots + 25$. Rather than performing the addition one term at a time, let's write the sum twice: one increasing, and one decreasing, then add vertically. A cool thing happens:

$$\begin{array}{r}
 1 + 2 + 3 + \dots + 24 + 25 \\
 + 25 + 24 + 23 + \dots + 2 + 1 \\
 \hline
 = 26 + 26 + 26 + \dots + 26 + 26
 \end{array}$$

Therefore,

$$\begin{array}{rcl}
 a_{25} + a_{25} & = & 25 \cdot 26 \\
 2a_{25} & = & 650 \\
 \frac{2a_{25}}{2} & = & \frac{650}{2} \\
 a_{25} & = & 325
 \end{array}$$

Adding Consecutive Numbers: 1 to n:

The previous example (Example 8.4.2) demonstrated a method for finding the sum of the first 25 consecutive numbers.

$$1 + 2 + 3 + \dots + 25 = \frac{25 \cdot 26}{2}$$

Let's generalize finding s_n , the sum of the first n consecutive natural numbers. Again, using the method of Example 8.4.2:

$$\begin{array}{r}
 1 + 2 + 3 + \dots + n-2 + n-1 + n \\
 + n + n-1 + n-2 + \dots + 3 + 2 + 1 \\
 \hline
 = (n+1) + (n+1) + (n+1) + \dots + (n+1) + (n+1) + (n+1)
 \end{array}$$

Therefore,

$$\begin{array}{rcl}
 s_n + s_n & = & n(n+1) \\
 2s_n & = & n(n+1) \\
 \frac{2s_n}{2} & = & \frac{n(n+1)}{2} \\
 s_n & = & \frac{n(n+1)}{2}
 \end{array}$$

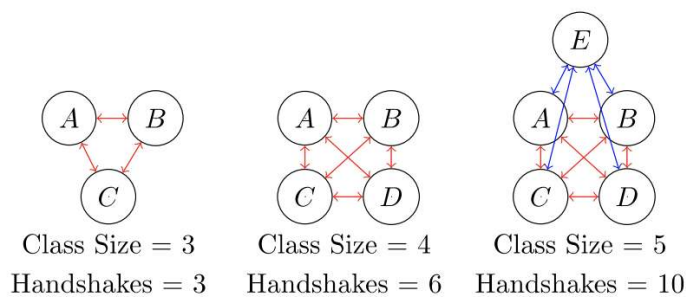
✓ Example 8.4.3

On his first day of class, Jackson's teacher asks that student shake hands and introduce themselves to each other. There are 25 students in the class. How many different handshakes can occur during this icebreaker if every student shakes hands with every other student?

Solution

An important step to problem solving is to understand the problem. Let's start small. Suppose there are only 2 people in a class. If Chloe shakes hands with Jackson, one handshake has occurred, and that's it! Now let's add another person to the room and consider a class of 3 students. Then add another for a class size of 4. Then another...and create a chart of results.

n = class size	b_n = handshakes
2	1
3	3
4	6
5	10



A pattern emerges much like the previous example (Example 8.4.2). However, the n -values (class size) are one more than the n -values (the figure number) in Example 8.4.2. We can easily adjust for the difference. Compare the tables between the two examples and adjust accordingly

From Example 8.4.2	
n = figure number	a_n = # of circles
1	1
2	3
3	6
4	10
n	$\frac{n(n+1)}{2}$

Use Example 8.4.2 as a model for finding the formula for Example 8.4.3. The n -values of Example 8.4.2 are one less than the n -values of Example 8.4.3.

From Example 8.4.3	
n = class size	b_n = handshakes
2	1
3	3
4	6
5	10
n	Formula?

Since we know the formula for Example 8.4.2, let's adjust the n -values of Example 8.4.3 to match Example 8.4.2. Since we are matching Example 8.4.3 to Example 8.4.2, we need to subtract one from the n -value. In essence, the value of b_n of Example 8.4.3 (handshakes) matches the value of a_{n-1} for Example 8.4.2 (number of circles).

b_{25} of Example 8.4.3 is the same as a_{24} of Example 8.4.2.

An important skill for problem solving is to use work that has already been achieved. Not every problem is a fresh slate.

$$\begin{aligned}
 a_{24} &= \frac{24(24+1)}{2} && \text{Substitute } n = 24 \text{ in the formula for Example 8.4.2.} \\
 a_{24} &= \frac{24(25)}{2} && \text{Simplify the parentheses first.} \\
 a_{24} &= \frac{600}{2} && \text{Simplify the numerator. Then divide.} \\
 a_{24} &= 300 && \text{But now apply this answer to Example 8.4.3: } b_{25} = 300.
 \end{aligned}$$

To find the formula, a_n , for Example 8.4.3, substitute $(n - 1)$ for n in Example 8.4.2.

Formula for Example 8.4.3:

$$\frac{(n-1)[(n-1)+1]}{2}$$

Replace each n with $(n - 1)$ in the formula of Example 8.4.2.

$$\frac{(n-1)n}{2}$$

Simplify within parentheses.

$$\frac{n(n-1)}{2}$$

This is the formula for Example 8.4.3.

$$b_{25} = \frac{25(24)}{2}$$

Verify that we get the same answer, 300, as above.

$$b_{25} = 300$$

Answer In Jerome's class of 25 students, there would be 300 different handshakes if each student shook hands with every other student.

Try It! (Exercises)

For #1-3, the general term of the sequence is given.

- State the first five terms, starting with $n = 1$.
- Find the value a_{12} .

- $a_n = n^2 + n - 1$
- $a_n = 2(n + 1)2$
- $a_n = 2n^2 + 2$

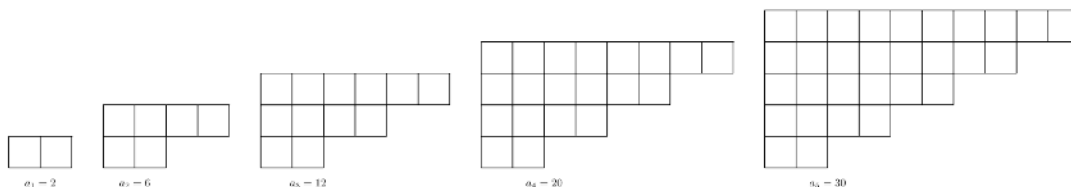
For #4-8, Let s_n = The sum of the first n consecutive natural numbers. Find the value indicated.

- s_{32}
- s_{80}
- s_{100}
- s_{201}
- s_{496}

For #9-13, A room has n people, all of whom will shake hands once with every other person in the room. Let b_n = number of different handshakes. Find the value indicated.

- b_{16}
- b_{36}
- b_{52}
- b_{100}
- b_{229}
- Find the sum of the first 30 odd numbers.
- Use the fact that $2 + 4 + 6 + 8 + \dots = 2(1 + 2 + 3 + 4 + \dots)$ to find the sum of the first 50 even numbers.
- Write a formula to describe the sum of the first n number of even numbers.

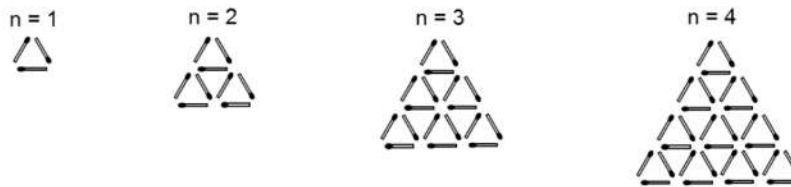
For #17-21, A pattern of figures is shown below. Let a_n = the number of small square tiles used to create figure n . Find the value indicated.



- a_6
- a_8

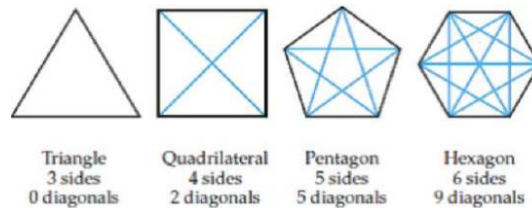
19. a_{22}
 20. a_{60}
 21. The formula a_n .

For #22-24, first determine the number of matchsticks in each Figure 1-4. Let a_n = the number of matchsticks in figure n . Find the value indicated.



22. a_5
 23. Find a_n given that $a_n = \frac{3n(n+k)}{2}$ for some k .
 24. a_{45}

For #25-27, the 4 polygons below demonstrate how to count diagonals within each polygon. A diagonal (shown in blue) is the connection of non-neighboring vertices. Let p_n = the number of diagonals in any n -sided figure. Find the value indicated.



25. p_7
 26. Given that $p_n = \frac{n(n-k)}{2}$ for some k , Find k and p_n .
 27. p_{40}

For #28-30, refer to the experiment with gravity described below:

You are curious if there is a relationship between the number of feet a ball drops from a balcony 144 feet above ground. You set up a camera to take a burst of photographs every second as a ball falls in front of a height chart. You create the following table based on the photographs:

Time (sec)	0	1	2	3
Distance (feet)	0	16	64	144

28. You reason that the ball's distance is a quadratic sequence. How did you know?
 29. After graphing the points, you reason that the sequence has general term $d_t = kt^2$ where t = time in seconds and k is a real number. Determine the value of k , then find d_t .
 30. Using your answers to #28 and #29, suppose you were standing at the top of a cliff for which you did not know the height. You dropped a rock from the top. 6 seconds later, you heard the rock hit the ground. How high above the ground are you?

For #31-33, use the table to describe the lift of airplanes:

For a given wing area, the lift of an airplane is proportional to the square of its speed.

Let s = speed of airplane. Then $l_s = ks^2$ for some constant, k .

The table below shows the lift for a Boeing 737 airplane at various speeds.

Speed (mph)	0	75	150	225	300	375

Lift (1000s of pound)	0	25	100	225	400	625
-----------------------	---	----	-----	-----	-----	-----

31. You reason that the lift of an airplane is a quadratic sequence. How did you know?
32. After graphing the points, you reason that the sequence has general term $l_s = ks^2$ where s = speed in mph and k is a constant. Determine the value of k , then find l_s .
33. Use your answers to #31 and #32 to find the lift for an airplane that is traveling 425 mph.

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